Astérisque

DAVID R. MORRISON

Compactifications of moduli spaces inspired by mirror symmetry

Compactifications of moduli spaces
inspired by mirror symmetry

David R. Morrison

The study of moduli spaces by means of the period mapping has found its greatest success for moduli spaces of varieties with trivial canonical bundle, or more generally, varieties with Kodaira dimension zero. Now these moduli spaces play a pivotal rôle in the classification theory of algebraic varieties, since varieties with nonnegative Kodaira dimension which are not of general type admit birational fibrations by varieties of Kodaira dimension zero. Since such fibrations typically include singular fibers as well as smooth ones, it is important to understand how to compactify the corresponding moduli spaces (and if possible, to give geometric interpretations to the boundary of the compactification). Note that because of the possibility of blowing up along the boundary, abstract compactifications of moduli spaces are far from unique.

The hope that the period mapping could be used to construct compactifications of moduli spaces was given concrete expression in some conjectures of Griffiths [25, §9] and others in the late 1960's. In particular, Griffiths conjectured that there would be an analogue of the Satake-Baily-Borel compactifications of arithmetic quotients of bounded symmetric domains, with some kind of "minimality" property among compactifications. Although there has been much progress since [25] in understanding the behavior of period mappings near the boundary of moduli, compactifications of this type have not been constructed, other than in special cases.

In the case of algebraic K3 surfaces, the moduli spaces themselves are arithmetic quotients of bounded symmetric domains, so each has a minimal (Satake-Baily-Borel) compactification. In studying the moduli spaces for K3 surfaces of low degree in the early 1980's, Looijenga [35] found that the Satake-Baily-Borel compactification needed to be blown up slightly in order to give a good geometric interpretation to the boundary. He introduced a class of compactifications, the *semi-toric compactifications*, which includes the ones with a good geometric interpretation.

In higher dimension, the moduli spaces are not expected to be arithmetic quotients of symmetric domains, so different techniques are needed. The study of these moduli spaces has received renewed attention recently, due to the discovery by theoretical physicists of a phenomenon called "mirror symmetry". One of the predictions of mirror symmetry is that the moduli space for a variety with trivial canonical bundle, which parameterizes the possible complex structures on the underlying differentiable manifold, should *also* serve as the parameter space for a very different kind of structure on a "mirror partner"—another variety with trivial canonical bundle. This alternate description of the moduli space turns out to be well-adapted to analysis by Looijenga's techniques; we carry out that analysis here.

In the physicists' formulation, one fixes a differentiable manifold X which admits complex structures with trivial canonical bundle (a "Calabi-Yau manifold"), and studies something called nonlinear sigma-models on X. Such an object can be determined by specifying both a complex structure on X, and some "extra structure" (cf. [40]); the moduli space of interest to the physicists parameterizes the choice of both. The rôles of the "complex structure" and "extra structure" subspaces of this parameter space are reversed when X is replaced by a mirror partner.

Most aspects of mirror symmetry must be regarded as conjectural by mathematicians at the moment, and in this paper we conjecture much more than we prove. In a companion paper [41], we consider formally degenerating variations of Hodge structure near normal crossing boundary points of the moduli space, and describe a conjectural link to the numbers of rational curves of various degrees on a mirror partner. In the present paper, we extend these considerations to boundary points which are *not* of normal crossing type, and formulate a mathematical mirror symmetry conjecture in greater generality. In addition, we find that when studied from the mirror perspective, a "minimal" partial compactification of the moduli space—analogous to the Satake-Baily-Borel compactification—appears very natural, provided that several conjectures about the mirror partner hold.

One of our conjectures is a simple and compelling statement about the Kähler cone of Calabi-Yau varieties. If true, it clarifies the rôle of some of the "infinite discrete" structures on such a variety, which nevertheless seem to be finite modulo automorphisms. We have verified this conjecture in a nontrivial case in joint work with A. Grassi [21].

The plan of the paper is as follows. In the first several sections, we review Looijenga's compactifications, describe a concrete example, and add a refinement to the theory in the form of a flat connection on the holomorphic cotangent bundle of the moduli space. We then turn to the description of the larger moduli spaces of interest to physicists, and analyze certain boundary

points of those spaces. Towards the end of the paper, we explore the mathematical implications of mirror symmetry in constructing compactifications of moduli spaces. We close by discussing some evidence for mirror symmetry which (in hindsight) was available in 1979.

1 Semi-toric compactifications

The first methods for compactifying arithmetic quotients of bounded symmetric domains were found by Satake [46] and Baily-Borel [5]. The compactification produced by their methods, often called the *Satake-Baily-Borel compactification*, adds a "minimal" amount to the quotient space in completing it to a compact complex analytic space. This minimality can be made quite precise, thanks to the Borel extension theorem [10] which guarantees that for a given quotient of a bounded symmetric domain by an arithmetic group, any compactification whose boundary is a divisor with normal crossings will map to the Satake-Baily-Borel compactification (provided that the arithmetic group is torsion-free).

Satake-Baily-Borel compactifications have rather bad singularities on their boundaries, so they are difficult to study in detail. Explicit resolutions of singularities for these compactifications were constructed in special cases by Igusa [30], Hemperly [27], and Hirzebruch [28]; the general case was subsequently treated by Satake [47] and Ash et al. [1]. The methods of [1] produce what are usually called *Mumford compactifications*—these are smooth, and have a divisor with normal crossings on the boundary, but unfortunately many choices must be made in their construction. The Satake-Baily-Borel compactification, on the other hand, is canonical.

Some years later, Looijenga [35] generalized both the Satake-Baily-Borel and the Mumford compactifications by means of a construction which can be applied widely, not just in the case of arithmetic quotients of bounded symmetric domains. Looijenga's construction gives partial compactifications of certain quotients of tube domains by discrete group actions. A *tube domain* is the set of points in a complex vector space whose imaginary parts are constrained to lie in a specified cone. Whereas Ash et al. [1] had only considered homogeneous self-adjoint cones, Looijenga showed that analogous constructions could be made in a more general context.

The starting point is a free $\mathbb{Z}$-module L of finite rank, and the real vector space $L_\mathbb{R} := L \otimes \mathbb{R}$ which it spans. A convex cone σ in $L_\mathbb{R}$ is *strongly convex* if $\sigma \cap (-\sigma) \subset \{0\}$. A convex cone is *generated* by the set S if every element in the cone can be written as a linear combination of the elements of S with nonnegative coefficients. And a convex cone is *rational polyhedral* if it is generated by a finite subset of the rational vector space $L_\mathbb{Q} := L \otimes \mathbb{Q}$.

Let $\mathcal{C} \subset L_\mathbb{R}$ be an open strongly convex cone, and let $\Gamma \subset \mathrm{Aff}(L)$ be a group of affine-linear transformations of L which contains the translation subgroup L of $\mathrm{Aff}(L)$. If the linear part $\Gamma_0 := \Gamma/L \subset \mathrm{GL}(L)$ of Γ preserves the cone $\mathcal{C}$, then the group Γ acts on the tube domain $\mathcal{D} := L_\mathbb{R} + i\,\mathcal{C}$. We wish to partially compactify the quotient space $\mathcal{D}/\Gamma$, including limit points for all paths moving out towards infinity in the tube domain.

Looijenga formulated a condition which guarantees the existence of partial compactifications of this kind. Let $\mathcal{C}_+$ be the convex hull of $\overline{\mathcal{C}} \cap L_\mathbb{Q}$. Following [35], we say that $(L_\mathbb{Q}, \mathcal{C}, \Gamma_0)$ is *admissible* if there exists a rational polyhedral cone $\Pi \subset \mathcal{C}_+$ such that $\Gamma_0.\Pi = \mathcal{C}_+$. Given an admissible triple $(L_\mathbb{Q}, \mathcal{C}, \Gamma_0)$, the (somewhat cumbersome) data needed to specify one of Looijenga's partial compactifications is as follows.[1]

DEFINITION 1 [35] *A locally rational polyhedral decomposition of $\mathcal{C}_+$ is a collection $\mathcal{P}$ of strongly convex cones such that*

(i) *$\mathcal{C}_+$ is the disjoint union of the cones belonging to $\mathcal{P}$,*

(ii) *for every $\sigma \in \mathcal{P}$, the $\mathbb{R}$-span of σ is defined over $\mathbb{Q}$,*

(iii) *if $\sigma \in \mathcal{P}$, if τ is the relative interior of a nonempty face of the closure of σ, and if $\tau \subset \mathcal{C}_+$, then $\tau \in \mathcal{P}$, and*

(iv) *if Π is a rational polyhedral cone in $\mathcal{C}_+$, then Π meets only finitely many members of $\mathcal{P}$.*

(The decomposition $\mathcal{P}$ is called *rational polyhedral* if all the cones in $\mathcal{P}$ are relative interiors of rational polyhedral cones. This is the same notion which appears in toric geometry [19, 43], except that the cones appearing in $\mathcal{P}$ as formulated here are the relative interiors of the cones appearing in that theory.)

For each Γ_0-invariant locally rational polyhedral decomposition $\mathcal{P}$ of $\mathcal{C}_+$, there is a partial compactification of $\mathcal{D}/\Gamma$ called the *semi-toric (partial) compactification associated to $\mathcal{P}$*. This partial compactification has the form $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$, where $\widehat{\mathcal{D}}(\mathcal{P})$ is the disjoint union of certain strata $\mathcal{D}(\sigma)$ associated to the cones σ in the decomposition. The complex dimension of the stratum $\mathcal{D}(\sigma)$ coincides with the real codimension of the cone σ in $L_\mathbb{R}$; in particular, the open cones in $\mathcal{P}$ correspond to the 0-dimensional strata in $\widehat{\mathcal{D}}(\mathcal{P})$. The delicate points in the construction are the specification of a topology on $\widehat{\mathcal{D}}(\mathcal{P})$,

---

[1]We have modified Looijenga's definition slightly, so that the use of the term "face" is the standard one (cf. [45]): a subset $\mathcal{F}$ of a convex set $\mathcal{S}$ is a *face* of $\mathcal{S}$ if every closed line segment in $\mathcal{S}$ which has one of its relative interior points lying in $\mathcal{F}$ also has both endpoints lying in $\mathcal{F}$.

and the proof that the quotient space $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$ has a natural structure of a normal complex analytic space. For more details, we refer the reader to [35] or [50].

