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Higher Castelnuovo Theory

David Eisenbud, Mark Green, Joe Harris[1]

0. Introduction.

In this paper and others to follow, we intend to set out a series of conjectures concerning the Hilbert functions of points (or more generally, zero-dimensional subschemes) in projective space; or, more generally still, the Hilbert functions of graded Artinian rings. We were first led to make some of these conjectures in Eisenbud-Harris [1982] in the course of our work on Castelnuovo theory. A special case of these was proved independently by us in that paper and by Miles Reid – though as Ciliberto later noted [1987] we were both anticipated by G. Fano [1894]. Recently, we saw how our conjectures might be generalized; and in this form they relate to a number of other areas: for example, another special case is equivalent to a conjectured generalization of the classical Cayley-Bacharach theorem (as we will also discuss here); another to the Kruskal-Katona and Clements-Lindström theorems of combinatorics (see, for example, Kleitman-Green [1978]); and still others, which we intend to describe in a later paper, to questions about the existence of exceptional linear series on complete intersection curves.

Good references for unexplained terminology are Arbarello-Cornalba-Griffiths-Harris [1985] or Eisenbud-Harris [1982].

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1. Castelnuovo theory.

Recall that a set of points in projective space is in *uniform position* if the Hilbert function (= postulation) of a subset depends only on the cardinality of the subset. Castelnuovo theory is concerned with the possible Hilbert functions of points in uniform position. Its origins are classical: Castelnuovo first used estimates on the Hilbert functions of points to derive his upper bound on the genus of an irreducible nondegenerate curve C in projective space $\mathbf{P}^r$ in terms of the degree d of C. Castelnuovo's argument has been reproduced too many times to repeat in detail here (see, for example, Eisenbud-Harris [1982] or Arbarello-Cornalba-Griffiths-Harris [1985]), but briefly what he shows first, by completely elementary means, is that if $\Gamma \subset \mathbf{P}^{n-1}$ is a general hyperplane section of C then

$$g(C) \leq \sum_{\ell=1}^{\infty} h^1(\mathbf{P}^{n-1}, \mathcal{I}_\Gamma(\ell))$$

or, in other words, the genus of C is bounded by the sum over all ℓ of the failure of Γ to impose independent conditons on hypersurfaces of degree ℓ. Curves of maximal genus for their degree therefore are likely to be those whose hyperplane sections Γ have the smallest possible Hilbert function h_Γ. Next, Castelnuovo shows that among all configurations Γ of $d \geq 2n+1$ points in uniform position in $\mathbf{P}^{n-1}$, the ones with minimal Hilbert function are exactly those lying on rational normal curves; he calculates his bound $\pi(d,n)$ on the genus of a curve accordingly. Finally, since if Γ is a subset of a rational normal curve any quadric containing Γ will contain the rational normal curve, he shows that if C is a curve achieving his bound the quadrics containing C must cut out in $\mathbf{P}^n$ a surface whose hyperplane section is a rational normal curve (in particular, a surface of degree $n-1$, the minimum possible degree for a nondegenerate surface in $\mathbf{P}^n$).

In Eisenbud-Harris [1982], we undertook to extend the results of Castelnuovo – in particular, his characterization of curves of maximal genus for their degree as lying on rational normal scrolls – to curves of high, but not maximal genus. This involved asking, for example, "What is the second smallest possible Hilbert function of a collection of points?" and in general, "What configurations of points have small Hilbert function?" What emerged was the following philosophy: The way to achieve a configuration $\Gamma \subset \mathbf{P}^r$ in uniform position having small Hilbert function is to put Γ on a positive-dimensional variety with small Hilbert function – in effect, on a curve of smallest possible degree, and of largest possible genus given that degree – which is the intersection of the hypersurfaces of low degree containing Γ.

