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REPRESENTATIONS OF QUANTUM GROUPS AT A *p-TH* ROOT OF UNITY AND OF SEMISIMPLE GROUPS IN CHARACTERISTIC p: INDEPENDENCE OF p

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Introduction

Let R be a finite root system with Weyl group W. For any prime p and any algebraically closed field k of characteristic p consider the connected, simply connected semisimple algebraic group G_k over k with root system R. Denote the Lie algebra of G_k by $\mathfrak{g}_k$. This is a p-Lie algebra, i.e., it has an additional structure, the p-th power map $X \mapsto X^{[p]}$. In this case it is the ordinary p-th power, if we think of elements of $\mathfrak{g}_k$ as derivations. We shall consider representations of $\mathfrak{g}_k$ as a p–Lie algebra. This means that the action of any $X^{[p]}$ on the module is the p-th power of the action of X. The corresponding module is then called a restricted $\mathfrak{g}_k$–module. It is a module for the restricted enveloping algebra $U^{[p]}(\mathfrak{g}_k)$, which is the quotient of the universal enveloping algebra of $\mathfrak{g}_k$ by the ideal generated by all $X^p - X^{[p]}$ with $X \in \mathfrak{g}_k$. The algebra $U^{[p]}(\mathfrak{g}_k)$ has finite dimension equal to p^m where $m = \dim \mathfrak{g}_k$.

The representation theory of the $U^{[p]}(\mathfrak{g}_k)$ turns out to have many features that are (conjecturally) independent of p. Let us mention first the one most easily described. Since $U^{[p]}(\mathfrak{g}_k)$ has finite dimension, it is the direct product of indecomposable algebras, the blocks of $U^{[p]}(\mathfrak{g}_k)$. Each indecomposable restricted $\mathfrak{g}_k$–module M belongs to exactly one of these blocks; it is the unique block not annihilating M. Denote by $\mathcal{B}_k$ the block of the trivial one dimensional $\mathfrak{g}_k$–module. Work ([Hu2]) by Humphreys from 1971 showed: If p is greater than the Coxeter number h of R, then the simple modules belonging to $\mathcal{B}_k$ are indexed by the Weyl group W. The Cartan matrix of $\mathcal{B}_k$ is therefore a $(W \times W)$–matrix. In the cases known at that time (and in the cases known today) this matrix is independent of p (as long as $p > h$). So one might conjecture that this independence should hold in general. (This conjecture is implicitly contained in Verma's last conjecture in [Ver] to be discussed below.) We shall prove:

Theorem 1: *There is a* $\mathbf{Z}$*–algebra* $\mathcal{B}$ *(finitely generated as a* $\mathbf{Z}$*–module) such that for all* k *with* $\operatorname{char}(k) > h$ *the block* $\mathcal{B}_k$ *is Morita equivalent to* $\mathcal{B} \otimes_{\mathbf{Z}} k$.

(More precisely, we construct a $\mathcal{B}$ such that $\mathcal{B} \otimes_{\mathbf{Z}} \mathbf{Z}[((h-1)!)^{-1}]$ is free of finite rank over $\mathbf{Z}[((h-1)!)^{-1}]$.) The theorem implies that for $p \gg 0$ the Cartan matrix above is indeed independent of p. Our methods do not yield reasonable bounds. These bounds arise from conditions that certain algebraic numbers should not have p in their denominators.

The algebra $\mathcal{B}$ has also an interpretation in characteristic 0. Take an odd integer $p > 1$ (prime to 3 if R has a G_2 component) and consider the quantized enveloping algebra U_p at a p–th root of unity. Here we take Lusztig's version constructed via divided powers. It contains a finite dimensional analogue $\mathbf{u}_p$ of the restricted enveloping algebra. (This was discovered by Lusztig, cf. [Lu6] and [Lu7].) Then:

Theorem 2: *If $p > h$, then $\mathcal{B} \otimes_{\mathbf{Z}} \mathbf{Q}(\sqrt[p]{1})$ is Morita equivalent to the block of $\mathbf{u}_p$ containing the trivial one dimensional module.*

Let us return to the characteristic $p > 0$ situation. One of the main tools of Humphreys in [Hu2] was the use of the analogues of the Verma modules: Consider the Lie algebra $\mathfrak{b}_k$ of a Borel subgroup of G_k, take a one dimensional $U^{[p]}(\mathfrak{b}_k)$–module and induce to $U^{[p]}(\mathfrak{g}_k)$. Suppose that $p > h$. Then there are $|W|$ induced modules $(Z_w)_{w \in W}$ of this type belonging to $\mathcal{B}_k$. Each Z_w has a unique simple homomorphic image L_w; the L_w with $w \in W$ are exactly the simple modules belonging to $\mathcal{B}_k$ mentioned above. One of the main results in [Hu2] is the following: The projective cover Q_w of L_w has a filtration with factors of the form $Z_{w'}$ with $w' \in W$; any $Z_{w'}$ occurs exactly $d_k(w', w)$ times where $d_k(w', w)$ is the multiplicity of the simple module L_w as a composition factor of $Z_{w'}$. This implies especially that the Cartan matrix of $\mathcal{B}_k$ is determined by the decomposition matrix, i.e., the matrix of all $d_k(w', w)$, and we can now replace the previous conjecture on the Cartan matrix by one on the decomposition matrix (for $p > h$). And that is indeed part of Verma's Conjecture V in [Ver]. We can show:

Theorem 3: *Let $w, w' \in W$. There is an integer $d(w', w) \geq 0$ such that $d_k(w', w) = d(w', w)$ for all k with $char(k) \gg 0$.*

Again, there is an interpretation in characteristic 0. The $\mathbf{u}_p$ have analogous modules Z_w and L_w, and then $d(w', w)$ is equal to the multiplicity of L_w in $Z_{w'}$ whenever $p > h$.

The remaining part of Verma's Conjecture V is concerned with multiplicities for the algebraic group G_k. (Both parts are in fact closely related, since Verma's (proven) Conjecture IV tells us how to express the $d(w', w)$ in terms of multiplicities for G_k.) At this point we need more notations. Let X be the group of weights of the root system R. Let W_a be the affine Weyl group of R, i.e., the semidirect product of W and the group of translations by elements in $\mathbf{Z}R$. Set ρ equal to the sum of the fundamental weights.

Let T_k be a maximal torus in G_k and set $\mathfrak{h}_k$ equal to the Lie algebra of T_k. We assume that T_k is contained in the Borel subgroup with Lie algebra $\mathfrak{b}_k$ so that $\mathfrak{h}_k \subset \mathfrak{b}_k$. We identify the group of characters of T_k with X. To each dominant weight $\lambda \in X$ there correspond a simple module and a Weyl module with highest weight λ. Denote by $b_k(\mu, \lambda)$ the multiplicity of the simple module with highest weight λ as a composition factor of the Weyl module with highest weight μ. The linkage principle (first conjectured by Verma) states that $b_k(\mu, \lambda) \neq 0$ implies $\lambda \in W_a {\bullet} \mu$ where any $w \in W$ acts via

$w{\bullet}\mu = w(\mu+\rho)-\rho$ and any translation by some $\nu \in \mathbf{Z}R$ as translation by $p\nu$. (Perhaps we should write $w{\bullet_p}\lambda$ for $w \in W_a$ and $\lambda \in X$ to denote this action. But it should always be clear what we mean.) The linkage principle together with the translation principle implies that we know (for $p \geq h$) all $b_k(\mu,\lambda)$ if we know all $b_k(w',w) = b_k(w'{\bullet}0, w{\bullet}0)$ with $w, w' \in W_a$ such that $w{\bullet}0$ and $w'{\bullet}0$ are dominant.

The second part of Verma's Conjecture V says that the $b_k(w',w)$ should be independent of k for all $w, w' \in W_a$ with $w{\bullet}0$ and $w'{\bullet}0$ dominant. (This condition is independent of k if $p \geq h$.) Steinberg's tensor product theorem shows easily that this conjecture will not work if $w'{\bullet}0$ is "large" with respect to p^2. So one should modify the conjecture and impose an upper bound on w'. In [Lu1] Lusztig has made a conjecture on the $b_k(w',w)$ (or rather for the inverse matrix) that would imply Verma's (modified) Conjecture V. Set W_a^+ equal to the set of all $w \in W_a$ with $w{\bullet}0$ dominant whenever $p \geq h$. Our results imply:

Theorem 4: *For all $w, w' \in W_a^+$ there is an integer $b(w',w)$ such that $b(w',w) = b_k(w',w)$ for all k with $char(k) \gg 0$.*

In characteristic 0 consider an odd integer p and the algebra U_p as above. We have simple modules and Weyl modules for U_p indexed by the dominant weights, and we have analogous multiplicities $b_p(\mu,\lambda)$. There is again a linkage principle involving the action of W_a where again the translation by a weight $\nu \in \mathbf{Z}R$ acts as translation by $p\nu$. We get now:

Theorem 5: *If $p \geq h$, then $b(w',w) = b_p(w'{\bullet}0, w{\bullet}0)$ for all $w, w' \in W_a^+$.*

These two results imply that one gets for $\mathrm{char}(k) = p \gg 0$ each irreducible G_k–module with "restricted" highest weight by reduction modulo p from a simple U_p–module as conjectured by Lusztig. It also implies that his conjecture in [Lu1] follows for $\mathrm{char}(k) \gg 0$ from its quantum analogue in [Lu4], 8.2.

Kazhdan and Lusztig have recently shown that there is an equivalence of categories between certain U_p–modules and certain representations of affine Kac–Moody Lie algebras. This result was announced in [KL1], proved in the simply laced case in [KL2] – [KL5], and in general in [Lu11]. (Recall that our p is always prime to the entries in the Cartan matrix.) This equivalence implies that Lusztig's conjecture in the quantum case is equivalent to a similar conjecture in the Kac–Moody case. In the latter case Kashiwara and Tanisaki have recently announced ([KT]) a proof of the conjecture; an earlier manuscript by Casian ([Cas]) has not convinced all of its readers.

In order to get our results we work mainly not in the categories of restricted $\mathfrak{g}_k$–modules or of G_k–modules (or their analogues in characteristic 0), but in the category of $\mathfrak{g}_k$-T_k–modules and a characteristic 0 analogue. A $\mathfrak{g}_k$-T_k–module is a restricted $\mathfrak{g}_k$–module M that is also a T_k–module such that (obvious) compatibility conditions hold. Giving a representation of T_k on M is the same as giving a grading $M = \bigoplus_{\nu \in X} M_\nu$ of M by X. The compatibility conditions say that every root vector E_α in $\mathfrak{g}_k$ maps each M_ν to $M_{\nu+\alpha}$ and that every $H \in \mathfrak{h}_k$ acts on each M_ν as multiplication by $\nu(H)$. (We write

$\nu(H)$ by abuse of notation. We really mean the differential of $\nu : T_k \to k$ on the Lie algebra $\mathfrak{h}_k$ of T_k.) Note that $\mathfrak{g}_k$-T_k-modules were called $\mathbf{u}_1$-T-modules in [Ja3] and that they are usually called $(G_k)_1T_k$-modules nowadays.

One can define $\mathfrak{b}_k$-T_k-modules similarly. One associates to each $\lambda \in X$ a one dimensional $\mathfrak{b}_k$-T_k-module, then an induced $\mathfrak{g}_k$-T_k-module $Z_k(\lambda)$. It has a simple head $L_k(\lambda)$. Denote the multiplicity of $L_k(\lambda)$ as a composition factor of $Z_k(\mu)$ by $d'_k(\mu, \lambda)$. If $\mathrm{char}(k) \geq h$, then all these multiplicities are known as soon as we know all $d'_k(w', w) = d'_k(w' \bullet 0, w \bullet 0)$ with $w, w' \in W_a$.

Theorem 6: *For all $w, w' \in W_a$ there is an integer $d'(w', w)$ such that $d'(w', w) = d'_k(w', w)$ for all k with* $\mathrm{char}(k) \gg 0$.

The $d'(w', w)$ have again an interpretation in characteristic 0. One defines a suitable category of $\mathbf{u}_p$-modules with an X-grading satisfying similar compatibility conditions. Then the $d'(w', w)$ are the corresponding multiplicities in this category whenever $p \geq h$.

* * *

We want to give an idea of how these results are reached. For the sake of simplicity let us concentrate on the prime characteristic case and just say that there are analogous results or constructions in the quantum case. Let us also assume that $\mathrm{char}(k) = p \geq h$.

In the category of $\mathfrak{g}_k$-T_k-modules each simple module $L_k(\lambda)$ has a projective cover $Q_k(\lambda)$. It has a filtration with factors of the form $Z_k(\mu)$ and each $Z_k(\mu)$ occurs exactly $d'_k(\mu, \lambda)$ times. One can therefore translate Theorem 6 into a statement about the characters of the $Q_k(w \bullet 0)$. In one case this character is well understood: Take the unique element $w_0 \in W$ that maps all positive roots into negative roots. One knows that $Q_k = Q_k(w_0 \bullet 0)$ has a filtration with factors $Z_k(w \bullet 0)$, $w \in W$ each w occurring once. We have now for our category the so-called wall crossing functors. If we apply a sequence I of these wall crossing functors to Q_k, we get a projective module $Q_{k,I}$. If one knows for all I and $w \in W_a$ the multiplicity of $Q_k(w \bullet 0)$ as a direct summand of $Q_{k,I}$, then one knows all $d'_k(w', w'')$. In fact, one can find a finite family $\mathfrak{I}$ of these sequences such that the decomposition of the $Q_{k,I}$ with $I \in \mathfrak{I}$ alone determines all multiplicities.

Theorem 7: *There is an algebra $\mathcal{E}$ over $\mathbf{Z}$ (finitely generated as a $\mathbf{Z}$-module) such that $\mathcal{E} \otimes_{\mathbf{Z}} k$ is isomorphic to the endomorphism ring of the $\mathfrak{g}_k$-T_k-module $\bigoplus_{I \in \mathfrak{I}} Q_{k,I}$ for all k. Moreover, we have a decomposition $\mathcal{E} = \bigoplus_{I,J \in \mathfrak{I}} \mathcal{E}_{I,J}$ such that the isomorphism takes each $\mathcal{E}_{I,J} \otimes_{\mathbf{Z}} k$ to* $\mathrm{Hom}_{\mathfrak{g}_k, T_k}(Q_{k,I}, Q_{k,J})$.

This theorem (together with some alcove geometry and elementary facts on idempotents) yields Theorem 6. From that Theorems 3 and 4 follow by known results. Considered as a $U^{[p]}(\mathfrak{g}_k)$-module $\bigoplus_{I \in \mathfrak{I}} Q_{k,I}$ is a projective generator for the block $\mathcal{B}_k$. A small modification of the construction of $\mathcal{E}$ from Theorem 7 yields an algebra $\mathcal{B}$ over $\mathbf{Z}$ such that $\mathcal{B} \otimes_{\mathbf{Z}} k$ is (for $\mathrm{char}(k) \gg 0$) isomorphic to the algebra of all endomorphisms of $\bigoplus_{I \in \mathfrak{I}} Q_{k,I}$ as a $\mathfrak{g}_k$-module. Then $\mathcal{B}$ satisfies the claim of Theorem 1.

These indications should convince you that Theorem 7 and its quantum analogue are really the crucial results. So we should now answer the question: How does one get a characteristic free approach to these endomorphism algebras? Well, one essential step is not to work just with these $\mathfrak{g}_k$-T_k–modules, but also to lift them to suitable local rings. This approach was inspired by the success of a similar method in [So2] and [So3].

For each commutative algebra A over the symmetric algebra $S(\mathfrak{h}_k)$ we define a category $\mathcal{C}_A$ generalizing the $\mathfrak{g}_k$-T_k–modules. An object in $\mathcal{C}_A$ is an A–module M with a structure as a $\mathfrak{g}_k$–module and with a grading $M = \bigoplus_{\nu \in X} M_\nu$ by X. Furthermore every root vector $E_\alpha \in \mathfrak{g}_k$ maps each M_ν to $M_{\nu+\alpha}$ and E_α^p annihilates M. Finally, every $H \in \mathfrak{h}_k$ acts on each M_ν as multiplication by $H + \nu(H)$, or rather by its image under the structural map $S(\mathfrak{h}_k) \to A$. If we take $A = k$ with the augmentation map $S(\mathfrak{h}_k) \to k$ where $H \mapsto 0$ for all $H \in \mathfrak{h}_k$, then $\mathcal{C}_k$ is just the category of all $\mathfrak{g}_k$-T_k–modules. (In our "real" definition in 2.3 we add a finiteness condition and assume additionally that A is Noetherian.)

The first seven sections of this paper contain the basic theory of the category $\mathcal{C}_A$ and its quantum analogue. We discuss topics such as induction, projective modules, Ext groups, base change, filtrations, the linkage principle, blocks, translation functors. All of this is quite parallel to the corresponding theories for algebraic groups (cf. [Ja6]) and quantum groups (cf. [APW1]). It should be noted that many of the special aspects of this type of theory over a ring were first dealt with in [GJ]. In the case where A is a field, we reprove some results from [VK], [FP1], [DCK1] and [DCK2].

The algebra $S(\mathfrak{h}_k)$ is a polynomial ring over k with generators H_α, α a simple root. Take for A especially the local ring at the maximal ideal generated by all H_α. (We should denote this algebra by A_k, but write A to avoid double indices.) It turns out that we can lift the $Q_{k,I}$ to projective objects $Q_{A,I}$ in $\mathcal{C}_A$ and that the homomorphisms behave well under base change:

$$\mathrm{Hom}_{\mathcal{C}_A}(Q_{A,I}, Q_{A,J}) \otimes_A k \simeq \mathrm{Hom}_{\mathcal{C}_k}(Q_{k,I}, Q_{k,J}).$$

There is a Lie algebra $\mathfrak{g}_{\mathbf{Z}}$ over $\mathbf{Z}$ with $\mathfrak{g}_{\mathbf{Z}} \otimes_{\mathbf{Z}} k \simeq \mathfrak{g}_k$ for all k. We can find a Cartan subalgebra $\mathfrak{h}_{\mathbf{Z}}$ of $\mathfrak{g}_{\mathbf{Z}}$ with $\mathfrak{h}_{\mathbf{Z}} \otimes_{\mathbf{Z}} k \simeq \mathfrak{h}_k$ for all k. Set $S = S(\mathfrak{h}_{\mathbf{Z}})$ equal to the symmetric algebra of $\mathfrak{h}_{\mathbf{Z}}$ over $\mathbf{Z}$. Then $S \otimes_{\mathbf{Z}} k \simeq S(\mathfrak{h}_k)$ for all k, and we can regard A as an S–algebra. Now Theorem 7 is an easy consequence of:

Theorem 7′: *There is an algebra $\widehat{\mathcal{E}}$ over S that is finitely generated as an S–module such that $\widehat{\mathcal{E}} \otimes_S A$ is isomorphic to the endomorphism ring of $\bigoplus_{I \in \mathfrak{I}} Q_{A,I}$ for all k. (Here $A = A_k$.) Moreover, we have a decomposition $\widehat{\mathcal{E}} = \bigoplus_{I,J \in \mathfrak{I}} \widehat{\mathcal{E}}_{I,J}$ such that the isomorphism takes each $\widehat{\mathcal{E}}_{I,J} \otimes_S A$ to* $\mathrm{Hom}_{\mathcal{C}_A}(Q_{A,I}, Q_{A,J})$.

Well, this just replaces the previous question by a modified one: How does one get a characteristic free approach to the homomorphisms of the

$Q_{A,I}$? Consider inside the fraction field of A the subring

$$A^{\emptyset} = A[H_{\alpha}^{-1} \mid \alpha \text{ a positive root }]$$

and set $Q_I^{\emptyset} = Q_{A,I} \otimes_A A^{\emptyset}$ for all I. Now it is easy to get a characteristic free approach to the $\mathrm{Hom}_{\mathcal{C}_{A^{\emptyset}}}(Q_I^{\emptyset}, Q_J^{\emptyset})$: Each $Q_I^{\emptyset}$ splits into a direct sum of certain $Z^{\emptyset}(w{\bullet}0)$ with $w \in W_a$. (These are the analogues of the $Z_k(w{\bullet}0)$ over $A^{\emptyset}$.) The multiplicity of $Z^{\emptyset}(w{\bullet}0)$ in $Q_I^{\emptyset}$ is independent of k. The module of homomorphisms from $Z^{\emptyset}(w{\bullet}0)$ to $Z^{\emptyset}(w'{\bullet}0)$ is equal to 0 for $w \neq w'$ and equal to $A^{\emptyset}$ for $w = w'$. So the module of homomorphisms from $Q_I^{\emptyset}$ to $Q_J^{\emptyset}$ can be described independently of k as a direct sum of matrix spaces over $A^{\emptyset}$.

Of course, the homomorphisms from $Q_{A,I}$ to $Q_{A,J}$ are exactly the homomorphisms from $Q_I^{\emptyset}$ to $Q_J^{\emptyset}$ that map $Q_{A,I} \subset Q_I^{\emptyset}$ to $Q_{A,J} \subset Q_J^{\emptyset}$. In order to find them we consider intermediate rings. For each positive root β set

$$A^{\beta} = A[H_{\alpha}^{-1} \mid \alpha \text{ a positive root}, \alpha \neq \beta] \subset A^{\emptyset}$$

and $Q_I^{\beta} = Q_{A,I} \otimes_A A^{\beta}$ for all I. We have then $A = \bigcap_{\beta>0} A^{\beta}$ and $Q_{A,I} = \bigcap_{\beta>0} Q_I^{\beta}$ and

$$\mathrm{Hom}_{\mathcal{C}_A}(Q_{A,I}, Q_{A,J}) = \bigcap_{\beta>0} \mathrm{Hom}_{\mathcal{C}_{A^{\beta}}}(Q_I^{\beta}, Q_J^{\beta}). \tag{$*$}$$

The terms on the right hand side of $(*)$ can be described independently of k: Denote the analogue of $Z_k(\lambda)$ over A^{β} by $Z^{\beta}(\lambda)$. Any $Z^{\beta}(w{\bullet}0)$ has a projective cover $Q^{\beta}(w{\bullet}0)$ in $\mathcal{C}_{A^{\beta}}$; it is an extension of $Z^{\beta}(w{\bullet}0)$ by another module of the form $Z^{\beta}(w_1{\bullet}0)$. Each Q_I^{β} is a direct sum of certain $Q^{\beta}(w{\bullet}0)$; the multiplicity of each $Q^{\beta}(w{\bullet}0)$ in each Q_I^{β} is independent of k. Furthermore, we can describe the module of homomorphisms between $Q^{\beta}(w{\bullet}0)$ and $Q^{\beta}(w'{\bullet}0)$ for all w, w' independently of k.

So we have described each term on the right hand side in $(*)$ independently of k. We take their intersection inside the corresponding Hom space over $A^{\emptyset}$ that we can describe independently of k. So, what we need is to have the terms over each A^{β} embedded into the terms over $A^{\emptyset}$ independently of k. This is done easily for each β separately, but it has to be done for all β simultaneously. And that turns out to be an unpleasant problem, but in Sections 8–15 we show that it can be done.

We do not want to discuss the details at this point. Let us just say that we discuss (in Section 8) the Ext groups over A^{β} of the $Z^{\beta}(w'{\bullet}0)$ involved in a $Q^{\beta}(w{\bullet}0)$. We describe in Section 9 how an arbitrary choice of bases for these Ext groups leads to a fully faithful embedding of a subcategory of $\mathcal{C}_A$ into a combinatorial category. This subcategory contains all projective modules. We show then in Section 10 how translation functors behave under this embedding. It turns out that we need to know precisely how the bases for the Ext groups behave under the translation functors. We investigate this in the sections 11 and 12, and we choose in Section 13 specific bases that behave nicely under the translation maps. We describe in Section 14 similar combinatorial categories over the integers and see in Section 15 that they can be given a graded structure so that the combinatorial translation functors are well behaved. We prove then in Section 16 our main results. In the sections 17 and 18 we show: If Lusztig's conjecture (on characters of irreducible modules) holds for k, then the block $\mathcal{B}_k$ is (for a suitable grading) a Koszul algebra and the coefficients of the Kazhdan–Lusztig polynomials have an interpretation in terms of our graded structure. In Section 19 we apply our theory to a few explicit examples. Four appendices contain several computations that are logically independent of our main theory. Two additional appendices discuss general properties of gradings and Koszulity.

Dear Reader: On a first reading we suggest that you — having acquainted yourself with the basic theory of the sections 1–7 — go on to understand the propositions 8.6, 9.4, and 10.11. If you are then willing to believe in Theorem 13.4 (which has a rather long computational proof), you can proceed directly to the main sections 14–16.

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Basic Notations

We work in this paper with a fixed (finite) root system R (in some real vector space). Before we start let us fix some notations involving the root system data only. These notations will be used without further explanation. We set:

R^+	a fixed system of positive roots in R,
Σ	the simple roots in R with respect to R^+,
ω_α	the fundamental weight corresponding to $\alpha \in \Sigma$,
X	$= \sum_{\alpha\in\Sigma} \mathbf{Z}\omega_\alpha$, the weight lattice of R,
ρ	$= \sum_{\alpha\in\Sigma} \omega_\alpha = \frac{1}{2}\sum_{\beta\in R^+} \beta$,
$\beta^\vee$	the coroot corresponding to a root $\beta \in R$,
s_β	the reflection with respect to $\beta \in R$, given by $s_\beta(\lambda) = \lambda - \langle\lambda, \beta^\vee\rangle\beta$ for all $\lambda \in X$,
W	the Weyl group of R,
$l(w)$	the length of $w \in W$ with respect to the generators $(s_\beta)_{\beta\in\Sigma}$,
w_0	the element in W with $w_0(R^+) = -R^+$,
$s_{\beta,m}$	for $\beta \in R$ and $m \in \mathbf{Z}$ the affine reflection with $s_{\beta,m}(\lambda) = \lambda - (\langle\lambda, \beta^\vee\rangle - m)\beta$ for all $\lambda \in X$,
W_a	the affine Weyl group of R (generated by all $s_{\beta,m}$),
$w{\scriptstyle\bullet}\lambda$	$= w(\lambda + \rho) - \rho$ (the dot action),
$\lambda \leq \mu$	$\iff \mu - \lambda \in \sum_{\alpha\in\Sigma} \mathbf{N}\alpha$,
$(d_\alpha)_{\alpha\in\Sigma}$	integers in $\{1,2,3\}$ such that the matrix $(d_\alpha\langle\beta, \alpha^\vee\rangle)_{\alpha,\beta\in\Sigma}$ is symmetric and such that each indecomposable component of R contains an $\alpha \in \Sigma$ with $d_\alpha = 1$,

and, if R is indecomposable,

α_0	the largest short root in R,
h	the Coxeter number of R.

All our rings will have an identity. If A is a ring and if $n \geq 0$ is an integer such that $n!$ is invertible in A, then we set

$$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}$$

for all $a \in A$.

A list of frequently used notations introduced in the text can be found at the end of the paper.

1. The General Setup

1.1. As pointed out above, we consider a root system R and shall use the basic notations introduced there. Let k be a Noetherian commutative ring. If M is an X-graded k-module, denote its homogeneous component of degree ν by M_ν (for all $\nu \in X$). We shall consider X-graded algebras U with graded subalgebras U^0, U^+, and U^- satisfying certain conditions. First of all, multiplication should induce an isomorphism of k-modules

$$U^- \otimes U^0 \otimes U^+ \xrightarrow{\sim} U. \tag{1}$$

(A notation like $M \otimes N$, where $\otimes$ appears without an index, will always denote a tensor product of k-modules. For tensor products of elements we use $\otimes$ without an index quite generally.)

We require further that

$$U^0 \subset U_0 \tag{2}$$

and

$$(U^+)_0 = (U^-)_0 = k \cdot 1, \tag{3}$$

that for all $\nu \in X$

$$(U^+)_\nu \neq 0 \qquad \Longrightarrow \qquad \nu \geq 0 \tag{4}$$

and

$$(U^-)_\nu \neq 0 \qquad \Longrightarrow \qquad \nu \leq 0. \tag{5}$$

There are more conditions that we shall introduce as we go along. Before we do that, we shall describe the two examples that we have in mind.

1.2. Case 1: Let k be an algebraically closed field of prime characteristic p, let G be a connected, simply connected semisimple algebraic group over k. Choose a maximal torus T in G. Suppose that the root system of G with respect to T has the same type as R. Then X can be identified with the group of characters $X(T)$ on T such that R is mapped to the root system of G.

Denote the Lie algebra of G resp. of T by $\mathfrak{g}$ resp. by $\mathfrak{h}$. We have a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^+$ resp. $\mathfrak{n}^-$ is the direct sum of the root spaces for positive resp. for negative roots. For each $\alpha \in R$

let E_α be a root vector for α. We shall suppose that we get these root vectors by base change from a Chevalley system in a suitable semisimple Lie algebra over $\mathbf{C}$. Set $H_\alpha = [E_\alpha, E_{-\alpha}] \in \mathfrak{h}$ for all $\alpha \in R$.

The Lie algebra $\mathfrak{g}$ is a restricted Lie algebra. Denote its p–th power map by $X \mapsto X^{[p]}$. One has $E_\alpha^{[p]} = 0$ and $H_\alpha^{[p]} = H_\alpha$ for all $\alpha \in R$. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. For any $X \in \mathfrak{g}$ the element $X^p - X^{[p]} \in U(\mathfrak{g})$ is central in $U(\mathfrak{g})$. The map $X \mapsto X^p - X^{[p]}$ from $\mathfrak{g}$ to $U(\mathfrak{g})$ is semilinear.

Let I be the (twosided) ideal in $U(\mathfrak{g})$ generated by all $X^p - X^{[p]}$ with $X \in \mathfrak{n}^- \cup \mathfrak{n}^+$. By the semilinearity just mentioned, it is also generated by all E_α^p with $\alpha \in R$. Set $U = \overline{U}(\mathfrak{g}) = U(\mathfrak{g})/I$. For any Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ let $\overline{U}(\mathfrak{g}')$ be the image of $U(\mathfrak{g}')$ in $\overline{U}(\mathfrak{g})$. We shall look especially at $U^0 = \overline{U}(\mathfrak{h})$ and $U^+ = \overline{U}(\mathfrak{n}^+)$ and $U^- = \overline{U}(\mathfrak{n}^-)$.

By the PBW–theorem we get a basis for $U(\mathfrak{g})$ by taking monomials in the E_α and in a basis of $\mathfrak{h}$ (for any fixed order of the product). Because the E_α^p are central in $U(\mathfrak{g})$, such a monomial is in I if and only if the exponent of one E_α is greater or equal to p. So we get a PBW–type basis for $\overline{U}(\mathfrak{g})$ by taking all those monomials where the exponents of the E_α are less than p. We get similar bases for U^0, U^+, and U^- by taking the monomials involving only terms from $\mathfrak{h}$, resp. $\mathfrak{n}^+$, $\mathfrak{n}^-$. A look at these bases shows that U^0 is isomorphic to the enveloping algebra $U(\mathfrak{h})$ — which coincides with the symmetric algebra $S(\mathfrak{h})$ — and that U^+ resp. U^- is isomorphic to the restricted enveloping algebra of $\mathfrak{n}^+$ resp. of $\mathfrak{n}^-$.

It is clear from the bases that 1.1(1) is satisfied in this case. The adjoint action of T on $U(\mathfrak{g})$ stabilizes I and induces an action on $U = \overline{U}(\mathfrak{g})$. If we set U_ν equal to the ν–weight space for this action (for all $\nu \in X = X(T)$), we get an X-grading on U, for which U^0, U^+, and U^- are graded subalgebras and for which the conditions 1.1(2)–(5) are obviously satisfied.

1.3. **Case 2:** Corresponding to the root system R there is a quantized enveloping algebra U_1 over $\mathbf{Q}(v)$ where v is an indeterminate over $\mathbf{Q}$. It is defined by generators E_α, F_α, K_α, and K_α^{-1} for all $\alpha \in \Sigma$ with certain relations. For example, each K_α^{-1} is indeed the inverse of K_α. We shall not write down these relations that can be found in [Lu7], p. 90.

Let $p > 1$ be a positive integer prime to the nonzero entries of the Cartan matrix of R, hence odd. Choose a primitive p–th root of unity ζ and set $k = \mathbf{Q}(\zeta)$. Define a k–algebra U_2 with generators E_α, F_α, K_α, and K_α^{-1} for all $\alpha \in \Sigma$ with the same relations used for U_1 but with v replaced by ζ. This algebra is a version over k of the $\mathbf{C}$–algebra U_ζ studied by DeConcini and Kac, cf. [DCK1], 1.5. The results that we are going to quote from [DCK1] can be proved over k with the same proofs as in [DCK1] over $\mathbf{C}$.

Denote the subalgebra of U_2 generated by all E_α (resp. by all F_α, resp. by all K_α and K_α^{-1}) by U_2^+ (resp. by U_2^-, resp. by U_2^0). Then U_2^0 is the algebra of Laurent polynomials in the commuting variables K_α. All monomials in the K_α with arbitrary integer exponents (positive and negative) are a basis of U_2^0.

There is a unique X-grading of U_2 such that each E_α has degree α, each F_α degree $-\alpha$, and each K_α degree 0. (The relations defining U_2 are obviously homogeneous for this grading.) It is then clear that 1.1(2)–(5) are satisfied by U_2. Also 1.1(1) can be shown to hold, cf. [DCK1], Prop. 1.7. (However, the finiteness condition to be introduced in 2.1 will not be satisfied.)

One can define for each $\beta \in R^+$ a root vector $E_\beta \in (U_2^+)_\beta$ and a root vector $F_\beta \in (U_2^-)_{-\beta}$. This involves certain automorphisms T_w. They were constructed on U_1 by Lusztig in [Lu7], 3.1; his construction carries over to U_2. For any $\beta \in R^+$ one chooses a simple root $\alpha \in \Sigma$ and $w \in W$ with $\beta = w(\alpha)$; one then sets $E_\beta = T_w(E_\alpha)$ and $F_\beta = T_w(F_\alpha)$. Here E_α and F_α are (some of) the generators of U_2 introduced above. For $\beta \in \Sigma$ this definition is compatible with our previous convention, cf. [Lu6], Prop. 1.8(d) that generalizes to the non-simply laced case. Note that these root vectors are indeed in $(U_2)^+$ resp. in $(U_2)^-$ by [Lu7], 4.2. We shall usually write $E_{-\beta}$ instead of F_β.

In some cases we have to assume that the root vectors have been chosen using a reduced decomposition of w_0 as in [Lu7], Appendix. This applies whenever we work with PBW-type bases. There are such bases for U_2^+ and U_2^- consisting of all monomials in the E_β with $\beta > 0$ resp. with $\beta < 0$ with nonnegative exponents. Here (and in (1) below) the product has to be carried out in a specific order described in [Lu7], Appendix. Now 1.1(1) says (and this is really Prop. 1.7 in [DCK1]) that we get a basis of U_2 consisting of all products

$$\prod_{\beta\in R^+} E_{-\beta}^{n(\beta)} \prod_{\alpha\in\Sigma} K_\alpha^{r(\alpha)} \prod_{\beta\in R^+} E_\beta^{m(\beta)} \tag{1}$$

with $n(\beta)$ and $m(\beta)$ nonnegative and $r(\alpha)$ arbitrary integers.

All E_β^p with $\beta \in R$ and all K_α^p with $\alpha \in \Sigma$ are central in U_2, cf. [DCK1], Cor. 3.1. Suppose that we have chosen the E_β using a fixed decomposition of w_0. Let I^+ resp. I^- be the ideal in U_2^+ resp. in U_2^- generated by all E_β^p resp. by all $E_{-\beta}^p$ with $\beta \in R^+$, and let I be the ideal in U_2 generated by I^+ and I^-. It is then clear that I^+ resp. I^- has a basis consisting of all monomials in the E_β resp. the $E_{-\beta}$ with $\beta \in R^+$ where at least one exponent is greater or equal to p, and that I has as basis all products as in (1) with at least one $m(\beta)$ or $n(\beta)$ greater or equal to p. Obviously $I^+ = I \cap U_2^+$ and $I^- = I \cap U_2^-$.

Set $U = U_2/I$ and let U^0, U^+, and U^- be the images of U_2^0, U_2^+, and U_2^- in U. We denote the images of the generators K_α and of the root vectors E_β again by K_α and E_β. We get a basis for U consisting of products as in (1) with $0 \le m(\beta), n(\beta) < p$ for all $\beta \in R^+$ and $r(\alpha) \in \mathbf{Z}$ arbitrary. Taking the products involving only K_α resp. only E_β resp. only $E_{-\beta}$ one gets bases for U^0 resp. for U^+ resp. for U^-. One has obviously isomorphisms $U_2^0 \simeq U^0$ and $U_2^+/I^+ \simeq U^+$ and $U_2^-/I^- \simeq U^-$. The conditions 1.1(1)–(5) are satisfied.

There is a different version U_3 of a quantized enveloping algebra over k due to Lusztig, cf. [Lu7], 8.1. It is constructed from U_1 via a certain form over

$\mathbf{Z}[v, v^{-1}]$ involving divided powers. There are elements E_α, F_α, K_α, and K_α^{-1} in U_3 (for all $\alpha \in \Sigma$) that satisfy the defining relations of the generators of U_2. So there is a homomorphism $f : U_2 \to U_3$ mapping the generators of U_2 to the analogous elements in U_3. The image is the finite dimensional subalgebra $\mathbf{u}$ of U_3 introduced in [Lu7], 8.2. The kernel contains all E_β^p, hence I, and we get thus a surjective homomorphism $\bar{f} : U \to \mathbf{u}$.

The algebra $\mathbf{u}$ has a triangular decomposition $\mathbf{u} = \mathbf{u}^- \mathbf{u}^0 \mathbf{u}^+$. The three factors have PBW–type bases and are the images under $\bar{f}$ of U^-, U^0, and U^+. A look at the bases shows that $\bar{f}$ induces isomorphisms $U^+ \simeq \mathbf{u}^+$ and $U^- \simeq \mathbf{u}^-$, whereas $\mathbf{u}^0$ is identified with U^0 divided by the ideal generated by all $K_\alpha^{2p} - 1$. We see especially that I^+ is the kernel of the natural map $U_2^+ \to U_3$, hence that it is independent of the choice of the reduced decomposition of w_0. A similar remark applies to I^-. Therefore I is independent of this choice.

1.4. Suppose for the rest of section 1 that we are in one of the cases from 1.2 or 1.3.

Lemma: *There is a group homomorphism* $X \to \mathrm{Aut}_{k-alg}(U^0)$ *denoted by* $\mu \mapsto \widetilde{\mu}$ *such that*

$$su = u\widetilde{\mu}(s) \tag{1}$$

for all $\mu \in X$, $u \in U_\mu$, *and* $s \in U^0$.

Proof: In Case 1 one takes for $\widetilde{\mu}$ the unique automorphism of $U^0 \simeq S(\mathfrak{h})$ with $\widetilde{\mu}(H) = H + \mu(H)$ for all $H \in \mathfrak{h}$. (One should really take the differential of μ when evaluating it at H.)

In Case 2 one sets

$$\widetilde{\lambda}(K_\alpha) = \zeta^{d_\alpha \langle \lambda, \alpha^\vee \rangle} K_\alpha$$

for all $\alpha \in \Sigma$. Then (1) is obvious for the generators by the defining relations, hence in general.

Remark: We could also consider a Case 2′ where we take as our algebra $\widetilde{\mathbf{u}} = \mathbf{u}^- U_3^0 \mathbf{u}^+$, cf. [APW2], 0.2. In that case one would define

$$\widetilde{\lambda}(\begin{bmatrix} K_\alpha; c \\ m \end{bmatrix}) = \begin{bmatrix} K_\alpha; c + \langle \lambda, \alpha^\vee \rangle \\ m \end{bmatrix}$$

for certain additional generators of U_3^0, cf. [Lu7], 6.6 or 5.1 below.

1.5. Lemma: *There is for each* $w \in W$ *an automorphism* $T_w : U \to U$ *with*

$$T_w(U^0) = U^0 \qquad \textit{and} \qquad T_w(U_\nu) = U_{w\nu} \qquad \textit{for all } \nu \in X, \tag{1}$$

such that

$$T_w \circ \widetilde{\lambda} \circ T_w^{-1}\big|_{U^0} = \widetilde{w(\lambda)} \qquad \textit{for all } \lambda \in X. \tag{2}$$

Proof: In Case 1 we consider a representative of w in the normalizer of T in G and its adjoint action on $\mathfrak{g}$ and $U(\mathfrak{g})$. It maps each E_α to a nonzero multiple of $E_{w\alpha}$. Therefore it stabilizes the ideal I as in 1.2 and induces an automorphism of $U = \overline{U}(\mathfrak{g})$. It has obviously the required properties.

We have already mentioned that in Case 2 Lusztig's construction of the T_w from [Lu7], 3.1 generalizes from U_1 to U_2. It is easy to see that they have the properties (1)–(2) for U_2 instead of U. We want to prove that each T_w stabilizes the ideal I as in 1.3. Then it will induce an automorphism of U having the desired properties. Of course, it is enough to show $T_\alpha(I^+) \subset I$ and $T_\alpha(I^-) \subset I$ for all $\alpha \in \Sigma$ where $T_\alpha = T_{s_\alpha}$.

The argument at the end of the proof of Prop. 3.3 in [DCK1] shows that there are choices of root vectors $(E_\beta)_{\beta \in R}$ and $(E'_\beta)_{\beta \in R}$ with $T_\alpha(E_\beta) = E'_{s_\alpha(\beta)}$ for all $\beta \in R - \{\alpha\}$. Because I^+ is independent of the choice of the root vectors, we see that $T_\alpha(E_\beta^p) \in I^+ \subset I$ for all these β. On the other hand, we have $T_\alpha(E_\alpha^p) = -K_\alpha^p F_\alpha^p \in I$ by formula (3.3.3) in [DCK1]. So indeed $T_\alpha(I^+) \subset I$. The case of I^- is symmetric.

1.6. Lemma: *There is an involutory antiautomorphism τ of U with $\tau(E_\alpha) = F_\alpha$ and $\tau(F_\alpha) = E_\alpha$ for all $\alpha \in \Sigma$ and with $\tau(s) = s$ for all $s \in U^0$.*

Proof: In Case 1 the properties of a Chevalley system imply that there exists an antiautomorphism of $\mathfrak{g}$ mapping E_β to $E_{-\beta}$ for all $\beta \in R$ and H to H for each $H \in \mathfrak{h}$. It induces an antiautomorphism of $U(\mathfrak{g})$ that maps each E_β^p to $E_{-\beta}^p$, hence stabilizes I, and yields an antiautomorphism of U with the desired properties.

In Case 2, a look at the defining relations shows that there are involutory antiautomorphisms of U_1 and U_2 interchanging E_α and F_α for all $\alpha \in \Sigma$ and fixing each K_α. Going through Lusztig's construction one checks that there is a similar antiautomorphism on U_3. Call all these involutions τ.

We want to show that the τ on U_2 stabilizes the ideal I. Then we can define it on $U = U_2/I$ as desired. The map $f : U_2 \to U_3$ commutes obviously with τ. This implies $\tau(\ker f) = \ker f$. Because of $\tau(U_2^+) = U_2^-$ we get also $\tau(\ker f \cap U_2^+) = \ker f \cap U_2^-$. These intersections are just I^+ and I^-, so we get $\tau(I^+) = I^-$, hence $\tau(I) = I$ as claimed.

1.7. We shall also need a comultiplication on U. It can be constructed easily in Case 1. In Case 2, however, it will require a nontrivial argument that we shall give in 7.1.

2. Module Categories

Keep all notations and assumptions from 1.1 through the whole paper. Later on (from Section 4 on) we shall assume that we are in Case 1 or 2 from 1.2/3. However, in the next two sections we need only weaker conditions that we shall describe in 2.1.

2.1. We want to make the following additional assumptions on U and its subalgebras. We require:

(A) *The set of $\nu \in X$ with $(U^+)_\nu \neq 0$ or $(U^-)_\nu \neq 0$ is finite.*

and:

(B) *Each $(U^+)_\nu$ and each $(U^-)_\nu$ is a free k-module of finite rank.*

These conditions are obvious in the cases from 1.2/3. They imply in general:

(C) *U^+ and U^- are free k-modules of finite rank.*

We require also the analogue of Lemma 1.4:

(D) *There is a group homomorphism $X \to \mathrm{Aut}_{k-alg}(U^0)$ denoted by $\mu \mapsto \widetilde{\mu}$ such that*

$$su = u\widetilde{\mu}(s) \tag{1}$$

for all $\mu \in X$, $u \in U_\mu$, and $s \in U^0$.

Because $\widetilde{0}$ is the identity map, this condition implies obviously:

(E) *The algebra U^0 is commutative.*

2.2. The following lemma contains some easy consequences of our assumptions.

Lemma: *The submodules U^0U^+ and U^-U^0 are graded subalgebras of U. Any basis of U^+ resp. of U^- as a k-module is a basis of U^0U^+ resp. of U^-U^0 as a U^0-module under left and right multiplication, and it is a basis of U as a module over U^-U^0 under left multiplication resp. over U^0U^+ under right multiplication.*

Proof: Because U^+ and U^- are graded, Condition 2.1(D) implies $U^0U^+ = U^+U^0$ and $U^0U^- = U^-U^0$. Therefore these submodules are (obviously graded) subalgebras. We get from 1.1(1) that also the multiplication maps $U^0 \otimes U^+ \to U^0U^+$ and $U^- \otimes U^0 \to U^-U^0$ are bijective. Condition 2.1(D) yields that the same is true for the multiplication maps on $U^+ \otimes U^0$ and on $U^0 \otimes U^-$. These bijectivity statements imply immediately the results on bases as U^0-modules. The claims on U follow from the bijectivity in 1.1(1).

Remark: We get also that the multiplication maps from $U^- \otimes U^+ \otimes U^0$ and from $U^0 \otimes U^- \otimes U^+$ to U are bijective. Therefore U is free as a U^0-module under left and right multiplication. One gets a basis of U over U^0, if one chooses bases, say $(u_i)_i$ resp. $(u'_j)_j$, of U^- resp. of U^+, and takes then all products $u_iu'_j$. Each U_ν is a free U^0-submodule of U.

2.3. Let A be a Noetherian commutative algebra over U^0. Denote the structural homomorphism by $\pi : U^0 \to A$.

We want to define three categories: $\mathcal{C}_A$, $\mathcal{C}'_A$, and $\mathcal{C}''_A$. An object of $\mathcal{C}_A$ is a $U \otimes A$-module M with an X-grading as a k-module satisfying certain properties. So M is a k-module with a direct sum decomposition

$$M = \bigoplus_{\mu \in X} M_\mu. \tag{1}$$

Furthermore there are commuting actions of U and A on M that are compatible with the k-structure. We shall write the action of U on the left and the action of A on the right, i.e., uma should mean the same as $(u \otimes a)m$ for all $u \in U$ and $a \in A$. We impose a finiteness condition:

(A) *M is finitely generated over A.*

Next we require that the action of A preserves the grading (1):

(B) $$M_\mu A \subset M_\mu \qquad \text{for all } \mu \in X(T).$$

In other words, any M_μ is an A-submodule of M. It is now clear that (A) is equivalent to:

(A′) *The set of μ with $M_\mu \neq 0$ is finite, and each M_μ is finitely generated over A.*

We also require that each homogeneous part U_ν of U shifts the grading by ν:

(C) $$U_\nu M_\mu \subset M_{\mu+\nu} \qquad \text{for all } \mu, \nu \in X(T).$$

The algebra U^0 acts in two ways on M. One action arises from the embedding of U^0 into U, the other one from the homomorphism $\pi : U^0 \to A$. Both actions preserve the grading. We require finally that they are related by

(D) $$sm = m\pi(\widetilde{\mu}(s)) \qquad \text{for all } \mu \in X(T), m \in M_\mu, s \in U^0.$$

A morphism between two objects in $\mathcal{C}_A$ is a homomorphism of $U \otimes A$–modules that preserves the gradings. There are obvious notions of submodules and quotient modules, we have kernels, images, and cokernels for homomorphisms. (In order to get kernels we assumed A to be Noetherian.)

By abuse of notation, we shall usually call an object M of $\mathcal{C}_A$ simply a *module* in $\mathcal{C}_A$ and the M_λ its *weight spaces*. If each M_μ is free over A, the formal character $\mathrm{ch}(M)$ of M has the obvious definition.

Replacing U in the definition of $\mathcal{C}_A$ by U^0U^+ resp. by U^0 we get categories that we shall denote by $\mathcal{C}'_A$ resp. $\mathcal{C}''_A$.

2.4. Suppose that we are in Case 1. The left action of an H_α with $\alpha \in R^+$ on an M_λ (with M in $\mathcal{C}_A$ and $\lambda \in X$) is equal to the multiplication on the right by $\lambda(H_\alpha) + \pi(H_\alpha)$. So H_α^p (the ordinary p–th power in $U(\mathfrak{g})$ and U^0) acts on the left as right multiplication by $\lambda(H_\alpha)^p + \pi(H_\alpha)^p = \lambda(H_\alpha) + \pi(H_\alpha)^p$. (Note that $\lambda(H_\alpha)$ is the reduction modulo p of the integer $\langle \lambda, \alpha^\vee \rangle$, hence in $\mathbf{F}_p$.) We have $H_\alpha^{[p]} = H_\alpha$ in $\mathfrak{g}$, so the central element $H_\alpha^p - H_\alpha^{[p]}$ of $U(\mathfrak{g})$ (and of U^0) acts on M on the left as right multiplication by $\pi(H_\alpha)^p - \pi(H_\alpha)$.

Suppose that A is a field. There is a linear form f on $\mathfrak{g}$ with values in some purely inseparable extension of A such that $f(E_\alpha) = 0$ for all $\alpha \in R$ and $f(H_\beta)^p = \pi(H_\beta)^p - \pi(H_\beta)$ for all $\beta \in \Sigma$. Then each $X^p - X^{[p]} \in U(\mathfrak{g})$ with $X \in \mathfrak{g}$ acts on M in $\mathcal{C}_A$ on the left as right multiplication by $f(X)^p$. A module in $\mathcal{C}_A$ is therefore the same thing as a finite dimensional $U(\mathfrak{g} \otimes A)$–module together with an X–grading such that each $X^p - X^{[p]}$ acts as multiplication by $f(X)^p$ and such that the compatibility conditions (C) and (D) in 2.3 hold.

As an example, consider $A = k$ where we regard k as a U^0–algebra via the augmentation map (sending each $H \in \mathfrak{h}$ to 0). Then the linear map f as above is just 0. The condition that each $X^p - X^{[p]}$ act as 0 on M in $\mathcal{C}_k$ means that M is a restricted $\mathfrak{g}$–module (or a G_1–module). Furthermore an X–grading of a k–module is the same as a structure as a T–module. (The homogeneous parts of the grading correspond to the weight spaces.) So a module in $\mathcal{C}_k$ is a G_1–module that is also a T–module such that a certain compatibility condition holds. This condition amounts just to the condition that M is a G_1T–module. So we can identify $\mathcal{C}_k$ with the category of all finite dimensional G_1T–modules.

Similarly, one identifies $\mathcal{C}'_k$ resp. $\mathcal{C}''_k$ with the categories of all finite dimensional B_1^+T–modules resp. T–modules. (Here B^+ is the Borel subgroup containing T corresponding to the positive roots.)

Consider Case 2 with $A = k$ and $\pi : U^0 \to k$ given by $\pi(K_\alpha) = 1$ for all $\alpha \in \Sigma$. Then each K_α acts on any M_μ on the left as right multiplication by $\zeta^{d_\alpha \langle \lambda, \alpha^\vee \rangle}$, so $K_\alpha^p - 1$ annihilates M. So M can be regarded as a module of type $\mathbf{1}$ (in the sense of [Lu4], 4.6) for the homomorphic image $\mathbf{u}$ of U. We can extend it to a $\widetilde{\mathbf{u}}$–module by letting U_3^0 act on each M_μ via the character denoted by χ_μ in [APW2], 0.3. Thus M is made into an object of the category $\mathcal{C}_{\widetilde{\mathbf{u}}}$ (over

k) introduced in [APW2], 0.4. One can check that this is an equivalence of categories.

In this case $\mathcal{C}'_k$ and $\mathcal{C}''_k$ correspond to similar categories involving $U_3^0\mathbf{u}^+$ and U_3^0 instead of $\widetilde{\mathbf{u}}$.

2.5. Lemma: *The forgetful functor induces (for an arbitrary A) an equivalence of categories between $\mathcal{C}''_A$ and the category of all X-graded A-modules that are finitely generated over A.*

Proof: On one side, condition 2.3(D) says that the action of U^0 on an M in $\mathcal{C}''_A$ is determined by the grading and the A-module structure: Any $s \in U^0$ has to act on any M_μ as the element $\pi(\widetilde{\mu}(s)) \in A$. On the other hand, we can use exactly this formula to define for each X-graded A-module M an action of U^0 first on each M_μ and then on M. We get obviously an object in $\mathcal{C}''_A$.

Remark: This discussion shows especially that the category $\mathcal{C}''_A$ contains enough projectives: An M in $\mathcal{C}''_A$ is projective, if and only if all M_μ are projective as A-modules, that is, if and only if M is projective as an A-module.

2.6. We have obvious forgetful functors from $\mathcal{C}_A$ and $\mathcal{C}'_A$ to $\mathcal{C}''_A$, and from $\mathcal{C}_A$ to $\mathcal{C}'_A$. We shall construct (left) adjoint induction functors using tensor products. Consider first an object M in $\mathcal{C}''_A$. We want to make

$$\Phi'_A(M) = U^0U^+ \otimes_{U^0} M \tag{1}$$

into an object in $\mathcal{C}'_A$. We let any $u \in U^0U^+$ act via left multiplication on the first factor, and any $a \in A$ via the given action of a on the second factor. (This makes sense because both maps are U^0-linear, where we regard U^0U^+ as a U^0-module via right multiplication.) These actions of U^0U^+ and A commute obviously. The grading is defined via

$$\left(U^0U^+ \otimes_{U^0} M\right)_\mu = \bigoplus_{\nu\in X} (U^0U^+)_\nu \otimes_{U^0} M_{\mu-\nu}. \tag{2}$$

This makes sense because all $(U^0U^+)_\nu$ and all $M_{\nu'}$ are U^0-submodules. It is now easy to check that the conditions in 2.3 are satisfied, or rather their analoga for $\mathcal{C}'_A$. In the case of 2.3(D) observe that one has for $s \in U^0$ and $m \in M_\mu$ and $u \in (U^0U^+)_\nu$:

$$\begin{aligned} s(u\otimes m) &= (su)\otimes m = (u\widetilde{\nu}(s))\otimes m = u\otimes(\widetilde{\nu}(s)m) \\ &= u\otimes m\widetilde{\mu}(\widetilde{\nu}(s)) = (u\otimes m)(\widetilde{\mu}\circ\widetilde{\nu}(s)) = (u\otimes m)(\widetilde{\mu+\nu})(s). \end{aligned}$$

Considered as an A-module $\Phi'_A(M)$ is a finite direct sum of copies of M, because U^0U^+ is free of finite rank over U^0. This yields the required finiteness condition. It is obvious how to define Φ'_A on morphisms.

Remark: We can carry over the construction and the arguments above more or less verbatim (replacing U^0U^+ by U) to get a functor $\Phi_A : \mathcal{C}''_A \to \mathcal{C}_A$ with

$$\Phi_A(M) = U \otimes_{U^0} M. \tag{3}$$

2.7. Lemma: *a) The functor Φ_A resp. Φ'_A is exact and is left adjoint to the forgetful functor $\mathcal{C}_A \to \mathcal{C}''_A$ resp. $\mathcal{C}'_A \to \mathcal{C}''_A$.*

b) The functors Φ_A and Φ'_A map projective objects to projective objects.

c) The categories $\mathcal{C}_A$ and $\mathcal{C}'_A$ have enough projective objects. Any projective object in these categories is projective as an A–module.

Proof: a) The exactness follows from the freeness of U resp. of U^0U^+ over U^0, cf. 2.2. The proof of the adjointness is similar in both cases. We give the details only for Φ'_A. Let M be in $\mathcal{C}''_A$. The map

$$f_0 : M \to \Phi'_A(M), \qquad m \mapsto 1 \otimes m$$

is a morphism in $\mathcal{C}''_A$. It induces for any N in $\mathcal{C}'_A$ an isomorphism (via $f \mapsto f \circ f_0$):

$$\mathrm{Hom}_{\mathcal{C}'_A}(\Phi'_A(M), N) \simeq \mathrm{Hom}_{\mathcal{C}''_A}(M, N). \tag{1}$$

The inverse map associates to any $g : M \to N$ the map $\hat{g}$ with $\hat{g}(u \otimes m) = u\, g(m)$ for all u, m. The only difference from the standard situation is that we have to take the grading into account.

b) This claim is obvious because each functor is left adjoint to an exact functor.

c) For any N in $\mathcal{C}_A$ choose a projective P in $\mathcal{C}''_A$ with an epimorphism $P \to N$, cf. 2.5. We get by adjunction a map $\Phi_A(P) \to N$ that is again surjective. This shows that $\mathcal{C}_A$ has enough projectives. If N is projective in $\mathcal{C}_A$, this surjection has to split. This shows that the projective objects in $\mathcal{C}_A$ are exactly the direct summands of the $\Phi_A(P)$ with P projective in $\mathcal{C}''_A$. These P are projective as A–modules. Furthermore $\Phi_A(P)$ is a direct sum of copies of P when regarded as an A–module. So any direct summand of $\Phi_A(P)$ is projective over A. The argument for $\mathcal{C}'_A$ is analogous.

Remark: It will be convenient at one point to work with an enlarged version of $\mathcal{C}_A$ and to replace 2.3(A) by the weaker condition:

(F) *The set of $\mu \in X$ with $M_\mu \neq 0$ is finite.*

Let us denote this enlarged category by $\mathcal{GC}_A$. If M is a module in $\mathcal{GC}_A$, then there is for all $x \in M$ a submodule M' of M with $x \in M'$ and M' in $\mathcal{C}_A$. Indeed, if $x = \sum_\mu x_\mu$ with $x_\mu \in M_\mu$, then $M' = \sum_\mu U x_\mu$ works. This property implies easily: *If Q is a projective object in $\mathcal{C}_A$, then Q is projective also in $\mathcal{GC}_A$.*

Similar remarks apply to $\mathcal{C}'_A$ and $\mathcal{C}''_A$.

2.8. We usually call projective objects in $\mathcal{C}_A$ projective modules in $\mathcal{C}_A$. Having enough projective objects in our categories we can use projective resolutions to compute Ext–groups.

Lemma: *Each* $\mathrm{Ext}^i_{\mathcal{C}_A}(M,N)$ *with* M *and* N *in* $\mathcal{C}_A$ *and* $i \geq 0$ *is a finitely generated* A*–module.*

Proof: Since A is Noetherian and since each Ext group is a subquotient of a suitable Hom group, it is enough to consider that case. The Hom group in $\mathcal{C}_A$ is a submodule of the Hom group between A–modules, hence is Noetherian since both modules and A are Noetherian.

Remarks: 1) Similar results hold in $\mathcal{C}'_A$ and $\mathcal{C}''_A$.

2) The functors Φ_A and Φ'_A send projective resolutions to projective resolutions. We get therefore formulas similar to the Shapiro lemma. For example, one has for all M in $\mathcal{C}''_A$ and all N in $\mathcal{C}_A$ isomorphisms

$$\mathrm{Ext}^i_{\mathcal{C}_A}(\Phi_A(M),N) \simeq \mathrm{Ext}^i_{\mathcal{C}''_A}(M,N) \tag{1}$$

for all i. There is a similar formula involving Φ'_A.

2.9. Proposition: *Let* M *be a module in* $\mathcal{C}'_A$ *that is projective as an* A*–module. Then there exists a projective resolution* $P_\bullet$ *of* M *in* $\mathcal{C}'_A$ *such that there is for each* N *in* $\mathcal{C}'_A$ *an integer* $r = r(N) \geq 0$ *with*

$$\mathrm{Hom}_{\mathcal{C}'_A}(P_i,N) = 0 \qquad \textit{for all } i > r. \tag{1}$$

Proof: Fix a group homomorphism $h : X \to \mathbf{Z}$ with $h(\alpha) > 0$ for all positive roots α. For any X–graded module L set $\min(L)$ resp. $\max(L)$ equal to the minimum resp. maximum of all $h(\mu)$ with $L_\mu \neq 0$.

Consider the natural map

$$f_1 : \Phi'_A(M) = U^0U^+ \otimes_{U^0} M \to M$$

with $f_1(u \otimes m) = um$. Denote the kernel of f_1 by M_1. The embedding f_0 as in 2.7 splits f_1. This shows especially that M_1 is a projective A–module. Regarded as a k–module $\Phi'_A(M)$ is the direct sum of all $(U^+)_\nu \otimes M_\mu$. We have $(U^+)_0 = k$ and $h(\nu) > 0$ for all other ν with $(U^+)_\nu \neq 0$, by 1.1. This implies easily

$$\min(M_1) > \min(\Phi'_A(M)) = \min(M). \tag{2}$$

We can construct a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to M$$

of M in $\mathcal{C}'_A$ as follows: Take $P_0 = \Phi'_A(M)$ with the map f_1 onto M. This works because M is projective in $\mathcal{C}''_A$ and because Φ'_A preserves projectives. Because M_1 is again projective over A, we can now take $P_1 = \Phi'_A(M_1)$ with the natural map onto $M_1 \subset P_0$. Now iterate. Formula (2) yields

$$\min(M) = \min(P_0) < \min(P_1) < \min(P_2) < \cdots. \tag{3}$$

For any N in $\mathcal{C}'_A$ there exists an index r with $\min(P_i) > \max(N)$ for all $i > r$. Then (1) holds obviously.

Remark: The proposition implies of course that

$$\mathrm{Ext}^i_{\mathcal{C}'_A}(M,N) = 0 \qquad \text{for all } i > r \tag{4}$$

with N and r as in (1).

2.10. We want to introduce an induction functor $Z_A : \mathcal{C}'_A \to \mathcal{C}_A$ with

$$Z_A(M) = U \otimes_{U^0U^+} M \tag{1}$$

for all M in $\mathcal{C}'_A$. Most of the arguments in 2.6 generalize to this construction. Only the definition of the grading has to be modified, because the homogeneous parts in U and M are not U^0U^+-stable. Multiplication induces an isomorphism of k-modules

$$U^- \otimes U^0U^+ \xrightarrow{\sim} U. \tag{2}$$

So we get an isomorphism as k-modules (in fact: as $U^- \otimes A$-modules)

$$Z_A(M) \simeq U^- \otimes M. \tag{3}$$

We get now an X-grading of $Z_A(M)$ by A-submodules via

$$Z_A(M)_\mu = \bigoplus_{\nu\in X} (U^-)_\nu \otimes M_{\mu-\nu}. \tag{4}$$

The compatibility of the U^0U^+-action on M with the grading implies that the canonical map $U \otimes M \to U \otimes_{U^0U^+} M$ induces a surjection

$$\bigoplus_{\nu\in X} U_\nu \otimes M_{\mu-\nu} \longrightarrow Z_A(M)_\mu.$$

This implies easily that the U-action on $Z_A(M)$ is compatible with the grading.

It is now easy to check that Z_A is left adjoint to the forgetful functor $\mathcal{C}_A \to \mathcal{C}'_A$, that it is exact and maps projectives to projectives. Furthermore, it is clear that we can identify $Z_A \circ \Phi'_A$ with Φ_A. One has (as in 2.8(1)) for all M in $\mathcal{C}'_A$ and all N in $\mathcal{C}_A$ isomorphisms

$$\mathrm{Ext}^i_{\mathcal{C}_A}(Z_A(M),N) \simeq \mathrm{Ext}^i_{\mathcal{C}'_A}(M,N) \tag{5}$$

for all i.

In the situation considered in 2.4 (Case 1 with $A = k$) the functors Φ_k, Φ'_k, and Z_k are just the usual coinduction functors between the categories of T-, B_1^+T-, and G_1T-modules, cf. [Ja6], I.8.20, II.9.1.

2.11. Because the multiplication map is an isomorphism $U^0 \otimes U^+ \to U^0U^+$ the conditions 1.1(2)–(4) imply that $(U^0U^+)_0 = U^0$ and that

$$U^0U^+ = U^0 \oplus I^+, \qquad \text{where} \qquad I^+ = \bigoplus_{\nu>0}(U^0U^+)_\nu. \tag{1}$$

Obviously I^+ is a two–sided ideal in U^0U^+, and we get an isomorphism

$$f : U^0U^+/I^+ \xrightarrow{\sim} U^0 \tag{2}$$

induced by the projection from U^0U^+ onto U^0 corresponding to the decomposition (1).

We can use (2) to regard any M in $\mathcal{C}''_A$ as an object in $\mathcal{C}'_A$: We keep the grading and the A–module structure and get the action of U^0U^+ from that of U^0 via f. This action still commutes with that of A and it is still compatible with the grading because all $(U^0U^+)_\mu$ with $\mu \neq 0$ are contained in the kernel of f.

We can consider for all $\mu \in X(T)$ an object M of $\mathcal{C}''_A$ such that $M_\mu = A$ and $M_\nu = 0$ for all $\nu \neq \mu$, cf. 2.5. We denote this object by A^μ. Using the remarks above we can regard A^μ as an object in $\mathcal{C}'_A$ that is annihilated by $\bigoplus_{\nu>0}(U^+)_\nu$. We can then construct the induced module

$$Z_A(\mu) = Z_A(A^\mu). \tag{3}$$

In the situation of 2.4 (Case 1 with $A = k$) any k^μ is just the one dimensional B_1^+T–module where T acts via μ. So $Z_k(\mu)$ is just the module $\widehat{Z}_1(\mu)$ as in [Ja6], II.9.1(5). In Case 2 one gets for $A = k$ the modules $\overline{M}_\zeta(\lambda)$ from [DCK1], 3.2.

We can apply Φ_A and Φ'_A to A^μ regarded as a module in $\mathcal{C}''_A$ and shall use an analogous notation:

$$\Phi_A(\mu) = \Phi_A(A^\mu) \qquad \text{and} \qquad \Phi'_A(\mu) = \Phi'_A(A^\mu). \tag{4}$$

2.12. A Z–filtration of a module M in $\mathcal{C}_A$ is a chain

$$M_0 = 0 \subset M_1 \subset \ldots \subset M_r = M \tag{1}$$

of submodules such that each M_i/M_{i-1} is isomorphic to $Z_A(\lambda_i)$ for some $\lambda_i \in X$.

If M has a Z–filtration, then all M_μ are free over A. For M as in (1) the formal character $\operatorname{ch} M$ is the sum of the $\operatorname{ch} Z_A(\lambda_i)$. It is obvious that the $\operatorname{ch} Z_A(\lambda)$ are linearly independent (cf. also 4.7(1)). Therefore the number of factors isomorphic to $Z_A(\lambda)$ in a Z–filtration of M depends only on M, not on the choice of the special Z–filtration.

Lemma: *Let M be a module in $\mathcal{C}'_A$ resp. in $\mathcal{C}''_A$ such that each M_μ is free over A. Then $Z_A(M)$ resp. $\Phi_A(M)$ has a Z–filtration.*

Proof: In the case where M is in $\mathcal{C}'_A$, there is a filtration of M with factors of the form A^μ with $\mu \in X$. The exactness of Z_A transforms this filtration of M into a Z–filtration of $Z_A(M)$. For M in $\mathcal{C}''_A$ one observes that the weight spaces of $\Phi'_A(M)$ are free. By the part already proved we get a Z–filtration of $Z_A(\Phi'_A(M)) \simeq \Phi_A(M)$.

2.13. Lemma: *For any M in $\mathcal{C}_A$ there is a projective module Q in $\mathcal{C}_A$ such that Q has a Z-filtration and such that M is a homomorphic image of Q.*

Proof: We can proceed as in the proof of 2.7.c. We choose an epimorphism $P \to M$ where P is projective in $\mathcal{C}''_A$. We can assume that P is free over A. Then $Q = \Phi_A(P)$ has a Z-filtration by the last lemma and satisfies the claim of the lemma.

Remark: In some cases we can be more precise. For example, consider $M = Z_A(\lambda)$ with $\lambda \in X$. We have an obvious surjection $\Phi'_A(\lambda) \to A^\lambda$ in $\mathcal{C}'_A$. Therefore we can take $Q = Z_A(\Phi'_A(\lambda)) = \Phi_A(\lambda)$. By 1.1 and 2.6 each weight μ of $\Phi'_A(\lambda)$ satisfies $\mu \geq \lambda$, and $\Phi'_A(\lambda)_\lambda$ is free of rank 1. So any Z-filtration of Q involves only $Z_A(\mu)$ with $\mu \geq \lambda$, and $Z_A(\lambda)$ occurs exactly once (at the top).

2.14. Lemma: *a) If $\mathrm{Ext}^1_{\mathcal{C}_A}(Z_A(\lambda), M) \neq 0$ for some M in $\mathcal{C}_A$ and $\lambda \in X$, then M has a weight μ with $\mu > \lambda$.*

b) Let $\lambda, \mu \in X$. If $\mathrm{Ext}^1_{\mathcal{C}_A}(Z_A(\lambda), Z_A(\mu)) \neq 0$, then $\mu > \lambda$.

c) If a module M in $\mathcal{C}_A$ has a Z-filtration, then one can find a (possibly different) Z-filtration of M with the following property: Whenever $\lambda_i > \lambda_j$ (in the notations of 2.12(1)), then $i < j$.

Proof: a) Consider in $\mathcal{C}_A$ an exact sequence

$$0 \to M \to N \to Z_A(\lambda) \to 0. \tag{1}$$

Let $v \in N_\lambda$ be an inverse image of the standard generator $v_0 = 1 \otimes 1$ of $Z_A(\lambda)$. If $E_\alpha v = 0$ for all $\alpha \in R^+$, then there is a homomorphism from $Z_A(\lambda)$ to N that maps v_0 to v and then splits (1). So if (1) does not split, there is $\alpha > 0$ with $E_\alpha v \neq 0$. We get especially that $\lambda + \alpha$ is a weight of M, hence a).

Now b) is an obvious consequence of a), and c) is one of b).

2.15. Lemma: *Let M be a module in $\mathcal{C}_A$ with a Z-filtration. Then there exists a projective resolution $P_\bullet$ of M in $\mathcal{C}_A$ such that there is for each N in $\mathcal{C}_A$ an integer $r = r(N) \geq 0$ with*

$$\mathrm{Hom}_{\mathcal{C}_A}(P_i, N) = 0 \qquad \textit{for all } i > r. \tag{1}$$

Proof: Suppose first that $M = Z_A(\mu)$ for some $\mu \in X$. Then there is a projective resolution $P'_\bullet$ of A^μ in $\mathcal{C}'_A$ with the property required in Proposition 2.9. We apply the functor Z_A to $P'_\bullet$ and get according to 2.10 a projective resolution of M with the desired property.

For general M we use induction on the length of a Z-filtration. If M' is a submodule of M, such that M' and M/M' have projective resolutions as desired, we use the algebraic mapping cone to get a resolution of M with the desired property.

Remark: The lemma implies of course that

$$\mathrm{Ext}^i_{\mathcal{C}_A}(M, N) = 0 \qquad \text{for all } i > r \tag{2}$$

with N and r as in (1).

2.16. Lemma: *Suppose that any finitely generated projective A-module is free.*

a) Any direct summand of a module with a Z-filtration has a Z-filtration.

b) Any projective module in $\mathcal{C}_A$ has a Z-filtration.

Proof: a) One can use the standard arguments, e.g. from [BGG1].
b) If M is projective in $\mathcal{C}_A$, then any epimorphism $Q \to M$ as in Lemma 2.13 has to split. Now apply a).

Remark: The assumption of this lemma will certainly be satisfied, if A is a field or a local ring.

3. Projective Modules

As in the last sections, we shall always denote by A a (Noetherian and commutative) U^0–algebra with structural map $\pi : U^0 \to A$. Any A–algebra is assumed to be Noetherian and commutative.

3.1. Let A' be an A–algebra. If A' is finitely generated as an A–module, then there is an obvious (exact) restriction functor $\mathcal{C}_{A'} \to \mathcal{C}_A$: We take the same U–module with the same grading, and make an $a \in A$ act via its canonical image in A'. If A' is arbitrary, then this construction yields a functor $\mathcal{GC}_{A'} \to \mathcal{GC}_A$, where $\mathcal{GC}_A$ is the enlarged category considered in the remark to 2.7. We get a functor in the opposite direction, called *extension of scalars:* Map any M in $\mathcal{GC}_A$ to $M \otimes_A A'$ with the obvious $U \otimes A'$–action and the grading given by

$$\left(M \otimes_A A'\right)_\mu = M_\mu \otimes_A A'. \tag{1}$$

This functor maps $\mathcal{C}_A$ to $\mathcal{C}_{A'}$ for arbitrary A'. It is easy to check for any N in $\mathcal{GC}_{A'}$ that the standard isomorphism $\mathrm{Hom}_A(M,N) \xrightarrow{\sim} \mathrm{Hom}_{A'}(M \otimes_A A', N)$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{GC}_A}(M,N) \simeq \mathrm{Hom}_{\mathcal{GC}_{A'}}(M \otimes_A A', N). \tag{2}$$

In other words, we have constructed a functor left adjoint to the restriction functor. There are obvious similar constructions for $\mathcal{C}'_A$ and $\mathcal{C}''_A$. The associativity of the tensor product implies that one has for each M in $\mathcal{C}''_A$ a canonical isomorphism

$$\Phi_A(M) \otimes_A A' \simeq \Phi_{A'}(M \otimes_A A'). \tag{3}$$

There are similar statements for Φ' and Z instead of Φ. Let us mention explicitly that

$$Z_A(\mu) \otimes_A A' \simeq Z_{A'}(\mu) \tag{4}$$

for all $\mu \in X$. In many cases we shall use the simplified notation

$$M_{A'} = M \otimes_A A'. \tag{5}$$

Lemma: *Let M be a module in $\mathcal{C}_A$.*

a) If M is projective in $\mathcal{C}_A$, then $M \otimes_A A'$ is projective in $\mathcal{C}_{A'}$.

b) If M has a Z–filtration, then so has $M \otimes_A A'$.

Proof: The first claim follows from the adjointness property (2) together with the remark in 2.7. Since modules in $\mathcal{C}_A$ with a Z–filtration are free over A, short exact sequences involving only modules of that type split over A. Therefore (4) implies b).

3.2. Lemma: *Let A' be a flat A-algebra. We have for all M, N in $\mathcal{C}_A$ and all $i \geq 0$ a canonical isomorphism*

$$\operatorname{Ext}^i_{\mathcal{C}_A}(M,N) \otimes_A A' \xrightarrow{\sim} \operatorname{Ext}^i_{\mathcal{C}_{A'}}(M_{A'}, N_{A'}). \tag{1}$$

Proof: First of all, we have an isomorphism

$$\operatorname{Hom}_A(M,N) \otimes_A A' \xrightarrow{\sim} \operatorname{Hom}_{A'}(M_{A'}, N_{A'}),$$

since M is finitely presented as an A–module. This isomorphism is compatible with the additional structure and induces an isomorphism

$$\operatorname{Hom}_{\mathcal{C}_A}(M,N) \otimes_A A' \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}_{A'}}(M_{A'}, N_{A'}). \tag{2}$$

Because tensoring with A' is exact, it takes projective resolutions to projective resolutions, and we get the more general claim for all $i \geq 0$.

Remark: Similar results hold in $\mathcal{C}'_A$ and $\mathcal{C}''_A$.

3.3. Proposition: *Let M be a projective module in $\mathcal{C}_A$ and let N be any module in $\mathcal{C}_A$. Then $\operatorname{Hom}_{\mathcal{C}_A}(M,N)$ is a finitely generated A–module and one has for any A–algebra A':*

$$\operatorname{Hom}_{\mathcal{C}_A}(M,N) \otimes_A A' \simeq \operatorname{Hom}_{\mathcal{C}_{A'}}(M_{A'}, N_{A'}). \tag{1}$$

If N is projective as an A–module, then so is $\operatorname{Hom}_{\mathcal{C}_A}(M,N)$.

Proof: We know (cf. 2.7) that there is a projective M' in $\mathcal{C}''_A$ such that M is a direct summand of $\Phi_A(M')$. It is obviously enough to consider the case $M = \Phi_A(M')$. By the adjointness property of Φ_A we get isomorphisms

$$\operatorname{Hom}_{\mathcal{C}_A}(M,N) \simeq \operatorname{Hom}_{\mathcal{C}''_A}(M',N) \simeq \bigoplus_{\mu \in X} \operatorname{Hom}_A(M'_\mu, N_\mu). \tag{2}$$

All M'_μ and N_μ are finitely generated, almost all of them are 0. This implies the first claim.

If N is projective over A, then so are all N_μ. Since the M'_μ are projective over A anyhow, this proves our last claim.

Using 3.1(3) and again adjointness one sees that the right hand side in (1) can be identified with

$$\operatorname{Hom}_{\mathcal{C}''_{A'}}(M' \otimes_A A', N \otimes_A A').$$

This module can be decomposed as in (2). In this way we can reduce (1) to the corresponding base change property for finitely generated projective A–modules M' which is obvious.

Remark: Similar results hold (with the same proof) in $\mathcal{C}'_A$ and (of course, by our proof) in $\mathcal{C}''_A$.

3.4. Proposition: *Let M be a module in $\mathcal{C}_A$ with a Z-filtration, and let N be a module in $\mathcal{C}_A$ that is projective over A. Suppose that*

$$\mathrm{Ext}^i_{\mathcal{C}_{A/\mathfrak{m}}}(M_{A/\mathfrak{m}}, N_{A/\mathfrak{m}}) = 0 \tag{1}$$

for all maximal ideals $\mathfrak{m}$ of A and all $i > 0$. Then the A-module $\mathrm{Hom}_{\mathcal{C}_A}(M, N)$ *is projective, and we have for all A-algebras A'*

$$\mathrm{Ext}^i_{\mathcal{C}_{A'}}(M_{A'}, N_{A'}) = 0 \qquad \textit{for all } i > 0 \tag{2}$$

and

$$\mathrm{Hom}_{\mathcal{C}_A}(M, N) \otimes_A A' \simeq \mathrm{Hom}_{\mathcal{C}_{A'}}(M_{A'}, N_{A'}). \tag{3}$$

Proof: We take a projective resolution $P_\bullet$ of M in $\mathcal{C}_A$ as in Lemma 2.15 and we set $Q_i = \mathrm{Hom}_{\mathcal{C}_A}(P_i, N)$. There is an integer r with $Q_i = 0$ for all $i > r$. For each A-algebra A' the complex $P_\bullet \otimes_A A'$ is a projective resolution of $M_{A'}$. (The $P_i \otimes_A A'$ are projective by 3.1.a. The exactness follows from the fact that the resolution $P_\bullet$ splits over A because M and all P_i are projective A-modules.) Furthermore, Proposition 3.3 implies that we can identify $(Q_i)_{A'} = Q_i \otimes_A A'$ with $\mathrm{Hom}_{\mathcal{C}_{A'}}(P_i \otimes_A A', N_{A'})$. Therefore the complex

$$\mathfrak{C}_{A'} :\ 0 \to \mathrm{Hom}_{\mathcal{C}_{A'}}(M_{A'}, N_{A'}) \to (Q_0)_{A'} \to (Q_1)_{A'} \to \cdots \to (Q_r)_{A'} \to 0$$

is exact at the Hom term and at the Q_0 term and has higher cohomology equal to $\mathrm{Ext}^i_{\mathcal{C}_{A'}}(M_{A'}, N_{A'})$. Our assumption (1) says that $\mathfrak{C}_{A/\mathfrak{m}}$ is exact for all maximal ideals $\mathfrak{m}$ of A. We claim that the complex $\mathfrak{C}_A$ is split exact. This will imply the proposition.

To prove this claim we first observe that by Proposition 3.3 the Q_i are all finitely generated projective A-modules. Assume for the moment that A is a local ring with maximal ideal $\mathfrak{m}$. The exactness of $\mathfrak{C}_{A/\mathfrak{m}}$ implies easily that of $\mathfrak{C}_A$ using the Nakayama lemma and induction on r.

For general A denote for each maximal ideal $\mathfrak{m}$ of A the local ring at $\mathfrak{m}$ by $A_\mathfrak{m}$. We can apply the proof above to each $A_\mathfrak{m}$ since $A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m} \simeq A/\mathfrak{m}$. The exactness of $\mathfrak{C}_{A_\mathfrak{m}}$ for all $\mathfrak{m}$ implies the exactness of $\mathfrak{C}_A$. Then the splitting follows since all Q_i are projective.

Remark: Let M, N be modules in $\mathcal{C}_A$ that are projective over A. Let A' be an A-algebra with a finite projective resolution as an A-module. In this situation one can show that there is a spectral sequence with

$$E_2^{i,-j} = \mathrm{Tor}_j^A(\mathrm{Ext}^i_{\mathcal{C}_A}(M, N), A') \Rightarrow \mathrm{Ext}^{i-j}_{\mathcal{C}_{A'}}(M_{A'}, N_{A'}).$$

If A is a regular local ring with maximal ideal $\mathfrak{m}$, this can be applied to $A = A/\mathfrak{m}$. This yields a different approach to Proposition 3.4 for regular rings.

3.5. Corollary: *Let M be a module in $\mathcal{C}_A$ with a Z-filtration. Then M is projective in $\mathcal{C}_A$, if and only if $M_{A/\mathfrak{m}}$ is projective in $\mathcal{C}_{A/\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of A.*

Proof: If M is projective, then each $M_{A/\mathfrak{m}}$ is projective by Lemma 3.1.a. Suppose now that each $M_{A/\mathfrak{m}}$ is projective. There is (by 2.13) an exact sequence

$$0 \to N \to Q \to M \to 0 \tag{1}$$

in $\mathcal{C}_A$ where Q is projective and has a Z-filtration. Since M is free over A, the sequence splits over A. Therefore N is projective over A. Since each $M_{A/\mathfrak{m}}$ is projective, the assumptions of Proposition 3.4 are satisfied. We get now from 3.4(2) especially that $\mathrm{Ext}^1_{\mathcal{C}_A}(M, N) = 0$, hence that (1) splits and M is projective as a direct summand of the projective module Q.

3.6. Fix $\nu \in X$. Denote by $\mathcal{C}_A(\leq \nu)$ the full subcategory of all M in $\mathcal{C}_A$ such that $M_\lambda \neq 0$ implies $\lambda \leq \nu$. This subcategory is closed under taking subquotients and extensions. All $Z_A(\lambda)$ with $\lambda \leq \nu$ belong to $\mathcal{C}_A(\leq \nu)$.

Let M be a module in $\mathcal{C}_A$. Suppose that $(M_i)_{i \in I}$ is a family of submodules of M such that each M/M_i is in $\mathcal{C}_A(\leq \nu)$. Then $M/\bigcap_i M_i$ is in $\mathcal{C}_A(\leq \nu)$. Indeed, if $\mu \in X$ satisfies $\mu \not\leq \nu$, then one has $M_\mu = (M_i)_\mu$ for all i, hence $M_\mu = (\bigcap_i M_i)_\mu$. Denote the intersection of all submodules M' of M with M/M' in $\mathcal{C}_A(\leq \nu)$ by $O^\nu M$, and set $\Gamma^\nu M = M/O^\nu M$. The argument above implies that $\Gamma^\nu M$ is in $\mathcal{C}_A(\leq \nu)$. If N is in $\mathcal{C}_A(\leq \nu)$ and if $f : M \to N$ is a homomorphism, then $M/\ker(f)$ is in $\mathcal{C}_A(\leq \nu)$, so $O^\nu M$ is contained in $\ker(f)$. This shows that the natural map $M \to \Gamma^\nu M$ induces for all N in $\mathcal{C}_A(\leq \nu)$ an isomorphism

$$\mathrm{Hom}_{\mathcal{C}_A}(\Gamma^\nu M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_A}(M, N). \tag{1}$$

It is now clear that $\Gamma^\nu M$ is functorial in M and that the functor Γ^ν is left adjoint to the inclusion. (This construction is of course in some sense dual to Donkin's O_π, cf. [Don], 12.1.6.)

3.7. Lemma: *Let M be a module in $\mathcal{C}_A$.*

a) If M is projective in $\mathcal{C}_A$, then $\Gamma^\nu M$ is projective in $\mathcal{C}_A(\leq \nu)$.

b) If M has a Z-filtration, then $O^\nu M$ and $\Gamma^\nu M$ have a Z-filtration and one has for any A-algebra A':

$$\Gamma^\nu(M) \otimes_A A' \simeq \Gamma^\nu(M \otimes_A A'). \tag{1}$$

Proof: The claim in a) is an immediate consequence of 3.6(1), since the inclusion of $\mathcal{C}_A(\leq \nu)$ into $\mathcal{C}_A$ is an exact functor.

Assume now that M has a Z-filtration. Lemma 2.14.a implies that we can find a Z-filtration of M as in 2.12(1) such that there is an index j with $\lambda_i \not\leq \nu$ for all $i \leq j$ and with $\lambda_i \leq \nu$ for all $i > j$. We claim that $O^\nu M = M_j$.

By construction M/M_j is in $\mathcal{C}_A(\leq \nu)$, so we have $O^\nu M \subset M_j$. On the other hand, one has for all N in $\mathcal{C}_A(\leq \nu)$ and all $\lambda \in X$:

$$\text{If } \lambda \not\leq \nu, \text{ then } \operatorname{Hom}_{\mathcal{C}_A}(Z_A(\lambda), N) = 0. \tag{2}$$

Indeed, a generator of weight λ in $Z_A(\lambda)$ has to be mapped to $N_\lambda = 0$. Now (2) implies $\operatorname{Hom}_{\mathcal{C}_A}(M_j, M/O^\nu M) = 0$, hence $M_j = O^\nu M$. Now the claim in b) about Z–filtrations is clear. The formula (1) follows from this construction and 3.1(4).

3.8. Lemma: *For any M in $\mathcal{C}_A(\leq \nu)$ there exists a projective module Q in $\mathcal{C}_A(\leq \nu)$ such that Q has a Z–filtration and such that M is a homomorphic image of Q.*

Proof: We can regard M as a module in $\mathcal{C}_A$ and get a projective module P in $\mathcal{C}_A$ with analogous properties from Lemma 2.13. Then $Q = \Gamma^\nu P$ satisfies our claim by Lemma 3.7.

Remark: The proof shows that the projective modules in $\mathcal{C}_A(\leq \nu)$ are the direct summands of all $\Gamma^\nu P$ with P projective in $\mathcal{C}_A$.

3.9. We define analogous categories $\mathcal{C}'_A(\leq \nu)$. For any M in $\mathcal{C}'_A$ the direct sum of all M_μ with $\mu \not\leq \nu$ is a submodule $O'^\nu M$ of M. (Use 1.1(4) and 2.3(C)!) Set $\Gamma'^\nu M = M/O'^\nu M$. This is a module in $\mathcal{C}'_A(\leq \nu)$ and it is clear that the natural map from M to $\Gamma'^\nu M$ induces for all N in $\mathcal{C}'_A(\leq \nu)$ an isomorphism

$$\operatorname{Hom}_{\mathcal{C}'_A}(\Gamma'^\nu M, N) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}'_A}(M, N). \tag{1}$$

This shows that Γ'^ν is an exact functor from $\mathcal{C}'_A$ to $\mathcal{C}'_A(\leq \nu)$ left adjoint to the inclusion. It takes projective resolutions to projective resolutions. This implies for all M, N in $\mathcal{C}'_A(\leq \nu)$ and all $i \geq 0$:

$$\operatorname{Ext}^i_{\mathcal{C}'_A(\leq\nu)}(M, N) \simeq \operatorname{Ext}^i_{\mathcal{C}'_A}(M, N). \tag{2}$$

Formulas 2.10(4) and 1.1(5) show that Z_A maps $\mathcal{C}'_A(\leq \nu)$ to $\mathcal{C}_A(\leq \nu)$.

3.10. Lemma 3.1.a and the results in the subsections 3.2 – 3.5 can be generalized to the categories $\mathcal{C}_A(\leq \nu)$.

For example, since these are full subcategories of $\mathcal{C}_A$, it is clear that we have still the adjointness property leading to 3.1.a and that 3.2(1) holds in $\mathcal{C}_A(\leq \nu)$ for $i = 0$, i.e., $\operatorname{Ext}^0 = \operatorname{Hom}$. It follows then for arbitrary i as in the proof of Lemma 3.2.

We can extend 3.3(1) to $\mathcal{C}_A(\leq \nu)$ as follows: If M is of the form $\Gamma^\nu Q$ with Q projective in $\mathcal{C}_A$, then the claim follows from 3.8(1) and 3.7(1). In general M is a direct summand of a module of this type.

In order to get the analogues of 3.4 and 3.5 we have to extend some results from Section 2. That is trivial for Lemma 2.8. If we take a projective resolution $P_\bullet$ of M as in Proposition 2.9 and apply Γ'^ν to it, then we get a projective resolution $P'_\bullet$ of M in $\mathcal{C}'_A(\leq \nu)$ satisfying the analogue of 2.9 for $\mathcal{C}'_A(\leq \nu)$. Finally, 2.10(5) and Lemma 2.15 generalize since Z_A maps $\mathcal{C}'_A(\leq \nu)$ to $\mathcal{C}_A(\leq \nu)$.

4. Lifting Projectives

We assume from now on that we are in Case 1 or 2 from 1.2/3. Certain subsections work still in a larger generality.

4.1. For any k–algebra A there is a natural X–grading on $U^- \otimes A$ such that $(U^- \otimes A)_\nu = (U^-)_\nu \otimes A$ for all $\nu \in X$. The properties 1.1(3),(5) and 2.1(A) imply that

$$(U^- \otimes A)_{<0} = \bigoplus_{\nu<0} (U^- \otimes A)_\nu \tag{1}$$

is a two–sided ideal in $U^- \otimes A$ consisting of nilpotent elements such that

$$(U^- \otimes A)/(U^- \otimes A)_{<0} \simeq A. \tag{2}$$

If $A = F$ is a field, this implies that $(U^- \otimes F)_{<0}$ is the radical of the algebra $U^- \otimes F$.

For each U^0–algebra A any $Z_A(\mu)$ with $\mu \in X$ is isomorphic to $U^- \otimes A$ as a $U^- \otimes A$–module. Under this isomorphism $(U^- \otimes A)_{<0}$ is mapped to $\bigoplus_{\nu<\mu} Z_A(\mu)_\nu$. In the case where $A = F$ is a field, this subspace of codimension 1 has to contain every proper $U^- \otimes F$–submodule, hence every proper $U \otimes F$–submodule of $Z_F(\mu)$. So the sum of all $U \otimes F$–submodules contained in $\bigoplus_{\nu<\mu} Z_F(\mu)_\nu$ is the radical of $Z_F(\mu)$ and the corresponding factor module (the head of $Z_F(\mu)$)

$$L_F(\mu) = Z_F(\mu)/\mathrm{rad} Z_F(\mu) \tag{3}$$

is a simple $U \otimes F$–module, called the simple $U \otimes F$–module with highest weight μ. Now $\mathrm{rad} Z_F(\mu)$ is the direct sum of its weight spaces, hence a module in $\mathcal{C}_F$. (Observe: If M' is a $U \otimes F$–submodule, then the direct sum of the projections of M' on all weight spaces is a submodule in $\mathcal{C}_F$.) Therefore also $L_F(\mu)$ is a module in $\mathcal{C}_F$. One can characterize $L_F(\mu)$ as the unique (up to isomorphism) simple module M in $\mathcal{C}_F$ generated by a highest weight vector of weight μ, i.e., by some $v \in M_\mu$ with $(U^+)_\nu v = 0$ for all weights $\nu > 0$. Each simple module in $\mathcal{C}_F$ is isomorphic to exactly one $L_F(\mu)$.

4.2. Let $\lambda \in X$. We can twist the structural homomorphism $\pi : U^0 \to A$ (for any U^0–algebra A) by $\widetilde{(-\lambda)}$ and get a new homomorphism

$$\pi' = \pi \circ \widetilde{(-\lambda)} : U^0 \to A.$$

This yields a new algebra structure on A, denoted by $A[\lambda]$. For any M in $\mathcal{C}_A$ we construct $M[\lambda]$ in $\mathcal{C}_{A[\lambda]}$ as follows: Take the old M as a $U \otimes A$–module. Define the grading on $M[\lambda]$ by

$$M[\lambda]_\mu = M_{\mu-\lambda}$$

for all $\mu \in X$. It is obvious that $M[\lambda]$ satisfies 2.3(A)–(C). In order to see 2.3(D) observe that for all $s \in U^0$ and $m \in M[\lambda]_\mu$:

$$sm = m\pi(\widetilde{(\mu-\lambda)}(s)) = m(\pi \circ \widetilde{(-\lambda)} \circ \widetilde{\mu})(s) = m\pi'(\widetilde{\mu}(s)).$$

It is obvious that we get an equivalence of categories from $\mathcal{C}_A$ to $\mathcal{C}_{A[\lambda]}$. If M is free over A, then the construction implies

$$\mathrm{ch}(M[\lambda]) = \mathrm{ch}(M)e(\lambda). \tag{1}$$

One has for all $\mu \in X$:

$$Z_A(\mu)[\lambda] \simeq Z_{A[\lambda]}(\mu+\lambda) \tag{2}$$

and (if $A = F$ is a field)

$$L_F(\mu)[\lambda] \simeq L_{F[\lambda]}(\mu+\lambda). \tag{3}$$

We have $A[0] = A$ and $A[\lambda][\lambda'] = A[\lambda+\lambda']$ for all $\lambda, \lambda' \in X$ since $\mu \mapsto \widetilde{\mu}$ is a group homomorphism. We have then also

$$M[0] = M \qquad \text{and} \qquad M[\lambda][\lambda'] = M[\lambda+\lambda'] \qquad \text{for all } M \text{ in } \mathcal{C}_A. \tag{4}$$

The construction in 1.4 shows in both cases that $\widetilde{\lambda} = \mathrm{id}$ for all $\lambda \in pX$. We get thus $A[p\nu] = A$ for all $\nu \in X$; the formulas (2) and (3) take then the form (for all $\mu \in X$)

$$Z_A(\mu)[p\nu] \simeq Z_A(\mu+p\nu) \qquad \text{and} \qquad L_F(\mu)[p\nu] \simeq L_F(\mu+p\nu). \tag{5}$$

Furthermore (4) shows that $\mathcal{C}_A$ is a pX–category in the sense of Appendix E.3.

Consider the special case $A = k$ where k is regarded as a U^0–algebra via the augmentation map (as in 2.4). Each $M[p\nu]$ with M in $\mathcal{C}_k$ and $\nu \in X$ can be identified with the tensor product $M \otimes L_k(p\nu)$ in the category of G_1T–modules resp. in the quantum analogue. Here $L_k(p\nu)$ has dimension 1, and $\mathfrak{g}$ resp. $\mathbf{u}$ acts trivially on this module. The tensor product $M \otimes L_k(p\nu)$ is often denoted by $M \otimes p\nu$, cf. (in Case 1) [Ja6], Section II.9. We have for M, N in $\mathcal{C}_k$

$$\bigoplus_{\nu \in X} \mathrm{Hom}_{\mathcal{C}_k}(M \otimes p\nu, N) \simeq \begin{cases} \mathrm{Hom}_{\mathfrak{g}}(M,N), & \text{in Case 1,} \\ \mathrm{Hom}_{\mathbf{u}}(M,N), & \text{in Case 2.} \end{cases} \tag{6}$$

(Cf. [Ja4], 2.5(2), in Case 1; one argues similarly in Case 2.) Compare this to the definition of $\mathrm{Hom}^\sharp$ in the pX–category $\mathcal{C}_k$, i.e., with (cf. E.3(1))

$$\mathrm{Hom}^\sharp_{\mathcal{C}_k}(M,N) = \bigoplus_{\nu \in X} \mathrm{Hom}_{\mathcal{C}_k}(M[p\nu], N). \tag{7}$$

We get clearly

$$\mathrm{Hom}^\sharp_{\mathcal{C}_k}(M,N) \simeq \begin{cases} \mathrm{Hom}_{\mathfrak{g}}(M,N), & \text{in Case 1,} \\ \mathrm{Hom}_{\mathbf{u}}(M,N), & \text{in Case 2,} \end{cases} \tag{8}$$

as long as we consider $\mathcal{C}_k$ as a pX–category. (Of course, if Y is any subgroup of pX, then $\mathcal{C}_k$ is a Y–category by restriction. If we consider that structure, then $\mathrm{Hom}^\sharp$ gets a new meaning and (8) will be false in general.)

4.3. The subalgebras $T_w(U^-)$, $T_w(U^0) = U^0$, and $T_w(U^+)$ of U satisfy the conditions in 1.1 and 2.1, if we work with $w(R^+)$ instead of R^+. Therefore the whole theory developed so far can be carried out in this situation as well. The categories $\mathcal{C}_A$ and $\mathcal{C}''_A$ do not change, and the category $\mathcal{C}'_A$ is replaced by an equivalent one. We have induction functors similar to Φ'_A and Z_A. We denote the analogue to Z_A by Z^w_A and get especially modules

$$Z^w_A(\mu) = U \otimes_{U^0 T_w(U^+)} A^\mu \tag{1}$$

for all $\mu \in X$. Note that $Z^1_A(\mu)$ is our old $Z_A(\mu)$.

If $A = F$ is a field, then the discussion in 4.1 generalizes to all $Z^w_F(\mu)$ with $w \in W$. One works with $T_w(U^-)$ instead of U^-. So for all $w \in W$ and $\mu \in X$ the radical of $Z^w_F(\mu)$ as a $U \otimes F$–module is in $\mathcal{C}_F$ and

$$L^w_F(\mu) = Z^w_F(\mu)/\mathrm{rad} Z^w_F(\mu) \tag{2}$$

is a simple module in $\mathcal{C}_F$.

By 4.1, there is for all $w \in W$ and $\mu \in X$ a weight μ_w with

$$L^w_F(\mu) \simeq L_F(\mu_w). \tag{3}$$

The results in Section 5 (cf. 5.12) will show, how to determine μ_w in principle. But they will not yield a closed formula.

Suppose that A has the property that each finitely generated projective A–module is free. Working with $w(R^+)$ instead of R^+ we see now that Lemma 2.16 implies: Every projective module in $\mathcal{C}_A$ has a filtration with factors of the form $Z^w_A(\mu)$.

4.4. Let $w \in W$. We can twist any structural homomorphism $\pi : U^0 \to A$ with T_w and consider $\pi' = \pi \circ T_w^{-1} : U^0 \to A$ to get a new algebra structure on A that we denote by $A[w]$.

We can define for any M in $\mathcal{C}_A$ a module $M[w]$ in $\mathcal{C}_{A[w]}$ as follows: Take M with the old structure as an A–module, let any $u \in U$ act as $T_w^{-1}(u)$ on M, and set $M[w]_\mu = M_{w^{-1}(\mu)}$ for all $\mu \in X$. Then 2.3(A),(B) are obviously satisfied. It is easy to check (and left to the reader) that 2.3(C),(D) hold. It is clear that we get an equivalence of categories between $\mathcal{C}_A$ and $\mathcal{C}_{A[w]}$. It satisfies

$$\mathrm{ch}(M[w]) = w(\mathrm{ch}(M)) \tag{1}$$

for all M that are free over A.

One has in this case for all $x, w \in W$ and $\mu \in X$:

$$Z_A^x(\mu)[w] \simeq Z_{A[w]}^{wx}(w\mu). \tag{2}$$

Indeed, let v_1 resp. v_2 be the standard generators of $Z_A^x(\mu)$ resp. of $Z_{A[w]}^{wx}(w\mu)$. By construction, $\mu + x\beta$ is not a weight of $Z_A^x(\mu)$ for any $\beta \in R^+$, hence $w\mu + wx\beta$ not a weight of $Z_A^x(\mu)[w]$. We get $T_{wx}(E_\beta)v_1 = 0$ in $Z_A^x(\mu)[w]$ for all $\beta \in R^+$. The universal property of $Z_{A[w]}^{wx}(w\mu)$ yields a homomorphism $Z_{A[w]}^{wx}(w\mu) \to Z_A^x(\mu)[w]$ mapping v_2 to v_1. One gets similarly a homomorphism $Z_A^x(\mu) \to Z_{A[w]}^{wx}(w\mu)[w^{-1}]$ mapping v_1 to v_2. It can be regarded as a homomorphism $Z_A^x(\mu)[w] \to Z_{A[w]}^{wx}(w\mu)$. These two maps are then inverse isomorphisms.

If $A = F$ is a field, then (2) implies for all $\mu \in X$ and $x, w \in W$.

$$L_F^x(\mu)[w] \simeq L_{F[w]}^{wx}(w\mu). \tag{3}$$

4.5. Recall the antiautomorphism τ from 1.6. For any $U \otimes A$–module M we get a new $U \otimes A$–module M^τ, the *contravariant dual* of M, by setting

$$M^\tau = \mathrm{Hom}_A(M, A) \tag{1}$$

as an A–module and by letting $u \in U$ act as follows:

$$(uf)(m) = f(\tau(u)m) \qquad \text{for all } f \in M^\tau \text{ and } m \in M. \tag{2}$$

If M is a module in $\mathcal{C}_A$, then we make M^τ into a module in $\mathcal{C}_A$ setting

$$M_\lambda^\tau = \{f \in M^\tau \mid f(M_\mu) = 0 \text{ for all } \mu \neq \lambda\}. \tag{3}$$

So M_λ^τ can be identified with the dual of M_λ. It is easy to check 2.3(A)–(D). (Use that A is Noetherian to get that M^τ is finitely generated.)

If M is free over A, then obviously

$$\mathrm{ch}(M^\tau) = \mathrm{ch}(M) \tag{4}$$

and (using our finiteness assumption)

$$(M^\tau)^\tau \simeq M \tag{5}$$

and

$$(M \otimes_A A')^\tau \simeq M^\tau \otimes_A A' \tag{6}$$

for any A–algebra A'.

Suppose now that $A = F$ is a field. For any μ the contravariant dual of the simple module $L_F(\mu)$ is simple and has the same formal character. We get

$$L_F(\mu)^\tau \simeq L_F(\mu) \tag{7}$$

for all $\mu \in X(T)$.

Because (5) holds for all M in $\mathcal{C}_F$ one has

$$\operatorname{Ext}^i_{\mathcal{C}_F}(M^\tau, M'^\tau) \simeq \operatorname{Ext}^i_{\mathcal{C}_F}(M', M) \tag{8}$$

for all i and all M, M' in $\mathcal{C}_F$.

4.6. Proposition: *Suppose that $A = F$ is a field. One has*

$$\operatorname{Ext}^i_{\mathcal{C}_F}(L_F(\mu), L_F(\lambda)) \simeq \operatorname{Ext}^i_{\mathcal{C}_F}(L_F(\lambda), L_F(\mu)) \tag{1}$$

for all i and all $\lambda, \mu \in X$. Furthermore:

$$\textit{If } \mu \not> \lambda, \textit{ then} \quad \operatorname{Hom}_{\mathcal{C}_F}(\operatorname{rad} Z_F(\lambda), L_F(\mu)) \simeq \operatorname{Ext}^1_{\mathcal{C}_F}(L_F(\lambda), L_F(\mu)). \tag{2}$$

Proof: The first claim is an immediate consequence of 4.5(7) and (8). As in the classical case, cf. [Ja6], II.2.14 or II.9.16, the exact sequence

$$0 \to \operatorname{rad} Z_F(\lambda) \longrightarrow Z_F(\lambda) \longrightarrow L_F(\lambda) \to 0$$

induces for all μ an exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_{\mathcal{C}_F}(\operatorname{rad} Z_F(\lambda), L_F(\mu)) &\longrightarrow \operatorname{Ext}^1_{\mathcal{C}_F}(L_F(\lambda), L_F(\mu)) \\ &\longrightarrow \operatorname{Ext}^1_{\mathcal{C}_F}(Z_F(\lambda), L_F(\mu)). \end{aligned}$$

In case $\mu \not> \lambda$ the last term is 0 by Lemma 2.14.a.

4.7. We have by construction for all $\mu \in X$ and $w \in W$:

$$\operatorname{ch} Z_A(\mu) = e(\mu) \operatorname{ch} U^-$$

and

$$\operatorname{ch} Z_A^w(\mu) = e(\mu) \operatorname{ch}(T_w(U^-)) = e(\mu) w(\operatorname{ch} U^-). \tag{1}$$

Using the PBW–type basis of U^- we get

$$\operatorname{ch} U^- = \prod_{\beta \in R+} \frac{1 - e(-p\beta)}{1 - e(-\beta)}. \tag{2}$$

One knows that $\rho - w\rho$ is the sum of all positive roots α with $w^{-1}(\alpha) < 0$. It is now easy to see that

$$w(\operatorname{ch} U^-) = e((p-1)(\rho - w\rho)) \operatorname{ch} U^-. \tag{3}$$

Set

$$\mu\langle w \rangle = \mu + (p-1)(w\rho - \rho) \tag{4}$$

for all $\mu \in X$ and $w \in W$. With this notation (3) implies (for all μ, w)

$$\operatorname{ch} Z_A(\mu) = \operatorname{ch} Z_A^w(\mu\langle w \rangle) \tag{5}$$

Lemma: *Let $\mu \in X$. One has for all $w, x \in W$:*

$$\operatorname{Hom}_{\mathcal{C}_A}\big(Z_A^x(\mu\langle x \rangle), Z_A^w(\mu\langle w \rangle)\big) \simeq A \tag{6}$$

and

$$\operatorname{Hom}_{\mathcal{C}_A}\big(Z_A^x(\mu\langle x \rangle), Z_A^w(\mu\langle w \rangle)^\tau\big) \simeq A. \tag{7}$$

Proof: Set $M^x = Z_A^x(\mu\langle x \rangle)$ for all $x \in W$. By construction $M^x_{\mu\langle x \rangle}$ is free over A of rank 1, and $\mu\langle x \rangle + x\beta$ is not a weight of M^x for any $\beta \in R^+$. By (5) the same is true for M^w. So we can find v with $M^w_{\mu\langle x \rangle} = Av$. Furthermore, any $\mu\langle x \rangle + x\beta$ with $\beta \in R^+$ is not a weight of M^w. This yields $T_x(E_\beta)v = 0$ for all these β. Therefore there is a homomorphism $M^x \to M^w$ mapping the standard generator to v. Obviously any other homomorphism from M^x to M^w has to be a multiple. This proves (6). This argument uses only the fact that M^w has the same character as M^x. It works equally well for $(M^w)^\tau$, since that module has the same character as M^w by 4.5(4).

4.8. Note that $\mu\langle w_0\rangle = \mu - 2(p-1)\rho$ for all $\mu \in X$.

Lemma: *Suppose that $A = F$ is a field.*

a) The socle of $U^- \otimes F$ as a module over itself under left multiplication has dimension 1 and is equal to $(U^- \otimes F)_{-2(p-1)\rho}$.

b) For all $\mu \in X$ the socle of $Z_F(\mu)$ is simple and equal to

$$\operatorname{soc} Z_F(\mu) = U Z_F(\mu)_{\mu-2(p-1)\rho}. \tag{1}$$

Proof: a) The weight $-2(p-1)\rho$ is the smallest weight of U^-, so $(U^- \otimes F)_{-2(p-1)\rho}$ is annihilated by $(U^- \otimes F)_{<0}$, hence a one dimensional simple submodule of $U^- \otimes F$. So it is contained in the socle, and we claim that it is the whole socle.