The construction has the property that if $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then there is a dominant morphism $\widehat{\mathcal{D}}(\mathcal{P}')/\Gamma \to \widehat{\mathcal{D}}(\mathcal{P})/\Gamma$. Blowups of the boundary can be realized in this way.

A bit more generally, we can partially compactify finite covers $\mathcal{D}/\Gamma'$ of $\mathcal{D}/\Gamma$, built from $L' \subset L$ of finite index, $\Gamma'_0 \subset \mathrm{GL}(L') \cap \Gamma_0$ of finite index in Γ_0, and $\Gamma' := L' \rtimes \Gamma'_0$, by specifying a Γ'_0-invariant locally rational polyhedral decomposition $\mathcal{P}'$ of $\mathcal{C}_+$.

There are two extreme cases of a semi-toric compactification. The *Satake-Baily-Borel decomposition* $\mathcal{P}_{\mathrm{SBB}}$ consists of all relative interiors of nonempty faces of $\mathcal{C}_+$. The resulting (partial) compactification $\widehat{\mathcal{D}}(\mathcal{P}_{\mathrm{SBB}})/\Gamma$ is the *Satake-Baily-Borel-type compactification of $\mathcal{D}/\Gamma$*. This is "minimal" among semi-toric compactifications in an obvious combinatorial sense; I do not know whether a more precise analogue of the Borel extension theorem holds in this context. The strata added to $\mathcal{D}/\Gamma$ include a unique 0-dimensional stratum $\mathcal{D}(\mathcal{C})$, which serves as a distinguished boundary point.

At the other extreme, if every cone $\sigma \in \mathcal{P}$ is the relative interior of a rational polyhedral cone $\overline{\sigma}$ which is generated by a subset of a basis of L, then the associated partial compactification is smooth, and the compactifying set is a divisor with normal crossings. We call this a *Mumford-type* semi-toric compactification. We will spell out the structure of the compactification more explicitly in this case, giving an alternative description of $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$.

We can think of producing a Mumford-type semi-toric compactification in two steps. In the first step, we construct a partial compactification $\widehat{\mathcal{D}}(\mathcal{P})/L$ of $\mathcal{D}/L$ which is Γ_0-equivariant; in the second step we recover $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$ as the quotient of $\widehat{\mathcal{D}}(\mathcal{P})/L$ by Γ_0.

The first step is done one cone at a time. Given $\sigma \in \mathcal{P}$, there is a basis $\ell^1, \ldots, \ell^r$ of L such that

$$\sigma = \mathbb{R}_{>0}\ell^1 + \cdots + \mathbb{R}_{>0}\ell^k \quad \text{for some } k \leq r.$$

Let $\{z_j\}$ be complex coordinates dual to $\{\ell^j\}$, so that $z = \sum z_j \ell^j$ represents a general element of $L_{\mathbb{C}}$. Consider the set $\mathcal{D}_\sigma := L_{\mathbb{R}} + i\sigma$. Translations by the lattice L preserve $\mathcal{D}_\sigma$, and coordinates on the quotient $\mathcal{D}_\sigma/L \subset L_{\mathbb{C}}/L$ can be given by $w_j = \exp(2\pi i z_j)$. In terms of those coordinates, $\mathcal{D}_\sigma/L$ can be described as

$$\mathcal{D}_\sigma/L = \{w \in \mathbb{C}^r : 0 < |w_j| < 1 \text{ for } j \leq k, |w_j| = 1 \text{ for } j > k\}.$$

We partially compactify this to

$$(\mathcal{D}_\sigma/L)^- := \{w \in \mathbb{C}^r : 0 \leq |w_j| < 1 \text{ for } j \leq k, |w_j| = 1 \text{ for } j > k\}.$$

(We have suppressed the σ-dependence of ℓ^j, z_j, w_j to avoid cluttering up the notation.) We call any $w \in (\mathcal{D}_\sigma/L)^-$ with $w_j = 0$ for $j \leq k$ a *distinguished limit point* of $\mathcal{D}_\sigma/L$. Note that any path in $\mathcal{D}_\sigma$ along which $\mathrm{Im}(z_j) \to \infty$ for all $j \leq k$, maps to a path in $\mathcal{D}_\sigma/L$ which approaches such a distinguished limit point. The set $\mathrm{DLP}(\sigma)$ of distinguished limit points is a subset of the stratum $\widehat{\mathcal{D}}(\sigma)$, and is a compact real torus of dimension $\dim_\mathbb{R} \mathrm{DLP}(\sigma) = r - k = \dim_\mathbb{C} \widehat{\mathcal{D}}(\sigma)$. When $k = r$, the distinguished limit point is unique, and it coincides with the 0-dimensional stratum $\mathcal{D}(\sigma)$ of $\widehat{\mathcal{D}}(\mathcal{P})$.

The partial compactification $\widehat{\mathcal{D}}(\mathcal{P})/L$ can now be described as a disjoint union of the $(\mathcal{D}_\sigma/L)^-$'s, with $(\mathcal{D}_\tau/L)^-$ lying in the closure of $(\mathcal{D}_\sigma/L)^-$ whenever τ is the relative interior of a face of $\overline{\sigma}$. This space $\widehat{\mathcal{D}}(\mathcal{P})/L$ is smooth and simply-connected, and the induced action of Γ_0 on it has no fixed points. The action of Γ_0 permutes the various $(\mathcal{D}_\sigma/L)^-$'s, a finite number of which serve to cover $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$ after we take the quotient by Γ_0. We have thus achieved an alternative description of Mumford-type compactifications.

Later in this paper, we will be concerned with recognizing when a complex analytic space has the structure of a semi-toric compactification. We can take a first step in that direction by formalizing the structure of $(\mathcal{D}_\sigma/L)^-$ near the distinguished limit point when $k = r$ in the following way. For a complex manifold T, we say that p is a *maximal-depth normal crossing point of $B \subset T$* if there is an open neighborhood U of p in T and an isomorphism $\varphi : U \to \Delta^r$ such that $\varphi(U \cap (T-B)) = (\Delta^*)^r$ and $\varphi(p) = (0, \ldots, 0)$, where Δ is the unit disk, and $\Delta^* := \Delta - \{0\}$. There are thus r local components $B_j := \varphi^{-1}(\{v_j = 0\})$ of $B \cap U$, with $p = B_1 \cap \cdots \cap B_r$, where v_j is a coordinate on the jth disk.

2 Cusps of Hilbert modular surfaces

We now give an example to illustrate the construction in the previous section: the cusps of Hilbert modular surfaces, as analyzed by Hirzebruch [28] and by Mumford in the first chapter of [1]. Let $\mathrm{PGL}^+(2, \mathbb{R}) = \mathrm{PSL}(2, \mathbb{R})$ act by fractional linear transformations on the upper half plane $\mathfrak{H}$. Let K be a real quadratic field with ring of integers $\mathcal{O}_K$, and let $\mathrm{PGL}^+(2, K)$ be the group of invertible 2×2 matrices with entries in K whose determinant is mapped to a positive number under both embeddings of K into $\mathbb{R}$, modulo scalar multiples of the identity matrix. The map $\Phi : K \to \mathbb{R}^2$ given by the two embeddings of K into $\mathbb{R}$ induces an action of $\mathrm{PGL}^+(2, K)$ on $\mathfrak{H} \times \mathfrak{H}$.

A *Hilbert modular surface* is an algebraic surface of the form $\mathfrak{H} \times \mathfrak{H}/\Gamma$ for some arithmetic group $\Gamma \subset \mathrm{PGL}^+(2, K)$ (that is, a group commensurable with $\mathrm{PGL}^+(2, \mathcal{O}_K)$), often assumed to be torsion-free. The Satake-Baily-

$\sum$ = convex hull of $\mathbb{R}_+ \times \mathbb{R}_+ \cap \Phi(\mathfrak{A})$.

Ex.:

Figure 1.
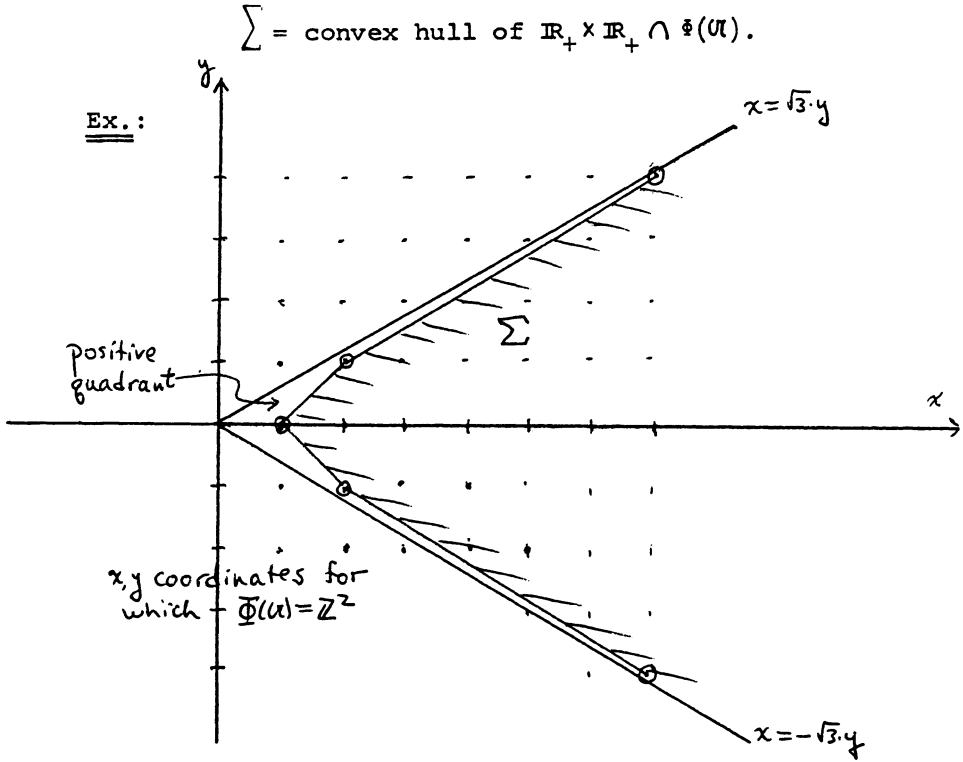
Borel compactification of a Hilbert modular surface adds a finite number of compactification points, called *cusps*. Small deleted neighborhoods of such points have inverse images in $\mathfrak{H} \times \mathfrak{H}$ whose Γ-stabilizer is a parabolic subgroup Γ_{par} of the form

$$\Gamma_{par} = \{ \begin{pmatrix} \varepsilon^k & a \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z}, a \in \mathfrak{A} \},$$

where $\mathfrak{A} \subset \mathfrak{O}_K$ is an ideal, and $\varepsilon \in \mathfrak{O}_K^\times$ is a totally positive unit such that $\varepsilon \mathfrak{A} = \mathfrak{A}$. We can analyze a neighborhood of a cusp by studying appropriate partial compactifications of $\mathfrak{H} \times \mathfrak{H}/\Gamma_{par}$.