To be specific, let $\Gamma \subset \mathbf{P}^r$ be a nondegenerate collection of d points in uniform position; let h_Γ be its Hilbert function, so that for example $h_\Gamma(2)$ is the number of conditions imposed by Γ on quadrics. Castelnuovo says that if $d \geq 2r+3$, then Γ must impose at least $2r+1$ conditions on quadrics; and if $h_\Gamma(2) = 2r+1$ exactly, then Γ must lie on a rational normal curve. Extending this, it turned out that if $d \geq 2r+5$ and if $h_\Gamma \geq 2r+2$ then necessarily Γ had to lie on an elliptic normal curve (Fano [1894],Eisenbud-Harris [1982], Reid [unpublished]). We deduced in particular that if a curve $C \subset \mathbf{P}^n$ had genus exceeding a bound $\pi_1(d,n)$ (substantially lower than $\pi(d,n)$), then the quadrics containing C have to cut out a surface of degree n in $\mathbf{P}^n$, which allowed us to classify such curves. Both we and Miles Reid went on to conjecture that this pattern would persist, at least for a while: for $\alpha < r$, we conjectured, under the hypothesis $d \geq 2r+2\alpha+1$ we could conclude that either $h_\Gamma \geq 2r+\alpha+1$ or Γ lay on a curve of degree $r+\alpha-1$ or less in $\mathbf{P}^r$.

In all of these cases, the latter conclusion – that Γ lay on a curve of small degree – would follow immediately if one knew that the intersection of the quadrics containing Γ was in fact positive dimensional. This observation last year suggested to us a seemingly trivial restatement. If we hypothesize that Γ is cut out by quadrics, we can ask: given $h_\Gamma(2)$, what is the largest possible d? In other words, *What is the largest number $d(h)$ of points of intersection of a linear system of quadrics of codimension h in the space of all quadrics in $\mathbf{P}^r$, given that the intersection of those quadrics is zero-dimensional?* In these terms, we may summarize the state of our knowledge as of 1981 (and its origins) as follows:

$$
\begin{array}{ll}
d(r+1) = r+1 & \text{(elementary)} \\
d(r+2) = r+2 & \text{(elementary)} \\
\vdots & \\
d(2r-1) = 2r-1 & \text{(elementary)} \\
d(2r) = 2r & \text{(elementary)} \\
d(2r+1) = 2r+2 & \text{(Castelnuovo)} \\
d(2r+2) = 2r+4 & \text{(Fano, Eisenbud-Harris, Reid)}
\end{array}
$$

The conjectures mentioned above extend this pattern to:

$$d(2r+3) = 2r+6$$

$$\vdots$$

$$d(3r-3) = 4r-6$$

$$d(3r-2) = 4r-4.$$

Note that this conjectured bound on the number of points is sharp, if it holds: for $h \leq 2r$, of course, any configuration of h points in linear general position will be cut out by quadrics and will impose independent conditions on quadrics; and for $2r+2 \leq h = 2r+\alpha \leq 3r-2$ we can take Γ the interseciton of a linearly normal curve of degree $r+\alpha$ – that is, a curve of degree $r+\alpha$ and (maximal) genus α – with another quadric. Note, moreover, that in the last case – $d(3r-2) = 4r-4$ – there is also another example we can use to show that the bounds is sharp: we can take Γ the intersection of a rational normal scroll $X \subset \mathbf{P}^r$ with two more quadrics.

This last example suggests that at this point the pattern of $d(h)$ increasing by 2 each time stops. Indeed, corresponding to the two examples above in case $h = 3r-2$ there are two examples to suggest that the next value of d should be

$$d(3r-1) = 4r.$$

On the one hand, the maximal genus of a curve of degree $r+\alpha$ in $\mathbf{P}^r$ increases by 2 from $\alpha = r-1$ to $\alpha = r$, with the result that a curve of degree $2r-1$ and genus $r-1$ in $\mathbf{P}^r$ will lie on the same number of quadrics as a curve of degree $2r$ and genus $r+1$ (that is, a canonical curve). Thus we can take Γ the intersection of a canonical curve in $\mathbf{P}^r$ with a quadric to arrive at a configuration of $4r$ points imposing only $3r-1$ conditions on quadrics. On the other hand, in the latter example, if we replace the rational normal surface scroll S, which has degree $r-1$, with a linearly normal surface of one larger degree r (for example, a del Pezzo surface or a cone over an elliptic normal curve), the intersection of our surface with two quadrics will again have degree $4r$ and impose $3r-1$ conditions on quadrics.