In Case 1 the algebra $U^- \otimes F$ is isomorphic to the restricted enveloping algebra $U^{[p]}(\mathfrak{n}^- \otimes F)$. This algebra has a simple socle, cf. [Ja6], I.2.14(9), I.8.5/7.

In Case 2 one uses the nondegenerate bilinear form on $\mathbf{u}$ considered by Xi in [Xi1] when he proved the symmetry of $\mathbf{u}$, cf. [Xi2], 2.9. It restricts to a non degenerate bilinear form on $\mathbf{u}^- \simeq U^-$ that can be extended to $U^- \otimes F$. The socle is the space perpendicular to the radical, hence has dimension 1. (Compare also the argument in [PW1], 9.5.)

b) The isomorphism between $U^- \otimes F$ and $Z_F(\mu)$ maps the socle determined in a) to $Z_F(\mu)_{\mu-2(p-1)\rho}$. Therefore this one dimensional weight space is contained in every nonzero $U^-\otimes F$–submodule. This implies that $Z_F(\mu)$ regarded as a $U \otimes F$–module has a simple socle given by (1). It is obvious from that formula that this socle is the direct sum of its weight spaces, hence in $\mathcal{C}_F$ and thus also the socle of $Z_F(\mu)$ as a module in $\mathcal{C}_F$.

Remark: More generally, each $Z_F^w(\mu)$ has a simple socle. For example, the socle of $Z_F^{w_0}(\mu)$ is generated by $Z_F^{w_0}(\mu)_{\mu+2(p-1)\rho}$.

4.9. Lemma: *Suppose that $A = F$ is a field. Let $\mu \in X$. One has for any nonzero homomorphism $\varphi : Z_F^{w_0}(\mu - 2(p-1)\rho) \to Z_F(\mu)$:*

$$\operatorname{soc} Z_F(\mu) = \operatorname{im}(\varphi) \simeq L_F^{w_0}(\mu - 2(p-1)\rho) \tag{1}$$

and for any nonzero homomorphism $\varphi' : Z_F(\mu) \to Z_F^{w_0}(\mu - 2(p-1)\rho)$:

$$\operatorname{soc} Z_F^{w_0}(\mu - 2(p-1)\rho) = \operatorname{im}(\varphi') \simeq L_F(\mu). \tag{2}$$

Proof: The image of φ is generated by

$$\varphi(Z_F^{w_0}(\mu - 2(p-1)\rho)_{\mu-2(p-1)\rho}) \subset Z_F(\mu)_{\mu-2(p-1)\rho}. \tag{3}$$

Since φ is nonzero and since the right hand side in (3) has dimension 1, we have equality in (3). Now (1) is an immediate consequence of 4.8(1). The proof of (2) is similar.

Remark: Lemma 4.7 implies that there exist φ and φ' as above.

4.10. Lemma: *One has for all $\mu \in X$ isomorphisms*

$$Z_A^{w_0}(\mu - 2(p-1)\rho) \xrightarrow{\sim} Z_A(\mu)^\tau \tag{1}$$

and

$$Z_A(\mu + 2(p-1)\rho) \xrightarrow{\sim} Z_A^{w_0}(\mu)^\tau. \tag{2}$$

Proof: Let u be a basis for the k–module $(U^-)_{-2(p-1)\rho}$. Then $u \otimes 1$ is a basis of $Z_A(\mu)_{\mu-2(p-1)\rho}$ and we can choose a basis f of $(Z_A(\mu)^\tau)_{\mu-2(p-1)\rho}$ with $f(u \otimes 1) = 1$. By 4.7(7) we get a homomorphism $\psi : Z_A^{w_0}(\mu - 2(p-1)\rho) \to Z_A(\mu)^\tau$ with $\psi(1 \otimes 1) = f$. We have $\psi(\tau(u) \otimes 1) = \tau(u)f = f \circ u$, hence $\psi(\tau(u)\otimes 1)(1 \otimes 1) = f(u \otimes 1) = 1$ and $\psi(\tau(u)\otimes 1) \neq 0$. In the case where $A = F$ is a field, this implies that ψ is nonzero on the simple socle of $Z_F^{w_0}(\mu - 2(p-1)\rho)$, hence injective. By dimension comparison it has to be bijective over a field, hence in general and we get (1). Now (2) follows by applying τ to (1) with μ replaced by $\mu + 2(p-1)\rho$ and from 4.5(5).

4.11. Proposition: *One has* $\mathrm{Ext}^n_{\mathcal{C}_A}(Z_A(\lambda), Z_A^{w_0}(\mu)) = 0$ *for all* $\lambda, \mu \in X$ *and all* $n \geq 1$.

Proof: Consider at first the case $n = 1$. We can interpret Ext^1 via short exact sequences. From that point of view it is obvious that

$$\mathrm{Ext}^1_{\mathcal{C}_A}(Z_A(\lambda), Z_A^{w_0}(\mu)) \simeq \mathrm{Ext}^1(Z_A^{w_0}(\mu)^\tau, Z_A(\lambda)^\tau), \tag{1}$$

since these modules are free over A, cf. 4.5(5). Using Lemma 4.10 we can rewrite this as

$$\mathrm{Ext}^1_{\mathcal{C}_A}(Z_A(\lambda), Z_A^{w_0}(\mu)) \simeq \mathrm{Ext}^1(Z_A(\mu + 2(p-1)\rho), Z_A^{w_0}(\lambda - 2(p-1)\rho)). \tag{2}$$

The largest weight of $Z_A^{w_0}(\mu)$ resp. of $Z_A^{w_0}(\lambda - 2(p-1)\rho)$ is $\mu + 2(p-1)\rho$ resp. λ. So Lemma 2.14.a implies: If the left resp. right hand side in (2) is nonzero, then $\mu + 2(p-1)\rho > \lambda$ resp. $\lambda > \mu + 2(p-1)\rho$. We cannot have both inequalities at the same time, so the Ext^1 groups in (2) have to be zero.

In general, we use induction on n. The remark in 2.13 yields a short exact sequence $0 \to N \to Q \to Z_A(\lambda) \to 0$ such that Q is projective and such that N has a Z–filtration. We get for all $n > 1$:

$$\mathrm{Ext}^n_{\mathcal{C}_A}(Z_A(\lambda), Z_A^{w_0}(\mu)) \simeq \mathrm{Ext}^{n-1}_{\mathcal{C}_A}(N, Z_A^{w_0}(\mu)). \tag{3}$$

The right hand side in (3) vanishes by induction, since it vanishes for each of the factors in a Z–filtration of N. The largest weight of $Z_A^{w_0}(\mu)$ resp. of $Z_A^{w_0}(\lambda - 2(p-1)\rho)$ is $\mu + 2(p-1)\rho$ resp. λ. So Lemma 2.14.a implies: If the left resp. right hand side in (2) is nonzero, then $\mu + 2(p-1)\rho > \lambda$ resp. $\lambda >$

$\mu+2(p-1)\rho$. We cannot have both inequalities at the same time, so the Ext^1 groups in (2) have to be zero.

In general, we use induction on n. The remark in 2.13 yields a short exact sequence $0 \to N \to Q \to Z_A(\lambda) \to 0$ such that Q is projective and such that N has a Z-filtration. We get for all $n > 1$:

$$\mathrm{Ext}^n_{\mathcal{C}_A}(Z_A(\lambda), Z_A^{w_0}(\mu)) \simeq \mathrm{Ext}^{n-1}_{\mathcal{C}_A}(N, Z_A^{w_0}(\mu)). \tag{3}$$

The right hand side in (3) vanishes by induction, since it vanishes for each of the factors in a Z-filtration of N.

4.12. Proposition: *Let $\lambda, \mu \in X$. One has*

$$\mathrm{Hom}_{\mathcal{C}_A}(Z_A(\lambda), Z_A^{w_0}(\mu)) \simeq \begin{cases} A, & \textit{if } \mu = \lambda - 2(p-1)\rho; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof: If this space of homomorphisms is nonzero, then λ is a weight of $Z_A^{w_0}(\mu)$, hence $\lambda \leq \mu + 2(p-1)\rho$. Using τ we see that also

$$\mathrm{Hom}_{\mathcal{C}_A}(Z_A^{w_0}(\mu)^\tau, Z_A(\lambda)^\tau) \simeq \mathrm{Hom}_{\mathcal{C}_A}(Z_A(\mu+2(p-1)\rho), Z_A^{w_0}(\lambda-2(p-1)\rho)) \neq 0.$$

This implies $\mu + 2(p-1)\rho \leq \lambda$, hence $\mu = \lambda - 2(p-1)\rho$. In that case the claim follows from 4.7(6).

Remark: Let $w \in W$. Consider the equivalence of categories $M \mapsto M[w^{-1}]$ as in 4.4. By 4.4(2) it induces isomorphisms (for all λ, μ and all $n \geq 0$)

$$\mathrm{Ext}^n_{\mathcal{C}_A}(Z_A^w(\lambda), Z_A^{ww_0}(\mu)) \simeq \mathrm{Ext}^n_{\mathcal{C}_{A'}}(Z_{A'}(w^{-1}\lambda), Z_{A'}^{w_0}(w^{-1}\mu))$$

where $A' = A[w^{-1}]$. Therefore this and the previous proposition imply for all λ and μ

$$\mathrm{Ext}^n_{\mathcal{C}_A}(Z_A^w(\lambda), Z_A^{ww_0}(\mu)) = 0 \qquad \text{for all } n > 0 \tag{1}$$

and

$$\mathrm{Hom}_{\mathcal{C}_A}(Z_A^w(\lambda), Z_A^{ww_0}(\mu)) = \begin{cases} A, & \text{if } \mu = \lambda - 2(p-1)w\rho; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

4.13. Let M, N be modules in $\mathcal{C}_A$ such that M has a Z-filtration and N has a filtration with factors of the form $Z_A^{w_0}(\mu)$. Then Proposition 4.11 implies that

$$\mathrm{Ext}^n_{\mathcal{C}_A}(M, N) = 0 \qquad \text{for all } n \geq 1, \tag{1}$$

and Proposition 4.12 implies that $\mathrm{Hom}_{\mathcal{C}_A}(M, N)$ is a free A-module. The following corollary is an important special case:

Corollary: *Let M be a module in $\mathcal{C}_A$ with a Z-filtration. The A-module $\mathrm{Hom}_{\mathcal{C}_A}(M, Z_A^{w_0}(\lambda - 2(p-1)\rho))$ is free for all $\lambda \in X$, and its rank is equal to the number of factors isomorphic to $Z_A(\lambda)$ in a Z-filtration of M.*

4.14. Lemma: *Let M, Q be modules in $\mathcal{C}_A$ such that M has a Z-filtration and Q is projective. Then*

$$\mathrm{Ext}^n_{\mathcal{C}_A}(M,Q) = 0 \qquad \textit{for all } n > 0 \tag{1}$$

and $\mathrm{Hom}_{\mathcal{C}_A}(M,Q)$ *is a projective A-module.*

Proof: If A is a field (or local), then the last statement in 4.3 implies that Q has a filtration with factors of the form $Z_A^{w_0}(\mu)$. Then the claims follow from 4.12. In general we apply Proposition 3.4.

4.15. Suppose for the rest of Section 4 that F is a U^0-algebra that is a field.

The category $\mathcal{C}_F$ has enough projectives and all modules have finite length. Therefore the usual arguments show that there is a one to one correspondence between simple modules and projective indecomposable modules. Denote by $Q_F(\lambda)$ the unique (up to isomorphism) projective module with

$$Q_F(\lambda)/\mathrm{rad} Q_F(\lambda) \simeq L_F(\lambda). \tag{1}$$

For any M in $\mathcal{C}_F$ the multiplicity $[M : L_F(\lambda)]$ of $L_F(\lambda)$ as a composition factor of M is given by

$$[M : L_F(\lambda)] = \dim \mathrm{Hom}_{\mathcal{C}_F}(Q_F(\lambda), M). \tag{2}$$

Proposition: *Any $Q_F(\lambda)$ with $\lambda \in X$ has a Z-filtration. Any $Z_F(\mu)$ with $\mu \in X$ occurs exactly $[Z_F(\mu) : L_F(\lambda)]$ times in such a Z-filtration.*

Proof: The existence follows from 2.15.b. By 4.13, we know that $Z_F(\mu)$ occurs as a factor exactly

$$\dim \mathrm{Hom}_{\mathcal{C}_F}(Q_F(\lambda), Z_F^{w_0}(\mu - 2(p-1)\rho))$$

times. This number is equal to $[Z_F^{w_0}(\mu - 2(p-1)\rho) : L_F(\lambda)]$ by (2). Because the formal characters of the simple modules in $\mathcal{C}_F$ are linearly independent, the multiplicities in a module depend only on the formal character of the module. We know by 4.7(5) that $Z_F^{w_0}(\mu - 2(p-1)\rho)$ and $Z_F(\mu)$ have the same formal character. Therefore we get the claim.

Remark: One can prove the reciprocity formula also using the approach in [Ja3], 3.8.

4.16. Let F' be an extension field of F. We have then

$$L_F(\lambda) \otimes_F F' \simeq L_{F'}(\lambda) \qquad \text{for all } \lambda \in X, \tag{1}$$

since obviously $\mathrm{End}_{\mathcal{C}_F} L_F(\lambda) = F$. This implies that extension of scalars from $\mathcal{C}_F$ to $\mathcal{C}_{F'}$ take composition series to composition series, especially that

$$[Z_F(\mu) : L_F(\lambda)] = [Z_{F'}(\mu) : L_{F'}(\lambda)] \qquad \text{for all } \lambda, \mu \in X. \tag{2}$$

By Lemma 3.1 each $Q_F(\lambda) \otimes_F F'$ is projective in $\mathcal{C}_{F'}$; by (1) it maps onto $L_{F'}(\lambda)$. So (2) and Proposition 4.15 imply that

$$Q_F(\lambda) \otimes_F F' \simeq Q_{F'}(\lambda) \qquad \text{for all } \lambda \in X. \tag{3}$$

4.17. Fix $\nu \in X$. Recall the discussion of $\mathcal{C}_F(\leq \nu)$ in 3.6 – 3.10.

Lemma: *There is for all $\lambda \in X$ with $\lambda \leq \nu$ a projective cover $Q_F^\nu(\lambda)$ of $L_F(\lambda)$ in $\mathcal{C}_F(\leq \nu)$. It has a Z-filtration with factors $Z_F(\mu)$ where $\mu \in X$ with $\mu \leq \nu$. Any such $Z_F(\mu)$ occurs exactly $[Z_F(\mu) : L_F(\lambda)]$ times. If $\lambda \leq \nu - 2(p-1)\rho$, then $Q_F^\nu(\lambda) = Q_F(\lambda)$.*

Proof: The module $Q_F^\nu(\lambda) = \Gamma^\nu Q_F(\lambda)$ is projective in $\mathcal{C}_F(\leq \nu)$. Since it is a homomorphic image of $Q_F(\lambda)$, one gets $Q_F^\nu(\lambda)/\mathrm{rad} Q_F^\nu(\lambda) \simeq L_F(\lambda)$. Therefore $Q_F^\nu(\lambda)$ is a projective cover of $L_F(\lambda)$ in $\mathcal{C}_F(\leq \nu)$. The claim about the Z-filtration follows from the construction in 3.7 and from Proposition 4.15. (One could also argue directly using 3.8 and imitating the proof in 4.15.) If $[Z_F(\mu) : L_F(\lambda)] \neq 0$ for an arbitrary $\mu \in X$, then λ is a weight of $Z_F(\mu)$. This implies $\lambda \geq \mu - 2(p-1)\rho$, hence $\mu \leq \lambda + 2(p-1)\rho$. For $\lambda \leq \nu - 2(p-1)\rho$ we get then $\mu \leq \nu$ for all $Z_F(\mu)$ occurring in $Q_F(\lambda)$, hence the last claim.

4.18. Suppose in 4.18–19 that A is a local ring with residue field F.

Proposition: *There is for all $\lambda \in X$ with $\lambda \leq \nu$ a projective module $Q_A^\nu(\lambda)$ in $\mathcal{C}_A(\leq \nu)$ with $Q_A^\nu(\lambda) \otimes_A F \simeq Q_F^\nu(\lambda)$.*

Proof: We use induction on λ from above. Recall from the remark in 2.13 that $\Phi_A(\lambda)$ is projective in $\mathcal{C}_A$ and has a Z-filtration where only $Z_A(\mu)$ with $\mu \geq \lambda$ occur. The module $Z_A(\lambda)$ occurs exactly once and is the top factor. Set $Q = \Gamma^\nu \Phi_A(\lambda)$. This is a projective module in $\mathcal{C}_A(\leq \nu)$ with a Z-filtration and a surjection $Q \to Z_A(\lambda)$.

Now Q_F is projective in $\mathcal{C}_F(\leq \nu)$ by the generalization of 3.1.a to $\mathcal{C}_A(\leq \nu)$, cf. 3.10. It is therefore a direct sum of certain $Q_F^\nu(\mu)$. The statement on the Z-filtration implies that only weights $\mu \geq \lambda$ can occur. The module $Q_F^\nu(\lambda)$ occurs because of the surjection $Q_F \to Z_F(\lambda)$, and it occurs only once because $Z_F(\lambda)$ occurs only once in a Z-filtration. So Q_F has a decomposition

$$Q_F \simeq Q_F^\nu(\lambda) \oplus \bigoplus_{\lambda < \mu \leq \nu} Q_F^\nu(\mu)^{m(\mu)} \tag{1}$$

for suitable integers $m(\mu) \geq 0$.

For $\lambda = \nu$ we can take $Q_A^\nu(\lambda) = Q$. In general, we know by induction that there exist $Q_A^\nu(\mu)$ for all μ with $\lambda < \mu \leq \nu$, and we have to show that we can find a decomposition

$$Q \simeq Q' \oplus \bigoplus_{\lambda < \mu \leq \nu} Q_A^\nu(\mu)^{m(\mu)}. \tag{2}$$

Then $Q' \otimes_A F$ has to be isomorphic to $Q_F(\lambda)$ by the Krull–Schmidt theorem, i.e., we can take $Q_A^\nu(\lambda) = Q'$. Set $M = \bigoplus_{\lambda < \mu \leq \nu} Q_A^\nu(\mu)^{m(\mu)}$. In order to get (2), consider the projection $f : Q_F \to M_F$ coming from (1). We have by 3.3 (generalized to $\mathcal{C}_A(\leq \nu)$, cf. 3.10)

$$\mathrm{Hom}_{\mathcal{C}_F}(Q_F, M_F) \simeq \mathrm{Hom}_{\mathcal{C}_A}(Q, M) \otimes_A F.$$

So there is a homomorphism $\widetilde{f} : Q \to M$ that lifts f. The Nakayama lemma implies that $\widetilde{f}$ is surjective. Because M is projective, the epimorphism $\widetilde{f}$ has to split. We get a decomposition $Q \simeq Q_1 \oplus M$, i.e., (2).

Remark: The argument used in the last part of the proof says: If Q and M are projective in $\mathcal{C}_A(\leq \nu)$ and if M_F is a direct summand of Q_F, then M is a direct summand of Q. The same argument works in $\mathcal{C}_A$.

4.19. Theorem: *a) There is for all $\lambda \in X$ a projective module $Q_A(\lambda)$ in $\mathcal{C}_A$ such that $Q_A(\lambda) \otimes_A F \simeq Q_F(\lambda)$. This module is unique up to isomorphism.*

b) Any projective module in $\mathcal{C}_A$ is isomorphic to a direct sum of certain $Q_A(\lambda)$.

Proof: Let us first show the existence of the $Q_A(\lambda)$. Choose an arbitrary $\nu \geq \lambda + 2(p-1)\rho$. The module $\Phi_A(\lambda)$ is then contained in $\mathcal{C}_A(\leq \nu)$ since all weights of $\Phi_A(\lambda)$ are less or equal $\lambda + 2(p-1)\rho$. Therefore it is equal to the module $Q = \Gamma^\nu \Phi_A(\lambda)$ in the proof of 4.18. So Q and its direct summand $Q_A^\nu(\lambda)$ are projective in $\mathcal{C}_A$. We have $Q_F(\lambda) = Q_F^\nu(\lambda)$ by Lemma 4.17. So we can take $Q_A(\lambda) = Q_A^\nu(\lambda)$.

The uniqueness (up to isomorphism) of the $Q_A(\lambda)$ as well as the claim in b) follow from the remark in 4.18. (Note: If M in $\mathcal{C}_A$ is projective and satisfies $M_F = 0$, then $M = 0$ since M is free over A by Lemma 2.7.c.)

Remark: The functors $M \mapsto M[p\nu]$ with $\nu \in X$ are equivalences of categories on $\mathcal{C}_F$ and on $\mathcal{C}_A$, cf. 4.2. They take (indecomposable) projective objects to (indecomposable) projective objects. Therefore 4.2(3) implies for all λ and ν in X

$$Q_F(\lambda)[p\nu] \simeq Q_F(\lambda + p\nu) \qquad \text{and} \qquad Q_A(\lambda)[p\nu] \simeq Q_A(\lambda + p\nu). \tag{1}$$

5. Homomorphisms and Simple Modules

5.1. In Case 2 we define K_λ for all $\lambda \in \mathbf{Z}R$ as

$$K_\lambda = \prod_{\alpha\in\Sigma} K_\alpha^{m(\alpha)} \in U^0 \qquad \text{if} \quad \lambda = \sum_{\alpha\in\Sigma} m(\alpha)\alpha.$$

One gets then $T_w(K_\lambda) = K_{w\lambda}$ for all $w \in W$ and $\lambda \in \mathbf{Z}R$. (It is enough to check this when w is a simple reflection and λ a simple root. Then it is obvious by the definitions.) In Case 1 one has $T_w(H_\beta) = H_{w\beta}$ for all $w \in W$ and $\beta \in R$.

We define d_β for all roots β via $d_\beta = d_\alpha$ for all $\alpha \in \Sigma$ with $\beta \in W\alpha$. This is well defined, because d_α has the same value for simple roots conjugate under W.

Let $\alpha \in R$. In Case 1 we set $E_\alpha^{(m)} = E_\alpha^m/(m!)$ for all integers m with $0 \leq m < p$. In Case 2 set first

$$[m]_d = (\zeta^{dm} - \zeta^{-dm})/(\zeta^d - \zeta^{-d}) \qquad \text{and} \qquad [m]_d^! = \prod_{j=1}^{m} [j]_d$$

(where $d = d_\alpha$) for all integers m resp. for all integers $m \geq 0$. Then $[m]_d^! \neq 0$ for all m with $0 \leq m < p$. So we can define $E_\alpha^{(m)} = E_\alpha^m/[m]_d^!$ for all these m, and we can set

$$\begin{bmatrix} m \\ j \end{bmatrix}_d = \frac{[m]_d^!}{[j]_d^![m-j]_d^!}$$

for all integers j and m with $0 \leq j \leq m < p$. (The left hand side can be defined more generally, but we shall need it only in this special case.)

In Case 2 we define (with d as above) elements

$$\begin{bmatrix} K_\alpha; a \\ m \end{bmatrix} = \prod_{j=1}^{m} \frac{K_\alpha \zeta^{d(a-j+1)} - K_\alpha^{-1}\zeta^{-d(a-j+1)}}{\zeta^{dj} - \zeta^{-dj}}$$

for all integers a, m with $0 \leq m < p$. We shall use the abbreviation

$$[K_\alpha; a] = \begin{bmatrix} K_\alpha; a \\ 1 \end{bmatrix} = \frac{K_\alpha \zeta^{ad} - K_\alpha^{-1} \zeta^{-ad}}{\zeta^d - \zeta^{-d}}$$

and get for $0 \leq m < p$

$$\begin{bmatrix} K_\alpha; a \\ m \end{bmatrix} = ([m]_d^!)^{-1} \prod_{j=0}^{m-1} [K_\alpha; a - j].$$

Note that $K_\alpha^2 \zeta^{2ad} - 1$ from $[K_\alpha; a]$ by the factor $K_\alpha(\zeta^d - \zeta^{-d})\zeta^{ad}$ which is a unit in U^0.

5.2. Let A be an algebra over U^0 with structural map $\pi : U^0 \to A$. Set

$$R_\pi = \{\beta \in R \mid \pi(H_\beta)^p - \pi(H_\beta) = \textstyle\prod_{j=1}^{p}(\pi(H_\beta) + j) \text{ is not a unit in } A\}$$

in Case 1, resp.

$$\begin{aligned} R_\pi &= \{\beta \in R \mid \pi(K_\beta)^{2p} - 1 = \textstyle\prod_{j=1}^{p}(\pi(K_\beta)^2 \zeta^{2j} - 1) \text{ is not a unit in } A\} \\ &= \{\beta \in R \mid \textstyle\prod_{j=1}^{p}(\pi[K_\beta; j]) \text{ is not a unit in } A\} \end{aligned}$$

in Case 2. If $A = F$ is a field, then these conditions mean

$$R_\pi = \{\beta \in R \mid \pi(H_\beta) \in \mathbf{F}_p\} \qquad \text{resp.} \qquad R_\pi = \{\beta \in R \mid \pi(K_\beta)^{2p} = 1\}.$$

Set W_π equal to the subgroup of W generated by all s_β with $\beta \in R_\pi$ and set $R_\pi^+ = R_\pi \cap R^+$. If $A = F$ is a field, then $\mathbf{Z}R_\pi^\vee \cap R^\vee = R_\pi^\vee$ in Case 1, and $\mathbf{Z}R_\pi \cap R = R_\pi$ in Case 2. So then R_π is a root system with Weyl group W_π and we can choose R_π^+ as a system of positive roots in R_π. Denote the corresponding set of simple roots in R_π by Σ_π.

In general, R_π will not be a root system. Take for example R of type A_2 and denote the two simple roots by α and β. Assume that we are in Case 1 and set A equal to the subalgebra of the field of fractions of U^0 generated by U^0 and by $\prod_{j=1}^{p}(H_{\alpha+\beta} + j)^{-1}$. Then $R_\pi = \{\pm\alpha, \pm\beta\}$ is not a root system. (More precisely, it is not a root system with the given choice of coroots.)

5.3. Set B equal to the subalgebra of the field of fractions of U^0 generated by U^0 and all

$$\prod_{j=1}^{p-1}(H_\alpha + j)^{-1} = (H_\alpha^{p-1} - 1)^{-1} \qquad \text{resp.} \qquad \prod_{j=1}^{p-1}[K_\alpha; j]^{-1} \tag{1}$$

with $\alpha \in R^+$. Define subalgebras B^β (for each $\beta \in R^+$) and $B^\emptyset$ of the fraction field of U^0 via

$$B^\beta = B[H_\alpha^{-1} \mid \alpha \in R^+, \alpha \neq \beta] \tag{2}$$

resp.

$$B^\beta = B[\,[K_\alpha; 0]^{-1} \mid \alpha \in R^+, \alpha \neq \beta] \tag{3}$$

and

$$B^\emptyset = B[H_\alpha^{-1} \mid \alpha \in R^+] \qquad \text{resp.} \qquad B^\emptyset = B[\,[K_\alpha; 0]^{-1} \mid \alpha \in R^+]. \tag{4}$$

The following statements are more or less obvious:

Lemma: *Let A be a U^0-algebra with structural map $\pi : U^0 \to A$.*

a) A is a $B^\emptyset$-algebra, if and only if $R_\pi = \emptyset$.

b) Let $\beta \in R^+$. Suppose that A is a B-algebra. Then A is a B^β-algebra, if and only if $R_\pi \subset \{\pm\beta\}$.

c) Suppose that A is a field. Then A is a B-algebra, if and only if

$$R_\pi = \{\alpha \in R \mid \pi(H_\alpha) = 0\} \qquad \textit{resp.} \qquad R_\pi = \{\alpha \in R \mid \pi(K_\alpha)^2 = 1\}. \tag{5}$$

5.4. For the next subsections (until 5.12) we fix a simple root $\alpha \in \Sigma$, we set $s = s_\alpha$ and $d = d_\alpha$.

Let $U(\alpha)$ resp. $U(-\alpha)$ be the subalgebra of U generated by E_α resp. by $E_{-\alpha}$. It has as basis all E_α^i resp. all $E_{-\alpha}^i$ with $0 \leq i < p$. We can also take all $E_\alpha^{(i)}$ resp. $E_{-\alpha}^{(i)}$.

Set

$$P(\alpha) = U(-\alpha)U^0U^+. \tag{1}$$

We claim that this is a subalgebra of U. In Case 1 this is clear, because $P(\alpha)$ is the image in U of the enveloping algebra $U(\mathfrak{p}_\alpha)$ of the minimal parabolic subalgebra

$$\mathfrak{p}_\alpha = \mathfrak{b}^+ + kX_{-\alpha} = \mathfrak{b}^+ + s(\mathfrak{b}^+)$$

of $\mathfrak{g}$. In Case 2 the defining relations imply $E_{-\alpha}E_\beta - E_\beta E_{-\alpha} \in U^0$ for all $\beta \in \Sigma$, hence $E_{-\alpha}u - uE_{-\alpha} \in U^0U^+$ for all $u \in U^+$. This implies $U^+U(-\alpha) \subset P(\alpha)$, hence the claim by Lemma 1.4.

In both cases the PBW-type bases for U and its subalgebras show that $P(\alpha)$ is free of rank p as a right module over U^0U^+, and that U is free of rank p^{N-1} as a right module over $P(\alpha)$. (N is the number of positive roots.) One has

$$P(\alpha) \;=\; U(-\alpha)U^0U(\alpha) \;\oplus\; Q(\alpha) \tag{2}$$

where

$$Q(\alpha) = \bigoplus_{\nu \notin \mathbf{Z}\alpha} P(\alpha)_\nu = \bigoplus_{\nu \notin \mathbf{Z}\alpha} U(-\alpha)U^0U(\alpha)(U^+)_\nu.$$

Obviously, $Q(\alpha)$ is a two-sided ideal in $P(\alpha)$, and $U(-\alpha)U^0U(\alpha)$ is a subalgebra isomorphic to $P(\alpha)/Q(\alpha)$.

The automorphism $T_s = T_\alpha$ stabilizes $P(\alpha)$, $Q(\alpha)$, and $U(-\alpha)U^0U(\alpha)$. This is obvious for Case 1. In Case 2 one uses the explicit formulas for T_α on the generators and (for $Q(\alpha)$) the fact that $T_s(P(\alpha)_\nu) = P(\alpha)_{s(\nu)}$.

5.5. For all $\mu, \mu' \in X$ we can regard

$$\Psi_A(\mu) = P(\alpha) \otimes_{U^0U^+} A^\mu \tag{1}$$

and

$$\Psi_A^s(\mu') = P(\alpha) \otimes_{U^0T_s(U^+)} A^{\mu'} \tag{2}$$

as modules in a category analogous to $\mathcal{C}_A$, $\mathcal{C}'_A, \ldots$ involving $P(\alpha)$ instead of U, $U^0U^+, \ldots$.

Because the multiplication map is an isomorphism $U(-\alpha) \otimes U^0U^+ \xrightarrow{\sim} P(\alpha)$, all $E_{-\alpha}^{(i)} \otimes 1$ with $0 \leq i < p$ are a basis of $\Psi_A(\mu)$. We denote this basis by $v_0, v_1, \ldots, v_{p-1}$ where

$$v_i = E_{-\alpha}^{(i)} \otimes 1 \in \Psi_A(\mu)_{\mu - i\alpha}. \tag{3}$$

One has then for all i — using the convention $v_p = 0 = v_{-1}$ —

$$E_{-\alpha} v_i = (i+1) v_{i+1} \qquad \text{resp.} \qquad E_{-\alpha} v_i = [i+1]_d v_{i+1} \tag{4}$$

and

$$E_\alpha v_i = v_{i-1}(\pi(H_\alpha) + \mu(H_\alpha) - i + 1) \tag{5}$$

resp.

$$E_\alpha v_i = v_{i-1} \pi[K_\alpha; \langle \mu, \alpha^\vee \rangle - i + 1]. \tag{6}$$

Similarly, any $\Psi_A^s(\mu')$ has a basis $v'_0, v'_1, \ldots, v'_{p-1}$ where

$$v'_i = E_\alpha^{(i)} \otimes 1 \in \Psi_A^s(\mu')_{\mu' + i\alpha}. \tag{7}$$

One has

$$E_\alpha v'_i = (i+1) v'_{i+1} \qquad \text{resp.} \qquad E_\alpha v'_i = [i+1]_d v'_{i+1} \tag{8}$$

and

$$E_{-\alpha} v'_i = -v'_{i-1}(\pi(H_\alpha) + \mu'(H_\alpha) + i - 1) \tag{9}$$

resp.

$$E_{-\alpha} v'_i = -v'_{i-1} \pi[K_\alpha; \langle \mu', \alpha^\vee \rangle + i - 1]. \tag{10}$$

The action of U^0 on these bases is determined by the weights. The ideal $Q(\alpha)$ annihilates $\Psi_A(\mu)$ because $\mu - i\alpha + \nu$ is not a weight of this module for any $\nu \notin \mathbf{Z}\alpha$. Similarly it annihilates $\Psi_A^s(\mu')$.

5.6. Let $\mu \in X$ and keep the notations from 5.5 with $\mu' = \mu - (p-1)\alpha$. The universal property of the induced modules $\Psi_A(\mu)$ and $\Psi_A^s(\mu - (p-1)\alpha)$ yields unique homomorphisms

$$\varphi_\alpha : \Psi_A(\mu) \to \Psi_A^s(\mu - (p-1)\alpha) \qquad \text{and} \qquad \varphi'_\alpha : \Psi_A^s(\mu - (p-1)\alpha) \to \Psi_A(\mu) \tag{1}$$

with $\varphi_\alpha(v_0) = v'_{p-1}$ resp. $\varphi'_\alpha(v'_0) = v_{p-1}$. An argument as in 4.7 shows that φ_α and φ'_α are bases for the corresponding Hom spaces. The formulas 5.5(3)–(10) imply for all i

$$\varphi_\alpha(v_i) = (-1)^i v'_{p-1-i} \binom{\pi(H_\alpha) + \mu(H_\alpha)}{i} \tag{2}$$

resp.

$$\varphi_\alpha(v_i) = (-1)^i v'_{p-1-i} \pi\left(\begin{bmatrix} K_\alpha; \langle \mu, \alpha^\vee \rangle \\ i \end{bmatrix}\right) \tag{3}$$

and

$$\varphi'_\alpha(v'_i) = v_{p-1-i} \binom{\pi(H_\alpha) + \mu(H_\alpha) + i + 1}{i} \tag{4}$$

resp.

$$\varphi'_\alpha(v'_i) = v_{p-1-i} \pi\left(\begin{bmatrix} K_\alpha; i + 1 + \langle \mu, \alpha^\vee \rangle \\ i \end{bmatrix}\right). \tag{5}$$

5.7. We get for all $\mu, \mu' \in X$ the induced modules $Z_A(\mu)$ and $Z_A^s(\mu')$ by a two–step induction as

$$Z_A(\mu) \simeq U \otimes_{P(\alpha)} \left(P(\alpha) \otimes_{U^0 U^+} A^\mu\right) = U \otimes_{P(\alpha)} \Psi_A(\mu) \tag{1}$$

and

$$Z_A^s(\mu') \simeq U \otimes_{P(\alpha)} \left(P(\alpha) \otimes_{U^0 T_s(U^+)} A^{\mu'}\right) = U \otimes_{P(\alpha)} \Psi_A^s(\mu). \tag{2}$$

Using induction from $P(\alpha)$ to U we get from 5.6(1) homomorphisms

$$\varphi = 1 \otimes \varphi_\alpha : Z_A(\mu) \to Z_A^s(\mu - (p-1)\alpha) \tag{3}$$

and

$$\varphi' = 1 \otimes \varphi'_\alpha : Z_A^s(\mu - (p-1)\alpha) \to Z_A(\mu). \tag{4}$$

Each of them maps the standard generator to an element that is a basis of its weight space. So φ resp. φ' is a basis of the corresponding Hom space, cf. 4.7.

Recall that U is free over $P(\alpha)$ of rank p^{N-1}. This implies, for example: One has $\ker \varphi = U \otimes_{P(\alpha)} \ker \varphi_\alpha$; so, if $\ker \varphi_\alpha$ is free over A of rank say m, then $\ker \varphi$ is free of rank $p^{N-1} m$.

5.8. Let $\mu \in X$ and let φ and φ' be the homomorphisms from 5.7(3),(4).

Lemma: *If $\alpha \notin R_\pi$, then φ and φ' are isomorphisms.*

Proof: By the discussion in 5.7 it is enough to prove the corresponding statement for φ_α and φ'_α as in 5.6. Under our asssumption on α the coefficients on the right hand side in 5.6(2)–(5) are units in A. So φ_α and φ'_α take a basis to a basis, hence are bijective.

5.9. Keep the assumptions and notations from 5.8. Suppose that $A = F$ is a field. Consider the case where $\alpha \in R_\pi$. There is a unique integer n with $0 < n \leq p$ and

$$\pi(H_\alpha) + \langle \mu + \rho, \alpha^\vee \rangle = n \cdot 1 \in \mathbf{F}_p \subset F \tag{1}$$

in Case 1, resp.

$$\pi(K_\alpha)^2 \zeta^{2d\langle \mu+\rho, \alpha^\vee \rangle} = \zeta^{2dn} \tag{2}$$

in Case 2. (Note that $\langle \rho, \alpha^\vee \rangle = 1$, because α is simple, and that $\mu(H_\alpha)$ is the reduction modulo p of $\langle \mu, \alpha^\vee \rangle$.)

Lemma: *a) If $n = p$, then φ and φ' are isomorphisms.*

b) For $n < p$ one has

$$\mathrm{im}(\varphi) = \ker(\varphi') \qquad \textit{and} \qquad \mathrm{im}(\varphi') = \ker(\varphi)$$

and

$$\dim \mathrm{im}(\varphi) = p^{N-1} n \qquad \textit{and} \qquad \dim \ker(\varphi) = p^{N-1}(p-n).$$

There is a homomorphism

$$\psi : Z_F(\mu - n\alpha) \to Z_F(\mu) \tag{3}$$

with $\mathrm{im}(\psi) = \ker(\varphi)$ *and a long exact sequence*

$$\cdots \to Z_F(\mu - (p+n)\alpha) \to Z_F(\mu - p\alpha) \to Z_F(\mu - n\alpha) \to Z_F(\mu) \to \cdots. \tag{4}$$

Proof: By the discussion in 5.7 it is enough to prove the corresponding statements for the maps φ_α and φ'_α from 5.6. In Case 1 the formulas (1) and 5.6(2) yield $\varphi_\alpha(v_i) = (-1)^i \binom{n-1}{i} v'_{p-1-i}$. Therefore the kernel of φ_α is spanned by all v_i with $n \leq i \leq p-1$ and is zero for $n = p$. One gets the same result from (2) and 5.6(3) in Case 2 and shows similarly in both cases that the kernel of φ'_α is spanned by all v'_i with $p - n \leq i \leq p-1$ for $n < p$ whereas it is 0 for $n = p$. This implies a) and the first claims in b).

Assume from now on $n < p$. Let $\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_{p-1}$ be the basis of $\Psi_F(\mu - n\alpha)$ analogous to $v_0, v_1, \ldots, v_{p-1}$. We have $E_\alpha v_n = 0$ by 5.5(5),(6). So the universal property of induced modules yields a homomorphism $\psi_\alpha : \Psi_F(\mu - n\alpha) \to \Psi_F(\mu)$ with $\psi_\alpha(\hat{v}_0) = v_n$. It satisfies

$$\psi_\alpha(\hat{v}_i) = \binom{n+i}{i} v_{n+i} \qquad \text{resp.} \qquad \psi_\alpha(\hat{v}_i) = \begin{bmatrix} n+i \\ i \end{bmatrix}_d v_{n+i} \tag{5}$$

for $0 \le i \le p-n-1$ and $\psi_\alpha(\hat{v}_i) = 0$ for $i > p-n-2$. It it now clear that $\mathrm{im}(\psi_\alpha) = \ker(\varphi_\alpha)$ and that the kernel of ψ_α is equal to the image of the analogue of φ'_α for $\Psi_F(\mu - n\alpha)$. (Note that $p-n$ plays the role of n for $\mu - n\alpha$.) We get ψ as in (3) by inducing ψ_α from $P(\alpha)$ to U. We have then $\mathrm{im}\psi = \ker\varphi$. By repeating the construction we get the sequence (4).

Remark: We get also an infinite exact sequence

$$\cdots \to Z_F(\mu) \to Z_F^s(\mu - (p-1)\alpha) \to Z_F(\mu) \to Z_F^s(\mu - (p-1)\alpha) \to \cdots \quad (6)$$

where the maps are φ or φ' alternatingly.

5.10. Keep the assumptions and notations from 5.9.

4.4(2) implies that we can regard $Z_A^w(\mu)$ as $Z_{A'}(w^{-1}\mu)[w]$ where $A' = A[w^{-1}]$, and $Z_A^{ws}(\mu')$ as $Z_{A'}^s(w^{-1}\mu')[w]$. Therefore the homomorphisms between $Z_{A'}(w^{-1}\mu)$ and $Z_{A'}^s(w^{-1}\mu-(p-1)\alpha)$ as in 5.7(3),(4) can be interpreted as homomorphisms

$$\varphi : Z_A^w(\mu) \to Z_A^{ws}(\mu - (p-1)w\alpha) \quad (1)$$

and

$$\varphi' : Z_A^{ws}(\mu - (p-1)w\alpha) \to Z_A^w(\mu). \quad (2)$$

The restrictions of T_w^{-1} and $T_{w^{-1}}$ to U^0 coincide. Therefore the structural map of A' is

$$\pi' = \pi \circ T_{w^{-1}}^{-1} = \pi \circ T_w.$$

The behaviour of φ and φ' depends therefore on $\pi'(H_\alpha) = \pi(H_{w\alpha})$ resp. on $\pi'(K_\alpha) = \pi(K_{w\alpha})$. Now the results from 5.8/9 translate as follows:

Lemma: *a) If $w\alpha \notin R_\pi$, then φ and φ' are isomorphisms for all $\mu \in X$. Furthermore, if $A = F$ is a field, then*

$$L_F^w(\mu) \simeq L_F^{ws}(\mu - (p-1)w\alpha) \qquad \text{and} \qquad L_F^{ws}(\mu) \simeq L_F^w(\mu + (p-1)w\alpha) \quad (3)$$

for all $\mu \in X$.

b) Suppose that $w\alpha \in R_\pi$ and that $A = F$ is a field. Let n be the integer with $0 < n \le p$ and

$$n \cdot 1 = \pi'(H_\alpha) + \langle w^{-1}\mu + \rho, \alpha^\vee \rangle = \pi(H_{w\alpha}) + \langle \mu + w\rho, (w\alpha)^\vee \rangle \quad (4)$$

resp.

$$\zeta^{2dn} = \pi'(K_\alpha)^2 \zeta^{2d\langle w^{-1}\mu+\rho,\alpha^\vee\rangle} = \pi(K_{w\alpha})^2 \zeta^{2d\langle \mu+w\rho,(w\alpha)^\vee\rangle}. \quad (5)$$

If $n = p$, then φ and φ' are isomorphisms. For $n < p$ one has $\mathrm{im}(\varphi) = \ker(\varphi')$ *and* $\mathrm{im}(\varphi') = \ker(\varphi)$ *and*

$$\dim \mathrm{im}(\varphi) = p^{N-1} n \qquad \text{and} \qquad \dim \ker(\varphi) = p^{N-1}(p-n). \quad (6)$$

There is (for $n < p$) a homomorphism $\psi : Z_F^w(\mu - nw\alpha) \to Z_F^w(\mu)$ with $\mathrm{im}(\psi) = \ker(\varphi)$.

Remark: We get also an exact sequence as in 5.9(4). It leads to the following character formula:

$$\operatorname{ch}(\ker\varphi) = \sum_{i\geq 0} \operatorname{ch} Z_F^w(\mu - (ip+n)w\alpha) - \sum_{i\geq 1} \operatorname{ch} Z_F^w(\mu - ipw\alpha). \tag{7}$$

Note that this formula works also for $n = p$ where it yields 0.

5.11. Keep the assumptions and notations from 5.10 with $A = F$ a field.

Lemma: *If $w\alpha \notin R_\pi$ or if $w\alpha > 0$, then $\mu + (p-1)(\rho - w\rho)$ is not a weight of* $\ker(\varphi)$.

Proof: We can assume that $w\alpha \in R_\pi^+$ and that $n < p$ in the notations of 5.10. Any weight of $Z_F^w(\mu)$ has the form

$$\mu - \sum_{\beta\in R^+} m_{w\beta} w\beta = \mu - \sum_{\beta\in R'(w)} m_\beta \beta + \sum_{\beta\in R(w)} m_{-\beta}\beta \tag{1}$$

with $0 \leq m_\beta < p$ for all β where

$$R(w) = \{\beta \in R^+ \mid w^{-1}\beta < 0\} \qquad \text{and} \qquad R'(w) = \{\beta \in R^+ \mid w^{-1}\beta > 0\}.$$

We have $w(\alpha) \in R'(w)$. The weights of $\ker(\varphi)$ are the sums as in (1) with $m_{w\alpha} \geq n$. So $\mu + \sum_{\beta\in R(w)}(p-1)\beta$ is not a weight of $\ker(\varphi)$. The claim follows because $\sum_{\beta\in R(w)} \beta = \rho - w\rho$.

Remark: The same kind of argument shows also:

If $w\alpha \in R_\pi^+$ and $n < p$, then $\mu - (p-1)(\rho + w\rho)$ is a weight of $\ker(\varphi)$. (2)

Use that $\mu - \sum_{\beta\in R'(w)}(p-1)\beta$ is a weight of $\ker(\varphi)$ if $n < p$.

5.12. Lemma: *Let $w \in W$ with $w\alpha \in R_\pi^+$. Then one has for all $\nu \in X$:*

$$L_F^{ws_\alpha}(\nu) \simeq L_F^w(\nu + (p-m)w\alpha) \tag{1}$$

where m is the integer with $0 < m \leq p$ and

$$m \cdot 1 = \pi(H_{w\alpha}) + \langle \nu, (w\alpha)^\vee\rangle \qquad \textit{resp.} \qquad \zeta^{2dm} = \pi(K_{w\alpha})^2 \zeta^{2d\langle \nu, (w\alpha)^\vee\rangle}.$$

Proof: Apply 5.10 to $\mu = \nu + (p-1)w\alpha$. The corresponding n is $n = m - 1$ for $m > 1$, and $n = p$ for $m = 1$. For $n < p$ the modules $Z_F^w(\mu - nw\alpha)$ and $Z_F^{ws}(\nu)$ have the same nonzero image in $Z_F^w(\mu)$, so their simple heads have to be isomorphic. This and the fact that φ is an isomorphism for $n = p$ yield the claim.

Remark: The lemma and 5.10(3) show (in principle) how to find the weight μ_w in 4.3(3).

5.13. Let $w, x \in W$. Choose a reduced decomposition $x^{-1}w = s_1 s_2 \cdots s_r$ of $x^{-1}w$, where s_i is the reflection with respect to a simple root α_i. Set $x_i = x s_1 s_2 \cdots s_{i-1}$ for $1 \le i \le r+1$. So $x_1 = x$ and $x_{r+1} = w$.

Fix $\mu \in X$. We have for $1 \le i \le r$ homomorphisms (recall the notation 4.7(4))

$$\varphi_i : Z_A^{x_i}(\mu\langle x_i \rangle) \to Z_A^{x_i s_i}(\mu\langle x_i \rangle - (p-1)x_i(\alpha_i))$$

as in 5.10(1). Because of $x_i s_i = x_{i+1}$ and $s_i \rho = \rho - \alpha_i$ we can write this as

$$\varphi_i : Z_A^{x_i}(\mu\langle x_i \rangle) \to Z_A^{x_{i+1}}(\mu\langle x_{i+1} \rangle). \tag{1}$$

The composition

$$\varphi = \varphi_r \circ \cdots \circ \varphi_2 \circ \varphi_1$$

is then a homomorphism

$$\varphi : Z_A^{x}(\mu\langle x \rangle) \to Z_A^{w}(\mu\langle w \rangle). \tag{2}$$

Lemma: *Suppose for all i that $x_i\alpha_i \notin R_\pi$ or $x_i\alpha_i > 0$. Then φ induces an isomorphism on the μ-weight spaces. It is a basis for the corresponding* Hom *space. If*

$$x^{-1}(R_\pi) \cap \{\alpha \in R^+ \mid w^{-1}x\alpha < 0\} = \emptyset, \tag{3}$$

then φ is an isomorphism.

Proof: Recall from 4.7(5) that all $Z_A^y(\mu\langle y \rangle)$ with $y \in W$ have the same character. Their μ–weight spaces are free of rank 1. Our first claim (and then also the second one via 4.7) will follow, if we show that each φ_i induces an isomorphism of the μ–weight spaces. It is enough to look at the case where A is a field and to show that μ is not a weight of $\ker(\varphi_i)$. That claim, however, follows from Lemma 5.11 applied to $\mu\langle x_i \rangle$ instead of μ.

We know by Lemma 5.10.a that φ_i is an isomorphism, if $x_i\alpha_i \notin R_\pi$. So φ is an isomorphism, if

$$R_\pi \cap \{x_i\alpha_i \mid 1 \le i \le r\} = \emptyset.$$

Since

$$\{\alpha \in R^+ \mid w^{-1}x\alpha < 0\} = \{s_1 s_2 \cdots s_{i-1}\alpha_i \mid 1 \le i \le r\},$$

the last claim follows.

Remark: We have $x_i\alpha_i > 0$ for all i if and only if $l(w) = l(x) + l(x^{-1}w)$. For example, this holds for $x = 1$ and w arbitrary, and it holds for $w = w_0$ and x arbitrary. If $l(w) = l(x) + l(x^{-1}w)$ or if (3) holds, then the assumption in the lemma is satisfied for each reduced decomposition of $x^{-1}w$. Our second claim implies then that a different reduced decomposition of $x^{-1}w$ multiplies φ by a unit in A.

5.14. Lemma: *Let $\mu \in X$ and $w \in W$.*

a) If $w^{-1}\alpha > 0$ for all $\alpha \in R_\pi^+$, then $Z_A(\mu) \xrightarrow{\sim} Z_A^w(\mu\langle w\rangle)$.

b) If $w^{-1}\alpha < 0$ for all $\alpha \in R_\pi^+$, then $Z_A^w(\mu\langle w\rangle) \xrightarrow{\sim} Z_A^{w_0}(\mu\langle w_0\rangle)$.

Proof: We get a) applying the last claim of Lemma 5.13 in the case $x = 1$. We get b) applying it to (w_0, w) instead of (w, x).

Remark: Lemma 5.3 implies now:

If A is a $B^\emptyset$-algebra, then $Z_A(\mu) \xrightarrow{\sim} Z_A^w(\mu\langle w\rangle)$ for all $w \in W$.

If A is a B^β-algebra for some $\beta \in R^+$, then $Z_A^w(\mu\langle w\rangle) \xrightarrow{\sim} Z_A(\mu)$ for all $w \in W$ with $w^{-1}\beta > 0$, and $Z_A^w(\mu\langle w\rangle) \xrightarrow{\sim} Z_A^{w_0}(\mu\langle w_0\rangle)$ for all $w \in W$ with $w^{-1}\beta < 0$.

5.15. We assume in the remaining subsections of this section that F is a field that is a U^0–algebra, and that π is the structural map.

Lemma: *Let $\beta \in R$ be a root with $\pi(H_\beta) = 0$ resp. with $\pi(K_\beta)^2 = 1$. One has then for all $\mu \in X$:*

$$\operatorname{ch} L_F^{s_\beta w}(s_\beta\mu) = s_\beta \operatorname{ch} L_F^w(\mu). \tag{1}$$

Proof: Set $s = s_\beta$. On U^0 the operator T_s is given by

$$T_s(H_\alpha) = H_\alpha - \beta(H_\alpha)H_\beta \qquad \text{resp.} \qquad T_s(K_\alpha) = K_\alpha K_\beta^{-\langle\alpha,\beta^\vee\rangle} \tag{2}$$

for all $\alpha \in R$. This implies: If $\pi(H_\beta) = 0$ resp. if $\pi(K_\beta) = 1$, then $\pi \circ T_s$ and π coincide on U^0. Then $A = A[s]$, and the functor $M \mapsto M[s]$ as in 4.4 maps $\mathcal{C}_A$ to itself. We have then $L_F^w(\mu)[s] \simeq L_F^{sw}(s\mu)$ for all $w \in W$ and $\mu \in X$ by 4.4(3), hence (1) in that case.

The situation where $\pi(K_\beta) = -1$ in Case 2 requires more preparation. One has for any choice of integers $r(\alpha)$ for $\alpha \in \Sigma$ an involutory automorphism σ of U_2 with $\sigma(K_\alpha) = (-1)^{r(\alpha)}K_\alpha$, $\sigma(E_\alpha) = (-1)^{r(\alpha)}E_\alpha$, and $\sigma(F_\alpha) = F_\alpha$ for all $\alpha \in \Sigma$, cf. [Lu4], 4.6. Each weight vector in U_2^+ and U_2^- is then multiplied by 1 or by -1 under σ. Therefore σ maps I to itself and thus induces a similar involutory automorphism (also denoted by σ) of U. We can twist π by σ and get $\pi' = \pi \circ \sigma : U^0 \to A$. Denote A with this algebra stucture by $A[\sigma]$. We get a functor $M \mapsto M[\sigma]$ from $\mathcal{C}_A$ to $\mathcal{C}_{A[\sigma]}$ where $M[\sigma]$ is M with the old structure as an A–module and the old grading, where any $u \in U$ acts as $\sigma(u)$ acts on M. It is easy to see that this functor is an equivalence of categories, that it satisfies $\operatorname{ch}(M[\sigma]) = \operatorname{ch}(M)$ whenever this makes sense and (for $A = F$ a field)

$$L_F^w(\mu)[\sigma] \simeq L_{F[\sigma]}^w(\mu) \tag{3}$$

for all $\mu \in X$ and $w \in W$.

Suppose now that $\pi(K_\beta) = -1$. We can choose σ such that $\sigma(K_\alpha) = (-1)^{\langle \alpha, \beta^\vee \rangle} K_\alpha$ for all $\alpha \in \Sigma$, hence for all $\alpha \in R$. Then $\pi \circ \sigma(K_\alpha) = \pi \circ T_s(K_\alpha)$ for all $\alpha \in R$. This implies $F[\sigma] = F[s]$. We get therefore

$$L_F^w(\mu)[s] \simeq L_{F[s]}^{sw}(s\mu) \simeq L_{F[\sigma]}^{sw}(s\mu) \simeq L_F^{sw}(s\mu)[\sigma],$$

hence (1) in this case.

5.16. The following result is an immediate consequence of Lemma 5.15:

Lemma: *If F is a B-algebra, then one has for all $w \in W$, $x \in W_\pi$, and $\mu \in X$:*

$$\operatorname{ch} L_F^{xw}(x\mu) = x \operatorname{ch} L_F^w(\mu).$$

5.17. Set

$$X_1^\pi = \{\mu \in X(T) \mid 0 \leq \langle \mu, \alpha^\vee \rangle < p \qquad \text{for all } \alpha \in \Sigma_\pi\}.$$

Proposition: *If F is a B-algebra, then one has for all $\nu \in X_1^\pi$ and $w \in W_\pi$:*

$$L_F((p-1)\rho - \nu) \simeq L_F^w((p-1)w\rho - w\nu).$$

Proof: We want to use induction on the length of w as an element of W_π. The case $w = 1$ is trivial. It is enough to show for all $w \in W_\pi$ and $\alpha \in \Sigma_\pi$ with $w(\alpha) > 0$ that

$$L_F^{ws_\alpha}((p-1)ws_\alpha\rho - ws_\alpha\nu) \simeq L_F^w((p-1)w\rho - w\nu). \tag{1}$$

Let $s_\alpha = s_1 s_2 \cdots s_r$ be a reduced decomposition of s_α in W. So each s_i is the reflection with respect to a simple root α_i in R, and one has

$$R(s_\alpha) = \{\beta \in R^+ \mid s_\alpha(\beta) < 0\} = \{w_i(\alpha_i) \mid 1 \leq i \leq r\},$$

where $w_i = s_1 \cdots s_{i-1}$ for $1 \leq i \leq r+1$. One has

$$R(s_\alpha) \cap R_\pi = \{\alpha\}$$

because α is simple in R_π and hence s_α stabilizes $R_\pi^+ - \{\alpha\}$. So there is an index j with $w_j(\alpha_j) = \alpha$, and one has $w_i(\alpha_i) \notin R_\pi$, hence $ww_i(\alpha_i) \notin R_\pi$ for all $i \neq j$. Therefore 5.10(3) yields for all ν' and all $i \neq j$ an isomorphism

$$L_F^{ww_i}((p-1)ww_i\rho + \nu') \simeq L_F^{ww_{i+1}}((p-1)ww_{i+1}\rho + \nu')$$

Using this repeatedly, we see: The left hand side in (1) is isomorphic to

$$L_F^{ww_j s_j}(\mu_1) \qquad \text{with} \qquad \mu_1 = (p-1)ww_j s_j \rho - w s_\alpha \nu,$$

and the right hand side in (1) is isomorphic to

$$L_F^{ww_j}(\mu_2) \qquad \text{with} \qquad \mu_2 = (p-1)ww_j \rho - w\nu.$$

We have $ww_j(\alpha_j) = w(\alpha) > 0$, so Lemma 5.12 implies that

$$L_F^{ww_j s_j}(\mu_1) \simeq L_F^{w_j s_j}(\mu_1 + (p-1-n)w\alpha),$$

where n is the integer with $0 \leq n < p$ and

$$\langle \mu_1, w(\alpha)^\vee \rangle \equiv n+1 \pmod{p}.$$

So we have to show that μ_2 is equal to

$$\mu_1 + (p-1-n)w\alpha = (p-1)(ww_j s_j \rho + w\alpha) - w s_\alpha \nu - n w\alpha.$$

Because

$$ww_j s_j \rho + w\alpha = ww_j(s_j \rho + \alpha_j) = ww_j \rho$$

we have to show that

$$-s_\alpha \nu - n\alpha = -\nu,$$

i.e., that

$$n = \langle \nu, \alpha^\vee \rangle.$$

By definition

$$\begin{aligned} n &\equiv \langle (p-1)ww_j s_j \rho - w s_\alpha \nu, w(\alpha)^\vee \rangle - 1 \equiv -\langle w_j s_j \rho, \alpha^\vee \rangle - \langle s_\alpha \nu, \alpha^\vee \rangle - 1 \\ &= -\langle s_j \rho, \alpha_j^\vee \rangle + \langle \nu, \alpha^\vee \rangle - 1 = \langle \nu, \alpha^\vee \rangle \pmod{p}. \end{aligned}$$

This shows that n and $\langle \nu, \alpha^\vee \rangle$ are congruent modulo p. Because both are between 0 and $p-1$ (by definition resp. by the assumption that $\nu \in X_1^\pi$), they are equal.

5.18. Corollary: *If F is a B-algebra, then for all $\nu \in X_1^\pi$ the formal character of $L_F((p-1)\rho - \nu)$ is W_π-invariant.*

This is an obvious consequence of Proposition 5.17 and Lemma 5.16.

6. Filtrations, Linkage, and Blocks

6.1. Choose a reduced decomposition $w_0 = s_1 s_2 \cdots s_N$ of w_0 and fix $\mu \in X$. Apply the construction from 5.13 with $x = 1$ and $w = w_0$. Write now $w_i = s_1 s_2 \cdots s_{i-1}$ for $1 \leq i \leq N+1$. We get homomorphisms

$$\varphi_i : Z_A^{w_i}(\mu\langle w_i \rangle) \to Z_A^{w_{i+1}}(\mu\langle w_{i+1} \rangle)$$

and their composition

$$\varphi : Z_A(\mu) \to Z_A^{w_0}(\mu - 2(p-1)\rho).$$

We assume in the next subsections (until 6.6) that $A = F$ is a field with structural homomorphism π.

Lemma 4.9 implies

$$L_F(\mu) \simeq \varphi(Z_F(\mu)) = \operatorname{soc} Z_F^{w_0}(\mu - 2(p-1)\rho). \tag{1}$$

For all $\beta \in R_\pi$ let $n_\beta = n_\beta(\mu)$ be the integer with $0 < n_\beta \leq p$ and

$$\pi(H_\beta) + \langle \mu + \rho, \beta^\vee \rangle = n_\beta \cdot 1 \qquad \text{resp.} \qquad \pi(K_\beta)^2 \zeta^{2d\langle \mu+\rho, \beta^\vee \rangle} = \zeta^{2dn_\beta} \tag{2}$$

where $d = d_\beta$.

Lemma: *Let $\beta = w_i(\alpha_i)$ for some i with $1 \leq i \leq N$. Then φ_i is an isomorphism if and only if $\beta \notin R_\pi$ or if $\beta \in R_\pi$ and $n_\beta = p$. Otherwise* $\ker(\varphi_i)$ *is a homomorphic image of $Z_F^{w_i}(\mu\langle w_i \rangle - n_\beta \beta)$.*

Proof: We have

$$\langle \mu\langle w_i \rangle + w_i \rho, \beta^\vee \rangle \equiv \langle \mu + \rho, \beta^\vee \rangle \pmod{p}.$$

Therefore the claim is an immediate consequence of Lemma 5.10.

6.2. Lemma: *Let $\lambda, \mu \in X$. If $L_F(\lambda)$ is a composition factor of $Z_F(\mu)$, then $\mu = \lambda$ or $L_F(\lambda)$ is a composition factor of $Z_F(\mu - n_\beta\beta)$ for some $\beta \in R_\pi^+$ with $n_\beta \neq p$.*

Proof: Use the notations from 6.1. Suppose that $L_F(\lambda)$ is a composition factor of $Z_F(\mu)$ with $\lambda \neq \mu$. Then 6.1(1) implies that $L_F(\lambda)$ is a composition factor of $\ker(\varphi)$, hence of $\ker(\varphi_i)$ for some i, therefore (by 6.1) of $Z_F^{w_i}(\mu\langle w_i\rangle - n_\beta\beta)$, where $\beta = w_i(\alpha_i) \in R_\pi^+$ and $n_\beta \neq p$. This module has (by 4.7(5)) the same character as $Z_F(\mu - n_\beta\beta)$, so $L_F(\lambda)$ is a composition factor of this second module also.

Remark: This lemma implies a *strong linkage principle.* First a notation: For μ and β as in 6.1(2) set $\beta\downarrow\mu = \mu - n_\beta\beta$ if $n_\beta < p$, and $\beta\downarrow\mu = \mu$ otherwise. We can now state the principle as follows: *If $L_F(\lambda)$ is a composition factor of $Z_F(\mu)$, then there is a chain $\lambda_0 = \lambda, \lambda_1, \ldots, \lambda_r = \mu$ and $\beta_i \in R_\pi^+$ with $\beta_i\downarrow\lambda_i = \lambda_{i-1}$ for $1 \leq i \leq r$.*

6.3. Lemma: *Let $\mu \in X$. The following are equivalent:*

(i) $Z_F(\mu)$ is irreducible.

(ii) One has $n_\beta(\mu) = p$ for all $\beta \in R_\pi^+$.

(iii) $L_F(\mu)$ is projective and injective in $\mathcal{C}_F$.

Proof: Use the notation from 6.1. Formula 6.1(1) implies that $Z_F(\mu)$ is simple if and only if φ is bijective. Because all $Z_F^w(\nu)$ have the same dimension, this holds if and only if each φ_i is an isomorphism. By Lemma 6.1 this is equivalent to $n_\beta = p$ for all $\beta \in R_\pi^+$, i.e. to (ii).

If $Z_F(\mu)$ is irreducible, then a dimension argument shows that $L_F(\mu)$ is not a composition factor of any $Z_F(\lambda)$ with $\lambda \neq \mu$. So Proposition 4.6 implies

$$\mathrm{Ext}^1_{\mathcal{C}_F}(L_F(\mu), M) = 0 = \mathrm{Ext}^1_{\mathcal{C}_F}(M, L_F(\mu)) \tag{1}$$

first for all simple M, then for all M in $\mathcal{C}_F$. Therefore $L_F(\mu)$ is both projective and injective in $\mathcal{C}_F$, cf. the argument in [Ja6], II.10.2.

On the other hand, if $L_F(\mu)$ is projective, then the exact sequence

$$0 \to \mathrm{rad}\, Z_F(\mu) \to Z_F(\mu) \to L_F(\mu) \to 0$$

has to split. The radical of a module can split off only if it is 0, so we get $\mathrm{rad} Z_F(\mu) = 0$ and (i).

Remark: If $R_\pi = \emptyset$, then (ii) is always satisfied. So all $Z_F(\mu)$ are irreducible, hence all simple modules in $\mathcal{C}_F$ projective and injective:

$$Q_F(\mu) = Z_F(\mu) = L_F(\mu) \qquad \text{for all } \mu \in X. \tag{2}$$

All modules in $\mathcal{C}_F$ are semisimple. These facts as well as the results in 6.3 follow in Case 1 easily from the much more general Theorem 2 in [VK], see also [FP1]. Similarly, in Case 2 they follow from [DCK1] and [DCK2].

6.4. As an example, consider the case where there is exactly one positive root β in R_π. In the set up of 6.1 there is a unique i with $1 \leq i \leq N$ and $\beta = w_i(\alpha_i)$. All φ_j with $j \neq i$ are isomorphisms. The image of φ is isomorphic to the image of φ_i. Let n_β be the integer as in 6.1(2). For $n_\beta = p$ also φ_i is an isomorphism and $Z_F(\mu)$ is irreducible and one has

$$Q_F(\mu) = Z_F(\mu) = L_F(\mu). \tag{1}$$

Suppose now that $0 < n_\beta < p$. Then the image of φ_i, hence of φ, has dimension $p^{N-1}n_\beta$ by 5.10(6), so we get

$$\dim L_F(\mu) = p^{N-1}n_\beta. \tag{2}$$

The kernel of φ_i is equal to a homomorphic image of $Z_F^{w_i}(\mu + (p-1)(w_i\rho - \rho) - n_\beta\beta)$. This module is isomorphic to $Z_F(\mu - n_\beta\beta)$ under the composition of analogues of the φ_j with $j < i$. So $\ker(\varphi_i)$ is a homomorphic image of $Z_F(\mu - n_\beta\beta)$ and has dimension $p^{N-1}(p - n_\beta)$. This is equal to the dimension of $L_F(\mu - n_\beta\beta)$ by (2) applied to $\mu - n_\beta\beta$. So $\ker(\varphi_i)$ is isomorphic to $L_F(\mu - n_\beta\beta)$ and we have an exact sequence

$$0 \to L_F(\mu - n_\beta\beta) \to Z_F(\mu) \to L_F(\mu) \to 0. \tag{3}$$

It is now obvious that $L_F(\mu)$ is a composition factor of $Z_F(\mu)$ and of $Z_F(\mu + (p - n_\beta)\beta)$ and it does not occur in any other $Z_F(\lambda)$. Therefore we have by 4.15 an exact sequence

$$0 \to Z_F(\mu + (p - n_\beta)\beta) \to Q_F(\mu) \to Z_F(\mu) \to 0. \tag{4}$$

It leads to a projective resolution

$$\cdots \to Q_F(\mu + p\beta) \to Q_F(\mu + (p - n_\beta)\beta) \to Q_F(\mu) \to Z_F(\mu) \to 0. \tag{5}$$

Using this resolution one can compute easily all Ext groups between two $Z_F(\mu')$. One gets especially

$$\operatorname{Ext}^1_{\mathcal{C}_F}(Z_F(\mu), Z_F(\mu + (p - n_\beta)\beta)) \simeq F. \tag{6}$$

This implies that for any nonsplit extension $0 \to Z_F(\mu + (p - n_\beta)\beta) \to M \to Z_F(\mu) \to 0$ the module M is isomorphic to $Q_F(\mu)$.

6.5. We shall use homomorphisms like the φ from 5.5(3) to construct filtrations of all $Z_F(\mu)$. Let $F[X]$ be the polynomial ring over F in one indeterminate X.

Lemma: *a) In Case 1 set $\widehat{A} = F[X]$. There exists a homomorphism of k-algebras $\tilde{\pi} : U^0 \to \widehat{A}$ such that there is for each root $\beta \in R$ an element $c_\beta \in k$ with*

$$\tilde{\pi}(H_\beta) = c_\beta X + \pi(H_\beta) \qquad \textit{and} \qquad c_\beta \neq 0. \tag{1}$$

b) In Case 2 set $\widehat{A} = F[X, (X+1)^{-1}]$ and let $\tilde{\pi} : U^0 \to \widehat{A}$ be the homomorphism with $\tilde{\pi}(K_\alpha) = (X+1)\pi(K_\alpha)$ for all $\alpha \in \Sigma$. One has for all $\beta \in R^+$:

$$\tilde{\pi}(K_\beta) \equiv (c_\beta X + 1)\pi(K_\beta) \pmod{X^2} \tag{2}$$

for some integer $c_\beta > 0$.

Proof: a) We can regard U^0 as a polynomial algebra over k in the indeterminates H_α with α simple. So for any choice of the c_α with α simple there is a unique $\tilde{\pi}$ with $\tilde{\pi}(H_\alpha) = c_\alpha X + \pi(H_\alpha)$ for all α simple. For an arbitrary root β we can express H_β as a linear combination $H_\beta = \sum_\alpha m_{\beta\alpha} H_\alpha$ of the H_α with α simple with coefficients $m_{\beta\alpha} \in \mathbf{F}_p$. We get then $\tilde{\pi}(H_\beta) = c_\beta X + \pi(H_\beta)$ with $c_\beta = \sum_\alpha m_{\beta\alpha} c_\alpha$. If we choose the c_α linearly independent over $\mathbf{F}_p$, then $c_\beta \neq 0$ for all β. (Note that $H_\beta \neq 0$, because there is a $w \in W$ with $w(\beta)$ simple, hence $H_\beta = w^{-1} H_{w(\beta)}$.) If p is greater than the Coxeter number of R, we can take $\pi(H_\alpha) = X$ for all α simple, but that does not work for small primes.

b) If $\beta = \sum_{\alpha \in \Sigma} m(\alpha)\alpha$ then

$$\tilde{\pi}(K_\beta) = \tilde{\pi}(\prod_{\alpha \in \Sigma} K_\alpha^{m(\alpha)}) = (X+1)^{\sum m(\alpha)} \pi(K_\beta) \equiv (1 + \sum m(\alpha)X)\pi(K_\beta),$$

so $c_\beta = \sum_{\alpha \in \Sigma} m(\alpha) > 0$ will work.

6.6. Choose $\widehat{A}$ and $\tilde{\pi}$ as in Lemma 6.5. Regard F as an algebra over $\widehat{A}$ via $X \mapsto 0$. Then

$$Z_F(\mu) \simeq Z_{\widehat{A}}(\mu) \otimes_{\widehat{A}} F \simeq Z_{\widehat{A}}(\mu) / X Z_{\widehat{A}}(\mu) \tag{1}$$

for all $\mu \in X$. Consider the homomorphisms φ and φ_i as in 6.1 over $\widehat{A}$. Define a filtration first on $Z_{\widehat{A}}(\mu)$ by

$$Z_{\widehat{A}}(\mu)^j = \{v \in Z_{\widehat{A}}(\mu) \mid \varphi(v) \in X^j Z_{\widehat{A}}^{w_0}(\mu - 2(p-1)\rho)\} \tag{2}$$

for all $j \geq 0$, and set then $Z_F(\mu)^j$ equal to the image of $Z_{\widehat{A}}(\mu)^j$ in $Z_F(\mu)$ under reduction modulo X.

Set

$$N(\mu) = \mid \{\beta \in R_\pi^+ \mid n_\beta \neq p\} \mid$$

with $n_\beta = n_\beta(\mu)$ as in 6.1(2).

Proposition: *One has*

$$Z_F(\mu)/Z_F(\mu)^1 \simeq L_F(\mu) \qquad \text{and} \qquad Z_F(\mu)^{N(\mu)} = \operatorname{soc} Z_F(\mu). \tag{3}$$

Furthermore, $Z_F(\mu)^{N(\mu)+1} = 0$ *and*

$$\sum_{j>0} \operatorname{ch} Z_F(\mu)^j = \sum_{\beta \in R_\pi^+} \Big(\sum_{i \geq 0} \operatorname{ch} Z_F(\mu - (ip + n_\beta)\beta) - \sum_{i>0} \operatorname{ch} Z_F(\mu - ip\beta)\Big). \tag{4}$$

Proof: By construction $Z_F(\mu)^1$ is the kernel of the reduction modulo X of φ. This reduction is the analogue of φ over F. So 6.1(1) implies the first part in (3).

Over the field of fractions of $\widehat{A}$ the analogue of R_π is empty. Therefore all φ_i and φ induce isomorphisms over this field, cf. 6.1. This implies that $Z_F(\mu)^j = 0$ for all $j \gg 0$ and that an expression like $\sum_{j>0} \operatorname{ch} Z_F(\mu)^j$ makes sense. This sum can now be computed using the ideas from [And], cf. [Ja6], ch. II.8.

Using φ_i we can define a filtration of each $Z_i = Z_F^{w_i}(\mu\langle w_i \rangle)$ analogous to the one on $Z_F(\mu)$. This filtration can be described explicitly. We can regard Z_i as a suitable $Z_{F'}(\mu')[w_i]$, cf. 5.10. We get φ_i, hence also the corresponding filtration from a map on $Z_{\widehat{A}'}(\mu')$. That map is induced from a map over $P(\alpha_i)$ for which we have explicit formulas as in 5.6(2),(3). A basis element of the form $\prod_\beta E_{-\beta}^{m(\beta)} v_j$ (where the product is over $\beta \in R^+ - \{\alpha_i\}$) is mapped to c_j times the basis element $\prod_\beta E_{-\beta}^{m(\beta)} v'_{p-j+1}$ for some $c_j \in \widehat{A}$. Now 6.5(1),(2) and the formulas in 5.6 imply that the X-adic evaluation of the c_j is either 0 or 1. This implies that $(Z_i)^2 = 0$ and that $(Z_i)^1$ is the kernel of the reduction modulo X of φ_i. So $\operatorname{ch}(Z_i)^1$ is given by a formula like 5.10(7). It involves characters of certain $Z_F^{w_i}(\mu_1)$ that can be replaced by characters of suitable $Z_F^1(\mu_2)$ using 4.7(5).

Now $\sum_{j>0} \operatorname{ch} Z_F(\mu)^j$ is equal to the sum of all $\operatorname{ch}(Z_i)^1$. This leads easily to (4). (Details are left to the reader. Notice that the contribution from β is 0 if $n_\beta = p$.)

The $\mu - 2(p-1)\rho$-weight space in each Z_i has dimension 1. By 5.11(2) it is contained in the kernel of φ_i whenever $\beta = w_i(\alpha_i) \in R_\pi$ and $n_\beta < p$. This fact together with the discussion above implies easily that $Z_F(\mu)_{\mu-2(p-1)\rho}$ is contained in $Z_F(\mu)^{N(\mu)}$ and not in $Z_F(\mu)^{N(\mu)+1}$. Because this weight space generates the socle of $Z_F(\mu)$, we see that $Z_F(\mu)^{N(\mu)}$ is the last nonzero term in the filtration.

Composing φ with an isomorphism as in Lemma 4.10 we get a homomorphism

$$f : Z_{\widehat{A}}(\mu) \to Z_{\widehat{A}}(\mu)^\tau.$$

We can use it to define a contravariant bilinear form on $Z_{\widehat{A}}(\mu)$ via $(x, y) = (f(x))(y)$. As usual, we get then a non-degenerate contravariant form on each $Z_F(\mu)^j / Z_F(\mu)^{j+1}$, i.e., an isomorphism

$$Z_F(\mu)^j / Z_F(\mu)^{j+1} \simeq \left(Z_F(\mu)^j / Z_F(\mu)^{j+1}\right)^\tau. \tag{5}$$

We get especially an isomorphism

$$Z_F(\mu)^{N(\mu)} \simeq \left(Z_F(\mu)^{N(\mu)}\right)^\tau.$$

Because the simple socle of $Z_F(\mu)$ occurs with multiplicity 1 in this term — the $\mu - 2(p-1)\rho$–weight space has dimension 1 — it splits off. Therefore it has to be equal to this term.

6.7. Recall the notations $s_{\beta,m}$ and $w{\bullet}\lambda$ from the list of basic notations. Denote by W_p the group generated by all $s_{\beta,mp}$ with $\beta \in R$ and $m \in \mathbf{Z}$. This group is isomorphic to the affine Weyl group W_a of R (under $s_{\beta,mp} \mapsto s_{\beta,m}$).

For any U^0–algebra A with structural map π denote by $W_{\pi,p}$ the group generated by all $s_{\beta,mp}$ with $\beta \in R_\pi$ and $m \in \mathbf{Z}$. If R_π is a root system (for example, if A is field, cf. 5.2), then $W_{\pi,p}$ is isomorphic to the affine Weyl group of R_π.

As observed in 6.2, there is a strong linkage principle for the $Z_F(\mu)$. Let us now state the (weaker) linkage principle in the special case of a B–algebra.

Lemma: *Suppose that the field F is a B–algebra. Let $\lambda, \mu \in X$. If $L_F(\lambda)$ is a composition factor of $Z_F(\mu)$, then $\lambda \in W_{\pi,p}{\bullet}\mu$.*

Proof: Under our assumption the integers $n_\beta = n_\beta(\mu)$ in 6.1(2) are determined by $0 < n_\beta \leq p$ and

$$\langle \mu + \rho, \beta^\vee \rangle \equiv n_\beta \pmod{p} \tag{1}$$

for all $\beta \in R_\pi$. This implies for each $\beta \in R_\pi^+$:

$$\mu - n_\beta \beta = s_{\beta,mp}{\bullet}\mu \qquad \text{where } mp = \langle \mu + \rho, \beta^\vee \rangle - n_\beta.$$

So the claim follows from Lemma 6.2 by induction on $\mu - \lambda$ for fixed λ.

6.8. The group W_p acts on the Euclidean space $X_{\mathbf{R}} = X \otimes_{\mathbf{Z}} \mathbf{R}$ as an affine reflection group. It defines a system of facets, alcoves and walls in $X_{\mathbf{R}}$, cf. [Bou2], chap. VI, §2, and chap. V, §1 and §3. For an explicit description of a facet and of its closure one may compare [Ja6], II.6.2(1), (2). Inside the closure of a facet F there is a special subset that is called the upper closure of F, cf. [Ja6], II.6.2(3). One defines symmetrically a lower closure of F.

Suppose that R' is a subset of R with $s_\beta(R') = R'$ for all $\beta \in R'$. Then R' is a root system in the vector space generated by R'. Set W' equal to the subgroup of W_p generated by all $s_{\beta,mp}$ with $\beta \in R'$ and $m \in \mathbf{Z}$. Then also W' acts on $X_{\mathbf{R}}$ as an affine reflection group. We get a system of facets with respect to W' and we can define upper and lower closures as before.

Let us look especially at the case where $R' = \{\pm\beta\}$ for some $\beta \in R^+$. Suppose that $\lambda \in X_{\mathbf{R}}$ satisfies

$$np < \langle \lambda + \rho, \beta^\vee \rangle < (n+1)p \tag{1}$$

for some integer n. The facet of λ with respect to $W' = < s_{\beta,mp} \mid m \in \mathbf{Z} >$ consists of all μ with

$$np < \langle \mu + \rho, \beta^\vee \rangle < (n+1)p, \tag{2}$$

the closure of that facet consists of all μ with

$$np \leq \langle \mu + \rho, \beta^\vee \rangle \leq (n+1)p, \tag{3}$$

its upper closure consists of all μ with

$$np < \langle \mu + \rho, \beta^\vee \rangle \leq (n+1)p, \tag{4}$$

and its lower closure of all μ with

$$np \leq \langle \mu + \rho, \beta^\vee \rangle < (n+1)p. \tag{5}$$

If $\lambda \in X_{\mathbf{R}}$ satisfies $np = \langle \lambda + \rho, \beta^\vee \rangle$ for some integer n, then the facet of λ with respect to W' consists of all μ with $np = \langle \mu + \rho, \beta^\vee \rangle$, and this facet is equal to its closure as well as to its upper and lower closures.

6.9. We define a partition of X into disjoint subsets that are called the *blocks* over A. We require that λ and μ belong to the same block, if $\mathrm{Hom}_{\mathcal{C}_A}(Z_A(\lambda), Z_A(\mu)) \neq 0$ or if $\mathrm{Ext}^1_{\mathcal{C}_A}(Z_A(\lambda), Z_A(\mu)) \neq 0$. We take then the finest possible partition with this property.

Denote by $\mathcal{D}_A$ the full subcategory of $\mathcal{C}_A$ containing exactly all objects with a Z–filtration. If b is a block over A, denote by $\mathcal{D}_A(b)$ the full subcategory of all N in $\mathcal{D}_A$ where the factors in a Z–filtration involve only $Z_A(\mu)$ with $\mu \in b$.

Proposition: *a) If b, b' are blocks with $b \neq b'$, then* $\mathrm{Hom}_{\mathcal{D}_A}(M, M') = 0$ *for all M in $\mathcal{D}_A(b)$ and M' in $\mathcal{D}_A(b')$.*

b) Each M in $\mathcal{D}_A$ has a unique decomposition $M = \bigoplus_b M_b$, where the sum is over all blocks b over A, such that each M_b is in $\mathcal{D}_A(b)$.

Proof: This follows directly from the definition of a block.

6.10. Set $\mathcal{C}_A(b)$ equal to the full subcategory of all modules M in $\mathcal{C}_A$ such that M is a homomorphic image of a module in $\mathcal{D}_A(b)$.

Any N in $\mathcal{D}_A(b)$ is by Lemma 2.13 a homomorphic image of a projective module Q in $\mathcal{D}_A$. By Proposition 6.9 it is then also the image of Q_b, so we may assume that Q is in $\mathcal{D}_A(b)$. This implies now that also any M in $\mathcal{C}_A(b)$ is a homomorphic image of a projective module in $\mathcal{D}_A(b)$.

Theorem: *a) If b, b' are blocks with $b \neq b'$, then $\mathrm{Hom}_{\mathcal{C}_A}(M, M') = 0$ for all M in $\mathcal{C}_A(b)$ and M' in $\mathcal{C}_A(b')$.*

b) Any M in $\mathcal{C}_A$ has a unique decomposition $M = \bigoplus_b M_b$, where the sum is over all blocks b over A, such that each M_b is in $\mathcal{C}_A(b)$.

c) Each $\mathcal{C}_A(b)$ is closed under homomorphic images, submodules, extensions, and finite direct sums.

Proof: a) Let $f : Q \to M$ and $f' : Q' \to M'$ be surjective homomorphisms with Q projective in $\mathcal{D}_A(b)$ and Q' projective in $\mathcal{D}_A(b')$. If $g : M \to M'$ is a homomorphism, then the projectivity of Q yields a homomorphism $g' : Q \to Q'$ with $f' \circ g' = g \circ f$. Now $g' = 0$ by 6.9.a, hence $g = 0$ by the surjectivity of f.

b) There is a projective module Q in $\mathcal{D}_A$ with a surjection $f : Q \to M$. We have $Q = \bigoplus_b Q_b$ as in 6.9, hence $M = \sum_b M_b$ where $M_b = f(Q_b)$ for all blocks b. Obviously each M_b is in $\mathcal{C}_A(b)$. The directness of the decomposition of M as well as its uniqueness follow from part a).

c) This is clear.

Remark: Any M in $\mathcal{C}_A(b)$ for some block b has a projective resolution completely contained in $\mathcal{C}_A(b)$. This implies for all M, M' as in a) that $\mathrm{Ext}^i_{\mathcal{C}_A}(M, M') = 0$ for all $i \geq 0$. Therefore we could admit arbitrary Ext groups in the definition of a block, not just $\mathrm{Ext}^0 = \mathrm{Hom}$ and Ext^1.

6.11. We say that a module M in $\mathcal{C}_A$ belongs to a block b if M is in $\mathcal{C}_A(b)$.

Lemma: *Let $\lambda \in X$. Each $Z^w_A(\lambda\langle w\rangle)$ with $w \in W$ belongs to the block of λ.*

Proof: By 4.7(6) the endomorphism algebra of $M = Z^w_A(\lambda\langle w\rangle)$ is A. Therefore each nonzero summand M_b as in 6.10.b has the form aM with $a \in A$ idempotent, $a \neq 0$. If f is a basis of the A–module $\mathrm{Hom}_{\mathcal{C}_A}(Z_A(\lambda), M)$, cf. 4.7(6), then af is a nonzero homomorphism from $Z_A(\lambda)$ to M_b. So 6.10.a implies that b is the block of λ. The claim follows.

6.12. In the case of a field one defines usually blocks via simple modules. We show now that our definition coincides with the usual one.

Lemma: *Suppose that $A = F$ is a field. The blocks over F can also be characterized as giving the finest partition of X such that λ, μ belong to the same block if $L_F(\lambda)$ and $L_F(\mu)$ have a non trivial extension.*

Proof: Let us call Blocks (with a capital B) the parts of the partition defined in the lemma. If λ, λ' belong to blocks b and b', then $L_F(\lambda)$ is in $\mathcal{C}_F(b)$ and $L_F(\lambda')$ is in $\mathcal{C}_F(b')$. If $b \neq b'$, then $\mathrm{Ext}^1_{\mathcal{C}_F}(L_F(\lambda), L_F(\lambda')) = 0$. Therefore each Block is contained in a block.

In order to get the other direction, note that Theorem 6.10 generalizes obviously to Blocks: For every Block b let $\mathcal{C}_F(b)'$ consist of all M in $\mathcal{C}_F$ with composition factors only among the $L_F(\lambda)$ with $\lambda \in b$. If M in $\mathcal{C}_F(b)'$ and M' in $\mathcal{C}_F(b')'$ with $b \neq b'$, then obviously $\mathrm{Hom}_{\mathcal{C}_F}(M, M') = 0 = \mathrm{Ext}^1_{\mathcal{C}_F}(M, M')$. Furthermore every M in $\mathcal{C}_F$ admits a unique decomposition $M = \bigoplus_b M_{(b)}$ with b running over all Blocks such that $M_{(b)}$ is in $\mathcal{C}_F(b)'$.

If λ belongs to a Block b, then $Z_F(\lambda)$ being indecomposable is in $\mathcal{C}_F(b)'$. If λ' belongs to a different Block b', then the groups $\mathrm{Hom}_{\mathcal{C}_F}(Z_F(\lambda), Z_F(\lambda'))$ and $\mathrm{Ext}^1_{\mathcal{C}_F}(Z_F(\lambda), Z_F(\lambda'))$ vanish. Therefore each block is contained in a Block. This proves the lemma.

6.13. Proposition: *If A is a B–algebra, then the block of each $\mu \in X$ is contained in $W_{\pi,p} \bullet \mu$.*

Proof: If A is a field, then the claim follows from Lemma 6.12 together with 4.6(2) and Lemma 6.7.

Consider now arbitrary A; let $\lambda, \mu \in X$. For each A–algebra A' and each i use the abbreviation

$$E^i_{A'} = \mathrm{Ext}^i_{\mathcal{C}_{A'}}(Z_{A'}(\lambda), Z_{A'}(\mu)).$$

Suppose that there is an i with $E^i_A \neq 0$. We have to show that $\lambda \in W_{\pi,p} \bullet \mu$. It will be enough to find a maximal ideal $\mathfrak{m}$ of A and an index $j \geq 0$ with $E^j_{A/\mathfrak{m}} \neq 0$. Then we can use the result in the case of a field, since each $A/\mathfrak{m}$ is a B–algebra and since the analogue of $W_{\pi,p}$ for $A/\mathfrak{m}$ is contained in $W_{\pi,p}$.

Well, suppose that $E^j_{A/\mathfrak{m}} = 0$ for all $j > 0$ and all $\mathfrak{m}$. Then Proposition 3.4 implies that $E^i_A = 0$ for all $i > 0$. So our assumption implies that $E^0_A \neq 0$. Now 3.4(3) shows that there is a maximal ideal $\mathfrak{m}$ in A with $E^0_{A/\mathfrak{m}} \neq 0$.

Remark: Suppose that A is a $B^{\emptyset}$–algebra. Then the block of each $\mu \in X$ is equal to $\{\mu\}$. (This is the case $R_\pi = \emptyset$ of the Proposition. It can be proved more directly using the remark in 6.3 instead of Lemma 6.7.) By Corollary 3.5 and 6.3(2) each $Z_A(\mu)$ is a projective module in $\mathcal{C}_A$. Any M in $\mathcal{C}_A(\{\mu\})$ is by definition a homomorphic image of $Z_A(\mu)^m$ for some integer $m \geq 0$. So $Z_A(\mu)$ is a projective generator for $\mathcal{C}_A(\{\mu\})$.

6.14. Proposition: *Let A' be an A-algebra. Then each block over A' is contained in some block over A.*

Proof: Let $\lambda \in X$ be a weight and b the block containing it. Using 6.10 we see that $Z_A(\lambda)$ has a resolution $P^\bullet \to Z_A(\lambda)$ by projective objects P^i that lie in $\mathcal{D}_A(b)$. Since $Z_A(\lambda)$ and all the P^i are free over A, this complex remains exact when we apply $\otimes_A A'$ to it. By Lemma 3.1 the $P^i \otimes_A A'$ are projective in $\mathcal{C}_{A'}$.

Let now $\mu \in X$ be another weight, belonging to the block b', and let $Q^\bullet \to Z_A(\mu)$ be a projective resolution with all Q^i in $\mathcal{D}_A(b')$. By standard homological algebra

$$\mathrm{Ext}^i_{\mathcal{C}_{A'}}(Z_{A'}(\lambda), Z_{A'}(\mu))$$

can be identified with the set of homotopy classes of chain maps from $P^\bullet \otimes_A A'$ to $Q^{\bullet+i} \otimes_A A'$. But if $b \neq b'$, then $\mathrm{Hom}_{\mathcal{C}_{A'}}(P^n \otimes_A A', Q^m \otimes_A A') = \mathrm{Hom}_{\mathcal{C}_A}(P^n, Q^m) \otimes_A A' = 0$ for all n, m, where we use first 3.3 and then 6.9.a. Hence $\mathrm{Ext}^i_{\mathcal{C}_{A'}}(Z_{A'}(\lambda), Z_{A'}(\mu)) = 0$ for all i and this proves the proposition.

6.15. Lemma: *Let $\beta \in R^+$. Suppose that A is a B^β-algebra, that is not a $B^\emptyset$-algebra. Let $\mu \in X$. If $\langle \mu + \rho, \beta^\vee \rangle \equiv 0 \pmod p$, then the block of μ is equal to $\{\mu\}$. Otherwise it is equal to*

$$\{\mu + ip\beta,\ \mu + (ip - \langle \mu + \rho, \beta^\vee \rangle)\beta \mid i \in \mathbf{Z}\}. \tag{1}$$

Proof: If A is a field, then the claim follows from the discussion in 6.4. For arbitrary A we argue as in the proof of 6.13. We get thus in the first case that the block of μ is equal to $\{\mu\}$ as desired, in the second case that it is contained in the set in (1). (Of course, that is also a special case of Proposition 6.13, since this set is equal to $W_{\pi,p} \bullet \mu$.) In order to see that the block is equal to this set (and not smaller), choose a maximal ideal $\mathfrak{m}$ of A containing the image of H_β resp. of $[K_\beta; 0]$. Then the block of μ over $A/\mathfrak{m}$ is equal to the set in (1) by the result for a field. The claim follows now from Proposition 6.14.

6.16. Suppose in this subsection that $A = F$ is a field that is a B–algebra. We know that R_π is a root system, cf. 5.2. For any $\lambda \in X$ let R_F^λ be the union of all irreducible components R_1 of R_π such that there is an $\alpha \in R_1$ with $\langle \lambda + \rho, \alpha^\vee \rangle \not\equiv 0 \pmod p$. Set $W_{F,p}^\lambda$ equal to the affine Weyl group for R_F^λ generated by all $s_{\alpha,mp}$ with $\alpha \in R_F^\lambda$ and $m \in \mathbf{Z}$. We have obviously $R_F^{\lambda+p\nu} = R_F^\lambda$ and $W_{F,p}^{\lambda+p\nu} = W_{F,p}^\lambda$ for all $\lambda, \nu \in X$.

Proposition: *For each $\lambda \in X$ the block of λ over F is equal to $W_{F,p}^\lambda \bullet \lambda$.*

Proof: Let us denote the block of any λ over F by $b_F(\lambda)$. We get the inclusion $b_F(\lambda) \subset W_{F,p}^\lambda \bullet \lambda$ arguing as in 6.13, since we can easily replace $W_{\pi,p}$ by $W_{F,p}^\mu$ in Lemma 6.7.

So we have to prove the other inclusion. We are going to use the notation $n_\beta(\mu)$ as in 6.1(2). So the $n_\beta(\mu)$ are integers between 1 and p with $\langle \mu+\rho, \beta^\vee \rangle \equiv$

$n_\beta(\mu) \pmod p$ for all $\beta \in R_\pi^+$, cf. 6.7(1). We want to show first for all $\beta \in R_\pi^+$ and all $\lambda \in X$:

$$\text{If } n_\beta(\lambda) < p, \text{ then } \lambda - n_\beta(\lambda)\beta \in b_F(\lambda). \tag{1}$$

If we have (1) for a given β and all λ, then we get for that β and all λ

$$\text{If } n_\beta(\lambda) < p, \text{ then } \lambda - (n_\beta(\lambda) + pm)\beta, \lambda - pm\beta \in b_F(\lambda) \text{ for all } m \in \mathbf{Z}. \tag{2}$$

(Note that $n_\beta(\lambda - n_\beta(\lambda)) = p - n_\beta(\lambda)$ in case $n_\beta(\lambda) < p$, and use induction on $|m|$.) We want to prove (1) by induction on the height of β and may assume (2) for all positive roots in R_π of smaller height. Well, if $n_\beta(\lambda) < p$, then ch $Z_F(\lambda - n_\beta(\lambda))$ is one of the summands in the sum formula 6.6(4). So we get $[Z_F(\lambda) : L_F(\lambda - n_\beta(\lambda)\beta)] \neq 0$ or there is an $\alpha \in R_\pi^+$ with $n_\alpha(\lambda) < p$ and $[Z_F(\lambda - ps\alpha) : L_F(\lambda - n_\beta\beta)] \neq 0$ for some integer $s > 0$. In the first case (1) is obvious. In the second one we get first $\lambda - n_\beta(\lambda)\beta \in b_F(\lambda - ps\alpha)$; since the height of α has to be smaller than that of β, we know by induction that $b_F(\lambda - ps\alpha) = b_F(\lambda)$, and (1) follows.

Let us now fix an irreducible component R_1 of R_π. We have to show that $s_{\alpha,mp} \bullet \lambda \in b_F(\lambda)$ for all $\lambda \in X$ with $R_1 \subset R_F^\lambda$ and all $\alpha \in R_1 \cap R^+$ and $m \in \mathbf{Z}$. If $n_\alpha(\lambda) < p$, then this follows from (2). If $n_\alpha(\lambda) = p$, then the set of all $s_{\alpha,mp} \bullet \lambda$ with $m \in \mathbf{Z}$ is equal to the set of all $\lambda - mp\alpha$ with $m \in \mathbf{Z}$. So we have to show for all $\lambda \in X$

$$\text{If } R_1 \subset R_F^\lambda, \text{ then } \lambda - p\alpha \in b_F(\lambda). \tag{3}$$

(We get then $\lambda - pm\alpha \in b_F(\lambda)$ for all $m \in \mathbf{Z}$ by induction on $|m|$.)

Choose a basis Σ_1 of the root system R_1 such that $R_1 \cap R^+$ is the corresponding positive system. It is enough to prove (3) for $\alpha \in \Sigma_1$. If there is some $\beta \in R_1$ with $\langle \lambda + \rho, \beta^\vee \rangle \not\equiv 0 \pmod p$, then there is also a $\beta \in \Sigma_1$ with this property. So, given $\alpha \in \Sigma_1$ and $\lambda \in X$ with $R_1 \subset R_F^\lambda$, we can find a sequence $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_s$ in Σ_1 with $\langle \lambda + \rho, \alpha_s^\vee \rangle \not\equiv 0 \pmod p$ and $\langle \alpha_i, \alpha_{i+1}^\vee \rangle < 0$ for all $i < s$. We may assume that s is minimal with this property (given λ and α); this implies in particular that $\langle \lambda + \rho, \alpha_i^\vee \rangle \equiv 0 \pmod p$ for all $i < s$ and $\langle \alpha_s, \alpha_i^\vee \rangle = 0$ for all $i < s - 1$. We use induction on s (for all possible λ). If $s = 1$, then $n_\alpha(\lambda) < p$ and (3) follows from (2). So suppose that $s > 1$. Set $b = n_{\alpha_s}(\lambda)$ and $\lambda' = \lambda - b\alpha_s$. We have $b_F(\lambda) = b_F(\lambda')$ and $b_F(\lambda - p\alpha) = b_F(\lambda' - p\alpha)$ by (2) applied to α_s. If $\langle \lambda' + \rho, \alpha_{s-1}^\vee \rangle \not\equiv 0 \pmod p$, then we can use the induction on s and get $b_F(\lambda') = b_F(\lambda' - p\alpha)$, hence $b_F(\lambda) = b_F(\lambda - p\alpha)$ as desired.

Suppose now that $\langle \lambda' + \rho, \alpha_{s-1}^\vee \rangle \equiv 0 \pmod p$; We have by construction $\langle \lambda' + \rho, \alpha_{s-1}^\vee \rangle \equiv -b\langle \alpha_s, \alpha_{s-1}^\vee \rangle \pmod p$, so we get $b\langle \alpha_s, \alpha_{s-1}^\vee \rangle \equiv 0 \pmod p$. Since $0 < b < p$, this is impossible if $\langle \alpha_s, \alpha_{s-1}^\vee \rangle = -1$. If $\langle \alpha_s, \alpha_{s-1}^\vee \rangle = -2$, then p has to be even and $b = p/2$. Then $(\alpha_1, \alpha_2, \dots, \alpha_s)$ is a basis of a root system of type C_s with α_s a long root. (Use the minimality of s.) We have then $\beta = 2\alpha_2 + \cdots + 2\alpha_{s-1} + \alpha_s \in R_1$ and $\gamma = 2\alpha_1 + \beta \in R_1$, and we have $\beta^\vee = \alpha_2^\vee + \cdots + \alpha_{s-1}^\vee + \alpha_s^\vee$ and $\gamma^\vee = \alpha_1^\vee + \beta^\vee$. We have then $n_\gamma(\lambda) = b = n_\beta(\lambda - p\alpha)$, hence — using (2) —

$$b_F(\lambda) = b_F(\lambda - b\gamma = \lambda - p\alpha - b\beta) = b_F(\lambda - p\alpha).$$

If $\langle \alpha_s, \alpha_{s-1}^\vee \rangle = -3$, then $s = 2$ and (α_1, α_2) is a basis of a root system of type G_2 with α_2 a long root. Furthermore p is divisible by 3 and $b \in \{p/3, 2p/3\}$. Take now $\beta = \alpha_2$ and $\gamma = 3\alpha_1 + \alpha_2$ in case $b = p/3$, resp. $\gamma = 3\alpha_1 + 2\alpha_2$ in case $b = 2p/3$. We have in both cases $n_\gamma(\lambda) = p/3$ and can argue as above.

Remark: The theorem implies (in the notation of its proof) that

$$W_{\pi,p} \bullet \lambda = b_F(\lambda) + p\mathbf{Z}R_\pi \qquad \text{for all } \lambda \in X. \tag{4}$$

In fact, we can derive this formula much faster: it follows already from (2).

6.17. If Ω is a union of blocks for A, we can define categories $\mathcal{D}_A(\Omega)$ and $\mathcal{C}_A(\Omega)$ extending the definitions in 6.7 and 6.9. For M in $\mathcal{C}_A$ set (in the notations from 6.10.b)

$$\mathrm{pr}_\Omega M = \bigoplus_{b \subset \Omega} M_b. \tag{1}$$

Then pr_Ω is an exact functor from $\mathcal{C}_A$ to $\mathcal{C}_A(\Omega)$, and we have $\mathrm{pr}_\Omega M = M$ if and only if M is in $\mathcal{C}_A(\Omega)$. If X is the disjoint union of subsets $(\Omega_i)_{i \in I}$ such that each Ω_i is a union of blocks, then one has for all M in $\mathcal{C}_A$

$$M = \bigoplus_{i \in I} \mathrm{pr}_{\Omega_i} M. \tag{2}$$

We get thus an isomorphism between $\mathcal{C}_A$ and the direct sum of all $\mathcal{C}_A(\Omega_i)$.

By abuse of notation we shall sometimes write pr_Ω for the projection map $M \to \mathrm{pr}_\Omega M$ with kernel equal to the direct sum of all M_b with $b \not\subset \Omega$.

If A is a B–algebra, then any $W_{\pi,p}$–orbit Ω in X is a union of blocks (by Proposition 6.13). We shall mainly work with Ω of this type. The family of all $W_{\pi,p}$–orbits is then a possible choice for the Ω_i in (2).

Any module in $\mathcal{C}_A(\Omega)$ has a projective resolution such that all terms in the resolution are in $\mathcal{D}_A(\Omega)$. So the Ext groups of two modules in $\mathcal{C}_A(\Omega)$ are the same, whether computed in $\mathcal{C}_A$ or in $\mathcal{C}_A(\Omega)$. If Ω' is another union of blocks for A with $\Omega \cap \Omega' = \emptyset$, then

$$\mathrm{Ext}^i_{\mathcal{C}_A}(M, M') = 0 \qquad \text{for all } i \geq 0 \tag{3}$$

whenever M in $\mathcal{C}_A(\Omega)$ and M' in $\mathcal{C}_A(\Omega')$.

Let A' be a A–algebra. By Proposition 6.14 any block for A is a union of blocks for A'. So for Ω as above, $\mathcal{C}_{A'}(\Omega)$ makes sense, and M in $\mathcal{C}_A(\Omega)$ implies $M \otimes_A A'$ in $\mathcal{C}_{A'}(\Omega)$. Similarly, M in $\mathcal{D}_A(\Omega)$ implies $M \otimes_A A'$ in $\mathcal{D}_{A'}(\Omega)$. (Note: If M is in $\mathcal{D}_A$, then $M \otimes_A A'$ is in $\mathcal{D}_{A'}$, cf. Lemma 3.1.b.) One has the functors pr_Ω also over A'. They commute obviously with base change: Given M in $\mathcal{C}_A$ one has

$$\mathrm{pr}_\Omega(M \otimes_A A') = (\mathrm{pr}_\Omega M) \otimes_A A'. \tag{4}$$

6.18 Let A be a B–algebra and let Ω be a union of orbits for $W_{\pi,p}$ in X. So Ω is nion of blocks over A; the categories $\mathcal{C}_A(\Omega)$ and $\mathcal{D}_A(\Omega)$ are defined.

For each $\nu \in \mathbf{Z}R_\pi$ the translation by $p\nu$ is an element of $W_{\pi,p}$. Therefore $\lambda \mapsto \lambda + p\nu$ maps Ω bijectively to itself. The functor $M \mapsto M[p\nu]$ as in 4.2 maps any $Z_A(\mu)$ to $Z_A(\mu + p\nu)$ and is clearly exact. So it preserves $\mathcal{D}_A(\Omega)$, hence also $\mathcal{C}_A(\Omega)$. Now 4.2(4) shows that $\mathcal{C}_A(\Omega)$ is a $(p\mathbf{Z}R_\pi)$–category in the sense of Appendix E.3.

Suppose that Ω is just one orbit for $W_{\pi,p}$ in X. Then the group of translations by elements in $p\mathbf{Z}R_\pi$ has only finitely many orbits in Ω, in other words, there are $\mu_1, \mu_2, \dots, \mu_r \in \Omega$ such that Ω is the union of all $\mu_i + p\mathbf{Z}R_\pi$. There is for each i a projective module Q_i in $\mathcal{C}_A(\Omega)$ that maps onto $Z_A(\mu_i)$. For each $\lambda \in \Omega$ there are an i and a $\nu \in \mathbf{Z}R_\pi$ with $\lambda = \mu_i + p\nu$; then $Z_A(\lambda)$ is a homomorphic image of $Q_i[p\nu]$. We can now deduce that each M in $\mathcal{D}_A(\Omega)$ is a homomorphic image of a (finite) direct sum of modules of the form $Q_i[p\nu_{ij}]$ with $1 \leq i \leq r$ and $\nu_{ij} \in p\mathbf{Z}R_\pi$. (Use induction on the length of a Z–filtration.) But then also each module in $\mathcal{C}_A(\Omega)$ is a homomorphic image of such a sum. So $P = \bigoplus_{i=1}^r Q_i$ is a projective $(p\mathbf{Z}R_\pi)$–generator of $\mathcal{C}_A(\Omega)$ in the sense of E.3.

We can carry out the constructions above with W_p instead of $W_{\pi,p}$. If Ω is an orbit of W_p in X, then $\mathcal{C}_A(\Omega)$ is in a natural way a $(p\mathbf{Z}R)$–category, and we can find a projective module P in $\mathcal{C}_A(\Omega)$ that is a $(p\mathbf{Z}R)$–generator of $\mathcal{C}_A(\Omega)$.

7. Translation Functors

7.1. In order to construct tensor products of modules we need a comultiplication on U. By construction, $U = U'/I$ where U' is an algebra that has already a comultiplication Δ and where I is a twosided ideal in U'. In Case 1 we have $U' = U(\mathfrak{g})$ and Δ is given by $\Delta(X) = X \otimes 1 + 1 \otimes X$ for all $X \in \mathfrak{g}$. In Case 2 we have $U' = U_2$ (cf. 1.2) and Δ is given by

$$\begin{aligned}
\Delta(K_\alpha) &= K_\alpha \otimes K_\alpha, \\
\Delta(E_\alpha) &= E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, \\
\Delta(F_\alpha) &= 1 \otimes F_\alpha + F_\alpha \otimes K_\alpha^{-1}
\end{aligned}$$

for all $\alpha \in \Sigma$.

Proposition: *Δ induces a comultiplication on U.*

Proof: This is easy in Case 1 where $\Delta(E_\beta^p) = E_\beta^p \otimes 1 + 1 \otimes E_\beta^p$ for all $\beta \in R$.