The elements in Γ_{par} with $k = 0$ form the translation subgroup, which we identify with $\mathfrak{A}$. This is a free abelian group of rank 2. Let (α, α'), (β, β') be a $\mathbb{Z}$-basis of $\Phi(\mathfrak{A})$. Define a map $\mathfrak{H} \times \mathfrak{H} \to \mathbb{C}^2$ by

$$(w_1, w_2) \mapsto \frac{1}{\alpha\beta' - \alpha'\beta}(\beta' w_1 - \beta w_2, -\alpha' w_1 + \alpha w_2),$$

and let $\mathcal{D}$ denote the image of $\mathfrak{H} \times \mathfrak{H}$ in $\mathbb{C}^2$. Under this map, $\Phi(\mathfrak{A})$ is sent to the standard lattice $L := \mathbb{Z}^2$, and $\Phi(\Gamma_{\mathrm{par}})$ is sent to a subgroup of $\mathrm{Aff}(L)$ with the translation subgroup $\mathfrak{A}$ of Γ_{par} mapped to the translation subgroup L of $\mathrm{Aff}(L)$. As in section 1, we form the quotient in two steps: first take the quotient $\mathfrak{H} \times \mathfrak{H}/\mathfrak{A} = \mathcal{D}/L$, and then take the quotient of the resulting space by the group $\Gamma_0 := \Gamma_{\mathrm{par}}/\mathfrak{A}$.

Mumford shows how to partially compactify the space $\mathcal{D}/L \subset L \otimes \mathbb{C}^* = (\mathbb{C}^*)^2$ in a Γ_0-equivariant way, so that the quotient by Γ_0 gives the desired partial compactification of $\mathfrak{H} \times \mathfrak{H}/\Gamma_{\mathrm{par}}$. The map of $\mathfrak{H} \times \mathfrak{H} \to \mathbb{C}^2$ was designed so that the image would be a tube domain $\mathcal{D} := \mathbb{R}^2 + i\mathcal{C}$, where $\mathcal{C}$ is the cone

$$\mathcal{C} = \{(y_1, y_2) : \alpha y_1 + \beta y_2 > 0, \alpha' y_1 + \beta' y_2 > 0\}.$$

The boundary lines of the closure $\overline{\mathcal{C}}$ have irrational slope, and in fact $\mathcal{C}_+ = \mathcal{C}$ is an open convex cone. To construct a Γ_0-invariant rational polyhedral decomposition $\mathcal{P}$, let Σ be the convex hull of $\mathcal{C} \cap \Phi(\mathfrak{A})$. The vertices of Σ form a countable set $\{v_j\}_{j \in \mathbb{Z}}$ which can be numbered so that the edges of Σ are exactly the line segments $\overline{v_j v_{j+1}}$. If we let σ_j be the relative interior of the cone on $\overline{v_j v_{j+1}}$, and let τ_j be the relative interior of the cone on v_j, then $\mathcal{P} := \{\sigma_j\}_{j \in \mathbb{Z}} \cup \{\tau_j\}_{j \in \mathbb{Z}}$ is a Γ_0-invariant rational polyhedral decomposition. An explicit example of this construction is illustrated on p. 52 of [1], reproduced as figure 1 of this paper.

The resulting partial compactification of $\mathcal{D}/L$ adds a point p_j for each σ_j, and a curve $B_j \cong \mathbb{P}^1$ for each τ_j, with $B_j \cap B_{j+1} = p_j$. This can be pictured as an "infinite chain" of $\mathbb{P}^1$'s, as in the top of figure 2 (which is also reproduced from [1], p. 46). The generator $[\mathrm{diag}(\varepsilon, 1)]$ of $\Gamma_0 = \Gamma_{\mathrm{par}}/\mathfrak{A}$ acts by sending v_j to v_{j+m} for some fixed m. Taking the quotient by Γ_0 leaves us with a "cycle" of rational curves, of length m (as depicted in the bottom of figure 2). We arrive at Hirzebruch's description of the resolution of the cusps.

Conversely, suppose we are given a normal surface singularity $p \in \overline{S}$ (with $\overline{S}$ a small neighborhood of p) which has a resolution of singularities $f : T \to \overline{S}$ such that $B := f^{-1}(p)$ is a cycle of rational curves, that is, $B = B_1 + \cdots + B_m$ is a divisor with normal crossings such that B_j only meets $B_{j\pm 1}$, with subscripts calculated mod m. Much of the structure above can be recovered from this information alone. In fact, by a theorem of Laufer [33] these singularities are *taut*, which means that the isomorphism type is determined by the resolution data. We will work out in detail some aspects of this tautness, in preparation for a general construction in the next section.

The starting point is Wagreich's calculation [54] of the local fundamental group $\pi_1(\overline{S} - p)$ for such singularities, which goes as follows. Let $S := \overline{S} - p = T - B$. The natural map $\iota : \pi_1(S) \to \pi_1(T)$ induced by the inclusion $S \subset T$
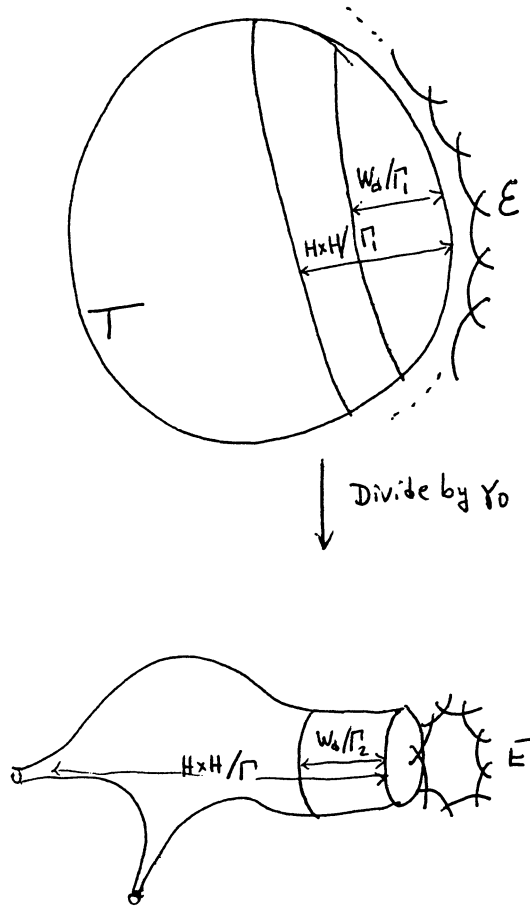
Figure 2.

is surjective. Since T retracts onto a cycle of $\mathbb{P}^1$'s, the group $\pi_1(T) \cong \pi_1(S^1)$ is infinite cyclic, and the universal cover $\widehat{T}$ of T contains an infinite chain $\widehat{B} = \cdots + \widehat{B}_j + \widehat{B}_{j+1} + \cdots$ of $\mathbb{P}^1$'s lying over the cycle B. The kernel of ι is $\pi_1(\widehat{T} - \widehat{B})$, and by a result of Mumford [42] this is a free abelian group generated by loops around any pair of adjacent components $\widehat{B}_j$, $\widehat{B}_{j+1}$ of $\widehat{B}$.

In this way, we recover the two steps of the quotient construction, and the compactification $\widehat{T}$ of the intermediate quotient $\widehat{T} - \widehat{B}$. Let $\widehat{S}$ be the universal cover of S (and of $\widehat{T} - \widehat{B}$). To complete the discussion of tautness, we should exhibit an isomorphism between $\widehat{S}$ and an open subset of $\mathfrak{H} \times \mathfrak{H}$,

which descends to a $\pi_1(T)$-equivariant map $(\widehat{T} - \widehat{B}) \to (\mathfrak{H} \times \mathfrak{H})/\mathfrak{A}$. The easiest way to do this is to consider an extra piece of structure on $p \in \overline{S}$: a flat connection on the holomorphic cotangent bundle $\Omega^1_{\overline{S}}$. We discuss this structure, and how to use it to determine the mapping from $\widehat{S}$ to $\mathfrak{H} \times \mathfrak{H} = \mathcal{D}$, in the next section. (To give a complete proof of Laufer's tautness result along these lines, we would also need to show how the connection is to be constructed; we will not attempt to do that here.)

3   The toric connection

Let $(L_{\mathbb{Q}}, \mathcal{C}, \Gamma_0)$ be an admissible triple, with associated tube domain $\mathcal{D} = L_{\mathbb{R}} + i\mathcal{C}$ and discrete group $\Gamma = L \rtimes \Gamma_0 \subset \mathrm{Aff}(L)$. We will define a flat connection on the holomorphic cotangent bundle of the quotient space $\mathcal{D}/\Gamma$.

The intermediate quotient space $\mathcal{D}/L$ is an open subset of the algebraic torus $L_{\mathbb{C}}/L = L \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^{\mathrm{rk}(L)}$. We identify the dual of the Lie algebra $\mathrm{Lie}(L_{\mathbb{C}}/L)^*$ of that torus with the space of right-invariant one-forms on the group $L_{\mathbb{C}}/L$. Any basis of $\mathrm{Lie}(L_{\mathbb{C}}/L)^*$, when regarded as a subset of the space of global sections of the sheaf $\Omega^1_{L_{\mathbb{C}}/L}$, freely generates that sheaf at any point. We can therefore define a connection ∇_{toric} on $\Omega^1_{L_{\mathbb{C}}/L}$, the *toric connection*, by the requirement that $\nabla_{\mathrm{toric}}(\alpha) = 0$ for every $\alpha \in \mathrm{Lie}(L_{\mathbb{C}}/L)^*$. Since the group $L_{\mathbb{C}}/L$ is abelian, the connection ∇_{toric} is flat.