Similar examples indicate that for the next $r-3$ steps $d(h)$ will increase by 4 each time we increase h: by way of an example, we can take Γ the intersection of a surface of degree $r-1+\beta$ with two further quadrics. When we get to the case $h = 4r-5$, however, we get a new example: the intersection of a threefold rational normal scroll X with three additional quadrics, and thereafter we can increase the degree of Γ by 8 while increasing $h_\Gamma(2)$ by only 1 by increasing the degree of X by 1.

By now the pattern seems relatively clear, and we may state the

CONJECTURE. *Starting with the value $d(r+1) = r+1$, the successive differences of the function $d(h)$ are:*

$$\begin{array}{ll}
1, 1, \ldots\ldots\ldots\ldots, 1 & (r-1 \textit{ times}); \\
2, 2, \ldots\ldots\ldots, 2 & (r-2 \textit{ times}); \\
4, 4, \ldots\ldots, 4 & (r-3 \textit{ times}); \\
\vdots & \\
2^{k-1}, \ldots, 2^{k-1} & (r-k \textit{ times}); \\
\vdots & \\
2^{r-3}, 2^{r-3}, & \\
2^{r-2}. &
\end{array}$$

Where do we wind up at the end of this string? Here we have our first surprise: the last predicted value of d is

$$d\left(\frac{r^2+r+2}{2}\right) = 2^r,$$

that is to say, the largest possible number of isolated points of intersectin of r quadrics in $\mathbf{P}^r$ is 2^r. The fact that the terminal case of the conjecture is simply the Bézout theorem is striking. But more intriguing is the next case:

$$d\left(\frac{r^2+r}{2}\right) = 2^r - 2^{r-2},$$

or, in other words,

(*)
The largest number of points of a complete intersection of quadrics in $\mathbf{P}^r$ that another independent quadric can contain is $2^r - 2^{r-2}$.

Let us express this conjecture in closed form. It will also be useful to replace the variable h, corresponding to the number of conditions imposed by a set of points Γ on quadrics, with the absolute number m of independent quadrics containing Γ – that is, $\binom{r+2}{2} - h$.

First, some notation: given r, any number $m \geq r+1$ can be uniquely written in the form

$$m = (r+1) + \binom{b}{2} + c, \qquad b > c \geq 0.$$

With this notation, we make the

CONJECTURE ($I_{m,r}$). *If Γ is any nondegenerate collection of d points in uniform position in $\mathbf{P}^r$ lying on m independent quadrics whose intersection is zero-dimensional, then*

$$d \leq (2b - c + 1) \cdot 2^{r-b-1}.$$

In particular, the statement (*) above is simply the case ($I_{r+1,r}$) of this conjecture.

As suggested above, examples show that this bound, if indeed it holds, is sharp: for m quadrics, we can take Γ the intersection of $r-b-1$ quadrics with a linearly normal variety of degree $2b-c+1$ and dimension $r-b-1$ in $\mathbf{P}^r$ (for example, the divisor residual to $c+1$ planes in the intersection of a rational normal $(r-b)$-fold scroll in $\mathbf{P}^r$ with a quadric).

Conjecture (I) remains an open problem in general, though we have been able to verify it for all $r \geq 5$ (note that all cases with $r \leq 4$ are covered by existing theorems of Castelnuovo, Fano, Eisenbud-Harris and Reid). We have also been able to verify the special case ($I_{r+1,r}$) for all $r \leq 6$; we will give a proof in §3 below.

2. The algebraic case.

What happens if we omit the hypothesis of uniform position from our basic Conjecture (I), or for that matter if we allow arbitrary (nondegenerate) zero-dimensional subschemes of $\mathbf{P}^r$? This problem is one that has a purely algebraic formulation. Passing to the homogeneous coordinate ring of the configuration $\Gamma \subset \mathbf{P}^r$ modulo a general linear form, it becomes the question: what is the largest possible length $e(m)$ of an Artinian ring of the form

$$R = k[x_1, \ldots, x_r]/I$$

where I is the ideal generated by an m-dimensional vector space of homogeneous quadric polynomials in $x_1, \ldots, x_r$?