The situation is more complicated in Case 2. In order to prove our claim we have to show that

$$\Delta(I) \subset I \otimes U_2 \; + \; U_2 \otimes I. \tag{1}$$

Since Δ is a homomorphism and I is generated by I^+ and I^- it will be enough to show that

$$\Delta(I^+) \subset I^+ \otimes U_2^+ \; + \; U_2^0 U_2^+ \otimes I^+ \tag{2}$$

and

$$\Delta(I^-) \subset I^- \otimes U_2^- U_2^0 \; + \; U_2^- \otimes I^-. \tag{3}$$

The proofs of these two claims are similar. We shall carry out only that of (3). One has by construction $\Delta(U_2^-) \subset U_2^- \otimes U_2^- U_2^0$. We want to show for any $\beta \in R^+$ that

$$\Delta(E_{-\beta}^p) \in I^- \otimes U_2^- U_2^0 \; + \; U_2^- \otimes I^-. \tag{4}$$

Let $(u_i)_{1 \le i \le s}$ be a basis of $\bigoplus_{\nu \ge -p\beta} (U_2^-)_\nu$ consisting of all monomials in a PBW–type basis of U_2^- having weights $\ge -p\beta$. We want to assume that the numbering is chosen such that all u_i with $i \leq r$ (for some index r) are exactly those monomials where all exponents are $\leq p-1$. Let $(t_j)_{j\in J}$ be a basis of $U_2^0 = U^0$. There are uniquely determined elements $u_{ij} \in U_2^-$ (almost all equal to 0) with

$$\Delta(E_{-\beta}^p) = \sum_{i=1}^{s} \sum_{j\in J} u_{ij} \otimes u_i t_j. \tag{5}$$

Denote the weight of u_i by μ_i. Then each u_{ij} has weight $-p\beta - \mu_i$. Claim (4) is equivalent to

$$\text{If } 1 \leq i \leq r, \text{ then } u_{ij} \in I^- \text{ for all } j \in J. \tag{6}$$

The counit ε on U_2 satisfies $(\varepsilon \otimes 1)\circ\Delta = \mathrm{id}$. This implies

$$E_{-\beta}^p = \sum_{i=1}^{s} \sum_{j\in J} \varepsilon(u_{ij}) u_i t_j.$$

One has $\varepsilon(1) = 1$ and ε maps each $(U_2^-)_\nu$ with $\nu \neq 0$ to 0. Now u_{ij} has weight 0, if and only if $\mu_i = -p\beta$. This implies $\varepsilon(u_{ij}) = 0$ whenever $\mu_i \neq -p\beta$. If $\mu_i = -p\beta$, however, then $u_{ij} \in k$ and $\varepsilon(u_{ij}) = u_{ij}$. Now $E_{-\beta}^p$ itself is one of the u_i with $i > r$. Therefore we get:

$$u_{ij} = 0 \text{ for all } i \leq r \text{ with } \mu_i = -p\beta \text{ and all } j \in J. \tag{7}$$

Suppose that (6) does not hold. Let $\nu \geq -p\beta$ be minimal for: There exist $i \leq r$ with $\mu_i = \nu$ and $u_{ij} \notin I^-$ for some $j \in J$. We have $\nu > -p\beta$ by (7).

We can consider any $\mathbf{u}$–module as a U_2–module via the homomorphism $f : U_2 \to \mathbf{u} \subset U_3$ as in 1.3. Take for example $M_1 = Z_k((p-1)\rho)$ with its standard generator m_1. Because $u \mapsto u m_1$ is a bijection from U^- to M_1, the annihilator of m_1 in U_2^- is equal to the kernel of f in U_2^-, i.e., to I^-.

Let K be the field of fractions of $U_2^0 = U^0$. Consider $M_2 = Z_K(0)$ in $\mathcal{C}_K$ and its standard generator m_2. Regard M_2 as a module over U_2. One has $u_i m_2 = 0$ for all $i > r$, because these u_i are in the kernel of $U_2 \to U$ and annihilate M_2. The $u_i m_2$ with $1 \leq i \leq r$ are linearly independent over K, hence the $u_i m_2 t_j$ with $i \leq r$ and $j \in J$ are linearly independent over k.

We can now regard $M_1 \otimes M_2$ as a module over U_2 using Δ. One has

$$E_{-\beta}^p(m_1 \otimes m_2) = \sum_{i=1}^{r} \sum_{j\in J} u_{ij} m_1 \otimes u_i m_2 t_j \tag{8}$$

since $t m_2 = m_2 t$ for all $t \in U^0$. All summands with $u_{ij} \in I^-$ are 0.

Because $E^p_{-\beta}$ is central in U_2, it commutes with all E_α with $\alpha \in \Sigma$. So $E_\alpha(m_1 \otimes m_2) = 0$ implies $0 = E_\alpha E^p_{-\beta}(m_1 \otimes m_2)$, hence

$$0 = \sum_{i=1}^{r} \sum_{j \in J} \left(E_\alpha u_{ij} m_1 \otimes u_i m_2 t_j + K_\alpha u_{ij} m_1 \otimes E_\alpha u_i m_2 t_j \right). \tag{9}$$

The choice of ν implies: The sum of all terms in (9) that belong to $Z_k((p-1)\rho) \otimes Z_K(0)_\nu$ is equal to

$$0 = \sum_{i} \sum_{j \in J} E_\alpha u_{ij} m_1 \otimes u_i m_2 t_j \tag{10}$$

where the sum over i is over all $i \leq r$ with $\mu_i = \nu$. Since the $u_i m_2 t_j$ are linearly independent over k, we get

$$0 = E_\alpha(u_{ij} m_1) \tag{11}$$

for all $j \in J$ and all i as above. This shows (for all these i and j) that $u_{ij} m_1$ is a vector of weight $(p-1)\rho - p\beta - \nu < (p-1)\rho$ in $Z_k((p-1)\rho)$ that is annihilated by all E_α with $\alpha \in \Sigma$. It therefore generates a proper submodule of $Z_k((p-1)\rho)$. On the other hand $Z_k((p-1)\rho)$ is irreducible by 6.3. Therefore $0 = u_{ij} m_1$ for all $j \in J$ and all $i \leq r$ with $\mu_i = \nu$. This implies now $u_{ij} \in I^-$ for all these i, j contradicting the choice of ν. So (6) has to be true.

7.2. It is clear by construction and by the proof in 7.1 that the comultiplication on U induces comultiplications on U^0, U^0U^+ and U^-U^0. In fact, we get on U and on these subalgebras structures of Hopf algebras. It is obvious that the counit factors through U. In Case 1 the antipode S on $U(\mathfrak{g})$ is given by $S(X) = -X$ for all $X \in \mathfrak{g}$. That induces clearly an antipode on U and those subalgebras. Again Case 2 is more complicated. On U_2 the antipode is given by

$$\begin{aligned} S(E_\alpha) &= -K_\alpha^{-1} E_\alpha, \\ S(F_\alpha) &= -F_\alpha K_\alpha, \\ S(K_\alpha) &= K_\alpha^{-1} \end{aligned}$$

for all $\alpha \in \Sigma$. Obviously U_2^0, $U_2^- U_2^0$, $U_2^0 U_2^+$ and all U_ν are stable under S. In order to show that S induces maps on U^0, U^-U^0, and U^0U^+ it suffices to prove that $S(I^+) \subset U_2^0 I^+$ and $S(I^-) \subset I^- U_2^0$. By symmetry it will be enough to treat the second case and to show

$$S(E^p_{-\beta}) \in I^- U_2^0 \tag{1}$$

for all $\beta \in R^+$. Consider the module M_2 and its generator m_2 as in the proof above. The annihilator of m_2 in $U_2^- U_2^0$ is exactly $I^- U_2^0$. So (1) is equivalent to:

$$S(E^p_{-\beta}) m_2 = 0. \tag{2}$$

Since S is an antiautomorphism of U_2 and since $E_{-\beta}^p$ is central in U_2, so is $S(E_{-\beta}^p)$. We get therefore for all $\alpha \in \Sigma$:

$$E_\alpha S(E_{-\beta}^p)m_2 = S(E_{-\beta}^p)E_\alpha m_2 = 0.$$

This implies (2) because $S(E_{-\beta}^p)m_2$ has weight $-p\beta$ and because $M_2 = Z_K(0)$ is simple by the remark in 6.3.

7.3. Consider k as an algebra over U^0 via the augmentation as in 2.4. For any module E in $\mathcal{C}_k$ and any module M in $\mathcal{C}_A$ we can make $E \otimes M$ into an object of $\mathcal{C}_A$ as follows: We let A act on the second factor only. As a U–action we take the tensor product of the two given representations using the comultiplication described above. This action commutes obviously with that of A. Finally we define the grading as usual giving any $E_\mu \otimes M_\nu$ degree $\mu+\nu$. This is obviously a grading by A–modules. One checks easily properties 2.3(C) and (D).

In the case of 2.4 this construction is the usual tensor product of G_1T–modules resp. of $\mathbf{u}$–modules. One has in general an obvious compatibility with base change:

$$(E \otimes M) \otimes_A A' \xrightarrow{\sim} E \otimes (M \otimes_A A'). \tag{1}$$

For all N in $\mathcal{C}_A$ the canonical isomorphism

$$\mathrm{Hom}_A(M, E \otimes N) \xrightarrow{\sim} \mathrm{Hom}_A(E^* \otimes M, N) \tag{2}$$

induces an isomorphism

$$\mathrm{Hom}_{\mathcal{C}_A}(M, E \otimes N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_A}(E^* \otimes M, N). \tag{3}$$

At this point we have to be careful in the quantum case with its complicated definition of the comultiplication. The trace map $\mathrm{tr} : E^* \otimes E \to k$ with $f \otimes e \mapsto f(e)$ is a homomorphism of U–modules. The map in (2) sends any $h : M \to E \otimes N$ to the composition of

$$\mathrm{id} \otimes h : E^* \otimes M \to E^* \otimes E \otimes N$$

with $\mathrm{tr} \otimes \mathrm{id}$, hence to a homomorphism of U–modules. The canonical map $E \otimes E^* \xrightarrow{\sim} \mathrm{End}_k(E)$ is an isomorphism of U–modules. Let ι be the inverse image of the identity map under this isomorphism; it is U–invariant. The inverse map in (2) sends any $h' : E^* \otimes M \to N$ to the composition

$$M \to E \otimes E^* \otimes M \to E \otimes N,$$

where the second map is $\mathrm{id} \otimes h'$ and the first one $v \mapsto \iota \otimes v$. The U–invariance of ι implies that this composition is a U–linear.

If we apply (3) to E^* instead of E, then we get an isomorphism

$$\mathrm{Hom}_{\mathcal{C}_A}(M, E^* \otimes N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_A}(E^{**} \otimes M, N).$$

This leads to

$$\mathrm{Hom}_{\mathcal{C}_A}(M, E^* \otimes N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_A}(E \otimes M, N) \tag{4}$$

as soon as we have an isomorphism $c : E \xrightarrow{\sim} E^{**}$. In Case 1 we can take the canonical map $c_1 : E \xrightarrow{\sim} E^{**}$ with $c_1(e)(f) = f(e)$. In Case 2, however, we have $uc_1(e) = c_1(S^2(u)e)$ for all $u \in U$ and $e \in E$. Now the square of the antipode S is given by

$$S^2(u) = K_{2\rho}^{-1} u K_{2\rho}.$$

So we can take in this case

$$c(e) = c_1(K_{2\rho}^{-1} e).$$

Both isomorphisms, (3) and (4), have the usual functorial properties in M and N.

7.4. One can define in the same way tensor products of modules in $\mathcal{C}'_k$ with modules in $\mathcal{C}'_A$, and of modules in $\mathcal{C}''_k$ with objects in $\mathcal{C}''_A$. One has then tensor identities: For example, if E is a module in $\mathcal{C}_k$ and M is in $\mathcal{C}'_A$, then there is a canonical isomorphism

$$Z_A(E \otimes M) \simeq E \otimes Z_A(M). \tag{1}$$

In order to prove this statement one checks that the right hand side has the universal property (cf. 2.10) of the left hand side using 7.3(4) and its analogue in $\mathcal{C}'_A$.

As usual one gets using the tensor identity:

If M in $\mathcal{C}_A$ has a Z-filtration, then so has each $E \otimes M$ with E in $\mathcal{C}_k$.

In other words, using the notation from 6.9, the subcategory $\mathcal{D}_A$ is stable under tensor products with finite dimensional modules in $\mathcal{C}_k$.

More precisely, each $E \otimes Z_A(\mu)$ has a filtration with factors $Z_A(\mu + \nu)$ each occurring $\dim E_\nu$ times. It can be constructed as in the classical case, see, e.g., [Ja2], 2.2: One takes a basis $(e_i)_{1 \leq i \leq n}$ of E such that e_i is in the ν_i weight space and such that $\nu_i < \nu_j$ implies $i > j$. Then each

$$V_i = \sum_{j=1}^{i} U_A(e_j \otimes Z_A(\mu))$$

is a submodule of $E \otimes Z_A(\mu)$ with $V_i/V_{i-1} \simeq Z_A(\mu + \nu_i)$. If v_μ spans the weight space $Z_A(\mu)_\mu$, then V_i/V_{i-1} is generated by the class of $e_i \otimes v_\mu$.

7.5. Suppose from now on that A is a B–algebra. Let W' be a subgroup of W_p that is generated by reflections and contains $W_{\pi,p}$. By Proposition 6.13 any orbit of W' in X is a union of blocks over A. (Here and below orbits are always orbits for the dot action.) We have thus for each orbit Ω of W' in X the subcategory $\mathcal{C}_A(\Omega)$ and the functor pr_Ω from $\mathcal{C}_A$ to $\mathcal{C}_A(\Omega)$ as in 6.17.

Let Ω, Γ be orbits for W' in X. Consider an alcove for the affine reflection group W'. Its closure contains exactly one element $\lambda \in \Omega$ and exactly one element $\mu \in \Gamma$. Then $W(\mu - \lambda)$ is independent of the choice of the alcove. Let ν be the unique dominant weight in $W(\mu - \lambda)$. Choose a simple module E with highest weight ν for G in Case 1, for U_3 in Case 2. Then E is a module in $\mathcal{C}_k$ with a W–invariant formal character.

For any M in $\mathcal{C}_A(\Omega)$ we can now define

$$T_\Omega^\Gamma M = \mathrm{pr}_\Gamma(E \otimes M). \tag{1}$$

This defines obviously an exact functor T_Ω^Γ from $\mathcal{C}_A(\Omega)$ to $\mathcal{C}_A(\Gamma)$ that we call the *translation functor* from Ω to Γ. It takes $\mathcal{D}_A(\Omega)$ to $\mathcal{D}_A(\Gamma)$, by 7.4 and 6.9.b.

If A' is an A–algebra, then we can define the translation functors also over A'. The natural isomorphism 7.3(1) induces then an isomorphism

$$T_\Omega^\Gamma(M \otimes_A A') \simeq T_\Omega^\Gamma(M) \otimes_A A' \tag{2}$$

for all M in $\mathcal{C}_A(\Omega)$.

Lemma: *Let $\lambda, \mu \in X$ be in the closure of a fixed alcove for W'. Then $T_\Omega^\Gamma Z_A(\lambda)$ has a Z–filtration with factors $Z_A(w_\bullet\mu)$ where w runs over a system of representatives of the stabilizer of λ in W' modulo its intersection with the stabilizer of μ. If v_λ spans the weight space $Z_A(\lambda)_\lambda$ and if e_w (for w as before) spans the weight space $E_{w\bullet\mu-\lambda} = E_{w(\mu-\lambda)}$, then the $\mathrm{pr}_\Gamma(e_w \otimes v_\lambda)$ generate $T_\Omega^\Gamma Z_A(\lambda)$.*

This is proved by the same arguments as in the classical case, e.g., in [Ja6], II.7.8.

7.6. Keep the notations from the last subsection. If we reverse the role of Γ and Ω, then we get a translation functor T_Γ^Ω. Its construction involves a simple module E' isomorphic to E^*. Fix an isomorphism $E' \xrightarrow{\sim} E^*$ (unique up to a scalar from k). Then we get from 7.3(4) an isomorphism (for M in $\mathcal{C}_A(\Omega)$ and N in $\mathcal{C}_A(\Gamma)$)

$$\mathrm{adj}_1 : \mathrm{Hom}_{\mathcal{C}_A}(M, T_\Gamma^\Omega N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_A}(T_\Omega^\Gamma M, N), \tag{1}$$

and from 7.3(3) (with M and N interchanged)

$$\mathrm{adj}_2 : \mathrm{Hom}_{\mathcal{C}_A}(N, T_\Omega^\Gamma M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_A}(T_\Gamma^\Omega N, M). \tag{2}$$

The isomorphisms in (1) and (2) are functorial in M and N. This shows that T_Ω^Γ is left and right adjoint to T_Γ^Ω (and vice versa) and that (1), (2) are possible adjunction maps.

The functorial properties of these maps can be expressed by the following formulas where we write $T = T_\Omega^\Gamma$ and $T' = T_\Gamma^\Omega$. One has for all maps f, g such that the formulas make sense

$$\begin{aligned} \mathrm{adj}_1(f \circ g) &= \mathrm{adj}_1(f) \circ T(g), & \mathrm{adj}_2(f \circ g) &= \mathrm{adj}_2(f) \circ T'(g), \\ \mathrm{adj}_1(T'(f) \circ g) &= f \circ \mathrm{adj}_1(g), & \mathrm{adj}_2(T(f) \circ g) &= f \circ \mathrm{adj}_2(g), \\ \mathrm{adj}_1^{-1}(f \circ g) &= T'(f) \circ \mathrm{adj}_1^{-1}(g), & \mathrm{adj}_2^{-1}(f \circ g) &= T(f) \circ \mathrm{adj}_2^{-1}(g), \\ \mathrm{adj}_1^{-1}(f \circ T(g)) &= \mathrm{adj}_1^{-1}(f) \circ g, & \mathrm{adj}_2^{-1}(f \circ T'(g)) &= \mathrm{adj}_2^{-1}(f) \circ g. \end{aligned}$$

The adjunction maps commute obviously with base change.

7.7. Keep the notations from the last subsection. Set $T = T_\Omega^\Gamma$.

If Q in $\mathcal{C}_A$ is projective, then $E \otimes Q$ is projective in $\mathcal{C}_A$ for any finite dimensional E in $\mathcal{C}_k$. That is a trivial consequence of the adjointness property 7.3(4). If Q is projective and in $\mathcal{C}_A(\Omega)$, then TQ is projective, since it is a direct summand of a suitable $E \otimes Q$.

The functor T induces for all M, M' in $\mathcal{C}_A(\Omega)$ maps (also denoted by T)

$$T : \mathrm{Ext}^n_{\mathcal{C}_A}(M, M') \longrightarrow \mathrm{Ext}^n_{\mathcal{C}_A}(TM, TM') \tag{1}$$

for all n.

Any $e \in \mathrm{Ext}^1(M, M')$ can be represented by a short exact sequence

$$0 \to M' \longrightarrow E \longrightarrow M \to 0. \tag{2}$$

Then E is in $\mathcal{C}_A(\Omega)$ and we can apply T to (2). We get a short exact sequence

$$0 \to TM' \longrightarrow TE \longrightarrow TM \to 0. \tag{3}$$

Then (3) is a representative of Te. This description of T can be generalized to classes in higher Ext–groups.

We use the following notation: If $f : M' \to M$ and $g : N \to N'$ are morphisms in a suitable abelian category, then we denote the induced map on the Ext groups (for all n) by

$$(f, g)^* : \mathrm{Ext}^n(M, N) \longrightarrow \mathrm{Ext}^n(M', N'). \tag{4}$$

Suppose that M, M' are in $\mathcal{C}_A(\Omega)$ and N, N' in $\mathcal{C}_A(\Gamma)$, and that we have isomorphisms $f : N \xrightarrow{\sim} TM$ and $f' : N' \xrightarrow{\sim} TM'$. Then we get an induced map

$$t[f, f'] : \mathrm{Ext}^n_{\mathcal{C}_A}(M, M') \longrightarrow \mathrm{Ext}^n_{\mathcal{C}_A}(N, N') \tag{5}$$

for all n setting

$$t[f,f'] = (f,(f')^{-1})^* \circ T. \tag{6}$$

(We shall use a similar notation for T_Γ^Ω instead of $T = T_\Omega^\Gamma$.) Note that we have for all units a, b in A

$$t[af,bf'] = ab^{-1}t[f,f']. \tag{7}$$

7.8. Before we go on, note the following obvious fact:
Observation: Suppose that $\lambda \in X$ and that M, M' are modules in $\mathcal{C}_A$ both isomorphic to $Z_A(\lambda)$. Then any generator f of the A-module $\mathrm{Hom}_{\mathcal{C}_A}(M,M')$ *is an isomorphism $f : M \xrightarrow{\sim} M'$.*

Keep the assumptions of the last subsection. Set $T = T_\Omega^\Gamma$ and $T' = T_\Gamma^\Omega$. Let us call Ω and Γ *equivalent* W'–orbits, if there is a facet for W' containing both an element of Ω and of Γ. Then there is for each $\lambda \in \Omega$ exactly one element $\lambda_\Gamma \in \Gamma$ such that λ and λ_Γ are in the same facet for W'. The map $\lambda \mapsto \lambda_\Gamma$ is a bijection $\Omega \to \Gamma$. One has for any $\lambda \in \Omega$ isomorphisms

$$TZ_A(\lambda) \simeq Z_A(\lambda_\Gamma) \qquad \text{and} \qquad T'Z_A(\lambda_\Gamma) \simeq Z_A(\lambda). \tag{1}$$

For any M in $\mathcal{C}_A(\Omega)$ we have an adjunction map

$$\mathrm{adj}_1 : \mathrm{Hom}_{\mathcal{C}_A}(M, T'TM) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_A}(TM,TM).$$

Set

$$i_M = \mathrm{adj}_1^{-1}(\mathrm{id}_{TM}) : M \longrightarrow T'TM. \tag{2}$$

The functorial properties of the adjunction maps, cf. 7.6, imply that we have for all homomorphisms $h : M \to M'$ in $\mathcal{C}_A(\Omega)$

$$T'T(h) \circ i_M = i_{M'} \circ h, \tag{3}$$

i.e., that the map $M \mapsto i_M$ is a natural transformation from the identity functor to $T'T$.

Lemma: *Suppose that Ω and Γ are equivalent W'-orbits.*

a) For all M in $\mathcal{C}_A(\Omega)$ the map i_M is an isomorphism.

b) Suppose that M is in $\mathcal{C}_A(\Omega)$ and N is in $\mathcal{C}_A(\Gamma)$ and that $f : N \xrightarrow{\sim} TM$ is an isomorphism. Then $T'(f^{-1}) \circ i_M$ is an isomorphism $M \xrightarrow{\sim} T'N$ and we have

$$T'(f^{-1}) \circ i_M = \mathrm{adj}_1^{-1}(f^{-1}). \tag{4}$$

Proof: a) For M of the form $Z_A(\lambda)$ with $\lambda \in \Omega$ we have $TM \simeq Z_A(\lambda_\Gamma)$. So id_{TM} is a basis of $\mathrm{Hom}_{\mathcal{C}_A}(TM,TM)$, hence its inverse image i_M under the isomorphism adj_1 a basis of $\mathrm{Hom}_{\mathcal{C}_A}(M,T'TM)$. Now $T'TM$ is isomorphic to $T'Z_A(\lambda_\Gamma) \simeq Z_A(\lambda) = M$. Therefore the observation above implies that i_M is an isomorphism. This extends by induction on the length of a Z–filtration to any M in $\mathcal{D}_A(\Omega)$. We get it for arbitrary M using a projective resolution contained in $\mathcal{D}_A(\Omega)$.

b) The first claim is obvious by a), the one in (4) follows by functoriality, cf. 7.6.

7.9. Suppose that Ω and Γ are equivalent W'–orbits; keep the notations of the last subsection. Lemma 7.8 implies that the identity functor on $\mathcal{C}_A(\Omega)$ and $T'T$ are naturally isomorphic. One can similarly construct a natural isomorphism between the identity functor on $\mathcal{C}_A(\Gamma)$ and TT'. So T and T' are equivalences of categories between $\mathcal{C}_A(\Omega)$ and $\mathcal{C}_A(\Gamma)$.

It follows that the maps induced by T on Ext groups as in 7.7(1) are isomorphisms. The same is true for T'. So we see for any M, M' in $\mathcal{C}_A(\Omega)$ that the map $e \mapsto T'Te$ is an isomorphism

$$\mathrm{Ext}^n_{\mathcal{C}_A}(M, M') \xrightarrow{\sim} \mathrm{Ext}^n_{\mathcal{C}_A}(T'TM, T'TM')$$

for all n. If we compose this map with those induced by i_M and $i_{M'}^{-1}$ we get an automorphism of $\mathrm{Ext}^n_{\mathcal{C}_A}(M, M')$.

Lemma: *Suppose that Ω and Γ are equivalent W'-orbits. Let M, M' be modules in $\mathcal{C}_A(\Omega)$ and N, N' modules in $\mathcal{C}_A(\Gamma)$.*

a) The map $(i_M, i_{M'}^{-1})^* \circ T'T$ is the identity on $\mathrm{Ext}^n_{\mathcal{C}_A}(M, M')$ for all n.

b) Let $f : N \to TM$ and $f' : N' \to TM'$ be isomorphisms. Set $g = \mathrm{adj}_1^{-1}(f^{-1})$ and $g' = \mathrm{adj}_1^{-1}(f'^{-1})$. Then $t[g, g'] \circ t[f, f']$ is the identity on $\mathrm{Ext}^n_{\mathcal{C}_A}(M, M')$ for all n.

Proof: a) By the functoriality of the i_M it is enough to look at the case $n = 0$. Here any $h : M \to M'$ is mapped to

$$i_{M'}^{-1} \circ T'T(h) \circ i_M = i_{M'}^{-1} \circ i_{M'} \circ h = h,$$

cf. 7.8(3).

b) By definition

$$t[g, g'] \circ t[f, f'] = (g, g'^{-1})^* \circ T' \circ (f, f'^{-1})^* \circ T.$$

By functoriality, the right hand side is equal to $(T'(f) \circ g, g'^{-1} \circ T'(f'^{-1}))^* \circ T'T$. Since $T'(f) \circ g = i_M$ by 7.8(4), and since similarly $g'^{-1} \circ T'(f'^{-1}) = i_{M'}^{-1}$, the claim follows from a).

7.10. Suppose that W'' is an affine reflection group with $W_{\pi,p} \subset W'' \subset W'$. We can carry out the constructions of 7.5/6 with W'' instead of W', i.e., with W''–orbits instead of W'–orbits.

Consider again two W'–orbits Ω and Γ. There are disjoint decompositions into W''-orbits

$$\Omega = \bigcup_{i \in I} \Omega_i \qquad \text{and} \qquad \Gamma = \bigcup_{j \in J} \Gamma_j \tag{1}$$

for suitable index sets I, J. Then the translation functor $T = T_\Omega^\Gamma$ has a decomposition $T = \bigoplus_{i \in I, j \in J} T_{ji}$ where T_{ji} is the restriction of T to $\mathcal{C}_A(\Omega_i)$

composed with pr_{Γ_j}. Similarly, there is a decomposition of $T' = T_\Gamma^\Omega$ as $T' = \bigoplus_{i\in I, j\in J} T'_{ij}$.

Fix $i \in I$ and $j \in J$. Choose $\lambda \in \Omega_i$ and choose an alcove for the larger group W' containing λ in its closure. Let μ be the unique element in Γ that is in the closure of the same alcove. If Γ_j contains one of the elements $w{\scriptstyle\bullet}\mu$ with w in the stabilizer of λ in W', then $T_{ji} = T_{\Omega_i}^{\Gamma_j}$ and $T'_{ij} = T_{\Gamma_j}^{\Omega_i}$. Otherwise one has $T_{ji} = 0$ and $T'_{ij} = 0$. (Use the Z–filtration of $TZ_A(\lambda)$ described in 7.5.)

Suppose we have M in $\mathcal{C}_A(\Omega_i)$ and N in $\mathcal{C}_A(\Gamma_j)$. We have then two adjunction maps

$$\mathrm{adj} : \mathrm{Hom}_{\mathcal{C}_A}(M, T'N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_A}(TM, N)$$

and

$$\mathrm{adj}' : \mathrm{Hom}_{\mathcal{C}_A}(M, T'_{ij}N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_A}(T_{ji}M, N).$$

They are related as follows: Any homomorphism $f : M \to T'N$ takes automatically values in $T'_{ij}N$ and $\mathrm{adj}'(f)$ is then the restriction of $\mathrm{adj}(f)$ to $T_{ji}M$:

$$\mathrm{adj}'(f) = \mathrm{adj}(f)_{|T_{ji}M}. \tag{2}$$

7.11. Let $w \in W$. If we work with the positive system $w(R^+)$, then 7.4 implies that each $E \otimes Z_A^w(\mu)$ has a filtration with factors $Z_A^w(\mu+\nu)$ each occurring $\dim E_\nu$ times. Consider W'–orbits Ω and Γ and weights λ, μ as in 7.5. Then $Z_A^w(\lambda\langle w\rangle)$ is in $\mathcal{C}_A(\Omega)$ — by Lemma 6.11 — and $T_\Omega^\Gamma Z_A^w(\lambda\langle w\rangle)$ has a filtration with factors $Z_A^w((x{\scriptstyle\bullet}\mu)\langle w\rangle)$, where x runs over a system of representatives of the stabilizer of λ in W' modulo its intersection with the stabilizer of μ. If v'_λ spans the weight space $Z_A^w(\lambda\langle w\rangle)_{\lambda\langle w\rangle}$ and if e_x (for x as above) spans the weight space $E_{x(\mu-\lambda)}$, then the $\mathrm{pr}_\Gamma(e_x \otimes v'_\lambda)$ generate $T_\Omega^\Gamma Z_A^w(\lambda\langle w\rangle)$.

7.12. The construction in 7.3 shows immediately for all E in $\mathcal{C}_k$ that

$$(E \otimes M)[p\nu] = E \otimes (M[p\nu]) \tag{1}$$

for all M in $\mathcal{C}_A$ and $\nu \in X$. In other words, tensoring with E is a pX–functor from $\mathcal{C}_A$ to itself (in the sense of E.3).

Let Ω and Γ be W'–orbits in X. The functors $M \mapsto M[p\nu]$ with $\nu \in \mathbf{Z}R_\pi$ commute with pr_Ω and pr_Γ, cf. 6.18. Together with (1) this implies that

$$T_\Omega^\Gamma(M[p\nu]) = (T_\Omega^\Gamma M)[p\nu] \tag{2}$$

for all M in $\mathcal{C}_A(\Omega)$ and $\nu \in \mathbf{Z}R_\pi$. So each T_Ω^Γ is a $(p\mathbf{Z}R_\pi)$–functor.

7.13. Suppose that $A = F$ is a field. We have in our present situation results on the translation of simple modules similar to those in [Ja6], II.7.15.

First, we need an analogue to Proposition II.6.23 in [Ja6]. Take the order relation $\uparrow$ as in [Ja6], II.6.4, and modify its definition: We allow only reflections $s_i \in W_{\pi,p}$. Denote this modified relation by $\uparrow_F$. We next modify the definition of *close* from [Ja6], II.6.22, and say that $\mu \in X$ is F–close to $\lambda \in X$ if $\mu \uparrow_F \lambda$ and if there is no $\alpha \in R^+_\pi$ with $\mu \uparrow_F (\lambda - p\alpha)$. The sum formula 6.6(4) implies immediately:

$$\text{If } \mu \text{ is } F\text{–close to } \lambda, \text{ then } [Z_F(\lambda) : L_F(\mu)] > 0. \tag{1}$$

Consider now orbits Ω and Γ for $W_{\pi,p}$ such that there is for each $\lambda \in \Omega$ exactly one $\lambda_\Gamma \in \Gamma$ in the closure of the facet (with respect to $W_{\pi,p}$) of λ. Set $T = T_\Omega^\Gamma$ and $T' = T_\Gamma^\Omega$. We have then (by 7.11) $TZ_F^w(\lambda\langle w\rangle) \simeq Z_F^w(\lambda_\Gamma\langle w\rangle)$ for all $\lambda \in \Omega$ and $w \in W$. Lemma 4.9 implies that $TL_F(\lambda)$ is either isomorphic to $L_F(\lambda_\Gamma)$ or equal to 0, cf. [Ja6], II.7.14. For each $\mu \in \Gamma$ there is a unique $\mu^- \in \Omega$ such that $\mu = (\mu^-)_\Gamma$ und such that μ is in the upper closure of the facet (with respect to $W_{\pi,p}$) of μ^-. Any other $\lambda \in \Omega$ with $\lambda_\Gamma = \mu$ has the form $\lambda = w{\bullet}\mu^-$ with w in the stabilizer of μ in $W_{\pi,p}$. One checks as in [Ja6], II.6.22, that μ^- is F–close to all these λ. We can now argue as in [Ja6], II.7.15 (using (1) instead of [Ja6], II.6.23) and get for all $\lambda \in \Omega$

$$TL_F(\lambda) \simeq \begin{cases} L_F(\lambda_\Gamma), & \text{if } \lambda = (\lambda_\Gamma)^-; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Arguing as in [Ja6], II.7.16 we get for all $\mu \in \Gamma$

$$T'Q_F(\mu) \simeq Q_F(\mu^-). \tag{3}$$

Introduction to the Sections 8–10

Let us stop a bit to consider where we are and where we want to go. We are interested in the structure of the categories $\mathcal{C}_k$ of G_1T–modules and their quantum analogues and want to study them by deformation. Let $\mathfrak{m} \in \mathrm{Spec}U^0$ be the annihilator of the trivial one dimensional representation, and let A be the completion of U^0 at $\mathfrak{m}$, so that Spec$A$ is a formal neighbourhood of $\mathfrak{m}$ in SpecU^0. We have seen that we can somehow *deform* the category $\mathcal{C}_k$ into a flat family $\mathcal{C}_A$ over Spec$A$: This is the intuitive meaning of our statements saying that

$$\otimes_A k : \mathcal{C}_A \to \mathcal{C}_k, \qquad P \mapsto P \otimes_A k$$

induces a bijection on isomorphism classes of projective objects (see 4.19), that for any projective objects P, Q in $\mathcal{C}_A$ the space $\mathrm{Hom}_{\mathcal{C}_A}(P,Q)$ is free of finite rank over A and the obvious map

$$\mathrm{Hom}_{\mathcal{C}_A}(P,Q) \otimes_A k \longrightarrow \mathrm{Hom}_{\mathcal{C}_k}(P \otimes_A k, Q \otimes_A k)$$

an isomorphism (see 3.3).

Now to describe a flat family it suffices to describe it up to codimension one. More precisely, recall from the general introduction our ring $A^\emptyset = A[H_\alpha^{-1} \mid \alpha \in R^+]$ and its subrings $A^\beta = A[H_\alpha^{-1} \mid \alpha \in R^+,\ \alpha \neq \beta]$. We have $A = \bigcap A^\beta$ if the characteristic of k is different from all $\langle \alpha, \beta^\vee \rangle$, cf. 9.1 below. Let us assume this for the rest of our introduction. We deduce that

$$\mathrm{Hom}_{\mathcal{C}}(P,Q) = \bigcap \mathrm{Hom}_{\mathcal{C}}(P \otimes_A A^\beta, Q \otimes_A A^\beta) \subset \mathrm{Hom}_{\mathcal{C}}(P \otimes_A A^\emptyset, Q \otimes_A A^\emptyset),$$

where we abbreviate $\mathrm{Hom}_{\mathcal{C}_{A'}}$ to $\mathrm{Hom}_{\mathcal{C}}$ for all $A' = A, A^\beta, A^\emptyset$ and use the quite elementary Lemma 3.2 to pull the localization inside the Hom.

Now SpecA^β is just Spec$A$ with all walls except the β-wall deleted, so it should not come as a surprise that $\mathcal{C}_{A^\beta}$ behaves like an $\mathfrak{sl}_2$-category. Recall from the general introduction the notations $Z^\beta(\lambda) = Z_A(\lambda) \otimes_A A^\beta$, $Z^\emptyset(\lambda) = Z_A(\lambda) \otimes_A A^\emptyset$ and abbreviate $\mathcal{C}_{A^\beta} = \mathcal{C}^\beta$. For $\lambda \in X$ and $\beta \in R^+$ let us define $\beta \uparrow \lambda$ to be $\lambda + n\beta$, where $n \geq 0$ is the smallest integer such that $\langle \lambda + \rho, \beta^\vee \rangle \equiv -n \pmod{p}$. We shall show in the next section

Theorem: *Let $\beta \in R^+$ and $\lambda \in X$. If $\beta \uparrow \lambda = \lambda$, then $Z^\beta(\lambda)$ is projective in $\mathcal{C}^\beta$. If $\beta \uparrow \lambda \neq \lambda$, then there exists an isomorphism $\operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta \uparrow \lambda)) \simeq A^\beta / H_\beta A^\beta$ of A^β-modules, and the term Q in a short exact sequence*

$$0 \to Z^\beta(\beta \uparrow \lambda) \longrightarrow Q \longrightarrow Z^\beta(\lambda) \to 0$$

is projective in $\mathcal{C}^\beta$ if and only if the sequence generates Ext^1 *over A^β.*

Since all our categories $\mathcal{C}_{A'}$ are generated by their modules with Z-filtrations (see 2.13), the projectives occurring in the theorem generate all of $\mathcal{C}^\beta$. We want now to somehow glue together all these $\mathfrak{sl}_2$-categories $\mathcal{C}^\beta$ to obtain the deformation category $\mathcal{C}_A$ of $\mathcal{C}_k$. To formalize this idea we introduce our *combinatorial* categories $\mathcal{K}(\Omega)$. They are defined for any W_p-orbit Ω in X. An object $\mathcal{M}$ in $\mathcal{K}(\Omega)$ is a family $(\mathcal{M}(\lambda))_{\lambda \in \Omega}$ of finitely generated $A^\emptyset$-modules, almost all of them zero, together with (for all $\beta \in R^+, \lambda \in \Omega$) a finitely generated A^β-submodule $\mathcal{M}(\lambda, \beta)$ of $\mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda)$ resp. of $\mathcal{M}(\lambda)$ if $\beta \uparrow \lambda \neq \lambda$ resp. $\beta \uparrow \lambda = \lambda$. The morphisms are the obvious ones.

Choose now for any $\beta \in R^+$ and $\lambda \in \Omega$ such that $\beta \uparrow \lambda \neq \lambda$ an element $e^\beta(\lambda) \in \operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta \uparrow \lambda))$. These choices determine a functor $\mathcal{V} = \mathcal{V}_\Omega : \mathcal{C}_A(\Omega) \to \mathcal{K}(\Omega)$. Namely for M in $\mathcal{C}_A(\Omega)$ put

$$\begin{aligned} \mathcal{V}M(\lambda) &= \operatorname{Hom}_{\mathcal{C}}(Z^\emptyset(\lambda), M \otimes_A A^\emptyset), \\ \mathcal{V}M(\lambda, \beta) &= \operatorname{Hom}_{\mathcal{C}}(Z^\beta(\lambda), M \otimes_A A^\beta) \qquad \text{if } \beta \uparrow \lambda = \lambda. \end{aligned}$$

If $\beta \uparrow \lambda \neq \lambda$ define $\mathcal{V}M(\lambda, \beta)$ as follows: Represent $e^\beta(\lambda)$ by a short exact sequence $Z^\beta(\beta \uparrow \lambda) \hookrightarrow Q^\beta(\lambda) \twoheadrightarrow Z^\beta(\lambda)$. After tensoring with $A^\emptyset$ it splits uniquely to determine an isomorphism $Q^\beta(\lambda) \otimes_A A^\emptyset \xrightarrow{\sim} Z^\emptyset(\lambda) \oplus Z^\emptyset(\beta \uparrow \lambda)$. Now let $\mathcal{V}M(\lambda, \beta)$ be the image of

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}(Q^\beta(\lambda), M \otimes_A A^\beta) &\longrightarrow \operatorname{Hom}_{\mathcal{C}}(Q^\beta(\lambda) \otimes_A A^\emptyset, M \otimes_A A^\emptyset) \\ &\xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(Z^\emptyset(\lambda) \oplus Z^\emptyset(\beta \uparrow \lambda), M \otimes_A A^\emptyset) \\ &= \mathcal{V}M(\lambda) \oplus \mathcal{V}M(\beta \uparrow \lambda). \end{aligned}$$

We obtain from the preceding considerations almost tautologically:

Theorem: *Choose all $e^\beta(\lambda)$ as generators of their* Ext *groups. Then $\mathcal{V}$ induces for all projective objects P and Q in $\mathcal{C}_A(\Omega)$ an isomorphism*

$$\operatorname{Hom}_{\mathcal{C}}(P, Q) \longrightarrow \operatorname{Hom}_{\mathcal{K}(\Omega)}(\mathcal{V}P, \mathcal{V}Q).$$

In Section 9 we shall prove this more generally for all P and Q in $\mathcal{C}_A(\Omega)$ that are flat as A–modules. From now on all $\mathcal{V}$ are supposed to be defined by a choice of generators $e^\beta(\lambda)$ so that the theorem applies.

Recall from the general introduction that what we need is a *characteristic free* description of the homomorphisms between certain projectives $Q_{A,I}$ in $\mathcal{C}_A(W_p{\bullet}0)$. Our theorem tells us that it suffices to describe the homomorphisms between the $\mathcal{V}Q_{A,I}$ in a characteristic free way. But it is more or less clear that the combinatorial category $\mathcal{K}(W_p{\bullet}0)$ can also be defined over $\mathbf{Z}$, and our strategy will then be to exhibit an object of this combinatorial category over $\mathbf{Z}$ that for all k specializes to $\mathcal{V}Q_{A,I}$ when we reduce scalars.

As a first step we should describe $\mathcal{V}Q_{A,I}$ for fixed k. Recall that to get $Q_{A,I}$ we first translate $Z_A(-\rho)$ out from all walls to $\mathcal{C}_A(W_p{\bullet}0)$ and then apply a sequence I of wall-crossings. To get $\mathcal{V}Q_{A,I}$, we first have to describe $\mathcal{V}Z_A(-\rho)$ — this is easy — and then have to understand how translations relate to combinatorics. More precisely, for any two W_p–orbits Ω and Γ we should construct a *combinatorial translation functor* $\mathcal{T} : \mathcal{K}(\Omega) \to \mathcal{K}(\Gamma)$ such that $\mathcal{V}_\Gamma TP \simeq \mathcal{T}\mathcal{V}_\Omega P$ for all projectives P in $\mathcal{C}_A(\Omega)$, where $T : \mathcal{C}_A(\Omega) \to \mathcal{C}_A(\Gamma)$ is our usual translation. This is done in Section 10. Well, not completely: We don't treat the most general translations but just translations out from or onto some walls, which suffices for our purposes.

Let us discuss the structure of our combinatorial translations. Suppose Γ lies on more walls than Ω, so for every $\lambda \in \Omega$ there is a unique $\lambda_\Gamma \in \Gamma$ in the closure of the alcove of λ. From 7.5 we know that $TZ_A(\lambda)$ is isomorphic to $Z_A(\lambda_\Gamma)$. To get our combinatorial translations, we have to actually choose isomorphisms $f_\lambda : Z_A(\lambda_\Gamma) \to TZ_A(\lambda)$ for all $\lambda \in \Omega$. These isomorphisms together with the choices $e^\beta(\lambda)$ and $e^\beta(\mu)$ we made to define $\mathcal{V}_\Omega$ and $\mathcal{V}_\Gamma$ will determine certain constants b_λ^β which in turn determine our combinatorial translations.

To give an idea of how this is done let us just explain the case where both Ω and Γ are regular. Then for all $\lambda \in \Omega$, $\beta \in R^+$ translation of extensions gives rise to an isomorphism

$$t[f_\lambda, f_{\beta\uparrow\lambda}] : \operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta\uparrow\lambda)) \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda_\Gamma), Z^\beta(\beta\uparrow\lambda_\Gamma))$$

(see 7.7) and we choose $b_\lambda^\beta \in A^\beta$ such that $b_\lambda^\beta t[f_\lambda, f_{\beta\uparrow\lambda}]e^\beta(\lambda) = e^\beta(\lambda_\Gamma)$. In this case we define the combinatorial translation $\mathcal{T}$ by

$$\begin{aligned}(\mathcal{T}\mathcal{M})(\lambda_\Gamma) &= \mathcal{M}(\lambda)\\ (\mathcal{T}\mathcal{M})(\lambda_\Gamma,\beta) &= (b_\lambda^\beta, 1)\mathcal{M}(\lambda,\beta) + (\mathcal{M}(\lambda,\beta) \cap \mathcal{M}(\lambda)).\end{aligned}$$

One may easily check that this works in case $b_\lambda^\beta = 1$. The case $\Omega = \Gamma$, $f_\lambda = id$ might also be instructive. It is almost equivalent to Lemma 9.10. Let us remark that although b_λ^β is only well-defined modulo H_β, our combinatorial translations do not depend on the choice of b_λ^β when we restrict them to the image of $\mathcal{V}$.

The general case is similar. We only have to take walls into account and this complicates our formulas. It will turn out that in case $\beta \uparrow \mu = \mu$ it is the space $H_\beta^{-1} A^\beta / A^\beta$ which takes over the role of the $\operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\mu), Z^\beta(\beta \uparrow \mu))$. More precisely, suppose that $\beta \uparrow \lambda \neq \lambda$ but $(\beta \uparrow \lambda)_\Gamma = \lambda_\Gamma$. Then we will define at the end of the next section an isomorphism

$$\theta[f_\lambda, f_{\beta\uparrow\lambda}] : \operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta \uparrow \lambda)) \to H_\beta^{-1} A^\beta / A^\beta$$

that will play a role similar to the maps $t[f_\lambda, f_{\beta\uparrow\lambda}]$ that appeared above. In particular, the combinatorial translations to and from walls will also involve constants $b_\lambda^\beta \in H_\beta A^\beta$ such that $b_\lambda^\beta \theta[f_\lambda, f_{\beta\uparrow\lambda}] e^\beta(\lambda) = 1$.

8. Extensions in Rank 1 Situations

8.1. In order to simplify notation we shall write in Case 2 for all $\alpha \in R$:

$$H_\alpha = [K_\alpha; 0].$$

Note that H_α differs from $K_\alpha^2 - 1$ by a unit in U^0.

For the next subsections (until 8.5) fix $\alpha \in \Sigma$ and suppose that A is an integral domain with $\pi(H_\alpha) \neq 0$. Denote the fraction field of A by K.

Consider $\lambda \in X$ such that

$$\langle \lambda + \rho, \alpha^\vee \rangle \equiv p - n \pmod{p} \tag{1}$$

for some n with $0 < n < p$. We want to study $\operatorname{Ext}^1_{\mathcal{C}_A}(Z_A(\lambda), Z_A(\lambda + n\alpha))$. Denote standard generators by $v_0 \in Z_A(\lambda + n\alpha)$ and $x_0 \in Z_A(\lambda)$. The congruence (1) implies by 5.5(5),(6)

$$E_\alpha E_{-\alpha}^{(n)} v_0 = E_{-\alpha}^{(n-1)} v_0 \pi(H_\alpha). \tag{2}$$

We can regard $Z_A(\lambda + n\alpha)$ resp. $Z_A(\lambda)$ as a $U \otimes A$–submodule of $Z_K(\lambda + n\alpha)$ resp. of $Z_K(\lambda)$. For any $b \in K$ set

$$x_b = x_0 + E_{-\alpha}^{(n)} v_0 b \in Z_K(\lambda) \oplus Z_K(\lambda + n\alpha). \tag{3}$$

and

$$Y(b) = U^- v_0 A + U^- x_b A \subset Z_K(\lambda) \oplus Z_K(\lambda + n\alpha). \tag{4}$$

The term $U^- v_0 A$ is equal to $Z_A(\lambda + n\alpha)$. Obviously, $Y(b)$ is a graded A–submodule of $Z_K(\lambda) \oplus Z_K(\lambda + n\alpha)$ with

$$Y(b)_{\lambda+\nu} = U^-_{\nu - n\alpha} v_0 A + U^-_\nu x_b A \tag{5}$$

for all $\nu \in X$. It is a module in $\mathcal{C}_A$ if and only if it is U–stable.

Lemma: *$Y(b)$ is U-stable if and only if $\pi(H_\alpha)b \in A$. If so, then $Y(b)$ is in $\mathcal{C}_A$, and there is an isomorphism $Y(b)/U^- v_0 A \xrightarrow{\sim} Z_A(\lambda)$ mapping the class of x_b to x_0.*

Proof: If $Y(b)$ is U-stable, then

$$E_\alpha x_b \in Y(b)_{\lambda+\alpha} = E_{-\alpha}^{(n-1)} v_0 A.$$

Now (2) implies

$$E_\alpha x_b = E_\alpha E_{-\alpha}^{(n)} v_0 b = E_{-\alpha}^{(n-1)} v_0 \pi(H_\alpha) b.$$

So, if $Y(b)$ is U-stable, then $\pi(H_\alpha)b \in A$.

On the other hand, if this condition is satisfied, then the computation above shows $E_\alpha x_b \in U^- v_0 A \subset Y(b)$, hence $E_\alpha^r x_b \in U^- v_0 A$ for all $r > 0$. Furthermore $\lambda + r\beta$ is not a weight of $Z_K(\lambda) \oplus Z_K(\lambda + n\alpha)$ for any $\beta \in R^+$ with $\beta \neq \alpha$ and any $r > 0$. This shows $U^+ x_b \subset x_b A + U^- v_0 A$. Since x_b is a weight vector, we have obviously $U^0 x_b = x_b U^0 \subset x_b A$. This implies that $U x_b A \subset Y(b)$, hence that $Y(b)$ is U-stable and therefore in $\mathcal{C}_A$. We see also that $Y(b)/U^- v_0 A$ is generated by the class of x_b. This class is annihilated by all E_β^r with $\beta \in R^+$ and $r > 0$ and is a weight vector of weight λ. Therefore we get a surjective homomorphism $Z_A(\lambda) \to Y(b)/U^- v_0 A$ that maps x_0 to the class of x_b. On the other hand, the projection from $Z_K(\lambda) \oplus Z_K(\lambda + n\alpha)$ onto the first factor induces a homomorphism from $Y(b)/U_0^- v_0 A$ to $U^- x_0 A = Z_A(\lambda)$ mapping the class of x_b to x_0. These two maps are inverse isomorphisms.

8.2. Each $Y(b)$ with $\pi(H_\alpha)b \in A$ yields an exact sequence

$$0 \to Z_A(\lambda + n\alpha) \xrightarrow{f} Y(b) \xrightarrow{g} Z_A(\lambda) \to 0 \tag{1}$$

where $f(v_0) = v_0$ and $g(x_b) = x_0$. Let $[b] \in \operatorname{Ext}^1_{\mathcal{C}_A}(Z_A(\lambda), Z_A(\lambda + n\alpha))$ denote the class of this extension for the moment.

Lemma: *One has $a[b] = [ab]$ for all $a \in A$.*

Proof: Consider the endomorphism h of $Z_K(\lambda) \oplus Z_K(\lambda + n\alpha)$ that is the identity on $Z_K(\lambda + n\alpha)$ and multiplication by a on $Z_K(\lambda)$. We have $h(v_0) = v_0$ and $h(x_0) = a x_0$, hence $h(x_{ab}) = a x_b$. This implies that h maps $Y(ab)$ into $Y(b)$. It is now clear that we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & Z_A(\lambda + n\alpha) & \longrightarrow & Y(ab) & \longrightarrow & Z_A(\lambda) & \to & 0 \\
 & & \big\downarrow \mathrm{id} & & \big\downarrow h & & \big\downarrow a\,\mathrm{id} & & \\
0 & \to & Z_A(\lambda + n\alpha) & \xrightarrow{f} & Y(b) & \xrightarrow{g} & Z_A(\lambda) & \to & 0
\end{array}$$

where the maps in the top row are the analogues of f and g. This proves the claim.

8.3. Lemma: *Let $b \in A$. For any A–algebra A' the extension $[b\pi(H_\alpha)^{-1}]$ splits over A', if and only if $b \in A'\pi(H_\alpha)$.*

Proof: Set $b' = b\pi(H_\alpha)^{-1}$. Now $[b']$ splits over A' if and only if there is an inverse image $x_0' \in Y(b')_\lambda \otimes_A A'$ of $x_0 \otimes 1$ with $E_\alpha x_0' = 0$. Any inverse image has the form $x_0' = x_{b'} \otimes 1 + E_{-\alpha}^{(n)} v_0 \otimes a'$ with $a' \in A'$. This inverse image satisfies $E_\alpha x_0' = E_{-\alpha}^{(n-1)} v_0 \otimes (b + a'\pi(H_\alpha))$. This expression is equal to 0 if and only if $b + a'\pi(H_\alpha) = 0$. This shows that we can find x_0' as required if and only if there is $a' \in A'$ with $b = -a'\pi(H_\alpha)$, i.e., $b \in A'\pi(H_\alpha)$.

8.4. Lemma: *The map $a \mapsto [a]$ induces an isomorphism*

$$A\pi(H_\alpha)^{-1}/A \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{C}_A}(Z_A(\lambda), Z_A(\lambda + n\alpha)).$$

Proof: By Lemma 8.2 the map is a homomorphism of A–modules from $A\pi(H_\alpha)^{-1} \subset K$ (the field of fractions of A) to the Ext group. Lemma 8.3 implies that its kernel is equal to A. We have to check surjectivity. Consider an arbitrary extension in $\mathcal{C}_A$:

$$0 \to Z_A(\lambda + n\alpha) \xrightarrow{f} M \xrightarrow{g} Z_A(\lambda) \to 0. \tag{1}$$

Denote the standard generators by v_0 and x_0 as in 8.1. We claim that it splits over K. It is enough to find a preimage in M_λ of x_0 that is annihilated by E_α. Let M' be the sum of all weight spaces $M_{\lambda+i\alpha}$ with $i \in \mathbf{Z}$. If we restrict to these weight spaces (1) yields an extension

$$0 \to \Psi_A(\lambda + n\alpha) \longrightarrow M' \longrightarrow \Psi_A(\lambda) \to 0 \tag{2}$$

with the $\Psi_A(\mu)$ as in 5.5(1). Because M_λ is contained in M', we see that (1) splits over K if and only if (2) splits over K. In this way one can reduce to the case where $R^+ = \{\alpha\}$. In this case $\pi(H_\alpha) \neq 0$ implies that K is an $B^\emptyset$–algebra. So the remark in 6.13 yields that λ and $\lambda + n\alpha$ belong to different blocks over K, so that (1) splits over K.

So we can assume that $M \otimes_A K = Z_K(\lambda) \oplus Z_K(\lambda + n\alpha)$, that f and g are induced by the embedding into the second factor resp. by projection onto the first factor. Choose $x_0' \in M_\lambda$ with $g(x_0') = x_0$. Because g is identified with the projection we see that x_0' has the form $x_0' = x_0 + E_{-\alpha}^{(n)} v_0 b$ for some $b \in K$. Since obviously $M = U^- v_0 A + U^- x_0' A$, we get $M = Y(b)$. Lemma 8.1 implies $b \in A\pi(H_\alpha)^{-1}$, so the sequence in (1) is in the image of our map.

8.5. Lemma: *If A is a B^α–algebra, then $Y(H_\alpha^{-1})$ is projective in $\mathcal{C}_A$.*

Proof: Set $Y = Y(H_\alpha^{-1})$. We want to apply Corollary 3.5. Consider a maximal ideal $\mathfrak{m}$ of A and set $F = A/\mathfrak{m}$. We have to show that Y_F is projective in $\mathcal{C}_F$. If $H_\alpha \notin \mathfrak{m}$, then the analogue of R_π for F is empty. In this case $Z_F(\lambda)$ and $Z_F(\lambda + n\alpha)$ are projective, hence so is their extension Y_F. If $H_\alpha \in \mathfrak{m}$, then the analogue of R_π^+ for F is equal to $\{\alpha\}$. By Lemma 8.3 the extension $[H_\alpha^{-1}]$ does not split over F. Therefore the discussion at the end of 6.4 shows that $Y_F \simeq Q_F(\lambda)$.

Remark: For A as above the same argument proves for any $\mu \in X$ with $\langle \mu + \rho, \alpha^\vee \rangle \equiv 0 \pmod p$ that $Z_A(\mu)$ is projective in $\mathcal{C}_A$.

8.6. We now want to generalize the preceding results from the case of a simple root α to that of an arbitrary positive root β. Until the end of this section we fix $\beta \in R^+$ and a B–algebra A. We set $A^\beta = A \otimes_B B^\beta$. We use the abbreviations

$$Z^\beta(\lambda) = Z_{A^\beta}(\lambda) = Z_{B^\beta}(\lambda) \otimes_{B^\beta} A^\beta \qquad \text{for all } \lambda \in X$$

and

$$\mathrm{Ext}^i_{\mathcal{C}}(M, N) = \mathrm{Ext}^i_{\mathcal{C}_{A^\beta}}(M, N)$$

for all M, N in $\mathcal{C}_{A^\beta}$ and all i.

Proposition: *Let $\lambda \in X$ and $n \in \mathbf{Z}$ with $0 < n \leq p$ such that $\langle \lambda + \rho, \beta^\vee \rangle \equiv p - n \pmod p$.*

a) If A^β is an integral domain such that $\pi(H_\beta)$ is nonzero in A^β, then

$$\mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda + n\beta)) \simeq \begin{cases} A^\beta \pi(H_\beta)^{-1}/A^\beta, & \text{if } n < p; \\ 0, & \text{if } n = p. \end{cases}$$

b) In general,

$$\mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda + n\beta)) \simeq \begin{cases} A^\beta/\pi(H_\beta)A^\beta, & \text{if } n < p; \\ 0, & \text{if } n = p. \end{cases}$$

c) If $n = p$, then $Z^\beta(\lambda)$ is projective in $\mathcal{C}_{A^\beta}$. If $n < p$, then there is a projective module Q^β in $\mathcal{C}_{A^\beta}$ with an exact sequence

$$0 \to Z^\beta(\lambda + n\beta) \longrightarrow Q^\beta \longrightarrow Z^\beta(\lambda) \to 0. \tag{1}$$

Proof: Suppose at first that A^β satisfies the assumption in a) and let as prove a) and c) in this case.

Choose $w \in W$ with $w^{-1}\beta$ simple. The remark in 5.14 implies that we have isomorphisms

$$Z^\beta(\lambda) \simeq Z^w_{A^\beta}(\lambda\langle w \rangle) \qquad \text{and} \qquad Z^\beta(\lambda + n\beta) \simeq Z^w_{A^\beta}(\lambda\langle w \rangle + n\beta); \tag{2}$$

they induce an isomorphism of the corresponding Ext–groups. We work now with the triangular decomposition $U = T_w(U^-)U^0T_w(U^+)$. So the $Z^w_{A^\beta}(\mu)$ play the role of the $Z^\beta(\mu)$, the positive roots are $w(R^+)$, the root $\beta \in w(\Sigma)$ is simple, and $w\rho$ is half the sum of the positive roots. We have

$$\langle\lambda\langle w\rangle + w\rho, \beta^\vee\rangle = \langle\lambda+\rho,\beta^\vee\rangle + \langle p(w\rho-\rho),\beta^\vee\rangle \equiv p-n \pmod p.$$

Now Lemma 8.4 yields the claim in a) for $n < p$. For $n = p$ the remark in 8.5 implies that $Z^\beta(\lambda)$ is projective in $\mathcal{C}_{A^\beta}$. So we have (for our present A) the first claim in c); the part of a) for $n = p$ follows. For the second claim in c) we take the analogue of $Y(H_\alpha^{-1})$ working with $w(R^+)$ instead of R^+ and apply Lemma 8.5.

We can apply the proof above especially to $A = B$. We get now b) and c) for general A using base change arguments. More precisely: Let A be an arbitrary B–algebra. For $n = p$ we know already that $Z_{B^\beta}(\lambda)$ is projective; so Lemma 3.1.a implies that $Z^\beta(\lambda)$ is projective and the vanishing of the Ext–group follows. Assume now that $n < p$. We get an exact sequence as in (1) over A^β by tensoring one over B^β with A^β. By Lemma 3.1.a the middle term is still projective. Apply the functor $\mathrm{Hom}_\mathcal{C}(?, Z^\beta(\lambda+n\beta))$ to (1). Use for any B^β–algebra A' the abbreviations

$$H_{A'} = \mathrm{Hom}_{\mathcal{C}_{A'}}(Q^\beta_{A'}, Z_{A'}(\lambda+n\beta))$$

and

$$E^1_{A'} = \mathrm{Ext}^1_{\mathcal{C}_{A'}}(Z_{A'}(\lambda), Z_{A'}(\lambda+n\beta)).$$

We get an exact sequence

$$H_{A'} \longrightarrow A' \longrightarrow E^1_{A'} \to 0$$

for all A'. We have $H_{A'} \simeq H_{B^\beta} \otimes_{B^\beta} A'$ by Proposition 3.3. This implies now also $E^1_{A'} \simeq E^1_{B^\beta} \otimes_{B^\beta} A'$. We have

$$E^1_{B^\beta} \simeq B^\beta H_\beta^{-1}/B^\beta \simeq B^\beta/H_\beta B^\beta,$$

where the second isomorphism is induced by $b \mapsto bH_\beta$ and where the first one follows from a). We get therefore $E^1_{A'} \simeq A'/\pi(H_\beta)A'$ for all A', especially for $A' = A^\beta$. This yields the first case in b).

Remark: Choose for all $\lambda \in X$ a projective module $Q^\beta(\lambda)$ in $\mathcal{C}_{A^\beta}$ with a surjection $Q^\beta(\lambda) \to Z^\beta(\lambda)$ as follows: If $n = p$ in the proposition, then take $Q^\beta(\lambda) = Z^\beta(\lambda)$, otherwise take the middle term in an exact sequence as in (1). If M in $\mathcal{C}_{A^\beta}$ has a Z–filtration, then we can find (using induction on the length of a Z–filtration) a surjective homomorphism of the form $\bigoplus_i Q^\beta(\lambda_i) \to M$. Now Lemma 2.13 implies that there is such a surjective homomorphism for every M in $\mathcal{C}_{A^\beta}$. In other words, the family of all $Q^\beta(\lambda)$ with $\lambda \in X$ generates the category $\mathcal{C}_{A^\beta}$.

8.7. Let λ and n be as in Proposition 8.6.

Corollary: *Let*

$$0 \to Z^\beta(\lambda + n\beta) \longrightarrow Y \longrightarrow Z^\beta(\lambda) \to 0 \tag{1}$$

be an exact sequence in $\mathcal{C}_{A^\beta}$. The following are equivalent:

(i) The class of (1) *generates* $\operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda + n\beta))$.

(ii) Y is projective in $\mathcal{C}_{A^\beta}$.

Proof: We can assume that $n < p$ and that $\pi(H_\beta)$ is not a unit in A^β. (Otherwise the Ext–group is 0 and all modules in sight are projective.) If A^β is a field, then (i) and (ii) are equivalent to the sequence not splitting, cf. 6.4. In general, the description of the Ext–groups in 8.6 shows that (1) generates the Ext–group over A^β if and only if it does so over $A^\beta/\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of A^β. On the other hand, Corollary 3.5 says that Y is projective, if and only if all $Y_{A^\beta/\mathfrak{m}}$ are projective. Now the claim is obvious.

8.8. Let λ and n be as in Proposition 8.6.

Proposition: *Suppose that $n < p$.*

a) One has $\operatorname{Ext}^2_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda + p\beta)) \simeq A^\beta/\pi(H_\beta)A^\beta$.

b) If ξ is a generator of the A^β-module $\operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda + n\beta), Z^\beta(\lambda + p\beta))$ *and if η is a generator of the A^β-module* $\operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda + n\beta))$, *then the cup product of ξ and η is a generator of the A^β-module* $\operatorname{Ext}^2_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda + p\beta))$.

Proof: Write $\lambda' = \lambda + n\beta$. Consider the exact sequence 8.6(1). We get a long exact sequence applying the functor $\operatorname{Hom}_{\mathcal{C}}(?, Z^\beta(\lambda + p\beta))$. Since Q^β is projective, we get an isomorphism

$$\operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda'), Z^\beta(\lambda + p\beta)) \xrightarrow{\sim} \operatorname{Ext}^2_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda + p\beta)). \tag{1}$$

Now a) follows from 8.6 applied to λ' instead of λ.

The same exact sequence yields also the second and third vertical map in the diagram (where we drop all indices $\mathcal{C}$ and all exponents β)

$$\begin{array}{ccccc}
\operatorname{Ext}^1(Z(\lambda'), Z(\lambda + p\beta)) & \times & \operatorname{Hom}(Z(\lambda'), Z(\lambda')) & \to & \operatorname{Ext}^1(Z(\lambda'), Z(\lambda + p\beta)) \\
\| & & \downarrow & & \downarrow \\
\operatorname{Ext}^1(Z(\lambda'), Z(\lambda + p\beta)) & \times & \operatorname{Ext}^1(Z(\lambda), Z(\lambda')) & \to & \operatorname{Ext}^2(Z(\lambda), Z(\lambda + p\beta))
\end{array}$$

Here both horizontal maps are given by the cup product. The commutativity follows from the functoriality of the cup product. The last vertical map is the isomorphism (1). The second vertical map is onto (by the long exact sequence). We can replace η by any other generator and can therefore assume that η is the image of the identity map id. Since the cup product with id is the identity, the claim follows from the commutativity of the diagram.

Remark: The proof yields easily that there is a commutative diagram (dropping indices)

$$\begin{array}{ccccc} A^\beta/\pi(H_\beta)A^\beta & \times & A^\beta/\pi(H_\beta)A^\beta & \to & A^\beta/\pi(H_\beta)A^\beta \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ext}^1(Z(\lambda'), Z(\lambda+p\beta)) & \times & \mathrm{Ext}^1(Z(\lambda), Z(\lambda')) & \to & \mathrm{Ext}^2(Z(\lambda), Z(\lambda+p\beta)) \end{array}$$

where the map in the first row is ordinary multiplication, the map in the second row is the cup product, and the vertical maps are isomorphisms.

8.9. In the remaining subsections of this section we assume in addition that A^β is an integral domain. Set W' equal to the group $W_{\beta,p}$ generated by all $s_{\beta,mp}$ with $m \in \mathbf{Z}$. Let Ω and Γ be orbits in X for W' such that any $\lambda \in \Omega$ has trivial stabilizer in W' and any $\mu \in \Gamma$ has stabilizer of order 2 in W'. We shall consider translation functcrs $T = T_\Omega^\Gamma$ and $T' = T_\Gamma^\Omega$, cf. 7.5.

For any $\lambda \in \Omega$ there is a unique weight $\lambda_\Gamma \in \Gamma$ that is in the closure of the alcove of λ. There is then an isomorphism

$$TZ^\beta(\lambda) \simeq Z^\beta(\lambda_\Gamma). \tag{1}$$

For any $\mu \in \Gamma$ there are exactly two weights $\lambda, \lambda' \in \Omega$ with $\mu = \lambda_\Gamma = (\lambda')_\Gamma$. One has then $\lambda' - \lambda \in \mathbf{Z}\beta$. If $\lambda' > \lambda$, then there is an exact sequence

$$0 \to Z^\beta(\lambda') \xrightarrow{f} T'Z^\beta(\mu) \xrightarrow{g} Z^\beta(\lambda) \to 0. \tag{2}$$

We have in this situation:

Lemma: *The module $T'Z^\beta(\mu)$ is projective in $\mathcal{C}_{A^\beta}$. The class of (2) generates the A^β-module $\mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda'))$.*

Proof: The case $n = p$ in Proposition 8.6 implies that $Z^\beta(\mu)$ is projective in $\mathcal{C}_{A^\beta}$, hence so is $T'Z_{A^\beta}(\mu)$, cf. 7.7. So Corollary 8.7 yields the second claim.

Remark: If $\mathfrak{m}$ is a maximal ideal in A^β, then $T'Z_{A^\beta/\mathfrak{m}}(\mu) \simeq (T'Z^\beta(\mu)) \otimes_{A^\beta} A^\beta/\mathfrak{m}$ maps onto $Z_{A^\beta/\mathfrak{m}}(\lambda)$ and onto $L_{A^\beta/\mathfrak{m}}(\lambda)$, hence contains the projective cover $Q_{A^\beta/\mathfrak{m}}(\lambda)$ of $L_{A^\beta/\mathfrak{m}}(\lambda)$. If $H_\beta \in \mathfrak{m}$, then 6.4(4) implies that $T'Z_{A/\mathfrak{m}}(\mu)$ is isomorphic to $Q_{A^\beta/\mathfrak{m}}(\lambda)$. If A^β is local and if H_β is not a unit in A^β, then $T'Z^\beta(\mu)$ is isomorphic to $Q_{A^\beta}(\lambda)$ as in 4.19.

8.10. Let us suppose in the next subsections that we have two weights $\lambda, \lambda' \in \Omega$ with $\lambda_\Gamma = (\lambda')_\Gamma$. We set $\mu = \lambda_\Gamma$ and assume that $\lambda' > \lambda$. So we are in the situation leading to 8.9(2), we have that exact sequence and the maps f, g occurring in it.

Lemma: *a) The maps f and g are bases of their* Hom *spaces:*

$$\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda'), T'Z^\beta(\mu)) = A^\beta f \tag{1}$$

and

$$\mathrm{Hom}_{\mathcal{C}}(T'Z^\beta(\mu), Z^\beta(\lambda)) = A^\beta g. \tag{2}$$

b) The A^β-modules $\mathrm{Hom}_{\mathcal{C}}(T'Z^\beta(\mu), Z^\beta(\lambda'))$ and $\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), T'Z^\beta(\mu))$ are free of rank 1. If f' is a basis of the former and g' a basis of the latter, then there are units $a_1, a_2 \in A^\beta$ with $f' \circ f = a_1 H_\beta \mathrm{id}$ and $g \circ g' = a_2 H_\beta \mathrm{id}$.

Proof: a) We have $\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda'), Z^\beta(\lambda)) = 0$, so the natural maps

$$\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda'), Z^\beta(\lambda')) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda'), T'Z^\beta(\mu)), \qquad h \mapsto f \circ h$$

and

$$\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(T'Z^\beta(\mu), Z^\beta(\lambda)), \qquad h \mapsto h \circ g$$

are isomorphisms and send the identity map to a basis.

b) We have isomorphisms given by adjunction

$$\mathrm{Hom}_{\mathcal{C}}(T'Z^\beta(\mu), Z^\beta(\lambda')) \simeq \mathrm{Hom}_{\mathcal{C}}(Z^\beta(\mu), TZ^\beta(\lambda')) \tag{3}$$

and

$$\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), T'Z^\beta(\mu)) \simeq \mathrm{Hom}_{\mathcal{C}}(TZ^\beta(\lambda), Z^\beta(\mu)). \tag{4}$$

The right hand side is in both cases isomorphic to $\mathrm{End}_{\mathcal{C}}(Z^\beta(\mu)) \simeq A^\beta$. So each left hand side is free over A^β of rank 1.

The proof of the two last claims is similar. We shall give it only for $f' \circ f$. We know that $\mathrm{End}_{\mathcal{C}}(Z^\beta(\lambda')) = A^\beta \cdot \mathrm{id}$, so there is $b \in A^\beta$ with $f' \circ f = b \cdot \mathrm{id}$. Then b annihilates the class of 8.9(2) in $\mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda'))$. This class generates the Ext group by 8.9, so 8.6 implies that its annihilator is generated by H_β. So there is $a_1 \in A^\beta$ with $b = a_1 H_\beta$.

If the image of H_β in A^β is zero, then $b = 0$. We get thus $f' \circ f = 0 = 1 \cdot H_\beta \mathrm{id}$ and are done. Assume now that the image of H_β in A^β is not equal to zero. Since H_β annihilates the Ext group, we can find a homomorphism $f'' : T'Z^\beta(\mu) \to Z^\beta(\lambda')$ with $f'' \circ f = H_\beta \mathrm{id}$. There has to be an $a \in A^\beta$ with $f'' = af'$. We get

$$H_\beta \mathrm{id} = a(f' \circ f) = a_1 a H_\beta \cdot \mathrm{id},$$

hence $a_1 a = 1$. So a_1 is a unit in A^β.

Remark: The following statements are now obvious:

If we choose f and g as arbitrary bases of the Hom spaces in (1) and (2), then the sequence 8.9(2) with these maps is exact.

For any choice of units a_1, a_2 in A^β there are maps f', g' satisfying b).

8.11. If we apply T to 8.9(2), then we get an exact sequence

$$0 \to Z^\beta(\mu) \longrightarrow TT'Z^\beta(\mu) \longrightarrow Z^\beta(\mu) \to 0 \tag{1}$$

that splits because each extension of $Z^\beta(\mu)$ with itself is trivial, cf. 2.14.b. So

$$TT'Z^\beta(\mu) \xrightarrow{\sim} Z^\beta(\mu) \oplus Z^\beta(\mu) \tag{2}$$

and

$$\mathrm{Hom}_{\mathcal{C}}(TT'Z^\beta(\mu), Z^\beta(\mu)) \simeq (A^\beta)^2. \tag{3}$$

Any basis h_1, h_2 over A^β of this Hom space yields an isomorphism as in (2) given by $x \mapsto (h_1(x), h_2(x))$.

We get by adjointness an isomorphism

$$\mathrm{adj}_1 : \mathrm{End}_{\mathcal{C}}(T'Z^\beta(\mu)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(TT'Z^\beta(\mu), Z^\beta(\mu)). \tag{4}$$

So also this space of endomorphisms is free of rank 2 over A^β.

Lemma: *Let $g : T'Z^\beta(\mu) \to Z^\beta(\lambda)$ and $g' : Z^\beta(\lambda) \to T'Z^\beta(\mu)$ be bases of their* Hom *spaces. Then the maps* id *and $g' \circ g$ are a basis of* $\mathrm{End}_{\mathcal{C}}(T'Z^\beta(\mu))$.

Proof: We may assume that g occurs in 8.9(2). That sequence yields an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), T'Z^\beta(\mu)) \xrightarrow{h} \mathrm{End}_{\mathcal{C}}(T'Z^\beta(\mu)) \xrightarrow{h'} \mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda'), T'Z^\beta(\mu)).$$

Here h maps the basis g' to $g' \circ g$, and h' maps id to the basis f. So the claim is clear.

Remark: One proves similarly: If $f : Z^\beta(\lambda') \to T'Z^\beta(\mu)$ and $f' : T'Z^\beta(\mu) \to Z^\beta(\lambda')$ are bases of their Hom spaces, then $\mathrm{End}_{\mathcal{C}}(T'Z^\beta(\mu))$ has basis id, $f \circ f'$.

8.12. Suppose that we have chosen for each $\nu \in \Omega$ an isomorphism

$$f_\nu : Z^\beta(\nu_{\mathrm{T}}) \xrightarrow{\sim} TZ^\beta(\nu). \tag{1}$$

Then

$$\mathrm{adj}_1^{-1}(f_\nu^{-1}) : Z^\beta(\nu) \to T'Z^\beta(\nu_{\mathrm{T}})$$

is a basis of its Hom space.

Return to the situation of the last subsections with λ, λ', and μ as in 8.9(2). Now choose

$$f = \mathrm{adj}_1^{-1}(f_{\lambda'}^{-1}) \qquad \text{and} \qquad g' = \mathrm{adj}_1^{-1}(f_\lambda^{-1}). \tag{2}$$

These elements are bases of their Hom spaces, and we can choose (by the remark at the end of 8.10) a basis g resp. f' for the Hom space in the other direction such that

$$g \circ g' = H_\beta \,\mathrm{id} \qquad \text{resp.} \qquad f' \circ f = H_\beta \,\mathrm{id}. \tag{3}$$

We have then the exact sequence 8.9(2) for these choices of f and g. If the image of H_β in A^β is non-zero, then g and f' are uniquely determined by g' and f, hence by f_λ and $f_{\lambda'}$.

If A^β is equal to B^β or, more generally, if the image of H_β in A^β is non-zero, then

$$\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda')) = 0, \tag{4}$$

since

$$0 = \mathrm{Hom}_{\mathcal{C}_K}(Z_K(\lambda), Z_K(\lambda')) = \mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda')) \otimes_{A^\beta} K,$$

cf. 3.2. This implies for any A^β that

$$f' \circ g' = 0 \tag{5}$$

since we get these maps (up to units in A^β) from analogous maps over B^β by extension of scalars.

Lemma: *There is an isomorphism $h : TT'Z^\beta(\mu) \xrightarrow{\sim} Z^\beta(\mu) \oplus Z^\beta(\mu)$ such that the diagram*

$$\begin{array}{ccccccccc}
0 \to & TZ^\beta(\lambda') & \xrightarrow{Tf} & TT'Z^\beta(\mu) & \xrightarrow{Tg} & TZ^\beta(\lambda) & \to 0 \\
& \Big\downarrow{\scriptstyle f_{\lambda'}^{-1}} & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle f_\lambda^{-1}} & \\
0 \to & Z^\beta(\mu) & \longrightarrow & Z^\beta(\mu) \oplus Z^\beta(\mu) & \longrightarrow & Z^\beta(\mu) & \to 0
\end{array}$$

is commutative, where the maps in the lower row are $x \mapsto (x, 0)$ and $(x, y) \mapsto y$, and such that the diagram

$$\begin{array}{ccccc}
TZ^\beta(\lambda') & \xleftarrow{Tf'} & TT'Z^\beta(\mu) & \xleftarrow{Tg'} & TZ^\beta(\lambda) \\
\Big\downarrow{\scriptstyle f_{\lambda'}^{-1}} & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle f_\lambda^{-1}} \\
Z^\beta(\mu) & \longleftarrow & Z^\beta(\mu) \oplus Z^\beta(\mu) & \longleftarrow & Z^\beta(\mu)
\end{array}$$

is commutative, where the maps in the lower row are $(x, y) \mapsto xH_\beta - y$ and $x \mapsto (x, xH_\beta)$.

Proof: Lemma 8.11 implies that $\mathrm{Hom}_{\mathcal{C}}(TT'Z^\beta(\mu), Z^\beta(\mu))$ has basis $\mathrm{adj}_1(\mathrm{id})$, $\mathrm{adj}_1(g' \circ g)$ and that

$$h : TT'Z^\beta(\mu) \longrightarrow Z^\beta(\mu) \oplus Z^\beta(\mu)$$

with
$$x \mapsto (\mathrm{adj}_1(\mathrm{id})(x), \mathrm{adj}_1(g' \circ g)(x))$$
is an isomorphism. It leads to two commutative diagrams as in the lemma. We just have to compute the maps in the lower row. For the first diagram they follow from

$$\begin{aligned} \mathrm{adj}_1(\mathrm{id}) \circ Tf &= \mathrm{adj}_1(f) = f_{\lambda'}^{-1}, \\ \mathrm{adj}_1(g' \circ g) \circ Tf &= \mathrm{adj}_1(g' \circ g \circ f) = 0, \\ f_\lambda^{-1} \circ Tg &= \mathrm{adj}_1(g') \circ Tg = \mathrm{adj}_1(g' \circ g). \end{aligned}$$

For the second diagram the second formula follows from

$$\begin{aligned} \mathrm{adj}_1(\mathrm{id}) \circ Tg' &= \mathrm{adj}_1(g') = f_\lambda^{-1}, \\ \mathrm{adj}_1(g' \circ g) \circ Tg' &= \mathrm{adj}_1(g' \circ g \circ g') = H_\beta \, \mathrm{adj}_1(g') = H_\beta f_\lambda^{-1}. \end{aligned}$$

The first formula follows from $Tf' \circ Tf = H_\beta \, \mathrm{id}$ and $Tf' \circ Tg' = 0$ which imply that $(x, 0) \mapsto xH_\beta$ and $(x, xH_\beta) \mapsto 0$.

Remark: The last claim follows also from

$$f \circ f' + g' \circ g = H_\beta \, \mathrm{id}. \tag{6}$$

We leave the proof of (6) to the reader.

8.13. Recall that we assume that A^β is an integral domain. Assume now in addition that the image of H_β in A^β is not zero. By abuse of notation we shall often write H_β for $\pi(H_\beta)$

Suppose that we have weights λ, λ', μ as in 8.12 and maps f_ν as in 8.12(1). We want to construct an isomorphism

$$\theta[f_\lambda, f_{\lambda'}] : \mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda')) \longrightarrow A^\beta H_\beta^{-1}/A^\beta. \tag{1}$$

(We know of course by 8.6 that there is such an isomorphism.) Choose a representative
$$0 \to Z^\beta(\lambda') \xrightarrow{i} Y \xrightarrow{i} jZ^\beta(\lambda) \to 0 \tag{2}$$
of a class ξ in the Ext group. Since $H_\beta \xi$ is 0 in the Ext–group, there is a homomorphism $j' : Z^\beta(\lambda) \to Y$ with $j \circ j' = H_\beta \cdot \mathrm{id}$. This homomorphism is unique since we have as in 8.12(4)

$$\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda')) = 0. \tag{3}$$

If we apply T to (2), we get an exact sequence

$$0 \to TZ^\beta(\lambda') \xrightarrow{Ti} TY \xrightarrow{Tj} TZ^\beta(\lambda) \to 0. \tag{4}$$

Under our assumption, both extreme modules in (4) are isomorphic to $Z^\beta(\mu)$, so this sequence splits (but not uniquely). Let $i' : TY \to TZ^\beta(\lambda')$ be a homomorphism with $i' \circ Ti = \mathrm{id}$. Now $f_{\lambda'}^{-1} \circ i' \circ Tj' \circ f_\lambda$ is an endomorphism of $Z^\beta(\mu)$, so there is a unique scalar $a \in A^\beta$ with

$$f_{\lambda'}^{-1} \circ i' \circ Tj' \circ f_\lambda = a \,\mathrm{id}.$$

Suppose that $i'_0 : TY \to TZ^\beta(\lambda')$ is another map with $i'_0 \circ Ti = \mathrm{id}$ and let $a' \in A^\beta$ be the analogue to a, constructed with i'_0 instead of i'. We have now $(i' - i'_0) \circ Ti = 0$, therefore $i' - i'_0$ factors through Tj. So there is $b \in A^\beta$ with

$$i' - i'_0 = b\, f_{\lambda'} \circ f_\lambda^{-1} \circ Tj.$$

This yields $a - a' = bH_\beta$ and proves that the class of aH_β^{-1} in $A^\beta H_\beta^{-1}/A^\beta$ is independent of the choice of i'.

If we replace (2) by another representative of ξ, we see easily that we can choose the analogue to i' so that we get the same element a as before. This shows that we can define

$$\theta[f_\lambda, f_{\lambda'}](\xi) = aH_\beta^{-1} + A^\beta. \tag{5}$$

Remark: We have obviously for all units c, d in A^β:

$$\theta[cf_\lambda, df_{\lambda'}] = cd^{-1}\theta[f_\lambda, f_{\lambda'}]. \tag{6}$$

8.14. Keep the assumptions from the last subsection.

Proposition: *The map $\theta[f_\lambda, f_{\lambda'}]$ is an isomorphism* $\mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\lambda')) \xrightarrow{\sim} A^\beta H_\beta^{-1}/A^\beta$.

Proof: Set $\theta = \theta[f_\lambda, f_{\lambda'}]$. Let us show first that $\theta(a\xi) = a\theta(\xi)$ for all $a \in A^\beta$ and ξ in the Ext group. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & Z_A(\lambda') & \xrightarrow{i_a} & Y_a & \xrightarrow{j_a} & Z_A(\lambda) & \to & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle a\,\mathrm{id}} & & \\
0 & \to & Z_A(\lambda') & \xrightarrow{i} & Y & \xrightarrow{j} & Z_A(\lambda) & \to & 0
\end{array}$$

where the top (resp. bottom) row is a representative of $a\xi$ (resp. of ξ). Take j' and i' as in 8.13, let j'_a be the analogue of j' for the top row. We have

$$j \circ h \circ j'_a = aj_a \circ j'_a = aH_\beta \,\mathrm{id} = j \circ (aj'),$$

hence $h \circ j'_a = aj'$ by 8.13(3). We have

$$\mathrm{id} = i' \circ Ti = i' \circ Th \circ Ti_a,$$

so we can take $i'_a = i' \circ Th$. We get then

$$i'_a \circ Tj'_a = i' \circ Th \circ Tj'_A = i' \circ (aTj') = a(i' \circ Tj'),$$

hence the claim.

Take now for ξ a generator of the Ext group that is represented by a sequence as in 8.9(2); choose f, g as in 8.12(2) and (3). Comparing the notations in 8.12 and 8.13 we have $i = f$, $j = g$ and $j' = g'$ (by 8.12(3)). Furthermore, the first diagram in Lemma 8.12 shows that we can take $i' = f_{\lambda'} \circ \mathrm{pr}_1 \circ h$ where pr_1 is the first projection $(x, y) \mapsto x$. Then

$$f_{\lambda'}^{-1} \circ i' \circ Tj' \circ f_\lambda = \mathrm{pr}_1 \circ h \circ Tg' \circ f_\lambda = \mathrm{id}$$

where the last equality follows from the second diagram in Lemma 8.12. This shows that $\theta(\xi) = H_\beta^{-1} + A^\beta$. Since the Ext group is free over $A^\beta/H_\beta A^\beta$ with basis ξ, the claim follows from the first part of the proof.

Remark: The second part of the proof shows: Let a be a unit in A^β. Then $\theta[f_\lambda, f_{\lambda'}]^{-1}(aH_\beta^{-1} + A^\beta)$ can be represented by a sequence as in 8.9(2) with $f = \mathrm{adj}_1^{-1}(f_{\lambda'}^{-1})$ and $g \circ \mathrm{adj}_1^{-1}(f_\lambda^{-1}) = aH_\beta \mathrm{id}$. Indeed, that sequence represents $a\xi$ with ξ as in the last part of the proof.

9. A Functor into Combinatorics

From now on we shall usually write $\mathrm{Hom}_{\mathcal{C}}$ and $\mathrm{Ext}_{\mathcal{C}}$ instead of $\mathrm{Hom}_{\mathcal{C}_{A'}}$ and $\mathrm{Ext}_{\mathcal{C}_{A'}}$ (where A' is a U^0–algebra) and shall add the index A' only in cases where confusion is likely. Usually A' will be A, $A^{\emptyset}$ or an A^{β} (see below), and it will be clear what is meant.

Throughout this section we suppose that A is a B–algebra that is an integral domain such that all H_α with $\alpha \in R$ are nonzero in A. We set

$$A^{\emptyset} = A \otimes_B B^{\emptyset} \qquad \text{and} \qquad A^{\beta} = A \otimes_B B^{\beta} \quad \text{for all } \beta \in R^+$$

and regard these algebras as subrings of the fraction field of A. From 9.5 on we shall assume in addition that A is flat as a B–module. Also from 9.5 on we shall impose a restriction on p. This will enable us to apply Lemma 9.1.

9.1. Lemma: *Suppose in Case 1 that $p \neq 2$ if R has two root lengths and that $p \neq 3$ if R has a component of type G_2. Then*

$$B = \bigcap_{\beta \in R^+} B^{\beta}$$

and, if A is flat as a B–module, $A = \bigcap_{\beta \in R^+} A^{\beta}$.

Proof: It is enough to prove the first claim; the second one follows immediately.

We get B and each B^{β} by localization from a polynomial ring over k (in the H_γ resp. in the K_γ with $\gamma \in \Sigma$). So they are unique factorisation domains. It is therefore enough to show: If b is a prime element in B, then there is a β with $b^{-1} \notin B^{\beta}$.

By construction, the prime elements $b \in B$ with $b^{-1} \in B^{\beta}$ are exactly the prime divisors of any H_α with $\alpha \in R^+$ and $\alpha \neq \beta$. In Case 1 each H_α with $\alpha \in R^+$ is prime in U^0 and in B, since it is a polynomial of degree 1 and since it is not a unit in B. Under our assumption on p distinct α lead to distinct H_α that are not even multiples of each other, cf. Lemma 5.2 in [Hu1] for $p > 3$. This implies $H_\beta^{-1} \notin B^{\beta}$, hence the claim.

In Case 2 all $K_\gamma + a$ with $\gamma \in \Sigma$ and $a \in k$, $a \neq 0$ are obviously irreducible in U^0. Each $\alpha \in R^+$ is conjugate to some $\gamma \in \Sigma$ under the Weyl group, hence each $K_\alpha + a$ to some $K_\gamma + a$ under an automorphism of U^0. Therefore all these $K_\alpha + a$ are prime in U^0 and distinct pairs (α, a) lead to irreducibles that are not multiples of each other. By construction each $K_\alpha + 1$ and each $K_\alpha - 1$ is still prime in B. We get B^β from B by adjoining each $(K_\alpha + 1)^{-1}$ and each $(K_\alpha - 1)^{-1}$ with $\alpha \in R^+$ and $\alpha \neq \beta$. We see now that $(K_\beta + 1)^{-1}$ and $(K_\beta - 1)^{-1}$ are not in B^β and get the claim.

9.2. For any A–module M set

$$M^\emptyset = M \otimes_A A^\emptyset \qquad \text{and} \qquad M^\beta = M \otimes_A A^\beta \tag{1}$$

for all $\beta \in R^+$. If M is in $\mathcal{C}_A$, then $M^\emptyset$ resp. M^β is in the corresponding category over $A^\emptyset$ resp. over A^β. We define similarly $M^\emptyset$ for a A^β–module M. We have

$$Z^\emptyset(\lambda) = Z_{A^\emptyset}(\lambda) = Z_A(\lambda)^\emptyset \qquad \text{and} \qquad Z^\beta(\lambda) = Z_{A^\beta}(\lambda) = Z_A(\lambda)^\beta \tag{2}$$

for all $\lambda \in R^+$.

Lemma: *Suppose that A is flat over B and that the restrictions on p as in Lemma 9.1 are satisfied. Let M be a flat A–module. Then the natural maps $M \to M^\beta$ and $M^\beta \to M^\emptyset$ are injective for all $\beta \in R^+$, and we have inside $M^\emptyset$*

$$M = \bigcap_{\beta \in R^+} M^\beta. \tag{3}$$

Proof: We can express Lemma 9.1 as follows: We have an exact sequence

$$0 \to A \longrightarrow \bigoplus_{\beta \in R^+} A^\beta \longrightarrow \bigoplus_{\beta,\gamma \in R^+} A^\emptyset$$

where the map on A is the diagonal embedding and where the last map takes a family $(a_\beta)_\beta$ to the family of all differences $a_\beta - a_\gamma$. This sequence remains exact when tensoring over A with the flat A–module M. The claim easily follows.

9.3. Denote by $\mathcal{FC}_A$ the full subcategory of all M in $\mathcal{C}_A$ that are flat as A–modules. This subcategory contains all projective modules in $\mathcal{C}_A$, cf. Lemma 2.7.c. For any M in $\mathcal{FC}_A$ and for any $\lambda \in X$ set

$$\mathcal{V}M(\lambda) = \mathrm{Hom}_{\mathcal{C}}(Z^\emptyset(\lambda), M^\emptyset). \tag{1}$$

This is (by 2.8) a finitely generated $A^\emptyset$–module.

For each $\beta \in R^+$ and any $\lambda \in X$ there is a unique integer n with $0 \leq n < p$ and $\langle \lambda + \rho, \beta^\vee \rangle \equiv p - n \pmod p$; then set

$$\beta \uparrow \lambda = \lambda + n\beta.$$

One has $\beta \uparrow \lambda \in s_\beta \bullet \lambda + p\mathbf{Z}\beta$, especially $\beta \uparrow \lambda \in W_p \bullet \lambda$.

Fix $\beta \in R^+$ and $\lambda \in X$. In case $\beta \uparrow \lambda = \lambda$ we set

$$\mathcal{V}M(\lambda, \beta) = \mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), M^\beta) \subset \mathcal{V}M(\lambda). \tag{2}$$

This is (again by 2.8) a finitely generated A^β–submodule of $\mathcal{V}M(\lambda)$ and the inclusion induces by 3.2 an isomorphism

$$\mathcal{V}M(\lambda, \beta) \otimes_{A^\beta} A^\emptyset \xrightarrow{\sim} \mathcal{V}M(\lambda). \tag{3}$$

In case $\beta \uparrow \lambda \neq \lambda$, we shall define for each $e \in \mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta \uparrow \lambda))$ and each M as above a finitely generated A^β–submodule

$$\mathcal{V}M(\lambda, \beta, e) \subset \mathcal{V}M(\lambda) \oplus \mathcal{V}M(\beta \uparrow \lambda) \tag{4}$$

such that the inclusion induces an isomorphism

$$\mathcal{V}M(\lambda, \beta, e) \otimes_{A^\beta} A^\emptyset \xrightarrow{\sim} \mathcal{V}M(\lambda) \oplus \mathcal{V}M(\beta \uparrow \lambda). \tag{5}$$

Let

$$0 \to Z^\beta(\beta \uparrow \lambda) \xrightarrow{f} Q \xrightarrow{g} Z^\beta(\lambda) \to 0 \tag{6}$$

be a representative of e. This sequence splits uniquely over $A^\emptyset$, so there is a unique morphism $g' : Z^\emptyset(\lambda) \to Q^\emptyset$ with $g \circ g' = \mathrm{id}$. The map

$$\varphi \mapsto (\varphi \circ g', \varphi \circ f)$$

is now an embedding of $\mathrm{Hom}_{\mathcal{C}}(Q, M^\beta)$ into $\mathcal{V}M(\lambda) \oplus \mathcal{V}M(\beta \uparrow \lambda)$. We want to set $\mathcal{V}M(\lambda, \beta, e)$ equal to the image of this map. The splitting of (6) over $A^\emptyset$ together with 3.2 will then imply that (5) holds, and the finite generation follows again from 2.8. We have to check that this image depends only on e and not on the special representative chosen above. Well, if

$$0 \to Z^\beta(\beta \uparrow \lambda) \xrightarrow{f_1} Q_1 \xrightarrow{g_1} Z^\beta(\lambda) \to 0$$

is another representative of e, then there is a commutative diagram

$$\begin{array}{ccccccc}
0 \to & Z^\beta(\beta \uparrow \lambda) & \xrightarrow{f} & Q & \xrightarrow{g} & Z^\beta(\lambda) & \to 0 \\
 & \| & & \downarrow h & & \| & \\
0 \to & Z^\beta(\beta \uparrow \lambda) & \xrightarrow{f_1} & Q_1 & \xrightarrow{g_1} & Z^\beta(\lambda) & \to 0.
\end{array}$$

We have then $g_1 \circ h \circ g' = g \circ g' = \mathrm{id}$, so $h \circ g'$ is the analogue to g' for the second sequence. Therefore the embedding for the second sequence is given by

$$\varphi \mapsto (\varphi \circ h \circ g', \varphi \circ f_1) = (\varphi \circ h \circ g', \varphi \circ h \circ f).$$

It has the same image as the earlier map, since $\varphi \mapsto \varphi \circ h$ is an isomorphism from $\mathrm{Hom}_{\mathcal{C}}(Q_1, M^\beta)$ onto $\mathrm{Hom}_{\mathcal{C}}(Q, M^\beta)$.

Lemma: *We have for all* $a \in A^\beta$

$$(a,1)\mathcal{V}M(\lambda,\beta,e) \subset \mathcal{V}M(\lambda,\beta,ae).$$

Proof: There is a commutative diagram

$$\begin{array}{ccccccc} 0 \to & Z^\beta(\beta\uparrow\lambda) & \xrightarrow{f_1} & Q_1 & \xrightarrow{g_1} & Z^\beta(\lambda) & \to 0 \\ & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle a\,\mathrm{id}} & \\ 0 \to & Z^\beta(\beta\uparrow\lambda) & \xrightarrow{f} & Q & \xrightarrow{g} & Z^\beta(\lambda) & \to 0 \end{array}$$

where the second row is a representative of e, the first one of $a\,e$. There are unique morphisms $g' : Z^\emptyset(\lambda) \to Q^\emptyset$ with $g \circ g' = \mathrm{id}$ and $g_1' : Z^\emptyset(\lambda) \to Q_1^\emptyset$ with $g_1 \circ g_1' = \mathrm{id}$. We have $g \circ h \circ g_1' = a\,\mathrm{id}$, hence

$$h \circ g_1' = a\,g_1'. \tag{7}$$

By definition, $\mathcal{V}M(\lambda,\beta,e)$ is the image of $\mathrm{Hom}_\mathcal{C}(Q, M^\beta)$ under $\varphi \mapsto (\varphi \circ g', \varphi \circ f)$. Then $\varphi \circ h$ is in $\mathrm{Hom}_\mathcal{C}(Q_1, M^\beta)$, so — using (7) for the first step —

$$(\varphi \circ h \circ g_1', \varphi \circ h \circ f_1) = (\varphi \circ a\,g', \varphi \circ f) = (a,1)(\varphi \circ g', \varphi \circ f)$$

is in $\mathcal{V}M(\lambda,\beta,ae)$. The claim follows.

Remark: If a is a unit in A^β, then one checks then easily that one has equality in the lemma. (Multiply by $(a^{-1},1)$.) We shall prove a more general result in 9.10.

9.4. Let $\Omega \subset X$ be an orbit for W_p (under the dot action, as always). We want to define a category $\mathcal{K}(\Omega)$ and a functor $\mathcal{V}_\Omega : \mathcal{FC}_A(\Omega) \to \mathcal{K}(\Omega)$.

An object $\mathcal{M}$ in $\mathcal{K}(\Omega)$ is a family $(\mathcal{M}(\lambda))_{\lambda\in\Omega}$ of finitely generated $A^\emptyset$–modules (almost all equal to 0) together with (for all $\beta \in R^+$ and $\lambda \in \Omega$) a finitely generated A^β–submodule $\mathcal{M}(\lambda,\beta)$ of $\mathcal{M}(\lambda) \oplus \mathcal{M}(\beta\uparrow\lambda)$ if $\beta\uparrow\lambda \neq \lambda$, of $\mathcal{M}(\lambda)$ if $\beta\uparrow\lambda = \lambda$. A morphism ψ between two such objects $\mathcal{M}$, $\mathcal{M}'$ is a family $(\psi_\lambda)_{\lambda\in\Omega}$ of $A^\emptyset$–linear maps $\psi_\lambda : \mathcal{M}(\lambda) \to \mathcal{M}'(\lambda)$ such that for all $\beta \in R^+$ and $\lambda \in \Omega$:

$$\begin{aligned} (\psi_\lambda \oplus \psi_{\beta\uparrow\lambda})\mathcal{M}(\lambda,\beta) &\subset \mathcal{M}'(\lambda,\beta) \qquad \text{in case } \beta\uparrow\lambda \neq \lambda, \\ \psi_\lambda\mathcal{M}(\lambda,\beta) &\subset \mathcal{M}'(\lambda,\beta) \qquad \text{in case } \beta\uparrow\lambda = \lambda. \end{aligned}$$

Suppose that we have fixed for each $\beta \in R^+$ and $\lambda \in \Omega$ with $\beta\uparrow\lambda \neq \lambda$ a generator $e^\beta(\lambda)$ of the A^β–module $\mathrm{Ext}^1_\mathcal{C}(Z^\beta(\lambda), Z^\beta(\beta\uparrow\lambda))$. For M in $\mathcal{FC}_A(\Omega)$ we set

$$\mathcal{V}_\Omega M(\lambda) = \mathcal{V}M(\lambda),$$

$$\mathcal{V}_\Omega M(\lambda,\beta) = \begin{cases} \mathcal{V}M(\lambda,\beta), & \text{if } \beta\uparrow\lambda = \lambda; \\ \mathcal{V}M(\lambda,\beta,e^\beta(\lambda)), & \text{if } \beta\uparrow\lambda \neq \lambda. \end{cases}$$

It is obvious that $\mathcal{V}_\Omega$ is a functor from $\mathcal{FC}_A(\Omega)$ to $\mathcal{K}(\Omega)$.

Proposition: *Suppose in Case 1 that $p \neq 2$ if R has two root lengths and that $p \neq 3$ if R has a component of type G_2. If A is flat over B, then the functor $\mathcal{V}_\Omega$ is fully faithful.*

Proof: Let M, N be modules in $\mathcal{FC}_A(\Omega)$. We have to show that $\mathcal{V}_\Omega$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(\Omega)}(\mathcal{V}_\Omega M, \mathcal{V}_\Omega N). \tag{1}$$

Over $A^\emptyset$ each $\{\lambda\}$ with $\lambda \in X$ is a block by itself and $Z^\emptyset(\lambda)$ is a projective generator of that block, cf. the remark in 6.13. Since each $\mathrm{End}_{\mathcal{C}}(Z^\emptyset(\lambda))$ is isomorphic to $A^\emptyset$, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(M^\emptyset, N^\emptyset) \xrightarrow{\sim} \bigoplus_{\lambda \in \Omega} \mathrm{Hom}_{A^\emptyset}(\mathcal{V}_\Omega M(\lambda), \mathcal{V}_\Omega N(\lambda)).$$

This map is part of a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(M, N) & \longrightarrow & \mathrm{Hom}_{\mathcal{K}(\Omega)}(\mathcal{V}_\Omega M, \mathcal{V}_\Omega N) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}}(M^\emptyset, N^\emptyset) & \xrightarrow{\sim} & \bigoplus_{\lambda \in \Omega} \mathrm{Hom}_{A^\emptyset}(\mathcal{V}_\Omega M(\lambda), \mathcal{V}_\Omega N(\lambda)). \end{array}$$

The two vertical maps are obvious inclusions. So we see that the map in (1) has to be injective.

In order to prove surjectivity consider a morphism $\psi : \mathcal{V}_\Omega M \to \mathcal{V}_\Omega N$. We know already that there is a homomorphism $h : M^\emptyset \to N^\emptyset$ inducing all ψ_λ. Fix $\beta \in R^+$. For λ with $\beta \uparrow \lambda \neq \lambda$ let $Q^\beta(\lambda)$ be the middle term in $e^\beta(\lambda)$, for $\beta \uparrow \lambda = \lambda$ set $Q^\beta(\lambda) = Z^\beta(\lambda)$. Now ψ induces for all λ a map

$$\mathrm{Hom}_{\mathcal{C}}(Q^\beta(\lambda), M^\beta) \to \mathrm{Hom}_{\mathcal{C}}(Q^\beta(\lambda), N^\beta).$$

If it maps a homomorphism f to f', then the definitions imply easily $f' = h \circ f$.

By the remark in 8.6 there is a surjective homomorphism of the form $\bigoplus_i Q^\beta(\lambda_i) \to M^\beta$. So we can find $\lambda_i \in \Omega$ and homomorphisms $h_i : Q^\beta(\lambda_i) \to M^\beta$ such that $M^\beta = \sum_i h_i(Q^\beta(\lambda_i))$. As seen above $h \circ h_i$ is a homomorphism from $Q^\beta(\lambda_i)$ to N^β. This proves $h(h_i Q^\beta(\lambda_i)) \subset N^\beta$ for all i, hence

$$h(M^\beta) = h(\sum_i h_i Q^\beta(\lambda_i)) \subset N^\beta.$$

Now

$$h(M) \subset \bigcap_{\beta > 0} N^\beta = N,$$

where we use for the last equality 9.2(3) and the fact that N is flat over A. The claim follows.

9.5. Assume from now on (in this section) that A is a flat B-module and the condition on p in Lemma 9.1 and Proposition 9.4 is satisfied. Fix an orbit $\Omega \subset X$ under W_p and suppose that we have chosen $e^\beta(\lambda)$ as in 9.4, hence chosen $\mathcal{V}_\Omega$.

For all $\lambda \in X$ and $\beta \in R^+$ let $\beta\downarrow\lambda \in X$ be the unique weight with $\beta\uparrow(\beta\downarrow\lambda) = \lambda$. (This is compatible with the notation in the remark to 6.2.) We have $\beta\downarrow\lambda \in W_p{\bullet}\lambda$.

Proposition: *Let $\lambda \in \Omega$ and $w \in W$. Then $\mathrm{Hom}_{\mathcal{C}}(Z_A(\lambda), Z_A^w(\lambda\langle w\rangle))$ is a free A-module of rank 1; choose a basis f_0. Then*

$$\mathcal{V}_\Omega Z_A^w(\lambda\langle w\rangle)(\lambda) = A^\emptyset f_0 \tag{1}$$

and $\mathcal{V}_\Omega Z_A^w(\lambda\langle w\rangle)(\mu) = 0$ for all $\mu \neq \lambda$. We have for all $\beta \in R^+$

$$\mathcal{V}_\Omega Z_A^w(\lambda\langle w\rangle)(\lambda,\beta) = \begin{cases} A^\beta f_0, & \text{if } \beta\uparrow\lambda = \lambda; \\ A^\beta(f_0,0), & \text{if } \beta\uparrow\lambda \neq \lambda, \end{cases} \tag{2}$$

and, if $\beta\uparrow\lambda \neq \lambda$,

$$\mathcal{V}_\Omega Z_A^w(\lambda\langle w\rangle)(\beta\downarrow\lambda,\beta) = \begin{cases} A^\beta(0,H_\beta f_0), & \text{if } w^{-1}\beta > 0; \\ A^\beta(0,f_0), & \text{if } w^{-1}\beta < 0; \end{cases} \tag{3}$$

all other $\mathcal{V}_\Omega Z_A^w(\lambda\langle w\rangle)(\mu,\beta)$ are equal to 0.

Proof: Set $M = Z_A^w(\lambda\langle w\rangle)$. The claim about the Hom space is a special case of Lemma 4.7; the basis vector f_0 is then also a basis for the corresponding Hom spaces over all A^β and over $A^\emptyset$, cf. 3.2. We get especially (1) and the first case in (2). Since $M^\emptyset \simeq Z^\emptyset(\lambda)$, it is clear that $\mathcal{V}M(\mu) = 0$ for all $\mu \neq \lambda$, and then also that $\mathcal{V}M(\mu,\beta) = 0$ for all $\mu \neq \lambda, \beta\downarrow\lambda$.

Assume from now on that $\beta\uparrow\lambda \neq \lambda$. Choose a representative

$$0 \to Z^\beta(\beta\uparrow\lambda) \xrightarrow{f} Q \xrightarrow{g} Z^\beta(\lambda) \to 0$$

of $e^\beta(\lambda)$. Let $g' : Z^\beta(\lambda) \to Q$ be the homomorphism with $g \circ g' = H_\beta \mathrm{id}$. Now $\mathcal{V}_\Omega M(\lambda,\beta)$ is the image of $\mathrm{Hom}_{\mathcal{C}}(Q, M^\beta)$ under $h \mapsto (H_\beta^{-1} h \circ g', h \circ f)$. Since $\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\beta\uparrow\lambda), M) = 0$, we get an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), M^\beta) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Q, M^\beta), \qquad h \mapsto h \circ g.$$

Therefore $f_0 \circ g$ is a basis of $\mathrm{Hom}_{\mathcal{C}}(Q, M^\beta)$, and $\mathcal{V}_\Omega M(\lambda,\beta) = A^\beta(f_0,0)$.

Choose now a representative

$$0 \to Z^\beta(\lambda) \xrightarrow{f} Q \xrightarrow{g} Z^\beta(\beta\downarrow\lambda) \to 0$$

of $e^\beta(\beta\downarrow\lambda)$. Let $g' : Z^\beta(\beta\downarrow\lambda) \to Q$ be the homomorphism with $g\circ g' = H_\beta \mathrm{id}$. Now $\mathcal{V}_\Omega M(\beta\downarrow\lambda,\beta)$ is the image of $\mathrm{Hom}_\mathcal{C}(Q, M^\beta)$ under $h \mapsto (H_\beta^{-1} h\circ g', h\circ f)$. There are two cases:
Case 1: $w^{-1}\beta > 0$. Then the remark in 5.14 implies that $Z^\beta(\lambda)$ and M^β are isomorphic, so the basis f_0 of their Hom space has to be an isomorphism. We have by 8.10 a basis f_1 of $\mathrm{Hom}_\mathcal{C}(Q, Z^\beta(\lambda))$ with $f_1 \circ f = H_\beta \mathrm{id}$. Then $f_0 \circ f_1$ is a basis of $\mathrm{Hom}_\mathcal{C}(Q, M^\beta)$, and we get: $\mathcal{V}_\Omega M(\beta\downarrow\lambda,\beta) = A^\beta(0, H_\beta f_0)$.
Case 2: $w^{-1}\beta < 0$. Then M^β is isomorphic to $Z_A^{w_0}(\lambda\langle w_0\rangle)^\beta$, again by the remark in 5.14. So Proposition 4.11 implies that $\mathrm{Ext}^1_\mathcal{C}(Z^\beta(\beta\downarrow\lambda), M^\beta) = 0$. We know already that $\mathrm{Hom}_\mathcal{C}(Z^\beta(\beta\downarrow\lambda), M^\beta) = 0$, so we get an isomorphism

$$\mathrm{Hom}_\mathcal{C}(Q, M^\beta) \xrightarrow{\sim} \mathrm{Hom}_\mathcal{C}(Z^\beta(\lambda), M^\beta), \qquad h \mapsto h\circ f.$$

Therefore there is a basis h of $\mathrm{Hom}_\mathcal{C}(Q, M^\beta)$ with $h\circ f = f_0$, and we get $\mathcal{V}_\Omega M(\beta\downarrow\lambda,\beta) = A^\beta(0, f_0)$.