The action of $\mathrm{Aff}(L)$ on $L_{\mathbb{C}}$ descends to an action of $\mathrm{GL}(L)$ on $L_{\mathbb{C}}/L$ which preserves the space of right-invariant one-forms. In particular, the $\mathrm{GL}(L)$-action will be compatible with the toric connection. Thus, if we restrict ∇_{toric} to $\mathcal{D}/L$, it commutes with the action of Γ_0 and induces a connection on the holomorphic cotangent bundle of $(\mathcal{D}/L)/\Gamma_0 = \mathcal{D}/\Gamma$, still denoted by ∇_{toric}.

Let $\sigma \subset L_{\mathbb{R}}$ be the relative interior of a rational polyhedral cone which is generated by a basis $\ell^1, \dots, \ell^r$ of L, and let $z_1, \dots, z_r$ be the coordinates on $L_{\mathbb{C}}$ dual to $\{\ell^j\}$. The one-forms $d\log w_j := 2\pi i \, dz_j$ are right-invariant one-forms on $L_{\mathbb{C}}/L$ which serve as a basis of $\mathrm{Lie}(L_{\mathbb{C}}/L)^*$. If we compactify the open set $\mathcal{D}_\sigma/L \subset L_{\mathbb{C}}/L$ to $U_\sigma := (\mathcal{D}_\sigma/L)^-$, then the forms $d\log w_j$ extend to meromorphic one-forms on U_σ with poles along the boundary $B_\sigma := (\mathcal{D}_\sigma/L)^- - (\mathcal{D}_\sigma/L)$. In fact, the forms $d\log w_1, \dots, d\log w_r$ freely generate the sheaf $\Omega^1_{U_\sigma}(\log B_\sigma)$ as an $\mathcal{O}_{U_\sigma}$-module. The flat connection ∇_{toric} therefore extends to a flat connection on $\Omega^1_{U_\sigma}(\log B_\sigma)$ for which the $d\log w_j$ are flat sections. Note that the connection does *not* acquire singularities along the boundary, but extends as a regular connection to the sheaf of logarithmic differentials.

If $\mathcal{P}$ is a rational polyhedral decomposition of $\mathcal{C}_+$, we get in this way an extension of the flat connection ∇_{toric} from $\Omega^1_{\mathcal{D}/L}$ to the sheaf of logarithmic

differentials on $\widehat{\mathcal{D}}(\mathcal{P})/L$ with poles on the boundary $(\widehat{\mathcal{D}}(\mathcal{P})/L) - (\mathcal{D}/L)$. As this extended connection still commutes with Γ_0, there is an induced extension of ∇_{toric} from $\Omega^1_{\mathcal{D}/\Gamma}$ to $\Omega^1_{\widehat{\mathcal{D}}(\mathcal{P})/\Gamma}(\log \mathcal{B})$, where $\mathcal{B} := (\widehat{\mathcal{D}}(\mathcal{P})/\Gamma) - (\mathcal{D}/\Gamma)$. This holds for any Mumford-type semi-toric compactification.

The existence of this toric connection on $\mathcal{D}/\Gamma$ depends in an essential way on Γ being a group of affine-linear transformations of L. If $\mathcal{D}$ admits an action by a larger group Γ_{big} which includes discrete symmetries that do not lie in $\text{Aff}(L)$, then ∇_{toric} may fail to descend to the quotient $\mathcal{D}/\Gamma_{\text{big}}$. For example, if $L = \mathbb{Z}$ acts on the upper half plane $\mathfrak{H}$ by translations, then the associated flat connection ∇_{toric} has the property that $\nabla_{\text{toric}}(d\tau) = 0$, where τ is the standard coordinate on $\mathfrak{H}$. The flat section $d\tau$ is invariant under translations $\tau \mapsto \tau + n$, but if we apply ∇_{toric} to the pullback of the flat section $d\tau$ under the inversion $\tau \mapsto -1/\tau$ we get

$$\nabla_{\text{toric}}(\tau^{-2}\, d\tau) = -2\tau^{-3}\, d\tau \otimes d\tau,$$

which is not 0. In particular, the connection ∇_{toric} does not descend to the j-line $\mathfrak{H}/\operatorname{SL}(2,\mathbb{Z})$.

We now want to explain how the abstract knowledge of the flat connection ∇_{toric} and of a Mumford-type semi-toric compactification of $\mathcal{D}/\Gamma$ can be used to recover the structure of $\mathcal{D}$ and of Γ. Suppose we are given a complex manifold T, a divisor with normal crossings B on T, and a flat connection ∇ on $\Omega^1_T(\log B)$. By the usual equivalence between flat connections and local systems [16], the flat sections of ∇ determine a local system $\mathbb{E}$ on T. Such a local system is specified by giving its fiber E at a fixed base point $\star$ (which we choose to lie in $T - B$), together with a representation of $\pi_1(T, \star)$ in $\operatorname{GL}(E)$.

We first restrict the connection and the local system to $T - B$. If we pass to the universal cover $\widehat{S}$ of $T - B$, the flat sections give a global trivialization of the bundle $\mathbb{E} \otimes \mathcal{O}_{\widehat{S}} = \Omega^1_{\widehat{S}}$. There is a natural map $\text{int}_\star : \widehat{S} \to E^*$ which sends $s \in \widehat{S}$ to the functional

$$\alpha \mapsto \int_\star^s \widehat{\alpha},$$

where $\widehat{\alpha}$ is the unique flat section of $\mathbb{E}$ (a holomorphic 1-form on $\widehat{S}$) such that $\widehat{\alpha}|_\star = \alpha \in E$. (Notice that if we vary the basepoint $\star$, we simply shift the image of the map by some constant vector in E^*.)

On the other hand, if we consider ∇ on T and pass to the universal cover $\widehat{T}$ of T, the flat sections of $\mathbb{E}$ will trivialize the bundle $\Omega^1_{\widehat{T}}(\log \widehat{B})$, where $\widehat{B}$ is a divisor with normal crossings in $\widehat{T}$, the inverse image of $B \subset T$. We

once again encounter the intermediate quotient space $\widehat{T} - \widehat{B}$, and its partial compactification $\widehat{T}$.

At any maximal-depth normal crossing point p of $\widehat{B} \subset \widehat{T}$, let $v_j = 0$ define the jth local component B_j of the boundary at p. There is a unique flat section $\widehat{\alpha}_j$ of $\Omega^1_{\widehat{T}}(\log \widehat{B})$, defined locally near p, such that $\widehat{\alpha}_j - d \log v_j$ vanishes at p. It follows that $\widehat{\alpha}_j - d \log v_j$ is a holomorphic one-form in a neighborhood of p, and so that $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_r$ is a basis for (flat) local sections of $\Omega^1_{\widehat{T}}(\log \widehat{B})$. Using the global trivialization, we may regard each $\alpha_j := \widehat{\alpha}_j|_*$ as an element of E. We let $L_p \subset E^*$ be the lattice spanned by the dual basis $\ell^1, \ldots, \ell^r$ to $\alpha_1, \ldots, \alpha_r$, and let $\sigma_p \subset L_p \otimes \mathbb{R}$ be the relative interior of the cone generated by $\ell^1, \ldots, \ell^r$.

If we are to recover the structure of the semi-toric compactification, we need a certain compatibility among the L_p's and the σ_p's: they should be related to a common lattice and a common cone, independent of p. We formalize this as follows.

DEFINITION 2 *We call (T, B, ∇) compatible provided that*

1. *each component of B contains at least one maximal-depth normal crossing point,*

2. *the lattices L_p for maximal-depth normal crossing points p all coincide with a common lattice $L \subset E^*$,*

3. *the natural map $\text{int}_* : \widehat{S} \to E^* = L_{\mathbb{C}}$ descends to a map $(\widehat{T} - \widehat{B}) \to (L_{\mathbb{C}}/L)$ which induces an isomorphism of fundamental groups, and*

4. *the collection $\mathcal{P}$ of relative interiors of faces of the σ_p's is a locally rational polyhedral decomposition of a strongly convex cone $\mathcal{C}_+$.*

Suppose that (T, B, ∇) is compatible, let $\mathcal{C}$ be the interior of $\mathcal{C}_+$, and let $\mathcal{D} = L_{\mathbb{R}} + i\mathcal{C}$. The action of $\pi_1(T)$ on $L_{\mathbb{C}}$ permutes the set of maximal-depth normal crossing points of $B \subset T$, and so preserves $\mathcal{P}$ and $\mathcal{C}$. Thus, $\Gamma := \pi_1(T - B)$ acts on $\mathcal{D}$, and there is an induced map $(T - B) \to (\mathcal{D}/\Gamma)$.

We can now recover the compactification T from this data (or at least its structure in codimension one). For any maximal-depth normal crossing boundary point p of $\widehat{B} \subset \widehat{T}$, there is a neighborhood U_p of p in $\widehat{T}$ and a natural extension of the induced map $U_p \cap (\widehat{T} - \widehat{B}) \to L_{\mathbb{C}}/L$ to a map $U_p \to \widehat{\mathcal{D}}(\mathcal{P})/L$. We cannot tell from the behavior of these extensions what happens at "interior" points of boundary components (those which do not lie in any U_p), but we *can* conclude that there is a meromorphic map $\widehat{T} \to \widehat{\mathcal{D}}(\mathcal{P})/L$ which does not blow down any boundary components. This map is $\pi_1(T)$-equivariant, so it descends to a map $T \to \widehat{\mathcal{D}}(\mathcal{P})/\Gamma$.

4   Moduli spaces of sigma-models

A *Calabi-Yau manifold* is a compact connected orientable manifold X of dimension $2n$ which admits Riemannian metrics whose (global) holonomy is contained in $SU(n)$.[2] Given such a metric, there exist complex structures on X for which the metric is Kähler. The holonomy condition is equivalent to requiring that this Kähler metric be Ricci-flat and that there exist a nonzero holomorphic n-form on X (cf. [7]). On the other hand, if we are given a complex structure on a Calabi-Yau manifold, then by the theorems of Calabi [11] and Yau [60], for each Kähler metric $\tilde{g}$ there is a unique Ricci-flat Kähler metric g whose Kähler form is in the same de Rham cohomology class as that of $\tilde{g}$. (We have implicitly used the topological consequence of Ricci-flatness: Calabi-Yau manifolds have vanishing first Chern class.)