The extreme cases $m = \binom{r+1}{2}$ and $m = r$ are exactly the same as before: the corresponding values of e are $r+1$ and 2^r. In between, though, the successive differences are quite different: they are expressed in the

CONJECTURE. *Starting with the value $e(r) = 2^r$, the (negative) successive differences of the function $e(m)$ are:*

$$\begin{aligned}
&2^{r-2}, 2^{r-3}, 2^{r-4}, \ldots\ldots, 4, 2, 1,\\
&2^{r-3}, 2^{r-4}, \ldots\ldots\ldots, 4, 2, 1,\\
&\quad\vdots\\
&4, 2, 1,\\
&2, 1,\\
&1.
\end{aligned}$$

In other words, they are the same successive differences as the function d, in a different order.

As strange as the conjecture may sound, it also has been completely verified for $r \leq 5$ (including cases where the value of e differs from that of d). It should also be noted that the conjectured last two values of the function e before the Bézout case ($e(r+1) = 2^r - 2^{r-2}$, $e(r+2) = 2^r - 2^{r-2} - 2^{r-3}$) are the same as for the function d; and these two values have also been verified for $r \leq 6$.

As in the geometric case, it will be useful to have a form of the conjecture that applies to individual values of the function e. To do this, we write an arbitrary $m \leq \binom{r+1}{2}$ in the form

$$m = \binom{r+1}{2} - \binom{u}{2} - v, \qquad u > v \geq 0.$$

With this notation, we make the

CONJECTURE ($II_{m,r}$). *Let $\Gamma \subset \mathbf{P}^r$ be any nondegenerate, zero-dimensional subscheme of degree d. If Γ lies on m quadrics whose intersection is zero-dimensional, then*

$$d \leq 2^u + 2^v + r - u - 1.$$

Equivalently, if R is an Artinian ring of the form

$$R = k[x_1, \dots, x_r]/I,$$

where I is the ideal generated by an m-dimensional vector space of homogeneous quadric polynomials in $x_1, \dots, x_r$, then

$$dim_k(R) \leq 2^u + 2^v + r - u - 1.$$

As in the case of Conjecture (I), this bound is sharp, if indeed it holds. To construct examples, let $\Lambda \subset \Omega \subset \mathbf{P}^r$ be subspaces of dimension v and $u+1$, respectively. Let Γ_1 be a 0-dimensional complete intersection of quadrics in Λ, consisting of 2^v points. Let C be a curve in Ω given as a complete intersection of quadrics and containing Γ_1 (that is, choose a regular sequence of quadrics in Ω restricting to the quadrics in Λ cutting out Γ_1 and add $u - v$ more quadrics in Ω containing Λ to form a regular sequence of length u. Let H be a hyperplane section of C. Let $p_1, \dots, p_{r-u-1}$ be $r - u - 1$ additional points in $\mathbf{P}^r$ that, together with Ω, span $\mathbf{P}^r$; and set

$$\Gamma = H \cup \Gamma_1 \cup \{p_1, \dots, p_{r-u-1}\}.$$

3. Cayley-Bacharach theory.

There is another way to interpret the statement (*) (equivalently, $(II_{r+1,r})$) above, which is as an extension of the classical Cayley-Bacharach theorem.

We start by reviewing the statement of the modern Cayley-Bacharach theorem. If Γ is a zero-dimensional scheme and $\Gamma' \subset \Gamma$ a closed subscheme, we define the ***residual subscheme*** of Γ' in Γ to be the subscheme Γ'' of Γ defined by the sheaf of ideals

$$\mathcal{I}_{\Gamma''} = Ann(\mathcal{I}_{\Gamma'}/\mathcal{I}_{\Gamma}).$$

In English: Γ'' is the smallest subscheme of Γ such that any product of functions vanishing on Γ' and Γ'' vanishes on Γ. For example, if Γ is reduced then Γ'' is the complement of Γ' in Γ.

In general, however, it is not true that the degree of Γ'' is the difference $\deg(\Gamma)-\deg(\Gamma')$ (nor is either inequality valid); and the residual of the residual of a subscheme $\Gamma' \subset \Gamma$ will not in general equal Γ'. One circumstance, however, in which residuation does behave well is if Γ is locally a complete intersection (or even locally Gorenstein): in this case, by liaison (Peskine-Szpiro [1974]) we do have

$$\deg(\Gamma') + \deg(\Gamma'') = \deg(\Gamma),$$

and the residual of the residual of Γ' is again Γ'. In this case, we say that Γ' and Γ'' are residual to each other in Γ.