9.6. Corollary: *Let $\lambda, \lambda' \in X$ with $\mathrm{Ext}^1_\mathcal{C}(Z_A(\lambda), Z_A(\lambda')) \neq 0$. Then there is $\beta \in R^+$ with $\lambda' = \beta\uparrow\lambda \neq \lambda$.*

Proof: We may assume that $\lambda \in \Omega$. We know by 2.14.b that $\lambda' > \lambda$, and by 6.13 that $\lambda' \in \Omega$. Suppose that

$$0 \to Z_A(\lambda') \longrightarrow M \longrightarrow Z_A(\lambda) \to 0$$

is a nontrivial extension. Then the corresponding sequence

$$0 \to \mathcal{V}_\Omega Z_A(\lambda') \longrightarrow \mathcal{V}_\Omega M \longrightarrow \mathcal{V}_\Omega Z_A(\lambda) \to 0$$

does not split. Therefore there are $\mu \in \Omega$ and $\beta \in R^+$ with

$$\mathcal{V}_\Omega Z_A(\lambda')(\mu,\beta) \neq 0 \neq \mathcal{V}_\Omega Z_A(\lambda)(\mu,\beta).$$

The explicit description of the $\mathcal{V}_\Omega Z_A(\nu)$ in 9.5 implies

$$\mu \in \{\lambda', \beta\downarrow\lambda'\} \cap \{\lambda, \beta\downarrow\lambda\}.$$

We have $\lambda' \neq \lambda$, hence $\beta\downarrow\lambda' \neq \beta\downarrow\lambda$. Furthermore $\lambda' > \lambda$ implies $\beta\downarrow\lambda \neq \lambda'$. Therefore $\mu = \lambda = \beta\downarrow\lambda'$ and $\lambda' = \beta\uparrow\lambda \neq \lambda$ as claimed.

9.7. Proposition: *Let $\lambda \in X$ with $\lambda+\rho \in pX$. We have for each B–algebra B' isomorphisms $Z^w_{B'}(\lambda\langle w\rangle) \simeq Z_{B'}(\lambda)$ for all $w \in W$; the module $Z_{B'}(\lambda)$ is projective in $\mathcal{C}_{B'}$.*

Proof: We may assume that $B' = B$. (That is obvious for the first claim; for the projectivity use Lemma 3.1.)

Set $\Omega = W_p{\scriptstyle\bullet}\lambda$. The condition $\lambda + \rho \in pX$ is equivalent to $\beta\uparrow\lambda = \lambda$ for all $\beta \in R^+$. So Proposition 9.5 implies $\mathcal{V}_\Omega Z_B^w(\lambda\langle w\rangle) = \mathcal{V}_\Omega Z_B(\lambda)$ for all $w \in W$. This yields an isomorphism $Z_B^w(\lambda\langle w\rangle) \simeq Z_B(\lambda)$ by Proposition 9.4. Corollary 9.6 implies that $Z_B(\lambda)$ does not extend any $Z_B(\lambda')$. By the remark in 2.13 there is a projective module Q in $\mathcal{C}_B$ with a Z–filtration such that $Z_B(\lambda)$ is a factor in this filtration. Now this factor has to split off, and $Z_B(\lambda)$ is projective as a direct summand of a projective module.

Remark: One can prove these results without the present theory: The isomorphisms can be constructed using maps as in 5.10(1), (2) and arguing as in 5.8. The projectivity follows in the case of a field from Lemma 6.3, in the general case then from Corollary 3.5.

9.8. Lemma: *Let $\lambda \in \Omega$ and $\beta \in R^+$ with $\beta\uparrow\lambda \neq \lambda$. Then we have for all M in $\mathcal{FC}_A(\Omega)$ and for all $e \in \mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta\uparrow\lambda))$*

$$\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), M^\beta) = \mathcal{V}M(\lambda,\beta,e) \cap \mathcal{V}M(\lambda). \tag{1}$$

Proof: Choose a representative

$$0 \to Z^\beta(\beta\uparrow\lambda) \xrightarrow{f} Q \xrightarrow{g} Z^\beta(\lambda) \to 0 \tag{2}$$

of e. Let $g' : Z^\beta(\lambda) \to Q$ be the homomorphism with $g \circ g' = H_\beta \mathrm{id}$. So $\mathcal{V}M(\lambda,\beta,e)$ is the image of

$$\mathrm{Hom}_{\mathcal{C}}(Q, M^\beta) \to \mathcal{V}M(\lambda) \oplus \mathcal{V}M(\beta\uparrow\lambda), \qquad h \mapsto (H_\beta^{-1} h \circ g', h \circ f).$$

Any $h_1 \circ g$ with $h_1 \in \mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), M^\beta)$ is mapped to $(h_1, 0)$. This shows that the left hand side in (1) is contained in the right hand side.

On the other hand, if a homomorphism $h : Q \to M^\beta$ has image in $\mathcal{V}M(\lambda)$, then $h \circ f = 0$. Then h factors via g and there is a homomorphism $h_1 : Z^\beta(\lambda) \to M^\beta$ with $h = h_1 \circ g$. Then the image of h is $(h_1, 0)$. This yields the other inclusion.

Remark: Keep the notations of the proof. If (x, y) is an element of $\mathcal{V}M(\lambda,\beta,e)$ then $(x,y) = (H_\beta^{-1} h \circ g', h \circ f)$ for some $h \in \mathrm{Hom}_{\mathcal{C}}(Q, M^\beta)$. This implies

$$H_\beta x \in \mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), M^\beta) \qquad \text{and} \qquad y \in \mathrm{Hom}_{\mathcal{C}}(Z^\beta(\beta\uparrow\lambda), M^\beta), \tag{3}$$

in particular, by the lemma,

$$(H_\beta x, 0) \in \mathcal{V}M(\lambda,\beta,e). \tag{4}$$

9.9. In the situation of 9.8 set

$$\mathcal{V}M(\lambda)_\beta = \mathcal{V}M(\lambda, \beta, e) \cap \mathcal{V}M(\lambda). \tag{1}$$

By 9.8(1) the right hand side is independent of e. Note that 9.8(4) implies

$$(H_\beta, 0)\mathcal{V}M(\lambda, \beta, e) \subset \mathcal{V}M(\lambda)_\beta. \tag{2}$$

For λ with $\beta \uparrow \lambda = \lambda$ we set

$$\mathcal{V}M(\lambda)_\beta = \mathcal{V}M(\lambda, \beta) \subset \mathcal{V}M(\lambda). \tag{3}$$

So we have in any case, by 9.8 or by definition

$$\mathcal{V}M(\lambda)_\beta = \mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), M^\beta). \tag{4}$$

So 3.2 implies that the inclusion induces an isomorphism

$$\mathcal{V}M(\lambda)_\beta \otimes_{A^\beta} A^\emptyset \xrightarrow{\sim} \mathcal{V}M(\lambda). \tag{5}$$

We set now

$$\mathcal{V}M(\lambda)_A = \bigcap_{\beta \in R^+} \mathcal{V}M(\lambda)_\beta \tag{6}$$

and get (since we assume the condition in Proposition 9.4)

$$\mathrm{Hom}_{\mathcal{C}}(Z_A(\lambda), M) = \mathcal{V}M(\lambda)_A. \tag{7}$$

If N is a second module in $\mathcal{FC}_A(\Omega)$ any morphism $\mathcal{V}_\Omega M \to \mathcal{V}_\Omega N$ has to map any $\mathcal{V}M(\lambda)_\beta$ to $\mathcal{V}N(\lambda)_\beta$, hence any $\mathcal{V}M(\lambda)_A$ to $\mathcal{V}N(\lambda)_A$. (This is clear also from Proposition 9.4 and from (7).)

Let $\mathcal{M}$ be an object in $\mathcal{K}(\Omega)$. Set (for all $\lambda \in \Omega$ and $\beta \in R^+$)

$$\mathcal{M}(\lambda)_\beta = \mathcal{M}(\lambda) \cap \mathcal{M}(\lambda, \beta) \qquad \text{and} \qquad \mathcal{M}(\lambda)_A = \bigcap_{\beta \in R^+} \mathcal{M}(\lambda)_\beta.$$

(So $\mathcal{M}(\lambda)_\beta = \mathcal{M}(\lambda, \beta)$ if $\beta \uparrow \lambda = \lambda$.) Each $\mathcal{M}(\lambda)_\beta$ is a finitely generated A^β-module since $\mathcal{M}(\lambda, \beta)$ is finitely generated and since we always assume A to be Noetherian. Note that these definitions are compatible with (1), (3), and (6). If $\mathcal{M}$ is in the image of $\mathcal{V}_\Omega$, then the inclusion induces by (5) isomorphisms

$$\mathcal{M}(\lambda)_\beta \otimes_{A^\beta} A^\emptyset \xrightarrow{\sim} \mathcal{M}(\lambda) \tag{8}$$

(for all $\lambda \in \Omega$ and $\beta \in R^+$) and (by (7) and 3.2) isomorphisms

$$\mathcal{M}(\lambda)_A \otimes_A A^\emptyset \xrightarrow{\sim} \mathcal{M}(\lambda) \tag{9}$$

for all $\lambda \in \Omega$. Note that 9.3(5) implies that (for all $\lambda \in \Omega$ and $\beta \in R^+$ with $\beta \uparrow \lambda \neq \lambda$)

$$\mathcal{M}(\lambda, \beta) \otimes_{A^\beta} A^\emptyset \xrightarrow{\sim} \mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda) \tag{10}$$

for $\mathcal{M}$ in the image of $\mathcal{V}_\Omega$. (It would be nice to have a characterization of that image.)

9.10. Lemma: *Let $\lambda \in \Omega$ and $\beta \in R^+$ with $\beta\uparrow\lambda \neq \lambda$. Let M be in $\mathcal{FC}_A(\Omega)$ and e in $\operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta\uparrow\lambda))$. We have then for all $a \in A^\beta$ such that the coset $a + H_\beta A^\beta$ is a unit in $A^\beta/H_\beta A^\beta$*

$$\mathcal{V}M(\lambda,\beta,a\,e) = (a,1)\mathcal{V}M(\lambda,\beta,e) + \mathcal{V}M(\lambda)_\beta. \tag{1}$$

Proof: We have by Lemma 9.3 and by the definition 9.9(1)

$$\mathcal{V}M(\lambda,\beta,c\,e) \supset (c,1)\mathcal{V}M(\lambda,\beta,e) + \mathcal{V}M(\lambda)_\beta \tag{2}$$

for all $c \in A^\beta$. If b, $c \in A^\beta$ are congruent modulo $A^\beta H_\beta$, then there is $d \in A^\beta$ with $b = c + dH_\beta$, hence

$$(b,1)(x,y) = (c,1)(x,y) + d(H_\beta,0)(x,y)$$

for all $(x,y) \in \mathcal{V}M(\lambda,\beta,e)$. This implies

$$\begin{aligned}(b,1)\mathcal{V}M(\lambda,\beta,e) &\subset (c,1)\mathcal{V}M(\lambda,\beta,e) + (H_\beta,0)\mathcal{V}M(\lambda,\beta,e)\\ &\subset (c,1)\mathcal{V}M(\lambda,\beta,e) + \mathcal{V}M(\lambda)_\beta\end{aligned}$$

using 9.9(2) for the second step. We get in particular, if $a \equiv 1 \pmod{H_\beta}$,

$$\begin{aligned}\mathcal{V}M(\lambda,\beta,e) &\subset (a,1)\mathcal{V}M(\lambda,\beta,e) + \mathcal{V}M(\lambda)_\beta\\ &\subset \mathcal{V}M(\lambda,\beta,ae) = \mathcal{V}M(\lambda,\beta,e).\end{aligned}$$

We get therefore equality here; this proves (1) for $a \equiv 1 \pmod{H_\beta}$.

In general, choose $b \in A^\beta$ with $ab \equiv 1 \pmod{H_\beta}$. Then

$$\begin{aligned}\mathcal{V}M(\lambda,\beta,e) &= \mathcal{V}M(\lambda,\beta,ba\,e)\\ &\supset (b,1)\mathcal{V}M(\lambda,\beta,ae) + \mathcal{V}M(\lambda)_\beta \qquad \text{— by (2) —}\\ &\supset (b,1)(a,1)\mathcal{V}M(\lambda,\beta,e) + \mathcal{V}M(\lambda)_\beta\\ &\qquad\qquad \text{since } (b,1)\mathcal{V}M(\lambda)_\beta \subset \mathcal{V}M(\lambda)_\beta\\ &= (ba,1)\mathcal{V}M(\lambda,\beta,e) + \mathcal{V}M(\lambda)_\beta = \mathcal{V}M(\lambda,\beta,e).\end{aligned}$$

where we use for the last step that (1) holds for ab by the first part of the proof. We get now equality at all steps, especially

$$\mathcal{V}M(\lambda,\beta,e) = (b,1)\mathcal{V}M(\lambda,\beta,ae) + \mathcal{V}M(\lambda)_\beta.$$

Apply $(a,1)$ to both sides and add $\mathcal{V}M(\lambda)_\beta$ to them. The claim follows.

Remark: If a is a unit in A^β, then we have for all $\mathcal{M}$ in $\mathcal{K}(\Omega)$

$$\mathcal{M}(\lambda)_\beta = (a,1)\mathcal{M}(\lambda)_\beta \subset (a,1)\mathcal{M}(\lambda,\beta). \tag{3}$$

This implies

$$(a,1)\mathcal{M}(\lambda,\beta) + \mathcal{M}(\lambda)_\beta = (a,1)\mathcal{M}(\lambda,\beta), \tag{4}$$

in particular

$$\mathcal{V}M(\lambda,\beta,a\,e) = (a,1)\mathcal{V}M(\lambda,\beta,e). \tag{5}$$

(One can prove (5) also directly, cf. the remark in 9.3.)

9.11. Consider an object $\mathcal{M}$ in $\mathcal{K}(\Omega)$ and a subobject $\mathcal{N}$ of $\mathcal{M}$, i.e., each $\mathcal{N}(\lambda)$ is an $A^{\emptyset}$–submodule of $\mathcal{M}(\lambda)$ and each $\mathcal{N}(\lambda,\beta)$ is an A^{β}–submodule of $\mathcal{M}(\lambda,\beta)$. We call $\mathcal{N}$ a *compatible subobject* of $\mathcal{M}$, if for all λ and β

$$\mathcal{N}(\lambda,\beta) = \begin{cases} \mathcal{M}(\lambda,\beta) \cap \mathcal{N}(\lambda), & \text{if } \beta\uparrow\lambda = \lambda; \\ \mathcal{M}(\lambda,\beta) \cap (\mathcal{N}(\lambda) \oplus \mathcal{N}(\beta\uparrow\lambda)), & \text{otherwise.} \end{cases} \tag{1}$$

If so, then we define an object $\mathcal{M}/\mathcal{N}$ in $\mathcal{K}(\Omega)$ with $(\mathcal{M}/\mathcal{N})(\lambda) = \mathcal{M}(\lambda)/\mathcal{N}(\lambda)$ for all λ and $(\mathcal{M}/\mathcal{N})(\lambda,\beta) = \mathcal{M}(\lambda,\beta)/\mathcal{N}(\lambda,\beta)$ for all λ and β. Note that $(\mathcal{M}/\mathcal{N})(\lambda,\beta)$ is — by (1) — embedded in $(\mathcal{M}/\mathcal{N})(\lambda)$ resp. in $(\mathcal{M}/\mathcal{N})(\lambda) \oplus (\mathcal{M}/\mathcal{N})(\beta\uparrow\lambda)$.

Lemma: *Let M be a module in $\mathcal{C}_A(\Omega)$ and N a submodule of M such that N and M/N are flat as A–modules. Then $\mathcal{V}_\Omega N$ is compatible in $\mathcal{V}_\Omega M$ and there is a natural isomorphism*

$$\mathcal{V}_\Omega M/\mathcal{V}_\Omega N \simeq \mathcal{V}_\Omega(M/N). \tag{2}$$

Proof: Our assumption implies that also M is flat over A. So M, N, and M/N are all in $\mathcal{FC}_A(\Omega)$ and $\mathcal{V}_\Omega M$, $\mathcal{V}_\Omega N$, $\mathcal{V}_\Omega(M/N)$ are defined. Recall from 9.2 that the natural maps $M \to M^{\beta}$ and $M^{\beta} \to M^{\emptyset}$ are injective; similarly for N and M/N. Applying this in the case of M/N we see that

$$N^{\beta} = M^{\beta} \cap N^{\emptyset}$$

for all $\beta \in R^{+}$. This implies

$$\mathrm{Hom}_{\mathcal{C}}(Q^{\beta}(\lambda), N^{\beta}) = \mathrm{Hom}_{\mathcal{C}}(Q^{\beta}(\lambda), M^{\beta}) \cap \mathrm{Hom}_{\mathcal{C}}(Q^{\beta}(\lambda)^{\emptyset}, N^{\emptyset})$$

for all $\lambda \in \Omega$, hence

$$\mathcal{V}_\Omega N(\lambda,\beta) = \mathcal{V}_\Omega M(\lambda,\beta) \cap \big(\mathcal{V}_\Omega N(\lambda) \oplus \mathcal{V}_\Omega N(\beta\uparrow\lambda)\big) \qquad \text{if } \beta\uparrow\lambda \neq \lambda, \tag{2}$$

resp.

$$\mathcal{V}_\Omega N(\lambda,\beta) = \mathcal{V}_\Omega M(\lambda,\beta) \cap \mathcal{V}_\Omega N(\lambda) \qquad \text{if } \beta\uparrow\lambda = \lambda. \tag{3}$$

This yields the claim on compatibility.

We have by projectivity exact sequences

$$0 \to \mathrm{Hom}_{\mathcal{C}}(Z^{\emptyset}(\lambda), N^{\emptyset}) \to \mathrm{Hom}_{\mathcal{C}}(Z^{\emptyset}(\lambda), M^{\emptyset}) \to \mathrm{Hom}_{\mathcal{C}}(Z^{\emptyset}(\lambda), (M/N)^{\emptyset}) \to 0$$

for all $\lambda \in \Omega$, and

$$0 \to \mathrm{Hom}_{\mathcal{C}}(Q^{\beta}(\lambda), N^{\beta}) \to \mathrm{Hom}_{\mathcal{C}}(Q^{\beta}(\lambda), M^{\beta}) \to \mathrm{Hom}_{\mathcal{C}}(Q^{\beta}(\lambda), (M/N)^{\beta}) \to 0$$

for all $\lambda \in \Omega$ and $\beta \in R^{+}$. Therefore the canonical map $M \to M/N$ induces isomorphisms

$$\mathcal{V}_\Omega M(\lambda)/\mathcal{V}_\Omega N(\lambda) \xrightarrow{\sim} \mathcal{V}_\Omega(M/N)(\lambda)$$

and

$$\mathcal{V}_\Omega M(\lambda,\beta)/\mathcal{V}_\Omega N(\lambda,\beta) \xrightarrow{\sim} \mathcal{V}_\Omega(M/N)(\lambda,\beta),$$

and (2) follows.

9.12. Let $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3$ be a chain of subobjects in $\mathcal{K}(\Omega)$. The following properties are more or less obvious:

If $\mathcal{M}_1$ is compatible in $\mathcal{M}_3$, then $\mathcal{M}_1$ is compatible in $\mathcal{M}_2$. If $\mathcal{M}_1$ is compatible in $\mathcal{M}_2$ and $\mathcal{M}_2$ is compatible in $\mathcal{M}_3$, then $\mathcal{M}_1$ is compatible in $\mathcal{M}_3$.

Suppose that $\mathcal{M}_1$ is compatible in $\mathcal{M}_3$. We can then construct $\mathcal{M}_3/\mathcal{M}_1$ and $\mathcal{M}_2/\mathcal{M}_1$ as in 9.11; we have a natural embedding $\mathcal{M}_2/\mathcal{M}_1 \subset \mathcal{M}_3/\mathcal{M}_1$. Then $\mathcal{M}_2$ is compatible in $\mathcal{M}_3$ if and only if $\mathcal{M}_2/\mathcal{M}_1$ is compatible in $\mathcal{M}_3/\mathcal{M}_1$. If so, then we have a natural isomorphism

$$\mathcal{M}_3/\mathcal{M}_2 \simeq (\mathcal{M}_3/\mathcal{M}_1)/(\mathcal{M}_2/\mathcal{M}_1).$$

Suppose again that $\mathcal{M}_1$ is compatible in $\mathcal{M}_3$. Let $\mathcal{Q}$ be a subobject of $\mathcal{M}_3/\mathcal{M}_1$. Let $\mathcal{N}$ be the inverse image of $\mathcal{Q}$ in $\mathcal{M}_3$, i.e., the subobject of $\mathcal{M}_3$ such that each $\mathcal{N}(\lambda)$ is the inverse image of $\mathcal{Q}(\lambda)$ in $\mathcal{M}_3(\lambda)$ and each $\mathcal{N}(\lambda,\beta)$ the inverse image of $\mathcal{Q}(\lambda,\beta)$ in $\mathcal{M}_3(\lambda,\beta)$. We have then inclusions $\mathcal{M}_1 \subset \mathcal{N} \subset \mathcal{M}_3$ and a natural isomorphism $\mathcal{N}/\mathcal{M}_1 \simeq \mathcal{Q}$. The discussion above (applied to $\mathcal{M}_2 = \mathcal{N}$) implies that $\mathcal{Q}$ is compatible in $\mathcal{M}_3/\mathcal{M}_1$ if and only if $\mathcal{N}$ is compatible in $\mathcal{M}_3$.

Let $M_1 \subset M_2 \subset M_3$ be a chain of modules in $\mathcal{FC}_A(\Omega)$ such that also M_2/M_1 and M_3/M_2 are flat over A. Then $\mathcal{V}_\Omega M_2 \subset \mathcal{V}_\Omega M_3$ is the inverse image (as above) of $\mathcal{V}_\Omega M_2/\mathcal{V}_\Omega M_1 \subset \mathcal{V}_\Omega M_3/\mathcal{V}_\Omega M_1$.

9.13. Recall from 6.18 that the functors $M \mapsto M[p\nu]$ with $\nu \in \mathbf{Z}R$ map $\mathcal{C}_A(\Omega)$ into itself and make it into a $(p\mathbf{Z}R)$–category. We define similar shift operators $\mathcal{M} \mapsto \mathcal{M}[p\nu]$ on $\mathcal{K}(\Omega)$: Set for all $\lambda \in \Omega$ and $\beta \in R^+$

$$\begin{aligned}\mathcal{M}[p\nu](\lambda) &= \mathcal{M}(\lambda - p\nu),\\ \mathcal{M}[p\nu](\lambda,\beta) &= \mathcal{M}(\lambda - p\nu,\beta).\end{aligned} \tag{1}$$

(Note that $\beta\uparrow(\lambda+p\nu) = (\beta\uparrow\lambda)+p\nu$ for all λ and β.) We have for all $\mathcal{M}$, $\mathcal{N}$ in $\mathcal{K}(\Omega)$ a natural isomorphism

$$\mathrm{Hom}_{\mathcal{K}(\Omega)}(\mathcal{M},\mathcal{N}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(\Omega)}(\mathcal{M}[p\nu],\mathcal{N}[p\nu]). \tag{2}$$

We map a family $(\psi_\lambda)_{\lambda\in\Omega}$ to the family $(\psi'_\mu)_{\mu\in\Omega}$ with $\psi'_\mu = \psi_{\mu-p\nu}$ for all μ. We have obviously $\mathcal{M}[0] = \mathcal{M}$ and

$$(\mathcal{M}[p\nu])[p\nu'] = \mathcal{M}[p(\nu+\nu')]. \tag{3}$$

for all $\mathcal{M}$ and all $\nu,\nu' \in \mathbf{Z}R$. So $\mathcal{K}(\Omega)$ gets the structure of a $(p\mathbf{Z}R)$–category.

For arbitrary $\nu \in X$ the map $\lambda \mapsto \lambda + p\nu$ takes Ω to another W_p–orbit Ω'. We can then define a functor $\mathcal{M} \mapsto M[p\nu]$ from $\mathcal{K}(\Omega)$ to $\mathcal{K}(\Omega')$ using the same formulas as above. The results above extend; we shall not have to use these generalizations.

We can identify $Z_A(\lambda + p\nu)$ with $Z_A(\lambda)[p\nu]$ for all $\lambda, \nu \in X$ by 4.2(5). Using this we get also an identification

$$\mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta \uparrow \lambda)) \simeq \mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda + p\nu), Z^\beta(\beta \uparrow \lambda + p\nu)). \tag{4}$$

Now the definitions imply easily:

If we have — under the identification in (4) — $e^\beta(\lambda + p\nu) = e^\beta(\lambda)$ *for all* $\lambda \in \Omega$, $\beta \in R^+$, *and* $\nu \in \mathbf{Z}R$, *then*

$$\mathcal{V}_\Omega(M[p\nu]) = (\mathcal{V}_\Omega M)[p\nu] \tag{5}$$

for all M *in* $\mathcal{FC}_A(\Omega)$ *and* $\nu \in \mathbf{Z}R$.

So $\mathcal{V}_\Omega$ is then a $(p\mathbf{Z}R)$–functor in the terminology of E.3. Our final choice of the e^β in 14.1 will satisfy the assumption above, cf. 14.12.

0. Translations and Combinatorics

Throughout this section we consider A as at the beginning of Section 9 and keep the notations from that beginning and from 9.2–3.

10.1. Consider two orbits Ω and Γ for W_p. We have then translation functors $T = T_\Omega^\Gamma$ from $\mathcal{C}_A(\Omega)$ to $\mathcal{C}_A(\Gamma)$, and $T' = T_\Gamma^\Omega$ in the other direction. These functors take the subcategories $\mathcal{FC}_A(\Omega)$ and $\mathcal{FC}_A(\Gamma)$ to each other. (If M is a flat A-module and E a vector space of dimension n over k, then $E \otimes M$ is isomorphic to M^n as an A-module, hence flat; so is every direct summand of $E \otimes M$.) We want to compare $\mathcal{V}_\Omega(T'N)$ to $\mathcal{V}_\Gamma(N)$ for N in $\mathcal{FC}_A(\Gamma)$ and $\mathcal{V}_\Gamma(TM)$ to $\mathcal{V}_\Omega(M)$ for M in $\mathcal{FC}_A(\Omega)$.

We have for all M in $\mathcal{C}_A(\Omega)$ and N in $\mathcal{C}_A(\Gamma)$ as in 7.6 adjunction isomorphisms

$$\mathrm{adj}_1 : \mathrm{Hom}_\mathcal{C}(M, T'N) \xrightarrow{\sim} \mathrm{Hom}_\mathcal{C}(TM, N), \tag{1}$$

and

$$\mathrm{adj}_2 : \mathrm{Hom}_\mathcal{C}(N, TM) \xrightarrow{\sim} \mathrm{Hom}_\mathcal{C}(T'N, M). \tag{2}$$

They satisfy the functorial properties listed in 7.6.

10.2. Suppose that Ω and Γ have the following property: For each $\lambda \in \Omega$ there is a unique $\mu \in \Gamma$ that is in the closure of the facet of λ. We denote then μ by λ_Γ. One has in this situation that $TZ_A(\lambda) \simeq Z_A(\lambda_\Gamma)$.

Suppose that we have chosen for each $\lambda \in \Omega$ a fixed isomorphism

$$f_\lambda : Z_A(\lambda_\Gamma) \xrightarrow{\sim} TZ_A(\lambda). \tag{1}$$

We have for all $\lambda \in \Omega$ and all N in $\mathcal{FC}_A(\Omega)$ an isomorphism

$$\begin{aligned}\mathcal{V}(T'N)(\lambda) = \mathrm{Hom}_\mathcal{C}(Z^\emptyset(\lambda), T'N^\emptyset) &\xrightarrow{\sim} \mathrm{Hom}_\mathcal{C}(TZ^\emptyset(\lambda), N^\emptyset)\\ &\xrightarrow{\sim} \mathrm{Hom}_\mathcal{C}(Z^\emptyset(\lambda_\Gamma), N^\emptyset) = \mathcal{V}N(\lambda_\Gamma),\end{aligned}$$

where the first map is adj_1 and the second one is composition with f_λ. Denote this map by

$$\mathcal{A}'_f(N)_\lambda : \mathcal{V}(T'N)(\lambda) \xrightarrow{\sim} \mathcal{V}N(\lambda_\Gamma). \tag{2}$$

Set

$$f'_\lambda = \mathrm{adj}_1^{-1}(f_\lambda^{-1}) : Z_A(\lambda) \to T'Z_A(\lambda_\Gamma) \tag{3}$$

for all $\lambda \in \Omega$. This map is a basis for its Hom space. For each $\mu \in \Gamma$ the f'_λ yield an isomorphism

$$\bigoplus_{\lambda_\Gamma=\mu} f'_\lambda : \bigoplus_{\lambda_\Gamma=\mu} Z^\emptyset(\lambda) \xrightarrow{\sim} T'Z^\emptyset(\mu). \tag{4}$$

(Observe that $T'Z_A(\mu)$ has a Z–filtration where exactly the $Z_A(\lambda)$ with $\lambda_\Gamma = \mu$ occur as factors. This filtration has to split over $A^\emptyset$, so there is some isomorphism as in (4). We can take the f'_λ as the components of the isomorphism, since we can take any bases of the relevant Hom spaces.)

For each $\lambda \in \Omega$ the map $\varphi \mapsto \mathrm{adj}_2(\varphi) \circ f'_\lambda$ is a homomorphism

$$\begin{aligned} \mathcal{A}_{f,\lambda}(M) : \mathcal{V}(TM)(\lambda_\Gamma) = \mathrm{Hom}_\mathcal{C}(Z^\emptyset(\lambda_\Gamma), TM^\emptyset) \xrightarrow{\sim} \mathrm{Hom}_\mathcal{C}(T'Z^\emptyset(\lambda_\Gamma), M^\emptyset) \\ \to \mathrm{Hom}_\mathcal{C}(Z^\emptyset(\lambda), M^\emptyset) = \mathcal{V}M(\lambda). \end{aligned} \tag{5}$$

Now (4) implies that the direct sum of the $\mathcal{A}_{f,\lambda}(M)$ is an isomorphism

$$\mathcal{A}_f(M)_\mu : \mathcal{V}(TM)(\mu) \xrightarrow{\sim} \bigoplus_{\lambda_\Gamma=\mu} \mathcal{V}M(\lambda). \tag{6}$$

We shall denote the maps in (2) resp. in (5), (6) simply by $\mathcal{A}'_f$ and $\mathcal{A}_f$ whenever there is no confusion likely.

10.3. For the next subsections (until 10.9), fix $\beta \in R^+$. We want to determine the effect of $\mathcal{A}'_f$ on A^β–modules of the form $\mathcal{V}(T'N)(\lambda, \beta, e)$ or $\mathcal{V}(T'N)(\lambda, \beta)$ with N in $\mathcal{FC}_A(\Gamma)$.

For any $\lambda \in \Omega$ we are in one of four cases:

(a)	$\lambda = \beta{\uparrow}\lambda,$	$\beta{\uparrow}\lambda_\Gamma = \lambda_\Gamma,$
(b)	$\lambda \neq \beta{\uparrow}\lambda,$	$\beta{\uparrow}\lambda_\Gamma = \lambda_\Gamma = (\beta{\uparrow}\lambda)_\Gamma,$
(c)	$\lambda \neq \beta{\uparrow}\lambda,$	$\beta{\uparrow}\lambda_\Gamma = \lambda_\Gamma \neq (\beta{\uparrow}\lambda)_\Gamma = \lambda_\Gamma + p\beta,$
(d)	$\lambda \neq \beta{\uparrow}\lambda,$	$\lambda_\Gamma \neq \beta{\uparrow}\lambda_\Gamma = (\beta{\uparrow}\lambda)_\Gamma.$

Indeed, if $\beta{\uparrow}\lambda = \lambda$, then necessarily $\beta{\uparrow}\lambda_\Gamma = \lambda_\Gamma$. If $\beta{\uparrow}\lambda \neq \lambda$ and $\beta{\uparrow}\lambda_\Gamma = \lambda_\Gamma$, then either $(\beta{\uparrow}\lambda)_\Gamma = \lambda_\Gamma$ or $(\beta{\uparrow}\lambda)_\Gamma = \lambda_\Gamma + p\beta$. Finally, $\beta{\uparrow}\lambda_\Gamma \neq \lambda_\Gamma$ implies $\beta{\uparrow}\lambda \neq \lambda$ and $(\beta{\uparrow}\lambda)_\Gamma = \beta{\uparrow}(\lambda_\Gamma)$).

In the case (a) the definitions imply obviously:

$$\text{If } \beta{\uparrow}\lambda = \lambda, \text{ then } \mathcal{A}'_f\Big(\mathcal{V}(T'N)(\lambda, \beta)\Big) = \mathcal{V}N(\lambda_\Gamma, \beta). \tag{1}$$

In the other cases we have to take $e \in \mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta\uparrow\lambda))$ and determine the effect of $\mathcal{A}'_f$ on $\mathcal{V}(T'N)(\lambda,\beta,e)$. Here $\mathcal{A}'_f$ denotes the isomorphism

$$\mathcal{A}'_f(N)_\lambda \oplus \mathcal{A}'_f(N)_{\beta\uparrow\lambda} : \mathcal{V}(T'N)(\lambda)\oplus\mathcal{V}(T'N)(\beta\uparrow\lambda) \xrightarrow{\sim} \mathcal{V}N(\lambda_\Gamma)\oplus\mathcal{V}N((\beta\uparrow\lambda)_\Gamma).$$

10.4. Let us now deal with the cases (c) and (d) from 10.3. In the last case our results will involve the map

$$t[f_\lambda, f_{\beta\uparrow\lambda}] : \mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta\uparrow\lambda)) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda_\Gamma), Z^\beta((\beta\uparrow\lambda)_\Gamma)) \tag{1}$$

defined in 7.7(6).

Lemma: *Suppose that* $(\beta\uparrow\lambda)_\Gamma \neq \lambda_\Gamma$. *Then*

$$\mathcal{A}'_f(\mathcal{V}(T'N)(\lambda,\beta,e)) = \begin{cases} \mathcal{V}N(\lambda_\Gamma,\beta,t[f_\lambda,f_{\beta\uparrow\lambda}](e)), & \text{if } \beta\uparrow\lambda_\Gamma \neq \lambda_\Gamma; \\ \mathcal{V}N(\lambda_\Gamma,\beta)\oplus\mathcal{V}N(\lambda_\Gamma+p\beta,\beta), & \text{if } (\beta\uparrow\lambda)_\Gamma = \lambda_\Gamma+p\beta. \end{cases}$$

Proof: Let

$$0 \to Z^\beta(\beta\uparrow\lambda) \xrightarrow{i} Q \xrightarrow{j} Z^\beta(\lambda) \to 0 \tag{2}$$

be a representative of e. Recall that there is a unique morphism $j' : Z^\emptyset(\lambda) \to Q^\emptyset$ with $j\circ j' = \mathrm{id}$ and that $\mathcal{V}(T'N)(\lambda,\beta,e)$ is the image of $\mathrm{Hom}_{\mathcal{C}}(Q, T'N^\beta)$ under the map $\varphi \mapsto (\varphi\circ j', \varphi\circ i)$. Therefore $\mathcal{A}'_f\mathcal{V}(T'N)(\lambda,\beta,e)$ is the image of $\mathrm{Hom}_{\mathcal{C}}(Q, T'N^\beta)$ under the map

$$\begin{aligned} \varphi \quad \mapsto \quad & (\mathrm{adj}_1(\varphi\circ j')\circ f_\lambda, \mathrm{adj}_1(\varphi\circ i)\circ f_{\beta\uparrow\lambda}) \\ =& (\mathrm{adj}_1(\varphi)\circ Tj'\circ f_\lambda, \mathrm{adj}_1(\varphi)\circ Ti\circ f_{\beta\uparrow\lambda}). \end{aligned}$$

Since adj_1 is an isomorphism, this is also the image of $\mathrm{Hom}_{\mathcal{C}}(TQ, N^\beta)$ under the map

$$\psi \mapsto (\psi\circ Tj'\circ f_\lambda, \psi\circ Ti\circ f_{\beta\uparrow\lambda}). \tag{3}$$

On the other hand, if we apply T to (2) and use f_λ and $f_{\beta\uparrow\lambda}$, we get an exact sequence

$$0 \to Z^\beta((\beta\uparrow\lambda)_\Gamma) \xrightarrow{i_1} TQ \xrightarrow{j_1} Z^\beta(\lambda_\Gamma) \to 0 \tag{4}$$

where $i_1 = Ti\circ f_{\beta\uparrow\lambda}$ and $j_1 = f_\lambda^{-1}\circ Tj$. We have

$$j_1\circ(Tj'\circ f_\lambda) = f_\lambda^{-1}\circ Tj\circ Tj'\circ f_\lambda = \mathrm{id}. \tag{5}$$

If $\beta\uparrow\lambda_\Gamma \neq \lambda_\Gamma$, then (4) is a representative of $t[f_\lambda, f_{\beta\uparrow\lambda}](e)$, cf. 7.7(3). Furthermore, (5) implies that $Tj'\circ f_\lambda$ is the analogue to j' for (4). Therefore $\mathcal{V}N(\lambda_\Gamma,\beta,t[f_\lambda,f_{\beta\uparrow\lambda}](e))$ is the image of $\mathrm{Hom}_{\mathcal{C}}(TQ, N^\beta)$ under the same map as in (3). The claim follows in this case.

If $(\beta\uparrow\lambda)_\Gamma = \lambda_\Gamma + p\beta$, then the sequence (4) splits uniquely already over A^β, and (5) implies that the right inverse of j_1 is equal to $Tj'\circ f_\lambda$. Therefore the map

$$\psi \mapsto (\psi\circ Tj'\circ f_\lambda, \psi\circ Ti\circ f_{\beta\uparrow\lambda})$$

is an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(TQ, N^\beta) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda_\Gamma), N^\beta)\oplus\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda_\Gamma+p\beta), N^\beta).$$

This is the same map as in (3), so the claim follows.

10.5. Suppose now that we are in the case (b) from 10.3. Neither $TZ^\beta(\lambda)$ nor $TZ^\beta(\beta\uparrow\lambda)$ change if we replace T by the translation functor over A^β with respect to $W_{\beta,p} = \ <s_{\beta,mp} \mid m \in \mathbf{Z}>$. Therefore we can apply 8.13/14 and get a map

$$\theta[f_\lambda, f_{\beta\uparrow\lambda}] : \operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta\uparrow\lambda)) \longrightarrow A^\beta H_\beta^{-1}/A^\beta. \tag{1}$$

Recall: Given $e \in \operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta\uparrow\lambda))$, we choose a representative

$$0 \to Z^\beta(\beta\uparrow\lambda) \xrightarrow{\ i\ } Q \xrightarrow{\ j\ } Z^\beta(\lambda) \to 0 \tag{2}$$

of e. Denote by j' the unique morphism $j' : Z^\emptyset(\lambda) \to Q^\emptyset$ with $j \circ j' = \mathrm{id}$. There is $i' : TQ \to TZ^\beta(\beta\uparrow\lambda)$ with $i' \circ Ti = \mathrm{id}$, and there is $b \in A^\beta H_\beta^{-1}$ with

$$f_{\beta\uparrow\lambda}^{-1} \circ i' \circ (Tj') \circ f_\lambda = b\,\mathrm{id}. \tag{3}$$

Then

$$\theta[f_\lambda, f_{\beta\uparrow\lambda}](e) = b + A^\beta. \tag{4}$$

Note that $(\beta\uparrow\lambda)_\Gamma = \lambda_\Gamma$ implies that $\mathcal{A}'_f$ maps each $\mathcal{V}(T'N)(\lambda,\beta,e)$ to a submodule of $\mathcal{V}N(\lambda_\Gamma) \oplus \mathcal{V}N(\lambda_\Gamma)$, cf. the last line in 10.3.

Lemma: *Suppose that $(\beta\uparrow\lambda)_\Gamma = \lambda_\Gamma$ and $\beta\uparrow\lambda \neq \lambda$. Let $e \in \operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta\uparrow\lambda))$ and let $b \in A^\beta H_\beta^{-1}$ be a representative of $\theta[f_\lambda, f_{\beta\uparrow\lambda}](e)$. Then*

$$\mathcal{A}'_f\Big(\mathcal{V}(T'N)(\lambda,\beta,e)\Big) = \{(x+by, y) \mid x, y \in \mathcal{V}N(\lambda_\Gamma, \beta)\}.$$

Proof: Here $\mathcal{A}'_f\mathcal{V}(T'N)(\lambda,\beta,e)$ is equal to the image of $\operatorname{Hom}_{\mathcal{C}}(TQ, N^\beta)$ under

$$\psi \mapsto (\psi \circ Tj' \circ f_\lambda, \psi \circ Ti \circ f_{\beta\uparrow\lambda}). \tag{5}$$

(Argue as in 10.4!) We have in this case an isomorphism

$$TQ \xrightarrow{\ \sim\ } Z^\beta(\lambda_\Gamma) \oplus Z^\beta(\lambda_\Gamma)$$

given by

$$v \mapsto (f_{\beta\uparrow\lambda}^{-1} \circ i'(v), f_\lambda^{-1} \circ Tj(v)).$$

It induces

$$\operatorname{Hom}_{\mathcal{C}}(Z^\beta(\lambda_\Gamma), N^\beta) \oplus \operatorname{Hom}_{\mathcal{C}}(Z^\beta(\lambda_\Gamma), N^\beta) \xrightarrow{\ \sim\ } \operatorname{Hom}_{\mathcal{C}}(TQ, N^\beta)$$

given by

$$(\psi_1, \psi_2) \mapsto \psi_1 \circ f_{\beta\uparrow\lambda}^{-1} \circ i' + \psi_2 \circ f_\lambda^{-1} \circ Tj.$$

The composition with (5) takes $(\psi_1, 0)$ to

$$(\psi_1 \circ f_{\beta\uparrow\lambda}^{-1} \circ i' \circ Tj' \circ f_\lambda, \psi_1 \circ f_{\beta\uparrow\lambda}^{-1} \circ i' \circ Ti \circ f_{\beta\uparrow\lambda}) = (\psi_1 b, \psi_1)$$

and $(0, \psi_2)$ to

$$(\psi_2 \circ f_\lambda^{-1} \circ Tj \circ Tj' \circ f_\lambda, \psi_2 \circ f_\lambda^{-1} \circ Tj \circ Ti \circ f_{\beta\uparrow\lambda}) = (\psi_2, 0).$$

The claim follows.

10.6. We now want to describe what $\mathcal{A}_f$ does to A^β-modules of the form $\mathcal{V}(TM)(\mu,\beta,e)$ or $\mathcal{V}(TM)(\mu,\beta)$ with M in $\mathcal{FC}_A(\Omega)$.

Set $W' = W_{\beta,p}$, cf. 10.5. The functors T and T' decompose over A^β into a direct sum of functors of the form $T_{W'\bullet\nu}^{W'\bullet\mu}$ resp. $T_{W'\bullet\mu}^{W'\bullet\nu}$ with $\nu \in \Omega$ and $\mu \in \Gamma$. For example, the restriction of T to $\mathcal{C}_{A^\beta}(W'\bullet\lambda)$ takes values in $\mathcal{C}_{A^\beta}(W'\bullet\lambda_\Gamma)$ and is equal to $T_{W'\bullet\lambda}^{W'\bullet\mu}$ where $\mu = \lambda_\Gamma$. For $\lambda' \in W'\bullet\lambda$ the maps $t[f_\lambda, f_{\beta\uparrow\lambda}]$ as in 10.4(1) can be constructed also using $T_{W'\bullet\lambda}^{W'\bullet\mu}$.

Fix $\mu \in \Gamma$. Let us look at first at the case where $\beta\uparrow\mu \neq \mu$. We have then for all $\lambda \in \Omega$ with $\lambda_\Gamma = \mu$ that $(\beta\uparrow\lambda)_\Gamma = \beta\uparrow\mu$, especially $\beta\uparrow\lambda \neq \lambda$. So all these λ belong to different orbits under W'. Now 7.10 implies that T' on $\mathcal{C}_{A^\beta}(W'\bullet\mu)$ is the direct sum of all $T_{W'\bullet\mu}^{W'\bullet\lambda}$ with $\lambda \in \Omega$, $\lambda_\Gamma = \mu$. On the other hand, $T_{W'\bullet\lambda}^{W'\bullet\mu}$ is just the restriction of T to $\mathcal{C}_{A^\beta}(W'\bullet\lambda)$.

For the moment, fix $\lambda \in \Omega$ with $\lambda_\Gamma = \mu$. Both λ and μ have trivial stabilizers in W', so 7.9 implies that $T_0 = T_{W'\bullet\lambda}^{W'\bullet\mu}$ and $T_0' = T_{W'\bullet\mu}^{W'\bullet\lambda}$ are equivalences of categories. For any $\nu \in W'\bullet\lambda$ the map f_ν' as in 10.2(3) is the inverse image of f_ν^{-1} also under the analogue to adj_1 for T_0 and T_0'. It is an isomorphism $Z^\beta(\nu) \xrightarrow{\sim} T_0' Z^\beta(\nu_\Gamma)$. As in 7.7(6) we can use the f_ν' to construct isomorphisms

$$t[f_\nu', f_{\nu'}'] : \mathrm{Ext}^n_{\mathcal{C}}(Z^\beta(\nu_\Gamma), Z^\beta(\nu'_\Gamma)) \xrightarrow{\sim} \mathrm{Ext}^n_{\mathcal{C}}(Z^\beta(\nu), Z^\beta(\nu'))$$

for all $\nu, \nu' \in W'\bullet\lambda$. The remark above (on the analogue to adj_1 for T_0 and T_0') allows us to apply Lemma 7.9.b. It implies

$$t[f_\nu, f_{\nu'}](t[f_\nu', f_{\nu'}']e) = e \qquad \text{for all } e \in \mathrm{Ext}^n_{\mathcal{C}}(Z^\beta(\nu_\Gamma), Z^\beta(\nu'_\Gamma)) \tag{1}$$

(and all $\nu, \nu' \in W'\bullet\lambda$).

10.7. Lemma: *If $\beta\uparrow\mu \neq \mu$, then one has*

$$\mathcal{A}_f\Big(\mathcal{V}(TM)(\mu,\beta,e)\Big) = \bigoplus_{\lambda_\Gamma=\mu} \mathcal{V}M(\lambda,\beta,t[f_\lambda', f_{\beta\uparrow\lambda}']e)$$

for all $e \in \mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\mu), Z^\beta(\beta\uparrow\mu))$

Proof: We proceed more or less as in 10.4. We choose a representative

$$0 \to Z^\beta(\beta\uparrow\mu) \xrightarrow{i} Q \xrightarrow{j} Z^\beta(\mu) \to 0$$

of e and take the right inverse j' to j over $A^\emptyset$. By definition, $\mathcal{V}(TM)(\mu,\beta,e)$ is the image of $\mathrm{Hom}_{\mathcal{C}}(Q, TM^\beta)$ under $h \mapsto (h\circ j', h\circ i)$. Such a pair is mapped under $\mathcal{A}_f$ to the family with $(\lambda, \beta\uparrow\lambda)$ components

$$(\mathrm{adj}_2(h\circ j')\circ f_\lambda', \mathrm{adj}_2(h\circ i)\circ f_{\beta\uparrow\lambda}') = (\mathrm{adj}_2(h)\circ T'j'\circ f_\lambda', \mathrm{adj}_2(h)\circ T'i\circ f_{\beta\uparrow\lambda}').$$

So our $\mathcal{A}_f$ image is also the image of $\mathrm{Hom}_{\mathcal{C}}(T'Q, M^\beta)$ under the map with $(\lambda, \beta\uparrow\lambda)$ component

$$h \mapsto (h \circ T'j' \circ f'_\lambda, h \circ T'i \circ f'_{\beta\uparrow\lambda}). \tag{1}$$

For the moment, fix λ and use the notation T'_0 as in 10.6. Arguing as in 10.4 one sees that $\mathcal{V}M(\lambda, \beta, t[f'_\lambda, f'_{\beta\uparrow\lambda}]e)$ is the image of the map

$$h \mapsto (h \circ T'_0 j' \circ f'_\lambda, h \circ T'_0 i \circ f'_{\beta\uparrow\lambda}).$$

This is also the restriction of (1) to $\mathrm{Hom}_{\mathcal{C}}(T'_0 Q, M^\beta)$. All $(\lambda', \beta\uparrow\lambda')$ components of (1) with $\lambda' \neq \lambda$ annihilate $\mathrm{Hom}_{\mathcal{C}}(T'_0 Q, M^\beta)$. Since $T'Q$ is the direct sum of all possible $T'_0 Q$, the claim follows.

10.8. Consider now the case where $\beta\uparrow\mu = \mu$. For any $\lambda \in \Omega$ with $\lambda_\Gamma = \mu$ we may have $\beta\uparrow\lambda = \lambda$ or $\beta\uparrow\lambda \neq \lambda$. In the first case λ is the only weight in $W'\bullet\lambda$ with $\lambda_\Gamma = \mu$. In the second case we have also $(\beta\uparrow\lambda)_\Gamma = \mu$ or $(\beta\uparrow\lambda - p\beta)_\Gamma = \mu$. Since $\beta\uparrow(\beta\uparrow\lambda - p\beta) = \lambda$, we can conclude:

The restriction of T' to $\mathcal{C}_{A^\beta}(W'\bullet\mu)$ is the direct sum of all $T_{W'\bullet\mu}^{W'\bullet\lambda}$ with $\lambda \in \Omega$, $\lambda_\Gamma = \mu$, $\beta\uparrow\lambda = \lambda$ and of all $T_{W'\bullet\mu}^{W'\bullet\lambda}$ with $\lambda \in \Omega$, $\lambda_\Gamma = \mu = (\beta\uparrow\lambda)_\Gamma$, $\beta\uparrow\lambda \neq \lambda$.

For λ of the second type in the last statement, the functor $T_{W'\bullet\mu}^{W'\bullet\lambda}$ leads to an exact sequence

$$0 \to Z^\beta(\beta\uparrow\lambda) \xrightarrow{i} Q \xrightarrow{j} Z^\beta(\lambda) \to 0. \tag{1}$$

with $Q = T_{W'\bullet\mu}^{W'\bullet\lambda} Z^\beta(\mu)$. This is just a special case of 8.9(2) with $\lambda' = \beta\uparrow\lambda$. We can choose $i = f'_{\beta\uparrow\lambda}$ and j with $j \circ f'_\lambda = H_\beta \mathrm{id}$, cf. 8.12(2), (3) and the remark in 8.10. Then the class of (1) in the appropriate Ext group is equal to $\theta[f_\lambda, f_{\beta\uparrow\lambda}]^{-1}(H_\beta^{-1} + A^\beta)$, cf. the remark in 8.14. We have by Lemma 8.12 a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & TZ^\beta(\beta\uparrow\lambda) & \xrightarrow{Ti} & TQ & \xrightarrow{Tj} & TZ^\beta(\lambda) & \to 0 \\
& \Big\downarrow f_{\beta\uparrow\lambda}^{-1} & & \Big\downarrow h & & \Big\downarrow f_\lambda^{-1} & \\
0 \to & Z^\beta(\mu) & \longrightarrow & Z^\beta(\mu) \oplus Z^\beta(\mu) & \longrightarrow & Z^\beta(\mu) & \to 0
\end{array}$$

where the maps in the lower row are $x \mapsto (x, 0)$ and $(x, y) \mapsto y$.

10.9. Lemma: *If $\beta\uparrow\mu = \mu$, then*

$$\mathcal{A}_f(\mathcal{V}(TM)(\mu,\beta)) = \bigoplus_{\substack{\beta\uparrow\lambda=\lambda, \\ \lambda_\Gamma=\mu}} \mathcal{V}M(\lambda,\beta) \oplus \bigoplus_{\substack{\beta\uparrow\lambda\neq\lambda, \\ \lambda_\Gamma=\mu=(\beta\uparrow\lambda)_\Gamma}} (H_\beta, 1)\mathcal{V}M(\lambda, \beta, \eta_f^\beta(\lambda))$$

where $\eta_f^\beta(\lambda) = \theta[f_\lambda, f_{\beta\uparrow\lambda}]^{-1}(H_\beta^{-1} + A^\beta)$.

Proof: $\mathcal{A}_f\mathcal{V}(TM)(\mu,\beta)$ is the image of $\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\mu), TM^\beta)$ under

$$\psi \mapsto (\mathrm{adj}_2(\psi) \circ f'_\nu \mid \nu \in \Omega, \nu_\Gamma = \mu),$$

hence of $\mathrm{Hom}_{\mathcal{C}}(T'Z^\beta(\mu), M^\beta)$ under

$$\psi \mapsto (\psi \circ f'_\nu \mid \nu \in \Omega, \nu_\Gamma = \mu).$$

For any λ as above each ν–component of this map with $\nu \notin W'{\bullet}\lambda$ annihilates $\mathrm{Hom}_{\mathcal{C}}(T_{W'\bullet\mu}^{W'\bullet\lambda} Z^\beta(\mu), M^\beta)$. So $\mathcal{A}_f\mathcal{V}(TM)(\mu,\beta)$ is the direct sum over all λ as above of the image of $\mathrm{Hom}_{\mathcal{C}}(T_{W'\bullet\mu}^{W'\bullet\lambda} Z^\beta(\mu), M^\beta)$ under

$$\psi \mapsto (\psi \circ f'_\nu \mid \nu \in W'{\bullet}\lambda, \nu_\Gamma = \mu).$$

For λ with $\beta\uparrow\lambda = \lambda$ only $\nu = \lambda$ occurs. In this case f'_λ is (by 7.10) an isomorphism between $Z^\beta(\lambda)$ and $T_{W'\bullet\mu}^{W'\bullet\lambda} Z^\beta(\mu)$. So the corresponding summand is $\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\lambda), M^\beta) = \mathcal{V}M(\lambda,\beta)$.

For λ with $\beta\uparrow\lambda \neq \lambda$ we use 10.8(1). The right inverse to j over $A^\emptyset$ is $H_\beta^{-1} f'_\lambda$, so $\mathcal{V}M(\lambda,\beta,\eta_f^\beta(\lambda))$ is the image of $\mathrm{Hom}_{\mathcal{C}}(Q, M^\beta)$ under

$$\psi \mapsto (\psi \circ (H_\beta^{-1} f'_\lambda), \psi \circ f'_{\beta\uparrow\lambda}).$$

If we apply $(H_\beta, 1)$ we get the corresponding summand of $\mathcal{A}_f\mathcal{V}(TM)(\mu,\beta)$. The claim follows.

10.10. Suppose that we have chosen for each $\beta \in R^+$ and each $\lambda \in \Omega$ with $\beta\uparrow\lambda \neq \lambda$ (resp. each $\mu \in \Gamma$ with $\beta\uparrow\mu \neq \mu$) a generator $e^\beta(\lambda)$ (resp. $e^\beta(\mu)$) of $\mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta\uparrow\lambda))$ (resp. of $\mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\mu), Z^\beta(\beta\uparrow\mu))$). Consider the functors $\mathcal{V}_\Omega$ and $\mathcal{V}_\Gamma$ as in 9.4 with values in $\mathcal{K}(\Omega)$ resp. in $\mathcal{K}(\Gamma)$ depending on these choices.

There are for each $\beta \in R^+$ and each $\lambda \in \Omega$ with $\beta\uparrow\lambda_\Gamma \neq \lambda_\Gamma$ (hence $\beta\uparrow\lambda \neq \lambda$) elements a_λ^β and b_λ^β in A^β such that

$$\begin{aligned} t[f_\lambda, f_{\beta\uparrow\lambda}]e^\beta(\lambda) &= a_\lambda^\beta e^\beta(\lambda_\Gamma), \\ t[f'_\lambda, f'_{\beta\uparrow\lambda}]e^\beta(\lambda_\Gamma) &= b_\lambda^\beta e^\beta(\lambda). \end{aligned} \tag{1}$$

The classes of a_λ^β and b_λ^β modulo H_β are uniquely determined by (1) and we have

$$a_\lambda^\beta b_\lambda^\beta \equiv 1 \pmod{H_\beta} \tag{2}$$

by 10.6(1).

There are for each $\beta \in R^+$ and each $\lambda \in \Omega$ with $\beta \uparrow \lambda \neq \lambda$ and $\beta \uparrow \lambda_\Gamma = \lambda_\Gamma = (\beta \uparrow \lambda)_\Gamma$ elements $a_\lambda^\beta \in A^\beta H_\beta^{-1}$ and $b_\lambda^\beta \in A^\beta H_\beta$ such that

$$\begin{aligned} \theta[f_\lambda, f_{\beta\uparrow\lambda}] e^\beta(\lambda) &= a_\lambda^\beta + A^\beta, \\ \theta[f_\lambda, f_{\beta\uparrow\lambda}]^{-1}(a + A^\beta) &= (b_\lambda^\beta a) e^\beta(\lambda) \end{aligned} \tag{3}$$

for all $a \in A^\beta H_\beta^{-1}/A^\beta$, where we regard $b_\lambda^\beta a$ as an element of $A^\beta / A^\beta H_\beta$. Now the classes of a_λ^β modulo A^β and of b_λ^β modulo $A^\beta H_\beta^2$ are uniquely determined by (2); we have obviously

$$a_\lambda^\beta b_\lambda^\beta \in 1 + A^\beta H_\beta.$$

We now define functors $\mathcal{T} : \mathcal{K}(\Omega) \to \mathcal{K}(\Gamma)$ and $\mathcal{T}' : \mathcal{K}(\Gamma) \to \mathcal{K}(\Omega)$. For all $\mathcal{M}$ in $\mathcal{K}(\Omega)$ set

$$\mathcal{T}\mathcal{M}(\mu) = \bigoplus_{\lambda_\Gamma = \mu} \mathcal{M}(\lambda) \tag{4}$$

for all $\mu \in \Gamma$, and (for all $\beta \in R^+$)

$$\mathcal{T}\mathcal{M}(\mu, \beta) = \bigoplus_{\substack{\beta\uparrow\lambda = \lambda, \\ \lambda_\Gamma = \mu}} \mathcal{M}(\lambda, \beta) \oplus \bigoplus_{\substack{\beta\uparrow\lambda \neq \lambda, \\ \lambda_\Gamma = \mu = (\beta\uparrow\lambda)_\Gamma}} \Big((b_\lambda^\beta, 1)\mathcal{M}(\lambda, \beta) + H_\beta \mathcal{M}(\lambda)_\beta \Big), \tag{5}$$

if $\beta \uparrow \mu = \mu$, resp.

$$\mathcal{T}\mathcal{M}(\mu, \beta) = \bigoplus_{\lambda_\Gamma = \mu} \Big((b_\lambda^\beta, 1)\mathcal{M}(\lambda, \beta) + \mathcal{M}(\lambda)_\beta \Big), \tag{6}$$

if $\beta \uparrow \mu \neq \mu$. The direct sums in (4)–(6) are finite. Our finiteness assumption in the definition of $\mathcal{K}(\Omega)$ says that each $\mathcal{M}(\lambda)$ is finitely generated over $A^\emptyset$, therefore so is each $\mathcal{T}\mathcal{M}(\mu)$. Similarly, all $\mathcal{M}(\lambda, \beta)$ and (cf. 9.9) all $\mathcal{M}(\lambda)_\beta$ are finitely generated over A^β. So are then their homomorphic images of the form $(b_\lambda^\beta, 1)\mathcal{M}(\lambda, \beta)$ and $H_\beta \mathcal{M}(\lambda)_\beta$, hence also each $\mathcal{T}\mathcal{M}(\mu, \beta)$. So $\mathcal{T}\mathcal{M}$ is indeed in $\mathcal{K}(\Omega)$.

For any $\mathcal{N}$ in $\mathcal{K}(\Gamma)$ we define $\mathcal{T}'\mathcal{N}$ by

$$\mathcal{T}'\mathcal{N}(\lambda) = \mathcal{N}(\lambda_\Gamma) \qquad \text{for all } \lambda \in \Omega, \tag{7}$$

and (for all $\beta \in R^+$ and $\lambda \in \Omega$):

$$\mathcal{T}'\mathcal{N}(\lambda, \beta) = \begin{cases} \mathcal{N}(\lambda_\Gamma, \beta), & \text{if } \beta \uparrow \lambda = \lambda; \\ (a_\lambda^\beta, 1)\mathcal{N}(\lambda_\Gamma, \beta) + \mathcal{N}(\lambda_\Gamma)_\beta, & \text{if } \beta \uparrow \lambda_\Gamma \neq \lambda_\Gamma; \\ \mathcal{N}(\lambda_\Gamma, \beta) \oplus \mathcal{N}(\lambda_\Gamma + p\beta, \beta), & \text{if } (\beta \uparrow \lambda)_\Gamma = \lambda_\Gamma + p\beta; \\ \{(x + a_\lambda^\beta y, y) \mid x, y \in \mathcal{N}(\lambda_\Gamma, \beta)\}, & \\ & \text{if } (\beta \uparrow \lambda)_\Gamma = \lambda_\Gamma, \beta \uparrow \lambda \neq \lambda. \end{cases} \tag{8}$$

In the last case we regard the right hand side as contained in

$$\mathcal{T}'\mathcal{N}(\lambda) \oplus \mathcal{T}'\mathcal{N}(\beta\uparrow\lambda) = \mathcal{N}(\lambda_\Gamma) \oplus \mathcal{N}(\lambda_\Gamma).$$

In order to show that $\mathcal{T}'\mathcal{N}$ is in $\mathcal{K}(\Gamma)$, we have to check some finiteness conditions; this is done using the same arguments as above for $\mathcal{T}\mathcal{M}$.

It is clear how to define $\mathcal{T}$ and $\mathcal{T}'$ on morphisms.

Remark: Our functors depend on the choice of the a_λ^β and b_λ^β in their classes modulo H_β (resp. modulo A^β and $A^\beta H_\beta^2$ in the situation of (3)). However, consider the full subcategory of all $\mathcal{M}$ in $\mathcal{K}(\Omega)$ with

$$(H_\beta, 0)\mathcal{M}(\lambda,\beta) \subset \mathcal{M}(\lambda)_\beta \tag{9}$$

for all λ and β with $\beta\uparrow\lambda \neq \lambda$. By 9.9(2) this subcategory contains the image of $\mathcal{V}_\Omega$. It is easy to check that the restriction of $\mathcal{T}$ to this subcategory is independent of the choices. A similar result holds for $\mathcal{T}'$.

10.11. Proposition: *There are natural isomorphisms*

$$\mathcal{A} : \mathcal{V}_\Gamma \circ T \xrightarrow{\sim} \mathcal{T} \circ \mathcal{V}_\Omega \qquad \textit{and} \qquad \mathcal{A}' : \mathcal{V}_\Omega \circ T' \xrightarrow{\sim} \mathcal{T}' \circ \mathcal{V}_\Gamma.$$

Proof: For any M in $\mathcal{FC}_A(\Omega)$ we define $\mathcal{A}(M)$ at first on each $\mathcal{V}_\Gamma(TM)(\mu)$ with $\mu \in \Gamma$ as the map

$$\mathcal{A}_f(TM)_\mu : \mathcal{V}_\Gamma(TM)(\mu) \xrightarrow{\sim} \bigoplus_{\lambda_\Gamma=\mu} \mathcal{V}M(\lambda) = \mathcal{T}\mathcal{V}_\Omega M(\lambda)$$

from 10.2(6). These maps induce isomorphisms at all possible (μ,β) by Lemma 10.7 or Lemma 10.9 together with Lemma 9.10.

For any N in $\mathcal{FC}_A(\Gamma)$ we define $\mathcal{A}'(N)$ at first on each $\mathcal{V}_\Omega(T'N)(\lambda)$ with $\lambda \in \Omega$ as the map

$$\mathcal{A}'_f(N)_\lambda : \mathcal{V}_\Omega(T'N)(\lambda) \xrightarrow{\sim} \mathcal{T}'\mathcal{V}_\Gamma N(\lambda) = \mathcal{V}_\Gamma N(\lambda_\Gamma)$$

from 10.2(2). It induces isomorphisms at all possible (λ,β) by 10.3(1), Lemma 10.4 and Lemma 10.5 together with Lemma 9.10.

Remark: Suppose that the a_λ^β in 10.10(1) are units in A^β. We can then choose $b_\lambda^\beta = (a_\lambda^\beta)^{-1}$. This allows us (by 9.10(4)) to replace 10.10(6) by

$$\mathcal{T}\mathcal{M}(\mu,\beta) = \bigoplus_{\lambda_\Gamma=\mu} ((a_\lambda^\beta)^{-1}, 1)\mathcal{M}(\lambda,\beta) \tag{1}$$

and the second case in 10.10(8) by

$$\mathcal{T}'\mathcal{N}(\lambda,\beta) = (a_\lambda^\beta, 1)\mathcal{N}(\lambda_\Gamma, \beta). \tag{2}$$

Suppose also that each a_λ^β in 10.10(3) is the product of H_β^{-1} with a unit in A^β. Again, we can choose $b_\lambda^\beta = (a_\lambda^\beta)^{-1}$. This allows us to replace 10.10(5) by

$$\mathcal{T}\mathcal{M}(\mu,\beta) = \bigoplus_{\substack{\beta\uparrow\lambda=\lambda,\\ \lambda_\Gamma=\mu}} \mathcal{M}(\lambda,\beta) \oplus \bigoplus_{\substack{\beta\uparrow\lambda\neq\lambda,\\ \lambda_\Gamma=\mu=(\beta\uparrow\lambda)_\Gamma}} ((a_\lambda^\beta)^{-1}, 1)\mathcal{M}(\lambda,\beta), \tag{3}$$

since

$$H_\beta\mathcal{M}(\lambda)_\beta = b_\lambda^\beta\mathcal{M}(\lambda)_\beta \subset (b_\lambda^\beta, 1)\mathcal{M}(\lambda,\beta).$$

We shall see later on that we can choose the a_λ^β as above if Ω is a regular orbit.

10.12. Suppose that M, M' are in $\mathcal{FC}_A(\Omega)$ and that $\varphi : M \to M'$ is a homomorphism. We get a translated homomorphism $T\varphi : TM \to TM'$. Now φ induces

$$\mathcal{V}\varphi(\lambda) : \mathcal{V}M(\lambda) \longrightarrow \mathcal{V}M'(\lambda) \qquad \text{for all } \lambda \in \Omega$$

taking any homomorphism $h : Z^\emptyset(\lambda) \to M^\emptyset$ to $\varphi \circ h$, and $T\varphi$ induces

$$\mathcal{V}(T\varphi)(\mu) : \mathcal{V}(TM)(\mu) \longrightarrow \mathcal{V}(TM')(\mu) \qquad \text{for all } \mu \in \Gamma$$

taking any homomorphism $h' : Z^\emptyset(\mu) \to TM^\emptyset$ to $T\varphi \circ h'$. If we use $\mathcal{A}_f$ as in 10.2(6) to identify

$$\mathcal{V}TM(\mu) \xrightarrow{\sim} \bigoplus_{\lambda_\Gamma=\mu} \mathcal{V}M(\lambda)$$

and similarly for M', then $\mathcal{V}(T\varphi)(\mu)$ is just the direct sum of all $\mathcal{V}\varphi(\lambda)$ with $\lambda_\Gamma = \mu$. This is an easy consequence of the functoriality properties of adj_2 involved in the definition of $\mathcal{A}_f$.

We can also consider N, N' in $\mathcal{FC}_A(\Gamma)$, a homomorphism $\psi : N \to N'$ and the translated homomorphism $T'\psi : T'N \to T'N'$. The map induced by $T'\psi$ on $\mathcal{V}(T'N)(\lambda)$ is the same map as the one induced by ψ on $\mathcal{V}N(\lambda_\Gamma)$ if we use the identifications $\mathcal{A}'_f(N)_\lambda$ and $\mathcal{A}'_f(N')_\lambda$ from 10.2(2). This follows from the functorial properties of adj_1.

10.13. Consider M in $\mathcal{FC}_A(\Omega)$ and N in $\mathcal{FC}_A(\Gamma)$. Look at a homomorphism $\varphi : N \to TM$ and its image $\mathrm{adj}_2(\varphi) : T'N \to M$ under adjunction. Now φ induces

$$\mathcal{V}\varphi(\lambda_\Gamma) : \mathcal{V}N(\lambda_\Gamma) \longrightarrow \mathcal{V}(TM)(\lambda_\Gamma)$$

taking any homomorphism $h' : Z^{\emptyset}(\lambda_\Gamma) \to N^{\emptyset}$ to $\varphi \circ h'$, and $\mathrm{adj}_2(\varphi)$ induces

$$\mathcal{V}\mathrm{adj}_2(\varphi)(\lambda) : \mathcal{V}(T'N)(\lambda) \longrightarrow \mathcal{V}M(\lambda)$$

taking any homomorphism $h : Z^{\emptyset}(\lambda) \to T'N^{\emptyset}$ to $\mathrm{adj}_2(\varphi) \circ h$. If we use $\mathcal{A}'_f$ to identify $\mathcal{V}(T'N)(\lambda)$ with $\mathcal{V}N(\lambda_\Gamma)$ and if we use $\mathcal{A}_f$ to identify $\mathcal{V}M(\lambda)$ with a direct summand of $\mathcal{V}(TM)(\lambda_\Gamma)$, then $\mathcal{V}\mathrm{adj}_2(\varphi)(\lambda)$ is identified with $\mathcal{V}\varphi(\lambda_\Gamma)$. Indeed, any h as above is mapped under $\mathcal{A}'_f$ to $\mathrm{adj}_1(h) \circ f_\lambda$, then under $\mathcal{V}\varphi(\lambda_\Gamma)$ to $\varphi \circ \mathrm{adj}_1(h) \circ f_\lambda$, finally under $\mathcal{A}_f$ to

$$\begin{aligned}\mathrm{adj}_2(\varphi \circ \mathrm{adj}_1(h) \circ f_\lambda) \circ f'_\lambda &= \mathrm{adj}_2(\varphi) \circ T'(\mathrm{adj}_1(h) \circ f_\lambda) \circ \mathrm{adj}_1^{-1}(f_\lambda^{-1})\\ &= \mathrm{adj}_2(\varphi) \circ \mathrm{adj}_1^{-1}(\mathrm{adj}_1(h) \circ f_\lambda \circ f_\lambda^{-1})\\ &= \mathrm{adj}_2(\varphi) \circ h.\end{aligned}$$

On the other hand consider a homomorphism $\psi : TM \to N$ and the corresponding $\psi' = \mathrm{adj}_1^{-1}(\psi) : M \to T'N$. It is not so straightforward to describe $\mathcal{V}\psi'(\lambda)$ in terms of $\mathcal{V}\psi(\lambda_\Gamma)$. In fact, it depends on the choice of the isomorphism $E' \xrightarrow{\sim} E^*$ in 7.6.

10.14. Consider as an example $\Omega = W_p{\bullet}\lambda$ with λ in the interior of the first dominant alcove, i.e., with

$$0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \qquad \text{for all } \alpha \in R^+,$$

and $\Gamma = W_p{\bullet}(-\rho)$. Set $\mathcal{Z} = \mathcal{V}_\Gamma Z_A(-\rho)$; we have obviously (cf. 9.5)

$$\mathcal{Z}(-\rho) = A^{\emptyset} \qquad \text{and} \qquad \mathcal{Z}(-\rho, \beta) = A^{\beta} \quad \text{for all } \beta \in R^+, \tag{1}$$

where the inclusions $\mathcal{Z}(-\rho,\beta) \subset \mathcal{Z}(-\rho)$ are the natural inclusions $A^\beta \subset A^{\emptyset}$. If $\mu \in W_p{\bullet}(-\rho)$ with $\mu \neq -\rho$, then

$$\mathcal{Z}(\mu) = 0 \qquad \text{and} \qquad \mathcal{Z}(\mu, \beta) = 0 \quad \text{for all } \beta \in R^+. \tag{2}$$

Set $Q = T'Z_A(-\rho)$; let us describe $\mathcal{Q} = \mathcal{T}'\mathcal{Z} \simeq \mathcal{V}_\Omega Q$ using the notation $a^\beta_{w\bullet\lambda}$ as in 10.10(3). We have by 10.10(7) for all $w \in W$ and $\nu \in \mathbf{Z}R$

$$\mathcal{Q}(w{\bullet}\lambda + p\nu) = \begin{cases} A^{\emptyset}, & \text{if } \nu = 0;\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

We have for all $\beta \in R^+$ and $w \in W$

$$\mathcal{Q}(w{\bullet}\lambda, \beta) = \begin{cases} A^\beta(1,0), & \text{if } w^{-1}\beta > 0;\\ A^\beta(1,0) \oplus A^\beta(a^\beta_{w\bullet\lambda}, 1), & \text{if } w^{-1}\beta < 0.\end{cases} \tag{4}$$

Furthermore, if $w^{-1}\beta > 0$, then

$$\mathcal{Q}(w{\bullet}\lambda - p\beta, \beta) = A^\beta(0,1). \tag{5}$$

All other $\mathcal{Q}(\lambda', \beta)$ are equal to 0. We shall use this description of $\mathcal{Q}$ to compute explicitly $\mathrm{End}_{\mathcal{C}}(\mathcal{Q})$ in 19.4.

Introduction to the Sections 11-13

Our goal is to actually choose isomorphisms f_λ and extensions $e^\beta(\lambda)$ and compute the constants a^β_λ needed for our combinatorics in the preceding section. The result of our computations is a "Theorem of good choices" that we want to explain now.

For $\lambda, \mu \in X$ let us abbreviate by $\lambda \to \mu$ the statement "μ is in the closure of the facet of λ". If A is a B-algebra and λ, μ are two elements of X, we let $T = T^\Gamma_\Omega$ denote the translation functor from $\Omega = W_p{\bullet}\lambda$ to $\Gamma = W_p{\bullet}\mu$. If $\lambda \to \mu$, then there is an isomorphism $Z_A(\mu) \xrightarrow{\sim} TZ_A(\lambda)$. If now $\beta \in R^+$ is a positive root and $(\beta \uparrow \lambda) \to (\beta \uparrow \mu)$, then we also get $Z_A(\beta \uparrow \mu) \xrightarrow{\sim} TZ_A(\beta \uparrow \lambda)$. Translation of extensions gives rise to a map

$$\operatorname{Ext}^1_{\mathcal{C}}(Z_A(\lambda), Z_A(\beta \uparrow \lambda)) \longrightarrow \operatorname{Ext}^1_{\mathcal{C}}(Z_A(\mu), Z_A(\beta \uparrow \mu)),$$

that depends on the choice of isomorphisms $f^\mu_\lambda : Z_A(\mu) \to TZ_A(\lambda)$ and $f^{\beta\uparrow\mu}_{\beta\uparrow\lambda} : Z_A(\beta \uparrow \mu) \to TZ_A(\beta \uparrow \lambda)$.

Suppose now $A, A^\beta, A^\emptyset$ are as at the beginning of Section 9 and use also the notation $Z^\beta(\lambda)$ introduced there. Translation of extensions gives also rise to a map

$$\operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta \uparrow \lambda)) \longrightarrow \operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\mu), Z^\beta(\beta \uparrow \mu)),$$

that again depends on the choice of f^μ_λ and of $f^{\beta\uparrow\mu}_{\beta\uparrow\lambda}$. In case $\mu \neq \beta \uparrow \mu$ this is just our $t[f^\mu_\lambda, f^{\beta\uparrow\mu}_{\beta\uparrow\lambda}]$ from the preceding section. In case $\mu = \beta \uparrow \mu$ this map is zero, since there are no self-extensions of $Z^\beta(\mu)$. But if $\mu = \beta \uparrow \mu$ and $\lambda \neq \beta \uparrow \lambda$ and $(\beta \uparrow \lambda) \to \mu$ we constructed in a more involved way (see 10.5) a map

$$\theta[f^\mu_\lambda, f^{\beta\uparrow\mu}_{\beta\uparrow\lambda}] : \operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta \uparrow \lambda)) \longrightarrow A^\beta H_\beta^{-1}/A^\beta$$

depending on the choice of $f^\mu_\lambda, f^{\beta\uparrow\mu}_{\beta\uparrow\lambda}$.

Suppose (for the sake of simplicity) that we take as A the completion of B with respect to the maximal ideal generated by all H_α resp. by all $K_\alpha - 1$.

(We shall work in greater generality in the sections 11–13, but this is really the situation that we shall use afterwards to get our main results.) Set $h_\alpha = d_\alpha H_\alpha$ in Case 1 and $h_\alpha = \log K_\alpha$ in Case 2. Assume in Case 1 that p does not divide any d_β. Each h_β differs from H_β by a unit in A^β (even in A).

To state our theorem, we further need some alcove geometry. Let $\mathcal{H}$ be the set of reflection hyperplanes for the dot-action of W_p on $X \otimes_{\mathbf{Z}} \mathbf{R}$. Each $H \in \mathcal{H}$ has an equation of the form $\langle \nu + \rho, \gamma^\vee \rangle = rp$ with $r \in \mathbf{Z}$ and $\gamma \in R^+$. Then we set $\gamma = \alpha(H)$; furthermore, for any $\nu \in X$ we shall write $\nu < H$ (resp. $\nu > H$) if and only if $\langle \nu + \rho, \gamma^\vee \rangle < rp$ (resp. $\langle \nu + \rho, \gamma^\vee \rangle > rp$). For all $\beta \in R^+$ let $\mathcal{H}(\beta)$ be the set of all $H \in \mathcal{H}$ with $s_\beta \alpha(H) < 0$. For $\beta \in R^+$ and $\lambda, \mu \in X$ with $\lambda \to \mu$ set

$$C^\beta(\lambda, \mu) = \prod_{\substack{H \in \mathcal{H}(\beta), \\ \mu \in H, \lambda > H}} h_{-\alpha(H)} \prod_{\substack{H \in \mathcal{H}(\beta), \\ \mu \in H, \lambda < H}} h^{-1}_{\alpha(H)};$$

this is an element of the fraction field of A. We have $\beta \uparrow \mu \neq \mu$ if and only if $\alpha(H) \neq \beta$ for all H with $\mu \in H$; in that case $C^\beta(\lambda, \mu)$ is a unit in A^β. If $\beta \uparrow \mu = \mu$, then there is exactly one $H \in \mathcal{H}(\beta)$ with $\mu \in H$; if now $\beta \uparrow \lambda \neq \lambda$, then either $\lambda < H$ (and $h_\beta C^\beta(\lambda, \mu)$ is a unit in A^β) or $\lambda > H$ (and $h_\beta^{-1} C^\beta(\lambda, \mu)$ is a unit in A^β). Note that in this case $\lambda < H$ is equivalent to $(\beta \uparrow \lambda) \to \mu$.

Now we can state our "Theorem of good choices".

Theorem: *Suppose that p is at least the Coxeter number and that in* Case 2 *all prime divisors of p are good for our root system. Then it is possible to choose simultaneously generators (over A^β)*

$$e^\beta(\mu) \in \operatorname{Ext}^1_{\mathcal{C}}(Z^\beta(\mu), Z^\beta(\beta \uparrow \mu)) \tag{1}$$

for all $\mu \in X$ and $\beta \in R^+$ with $\beta \uparrow \mu \neq \mu$ and isomorphisms

$$f^\mu_\lambda : Z_A(\mu) \to T Z_A(\lambda) \tag{2}$$

for all $\lambda, \mu \in X$ with λ regular and $\lambda \to \mu$ such that for all $\lambda, \mu \in X$ and $\beta \in R^+$ with λ regular and $\lambda \to \mu$ we have

$$t[f^\mu_\lambda, f^{\beta\uparrow\mu}_{\beta\uparrow\lambda}] e^\beta(\lambda) = C^\beta(\lambda, \mu)\, e^\beta(\mu), \qquad \text{if } \beta \uparrow \mu \neq \mu;$$

and

$$\theta[f^\mu_\lambda, f^{\beta\uparrow\mu}_{\beta\uparrow\lambda}] e^\beta(\lambda) = C^\beta(\lambda, \mu) + A^\beta, \qquad \text{if } \beta \uparrow \mu = \mu \text{ and } (\beta \uparrow \lambda) \to \mu.$$

Remarks: 1) We can identify $Z_A(\lambda)$ and $Z_A(\lambda + p\nu)$ for all $\lambda, \nu \in X$; this induces identifications of corresponding Ext groups. In our theorem we can make the choices in (1) and (2) such that each $e^\beta(\lambda + p\nu)$ is identified with $e^\beta(\lambda)$, and each f_λ^μ with $f_{\lambda+p\nu}^{\mu+p\nu}$.

2) Note that the $C^\beta(\lambda, \mu)$ are in some sense "independent of the characteristic". This is precisely what allows us to prove our "independence of characteristic" results from the introduction, since the a_λ^β from 10.10 appearing in our combinatorics can now be taken to be just the $C^\beta(\lambda, \lambda_\Gamma)$.

3) We hope that one might extend the theorem to non-regular λ and to small p. However already our proof of the above theorem involves so dreadful computations, that we were discouraged from trying to prove the generalization.

Our proof of the above theorem proceeds in two steps. In Section 12 we show how to choose extensions $e_0^\beta(\lambda), e_0^\beta(\mu)$ and isomorphisms f_λ^μ and compute for these choices the constants $a \in A^\beta/H_\beta A^\beta$ such that $t[f_\lambda^\mu, f_{\beta\uparrow\lambda}^{\beta\uparrow\mu}]e_0^\beta(\lambda) = ae_0^\beta(\mu)$ resp. $\theta[f_\lambda^\mu, f_{\beta\uparrow\lambda}^{\beta\uparrow\mu}]e_0^\beta(\lambda) = a(h_\beta^{-1} + A^\beta)$. (Observe that $A^\beta H_\beta^{-1}/A^\beta$ is a free module over $A^\beta/H_\beta A^\beta$ of rank 1 with basis $h_\beta^{-1} + A^\beta$.) In Section 13 we modify our choices by suitable units to get the nice constants of the theorem above. Section 11 contains preliminary calculations.

Let us discuss the contents of the Sections 11 and 12 in more detail. Let us fix a regular orbit Ω, an arbitrary orbit Γ and let L be the simple finite dimensional module used in the construction of $T = T_\Omega^\Gamma$ as in 10.2. For $\lambda \in \Omega$ and $\mu = \lambda_\Gamma$ we may choose $f_\lambda^\mu = f_\lambda : Z_A(\mu) \to TZ_A(\lambda)$ such that it takes the standard generator v_μ to $\mathrm{pr}(e \otimes v_\lambda)$ where v_λ is the standard generator of $Z_A(\lambda)$ and e is a nonzero vector in $L_{\mu-\lambda}$. Similarily, we can choose $f_{\beta\uparrow\lambda}^{\beta\uparrow\mu} = f_{\beta\uparrow\lambda}$ so that it takes $v_{\beta\uparrow\mu}$ to $\mathrm{pr}(e' \otimes v_{\beta\uparrow\lambda})$ where $e' \in L_{\beta\uparrow\mu-\beta\uparrow\lambda} = L_{s_\beta(\mu-\lambda)}$. Now the corresponding $t[f_\lambda, f_{\beta\uparrow\lambda}]$ resp. $\theta[f_\lambda, f_{\beta\uparrow\lambda}]$ depends on e and e'; denote it for the moment by $t^\beta[\mu, \lambda, e, e']$. Both $\mu - \lambda$ and $s_\beta(\mu - \lambda)$ are extremal weights of L; so there is an integer $r \geq 0$ and an element $a \in k$, $a \neq 0$ with $e' = aE_{-\beta}^{(r)}e$ or $e' = (-1)^r aE_\beta^{(r)}e$. (The term $(-1)^r$ will simplify some formulas.) Now $t^\beta[\mu, \lambda, e, e']$ will depend only on a; denote it by $t^\beta[\mu, \lambda, a]$. (Actually, it depends also on the choice of the root vectors $E_{\pm\beta}$.) Later on we shall choose a as follows: If e is part of a canonical basis of L, then so is e' (up to sign). For this choice we shall write $t_0^\beta[\mu, \lambda]$ instead of $t^\beta[\mu, \lambda, a]$.

Suppose that we have a third W_p-orbit Γ' such that there is for all $\omega \in \Gamma$ a unique $\omega_{\Gamma'} \in \Gamma'$ in the closure of the facet of ω. We have then translation functors also from Γ to Γ' and from Ω to Γ'. We can consider $t^\beta[\nu, \mu, a']$ and $t^\beta[\nu, \lambda, a'']$ where $\nu = \mu_{\Gamma'}$ for all a', a'' and then compare $t^\beta[\nu, \mu, a'] \circ t^\beta[\mu, \lambda, a]$ to $t^\beta[\nu, \lambda, a'']$. This is done in 12.11 and then specialized to a comparison of $t_0^\beta[\nu, \mu] \circ t_0^\beta[\mu, \lambda]$ with $t_0^\beta[\nu, \lambda]$ in 12.12. Using Γ' with $\nu + \rho \in pX$ — so that $t_0^\beta[\nu, \lambda]$ takes values in $A^\beta H_\beta^{-1}/A^\beta$ — we can make our preliminary choice

$e_0^\beta(\lambda)$ for $e^\beta(\lambda)$ and compute $t_0^\beta[\mu,\lambda]e_0^\beta(\lambda)$ for all λ.

In our calculations we have to work with translation functors not just over A between W_p–orbits, but also over A^β where we work with orbits for $W_{\beta,p} = < s_{\beta,rp} \mid r \in \mathbf{Z} >$. Let us disregard this complication for the purposes of this survey.

As mentioned above, we have to find explicitly a $b \in A^\beta$ with

$$t^\beta[\nu,\mu,a'] \circ t^\beta[\mu,\lambda,a] = b\, t^\beta[\nu,\lambda,a''].$$

(Of course, b is determined only modulo $H_\beta A^\beta$; our choice of b within its coset will have particularly nice properties.) If β is a simple root, then one can evaluate these maps (and determine b) using the explicit description of the extensions in 8.1. If β is not simple, then choose $w \in W$ such that $w^{-1}\beta$ is simple. We can then work with the positive system $w(R^+)$ and have an explicit description of the extensions. However, they involve new "standard" generators. For example, for $Z^\beta(\lambda)$ we no longer work with v_λ, but with a vector v_λ^w of weight $\lambda + (p-1)(w\rho - \rho)$. We use then an isomorphism $f_\lambda^w : Z^\beta(\mu) \xrightarrow{\sim} TZ^\beta(\lambda)$ with $f_\lambda^w(v_\mu^w) = \mathrm{pr}(e \otimes v_\lambda^w)$; similarly for $\beta \uparrow \lambda$. Denote the $t[f_\lambda, f_{\beta\uparrow\lambda}]$ resp. $\theta[f_\lambda, f_{\beta\uparrow\lambda}]$ corresponding to f_λ^w and $f_{\beta\uparrow\lambda}^w$ by $t^{\beta,w}[\mu,\lambda,a]$ with a as above. We can then evaluate $t^{\beta,w}[\mu,\lambda,a]$ on our extensions (12.5 and 12.7), hence find $b_w \in A^\beta$ with

$$t^{\beta,w}[\nu,\mu,a'] \circ t^{\beta,w}[\mu,\lambda,a] = b_w\, t^{\beta,w}[\nu,\lambda,a''].$$

Now we have to determine $b_w b^{-1}$. This requires lengthy explicit calculations that are carried out in Section 11. What we need is the formula 11.10(6).

11. Translations in Rank 1 Situations

11.1. Fix $\beta \in R^+$. Set $d = d_\beta$. Throughout this section we shall assume that we are in one of the following two cases:

(A) We consider a $B^\emptyset$–algebra A and arbitrary $\lambda, \mu \in X$. We apply the setup of 7.4/5 with $W' = \{1\}$, $\Omega = \{\lambda\}$, and $\Gamma = \{\mu\}$.

(B) We consider a B^β–algebra A and set W' equal to the group generated by all $s_{\beta,rp}$ with $r \in \mathbf{Z}$. Let $\lambda, \mu \in X$ such that μ is in the closure of the facet of λ. We apply the setup of 7.4/5 with this W', with $\Omega = W'{\bullet}\lambda$ and $\Gamma = W'{\bullet}\mu$.

We set $T = T_\Omega^\Gamma$ in both situations. It involves the tensor product with a module in $\mathcal{C}_k$ with extremal weight $\mu - \lambda$. Denote this module by L and choose a basis e of the weight space $L_{\mu-\lambda}$. For the sake of easy reference let us write down in the quantum case some rank one formulas for the action on e. They follow easily from the so-called Kac commutation formula, cf. [Lu7], 6.5(a2). We have for $\langle \mu - \lambda, \beta^\vee \rangle \geq 0$

$$E_\beta^{(r)} E_{-\beta}^{(r+s)} e = E_{-\beta}^{(s)} e \begin{bmatrix} \langle \mu - \lambda, \beta^\vee \rangle - s \\ r \end{bmatrix}_d$$

and for $\langle \lambda - \mu, \beta^\vee \rangle \geq 0$

$$E_{-\beta}^{(r)} E_\beta^{(r+s)} e = E_\beta^{(s)} e \begin{bmatrix} \langle \lambda - \mu, \beta^\vee \rangle - s \\ r \end{bmatrix}_d .$$

Furthermore

$$K_\beta E_\beta^{(i)} e = \zeta^{d(\langle \mu-\lambda,\beta^\vee\rangle+2i)} E_\beta^{(i)} e \qquad \text{and} \qquad K_\beta E_{-\beta}^{(i)} e = \zeta^{d(\langle \mu-\lambda,\beta^\vee\rangle-2i)} E_{-\beta}^{(i)} e.$$

11.2. Recall the notation $\omega\langle w \rangle = \omega + (p-1)(w\rho - \rho)$ for $\omega \in X$ and $w \in W$ from 4.7(4). We know by 4.7(5) that the $\omega\langle w \rangle$ weight space of $Z_A(\omega)$ is free of rank 1 over A (for all ω and w); choose a generator v_ω^w. Write $v_\omega = v_\omega^1$. We have $Z_A(\omega) \simeq Z_B(\omega) \otimes_B A$. We can and shall assume that the v_ω^w arise from corresponding elements over B.

The discussion in 7.5 and our assumptions imply that $TZ_A(\lambda)$ is generated by $\mathrm{pr}(e \otimes v_\lambda)$ where $\mathrm{pr} = \mathrm{pr}_\Gamma$. More precisely, there is an isomorphism

$$f_e : Z_A(\mu) \xrightarrow{\sim} TZ_A(\lambda) = \mathrm{pr}(L \otimes Z_A(\lambda)) \qquad \text{with } f_e(v_\mu) = \mathrm{pr}(e \otimes v_\lambda). \quad (1)$$

Let $w \in W$ with $w^{-1}\beta > 0$. Denote the standard generator of $Z_A^w(\lambda\langle w \rangle)$ (of weight $\lambda\langle w \rangle$) by v'_λ; define v'_μ analogously. We have by 5.14 an isomorphism $Z_A(\lambda) \xrightarrow{\sim} Z_A^w(\lambda\langle w \rangle)$ with $v_\lambda^w \mapsto v'_\lambda$; similarly for μ. We get from 7.11 an isomorphism

$$Z_A^w(\mu\langle w \rangle) \xrightarrow{\sim} TZ_A^w(\lambda\langle w \rangle) \qquad \text{with } v'_\mu \mapsto \mathrm{pr}(e \otimes v'_\lambda),$$

hence an isomorphism

$$f_e^w : Z_A(\mu) \xrightarrow{\sim} TZ_A(\lambda) \qquad \text{with } f_e^w(v_\mu^w) = \mathrm{pr}(e \otimes v_\lambda^w). \quad (2)$$

Of course, f_e and f_e^w differ by a factor that is a unit in A.

In the situation (A) we have

$$TZ_A(\lambda) \simeq \big(TZ_{B^\emptyset}(\lambda)\big) \otimes_{B^\emptyset} A,$$

cf. 7.5(2), and the maps f_e, f_e^w arise from the corresponding maps over $B^\emptyset$. In the situation (B) we have an analogous result with $B^\emptyset$ replaced by B^β. Furthermore, if we are in the situation (B) with $A = B^\beta$, then we can extend scalars to $B^\emptyset$ and get into a situation (A). Using constructions of this type, we can usually restrict to the case $A = B^\emptyset$ in our calculations in this section.

11.3. Consider $w \in W$ such that $w^{-1}\beta$ is a simple root. The $\lambda\langle w \rangle - i\beta$ weight space of $Z_A(\lambda) \simeq Z_A^w(\lambda\langle w \rangle)$ is spanned by $E_{-\beta}^{(i)} v_\lambda^w$ for $0 \le i < p$, and is zero for any other i. Our assumption on w implies:

$$\langle \lambda\langle w \rangle, \beta^\vee \rangle \equiv \langle \lambda + \rho - w\rho, \beta^\vee \rangle = \langle \lambda + \rho, \beta^\vee \rangle - 1 \pmod p$$

and

$$\langle \lambda\langle w \rangle + w\rho, \beta^\vee \rangle \equiv \langle \lambda + \rho, \beta^\vee \rangle \pmod p. \quad (1)$$

The calculations in 5.5 imply in Case 1:

$$E_\beta^{(i)} E_{-\beta}^{(i+j)} v_\lambda^w = E_{-\beta}^{(j)} v_\lambda^w \binom{H_\beta + \langle \lambda\langle w \rangle, \beta^\vee \rangle - j}{i}$$

for $0 \le i, j, i+j < p$. Using (1) we rewrite this as

$$E_\beta^{(i)} E_{-\beta}^{(i+j)} v_\lambda^w = E_{-\beta}^{(j)} v_\lambda^w \binom{H_\beta + \langle \lambda + \rho, \beta^\vee \rangle - j - 1}{i}. \quad (2)$$

In Case 2 we get similarly

$$E_{\beta}^{(i)} E_{-\beta}^{(i+j)} v_{\lambda}^{w} = E_{-\beta}^{(j)} v_{\lambda}^{w} ([i]_d^!)^{-1} \prod_{k=1}^{i} [K_\beta; \langle \lambda + \rho, \beta^\vee \rangle - j - k]. \tag{3}$$

If we take (2) and (3) for μ instead of λ and apply f_e^w, then we get

$$E_{\beta}^{(i)} E_{-\beta}^{(i+j)} \mathrm{pr}(e \otimes v_{\lambda}^{w}) = E_{-\beta}^{(j)} \mathrm{pr}(e \otimes v_{\lambda}^{w}) \binom{H_\beta + \langle \mu + \rho, \beta^\vee \rangle - j - 1}{i} \tag{4}$$

resp.

$$E_{\beta}^{(i)} E_{-\beta}^{(i+j)} \mathrm{pr}(e \otimes v_{\lambda}^{w}) = E_{-\beta}^{(j)} \mathrm{pr}(e \otimes v_{\lambda}^{w}) ([i]_d^!)^{-1} \prod_{k=1}^{i} [K_\beta; \langle \mu + \rho, \beta^\vee \rangle - j - k]. \tag{5}$$

The $\mu\langle w \rangle - i\beta$ weight space of $TZ_A(\lambda) \simeq Z_A(\mu)$ is spanned by $E_{-\beta}^{(i)} \mathrm{pr}(e \otimes v_{\lambda}^{w})$ for $0 \leq i < p$. So there are for all integers i, j with $0 \leq i, j < p$ elements $\kappa, \kappa' \in A$ with

$$\mathrm{pr}(E_{-\beta}^{(i)} e \otimes E_{-\beta}^{(j)} v_{\lambda}^{w}) = E_{-\beta}^{(i+j)} \mathrm{pr}(e \otimes v_{\lambda}^{w}) \kappa \tag{6}$$

resp. with

$$\mathrm{pr}(E_{\beta}^{(i)} e \otimes E_{-\beta}^{(j)} v_{\lambda}^{w}) = E_{-\beta}^{(j-i)} \mathrm{pr}(e \otimes v_{\lambda}^{w}) \kappa'. \tag{7}$$

Here the right hand side is to be interpreted as 0 if $i + j \geq p$ resp. if $j - i < 0$.

In 11.5, 11.6, and 11.7 we shall compute κ or κ' in certain situations. Here 11.5 and 11.7 will be auxiliary results needed to establish 11.6. In 11.8 we shall then state two special cases of 11.6 that will be applied in 12.3 to evaluate our translation functors on extensions. The formula in 11.5 will be used again to prove Proposition 11.10, the main result of this section.

11.4. We fix (until 11.8) an element $w \in W$ with $\alpha = w^{-1}\beta$ simple. In Case 2 we shall choose $E_\beta = T_w(E_\alpha)$ and $E_{-\beta} = T_w(F_\alpha)$. (Nothing will change, if we multiply E_β by a unit and $E_{-\beta}$ by the inverse of that unit.) In Case 2 the coproduct of E_β has the form

$$\Delta(E_\beta) = E_\beta \otimes 1 + K_\beta \otimes E_\beta + \sum_i u_i \otimes u_i'$$

where u_i' is a weight vector of weight say $\gamma_i > 0$ and where u_i is a weight vector of weight $\beta - \gamma_i > 0$ and where $w^{-1}\gamma_i > 0$ and $w^{-1}(\beta - \gamma_i) < 0$ for all i. This follows from C.5. We have also, cf. C.6,

$$\Delta(E_{-\beta}) = E_{-\beta} \otimes K_\beta^{-1} + 1 \otimes E_{-\beta} + \sum_i v_i \otimes v_i'$$

where v_i' has weight say $\gamma_i' < 0$ and where v_i has weight $-\beta - \gamma_i' < 0$ with $w^{-1}\gamma_i' > 0$ and $w^{-1}(-\beta - \gamma_i') < 0$ for all i.

If $w^{-1}\gamma > 0$ and if $w^{-1}\gamma \neq jw^{-1}\beta$ for all $j > 0$, then $\lambda\langle w\rangle - i\beta + \gamma$ is not a weight of $Z^w(\lambda\langle w\rangle)$ (for any $i \geq 0$). This implies $uE_{-\beta}^{(i)}v_\lambda^w = 0$ for all $i \geq 0$ and for any u of weight γ. If $0 < \gamma < \beta$ or if $0 > \gamma > -\beta$, then clearly $w^{-1}\gamma \neq jw^{-1}\beta$. So we see that in Case 2 for all $e' \in L$

$$E_\beta(e' \otimes E_{-\beta}^{(i)}v_\lambda^w) = (E_\beta \otimes 1 + K_\beta \otimes E_\beta)(e' \otimes E_{-\beta}^{(i)}v_\lambda^w) \tag{1}$$

and

$$E_{-\beta}(e' \otimes E_{-\beta}^{(i)}v_\lambda^w) = (E_{-\beta} \otimes K_\beta^{-1} + 1 \otimes E_{-\beta})(e' \otimes E_{-\beta}^{(i)}v_\lambda^w). \tag{2}$$

11.5. Lemma: *We have*

$$\mathrm{pr}(e \otimes E_{-\beta}^{(p-1)}v_\lambda^w) = E_{-\beta}^{(p-1)}\mathrm{pr}(e \otimes v_\lambda^w)\kappa \tag{1}$$

where κ is equal to 1 if $\langle \mu - \lambda, \beta^\vee\rangle \leq 0$, and equal to

$$\frac{H_\beta + \langle \mu + \rho, \beta^\vee\rangle}{H_\beta + \langle \lambda + \rho, \beta^\vee\rangle} \qquad \textit{resp.} \qquad \zeta^{\langle \lambda - \mu, \beta^\vee\rangle d}\frac{[K_\beta; \langle \mu + \rho, \beta^\vee\rangle]}{[K_\beta; \langle \lambda + \rho, \beta^\vee\rangle]}$$

if $\langle \mu - \lambda, \beta^\vee\rangle > 0$ in Case 1 *resp.* in Case 2.

Proof: We know by 11.3 that there is a $\kappa \in A$ such that (1) holds. If $\langle \mu - \lambda, \beta^\vee\rangle \leq 0$, then $E_{-\beta}e = 0$, and — using 11.4(2) in Case 2 —

$$E_{-\beta}^{(p-1)}\mathrm{pr}(e \otimes v_\lambda^w) = \mathrm{pr}(E_{-\beta}^{(p-1)}(e \otimes v_\lambda^w)) = \mathrm{pr}(e \otimes E_{-\beta}^{(p-1)}v_\lambda^w)$$

and $\kappa = 1$.

Suppose now that $\langle \mu - \lambda, \beta^\vee\rangle > 0$. Then $E_\beta e = 0$. We get in Case 1

$$E_\beta^{p-1}\mathrm{pr}(e \otimes E_{-\beta}^{(p-1)}v_\lambda^w) = \mathrm{pr}(E_\beta^{p-1}(e \otimes E_{-\beta}^{(p-1)}v_\lambda^w)) = \mathrm{pr}(e \otimes E_\beta^{p-1}E_{-\beta}^{(p-1)}v_\lambda^w).$$

We have by 11.3(2)

$$\begin{aligned} E_\beta^{p-1}E_{-\beta}^{(p-1)}v_\lambda^w &= v_\lambda^w \prod_{j=0}^{p-2}(H_\beta + \langle \lambda + \rho, \beta^\vee\rangle - 1 - j) \\ &= v_\lambda^w(H_\beta + \langle \lambda + \rho, \beta^\vee\rangle)^{-1}\prod_{j=0}^{p-1}(H_\beta + j) \end{aligned}$$

and similarly by 11.3(4)

$$E_\beta^{p-1}E_{-\beta}^{(p-1)}\mathrm{pr}(e\otimes v_\lambda^w)=\mathrm{pr}(e\otimes v_\lambda^w)(H_\beta+\langle\mu+\rho,\beta^\vee\rangle)^{-1}\prod_{j=0}^{p-1}(H_\beta+j).$$

The claim follows by plugging these formulas into

$$\mathrm{pr}(e\otimes E_\beta^{p-1}E_{-\beta}^{(p-1)}v_\lambda^w)=E_\beta^{p-1}E_{-\beta}^{(p-1)}\mathrm{pr}(e\otimes v_\lambda^w)\kappa.$$

In Case 2 we get first of all — using 11.4(1) —

$$\begin{aligned}E_\beta^{p-1}\mathrm{pr}(e\otimes E_{-\beta}^{p-1}v_\lambda^w)&=\mathrm{pr}(K_\beta^{p-1}e\otimes E_\beta^{p-1}E_{-\beta}^{p-1}v_\lambda^w)\\&=\mathrm{pr}(e\otimes E_\beta^{p-1}E_{-\beta}^{p-1}v_\lambda^w)\zeta^{(p-1)\langle\mu-\lambda,\beta^\vee\rangle d}.\end{aligned}$$

Furthermore,

$$\begin{aligned}E_\beta^{p-1}E_{-\beta}^{(p-1)}v_\lambda^w&=v_\lambda^w\prod_{j=0}^{p-2}[K_\beta;\langle\lambda\langle w\rangle,\beta^\vee\rangle-j]\\&=v_\lambda^w[K_\beta;\langle\lambda+\rho,\beta^\vee\rangle]^{-1}\prod_{j=0}^{p-1}[K_\beta;j].\end{aligned}$$

There is a similar formula for v_μ^w. The claim follows using $\zeta^{p-1}=\zeta^{-1}$.

Remark: Using the notations introduced in the appendix (A.2 and A.12) we can rewrite the formula as

$$\kappa=d(\lambda,\mu,\beta)^{-1}\zeta_\beta(\lambda-\mu). \tag{2}$$

(Recall that in Case 1 all ζ terms are equal to 1.)