Examples of Calabi-Yau manifolds are provided by the differentiable manifolds underlying smooth complex projective varieties with trivial canonical bundle. One can apply Yau's theorem to a Kähler metric coming from a projective embedding in order to produce a metric with holonomy contained in $SU(n)$, where n is the complex dimension of the variety. As explained in [7], if the Hodge numbers $h^{p,0}$ vanish for $0 < p < n$ and if the manifold is simply-connected, then the holonomy of this metric is precisely $SU(n)$.

Physicists have constructed a class of conformal field theories called nonlinear sigma-models on Calabi-Yau manifolds X (cf. [22, 29]). We consider here an approximation to those theories, which should be called "one-loop semiclassical nonlinear sigma-models". Such an object is determined by the data of a Riemannian metric g on X whose holonomy is contained in $SU(n)$ together with the de Rham cohomology class $[b] \in H^2(X, \mathbb{R})$ of a real closed 2-form b on X.

Two such pairs (g, b) and (g', b') will determine isomorphic conformal field theories if there is a diffeomorphism $\varphi : X \to X$ such that $\varphi^*(g') = g$, and $\varphi^*([b']) - [b] \in H^2(X, \mathbb{Z})$. It is therefore natural to regard the class of $[b]$ in $H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$ as the fundamental datum. We denote this class by $[b]$ mod $\mathbb{Z}$.

The set of all isomorphism classes of such pairs we call the *one-loop semiclassical nonlinear sigma-model moduli space*, or simply the *sigma-model moduli space* (for short). This may differ from the actual *conformal field theory moduli space*, for several reasons: first, there may be additional isomorphisms of conformal field theories which are not visible in this geometric interpretation, second, there may be deformations of the nonlinear sigma-model as

---

[2]There is some confusion in the literature about whether "Calabi-Yau" should mean that the holonomy is precisely $SU(n)$, or simply contained in $SU(n)$. In this paper, we adopt the latter interpretation.

a conformal field theory which do not have a sigma-model interpretation on X (cf. [2, 59]), and third, the putative conformal field theory may fail to converge for some values of the input data (g, b) (although it is believed to converge whenever the volume of the metric is sufficiently large). For our present purposes, we ignore these more delicate questions about the conformal field theory moduli space, and concentrate on the sigma-model moduli space we have defined above.

We focus attention in this paper on the case in which the holonomy of the metric g is precisely $SU(n)$, $n \neq 2$. For each such metric, there are exactly two complex structures on X for which the metric is Kähler (complex conjugates of each other).[3] Thus, there is a natural map from a double cover of the sigma-model moduli space to the usual "complex structure moduli space", given by assigning to (g, b) one of the two complex structures for which g is Kähler. The fibers of this map can be described as follows. If we fix a complex structure on X, then the corresponding fiber consists of all $B + i J \mod \mathbb{Z} \in H^2(X, \mathbb{C})/H^2(X, \mathbb{Z})$ (modulo diffeomorphism) with B denoting the class $[b]$, for which J is the cohomology class of a Kähler form. (The metric g is uniquely determined by J, by Calabi's theorem.) This quantity $B + i J \mod \mathbb{Z}$ describes the "extra structure" S which was alluded to in [40]. This is often called the *complexified Kähler structure* on X determined by (g, b).

The natural double cover of the sigma-model moduli space will be locally a product near (g, b), with the variations of complex structure and of complexified Kähler structure describing the factors in the product, provided that neither the Kähler cone nor the group of holomorphic automorphisms "jumps" when the complex structure varies. (The non-jumping of the Kähler cone was shown to hold by Wilson [55] in the case of holonomy SU(3), when the complex structure is generic.) We will tacitly assume this local product structure, and separately study the parameter spaces for the variations of complexified Kähler structure and of complex structure.

With a fixed complex structure on X, the parameter space for complexified Kähler structures on X can be described in terms of the Kähler cone $\mathcal{K}$ of X, and the lattice $L = H^2(X, \mathbb{Z})/(\text{torsion})$. We must identify any pair of complexified Kähler structures which differ by a diffeomorphism that fixes the complex structure, that is, by an element of the group $\Gamma_0 = \text{Aut}(X)$ of holomorphic automorphisms. The natural parameter space for pairs (g, b) such that g is Kähler for the given complex structure thus has the form $\mathcal{D}/\Gamma$, where $\mathcal{D} = \{B + i J : J \in \mathcal{K}\}$ and $\Gamma = L \rtimes \Gamma_0$ is the extension of Γ_0 by the lattice translations. This is exactly the kind of space encountered in the

---

[3] More generally, as we will show elsewhere, if $h^{2,0}(X) = 0$ there are only a finite number of complex structures for which g is Kähler. The number depends on the decomposition of the holonomy representation into irreducible pieces.

first part of this paper: a tube domain modulo a discrete symmetry group of affine-linear transformations which includes a lattice acting by translations.

A common technique in the physics literature is to consider what happens along paths $\{tz \bmod \Gamma\}_{t \to \infty}$, which go from $z \in \mathcal{D}$ out towards infinity in the tube domain. Many aspects of the conformal field theory can be analyzed perturbatively in t along such paths. It seems reasonable to hope that such limits can be described in a common framework, based on a single partial compactification of $\mathcal{D}/\Gamma$. This hope (together with a bit of evidence, discussed below) leads us to conjecture that $(L_{\mathbb{Q}}, \mathcal{K}, \operatorname{Aut}(X))$ is an admissible triple, in order that Looijenga's methods could be applied to construct compactifications of $\mathcal{D}/\Gamma$. We formulate this conjecture more explicitly as follows.

THE CONE CONJECTURE *Let X be a Calabi-Yau manifold on which a complex structure has been chosen, and suppose that $h^{2,0}(X) = 0$. Let $L :=$ $H^2(X, \mathbb{Z})/\text{torsion}$, let $\mathcal{K}$ be the Kähler cone of X, let $\mathcal{K}_+$ be the convex hull of $\overline{\mathcal{K}} \cap L_{\mathbb{Q}}$, and let $\operatorname{Aut}(X)$ be the group of holomorphic automorphisms of X. Then there exists a rational polyhedral cone $\Pi \subset \mathcal{K}_+$ such that $\operatorname{Aut}(X).\Pi = \mathcal{K}_+$.*

The Kähler cone of X can have a rather complicated structure, analyzed in the case $n = 3$ by Kawamata [31] and Wilson [55]. Away from classes of triple-self-intersection zero, the closed cone $\overline{\mathcal{K}}$ is locally rational polyhedral, but the rational faces may accumulate towards points with vanishing triple-self-intersection. The cone conjecture predicts that while the closed cone $\overline{\mathcal{K}}$ of X may have infinitely many edges, there will only be finitely many $\operatorname{Aut}(X)$-orbits of edges. Other finiteness predictions which follow from the cone conjecture include finiteness of the set of fiber space structures on X, modulo automorphisms.

Many of the large classes of examples, such as toric hypersurfaces, have Kähler cones $\mathcal{K}$ such that $\mathcal{K}_+ = \overline{\mathcal{K}}$ is a rational polyhedral cone. For these, the cone conjecture automatically holds. A nontrivial case of the cone conjecture—Calabi-Yau threefolds which are fiber products of generic rational elliptic surfaces with section (as studied by Schoen [49])—has been checked by Grassi and the author [21]. In addition, Borcea [9] has verified the finiteness of $\operatorname{Aut}(X)$-orbits of edges of $\overline{\mathcal{K}}$ in another nontrivial example, and Oguiso [44] has discussed finiteness of $\operatorname{Aut}(X)$-orbits of fiber space structures in yet another example.[4] All three examples involve cones with an infinite number of edges.

For any X for which the cone conjecture holds, the Kähler parameter space $\mathcal{D}/\Gamma$ will admit both a Satake-Baily-Borel-type "minimal" compactification,

---

[4]Neither of these constitutes a complete verification of the cone conjecture for the threefold in question.

and smooth compactifications of Mumford type built out of many cones $\sigma \subset \mathcal{K}$ as above.

5   Additional structures on the moduli spaces

Of particular interest to the physicists studying nonlinear sigma-models has been the "large radius limit" in the Kähler parameter space. This is typically analyzed in the physics literature as follows (cf. [56, 57]). The quantities of physical interest will be invariant under translation by L. Many such quantities vary holomorphically with parameters, and their Fourier expansions take the form

$$\sum_{\eta \in L^*} c_\eta \, e^{2\pi i \, z \cdot \eta}. \qquad (*)$$

The coefficients c_η for $\eta \neq 0$ are called *instanton contributions* to the quantity (*), and in many cases they can be given a geometric interpretation which shows that they vanish unless η is the class of an effective curve on X. A "large radius limit" should be a point at which instanton contributions to quantities like (*) are suppressed [24, 3].

If we pick a basis $\ell^1, \ldots, \ell^r$ of L consisting of vectors which lie in the closure of the Kähler cone, write $\eta = \sum \eta^j \ell_j$ in terms of the basis $\{\ell_j\}$ of L^* dual to $\{\ell^j\}$, and express (*) as a power series in $w_j := \exp(2\pi i \, z_j)$, where $\{z_j\}$ are coordinates dual to $\{\ell^j\}$, then the series expansion

$$\sum_{\eta \in L^*} c_\eta \, w_1^{\eta^1} \cdots w_r^{\eta^r} \qquad (**)$$

involves only terms with nonnegative exponents [4]. If convergent,[5] this will define a function on $(\mathcal{D}_\sigma/L)^-$, where σ is the relative interior of the cone generated by $\ell^1, \ldots, \ell^r$. Thus, approaching the distinguished limit point of $\mathcal{D}_\sigma/L$ (where all w_j's approach 0) suppresses the instanton contributions, so the distinguished limit point is a good candidate for the large radius limit. We can repeat this construction for any cone $\sigma \subset \mathcal{K}$ which is the relative interior of a cone generated by a basis of L, obtaining partial compactifications which include large radius limit points for paths that lie in various cones σ.

Among the "quantities of physical interest" to which this analysis is applied are a collection of multilinear maps of cohomology groups called *three-point functions*. These maps should depend on the data (g, b), and should vary holomorphically with both complex structure and complexified Kähler

---

[5]From a rigorous mathematical point of view, the Fourier coefficients c_η can often be defined and calculated, but no convergence properties of the series (*) or (**) are known.

structure parameters. Certain of these three-point functions (related to Witten's "*A*-model" [58]) would depend only on the complexified Kähler structure, while others (related to Witten's "*B*-model") would depend only on the complex structure. The *B*-model three-point functions can be mathematically interpreted in terms of the variation of Hodge structure, or period mapping, induced by varying the complex structure on the Calabi-Yau manifold [15, 40, 23].