Note also that if the ideal of Γ' in Γ is locally principal, then the equality on degrees holds (though it is not in general true in this case that the residual of the residual of Γ' is Γ').

With this, we may state a

MODERN CAYLEY-BACHARACH THEOREM. *(Davis-Geramita-Orecchia [1985]): Let $\Gamma \subset \mathbf{P}^r$ be a complete intersection of hypersurfaces $X_1, \ldots, X_r$ of degrees $d_1, \ldots, d_r$, and let $\Gamma', \Gamma'' \subset \Gamma$ be closed subschemes residual to one another. Set*

$$m = -r - 1 + \sum d_i.$$

Then, for any $\ell \geq 0$, we have

$$h^0(\mathbf{P}^r, \mathcal{I}_{\Gamma'}(\ell)) - h^0(\mathbf{P}^r, \mathcal{I}_{\Gamma}(\ell)) = h^1(\mathbf{P}^r, \mathcal{I}_{\Gamma''}(m-\ell)).$$

In English: the number of hypersurfaces of degree ℓ containing Γ' (modulo the ideal of Γ) is exactly the failure of Γ'' to impose independent conditions on hypersurfaces of degree $m - \ell$.

According to Semple and Roth [1949], p. 98, the classical Cayley-Bacharach Theorem concerns the special case of where Γ is a reduced complete intersection of points in the plane. It asserts that if

$$\text{degree } \Gamma'' = \binom{m - \ell + 2}{2}$$

and the right hand side of the above equality is 0, then the left hand side is as well. This was asserted by Cayley without the hypothesis that the right hand side is 0 (which is automatic if $m - \ell = 0$ and degree $\Gamma'' = 1$), and corrected by Bacharach (Math. Annalen 26, p. 275). The most commonly stated form of the Theorem is this:

CLASSICAL CAYLEY-BACHARACH THEOREM. *Let $\Gamma \subset \mathbf{P}^r$ be a reduced complete intersection of hypersurfaces $X_1, \ldots, X_r$ of degrees $d_1, \ldots, d_r$. Then any hypersurface of degree $m = \sum d_i - r - 1$ containing a closed subscheme of Γ of degree $\prod d_i - 1$ contains Γ.*

This Cayley-Bacharach theorem says in particular that if $\Omega \subset \mathbf{P}^r$ is a complete intersection of quadrics, then any hypersurface $X \subset \mathbf{P}^r$ of degree $r-1$ containing all but one point of Γ contains Γ. We could ask more generally: Suppose Ω is the complete intersection of r quadrics in $\mathbf{P}^r$. What is the largest degree $g(k)$ of a subscheme of Ω that a hypersurface of degree k, not containing Ω, can contain? By Bézout in $\mathbf{P}^{r-1}$, a hyperplane can contain at most 2^{r-1}, so that $g(1) = 2^{r-1}$, while Bézout in $\mathbf{P}^r$ says that $g(r-1) = 2^r - 2$ and $g(r) = 2^r - 1$. These two remarks are the cases $k = 1$ and $k = r - 1$ of

CONJECTURE $(III_{k,r})$. *(Generalized Cayley-Bacharach). Let $\Omega \subset \mathbf{P}^r$ be a complete intersection of quadrics. Any hypersurface of degree k that contains a subscheme $\Gamma \subset \Omega$ of degree strictly greater than $2^r - 2^{r-k}$ must contain Ω.*

There is an appealing boundary case:

CONJECTURE $(III_{k,r}$, BOUNDARY CASE$)$. *Moreover, if X is a hypersurface of degree k with $\deg(X \cap \Omega) = 2^r - 2^{r-k}$, the scheme residual to $X \cap \Omega$ in Ω is a complete intersection of quadrics in a subspace $\mathbf{P}^{r-k}$.*

Note that the inequality in case $(III_{2,r})$ of this conjecture is exactly the conjecture $(II_{r+1,r})$ above.