11.6. While in Case 1 a power $(E_\beta\otimes 1+1\otimes E_\beta)^r$ of $\Delta(E_\beta)$ is simply given by the binomial theorem, things are more complicated in Case 2. One can show by induction on r, cf. [Lu7], 1.3:

$$\frac{(E_\beta\otimes 1+K_\beta\otimes E_\beta)^r}{[r]_d^!}=\sum_{j=0}^{r}\zeta^{-j(r-j)d}K_\beta^jE_\beta^{(r-j)}\otimes E_\beta^{(j)}. \tag{1}$$

Lemma: *a) If $0\le r=\langle\mu-\lambda,\beta^\vee\rangle<p$, then for all i with $0\le i<p-r$*

$$\mathrm{pr}(E_{-\beta}^{(r)}e\otimes E_{-\beta}^{(i)}v_\lambda^w)=E_{-\beta}^{(i+r)}\mathrm{pr}(e\otimes v_\lambda^w)\kappa_i \tag{2}$$

where

$$\kappa_i=\begin{cases}(r+i)!(i!)^{-1}\prod_{j=1}^r(H_\beta+\langle\mu+\rho,\beta^\vee\rangle-j)^{-1}, & \textit{in Case 1;}\\ [r+i]_d^!([i]_d^!)^{-1}\prod_{j=1}^r[K_\beta;\langle\mu+\rho,\beta^\vee\rangle-j]^{-1}, & \textit{in Case 2.}\end{cases}$$

b) If $0 \le r = \langle \lambda - \mu, \beta^\vee \rangle < p$, *then for all* i *with* $r \le i < p$

$$\mathrm{pr}(E_\beta^{(r)} e \otimes E_{-\beta}^{(i)} v_\lambda^w) = E_{-\beta}^{(i-r)} \mathrm{pr}(e \otimes v_\lambda^w) \kappa_i' \tag{3}$$

where

$$\kappa_i' = \begin{cases} (-1)^r, & \textit{in } \text{Case 1;} \\ (-1)^r \zeta^{r(i - \langle \lambda + \rho, \beta^\vee \rangle) d} K_\beta^{-r}, & \textit{in } \text{Case 2.} \end{cases}$$

Proof: a) We are dealing with a special case of 11.3(6). So we know that there is some $\kappa_i \in A$ satisfying (2). Apply $E_\beta^{(r+i)}$ to both sides. We have $E_\beta^{(j)} E_{-\beta}^{(r)} e = 0$ for $j > r$ and $E_\beta^{(r)} E_{-\beta}^{(r)} e = e$, so we get on the left hand side in Case 1

$$\begin{aligned} E_\beta^{(r+i)} \mathrm{pr}(E_{-\beta}^{(r)} e \otimes E_{-\beta}^{(i)} v_\lambda^w) &= \sum_{j=0}^{r+i} \mathrm{pr}(E_\beta^{(j)} E_{-\beta}^{(r)} e \otimes E_\beta^{(r+i-j)} E_{-\beta}^{(i)} v_\lambda^w) \\ &= \mathrm{pr}(E_\beta^{(r)} E_{-\beta}^{(r)} e \otimes E_\beta^{(i)} E_{-\beta}^{(i)} v_\lambda^w) \\ &= \mathrm{pr}(e \otimes v_\lambda^w) \binom{H_\beta + \langle \lambda + \rho, \beta^\vee \rangle - 1}{i} \end{aligned}$$

(using 11.3(2) for the last step), and in Case 2

$$\begin{aligned} &E_\beta^{(r+i)} \mathrm{pr}(E_{-\beta}^{(r)} e \otimes E_{-\beta}^{(i)} v_\lambda^w) \\ &\quad = \sum_{j=0}^{r+i} \zeta^{-j(r+i-j)d} \mathrm{pr}(K_\beta^{r+i-j} E_\beta^{(j)} E_{-\beta}^{(r)} e \otimes E_\beta^{(r+i-j)} E_{-\beta}^{(i)} v_\lambda^w) \\ &\quad = \zeta^{-ird} \mathrm{pr}(K_\beta^i E_\beta^{(r)} E_{-\beta}^{(r)} e \otimes E_\beta^{(i)} E_{-\beta}^{(i)} v_\lambda^w) \\ &\quad = \mathrm{pr}(e \otimes v_\lambda^w) \zeta^{i(\langle \mu - \lambda, \beta^\vee \rangle - r)d} ([i]_d^!)^{-1} \prod_{j=1}^{i} [K_\beta; \langle \lambda + \rho, \beta^\vee \rangle - j] \end{aligned}$$

using now 11.3(3). We use 11.3(4),(5) to evaluate the effect of $E_\beta^{(r+i)}$ on the right hand side of (2); we obtain in Case 1

$$\kappa_i = \binom{H_\beta + \langle \lambda + \rho, \beta^\vee \rangle - 1}{i} \binom{H_\beta + \langle \mu + \rho, \beta^\vee \rangle - 1}{r+i}^{-1},$$

and in Case 2

$$\kappa_i = \frac{[r+i]_d^!}{[i]_d^!} \prod_{j=1}^{i} [K_\beta; \langle \lambda + \rho, \beta^\vee \rangle - j] \prod_{j=1}^{r+i} [K_\beta; \langle \mu + \rho, \beta^\vee \rangle - j]^{-1}.$$

Using $\langle \mu+\rho, \beta^\vee\rangle = \langle \lambda+\rho, \beta^\vee\rangle + r$ one rewrites κ_i and gets the claim in the lemma.

b) We are dealing with a special case of 11.3(7). So we know that there is some $\kappa_i' \in A$ satisfying (3). Apply $E_\beta^{(i-r)}$ to both sides. Now $E_\beta E_\beta^{(r)} e = 0$, so the left hand side yields in Case 1

$$\mathrm{pr}(E_\beta^{(r)} e \otimes E_\beta^{(i-r)} E_{-\beta}^{(i)} v_\lambda^w) = \mathrm{pr}(E_\beta^{(r)} e \otimes E_{-\beta}^{(r)} v_\lambda^w) \binom{H_\beta + \langle \lambda+\rho, \beta^\vee\rangle - r - 1}{i-r}$$

using 11.3(2), and in Case 2

$$\mathrm{pr}(K_\beta^{i-r} E_\beta^{(r)} e \otimes E_\beta^{(i-r)} E_{-\beta}^{(i)} v_\lambda^w) = \mathrm{pr}(E_\beta^{(r)} e \otimes E_{-\beta}^{(r)} v_\lambda^w) \zeta^{r(i-r)d}$$
$$([i-r]_d^!)^{-1} \prod_{j=1}^{i-r} [K_\beta; \langle \lambda+\rho, \beta^\vee\rangle - r - j]$$

using 11.3(3). When we apply $E_\beta^{(i-r)}$ to the right hand side of (3), then the following terms occur: In Case 1

$$E_\beta^{(i-r)} E_{-\beta}^{(i-r)} \mathrm{pr}(e \otimes v_\lambda^w) = \mathrm{pr}(e \otimes v_\lambda^w) \binom{H_\beta + \langle \mu+\rho, \beta^\vee\rangle - 1}{i-r}$$

and in Case 2

$$E_\beta^{(i-r)} E_{-\beta}^{(i-r)} \mathrm{pr}(e \otimes v_\lambda^w) = \mathrm{pr}(e \otimes v_\lambda^w) ([i-r]_d^!)^{-1} \prod_{j=1}^{i-r} [K_\beta; \langle \mu+\rho, \beta^\vee\rangle - j].$$

Since $\langle \mu+\rho, \beta^\vee\rangle = \langle \lambda+\rho, \beta^\vee\rangle - r$, we can cancel binomial terms resp. products. We get

$$\mathrm{pr}(e \otimes v_\lambda^w)\kappa_i' = \begin{cases} \mathrm{pr}(E_\beta^{(r)} e \otimes E_{-\beta}^{(r)} v_\lambda^w), & \text{in Case 1;} \\ \mathrm{pr}(E_\beta^{(r)} e \otimes E_{-\beta}^{(r)} v_\lambda^w)\zeta^{r(i-r)d}, & \text{in Case 2.} \end{cases}$$

Now the claim will follow if we can show that (in Case 1 resp. Case 2)

$$\mathrm{pr}(E_\beta^{(r)} e \otimes E_{-\beta}^{(r)} v_\lambda^w) = \begin{cases} (-1)^r \mathrm{pr}(e \otimes v_\lambda^w), \\ (-1)^r \mathrm{pr}(e \otimes v_\lambda^w) \zeta^{r(r - \langle \lambda+\rho, \beta^\vee\rangle)d} K_\beta^{-r}. \end{cases} \tag{4}$$

That is the special case $j = r$ of the following lemma.

11.7. Lemma: *If $0 \le r = \langle \lambda - \mu, \beta^\vee \rangle < p$, then for all j with $0 \le j \le r$*

$$\mathrm{pr}(E_\beta^{(j)} e \otimes E_{-\beta}^{(j)} v_\lambda^w) = \begin{cases} (-1)^j \binom{r}{j} \mathrm{pr}(e \otimes v_\lambda^w), \\ (-1)^j \mathrm{pr}(e \otimes v_\lambda^w) \begin{bmatrix} r \\ j \end{bmatrix}_d \zeta^{j(j-\langle \lambda+\rho, \beta^\vee \rangle)d} K_\beta^{-j}, \end{cases} \tag{1}$$

in Case 1 *resp. in* Case 2.

Proof: Since $\mu\langle w \rangle + \beta$ is not a weight of $TZ_A(\lambda) \simeq Z_A^w(\mu\langle w \rangle)$, we have for all $j \ge 0$:

$$\mathrm{pr}(E_\beta^{(j+1)} e \otimes E_{-\beta}^{(j)} v_\lambda^w) = 0.$$

Apply $E_{-\beta}$ to this equation. We get in Case 1

$$(r-j)\mathrm{pr}(E_\beta^{(j)} e \otimes E_{-\beta}^{(j)} v_\lambda^w) + (j+1)\mathrm{pr}(E_\beta^{(j+1)} e \otimes E_{-\beta}^{(j+1)} v_\lambda^w) = 0,$$

and in Case 2 — using 11.4(2) —

$$\mathrm{pr}(E_\beta^{(j)} e \otimes E_{-\beta}^{(j)} v_\lambda^w) \frac{[r-j]_d}{\zeta^{(\langle \lambda+\rho, \beta^\vee \rangle - 1 - 2j)d}} K_\beta^{-1} + \mathrm{pr}(E_\beta^{(j+1)} e \otimes E_{-\beta}^{(j+1)} v_\lambda^w)[j+1]_d = 0.$$

We can rewrite this as

$$\mathrm{pr}(E_\beta^{(j+1)} e \otimes E_{-\beta}^{(j+1)} v_\lambda^w) = -\frac{r-j}{j+1} \mathrm{pr}(E_\beta^{(j)} e \otimes E_{-\beta}^{(j)} v_\lambda^w),$$

resp.

$$\mathrm{pr}(E_\beta^{(j+1)} e \otimes E_{-\beta}^{(j+1)} v_\lambda^w) = -\mathrm{pr}(E_\beta^{(j)} e \otimes E_{-\beta}^{(j)} v_\lambda^w) \frac{[r-j]_d}{[j+1]_d} \zeta^{-(\langle \lambda+\rho, \beta^\vee \rangle - 1 - 2j)d} K_\beta^{-1}.$$

Now the claim follows by induction.

11.8. Let us state explicitly one special case of Lemma 11.6. Let m, n be the integers with $0 \le n, m < p$ such that

$$\langle \lambda + \rho, \beta^\vee \rangle \equiv n \pmod p \qquad \text{and} \qquad \langle \mu + \rho, \beta^\vee \rangle \equiv m \pmod p. \tag{1}$$

Suppose that

$$\langle \mu - \lambda, \beta^\vee \rangle = m - n. \tag{2}$$

This assumption is satisfied, if we are in the situation (B) with μ in the *lower* closure (cf. 6.8) of the facet of λ for W'.

We get then: If $m \ge n$, then

$$\mathrm{pr}(E_{-\beta}^{(m-n)} e \otimes E_{-\beta}^{(n)} v_\lambda^w) = E_{-\beta}^{(m)} \mathrm{pr}(e \otimes v_\lambda^w) \kappa$$

with
$$\kappa = \begin{cases} m!(n!)^{-1}\prod_{j=1}^{m-n}(H_\beta + m - j)^{-1}, & \text{in Case 1;} \\ [m]_d^!([n]_d^!)^{-1}\prod_{j=1}^{m-n}[K_\beta; m-j]^{-1}, & \text{in Case 2.} \end{cases} \tag{3}$$

If $m \leq n$, then
$$\mathrm{pr}(E_\beta^{(n-m)} e \otimes E_{-\beta}^{(n)} v_\lambda^w) = E_{-\beta}^{(m)} \mathrm{pr}(e \otimes v_\lambda^w)\kappa'$$
with
$$\kappa' = \begin{cases} (-1)^{n-m}, & \text{in Case 1;} \\ (-1)^{n-m} K_\beta^{-(n-m)}, & \text{in Case 2.} \end{cases} \tag{4}$$

11.9. Consider a third weight $\nu \in X$. If we are in the situation (B), assume that ν is in the closure of the facet of μ for W'. We shall write quite generally pr instead of $\mathrm{pr}_{W' \bullet \omega}$ whenever it is clear which W' orbit is to be taken.

We can apply the construction of the preceding subsections to (μ, ν) and to (λ, ν) instead of (λ, μ). We have translation functors $T' = T_{W' \bullet \mu}^{W' \bullet \nu}$ and $T'' = T_{W' \bullet \lambda}^{W' \bullet \nu}$. They are constructed using modules L' and L'', and we choose nonzero vectors $e' \in L'_{\nu - \mu}$ and $e'' \in L''_{\nu - \lambda}$. There are isomorphisms
$$f_{e'} : Z_A(\nu) \xrightarrow{\sim} T' Z_A(\mu) \qquad \text{with } f_{e'}(v_\nu) = \mathrm{pr}(e' \otimes v_\mu)$$
and
$$f_{e''} : Z_A(\nu) \xrightarrow{\sim} T'' Z_A(\lambda) \qquad \text{with } f_{e''}(v_\nu) = \mathrm{pr}(e'' \otimes v_\lambda).$$
Then
$$g = g_{e,e',e''} = f_{e''} \circ f_{e'}^{-1} \circ (T' f_e)^{-1} \tag{1}$$
is an isomorphism
$$g : T'TZ_A(\lambda) \xrightarrow{\sim} T'' Z_A(\lambda) \quad \text{with } g(\mathrm{pr}(e' \otimes \mathrm{pr}(e \otimes v_\lambda))) = \mathrm{pr}(e'' \otimes v_\lambda). \tag{2}$$
It is obviously independent of the choice of v_λ.

Let $w \in W$ with $w^{-1}\beta > 0$. We get then as in 11.2(2) isomorphisms $f_{e'}^w$ and $f_{e''}^w$ from $Z_A(\nu)$ to $T' Z_A(\mu)$ resp. to $T'' Z_A(\lambda)$ with
$$f_{e'}^w(v_\nu^w) = \mathrm{pr}(e' \otimes v_\mu^w) \qquad \text{resp.} \qquad f_{e''}^w(v_\nu^w) = \mathrm{pr}(e'' \otimes v_\lambda^w).$$
Then
$$g^w = g_{e,e',e''}^w = f_{e''}^w \circ (f_{e'}^w)^{-1} \circ (T' f_e^w)^{-1} \tag{3}$$
is an isomorphism
$$g^w : T'TZ_A(\lambda) \xrightarrow{\sim} T'' Z_A(\lambda) \text{ with } g^w(\mathrm{pr}(e' \otimes \mathrm{pr}(e \otimes v_\lambda^w))) = \mathrm{pr}(e'' \otimes v_\lambda^w). \tag{4}$$
Both $T'TZ_A(\lambda)$ and $T'' Z_A(\lambda)$ are isomorphic to $Z_A(\nu)$, so g and g^w differ by a unit in A.

11.10. Keep the assumptions and notations from 11.9. We shall use the notations c_w and ζ_w introduced in A.7 and A.14.

Proposition: *We have*

$$g(\mathrm{pr}(e' \otimes \mathrm{pr}(e \otimes v_\lambda^w))) = \mathrm{pr}(e'' \otimes v_\lambda^w) c_w(\nu, \mu, \lambda) \zeta_w(\nu, \mu, \lambda).$$

Proof: The argument at the end of 11.2 shows that we may assume $A = B^\emptyset$. We use induction on the length of w. The case $w = 1$ is trivial. It is enough to prove the following: Let $w \in W$ and α a simple root with $\gamma = w\alpha > 0$. If the formula holds for w, then it hold for ws_α.

For such w and α we have $\lambda\langle ws_\alpha\rangle = \lambda\langle w\rangle - (p-1)\gamma$ and we can take $v_\lambda^{ws_\alpha} = E_{-\gamma}^{(p-1)} v_\lambda^w$, since the claim is independent of the choice of $v_\lambda^{ws_\alpha}$. We get from Lemma 11.5 constants κ, κ', and κ'' such that

$$\mathrm{pr}(e \otimes v_\lambda^{ws_\alpha}) = \mathrm{pr}(e \otimes E_{-\gamma}^{(p-1)} v_\lambda^w) = E_{-\gamma}^{(p-1)} \mathrm{pr}(e \otimes v_\lambda^w)\kappa, \tag{1}$$

$$\mathrm{pr}(e' \otimes E_{-\gamma}^{(p-1)} v_\mu^w) = E_{-\gamma}^{(p-1)} \mathrm{pr}(e' \otimes v_\mu^w)\kappa', \tag{2}$$

and

$$\mathrm{pr}(e'' \otimes E_{-\gamma}^{(p-1)} v_\lambda^w) = E_{-\gamma}^{(p-1)} \mathrm{pr}(e'' \otimes v_\lambda^w)\kappa''. \tag{3}$$

Formula 11.5(2) and the definitions A.6(1) and A.12(3) show that

$$\kappa\kappa'(\kappa'')^{-1} = c_\gamma(\nu, \mu, \lambda)\zeta_\gamma(\nu, \mu, \lambda). \tag{4}$$

We can replace v_μ^w above by its multiple $\mathrm{pr}(e \otimes v_\lambda^w)$ in the equation involving κ'. We get now — writing $c_w = c_w(\nu, \mu, \lambda)$ and $\zeta_w = \zeta_w(\nu, \mu, \lambda)$ —

$$\begin{aligned} g(\mathrm{pr}(e' \otimes \mathrm{pr}(e \otimes E_{-\gamma}^{(p-1)} v_\lambda^w))) &= E_{-\gamma}^{(p-1)} g(\mathrm{pr}(e' \otimes \mathrm{pr}(e \otimes v_\lambda^w)))\kappa\kappa' \\ &\qquad \text{by (1) and (2)} \\ &= E_{-\gamma}^{(p-1)} \mathrm{pr}(e'' \otimes v_\lambda^w))\kappa\kappa' c_w \zeta_w \\ &\qquad \text{by induction} \\ &= \mathrm{pr}(e'' \otimes E_{-\gamma}^{(p-1)} v_\lambda^w)\kappa\kappa' c_w \zeta_w (\kappa'')^{-1} \\ &\qquad \text{by (3).} \end{aligned}$$

Now apply (4) and use that $c_{ws_\alpha} = c_w c_\gamma$ and $\zeta_{ws_\alpha} = \zeta_w \zeta_\gamma$ since

$$\{\alpha' > 0 \mid (ws_\alpha)^{-1}\alpha' < 0\} = \{\alpha' > 0 \mid w^{-1}\alpha' < 0\} \cup \{w\alpha = \gamma\}.$$

Remark: Comparing this result to 11.9(4) we see that

$$g = c_w(\nu, \mu, \lambda)\zeta_w(\nu, \mu, \lambda) g^w. \tag{5}$$

There are units $a, a', a'' \in A$ with

$$f_e^w = a f_e, \qquad f_{e'}^w = a' f_{e'}, \qquad f_{e''}^w = a'' f_{e''}.$$

Then

$$a a' a''^{-1} = c_w(\nu, \mu, \lambda)\zeta_w(\nu, \mu, \lambda). \tag{6}$$

12. Translations of Extensions

12.1. We shall assume throughout this section that we are in the situation (B) from Section 11. We shall keep the general assumptions from that section. In particular, we fix a positive root $\beta \in R^+$ and assume that A is a B^β–algebra. Assume in addition that A is an integral domain with fraction field K and that the image of H_β in A is nonzero.

In Case 1 we have for all $x \in B^\beta$ that $x \equiv s_\beta x \pmod{H_\beta}$. In Case 2 we get instead $x \equiv s_\beta x \pmod{K_\beta - 1}$. However, the Ext groups as in 8.6 involve congruences modulo $H_\beta = [K_\beta; 0]$. We shall have to get congruences modulo H_β from congruences modulo $K_\beta - 1$. Therefore we shall assume from 12.9 on that A is an algebra over $B^\beta[(K_\beta + 1)^{-1}]$. Then we have $s_\beta x \equiv x \pmod{H_\beta A}$ for all $x \in B^\beta$ in both cases.

We fix $w \in W$ with $w^{-1}\beta$ simple and suppose that $E_{\pm\beta} = T_w E_{\pm w^{-1}\beta}$. (There are a few statements that are independent of w or where $w^{-1}\beta > 0$ suffices.)

12.2. For any weight $\omega \in X$ there is a unique integer r with $0 \le r < p$ such that $\langle \omega + \rho, \beta^\vee \rangle \equiv p - r \pmod{p}$. Recall from 9.3 that we then set

$$\beta{\uparrow}\omega = \omega + r\beta. \tag{1}$$

We have $\beta{\uparrow}\omega = s_{\beta,lp}{\scriptstyle\bullet}\omega$ for some $l \in \mathbf{Z}$; obviously

$$\beta{\uparrow}\omega = \omega \iff \langle \omega + \rho, \beta^\vee \rangle \equiv 0 \pmod{p}. \tag{2}$$

As in Section 11 we consider weights $\lambda, \mu \in X$ such that μ is in the closure of the facet of λ for $W' = W_{\beta,p} = < s_{\beta,rp} \mid r \in \mathbf{Z} >$. So $\beta \uparrow \lambda = \lambda$ implies $\beta \uparrow \mu = \mu$. We shall make the stronger assumption that μ is in the *upper* closure (cf. 6.8) of the facet of λ.

Let m, n be the integers with $0 \le n, m < p$ and

$$\langle \lambda + \rho, \beta^\vee \rangle \equiv p - n \pmod{p} \qquad \text{and} \quad \langle \mu + \rho, \beta^\vee \rangle \equiv p - m \pmod{p},$$

i.e., with

$$\beta{\uparrow}\lambda = \lambda + n\beta \qquad \text{and} \qquad \beta{\uparrow}\mu = \mu + m\beta.$$

Our assumption on the facets implies that

$$\langle \mu - \lambda, \beta^\vee \rangle = n - m.$$

We use the notations T, L, e, f_e, and f_e^w as in 11.1/2. We have

$$\beta \uparrow \mu - \beta \uparrow \lambda = s_\beta(\mu - \lambda).$$

Denote by $\overline{e}$ a basis element of $L_{s_\beta(\mu-\lambda)}$. We have analogously to f_e an isomorphism

$$f_{\overline{e}} : Z_A(\beta \uparrow \mu) \xrightarrow{\sim} TZ_A(\beta \uparrow \lambda) = \mathrm{pr}(E \otimes Z_A(\beta \uparrow \lambda)) \tag{3}$$

with $f_{\overline{e}}(v_{\beta\uparrow\mu}) = \mathrm{pr}(\overline{e} \otimes v_{\beta\uparrow\lambda})$. Similarly, there is an isomorphism

$$f_{\overline{e}}^w : Z_A(\beta \uparrow \mu) \xrightarrow{\sim} TZ_A(\beta \uparrow \lambda) \qquad \text{with } f_{\overline{e}}^w(v_{\beta\uparrow\mu}^w) = \mathrm{pr}(\overline{e} \otimes v_{\beta\uparrow\lambda}^w). \tag{4}$$

12.3. There is a nonzero element $a_{\lambda\mu} \in k$ such that

$$\overline{e} = \begin{cases} a_{\lambda\mu}(-1)^{n-m} E_{-\beta}^{(n-m)} e, & \text{if } n \geq m; \\ a_{\lambda\mu} E_\beta^{(m-n)} e, & \text{if } m \geq n. \end{cases} \tag{1}$$

(Note that for $n = m$ both equations say $\overline{e} = a_{\lambda\mu} e$.)

The formulas 11.3(6), (7) applied to $\beta \uparrow \lambda$ and $\beta \uparrow \mu$ instead of λ and μ show that there is $\kappa(e, \overline{e}) \in A$ with

$$\mathrm{pr}(e \otimes E_{-\beta}^{(n)} v_{\beta\uparrow\lambda}^w) = E_{-\beta}^{(m)} \mathrm{pr}(\overline{e} \otimes v_{\beta\uparrow\lambda}^w) \kappa(e, \overline{e}). \tag{2}$$

Lemma: *We have*

$$\kappa(e, \overline{e}) \equiv a_{\lambda\mu}^{-1} d(\mu, \lambda, \beta) \quad \begin{cases} \pmod{H_\beta}, & \text{in Case 1;} \\ \pmod{K_\beta - 1}, & \text{in Case 2.} \end{cases}$$

Proof: Let us assume that $a_{\lambda\mu} = 1$. (The general case is an obvious consequence.) Abbreviate $\kappa = \kappa(e, \overline{e})$. We want to apply the formulas in 11.8 to $\beta \uparrow \lambda$, $\beta \uparrow \mu$, $\overline{e}$ instead of λ, μ, e. Note that our present n and m have exactly the meaning of n and m in 11.8 for $\beta \uparrow \lambda$ and $\beta \uparrow \mu$ instead of λ and μ. Note also that $\beta \uparrow \mu$ is in the lower closure of the facet of $\beta \uparrow \lambda$ for W'.

If $m \leq n$, then $e = (-1)^{n-m} E_\beta^{(n-m)} \overline{e}$. So in this case our κ is the κ' from 11.8(4) multiplied by $(-1)^{n-m}$. So it is equal to 1 in Case 1, to a power of K_β^{-1} in Case 2. Since obviously $K_\beta^{-1} \equiv 1 \pmod{K_\beta - 1}$, we have $\kappa \equiv 1$

in both cases. On the other hand, we have $0 \leq n - m = \langle \mu - \lambda, \beta^\vee \rangle$, hence $\overline{\beta}(\mu - \lambda) = 0$ and $d(\mu, \lambda, \beta) = 1$.

Suppose now that $m > n$. We have then $e = E_{-\beta}^{(m-n)} \overline{e}$. So 11.8(3) implies

$$\kappa = \begin{cases} m!(n!)^{-1} \prod_{j=1}^{m-n} (H_\beta + m - j)^{-1}, \\ [m]^!_d ([n]^!_d)^{-1} \prod_{j=1}^{m-n} [K_\beta; m - j]^{-1}. \end{cases}$$

We get in Case 1

$$\kappa \equiv \frac{m!}{n!} \prod_{j=1}^{m-n} (m - j)^{-1} = \frac{m}{n} = \frac{-m}{-n} \equiv \frac{H_\beta + \langle \mu + \rho, \beta^\vee \rangle}{H_\beta + \langle \lambda + \rho, \beta^\vee \rangle} \pmod{H_\beta}.$$

In Case 2 observe that

$$[K_\beta; i] = \frac{K_\beta \zeta^{id} - K_\beta^{-1} \zeta^{-id}}{\zeta^d - \zeta^{-d}} \equiv \frac{\zeta^{id} - \zeta^{-id}}{\zeta^d - \zeta^{-d}} = [i]_d \pmod{K_\beta - 1}.$$

This yields (modulo $K_\beta - 1$)

$$\kappa \equiv \frac{[m]^!_d}{[n]^!_d} \prod_{j=1}^{m-n} ([m - j]_d)^{-1} = \frac{[m]_d}{[n]_d} = \frac{[-m]_d}{[-n]_d} \equiv \frac{[K_\beta; \langle \mu + \rho, \beta^\vee \rangle]}{[K_\beta; \langle \lambda + \rho, \beta^\vee \rangle]}.$$

Finally $m > n$ implies $\overline{\beta}(\mu - \lambda) = 1$. So the claim follows from the definition of $d(\mu, \lambda, \beta)$, cf. A.2(1),(2).

Remark: Note that (under our assumptions) the term $d(\mu, \lambda, \beta)$ is a unit in B^β, hence in A. In the definition A.2(1), (2) we can get a factor H_β resp. $[K_\beta; 0]$ only if $\beta \uparrow \mu = \mu$. Since μ is in the upper closure of the facet of λ, we get in that case $\overline{\beta}(\mu - \lambda) = 0$.

12.4. We can apply the theory from 8.1–4 to the positive system $w(R^+)$. Note that $w\rho$ plays the role of ρ and that

$$\langle \omega \langle w \rangle + w\rho, \beta^\vee \rangle \equiv \langle \omega + \rho, \beta^\vee \rangle \pmod{p}$$

for all $\omega \in X$.

Suppose that $\beta \uparrow \lambda \neq \lambda$. For each $b \in AH_\beta^{-1}$ set $Y_\lambda^w(b)$ equal to the $U \otimes A$–submodule of $Z_K(\beta \uparrow \lambda) \oplus Z_K(\lambda)$ generated by $v_{\beta\uparrow\lambda}^w$ and z_λ, where

$$z_\lambda = v_\lambda^w + E_{-\beta}^{(n)} v_{\beta\uparrow\lambda}^w b.$$

(For β simple and $w = 1$ our present $Y_\lambda^1(b)$ is the $Y(b)$ from 8.1(4). We now emphasize the dependence on λ by a subscript.) We have an exact sequence

$$0 \to Z_A(\beta\uparrow\lambda) \longrightarrow Y_\lambda^w(b) \longrightarrow Z_A(\lambda) \to 0 \tag{1}$$

where the first map is the obvious inclusion and where the second map takes z_λ to v_λ^w. Denote the class of (1) in its Ext group by $y_\lambda^w(b)$. We know from Lemma 8.4 that y_λ^w induces an isomorphism

$$AH_\beta^{-1}/A \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{C}}(Z_A(\lambda), Z_A(\beta\uparrow\lambda)).$$

If also $\beta\uparrow\mu \neq \mu$, then we define analogously $Y_\mu^w(b)$, involving m instead of n, and y_μ^w.

12.5. Suppose that $\beta\uparrow\lambda \neq \lambda$ and $\beta\uparrow\mu \neq \mu$. For any choice of isomorphisms $f : Z_A(\mu) \xrightarrow{\sim} TZ_A(\lambda)$ and $f' : Z_A(\beta\uparrow\mu) \xrightarrow{\sim} TZ_A(\beta\uparrow\lambda)$ we get an isomorphism

$$t[f,f'] : \mathrm{Ext}^1_{\mathcal{C}}(Z_A(\lambda), Z_A(\beta\uparrow\lambda)) \to \mathrm{Ext}^1_{\mathcal{C}}(Z_A(\mu), Z_A(\beta\uparrow\mu)),$$

cf. 7.7(5), (6) and 7.9. We define $t[\mu,\lambda,e,\overline{e}]$ and $t^w[\mu,\lambda,e,\overline{e}]$ as two special cases:

$$t[\mu,\lambda,e,\overline{e}] = t[f_e, f_{\overline{e}}] = (f_e, (f_{\overline{e}})^{-1})^* \circ T, \tag{1}$$

$$t^w[\mu,\lambda,e,\overline{e}] = t[f_e^w, f_{\overline{e}}^w] = (f_e^w, (f_{\overline{e}}^w)^{-1})^* \circ T. \tag{2}$$

Proposition: *If $\beta\uparrow\lambda \neq \lambda$ and $\beta\uparrow\mu \neq \mu$, then*

$$t^w[\mu,\lambda,e,\overline{e}](y_\lambda^w(b)) = y_\mu^w(\kappa(e,\overline{e})b) \qquad \textit{for all } b \in AH_\beta^{-1}.$$

Proof: The exact sequence

$$0 \to TZ_A(\beta\uparrow\lambda) \longrightarrow TY_\lambda^w(b) \longrightarrow TZ_A(\lambda) \to 0$$

shows that $TY_\lambda^w(b)$ is generated over $U \otimes A$ by $\mathrm{pr}(\overline{e} \otimes v_{\beta\uparrow\lambda}^w)$, a generator of $TZ_A(\beta\uparrow\lambda)$, and $\mathrm{pr}(e \otimes z_\lambda)$, an inverse image of the generator $\mathrm{pr}(e \otimes v_\lambda^w)$ of $TZ_A(\lambda)$. The isomorphism (arising from the inclusion)

$$Y_\lambda^w(b) \otimes K \xrightarrow{\sim} Z_K(\beta\uparrow\lambda) \oplus Z_K(\lambda)$$

induces an isomorphism

$$TY_\lambda^w(b) \otimes K \xrightarrow{\sim} TZ_K(\beta\uparrow\lambda) \oplus TZ_K(\lambda).$$

Composing it with $(f^w_{\overline{e}})^{-1} \oplus (f^w_e)^{-1}$ we get an isomorphism

$$TY^w_\lambda(b) \otimes K \xrightarrow{\sim} Z_K(\beta\uparrow\mu) \oplus TZ_K(\mu).$$

It takes $\mathrm{pr}(\overline{e} \otimes v^w_{\beta\uparrow\lambda})$ to $v^w_{\beta\uparrow\mu}$ and — using 12.3(2) with $\kappa = \kappa(e,\overline{e})$ —

$$\begin{aligned}\mathrm{pr}(e \otimes z_\lambda) &= \mathrm{pr}(e \otimes v^w_\lambda) + \mathrm{pr}(e \otimes E^{(n)}_{-\beta} v^w_{\beta\uparrow\lambda})b \\ &= \mathrm{pr}(e \otimes v^w_\lambda) + E^{(m)}_{-\beta}\mathrm{pr}(\overline{e} \otimes v^w_{\beta\uparrow\lambda})b\kappa\end{aligned}$$

to $v^w_\mu + E^{(m)}_{-\beta} v^w_{\beta\uparrow\mu} b\kappa$. We get thus an isomorphism

$$f : TY^w_\lambda(b) \xrightarrow{\sim} Y^w_\mu(b\kappa)$$

such that the diagram

$$\begin{array}{ccccccccc} 0 \to & TZ_A(\beta\uparrow\lambda) & \longrightarrow & TY^w_\lambda(b) & \longrightarrow & TZ_A(\lambda) & \to 0 \\ & \Big\downarrow{\scriptstyle (f^w_{\overline{e}})^{-1}} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle (f^w_e)^{-1}} & \\ 0 \to & Z_A(\beta\uparrow\mu) & \longrightarrow & Y^w_\lambda(b\kappa) & \longrightarrow & Z_A(\mu) & \to 0 \end{array}$$

is commutative. The claim follows.

12.6. Consider the case where $\beta\uparrow\lambda \neq \lambda$ and $\beta\uparrow\mu = \mu$. We then set

$$t[\mu,\lambda,e,\overline{e}] : \mathrm{Ext}^1_{\mathcal{C}}(Z_A(\lambda), Z_A(\beta\uparrow\lambda)) \to AH^{-1}_\beta/A \tag{1}$$

equal to the isomorphism $\theta[f_e, f_{\overline{e}}]$ from 8.13/14. Recall: We choose a representative

$$0 \to Z_A(\beta\uparrow\lambda) \xrightarrow{i} Y \longrightarrow jZ_A(\lambda) \to 0 \tag{2}$$

of a class ξ in the Ext group. There is a unique homomorphism $j' : Z_A(\lambda) \to Y$ with $j \circ j' = H_\beta \cdot \mathrm{id}$. We apply T to (2) and get an exact sequence

$$0 \to TZ_A(\beta\uparrow\lambda) \xrightarrow{Ti} TY \xrightarrow{Tj} TZ_A(\lambda) \to 0. \tag{3}$$

Both extreme modules in (3) are isomorphic to $Z_A(\mu)$, so we can find a homomorphism $i' : TY \to TZ_A(\beta\uparrow\lambda)$ with $i' \circ Ti = \mathrm{id}$. Now $f^{-1}_{\overline{e}} \circ i' \circ Tj' \circ f_e$ is an endomorphism of $Z_A(\mu)$, so there is a unique scalar $a \in A$ with

$$f^{-1}_{\overline{e}} \circ i' \circ Tj' \circ f_e = a\,\mathrm{id}.$$

We set now

$$t[\mu,\lambda,e,\overline{e}](\xi) = aH^{-1}_\beta + A. \tag{4}$$

We define similarly $t^w[\mu,\lambda,e,\overline{e}]$ replacing f_e and $f_{\overline{e}}$ by f^w_e and $f^w_{\overline{e}}$.

12.7. Lemma: *Suppose that $\beta\uparrow\lambda \neq \lambda$ and $\beta\uparrow\mu = \mu$. Then*

$$t^w[\mu,\lambda,e,\overline{e}](y^w_\lambda(b)) = -\kappa(e,\overline{e})b + A \qquad \textit{for all } b \in AH_\beta^{-1}.$$

Proof: Consider the exact sequence 12.4(1). We use the notations j' and i' as in 12.6. We have to have

$$j'(v^w_\lambda) = v^w_\lambda H_\beta = z_\lambda H_\beta - E^{(n)}_{-\beta} v^w_{\beta\uparrow\lambda} b H_\beta.$$

The two generators $\mathrm{pr}(\overline{e}\otimes v^w_{\beta\uparrow\lambda})$ and $\mathrm{pr}(e\otimes z_\lambda)$ of $TY^w_\lambda(b)$ are highest weight vectors (with respect to wR^+) of the same weight $\mu\langle w\rangle$. So we can take for i' the map with $\mathrm{pr}(e\otimes z_\lambda)\mapsto 0$ and $\mathrm{pr}(\overline{e}\otimes v^w_{\beta\uparrow\lambda})\mapsto \mathrm{pr}(\overline{e}\otimes v^w_{\beta\uparrow\lambda})$. Then

$$\begin{aligned}
i'\circ Tj'\circ f^w_e(v^w_\mu) &= i'\circ Tj'(\mathrm{pr}(e\otimes v^w_\lambda))\\
&= i'(\mathrm{pr}(e\otimes (z_\lambda H_\beta - E^{(n)}_{-\beta} v^w_{\beta\uparrow\lambda} b H_\beta)))\\
&= i'(\mathrm{pr}(e\otimes z_\lambda))H_\beta - i'(\mathrm{pr}(e\otimes E^{(n)}_{-\beta} v^w_{\beta\uparrow\lambda} b H_\beta))\\
&= -\mathrm{pr}(\overline{e}\otimes v^w_{\beta\uparrow\lambda})\kappa(e,\overline{e}) b H_\beta
\end{aligned}$$

by 12.3(2) where $m=0$ since $\beta\uparrow\mu=\mu$. The claim follows.

12.8. We define $t[\mu,\lambda,e,\overline{e}]$ and $t^w[\mu,\lambda,e,\overline{e}]$ in the last missing case, i.e., when $\beta\uparrow\mu=\mu$ and $\beta\uparrow\lambda=\lambda$, by

$$t[\mu,\lambda,e,\overline{e}] = t^w[\mu,\lambda,e,\overline{e}] = a_{\lambda\mu}^{-1}\mathrm{id} : AH_\beta^{-1}/A \to AH_\beta^{-1}/A. \tag{1}$$

Observe that in this case $m=n=0$ and $\overline{e} = a_{\lambda\mu}e$.

It is easy to check (cf. 7.7(7) and 8.13(6)) that one has

$$t[\mu,\lambda,be,b\overline{e}] = t[\mu,\lambda,e,\overline{e}] \qquad \text{for all } b\in k, b\neq 0$$

in all three cases, 12.5(1), 12.6(4), and (1) above; similarly for t^w. Since e is unique up to a scalar, the map $t[\mu,\lambda,e,\overline{e}]$ (resp. $t^w[\mu,\lambda,e,\overline{e}]$) depends only on $a_{\lambda\mu}$ and we shall write

$$t[\mu,\lambda,a_{\lambda\mu}] = t[\mu,\lambda,e,\overline{e}], \tag{2}$$

similarly for t^w.

12.9. Suppose from now on in Case 2 that A is a $B^\beta[(K_\beta+1)^{-1}]$ algebra.

Fix for the next subsections (until 12.14) a weight $\nu \in X$ in the upper closure of the facet of μ with respect to W'. We can apply the constructions, notations, and results in 12.2–12.8 to (μ,ν) and (λ,ν) instead of (λ,μ). The calculations to follow will involve the terms c_α, c_w, ζ_α, ζ_w introduced in Appendix A, cf. A.6, A.7, A.12, A.14.

Proposition: *We have*

$$t^w[\nu,\lambda,a_{\lambda\nu}] = a_{\lambda\nu}^{-1}a_{\lambda\mu}a_{\mu\nu}c_\beta(\nu,\mu,\lambda)t^w[\nu,\mu,a_{\mu\nu}]t^w[\mu,\lambda,a_{\lambda\mu}].$$

Proof: Set $\kappa = d(\mu,\lambda,\beta)$. If $\beta\uparrow\lambda \neq \lambda$, $\beta\uparrow\mu\neq\mu$, then Proposition 12.5 and Lemma 12.3 imply that

$$t^w[\mu,\lambda,a_{\lambda\mu}]y_\lambda^w(b) = y_\mu^w(a_{\lambda\mu}^{-1}\kappa b)$$

for all $b \in AH_\beta^{-1}$. (In Case 2 we have to use here and in the next case our assumption on A to replace in Lemma 12.3 the congruence modulo $K_\beta - 1$ by one modulo H_β.) If $\beta\uparrow\lambda\neq\lambda$, $\beta\uparrow\mu=\mu$, then Lemma 12.7 and Lemma 12.3 imply that

$$t^w[\mu,\lambda,a_{\lambda\mu}]y_\lambda^w(b) = -a_{\lambda\mu}^{-1}\kappa b + A.$$

If, finally, $\beta\uparrow\lambda=\lambda$ and $\beta\uparrow\mu=\mu$, then $m=n=0$ and $\kappa=1$, so we get by the definition 12.8(1)

$$t^w[\mu,\lambda,a_{\lambda\mu}](b+A) = a_{\lambda\mu}^{-1}\kappa b + A.$$

We get similar descriptions for the maps corresponding to the pairs (λ,ν) and (λ,μ). They yield

$$t^w[\nu,\lambda,a_{\lambda\nu}] = c\,t^w[\nu,\mu,a_{\mu\nu}]t^w[\mu,\lambda,a_{\lambda\mu}]$$

where

$$c = a_{\lambda\mu}a_{\mu\nu}a_{\lambda\nu}^{-1}d(\mu,\lambda,\beta)^{-1}d(\nu,\mu,\beta)^{-1}d(\nu,\lambda,\beta).$$

Now the claim follows from A.6(2).

12.10. Recall that we assume that ν is in the upper closure of the facet of μ and that μ is in the upper closure of the facet of λ. This implies that there is one integer r with $\beta\uparrow\nu = s_\beta{\scriptstyle\bullet}\nu + pr\beta$ and $\beta\uparrow\mu = s_\beta{\scriptstyle\bullet}\mu + pr\beta$ and $\beta\uparrow\lambda = s_\beta{\scriptstyle\bullet}\lambda + pr\beta$. So A.6(4), (5) imply for all $\alpha\in R$:

$$c_\alpha(\beta\uparrow\nu,\beta\uparrow\mu,\beta\uparrow\lambda) = s_\beta\big(c_{s_\beta\alpha}(\nu,\mu,\lambda)\big). \tag{1}$$

Lemma: *Each $c_\alpha(\nu,\mu,\lambda)$ and each $c_\alpha(\beta\uparrow\nu,\beta\uparrow\mu,\beta\uparrow\lambda)$ is a unit in B^β.*

Proof: The claim is obvious for $\alpha \neq \pm\beta$. Using A.6(3) and (1) one restricts to the case $c_\beta = c_\beta(\nu, \mu, \lambda)$. We have to show that all terms equal to H_β cancel (where $H_\beta = [K_\beta; 0]$ in Case 2). We can assume that $\langle \nu + \rho, \beta^\vee \rangle = rp$ for some integer r, since otherwise no H_β will occur. If $\langle \mu + \rho, \beta^\vee \rangle \not\equiv 0 \pmod p$, then H_β occurs with exponent $\overline{\beta}(\mu - \nu) - \overline{\beta}(\lambda - \nu)$ in c_β. Under our general assumption $\langle \mu + \rho, \beta^\vee \rangle$ and $\langle \lambda + \rho, \beta^\vee \rangle$ lie strictly between $(r-1)p$ and rp. This implies $\overline{\beta}(\mu - \nu) = \overline{\beta}(\lambda - \nu)$. If $\langle \mu + \rho, \beta^\vee \rangle \equiv 0 \pmod p$ and $\langle \lambda + \rho, \beta^\vee \rangle \not\equiv 0 \pmod p$, then H_β occurs with exponent $\overline{\beta}(\lambda - \mu) - \overline{\beta}(\lambda - \nu)$ in c_β. This exponent is 0, since $\langle \mu, \beta^\vee \rangle = \langle \nu, \beta^\vee \rangle$. If finally also $\langle \lambda + \rho, \beta^\vee \rangle \equiv 0 \pmod p$, then $\langle \mu, \beta^\vee \rangle = \langle \lambda, \beta^\vee \rangle$, and $c_\beta = 1$.

Remark: Lemma A.10 and A.6(5) imply that modulo $H_\beta A$

$$c_w(\nu, \mu, \lambda) c_w(\beta\uparrow\nu, \beta\uparrow\mu, \beta\uparrow\lambda)^{-1} \equiv c_{s_\beta}(\nu, \mu, \lambda) c_\beta(\nu, \mu, \lambda)^{-1}. \qquad (2)$$

12.11. Proposition: *We have*

$$t[\nu, \lambda, a_{\lambda\nu}] = \frac{a_{\lambda\mu} a_{\mu\nu}}{a_{\lambda\nu}} \frac{\zeta_w(\nu, \mu, \lambda)}{\zeta_w(\beta\uparrow\nu, \beta\uparrow\mu, \beta\uparrow\lambda)} c_{s_\beta}(\nu, \mu, \lambda) t[\nu, \mu, a_{\mu\nu}] t[\mu, \lambda, a_{\lambda\mu}].$$

Proof: There are units $a, b \in A$ with

$$f_e^w = a f_e \qquad \text{and} \qquad f_{\overline{e}}^w = b f_{\overline{e}}$$

and one has then

$$t[\mu, \lambda, a_{\lambda\mu}] = a^{-1} b\, t^w[\mu, \lambda, a_{\lambda\mu}].$$

This is obvious except possibly in the case where $\beta\uparrow\lambda = \lambda$ and $\beta\uparrow\mu = \mu$, where the maps are defined in 12.8(1). But in that case $f_{\overline{e}} = a_{\lambda\mu} f_e$, similarly for f^w, hence $a = b$. The formula follows.

Similarly, there are units a', b' and a'', b'' having analogous properties for (ν, μ) and (ν, λ) instead of (μ, λ). Proposition 12.9 implies now

$$t[\nu, \lambda, a_{\lambda\nu}] = a''^{-1} b'' t^w[\nu, \lambda, a_{\lambda\nu}] = a a' a''^{-1} b^{-1} b'^{-1} b'' c\, t[\nu, \mu, a_{\mu\nu}] t[\mu, \lambda, a_{\lambda\mu}]$$

where

$$c = a_{\lambda\nu}^{-1} a_{\lambda\mu} a_{\mu\nu} c_\beta(\nu, \mu, \lambda).$$

Now 11.10(6) implies

$$a a' (a'')^{-1} = c_w(\nu, \mu, \lambda) \zeta_w(\nu, \mu, \lambda).$$

If we apply 11.10(6) to $\beta\uparrow\lambda$, $\beta\uparrow\mu$, and $\beta\uparrow\nu$, we get similarly

$$b b' (b'')^{-1} = c_w(\beta\uparrow\nu, \beta\uparrow\mu, \beta\uparrow\lambda) \zeta_w(\beta\uparrow\nu, \beta\uparrow\mu, \beta\uparrow\lambda).$$

The c_w factors in these two products together with the c_β term in c yield by 12.10(2) the c_{s_β} term in the proposition. Now the claim is obvious.

12.12. We are now going to make a specific choice of $a_{\lambda\mu}$ and shall write $t_0^\beta[\mu,\lambda] = t[\mu,\lambda,a_{\lambda\mu}]$ for this choice. We set $a_{\lambda\mu} = a'_{\lambda\mu}\varepsilon^\beta_{\lambda\mu}$ with

$$a'_{\lambda\mu} = \prod_{\substack{\gamma>0,s_\beta\gamma<0,\\ w^{-1}\gamma<0}} \zeta_\gamma(\mu-\lambda)^{-1} \prod_{\substack{\gamma>0,s_\beta\gamma<0,\\ \gamma\neq\beta,w^{-1}\gamma>0}} \zeta_\gamma(\lambda-\mu), \tag{1}$$

using the notation from A.12, and

$$\varepsilon^\beta_{\lambda\mu} = (-1)^{\langle\mu-\lambda,\beta^\vee\rangle\overline{\beta}(\mu-\lambda)}\varepsilon(\mu-\lambda,\beta,w) \tag{2}$$

using the notation from B.5. It should be noted that these elements depend not only on β, λ and μ, but also on the choice of the element $w\in W$ with $w^{-1}\beta$ simple.

Now A.12(3), (5) imply

$$a'_{\lambda\mu}a'_{\mu\nu}a'^{-1}_{\lambda\nu} = \prod_{\substack{\gamma>0,s_\beta\gamma<0,\\ w^{-1}\gamma<0}} \zeta_\gamma(\nu,\mu,\lambda)^{-1} \prod_{\substack{\gamma>0,s_\beta\gamma<0,\\ \gamma\neq\beta,w^{-1}\gamma>0}} \zeta_\gamma(\nu,\mu,\lambda), \tag{3}$$

and A.14(2) yields

$$a'_{\lambda\mu}a'_{\mu\nu}a'^{-1}_{\lambda\nu} = \zeta_w(\nu,\mu,\lambda)^{-1}\zeta_w(\beta\uparrow\nu,\beta\uparrow\mu,\beta\uparrow\lambda). \tag{4}$$

We have by B.7(3)

$$\varepsilon^\beta_{\lambda\mu} = \prod_{\gamma>0,s_\beta\gamma<0} (-1)^{\langle\mu-\lambda,\gamma^\vee\rangle\overline{\gamma}(\mu-\lambda)} \prod_{\substack{\gamma>0,s_\beta\gamma<0,\\ \gamma\neq\beta,w^{-1}\gamma>0}} (-1)^{\langle\mu-\lambda,\gamma^\vee\rangle}. \tag{5}$$

We get from this using the definitions in A.12(4) and A.14(1)

$$\varepsilon^\beta_{\lambda\mu}\varepsilon^\beta_{\mu\nu}\varepsilon^\beta_{\lambda\nu} = z_{s_\beta}(\lambda,\mu,\nu). \tag{6}$$

(Note that the factors from the second product in (5) cancel.) Now Proposition 12.11 implies:

Theorem: *We have*

$$t_0^\beta[\nu,\lambda] = z_{s_\beta}(\lambda,\mu,\nu)c_{s_\beta}(\nu,\mu,\lambda)t_0^\beta[\nu,\mu]t_0^\beta[\mu,\lambda].$$

Remarks: 1) Note that $a'_{\lambda\mu}$ is (in the notations from B.5) the specialization of $v(\mu-\lambda,\beta,w)^{-1}$ at $v=\zeta$, cf. B.7(1). So B.5(3) implies that $a'_{\lambda\mu}=\varepsilon^{\beta}_{\lambda\mu}=1$ in case $m=n$.

2) Remark 1 implies especially: If $\beta\uparrow\lambda=\lambda$ (and hence $\beta\uparrow\mu=\mu$), then $t_0^{\beta}[\mu,\lambda]$ is the identity on AH_{β}^{-1}/A, cf. 12.8(1). In the situation of the theorem we get then

$$c_{s_\beta}(\nu,\mu,\lambda)\equiv z_{s_\beta}(\lambda,\mu,\nu)\pmod{H_\beta A},$$

hence

$$c_{s_\beta}(\nu,\mu,\lambda)\equiv\pm1\pmod{H_\beta A}.\tag{7}$$

This can be seen directly as follows. We may assume that $A=B^{\beta}$ resp. $A=B^{\beta}[(K_\beta+1)^{-1}]$, hence that H_β resp. $K_\beta-1$ is a prime element in A (or a unit in the cases excluded in 9.1). The formulas A.8(1) and A.6(5) show that

$$s_\beta c_{s_\beta}(\nu,\mu,\lambda)=c_{s_\beta}(\nu,\mu,\lambda)^{-1},$$

hence

$$c_{s_\beta}(\nu,\mu,\lambda)^2\equiv1\pmod{H_\beta A}.$$

Now (7) follows, since $A/H_\beta A$ is a domain (or 0).

12.13. Choose a generator h_β of the ideal AH_β. Later on we shall want to work with different choices than the obvious one ($h_\beta=H_\beta$), since this will enable us to compare the quantum group case with the positive characteristic case. We allow some flexibility at this point so to simplify certain transition formulas later on.

For each $\lambda\in X$ there is an integer r with $(r-1)p<\langle\lambda+\rho,\beta^\vee\rangle\leq rp$. Since β is conjugate to a simple root, there is a weight $\omega\in X$ with $\omega+\rho\in pX$ such that $\langle\omega+\rho,\beta^\vee\rangle=rp$, i.e., such that ω is in the upper closure of the facet of λ for W'. So $t_0^{\beta}[\omega,\lambda]$ is defined; since it is an isomorphism, there is a unique element $e_0^{\beta}(\lambda)$ of $\mathrm{Ext}^1_{\mathcal{C}}(Z_A(\lambda),Z_A(\beta\uparrow\lambda))$ in case $\beta\uparrow\lambda\neq\lambda$, resp. of AH_β^{-1}/A in case $\beta\uparrow\lambda=\lambda$ such that

$$t_0^{\beta}[\omega,\lambda]e_0^{\beta}(\lambda)=\varepsilon^{\beta}_{\lambda\omega}d(\omega,\lambda,s_\beta)h_\beta^{-1}+A.\tag{1}$$

We claim that $e_0^{\beta}(\lambda)$ is independent of the choice of ω. Indeed, if ω' is another weight with $\omega'+\rho\in pX$ and $\langle\omega'+\rho,\beta^\vee\rangle=rp$, then $\varepsilon^{\beta}_{\omega'\omega}=1=a_{\omega'\omega}$ by Remark 1 in 12.12. Therefore

$$\begin{aligned}t_0^{\beta}[\omega',\lambda]e_0^{\beta}(\lambda)&=t_0^{\beta}[\omega,\omega']t_0^{\beta}[\omega',\lambda]e_0^{\beta}(\lambda)\\&=\varepsilon^{\beta}_{\lambda\omega}\varepsilon^{\beta}_{\lambda\omega'}\varepsilon^{\beta}_{\omega'\omega}c_{s_\beta}(\omega,\omega',\lambda)^{-1}t_0^{\beta}[\omega,\lambda]e_0^{\beta}(\lambda)\\&=\varepsilon^{\beta}_{\lambda\omega}\varepsilon^{\beta}_{\lambda\omega'}\varepsilon^{\beta}_{\omega'\omega}\frac{d(\omega,\omega',s_\beta)d(\omega',\lambda,s_\beta)}{d(\omega,\lambda,s_\beta)}\varepsilon^{\beta}_{\lambda\omega}d(\omega,\lambda,s_\beta)h_\beta^{-1}+A\\&=\varepsilon^{\beta}_{\lambda\omega'}d(\omega',\lambda,s_\beta)h_\beta^{-1}+A\end{aligned}$$

since $d(\omega',\omega,s_\beta)=1$ by A.5.(2).

Lemma: *We have for all* λ, μ *as in* 12.2

$$t_0^\beta[\mu, \lambda] e_0^\beta(\lambda) = \varepsilon_{\lambda\mu}^\beta d(\mu, \lambda, s_\beta) e_0^\beta(\mu).$$

Proof: Choose ω as above. Then ω is also in the upper closure of the facet of μ for W'. We have

$$\begin{aligned} t_0^\beta[\omega, \mu] t_0^\beta[\mu, \lambda] e_0^\beta(\lambda) &= \varepsilon_{\lambda\omega}^\beta \varepsilon_{\lambda\mu}^\beta \varepsilon_{\mu\omega}^\beta c_{s_\beta}(\omega, \mu, \lambda)^{-1} t_0^\beta[\omega, \lambda] e_0^\beta(\lambda) \\ &= \varepsilon_{\lambda\mu}^\beta \varepsilon_{\mu\omega}^\beta d(\omega, \mu, s_\beta) d(\mu, \lambda, s_\beta) h_\beta^{-1} + A \\ &= t_0^\beta[\omega, \mu] \big(\varepsilon_{\lambda\mu}^\beta d(\mu, \lambda, s_\beta) e_0^\beta(\mu)\big). \end{aligned}$$

The claim follows from the injectivity of $t_0^\beta[\omega, \mu]$.

12.14. Let $\lambda' \in X$ and $\nu \in X$. We identify $Z_A(\lambda' + p\nu)$ with $Z_A(\lambda')$ with the grading shifted by $p\nu$ and get as in 9.13(4) an identification

$$\operatorname{Ext}^1_{\mathcal{C}}(Z_A(\lambda'), Z_A(\beta\uparrow\lambda')) \simeq \operatorname{Ext}^1_{\mathcal{C}}(Z_A(\lambda' + p\nu), Z_A(\beta\uparrow\lambda' + p\nu)). \tag{1}$$

In the situation of 12.12(1), (2) we see that

$$a'_{\lambda+p\nu,\mu+p\nu} = a'_{\lambda,\mu} \qquad \text{and} \qquad \varepsilon^\beta_{\lambda+p\nu,\mu+p\nu} = \varepsilon^\beta_{\lambda,\mu}.$$

This means that the isomorphism

$$f_{\lambda+p\nu} : Z_A(\mu + p\nu) \xrightarrow{\sim} TZ_A(\lambda + p\nu) \tag{2}$$

used to construct $t_0^\beta[\mu + p\nu, \lambda + p\nu]$ is identified with the isomorphism

$$f_\lambda : Z_A(\mu) \xrightarrow{\sim} TZ_A(\lambda) \tag{3}$$

used to construct $t_0^\beta[\mu, \lambda]$. This implies (modulo our identifications) that

$$t_0^\beta[\mu + p\nu, \lambda + p\nu] = t_0^\beta[\mu, \lambda]. \tag{4}$$

In 12.13 we can take $\omega + p\nu$ to construct $e_0^\beta(\lambda + p\nu)$. We get

$$e_0^\beta(\lambda' + p\nu) = e_0^\beta(\lambda') \tag{5}$$

for all λ' and ν in X.

12.15. Suppose that $\beta \uparrow \lambda \neq \lambda$ and $\beta \uparrow \mu \neq \mu$. Recall from 8.8 that $\mathrm{Ext}^2_{\mathcal{C}}(Z_A(\lambda), Z_A(\lambda + p\beta))$ is isomorphic to $A/H_\beta A$, and that the cup product $e_0^\beta(\lambda) e_0^\beta(\beta \uparrow \lambda)$ generates this module over A. Denote this cup product by $e^{\beta,2}(\lambda)$. Analogous results hold for μ instead of λ. The isomorphisms as in 12.14(2), (3) for $\nu = \beta$ induce a map

$$t^{\beta,2}[\mu, \lambda] = (f_\lambda, (f_{\lambda+p\beta})^{-1})^* \circ T \tag{1}$$

between these two Ext^2 groups. This map is independent of the choice of the isomorphism f_λ as long as we make the identifications as in 12.14 and choose $f_{\lambda+p\beta}$ equal to f_λ modulo these identifications.

Lemma: *We have* $t^{\beta,2}[\mu, \lambda] e^{\beta,2}(\lambda) = (-1)^{\langle \lambda-\mu, \beta^\vee \rangle} e^{\beta,2}(\mu)$.

Proof: Our maps are compatible with the cup product, so the left hand side in the lemma is equal to the product of $t_0^\beta[\mu, \lambda] e_0^\beta(\lambda)$ with $t_0^\beta[\beta \uparrow \mu, \beta \uparrow \lambda] e^\beta(\beta \uparrow \lambda)$. By Lemma 12.13 this product is equal to $c\, e^{\beta,2}(\mu)$ where

$$c = \varepsilon^\beta_{\lambda\mu} \varepsilon^\beta_{\beta\uparrow\lambda, \beta\uparrow\mu} d(\mu, \lambda, s_\beta) d(\beta \uparrow \mu, \beta \uparrow \lambda, s_\beta).$$

We have $(\beta \uparrow \mu) - (\beta \uparrow \lambda) = s_\beta(\mu - \lambda)$, so B.5(4) implies easily that $\varepsilon^\beta_{\lambda\mu} = (-1)^{\langle \lambda-\mu, \beta^\vee \rangle} \varepsilon^\beta_{\beta\uparrow\lambda, \beta\uparrow\mu}$. On the other hand,

$$\begin{aligned} d(\beta \uparrow \mu, \beta \uparrow \lambda, s_\beta) &= d(s_\beta \bullet \mu, s_\beta \bullet \lambda, s_\beta) = s_\beta d(\mu, \lambda, s_\beta)^{-1} \\ &\equiv d(\mu, \lambda, s_\beta)^{-1} \pmod{H_\beta} \end{aligned}$$

by A.8. So $c \equiv (-1)^{\langle \lambda-\mu, \beta^\vee \rangle} \pmod{H_\beta}$, and the claim follows.

13. A Specific Choice of Extensions

We assume in this section that A is a B–algebra that is an integral domain such that all H_α with $\alpha \in R$ are nonzero in A. In Case 2 we assume additionally that all $K_\beta + 1$ with $\beta \in R$ are units in A. We use the notations $A^\emptyset$ and A^β (for all $\beta \in R^+$) as at the beginning of Section 9 and the notations $Z^\emptyset(\lambda)$ and $Z^\beta(\lambda)$ as in 9.2(1).

For each $\beta \in R^+$ we fix an element $w_\beta \in W$ such that $w_\beta^{-1}\beta$ is simple. The algebra A^β satisfies the assumptions of the case (B) in 11.1 and of Section 12. We can apply the results of Section 12 to A^β with $w = w_\beta$.

13.1. We suppose that we have fixed for all $\alpha \in R$ an element $h_\alpha \in A$ that generates the ideal AH_α. So each $H_\alpha^{-1}h_\alpha$ is a unit in A. We assume additionally that

$$h_{-\alpha} = -h_\alpha \qquad \text{for all } \alpha \in R \tag{1}$$

and that we can extend the action of W to A such that

$$w(h_\alpha) = h_{w\alpha} \qquad \text{for all } w \in W,\ \alpha \in R \tag{2}$$

and such that

$$a \equiv s_\beta(a) \pmod{h_\beta} \qquad \text{for all } a \in A \text{ and } \beta \in R. \tag{3}$$

We assume that the $e_0^\beta(\lambda)$ from 12.13 have been constructed using this h_β.

We shall use the abbreviation

$$[k_\alpha;\mu] = \begin{cases} (H_\alpha + \langle\mu,\alpha^\vee\rangle)H_\alpha^{-1}h_\alpha, & \text{in Case 1;} \\ [K_\alpha;\langle\mu,\alpha^\vee\rangle][K_\alpha;0]^{-1}h_\alpha, & \text{in Case 2} \end{cases} \tag{4}$$

for all $\mu \in X$ and $\alpha \in R$. We have obviously

$$\langle\mu,\alpha^\vee\rangle \equiv 0 \pmod{p} \implies [k_\alpha;\mu] = h_\alpha.$$

Using (1) and (2) one checks easily that (for all $\alpha \in R$ and $\mu \in X$)

$$[k_{-\alpha};\mu] = -[k_\alpha;\mu] \tag{5}$$

and

$$[k_{w\alpha}; w\mu] = w[k_\alpha; \mu] \qquad \text{for all } w \in W. \tag{6}$$

Using this notation we can rewrite the $d(\mu, \lambda, \alpha)$ from A.2 as

$$d(\mu, \lambda, \alpha) = \left(\frac{[k_\alpha; \mu + \rho]}{[k_\alpha; \lambda + \rho]}\right)^{\overline{\alpha}(\mu-\lambda)} \tag{7}$$

13.2. Suppose that p is greater or equal to the Coxeter number of R. We can now choose a weight λ in the interior of the "first" dominant alcove for W_p, i.e., with

$$0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \qquad \text{for all } \alpha \in R^+.$$

We keep λ fixed throughout Section 13 and set $\Omega = W_p{\bullet}\lambda$.

We have $\langle w(\lambda + \rho), \beta^\vee \rangle = \langle \lambda + \rho, (w^{-1}\beta)^\vee \rangle$ for all $\beta \in R$ and $w \in W$. So we see by the choice of λ that

$$\overline{\beta}(w(\lambda + \rho)) = 1 \iff w^{-1}\beta < 0. \tag{1}$$

We have (using the notation from A.7)

$$d(w{\bullet}\lambda, w{\bullet}(-\rho), s_\beta) = \prod_{\alpha>0, s_\beta\alpha<0} \left(\frac{[k_\alpha; w(\lambda+\rho)]}{h_\alpha}\right)^{\overline{\alpha}(w(\lambda+\rho))}$$

So (1) implies

$$d(w{\bullet}\lambda, -\rho, s_\beta) = \prod_{\substack{\alpha>0, s_\beta\alpha<0, \\ w^{-1}\alpha<0}} \frac{[k_\alpha; w(\lambda+\rho)]}{h_\alpha}. \tag{2}$$

For any $\lambda' \in \Omega$ there are unique $w \in W$ and $\nu \in \mathbf{Z}R$ with $\lambda' = w{\bullet}\lambda + p\nu$; then set for all $\beta \in R^+$:

$$b^\beta(\lambda') = \begin{cases} \varepsilon^\beta_{w\bullet\lambda,-\rho}\, d(w{\bullet}\lambda, -\rho, s_\beta)\kappa(\beta), & \text{if } w^{-1}\beta > 0; \\ \varepsilon^\beta_{w\bullet\lambda,-\rho}\, d(w{\bullet}\lambda, -\rho, s_\beta)h_\beta, & \text{if } w^{-1}\beta < 0, \end{cases} \tag{3}$$

where

$$\kappa(\beta) = \prod_{\substack{\alpha>0, s_\beta\alpha<0, \\ \alpha\neq\beta, w_\beta^{-1}\alpha<0}} h_\alpha^{-\langle \beta, \alpha^\vee \rangle}. \tag{4}$$

By (2) each $b^\beta(\lambda')$ is a unit in A^β. Set

$$e^\beta(\lambda') = b^\beta(\lambda')e_0^\beta(\lambda') \qquad \text{for all } \lambda' \in \Omega. \tag{5}$$

This element is a basis of $\mathrm{Ext}^1_{\mathcal{C}_{A^\beta}}(Z^\beta(\lambda'), Z^\beta(\beta \uparrow \lambda'))$.

13.3. For all $\mu \in X$ set

$$R_\mu = \{\alpha \in R \mid \langle \mu + \rho, \alpha^\vee \rangle \equiv 0 \pmod p\}. \tag{1}$$

The stabilizer of μ in W_p (for the dot action) is generated by all $s_{\alpha,\langle \mu+\rho,\alpha^\vee\rangle}$ with $\alpha \in R_\mu$. This is a reflection group with root system R_μ.

We call a W_p–orbit Γ *good* if there are for each $\mu \in \Gamma$ an element $w \in W$ and a subset $\Sigma' \subset \Sigma$ such that $R_\mu = w(\mathbf{Z}\Sigma' \cap R)$. It is enough to check this condition for one $\mu \in \Gamma$, since the R_μ for different $\mu \in \Gamma$ are conjugate under W. Clearly all regular orbits are good since there $R_\mu = \emptyset$ always. In Case 1 all W_p–orbits are good, if p is a good prime, cf. 2.7 in [Ja5]. Similarly, in Case 2 all W_p–orbits are good, if all prime divisors of p are good.

Consider a weight μ in the closure of the alcove of λ and set $\Gamma = W_p{\bullet}\mu$. Set $R' = R_\mu$. We have

$$0 < \langle \mu + \rho, \alpha^\vee \rangle < p \qquad \text{for all } \alpha \in R^+, \alpha \notin R', \tag{2}$$

since μ is in the closure of the alcove of λ. For any $\lambda' \in \Omega$ there is a unique $\mu' \in \Gamma$ such that μ' is in the closure of the alcove of λ'; denote this weight by λ'_Γ. If we write (uniquely) $\lambda' = w{\bullet}\lambda + p\nu$ with $w \in W$ and $\nu \in \mathbf{Z}R$, then $\lambda'_\Gamma = w{\bullet}\mu + p\nu$.

Let T be the translation functor from $\mathcal{C}_A(\Omega)$ to $\mathcal{C}_A(\Gamma)$. In order to prove our main result (13.4) on T we shall need (in 13.10) that Γ is good. Let L be the simple module with extreme weight $\mu - \lambda$ used in the construction of T. Choose for all $w \in W$ a basis e_w of $L_{w(\mu-\lambda)}$ such that (in the notations of the appendix B) $e_w = P(wv^{-1})e_v$ if $v \in W$ with $v(\mu - \lambda)$ dominant.

We have for any $\lambda' = w{\bullet}\lambda + p\nu \in \Omega$ (with $w \in W$ and $\nu \in \mathbf{Z}R$) an isomorphism

$$f'_{\lambda'} : Z_A(\mu') \xrightarrow{\sim} TZ_A(\lambda') \qquad \text{with } f'_{\lambda'}(v_{\mu'}) = \mathrm{pr}(e_w \otimes v_{\lambda'}), \tag{3}$$

where $\mu' = \lambda'_\Gamma = w{\bullet}\mu + p\nu$. The $f'_{\lambda'}$ induce analogous maps over each A^β by extension of scalars.

Let $\beta \in R^+$. If $n = \langle \mu' - \lambda', \beta^\vee \rangle \geq 0$, then B.6 implies (cf. Remark 1 in 12.12) that

$$E_{-\beta}^{(n)} e_w = (a'_{\lambda'\mu'})^{-1} \varepsilon(\lambda' - \mu', \beta, w_\beta) e_{s_\beta w}.$$

Similarly, if $n = -\langle \mu' - \lambda', \beta^\vee \rangle \geq 0$, then

$$E_\beta^{(n)} e_w = (a'_{\lambda'\mu'})^{-1} \varepsilon(\lambda' - \mu', \beta, w_\beta) e_{s_\beta w}.$$

If μ' is in the upper closure of the alcove of λ' for $W_{\beta,p} = < s_{\beta,rp} \mid r \in \mathbf{Z} >$, then the map $t_0^\beta[\mu', \lambda']$ is defined, see 12.5, 12.6, 12.8, 12.12. The formulas above and the choices in 12.12 imply by the definition in 12.3(1) that

$$t_0^\beta[\mu', \lambda'] = t[\mu', \lambda', e_w, e_{s_\beta w}]. \tag{4}$$

If $\beta \uparrow \mu' \neq \mu'$, then $t_0^\beta[\mu', \lambda'] = (f'_{\lambda'}, (f'_{\beta\uparrow\lambda'})^{-1})^* \circ T$.

13.4. Let $\mathcal{H}$ be the set of reflection hyperplanes for W_p. Each $H \in \mathcal{H}$ has an equation of the form $\langle \nu + \rho, \gamma^\vee \rangle = rp$ with $r \in \mathbf{Z}$ and $\gamma \in R^+$. Then we set $\gamma = \alpha(H)$; furthermore, for any $\nu \in X$ we shall write $\nu < H$ (resp. $\nu > H$) if and only if $\langle \nu + \rho, \alpha(H)^\vee \rangle < rp$ (resp. $\langle \nu + \rho, \alpha(H)^\vee \rangle > rp$). For all $\beta \in R^+$ let $\mathcal{H}(\beta)$ be the set of all $H \in \mathcal{H}$ with $s_\beta \alpha(H) < 0$. For all $\lambda' \in \Omega$ and $\beta \in R^+$ set

$$C^\beta(\lambda', \mu') = \prod_{\substack{H \in \mathcal{H}(\beta), \\ \mu' \in H, \lambda' > H}} h_{-\alpha(H)} \prod_{\substack{H \in \mathcal{H}(\beta), \\ \mu' \in H, \lambda' < H}} h_{\alpha(H)}^{-1} \tag{1}$$

where $\mu' = \lambda'_\Gamma$.

Suppose that we have chosen for all $\mu' \in \Gamma$ and all $\beta \in R^+$ a unit $b^\beta(\mu')$ in A^β. We can then set

$$e^\beta(\mu') = b^\beta(\mu') e_0^\beta(\mu') \tag{2}$$

for all μ' and β. This element is again a basis of the corresponding Ext group resp. — for $\beta \uparrow \mu' = \mu'$ — of $A^\beta h_\beta^{-1}/A^\beta = A^\beta H_\beta^{-1}/A^\beta$.

Suppose that we have chosen for all $\lambda' \in \Omega$ a unit $a_\Gamma(\lambda')$ in A. We can then set

$$f_{\lambda'} = a_\Gamma(\lambda') f'_{\lambda'} : Z_A(\mu') \xrightarrow{\sim} TZ_A(\lambda') \tag{3}$$

where $\mu' = \lambda'_\Gamma$. We get then maps

$$t^\beta[\mu', \lambda'] = t[\mu', \lambda', a_\Gamma(\lambda') e_w, a_\Gamma(\beta \uparrow \lambda') e_{s_\beta w}] \tag{4}$$

for all $\beta \in R^+$ such that μ' is in the upper closure of the alcove of λ' for $W_{\beta,p}$. If $\beta \uparrow \mu' \neq \mu'$, then $t^\beta[\mu', \lambda'] = (f_{\lambda'}, (f_{\beta\uparrow\lambda'})^{-1})^* \circ T$. We have obviously in any case

$$t^\beta[\mu', \lambda'] = a_\Gamma(\lambda') a_\Gamma(\beta \uparrow \lambda')^{-1} t_0^\beta[\mu', \lambda']. \tag{5}$$

Since this is a homomorphism of A^β–modules that are annihilated by h_β, 13.1(3) implies that

$$t^\beta[\mu', \lambda'] = a_\Gamma(\lambda') \big(s_\beta a_\Gamma(\beta \uparrow \lambda')\big)^{-1} t_0^\beta[\mu', \lambda']. \tag{6}$$

We shall prove in this section:

Theorem: *Suppose that* Γ *is good. One can choose the* $b^\beta(\mu')$ *and the* $a_\Gamma(\lambda')$ *such that for all* $\lambda' \in \Omega$ *and* $\beta \in R^+$ *(setting* $\mu' = \lambda'_\Gamma$*)*

$$t^\beta[\mu', \lambda'] e^\beta(\lambda') = \begin{cases} C^\beta(\lambda', \mu') e^\beta(\mu'), & \text{if } \beta \uparrow \mu' \neq \mu'; \\ C^\beta(\lambda', \mu') + A^\beta, & \text{if } \beta \uparrow \mu' = \mu' = (\beta \uparrow \lambda')_\Gamma. \end{cases} \tag{7}$$

13.5. We shall first prove Theorem 13.4 in the case where μ is in the lower closure of the "first" dominant alcove. Assume in the next subsections (until 13.9) that this holds. So we have

$$R' = \{\alpha \in R \mid \langle \mu + \rho, \alpha^\vee \rangle = 0\}. \tag{1}$$

The stabilizer of μ in W_p for the dot action is generated by all s_α with $\alpha \in R'$. Note that R' is the root subsystem of R generated by $\Sigma \cap R'$; so Γ is obviously good.

We have by 13.3(2) for all $\beta \in R$

$$\overline{\beta}(w(\mu + \rho)) = 1 \iff w^{-1}\beta < 0, w^{-1}\beta \notin R'. \tag{2}$$

This implies

$$d(w{\scriptstyle\bullet}\mu, -\rho, s_\beta) = \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha<0, w^{-1}\alpha\notin R'}} \frac{[k_\alpha; w(\mu+\rho)]}{h_\alpha}. \tag{3}$$

For all $\beta \in R^+$, $w \in W$, and $\nu \in \mathbf{Z}R$ set:

$$b^\beta(w{\scriptstyle\bullet}\mu + p\nu) = \begin{cases} \varepsilon^\beta_{w\bullet\mu,-\rho}\, d(w{\scriptstyle\bullet}\mu, -\rho, s_\beta)\kappa(\beta), & \text{if } w^{-1}\beta > 0, w^{-1}\beta \notin R'; \\ \varepsilon^\beta_{w\bullet\mu,-\rho}\, d(w{\scriptstyle\bullet}\mu, -\rho, s_\beta)h_\beta, & \text{if } w^{-1}\beta < 0, w^{-1}\beta \notin R'; \\ \varepsilon^\beta_{w\bullet\mu,-\rho}\, d(w{\scriptstyle\bullet}\mu, -\rho, s_\beta), & \text{if } w^{-1}\beta \in R'. \end{cases} \tag{4}$$

This is well defined, since w and ws_α yield the same result whenever $\alpha \in R'$. It is a unit in A^β by (3). So we can define $e^\beta(\mu')$ for all $\mu' \in \Gamma$ by 13.4(2). (Note that this definition is independent of the choice of Ω.)

Lemma: *Let $\beta \in R^+$ and $\mu' \in W_p{\scriptstyle\bullet}\mu$. If $\beta \uparrow \mu' = \mu'$, then $e^\beta(\mu') = h_\beta^{-1} + A^\beta$.*

Proof: There are $w \in W$ and $\nu \in \mathbf{Z}R$ with $\mu' = w{\scriptstyle\bullet}\mu + p\nu$. The assumption $\beta \uparrow \mu' = \mu'$ implies $w^{-1}\beta \in R'$. Since $\langle \mu' + \rho, \beta^\vee \rangle = \langle p\nu, \beta^\vee \rangle$ we can take $-\rho + p\nu$ as the weight playing the role of ω in 12.13(1) (applied to μ' instead of λ). Now $t_0^\beta[-\rho + p\nu, \mu']$ is the identity (by Remark 2 in 12.12) so we get

$$\begin{aligned} e^\beta(\mu') &= b^\beta(\mu')\varepsilon^\beta_{\mu',-\rho+p\nu} d(-\rho + p\nu, w{\scriptstyle\bullet}\mu + p\nu, s_\beta)h_\beta^{-1} + A^\beta \\ &= d(w{\scriptstyle\bullet}\mu, -\rho, s_\beta)d(-\rho, w{\scriptstyle\bullet}\mu, s_\beta)h_\beta^{-1} + A^\beta. \end{aligned}$$

Since $s_\beta{\scriptstyle\bullet}(w{\scriptstyle\bullet}\mu) = w{\scriptstyle\bullet}\mu$ and $s_\beta{\scriptstyle\bullet}(-\rho) = -\rho$, Lemma A.8 implies

$$d(w{\scriptstyle\bullet}\mu, -\rho, s_\beta)d(-\rho, w{\scriptstyle\bullet}\mu, s_\beta) = \frac{d(-\rho, w{\scriptstyle\bullet}\mu, s_\beta)}{s_\beta d(-\rho, w{\scriptstyle\bullet}\mu, s_\beta)}. \tag{5}$$

Under our assumptions, $d(-\rho, w{\scriptstyle\bullet}\mu, \beta) = 1$, so $d(-\rho, w{\scriptstyle\bullet}\mu, s_\beta)$ is a unit in A^β and the fraction in (5) is congruent to 1 modulo h_β. The claim follows.

13.6. Consider $\lambda' = w\bullet\lambda + p\nu \in \Omega$ where $w \in W$ and $\nu \in \mathbf{Z}R$. Let $\beta \in R^+$. Then $\lambda'_\Gamma = w\bullet\mu + p\nu$ is in the upper closure of the alcove of λ' for $W_{\beta,p}$, if and only if $w^{-1}\beta \notin R' \cap R^+$.

Lemma: *Let $\beta \in R^+$, $w \in W$, and $\nu \in \mathbf{Z}R$ with $w^{-1}\beta \notin R' \cap R^+$. Set $\lambda' = w\bullet\lambda + p\nu$ and $\mu' = w\bullet\mu + p\nu$. Then*

$$t_0^\beta[\mu', \lambda']e^\beta(\lambda') = z_{s_\beta}(w\bullet\lambda, w\bullet\mu, -\rho)c_{s_\beta}(w\bullet\mu, w\bullet\lambda, -\rho)^{-1}e^\beta(\mu'),$$

if $w^{-1}\beta \notin R'$; and

$$t_0^\beta[\mu', \lambda']e^\beta(\lambda') = z_{s_\beta}(w\bullet\lambda, w\bullet\mu, -\rho)(h_\beta c_{s_\beta}(w\bullet\mu, w\bullet\lambda, -\rho)^{-1})e^\beta(\mu'),$$

if $w^{-1}\beta \in R' \cap (-R^+)$.

Proof: The definitions 13.2(4) and 13.4(2) together with Lemma 12.13 imply that

$$t_0^\beta[\mu', \lambda']e^\beta(\lambda') = c\, e^\beta(\mu')$$

where

$$c = \varepsilon^\beta_{\lambda',\mu'} d(\mu', \lambda', s_\beta)\frac{b^\beta(\lambda')}{b^\beta(\mu')}. \tag{1}$$

The ε^β factors from the b^β terms together with the one from (1) yield

$$\varepsilon^\beta_{w\bullet\lambda, w\bullet\mu}\varepsilon^\beta_{w\bullet\lambda,-\rho}\varepsilon^\beta_{w\bullet\mu,-\rho} = z_{s_\beta}(w\bullet\lambda, w\bullet\mu, -\rho)$$

by 12.12(6). The d factors from the b^β terms together with the d factor in (1) yield

$$d(w\bullet\mu, w\bullet\lambda, s_\beta)d(w\bullet\lambda, -\rho, s_\beta)d(w\bullet\mu, -\rho, s_\beta)^{-1},$$

hence $c_{s_\beta}(w\bullet\mu, w\bullet\lambda, -\rho)^{-1}$ by A.7(3). The factors h_β and $\kappa(\beta)$ from the b^β terms cancel for $w^{-1}\beta \notin R'$, whereas they yield h_β for $w^{-1}\beta \in R'$.

Remark: Note that the proof also shows that the c_{s_β} factor (for $w^{-1}\beta \notin R'$) resp. h_β times the c_{s_β} factor (for $w^{-1}\beta \in R'$) is a unit in A^β.

13.7. We have $\langle \mu - \lambda, \alpha^\vee\rangle = -\langle \lambda + \rho, \alpha^\vee\rangle$ for all $\alpha \in R'$, hence

$$\overline{\alpha}(\mu - \lambda) = 1 \qquad \text{for all } \alpha \in R' \cap R^+. \tag{1}$$

Lemma: *Let $w \in W$. Then*

$$d(\mu, \lambda, w^{-1}) \prod_{\substack{\alpha>0, w\alpha<0,\\ \alpha\in R'}} h_\alpha^{-1}$$

is a unit in A.

Proof: Recall from A.7(1) and 13.1(7) that

$$d(\mu,\lambda,w^{-1}) = \prod_{\alpha>0, w\alpha<0} \left(\frac{[k_\alpha;\mu+\rho]}{[k_\alpha;\lambda+\rho]}\right)^{\overline{\alpha}(\mu-\lambda)}.$$

All factors in the denominator are units in A; so are those in the numerator for $\alpha \notin R'$. For $\alpha \in R'$ we get a factor of h_α in the numerator. Now the claim follows from (1).

13.8. Set for all $w \in W$:

$$a_\Gamma(w) = z_{w^{-1}}(\lambda,\mu,-\rho)\cdot w\big(d(\mu,\lambda,w^{-1})\big)\cdot \prod_{\substack{\alpha>0, w^{-1}\alpha<0,\\ w^{-1}\alpha\in R'}} h_\alpha^{-1}. \tag{1}$$

Since $h_{-\alpha} = -h_\alpha$ for all α, cf. 13.1(1), Lemma 13.7 implies that each $a_\Gamma(w)$ with $w \in W$ is a unit in A.

Lemma: *We have for all $w \in W$ and $\beta \in R^+$*

$$a_\Gamma(w)\cdot\big(s_\beta a_\Gamma(s_\beta w)\big)^{-1} = z_{s_\beta}(w_\bullet\lambda, w_\bullet\mu,-\rho)c_{s_\beta}(w_\bullet\mu,w_\bullet\lambda,-\rho)\kappa$$

where

$$\kappa = \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha>0, w^{-1}\alpha\in R'}} h_{-\alpha} \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha<0, w^{-1}\alpha\in R'}} h_\alpha^{-1}.$$

Proof: A look at (1) shows that we can write $a_\Gamma(w)/(s_\beta a_\Gamma(s_\beta w))$ in a natural way as a product of three factors. The first one is

$$z_{w^{-1}}(\lambda,\mu,-\rho)z_{(s_\beta w)^{-1}}(\lambda,\mu,-\rho)^{-1} = z_{s_\beta}(w_\bullet\lambda,w_\bullet\mu,-\rho)$$

by A.14(3). The second one is

$$w\big(d(\mu,\lambda,w^{-1})\cdot d(\mu,\lambda,w^{-1}s_\beta)^{-1}\big) = c_{s_\beta}(w_\bullet\mu,w_\bullet\lambda,-\rho)$$

by A.11(3). The last one is

$$\prod_{\substack{\alpha>0, w^{-1}\alpha<0,\\ w^{-1}\alpha\in R'}} h_\alpha^{-1} \prod_{\substack{\alpha>0, w^{-1}s_\beta\alpha<0,\\ w^{-1}s_\beta\alpha\in R'}} h_{s_\beta\alpha} = \prod_{\substack{\alpha>0, w^{-1}\alpha<0,\\ w^{-1}\alpha\in R'}} h_\alpha^{-1} \prod_{\substack{s_\beta\alpha>0, w^{-1}\alpha<0,\\ w^{-1}\alpha\in R'}} h_\alpha.$$

We cancel now the common factors (where $\alpha > 0$ and $s_\beta\alpha > 0$), we substitute $-\alpha$ for α in the second product, and we get now the κ in the lemma.

13.9. For any $\lambda' = w{\bullet}\lambda + p\nu \in \Omega$ with $w \in W$ and $\nu \in \mathbf{Z}R$ set $a_\Gamma(\lambda') = a_\Gamma(w)$. We claim that Theorem 13.4 holds (for Γ satisfying 13.5(1)) with this choice of $a_\Gamma(\lambda')$ and with the choice of the $b^\beta(\mu')$ as in 13.5(4).

Proof: Let $\beta \in R^+$, $w \in W$ and $\nu \in \mathbf{Z}R$. Set $\lambda' = w{\bullet}\lambda + p\nu$ and $\mu' = w{\bullet}\mu + p\nu$. Suppose that $w^{-1}\beta \notin R' \cap R^+$ so that $t^\beta[\mu', \lambda']$ is defined. Now 13.4(6), Lemma 13.6 and Lemma 13.8 show that $t^\beta[\mu', \lambda'] e^\beta(\lambda') = \kappa' e^\beta(\mu')$ where

$$\kappa' = \prod_{\substack{\alpha>0, s_\beta\alpha<0, \\ w^{-1}\alpha>0, w^{-1}\alpha\in R'}} h_{-\alpha} \prod_{\substack{\alpha>0, s_\beta\alpha<0, \alpha\neq\beta \\ w^{-1}\alpha<0, w^{-1}\alpha\in R'}} h_\alpha^{-1}. \tag{1}$$

Compare κ' to $C^\beta(\lambda', \mu')$ as in 13.4(1): The hyperplanes $H \in \mathcal{H}$ with $\mu' \in H$ are exactly those with an equation $\langle x + \rho, \alpha^\vee\rangle = \langle \nu, \alpha^\vee\rangle p$ with $\alpha \in R^+$ and $w^{-1}\alpha \in R'$. We have $\lambda' < H$ if and only if $w^{-1}\alpha < 0$, and $H \in \mathcal{H}(\beta)$ is equivalent to $s_\beta\alpha < 0$. So the only difference with the products in 13.4(1) is the additional condition $\alpha \neq \beta$ in the second product above. If $\beta \uparrow \mu' \neq \mu'$, then $\beta \neq \alpha(H)$ for all $H \in \mathcal{H}$ with $\mu' \in H$, hence $\kappa' = C^\beta(\lambda', \mu')$ and 13.4(7) follows in this case. If $\beta \uparrow \mu' = \mu'$, then $\beta = \alpha(H)$ occurs in the second product in 13.4(1), but not in the first one, since $w^{-1}\beta \notin R' \cap R^+$. This shows that $\kappa' = C^\beta(\lambda', \mu') h_\beta$ in this case. Now 13.4(7) follows from Lemma 13.5.

13.10. We now have to look at the case where μ is *not* in the lower closure of the first dominant alcove. Assume that Γ is good. Set

$$R'_i = \{\alpha \in R \mid \langle \mu + \rho, \alpha^\vee\rangle = ip\} \tag{1}$$

for all $i \in \mathbf{Z}$. Of course, R' is the disjoint union of the R'_i. Since μ is still in the closure of that alcove, we have $R_i = \emptyset$ for $|i| > 1$ and

$$R'_1 \subset R^+ \qquad \text{and} \qquad R'_{-1} = -R'_1 \subset -R^+. \tag{2}$$

Obviously

$$s_\gamma R'_i = R'_i \qquad \text{for all } \gamma \in R'_0 \text{ and all } i. \tag{3}$$

We have $R'_1 \neq \emptyset$ by our assumption.

Lemma: *There is a weight $\sigma \in X$ with*

$$\sigma + \rho \in pX \qquad \textit{and} \qquad \langle \sigma - \mu, \alpha^\vee\rangle = 0 \quad \textit{for all } \alpha \in R'. \tag{4}$$

Proof: Since Γ is good, there are a subset Σ' of Σ and an element $w' \in W$ such that $R' = w'(R \cap \mathbf{Z}\Sigma')$. We can find a weight $\mu' \in X$ with $\langle \mu' + \rho, \alpha^\vee\rangle = \langle w'^{-1}(\mu + \rho), \alpha^\vee\rangle$ for all $\alpha \in \Sigma'$ (hence for all $\alpha \in w'^{-1}R'$) and with $\langle \mu' + \rho, \alpha^\vee\rangle = 0$ for all $\alpha \in \Sigma$, $\alpha \notin \Sigma'$. Obviously $\mu' + \rho \in pX$. Then $\sigma = w'{\bullet}\mu'$ satisfies (4).

13.11. We fix from now on a weight σ satisfying Lemma 13.10. We have $\langle\sigma+\rho,\alpha^\vee\rangle \in \mathbf{Z}p$ and $\langle\mu+\rho,\alpha^\vee\rangle \notin \mathbf{Z}p$ for all $\alpha \in R$ with $\alpha \notin R'$, hence $\langle\sigma-\mu,\alpha^\vee\rangle \neq 0$. Therefore R is the disjoint union of R', $R(1)$ and $R(-1)$ where

$$R(1) = \{\alpha \in R \mid \langle\sigma-\mu,\alpha^\vee\rangle > 0\} \tag{1}$$

and

$$R(-1) = \{\alpha \in R \mid \langle\sigma-\mu,\alpha^\vee\rangle < 0\} = -R(1). \tag{2}$$

Obviously

$$s_\gamma R(1) = R(1) \qquad \text{and} \qquad s_\gamma R(-1) = R(-1) \qquad \text{for all } \gamma \in R'. \tag{3}$$

Lemma: *If $\overline{\alpha}(\sigma+\rho) = 1$, then $\alpha \in R(-1) \cup R'_{-1}$. Conversely, if $\alpha \in R(-1) \cup R'_{-1}$, then $\overline{\alpha}(\sigma+\rho) = 1$ or $\langle\sigma+\rho,\alpha^\vee\rangle = 0$.*

Proof: We have

$$\langle\sigma+\rho,\alpha^\vee\rangle > \langle\mu+\rho,\alpha^\vee\rangle > -p$$

for all $\alpha \in R(1)$. Since the first term is divisible by p we get $\langle\sigma+\rho,\alpha^\vee\rangle \geq 0$, hence $\overline{\alpha}(\sigma+\rho) = 0$ for all $\alpha \in R(1)$. On the other hand, we get for all $\alpha \in R(-1) = -R(1)$ that $\langle\sigma+\rho,\alpha^\vee\rangle \leq 0$.

13.12. Let $w \in W$. If $\nu \in \mathbf{Z}R$ and $\mu' = w_\bullet\mu + p\nu$, then the hyperplanes $H \in \mathcal{H}$ with $\mu' \in H$ correspond to the $\alpha \in R^+$ with $w^{-1}\alpha \in R'$. If $w^{-1}\alpha \in R'_i$, then the corresponding hyperplane has equation $\langle x+\rho,\alpha^\vee\rangle = p(\langle\nu,\alpha^\vee\rangle + i)$. If $\beta \in R^+$, then this hyperplane is in $\mathcal{H}(\beta)$ if and only if $\alpha \in R(\beta) \cup \{\beta\}$ where

$$R(\beta) = \{\alpha \in R^+ \mid s_\beta\alpha < 0, \alpha \neq \beta\}. \tag{1}$$

Set for all $w \in W$ and $\beta \in R^+$

$$\kappa_0(w,\beta) = \prod_{\substack{\alpha\in R(\beta), w^{-1}\alpha\in R'_0,\\ w^{-1}\alpha>0}} h_{-\alpha} \prod_{\substack{\alpha\in R(\beta), w^{-1}\alpha\in R'_0,\\ w^{-1}\alpha<0}} h_\alpha^{-1} \tag{2}$$

and

$$\kappa_1(w,\beta) = \prod_{\alpha\in R(\beta), w^{-1}\alpha\in R'_{-1}} h_{-\alpha} \prod_{\alpha\in R(\beta), w^{-1}\alpha\in R'_1} h_\alpha^{-1}. \tag{3}$$

The discussion above shows: If $\lambda' = w_\bullet\lambda + p\nu$ and $\mu' = w_\bullet\lambda + p\nu$, then

$$C^\beta(\lambda',\mu') = \begin{cases} \kappa_0(w,\beta)\kappa_1(w,\beta), & \text{if } w^{-1}\beta \notin R'; \\ \kappa_0(w,\beta)\kappa_1(w,\beta)h_\beta^{-1}, & \text{if } w^{-1}\beta \in (R'_0 \cap (-R^+)) \cup R'_1. \end{cases} \tag{4}$$

Note that the conditions on $w^{-1}\beta$ cover all cases where μ' is in the upper closure of the alcove of λ' for $W_{\beta,p}$.

13.13. Fix (until 13.22) $w \in W$ and $\nu \in \mathbf{Z}R$, set $\lambda' = w_\bullet\lambda + p\nu$ and $\mu' = w_\bullet\mu + p\nu = \lambda'_\Gamma$. For all $\beta \in R^+$ set — with $\kappa(\beta)$ as in 13.2(4) —

$$K(w,\beta) = \begin{cases} \kappa(\beta), & \text{if } w^{-1}\beta > 0; \\ h_\beta, & \text{if } w^{-1}\beta < 0. \end{cases} \tag{1}$$

So we can rewrite 13.2(3) uniformly as

$$b^\beta(\lambda') = \varepsilon^\beta_{w_\bullet\lambda,-\rho} d(w_\bullet\lambda, -\rho, s_\beta)K(w,\beta).$$

If we combine Lemma 12.13 with the formula (that follows from A.7(3))

$$c_{s_\beta}(w_\bullet\mu, w_\bullet\lambda, -\rho)d(w_\bullet\lambda, -\rho, s_\beta)d(w_\bullet\mu, w_\bullet\lambda, s_\beta) = d(w_\bullet\mu, -\rho, s_\beta),$$

we get

$$t_0^\beta[\mu', \lambda']e^\beta(\lambda') = b_0(w,\beta)e_0^\beta(\mu') \tag{2}$$

where

$$b_0(w,\beta) = \varepsilon^\beta_{w_\bullet\lambda, w_\bullet\mu}\varepsilon^\beta_{w_\bullet\lambda,-\rho}\frac{d(w_\bullet\mu, -\rho, s_\beta)K(w,\beta)}{c_{s_\beta}(w_\bullet\mu, w_\bullet\lambda, -\rho)} \tag{3}$$

whenever $t^\beta[\mu', \lambda']$ is defined.

Lemma: *We have*

$$\frac{d(w_\bullet\mu, -\rho, s_\beta)}{d(w_\bullet\mu, w_\bullet\sigma, s_\beta)} = \kappa_2(w,\beta) \prod_{\alpha\in M(w,\beta)} [k_\alpha; w(\mu+\rho)] \cdot \prod_{\alpha \in M'(w,\beta)} [k_\alpha; w(\mu+\rho)]^{-1}, \tag{4}$$

where

$$M(w,\beta) = \{\alpha > 0 \mid s_\beta\alpha < 0, w^{-1}\alpha < 0, w^{-1}\alpha \in R(-1)\} \tag{5}$$

and

$$M'(w,\beta) = \{\alpha > 0 \mid s_\beta\alpha < 0, w^{-1}\alpha > 0, w^{-1}\alpha \in R(1)\} \tag{6}$$

and

$$\kappa_2(w,\beta) = \prod_{\alpha\in M'(w,\beta)} h_\alpha \cdot \prod_{\alpha\in M(w,\beta)} h_\alpha^{-1}. \tag{7}$$

Proof: The definition of $R(1)$ implies

$$d(w_\bullet\mu, w_\bullet\sigma, s_\beta) = \prod_{\substack{\alpha>0, s_\beta\alpha<0, \\ w^{-1}\alpha\in R(1)}} \frac{[k_\alpha; w(\mu+\rho)]}{h_\alpha}. \tag{8}$$

On the other hand,

$$d(w_\bullet\mu, -\rho, s_\beta) = \prod_{\substack{\alpha>0, s_\beta\alpha<0, \\ w^{-1}\alpha<0, w^{-1}\alpha\notin R'}} \frac{[k_\alpha; w(\mu+\rho)]}{h_\alpha}, \tag{9}$$

since we get a factor equal to 1 for $w^{-1}\alpha \in R'$. When we take the quotient, all factors with $w^{-1}\alpha < 0, w^{-1}\alpha \in R(1)$ cancel and the claim follows.

13.14. Lemma: *Set $a_\Gamma(w)$ equal to*

$$\frac{z_{w^{-1}}(\lambda,\mu,\sigma)}{z_{w^{-1}}(\lambda,\sigma,-\rho)}\cdot wd(\mu,\lambda,w^{-1}) \prod_{\substack{\alpha>0,w^{-1}\alpha<0,\\ w^{-1}\alpha\in R'_0}} h_\alpha^{-1} \prod_{\substack{\alpha>0,w\alpha<0,\\ \alpha\in R(1)}} w([k_\alpha;\mu+\rho])^{-1}. \tag{1}$$

This product is a unit in A.

Proof: Recall that

$$d(\mu,\lambda,w^{-1}) = \prod_{\alpha>0,w\alpha<0} \left(\frac{[k_\alpha;\mu+\rho]}{[k_\alpha;\lambda+\rho]}\right)^{\overline{\alpha}(\mu-\lambda)}. \tag{2}$$

All factors in the denominator are units in A; so are those in the numerator for $\alpha\notin R'$. For $\alpha\in R'$ we get a factor of h_α in the numerator.

We have $\langle\mu-\lambda,\alpha^\vee\rangle = -\langle\lambda+\rho,\alpha^\vee\rangle$ for all $\alpha\in R'_0$, hence

$$\overline{\alpha}(\mu-\lambda)=1 \qquad \text{for all } \alpha\in R'_0\cap R^+.$$

On the other hand $\langle\mu-\lambda,\alpha^\vee\rangle = p-\langle\lambda+\rho,\alpha^\vee\rangle$ for all $\alpha\in R'_1$, hence

$$\overline{\alpha}(\mu-\lambda)=0 \qquad \text{for all } \alpha\in R'_1.$$

This shows that

$$d(\mu,\lambda,w^{-1}) \prod_{\substack{\alpha>0,w\alpha<0,\\ \alpha\in R'_0}} h_\alpha^{-1} \quad\text{and}\quad w\big(d(\mu,\lambda,w^{-1})\big)\cdot \prod_{\substack{\alpha>0,w^{-1}\alpha<0,\\ w^{-1}\alpha\in R'_0}} h_\alpha^{-1}$$

are units in A. So are obviously the remaining factors in $a_\Gamma(w)$.

13.15. We set now $a_\Gamma(\lambda')=a_\Gamma(w)$ and use these units to construct $t^\beta[\mu',\lambda']$ as decribed in 13.4. We get then (whenever $t^\beta[\mu',\lambda']$ is defined)

$$t^\beta[\mu',\lambda']e^\beta(\lambda') = b_1(w,\beta)e_0^\beta(\mu') \tag{1}$$

where — with $b_0(w,\beta)$ as in 13.13(3) —

$$b_1(w,\beta) = a_\Gamma(w)\big(s_\beta a_\Gamma(s_\beta w)\big)^{-1}b_0(w,\beta). \tag{2}$$

Lemma: *Suppose that $t^\beta[\mu',\lambda']$ is defined, i.e., that $w^{-1}\beta\notin R'_0\cap R^+, w^{-1}\beta\notin R'_{-1}$. If $w^{-1}\beta\notin R'_0$, then $b_1(w,\beta)$ is equal to*

$$(-1)^{|M(w,\beta)|}\varepsilon^\beta_{w\bullet\sigma,-\rho}\varepsilon^\beta_{w\bullet\mu,w\bullet\sigma}\kappa_0(w,\beta)\kappa_2(w,\beta)K(w,\beta)d(w\bullet\mu,w\bullet\sigma,s_\beta); \tag{3}$$

otherwise we get

$$b_1(w,\beta) = (-1)^{|M(w,\beta)|}\varepsilon^\beta_{w\bullet\sigma,-\rho}\varepsilon^\beta_{w\bullet\mu,w\bullet\sigma}\kappa_0(w,\beta)\kappa_2(w,\beta)d(w\bullet\mu,w\bullet\sigma,s_\beta). \tag{4}$$

Proof: We are going to decompose

$$a_\Gamma(w)\big(s_\beta a_\Gamma(s_\beta w)\big)^{-1} = a_1a_2a_3a_4$$

with the single factors as below in (5) and (7)–(9). First of all, the z terms from 13.14(1) yield a contribution

$$a_1 = z_{w^{-1}}(\lambda,\mu,\sigma)z_{(s_\beta w)^{-1}}(\lambda,\mu,\sigma)z_{w^{-1}}(\lambda,\sigma,-\rho)z_{(s_\beta w)^{-1}}(\lambda,\sigma,-\rho).$$

By A.14(3) this product is equal to

$$z_{s_\beta}(w{\bullet}\lambda, w{\bullet}\mu, w{\bullet}\sigma)z_{s_\beta}(w{\bullet}\lambda, w{\bullet}\sigma,-\rho),$$

so we get by 12.12(6)

$$a_1 = \varepsilon^\beta_{w\bullet\lambda,w\bullet\mu}\varepsilon^\beta_{w\bullet\lambda,-\rho}\varepsilon^\beta_{w\bullet\mu,w\bullet\sigma}\varepsilon^\beta_{w\bullet\sigma,-\rho}. \tag{5}$$

This implies, cf. 13.13(3),

$$a_1b_0(w,\beta) = \varepsilon^\beta_{w\bullet\mu,w\bullet\sigma}\varepsilon^\beta_{w\bullet\sigma,-\rho}\frac{d(w{\bullet}\mu,-\rho,s_\beta)K(w,\beta)}{c_{s_\beta}(w{\bullet}\mu,w{\bullet}\lambda,-\rho)}. \tag{6}$$

We have

$$w\big(d(\mu,\lambda,w^{-1})\cdot d(\mu,\lambda,w^{-1}s_\beta)^{-1}\big) = c_{s_\beta}(w{\bullet}\mu,w{\bullet}\lambda,-\rho)$$

by A.11(3). We merge this with a possible factor h_β from the product of the h_α^{-1} and set

$$a_2 = \begin{cases} c_{s_\beta}(w{\bullet}\mu,w{\bullet}\lambda,-\rho), & \text{if } w^{-1}\beta\notin R_0';\\ c_{s_\beta}(w{\bullet}\mu,w{\bullet}\lambda,-\rho)h_\beta^{-1}, & \text{if } w^{-1}\beta\in R_0'\cap(-R^+).\end{cases} \tag{7}$$

(Recall that we exclude the case $w^{-1}\beta\in R_0'\cap R^+$.) After having removed possible factors equal to $h_\beta^{\pm1}$ the h_α products yield a contribution a_3 equal to

$$\prod_{\substack{\alpha>0,w^{-1}\alpha<0,\\ w^{-1}\alpha\in R_0',\alpha\neq\beta}} h_\alpha^{-1}\prod_{\substack{\alpha>0,w^{-1}s_\beta\alpha<0,\\ w^{-1}s_\beta\alpha\in R_0'}} h_{s_\beta\alpha} = \prod_{\substack{\alpha>0,w^{-1}\alpha<0,\\ w^{-1}\alpha\in R_0',\alpha\neq\beta}} h_\alpha^{-1}\prod_{\substack{s_\beta\alpha>0,w^{-1}\alpha<0,\\ w^{-1}\alpha\in R_0'}} h_\alpha.$$

We cancel now the common factors (where $\alpha>0$ and $s_\beta\alpha>0$) and substitute $-\alpha$ for α in the second product. We get thus

$$a_3 = \prod_{\substack{\alpha\in R(\beta),\\ w^{-1}\alpha>0,w^{-1}\alpha\in R_0'}} h_{-\alpha}\prod_{\substack{\alpha\in R(\beta),\\ w^{-1}\alpha<0,w^{-1}\alpha\in R_0'}} h_\alpha^{-1} = \kappa_0(w,\beta). \tag{8}$$

Our last contribution is

$$a_4 = \prod_{\substack{\alpha>0, s_\beta w\alpha<0,\\ \alpha\in R(1)}} w[k_\alpha; \mu+\rho] \prod_{\substack{\alpha>0, w\alpha<0,\\ \alpha\in R(1)}} w[k_\alpha; \mu+\rho]^{-1}.$$

We can rewrite this product using 13.1(6) and get

$$a_4 = \prod_{\substack{w^{-1}\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha\in R(1)}} [k_\alpha; w(\mu+\rho)] \prod_{\substack{w^{-1}\alpha>0, \alpha<0,\\ w^{-1}\alpha\in R(1)}} [k_\alpha; w(\mu+\rho)]^{-1}.$$

All factors with $\alpha < 0, s_\beta\alpha < 0$ cancel. We get thus

$$a_4 = \prod_{\alpha\in M'(w,\beta)} [k_\alpha; w(\mu+\rho)] \prod_{-\alpha\in M(w,\beta)} [k_\alpha; w(\mu+\rho)]^{-1},$$

hence by 13.1(5)

$$a_4 = (-1)^{|M(w,\beta)|} \prod_{\alpha\in M'(w,\beta)} [k_\alpha; w(\mu+\rho)] \prod_{\alpha\in M(w,\beta)} [k_\alpha; w(\mu+\rho)]^{-1}.$$

Now Lemma 13.13 implies

$$a_4 d(w{\bullet}\mu, -\rho, s_\beta) = (-1)^{|M(w,\beta)|}\kappa_2(w,\beta) d(w{\bullet}\mu, w{\bullet}\sigma, s_\beta). \tag{9}$$

The claim follows now from (2) and (6)–(9). (Recall that $K(w,\beta) = h_\beta$ in case $w^{-1}\beta \in R'_0 \cap (-R^+)$, cf. 13.13(1).)

13.16. In case $w^{-1}\beta \notin R'$ set

$$b(w,\beta) = (-1)^{|M(w,\beta)|}\varepsilon^\beta_{w\bullet\sigma,-\rho}\kappa_2(w,\beta)\kappa_1(w,\beta)^{-1}K(w,\beta). \tag{1}$$

We have then by 13.15(1), (3)

$$t^\beta[\mu',\lambda']e^\beta(\lambda') = b(w,\beta)\varepsilon^\beta_{w\bullet\mu,w\bullet\sigma}\kappa_0(w,\beta)\kappa_1(w,\beta)d(w{\bullet}\mu, w{\bullet}\sigma, s_\beta)e^\beta_0(\mu'). \tag{2}$$

If $w^{-1}\beta \in R'$, then $\beta\uparrow\mu' = \mu'$. We can choose $w{\bullet}\sigma + p\nu$ as the weight playing the role of ω (for μ') in 12.13(1). We get

$$e^\beta_0(\mu') = d(w{\bullet}\sigma, w{\bullet}\mu, s_\beta)h_\beta^{-1} + A^\beta,$$

since $\varepsilon^{\beta}_{w\bullet\mu,w\bullet\sigma} = 1$, cf. Remark 1 in 12.12. We have $s_\beta\bullet(w\bullet\mu) = w\bullet\mu + p\beta$ and $s_\beta\bullet(w\bullet\sigma) = w\bullet\sigma + p\beta$, so Lemma A.8 implies

$$d(w\bullet\sigma, w\bullet\mu, s_\beta) = s_\beta d(w\bullet\mu, w\bullet\sigma, s_\beta)^{-1}.$$

This is a unit in A^β. We get therefore

$$e_0^\beta(\mu') = d(w\bullet\mu, w\bullet\sigma, s_\beta)^{-1} h_\beta^{-1} + A^\beta.$$

So 13.15(1)–(4) imply in this case

$$t^\beta[\mu', \lambda'] e^\beta(\lambda') = b(w,\beta)\kappa_0(w,\beta)\kappa_1(w,\beta) h_\beta^{-1} + A^\beta \tag{3}$$

where

$$b(w,\beta) = \begin{cases} (-1)^{|M(w,\beta)|}\varepsilon^{\beta}_{w\bullet\sigma,-\rho}\kappa_2(w,\beta)\kappa_1(w,\beta)^{-1}, \\ (-1)^{|M(w,\beta)|}\varepsilon^{\beta}_{w\bullet\sigma,-\rho}\kappa_2(w,\beta)\kappa_1(w,\beta)^{-1}\kappa(\beta), \end{cases} \tag{4}$$

if $w^{-1}\beta \in R'_0 \cap (-R^+)$ resp. if $w^{-1}\beta \in R'_1$. We claim that Theorem 13.4 will follow in our present case if we show:

Proposition: *a) If $w^{-1}\beta \notin R'$, then $b(ws_\gamma, \beta) \equiv b(w,\beta) \pmod{h_\beta A^\beta}$ for all $\gamma \in R'$.*

b) If $w^{-1}\beta \in (R'_0 \cap (-R^+)) \cup R'_1$, then $b(w,\beta) \equiv 1 \pmod{h_\beta A^\beta}$.