In [41], we discuss a mathematical version of the *A*-model three-point functions, expressed as formal power series near the distinguished limit point associated to the relative interior σ of a rational polyhedral cone generated by a basis of L. (The coefficients c_η of this power series are derived from the numbers of rational curves on X of various degrees.) The choice of σ is an additional piece of data in the construction which we call a *framing*.

These formal power series representations of *A*-model three-point functions can be regarded as defining a formal degenerating variation of Hodge structure, which we call the *framed A-variation of Hodge structure* with framing σ. Now there are manipulations of these formal series which suggest that the underlying convergent three-point functions (if they exist) will not depend on the choice of σ and will be invariant under the action of $\mathrm{Aut}(X)$.

We must refer the reader to [41] for the precise definition of framed *A*-variation of Hodge structure. But for reference, we would like to state here a conjecture which suggests how the various framed *A*-variations of Hodge structure will fit together, along the lines being discussed in this paper.

THE CONVERGENCE CONJECTURE *Suppose that X is a Calabi-Yau manifold with $h^{2,0}(X) = 0$, endowed with a complex structure, which satisfies the cone conjecture. Let $L := H^2(X, \mathbb{Z})/\mathrm{torsion}$, let $\mathcal{K}$ be the Kähler cone of X, let $\mathcal{D} := L_{\mathbb{R}} + i\mathcal{K}$ be the associated tube domain, and let $\Gamma := L \rtimes \mathrm{Aut}(X)$. Then there is a neighborhood U of the 0-dimensional stratum $\widehat{\mathcal{D}}(\mathcal{K})$ in the Satake-Baily-Borel-type compactification $\widehat{\mathcal{D}}(\mathcal{P}_{\mathrm{SBB}})/\Gamma$, and a variation of Hodge structure on $U \cap (\mathcal{D}/\Gamma)$, such that for any $\sigma \subset \mathcal{K}$ which is the relative interior of a rational polyhedral cone $\overline{\sigma} \subset \mathcal{K}_+$ generated by a basis of L, the induced formal degenerating variation of Hodge structure at the distinguished limit point of $\mathcal{D}_\sigma/L$ agrees with the framed A-variation of Hodge structure with framing σ.*

If this variation of Hodge structure exists, we call it the *A-variation of Hodge structure* associated to X.

6 Maximally unipotent boundary points

In the previous section, we discussed how to let the complexified Kähler parameter $B + iJ$ approach infinity, analyzing certain partial compactifications

and boundary points of the sigma-model moduli space in the $B + iJ$ directions. We now turn to compactifications and boundary points in the transverse directions—the directions obtained by varying the complex structure on the Calabi-Yau manifold. We consider what happens when the complex structure degenerates.

The local moduli spaces of complex structures on Calabi-Yau manifolds are particularly well-behaved, thanks to a theorem of Bogomolov [8], Tian [51], and Todorov [52], which guarantees that all first-order deformations are unobstructed. In particular, there will be a local family of deformations of a given complex structure for which the Kodaira-Spencer map is an isomorphism. More generally, we consider arbitrary families $\pi : \mathcal{Y} \to S$ of complex structures on a fixed Calabi-Yau manifold Y, by which we mean: π is a proper and smooth map between connected complex manifolds, and all fibers $Y_s := \pi^{-1}(s)$ are diffeomorphic to Y. We will often assume that the Kodaira-Spencer map is an isomorphism at every point $s \in S$, so that S provides good local moduli spaces for the fibers Y_s.

To study the behavior when the complex structure degenerates, we partially compactify the parameter space S to $\overline{S}$. There is a class of boundary points on $\overline{S}$ of particular interest from the perspective of conformal field theory. According to the interpretation of [40, 41], these points can be identified by the monodromy properties of the associated variation of Hodge structure[6] near $p \in \overline{S}$. We first review from [41] these monodromy properties for normal crossing boundary points, and then extend the definition to a wider class of compactifications and boundary points.

Let p be a maximal-depth normal crossing point of $B \subset \overline{S}$, where $B :=$ $\overline{S} - S$ is the boundary, assumed for the moment to be a divisor with normal crossings. Let U be a small neighborhood of p in $\overline{S}$, and write $B \cap U$ in the form $B_1 + \cdots + B_r$. If we fix a point $s \in U - B$, then each local divisor B_j gives rise to an monodromy transformation $T^{(j)} : H^n(Y_s, \mathbb{Q}) \to H^n(Y_s, \mathbb{Q})$, which is guaranteed to be quasi-unipotent by the monodromy theorem [32].

DEFINITION 3 *A maximal-depth normal crossing point p of $B \in \overline{S}$ is called a maximally unipotent point[7] under the following conditions.*

 1. *The monodromy transformations $T^{(j)}$ around local boundary components B_j near p are all unipotent.*

---

[6]The variation of Hodge structure in question is the usual geometric one (cf. [26]) associated to a variation of complex structure. These might be called "B-variations of Hodge structure" by analogy with the previous section.

[7]When $\dim(S) = 1$, this definition is equivalent to the one given in [40].

2. Let $N^{(j)} := \log T^{(j)}$, let $N := \sum a_j N^{(j)}$ for some $a_j > 0$, and define

$$
\begin{aligned}
W_0 &:= \operatorname{Im}(N^n) \\
W_1 &:= \operatorname{Im}(N^{n-1}) \cap \operatorname{Ker} N \\
W_2 &:= \left(\operatorname{Im}(N^{n-2}) \cap \operatorname{Ker}(N)\right) + \left(\operatorname{Im}(N^{n-1}) \cap \operatorname{Ker}(N^2)\right).
\end{aligned}
$$

Then $\dim W_0 = \dim W_1 = 1$ and $\dim W_2 = 1 + \dim(S)$.

3. Let $g^0, g^1, \ldots, g^r$ be a basis of W_2 such that g^0 spans W_0, and define m^{jk} by $N^{(j)} g^k = m^{jk} g^0$ for $1 \le j, k \le r$. Then $m := (m^{jk})$ is an invertible matrix.

(The spaces W_0 and W_2 are independent of the choice of coefficients $\{a_j\}$ [14, 17], and the invertibility of m is independent of the choice of basis $\{g^k\}$.)

Given a maximally unipotent point $p \in \overline{S}$, we define the *canonical logarithmic one-forms* $d \log q_j \in \Gamma(U, \Omega^1_{\overline{S}}(\log B))$ at p by

$$
\frac{1}{2\pi i} d \log q_j := d \left(\frac{\sum_{k=1}^r \langle g^k | \omega \rangle m_{kj}}{\langle g^0 | \omega \rangle} \right)
$$

where (m_{kj}) is the inverse matrix of (m^{jk}), and ω is a section of the sheaf $\Omega^n_{\mathcal{Y}/S}$ of relative holomorphic n-forms on the family of complex structures parameterized by S. The elements $g^k \in H^n(Y_s, \mathbb{Q})$ have been implicitly extended to multi-valued sections of the local system $R^n \pi^*(\mathbb{Q}_{\mathcal{Y}})$ in order to evaluate $\langle g^k | \omega \rangle$; the monodromy measures the multi-valuedness of the resulting (locally defined) holomorphic functions $\langle g^k | \omega \rangle$. The fact that each $d \log q_j$ as defined above has a single-valued meromorphic extension to U follows from the nilpotent orbit theorem [48]. In [41] we show that the canonical one-forms are independent of the choice of basis $\{g^k\}$, and also of the choice of relative n-form ω; that for any local defining equation $v_j = 0$ of B_j, the one-form $d \log q_j - d \log v_j$ extends to a regular one-form on U; and that $d \log q_1, \ldots, d \log q_r$ freely generate the locally free sheaf $\Omega^1_{\overline{S}}(\log B)$ near p.

The canonical logarithmic one-forms can be integrated to produce *quasi-canonical coordinates* $q_1, \ldots, q_r$ near p, but due to constants of integration, these coordinates are not unique. That is, if we attempt to define

$$
q_j = \exp\left(2\pi i \frac{\sum_{k=1}^r \langle g^k | \omega \rangle m_{kj}}{\langle g^0 | \omega \rangle} \right)
$$

we find that changing the basis $\{g^k\}$ will alter the q_j's by multiplicative constants (cf. [39]). To specify truly canonical coordinates, further conditions

on the basis $\{g^k\}$ must be imposed, as discussed in [40, 41]. For example, by demanding that g^0 span $W_0 \cap H^n(Y_s, \mathbb{Z})$/torsion and that $g^0, \ldots, g^r$ span $W_2 \cap H^n(Y_s, \mathbb{Z})$/torsion we can reduce the ambiguity in the q_j's to a finite number of choices.

With no ambiguity, we can use the canonical logarithmic one-forms to produce a (canonical) flat connection ∇ on the holomorphic vector bundle $\Omega^1_U(\log B)$ by declaring $d \log q_1, \ldots, d \log q_r$ to be a basis for the ∇-flat sections, that is, $\nabla(d \log q_j) = 0$. Notice that the connection ∇ is regular along the boundary divisor B. This connection is what we will use to extend the definition of maximally unipotent to a more general case.

We now consider partial compactifications $\overline{S}$ of S which are not necessarily smooth, and whose boundary is not necessarily a divisor with normal crossings.

DEFINITION 4 *Let $\Xi \subset \overline{S} - S$ be a connected subset of the boundary. We say that Ξ is* maximally unipotent *if there is a neighborhood V of Ξ in $\overline{S}$ and a flat connection ∇_{unip} on $\Omega^1_{V \cap S}$ such that for some resolution of singularities $f : U \to V$ which is an isomorphism over $V \cap S$, we have*

    1. the new boundary $B := U - f^{-1}(V \cap S)$ on U is a divisor with normal crossings,

    2. the flat connection ∇_{unip} extends to a connection on $\Omega^1_U(\log B)$ (also denoted by ∇_{unip}),

    3. for every maximal-depth normal crossing point p of $B \subset U$, we have $\nabla_{\text{unip}}(d \log q_j) = 0$ for each canonical logarithmic one-form $d \log q_j$ at p, and

    4. $(U, B, \nabla_{\text{unip}})$ is compatible in the sense of definition 2.