We will prove below the conjecture $(III_{k,r})$ for all k and $r \leq 6$. To do this, it will be useful to introduce yet another conjecture:

CONJECTURE (IV_m). *Let $\Gamma \subset \mathbf{P}^r$ be any subscheme of a zero-dimensional complete intersection of quadrics, let $d = \deg(\Gamma)$, and suppose that Γ fails to impose independent conditions on hypersurfaces of degree m – that is, $h^1(\mathbf{P}^r, \mathcal{I}_\Gamma(m)) \neq 0$. Then*

$$d \geq 2^{m+1}.$$

Note that this conjecture is independent of the dimension r of the ambient projective space (in particular, we do not assume that Γ spans $\mathbf{P}^r$).

CONJECTURE $(IV_m$, BOUNDARY CASE$)$. *Equality holds in Conjecture (IV_m) if and only if Γ is itself a complete intersection of quadrics in $\mathbf{P}^{m+1}$.*

THEOREM 1. *The following are equivalent:*

a. *$(III_{k,r})$ for all k and r;*

b. *(IV_m) for all m.*

In particular, either one implies $(III_{2,r})$, and hence $(II_{r+r,r})$, for all r. Moreover, for any value of m, (IV_m) implies $(III_{k,r})$ for all k and r with $r-k-1=m$ and in particular $(II_{r+1,r})$ for $r=m+3$. (The same is true for the boundary cases of (IV_m) and $(III_{k,r})$).

PROOF: We first prove that (IV_m) implies $(III_{r-m-1,r})$. We apply the modern Cayley-Bacharach Theorem. To begin with, assume (IV_m), and let $\Omega \subset \mathbf{P}^r$ be a complete intersection of quadrics. Let X be any hypersurface of degree $k=r-m-1$ not containing Ω, and let Γ be the subscheme of Ω residual to the intersection $\Omega \cap X$. By Cayley-Bacharach Γ must fail to impose independent conditions on hypersurfaces of degree $m=r-1-k$. By assumption, $\deg(\Gamma) \geq 2^{r-k}$ and correspondingly $\deg(X \cap \Omega) \leq 2^r - 2^{r-k}$. (The boundary case of (IV_m) easily implies the boundary case of $(III_{r-m-1,r})$ as well.)

Now assume $(III_{k,r})$ for all r. Let Γ be any subscheme of a complete intersection of quadrics and suppose that Γ fails to impose independent conditions on hypersurfaces of degree k. Assuming that Γ spans a projective space $\mathbf{P}^n$, take Ω a complete intersection of quadrics in $\mathbf{P}^n$ containing Γ, and let $\Gamma' \subset \Omega$ be the subscheme of Ω residual to Γ. By Cayley-Bacharach, Γ' lies on a hypersurface of degree $n-1-k$ not containing Ω; it follows that $\deg(\Gamma') \leq 2^n - 2^{n-1-k}$ and hence that $\deg(\Gamma) \geq 2^{k+1}$. Moreover, if we have equality in the last inequality, then Γ is itself a complete intersection of quadrics. ∎

As promised, we will prove $(III_{k,r})$ for all k and $r \leq 6$ by establishing:

THEOREM 2. *Conjecture (IV_m) holds for $m \leq 3$.*

We will make use of the following simple result several times.

LEMMA. *Let $\Omega \subset \mathbf{P}^r$ be a finite subscheme, and let m be a nonnegative integer. Suppose that every form of degree m vanishing on a codegree 1 subscheme of Ω (that is, on a subscheme of degree one less than Ω) vanishes on all of Ω. If $H \subset \mathbf{P}^r$ is any hypersurface of degree k, and Θ is the subscheme residual to $H \cap \Omega$, then any form of degree $m-k$ vanishing on a codegree 1 subscheme of Θ vanishes on all of Θ.*

PROOF OF THE LEMMA: To say that a form G vanishes on a codegree 1 subscheme of Θ is to say that $((G)+\mathcal{I}_\Theta)/\mathcal{I}_\Theta$ is a 1-dimensional vector space, or equivalently $G \in (\mathcal{I}_\Theta : \mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of $\mathcal{O}_\Theta$.