Indeed, in the situation of b) the claim follows then from (3), cf. 13.12(4). For $w^{-1}\beta \notin R'$ we want to define

$$e^\beta(\mu') = b(w,\beta)\, \varepsilon^{\beta}_{w\bullet\mu,w\bullet\sigma}\, d(w\bullet\mu, w\bullet\sigma, s_\beta)\, e_0^\beta(\mu').$$

Then 13.4(7) will follow from (2). However, we have to make sure that $e^\beta(\mu')$ depends only on μ', not on the special choice of w and ν with $\mu' = w\bullet\mu + p\nu$. We have by 13.10(4) and A.5(1)

$$\varepsilon^{\beta}_{ws_\gamma\bullet\mu,ws_\gamma\bullet\sigma}\, d(ws_\gamma\bullet\mu, ws_\gamma\bullet\sigma, s_\beta) = \varepsilon^{\beta}_{w\bullet\mu,w\bullet\sigma}\, d(w\bullet\mu, w\bullet\sigma, s_\beta) \tag{5}$$

for all $\gamma \in R'$. Recall the description of the stabilizer of μ in W_p in 13.3. It shows that the independence follows from a) and (5).

Remark: Note that (as in 13.5) the $e^\beta(\mu')$ are independent of the choice of Ω.

13.17. Since $s_\gamma = s_{-\gamma}$ it is enough to prove 13.16.a for $\gamma \in (R'_0 \cap (-R^+)) \cup R'_1$. In this subsection we want to show that 13.16.a holds for $\gamma \in R'_0 \cap (-R^+)$ and that 13.16.b holds in case $w^{-1}\beta \in R'_0 \cap (-R^+)$

Proof: Let $\gamma \in R'_0 \cap (-R^+)$. Since $\mu+\rho$ is dominant, s_γ permutes the positive roots not in R'_0. Combining this with 13.11(3) we see that $M(ws_\gamma, \beta) = M(w, \beta)$ and $M'(ws_\gamma, \beta) = M'(w, \beta)$, hence that

$$(-1)^{|M(ws_\gamma,\beta)|}\kappa_2(ws_\gamma, \beta) = (-1)^{|M(w,\beta)|}\kappa_2(w, \beta).$$

When dealing with 13.16.a, we assume $w^{-1}\beta \notin R'$; so the argument above implies that $w^{-1}\beta > 0$ if and only if $s_\gamma w^{-1}\beta > 0$, hence that $K(ws_\gamma, \beta) = K(w, \beta)$. Furthermore 13.10(3) shows that $\kappa_1(ws_\gamma, \beta) = \kappa_1(w, \beta)$. Finally, we have $s_\gamma \bullet \sigma = \sigma$, hence $\varepsilon^\beta_{ws_\gamma \bullet \sigma, -\rho} = \varepsilon^\beta_{w\bullet\sigma,-\rho}$. So 13.16.a is satisfied for $\gamma \in R'_0$.

Assume now that $w^{-1}\beta = \gamma$. We have then

$$w^{-1}(-s_\beta \alpha) = -s_\gamma w^{-1}\alpha \qquad \text{for all } \alpha \in R. \tag{1}$$

The map $\alpha \mapsto -s_\beta\alpha$ permutes $R(\beta)$. It takes $\{\alpha \in R(\beta) \mid w^{-1}\alpha \in R'_1\}$ to $\{\alpha \in R(\beta) \mid w^{-1}\alpha \in R'_{-1}\}$ by (1) and 13.10(3). This implies

$$\kappa_1(w, \beta) = \prod_{\alpha \in R(\beta), w^{-1}\alpha \in R'_1} (h_{s_\beta\alpha} \cdot h_\alpha^{-1}) \equiv 1 \pmod{h_\beta}.$$

Furthermore, (1) and 13.11(3) and the fact that s_γ permutes the positive roots not in R'_0 show that $\alpha \mapsto -s_\beta\alpha$ maps $M(w, \beta)$ to $M'(w, \beta)$, hence

$$(-1)^{|M(w,\beta)|}\kappa_2(w, \beta) = \prod_{\alpha \in M(w,\beta)} (h_{-s_\beta\alpha} \cdot h_{-\alpha}^{-1}) \equiv 1 \pmod{h_\beta}.$$

Finally, we have

$$\langle w(\sigma+\rho), \beta^\vee \rangle = \langle \sigma + \rho, \gamma^\vee \rangle = 0,$$

hence $\varepsilon^\beta_{w\bullet\sigma,-\rho} = 1$ by Remark 1 in 12.12. So 13.16.b holds for $w^{-1}\beta \in R'_0 \cap (-R^+)$.

13.18. It remains to prove 13.16.a for $\gamma \in R'_1$ and 13.16.b for $w^{-1}\beta \in R'_1$. This will be done in the next subsections. The proof in the first case will be concluded in 13.20, that in the second one in 13.22.

Let $\gamma \in R'_1$. We have $0 < \langle \mu + \rho, \alpha^\vee \rangle < p$ for all $\alpha \in R^+$, $\alpha \notin R'$; so $s_\gamma \alpha^\vee = \alpha^\vee - \langle \gamma, \alpha^\vee \rangle \gamma^\vee$ implies

$$-\langle \gamma, \alpha^\vee \rangle p < \langle \mu + \rho, s_\gamma \alpha^\vee \rangle < (1 - \langle \gamma, \alpha^\vee \rangle) p.$$

We have therefore either $\langle\gamma,\alpha^\vee\rangle = 0$ and $s_\gamma\alpha = \alpha > 0$ or $\langle\gamma,\alpha^\vee\rangle = 1$ and $s_\gamma\alpha < 0$. If $\alpha \in -R^+$, $\alpha \notin R'$, then we can apply this argument to $-\alpha$. We see that either $\langle\gamma,\alpha^\vee\rangle = 0$ and $s_\gamma\alpha = \alpha < 0$ or $\langle\gamma,\alpha^\vee\rangle = -1$ and $s_\gamma\alpha > 0$. So we get for all $\alpha \in R, \alpha \notin R'$:

$$\langle\gamma,\alpha^\vee\rangle = \begin{cases} 1, & \text{if } \alpha > 0, s_\gamma\alpha < 0; \\ -1, & \text{if } \alpha < 0, s_\gamma\alpha > 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Lemma: *a) If $w^{-1}\beta \notin R'$, then for all $\gamma \in R'_1$*

$$\frac{\kappa_2(ws_\gamma,\beta)K(ws_\gamma,\beta)}{\kappa_2(w,\beta)K(w,\beta)} = \kappa(\beta)^{-\langle w\gamma,\beta^\vee\rangle} \prod_{\alpha\in R(\beta), w^{-1}\alpha\notin R'} h_\alpha^{-\langle w\gamma,\alpha^\vee\rangle} \tag{2}$$

and

$$(-1)^{|M(ws_\gamma,\beta)|} = (-1)^{|M(w,\beta)|} \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha\in R(-1)}} (-1)^{\langle w\gamma,\alpha^\vee\rangle}. \tag{3}$$

b) If $w^{-1}\beta \in R'_1$, then

$$(-1)^{|M(w,\beta)|} = 1 \qquad \textit{and} \qquad \kappa_2(w,\beta) = \prod_{\alpha\in R(\beta), w^{-1}\alpha\in R(1)} h_\alpha^{\langle\beta,\alpha^\vee\rangle}. \tag{4}$$

Proof: In both situations — a) and b) — (1) yields for all $\gamma \in R'_1$

$$\begin{aligned} M'(w,\beta) = \{&\alpha > 0 \mid s_\beta\alpha < 0, w^{-1}\alpha \in R(1), w^{-1}\alpha > 0, \langle w\gamma,\alpha^\vee\rangle = 0\} \\ &\cup \{\alpha > 0 \mid s_\beta\alpha < 0, w^{-1}\alpha \in R(1), \langle w\gamma,\alpha^\vee\rangle = 1\} \end{aligned}$$

and — since $\langle ws_\gamma\gamma,\alpha^\vee\rangle = -\langle w\gamma,\alpha^\vee\rangle$ and since s_γ permutes $R(1)$ —

$$\begin{aligned} M'(ws_\gamma,\beta) = \{&\alpha > 0 \mid s_\beta\alpha < 0, w^{-1}\alpha \in R(1), w^{-1}\alpha > 0, \langle w\gamma,\alpha^\vee\rangle = 0\} \\ &\cup \{\alpha > 0 \mid s_\beta\alpha < 0, w^{-1}\alpha \in R(1), \langle w\gamma,\alpha^\vee\rangle = -1\}, \end{aligned}$$

hence

$$\prod_{\alpha\in M'(ws_\gamma,\beta)} h_\alpha = \prod_{\alpha\in M'(w,\beta)} h_\alpha \cdot \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha\in R(1)}} h_\alpha^{-\langle w\gamma,\alpha^\vee\rangle}. \tag{5}$$

We have similarly

$$\begin{aligned} M(w,\beta) = \{&\alpha > 0 \mid s_\beta\alpha < 0, w^{-1}\alpha \in R(-1), w^{-1}\alpha < 0, \langle w\gamma,\alpha^\vee\rangle = 0\} \\ &\cup \{\alpha > 0 \mid s_\beta\alpha < 0, w^{-1}\alpha \in R(-1), \langle w\gamma,\alpha^\vee\rangle = -1\} \end{aligned}$$

and

$$\begin{aligned} M(ws_\gamma,\beta) = \{\alpha > 0 \mid s_\beta\alpha < 0, w^{-1}\alpha \in R(-1), w^{-1}\alpha < 0, \langle w\gamma, \alpha^\vee\rangle = 0\} \\ \cup \{\alpha > 0 \mid s_\beta\alpha < 0, w^{-1}\alpha \in R(-1), \langle w\gamma, \alpha^\vee\rangle = 1\}, \end{aligned}$$

hence

$$\prod_{\alpha\in M(ws_\gamma,\beta)} h_\alpha = \prod_{\alpha\in M(w,\beta)} h_\alpha \cdot \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha\in R(-1)}} h_\alpha^{\langle w\gamma,\alpha^\vee\rangle}. \tag{6}$$

Also (3) follows immediately. Combining (5) and (6) we get

$$\kappa_2(ws_\gamma,\beta) = \kappa_2(w,\beta) \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha\notin R'}} h_\alpha^{-\langle w\gamma,\alpha^\vee\rangle}.$$

On the other hand, if we apply (1) to $\alpha = w^{-1}\beta$, we get

$$K(ws_\gamma,\beta) = K(w,\beta)(\kappa(\beta)^{-1}h_\beta)^{\langle w\gamma,\beta^\vee\rangle} \qquad \text{if } w^{-1}\beta \notin R', \tag{7}$$

hence (2).

It remains to show (4). So suppose now that $w^{-1}\beta \in R'_1$ and apply the formulas above with $\gamma = w^{-1}\beta$. We have $\langle\alpha,\beta^\vee\rangle > 0$ for all $\alpha > 0$ with $s_\beta\alpha < 0$, hence $\langle w\gamma,\alpha^\vee\rangle = \langle\beta,\alpha^\vee\rangle > 0$. Now these formulas show that $M(w,\beta) = \emptyset$ and that all $\alpha > 0$ with $s_\beta\alpha < 0$ and $w^{-1}\alpha \notin R'$ satisfy $\langle\beta,\alpha^\vee\rangle = 1$. This yields (4).

13.19. Set for all $\omega \in X$ and $\beta \in R^+$

$$z_a(\omega,\beta) = \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha\notin R'}} (-1)^{\langle\omega,\alpha^\vee\rangle\overline{\alpha}(\omega)}, \tag{1}$$

$$z_b(\omega,\beta) = \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha\in R'}} (-1)^{\langle\omega,\alpha^\vee\rangle\overline{\alpha}(\omega)}, \tag{1'}$$

and (recall w_β from the discussion preceding 13.1)

$$z_c(\omega,\beta) = \prod_{\alpha\in R(\beta), w_\beta^{-1}\alpha>0} (-1)^{\langle\omega,\alpha^\vee\rangle}. \tag{2}$$

Note that 12.12(5) implies for all $\omega,\ \omega' \in X$ that

$$\varepsilon^{\beta}_{\omega\omega'} = z_a(\omega-\omega',\beta) z_b(\omega-\omega',\beta) z_c(\omega-\omega',\beta). \tag{3}$$

We have obviously $z_c(\omega+\omega',\beta) = z_c(\omega,\beta)z_c(\omega',\beta)$ for all $\omega,\omega' \in X$. We want to apply (3) to $\varepsilon^{\beta}_{ws_\gamma\bullet\sigma,-\rho}$ and $\varepsilon^{\beta}_{w\bullet\sigma,-\rho}$. Note that Lemma 13.11 implies

$$z_a(w(\sigma+\rho),\beta) = \prod_{\substack{\alpha>0,s_\beta\alpha<0,\\ w^{-1}\alpha\in R(-1)}} (-1)^{\langle w(\sigma+\rho),\alpha^\vee\rangle} \tag{4}$$

and (since p is odd)

$$z_b(w(\sigma+\rho),\beta) = \prod_{\substack{\alpha>0,s_\beta\alpha<0,\\ w^{-1}\alpha\in R'_{-1}}} (-1). \tag{5}$$

Lemma: *We have for all* $\gamma \in R'_1$

$$\frac{(-1)^{|M(ws_\gamma,\beta)|}\,\varepsilon^{\beta}_{ws_\gamma\bullet\sigma,-\rho}\,\kappa_1(w,\beta)}{(-1)^{|M(w,\beta)|}\,\varepsilon^{\beta}_{w\bullet\sigma,-\rho}\,\kappa_1(ws_\gamma,\beta)} = z_c(-w\gamma,\beta) \prod_{\substack{\alpha\in R(\beta),\\ w^{-1}\alpha\in R'}} h_\alpha^{-\langle w\gamma,\alpha^\vee\rangle}. \tag{6}$$

Proof: First of all, since $s_\gamma(\sigma+\rho) = \sigma+\rho-p\gamma$, we get

$$\begin{aligned} z_c(ws_\gamma(\sigma+\rho),\beta) &= z_c(w(\sigma+\rho),\beta)z_c(-pw\gamma,\beta)\\ &= z_c(w(\sigma+\rho),\beta)z_c(-w\gamma,\beta). \end{aligned} \tag{7}$$

This leads to the factor $z_c(-w\gamma,\beta)$ on the right hand side of (6).

Furthermore, we have $s_\gamma R(-1) = R(-1)$ and $s_\gamma(\sigma+\rho) = \sigma+\rho-p\gamma$, so (4) yields

$$\begin{aligned} z_a(ws_\gamma(\sigma+\rho),\beta) &= \prod_{\substack{\alpha>0,s_\beta\alpha<0,\\ w^{-1}\alpha\in R(-1)}} (-1)^{\langle ws_\gamma(\sigma+\rho),\alpha^\vee\rangle}\\ &= \prod_{\substack{\alpha>0,s_\beta\alpha<0,\\ w^{-1}\alpha\in R(-1)}} (-1)^{\langle w(\sigma+\rho),\alpha^\vee\rangle-\langle w\gamma,\alpha^\vee\rangle}\\ &= z_a(w(\sigma+\rho),\beta) \prod_{\substack{\alpha>0,s_\beta\alpha<0,\\ w^{-1}\alpha\in R(-1)}} (-1)^{\langle w\gamma,\alpha^\vee\rangle}. \end{aligned}$$

Now 13.18(3) implies

$$(-1)^{|M(ws_\gamma,\beta)|} z_a(ws_\gamma(\sigma+\rho),\beta) = (-1)^{|M(w,\beta)|} z_a(w(\sigma+\rho),\beta). \tag{8}$$

Finally, we have by 13.19(5) (since we assume $w^{-1}\beta \notin R'_{-1}$)

$$\begin{aligned}\kappa_1(w,\beta) z_b(w(\sigma+\rho),\beta)^{-1} &= \prod_{\alpha\in R(\beta),w^{-1}\alpha\in R'_{-1}} h_\alpha \prod_{\alpha\in R(\beta),w^{-1}\alpha\in R'_1} h_\alpha^{-1} \\ &= \prod_{\alpha\in R(\beta),w^{-1}\alpha\in R'} h_\alpha^{-\langle w(\mu+\rho),\alpha^\vee\rangle/p}.\end{aligned} \tag{9}$$

Since $-\langle ws_\gamma(\mu+\rho),\alpha^\vee\rangle = -\langle w(\mu+\rho),\alpha^\vee\rangle + p\langle w\gamma,\alpha^\vee\rangle$, we get

$$\frac{\kappa_1(w,\beta) z_b(ws_\gamma(\sigma+\rho),\beta)}{\kappa_1(ws_\gamma,\beta) z_b(w(\sigma+\rho),\beta)} = \prod_{\alpha\in R(\beta),w^{-1}\alpha\in R'} h_\alpha^{-\langle w\gamma,\alpha^\vee\rangle}. \tag{10}$$

Now (7), (8), and (10) yield (6) by (3).

13.20. Set for all $\omega \in X$

$$h(\omega,\beta) = \prod_{\alpha\in R(\beta)} h_\alpha^{\langle\omega,\alpha^\vee\rangle}. \tag{1}$$

We have obviously $h(\omega+\omega',\beta) = h(\omega,\beta)h(\omega',\beta)$ for all $\omega,\omega' \in X$. Using this notation, 13.16(1) together with 13.18(2) and 13.19(6) yields for all $\gamma \in R'_1$

$$\frac{b(ws_\gamma,\beta)}{b(w,\beta)} = z_c(-w\gamma,\beta)\kappa(\beta)^{-\langle w\gamma,\beta^\vee\rangle} h(-w\gamma,\beta) \qquad \text{if } w^{-1}\beta \notin R'. \tag{2}$$

Note that $\kappa(\beta)$ and all $h(\omega,\beta)$ are units in A^β.

Lemma: *We have for all* $\omega \in X$

$$h(\omega,\beta) \equiv \kappa(\beta)^{-\langle\omega,\beta^\vee\rangle} z_c(\omega,\beta) \pmod{h_\beta A^\beta}. \tag{3}$$

Proof: The map $\alpha \mapsto -s_\beta\alpha$ permutes $R(\beta)$. We have $w_\beta^{-1}\alpha > 0$ if and only if $w_\beta^{-1}(-s_\beta\alpha) < 0$ (for $\alpha \neq \beta$). We can therefore rewrite (1) as

$$h(\omega,\beta) = \prod_{\alpha\in R(\beta),w_\beta^{-1}\alpha<0} h_\alpha^{\langle\omega,\alpha^\vee\rangle} h_{-s_\beta\alpha}^{-\langle\omega,s_\beta\alpha^\vee\rangle}.$$

We have $h_{-s_\beta\alpha} \equiv h_{-\alpha} = -h_\alpha$ modulo h_β, hence

$$h(\omega,\beta) \equiv \prod_{\alpha\in R(\beta), w_\beta^{-1}\alpha<0} h_\alpha^{\langle\omega,\alpha^\vee\rangle} h_\alpha^{-\langle\omega,s_\beta\alpha^\vee\rangle}(-1)^{-\langle\omega,s_\beta\alpha^\vee\rangle}$$

$$= \prod_{\alpha\in R(\beta), w_\beta^{-1}\alpha<0} h_\alpha^{\langle\omega,\alpha^\vee\rangle} h_\alpha^{-\langle s_\beta\omega,\alpha^\vee\rangle} \prod_{\alpha\in R(\beta), w_\beta^{-1}\alpha>0} (-1)^{\langle\omega,\alpha^\vee\rangle}.$$

The last product is the wanted $z_c(\omega,\beta)$. We have $s_\beta\omega = \omega - \langle\omega,\beta^\vee\rangle\beta$, hence

$$h_\alpha^{-\langle s_\beta\omega,\alpha^\vee\rangle} = h_\alpha^{-\langle\omega,\alpha^\vee\rangle}(h_\alpha^{\langle\beta,\alpha^\vee\rangle})^{\langle\omega,\beta^\vee\rangle}.$$

Plug this into the last equation and the claim follows.

Remarks: 1) In our first formula in the proof we might as well replace the condition $w_\beta^{-1}\alpha < 0$ by $w_\beta^{-1}\alpha > 0$. Then the same calculation as above shows that modulo $h_\beta A^\beta$

$$h(\omega,\beta) \equiv \Bigg(\prod_{\alpha\in R(\beta), w_\beta^{-1}\alpha>0} h_\alpha^{\langle\beta,\alpha^\vee\rangle}\Bigg)^{\langle\omega,\beta^\vee\rangle} \prod_{\alpha\in R(\beta), w_\beta^{-1}\alpha<0} (-1)^{\langle\omega,\alpha^\vee\rangle}. \qquad (4)$$

2) The lemma and (2) show that Proposition 13.16.a is satisfied for all $\gamma \in R_1'$.

13.21. Suppose that $w^{-1}\beta \in R_1'$.
Set for all $\omega \in X$

$$z_c'(\omega,\beta) = \prod_{\substack{\alpha\in R(\beta), w_\beta^{-1}\alpha>0,\\ w^{-1}\alpha\notin R'}} (-1)^{\langle\omega,\alpha^\vee\rangle} \qquad (1)$$

and

$$z_c''(\omega,\beta) = \prod_{\substack{\alpha\in R(\beta), w_\beta^{-1}\alpha>0,\\ w^{-1}\alpha\in R'}} (-1)^{\langle\omega,\alpha^\vee\rangle}. \qquad (2)$$

Obviously

$$z_c(\omega,\beta) = z_c'(\omega,\beta) z_c''(\omega,\beta). \qquad (3)$$

Lemma: *Suppose that $w^{-1}\beta \in R_1'$. Then (modulo h_β)*

$$\kappa_1(w,\beta) z_b(w(\sigma+\rho),\beta) z_c''(w(\sigma+\rho),\beta) \equiv \prod_{\substack{\alpha\in R(\beta), w^{-1}\alpha\in R',\\ w_\beta^{-1}\alpha<0}} h_\alpha^{-\langle\beta,\alpha^\vee\rangle}. \qquad (4)$$

Proof: Set $\gamma = w^{-1}\beta$. The map $\alpha \mapsto -s_\beta\alpha$ permutes $R(\beta)$. More precisely, it maps $\{\alpha \in R(\beta) \mid w^{-1}\alpha \in R'\}$ to itself and $\{\alpha \in R(\beta) \mid w^{-1}\alpha \in R(1)\}$ to $\{\alpha \in R(\beta) \mid w^{-1}\alpha \in R(-1)\}$, since $w^{-1}(-s_\beta\alpha) = -s_\gamma w^{-1}\alpha$ and since s_γ permutes each of R', $R(1)$ and $R(-1)$. We have $w_\beta^{-1}\alpha > 0$ if and only if $w_\beta^{-1}(-s_\beta\alpha) < 0$ (for $\alpha \neq \beta$), so we get using 13.19(9)

$$\kappa_1(w,\beta)z_b(w(\sigma+\rho),\beta) = \prod_{\substack{\alpha\in R(\beta),w^{-1}\alpha\in R',\\ w_\beta^{-1}\alpha>0}} h_\alpha^{-\langle w(\mu+\rho),\alpha^\vee\rangle/p}\, h_{-s_\beta\alpha}^{\langle w(\mu+\rho),s_\beta\alpha^\vee\rangle/p}.$$

We have $\langle w(\mu+\rho), s_\beta\alpha^\vee\rangle = \langle w(\mu+\rho),\alpha^\vee\rangle - p\langle\beta,\alpha^\vee\rangle$, so each of the second terms in this product is equal to

$$h_{-s_\beta\alpha}^{\langle w(\mu+\rho),\alpha^\vee\rangle/p}\, h_{-s_\beta\alpha}^{-\langle\beta,\alpha^\vee\rangle} = h_{s_\beta\alpha}^{\langle w(\mu+\rho),\alpha^\vee\rangle/p}\, h_{-s_\beta\alpha}^{\langle\beta,s_\beta\alpha^\vee\rangle}.$$

This implies that $\kappa_1(w,\beta)z_b(w(\sigma+\rho),\beta)$ is congruent to

$$\prod_{\substack{\alpha\in R(\beta),w^{-1}\alpha\in R',\\ w_\beta^{-1}\alpha>0}} (-1)^{\langle w(\mu+\rho),\alpha^\vee\rangle/p} \prod_{\substack{\alpha\in R(\beta),w^{-1}\alpha\in R',\\ w_\beta^{-1}\alpha<0}} h_\alpha^{-\langle\beta,\alpha^\vee\rangle}$$

modulo h_β. The first product is equal to $z_c''(w(\sigma+\rho),\beta)$: Apply 13.10(4) to $w^{-1}\alpha$ and recall that p is odd. The claim follows.

13.22. Lemma: *Suppose that* $w^{-1}\beta \in R_1'$. *Then*

$$\kappa_2(w,\beta)z_a(w(\sigma+\rho),\beta)z_c'(w(\sigma+\rho),\beta) \equiv \prod_{\substack{\alpha\in R(\beta),w^{-1}\alpha\notin R',\\ w_\beta^{-1}\alpha<0}} h_\alpha^{\langle\beta,\alpha^\vee\rangle} \pmod{h_\beta}.$$

Proof: By 13.19(4) and 13.21(1) the product of $z_a(w(\sigma+\rho),\beta)$ and $z_c'(w(\sigma+\rho),\beta)$ is equal to

$$\prod_{\substack{\alpha\in R(\beta),w_\beta^{-1}\alpha<0,\\ w^{-1}\alpha\in R(-1)}} (-1)^{\langle w(\sigma+\rho),\alpha^\vee\rangle} \prod_{\substack{\alpha\in R(\beta),w_\beta^{-1}\alpha>0,\\ w^{-1}\alpha\in R(1)}} (-1)^{\langle w(\sigma+\rho),\alpha^\vee\rangle},$$

since all terms with $w^{-1}\alpha \in R(-1), w_\beta^{-1}\alpha > 0$ cancel. Now $\alpha \mapsto -s_\beta\alpha$ takes one index set to the other one, and so — since p is odd — this product is equal to

$$\prod_{\substack{\alpha\in R(\beta),w^{-1}\alpha\in R(1),\\ w_\beta^{-1}\alpha>0}} (-1)^{\langle w(\sigma+\rho),\alpha^\vee\rangle-\langle w(\sigma+\rho),s_\beta\alpha^\vee\rangle}.$$

The exponent is equal to $p\langle\beta,\alpha^\vee\rangle$. This implies

$$z_a(w(\sigma+\rho),\beta)z'_c(w(\sigma+\rho),\beta) = \prod_{\substack{\alpha\in R(\beta), w^{-1}\alpha\in R(1),\\ w_\beta^{-1}\alpha>0}} (-1)^{\langle\beta,\alpha^\vee\rangle}. \tag{1}$$

We can rewrite the second part of 13.18(4) as

$$\begin{aligned}
\kappa_2(w,\beta) &= \prod_{\substack{\alpha\in R(\beta), w^{-1}\alpha\in R(1),\\ w_\beta^{-1}\alpha<0}} h_\alpha^{\langle\beta,\alpha^\vee\rangle} \prod_{\substack{\alpha\in R(\beta), w^{-1}\alpha\in R(1),\\ w_\beta^{-1}\alpha>0}} h_\alpha^{\langle\beta,\alpha^\vee\rangle}\\
&= \prod_{\substack{\alpha\in R(\beta), w^{-1}\alpha\in R(1),\\ w_\beta^{-1}\alpha<0}} h_\alpha^{\langle\beta,\alpha^\vee\rangle} \prod_{\substack{\alpha\in R(\beta), w^{-1}\alpha\in R(-1),\\ w_\beta^{-1}\alpha<0}} h_{-s_\beta\alpha}^{\langle\beta,\alpha^\vee\rangle}\\
&\equiv \prod_{\substack{\alpha\in R(\beta), w^{-1}\alpha\notin R',\\ w_\beta^{-1}\alpha<0}} h_\alpha^{\langle\beta,\alpha^\vee\rangle} \prod_{\substack{\alpha\in R(\beta), w^{-1}\alpha\in R(-1),\\ w_\beta^{-1}\alpha<0}} (-1)^{\langle\beta,\alpha^\vee\rangle}\\
&= \prod_{\substack{\alpha\in R(\beta), w^{-1}\alpha\notin R',\\ w_\beta^{-1}\alpha<0}} h_\alpha^{\langle\beta,\alpha^\vee\rangle} \prod_{\substack{\alpha\in R(\beta), w^{-1}\alpha\in R(1),\\ w_\beta^{-1}\alpha>0}} (-1)^{\langle\beta,\alpha^\vee\rangle}.
\end{aligned}$$

Combined with (1) this yields the claim.

Remark: We combine this lemma with 13.19(3) and 13.21(4), and get

$$\kappa_2(w,\beta)\kappa_1(w,\beta)^{-1}\varepsilon^\beta_{w\bullet\sigma,-\rho} \equiv \prod_{\alpha\in R(\beta), w_\beta^{-1}\alpha<0} h_\alpha^{\langle\beta,\alpha^\vee\rangle} = \kappa(\beta)^{-1} \pmod{h_\beta}.$$

Now the definition 13.16(4) together with the first part of 13.18(4) shows that $b(w,\beta)\equiv 1$, hence that 13.16.b holds.

13.23. We can apply Theorem 13.4 especially to $\mu=-\rho$. Then $R'=R$ (so Γ is good) and we get for all $\beta\in R^+$ and $w\in W$ with $w^{-1}\beta<0$

$$t^\beta[-\rho, w\bullet\lambda]e^\beta(w\bullet\lambda) = \Big(\prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha>0}} h_{-\alpha} \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha<0}} h_\alpha^{-1}\Big) + A^\beta. \tag{1}$$

If we reverse the calculation from the proof of Lemma 13.8, then we see that the term in parentheses on the right hand side of (1) is equal to

$$\prod_{\alpha>0, w^{-1}s_\beta\alpha<0} h_\alpha \prod_{\alpha>0, w^{-1}\alpha<0} h_\alpha^{-1}. \tag{2}$$

This shows: If A is graded such that each h_α with $\alpha\in R$ is homogeneous of degree 1, then the product in (1) is homogeneous of degree $l(s_\beta w)-l(w)$.

13.24. Consider now as an example the case where μ is semiregular, i.e., where there is one root $\gamma \in R^+$ with $R' = \{\pm\gamma\}$. Note that Γ is good, since any root is conjugate to a simple root.

If γ is a simple root, then Theorem 13.4 amounts to the following statements (where we use its notations):
If $w\gamma \neq \pm\beta$, then

$$t^\beta[\mu',\lambda']e^\beta(\lambda') = \begin{cases} h_{-w\gamma}e^\beta(\mu'), & \text{if } w\gamma > 0, s_\beta w\gamma < 0; \\ h_{-w\gamma}^{-1}e^\beta(\mu'), & \text{if } w\gamma < 0, s_\beta w\gamma > 0; \\ e^\beta(\mu'), & \text{otherwise.} \end{cases} \tag{1}$$

If $w\gamma = -\beta$, then

$$t^\beta[\mu',\lambda']e^\beta(\lambda') = h_\beta^{-1} + A^\beta. \tag{2}$$

If γ is not a simple root, then it is the short dominant root in an irreducible component of R, cf. [Ja6], II.6.3. In this case Theorem 13.4 says:
If $w\gamma \neq \pm\beta$, then

$$t^\beta[\mu',\lambda']e^\beta(\lambda') = \begin{cases} h_{w\gamma}e^\beta(\mu'), & \text{if } w\gamma < 0,\ s_\beta w\gamma > 0; \\ h_{w\gamma}^{-1}e^\beta(\mu'), & \text{if } w\gamma > 0,\ s_\beta w\gamma < 0; \\ e^\beta(\mu'), & \text{otherwise.} \end{cases} \tag{3}$$

If $w\gamma = \beta$, then

$$t^\beta[\mu',\lambda']e^\beta(\lambda') = h_\beta^{-1} + A^\beta. \tag{4}$$

13.25. Identify $Z_A(\lambda' + p\nu)$ and $Z_A(\lambda')$ for all λ' and ν in X as in 12.14. Identify the Ext groups as in 12.14(1). The definitions in 13.2, 13.5, and 13.16 show that we have for all λ' in $W_p\bullet\lambda$ or in $W_p\bullet\mu$ (in both situations)

$$e^\beta(\lambda' + p\nu) = e^\beta(\lambda') \qquad \text{for all } \beta \in R^+ \text{ and } \nu \in \mathbf{Z}R.$$

The definitions 13.8(1) and 13.14(1) show that

$$t^\beta[\mu' + p\nu, \lambda' + p\nu] = t^\beta[\mu',\lambda']$$

whenever these maps are defined.

Introduction to the Sections 14–16

Our main goal is to show that the categories $\mathcal{C}_k(\Omega)$ for regular orbits Ω are in some sense independent of the ground field k. To reach this goal we develop a combinatorial description of these categories in terms of the underlying root system only. In the preceding sections we have prepared the necessary tools. In the next sections we are going to apply them.

We want to explain now in a bit more detail how our combinatorial description works. Let for the moment Ω be an arbitrary orbit. To describe $\mathcal{C}_k(\Omega)$ we should describe the k–algebra End$P$ of endomorphisms for a projective generator P of $\mathcal{C}_k(\Omega)$, since then Hom$(P,\ )$ induces an equivalence of categories from $\mathcal{C}_k(\Omega)$ to the category of all finite dimensional right (End$P$)–modules.

Unfortunately $\mathcal{C}_k(\Omega)$ has no projective generator. The remedy is to better exploit the periodicity of our situation under the group $Y = p\mathbf{Z}R$. Let us define (for an arbitrary abelian group Y) a Y–category to be a category equipped with a collection of shift-functors $M \mapsto M[\nu]$ for all $\nu \in Y$ satisfying the obvious compatibility conditions, and for any object M of an additive Y–category $\mathcal{C}$ consider the Y–graded ring $\operatorname{End}^\sharp_{\mathcal{C}}(M) = \bigoplus_{\nu\in Y} \operatorname{Hom}_{\mathcal{C}}(M[\nu], M)$. By a Y–functor between Y–categories we mean a functor that commutes with all the shifts $[\nu]$. (Details can be found in Appendix E.)

Now let us take again $Y = p\mathbf{Z}R$. Clearly $\mathcal{C}_k(\Omega)$ is a Y–category, and it admits a projective Y–generator P, i.e., a projective object P in $\mathcal{C}_k(\Omega)$ such that the family of all $P[\nu]$ ($\nu \in Y$) generates $\mathcal{C}_k(\Omega)$. We can prove that then $\operatorname{Hom}^\sharp_{\mathcal{C}}(P,\ ) = \bigoplus_{\nu\in Y} \operatorname{Hom}_{\mathcal{C}}(P[\nu],\ )$ induces an equivalence of Y–categories from $\mathcal{C}_k(\Omega)$ to the category of all finite dimensional Y–graded right $(\operatorname{End}^\sharp_{\mathcal{C}}P)$–modules. So to combinatorially describe $\mathcal{C}_k(\Omega)$ we should combinatorially describe the Y–graded k–algebra $\operatorname{End}^\sharp_{\mathcal{C}}P$ for a suitable projective Y–generator P of $\mathcal{C}_k(\Omega)$.

Let us now assume again that Ω is regular. Then we can take for P an object of the from $Q_{k,I}$ obtained form $Z_k(-\rho)$ by translating out from all walls and applying a suitable sequence I of wall crossings. (Actually, we show only that we can take for P a direct sum of such objects, but let us neglect this complication for the moment.) Let $A = A(k)$ be the deformation ring and $Q_{A,I}$ the deformation of $Q_{k,I}$ as in the introduction to the sections 8–10.

Certainly also $\mathcal{C}_A(\Omega)$ is a Y–category, and by base change we have $\mathrm{End}^\sharp_{\mathcal{C}} Q_{k,I} = (\mathrm{End}^\sharp_{\mathcal{C}} Q_{A,I}) \otimes_A k$. Also $\mathcal{K}(\Omega, A)$ is a Y–category, and $\mathcal{V} = \mathcal{V}_\Omega$ is a Y–functor that is fully faithful on projectives, whence $\mathrm{End}^\sharp_{\mathcal{C}} Q_{A,I} = \mathrm{End}^\sharp_{\mathcal{K}} \mathcal{V} Q_{A,I}$.

Now let our sequence of walls I be given as $s_1, \ldots, s_r \in \Sigma'$, choose subregular orbits $\Gamma_1, \ldots, \Gamma_r$ on the respective walls and let T_i resp. T_i' be the translations onto resp. out from the walls relating $\mathcal{C}_A(\Omega)$ and $\mathcal{C}_A(\Gamma_i)$. Also let $T' : \mathcal{C}_A(Y - \rho) \to \mathcal{C}_A(\Omega)$ the translation out from all walls. Then by definition

$$Q_{A,I} = T_1' T_1 \cdots T_r' T_r T' \, Z_A(-\rho)$$

and hence

$$\mathcal{V} Q_{A,I} = \mathcal{T}_1' \mathcal{T}_1 \cdots \mathcal{T}_r' \mathcal{T}_r \mathcal{T}' \, \mathcal{Z}$$

where the $\mathcal{T}_i, \mathcal{T}_i', \mathcal{T}'$ are the corresponding combinatorial translation functors with $\mathcal{V} T_i = \mathcal{T}_i \mathcal{V}, \ldots$ and $\mathcal{Z} = \mathcal{V} Z_A(-\rho)$ in $\mathcal{K}(Y - \rho, A)$ is just given by $\mathcal{Z}(-\rho) = A^\emptyset$, $\mathcal{Z}(-\rho, \beta) = A^\beta$ for all $\beta \in R^+$, and $\mathcal{Z}(\lambda) = 0$ for $\lambda \neq -\rho$. We always assume the $\mathcal{V}$–functors and the combinatorial translations to be defined via "good choices", so that only the beautiful constants $C^\beta(\lambda, \mu)$ from the introduction to the sections 11–13 enter into our combinatorial translations. Summing up, we then find that the category $\mathcal{C}_k(\Omega)$ is equivalent to the category of all finite dimensional Y–graded right modules over the Y–graded ring

$$\mathcal{B}_k = \mathcal{B}_k(I) = (\mathrm{End}^\sharp_{\mathcal{K}} \mathcal{T}_1' \mathcal{T}_1 \cdots \mathcal{T}_r' \mathcal{T}_r \mathcal{T}' \mathcal{Z}) \otimes_A k.$$

We have to admit: This is a horrible description. However, it has two virtues and they are the very essence of our paper: First of all, the ring $\mathcal{B}_k$ can be defined "over $\mathbf{Z}$". Secondly it can be equipped with a $\mathbf{Z}$–grading in a natural way. To explain this in more detail we need more notations.

Let S be the symmetric algebra of the $\mathbf{Z}$–module $\mathbf{Z}R$. We write $h_\alpha \in S$ instead of $\alpha \in R$, put $S_k = S \otimes_{\mathbf{Z}} k$, and identify the completion $\widehat{S}_k$ of S_k at the maximal ideal generated by all h_α with the deformation ring $A = A(k)$ via $h_\alpha \mapsto d_\alpha H_\alpha$ resp. $h_\alpha \mapsto \log K_\alpha$ in Case 1 resp. Case 2.

We proceed to define combinatorial categories $\mathcal{K}(\Omega, S)$, $\mathcal{K}(\Gamma_i, S)$, $\mathcal{K}(Y - \rho, S)$, combinatorial translations $\mathcal{T}_i, \mathcal{T}_i', \mathcal{T}'$, between these, and an object $\mathcal{Z}$ in $\mathcal{K}(Y - \rho, S)$ as follows: We just copy our old definitions and replace A by S everywhere. We may now in addition identify our regular orbit Ω with the affine Weyl group W_a and thus arrive at a Y–graded S–algebra

$$\mathcal{B}_S = \mathcal{B}_S(I) = \mathrm{End}^\sharp_{\mathcal{K}(W_a, S)} (\mathcal{T}_1' \mathcal{T}_1 \cdots \mathcal{T}_r' \mathcal{T}_r \mathcal{T}' \mathcal{Z}),$$

which is completely independent of k and depends only on the underlying root system and our sequence I of walls. Put $\mathcal{B}_{\mathbf{Z}} = \mathcal{B}_S \otimes_S \mathbf{Z}$, where we use the augmentation $S \to \mathbf{Z}$, $h_\alpha \mapsto 0$. By our constructions it is almost clear that

$$\mathcal{B}_{\mathbf{Z}} \otimes_{\mathbf{Z}} k \simeq \mathcal{B}_k$$

in Case 2 or in Case 1 for $p \gg 0$. This proves that the Cartan matrix of regular blocks is the same in Case 1 and Case 2, if we assume $p \gg 0$ in Case 1. The same statement concerning the decomposition matrix is proved similarily, but with more technicalities. All this is explained in section 16.

As announced in the general introduction, we want to establish the isomorphism $\mathcal{B}_{\mathbf{Z}} \otimes_{\mathbf{Z}} k \simeq \mathcal{B}_k$ for *all* k with $p = \operatorname{char} k \geq h$. This is a much more subtle statement whose proof requires the consideration of our $\mathbf{Z}$–graded combinatorial categories $\widetilde{\mathcal{K}}(\Omega, S)$ resp. $\widetilde{\mathcal{K}}(\Omega, S_k)$ which are the subject matter of section 15. To define them we give S a $\mathbf{Z}$–grading such that $\deg h_\alpha = 2$ for all $\alpha \in R$ and let an object $\mathcal{M}$ in $\widetilde{\mathcal{K}}(\Omega, S)$ be a collection of $\mathbf{Z}$–graded $S^\emptyset$–modules $\mathcal{M}(\lambda)$ with a collection of homogeneous S^β–submodules $\mathcal{M}(\lambda, \beta) \subset \mathcal{M}(\lambda)$ resp. $\mathcal{M}(\lambda, \beta) \subset \mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda)$.

It turns out that the constants $C'^\beta(\lambda, \mu)$ appearing in the combinatorial translations have precisely the right degrees to determine "graded combinatorial translations" between our graded combinatorial categories. Thus $\mathcal{B}_S$ gets a $\mathbf{Z}$–grading, which descends to give a $\mathbf{Z}$–grading on $\mathcal{B}_k$. In Section 18 we study in detail the "$\mathbf{Z}$–graded representation categories" $\widetilde{\mathcal{C}}_k(\Omega)$ of all finitely generated $(Y \times \mathbf{Z})$–graded right $\mathcal{B}_k$–modules.

14. General Combinatorial Categories

We suppose in this section that (in Case 1) p satisfies the assumption of Lemma 9.1.

14.1. We set $A(k)$ equal to the completion of B with respect to the maximal ideal generated by all H_α resp. $K_\alpha - 1$. All results of the sections 9–13 can be applied to $A = A(k)$. We shall write $A^\beta(k)$ and $A^\emptyset(k)$ for the corresponding A^β and $A^\emptyset$. We make the following choices for the elements h_α as in 13.1. In Case 1 we take

$$h_\alpha = d_\alpha H_\alpha$$

for all α; this makes sense, since our assumption above makes sure that p is prime to each d_α. In Case 2 we set

$$h_\alpha = \log K_\alpha = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(K_\alpha - 1)^j$$

for all α. It is easy to check that 13.1(1)–(3) are satisfied.

Suppose for the moment that $p \geq h$. For each good W_p–orbit Γ (as in 13.3) and each $\mu \in \Gamma$ we choose all $e^\beta(\mu)$ as in 13.5 resp. as in 13.16. (Recall that these definitions were independent of the choice of Ω and note that they yield the same definition as in 13.2 in case $\Gamma = \Omega$.) We define the functors $\mathcal{V}_\Gamma$ on $\mathcal{FC}_{A(k)}(\Gamma)$ for each good Γ using this choice of e^β.

14.2. The following lemma generalizes to any A that is a complete local Noetherian ring.

Lemma: *a) A module M in $\mathcal{C}_{A(k)}$ is indecomposable if and only if* $\mathrm{End}_\mathcal{C}(M)$ *is a local ring.*

b) The Krull–Schmidt Theorem holds in the category $\mathcal{C}_{A(k)}$.

Proof: Set $A = A(k)$. Any M in $\mathcal{C}_A$ is a finitely generated A–module; therefore also $\mathrm{End}_A(M)$ is a finitely generated A–module. Consider the subalgebra $U(M)$ of $\mathrm{End}_A(M)$ generated by the image of U_A and by the projections to the M_ν with $\nu \in X$. It is again a finitely generated A–module. We have obviously

$$\mathrm{End}_{\mathcal{C}}(M) = \mathrm{End}_{U(M)}(M);$$

a $U(M)$–submodule of M is the same thing as a subobject of M in $\mathcal{C}_A$. Since A is a complete local Noetherian ring, our claims follow from standard results, cf. [CR], I.6.10(ii), I.6.12.

14.3. Set S equal to the symmetric algebra of the $\mathbf{Z}$–module $\mathbf{Z}R$. In order to have compatible notations, we shall write h_α instead of α (for all $\alpha \in R$), if we regard it as an element of S. So S can be identified with the polynomial ring over $\mathbf{Z}$ in the $h_\alpha, \alpha \in \Sigma$. Set

$$S^\emptyset = S[h_\alpha^{-1} \mid \alpha \in R^+] \qquad \text{and} \qquad S^\beta = S[h_\alpha^{-1} \mid \alpha \in R^+, \alpha \neq \beta] \tag{1}$$

(for all $\beta \in R^+$). The units in S are $\{\pm 1\}$, the units in $S^\emptyset$ are all products $\pm \prod_{\alpha \in R^+} h_\alpha^{m(\alpha)}$ with all $m(\alpha) \in \mathbf{Z}$.

Lemma: *For all $\beta \in R^+$ the $\mathbf{Z}$–module $S^\emptyset / S^\beta$ is torsion free.*

Proof: The ring S is a unique factorization domain. The h_γ with $\gamma \in R^+$ are prime elements in S and do not differ by units. Any nonzero element $c \in S^\emptyset$ can be written

$$c = m \cdot d \cdot \prod_{\gamma \in R^+} h_\gamma^{r(\gamma)}$$

with $m \in \mathbf{Z}, m \neq 0$, with all $r(\gamma) \in \mathbf{Z}$, and with d a product of prime elements in S of positive degree that are not multiples of any h_γ. The $r(\gamma)$ and $|m|$ are uniquely determined by c. We have $c \in S^\beta$ if and only if $r(\beta) \geq 0$. This implies the claim.

14.4. Set

$$S_k = S \otimes_{\mathbf{Z}} k, \qquad S_k^\emptyset = S^\emptyset \otimes_{\mathbf{Z}} k, \qquad S_k^\beta = S^\beta \otimes_{\mathbf{Z}} k \tag{1}$$

(for all $\beta \in R^+$). Set $\widehat{S_k}$ equal to the completion of S_k with respect to the maximal ideal generated by all h_α. Under our assumption on p, there is an isomorphism

$$\widehat{S_k} \xrightarrow{\sim} A(k) \tag{2}$$

such that each h_α in S (with $\alpha \in R$) is mapped to $h_\alpha = d_\alpha H_\alpha$ resp. to $h_\alpha = \log K_\alpha$ in $A(k)$. It induces isomorphisms

$$S_k^\emptyset \otimes_{S_k} A(k) \xrightarrow{\sim} A^\emptyset(k) \qquad \text{and} \qquad S_k^\beta \otimes_{S_k} A(k) \xrightarrow{\sim} A^\beta(k) \tag{3}$$

(for all $\beta \in R^+$). Note that $A(k)$ is flat over S_k, cf. [Bou1], III, §3, Th. 3.

14.5. Let A be any (commutative) S–algebra that is an integral domain and Noetherian such that the images of the h_β with $\beta \in R^+$ in A are nonzero. Set

$$A^\emptyset = S^\emptyset \otimes_S A \qquad \text{and} \qquad A^\beta = S^\beta \otimes_S A \tag{1}$$

(for all $\beta \in R^+$). We identify these rings with subrings of the field of fractions of A. (Note that these definitions are compatible with earlier ones, e.g., in 14.4(1).) We shall always assume that

$$A = \bigcap_{\beta \in R^+} A^\beta. \tag{2}$$

Recall that $A = A(k)$ satisfies (2) by Lemma 9.1. The same proof as for Lemma 9.1 shows that (2) is satisfied also by $A = S$ and $A = S_k$; it then follows for any A that is a flat S_k–module (or a flat S–module).

Let Ω be a W_p–orbit in X. We can define a combinatorial category $\mathcal{K}(\Omega) = \mathcal{K}(\Omega, A)$ generalizing the construction in 9.4 (that works, e.g., for $A = A(k)$). Let us repeat: An object $\mathcal{M}$ in $\mathcal{K}(\Omega)$ is a family $(\mathcal{M}(\lambda))_{\lambda \in \Omega}$ of finitely generated $A^\emptyset$–modules (almost all equal to 0) together with (for all $\beta \in R^+$ and $\lambda \in \Omega$) a finitely generated A^β–submodule $\mathcal{M}(\lambda, \beta)$ of $\mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda)$ if $\beta \uparrow \lambda \neq \lambda$, of $\mathcal{M}(\lambda)$ if $\beta \uparrow \lambda = \lambda$. A morphism ψ between two such objects $\mathcal{M}$, $\mathcal{M}'$ is a family $(\psi_\lambda)_{\lambda \in \Omega}$ of $A^\emptyset$–linear maps $\psi_\lambda : \mathcal{M}(\lambda) \to \mathcal{M}'(\lambda)$ such that for all $\beta \in R^+$ and $\lambda \in \Omega$:

$$\begin{aligned} (\psi_\lambda \oplus \psi_{\beta\uparrow\lambda})\mathcal{M}(\lambda, \beta) &\subset \mathcal{M}'(\lambda, \beta) \qquad && \text{in case } \beta \uparrow \lambda \neq \lambda, \\ \psi_\lambda \mathcal{M}(\lambda, \beta) &\subset \mathcal{M}'(\lambda, \beta) && \text{in case } \beta \uparrow \lambda = \lambda. \end{aligned}$$

We set (as in 9.9) — given $\mathcal{M}$ in $\mathcal{K}(\Omega)$ —

$$\mathcal{M}(\lambda)_\beta = \mathcal{M}(\lambda) \cap \mathcal{M}(\lambda, \beta) \tag{3}$$

for all $\lambda \in \Omega$ and $\beta \in R^+$. This is a finitely generated A^β-module; if $\beta \uparrow \lambda = \lambda$, then $\mathcal{M}(\lambda)_\beta = \mathcal{M}(\lambda, \beta)$. Similarly, we set

$$\mathcal{M}(\lambda)_A = \bigcap_{\beta \in R^+} \mathcal{M}(\lambda)_\beta. \tag{4}$$

For all λ and β set

$$\mathcal{M}(\lambda, \beta)^o = \begin{cases} (\mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda)) / \mathcal{M}(\lambda, \beta), & \text{if } \beta \uparrow \lambda \neq \lambda; \\ \mathcal{M}(\lambda)/\mathcal{M}(\lambda, \beta), & \text{if } \beta \uparrow \lambda = \lambda. \end{cases} \tag{5}$$

This is an A^β–module.

14.6. Assume from now on in Section 14 that A is an S–algebra as in 14.5. Assume until 14.13 that Ω is an arbitrary W_p–orbit in X.

Lemma: *Let $\mathcal{M}$ be in $\mathcal{K}(\Omega)$. Fix $\lambda \in \Omega$.*

a) If each $\mathcal{M}(\lambda)_\beta$ with $\beta \in R^+$ generates $\mathcal{M}(\lambda)$ over $A^\emptyset$, then $\mathcal{M}(\lambda)_A$ generates $\mathcal{M}(\lambda)$ over $A^\emptyset$.

b) If $\mathcal{M}(\lambda)$ is a torsion free $A^\emptyset$–module, then $\mathcal{M}(\lambda)_A$ is a finitely generated and torsion free A–module.

Proof: a) We have $A^\emptyset = A^\beta[h_\beta^{-1}]$. So the condition that $\mathcal{M}(\lambda)_\beta$ generates $\mathcal{M}(\lambda)$ over $A^\emptyset$ means that there is for each $x \in \mathcal{M}(\lambda)$ an integer $m = m(\beta, x) \geq 0$ such that $h_\beta^m x \in \mathcal{M}(\lambda)_\beta$. We can now take a product over the h_β^m for all β and get a unit $a \in A^\emptyset$ with $ax \in \mathcal{M}(\lambda)_\beta$ for all β, hence with $ax \in \mathcal{M}(\lambda)_A$. The claim follows.

b) By our assumption, we can embed $\mathcal{M}(\lambda)$ into a vector space, say V, over the fraction field of $A^\emptyset$. Choose a basis $(v_i)_i$ of V. Each $\mathcal{M}(\lambda)_\beta$ is finitely generated over A^β, so we can choose a finite set $(y_{\lambda,\beta,j})_j$ of generators for $\mathcal{M}(\lambda)_\beta$ over A^β. Write each $y_{\lambda,\beta,j}$ as a linear combination of the v_i with coefficients in the fraction field of A. There is then an element $a \in A$, $a \neq 0$ with

$$a\, y_{\lambda,\beta,j} \in \sum_i A v_i$$

for all β and j. This implies that

$$\mathcal{M}(\lambda)_\beta \subset \sum_i A^\beta a^{-1} v_i,$$

hence — using $A = \bigcap A^\beta$ —

$$\mathcal{M}(\lambda)_A \subset \sum_i A a^{-1} v_i.$$

The claim follows.

14.7. Lemma: *Let $\mathcal{M}$ and $\mathcal{N}$ be in $\mathcal{K}(\Omega)$ such that all $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$ with $\lambda \in \Omega$ are torsion free $A^\emptyset$–modules. If each $\mathcal{M}(\lambda)_\beta$ with $\beta \in R^+$ and $\lambda \in \Omega$ generates $\mathcal{M}(\lambda)$ over $A^\emptyset$, then $\mathrm{Hom}_{\mathcal{K}(\Omega)}(\mathcal{M}, \mathcal{N})$ is a finitely generated and torsion free A–module.*

Proof: We have a restriction map

$$\mathrm{Hom}_{\mathcal{K}(\Omega)}(\mathcal{M}, \mathcal{N}) \longrightarrow \prod_\lambda \mathrm{Hom}_A(\mathcal{M}(\lambda)_A, \mathcal{N}(\lambda)_A).$$

It is injective since each $\mathcal{M}(\lambda)$ is generated over $A^\emptyset$ by $\mathcal{M}(\lambda)_A$ by Lemma 14.6.a and by our assumption. Each factor on the right hand side is a finitely generated A–module (by Lemma 14.6.b). Now the claim follows, because there are only finitely many nonzero factors and because A is Noetherian.

14.8. Suppose that A' is an A–algebra that satisfies the assumptions on A from 14.5. We have a natural extension of scalars functor from $\mathcal{K}(\Omega, A)$ to $\mathcal{K}(\Omega, A')$: Map any $\mathcal{M}$ in $\mathcal{K}(\Omega, A)$ to $\mathcal{M}_{A'}$ with

$$\mathcal{M}_{A'}(\lambda) = \mathcal{M}(\lambda) \otimes_{A^\emptyset} A'^\emptyset = \mathcal{M}(\lambda) \otimes_A A' \tag{1}$$

and with $\mathcal{M}_{A'}(\lambda, \beta)$ equal to the canonical image of $\mathcal{M}(\lambda, \beta) \otimes_{A^\beta} A'^\beta = \mathcal{M}(\lambda, \beta) \otimes_A A'$ in $\mathcal{M}_{A'}(\lambda) \oplus \mathcal{M}_{A'}(\beta \uparrow \lambda)$ resp. in $\mathcal{M}_{A'}(\lambda)$ (for all $\beta \in R^+$).

Lemma: *Suppose that A' is a flat A–module. We have then for all $\mathcal{M}$ in $\mathcal{K}(\Omega, A)$ natural isomorphisms for all $\lambda \in \Omega$ and $\beta \in R^+$*

$$\mathcal{M}(\lambda, \beta) \otimes_{A^\beta} A'^\beta = \mathcal{M}(\lambda, \beta) \otimes_A A' \xrightarrow{\sim} \mathcal{M}_{A'}(\lambda, \beta) \tag{2}$$

and

$$\mathcal{M}(\lambda, \beta)^o \otimes_{A^\beta} A'^\beta = \mathcal{M}(\lambda, \beta)^o \otimes_A A' \xrightarrow{\sim} \mathcal{M}_{A'}(\lambda, \beta)^o. \tag{3}$$

We have for all $\mathcal{M}$, $\mathcal{N}$ in $\mathcal{K}(\Omega, A)$ a natural isomorphism

$$\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{M}, \mathcal{N}) \otimes_A A' \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(\Omega,A')}(\mathcal{M}_{A'}, \mathcal{N}_{A'}). \tag{4}$$

Proof: The flatness obviously yields (2) and (3). Consider (4). We can describe $\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{M}, \mathcal{N})$ as the kernel of a natural map

$$\bigoplus_\lambda \mathrm{Hom}_{A^\emptyset}(\mathcal{M}(\lambda), \mathcal{N}(\lambda)) \longrightarrow \bigoplus_\lambda \bigoplus_\beta \mathrm{Hom}_{A^\beta}(\mathcal{M}(\lambda, \beta), \mathcal{N}(\lambda, \beta)^o). \tag{5}$$

We get $\mathrm{Hom}_{\mathcal{K}(\Omega,A')}(\mathcal{M}_{A'}, \mathcal{N}_{A'})$ as the kernel of an analogous map. We claim that we get the analogue of (5) over A' from (5) by extending scalars from A to A'. Then (4) will follow because A' is flat over A.

Well, each $\mathcal{M}(\lambda)$ is finitely generated (hence finitely presented) over the (Noetherian) ring $A^\emptyset$, and $A^\emptyset$ is flat over $S^\emptyset$, so we get

$$\begin{aligned}\mathrm{Hom}_{A'^\emptyset}(\mathcal{M}_{A'}(\lambda), \mathcal{N}_{A'}(\lambda)) &\simeq \mathrm{Hom}_{A^\emptyset}(\mathcal{M}(\lambda), \mathcal{N}(\lambda)) \otimes_{A^\emptyset} A'^\emptyset \\ &= \mathrm{Hom}_{A^\emptyset}(\mathcal{M}(\lambda), \mathcal{N}(\lambda)) \otimes_A A'\end{aligned}$$

for all λ. We see similarly — using (3) —

$$\begin{aligned}\mathrm{Hom}_{A'^\beta}(\mathcal{M}_{A'}(\lambda, \beta), \mathcal{N}_{A'}(\lambda, \beta)^o) &\simeq \mathrm{Hom}_{A^\beta}(\mathcal{M}(\lambda, \beta), \mathcal{N}(\lambda, \beta)^o) \otimes_{A^\beta} A'^\beta \\ &= \mathrm{Hom}_{A^\beta}(\mathcal{M}(\lambda, \beta), \mathcal{N}(\lambda, \beta)^o) \otimes_A A'\end{aligned}$$

for all λ and β. Now the claim follows taking direct sums.

14.9. We can apply Lemma 14.8 always to $(A, A') = (S_k, A(k))$. Consider on the other hand $(A, A') = (S, S_k)$. In that situation we have $M \otimes_S S_k = M \otimes_{\mathbf{Z}} k$ for any S–module M; usually we shall write $\mathcal{M}_k$ instead of $\mathcal{M}_{S_k}$. In Case 2 we have $\mathrm{char}(k) = 0$ and S_k is flat over S; so we can apply Lemma 14.8. On the other hand, in Case 1 we have $\mathrm{char}(k) = p \neq 0$ and so S_k is not a flat module over S.

However, most of the $\mathcal{M}$ in $\mathcal{K}(\Omega, S)$ to be considered will satisfy the following condition:

(TF) *Each $\mathcal{M}(\lambda, \beta)^o$ has no p–torsion.*

This condition implies (in case $\beta \uparrow \lambda \neq \lambda$) that the short exact sequence

$$0 \to \mathcal{M}(\lambda, \beta) \longrightarrow \mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda) \longrightarrow \mathcal{M}(\lambda, \beta)^o \to 0$$

remains exact after tensoring over $\mathbf{Z}$ with k. This implies that

$$\mathcal{M}(\lambda, \beta) \otimes_{\mathbf{Z}} k = \mathcal{M}_k(\lambda, \beta) \qquad \text{and} \qquad \mathcal{M}(\lambda, \beta)^o \otimes_{\mathbf{Z}} k = \mathcal{M}_k(\lambda, \beta)^o. \qquad (1)$$

We get the same result for $\beta \uparrow \lambda = \lambda$. So we see that 14.8(2), (3) extend to this situation. We get for 14.8(4) a somewhat weaker result:

Lemma: *Suppose that we are in* Case 1. *Let $\mathcal{M}$ and $\mathcal{N}$ be objects in $\mathcal{K}(\Omega, S)$ satisfying* (TF). *Suppose that all $\mathcal{M}(\lambda)$ are free over $S^\emptyset$. Then the natural map*

$$\mathrm{Hom}_{\mathcal{K}(\Omega,S)}(\mathcal{M}, \mathcal{N}) \otimes_{\mathbf{Z}} k \longrightarrow \mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}_k, \mathcal{N}_k)$$

is injective.

Proof: We can describe $\mathrm{Hom}_{\mathcal{K}(\Omega,S)}(\mathcal{M}, \mathcal{N})$ (as in the proof of 14.8) as the kernel of a natural map

$$\bigoplus_\lambda \mathrm{Hom}_{S^\emptyset}(\mathcal{M}(\lambda), \mathcal{N}(\lambda)) \xrightarrow{\psi} \bigoplus_\lambda \bigoplus_\beta \mathrm{Hom}_{S^\beta}(\mathcal{M}(\lambda, \beta), \mathcal{N}(\lambda, \beta)^o). \qquad (2)$$

All summands in (2) have no p–torsion, therefore also the image of ψ has no p–torsion. This implies that the natural map

$$\mathrm{Hom}_{\mathcal{K}(\Omega,S)}(\mathcal{M}, \mathcal{N}) \otimes_{\mathbf{Z}} k \longrightarrow \bigoplus_\lambda \mathrm{Hom}_{S^\emptyset}(\mathcal{M}(\lambda), \mathcal{N}(\lambda)) \otimes_{\mathbf{Z}} k \qquad (3)$$

is injective. Our freeness assumption shows that we have natural isomorphisms

$$\mathrm{Hom}_{S^\emptyset}(\mathcal{M}(\lambda), \mathcal{N}(\lambda)) \otimes_{\mathbf{Z}} k \xrightarrow{\sim} \mathrm{Hom}_{S^\emptyset_k}(\mathcal{M}_k(\lambda), \mathcal{N}_k(\lambda)) \qquad (4)$$

for all λ. So we can rewrite (3) as

$$\operatorname{Hom}_{\mathcal{K}(\Omega,S)}(\mathcal{M},\mathcal{N})\otimes_{\mathbf{Z}} k \longrightarrow \bigoplus_{\lambda} \operatorname{Hom}_{S_k^{\emptyset}}(\mathcal{M}_k(\lambda),\mathcal{N}_k(\lambda)). \tag{5}$$

This map is the composition of the functorial map

$$\operatorname{Hom}_{\mathcal{K}(\Omega,S)}(\mathcal{M},\mathcal{N})\otimes_{\mathbf{Z}} k \longrightarrow \operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}_k,\mathcal{N}_k)$$

with the obvious embedding

$$\operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}_k,\mathcal{N}_k) \hookrightarrow \bigoplus_{\lambda} \operatorname{Hom}_{S_k^{\emptyset}}(\mathcal{M}_k(\lambda),\mathcal{N}_k(\lambda)).$$

Since the composition is injective, so is the first map — and that is our claim.

Remark: We can replace the assumption that all $\mathcal{M}(\lambda)$ are free over $S^{\emptyset}$ by the weaker condition that all $\mathcal{M}(\lambda)$ are projective over $S^{\emptyset}$.

14.10. We define for all $\lambda \in \Omega$ and $w \in W$ an object $\mathcal{Z}_\lambda^w = \mathcal{Z}_\lambda^w(A)$ in $\mathcal{K}(\Omega)$ as follows: Set for all $\mu \in X$

$$\mathcal{Z}_\lambda^w(\mu) = \begin{cases} A^{\emptyset}, & \text{if } \mu = \lambda; \\ 0, & \text{if } \mu \neq \lambda; \end{cases} \tag{1}$$

for all $\beta \in R^+$

$$\mathcal{Z}_\lambda^w(\lambda,\beta) = \begin{cases} A^{\beta}, & \text{if } \beta\uparrow\lambda = \lambda; \\ A^{\beta}(1,0), & \text{if } \beta\uparrow\lambda \neq \lambda; \end{cases} \tag{2}$$

and, if $\beta\uparrow\lambda \neq \lambda$, (with $\beta\downarrow\lambda$ as in 9.5)

$$\mathcal{Z}_\lambda^w(\beta\downarrow\lambda,\beta) = \begin{cases} A^{\beta}(0,h_\beta), & \text{if } w^{-1}\beta > 0; \\ A^{\beta}(0,1), & \text{if } w^{-1}\beta < 0; \end{cases} \tag{3}$$

set all other $\mathcal{Z}_\lambda^w(\mu,\beta)$ equal to 0.

Note that Proposition 9.5 says

$$\mathcal{V}_\Omega Z^w_{A(k)}(\lambda\langle w\rangle) \simeq \mathcal{Z}_\lambda^w(A(k)). \tag{4}$$

We have obviously

$$\mathcal{Z}_\lambda^w(A)_{A'} \simeq \mathcal{Z}_\lambda^w(A') \tag{5}$$

for any A–algebra A' as in 14.8; similarly in the situation of 14.9.

Lemma: *For $A = S$ each $\mathcal{Z}_\lambda^w$ satisfies the condition* 14.9(TF).

Proof: For any A the A^{β}–module $A^{\emptyset}/h_\beta A^{\beta}$ is isomorphic to $A^{\emptyset}/A^{\beta}$ (via $x + h_\beta A^{\beta} \mapsto h_\beta^{-1}x + A^{\beta}$); so each nonzero $\mathcal{Z}_\lambda(\mu,\beta)^o$ is isomorphic to $A^{\emptyset}/A^{\beta}$. Therefore the claim follows from Lemma 14.3.

14.11. Recall the definition of compatible subobjects from 9.11. We make the same definition in $\mathcal{K}(\Omega, A)$ for any A as in 14.5. As in 9.11 we define for each $\mathcal{M}$ in $\mathcal{K}(\Omega, A)$ and each compatible subobject $\mathcal{N}$ of $\mathcal{M}$ the quotient $\mathcal{M}/\mathcal{N}$. Furthermore, we define the inverse image in $\mathcal{M}$ of a subobject in $\mathcal{M}/\mathcal{N}$ as in 9.12. The remarks from 9.12 generalize to this situation.

Consider for A as above an A–algebra A' as in 14.8.

Lemma: *Suppose that A' is a flat A–module. If $\mathcal{N}$ is a compatible subobject of an $\mathcal{M}$ in $\mathcal{K}(\Omega, A)$, then $\mathcal{N}_{A'}$ is a compatible subobject of $\mathcal{M}_{A'}$ and there is a natural isomorphism $(\mathcal{M}/\mathcal{N})_{A'} \xrightarrow{\sim} \mathcal{M}_{A'}/\mathcal{N}_{A'}$.*

Proof: The flatness implies easily that $\mathcal{N}_{A'}$ is a subobject of $\mathcal{M}_{A'}$. The compatibility condition 9.11(1) means that $\mathcal{N}(\lambda, \beta)$ is the kernel of the natural map

$$\mathcal{M}(\lambda, \beta) \longrightarrow (\mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda))/(\mathcal{N}(\lambda) \oplus \mathcal{N}(\beta \uparrow \lambda)) \tag{1}$$

(with an obvious modification in case $\beta \uparrow \lambda = \lambda$). The flatness of A' over A implies that $\mathcal{N}_{A'}$ has the same property inside $\mathcal{M}_{A'}$, hence is is a compatible subobject; also the last claim follows.

14.12. As in 9.13 we can make $\mathcal{K}(\Omega)$ into a $(p\mathbf{Z}R)$–category: We define the functors $\mathcal{M} \mapsto \mathcal{M}[p\nu]$ with $\nu \in \mathbf{Z}R$ by the formula 9.13(1). The statements in 9.13(2), (3) extend. The definition in 14.10 implies for all λ and w

$$\mathcal{Z}_\lambda^w[p\nu] = \mathcal{Z}_{\lambda+p\nu}^w. \tag{1}$$

Each functor $\mathcal{M} \mapsto \mathcal{M}[p\nu]$ commutes obviously with any extension of scalars as in 14.8.

For $A = A(k)$ the choice of e^β in Section 13 satisfies $e^\beta(\mu + p\nu) = e^\beta(\nu)$ for all μ, see 13.25. So 9.13(5) implies that $\mathcal{V}_\Omega$ is a $(p\mathbf{Z}R)$–functor.

14.13. Following the general convention in E.3(1) we set for all $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{K}(\Omega)$

$$\mathrm{Hom}^\sharp_{\mathcal{K}(\Omega)}(\mathcal{M}, \mathcal{N}) = \bigoplus_{\nu \in \mathbf{Z}R} \mathrm{Hom}_{\mathcal{K}(\Omega)}(\mathcal{M}[p\nu], \mathcal{N}). \tag{1}$$

There are only finitely many nonzero summands on the right hand side: Indeed, if $\mathrm{Hom}_{\mathcal{K}(\Omega)}(\mathcal{M}[p\nu], \mathcal{N}) \neq 0$, then there are λ and μ with $\mathcal{M}(\lambda) \neq 0$ and $\mathcal{N}(\mu) \neq 0$ and $\mu - \lambda = p\nu$; this leaves only finitely many possibilities for ν.

Consider the case $A = A(k)$; take M, N in $\mathcal{FC}_A(\Omega)$ and set $\mathcal{M} = \mathcal{V}_\Omega M$, $\mathcal{N} = \mathcal{V}_\Omega N$. Since $\mathcal{V}_\Omega$ is a $(p\mathbf{Z}R)$–functor (and fully faithful) we have

$$\mathrm{Hom}^\sharp_{\mathcal{K}(\Omega)}(\mathcal{M}, \mathcal{N}) \simeq \bigoplus_{\nu \in \mathbf{Z}R} \mathrm{Hom}_{\mathcal{C}}(M[p\nu], N). \tag{2}$$

Regard k as an $A(k)$–algebra via the augmentation map (taking all h_γ to 0). If M is projective in $\mathcal{C}_{A(k)}$ and if N is projective over $A(k)$, then Proposition 3.3 and (3) imply

$$\mathrm{Hom}^\sharp_{\mathcal{K}(\Omega)}(\mathcal{M},\mathcal{N}) \otimes_{A(k)} k \simeq \bigoplus_{\nu\in\mathbf{Z}R} \mathrm{Hom}_{\mathcal{C}_k}(M_k[p\nu], N_k). \tag{3}$$

Claim: *If p is prime to the index of connection of R, then*

$$\mathrm{Hom}^\sharp_{\mathcal{K}(\Omega)}(\mathcal{M},\mathcal{N}) \otimes_{A(k)} k \simeq \begin{cases} \mathrm{Hom}_{\mathfrak{g}}(M,N), & \text{in Case 1,} \\ \mathrm{Hom}_{\mathbf{u}}(M,N), & \text{in Case 2.} \end{cases} \tag{4}$$

Proof: Compare (4) to 4.2(6)–(8). We see that we have to show: If $\nu \in X$ with $\mathrm{Hom}_{\mathcal{C}_k}(M_k[p\nu], N_k) \neq 0$, then $\nu \in \mathbf{Z}R$. Well, if that Hom space is nonzero, then there are weights λ of M and μ of N with $\mu - \lambda = p\nu$. If λ_1 is an element of Ω, then all weights of a module in $\mathcal{C}_A(\Omega)$ are in the coset $\lambda_1 + \mathbf{Z}R$. So we get now $p\nu \in (\mathbf{Z}R) \cap pX$. By assumption p is prime to the index of connection $[X : \mathbf{Z}R]$, and so $\nu \in \mathbf{Z}R$ as desired.

14.14. Assume from now on that p is greater than or equal to the Coxeter number. Fix a regular W_p–orbit Ω. This means that the stabilizer in W_p of any element in Ω is trivial. So Ω is the orbit of an element such as the λ in 13.2. Let Γ be a good W_p–orbit. As in 10.2 we denote for each $\lambda \in \Omega$ by λ_Γ the unique element in Γ contained in the closure of the alcove of λ. Denote for all $\lambda \in \Omega$ and $\beta \in R^+$ the element $C^\beta(\lambda, \lambda_\Gamma)$ from 13.4(1) by a^β_λ and set $b^\beta_\lambda = (a^\beta_\lambda)^{-1}$.

We define functors $\mathcal{T} : \mathcal{K}(\Omega) \to \mathcal{K}(\Gamma)$ and $\mathcal{T}' : \mathcal{K}(\Gamma) \to \mathcal{K}(\Omega)$ using the formulas 10.10(5)–(8) with the a^β_λ and b^β_λ as above. There are a few simplifications since $\beta \uparrow \lambda \neq \lambda$ for all $\lambda \in \Omega$ and all β (because Ω is regular). Furthermore, we can apply the remark to 10.11 since our a^β_λ satisfy the assumptions there. Let us state the definition explicitly: We set for all $\mathcal{M}$ in $\mathcal{K}(\Omega)$

$$\mathcal{T}\mathcal{M}(\mu) = \bigoplus_{\lambda_\Gamma=\mu} \mathcal{M}(\lambda) \tag{1}$$

for all $\mu \in \Gamma$, and (for all $\beta \in R^+$)

$$\mathcal{T}\mathcal{M}(\mu,\beta) = \bigoplus_{\lambda_\Gamma=\mu=(\beta\uparrow\lambda)_\Gamma} (b^\beta_\lambda, 1)\mathcal{M}(\lambda,\beta), \tag{2}$$

if $\beta\uparrow\mu = \mu$, resp.

$$\mathcal{T}\mathcal{M}(\mu,\beta) = \bigoplus_{\lambda_\Gamma=\mu} (b^\beta_\lambda, 1)\mathcal{M}(\lambda,\beta), \tag{3}$$

if $\beta \uparrow \mu \neq \mu$.

For any $\mathcal{N}$ in $\mathcal{K}(\Gamma)$ we define $\mathcal{T}'\mathcal{N}$ by

$$\mathcal{T}'\mathcal{N}(\lambda) = \mathcal{N}(\lambda_\Gamma) \qquad \text{for all } \lambda \in \Omega, \tag{4}$$

and (for all $\beta \in R^+$ and $\lambda \in \Omega$):

$$\mathcal{T}'\mathcal{N}(\lambda, \beta) = \begin{cases} (a_\lambda^\beta, 1)\mathcal{N}(\lambda_\Gamma, \beta), & \text{if } \beta \uparrow \lambda_\Gamma \neq \lambda_\Gamma; \\ \mathcal{N}(\lambda_\Gamma, \beta) \oplus \mathcal{N}(\lambda_\Gamma + p\beta, \beta), & \text{if } (\beta \uparrow \lambda)_\Gamma = \lambda_\Gamma + p\beta; \\ \{(x + a_\lambda^\beta y, y) \mid x, y \in \mathcal{N}(\lambda_\Gamma, \beta)\}, & \\ & \text{if } (\beta \uparrow \lambda)_\Gamma = \lambda_\Gamma, \beta \uparrow \lambda \neq \lambda. \end{cases} \tag{5}$$

One checks as in 10.10 that $\mathcal{T}$ and $\mathcal{T}'$ preserve the finiteness conditions in the definition of the combinatorial categories. It is clear how to define $\mathcal{T}$ and $\mathcal{T}'$ on morphisms.

It is easy to see that $\mathcal{T}$ and $\mathcal{T}'$ commute with the extension of scalars functors defined in 14.8. We have $C^\beta(\lambda + p\nu, (\lambda + p\nu)_\Gamma) = C^\beta(\lambda, \lambda_\Gamma)$, hence $a_\lambda^\beta = a_{\lambda+p\nu}^\beta$ and $b_\lambda^\beta = b_{\lambda+p\nu}^\beta$ for all $\lambda \in \Omega$, $\beta \in R^+$, and $\nu \in \mathbf{Z}R$. So $\mathcal{T}$ and $\mathcal{T}'$ are compatible with the functors $\mathcal{Q} \mapsto \mathcal{Q}[p\nu]$ on $\mathcal{K}(\Omega)$ and on $\mathcal{K}(\Gamma)$ (for each $\nu \in \mathbf{Z}R$), i.e., they are $(p\mathbf{Z}R)$-functors in the sense of E.3.

In the case $A = A(k)$ we have isomorphisms

$$\mathcal{V}_\Gamma \circ T \xrightarrow{\sim} \mathcal{T} \circ \mathcal{V}_\Omega \qquad \text{and} \qquad \mathcal{V}_\Omega \circ T' \xrightarrow{\sim} \mathcal{T}' \circ \mathcal{V}_\Gamma \tag{6}$$

with $T = T_\Omega^\Gamma$ and $T' = T_\Gamma^\Omega$ as in 10.1. This follows from Proposition 10.11 and Theorem 13.4. (Recall the choice of $\mathcal{V}_\Omega$ and $\mathcal{V}_\Gamma$ in 14.1. The construction of the isomorphism in 10.11 involves the choice of certain f_λ in 10.12. We choose them as in 13.9 resp. as in 13.15. The maps $t[f_\lambda, f_{\beta\uparrow\lambda}]$ and $\theta[f_\lambda, f_{\beta\uparrow\lambda}]$ from 10.3(1) and 10.5(1) are then equal to certain $t^\beta[(\lambda')_\Gamma, \lambda']$, cf. 13.4(4). We get then that 10.10(1),(3) are satisfied for the a_λ^β as above.)

14.15. Let (Ω_1, Ω_2) be either (Ω, Γ) or (Γ, Ω) with Ω and Γ as in 14.14. Denote the corresponding functor ($\mathcal{T}$ or $\mathcal{T}'$, as constructed in 14.14) by $\mathcal{T}_1 : \mathcal{K}(\Omega_1) \to \mathcal{K}(\Omega_2)$.

Lemma: *Let $\mathcal{M}$ be an object in $\mathcal{K}(\Omega_1)$.*

a) Each $A^\emptyset$-module $\mathcal{T}_1\mathcal{M}(\lambda)$ with $\lambda \in \Omega_2$ is isomorphic to a finite direct sum of certain $\mathcal{M}(\mu)$ with $\mu \in \Omega_1$.

b) Let $\beta \in R^+$. Each A^β-module $\mathcal{T}_1\mathcal{M}(\lambda, \beta)$ with $\lambda \in \Omega_2$ is isomorphic to a finite direct sum of certain $\mathcal{M}(\mu, \beta)$ with $\mu \in \Omega_1$.

c) Let $\beta \in R^+$. Each A^β-module $\mathcal{T}_1\mathcal{M}(\lambda, \beta)^o$ with $\lambda \in \Omega_2$ is isomorphic to a finite direct sum of certain $\mathcal{M}(\mu, \beta)^o$ with $\mu \in \Omega_1$.

Proof: The claim in a) is obvious from 14.14(1), (4). For b) and c) note: Let b be a unit in $A^\emptyset$. For each $\mu \in \Omega_1$ with $\beta \uparrow \mu \neq \mu$ the map $(x,y) \mapsto (bx,y)$ is an automorphism of the $A^\emptyset$–module $\mathcal{M}(\mu) \oplus \mathcal{M}(\beta \uparrow \mu)$. In induces isomorphisms of A^β–modules

$$\mathcal{M}(\mu,\beta) \xrightarrow{\sim} (b,1)\mathcal{M}(\mu,\beta)$$

and

$$\begin{aligned}\mathcal{M}(\mu,\beta)^o &= \big(\mathcal{M}(\mu) \oplus \mathcal{M}(\beta \uparrow \mu)\big)/\mathcal{M}(\mu,\beta)\\ &\xrightarrow{\sim} \big(\mathcal{M}(\mu) \oplus \mathcal{M}(\beta \uparrow \mu)\big)/(b,1)\mathcal{M}(\mu,\beta).\end{aligned}$$

For each $\mu \in \Omega_1$ with $\beta \uparrow \mu = \mu$ the map $(x,y) \mapsto (x+by,y)$ is an isomorphism of A^β–modules

$$\mathcal{M}(\mu,\beta) \oplus \mathcal{M}(\mu,\beta) \xrightarrow{\sim} \{(x+by,y) | x,y \in \mathcal{M}(\mu,\beta)\}$$

and induces an isomorphism of A^β–modules

$$\begin{aligned}&\mathcal{M}(\mu)/\mathcal{M}(\mu,\beta) \oplus \mathcal{M}(\mu)/\mathcal{M}(\mu,\beta)\\ &\qquad\xrightarrow{\sim} \big(\mathcal{M}(\mu) \oplus \mathcal{M}(\mu)\big)/\{(x+by,y) \mid x,y \in \mathcal{M}(\mu,\beta)\}.\end{aligned}$$

Now the claims follow easily from 14.14(2), (3), (5).

14.16. Keep the notations from 14.15. That lemma says that $\mathcal{T}_1$ preserves (i.e., that $\mathcal{T}$ and $\mathcal{T}'$ preserve) many properties. In particular, if each $\mathcal{M}(\mu)$ with $\mu \in \Omega_1$ is free (resp. torsion free, resp. a projective module) over $A^\emptyset$, then so is each $\mathcal{T}_1\mathcal{M}(\lambda)$ with $\lambda \in \Omega_2$. There is a similar statement for the $\mathcal{M}(\mu,\beta)$ with A^β replacing $A^\emptyset$.

We see also in case $A = S$: If $\mathcal{M}$ satisfies 14.9(TF), then so does $\mathcal{T}_1\mathcal{M}$.

14.17. Lemma: *Suppose that $\mathcal{M}$ has the following property: Each $\mathcal{M}(\mu,\beta)$ with $\beta \in R^+$ and $\mu \in \Omega_1$ generates $\mathcal{M}(\mu)$ (in case $\beta \uparrow \mu = \mu$) resp. $\mathcal{M}(\mu) \oplus \mathcal{M}(\beta \uparrow \mu)$ (in case $\beta \uparrow \mu \neq \mu$) over $A^\emptyset$. Then $\mathcal{T}_1\mathcal{M}$ has the analogous property.*

Proof: The condition that $\mathcal{M}(\mu,\beta)$ generates $\mathcal{M}(\mu) \oplus \mathcal{M}(\beta \uparrow \mu)$ (in case $\beta \uparrow \mu \neq \mu$) means that there is for each $x \in \mathcal{M}(\mu) \oplus \mathcal{M}(\beta \uparrow \mu)$ an integer $m \geq 0$ with $h_\beta^m x \in \mathcal{M}(\mu,\beta)$, cf. the proof of 14.6.a. In other words, it means that there is for each $\overline{x} \in \mathcal{M}(\mu,\beta)^o$ an integer $m \geq 0$ with $h_\beta^m \overline{x} = 0$. This last version works equally well for $\beta \uparrow \mu = \mu$. Now it is clear by Lemma 14.15.c that this property is preserved under $\mathcal{T}_1$.

Remark: Note that the condition in this lemma implies the condition in Lemma 14.6 (also used in 14.7): Each $\mathcal{M}(\mu)_\beta$ generates $\mathcal{M}(\mu)$ over $A^\emptyset$. Indeed, this is obvious for $\beta \uparrow \mu = \mu$. For $\beta \uparrow \mu \neq \mu$ take any $x \in \mathcal{M}(\mu)$. As we saw in the proof, there is an integer $m \geq 0$ such that

$$h_\beta^m(x,0) = (h_\beta^m x, 0) \in \mathcal{M}(\mu,\beta).$$

Then $h_\beta^m x \in \mathcal{M}(\mu)_\beta$. This shows that $\mathcal{M}(\mu)_\beta$ generates $\mathcal{M}(\mu)$ as claimed.

15. Graded Combinatorial Categories

15.1. In this section we are going to look at certain graded rings and modules. We are going to use the general conventions from E.1. Unless explicitly stated otherwise, *graded* means **Z**–*graded* in this section.

The algebras S and S_k have a natural grading. We change this grading such that each h_α with $\alpha \in R$ has degree 2; so we have $S = \bigoplus_{d \geq 0} S_{2d}$; similarly for S_k.

Suppose until 15.4 that A is a graded S–algebra that satisfies the assumptions of 14.5. We get each A^β and $A^\emptyset$ from A by localizing with respect to a multiplicative set consisting of homogeneous elements. So we get induced gradings on the A^β and on $A^\emptyset$.

15.2. Let Ω be an orbit of W_p in X. We define a graded version $\widetilde{\mathcal{K}}(\Omega) = \widetilde{\mathcal{K}}(\Omega, A)$ of the category $\mathcal{K}(\Omega, A)$. An object in $\widetilde{\mathcal{K}}(\Omega, A)$ is an object $\mathcal{M}$ in $\mathcal{K}(\Omega, A)$ with a grading

$$\mathcal{M}(\lambda) = \bigoplus_{i \in \mathbf{Z}} \mathcal{M}(\lambda)_i \tag{1}$$

(for each $\lambda \in \Omega$) as an $A^\emptyset$–module such that each $\mathcal{M}(\lambda, \beta)$ is a homogeneous subgroup (in fact: a homogeneous A^β–submodule) of $\mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda)$ resp. of $\mathcal{M}(\lambda)$. Morphisms are morphisms in $\mathcal{K}(\Omega, A)$ that respect the gradings, i.e., with $\psi_\lambda \mathcal{M}(\lambda)_i \subset \mathcal{M}'(\lambda)_i$ for all λ and i (using the notations from 14.5).

We have an obvious forgetful functor from $\widetilde{\mathcal{K}}(\Omega, A)$ to $\mathcal{K}(\Omega, A)$. We shall often use the same notation for an object in $\widetilde{\mathcal{K}}(\Omega, A)$ and for its image in $\mathcal{K}(\Omega, A)$; it should then be clear from the context what we mean. If $\mathcal{M}$ is an object in $\widetilde{\mathcal{K}}(\Omega, A)$ and if r is an integer, then we define $\mathcal{M}\langle r \rangle$ as $\mathcal{M}$ with the grading shifted by r, i.e., with

$$\mathcal{M}\langle r \rangle(\lambda) = \mathcal{M}(\lambda)\langle r \rangle \tag{2}$$

for all $\lambda \in \Omega$. Obviously $\mathcal{M}\langle r \rangle$ and $\mathcal{M}$ define the same object in $\mathcal{K}(\Omega, A)$.

If $\mathcal{M}$ and $\mathcal{N}$ are objects in $\widetilde{\mathcal{K}}(\Omega, A)$, then set for all integers r

$$\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{M}, \mathcal{N})_r = \{\varphi \in \mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{M}, \mathcal{N}) \mid \varphi \mathcal{M}(\lambda)_i \subset \mathcal{N}(\lambda)_{i+r} \ \forall i, \lambda\}.$$

We have

$$\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{M},\mathcal{N})_r = \mathrm{Hom}_{\widetilde{\mathcal{K}}(\Omega,A)}(\mathcal{M}\langle r\rangle,\mathcal{N}) = \mathrm{Hom}_{\widetilde{\mathcal{K}}(\Omega,A)}(\mathcal{M},\mathcal{N}\langle -r\rangle) \quad (3)$$

and

$$\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{M},\mathcal{N}) = \bigoplus_{r\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{M},\mathcal{N})_r. \quad (4)$$

Note that this is a grading of $\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{M},\mathcal{N})$ as an A–module. In the case $\mathcal{M} = \mathcal{N}$ it makes $\mathrm{End}_{\mathcal{K}(\Omega,A)}(\mathcal{M})$ into a graded A–algebra. We have as in E.1(3) for all $s \in \mathbf{Z}$

$$\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{M}\langle -s\rangle,\mathcal{N}) = \mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{M},\mathcal{N})\langle s\rangle = \mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{M},\mathcal{N}\langle s\rangle). \quad (5)$$

If A' is a graded A–algebra that also satisfies the assumption in 14.5, then we have a natural extension of scalars functor from $\widetilde{\mathcal{K}}(\Omega,A)$ to $\widetilde{\mathcal{K}}(\Omega,A')$. (For $\mathcal{M}$ in $\widetilde{\mathcal{K}}(\Omega,A)$ define $\mathcal{M}_{A'}$ as an object in $\mathcal{K}(\Omega,A')$ as in 14.8 and take then the obvious grading on each $\mathcal{M}_{A'}(\lambda)$, i.e., each $x\otimes b$ with $x \in \mathcal{M}(\lambda)_i$ and $b \in A'_j$ gets degree $i+j$.) The functorial maps

$$\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{M},\mathcal{N}) \otimes_A A' \to \mathrm{Hom}_{\mathcal{K}(\Omega,A')}(\mathcal{M}_{A'},\mathcal{N}_{A'}) \quad (6)$$

(for $\mathcal{M}$ and $\mathcal{N}$ in $\widetilde{\mathcal{K}}(\Omega,A)$) preserve the grading.

15.3. Consider as an example the $\mathcal{Z}^w_\lambda$ introduced in 14.10. We make them into objects in $\widetilde{\mathcal{K}}(\Omega,A)$ by choosing the given grading on $\mathcal{Z}^w_\lambda(\lambda) = A^\emptyset$; so the generator 1 of this module gets degree 0. It is then obvious from 14.10(2), (3) that all $\mathcal{Z}^w_\lambda(\mu,\beta)$ are homogeneous submodules. We shall always take this grading, when we regard $\mathcal{Z}^w_\lambda$ as a graded object.

Take w, $x \in W$ and $\lambda \in \Omega$ and consider $\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{Z}^w_\lambda,\mathcal{Z}^x_\lambda)$ as a graded A–module. Any homomorphism $\psi : \mathcal{Z}^w_\lambda \to \mathcal{Z}^x_\lambda$ takes the generator 1 of $\mathcal{Z}^w_\lambda(\lambda)$ to an element of $\mathcal{Z}^x_\lambda(\lambda) = A^\emptyset$. So there is an element $a \in A^\emptyset$ with $\psi(1) = a$, and ψ is uniquely determined by a. The condition that ψ maps each $\mathcal{Z}^w_\lambda(\lambda,\beta)$ to $\mathcal{Z}^x_\lambda(\lambda,\beta)$ is equivalent to $a \in A^\beta$ for all β, hence to $a \in \bigcap_\beta A^\beta = A$. We get additional conditions from the β with $\beta\uparrow\lambda \neq \lambda$. If $w^{-1}\beta < 0$, then the condition is that $ah_\beta \in A^\beta h_\beta$ (if $x^{-1}\beta < 0$) resp. that $ah_\beta \in A^\beta$ (if $x^{-1}\beta > 0$). This is automatically satisfied for $a \in A$. If $w^{-1}\beta > 0$, then the condition is that $a \in A^\beta h_\beta$ (if $x^{-1}\beta < 0$) resp. that $a \in A^\beta$ (if $x^{-1}\beta > 0$). In the second case this condition follows from $a \in A$, but in the first case we have to satisfy $a \in A^\beta h_\beta \cap A$; this shows that we have an isomorphism

$$\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{Z}^w_\lambda,\mathcal{Z}^x_\lambda) \xrightarrow{\sim} A \cap \bigcap_{\substack{\beta>0,\beta\uparrow\lambda\neq\lambda\\ w^{-1}\beta<0,x^{-1}\beta>0}} A^\beta h_\beta. \quad (1)$$

The right hand side is graded and the isomorphism is compatible with the gradings.

We can do better if A is equal to S or to S_k. Then A is a unique factorization domain and h_β is irreducible in A. This implies that $A^\beta h_\beta \cap A = Ah_\beta$ for all β. Furthermore, the h_β are not proportional for distinct β (under our assumptions on the characteristic of k in Case 1), so we can rewrite (1) as

$$\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{Z}_\lambda^w, \mathcal{Z}_\lambda^x) \xrightarrow{\sim} A \cdot \prod_{\substack{\beta>0,\beta\uparrow\lambda\neq\lambda \\ w^{-1}\beta<0,x^{-1}\beta>0}} h_\beta. \tag{2}$$

For arbitrary A this works still in those cases where the product is empty, e.g., if $w = x$ or if $w = 1$.

15.4. Let us restate the most important cases of 15.3(1), (2). In order to simplify notation we set

$$\mathcal{Z}_\lambda = \mathcal{Z}_\lambda^1 \qquad \text{and} \qquad \mathcal{Z}'_\lambda = \mathcal{Z}_\lambda^{w_0} \tag{1}$$

for all $\lambda \in X$. The number

$$\begin{aligned} N_\lambda &= |\{\beta \in R^+ \mid \beta\uparrow\lambda = \lambda\}| \\ &= |\{\beta \in R^+ \mid \langle\lambda+\rho, \beta^\vee\rangle \equiv 0 \pmod p\}| \\ &= |\{H \in \mathcal{H} \mid \lambda \in H\}| \end{aligned}$$

(using the notation from 13.4) is constant on W_p-orbits. Set

$$N_\Omega = N_\lambda \qquad \text{for all } \lambda \in \Omega. \tag{2}$$

Now 15.3(1), (2) yield obviously (since each h_β has degree 2!)

Lemma: *We have for all $\lambda \in \Omega$ an isomorphism of graded A-modules*

$$\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{Z}_\lambda, \mathcal{Z}'_\lambda) \simeq A. \tag{3}$$

If A is equal to S or S_k, then we have also an isomorphism of graded A-modules

$$\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{Z}'_\lambda, \mathcal{Z}_\lambda) \simeq A\langle 2(|R^+| - N_\Omega)\rangle. \tag{4}$$

Remark: If $\mu \neq \lambda$, then there is not a weight ν with $\mathcal{Z}_\lambda^w(\nu) \neq 0$ and $\mathcal{Z}_\mu^x(\nu) \neq 0$; so we get (for all w, x and arbitrary A)

$$\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{Z}_\lambda^w, \mathcal{Z}_\mu^x) = 0. \tag{5}$$

15.5. In the next sections (until 15.12) we shall mainly look at the case $A = S_k$ and the relationship between $\widetilde{\mathcal{K}}(\Omega, S_k)$ and $\mathcal{K}(\Omega, A(k))$. If $\mathcal{M}$ is an object in $\mathcal{K}(\Omega, A(k))$, then an S_k*-form* of $\mathcal{M}$ is an object $\mathcal{M}'$ in $\mathcal{K}(\Omega, S_k)$ with $(\mathcal{M}')_{A(k)} = \mathcal{M}$; or, more rigorously, with a fixed isomorphism $(\mathcal{M}')_{A(k)} \xrightarrow{\sim} \mathcal{M}$. Similarly, a *graded* S_k*-form* of $\mathcal{M}$ is an object $\widetilde{\mathcal{M}}$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$ with $\widetilde{\mathcal{M}}_{A(k)} = \mathcal{M}$. (Note that $A(k)$ is not a graded ring; so the last identity is in $\mathcal{K}(\Omega, A(k))$, not in a — nonexistent — $\widetilde{\mathcal{K}}(\Omega, A(k))$.)

If $\mathcal{M}$ and $\mathcal{N}$ are objects in $\mathcal{K}(\Omega, A(k))$ and if $\mathcal{M}'$ resp. $\mathcal{N}'$ is an S_k–form of $\mathcal{M}$ resp. of $\mathcal{N}$, then 14.8(4) implies

$$\mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}', \mathcal{N}') \otimes_{S_k} A(k) \simeq \mathrm{Hom}_{\mathcal{K}(\Omega,A(k))}(\mathcal{M}, \mathcal{N}). \tag{1}$$

If $\widetilde{\mathcal{M}}$ is a graded S_k–form of an object $\mathcal{M}$ in $\mathcal{K}(\Omega, A(k))$, then so are all $\widetilde{\mathcal{M}}\langle r \rangle$ with $r \in \mathbf{Z}$.

We shall be mainly interested in graded S_k–forms of certain $\mathcal{V}_\Omega M$ with M in $\mathcal{FC}_{A(k)}(\Omega)$. For example, each $\mathcal{Z}^w_\lambda$ is a graded S_k–form of $\mathcal{V}_\Omega Z^w_{A(k)}(\lambda\langle w \rangle)$, cf. 14.10(4), (5).

15.6. Let $\mathcal{M}$, $\mathcal{N}$ be objects in $\mathcal{K}(\Omega, A(k))$, let $\widetilde{\mathcal{M}}$ resp. $\widetilde{\mathcal{N}}$ be a graded S_k–form of $\mathcal{M}$ resp. $\mathcal{N}$. Suppose that $\mathrm{Hom}_{\mathcal{K}(\Omega,A(k))}(\mathcal{M}, \mathcal{N})$ is a finitely generated $A(k)$–module. Then 15.5(1) and Lemma E.8.c imply that $\mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ is a finitely generated S_k–module. Since S_k lives only in degrees ≥ 0, there is then an integer r with $\mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})_i = 0$ for all $i < r$. We get therefore

$$\mathrm{Hom}_{\mathcal{K}(\Omega,A(k))}(\mathcal{M}, \mathcal{N}) \xrightarrow{\sim} \prod_{i \geq r} \mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})_i. \tag{1}$$

Note: If $\mathcal{M} = \mathcal{V}_\Omega M$ and $\mathcal{N} = \mathcal{V}_\Omega N$ for some M, N in $\mathcal{FC}_{A(k)}(\Omega)$, then $\mathrm{Hom}_{\mathcal{K}(\Omega,A(k))}(\mathcal{M}, \mathcal{N})$ is finitely generated over $A(k)$ by 9.4 and 2.8. So we can apply (1) in that case. On the other hand, in some cases we can deduce finite generation of $\mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ from Lemma 14.7.

Lemma: *Let M be a nonzero module in $\mathcal{FC}_{A(k)}(\Omega)$ and let $\mathcal{M}$ be a graded S_k-form of $\mathcal{V}_\Omega M$.*

a) $\mathcal{M}$ is not isomorphic to any $\mathcal{M}\langle i \rangle$ with $i \neq 0$.

b) If M is indecomposable, then $\mathrm{End}_{\widetilde{\mathcal{K}}(\Omega,S_k)}(\mathcal{M})$ is a local ring and $\mathcal{M}$ is indecomposable in $\widetilde{\mathcal{K}}(\Omega, S_k)$.

c) If M is indecomposable and if $\mathcal{M}'$ is another graded S_k-form of $\mathcal{V}_\Omega M$, then there is an integer n with $\mathcal{M}' \simeq \mathcal{M}\langle n \rangle$.

Proof: The discussion above implies that $\mathrm{End}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M})$ is a finitely generated graded S_k–module and that there is an integer m with $\mathrm{End}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M})_i = 0$ for all $i < m$. This implies that all $f \in \bigoplus_{i<0} \mathrm{End}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M})_i$ are nilpotent. If $f \in \mathrm{End}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M})_i$ (for arbitrary $i \in \mathbf{Z}$) is invertible, then f^{-1} has to belong to $\mathrm{End}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M})_{-i}$. We see that any $f \in \mathrm{End}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M})_i$ with $i \neq 0$ is not invertible; this yields a). If we take $i = 0$, we get from Lemma 14.2.a the first claim in b); the second one is then obvious.

Consider now $\mathcal{M}'$ as in c). We can regard the identity map on $\mathcal{V}_\Omega M$ as an element in the completion of $\mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M},\mathcal{M}')$, cf. (1). So we can decompose $\mathrm{id} = \sum_{r=-s}^{\infty} i_r$ with each $i_r \in \mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M},\mathcal{M}')_r$. Similarly, we can regard it as an element in the completion of $\mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}',\mathcal{M})$ and decompose $\mathrm{id} = \sum_{r=-s}^{\infty} j_r$. (We may assume that we start with the same index $-s \leq 0$.) We get then $\mathrm{id} = \sum_{r=-s}^{s} j_{-r} \circ i_r$ in

$$\mathrm{End}_{\widetilde{\mathcal{K}}(\Omega,S_k)}(\mathcal{M}) = \mathrm{End}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M})_0 \subset \mathrm{End}_{\mathcal{K}(\Omega,A(k))}(\mathcal{V}_\Omega M),$$

so at least one summand $j_{-n} \circ i_n$ has to be a unit in $\mathrm{End}_{\mathcal{K}(\Omega,A(k))}(\mathcal{V}_\Omega M)$. Denote its inverse by f. Then f has to be contained in $\mathrm{End}_{\widetilde{\mathcal{K}}(\Omega,S_k)}(\mathcal{M})$. We get a direct sum decomposition $\mathcal{M}' = i_n(\mathcal{M}) \oplus \ker(f \circ j_{-n})$. Since $\mathcal{M}'$ is indecomposable, this yields $\mathcal{M}' = i_n(\mathcal{M}) \simeq \mathcal{M}\langle n\rangle$ as desired.

Remark: Part c) of the lemma implies for all $\lambda \in X$ that the $\mathcal{Z}_\lambda^w\langle r\rangle$ with $r \in \mathbf{Z}$ are the only graded S_k–forms of $\mathcal{V}_\Omega Z_{A(k)}^w(\lambda\langle w\rangle)$ (for any λ and w).

15.7. Proposition: *Let M_1 and M_2 be modules in $\mathcal{FC}_{A(k)}(\Omega)$ with M_1 indecomposable. Suppose that $\mathcal{M}$ is a graded S_k-form of $\mathcal{V}_\Omega(M_1 \oplus M_2)$ and that $\mathcal{M}_1$ is a graded S_k-form of $\mathcal{V}_\Omega M_1$. Then there exists a graded S_k-form $\mathcal{M}_2$ of $\mathcal{V}_\Omega M_2$ and an integer n with $\mathcal{M} \simeq \mathcal{M}_1\langle n\rangle \oplus \mathcal{M}_2$.*

Proof: Set $M = M_1 \oplus M_2$, let $\iota : M_1 \to M$ and $\pi : M \to M_1$ be the inclusion and the projection. We can decompose, cf. 15.6(1),

$$\iota = \sum_{r=-s}^{\infty} \iota_r \qquad \text{with} \qquad \iota_r \in \mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}_1,\mathcal{M})_r$$

and

$$\pi = \sum_{r=-s}^{\infty} \pi_r \qquad \text{with} \qquad \pi_r \in \mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M},\mathcal{M}_1)_r.$$

Then $\pi \circ \iota = \sum_{r=-s}^{s} \pi_{-r} \circ \iota_r$ is the identity on $\mathcal{M}_1$. Since $\mathrm{End}_{\widetilde{\mathcal{K}}(\Omega,S_k)}(\mathcal{M}_1)$ is a local ring by Lemma 15.6.b, there is an integer n such that $\pi_{-n} \circ \iota_n$ is a unit in this ring; denote the inverse by f. Then $e = \iota_n \circ f \circ \pi_{-n}$ is an idempotent in $\operatorname{End}_{\widetilde{\mathcal{K}}(\Omega,S_k)}(\mathcal{M})$ and we have a decomposition $\mathcal{M} = e\mathcal{M} \oplus (1-e)\mathcal{M}$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$. Furthermore $\iota_n = e \circ \iota_n$ is an isomorphism $\mathcal{M}_1\langle n\rangle \xrightarrow{\sim} e\mathcal{M}$ with inverse $f \circ \pi_{-n}$. We want to take $\mathcal{M}_2 = (1-e)\mathcal{M}$ and have to show that $((1-e)\mathcal{M})_{A(k)}$ is isomorphic to $\mathcal{V}_\Omega M_2$ in $\mathcal{K}(\Omega, A(k))$.

Regard e as an endomorphism of $\mathcal{V}_\Omega M$ and identify it with its inverse image under $\mathcal{V}_\Omega$. We have $\mathcal{V}_\Omega(eM) = e(\mathcal{M}_{A(k)}) = (e\mathcal{M})_{A(k)} \simeq \mathcal{V}_\Omega M_1$, hence $eM \simeq M_1$, and $\mathcal{V}_\Omega((1-e)M) = (1-e)(\mathcal{M}_{A(k)}) = ((1-e)\mathcal{M})_{A(k)}$. We get also $M = eM \oplus (1-e)M \simeq M_1 \oplus (1-e)M$, hence $(1-e)M \simeq M_2$ by Lemma 14.2.b. So we get indeed $((1-e)\mathcal{M})_{A(k)} \simeq \mathcal{V}_\Omega M_2$.

15.8. Consider for the moment an arbitrary (graded) A as in 15.1. If $\mathcal{M}$ is an object in $\widetilde{\mathcal{K}}(\Omega, A)$ and if $\mathcal{N}$ is a compatible subobject that is also a homogeneous subobject (this amounts to: each $\mathcal{N}(\lambda)$ is a homogeneous submodule of $\mathcal{M}(\lambda)$), then we get a natural grading on $\mathcal{M}/\mathcal{N}$. The inverse image in $\mathcal{M}$ of a homogeneous subobject of $\mathcal{M}/\mathcal{N}$ is then homogeneous.

Lemma : *a) Let $\mathcal{M}$ be an object in $\widetilde{\mathcal{K}}(\Omega, S_k)$ and $\mathcal{N}$ a homogeneous subobject of $\mathcal{M}$. If $\mathcal{N}_{A(k)}$ is a compatible subobject of $\mathcal{M}_{A(k)}$, then $\mathcal{N}$ is a compatible subobject of $\mathcal{M}$.*

b) If M and $N \subset M$ are modules in $\mathcal{FC}_{A(k)}(\Omega)$ with M/N flat over $A(k)$, and if $\mathcal{M}$ and $\mathcal{N} \subset \mathcal{M}$ are graded S_k-forms of $\mathcal{V}_\Omega M$ resp. of $\mathcal{V}_\Omega N$, then $\mathcal{N}$ is a compatible subobject of $\mathcal{M}$ and $\mathcal{M}/\mathcal{N}$ is a graded S_k-form of $\mathcal{V}_\Omega(M/N)$.

Proof: a) If L is the kernel of a map as in 14.11(1), then $\mathcal{N}(\lambda,\beta) \subset L$. Both L and $\mathcal{N}(\lambda,\beta)$ are homogeneous. The assumption on $\mathcal{N}_{A(k)}$ in $\mathcal{M}_{A(k)}$ implies that $\mathcal{N}(\lambda,\beta) \otimes_{S_k} A(k) = L \otimes_{S_k} A(k)$ inside $\mathcal{M}(\lambda,\beta) \otimes_{S_k} A(k)$, hence that $(L/\mathcal{N}(\lambda,\beta)) \otimes_{S_k} A(k) = 0$. Since $L/\mathcal{N}(\lambda,\beta)$ is graded, this implies $L/\mathcal{N}(\lambda,\beta) = 0$ by Lemma E.8.a. We get thus $\mathcal{N}(\lambda,\beta) = L$ as claimed.

Now b) follows easily from a).

Remark: Suppose in b) that Q is a submodule of M containing N with M/Q and Q/N flat over $A(k)$. Suppose that $\mathcal{Q}' \subset \mathcal{M}/\mathcal{N}$ is a graded S_k-form of $\mathcal{V}_\Omega(Q/N)$. Then the inverse image $\mathcal{Q} \subset \mathcal{M}$ of $\mathcal{Q}'$ is a graded S_k-form of $\mathcal{V}_\Omega Q$. (Note that $\mathcal{Q}_{A(k)} \subset \mathcal{M}_{A(k)}$ is the inverse image of $\mathcal{Q}'_{A(k)}$ and apply the final remark in 9.12.)

15.9. Suppose that M is in $\mathcal{FC}_{A(k)}(\Omega)$ and that $\mathcal{M} \in \widetilde{\mathcal{K}}(\Omega, S_k)$ is a graded S_k-form of $\mathcal{V}_\Omega M$. We call a filtration

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}_r = \mathcal{M} \tag{1}$$

of $\mathcal{M}$ *permissible* if all $\mathcal{M}_i$ are homogeneous subobjects of $\mathcal{M}$ and if there are submodules M_i of M with M/M_i (and hence M_i) flat over $A(k)$ such that $\mathcal{V}_\Omega M_i = (\mathcal{M}_i)_{A(k)}$ for all i. The inclusions $(\mathcal{M}_i)_{A(k)} \subset (\mathcal{M}_j)_{A(k)}$ for $i < j$

yield then inclusions $M_i \subset M_j$. Each M_j/M_i is flat over $A(k)$, so $(\mathcal{M}_i)_{A(k)}$ is a compatible subobject of $(\mathcal{M}_j)_{A(k)}$. Now 15.8 implies that also each $\mathcal{M}_i$ is a compatible subobject of $\mathcal{M}_j$ and that each $\mathcal{M}_j/\mathcal{M}_i$ is a graded S_k-form of $\mathcal{V}_\Omega(M_j/M_i)$.