We call ∇_{unip} the maximally unipotent connection *determined by Ξ.*

Note that $d \log q_1, \ldots, d \log q_r$ is a basis for the vector space of local solutions of $\nabla_{\text{unip}} e = 0$ near p. By analytic continuation of solutions, the connection ∇_{unip} is unique if it exists. The requirement of compatibility is quite strong, essentially guaranteeing that the structure of $\overline{S}$ near Ξ resembles that of a semi-toric compactification.

7   Implications of mirror symmetry

Mirror symmetry [18, 34, 13, 24] predicts that Calabi-Yau manifolds should come in pairs,[8] with the rôles of variation of complex structure and of com-

---

[8]The most recent results [2, 59] suggest that it is *birational equivalence classes* of Calabi-Yau manifolds which come in pairs.

plexified Kähler structure being reversed between mirror partners. We wish to formulate a precise mathematical version of these mirror symmetry predictions, taking into account the semi-toric compactification structure we have studied in this paper. The resulting statements are unfortunately rather technical, but they appear to be completely general. We hope the reader will bear with the technicalities.

Our conjectures involve the A-variations of Hodge structure introduced in [41], whose essential ingredients are the numbers of rational curves of various degrees on a Calabi-Yau manifold. Our first mathematical conjecture about mirror symmetry is carefully formulated in [41], and can be stated as follows.

THE MATHEMATICAL MIRROR SYMMETRY CONJECTURE (Normal Crossings Case) *Let Y be a Calabi-Yau manifold with $h^{2,0}(Y) = 0$, and let $\pi : \mathcal{Y} \to S$ be a family of complex structures on Y such that the Kodaira-Spencer map is an isomorphism at every point. Let $S \subset \overline{S}$ be a partial compactification whose boundary is a divisor with normal crossings. To each maximally unipotent normal crossing boundary point p in $\overline{S}$ there is associated the following:*

1. *a Calabi-Yau manifold X with $h^{2,0}(X) = 0$,*

2. *a lattice L of finite index[9] in $H^2(X, \mathbb{Z})/\text{torsion}$,*

3. *the relative interior $\sigma \subset H^2(X, \mathbb{R})$ of a rational polyhedral cone $\overline{\sigma}$ which is generated by a basis $\ell^1, \ldots, \ell^r$ of L, and*

4. *a map μ from a neighborhood of p in $\overline{S}$ to $((H^2(X, \mathbb{R}) + i\,\sigma)/L)^-$, determined up to constants of integration by the requirement that $\mu^*(d \log w_j)$ is the canonical logarithmic one-form $d \log q_j$ on $\overline{S}$ at p (as defined in section 6), where $z_1, \ldots, z_r$ are coordinates dual to $\ell^1, \ldots, \ell^r$, and $w_j := \exp(2\pi i\, z_j)$,*

such that

a. *σ is contained in the Kähler cone for some complex structure on X, and*

b. *μ induces an isomorphism between the formally degenerating geometric variation of Hodge structure at p and the A-variation of Hodge structure with framing σ associated to X.*

---

[9]The reason for allowing such an L rather than insisting on $H^2(X, \mathbb{Z})/\text{torsion}$ itself is that our basic defining condition on the family S—that the Kodaira-Spencer map be an isomorphism at every point—is invariant under finite unramified base change. So we must allow finite unramified covers of the parameter spaces.

Put more concretely, if we calculate the geometric variation of Hodge structure near $p \in \overline{S}$ using appropriate quasi-canonical coordinates q_j, we should produce power series expansions for B-model three-point functions (for Y) whose coefficients agree with the c_η which are derived from the numbers of rational curves on X. This is precisely the type of calculation pioneered by Candelas, de la Ossa, Green, and Parkes [12] in the case of the quintic threefold.

This first version of our mathematical mirror symmetry conjecture depends rather explicitly on the choice of a maximally unipotent (normal crossing) boundary point. And unfortunately, if we move from point to point along the boundary of $\overline{S}$, or if we vary the compactification $\overline{S}$ by blowing up the boundary, we can produce many such boundary points. On the other hand, if X is a mirror partner of Y for which the cone and convergence conjectures hold, there are many framed A-variations of Hodge structure (with different framings) associated to X. In fact, given framings σ and σ' which belong to rational polyhedral decompositions $\mathcal{P}$ and $\mathcal{P}'$, respectively, there is always a common refinement $\mathcal{P}''$ of these decompositions. Geometrically, the corresponding compactification $\widehat{\mathcal{D}}(\mathcal{P}'')/\Gamma$ is a blowup of both $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$ and $\widehat{\mathcal{D}}(\mathcal{P}')/\Gamma$. Analytic continuation on the common blowup $\widehat{\mathcal{D}}(\mathcal{P}'')/\Gamma$ from a point in the inverse image of $\mathcal{D}(\sigma)$ to one in the inverse image of $\mathcal{D}(\sigma')$ will give an isomorphism of the A-variations of Hodge structure.

Each of the various maximally unipotent normal crossing boundary points will conjecturally lead to a mirror isomorphism. We wish to fit these various mirror isomorphisms together, thus removing the dependence of the conjectures on an arbitrary choice of boundary point. In fact, the mirror symmetry isomorphism is expected by the physicists to extend to an isomorphism between the full conformal field theory moduli spaces, and so, presumably, to compactifications as well. Thus, the structure of the semi-toric compactifications which is natural from the point of view of variation of complexified Kähler structure on X should be reflected in the structure of compactifications of the complex structure moduli space $\mathcal{M}_Y$ of Y.

This philosophy suggests two things about the compactified parameter spaces $\overline{S}$ of complex structures on Y. First, there should be a compatibility between compactification points whose mirror families are associated to the *same* space X, and the *same* Kähler cone $\mathcal{K}$. In fact, we should be able to extend our mathematical mirror symmetry conjecture to arbitrary maximally unipotent subsets of the boundary for *any* compactification, not just ones whose boundary is a divisor with normal crossings. And second, there should be some kind of *minimal* compactification of the coarse moduli space $\mathcal{M}_Y$ of complex structures on Y, whose mirror compactified family would be the

Satake-Baily-Borel-type compactification of $\mathcal{D}/\Gamma$.

The compatibility between compactifications can be recognized by means of the flat connection ∇_{unip} which we used to identify maximally unipotent subsets of the boundary. We extend our mirror symmetry conjecture to the general case as follows.

THE MATHEMATICAL MIRROR SYMMETRY CONJECTURE (General Case) *Let Y be a Calabi-Yau manifold with $h^{2,0}(Y) = 0$, let $\pi : \mathcal{Y} \to S$ be a family of complex structures on Y such that the Kodaira-Spencer map is an isomorphism at every point, and let $S \subset \overline{S}$ be a partial compactification. To each maximally unipotent connected subset Ξ of the boundary $\overline{S} - S$ there is associated the following:*

1. *a Calabi-Yau manifold X satisfying the cone and convergence conjectures,*

2. *a subgroup $\Gamma \subset \text{Aff}(H^2(X,\mathbb{R}))$ whose translation subgroup L is a lattice of finite index in $H^2(X,\mathbb{Z})$/torsion,*

3. *a locally rational polyhedral decomposition $\mathcal{P}$ of a cone $\mathcal{C}_+$ (which coincides with the convex hull of $\overline{\mathcal{C}_+} \cap L_\mathbb{Q}$) that is invariant under the group $\Gamma_0 := \Gamma/L$, and*

4. *a map μ from a neighborhood U of Ξ in $\overline{S}$ to $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$, determined up to constants of integration by the requirement that the flat connection ∇_{toric} on $\mathcal{D}/\Gamma$ pulls back to ∇_{unip} on $U \cap S$, where ∇_{unip} is the maximally unipotent connection determined by Ξ,*

such that

a. *for some complex structure on X, the interior $\mathcal{C}$ of $\mathcal{C}_+$ is contained in the Kähler cone and Γ_0 is contained in the group of holomorphic automorphisms, and*

b. *μ induces an isomorphism between the geometric variation of Hodge structure over $U \cap S$ and the A-variation of Hodge structure associated to X.*

A priori, the map μ determined by compatibility of the connections would only be a meromorphic map; we are asserting that it is in fact regular, and a local isomorphism.

There is one further refinement of this conjecture which could be made: we could demand that the map μ also respect the quasi-canonical coordinates

determined by choosing integral bases $g^0, \ldots, g^r$. This would reduce the ambiguity in the choice of μ to a finite number of choices, but would require a compatibility among such integral quasi-canonical coordinates at various boundary points.

Finally, suppose that $\mathcal{M}_Y$ is the coarse moduli space for complex structures on a Calabi-Yau variety Y such that $h^{2,0}(Y){=}0$. (This coarse moduli space is known to exist as a quasi-projective variety, once we have specified a polarization, thanks to a theorem of Viehweg [53].) In this case, we conjecture the existence of a Satake-Baily-Borel-style compactification, as follows.

THE MINIMAL COMPACTIFICATION CONJECTURE *There is a partial compactification* $(\overline{\mathcal{M}_Y})_{\mathrm{SBB}}$ *of the coarse moduli space* $\mathcal{M}_Y$ *with distinguished boundary points* $p_1, \ldots, p_k$ *which are maximally unipotent, such that the data associated by the mathematical mirror symmetry conjecture to* p_j *consists of: (1) a Calabi-Yau manifold* X_j *(with a complex structure specified that determines the group* $\mathrm{Aut}(X_j)$ *of holomorphic automorphisms and the Kähler cone* $\mathcal{K}_j$ *of* X_j*), (2) the group*

$$\Gamma_j := (H^2(X_j, \mathbb{Z})/\mathrm{torsion}) \rtimes \mathrm{Aut}(X_j),$$

and (3) the locally rational polyhedral decomposition $\mathcal{P}_j$ *which is the Satake-Baily-Borel decomposition* $\mathcal{P}_{\mathrm{SBB}}$ *of the cone* $(\mathcal{K}_j)_+$ *(the convex hull of* $\overline{\mathcal{K}}_j \cap H^2(X_j, \mathbb{Q})$*).*

A related conjecture has been made independently by Batyrev [6].