Now let F be the form of degree k defining H, let $\Gamma = H \cap \Omega$, and let Θ be the subscheme residual in Ω to Γ. If G is a form of degree $m-k$ vanishing on a codegree 1 subscheme of Θ, then $G \in (\mathcal{I}_\Theta : \mathfrak{m})$, so $FG \in (\mathcal{I}_\Theta : \mathfrak{m})$ and FG vanishes on a codegree 1 subscheme of Ω. Since FG has degree m it follows from our hypothesis on Ω that FG vanishes on Ω, and thus $G \in (\mathcal{I}_\Theta : F) = \mathcal{I}_\Theta$ – that is, G vanishes on Θ. ∎

PROOF OF THEOREM 2: First we show that (IV_m) holds for any m in case the linear span of the scheme Γ is a projective space of dimension $n \leq m+2$. The modern Cayley-Bacharach Theorem implies that a complete intersection of quadrics in $\mathbf{P}^n$ imposes independent conditions of hypersurfaces of degree n, and any proper subscheme of it imposes independent conditions on hypersurfaces of degree $n-1$, from which we get the case $n \leq m+1$. If, on the other hand, $n = m+2$, let Ω be a complete intersection of quadrics in $\mathbf{P}^{m+2}$ containing Γ, and let $\Gamma' \subset \Omega$ be the subscheme residual to Γ in Ω. By Cayley-Bacharach the subscheme Γ' lies in a hyperplane $\mathbf{P}^{m+1} \subset \mathbf{P}^{m+2}$. We thus have

$$\deg(\Gamma') \leq 2^{m+1}$$

and hence

$$\deg(\Gamma) \geq 2^{m+1}.$$

Note, moreover, that if equality holds in the last inequality, then Γ' must be a complete intersection of $m+1$ quadrics in $\mathbf{P}^{m+1}$. It follows that the restriction map

$$H^0(\mathbf{P}^{m+2}, \mathcal{I}_\Omega(2)) \longrightarrow H^0(\mathbf{P}^{m+1}, \mathcal{I}_{\Gamma'}(2))$$

must have a kernel. In other words, the linear system of quadrics cutting out Ω contains a reducible element $Q_0 = H_0 \cup L_0$, with $L_0 = \mathbf{P}^{m+1} \supset \Gamma'$. Since Ω is a complete intersection, Γ is residual to Γ', and thus L_1 vanishes on Γ, contradicting the hypothesis that Γ spanned $\mathbf{P}^{m+2}$.

Conjecture (IV_m) is immediate for $m = 0$ or 1; we will deal with the remaining two cases in turn. By what we have just done we may assume that Γ spans a space of dimension $n > m+2$, and we wish to show that $\deg \Gamma > 2^{m+1}$. We may as well assume that Γ is minimal among schemes failing to impose independent conditions on hypersurfaces of degree m and thus that any hypersurface of degree m containing a subscheme of Γ of codegree 1 in Γ must in fact contain Γ.

Case i. $m = 2$. Suppose that Γ spans a space of dimension $r \geq 5$. Then we can find a proper subscheme of Γ of degree at least r contained a hyperplane H in $\overline{\Gamma} = \mathbf{P}^r$; let $\Gamma' = H \cap \Gamma$ be the degree of Γ'. Let Θ be the residual scheme to Γ' in Γ.

By the Lemma, Θ fails to impose independent conditions on hyperplanes. By the case $m = 1$ of our conjecture we have

$$\deg(\Theta) \geq 4,$$

so

$$\begin{aligned} d = \deg(\Gamma) &\geq \deg(\Gamma') + 4 \\ &\geq r + 4 \\ &\geq 9, \end{aligned}$$

as desired.

Case ii. $m = 3$. Say Γ spans a linear space $\mathbf{P}^r$ of dimension $r \geq 6$ and fails to impose independent conditions on cubics. We must show that $\deg \Gamma \geq 17$. By Castelnuovo theory for schemes (see Eisenbud-Harris [1992]) any subscheme of $\mathbf{P}^r$ in linearly general position imposes independent conditions on m-ics if $d \leq mr + 1$. If Γ were in linearly general position, then taking $m = 3$ and $r = 6$ we find $\deg \Gamma > 3r + 1 = 19$, and we would be done. Thus we may assume that there is a hyperplane $H \subset \mathbf{P}^r$ intersecting Γ in a subscheme $\Gamma' = H \cap \Gamma$ of degree $s \geq r + 1 \geq 7$; we suppose that s is the maximal degree of such a subscheme. Let $\Theta \subset \Gamma$ be the subscheme residual to Γ' in Γ. Since the ideal of Γ' in Γ is principal, we have $\deg \Theta + \deg \Gamma' = \deg \Gamma$, so we must show that $s +$ degree $\Theta \geq 17$. Thus we may assume that degree $\Theta \leq 9$.