Suppose that (1) is a permissible filtration and that we are given for each $i > 0$ a permissible filtration of $\mathcal{M}_i/\mathcal{M}_{i-1}$. We can then refine (1) by inserting (for each $i > 0$) the inverse images in $\mathcal{M}_i$ of the terms in the given filtration of $\mathcal{M}_i/\mathcal{M}_{i-1}$. The final remark in 15.8 shows that this refinement is again a permissible filtration.

If a filtration as in (1) is permissible and if there is for each $i > 0$ a weight $\mu_i \in \Omega$ and an integer m_i such that in $\widetilde{\mathcal{K}}(\Omega, S_k)$

$$\mathcal{M}_i/\mathcal{M}_{i-1} \simeq \mathcal{Z}_{\mu_i}\langle m_i\rangle, \tag{2}$$

then we call it a *permissible $\mathcal{Z}$-filtration.* If so, then

$$\mathcal{V}_\Omega M_i/\mathcal{V}_\Omega M_{i-1} \simeq (\mathcal{M}_i/\mathcal{M}_{i-1})_{A(k)} \simeq \mathcal{V}_\Omega Z_{A(k)}(\mu_i),$$

hence $M_i/M_{i-1} \simeq Z_{A(k)}(\mu_i)$ for all i. On the other hand, if there are $\mu_i \in \Omega$ such that $M_i/M_{i-1} \simeq Z_{A(k)}(\mu_i)$ for $1 \leq i \leq r$, then each $\mathcal{M}_i/\mathcal{M}_{i-1}$ is a graded S_k-form of $\mathcal{V}_\Omega Z_{A(k)}(\mu_i)$. There are then integers m_i such that (2) holds for all i; so (1) is a permissible $\mathcal{Z}$-filtration.

We define analogously a *permissible $\mathcal{Z}'$-filtration* of $\mathcal{M}$ to be a permissible filtration as in (1) such that there are $\mu_i \in \Omega$ and integers m_i such that $\widetilde{\mathcal{K}}(\Omega, S_k)$

$$\mathcal{M}_i/\mathcal{M}_{i-1} \simeq \mathcal{Z}'_{\mu_i}\langle m_i\rangle. \tag{3}$$

Arguing as above we see that this is equivalent to $M_i/M_{i-1} \simeq Z'_{A(k)}(\mu_i)$ using the notation

$$Z'_A(\mu) = Z_A^{w_0}(\mu\langle w_0\rangle) \tag{4}$$

for any U^0-algebra A and all $\mu \in X$.

15.10. Proposition: *Let M and N be modules in $\mathcal{FC}_{A(k)}$, let $\mathcal{M}$ (resp. $\mathcal{N}$) be a graded S_k-form of $\mathcal{V}_\Omega M$ (resp. of $\mathcal{V}_\Omega N$). Suppose that $\mathcal{M}$ has a permissible $\mathcal{Z}$-filtration with factors $\mathcal{Z}_{\mu_i}\langle m_i\rangle$, $1 \leq i \leq r$, and that $\mathcal{N}$ has a permissible $\mathcal{Z}'$-filtration with factors $\mathcal{Z}'_{\nu_j}\langle n_j\rangle$, $1 \leq j \leq r$. Then we have isomorphisms of graded S_k-modules*

$$\mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M},\mathcal{N}) \simeq \bigoplus_{i,j;\mu_i=\nu_j} S_k\langle n_j - m_i\rangle \tag{1}$$

and

$$\mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{N},\mathcal{M}) \simeq \bigoplus_{i,j;\mu_i=\nu_j} S_k\langle m_i - n_j + 2(|R^+| - N_\Omega)\rangle. \tag{2}$$

Proof: Write $A = A(k)$. Fix a filtration of $\mathcal{M}$ as in 15.9(1) and consider the submodules M_i of M as in 15.9. So we have $M_i/M_{i-1} \simeq Z_A(\mu_i)$ and $\mathcal{M}_i/\mathcal{M}_{i-1} \simeq \mathcal{Z}_{\mu_i}\langle m_i\rangle$ for all i. We consider first (1) in the case where $N = Z'_A(\nu)$ for some $\nu \in X$. By 4.13(1) each

$$0 \to \mathrm{Hom}_{\mathcal{C}}(M_i/M_{i-1}, Z'_A(\nu)) \to \mathrm{Hom}_{\mathcal{C}}(M_i, Z'_A(\nu)) \\ \to \mathrm{Hom}_{\mathcal{C}}(M_{i-1}, Z'_A(\nu)) \to 0$$

is exact, hence so is each

$$0 \to \mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{V}_\Omega(M_i/M_{i-1}), \mathcal{V}_\Omega Z'_A(\nu)) \to \mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{V}_\Omega M_i, \mathcal{V}_\Omega Z'_A(\nu)) \\ \to \mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{V}_\Omega M_{i-1}, \mathcal{V}_\Omega Z'_A(\nu)) \to 0.$$

By our assumption, we can rewrite this as

$$0 \to \mathrm{Hom}_{\mathcal{K}(\Omega,A)}((\mathcal{M}_i/\mathcal{M}_{i-1})_A, (\mathcal{Z}'_\nu)_A) \to \mathrm{Hom}_{\mathcal{K}(\Omega,A)}((\mathcal{M}_i)_A, (\mathcal{Z}'_\nu)_A) \\ \to \mathrm{Hom}_{\mathcal{K}(\Omega,A)}((\mathcal{M}_{i-1})_A, (\mathcal{Z}'_\nu)_A) \to 0.$$

The maps in this sequence arise from the corresponding maps over S_k by extension of scalars from S_k to $A(k)$. Lemma E.8.b implies that the corresponding sequence over S_k (of graded S_k–modules) is again exact:

$$0 \to \mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}_i/\mathcal{M}_{i-1}, \mathcal{Z}'_\nu) \to \mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}_i, \mathcal{Z}'_\nu) \\ \to \mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}_{i-1}, \mathcal{Z}'_\nu) \to 0.$$

The first term is by 15.4(5) equal to 0 if $\mu_i \neq \nu$, otherwise it is isomorphic to $S_k\langle -m_i\rangle$ by 15.4(3) and 15.2(5). This shows that (as graded S_k–modules)

$$\mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}, \mathcal{Z}'_\nu) \simeq \bigoplus_{\mu_i=\nu} S_k\langle -m_i\rangle$$

and (slightly more generally, cf. 15.2(5)) for all $n \in \mathbf{Z}$

$$\mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}, \mathcal{Z}'_\nu\langle n\rangle) \simeq \bigoplus_{\mu_i=\nu} S_k\langle n - m_i\rangle. \tag{3}$$

Consider now arbitrary $\mathcal{N}$ with a permissible filtration as in 15.9(1) with terms now denoted by $\mathcal{N}_j$ and with corresponding submodules N_j of N. Suppose now that $N_j/N_{j-1} \simeq Z'_A(\nu_j)$ and $\mathcal{N}_j/\mathcal{N}_{j-1} \simeq \mathcal{Z}'_{\nu_j}\langle n_j\rangle$ for all j. By 4.13(1) each

$$0 \to \mathrm{Hom}_{\mathcal{C}}(M, N_{j-1}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M, N_j) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M, N_j/N_{j-1}) \to 0$$

is exact. Arguing as above we see that each

$$0 \to \mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}, \mathcal{N}_{j-1}) \longrightarrow \mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}, \mathcal{N}_j) \\ \longrightarrow \mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}, \mathcal{Z}'_{\nu_j}\langle n_j\rangle) \to 0$$

is exact. The last term is by (3) isomorphic to the direct sum of all $S_k\langle n_j - m_i\rangle$ with $\mu_i = \nu_j$. This yields (1) by induction. The proof of (2) is similar and left to the reader. (One has to use 15.4(4) instead of 15.4(3) and 4.12(1) for $w = w_0$ instead of 4.13(1).)

15.11. For a graded S_k–module M of the form

$$M \simeq \bigoplus_{i=1}^{s} S_k\langle r_i\rangle \tag{1}$$

with $r_i \in \mathbf{Z}$ the r_i are uniquely determined by M (with its grading). This follows, for example, by looking at the Poincaré series of M as a graded vector space over $k = (S_k)_0$. We can therefore define the graded rank of M as the element

$$\operatorname{rk} M = \sum_{i=1}^{s} t^{-r_i} \in \mathbf{Z}[t, t^{-1}] \tag{2}$$

in the ring $\mathbf{Z}[t,t^{-1}]$ of Laurent polynomials over $\mathbf{Z}$. (The minus sign in the exponent will simplify some formulas later on.) We have obviously

$$\operatorname{rk}(M\langle r\rangle) = t^{-r}\operatorname{rk} N \qquad \text{for all } r \in \mathbf{Z}. \tag{3}$$

Proposition 15.10 yields a formula for $\operatorname{rk}\operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M},\mathcal{N})$. Let us state explicitly two special cases: First take $\mathcal{N} = \mathcal{Z}'_\mu$ with $\mu \in \Omega$. Then

$$\operatorname{rk}\operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}, \mathcal{Z}'_\mu) = \sum_{m\in\mathbf{Z}} [\mathcal{M} : \mathcal{Z}_\mu\langle m\rangle] t^m \tag{4}$$

where $[\mathcal{M} : \mathcal{Z}_\mu\langle m\rangle]$ denotes the multiplicity of $\mathcal{Z}_\mu\langle m\rangle$ as a factor in a permissible $\mathcal{Z}$–filtration of $\mathcal{M}$. Note that this formula together with the uniqueness in (1) shows that $[\mathcal{M} : \mathcal{Z}_\mu\langle m\rangle]$ is independent of the choice of the permissible $\mathcal{Z}$–filtration. We have similarly

$$\operatorname{rk}\operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{Z}_\mu, \mathcal{N}) = \sum_{n\in\mathbf{Z}} [\mathcal{N} : \mathcal{Z}'_\mu\langle n\rangle] t^{-n} \tag{5}$$

where now $[\mathcal{N} : \mathcal{Z}'_\mu\langle n\rangle]$ denotes the multiplicity of $\mathcal{Z}'_\mu\langle n\rangle$ as a factor in a permissible $\mathcal{Z}'$–filtration of $\mathcal{N}$. As before, this number turns out to be independent of the choice of the permissible $\mathcal{Z}'$–filtration.

Note that the uniqueness result above does not extend to $S_k^\emptyset$ and the S_k^β. For example, multiplication by any h_α is an isomorphism $S_k^\emptyset \xrightarrow{\sim} S_k^\emptyset\langle -2\rangle$. On the other hand, it does extend to S.

15.12. Lemma: *Let M be a module in $\mathcal{FC}_{A(k)}(\Omega)$. Suppose that $\mathcal{M}$ is a graded S-form of $\mathcal{V}_\Omega M$. If M has a Z-filtration, then there exists a permissible $\mathcal{Z}$-filtration on $\mathcal{M}$.*

Proof: Write $A = A(k)$. Let $\lambda \in \Omega$ be maximal for $M_\lambda \neq 0$. Then there is (by 2.14) a submodule N of M such that $N \simeq Z_A(\lambda)^r$ for some integer $r > 0$ and such that M/N has a Z-filtration involving only factors $Z_A(\nu)$ with $\nu \neq \lambda$, in fact with $\nu \not\geq \lambda$. We have

$$\begin{aligned} A^r \simeq \mathrm{Hom}_{\mathcal{C}}(Z_A(\lambda), N) &= \mathrm{Hom}_{\mathcal{C}}(Z_A(\lambda), M) \\ &\simeq \mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{V}_\Omega Z_A(\lambda), \mathcal{V}_\Omega M) \\ &\simeq \mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{Z}_\lambda, \mathcal{M}) \otimes_{S_k} A. \end{aligned}$$

Lemma E.8.d implies that $\mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{Z}_\lambda, \mathcal{M})$ is free over S_k and that we can find a basis consisting of homogeneous elements. So there a basis $f_1, f_2, \ldots, f_r$ of $\mathrm{Hom}_{\mathcal{C}}(Z_A(\lambda), M)$ such that their images $\mathcal{V}f_i$ in $\mathrm{Hom}_{\mathcal{K}(\Omega,A)}(\mathcal{V}_\Omega Z_A(\lambda), \mathcal{V}_\Omega M)$ are a basis of $\mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{Z}_\lambda, \mathcal{M})$ consisting of homogeneous elements. We have now $N = \bigoplus_{i=1}^r f_i(Z_A(\lambda))$ and

$$\mathcal{V}_\Omega(f_i Z_A(\lambda)) = (\mathcal{V}f_i)\mathcal{V}_\Omega Z_A(\lambda) = ((\mathcal{V}f_i)\mathcal{Z}_\lambda)_A$$

for all i. Each $(\mathcal{V}f_i)\mathcal{Z}_\lambda$ is homogeneous. Set

$$\mathcal{M}_i = \bigoplus_{j \leq i} (\mathcal{V}f_j)\mathcal{Z}_\lambda \qquad \text{and} \qquad M_i = \bigoplus_{j \leq i} f_j Z_A(\lambda)$$

for all i, then each $\mathcal{M}_i$ is homogeneous and satisfies $(\mathcal{M}_i)_A \simeq \mathcal{V}_\Omega M_i$. So the $\mathcal{M}_i$ are a permissible $\mathcal{Z}$-filtration of $\mathcal{N} = \mathcal{M}_r$ and $\mathcal{N}$ is a graded S_k-form of $\mathcal{V}_\Omega N$. Now $\mathcal{M}/\mathcal{N}$ is a graded S_k-form of $\mathcal{V}_\Omega(M/N)$; we can apply induction to M/N, and then take inverse images in $\mathcal{M}$ to complete the proof.

Remark: A similar proof shows: If M has a Z'-filtration (i.e., a filtration with factors of the form $Z'_{A(k)}(\mu)$), then there exists a permissible $\mathcal{Z}'$-filtration on $\mathcal{M}$.

The only change in the proof is that we now take λ minimal among the weights in Ω such that $Z'_{A(k)}(\lambda)$ occurs as a factor in a Z'-filtration of M.

15.13. Consider as in 14.14 a regular orbit Ω and a good orbit Γ of W_p in X. For any $\lambda \in \Omega$ denote by λ_Γ the unique element in Γ in the closure of the alcove of λ. Set

$$o(\lambda) = o(\lambda, \Gamma) = |\{H \in \mathcal{H} \mid \lambda_\Gamma \in H, \lambda > H\}|, \tag{1}$$

$$u(\lambda) = u(\lambda, \Gamma) = |\{H \in \mathcal{H} \mid \lambda_\Gamma \in H, \lambda < H\}|, \tag{2}$$

and

$$r(\lambda) = r(\lambda, \Gamma) = o(\lambda, \Gamma) - u(\lambda, \Gamma) \tag{3}$$

using the notations from 13.4. Note that obviously (for all $\lambda \in \Omega$)

$$o(\lambda, \Gamma) + u(\lambda, \Gamma) = N_\Gamma. \tag{4}$$

Choose (for all $\beta \in R^+$ and all $\lambda \in \Omega$) the a_λ^β and b_λ^β as in 14.14. All are homogeneous elements in S^β or in $S^\beta h_\beta^{-1}$.

Lemma: *We have*

$$\deg a_\lambda^\beta = r(\lambda) - r(\beta \uparrow \lambda) \qquad \text{and} \qquad \deg b_\lambda^\beta = r(\beta \uparrow \lambda) - r(\lambda). \tag{5}$$

Proof: Recall that we work here with the grading where $\deg h_\alpha = 2$ for all α. Of course it is enough to consider a_λ^β since we take $b_\lambda^\beta = (a_\lambda^\beta)^{-1}$. Set $\mu = \lambda_\Gamma$. Consider the reflection hyperplanes H with $\mu \in H$. Let $\beta \uparrow H$ be the reflection hyperplane with $\beta \uparrow \mu \in \beta \uparrow H$ and $\alpha(\beta \uparrow H) = \pm s_\beta(\alpha(H))$. We shall compare the contribution of H to the degree of a_λ^β to its contribution to $r(\lambda)$ minus the contribution of $\beta \uparrow H$ to $r(\beta \uparrow \lambda)$.

Suppose first that $s_\beta \alpha(H) > 0$. Then $\lambda > H$ if and only if $\beta \uparrow \lambda > \beta \uparrow H$; therefore H contributes the same number ($+1$ or -1) to $r(\lambda)$ as $\beta \uparrow H$ contributes to $r(\beta \uparrow \lambda)$. So the contribution to the difference is 0; on the other hand H contributes nothing to a_λ^β.

Suppose now that $s_\beta \alpha(H) < 0$. Then either $\lambda > H$ and $\beta \uparrow \lambda < \beta \uparrow H$ or $\lambda < H$ and $\beta \uparrow \lambda > \beta \uparrow H$. In the first case we get a contribution of $+2$ to $r(\lambda) - r(\beta \uparrow \lambda)$, in the second one we get -2. On the other hand, in the first case H contribute $h_{-\alpha(H)}$ to a_λ^β, in the second case it contributes $h_{\alpha(H)}^{-1}$. So the degree of this factor is equal to the contribution to $r(\lambda) - r(\beta \uparrow \lambda)$.

15.14. In the remaining subsections of Section 15 we suppose (as in 15.1–3) that A is a graded S–algebra satisfying the conditions in 14.5. Keep the assumption on Ω and Γ from 15.13. Consider the functors $\mathcal{T}$ and $\mathcal{T}'$ between $\mathcal{K}(\Omega, A)$ and $\mathcal{K}(\Gamma, A)$ as in 14.14. We want to define functors between $\widetilde{\mathcal{K}}(\Omega, A)$ and $\widetilde{\mathcal{K}}(\Gamma, A)$ that yield $\mathcal{T}$ and $\mathcal{T}'$ when we forget the grading.

For $\mathcal{M}$ in $\widetilde{\mathcal{K}}(\Omega, A)$ define a grading on each $\mathcal{T}\mathcal{M}(\mu)$ with $\mu \in \Gamma$ by

$$\mathcal{T}\mathcal{M}(\mu)_i = \bigoplus_{\lambda_\Gamma = \mu} \mathcal{M}(\lambda)_{i - r(\lambda)} \qquad \text{for all } i \in \mathbf{Z}. \tag{1}$$

For $\mathcal{N}$ in $\widetilde{\mathcal{K}}(\Gamma, A)$ define a grading on each $\mathcal{T}'\mathcal{N}(\lambda)$ with $\lambda \in \Omega$ by

$$\mathcal{T}'\mathcal{N}(\lambda)_i = \mathcal{N}(\lambda_\Gamma)_{i + r(\lambda)} \qquad \text{for all } i \in \mathbf{Z}. \tag{2}$$

Proposition: *We get on $\mathcal{T}\mathcal{M}$ resp. on $\mathcal{T}'\mathcal{N}$ via* (1) *resp.* (2) *a structure as an element of $\widetilde{\mathcal{K}}(\Gamma, A)$ resp. of $\widetilde{\mathcal{K}}(\Omega, A)$.*

Proof: We have to show that each $\mathcal{T}\mathcal{M}(\mu, \beta)$ resp. each $\mathcal{T}'\mathcal{N}(\lambda, \beta)$ is homogeneous. Let us begin with the first case. Each $\mathcal{T}\mathcal{M}(\mu, \beta)$ is the direct sum of certain $(b_\lambda^\beta, 1)\mathcal{M}(\lambda, \beta)$ with $\lambda_\Gamma = \mu$. Now

$$\begin{aligned}(b_\lambda^\beta, 1)\mathcal{M}(\lambda, \beta)_i &\subset \mathcal{M}(\lambda)_{i + r(\beta\uparrow\lambda) - r(\lambda)} \oplus \mathcal{M}(\beta \uparrow \lambda)_i \\ &\subset \mathcal{T}\mathcal{M}(\mu)_{i + r(\beta\uparrow\lambda)} \oplus \mathcal{T}\mathcal{M}(\beta \uparrow \mu)_{i + r(\beta\uparrow\lambda)};\end{aligned}$$

this shows that $\mathcal{T}\mathcal{M}(\mu,\beta)$ is homogeneous.

Consider now $\mathcal{T}'$. If $\beta\uparrow\lambda_\Gamma\neq\lambda_\Gamma$, then $\mathcal{T}'\mathcal{N}(\lambda,\beta)=(a_\lambda^\beta,1)\mathcal{N}(\lambda_\Gamma,\beta)$ and

$$\begin{aligned}(a_\lambda^\beta,1)\mathcal{N}(\lambda_\Gamma,\beta)_i &\subset \mathcal{N}(\lambda_\Gamma)_{i+r(\lambda)-r(\beta\uparrow\lambda)}\oplus\mathcal{N}(\beta\uparrow\lambda_\Gamma)_i\\ &\subset \mathcal{T}'\mathcal{N}(\lambda)_{i-r(\beta\uparrow\lambda)}\oplus\mathcal{T}'\mathcal{N}(\beta\uparrow\lambda)_{i-r(\beta\uparrow\lambda)};\end{aligned}$$

so $\mathcal{T}'\mathcal{N}(\lambda,\beta)$ is homogeneous. The homogeneity is obvious in the case where $(\beta\uparrow\lambda)_\Gamma=\lambda_\Gamma+p\beta$. If $(\beta\uparrow\lambda)_\Gamma=\lambda_\Gamma$, then $\{(x,0)\mid x\in\mathcal{N}(\lambda_\Gamma,\beta)\}$ is clearly homogeneous; since

$$\begin{aligned}\{(a_\lambda^\beta y,y)\mid y\in\mathcal{N}(\lambda_\Gamma,\beta)_i\} &\subset \mathcal{N}(\lambda_\Gamma)_{i+r(\lambda)-r(\beta\uparrow\lambda)}\oplus\mathcal{N}(\lambda_\Gamma)_i\\ &\subset \mathcal{T}'\mathcal{N}(\lambda)_{i-r(\beta\uparrow\lambda)}\oplus\mathcal{T}'\mathcal{N}(\beta\uparrow\lambda)_{i-r(\beta\uparrow\lambda)},\end{aligned}$$

we see that $\mathcal{T}'\mathcal{N}(\lambda,\beta)$ is homogeneous also in this case.

15.15. It is clear that we get now functors between $\widetilde{\mathcal{K}}(\Omega,A)$ and $\widetilde{\mathcal{K}}(\Gamma,A)$. We shall denote them again by $\mathcal{T}$ and $\mathcal{T}'$ and call them graded translation functors. They commute with extensions of scalars. We can take in particular $A=S_k$ and get:

Lemma: *Let M be a module in $\mathcal{FC}_{A(k)}(\Omega)$. If $\mathcal{M}$ is an S_k-form of $\mathcal{V}_\Omega M$, then $\mathcal{T}\mathcal{M}$ is an S_k-form of $\mathcal{V}_\Omega TM$. If $\mathcal{M}$ is a graded S_k-form of $\mathcal{V}_\Omega M$, then $\mathcal{T}\mathcal{M}$ is a graded S_k-form of $\mathcal{V}_\Omega TM$.*

Proof: The first claim could have beeen stated already in Section 14, since it follows from 14.14(6) and 14.16. (Note that we need 14.16 to get $\mathcal{T}\mathcal{M}$ of finite type.) The extension to the graded case is now obvious.

Remarks: 1) If we have a permissible filtration $(\mathcal{M}_i)_{0\le i\le r}$ of $\mathcal{M}$ as in 15.9(1), then the $\mathcal{T}\mathcal{M}_i$ are a permissible filtration of $\mathcal{T}\mathcal{M}$.
2) Analogous statements hold for $\mathcal{T}'$ instead of $\mathcal{T}$.

15.16. Lemma: *We have for all $\lambda\in\Omega$ isomorphisms in $\widetilde{\mathcal{K}}(\Gamma,A)$*

$$\mathcal{T}\mathcal{Z}_\lambda\simeq\mathcal{Z}_{\lambda_\Gamma}\langle 2o(\lambda,\Gamma)+N_\Gamma\rangle \tag{1}$$

and

$$\mathcal{T}\mathcal{Z}'_\lambda\simeq\mathcal{Z}'_{\lambda_\Gamma}\langle N_\Gamma\rangle. \tag{2}$$

Proof: Set $\mu=\lambda_\Gamma\in\Gamma$ and $\mathcal{N}=\mathcal{T}\mathcal{Z}_\lambda$. We have

$$\mathcal{N}(\mu)=\bigoplus_{\lambda'_\Gamma=\mu}\mathcal{Z}_\lambda(\lambda')=\mathcal{Z}_\lambda(\lambda)=A^\emptyset$$

and $\mathcal{N}(\mu') = 0$ for all $\mu' \neq \mu$. Consider $\beta \in R^+$. If $\beta \uparrow \mu \neq \mu$, then

$$\mathcal{N}(\mu,\beta) = (b_\lambda^\beta, 1)\mathcal{Z}_\lambda(\lambda,\beta) = A^\beta(b_\lambda^\beta, 0) = A^\beta(1,0)$$

— since b_λ^β is a unit in A^β — and

$$\mathcal{N}(\beta\downarrow\mu,\beta) = (b_{\beta\downarrow\lambda}^\beta, 1)\mathcal{Z}_\lambda(\beta\downarrow\lambda,\beta) = A^\beta(0,h_\beta).$$

If $\beta\uparrow\mu = \mu$, denote by H_0 the hyperplane with $\alpha(H_0) = \beta$ and $\mu \in H_0$. If $\lambda < H_0$, then

$$\mathcal{N}(\mu,\beta) = (b_\lambda^\beta, 1)\mathcal{Z}_\lambda(\lambda,\beta) = A^\beta h_\beta,$$

since $b_\lambda^\beta h_\beta^{-1}$ is a unit in A^β. If $\lambda > H_0$, then

$$\mathcal{N}(\mu,\beta) = (b_{\beta\downarrow\lambda}^\beta, 1)\mathcal{Z}_\lambda(\beta\downarrow\lambda,\beta) = A^\beta h_\beta.$$

Set

$$\kappa_\mu = \prod_{\mu\in H} h_{\alpha(H)}.$$

This is a unit in $A^\emptyset$, so κ_μ is a basis of $\mathcal{N}(\mu)$ and we have an isomorphism $\psi : \mathcal{Z}_\mu(\mu) \xrightarrow{\sim} \mathcal{N}(\mu)$ with $\psi(1) = \kappa_\mu$. We claim that ψ induces an isomorphism $\mathcal{Z}_\mu \xrightarrow{\sim} \mathcal{N}$ extending it by 0 to all $\mathcal{Z}_\mu(\mu') = 0$ with $\mu' \neq \mu$. We have to check that it induces isomorphisms $\mathcal{Z}_\mu(\mu',\beta) \xrightarrow{\sim} \mathcal{N}(\mu',\beta)$ for all $\mu' \in \Gamma$ and $\beta \in R^+$. If $\beta\uparrow\mu \neq \mu$, then κ_μ is a unit in A^β; so ψ takes the basis $(1,0)$ of $\mathcal{Z}_\mu(\mu,\beta)$ to the basis $(\kappa_\mu, 0)$ of $\mathcal{N}(\mu,\beta)$ and the basis $(0,h_\beta)$ of $\mathcal{Z}_\mu(\beta\downarrow\mu,\beta)$ to the basis $(0,h_\beta\kappa_\mu)$ of $\mathcal{N}(\beta\downarrow\mu,\beta)$. If $\beta\uparrow\mu = \mu$, then $\kappa_\mu h_\beta^{-1}$ is a unit in A^β, so $\kappa_\mu = (\kappa_\mu h_\beta^{-1})h_\beta$ is a basis of $\mathcal{N}(\mu,\beta)$. It is the image under ψ of the basis 1 of $\mathcal{Z}_\mu(\mu,\beta)$, so again ψ induces an isomorphism. We have thus proved our claim. So far we have not yet taken the grading into account. By 15.14(1) the element 1 in $\mathcal{N}(\mu)$ has degree $r(\lambda)$ when regarded as an element of $\mathcal{N}(\mu)$. Furthermore κ_μ has degree $2N_\Gamma$, so ψ is homogeneous of degree $r(\lambda) + 2N_\Gamma = 2o(\lambda) + N_\Gamma$. In other words, we have the isomorphism of graded objects as claimed in (1).

Consider now $\mathcal{Z}'_\lambda$. We can go through the same series of steps as for the $\mathcal{Z}_\lambda$ above. Let us point out only the changes. In the general description at the beginning we get at the end

$$\mathcal{Z}'_\lambda(\beta\downarrow\lambda,\beta) = A^\beta(0,1) \qquad \text{if } \beta\uparrow\lambda \neq \lambda.$$

Correspondingly we get later on for $\mathcal{N} = \mathcal{T}\mathcal{Z}'_\lambda$ that

$$\mathcal{N}(\beta\downarrow\mu,\beta) = A^\beta(0,1) \qquad \text{if } \beta\uparrow\mu \neq \mu,$$

whereas we get for $\beta\uparrow\mu=\mu$

$$\mathcal{N}(\mu,\beta)=\begin{cases} A^\beta h_\beta, & \text{if } \lambda<H_0;\\ A^\beta, & \text{if } \lambda>H_0.\end{cases}$$

We consider now $\psi:\mathcal{Z}'_\mu(\mu)\xrightarrow{\sim}\mathcal{N}(\mu)$ with $\psi(1)=\kappa'_\mu$ where

$$\kappa'_\mu=\prod_{\mu\in H,\lambda<H} h_{\alpha(H)}.$$

Now κ'_μ is a unit in $A^\emptyset$, in all A^β with $\beta\uparrow\mu\neq\mu$, and in those A^β with $\beta\uparrow\mu=\mu$ and $\lambda>H_0$. In the remaining cases $\kappa'_\mu h_\beta^{-1}$ is a unit in A^β. Going through the different cases we see that ψ induces isomorphisms at all (μ,β). Now κ'_μ has degree $2u(\lambda)$ so ψ is homogeneous of degree $r(\lambda)+2u(\lambda)=N_\Gamma$. So ψ induces an isomorphism of graded objects as in (2).

15.17. Lemma: *Suppose that $A=S_k$. Let $\mu\in\Gamma$. Then $\mathcal{T}'\mathcal{Z}_\mu$ (resp. $\mathcal{T}'\mathcal{Z}'_\mu$) has a permissible filtration with factors $\mathcal{Z}_\lambda\langle -N_\Gamma\rangle$ (resp. $\mathcal{Z}'_\lambda\langle -r(\lambda,\Gamma)\rangle$) with λ running over all weights in Ω with $\lambda_\Gamma=\mu$.*

Proof: We look first at $\mathcal{M}=\mathcal{T}'\mathcal{Z}'_\mu$. We have $\mathcal{M}(\lambda)=0$ for $\lambda\in\Omega$ with $\lambda_\Gamma\neq\mu$, and $\mathcal{M}(\lambda)$ is free over $S_k^\emptyset$ of rank 1, if $\lambda_\Gamma=\mu$. In the second case we denote by g_λ the basis of $\mathcal{M}(\lambda)$ that corresponds to the basis element 1 of $\mathcal{Z}'_\mu(\mu)=S_k^\emptyset$ under the obvious identification. For our choice of grading g_λ has degree $-r(\lambda)=-r(\lambda,\Gamma)$. We have for $\beta\in R^+$ with $\beta\uparrow\mu\neq\mu$ for all λ with $\lambda_\Gamma=\mu$

$$\mathcal{M}(\lambda,\beta)=S_k^\beta(g_\lambda,0)\qquad\text{and}\qquad \mathcal{M}(\beta\downarrow\lambda,\beta)=S_k^\beta(0,g_\lambda).$$

(This uses the fact that a_λ^β is a unit in S_k^β.) If $\beta\uparrow\mu=\mu$, then we get for all λ with $\mu=\lambda_\Gamma=(\beta\uparrow\lambda)_\Gamma$

$$\mathcal{M}(\lambda,\beta)=S_k^\beta(g_\lambda,0)\oplus S_k^\beta(a_\lambda^\beta g_\lambda,g_{\beta\uparrow\lambda}),$$

$$\mathcal{M}(\beta\uparrow\lambda,\beta)=S_k^\beta(g_{\beta\uparrow\lambda},0)\qquad\text{and}\qquad \mathcal{M}(\beta\downarrow\lambda,\beta)=S_k^\beta(0,g_\lambda).$$

We choose now a numbering $\lambda_1,\lambda_2,\dots,\lambda_r$ of $\{\lambda\in\Omega\mid\lambda_\Gamma=\mu\}$ such that $\lambda_i<\lambda_j$ implies $i<j$. We have especially: If $\beta\uparrow\lambda_i=\lambda_j$, then $i<j$, and if $\beta\downarrow\lambda_i=\lambda_j$, then $j<i$.

We define a filtration

$$0=\mathcal{M}_0\subset\mathcal{M}_1\subset\mathcal{M}_2\subset\ldots\subset\mathcal{M}_r=\mathcal{M}$$

of $\mathcal{M}$ by

$$\mathcal{M}_i(\lambda_j) = \begin{cases} \mathcal{M}(\lambda_j), & \text{if } j \leq i; \\ 0, & \text{if } j > i \end{cases}$$

for all i and j and by

$$\mathcal{M}_i(\lambda, \beta) = \mathcal{M}(\lambda, \beta) \cap (\mathcal{M}_i(\lambda) \oplus \mathcal{M}_i(\beta \uparrow \lambda))$$

for all i and all $\lambda \in \Omega$ and $\beta \in R^+$. Each $\mathcal{M}(\lambda, \beta)$ is obviously homogeneous, hence so is each $\mathcal{M}_i(\lambda, \beta)$ as an intersection of homogeneous submodules. So each $\mathcal{M}_i$ is a homogeneous and compatible subobject of $\mathcal{M}$.

On the other hand, the module $M = T'Z'_{A(k)}(\mu)$ has a filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M$$

with $M_i/M_{i-1} \simeq Z'_{A(k)}(\lambda_i)$ for all $i > 0$, cf. 7.11. We have obviously

$$\mathrm{Hom}_{\mathcal{C}}(Z^\emptyset(\lambda_j), M_i^\emptyset) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(Z^\emptyset(\lambda_j), M^\emptyset), & \text{if } j \leq i; \\ 0, & \text{if } j > i. \end{cases}$$

This implies for all i and λ that $(\mathcal{M}_i)_{A(k)}(\lambda) = \mathcal{V}_\Omega M_i(\lambda)$. Since both $(\mathcal{M}_i)_{A(k)}$ and $\mathcal{V}_\Omega M_i(\lambda)$ are compatible subobjects of $\mathcal{M} = \mathcal{V}_\Omega M$, we get $(\mathcal{M}_i)_{A(k)} = \mathcal{V}_\Omega M_i(\lambda)$ for all i. So the $\mathcal{M}_i$ are a permissible filtration of $\mathcal{M}$.

Set now $\overline{\mathcal{M}_i} = \mathcal{M}_i/\mathcal{M}_{i-1}$ for all $i > 0$. Let us use the notation $g_j = g_{\lambda_j}$ to avoid double indices. We have obviously

$$\overline{\mathcal{M}_i}(\lambda) = \begin{cases} S_k^\emptyset g_i \simeq S_k^\emptyset, & \text{if } \lambda = \lambda_i; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\beta \in R^+$. We have $\mathcal{M}_i(\beta \uparrow \lambda_i) = 0$, hence in each case ($\beta \uparrow \mu \neq \mu$ and $\beta \uparrow \mu = \mu$)

$$\mathcal{M}_i(\lambda_i, \beta) = S_k^\beta(g_i, 0) = \overline{\mathcal{M}_i}(\lambda_i, \beta).$$

If $\beta \uparrow \mu \neq \mu$, then clearly

$$\mathcal{M}_i(\beta \downarrow \lambda_i, \beta) = S_k^\beta(0, g_i) = \overline{\mathcal{M}_i}(\beta \downarrow \lambda_i, \beta).$$

We get easily the same result, if $\beta \uparrow \mu = \mu$ and $(\beta \downarrow \lambda_i)_\Gamma \neq \mu$. On the other hand, if $\beta \uparrow \mu = \mu$ and $(\beta \downarrow \lambda_i)_\Gamma = \mu$, then $\beta \downarrow \lambda_i = \lambda_j$ for some $j < i$. Furthermore

$$\mathcal{M}_i(\lambda_j, \beta) = S_k^\beta(g_j, 0) \oplus S_k^\beta(a_{\lambda_j}^\beta g_j, g_i)$$

and

$$\mathcal{M}_{i-1}(\lambda_j, \beta) = S_k^\beta(g_j, 0).$$

So $\overline{\mathcal{M}_i}(\beta\downarrow\lambda_i,\beta)$ is generated freely (over S_k^β) by the image of $(a_{\lambda_j}^\beta g_j, g_i)$. If we embed $\overline{\mathcal{M}_i}(\beta\downarrow\lambda_i,\beta)$ canonically into $\overline{\mathcal{M}_i}(\beta\downarrow\lambda_i)\oplus\overline{\mathcal{M}_i}(\lambda_i)$ this generator is mapped to $(0,g_i)$. So we get also in this case that

$$\overline{\mathcal{M}_i}(\beta\downarrow\lambda_i) = S_k^\beta(0,g_i).$$

It is now clear that there is an isomorphism $\mathcal{Z}'_{\lambda_i} \xrightarrow{\sim} \overline{\mathcal{M}_i}$ taking the standard generator 1 to g_i. If we take the grading into account, we get an isomorphism

$$\mathcal{M}_i/\mathcal{M}_{i-1} \simeq \mathcal{Z}'_{\lambda_i}\langle -r(\lambda_i)\rangle.$$

The second claim in the proposition follows.

Let us now look at the first claim and set $\mathcal{M} = \mathcal{T}'\mathcal{Z}_\mu$. So we have

$$\mathcal{M}(\lambda) = \begin{cases} S_k^\emptyset g_\lambda \simeq S_k^\emptyset, & \text{if } \lambda_\Gamma = \mu; \\ 0, & \text{otherwise} \end{cases}$$

where the basis element g_λ has degree $-r(\lambda)$. We have now for $\beta\in R^+$ with $\beta\uparrow\mu\neq\mu$ for all λ with $\lambda_\Gamma=\mu$

$$\mathcal{M}(\lambda,\beta) = S_k^\beta(g_\lambda,0) \qquad\text{and}\qquad \mathcal{M}(\beta\downarrow\lambda,\beta) = S_k^\beta(0,h_\beta g_\lambda).$$

If $\beta\uparrow\mu=\mu$, then we get (as before) for all λ with $\mu=\lambda_\Gamma=(\beta\uparrow\lambda)_\Gamma$

$$\mathcal{M}(\lambda,\beta) = S_k^\beta(g_\lambda,0)\oplus S_k^\beta(a_\lambda^\beta g_\lambda, g_{\beta\uparrow\lambda}),$$

$$\mathcal{M}(\beta\uparrow\lambda,\beta) = S_k^\beta(g_{\beta\uparrow\lambda},0) \qquad\text{and}\qquad \mathcal{M}(\beta\downarrow\lambda,\beta) = S_k^\beta(0,g_\lambda).$$

We choose now a numbering $\lambda_1,\lambda_2,\ldots,\lambda_r$ of $\{\lambda\in\Omega \mid \lambda_\Gamma=\mu\}$ such that $\lambda_i<\lambda_j$ implies $i>j$. For example, we can take the reverse of the previous ordering. We have especially: If $\beta\uparrow\lambda_i=\lambda_j$, then $i>j$, and if $\beta\downarrow\lambda_i=\lambda_j$, then $j>i$.

We define again a filtration

$$0=\mathcal{M}_0\subset\mathcal{M}_1\subset\mathcal{M}_2\subset\ldots\subset\mathcal{M}_r=\mathcal{M}$$

of $\mathcal{M}$ by

$$\mathcal{M}_i(\lambda_j) = \begin{cases} \mathcal{M}(\lambda_j), & \text{if } j\le i; \\ 0, & \text{if } j>i \end{cases}$$

for all i and j and by

$$\mathcal{M}_i(\lambda,\beta) = \mathcal{M}(\lambda,\beta)\cap(\mathcal{M}_i(\lambda)\oplus\mathcal{M}_i(\beta\uparrow\lambda))$$

for all i and all $\lambda \in \Omega$ and $\beta \in R^+$. As before, each $\mathcal{M}_i$ is a homogeneous and compatible subobject of $\mathcal{M}$.

The module $M = T' Z_{A(k)}(\mu)$ has a filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M$$

with $M_i/M_{i-1} \simeq Z_{A(k)}(\lambda_i)$ for all $i > 0$, cf. 7.5. We see as in the other case that $(\mathcal{M}_i)_{A(k)} = \mathcal{V}_\Omega M_i$ for all i, hence that the $\mathcal{M}_i$ are a permissible filtration of $\mathcal{M}$. Set now $\overline{\mathcal{M}_i} = \mathcal{M}_i/\mathcal{M}_{i-1}$ for all $i > 0$. Again use the notation $g_j = g_{\lambda_j}$. We have obviously

$$\overline{\mathcal{M}_i}(\lambda) = \begin{cases} S_k^{\emptyset} g_i \simeq S_k^{\emptyset}, & \text{if } \lambda = \lambda_i; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\beta \in R^+$. If $\beta \uparrow \mu \neq \mu$, then obviously

$$\mathcal{M}_i(\lambda_i, \beta) = S_k^\beta(g_i, 0) = \overline{\mathcal{M}_i}(\lambda_i, \beta)$$

and

$$\mathcal{M}_i(\beta \downarrow \lambda_i, \beta) = S_k^\beta(0, g_i) = \overline{\mathcal{M}_i}(\beta \downarrow \lambda_i, \beta).$$

Suppose now that $\beta \uparrow \mu = \mu$. Let H_0 be the reflection hyperplane with $\mu \in H_0$ and $\alpha(H_0) = \beta$. Consider first the case where $\lambda_i < H_0$, hence $\beta \uparrow \lambda_i = \lambda_j$ for some $j < i$. Then

$$\mathcal{M}_i(\lambda_i, \beta) = S_k^\beta(g_i, 0) \oplus S_k^\beta(a_{\lambda_i}^\beta g_i, g_j)$$

and $\mathcal{M}_{i-1}(\lambda_i, \beta)$ is the intersection of this module with $S_k^{\emptyset}(0, g_j)$, hence

$$\mathcal{M}_{i-1}(\lambda_i, \beta) = S_k^\beta(0, h_\beta g_j).$$

So $\overline{\mathcal{M}_i}(\lambda_i, \beta)$ is generated freely (over S_k^β) by the image of $(a_{\lambda_i}^\beta g_i, g_j)$. (*Think!*) If we embed $\overline{\mathcal{M}_i}(\lambda_i, \beta)$ canonically into $\overline{\mathcal{M}_i}(\lambda_i) \oplus \overline{\mathcal{M}_i}(\beta \uparrow \lambda_i)$ this generator is mapped to $(a_{\lambda_i}^\beta g_i, 0)$. Since $a_{\lambda_i}^\beta h_\beta$ is a unit in S_k^β, we get

$$\overline{\mathcal{M}_i}(\lambda_i, \beta) = S_k^\beta(h_\beta^{-1} g_i, 0).$$

On the other hand, since $(\beta \downarrow \lambda_i)_\Gamma \neq \mu$, we have

$$\mathcal{M}_i(\beta \downarrow \lambda_i, \beta) = S_k^\beta(0, g_i) = \overline{\mathcal{M}_i}(\beta \downarrow \lambda_i, \beta).$$

Consider now the case where $\lambda_i > H_0$. Then $(\beta \uparrow \lambda_i)_\Gamma \neq \mu$, hence

$$\mathcal{M}_i(\lambda_i, \beta) = S_k^\beta(g_i, 0) = \overline{\mathcal{M}_i}(\lambda_i, \beta).$$

Furthermore, we see

$$\mathcal{M}_i(\beta \downarrow \lambda_i, \beta) = S_k^\beta(0, h_\beta g_i) = \overline{\mathcal{M}_i}(\beta \downarrow \lambda_i, \beta),$$

cf. the determination of $\mathcal{M}_{i-1}(\lambda_i, \beta)$ a moment ago in the other case.

These formulas for $\overline{\mathcal{M}_i}$ imply that there is an isomorphism $\mathcal{Z}_{\lambda_i} \xrightarrow{\sim} \overline{\mathcal{M}_i}$ taking the standard generator 1 to $(\kappa'_\mu)^{-1} g_i$, with κ'_μ as at the end of 15.16. This map is homogeneous of degree $-r(\lambda_i) - 2u(\lambda_i) = -N_\Gamma$. The claim follows.

Remark: The lemma holds for arbitrary A if we drop the word "permissible" from its statement.

15.18. Consider especially the case where Γ is semiregular, cf. 13.24. For each $\mu \in \Omega$ there is exactly one weight $\mu' \in \Omega$ with $(\mu')_\Gamma = \mu_\Gamma$ and $\mu' \neq \mu$. One has then either $\mu' < \mu$ or $\mu < \mu'$.

Lemma: *Suppose that Γ is semiregular. Let $\mathcal{M}$ be a graded S_k-form of an $\mathcal{V}_\Omega M$ with M in $\mathcal{FC}_{A(k)}(\Omega)$. If $\mathcal{M}$ has a permissible $\mathcal{Z}$-filtration (resp. $\mathcal{Z}'$-filtration), then so has $\mathcal{T}_s'\mathcal{T}_s\mathcal{M}$. One has then for all $\mu, \mu' \in \Omega$ with $\mu_\Gamma = (\mu')_\Gamma$ and $\mu < \mu'$ and for all $r \in \mathbf{Z}$*

$$[\mathcal{T}'\mathcal{T}\mathcal{M} : \mathcal{Z}_\mu\langle r\rangle] = [\mathcal{M} : \mathcal{Z}_\mu\langle r\rangle] + [\mathcal{M} : \mathcal{Z}_{\mu'}\langle r-2\rangle], \tag{1}$$

$$[\mathcal{T}'\mathcal{T}\mathcal{M} : \mathcal{Z}_{\mu'}\langle r\rangle] = [\mathcal{M} : \mathcal{Z}_\mu\langle r\rangle] + [\mathcal{M} : \mathcal{Z}_{\mu'}\langle r-2\rangle], \tag{2}$$

resp.

$$[\mathcal{T}'\mathcal{T}\mathcal{M} : \mathcal{Z}'_\mu\langle r\rangle] = [\mathcal{M} : \mathcal{Z}'_\mu\langle r-2\rangle] + [\mathcal{M} : \mathcal{Z}'_{\mu'}\langle r-2\rangle], \tag{3}$$

$$[\mathcal{T}'\mathcal{T}\mathcal{M} : \mathcal{Z}'_{\mu'}\langle r\rangle] = [\mathcal{M} : \mathcal{Z}'_\mu\langle r\rangle] + [\mathcal{M} : \mathcal{Z}'_{\mu'}\langle r\rangle]. \tag{4}$$

Proof: We know by 15.15 that our translation functor takes permissible filtrations to permissible filtrations. The lemmas 15.16 and 15.17 show that in our situation $\mathcal{T}'\mathcal{T}\mathcal{M}$ has a permissible filtration where each factor has a permissible $\mathcal{Z}$-filtration (resp. $\mathcal{Z}'$-filtration). The discussion in 15.9 (second paragraph) shows that we can refine the filtration of $\mathcal{T}'\mathcal{T}\mathcal{M}$ to a filtration of the desired type. This construction shows also that it is enough to prove (1),(2) for $\mathcal{M} = \mathcal{Z}_\lambda$ and to prove (3),(4) for $\mathcal{M} = \mathcal{Z}'_\lambda$ (for all $\lambda \in \Omega$).

Let $\lambda' \in \Omega$ be the weight with $(\lambda')_\Gamma = \lambda_\Gamma$ and $\lambda' \neq \lambda$. We have $N_\Gamma = 1$, and $o(\lambda) = 1 = r(\lambda)$ in case $\lambda > \lambda'$, and $o(\lambda) = 0$ and $r(\lambda) = -1$ in case $\lambda < \lambda'$. So the lemmas 15.16 and 15.17 yield in the second case that $\mathcal{T}'\mathcal{T}\mathcal{Z}_\lambda$ has a permissible filtration with factors $\mathcal{Z}_{\lambda'}$ and $\mathcal{Z}_\lambda$. Similarly, $\mathcal{T}'\mathcal{T}\mathcal{Z}'_\lambda$ has a permissible filtration with factors $\mathcal{Z}'_\lambda\langle 2\rangle$ and $\mathcal{Z}'_{\lambda'}$. In the first case $\mathcal{T}'\mathcal{T}\mathcal{Z}_\lambda$ has a permissible filtration with factors $\mathcal{Z}_\lambda\langle 2\rangle$ and $\mathcal{Z}_{\lambda'}\langle 2\rangle$, whereas $\mathcal{T}'\mathcal{T}\mathcal{Z}'_\lambda$ has a permissible filtration with factors $\mathcal{Z}'_{\lambda'}\langle 2\rangle$ and $\mathcal{Z}'_\lambda$. This yields easily the formulas (1)–(4).

16. The Main Results

16.1. So far we have been working with a fixed field k that is either algebraically closed of characteristic $p \neq 0$ (in Case 1), or the p-th cyclotomic field for some odd number p (in Case 2), cf. 1.2, 1.3. In this section we regard k as variable, but fix R, X, W, Σ. We add (k) to certain notations introduced in the preceding sections in order to indicate their dependence on k. For example, we shall denote by $B(k)$ the algebra called B before. Note that the algebra S introduced in 14.3 depends only on R, not on k.

By our conventions $p(k)$ is the characteristic of k in Case 1, it is the (multiplicative) order of ζ in Case 2. We consider only k with $p(k) \geq h$. Furthermore, we assume that $p(k)$ is odd; if R has a component of type G_2, we assume also that $p(k)$ is prime to 3.

By our assumption on $p(k)$ we can find (for each k) a weight $\lambda_0(k)$ in the interior of the first dominant alcove for $W_{p(k)}$, i.e., with

$$0 < \langle \lambda_0(k) + \rho, \alpha^\vee \rangle < p(k) \qquad \text{for all } \alpha \in R^+.$$

Set $\Omega_k = W_{p(k)} \bullet \lambda_0(k)$; this is then a regular orbit of $W_{p(k)}$ in X. Set $\Sigma_a(k)$ equal to the set of reflections with respect to the walls of the alcove of $\lambda_0(k)$. So $\Sigma_a(k)$ consists of all s_α with $\alpha \in \Sigma$ and of all $s_{\alpha,p}$ with α the dominant short root in an irreducible component of R. Then $\Sigma_a(k)$ generates $W_{p(k)}$.

We shall always identify the affine Weyl group W_a with $W_{p(k)}$ via $s_{\alpha,m} \mapsto s_{\alpha,mp}$. We have a generating set Σ_a for W_a that identifies with $\Sigma_a(k) \subset W_{p(k)}$. For all $\beta \in R^+$ and $w \in W_a$ we define $\beta \uparrow w \in W_a$ such that $(\beta \uparrow w) \bullet \lambda_0(k) = \beta \uparrow (w \bullet \lambda_0(k))$ for all k (modulo the identification $W_a \xrightarrow{\sim} W_{p(k)}$).

16.2. For each subset $\Sigma' \subset \Sigma_a$ let $W_{\Sigma'}$ be the subgroup of W_a generated by Σ'. We define a category $\mathcal{K}(W_a/W_{\Sigma'}, S)$ analogous to the categories $\mathcal{K}(\Gamma, S)$. An object $\mathcal{M}$ in $\mathcal{K}(W_a/W_{\Sigma'}, S)$ is a family $\mathcal{M}(wW_{\Sigma'})$ $(wW_{\Sigma'} \in W_a/W_{\Sigma'})$ of $S^\emptyset$-modules (almost all equal to 0) together with (for all $\beta \in R^+$ and $wW_{\Sigma'} \in W_a/W_{\Sigma'}$) an S^β-submodule $\mathcal{M}(wW_{\Sigma'}, \beta)$ of $\mathcal{M}(wW_{\Sigma'}) \oplus \mathcal{M}((\beta \uparrow w)W_{\Sigma'})$ if $(\beta \uparrow w)W_{\Sigma'} \neq wW_{\Sigma'}$, of $\mathcal{M}(wW_{\Sigma'})$ if $(\beta \uparrow w)W_{\Sigma'} = wW_{\Sigma'}$. The definition of morphisms should be obvious.

For all k each $W_{p(k)}$–orbit Γ in X has a representative μ in the closure of the first dominant alcove. Then the stabilizer of μ in W_a has the form $W_{\Sigma'}$ for some Σ' as above. The map $wW_{\Sigma'} \mapsto w{\scriptstyle\bullet}\mu$ induces a bijection $W_a/W_{\Sigma'} \xrightarrow{\sim} \Gamma$. We now identify $\mathcal{K}(W_a/W_{\Sigma'}, S)$ with $\mathcal{K}(\Gamma, S)$: We map any $\mathcal{N}$ in $\mathcal{K}(\Gamma, S)$ to the $\mathcal{M}$ in $\mathcal{K}(W_a/W_{\Sigma'}, S)$ with $\mathcal{M}(wW_{\Sigma'}) = \mathcal{N}(w{\scriptstyle\bullet}\mu)$ and $\mathcal{M}(wW_{\Sigma'}, \beta) = \mathcal{N}(w{\scriptstyle\bullet}\mu, \beta)$ for all $w \in W_a$ and all $\beta \in R^+$. It is clear that this is an isomorphism of categories.

We define similarly graded categories $\widetilde{\mathcal{K}}(W_a/W_{\Sigma'}, S)$. The construction above yields also an isomorphism of categories between $\widetilde{\mathcal{K}}(\Gamma, S)$ and $\widetilde{\mathcal{K}}(W_a/W_{\Sigma'}, S)$. In particular, we get thus isomorphisms between $\mathcal{K}(\Omega_k, S)$ and $\mathcal{K}(W_a, S)$, between $\widetilde{\mathcal{K}}(\Omega_k, S)$ and $\widetilde{\mathcal{K}}(W_a, S)$.

For Γ and μ as above the root subsystem R_μ as in 13.3(1) is determined by Σ'; so it depends on Σ' only, whether Γ is good. If so, then we define translation functors $\mathcal{T}$ from $\mathcal{K}(W_a, S)$ to $\mathcal{K}(W_a/W_{\Sigma'}, S)$ and $\mathcal{T}'$ in the opposite direction that correspond (for all k and all Γ as above) to the functors $\mathcal{T}$ and $\mathcal{T}'$ from 14.14 between $\mathcal{K}(\Omega_k, S)$ and $\mathcal{K}(\Gamma, S)$ (under our identifications as above). The main point is to note that the $a_\lambda^\beta = C^\beta(\lambda, \lambda_\Gamma)$ depend only on the $w \in W_a$ with $\lambda = w{\scriptstyle\bullet}\lambda_0(k)$ and on Σ', not on k. Details are left to the reader.

Also graded versions of $\mathcal{T}$ and $\mathcal{T}'$ can be defined that correspond to the graded translation functors from 15.14. Now the main point is to observe that the numbers $o(w{\scriptstyle\bullet}\lambda_0(k))$ and $u(w{\scriptstyle\bullet}\lambda_0(k))$ from 15.13(1),(2) depend on w and Σ' only.

We can define for each $\mu \in \mathbf{Z}R$ a functor $\mathcal{M} \mapsto \mathcal{M}[\mu]$ on $\mathcal{K}(W_a, S)$ that yields for each k the functor $\mathcal{M} \mapsto \mathcal{M}[p(k)\mu]$ on $\mathcal{K}(\Omega_k, S)$ (as in 14.12) under our identifications from above. In this way $\mathcal{K}(W_a, S)$ is a $(\mathbf{Z}R)$–category that we can identify with each $(p(k)\mathbf{Z}R)$–category $\mathcal{K}(\Omega_k, S)$ modulo the identification $\mathbf{Z}R \xrightarrow{\sim} p(k)\mathbf{Z}R$, $\lambda \mapsto p(k)\lambda$. The corresponding $\mathrm{Hom}^\sharp$ on $\mathcal{K}(W_a, S)$ yields then the $\mathrm{Hom}^\sharp$ on each $\mathcal{K}(\Omega_k, S)$ under our identifications.

16.3. Let $s \in \Sigma_a$. Choose for all k a weight $\mu(s)_k \in X$ in the closure of the alcove of $\lambda_0(k)$ such that the stabilizer in $W_{p(k)}$ of $\mu(s)_k$ is equal to $\{1, s\}$ (modulo our identification). The orbit

$$\Gamma(s)_k = W_{p(k)}{\scriptstyle\bullet}\mu(s)_k \tag{1}$$

is then a semiregular orbit, hence good, cf. 13.24. There is for each $\lambda \in \Omega_k$ a unique weight $\lambda' \in \Omega_k$ with $(\lambda')_\Gamma = \lambda_\Gamma$ (where $\Gamma = \Gamma(s)_k$) and $\lambda' \neq \lambda$. Denote this weight by λs; if $\lambda = w{\scriptstyle\bullet}\lambda_0(k)$ with $w \in W_{p(k)}$, then $\lambda s = (ws){\scriptstyle\bullet}\lambda_0(k)$.

We have for each $B(k)$–algebra B' a translation functor T from $\mathcal{C}_{B'}(\Omega_k)$ to $\mathcal{C}_{B'}(\Gamma(s)_k)$ and a translation functor T' in the other direction. We denote their composition by $\Theta_s = T' \circ T$. This functor from $\mathcal{C}_{B'}(\Omega_k)$ to itself is usually called the translation functor *through the s–wall.* (The $B(k)$–algebras that we are mainly interested in are $B' = A(k)$ and $B' = k$ via the augmentation map.)

We have for each A as in 14.5 combinatorial translation functors $\mathcal{T}$ and $\mathcal{T}'$ as in 14.14 between $\mathcal{K}(\Omega_k, A)$ and $\mathcal{K}(\Gamma(s)_k, A)$. We denote their composition by $\widetilde{\Theta}_s$. So this is a functor from $\mathcal{K}(\Omega_k, A)$ to itself. If A is graded, then we take for $\mathcal{T}$ and $\mathcal{T}'$ their graded versions as in 15.14 and get a functor on $\widetilde{\mathcal{K}}(\Omega_k, A)$ also denoted by $\widetilde{\Theta}_s$.

We can carry out the construction of $\widetilde{\Theta}_s$ especially for $A = S$. Using the identification of $\widetilde{\mathcal{K}}(W_a, S)$ with each $\widetilde{\mathcal{K}}(\Omega_k, S)$ we get then for each k a translation functor on $\widetilde{\mathcal{K}}(\Omega_k, S)$. It is independent of k (by the discussion in 16.2); we denote this functor again by $\widetilde{\Theta}_s$.

16.4. Let $\nu \in X$. Consider for each k the special point $p(k)\nu - \rho$ (special for $W_{p(k)}$) and its (good) orbit

$$\Gamma_k = W_{p(k)} \bullet (p(k)\nu - \rho) = p(k)\nu - \rho + p(k)\mathbf{Z}R.$$

Consider for each $B(k)$–algebra B' the translation functor T' from $\mathcal{C}_{B'}(\Gamma_k)$ to $\mathcal{C}_{B'}(\Omega_k)$ and set

$$Q_{\emptyset,\nu}(B') = T' Z_{B'}(p(k)\nu - \rho) = T' Z'_{B'}(p(k)\nu - \rho). \tag{1}$$

(Recall from the remark in 9.5 that $Z_{B'}(\mu) \simeq Z'_{B'}(\mu)$ for any special point μ.)

Define similarly for each A as in 14.5

$$\mathcal{Q}_{\emptyset,\nu}(A) = \mathcal{T}' \mathcal{Z}_{p(k)\nu-\rho}(A) = \mathcal{T}' \mathcal{Z}'_{p(k)\nu-\rho}(A) \tag{2}$$

where $\mathcal{T}'$ is the combinatorial analogue to T'. If A is graded, then we take the graded $\mathcal{T}'$ and regard $\mathcal{Q}_{\emptyset,\nu}(A)$ as an element in $\widetilde{\mathcal{K}}(\Omega_k, A)$.

By our identifications each k gives rise to an object $\mathcal{Q}_{\emptyset,\nu}(S)$ in $\widetilde{\mathcal{K}}(W_a, S)$. This object is independent of k. (Well, there is $\Sigma' \subset \Sigma_a$ such that $W_a/W_{\Sigma'}$ is identified with Γ_k for all k. Then there is $\mathcal{Z}$ in $\widetilde{\mathcal{K}}(W_a/W_{\Sigma'}, S)$ corresponding to each $\mathcal{Z}_{p(k)\nu-\rho}(S)$. Finally the analogue to $\mathcal{T}'$ on $\widetilde{\mathcal{K}}(W_a/W_{\Sigma'}, S)$ is independent of k.)

16.5. For each pair (I, ν), where I is a (finite) sequence $I = (i_1, i_2, \dots, i_r)$ of elements in Σ_a and where $\nu \in X$, set (for each $B(k)$–algebra B')

$$Q_{I,\nu}(B') = \Theta_{i_1} \Theta_{i_2} \cdots \Theta_{i_r} Q_{\emptyset,\nu}(B'). \tag{1}$$

Proposition 9.7 implies that $Z_{B'}(p(k)\nu - \rho)$ is projective in $\mathcal{C}_{B'}$. Since the translation functors take projective modules to projective modules (7.7), also each $Q_{I,\nu}(B')$ is a projective module in $\mathcal{C}_{B'}(\Omega_k)$. The translation functors commute with extension of scalars, cf. 7.5(2). Therefore there are natural isomorphisms

$$Q_{I,\nu}(A(k)) \otimes_{A(k)} k \simeq Q_{I,\nu}(k). \tag{2}$$

Lemma 3.3 yields for all pairs (I, ν) and (J, μ) as above an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(Q_{I,\nu}(A(k)), Q_{J,\mu}(A(k))) \otimes_{A(k)} k \simeq \mathrm{Hom}_{\mathcal{C}}(Q_{I,\nu}(k), Q_{J,\mu}(k)). \tag{3}$$

Similarly, set for all (I, ν) as in (1)

$$\mathcal{Q}_{I,\nu}(A) = \widetilde{\Theta}_{i_1} \widetilde{\Theta}_{i_2} \cdots \widetilde{\Theta}_{i_r} \mathcal{Q}_{\emptyset,\nu}(A) \tag{4}$$

for each algebra A as in 14.5. If A is a graded algebra, then we regard (4) as a definition in $\widetilde{\mathcal{K}}(\Omega_k, A)$ using the graded translation functors as in 15.14. The statements in 16.3 and 16.4 about independence from k show that we get an object $\mathcal{Q}_{I,\nu}(S)$ in $\widetilde{\mathcal{K}}(W_a, S)$ that is independent of k.

By 15.15 each $\mathcal{Q}_{I,\nu}(S_k)$ is a graded S_k–form of $\mathcal{V}_{\Omega_k} Q_{I,\nu}(A(k))$; it admits a permissible $\mathcal{Z}$–filtration and a permissible $\mathcal{Z}'$–filtration. Also the combinatorial translation functors commute with extension of scalars; so we have isomorphisms (of graded objects)

$$\mathcal{Q}_{I,\nu}(S) \otimes_S S_k \simeq \mathcal{Q}_{I,\nu}(S_k). \tag{5}$$

The compatibility of the translation functors with the functors $\mathcal{Q} \mapsto \mathcal{Q}[\omega]$ for $\omega \in \mathbf{Z}R$ (cf. 14.14) and 14.12(1) imply that for all A and (I, ν)

$$\mathcal{Q}_{I,\nu}(A)[\omega] = \mathcal{Q}_{I,\nu+\omega}(A). \tag{6}$$

Proposition: *a) For each S-algebra A as in* 14.5 *and each pair (I, ν) as above $\mathcal{Q}_{I,\nu}(A)$ is of finite type. Each $\mathcal{Q}_{I,\nu}(A)(w)$ with $w \in W_a$ is free of finite rank over $A^{\emptyset}$. Each* $\mathrm{Hom}_{\mathcal{K}(\Omega_k,A)}(\mathcal{Q}_{I,\nu}(A), \mathcal{Q}_{J,\mu}(A))$ *(where (I, ν) and (J, μ) are two sequences as above) is a finitely generated and torsion free A–module.*

b) Each $\mathcal{Q}_{I,\nu}(S)$ satisfies 14.9(TF) *for all primes p.*

c) For all (I, ν) and (J, μ) as above there are isomorphisms

$$\mathrm{Hom}_{\mathcal{K}(\Omega_k,S_k)}(\mathcal{Q}_{I,\nu}(S_k), \mathcal{Q}_{J,\mu}(S_k)) \otimes_{S_k} k \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Q_{I,\nu}(k), Q_{J,\mu}(k))$$

and (in Case 1 *resp. in* Case 2*)*

$$\mathrm{Hom}^{\sharp}_{\mathcal{K}(\Omega_k,S_k)}(\mathcal{Q}_{I,\nu}(S_k), \mathcal{Q}_{J,\mu}(S_k)) \otimes_{S_k} k \xrightarrow{\sim} \begin{cases} \mathrm{Hom}_{\mathfrak{g}}(Q_{I,\nu}(k), Q_{J,\mu}(k)), \\ \mathrm{Hom}_{\mathbf{u}}(Q_{I,\nu}(k), Q_{J,\mu}(k)). \end{cases}$$

We regard here k as an S_k-algebra via the augmentation map.

Proof: By 14.16 the $\mathcal{Q}_{I,\nu}(A)$ satisfy the first two claims in a) as well as (for $A = S$) the claim in b), since they are obviously satisfied by $\mathcal{Z}_{p(k)\nu-\rho}$. Also the property in Lemma 14.17 is inherited, hence (see the remark in 14.17) each $\mathcal{Q}_{I,\nu}(A)(w)_\beta$ (with $w \in W_a$ and $\beta \in R^+$) generates $\mathcal{Q}_{I,\nu}(A)(w)$ over $A^{\emptyset}$. Therefore Lemma 14.7 yields the last claim in a).

We can also apply 14.8(4) to the flat ring extension $A(k) \supset S_k$ for the $\mathcal{Q}_{I,\nu}$. Combine this with the fact that each $\mathcal{Q}_{I,\nu}(S_k)$ is an S_k–form of $\mathcal{V}_{\Omega_k} Q_{I,\nu}(A(k))$. We get thus by Theorem 9.4 isomorphisms

$$\mathrm{Hom}_{\mathcal{K}(\Omega_k,S_k)}(\mathcal{Q}_{I,\nu}(S_k), \mathcal{Q}_{J,\mu}(S_k)) \otimes_{S_k} A(k) \simeq \mathrm{Hom}_{\mathcal{C}}(Q_{I,\nu}(A(k)), Q_{J,\mu}(A(k))).$$

The first claim in c) now follows from (3), the second one from 14.13(4), the first claim, and (6).

16.6. Lemma: *For all pairs* (I,ν) *and* (J,μ) *as in* 16.5 *the element*

$$\operatorname{rk} \operatorname{Hom}_{\mathcal{K}(\Omega_k,S_k)}(\mathcal{Q}_{I,\nu}(S_k), \mathcal{Q}_{J,\mu}(S_k)) \in \mathbf{Z}[t,t^{-1}]$$

is independent of k.

Proof: Each factor in a permissible $\mathcal{Z}$–filtration of $\mathcal{Q}_{I,\nu}(S_k)$ has the form $\mathcal{Z}_{w\bullet\lambda_0(k)}(S_k)\langle r\rangle$ with $w \in W_a$ and $r \in \mathbf{Z}$. The multiplicity

$$[\mathcal{Q}_{I,\nu}(S_k) : \mathcal{Z}_{w\bullet\lambda_0(k)}(S_k)\langle r\rangle]$$

depends only on I, ν and w. For $I = \emptyset$ this follows from Lemma 15.17. For arbitrary I use induction on the length of I and Lemma 15.18 (where $\nu' = \nu s$ and $\nu' s = \nu$ in our present notation). There is a similar result for the factors in a permissible $\mathcal{Z}'$–filtration of $Q_{J,\mu}(S_k)$. Now the claim follows from Proposition 15.10.

16.7. Theorem: *For all pairs* (I,ν) *and* (J,μ) *as in* 16.5 *the natural maps arising from extension of scalars are isomorphisms*

$$\operatorname{Hom}_{\mathcal{K}(W_a,S)}(\mathcal{Q}_{I,\nu}(S), \mathcal{Q}_{J,\mu}(S)) \otimes_S S_k \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{K}(\Omega_k,S_k)}(\mathcal{Q}_{I,\nu}(S_k), \mathcal{Q}_{J,\mu}(S_k)) \tag{1}$$

and

$$\operatorname{Hom}^\sharp_{\mathcal{K}(W_a,S)}(\mathcal{Q}_{I,\nu}(S), \mathcal{Q}_{J,\mu}(S)) \otimes_S S_k \xrightarrow{\sim} \operatorname{Hom}^\sharp_{\mathcal{K}(\Omega_k,S_k)}(\mathcal{Q}_{I,\nu}(S_k), \mathcal{Q}_{J,\mu}(S_k)).$$

Proof: The second claim will follow from the first one using 16.5(6). So we shall deal with (1) only.

Note that we can rewrite the left hand side in (1) as

$$\operatorname{Hom}_{\mathcal{K}(\Omega_k,S)}(\mathcal{Q}_{I,\nu}(S), \mathcal{Q}_{J,\mu}(S)) \otimes_{\mathbf{Z}} k.$$

In Case 2 our claim follows from 14.8(4) and Proposition 16.5. Suppose that we are in Case 1. We get from Lemma 14.9 that the map in (1) is injective. By Proposition 16.5 and by 15.2(4) each $\operatorname{Hom}_{\mathcal{K}(\Omega_k,S)}(\mathcal{Q}_{I,\nu}(S), \mathcal{Q}_{J,\mu}(S))$ is a finitely generated and torsion free graded S–module. Therefore each $\operatorname{Hom}_{\mathcal{K}(\Omega_k,S)}(\mathcal{Q}_{I,\nu}(S), \mathcal{Q}_{J,\mu}(S))_r$ with $r \in \mathbf{Z}$ is a finitely generated and torsion free $\mathbf{Z}$–module, hence it is free of finite rank over $\mathbf{Z}$. Denote its rank by $n(r)$. Similarly, $\operatorname{Hom}_{\mathcal{K}(\Omega_k,S_k)}(\mathcal{Q}_{I,\nu}(S_k), \mathcal{Q}_{J,\mu}(S_k))$ is a finitely generated and graded S_k–module, hence each $\operatorname{Hom}_{\mathcal{K}(\Omega_k,S_k)}(\mathcal{Q}_{I,\nu}(S_k), \mathcal{Q}_{J,\mu}(S_k))_r$ a finite dimensional vector space over k. Denote its dimension by $m(r)$. The map in (1) respects the grading and induces injective maps

$$\operatorname{Hom}_{\mathcal{K}(\Omega_k,S)}(\mathcal{Q}_{I,\nu}(S), \mathcal{Q}_{J,\mu}(S))_r \otimes_{\mathbf{Z}} k \to \operatorname{Hom}_{\mathcal{K}(\Omega_k,S_k)}(\mathcal{Q}_{I,\nu}(S_k), \mathcal{Q}_{J,\mu}(S_k))_r. \tag{2}$$

In order to see that (1) is an isomorphism it is enough to show that both sides in (2) have the same dimension, i.e., that $n(r) = m(r)$ for all $r \in \mathbf{Z}$.

Let k' be the cyclotomic field that we get from $\mathbf{Q}$ by adjoining a primitive p–th root of unity. We can study for our root system R also the category $\mathcal{C}_{k'}$ working in the quantum case with a primitive p–th root of unity. So there are also $\mathcal{Q}_{I,\nu}(S_{k'})$ and $\mathcal{Q}_{J,\mu}(S_{k'})$ in $\widetilde{\mathcal{K}}(\Omega_{k'}, S_{k'})$. Since we know that the map in (1) is an isomorphism in Case 2, we have

$$\mathrm{Hom}_{\mathcal{K}(W_a,S)}(\mathcal{Q}_{I,\nu}(S), \mathcal{Q}_{J,\mu}(S)) \otimes_{\mathbf{Z}} k' \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(\Omega_{k'},S_{k'})}(\mathcal{Q}_{I,\nu}(S_{k'}), \mathcal{Q}_{J,\mu}(S_{k'})). \tag{3}$$

(Recall that $\mathcal{Q}_{I,\nu}(S)$ and $\mathcal{Q}_{J,\mu}(S)$ are independent of k.) Since the map in (3) respects the grading we get especially

$$n(r) = \dim_{k'} \mathrm{Hom}_{\mathcal{K}(\Omega_{k'},S_{k'})}(\mathcal{Q}_{I,\nu}(S_{k'}), Q_{J,\mu}(S_{k'}))_r. \tag{4}$$

These dimensions are determined by $\mathrm{rk}\,\mathrm{Hom}_{\mathcal{K}(\Omega_{k'},S_{k'})}(\mathcal{Q}_{I,\nu}(S_{k'}), \mathcal{Q}_{J,\mu}(S_{k'}))$; recall the definition in 15.11. So Lemma 16.6 implies that $n(r) = m(r)$ for all r as desired.

16.8. For all pairs (I, ν) and (J, μ) as in 16.5 set

$$\mathcal{E}[(I,\nu),(J,\mu)] = \mathrm{Hom}_{\mathcal{K}(W_a,S)}(\mathcal{Q}_{I,\nu}(S), \mathcal{Q}_{J,\mu}(S)) \otimes_S \mathbf{Z} \tag{1}$$

and

$$\mathcal{E}^\sharp[(I,\nu),(J,\mu)] = \mathrm{Hom}^\sharp_{\mathcal{K}(W_a,S)}(\mathcal{Q}_{I,\nu}(S), \mathcal{Q}_{J,\mu}(S)) \otimes_S \mathbf{Z} \tag{2}$$

where we regard $\mathbf{Z}$ as an S–algebra via the augmentation map. By Lemma 14.7 each $\mathcal{E}[(I,\nu),(J,\mu)]$ and each $\mathcal{E}^\sharp[(I,\nu),(J,\mu)]$ is finitely generated as a $\mathbf{Z}$–module. (In the second case recall that direct sums as in 14.13(1) involve only finitely many non-zero terms.)

Corollary: *We have for all (I,ν) and (J,μ) as above natural isomorphisms*

$$\mathcal{E}[(I,\nu),(J,\mu)] \otimes_{\mathbf{Z}} k \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Q_{I,\nu}(k), Q_{J,\mu}(k)) \tag{3}$$

and

$$\mathcal{E}^\sharp[(I,\nu),(J,\mu)] \otimes_{\mathbf{Z}} k \xrightarrow{\sim} \begin{cases} \mathrm{Hom}_{\mathfrak{g}}(Q_{I,\nu}(k), Q_{J,\mu}(k)), & \textit{in } \text{Case 1;} \\ \mathrm{Hom}_{\mathbf{u}}(Q_{I,\nu}(k), Q_{J,\mu}(k)), & \textit{in } \text{Case 2.} \end{cases} \tag{4}$$

Proof: Well, Theorem 16.7 combined with Proposition 16.5.c yields an isomorphism

$$\mathrm{Hom}_{\mathcal{K}(W_a,S)}(\mathcal{Q}_{I,\nu}(S), \mathcal{Q}_{J,\mu}(S)) \otimes_S k \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Q_{I,\nu}(k), Q_{J,\mu}(k)); \tag{5}$$

here we regard k as an S–algebra via the composition $S \to S_k \to k$ where the last map is the augmentation map. This map is equal to the composition $S \to \mathbf{Z} \to k$, where the first map is the augmentation map. Therefore we can rewrite the left hand side in (5) as

$$\mathrm{Hom}_{\mathcal{K}(W_a,S)}(\mathcal{Q}_{I,\nu}(S), \mathcal{Q}_{J,\mu}(S)) \otimes_S \mathbf{Z} \otimes_{\mathbf{Z}} k = \mathcal{E}[(I,\nu),(J,\mu)] \otimes_{\mathbf{Z}} k.$$

So (3) follows. To get (4) replace all Hom by $\mathrm{Hom}^\sharp$.

16.9. For three pairs (I,ν), (J,μ), and (K,ξ) as in 16.5 the composition of morphisms in $\mathcal{K}(W_a,S)$ induces bilinear maps

$$\mathcal{E}[(I,\nu),(J,\mu)] \times \mathcal{E}[(J,\mu),(K,\xi)] \longrightarrow \mathcal{E}[(I,\nu),(K,\xi)] \tag{1}$$

with the usual associativity properties. Under isomorphisms as in 16.8(3) they correspond to the composition of morphisms in $\mathcal{C}$. A similar remark holds for the $\mathcal{E}^\sharp$, cf. E.3(2).

For any finite set $\mathcal{I}$ of pairs (I,ν) as above set $Q_\mathcal{I}(k)$ resp. $\mathcal{Q}_\mathcal{I}(S)$ equal to the direct sum of all $Q^{(I,\nu)}(k)$ resp. $\mathcal{Q}^{(I,\nu)}(S)$ with $(I,\nu)\in\mathcal{I}$. Set $\mathcal{E}_\mathcal{I}$ resp. $\mathcal{E}^\sharp_\mathcal{I}$ equal to the direct sum of all $\mathcal{E}[(I,\nu),(J,\mu)]$ resp. $\mathcal{E}^\sharp[(I,\nu),(J,\mu)]$ with $(I,\nu),(J,\mu)\in\mathcal{I}$. Each $\mathcal{E}_\mathcal{I}$ and each $\mathcal{E}^\sharp_\mathcal{I}$ is a finitely generated and graded $\mathbf{Z}$–module. We have isomorphisms

$$\begin{aligned}\mathrm{End}_{\mathcal{K}(W_a,S)}\mathcal{Q}_\mathcal{I}(S)\otimes_S \mathbf{Z} &\xrightarrow{\sim} \mathcal{E}_\mathcal{I},\\ \mathrm{End}^\sharp_{\mathcal{K}(W_a,S)}\mathcal{Q}_\mathcal{I}(S)\otimes_S \mathbf{Z} &\xrightarrow{\sim} \mathcal{E}^\sharp_\mathcal{I},\end{aligned} \tag{2}$$

and

$$\begin{aligned}\mathcal{E}_\mathcal{I}\otimes_\mathbf{Z} k &\xrightarrow{\sim} \mathrm{End}_\mathcal{C} Q_\mathcal{I}(k),\\ \mathcal{E}^\sharp_\mathcal{I}\otimes_\mathbf{Z} k &\xrightarrow{\sim} \begin{cases}\mathrm{End}_\mathfrak{g} Q_\mathcal{I}(k), & \text{in Case 1;}\\ \mathrm{End}_\mathfrak{u} Q_\mathcal{I}(k), & \text{in Case 2.}\end{cases}\end{aligned} \tag{3}$$

We get from (2) a structure of a $\mathbf{Z}$–algebra on $\mathcal{E}_\mathcal{I}$ resp. on $\mathcal{E}^\sharp_\mathcal{I}$ that also arises from maps as in (1). We get in (3) now algebra isomorphisms.

16.10. For each prime number p fix an algebraically closed field $k(p)$ of characteristic p. Consider (in this subsection) a positive integer $d>0$ and set $\mathbf{Z}'=\mathbf{Z}[d^{-1}]$ and $S'=S[d^{-1}]$.

Proposition: *Let M be a finitely generated graded S'–module that is torsion free as a $\mathbf{Z}'$–module. Suppose that $M\otimes_{\mathbf{Z}'}k(p)$ is a free graded $S_{k(p)}$–module for all prime numbers p not dividing d, and that $\mathrm{rk}\, M\otimes_{\mathbf{Z}'}k(p)$ is independent of p (for these p). Then M is a free graded S'–module.*

Proof: Note that we use $\mathrm{rk}\, M\otimes_{\mathbf{Z}'}k(p)$ here in the sense of 15.11(2). We prove the claim by induction on the (ordinary) rank of this free $S_{k(p)}$–module. Assume that $M\neq 0$. Each M_s is a free $\mathbf{Z}'$–module of finite rank. Let $r\in\mathbf{Z}$ be minimal for $M_r\neq 0$. Choose $v\in M_r$ that is part of a basis of M_r over $\mathbf{Z}'$. Then

$$v\otimes 1\in M_r\otimes_{\mathbf{Z}'}k(p)\subset M\otimes_{\mathbf{Z}'}k(p)$$

is nonzero (for all p as above). Since $M\otimes_{\mathbf{Z}'}k(p)$ is a free graded $S_{k(p)}$–module and since r is minimal, the $S_{k(p)}$–submodule $S_{k(p)}(v\otimes 1)$ generated by $v\otimes 1$

is isomorphic to $S_{k(p)}\langle r\rangle$ and the factor module $M \otimes_{\mathbf{Z}'} k(p)/S_{k(p)}(v\otimes 1)$ is graded free with

$$\mathrm{rk}\,\big(M \otimes_{\mathbf{Z}'} k(p)/S_{k(p)}(v\otimes 1)\big) \;=\; \mathrm{rk}\,(M\otimes_{\mathbf{Z}'} k(p)) \;-\; t^{-r}. \tag{1}$$

This implies that the graded S'–module $S'v$ is isomorphic to $S'\langle r\rangle$ and that the inclusion $S'v \hookrightarrow M$ induces an injection after reduction modulo p. So the short exact sequence

$$0 \to S'v \longrightarrow M \longrightarrow M/S'v \to 0$$

induces exact sequences (for all $n \in \mathbf{Z}$)

$$0 \to S_{k(p)}(v\otimes 1)_n \longrightarrow M_n \otimes_{\mathbf{Z}'} k(p) \longrightarrow (M/S'v)_n \otimes_{\mathbf{Z}'} k(p) \to 0.$$

We have

$$(M/S'v)_n \otimes_{\mathbf{Z}'} k(p) \simeq \big(M\otimes_{\mathbf{Z}'} k(p)/S_{k(p)}(v\otimes 1)\big)_n .$$

Now (1) and our assumption on M (on the independence of $\mathrm{rk}\, M\otimes_{\mathbf{Z}'} k(p)$) imply that the dimension of this vector space over $k(p)$ is independent of p. This shows that the finitely generated $\mathbf{Z}'$–module $(M/S'v)_n$ has no p–torsion for any p. Therefore $M/S'v$ is a torsion free $\mathbf{Z}'$–module. It satisfies the assumptions of the lemma and we can apply induction to it. Then the claim for M follows.

16.11. Corollary: *Let d be the product of all prime numbers less than h. All $\mathcal{E}[(I,\nu),(J,\mu)]\otimes_{\mathbf{Z}}\mathbf{Z}[d^{-1}]$ and all $\mathcal{E}^\sharp[(I,\nu),(J,\mu)]\otimes_{\mathbf{Z}}\mathbf{Z}[d^{-1}]$ are free graded $\mathbf{Z}[d^{-1}]$–modules of finite rank.*

Proof: All $\mathrm{Hom}_{\mathcal{K}(W_a,S)}(\mathcal{Q}_{I,\nu}(S),\mathcal{Q}_{J,\mu}(S))\otimes_{\mathbf{Z}}\mathbf{Z}[d^{-1}]$ are by Proposition 16.10 free graded $(S\otimes_{\mathbf{Z}}\mathbf{Z}[d^{-1}])$–modules of finite rank, since we can take in 16.6 all $k(p)$ with $p \geq h$. The same argument works for Hom replaced by $\mathrm{Hom}^\sharp$. If we reduce these modules modulo the augmentation ideal, we get the modules in the corollary and the claim follows.

Remark: Similarly all $\mathcal{E}_{\mathcal{I}}\otimes_{\mathbf{Z}}\mathbf{Z}[d^{-1}]$ and $\mathcal{E}^\sharp_{\mathcal{I}}\otimes_{\mathbf{Z}}\mathbf{Z}[d^{-1}]$ are free graded $\mathbf{Z}[d^{-1}]$–modules of finite rank.

16.12. Set

$$X_{p(k)} = \{\mu\in X \mid 0 \leq \langle\mu,\alpha^\vee\rangle < p(k) \quad \text{for all simple roots } \alpha\}, \tag{1}$$

the restricted region for $p(k)$. Its translates of the form $p(k)\nu + X_{p(k)}$ with $\nu \in X$ are called the boxes for $W_{p(k)}$. Each $\lambda\in X$ is contained in exactly one of these boxes (for a given k).

For each $w \in W_a$ there is a unique $\nu = \nu(w) \in X$ such that $w{\scriptstyle\bullet}\lambda_0(k)$ is contained in the box $p(k)(\nu - \rho) + X_{p(k)}$ for all k. We want to associate to w a sequence $I = I(w)$ as in 16.5 and set

$$Q^{(w)}(B') = Q_{I(w),\nu(w)}(B') \qquad \text{and} \qquad \mathcal{Q}^{(w)}(A) = \mathcal{Q}_{I(w),\nu(w)}(A) \tag{2}$$

for all $B(k)$–algebras B' and for all A as in 16.5(4). We do this by induction on the number of reflection hyperplanes separating $w{\scriptstyle\bullet}\lambda_0(k)$ from the special point $p(k)\nu - \rho$. If that number is 0, then $w{\scriptstyle\bullet}\lambda_0(k)$ is in the top alcove of the box $p(k)(\nu - \rho) + X_{p(k)}$; in that case we take $I = \emptyset$. Otherwise we can find $s \in \Sigma_a$ independently of k such that $w{\scriptstyle\bullet}\lambda_0(k) < ws{\scriptstyle\bullet}\lambda_0(k)$ and such that $ws{\scriptstyle\bullet}\lambda_0(k)$ is in the same box as $w{\scriptstyle\bullet}\lambda_0(k)$. Then the alcoves of $w{\scriptstyle\bullet}\lambda_0(k)$ and $ws{\scriptstyle\bullet}\lambda_0(k)$ are separated only by one reflection hyperplane, namely by the one corresponding to wsw^{-1}, cf. [Hu3], 4.4. (The discussion there extends to the case where R is not irreducible.) So we can apply induction to ws and assume that $I(ws)$ has already been defined. Now set $I(w) = (s, I(ws))$. So we have (for all B' and A as above)

$$Q^{(w)}(B') = \Theta_s Q^{(ws)}(B') \qquad \text{and} \qquad \mathcal{Q}^{(w)}(A) = \widetilde{\Theta}_s \mathcal{Q}^{(ws)}(A). \tag{3}$$

Note that we have especially defined $\mathcal{Q}^{(w)}(S)$ in $\widetilde{\mathcal{K}}(W_a, S)$ for all $w \in W_a$. We set for all $w, w' \in W_a$

$$\begin{aligned} \mathcal{E}_{(w,w')} &= \mathcal{E}[(I(w), \nu(w)), (I(w'), \nu(w'))], \\ \mathcal{E}^{\sharp}_{(w,w')} &= \mathcal{E}^{\sharp}[(I(w), \nu(w)), (I(w'), \nu(w'))] \end{aligned} \tag{4}$$

using the notation from 16.8.

16.13. For the moment fix k. For each $\lambda \in \Omega_k$ there is a unique $w \in W_a$ such that $\lambda = w{\scriptstyle\bullet}\lambda_0(k)$. We shall often write $Q^{(\lambda)}(B')$ instead of $Q^{(w)}(B')$ and $\mathcal{Q}^{(\lambda)}(A)$ instead of $\mathcal{Q}^{(w)}(A)$.

Any $\lambda \in X$ can be decomposed uniquely $\lambda = \lambda^0 + p(k)\lambda^1$ with $\lambda^0 \in X_{p(k)}$ and $\lambda^1 \in X$. Set then

$$\widehat{\lambda} = w_0{\scriptstyle\bullet}\lambda^0 + p(k)(\lambda^1 + 2\rho). \tag{1}$$

Obviously

$$\widehat{\lambda} = w_0{\scriptstyle\bullet}\lambda + p(k)(\lambda^1 - w_0\lambda^1 + 2\rho) \in W_{p(k)}{\scriptstyle\bullet}\lambda.$$

If $\lambda \in \Omega_k$, then $\widehat{\lambda} \in \Omega_k$. So there is for all $w \in W_{p(k)}$ an element $\widehat{w} \in W_{p(k)}$ with $\widehat{w{\scriptstyle\bullet}\lambda_0(k)} = \widehat{w}{\scriptstyle\bullet}\lambda_0(k)$. If we identify (as usual) W_a with $W_{p(k)}$, then the resulting map $(w \mapsto \widehat{w})$ on W_a is independent of k. (Note that the map $\lambda \mapsto \widehat{\lambda}$ on X does depend on k and that this fact is not reflected in the notation. However, whenever we are going to use the notation $\widehat{\lambda}$ it will be clear over which k we work.)

Our next statements involve the order relation $\uparrow$ on X. It depends on $p(k)$. We refer to [Ja6], II.6.4 for its definition and basic properties.

Lemma: *Let $\lambda \in \Omega_k$. Let B' be a $B(k)$-algebra. If $Z_{B'}(\mu)$ with $\mu \in \Omega_k$ occurs as a factor in a Z-filtration of $Q^{(\lambda)}(B')$, then $\lambda \uparrow \mu \uparrow \widehat{\lambda}$. Both $Z_{B'}(\lambda)$ and $Z_{B'}(\widehat{\lambda})$ occur with multiplicity 1 as factors in a Z-filtration of $Q^{(\lambda)}(B')$.*

Proof: Let $\nu \in X$ be the weight such that λ is contained in the box $p(k)(\nu - \rho) + X_{p(k)}$. We are going to use the same type of induction as in the construction of the $Q^{(\lambda)}(B')$ in 16.12. If λ is in the top alcove of that box, then $Q^{(\lambda)}(B') = Q_{\emptyset,\nu}(B')$ has a Z-filtration with factors $Z_{B'}(p(k)\nu + w{\bullet}\mu)$ where $\mu = w_0{\bullet}(\lambda - p(k)\nu) = \widehat{\lambda} - p(k)\nu$ and where w runs over the ordinary Weyl group W, cf. 15.17. Each of these factors occurs with multiplicity 1 and we have

$$\lambda = p(k)\nu + w_0{\bullet}\mu \uparrow p(k)\nu + w{\bullet}\mu \uparrow p(k)\nu + \mu = \widehat{\lambda}$$

for all $w \in W$ since μ is dominant. So the claim follows in this case.

If λ is not in the top alcove of its box, then there is $s \in \Sigma_a$ with $\lambda < \lambda s$, with λs in the same box as λ and with $Q^{(\lambda)}(B') = \Theta_s Q^{(\lambda s)}(B')$. Then $Z_{B'}(\mu)$ occurs in a Z-filtration of $Q^{(\lambda)}(B')$ if and only if $Z_{B'}(\mu)$ or $Z_{B'}(\mu s)$ occurs in a Z-filtration of $Q^{(\lambda s)}(B')$, cf. 15.17. We get then by induction $\lambda s \uparrow \mu \uparrow \widehat{\lambda s}$ resp. $\lambda s \uparrow \mu s \uparrow \widehat{\lambda s}$. We have $\lambda \uparrow \lambda s$ and $\widehat{\lambda s} = \widehat{\lambda} s \uparrow \widehat{\lambda}$. Now the claim is obvious in the first case ($\lambda s \uparrow \mu \uparrow \widehat{\lambda s}$); in the second one it follows from [Ja6], Lemma II.6.7.a. (In order to get $\lambda \uparrow \mu$ one has to apply that lemma to the analogue of $\uparrow$ defined using $-R^+$ as the positive system.)

Remarks: 1) Any $Z_{B'}(\mu)$ with $\mu \in \Omega_k$ occurs as a factor in a Z-filtration of $Q_{I,\nu}(B')$, if and only if some $\mathcal{Z}_\mu(S_k)\langle r\rangle$ occurs as a factor in a permissible $\mathcal{Z}$-filtration of $\mathcal{Q}_{I,\nu}(S_k)$ (for any (I,ν) as in 16.5). So the first claim of the lemma says also: If $\mathcal{Z}_\mu(S_k)\langle r\rangle$ occurs as a factor in a permissible $\mathcal{Z}$-filtration of $\mathcal{Q}^{(\lambda)}(S_k)$, then $\lambda \uparrow \mu \uparrow \widehat{\lambda}$.
2) The lemma can be extended to Z'-filtrations. Any $Z'_{B'}(\mu)$ with $\mu \in \Omega_k$ occurs as a factor in a Z'-filtration of $Q_{I,\nu}(B')$ with the same multiplicity with which $Z_{B'}(\mu)$ occurs as a factor in a Z-filtration of $Q_{I,\nu}(B')$, since these multiplicities are completely determined by the formal character of $Q_{I,\nu}(B')$.

16.14. We continue to fix k.

Lemma: *For each $\lambda \in \Omega_k$ the module $Q^{(\lambda)}(k)$ is the direct sum of one copy of the projective indecomposable module $Q_k(\lambda)$ and of certain $Q_k(\nu)$ with $\nu \in \Omega_k$, $\nu \neq \lambda$ and $\lambda \uparrow \nu \uparrow \widehat{\nu} \uparrow \widehat{\lambda}$. Similarly, $Q^{(\lambda)}(A(k))$ is the direct sum of one copy of $Q_{A(k)}(\lambda)$ and of certain $Q_{A(k)}(\nu)$ with $\nu \in \Omega_k$, $\nu \neq \lambda$ and $\lambda \uparrow \nu \uparrow \widehat{\nu} \uparrow \widehat{\lambda}$.*

Proof: It is known that both $Z_k(\lambda)$ and $Z_k(\widehat{\lambda})$ occur with multiplicity 1 as factors in a Z-filtration of $Q_k(\lambda)$ and that all weights μ of $Q_k(\lambda)$ satisfy $\mu \leq \widehat{\lambda}$. (In Case 1 see [Ja6], II.11.6; the argument there extends to Case 2.) Since $Q^{(\lambda)}(k)$ is projective in $\mathcal{C}_k$, it is the direct sum of certain projective indecomposable modules. If $Q_k(\nu)$ occurs, then $Z_k(\lambda)$ and $Z_k(\widehat{\lambda})$ occur in a Z–filtration of $Q^{(\lambda)}(k)$. So Lemma 16.13 implies that $\lambda \uparrow \nu \uparrow \widehat{\nu} \uparrow \widehat{\lambda}$. Distinct $Q_k(\nu)$ have distinct highest weights since $\nu \mapsto \widehat{\nu}$ is bijective. Lemma 16.13 implies that $\widehat{\lambda}$ is the highest weight of $Q^{(\lambda)}(k)$ and that it occurs with multiplicity 1. This implies now that $Q_k(\lambda)$ occurs with multiplicity 1 as a direct summand of $Q^{(\lambda)}(k)$.

The statements over $A(k)$ follow now from Theorem 4.19.

16.15. Proposition: *For all k each $\mathcal{V}_{\Omega_k} Q_{A(k)}(\lambda)$ with $\lambda \in \Omega_k$ has a graded S_k–form. It is unique up to shift in the grading and admits both a permissible $\mathcal{Z}$–filtration and a permissible $\mathcal{Z}'$–filtration.*

Proof: We want to use induction on $\widehat{\lambda} - \lambda$. Suppose that we know the result already for all $\nu \in \Omega_k$ with $\widehat{\nu} - \nu < \widehat{\lambda} - \lambda$. Now $\mathcal{Q}^{(\lambda)}(S_k)$ is a graded S_k–form of $\mathcal{V}_{\Omega_k} Q^{(\lambda)}(A(k))$. By induction and Lemma 16.14 all indecomposable summands of $\mathcal{V}_{\Omega_k} Q^{(\lambda)}(A(k))$ except for $\mathcal{V}_{\Omega_k} Q_{A(k)}(\lambda)$ are known to have graded S_k–forms. Now repeated application of Proposition 15.7 yields eventually the desired graded S_k–form of $\mathcal{V}_{\Omega_k} Q_{A(k)}(\lambda)$.

The uniqueness claim follows from 15.6.d, the existence of the filtrations from 15.12 and the analogous result for the $Q_{A(k)}(\lambda)$, cf. 2.16.

Remarks: 1) The proof shows that we can construct the graded S_k–form of $\mathcal{V}_{\Omega_k} Q_{A(k)}(\lambda)$ as a direct summand of $\mathcal{Q}^{(\lambda)}(S_k)$.
2) We can extend the proposition to all λ in a good W_p–orbit: There is a unique weight $\mu \in \Omega_k$ such that λ is in the lower closure of the alcove of μ. Then $TQ_{A(k)}(\mu)$ contains $Q_{A(k)}(\lambda)$ as a direct summand with multiplicity 1 for a suitable translation functor T. We can now apply an induction similar to the one above. (Note that we assume $p \geq h$.)

16.16. Corollary: *Let M be a projective module in $\mathcal{C}_{A(k)}(\Omega_k)$ and $\mathcal{M}$ a graded S_k–form of $\mathcal{V}_{\Omega_k} M$. Then there are $\lambda_i \in \Omega_k$ and a direct sum decomposition $\mathcal{M} = \bigoplus_{i=1}^r \mathcal{Q}_i$ in $\widetilde{\mathcal{K}}(\Omega_k, S_k)$ such that each $\mathcal{Q}_i$ is a graded S_k–form of $\mathcal{V}_{\Omega_k} Q_{A(k)}(\lambda_i)$.*

Proof: We have a direct sum decomposition $M = \bigoplus_{i=1}^r Q_{A(k)}(\lambda_i)$ by Theorem 4.19.b. Now apply 15.7 repeatedly.

16.17. Pick a system $\nu_1, \nu_2, \ldots, \nu_r$ of representatives for the cosets in $X/\mathbf{Z}R$. Set V equal to the set of $v \in W_a$ with $v{\bullet}\lambda_0(k) \in p(k)\nu_i + X_{p(k)}$. This set is independent of k. There are for each $w \in W_a$ a unique $v \in V$ and a unique $\mu \in \mathbf{Z}R$ with $w{\bullet}\lambda_0(k) = v{\bullet}\lambda_0(k) + p(k)\mu$.

Set

$$Q'(B') = \bigoplus_{v \in V} Q^{(v)}(B') \qquad \text{and} \qquad \mathcal{Q}'(A) = \bigoplus_{v \in V} \mathcal{Q}^{(v)}(A)$$

for all $B(k)$–algebras B' resp. all A as in 16.5(4).

Proposition: *For each k the module $Q'(A(k))$ is a projective $(p(k)\mathbf{Z}R)$-generator for $\mathcal{C}_{A(k)}(\Omega_k)$ and $Q'(k)$ is a projective generator for the block of $L_k(\lambda_0(k))$ as a restricted $\mathfrak{g}$-module resp. as a $\mathbf{u}$-module.*

Proof: Set $\lambda_0 = \lambda_0(k)$ and $p = p(k)$. The direct sum $Q = \bigoplus_{v \in V} Q_{A(k)}(v{\scriptstyle\bullet}\lambda_0)$ is by 6.18 (and construction) a projective $(p\mathbf{Z}R)$-generator for $\mathcal{C}_{A(k)}(\Omega_k)$. This extends to $Q'(A(k))$, which contains Q as a direct summand (by Lemma 16.14) and is still projective in $\mathcal{C}_{A(k)}(\Omega_k)$.

We know that $Q'(k)$ is a direct sum of certain $Q_k(\nu)$ with $\nu \in \Omega_k$. Each $Q_k(\nu)$ is still projective when considered as a $\mathfrak{g}$-module resp. as a $\mathbf{u}$-module, in fact it is still the projective cover of $L_k(\nu)$ in that category, cf. [Ja6], II.11.3(3) in Case 1. (Case 2 is similar). So $Q'(k)$ is projective also as a restricted $\mathfrak{g}$-module resp. as a $\mathbf{u}$-module.

The simple $\mathfrak{g}$-modules (resp. $\mathbf{u}$-modules) in the block of $L_k(\lambda_0)$ are exactly the $L_k(w{\scriptstyle\bullet}\lambda_0)$ with $w \in W$, cf. [Hu2] or [Ja6], II.9.19 in Case 1. (In Case 2 this can be proved similarly.) Since all $\nu \in \Omega_k$ have the form $\nu = w{\scriptstyle\bullet}\lambda_0 + p\mu$ with $\mu \in \mathbf{Z}R$ and since then $L_k(\nu) \simeq L_k(w{\scriptstyle\bullet}\lambda_0)$ over $\mathfrak{g}$ resp. over $\mathbf{u}$, we see that all indecomposable summands of $Q'(k)$ belong to the right block.

For each $w \in W$ there are $v \in V$ and $\mu \in \mathbf{Z}R$ with $w{\scriptstyle\bullet}\lambda_0 = v{\scriptstyle\bullet}\lambda_0 + p\mu$. We have then $Q_k(w{\scriptstyle\bullet}\lambda_0) \simeq Q_k(v{\scriptstyle\bullet}\lambda_0)[p\mu]$, cf. 4.19(1). So, considered as a $\mathfrak{g}$-module (resp. as a $\mathbf{u}$-module) $Q_k(v{\scriptstyle\bullet}\lambda_0)$ is isomorphic to $Q_k(w{\scriptstyle\bullet}\lambda_0)$, hence to the projective cover of $L_k(w{\scriptstyle\bullet}\lambda_0)$ as a $\mathfrak{g}$-module resp. as a $\mathbf{u}$-module. Lemma 16.14 implies that $Q_k(v{\scriptstyle\bullet}\lambda_0)$ is a direct summand of $Q'(k)$. Therefore $Q'(k)$ contains (as a direct summand) a projective cover for each simple module in the block of $L_k(\lambda_0)$. The second claim follows.

16.18. Set $\mathcal{E}^\sharp$ equal to the direct sum of all $\mathcal{E}^\sharp_{(v,v')}$ with $v, v' \in V$, cf. 16.12(4). So $\mathcal{E}^\sharp$ is a particular case of an $\mathcal{E}^\sharp_{\mathcal{I}}$ as in 16.9. It is finitely generated as a $\mathbf{Z}$-module; we make it into a $\mathbf{Z}$-algebra via the isomorphism

$$\mathrm{End}^\sharp_{\mathcal{K}(W_a,S)} \mathcal{Q}'(S) \otimes_S \mathbf{Z} \xrightarrow{\sim} \mathcal{E}^\sharp, \tag{1}$$

cf. 16.9(2).

We are now ready to prove Theorems 1 and 2 from the introduction:

Theorem: *For all k the opposite algebra of $\mathcal{E}^\sharp \otimes_{\mathbf{Z}} k$ is Morita equivalent to the block of $\mathbf{u}(k)$ resp. of $U^{[p]}(\mathfrak{g}(k))$ belonging to $L_k(\lambda_0(k))$.*

Proof: This follows from 16.9(3), Proposition 16.17, and general results on Morita equivalence, cf. [Ben], 2.2.

16.19. Let $\mathcal{E}$ be a ring that is finitely generated as a module over $\mathbf{Z}$. Choose an integer $N_1 > 0$ such that

$$\mathcal{E} \otimes_{\mathbf{Z}} \mathbf{Z}[N_1^{-1}] \qquad \text{is free over } \mathbf{Z}[N_1^{-1}]. \tag{1}$$

(For example, take the product of all primes l such that $\mathcal{E}$ has l–torsion.)

Let us write

$$\mathcal{E}_A = \mathcal{E} \otimes_{\mathbf{Z}} A$$

for any commutative ring A. Recall (e.g., from [Ben], p. 6, Remark ii) that a field F is a splitting field of a finite dimensional F–algebra E, if $\operatorname{End}_E(M) = F$ for every simple E–module M. There is a finite extension F of $\mathbf{Q}$ that is a splitting field of the finite dimensional algebra $\mathcal{E}_{\mathbf{Q}}$, i.e., of $\mathcal{E}_{\mathbf{Q}} \otimes_{\mathbf{Q}} F \simeq \mathcal{E}_F$, cf. [Ben], p. 6, Exercise. Denote the ring of algebraic integers in F by $\mathfrak{o}_F$. Let

$$1 = \sum_{i \in I} e_i \tag{2}$$

be a decomposition of 1 into orthogonal primitive idempotents in $\mathcal{E}_F$. Here I is a suitable finite index set. There is an integer $N_2 > 0$ such that all $N_2 e_i$ are in the image of $\mathcal{E} \otimes_{\mathbf{Z}} \mathfrak{o}_F$ in $\mathcal{E}_F$. Replace N_2 by its least common multiple with N_1. We can then regard $\mathcal{E} \otimes_{\mathbf{Z}} \mathfrak{o}_F[N_2^{-1}]$ as a subring of $\mathcal{E}_F$ containing all e_i. We can choose a multiple N of N_2 so large that two idempotents e_i, e_j are conjugate by a unit in $\mathcal{E} \otimes_{\mathbf{Z}} \mathfrak{o}_F[N^{-1}]$ if they are conjugate by a unit in $\mathcal{E}_F$. Set

$$\mathfrak{o} = \mathfrak{o}_F[N^{-1}]. \tag{3}$$

Recall that $\mathfrak{o}_F$ is a finite $\mathbf{Z}$–module, cf. [Boul], chap. V, §1, n° 6, prop. 18. We get by construction:

a) $\mathfrak{o}$ is finitely generated over $\mathbf{Z}[N^{-1}]$.

b) $\mathcal{E}_{\mathfrak{o}}$ is a free $\mathfrak{o}$-module of finite rank and a lattice in $\mathcal{E}_F$.

c) All e_i with $i \in I$ are in $\mathcal{E}_{\mathfrak{o}}$.

d) e_i, e_j with $i, j \in I$ are conjugate in $\mathcal{E}_{\mathfrak{o}}$ if and only if they are conjugate in $\mathcal{E}_F$.

Each $\mathcal{E}_F e_i$ is the projective cover of a simple $\mathcal{E}_F$–module L_i. Consider an extension field F' of F. Then each $L_i \otimes_F F'$ is a simple $\mathcal{E}_{F'}$–module, since F is a splitting field. This implies that $(\mathcal{E}_F e_i) \otimes_F F'$ is the projective cover of $L_i \otimes_F F'$, hence that e_i is still primitive in $\mathcal{E}_{F'}$. Furthermore, we have for two indices i, j that L_i is isomorphic to L_j if and only if $L_i \otimes_F F'$ is isomorphic to $L_j \otimes_F F'$. So e_i and e_j are conjugate in $\mathcal{E}_F$ if and only if they are conjugate in $\mathcal{E}_{F'}$. (For the equivalence of conjugacy and generating isomorphic left ideals, cf. [Ben], 1.7.2.)

Consider a maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$. Denote the $\mathfrak{m}$–adic completion of $\mathfrak{o}$ by $\mathfrak{o}'$; this is a domain since the local ring $\mathfrak{o}_{\mathfrak{m}}$ is a principal ideal domain. Set F' equal to the fraction field of $\mathfrak{o}'$ and $\mathfrak{m}' = \mathfrak{o}'\mathfrak{m}$. The e_i remain primitive in $\mathcal{E}_{\mathfrak{o}'}$, and e_i is conjugate to e_j in $\mathcal{E}_{\mathfrak{o}'}$ if and only if it is so in $\mathcal{E}_{\mathfrak{o}}$. (This follows from the analogous result for F'.) Now standard results, cf. [Ben], 1.9.4, show that the images $\overline{e}_i$ of the e_i in $\mathcal{E}_{\mathfrak{o}/\mathfrak{m}} \simeq \mathcal{E}_{\mathfrak{o}'/\mathfrak{m}'}$ are still primitive idempotents, and that $\overline{e}_i$ and $\overline{e}_j$ are conjugate in $\mathcal{E}_{\mathfrak{o}/\mathfrak{m}}$ if and only if e_i and e_j are conjugate in $\mathcal{E}_{\mathfrak{o}}$. Furthermore, the $\overline{e}_i$ remain primitive over each extension field of $\mathfrak{o}/\mathfrak{m}$.