8 Mumford cones and Mori cones

In the fall of 1979, Mori lectured at Harvard on his then-new results [36] on the cone of effective curves. In order to show that his theorem about local finiteness of extremal rays fail when the canonical bundle is numerically effective, he gave an example. (A similar example appears in a Japanese expository paper he wrote a few years later, which has since been translated into English [37].) The example was of an abelian surface with real multiplication, that is, one whose endomorphism algebra contains the ring of integers $\mathfrak{O}_K$ of a real quadratic field K. For such a surface X, the Néron-Severi group $L := H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ is a lattice of rank 2. The Kähler cone of X lies naturally in $L_{\mathbb{R}}$, and is an open cone $\mathcal{K}$ bounded by two rays whose slopes are irrational numbers in the field K (cf. [38, 20]). Rays through classes of ample divisors $[D] \in L \cap \mathcal{K}$ can be found which are arbitrarily close to the boundary, but the boundary is never reached. This phenomenon indicated

that Mori's results on the structure of the dual cone $\mathcal{K}^\vee \subset H_2(X, \mathbb{R})$ could not be extended to the case of abelian surfaces. The picture Mori drew for this example was remarkably similar to figure 1.

The Hilbert modular surfaces in fact serve as moduli spaces for abelian surfaces with endomorphisms of this type (cf. [20, Chap. IX]), although a bit more data must be specified, which determines the group Γ. Now Mumford's figure 1 was drawn in some auxiliary space being used to describe this "complex structure moduli space", while Mori's version of figure 1 depicted the Kähler cone in $H^{1,1}$, and so is related to "complexified Kähler moduli" of the surfaces. The setting is not quite the same as the one in the present paper, since $h^{2,0} \neq 0$. However, mirror symmetry for complex tori does predict that each cusp in the complex structure moduli space will be related to the Kähler moduli space for the abelian varieties parametrized by some $\mathfrak{H} \times \mathfrak{H}/\Gamma$, with the Γ determined by the cusp. (This is not completely clear from the literature; I will return to this point in a subsequent paper.) In fact, under this association the Mumford cone from figure 1 corresponds precisely to the (dualized) Mori cone.

Mirror symmetry might have been anticipated by mathematicians had anyone noticed the striking similarity between these two pictures back in 1979!

Acknowledgments

References

1. A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth compactifications of locally symmetric varieties*, Lie Groups: History, Frontiers and Applications, vol. IV, Math Sci Press, Brookline (Mass.), 1975.

2. P. S. Aspinwall, B. R. Greene, and D. R. Morrison, *Multiple mirror manifolds and topology change in string theory*, Phys. Lett. B, **303** (1993), 249–259.

3. P. S. Aspinwall and C. A. Lütken, *Quantum algebraic geometry of superstring compactifications*, Nuclear Phys. B **355** (1991), 482–510.

4. P. S. Aspinwall and D. R. Morrison, *Topological field theory and rational curves*, Comm. Math. Phys. **151** (1993), 245–262.

5. W. L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Annals of Math. (2) **84** (1966), 442–528.

6. V. V. Batyrev, *Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori*, Duke. Math. J. **69** (1993), 349–409.

7. A. Beauville, *Variétés Kählereinnes dont la première classe de Chern est nulle*, J. Diff. Geom. **18** (1983), 755–782.

8. F. A. Bogomolov, *Hamiltonian Kähler manifolds*, Dokl. Akad. Nauk SSSR **243** (1978), no. 5, 1101–1104.

9. C. Borcea, *On desingularized Horrocks-Mumford quintics*, J. Reine Angew. Math. **421** (1991), 23–41.

10. A. Borel, *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, J. Diff. Geom. **6** (1972), 543–560.

11. E. Calabi, *On Kähler manifolds with vanishing canonical class*, Algebraic Geometry and Topology, A Symposium in Honor of S. Lefschetz (R. H. Fox et al., eds.), Princeton University Press, Princeton, 1957, pp. 78–89.

12. P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B **359** (1991), 21–74.

13. P. Candelas, M. Lynker, and R. Schimmrigk, *Calabi-Yau manifolds in weighted* $\mathbb{P}_4$, Nuclear Phys. B **341** (1990), 383–402.

14. E. Cattani and A. Kaplan, *Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structure*, Invent. Math. **67** (1982), 101–115.

15. S. Cecotti, $N = 2$ *Landau-Ginzburg vs. Calabi-Yau* σ-*models: Non-perturbative aspects*, Internat. J. Modern Phys. A **6** (1991), 1749–1813.

16. P. Deligne, *Equations différentielles à points singuliers réguliers*, Lecture Notes in Math., vol. 163, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

17. _____, *La conjecture de Weil, II*, Inst. Hautes Études Sci. Publ. Math. **52** (1980), 137–252.

18. L. J. Dixon, *Some world-sheet properties of superstring compactifications, on orbifolds and otherwise*, Superstrings, Unified Theories, and Cosmology 1987 (G. Furlan et al., eds.), World Scientific, Singapore, New Jersey, Hong Kong, 1988, pp. 67–126.

19. W. Fulton, *Introduction to toric varieties*, Annals of Math. Studies, vol. 131, Princeton University Press, Princeton, 1993.

20. G. van der Geer, *Hilbert modular surfaces*, Ergeb. Math. Grenzgeb. (3) **16**, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1988.

21. A. Grassi and D. R. Morrison, *Automorphisms and the Kähler cone of certain Calabi-Yau manifolds*, Duke Math. J. **71** (1993), 831–838.

22. M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory*, Cambridge University Press, Cambridge, 1987.

23. B. R. Greene, D. R. Morrison, and M. R. Plesser, *Mirror manifolds in higher dimension*, preprint, 1993.

24. B. R. Greene and M. R. Plesser, *Duality in Calabi-Yau moduli space*, Nuclear Phys. B **338** (1990), 15–37.

25. P. A. Griffiths, *Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems*, Bull. Amer. Math. Soc. **76** (1970), 228–296.

26. _____, ed., *Topics in transcendental algebraic geometry*, Annals of Math. Studies, vol. 106, Princeton University Press, Princeton, 1984.

27. J. C. Hemperly, *The parabolic contribution to the number of linearly independent automorphic forms on a certain bounded domain*, Amer. J. Math. **94** (1972), 1078–1100.

28. F. E. P. Hirzebruch, *Hilbert modular surfaces*, Enseign. Math. (2) **19** (1973), 183–282.

29. T. Hübsch, *Calabi-Yau manifolds: A bestiary for physicists*, World Scientific, Singapore, New Jersey, London, Hong Kong, 1992.

30. J. Igusa, *A desingularization problem in the theory of Siegel modular functions*, Math. Ann. **168** (1967), 228–260.

31. Y. Kawamata, *Crepent blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Annals of Math. (2) **127** (1988), 93–163.

32. A. Landman, *On the Picard-Lefschetz transformations*, Trans. Amer. Math. Soc. **181** (1973), 89–126.

33. H. B. Laufer, *Taut two-dimensional singularities*, Math. Ann. **205** (1973), 131–164.

34. W. Lerche, C. Vafa, and N. P. Warner, *Chiral rings in $N = 2$ superconformal theories*, Nuclear Phys. B **324** (1989), 427–474.

35. E. Looijenga, *New compactifications of locally symmetric varieties*, Proceedings of the 1984 Vancouver Conference in Algebraic Geometry (J. Carrell et al., eds.), CMS Conference Proceedings, vol. 6, American Mathematical Society, Providence, 1986, pp. 341–364.

36. S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Annals of Math. (2) **116** (1982), 133–176.

37. _____, *Hartshorne conjecture and extremal ray*, Sugaku Expositions **1** (1988), 15–37.

38. D. R. Morrison, *The Kuga-Satake variety of an abelian surface*, J. Algebra **92** (1985), 454–476.

39. _____, *Picard-Fuchs equations and mirror maps for hypersurfaces*, Essays on Mirror Manifolds (S.-T. Yau, ed.), International Press Co., Hong Kong, 1992, pp. 241–264.

40. _____, *Mirror symmetry and rational curves on quintic threefolds: A guide for mathematicians*, J. Amer. Math. Soc. **6** (1993), 223–247.

41. _____, *Hodge-theoretic aspects of mirror symmetry*, in preparation.

42. D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Études Sci. Publ. Math. **9** (1961), 5–22.

43. T. Oda, *Convex bodies and algebraic geometry: An introduction to the theory of toric varieties*, Ergeb. Math. Grenzgeb. (3) **15**, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1988.

44. K. Oguiso, *On algebraic fiber space structures on a Calabi-Yau 3-fold*, Int. J. Math. **4** (1993), 439–465.

45. R. T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton, 1970.

46. I. Satake, *On the compactification of the Siegel space*, J. Indian Math. Soc. (N.S.) **20** (1956), 259–281.

47. _____, *On the arithmetic of tube domains (blowing-up of the point at infinity)*, Bull. Amer. Math. Soc. **79** (1973), 1076–1094.

48. W. Schmid, *Variation of Hodge structure: The singularities of the period mapping*, Invent. Math. **22** (1973), 211–319.

49. C. Schoen, *On fiber products of rational elliptic surfaces with section*, Math. Z. **197** (1988), 177–199.

50. H. J. M. Sterk, *Compactifications of the period space of Enriques surfaces: Arithmetic and geometric aspects*, Ph.D. thesis, Katholiecke Universiteit Nijmegen, 1988.

51. G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Peterson-Weil metric*, Mathematical Aspects of String Theory (S.-T. Yau, ed.), World Scientific, Singapore, 1987, pp. 629–646.

52. A. N. Todorov, *The Weil-Petersson geometry of the moduli space of $SU(n \geq 3)$ (Calabi-Yau) manifolds, I*, Comm. Math. Phys. **126** (1989), 325–246.

53. E. Viehweg, *Weak positivity and the stability of certain Hilbert points, III*, Invent. Math. **101** (1990), 521–543.

54. P. Wagreich, *Singularities of complex surfaces with solvable local fundamental group*, Topology **11** (1972), 51–72.

55. P. M. H. Wilson, *The Kähler cone on Calabi-Yau threefolds*, Invent. Math. **107** (1992), 561–583.

56. E. Witten, *Topological sigma models*, Comm. Math. Phys. **118** (1988), 411–449.

57. _____, *On the structure of the topological phase of two-dimensional gravity*, Nuclear Phys. B **340** (1990), 281–332.

58. _____, *Mirror manifolds and topological field theory*, Essays on Mirror Manifolds (S.-T. Yau, ed.), International Press Co., Hong Kong, 1992, pp. 120–159.

59. _____ , *Phases of N=2 theories in two dimensions*, Nuclear Phys. B **403** (1993), 159–222.

60. S.-T. Yau, *On Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), 1798–1799.

David R. Morrison

Department of Mathematics
Duke University
Durham, NC 27708-0320, USA

and

School of Mathematics
Institute for Advanced Study
Princeton, NJ 08540, USA