By the Lemma, Θ fails to impose independent conditions on quadrics. By the case $m = 2$ we must have $\deg(\Theta) \geq 8$. If degree $\Theta = 8$, then by case $m = 2$, Θ must be contained in a $\mathbf{P}^3$. It follows that some subscheme of length > 8 containing Θ is contained in a hyperplane in $\mathbf{P}^r$. Thus $s \geq 9$, and we are done.

It remains to treat the case where degree $\Theta = 9$. If Θ lies in a hyperplane, then $s \geq$ degree $\Theta = 9$, so we are done. If Θ were in linearly general position in $\mathbf{P}^r$ then since Θ imposes dependent conditions on quadrics it follows as above that $\deg \Theta > 2r + 1$. Since $r \geq 6$, this contradicts the assumption $\deg \Theta = 9$. Thus we may find a hyperplane section $\Theta' = H' \cap \Theta \subset \Theta$ of degree $t \geq r + 1 \geq 7$.

Let Ξ be the subscheme of Θ residual to Θ'. By the Lemma, Ξ fails to impose independent conditions on hyperplanes, so degree $\Xi \geq 3$. Since Θ' is cut out in Θ by just one equation, deg $\Theta = \deg \Theta' + \deg \Xi \geq 7 + 3 = 10$. ∎

4. A stepwise formulation.

Another way of approaching Hilbert functions is to ask, simply: suppose we know the value $h(m)$ of the Hilbert function of a graded ring in degree m. What can we say about the value in degree $m+1$? In this generality, the answer was supplied by Macaulay, who proved that if we wrote

$$h(m) = \binom{a_m}{m} + \binom{a_{m-1}}{m-1} + \ldots + \binom{a_1}{1}$$

with $a_m > a_{m-1} > \ldots > a_1 \geq 0$, then $h(m+1)$ satisfied the inequality

$$h(m+1) = \binom{a_m + 1}{m+1} + \binom{a_{m-1} + 1}{m} + \ldots + \binom{a_1 + 1}{2}.$$

Moreover, this bound is sharp. In line with what we have suggested above, however, we now ask what the estimate should be if we assume in addition that the ring is of the form

$$R = k[x_1, \ldots, x_r]/I$$

where I contains a regular sequence of length r in degree 2. Based on examples and some partial proofs, we make the

Conjecture (V_m). *Under this hypothesis, if $h(m)$ is as above, the value $h(m+1)$ of the Hilbert function of R satisfies the inequality*

$$h(m+1) = \binom{a_m}{m+1} + \binom{a_{m-1}}{m} + \ldots + \binom{a_1}{2}.$$

This is sharp, if true; an example would be the ideal generated by the squares of the variables together with the lexicographical ideal of appropriate size in degree m. Moreover, if we sum up the estimates for $h(m)$ over all m, we arrive at the same estimate for the length of R in terms of $h(2)$ given in Conjecture III; thus Conjecture (V) in general implies Conjecture (III).

Moreover, Conjecture (V) is true if the ideal of I contains the squares of the variables. This follows from the Kruskal-Katona Theorem (see Kleitman-Green [1978]), which is equivalent to the monomial case, and a deformation argument. Of course it follows in turn from this that the Theorem is true if I contains a "sufficiently general" regular sequence of quadrics.

In this setting, the hypothesis that the ideal $I \subset k[x_1, \ldots, x_r]$ defining R contains a regular sequence specifically of quadrics is artificial. Conjecture (V) generalizes directly to the case where we assume just that I contains a regular sequence $(f_1, \ldots, f_2)$ of homogeneous polynomials of arbitrary degrees. The case where the f_i are powers of the variables then follows from the Clements-Lindström Theorem, also treated in Kleitman-Green [1978]; we intend to devote a future paper to this and the cases of it that we can prove.

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