(Otherwise they would split over some finite extension. Then one could lift such a splitting to some extension of the complete local ring $\mathfrak{o}'$. This would contradict the assumption that F is a splitting field.)

The Cartan matrix of $\mathcal{E}_F$ is the matrix of all $\dim_F e_i \mathcal{E}_F e_j$ with e_i running over representatives for the conjugacy classes. For $\mathfrak{m}$ as above, the Cartan matrix of $\mathcal{E}_{\mathfrak{o}/\mathfrak{m}}$ is the matrix of all $\dim_{\mathfrak{o}/\mathfrak{m}} \overline{e}_i \mathcal{E}_{\mathfrak{o}/\mathfrak{m}} \overline{e}_j$ where we may take the same e_i as before. Both dimensions are equal to the rank of the projective $\mathfrak{o}$–module $e_i \mathcal{E}_{\mathfrak{o}} e_j$. This shows that $\mathcal{E}_F$ and all $\mathcal{E}_{\mathfrak{o}/\mathfrak{m}}$ have the same Cartan matrix.

Morita equivalent rings have the same Cartan matrix. If we apply our discussion above to the algebra $\mathcal{E}^\sharp$ from 16.18, we get therefore the following result

Corollary: *There is a matrix C such that the Cartan matrix of the block of $L_k(\lambda_0(k))$ as a $\mathbf{u}(k)$–module is equal to C for all k in Case 2. It is also equal to the Cartan matrix of the block of $L_k(\lambda_0(k))$ as a $U^{[p]}(\mathfrak{g}(k))$–module for all k in Case 1 with char$(k) \gg 0$.*

Remark: The condition "char$(k) \gg 0$" in this result as well as in 16.22–24 below means: There is an integer $n(R)$ depending on R such that for all k with char$(k) > n(R)$ ...

16.20. The set of $w \in W_a$ with

$$\lambda_0(k) \uparrow w{\scriptstyle\bullet}\lambda_0(k) \uparrow w{\scriptstyle\bullet}\widehat{\lambda_0(k)} \uparrow \widehat{\lambda_0(k)}$$

is finite and independent of k. Enumerate these elements $w_1, w_2, \ldots, w_r$ such that (for all k)

$$w_i{\scriptstyle\bullet}\lambda_0(k) \uparrow w_j{\scriptstyle\bullet}\lambda_0(k) \uparrow w_j{\scriptstyle\bullet}\widehat{\lambda_0(k)} \uparrow w_i{\scriptstyle\bullet}\widehat{\lambda_0(k)} \implies j \leq i. \tag{1}$$

Set

$$Q^{[i]}(B') = Q^{(w_i)}(B') \qquad \text{and} \qquad \mathcal{Q}^{[i]}(A) = \mathcal{Q}^{(w_i)}(A)$$

for all $B(k)$–algebras B' resp. all A as in 16.5(4). For all i and j set

$$\mathcal{E}_{[i],[j]} = \mathcal{E}_{(w_i,w_j)}, \tag{2}$$

cf. 16.12(4). We have for all i and j natural isomorphisms (by 16.8)

$$\mathcal{E}_{[i],[j]} \otimes_{\mathbf{Z}} k \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Q^{[i]}(k), Q^{[j]}(k)). \tag{3}$$

For any finite subset $\mathcal{I}$ of $\{1, 2, \ldots, r\}$ set $Q_{\mathcal{I}}(k)$ resp. $\mathcal{Q}_{\mathcal{I}}(S)$ equal to the direct sum of all $Q^{[i]}(k)$ resp. $\mathcal{Q}^{[i]}(S)$ with $i \in \mathcal{I}$. Set $\mathcal{E}_{\mathcal{I}}$ equal to the direct sum of all $\mathcal{E}_{[i],[j]}$ with $(i,j) \in \mathcal{I}$. We have then (as in 16.9) isomorphisms

$$\mathrm{End}_{\mathcal{K}(W_a,S)} \mathcal{Q}_{\mathcal{I}}(S) \otimes_S \mathbf{Z} \xrightarrow{\sim} \mathcal{E}_{\mathcal{I}} \tag{4}$$

and

$$\mathcal{E}_{\mathcal{I}} \otimes_{\mathbf{Z}} k \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}} Q_{\mathcal{I}}(k). \tag{5}$$

We get from (4) on $\mathcal{E}_{\mathcal{I}}$ a structure of an algebra over $\mathbf{Z}$; the map in (5) is then an algebra isomorphism.

Each $\mathcal{E}_{[i],[j]}$ and each $\mathcal{E}_{\mathcal{I}}$ is a finitely generated and graded $\mathbf{Z}$–module. If we choose d as in Corollary 16.11, then each $\mathcal{E}_{[i],[j]} \otimes_{\mathbf{Z}} \mathbf{Z}[d^{-1}]$ and each $\mathcal{E}_{\mathcal{I}} \otimes_{\mathbf{Z}} \mathbf{Z}[d^{-1}]$ is a graded module that is free of finite rank over $\mathbf{Z}[d^{-1}]$. We can factor for all k the isomorphism in (3) via

$$(\mathcal{E}_{[i],[j]} \otimes_{\mathbf{Z}} \mathbf{Z}[d^{-1}]) \otimes_{\mathbf{Z}[d^{-1}]} k \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Q^{[i]}(k), Q^{[j]}(k)),$$

similarly for (5).

16.21. We know by 15.17/18 how to determine the terms in the Z–filtration of each $Q^{[i]}(k)$, cf. the proof of Lemma 16.6. Lemma 16.14 implies for each i that $Q^{[i]}(k)$ decomposes into a direct sum of certain $Q_k(w_j{\scriptstyle\bullet}\lambda_0(k))$ with $j \leq i$ and that $Q_k(w_i{\scriptstyle\bullet}\lambda_0(k))$ occurs with multiplicity one. The decomposition of all $Q^{[i]}(k)$ into indecomposables determines the factors in the Z–filtration of each $Q_k(w_i{\scriptstyle\bullet}\lambda_0(k))$, hence (by the Brauer–Humphreys reciprocity 4.15) all multiplicities $[Z_k(\mu) : L_k(w_i{\scriptstyle\bullet}\lambda_0(k))]$ with $\mu \in \Omega_k$. Since these multiplicities are invariant under shifts by weights in $p(k)X$ and since $W_{p(k)}{\scriptstyle\bullet}\lambda_0(k) \cap (-p(k)\rho + X_{p(k)})$ is contained in the set of all $w_i{\scriptstyle\bullet}\lambda_0(k)$, more generally all $[Z_k(\mu) : L_k(\lambda)]$ with $\lambda, \mu \in \Omega_k$ are determined. It is well known how to get from these multiplicities then the characters of the simple modules $L_k(\lambda)$, cf. [Ja6], II.9.11(1). So, in order to show that these characters are independent of k (in a suitable sense), we shall show that the decompositions of the $Q^{[i]}(k)$ into indecomposables are independent of k (in a suitable sense). In fact, that is our next result:

16.22. Theorem: *There are integers $m(j,i) \geq 0$ for all i, j with $1 \leq j \leq i \leq r$ with*

$$Q^{[i]}(k) \simeq \bigoplus_{j \leq i} Q_k(w_j{\scriptstyle\bullet}\lambda_0(k))^{m(j,i)} \tag{1}$$

for all i and all k with $char(k) = 0$ or with $char(k) \gg 0$.

Proof: Of course, Lemma 16.14 shows that there is for each k a decomposition as in (1). However, the multiplicities $m(i,j)$ might depend on k; denote them for the moment by $m_k(i,j)$. Lemma 16.14 implies especially that $m_k(i,i) = 1$ for all i.

We can apply the discussion from 16.19 to each $\mathcal{E}_{\mathcal{I}}$, especially to each $\mathcal{E}_{[i],[i]}$ and to $\mathcal{E} = \bigoplus_{i,j} \mathcal{E}_{[i],[j]}$. We can choose $\mathfrak{o}$ in 16.19 so that it works for all these algebras. For all i take an index set $E(i)$ such that $1 = \sum_{r \in E(i)} e_r^{(i)}$ is a decomposition into orthogonal primitive idempotents in $\mathcal{E}_{[i],[i]} \otimes_{\mathbf{Z}} \mathfrak{o}$. One gets a similar decomposition for $\mathcal{E}_{\mathfrak{o}}$ by adding the decompositions for all i.

In Case 2 the field k need not contain the fraction field of $\mathfrak{o}$. Write $k[\mathfrak{o}]$ for k adjoined that fraction field. We get then for all k in Case 2 that

$$Q^{[i]}(k[\mathfrak{o}]) = \bigoplus_{r\in E(i)} e_r^{(i)} Q^{[i]}(k[\mathfrak{o}]) \tag{2}$$

is a decomposition into indecomposables. On the other hand, we get such a decomposition from that in (1) by field extension, since 4.16(3) implies $Q_{k[\mathfrak{o}]}(\mu) = Q_k(\mu) \otimes k[\mathfrak{o}]$ for all $\mu \in X$. So the Krull-Schmidt theorem implies that each summand in (2) is isomorphic to some $Q_{k[\mathfrak{o}]}(w_j{\scriptstyle\bullet}\lambda_0(k))$ with $j \leq i$ and that $Q_{k[\mathfrak{o}]}(w_i{\scriptstyle\bullet}\lambda_0(k))$ occurs exactly once. Then the idempotent corresponding to the unique summand isomorphic to $Q_{k[\mathfrak{o}]}(w_i{\scriptstyle\bullet}\lambda_0(k))$ is not conjugate to any idempotent in an $E(j)$ with $j < i$, whereas all the other idempotents are. This is at first a statement about conjugacy over $k[\mathfrak{o}]$, but by the construction in 16.19 it is also true over $\mathfrak{o}$. After changing the index set we may assume that $0 \in E(i)$ and $e_0^{(i)}$ is the unique idempotent in $E(i)$ not conjugate to an idempotent in $E(j)$ for all $j < i$. We have then for all k in Case 2

$$e_0^{(i)} Q^{[i]}(k[\mathfrak{o}]) \simeq Q_{k[\mathfrak{o}]}(w_i{\scriptstyle\bullet}\lambda_0(k)).$$

We see now in Case 2 that each $m_k(j,i)$ is equal to the number of $r \in E(i)$ with $e_r^{(i)}$ conjugate to $e_0^{(j)}$ in $\mathcal{E}_\mathfrak{o}$. In particular, that number is independent of k (in Case 2); denote it by $m(j,i)$.

Consider now Case 1. We have to take $\mathrm{char}(k) = p(k)$ large enough so that $p(k)$ is not a unit in $\mathfrak{o}$, i.e., so that k contains a residue field of $\mathfrak{o}$. We have then $\mathcal{E} \otimes_{\mathbf{Z}} k \simeq \mathcal{E}_\mathfrak{o} \otimes_\mathfrak{o} k$. Write $\overline{e}_r^{(i)} = e_r^{(i)} \otimes 1$. The discussion in 16.19 shows that the $\overline{e}_r^{(i)}$ are primitive idempotents in $\mathcal{E}_\mathfrak{o} \otimes_\mathfrak{o} k$, and that $\overline{e}_r^{(i)}$, $\overline{e}_s^{(j)}$ are conjugate if and only if $e_r^{(i)}$, $e_s^{(j)}$ are conjugate. So for all i

$$Q^{[i]}(k) = \bigoplus_{r\in E(i)} \overline{e}_r^{(i)} Q^{[i]}(k) \tag{3}$$

is a a decomposition into indecomposables, hence has the form of (1) with possibly different multiplicities. The statement about the conjugacy implies first that $\overline{e}_0^{(i)} Q^{[i]}(k) \simeq Q_k(w_i{\scriptstyle\bullet}\lambda_0(k))$ for all i, and then yields the claim about the multiplicities.

16.23. Corollary: *There is for all $w, w' \in W_a$ an integer $d'(w,w')$ such that*

$$[Z_k(w{\scriptstyle\bullet}\lambda_0(k)) : L_k(w'{\scriptstyle\bullet}\lambda_0(k))] = d'(w,w')$$

for all k with $char(k) = 0$ or $char(k) \gg 0$.

Proof: This follows from Theorem 16.22 using the discussion in 16.21.

Remark: The corollary yields Theorem 6 from the introduction; it is now easy (and left to the reader) to deduce Theorem 3 from the introduction and its analogue for $\mathbf{u}(k)$.

16.24. Consider in Case 1 an algebraic group G_k as in the introduction. In Case 2 consider Lusztig's version of the quantized enveloping algebra at a $p(k)$-th root of unity. We have in both cases for each dominant weight $\lambda \in X$ a Weyl module $V(\lambda)_k$ and a simple module $L(\lambda)_k$ with highest weight λ, cf. Section II.2 in [Ja6] and [Lu4] or [APW1].

Set W_a^+ equal to the set of all $w \in W_a$ with $w{\scriptstyle\bullet}\lambda_0(k)$ dominant. This set is independent of k. Also the condition "$w{\scriptstyle\bullet}\lambda_0(k) \in X_{p(k)}$" defines a subset of W_a that is independent of k.

Corollary: *There is for all $w, w' \in W_a^+$ with $w{\scriptstyle\bullet}\lambda_0(k) \in X_{p(k)}$ an integer $b(w, w')$ with*

$$[V(w{\scriptstyle\bullet}\lambda_0(k))_k : L(w'{\scriptstyle\bullet}\lambda_0(k))_k] = b(w, w')$$

for all k with $char(k) = 0$ or $char(k) \gg 0$.

Proof: This follows from the last corollary, the formula II.9.11(1) in [Ja6], and Steinberg's tensor product theorem (cf. [Ja6], II.3.17) resp. Lusztig's analogue (cf. [Lu4], 7.4 for the simply laced case; the result extends to the general case by the method in [Lu7]).

Remarks: 1) If we drop the condition that $w{\scriptstyle\bullet}\lambda_0(k) \in X_{p(k)}$, then the bound on char$(k)$ in Case 1 will depend not only on R, but also on w.
2) We get now Theorems 4 and 5 from the introduction; there one can find more discussion of this result.

Introduction to the Sections 17 and 18

Let us use the notations of our general introduction. Analogous results for the category $\mathcal{O}$ in [BGS] lead to the following

Conjecture 1: *Suppose we are in* Case 1 *and the characteristic p of our field k is at least the Coxeter number, $p \geq h$. Then the restricted enveloping algebra $U^{[p]}(\mathfrak{g}_k)$ admits a T_k-stable Koszul grading.*

Certainly there also is a quantum version.

Conjecture 2: *Suppose we are in* Case 2, *that $p > 1$ is an odd integer, and that all prime divisors of p are good for our root system. Then the quantum version $\mathbf{u}_p$ of the restricted enveloping algebra admits a Koszul grading compatible with its natural X-grading.*

We give the precise definition of a "Koszul grading" in Appendix F. Let us say here only that a Koszul grading on a ring A is a positive $\mathbf{Z}$–grading $A = \bigoplus_{i \geq 0} A^i$ with particularly good homological properties. Appendix F contains also a discussion of the general properties of Koszul gradings. For example, we prove there that a finite dimensional algebra admits a Koszul grading if and only if all of its blocks do. So we can attack the conjectures block by block.

In the next two sections we show how the validity of Lusztig's conjecture implies that all regular blocks of $U^{[p]}(\mathfrak{g}_k)$ resp. of $\mathbf{u}_p$ admit a Koszul grading. So to establish the conjecture it remains to treat the singular blocks. The main obstacle there is that we do not know how to prove parity vanishing for the extensions between standard objects and simple objects. The analogous problem for the category $\mathcal{O}$ was solved in [So1] using geometric methods, but we do not see how to generalize this to our present situation.

Let us explain how we proceed in the case of regular blocks. We first show how our graded combinatorics from Section 15 determines a $\mathbf{Z}$–grading on the blocks. This much is independent of Lusztig's conjecture and works (under our general assumption $p \geq h$) for any block corresponding to an orbit Ω that is good in the sense of 13.3. Considering categories of $\mathbf{Z}$–graded modules over these blocks with their $\mathbf{Z}$–grading we construct the "graded" representation categories $\widetilde{\mathcal{C}}_k(\Omega)$, define graded translation functors relating them, and study their properties.

Assuming now Lusztig's conjecture, we prove that the $\mathbf{Z}$–grading on the blocks can be choosen such that there are no components of negative degrees and that the part of degree zero is semisimple — this is a consequence of the restrictions on the degrees appearing in the definition of Lusztig's polynomials. We can even calculate the Poincaré polynomials of the blocks with respect to this $\mathbf{Z}$–grading from our graded combinatorics.

If our block is in addition regular and $p > h$, then the dimensions of the higher Ext groups between simple objects of $\mathcal{C}_k(\Omega)$ can be found in [CPS] assuming Lusztig's conjecture. The numerical Koszulity criterion of [BGS] implies then that our grading makes the regular blocks into Koszul rings.

Let us just warn the reader that in the next two sections we treat these matters in a different order than in this introduction. Recall that Lusztig's conjecture predicts the multiplicity $[Q_k(\mu) : Z_k(\lambda)]$ of $Z_k(\lambda)$ as a subquotient in a Z–filtration of the indecomposable projective $Q_k(\mu)$ to be just the value at 1 of a certain polynomial. In Section 17 we show that the individual coefficients of this polynomial are analogous multiplicities in the graded category $\widetilde{\mathcal{C}}_k(\Omega)$. This, however, will only become clear in Section 18, where we discuss the graded categories $\widetilde{\mathcal{C}}_k(\Omega)$ along with their deformed versions $\widetilde{\mathcal{C}}(\Omega, S_k)$ and the (deformed) graded translation functors relating them. We then use the results from 17 to deduce Koszulity of the regular blocks from Lusztig's conjecture.

17. Gradings and Lusztig's Conjectures

17.1. In this section we work again with a fixed k; however, we assume that k satisfies the assumptions in 16.1. We drop the extra k from notations such as $p(k)$ or $B(k)$.

We choose a length function δ as in [Lu2], 2.11. However, we regard it not as a function on the set of alcoves for W_p, but on the set of regular elements λ in X: Set $\delta(\lambda)$ equal to the value of δ on the unique alcove containing λ.

Choose (as in 16.1) a weight λ_0 in the interior of the first dominant alcove and set $\Omega = W_p{\scriptstyle\bullet}\lambda_0$. For each $\lambda \in \Omega$ there is a unique $w \in W_p$ with $\lambda = w{\scriptstyle\bullet}\lambda_0$; we set then (as in 16.3) $\lambda s = ws{\scriptstyle\bullet}\lambda_0$ for all $s \in \Sigma_a$. We can apply δ to all elements in Ω; it satisfies for all $\lambda \in \Omega$ and $s \in \Sigma_a$

$$\delta(\lambda s) = \begin{cases} \delta(\lambda)+1, & \text{if } \lambda < \lambda s; \\ \delta(\lambda)-1, & \text{if } \lambda > \lambda s. \end{cases} \tag{1}$$

(This property determines δ up to a constant.) One has (cf. [Kat], 1.12) for all $\lambda \in \Omega$ and $\nu \in \mathbf{Z}R$

$$\delta(\lambda + p\nu) - \delta(\lambda) = 2\,\mathrm{ht}(\nu), \tag{2}$$

where ht is the height function defined by $\mathrm{ht}(\sum_{\beta\in\Sigma} m_\beta \beta) = \sum_{\beta\in\Sigma} m_\beta$.

Let Γ be an arbitrary orbit of W_p in X. For each $\mu \in \Gamma$ there is a unique $\lambda_1 \in \Omega$ such that μ is in the upper closure (cf. 6.8) of the alcove of λ_1. Denote this element λ_1 by μ^-. We have then for all $\lambda \in \Omega$ with $\lambda_\Gamma = \mu$ (using the notation from 15.13)

$$\delta(\lambda) = \delta(\mu^-) + o(\lambda, \Gamma). \tag{3}$$

Indeed, the definition of the upper closure implies that $o(\mu^-, \Gamma) = 0$. If $\lambda \neq \mu^-$, then there is $s \in \Sigma_a$ with $\lambda s < \lambda$ and $(\lambda s)_\Gamma = \mu$; we have then $o(\lambda s, \Gamma) = o(\lambda, \Gamma) - 1$ and get (3) by induction.

17.2. Consider the group algebra $\mathbf{Z}[t,t^{-1}][X]$ and denote its standard basis by $(e(\mu))_{\mu\in X}$. Denote by $\mathfrak{M}$ the submodule of $\mathbf{Z}[t,t^{-1}][X]$ generated freely over $\mathbf{Z}[t,t^{-1}]$ by all $e(\mu)$ with $\mu \in \Omega$.

By 15.10 we can define for any $\mathcal{M}$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$ with a permissible $\mathcal{Z}$–filtration

$$h_Z\mathcal{M} = \sum_{\mu\in\Omega} \operatorname{rk}\operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}, \mathcal{Z}'_\mu)e(\mu) \in \mathfrak{M} \tag{1}$$

and for any $\mathcal{M}$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$ with a permissible $\mathcal{Z}'$–filtration

$$h'_Z\mathcal{M} = \sum_{\mu\in\Omega} \operatorname{rk}\operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{Z}_\mu, \mathcal{M})t^{-2\delta(\mu)}e(\mu) \in \mathfrak{M}. \tag{2}$$

Note that 15.11(4),(5) imply

$$h_Z\mathcal{M} = \sum_{\mu\in\Omega}\sum_{r\in\mathbf{Z}}[\mathcal{M} : \mathcal{Z}_\mu\langle r\rangle]t^r e(\mu) \tag{3}$$

resp.

$$h'_Z\mathcal{M} = \sum_{\mu\in\Omega}\sum_{r\in\mathbf{Z}}[\mathcal{M} : \mathcal{Z}'_\mu\langle r\rangle]t^{-r-2\delta(\mu)} e(\mu). \tag{4}$$

We have obviously

$$h_Z\mathcal{Z}_\lambda = e(\lambda) \qquad \text{and} \qquad h'_Z\mathcal{Z}'_\lambda = t^{-2\delta(\lambda)}e(\lambda) \tag{5}$$

for all $\lambda \in \Omega$. It is clear that h_Z and h'_Z are additive on permissible filtrations if defined on all quotients. We have also for all $r \in \mathbf{Z}$ and all $\mathcal{M}$ (such that $h_Z\mathcal{M}$ resp. $h'_Z\mathcal{M}$ is defined)

$$h_Z(\mathcal{M}\langle r\rangle) = t^r h_Z(\mathcal{M}) \qquad \text{resp.} \qquad h'_Z(\mathcal{M}\langle r\rangle) = t^{-r}h'_Z(\mathcal{M}). \tag{6}$$

Proposition 15.10 shows how we can determine $\operatorname{rk}\operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M},\mathcal{N})$ and $\operatorname{rk}\operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{N},\mathcal{M})$ for $\mathcal{M}$ and $\mathcal{N}$ as in that proposition from a knowledge of $h_Z\mathcal{M}$ and $h'_Z\mathcal{N}$.

17.3. Let $\mathfrak{H}$ be the Hecke algebra of W_a over $\mathbf{Z}[t, t^{-1}]$. This is an algebra with generators T_s with $s \in \Sigma_a$. They satisfy $(T_s + 1)(T_s - t^2) = 0$ for all $s \in \Sigma_a$ (and other relations). There is an action of $\mathfrak{H}$ on $\mathfrak{M}$ given by

$$T_s e(\lambda) = \begin{cases} e(\lambda s), & \text{if } \lambda < \lambda s; \\ t^2 e(\lambda s) + (t^2 - 1)e(\lambda), & \text{if } \lambda > \lambda s; \end{cases} \tag{1}$$

cf. [Lu2], 1.6. (Note that we identify Ω with the set of alcoves considered by Lusztig and that we take $t = q^{1/2}$ in Lusztig's paper.)

Lemma: *Let $s \in \Sigma_a$ and $\mathcal{M}$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$.*

a) If $\mathcal{M}$ has a permissible $\mathcal{Z}$-filtration, then $h_Z(\widetilde{\Theta}_s\mathcal{M}) = (T_s + 1)h_Z(\mathcal{M})$.

b) If $\mathcal{M}$ has a permissible $\mathcal{Z}'$-filtration, then $h'_Z(\widetilde{\Theta}_s\mathcal{M}\langle -2\rangle) = (T_s+1)h'_Z(\mathcal{M})$.

Proof: This is an easy consequence of Lemma 15.18 and of 17.1(1). Details are left to the reader.

17.4. Consider an orbit Γ for W_p consisting of special points. We have then $N_\Gamma = |R^+|$ in the notation from 15.4(2); each $\mu \in \Gamma$ satisfies $Z_{B'}(\mu) \simeq Z'_{B'}(\mu)$ for any B–algebra B' and $\mathcal{Z}_\mu(A) = \mathcal{Z}'_\mu(A)$ for any A as in 14.5.

For any $\mu \in \Gamma$ the projective indecomposable module $Q_k(\mu^-)$ in $\mathcal{C}_k$ is given by

$$Q_k(\mu^-) = T_\Gamma^\Omega Z_k(\mu); \tag{1}$$

cf. [Ja6], II.11.10 for $\mu = (p-1)\rho$ in Case 1. The argument there extends to arbitrary μ and to Case 2. In $\mathcal{C}_{A(k)}$ the module $Q_{A(k)}(\mu)$ is isomorphic to $Z_{A(k)}(\mu)$ since it has a Z–filtration (2.16) and since it is a lift of $Q_k(\mu) = Z_k(\mu)$. Therefore 7.7 shows that also $T_\Gamma^\Omega Z_{A(k)}(\mu)$ is projective. It is a lift of $T_\Gamma^\Omega Z_k(\mu)$; so we have

$$Q_{A(k)}(\mu^-) = T_\Gamma^\Omega Z_{A(k)}(\mu). \tag{2}$$

If $\nu \in X$ is the weight with $\mu = p\nu - \rho$, then (2) shows that $Q_{A(k)}(\mu^-) = Q_{\emptyset,\nu}(A(k))$ in the notation from 16.4. Set

$$\mathcal{Q}(\mu^-) = \mathcal{Q}_{\emptyset,\nu}(S_k)\langle |R^+| - \delta(\mu^-)\rangle. \tag{3}$$

(The reason for choosing this specific shift should become clear later on.) Now $\mathcal{Q}(\mu^-)$ is a graded S_k–form of $\mathcal{V}_\Omega Q_{A(k)}(\mu^-)$. Lemma 15.16 implies that $\mathcal{Q}(\mu^-)$ has a permissible $\mathcal{Z}$–filtration with factors $\mathcal{Z}_\lambda\langle -\delta(\mu^-)\rangle$ where $\lambda \in \Omega$ with $\lambda_\Gamma = \mu$. So we have

$$h_Z(\mathcal{Q}(\mu^-)) = t^{-\delta(\mu^-)} \sum_{\lambda_\Gamma = \mu} e(\lambda). \tag{4}$$

Because of $\mathcal{Z}_\mu = \mathcal{Z}'_\mu$ there is (by 15.17) also a permissible $\mathcal{Z}'$–filtration of $\mathcal{Q}(\mu^-)$; its factors are the $\mathcal{Z}'_\lambda\langle |R^+| - \delta(\mu^-) - r(\lambda)\rangle$ with λ as above. Using 17.1(3) and $r(\lambda) = 2o(\lambda) - |R^+|$, cf. 15.13(3),(4), we can rewrite this factor as $\mathcal{Z}'_\lambda\langle 2|R^+| + \delta(\mu^-) - 2\delta(\lambda)\rangle$. We get therefore

$$h'_Z(\mathcal{Q}(\mu^-)) = t^{-2|R^+| - \delta(\mu^-)} \sum_{\lambda_\Gamma = \mu} e(\lambda), \tag{5}$$

hence

$$h_Z(\mathcal{Q}(\mu^-)) = t^{2|R^+|} h'_Z(\mathcal{Q}(\mu^-)). \tag{6}$$

17.5. Lusztig introduces in [Lu2], 2.7 an $\mathfrak{H}$–submodule $\mathfrak{M}^0$ of $\mathfrak{M}$. One takes all W_p–orbits Γ consisting of special points and considers for all $\mu \in \Gamma$ the sum $\sum_{\lambda_\Gamma = \mu} e(\lambda)$. Then $\mathfrak{M}^0$ is generated by all these sums (for all μ and all Γ).

Lusztig then goes on to prove (in [Lu2], 2.12) the existence of an involutory and $\mathfrak{H}$–antilinear map $\Phi_\delta : \mathfrak{M}^0 \to \mathfrak{M}^0$ with

$$\Phi_\delta\Big(\sum_{\lambda_\Gamma=\mu} e(\lambda)\Big) = t^{-2|R^+|-2\delta(\mu^-)} \sum_{\lambda_\Gamma=\mu} e(\lambda). \tag{1}$$

Here antilinear refers to the involutory antiautomorphism of $\mathfrak{H}$ with

$$t \mapsto t^{-1} \qquad \text{and} \qquad T_s \mapsto T_s^{-1} = t^{-2}T_s + t^{-2} - 1$$

for all $s \in \Sigma_a$. This antilinearity implies especially

$$\Phi_\delta((T_s+1)u) = t^{-2}(T_s+1)\Phi_\delta(u) \qquad \text{for all } u \in \mathfrak{M}^0 \text{ and } s \in \Sigma_a. \tag{2}$$

Recall the notation $\widehat{\lambda}$ from 16.13(1).

Lemma: *For all $\lambda \in \Omega$ the elements $h_Z \mathcal{Q}^{(\lambda)}(S_k)$ and $h'_Z \mathcal{Q}^{(\lambda)}(S_k)$ are in $\mathfrak{M}^0$ and satisfy $h'_Z \mathcal{Q}^{(\lambda)}(S_k) = \Phi_\delta(h_Z \mathcal{Q}^{(\lambda)}(S_k))$. We have (with $N = |R^+|$)*

$$[\mathcal{Q}^{(\lambda)}(S_k)\langle -\delta(\lambda)\rangle : \mathcal{Z}_\lambda\langle m\rangle] = \delta_{m,-N-\delta(\lambda)}, \tag{1}$$

$$[\mathcal{Q}^{(\lambda)}(S_k)\langle -\delta(\lambda)\rangle : \mathcal{Z}'_\lambda\langle m\rangle] = \delta_{m,N-\delta(\lambda)}, \tag{2}$$

$$[\mathcal{Q}^{(\lambda)}(S_k)\langle -\delta(\lambda)\rangle : \mathcal{Z}_{\widehat{\lambda}}\langle m\rangle] = \delta_{m,-\delta(\widehat{\lambda})}, \tag{3}$$

$$[\mathcal{Q}^{(\lambda)}(S_k)\langle -\delta(\lambda)\rangle : \mathcal{Z}'_{\widehat{\lambda}}\langle m\rangle] = \delta_{m,-\delta(\widehat{\lambda})}. \tag{4}$$

Proof: We use induction on the number of reflection hyperplanes separating λ from the top alcove of its box. If λ is in that top alcove, then the first claim follows from 17.4(4),(5) and the second one from (1) and 17.4(6). We have $\mathcal{Q}^{(\lambda)}(S_k)\langle -\delta(\lambda)\rangle = \mathcal{Q}(\lambda)(S_k)\langle -N\rangle$ in the notations from 17.4. Therefore the formulas (1)–(4) follow from 17.4(4),(5) together with the fact that in this case $\delta(\widehat{\lambda}) = \delta(\lambda) + N$.

If λ is not in that top alcove, then there is an $s \in \Sigma_a$ with $\mathcal{Q}^{(\lambda)}(S_k) = \widetilde{\Theta}_s \mathcal{Q}^{(\lambda s)}(S_k)$ and $\lambda < \lambda s$. Then $\mathcal{Q}^{(\lambda s)}(S_k)$ satisfies the claims by induction. We have $\delta(\lambda) = \delta(\lambda s) - 1$ and $(\widehat{\lambda})s = \widehat{\lambda s} < \widehat{\lambda}$ and $\delta(\widehat{\lambda}) = \delta(\widehat{\lambda s}) + 1$. Now the claim follows from Lemma 17.3; for (1)–(4) we can also use 15.18.

17.6. Recall (from Remark 1 in 16.15) that $\mathcal{Q}^{(\lambda)}(S_k)$ has a direct summand that is a graded S_k–form of $\mathcal{V}_\Omega Q_{A(k)}(\lambda)$. The same is true for any $\mathcal{Q}^{(\lambda)}(S_k)\langle m\rangle$. So we can find a graded S_k–form $\mathcal{Q}(\lambda)$ of $\mathcal{V}_\Omega Q_{A(k)}(\lambda)$ that is a direct summand of $\mathcal{V}_\Omega Q_{A(k)}(\lambda)\langle N-\delta(\lambda)\rangle$. The formulas in 17.5 together with Lemma 16.15 imply

$$[\mathcal{Q}(\lambda):\mathcal{Z}_\lambda\langle m\rangle]=\delta_{m,-\delta(\lambda)}, \tag{1}$$
$$[\mathcal{Q}(\lambda):\mathcal{Z}'_\lambda\langle m\rangle]=\delta_{m,2N-\delta(\lambda)}, \tag{2}$$
$$[\mathcal{Q}(\lambda):\mathcal{Z}_{\widehat{\lambda}}\langle m\rangle]=\delta_{m,N-\delta(\widehat{\lambda})}, \tag{3}$$
$$[\mathcal{Q}(\lambda):\mathcal{Z}'_{\widehat{\lambda}}\langle m\rangle]=\delta_{m,N-\delta(\widehat{\lambda})}, \tag{4}$$

where $N=|R^+|$. Each of these formulas together with the fact that $\mathcal{Q}(\lambda)$ is a graded S_k-form of $\mathcal{V}_\Omega Q_{A(k)}(\lambda)$ determine it uniquely up to isomorphism, cf. Lemma 15.6.c.

Note that for λ in the top alcove of its box this notation is compatible with the one from 17.4(3), cf. 17.4(4).

17.7. Lemma: *If $\mathcal{M}$ is a graded S_k-form of $\mathcal{V}_\Omega M$ for a projective module M in $\mathcal{C}_{A(k)}(\Omega)$, then $h_Z\mathcal{M}$ and $h'_Z\mathcal{M}$ are in $\mathfrak{M}^0$ and satisfy $h'_Z\mathcal{M}=\Phi_\delta(h_Z\mathcal{M})$.*

Proof: Note first: If $\mathcal{M}$ satisfies the claim, so does each $\mathcal{M}\langle r\rangle$. In general, Corollary 16.16 implies that it is enough to look at $M=Q_{A(k)}(\lambda)$ with $\lambda\in\Omega$. So we may assume that $\mathcal{M}=\mathcal{Q}(\lambda)$. Now we use induction on $\widehat{\lambda}-\lambda$. We can decompose $\mathcal{Q}^{(\lambda)}(S_k)\langle-\delta(\lambda)\rangle$ into the direct sum of $\mathcal{Q}(\lambda)$ and of certain $\mathcal{Q}(\nu)(S_k)\langle m\rangle$ with $\widehat{\nu}-\nu<\widehat{\lambda}-\lambda$. By induction each of these $\mathcal{Q}(\nu)\langle m\rangle$ satisfies the claim. So does $\mathcal{Q}^{(\lambda)}(S_k)\langle-\delta(\lambda)\rangle$ by Lemma 17.5. Therefore it holds also for the remaining summand $\mathcal{Q}(\lambda)$.

17.8. Lusztig introduces in [Lu2], 5.2 and 8.9, certain elements D_C and $\widetilde{D}_C$ in $\mathfrak{M}$ for each alcove for W_p. Identifying each $\lambda\in\Omega$ with its alcove, we can regard these elements as parametrized by Ω. Set $\mathbf{D}(\lambda)=\widetilde{D}_\lambda$ for all $\lambda\in\Omega$. By [Lu2], 7.3 and 8.9 the coefficient of $e(\lambda)$ in $\mathbf{D}(\lambda)$ is equal to $t^{-\delta(\lambda)}$; if the coefficient of $e(\nu)$ with $\nu\neq\lambda$ in $\mathbf{D}(\lambda)$ is not equal to 0, then $\nu<\lambda$ and the coefficient has degree at most $-\delta(\nu)-1$. (Recall that our t is the $q^{1/2}$ in [Lu2].) We have furthermore $\Phi_\delta\mathbf{D}(\lambda)=\mathbf{D}(\lambda)$ for all λ.

If μ is a special point for W_p, then

$$\mathbf{D}(\widehat{\mu^-})=t^{-|R^+|-\delta(\mu^-)}\sum_{\nu\Gamma=\mu}e(\nu) \tag{1}$$

where $\Gamma=W_p{\scriptstyle\bullet}\mu$, cf. [Lu2], proof of 8.3. (Recall that $\delta(\widehat{\mu^-})=\delta(\mu^-)+|R^+|$ by 17.1(3).)

Lusztig has made in [Lu1] a conjecture on the characters of irreducible G_k–modules in Case 1 and in [Lu4] a conjecture on the characters of irreducible U_3–modules in Case 2. We get then conjectures also for the characters of simple modules in $\mathcal{C}_k$, since these modules lift to G_k resp. to U_3 if the highest weight is in X_p. (For the other weights apply 4.2(5).) However, there is in Case 1 in [Lu1] a restriction on the highest weight of the modules. For small p (more precisely, for $p < 2h-3$) Lusztig's conjecture does not cover all weights in X_p. When we say below (and in 18.17ff.) "If Lusztig's conjecture holds for k" we mean that for all p–regular weights $\lambda \in X_p$ the character of $L_k(\lambda)$ is given by the formula in Lusztig's conjecture. So this is in Case 1 for $h \le p < 2h-3$ somewhat stronger than Lusztig's conjecture; it is in that case the conjecture 5.5 in [Kat].

Theorem: *If Lusztig's conjecture holds for k, then*

$$h_Z \mathcal{Q}(\lambda) = t^{|R^+|}\mathbf{D}(\widehat{\lambda}) \qquad \textit{for all } \lambda \in \Omega.$$

Proof: Let us use the abbreviation $N = |R^+|$. Note that the claim is (by Lemma 17.7) equivalent to $h'_Z\mathcal{Q}(\lambda) = t^{-N}\mathbf{D}(\widehat{\lambda})$. We prove the theorem again using induction on $\widehat{\lambda} - \lambda$. If $\lambda = \mu^-$ for a special point μ, then the claim follows from (1) and 17.4(4).

Suppose now that there is a reflection $s \in \Sigma_a$ with $\lambda < \lambda s$ and λs in the same box as λ. We assume inductively

$$h_Z \mathcal{Q}(\lambda s) = t^N \mathbf{D}(\widehat{\lambda s}) \qquad \text{and} \qquad h'_Z \mathcal{Q}(\lambda s) = t^{-N}\mathbf{D}(\widehat{\lambda s}).$$

So the degree estimate for the coefficients of $\mathbf{D}(\widehat{\lambda s})$ implies for all $\nu \neq \widehat{\lambda s}$

$$[\mathcal{Q}(\lambda s) : \mathcal{Z}_\nu\langle i\rangle] \neq 0 \implies i \le N - \delta(\nu) - 1 \text{ and } \nu < \widehat{\lambda s} \tag{2}$$

and

$$[\mathcal{Q}(\lambda s) : \mathcal{Z}'_\nu\langle i\rangle] \neq 0 \implies i \ge N - \delta(\nu) + 1 \text{ and } \nu < \widehat{\lambda s}. \tag{3}$$

Consider the graded S_k–form $\mathcal{M} = \widetilde{\Theta}_s\mathcal{Q}(\lambda s)\langle -1\rangle$ of $\mathcal{V}_\Omega\Theta_s Q_{A(k)}(\lambda s)$. The formulas in 15.18 imply (for all $\nu \in \Omega$ and $m \in \mathbf{Z}$):
If $\nu > \nu s$, then

$$[\mathcal{M} : \mathcal{Z}_\nu\langle m\rangle] = [\mathcal{Q}(\lambda s) : \mathcal{Z}_\nu\langle m-1\rangle] + [\mathcal{Q}(\lambda s) : \mathcal{Z}_{\nu s}\langle m+1\rangle] \tag{4}$$

and

$$[\mathcal{M} : \mathcal{Z}'_\nu\langle m\rangle] = [\mathcal{Q}(\lambda s) : \mathcal{Z}'_\nu\langle m+1\rangle] + [\mathcal{Q}(\lambda s) : \mathcal{Z}'_{\nu s}\langle m+1\rangle]. \tag{5}$$

If $\nu < \nu s$, then

$$[\mathcal{M} : \mathcal{Z}_\nu\langle m\rangle] = [\mathcal{Q}(\lambda s) : \mathcal{Z}_\nu\langle m+1\rangle] + [\mathcal{Q}(\lambda s) : \mathcal{Z}_{\nu s}\langle m-1\rangle] \tag{6}$$

and

$$[\mathcal{M} : \mathcal{Z}'_\nu\langle m\rangle] = [\mathcal{Q}(\lambda s) : \mathcal{Z}'_\nu\langle m-1\rangle] + [\mathcal{Q}(\lambda s) : \mathcal{Z}'_{\nu s}\langle m-1\rangle]. \tag{7}$$

There is a decomposition of $\mathcal{M}$ into indecomposables of the form $\mathcal{Q}(\nu)\langle i\rangle$ with $\nu \in \Omega$ and $i \in \mathbf{Z}$, cf. 16.16. Since $\widehat{\lambda}$ is the largest weight of $\mathcal{M}$ (or rather of $\Theta_s Q_{A(k)}(\lambda s)$) and occurs with multiplicity 1, there is exactly one summand of the form $\mathcal{Q}(\lambda)\langle n\rangle$. The formulas (4)–(7) show especially (since $\delta(\lambda s) = \delta(\lambda) + 1$)

$$[\mathcal{M} : \mathcal{Z}_\lambda\langle m\rangle] = [\mathcal{Q}(\lambda s) : \mathcal{Z}_{\lambda s}\langle m-1\rangle] = \delta_{m-1,-\delta(\lambda s)} = \delta_{m,-\delta(\lambda)};$$

we get here the first equality, because $Z_k(\lambda)$ does not occur in $Q_k(\lambda s)$, and the second equality from the normalization of $\mathcal{Q}(\lambda s)$. On the other hand, if $\mathcal{Q}(\lambda)\langle n\rangle$ is a direct summand of $\mathcal{M}$, we have $[\mathcal{M} : \mathcal{Z}_\lambda\langle n-\delta(\lambda)\rangle] \neq 0$, hence $n = 0$. This shows that that $\mathcal{Q}(\lambda)\langle n\rangle$ occurs if and only if $n = 0$, and then it occurs exactly once.

Consider now summands with $\nu \neq \lambda$. If $\mathcal{Q}(\nu)\langle i\rangle$ occurs as a summand of $\mathcal{M}$, then

$$[\mathcal{M} : \mathcal{Z}_{\widehat{\nu}}\langle i - \delta(\widehat{\nu}) + N\rangle] > 0 \qquad \text{and} \qquad [\mathcal{M} : \mathcal{Z}'_{\widehat{\nu}}\langle i - \delta(\widehat{\nu}) + N\rangle] > 0$$

by 17.6(3), (4). Suppose first that $\nu < \nu s$, hence $\widehat{\nu} > \widehat{\nu} s$. Then (4),(5) imply

$$[\mathcal{Q}(\lambda s) : \mathcal{Z}_{\widehat{\nu}}\langle i - \delta(\widehat{\nu}) + N - 1\rangle] + [\mathcal{Q}(\lambda s) : \mathcal{Z}_{\widehat{\nu s}}\langle i - \delta(\widehat{\nu}) + N + 1\rangle] > 0$$

and

$$[\mathcal{Q}(\lambda s) : \mathcal{Z}'_{\widehat{\nu}}\langle i - \delta(\widehat{\nu}) + N + 1\rangle] + [\mathcal{Q}(\lambda s) : \mathcal{Z}'_{\widehat{\nu s}}\langle i - \delta(\widehat{\nu}) + N + 1\rangle] > 0.$$

Now (2) implies $i \leq 0$ and (3) implies $i \geq 0$. So only $i = 0$ can occur. We see also that the number of summands $\mathcal{Q}(\nu)$ is bounded above by $[\mathcal{Q}(\lambda s) : \mathcal{Z}_{\widehat{\nu}}\langle N - \delta(\widehat{\nu}) - 1\rangle]$; by our induction hypothesis this is just the coefficient of $t^{-\delta(\widehat{\nu})-1}e(\widehat{\nu})$ in $\mathbf{D}(\lambda s)$ that is denoted by $\mu(\widehat{\nu}, \widehat{\lambda s})$ in [Lu2], 8.1. For $\nu > \nu s$ (i.e. $\widehat{\nu} > \widehat{\nu} s$), the same argument as above (with a small change for $\nu = \lambda$) yields $i > 0$ and $i < 0$; so these ν cannot occur. This shows that we have a decomposition of the form

$$\mathcal{M} \;=\; \mathcal{Q}(\lambda) \,\oplus\, \bigoplus_{\nu\,\ldots} \mathcal{Q}(\nu)^{m(\lambda s,\nu)} \tag{8}$$

with all $m(\lambda s, \nu) \leq \mu(\widehat{\nu}, \widehat{\lambda s})$. Here "$\nu \ldots$" is short for "$\nu < \nu s, \widehat{\nu} < \widehat{\lambda s}, \delta(\widehat{\lambda s}) - \delta(\widehat{\nu})$ odd". (The last condition follows from $\mu(\widehat{\nu}, \widehat{\lambda s}) \neq 0$, cf. [Lu2].) We get

$$h_Z\mathcal{M} = h_Z\mathcal{Q}(\lambda) + \sum_{\nu\,\ldots} m(\lambda s, \nu) h_Z\mathcal{Q}(\nu).$$

Each ν in this sum satisfies $\widehat{\nu} - \nu < \widehat{\lambda} - \lambda$. So we can apply induction to them. We can also apply induction to $\mathcal{Q}(\lambda s)$ and then compute $h_Z\mathcal{M}$ using 17.3. We get thus

$$t^N h_Z \mathcal{Q}(\lambda) = t^{-1}(T_s+1)\mathbf{D}(\widehat{\lambda s}) - \sum_{\nu \ldots} m(\lambda s, \nu)\mathbf{D}(\widehat{\nu}).$$

On the other hand, the formula in [Lu2], 10.7 for the D_C can be rewritten as

$$\mathbf{D}(\widehat{\lambda}) = t^{-1}(T_s+1)\mathbf{D}(\widehat{\lambda s}) - \sum_{\nu \ldots} \mu(\widehat{\nu}, \widehat{\lambda s})\mathbf{D}(\widehat{\nu}).$$

The $\mathbf{D}(\widehat{\nu})$ with $\nu \in \Omega$ are linearly independent, cf. [Lu2], 8.3. So we see that $h_Z\mathcal{Q}(\lambda) = t^{-N}\mathbf{D}(\widehat{\lambda})$ if and only if $m(\lambda s, \nu) = \mu(\widehat{\nu}, \widehat{\lambda s})$ for all ν as above. Since the $m(\lambda s, \nu)$ are determined by

$$T_s' T_s Q_k(\lambda s) \;\simeq\; Q_k(\lambda) \,\oplus\, \bigoplus_{\nu \ldots} Q_k(\nu)^{m(\lambda s, \nu)},$$

this condition is equivalent to Lusztig's conjecture, cf. (e.g.) [Cli], 3.1 in Case 1. (The same argument works in the quantum case.)

17.9. Let us introduce a notation for the coefficients of $\mathbf{D}(\lambda)$ and write

$$\mathbf{D}(\lambda) = \sum_{\nu} D_{\nu,\lambda}(t)e(\nu) = \sum_{\nu,i} D^i_{\nu,\lambda}t^i e(\nu). \tag{1}$$

If Lusztig's conjecture (as above) holds for our k, then we have by the last theorem $h_Z\mathcal{Q}(\lambda) = t^N\mathbf{D}(\widehat{\lambda})$ where $N = |R^+|$, and also $h'_Z\mathcal{Q}(\lambda) = t^{-N}\mathbf{D}(\widehat{\lambda})$, see the beginning of the proof. So we have for all $\lambda, \mu \in \Omega$ and $n \in \mathbf{Z}$

$$[\mathcal{Q}(\lambda) : \mathcal{Z}_\mu\langle n\rangle] = D^{n-N}_{\mu,\widehat{\lambda}} \tag{L2}$$

and

$$[\mathcal{Q}(\lambda) : \mathcal{Z}'_\mu\langle n\rangle] = D^{-n+N-2\delta(\mu)}_{\mu,\widehat{\lambda}}, \tag{L3}$$

cf. 17.2(3),(4). We add here an L to the equation number to indicate that the result depends on the truth of Lusztig's conjecture.

Proposition 15.10 shows how to determine $\mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{Q}(\lambda), \mathcal{Q}(\mu))$ for $\lambda, \mu \in \Omega$: We have to look at all $\nu \in \Omega$. Each occurrence of a $\mathcal{Z}_\nu\langle m\rangle$ (with $m \in \mathbf{Z}$) as a factor in a permissible $\mathcal{Z}$-filtration of $\mathcal{Q}(\lambda)$ together with an occurrence of a $\mathcal{Z}'_\nu\langle n\rangle$ (with $n \in \mathbf{Z}$) as a factor in a permissible $\mathcal{Z}'$-filtration of $\mathcal{Q}(\nu)$ leads to a direct summand $S_k\langle n-m\rangle$ in this Hom space; and we get the total Hom space summing over all pairs as above. If we assume $(L2)$ and $(L3)$ for all weights in Ω, then an easy calculation shows

$$\operatorname{rk} \operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{Q}(\lambda),\mathcal{Q}(\mu)) = \sum_{\nu\in\Omega} D_{\nu,\widehat{\lambda}} D_{\nu,\widehat{\mu}}\, t^{2\delta(\nu)}. \tag{L4}$$

(Recall that the coefficient of t^i is equal to $\dim_k \operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{Q}(\lambda),\mathcal{Q}(\mu))_{-i}$.) The highest power of t occurring in any $D_{\nu,\nu'}$ is less or equal to $-\delta(\nu)-1$ in case $\nu \neq \nu'$, whereas $D_{\nu,\nu'} = t^{-\delta(\nu)}$ in case $\nu = \nu'$. So $(L4)$ shows: If Lusztig's conjecture holds for our k, then $\operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{Q}(\lambda),\mathcal{Q}(\mu))_i = 0$ for all $i < 0$; in case $\lambda \neq \mu$ this vanishing holds also for $i = 0$, whereas in case $\lambda = \mu$ that degree 0 part has dimension 1.

Independently of Lusztig's conjecture, these arguments show:

Lemma: *Let M and N be modules in $\mathcal{FC}_{A(k)}(\Omega)$, let $\mathcal{M}$ (resp. $\mathcal{N}$) be a graded S_k-form of $\mathcal{V}_\Omega M$ (resp. of $\mathcal{V}_\Omega N$). Suppose that $\mathcal{M}$ has a permissible $\mathcal{Z}$-filtration and that $\mathcal{N}$ has a permissible $\mathcal{Z}'$-filtration. Suppose that there are weights λ_i and μ_j in Ω with*

$$h_Z\mathcal{M} = t^{|R^+|}\sum_{i=1}^{r}\mathbf{D}(\lambda_i) \qquad \text{and} \qquad h'_Z\mathcal{N} = t^{-|R^+|}\sum_{j=1}^{s}\mathbf{D}(\mu_j).$$

Then we have $\operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M},\mathcal{N})_m = 0$ for all $m < 0$; the dimension of $\operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M},\mathcal{N})_0$ over k is equal to the number of pairs (i,j) with $\lambda_i = \mu_j$.

18. Graded Representation Categories and Koszulity

In this section we shall work over a fixed field k. Let us abbreviate $p\mathbf{Z}R = Y$.

18.1. Let A be a B–algebra. Let Ω be an orbit of W_p in X. We know by 6.18 that $\mathcal{C}_A(\Omega)$ is a Y–category and that we can find a projective Y–generator P of $\mathcal{C}_A(\Omega)$. Set

$$E(\Omega) = E(\Omega, P) = (\mathrm{End}^{\sharp}_{\mathcal{C}} P)^{opp}. \tag{1}$$

This is a Y–graded A–algebra that is finitely generated as an A–module. (Note that there are only finitely many $\tau \in Y$ with $\mathrm{Hom}_{\mathcal{C}}(P[\tau], P) \neq 0$. Then use Lemma 2.8.) So $E(\Omega)$ is a Noetherian ring. By Proposition E.4 the functor

$$H_\Omega : N \mapsto \mathrm{Hom}^{\sharp}_{\mathcal{C}}(P, N) \tag{2}$$

induces an equivalence of categories from $\mathcal{C}_A(\Omega)$ to the (abelian) category of all finitely generated Y–graded $E(\Omega)$–modules.

It is not hard to construct an explicit inverse for this functor. Recall that by definition an object of $\mathcal{C}_A(\Omega)$ is just an X–graded $(U \otimes A)$–module or, equivalently, an X–graded (U, A)–bimodule with some properties. Certainly we can regard any Y–graded space as an X–graded space. So we can view $E(\Omega)$ as an X–graded ring, and then P is an X–graded $(U, E(\Omega))$–bimodule. Hence we can form for any finitely generated Y–graded $E(\Omega)$–module M the X–graded $(U \otimes A)$–module $P \otimes_{E(\Omega)} M$. For $M = E(\Omega)$ we just find $P \otimes_{E(\Omega)} E(\Omega) = P$, and since our tensor product is right exact, we really get a functor

$$D_\Omega : M \mapsto P \otimes_{E(\Omega)} M \tag{3}$$

in the opposite direction. By E.2 our functor D_Ω is adjoint to H_Ω, so it has to be an equivalence of categories inverse to H_Ω.

18.2. Let Γ be another W_p–orbit and let Q be a projective Y–generator of $\mathcal{C}_A(\Gamma)$. We can carry out the constructions in 18.1 for (Γ, Q) instead of (Ω, P). Recall the translation functors

$$T : \mathcal{C}_A(\Omega) \to \mathcal{C}_A(\Gamma) \qquad \text{and} \qquad T' : \mathcal{C}_A(\Gamma) \to \mathcal{C}_A(\Omega)$$

from Section 7. We can use the equivalences of categories H_Ω and H_Γ to transport T and T' to the categories of finitely generated Y–graded modules over $E(\Omega)$ resp. $E(\Gamma)$. More explicitly, set

$$ {}_\Omega\mathbf{T}_\Gamma = \mathrm{Hom}^\sharp_{\mathcal{C}}(P, T'Q) \quad \text{and} \quad {}_\Gamma\mathbf{T}_\Omega = \mathrm{Hom}^\sharp_{\mathcal{C}}(Q, TP). \tag{1} $$

Thus ${}_\Omega\mathbf{T}_\Gamma$ is a Y–graded $(E(\Omega), E(\Gamma))$–bimodule, and ${}_\Gamma\mathbf{T}_\Omega$ is a Y–graded $(E(\Gamma), E(\Omega))$–bimodule. We get now functors $\mathbf{T}$ and $\mathbf{T}'$ given by

$$ \mathbf{T}(M) = {}_\Gamma\mathbf{T}_\Omega \otimes_{E(\Omega)} M \quad \text{resp.} \quad \mathbf{T}'(N) = {}_\Omega\mathbf{T}_\Gamma \otimes_{E(\Gamma)} N \tag{2} $$

from the category of finitely generated Y–graded modules over $E(\Omega)$ to that over $E(\Gamma)$ resp. in the opposite direction. We have by Proposition E.5 up to natural equivalence

$$ \mathbf{T} \circ H_\Omega \equiv H_\Gamma \circ T \quad \text{resp.} \quad \mathbf{T}' \circ H_\Gamma \equiv H_\Omega \circ T'. \tag{3} $$

The functor $\mathbf{T}'$ has by E.2 a right adjoint given by $M \mapsto \mathrm{Hom}_{E(\Omega)}({}_\Omega\mathbf{T}_\Gamma, M)$. On the other hand T is right adjoint to T', so also $\mathbf{T}$ is right adjoint to $\mathbf{T}'$ by (3). Since the right adjoint is unique, we see that there is for each finitely generated Y–graded $E(\Omega)$–modules M an isomorphism

$$ {}_\Gamma\mathbf{T}_\Omega \otimes_{E(\Omega)} M \xrightarrow{\sim} \mathrm{Hom}_{E(\Omega)}({}_\Omega\mathbf{T}_\Gamma, M) \tag{4} $$

that is functorial in M. If we take in particular $M = E(\Omega)$, then we get an isomorphism of Y–graded $(E(\Gamma), E(\Omega))$–bimodules

$$ {}_\Gamma\mathbf{T}_\Omega \xrightarrow{\sim} \mathrm{Hom}_{E(\Omega)}({}_\Omega\mathbf{T}_\Gamma, E(\Omega)). \tag{5} $$

We deduce also that $\mathrm{Hom}_{E(\Omega)}({}_\Omega\mathbf{T}_\Gamma, \quad)$ is an exact functor, hence for all finitely generated Y–graded $E(\Omega)$–modules M the obvious map

$$ \mathrm{Hom}_{E(\Omega)}({}_\Omega\mathbf{T}_\Gamma, E(\Omega)) \otimes_{E(\Omega)} M \longrightarrow \mathrm{Hom}_{E(\Omega)}({}_\Omega\mathbf{T}_\Gamma, M) \tag{6} $$

is an isomorphism.

18.3. Let A' be an A–algebra. If P is a projective Y–generator of $\mathcal{C}_A(\Omega)$, then $P \otimes_A A'$ is a projective Y–generator of $\mathcal{C}_{A'}(\Omega)$. (Indeed, $P \otimes_A A'$ is projective by 3.1.a. Clearly each $Z_{A'}(\lambda)$ with $\lambda \in \Omega$ is a homomorphic image of some $(P \otimes_A A')[\tau] \simeq (P[\tau]) \otimes_A A'$ with $\tau \in Y$. Now argue as in 6.18.)

Proposition 3.3 implies for the algebras as in 18.1(1)

$$ E(\Omega, P \otimes_A A') \simeq E(\Omega, P) \otimes_A A'. \tag{1} $$

Moreover, the functors H_Ω as in 18.1(2) — constructed with P and $P \otimes_A A'$ — commute with extension of scalars.

Consider, as in 18.2, a second pair (Γ, Q). Then ${}_\Omega\mathbf{T}_\Gamma \otimes_A A'$ resp. ${}_\Gamma\mathbf{T}_\Omega \otimes_A A'$ are the analogues of ${}_\Omega\mathbf{T}_\Gamma$ resp. of ${}_\Gamma\mathbf{T}_\Omega$ [with the bimodule structures arising from (1) and its analogue for Γ]. We get therefore the analogue of $\mathbf{T}$ over A' from $\mathbf{T}$ by extension of scalars; similarly for $\mathbf{T}'$.

18.4. Let Ω and Γ be two orbits of W_p in X as in 10.2; so there is for each $\lambda \in \Omega$ a unique $\lambda_\Gamma \in \Gamma$ in the closure of the facet of λ.

Lemma: *Let A be a B-algebra such that 0 and 1 are the only idempotents of A and such that the image of each H_β in A is not a unit in A. Then the Y-graded bimodules ${}_\Gamma\mathbf{T}_\Omega$ and ${}_\Omega\mathbf{T}_\Gamma$ are indecomposable.*

Proof: Formula 18.2(4) shows that it is enough to deal with ${}_\Gamma\mathbf{T}_\Omega$. So, suppose ${}_\Gamma\mathbf{T}_\Omega \simeq M_1 \oplus M_2$ is a decomposition of ${}_\Gamma\mathbf{T}_\Omega$. By 18.2(2) this decomposition leads to a decomposition of functors $\mathbf{T} \simeq \mathbf{T}_1 \oplus \mathbf{T}_2$ where $\mathbf{T}_i$ corresponds to $M_i \otimes_{E(\Omega)}$. More generally, by 18.3 the analogue of $\mathbf{T}$ over each A-algebra A' splits, and this splitting is compatible with extension of scalars from A to A'. Using 18.2(3) we see that also the functor T (over A and each A') splits $T = T_1 \oplus T_2$ where both summands are Y-functors, and that also this splitting is compatible with extension of scalars. (Recall that the H_Ω and H_Γ are equivalences of categories.)

Furthermore, the right adjoint T' of T corresponds to the right adjoint $\mathrm{Hom}_{E(\Gamma)}({}_\Gamma\mathbf{T}_\Omega, \ )$ of ${}_\Gamma\mathbf{T}_\Omega \otimes_{E(\Omega)}$. Arguing as above we see that also T' breaks up as $T' \simeq T_1' \oplus T_2'$ [into Y-functors, over each A', compatibly with extension of scalars] where T_i' is right adjoint to T_i.

Now $TZ_A(\lambda) \simeq Z_A(\lambda_\Gamma)$ is indecomposable for all $\lambda \in \Omega$, since 0 and 1 are the only idempotents of A; so we get a partition $\Omega = \Omega_1 \cup \Omega_2$ with

$$\Omega_i = \{\lambda \in \Omega \mid T_i Z_A(\lambda) \neq 0\}.$$

Clearly Ω_1 and Ω_2 are stable under translations by elements from Y. We may assume that $\Omega_1 \neq \emptyset$. We want to show that $\Omega_2 = \emptyset$. If so, then T_2 kills every $Z_A(\lambda)$, hence kills every projective, hence is the zero functor, in other words $M_2 = 0$. So ${}_\Gamma\mathbf{T}_\Omega$ is indecomposable as claimed.

In order to show that $\Omega_2 = \emptyset$, we look at all residue fields F of A modulo a maximal ideal. Fix such an F and let $W_{\pi,p}$ be the corresponding affine Weyl group as in 6.7. Fix an arbitrary $\lambda_0 \in \Omega_1$ and set $\Omega' = W_{\pi,p} \cdot \lambda_0$. Then $\Gamma' = \{\lambda_\Gamma \mid \lambda \in \Omega'\}$ is also an orbit for $W_{\pi,p}$. The results in 7.10 on the decomposition of T and T' for orbits under $W_{\pi,p}$ together with 7.13(3) show for all $\mu \in \Gamma'$ that $(\mathrm{pr}_{\Omega'} \circ T')Q_F(\mu) \simeq Q_F(\mu^-)$ where μ^- is the unique weight in Ω' such that μ is in the upper closure of the facet of μ^- for $W_{\pi,p}$. The decomposition of T' leads to a decomposition

$$\mathrm{pr}_{\Omega'} \circ T' = (\mathrm{pr}_{\Omega'} \circ T_1') \oplus (\mathrm{pr}_{\Omega'} \circ T_2'),$$

where both summands are again exact. Since each $Q_F(\mu^-)$ is indecomposable, we get a disjoint decomposition $\Gamma' = \Gamma_1' \cup \Gamma_2'$ where

$$\Gamma_i' = \{\mu \in \Gamma' \mid (\mathrm{pr}_{\Omega'} \circ T_i')Q_F(\mu) = (\mathrm{pr}_{\Omega'} \circ T')Q_F(\mu)\}.$$

If (e.g.) $\mu \in \Gamma_1'$, then $(\mathrm{pr}_{\Omega'} \circ T_2')Q_F(\mu) = 0$, hence also $(\mathrm{pr}_{\Omega'} \circ T_2')Z_F(\mu') = 0$ for all $Z_F(\mu')$ in a Z-filtration of $Q_F(\mu)$ (since $\mathrm{pr}_{\Omega'} \circ T_2'$ is exact). We can take in particular $\mu' = \mu$ and see thus that

$$\Gamma_i' = \{\mu \in \Gamma' \mid (\mathrm{pr}_{\Omega'} \circ T_i')Z_F(\mu) \neq 0\} \qquad (\text{for } i = 1, 2).$$

All $Z_F(\mu')$ with $[Z_F(\mu') : L_F(\mu)] \neq 0$ occur in a Z–filtration of $Q_F(\mu)$, so all these μ' are in the same Γ'_i as μ. Using Proposition 4.6 we see now that all μ' with $\mathrm{Ext}^1(L_F(\mu'), L_F(\mu)) \neq 0$ are in the same Γ'_i as μ, hence that the whole block of μ over F is contained in that Γ'_i. So each Γ'_i is a union of blocks over F. On the other hand, the T'_i commute with the shift functors $[p\nu]$ with $\nu \in \mathbf{Z}R$, hence the $(\mathrm{pr}_{\Omega'} \circ T'_i)$ with the $[p\nu]$ with $\nu \in \mathbf{Z}R_\pi$. So each Γ'_i is closed under translation by elements from $p\mathbf{Z}R_\pi$. Now 6.16(4) implies that one of the Γ'_i has to be equal to Γ', the other one empty. If $\Gamma' = \Gamma'_2$, then we have for all $\lambda \in \Omega'$

$$\mathrm{Hom}_{\mathcal{C}}(Q_F(\lambda_\Gamma), T_1 Z_F(\lambda)) \simeq \mathrm{Hom}_{\mathcal{C}}((\mathrm{pr}_{\Omega'} \circ T'_1)Q_F(\lambda_\Gamma), Z_F(\lambda)) = 0,$$

hence $T_1 Z_F(\lambda) \not\simeq Z_F(\lambda_\Gamma)$. This yields $\Omega' \subset \Omega_2$ contradicting the choice of $\lambda_0 \in \Omega' \cap \Omega_1$. Therefore we have $\Gamma' = \Gamma'_1$ and get now as above $\Omega' \subset \Omega_1$.

We can find for each $\beta \in R^+$ a maximal ideal in A that contains H_β and get a residue field F of A such that the image of H_β in F is equal to 0. Now the discussion above shows that Ω_1 is stable under all $s_{\beta,mp}$ with $m \in \mathbf{Z}$. Since this works for all β, we see that Ω_1 is stable under W_p, hence $\Omega_1 = \Omega$ and $\Omega_2 = \emptyset$ as desired.

18.5. We can apply the results of the preceding subsections to $A = A(k) = \widehat{S}_k$. We now want to construct a graded version (over S_k) of the $E(\Omega)$ and the $\mathbf{T}$ from above. First we have to make the graded combinatorial categories $\widetilde{\mathcal{K}}(\Omega, S_k)$ into Y–categories by introducing a graded version of the shift operators from 9.13.

Denote by $l : Y \to 2\mathbf{Z}$ the character with (in the notations from 17.1)

$$l(\tau) = \delta(\lambda + \tau) - \delta(\lambda) \tag{1}$$

for all $\lambda \in X$ in a regular W_p–orbit, i.e., we set $l(\tau) = 2\,\mathrm{ht}(\tau/p)$, cf. 17.1(2).

Let Ω be an arbitrary W_p–orbit in X and consider the $\mathbf{Z}$–graded combinatorial category $\widetilde{\mathcal{K}}(\Omega, S_k)$. For $\mathcal{M}$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$ and $\tau \in Y$ we define $\mathcal{M}[\tau]$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$ by the formulas

$$\begin{aligned} \mathcal{M}[\tau](\lambda) &= \mathcal{M}(\lambda - \tau)\langle -l(\tau)\rangle, \\ \mathcal{M}[\tau](\lambda, \beta) &= \mathcal{M}(\lambda - \tau, \beta)\langle -l(\tau)\rangle. \end{aligned} \tag{2}$$

If $\mathcal{M}$ has a permissible $\mathcal{Z}$–filtration, then 17.2(3) shows

$$h_Z(\mathcal{M}[\tau]) = (h_Z\mathcal{M})t^{-l(\tau)}e(\tau), \tag{3}$$

and if $\mathcal{M}$ has a permissible $\mathcal{Z}'$–filtration, then 17.2(4) shows

$$h'_Z(\mathcal{M}[\tau]) = (h'_Z\mathcal{M})t^{l(\tau)}e(\tau). \tag{4}$$

Note that the graded S_k–forms $\mathcal{Q}(\lambda)$ of $\mathcal{V}_\Omega Q_{\widehat{S}_k}(\lambda)$ in 17.6 are chosen such that $\mathcal{Q}(\lambda+\tau) \simeq \mathcal{Q}(\lambda)[\tau]$.

For $\mathcal{M}, \mathcal{N}$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$ we denote by $\mathrm{Hom}^\sharp_{\widetilde{\mathcal{K}}}(\mathcal{M},\mathcal{N})$ the $(Y \times \mathbf{Z})$–graded space with homogeneous components

$$\mathrm{Hom}^\sharp_{\widetilde{\mathcal{K}}}(\mathcal{M},\mathcal{N})_{\tau,i} = \mathrm{Hom}_{\widetilde{\mathcal{K}}(\Omega,S_k)}(\mathcal{M}[\tau]\langle i\rangle,\mathcal{N}) = \mathrm{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{M}[\tau],\mathcal{N})_i. \quad (5)$$

18.6. All W_p–orbits to be considered below are supposed to be good in the sense of 13.3. In order to avoid too many indices we shall write $\mathcal{C}(\Omega, \widehat{S}_k)$ instead of $\mathcal{C}_{\widehat{S}_k}(\Omega)$.

Choose a projective Y–generator P of $\mathcal{C}(\Omega, \widehat{S}_k)$. Then P is flat over $\widehat{S}_k$ by 2.7, i.e., in $\mathcal{FC}(\Omega, \widehat{S}_k)$. Choose a graded S_k–form $\mathcal{P}$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$ of $\mathcal{V}_\Omega P$. This is always possible by 16.15. Now consider the $(Y \times \mathbf{Z})$–graded S_k–algebra

$$E_\Omega = E_\Omega(P,\mathcal{P}) = (\mathrm{End}^\sharp_{\widetilde{\mathcal{K}}}\mathcal{P})^{opp} \quad (1)$$

and denote the category of finitely generated $(Y \times \mathbf{Z})$–graded E_Ω–modules by $\widetilde{\mathcal{C}}(\Omega, S_k) = \widetilde{\mathcal{C}}(\Omega; P, \mathcal{P})$. It follows from 15.10 and 16.15 that E_Ω is a graded free module of finite rank over S_k. So it satisfies the assumptions on A in E.6, and we can apply the results in E.6 and E.9–11. Each module in $\widetilde{\mathcal{C}}(\Omega)$ is finitely generated as an S_k–module, the ring E_Ω is Noetherian, and $\widetilde{\mathcal{C}}(\Omega, S_k)$ is an abelian category. We call it a *graded deformation category*.

Remark: Another pair of choices $(P', \mathcal{P}')$ leads to an algebra $E_\Omega(P', \mathcal{P}')$ that turns out to be Morita equivalent to $E_\Omega(P, \mathcal{P})$. We get thus an equivalent graded deformation category $\widetilde{\mathcal{C}}(\Omega; P', \mathcal{P}') \simeq \widetilde{\mathcal{C}}(\Omega; P, \mathcal{P})$. This will follow from our results in subsection 18.13 on graded translation functors, applied to the case $\Omega = \Gamma$. However we don't need this independence, so we omit the details and just suppose a pair $(P, \mathcal{P})$ chosen when we speak of $\widetilde{\mathcal{C}}(\Omega, S_k)$.

18.7. Since $\widetilde{\mathcal{C}}(\Omega, S_k)$ is just a category of graded modules over a graded ring, we can shift the grading and form for M in $\widetilde{\mathcal{C}}(\Omega)$ and $i \in \mathbf{Z}$ resp. $\nu \in Y$ the shifted objects $M\langle i\rangle$ resp. $M\langle\nu\rangle$ in $\widetilde{\mathcal{C}}(\Omega, S_k)$. However we shall write $M[\nu]$ instead of $M\langle\nu\rangle$ for $\nu \in Y$ so to stay coherent in our notations.

The completion $\widehat{E}_\Omega = E_\Omega \otimes_{S_k} \widehat{S}_k$ of E_Ω along the $\mathbf{Z}$–grading is by 15.5(1) isomorphic to the algebra $E(\Omega)$ as in 18.1(1) where $A = \widehat{S}_k$. Recall the equivalences of categories H_Ω and D_Ω from 18.1 that are supposed to be constructed with our present P.

Let us now define the functor "completion along the $\mathbf{Z}$–grading"

$$v : \widetilde{\mathcal{C}}(\Omega, S_k) \longrightarrow \mathcal{C}(\Omega, \widehat{S}_k), \qquad M \mapsto P \otimes_{E_\Omega} M. \quad (1)$$

We have for all M in $\widetilde{\mathcal{C}}(\Omega, S_k)$ a natural isomorphism

$$D_\Omega(M \otimes_{S_k} \widehat{S}_k) = P \otimes_{E_\Omega \otimes_{S_k} \widehat{S}_k} (M \otimes_{S_k} \widehat{S}_k) \xrightarrow{\sim} P \otimes_{E_\Omega} M = vM, \quad (2)$$

hence an isomorphism of functors

$$D_\Omega \circ (\ \otimes_{S_k} \widehat{S}_k) \xrightarrow{\sim} v. \quad (3)$$

So v has the same properties as the completion $\otimes_{S_k} \widehat{S}_k$, since D_Ω is an equivalence of categories.

18.8. By a "graded form" of an object M in $\mathcal{C}(\Omega, \widehat{S}_k)$ we mean an object $\widetilde{M}$ in $\widetilde{\mathcal{C}}(\Omega, S_k)$ along with an isomorphism $v\widetilde{M} \xrightarrow{\sim} M$. Equivalently, we might ask for an isomorphism $\widetilde{M} \otimes_{S_k} \widehat{S}_k \xrightarrow{\sim} H_\Omega M$ of Y-graded $\widehat{E}_\Omega$-modules. By Lemma E.9.c each indecomposable object has up to shift and isomorphism at most one graded form. We shall construct for each $\lambda \in \Omega$ a graded form $\widetilde{Z}_{S_k}(\lambda)$ resp. $\widetilde{Z}'_{S_k}(\lambda)$ of $Z_{\widehat{S}_k}(\lambda)$ resp. $Z'_{\widehat{S}_k}(\lambda)$ and prove (with N_Ω as in 15.4):

Theorem: *a) There are isomorphisms of* $\mathbf{Z}$*-graded* S_k*-modules:*

$$\operatorname{Ext}^i_{E_\Omega, Y}(\widetilde{Z}_{S_k}(\lambda), \widetilde{Z}'_{S_k}(\mu)) \xrightarrow{\sim} \begin{cases} S_k, & \text{if } \lambda = \mu, i = 0; \\ 0, & \text{else;} \end{cases}$$

$$\operatorname{Ext}^i_{E_\Omega, Y}(\widetilde{Z}'_{S_k}(\lambda), \widetilde{Z}_{S_k}(\mu)) \xrightarrow{\sim} \begin{cases} S_k\langle 2(|R^+| - N_\Omega)\rangle, & \text{if } \lambda = \mu, i = 0; \\ 0, & \text{else.} \end{cases}$$

b) Each $\widetilde{Z}_{S_k}(\lambda)$ *has a unique simple quotient* $\widetilde{L}_k(\lambda)$*. This* $\widetilde{L}_k(\lambda)$ *is a graded form of* $L_k(\lambda)$*. Every simple object of* $\widetilde{\mathcal{C}}(\Omega, S_k)$ *is isomorphic to some* $\widetilde{L}_k(\lambda)\langle i\rangle$ *for unique* $\lambda \in \Omega$, $i \in \mathbf{Z}$.

c) Each projective object of $\mathcal{C}(\Omega, \widehat{S}_k)$ *admits a graded form. An object* $\widetilde{M}$ *in* $\widetilde{\mathcal{C}}(\Omega, S_k)$ *is projective if and only if* $v\widetilde{M}$ *in* $\mathcal{C}(\Omega, \widehat{S}_k)$ *is projective. Any projective object in* $\widetilde{\mathcal{C}}(\Omega, S_k)$ *admits a filtration with all subquotients among the* $\widetilde{Z}_{S_k}(\lambda)\langle i\rangle$*. It also admits a filtration with all subquotients among the* $\widetilde{Z}'_{S_k}(\lambda)\langle i\rangle$.

18.9. We shall construct the modules in 18.10; the proof of the theorem will be concluded in 18.12. First we need a better link between our graded deformation category $\widetilde{\mathcal{C}}(\Omega, S_k)$ and the graded combinatorial category $\widetilde{\mathcal{K}}(\Omega, S_k)$.

Consider the functor r from $\widetilde{\mathcal{K}}(\Omega, S_k)$ to $\widetilde{\mathcal{C}}(\Omega, S_k)$ given by

$$r\mathcal{M} = \operatorname{Hom}^\sharp_{\widetilde{\mathcal{K}}}(\mathcal{P}, \mathcal{M}), \quad (1)$$

and the functor $\widehat{r}$ from $\mathcal{K}(\Omega, \widehat{S}_k)$ to the category of all finitely generated Y–graded $\widehat{E}_\Omega$–modules given by

$$\widehat{r}\mathcal{N} = \mathrm{Hom}^\sharp_{\mathcal{K}}(\mathcal{V}_\Omega P, \mathcal{N}). \tag{2}$$

We have by 14.8(4) and the choice of $\mathcal{P}$ natural isomorphisms

$$\mathrm{Hom}^\sharp_{\widetilde{\mathcal{K}}}(\mathcal{P}, \mathcal{M}) \otimes_{S_k} \widehat{S}_k \xrightarrow{\sim} \mathrm{Hom}^\sharp_{\mathcal{K}}(\mathcal{V}_\Omega P, M \otimes_{S_k} \widehat{S}_k) \tag{3}$$

for all $\mathcal{M}$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$, hence up to natural equivalence

$$(\ \otimes_{S_k} \widehat{S}_k) \circ r \ \equiv\ \widehat{r} \circ (\ \otimes_{S_k} \widehat{S}_k). \tag{4}$$

On the other hand, we have natural isomorphisms

$$\mathrm{Hom}^\sharp_{\mathcal{C}}(P, M) \xrightarrow{\sim} \mathrm{Hom}^\sharp_{\mathcal{K}}(\mathcal{V}_\Omega P, \mathcal{V}_\Omega M) \tag{5}$$

for all M in $\mathcal{FC}(\Omega, \widehat{S}_k)$, hence up to natural equivalence

$$H_\Omega \ [\text{on } \mathcal{FC}(\Omega, \widehat{S}_k)] \ \equiv\ \widehat{r} \circ \mathcal{V}_\Omega. \tag{6}$$

Lemma: *Let M be in $\mathcal{FC}(\Omega, \widehat{S}_k)$ and let $\mathcal{M}$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$ be a graded S_k-form of $\mathcal{V}_\Omega M$.*

a) Then $r\mathcal{M}$ in $\widetilde{\mathcal{C}}(\Omega, S_k)$ is a graded form of M.

b) Let $\mathcal{N}$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$ be a graded S_k-form of $\mathcal{V}_\Omega N$ for some $N \in \mathcal{FC}(\Omega, \widehat{S}_k)$. Then the obvious map

$$\mathrm{Hom}_{\widetilde{\mathcal{K}}(\Omega, S_k)}(\mathcal{M}, \mathcal{N}) \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{C}}(\Omega, S_k)}(r\mathcal{M}, r\mathcal{N})$$

is an isomorphism.

c) Suppose that $0 \subset \mathcal{N} \subset \mathcal{M}$ is a permissible two-step filtration of $\mathcal{M}$. Then the maps $\mathcal{N} \to \mathcal{M} \to \mathcal{M}/\mathcal{N}$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$ induce a short exact sequence

$$0 \to r\mathcal{N} \longrightarrow r\mathcal{M} \longrightarrow r(\mathcal{M}/\mathcal{N}) \to 0$$

of $(Y \times \mathbf{Z})$-graded E_Ω-modules.

Proof: a) follows immediately from (4) and (6) and the definitions. For b) observe that our map is just the degree zero part of a map

$$\mathrm{Hom}_{\mathcal{K}(\Omega, S_k)}(\mathcal{M}, \mathcal{N}) \longrightarrow \mathrm{Hom}_{E_\Omega, Y}(r\mathcal{M}, r\mathcal{N}) \tag{7}$$

of graded S_k–modules. So we need just to show that this map is an isomorphism or, equivalently, that it becomes an isomorphism under $\otimes_{S_k} \widehat{S}_k$. Now if we apply $\otimes_{S_k} \widehat{S}_k$ to (7) we get just the map

$$\mathrm{Hom}_{\mathcal{K}(\Omega, \widehat{S}_k)}(\mathcal{V}_\Omega M, \mathcal{V}_\Omega N) \longrightarrow \mathrm{Hom}_{\widehat{E}_\Omega, Y}(\widehat{r}\mathcal{V}_\Omega M, \widehat{r}\mathcal{V}_\Omega N)$$

induced from our functor $\widehat{r}$. But $\widehat{r}$ is fully faithful on the image of $\mathcal{V}_\Omega$ by (6), since H_Ω is an equivalence of categories and $\mathcal{V}_\Omega$ is fully faithful.

For c) just notice that by definition of a permissible filtration (15.9) our sequence becomes short exact under completion $\otimes_{S_k} \widehat{S}_k$.

18.10. Recall the objects $\mathcal{Z}_\mu^w$ in $\widetilde{\mathcal{K}}(\Omega, S_k)$ from 15.3. Set for all $\mu \in X$ and $w \in W$

$$\widetilde{Z}_{S_k}^w(\mu) = r\mathcal{Z}_\mu^w\langle N_\Omega - \delta(\mu^-)\rangle \tag{1}$$

using the notation μ^- from 17.1. This is a module in $\widetilde{\mathcal{C}}(\Omega, S_k)$. We have obviously

$$\widetilde{Z}_{S_k}^w(\mu)[\tau] \simeq \widetilde{Z}_{S_k}^w(\mu+\tau) \qquad \text{for all } \tau \in Y. \tag{2}$$

Note that 18.9(4), (6) yield

$$\widetilde{Z}_{S_k}^w(\mu) \otimes_{S_k} \widehat{S}_k \simeq H_\Omega Z_{\widehat{S}_k}^w(\mu). \tag{3}$$

We shall use the abbreviations

$$\widetilde{Z}_{S_k}(\mu) = \widetilde{Z}_{S_k}^1(\mu) \qquad \text{and} \qquad \widetilde{Z}'_{S_k}(\mu) = \widetilde{Z}_{S_k}^{w_0}(\mu). \tag{4}$$

Let us now prove part a) of Theorem 18.8. Observe first that

$$\begin{aligned}
\mathrm{Ext}^i_{E_\Omega,Y}(\widetilde{Z}_{S_k}(\lambda), \widetilde{Z}'_{S_k}(\mu)) \otimes_{S_k} \widehat{S}_k &= \mathrm{Ext}^i_{\widehat{E}_\Omega,Y}(H_\Omega Z_{\widehat{S}_k}(\lambda), H_\Omega Z'_{\widehat{S}_k}(\mu)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{[by E.7(2)]} \\
&\simeq \mathrm{Ext}^i_{\mathcal{C}}(Z_{\widehat{S}_k}(\lambda), Z'_{\widehat{S}_k}(\mu)) \\
&= 0 \quad \text{unless } \lambda = \mu,\ i = 0 \qquad\qquad \text{[by 4.11/12]}.
\end{aligned}$$

We get similarly $\mathrm{Ext}^i_{E_\Omega,Y}(\widetilde{Z}'_{S_k}(\lambda), \widetilde{Z}_{S_k}(\mu)) = 0$ unless $\lambda = \mu$, $i = 0$. So, to finish the proof of part a) of our theorem, it remains to treat the cases with $i = 0$. But there the claim follows from 15.4 via Lemma 18.9.b.

18.11. We next go for 18.8.b. Note that we can regard modules in $\mathcal{C}_k(\Omega)$ also as modules in $\mathcal{C}(\Omega, \widehat{S}_k)$ via the natural map $\widehat{S}_k \twoheadrightarrow k$, cf. 3.1. This applies in particular to all $M \otimes_{\widehat{S}_k} k$ with M in $\mathcal{C}(\Omega, \widehat{S}_k)$.

Lemma: *Suppose $\widetilde{M}$ in $\widetilde{\mathcal{C}}(\Omega, S_k)$ is a graded form of M in $\mathcal{C}(\Omega, \widehat{S}_k)$. Then $\widetilde{M} \otimes_{S_k} k$ is a graded form of $M \otimes_{\widehat{S}_k} k$.*

Proof: Apply v to the exact sequence

$$\bigoplus_{\alpha\in\Sigma} \widetilde{M}\langle 2\rangle \longrightarrow \widetilde{M} \longrightarrow \widetilde{M} \otimes_{S_k} k \to 0,$$

where the first map is given by the row matrix $(h_\alpha)_{\alpha\in\Sigma}$.

18.12. Set $\widetilde{Z}_k(\lambda) = \widetilde{Z}_{S_k}(\lambda) \otimes_{S_k} k$; this is by Lemma 18.9 a graded form of $Z_k(\lambda)$. Clearly $\widetilde{Z}_k(\lambda)$ does not change under completion, so we have

$$H_\Omega Z_k(\lambda) \simeq \widetilde{Z}_k(\lambda) \otimes_{S_k} \widehat{S}_k = \widetilde{Z}_k(\lambda).$$

We know from 4.1 that $Z_k(\lambda)$ has a unique maximal proper subobject in $\mathcal{C}_k(\Omega)$, hence also in $\mathcal{C}(\Omega, \widehat{S}_k)$. Therefore also $\widetilde{Z}_k(\lambda)$, considered as a Y–graded $\widehat{E}_\Omega$–module, has a unique maximal subobject equal to $\mathrm{rad}_{\widehat{E}_\Omega, Y} \widetilde{Z}_k(\lambda)$. Since $\widetilde{Z}_k(\lambda)$ is annihilated by the maximal ideal $\widehat{\mathfrak{m}} \subset \widehat{S}_k$, this is also the maximal subobject of $\widetilde{Z}_k(\lambda)$ considered as a Y–graded E_Ω–module, i.e.,

$$\mathrm{rad}_{\widehat{E}_\Omega, Y} \widetilde{Z}_k(\lambda) = \mathrm{rad}_{E_\Omega, Y} \widetilde{Z}_k(\lambda).$$

By Proposition E.11 this radical is $\mathbf{Z}$–homogeneous. Thus we can define the object $\widetilde{L}_k(\lambda)$ in $\widetilde{\mathcal{C}}(\Omega, S_k)$ to be the unique simple quotient of $\widetilde{Z}_k(\lambda)$,

$$\widetilde{L}_k(\lambda) = \widetilde{Z}_k(\lambda) / \mathrm{rad}_{E_\Omega, Y} \widetilde{Z}_k(\lambda),$$

and it is clear that $\widetilde{L}_k(\lambda)$ is a graded form of $L_k(\lambda)$ in $\mathcal{C}(\Omega, \widehat{S}_k)$. Observe next that every object $\widetilde{M}$ in $\widetilde{\mathcal{C}}(\Omega, S_k)$ is a finitely generated graded S_k–module. Hence $(S_k)_{>0} \widetilde{M} = \widetilde{M}$ implies $\widetilde{M} = 0$. Thus every simple object in $\widetilde{\mathcal{C}}(\Omega, S_k)$ is annihilated by $(S_k)_{>0}$. Hence the simple quotients of $\widetilde{Z}_{S_k}(\lambda)$ and $\widetilde{Z}_k(\lambda)$ are the same, and hence $\widetilde{L}_k(\lambda)$ is the unique simple quotient of $\widetilde{Z}_{S_k}(\lambda)$.

If $\widetilde{L}$ is any simple object in $\widetilde{\mathcal{C}}(\Omega, S_k)$, we know $(S_k)_{>0} \widetilde{L} = 0$ and in particular $\widetilde{L}$ is of finite dimension and does not change under completion. Since we know that the $H_\Omega L_k(\lambda)$ are all the simple Y–graded $\widehat{E}_\Omega$–modules, there has to exist (for some $\lambda \in \Omega$) a surjection $\widetilde{L} \twoheadrightarrow \widetilde{L}_k(\lambda)$ of Y–graded $\widehat{E}_\Omega$–modules. This surjection can also be considered as a map of Y–graded E_Ω–modules, and one homogeneous component of this surjection gives necessarily a nonzero map $\widetilde{L} \to \widetilde{L}_k(\lambda)\langle i \rangle$ in $\widetilde{\mathcal{C}}(\Omega, S_k)$, which then has to be an isomorphism. This settles part b) of our theorem.

We finally treat part c). Lemma E.9.a says that $v\widetilde{M}$ is projective if and only if $\widetilde{M}$ is so. The remaining statements follow from 16.15 using Lemma 18.9. This concludes the proof of Theorem 18.8.

18.13. Assume from now on that Ω is a regular W_p–orbit, and that Γ is an arbitrary orbit. Suppose that we have chosen $(P, \mathcal{P}, E_\Omega)$ as above and that $(Q, \mathcal{Q}, E_\Gamma)$ are analogous choices for Γ. We can imitate the construction from 18.2 for our graded deformation categories. Set now

$${}_\Omega T_\Gamma = \mathrm{Hom}^\sharp_{\widetilde{\mathcal{K}}}(\mathcal{P}, \mathcal{T}'\mathcal{Q}) \quad \text{and} \quad {}_\Gamma T_\Omega = \mathrm{Hom}^\sharp_{\widetilde{\mathcal{K}}}(\mathcal{Q}, \mathcal{T}\mathcal{P}). \tag{1}$$

So ${}_\Omega T_\Gamma$ is a $(Y \times \mathbf{Z})$–graded (E_Ω, E_Γ)–bimodule, and ${}_\Gamma T_\Omega$ is a $(Y \times \mathbf{Z})$–graded (E_Γ, E_Ω)–bimodule. By 15.5(1) and 15.15 the $\mathbf{Z}$–completion of ${}_\Omega T_\Gamma$ resp. of ${}_\Gamma T_\Omega$ can be identified with the bimodule ${}_\Omega \mathbf{T}_\Gamma$ resp. with ${}_\Gamma \mathbf{T}_\Omega$ from 18.2(1). Next we define the *graded translation functors* $T_!$ and T^* by

$$T_! : \widetilde{\mathcal{C}}(\Omega, S_k) \longrightarrow \widetilde{\mathcal{C}}(\Gamma, S_k), \quad M \mapsto {}_\Gamma T_\Omega \otimes_{E_\Omega} M, \tag{2}$$

and

$$T^* : \widetilde{\mathcal{C}}(\Gamma, S_k) \longrightarrow \widetilde{\mathcal{C}}(\Omega, S_k), \quad N \mapsto {}_\Omega T_\Gamma \otimes_{E_\Gamma} N. \tag{3}$$

Proposition: *The following functorial diagrams commute up to natural equivalence:*

$$\begin{array}{ccc} \widetilde{\mathcal{C}}(\Omega, S_k) & \xrightarrow{v} & \mathcal{C}(\Omega, \widehat{S}_k) \\ \downarrow{\scriptstyle T_!} & & \downarrow{\scriptstyle T} \\ \widetilde{\mathcal{C}}(\Gamma, S_k) & \xrightarrow{v} & \mathcal{C}(\Gamma, \widehat{S}_k) \end{array} \qquad \begin{array}{ccc} \widetilde{\mathcal{C}}(\Gamma, S_k) & \xrightarrow{v} & \mathcal{C}(\Gamma, \widehat{S}_k) \\ \downarrow{\scriptstyle T^*} & & \downarrow{\scriptstyle T'} \\ \widetilde{\mathcal{C}}(\Omega, S_k) & \xrightarrow{v} & \mathcal{C}(\Omega, \widehat{S}_k). \end{array}$$

In particular, our graded translation functors are exact.

Proof: We only treat the first diagram. If we compose $v \circ T_!$ with H_Γ, then we get by 18.7(3) up to natural equivalence the functor

$$M \mapsto ({}_\Gamma T_\Omega \otimes_{E_\Omega} M) \otimes_{S_k} \widehat{S}_k \simeq {}_\Gamma \mathbf{T}_\Omega \otimes_{\widehat{E}_\Omega} (M \otimes_{S_k} \widehat{S}_k),$$

i.e., $M \mapsto \mathbf{T}(M \otimes_{S_k} \widehat{S}_k)$. By definition of v this functor is equal (up to natural equivalence) to $\mathbf{T} \circ H_\Omega \circ v$, hence to $H_\Gamma \circ T \circ v$ by 18.2(3). This yields the claim in the proposition since H_Γ is an equivalence.

18.14. Consider the functorial diagram

$$\begin{array}{ccc} \widetilde{\mathcal{K}}(\Omega, S_k) & \xrightarrow{\mathcal{T}} & \widetilde{\mathcal{K}}(\Gamma, S_k) \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ \widetilde{\mathcal{C}}(\Omega, S_k) & \xrightarrow{T_!} & \widetilde{\mathcal{C}}(\Gamma, S_k) \end{array}$$

and the natural transformation $T_! \circ r \to r \circ \mathcal{T}$ given as the composition

$$\begin{aligned} {}_\Gamma T_\Omega \otimes_{E_\Omega} \operatorname{Hom}^\sharp_{\widetilde{\mathcal{K}}}(\mathcal{P}, \mathcal{M}) \longrightarrow & \operatorname{Hom}^\sharp_{\widetilde{\mathcal{K}}}(\mathcal{Q}, \mathcal{T}\mathcal{P}) \otimes_{E_\Omega} \operatorname{Hom}^\sharp_{\widetilde{\mathcal{K}}}(\mathcal{T}\mathcal{P}, \mathcal{T}\mathcal{M}) \\ \longrightarrow & \operatorname{Hom}^\sharp_{\widetilde{\mathcal{K}}}(\mathcal{Q}, \mathcal{T}\mathcal{M}). \end{aligned}$$

Lemma: *If $\mathcal{M}$ is a graded S_k-form of $\mathcal{V}_\Omega M$ for some $M \in \mathcal{FC}(\Omega, \widehat{S}_k)$, then our natural transformation gives an isomorphism $T_! r\mathcal{M} \xrightarrow{\sim} r\mathcal{T}\mathcal{M}$.*

Proof: Indeed, we just have to show that our map becomes an isomorphism under completion. But then our map gets transformed into the composition

$$
\begin{aligned}
{}_\Gamma\mathbf{T}_\Omega \otimes_{\widehat{E}_\Omega} \mathrm{Hom}^\sharp_{\mathcal{C}}(P, M) \longrightarrow & \mathrm{Hom}^\sharp_{\mathcal{C}}(Q, TP) \otimes_{\widehat{E}_\Omega} \mathrm{Hom}^\sharp_{\mathcal{C}}(TP, TM) \\
& \longrightarrow \mathrm{Hom}^\sharp_{\mathcal{C}}(Q, TM),
\end{aligned}
$$

and this is an isomorphism by Proposition E.5 or, closer to the truth, by its proof.

Remark: We can construct similarly a natural transformation $T^* \circ r \to r \circ \mathcal{T}'$ that induces an isomorphism $T^* r\mathcal{N} \xrightarrow{\sim} r\mathcal{T}'\mathcal{N}$ whenever $\mathcal{N}$ is a graded S_k–form of $\mathcal{V}_\Omega N$ for some $N \in \mathcal{FC}(\Gamma, \widehat{S}_k)$.

18.15. By E.2 our graded translation functors $T_!$ and T^* have right adjoints $T^!$ and T_* given by

$$T^! = \mathrm{Hom}_{E_\Gamma}({}_\Gamma T_\Omega, \ ) \qquad \text{and} \qquad T_* = \mathrm{Hom}_{E_\Omega}({}_\Omega T_\Gamma, \ ).$$

Recall the notation $o(\lambda, \Gamma)$ from 15.13.

Theorem: *a) We have $T_! \simeq T_*$ and $T^! \simeq T^*\langle -2N_\Gamma\rangle$.*

b) We have $T_* \widetilde{Z}_{S_k}(\lambda) \simeq \widetilde{Z}_{S_k}(\lambda_\Gamma)\langle o(\lambda, \Gamma)\rangle$ and $T_! \widetilde{Z}'_{S_k}(\lambda) \simeq \widetilde{Z}'_{S_k}(\lambda_\Gamma)\langle -o(\lambda, \Gamma)\rangle$ for all $\lambda \in \Omega$.

c) For all $\mu \in \Gamma$ the translated module $T^* \widetilde{Z}_{S_k}(\mu)$ has a filtration with subquotients $\widetilde{Z}_{S_k}(\lambda)\langle o(\lambda, \Gamma)\rangle$, where λ runs over all $\lambda \in \Omega$ such that $\lambda_\Gamma = \mu$. Similarly $T^! \widetilde{Z}'_{S_k}(\mu)$ has a filtration with subquotients $\widetilde{Z}'_{S_k}(\lambda)\langle -o(\lambda, \Gamma)\rangle$, where λ runs over all $\lambda \in \Omega$ such that $\lambda_\Gamma = \mu$.

Proof: After completion ${}_\Gamma T_\Omega$ and $\mathrm{Hom}_{E_\Omega}({}_\Omega T_\Gamma, E_\Omega)$ get isomorphic by 18.2(5). Since they are indecomposable by Lemma 18.4, they have to be isomorphic up to shift:

$${}_\Gamma T_\Omega\langle i\rangle \simeq \mathrm{Hom}_{E_\Omega}({}_\Omega T_\Gamma, E_\Omega)$$

for some $i \in \mathbf{Z}$. For each M in $\widetilde{\mathcal{C}}(\Omega, S_k)$ the canonical map f in

$$T_! M\langle i\rangle \simeq \mathrm{Hom}_{E_\Omega}({}_\Omega T_\Gamma, E_\Omega) \otimes_{E_\Omega} M \xrightarrow{f} \mathrm{Hom}_{E_\Omega}({}_\Omega T_\Gamma, M) = T_* M$$

is an isomorphism, since it becomes the isomorphism 18.2(6) under completion. So there is an $i \in \mathbf{Z}$ such that $T_* \simeq T_!\langle i\rangle$. Similarly we find a $j \in \mathbf{Z}$ such that $T^! \simeq T^*\langle j\rangle$.

In order to prove a) we have to show that $i = 0$ and $j = -2N_\Gamma$. We postpone this and look first at b) and c). We have for all $\lambda \in \Omega$

$$\begin{aligned}
T_! \widetilde{Z}'_{S_k}(\lambda) &\simeq T_! r \mathcal{Z}'_\lambda \langle -\delta(\lambda)\rangle && [\text{by definition of } \widetilde{Z}'_{S_k}(\lambda)] \\
&\simeq r \mathcal{T} \mathcal{Z}'_\lambda \langle -\delta(\lambda)\rangle && [\text{by } 18.14] \\
&\simeq r \mathcal{Z}'_{\lambda_\Gamma} \langle N_\Gamma - \delta(\lambda)\rangle && [\text{by } 15.16] \\
&\simeq \widetilde{Z}'_{S_k}(\lambda_\Gamma)\langle -o(\lambda,\Gamma)\rangle && [\text{by definition of } \widetilde{Z}'_{S_k}(\lambda_\Gamma)].
\end{aligned}$$

For the last step we need also that $\delta((\lambda_\Gamma)^-) = \delta(\lambda) - o(\lambda, \Gamma)$, cf. 17.1(3). This proves the second claim in b); we get similarly $T_! \widetilde{Z}_{S_k}(\lambda) \simeq \widetilde{Z}_{S_k}(\lambda_\Gamma)\langle o(\lambda,\Gamma)\rangle$. This will yield the first claim in b) as soon as we have shown that $i = 0$.

Using similar arguments and 18.8.c we deduce from 15.17 that each $T^* \widetilde{Z}_{S_k}(\mu)$ (resp. each $T^* \widetilde{Z}'_{S_k}(\mu)$) with $\mu \in \Gamma$ has a filtration with subquotients $\widetilde{Z}_{S_k}(\lambda)\langle o(\lambda,\Gamma)\rangle$ (resp $\widetilde{Z}'_{S_k}(\lambda)\langle 2N_\Gamma - o(\lambda,\Gamma)\rangle$), where λ runs over all $\lambda \in \Omega$ such that $\lambda_\Gamma = \mu$. So we get the first claim in c); the second one will follow when we know that $j = -2N_\Gamma$.

So let us return to a). Choose $\lambda \in \Omega$. We have isomorphisms (by 18.8 and the results on $T_!$ and T^* above)

$$\begin{aligned}
S_k\langle -o(\lambda,\Gamma)\rangle &\simeq \operatorname{Hom}_{E_\Gamma, Y}(\widetilde{Z}_{S_k}(\lambda_\Gamma)\langle o(\lambda,\Gamma)\rangle, \widetilde{Z}'_{S_k}(\lambda_\Gamma)) \\
&\simeq \operatorname{Hom}_{E_\Gamma, Y}(T_! \widetilde{Z}_{S_k}(\lambda), \widetilde{Z}'_{S_k}(\lambda_\Gamma)) \\
&\simeq \operatorname{Hom}_{E_\Omega, Y}(\widetilde{Z}_{S_k}(\lambda), T^! \widetilde{Z}'_{S_k}(\lambda_\Gamma)) \\
&\simeq \operatorname{Hom}_{E_\Omega, Y}(\widetilde{Z}_{S_k}(\lambda), T^* \widetilde{Z}'_{S_k}(\lambda_\Gamma)\langle j\rangle) \\
&\simeq S_k\langle j - o(\lambda,\Gamma) + 2N_\Gamma\rangle.
\end{aligned}$$

We deduce $j = -2N_\Gamma$. Similarly we calculate i from the isomorphisms

$$\begin{aligned}
S_k\langle -o(\lambda,\Gamma)\rangle &\simeq \operatorname{Hom}_{E_\Omega, Y}(T^* \widetilde{Z}_{S_k}(\lambda_\Gamma), \widetilde{Z}'_{S_k}(\lambda)) \\
&\simeq \operatorname{Hom}_{E_\Gamma, Y}(\widetilde{Z}_{S_k}(\lambda), T_* \widetilde{Z}'_{S_k}(\lambda)) \\
&\simeq \operatorname{Hom}_{E_\Gamma, Y}(\widetilde{Z}_{S_k}(\lambda), T_! \widetilde{Z}'_{S_k}(\lambda)\langle i\rangle) \\
&\simeq \operatorname{Hom}_{E_\Gamma, Y}(\widetilde{Z}_{S_k}(\lambda), \widetilde{Z}'_{S_k}(\lambda)\langle -o(\lambda,\Gamma) + i\rangle) \\
&\simeq S_k\langle i - o(\lambda,\Gamma)\rangle.
\end{aligned}$$

We deduce $i = 0$. The theorem follows.

18.16. Set for all $\lambda \in \Omega$

$$\widetilde{Q}_{S_k}(\lambda) = r\mathcal{Q}(\lambda). \tag{1}$$

This module in $\widetilde{\mathcal{C}}(\Omega, S_k)$ is (by Lemma 18.9.a) a graded form of $Q_{\widehat{S}_k}(\lambda)$. By Lemma 18.9.c the permissible $\mathcal{Z}$–filtration of $\mathcal{Q}(\lambda)$ yields a filtration of $\widetilde{Q}_{S_k}(\lambda)$ with factors of the form $\widetilde{Z}_{S_k}(\mu)\langle n\rangle$ with $\mu \in \Omega$ and $n \in \mathbf{Z}$. The top factor $\mathcal{Z}_\lambda\langle -\delta(\lambda)\rangle$ in the first filtration leads to an epimorphism $\widetilde{Q}_{S_k}(\lambda) \twoheadrightarrow \widetilde{Z}_{S_k}(\lambda)$, cf. 18.10(1), hence to an epimorphism $\widetilde{Q}_{S_k}(\lambda) \twoheadrightarrow \widetilde{L}_k(\lambda)$.

We get from Lemma 18.9.b for all $\lambda, \mu \in \Omega$ an isomorphism of $(Y \times \mathbf{Z})$–graded S_k–modules

$$\operatorname{Hom}^\sharp_{\widetilde{\mathcal{C}}}(\widetilde{Q}_{S_k}(\lambda), \widetilde{Q}_{S_k}(\mu)) \simeq \operatorname{Hom}^\sharp_{\mathcal{K}(\Omega,S_k)}(\mathcal{Q}(\lambda), \mathcal{Q}(\mu)). \tag{2}$$

The right hand side can be described explicitly, if we know that Lusztig's conjecture (as in 17.8) holds for our k. Well, we have by definition

$$\operatorname{rk} \operatorname{Hom}^\sharp_{\mathcal{K}(\Omega,S_k)}(\mathcal{Q}(\lambda), \mathcal{Q}(\mu)) = \sum_{\tau \in Y} \operatorname{rk} \operatorname{Hom}_{\mathcal{K}(\Omega,S_k)}(\mathcal{Q}(\lambda+\tau), \mathcal{Q}(\mu)). \tag{3}$$

So 17.9($L4$) implies

$$\operatorname{rk} \operatorname{Hom}^\sharp_{\mathcal{K}(\Omega,S_k)}(\mathcal{Q}(\lambda), \mathcal{Q}(\mu)) = \sum_{\nu \in \Omega} \sum_{\tau \in Y} D_{\nu,\widehat{\lambda}+\tau} D_{\nu,\widehat{\mu}}\, t^{2\delta(\nu)}, \tag{$L4$}$$

where we use (as in 17.9) an L to indicate that the formula depends on the truth of Lusztig's conjecture. Some calculations later on will simplify if we replace the D–polynomials by Lusztig's Q–polynomials (again regarded as indexed by elements in Ω). So they are given by

$$Q_{\lambda,\mu} = t^{\delta(\mu)} D_{\lambda,\mu}, \tag{5}$$

cf. [Lu2], 8.9. Recall that our t is the $q^{1/2}$ in [Lu2]; so the $Q_{\lambda,\mu}$ involve only even powers of t (cf. [Lu2], 5.2). Using (5) we can rewrite ($L4$) as

$$\operatorname{rk} \operatorname{Hom}^\sharp_{\mathcal{K}(\Omega,S_k)}(\mathcal{Q}(\lambda), \mathcal{Q}(\mu)) = \sum_{\nu \in \Omega} \sum_{\tau \in Y} Q_{\nu,\widehat{\lambda}+\tau} Q_{\nu,\widehat{\mu}}\, t^{2\delta(\nu)-\delta(\widehat{\lambda}+\tau)-\delta(\widehat{\mu})}. \tag{$L6$}$$

The Q–polynomials satisfy (by [Lu2], 7.4)

$$Q_{\lambda+\tau,\mu+\tau} = Q_{\lambda,\mu} \tag{7}$$

for all $\lambda, \mu \in \Omega$ and $\tau \in Y$. There are $\tau_1, \tau_2 \in Y$ with $\widehat{\lambda} = w_0 \bullet \lambda + \tau_1$ and $\widehat{\mu} = w_0 \bullet \mu + \tau_2$. If we now substitute $\nu + \tau_2$ for ν and $\tau - \tau_1 + \tau_2$ for τ, then we can rewrite the right hand side of ($L6$) using (7) as

$$\sum_{\nu \in \Omega} \sum_{\tau \in Y} Q_{\nu, w_0\bullet\lambda+\tau} Q_{\nu, w_0\bullet\mu}\, t^{2\delta(\nu)-\delta(w_0\bullet\lambda+\tau)-\delta(w_0\bullet\mu)}. \tag{$L8$}$$

18.17. Let $\Omega_1 \subset \Omega$ be a system of representatives for the orbits of Y in Ω acting by translation. Then $P = \bigoplus_{\lambda\in\Omega_1} Q_k(\lambda)$ is a projective Y–generator of $\mathcal{C}_k(\Omega)$. From 18.9 we deduce an isomorphism of Y–graded rings

$$\big((\mathrm{End}^\sharp_{\widetilde{\mathcal{C}}}(\bigoplus_{\lambda\in\Omega_1} \widetilde{Q}_{S_k}(\lambda))) \otimes_{S_k} k\big)^{opp} \xrightarrow{\sim} (\mathrm{End}^\sharp P)^{opp}. \tag{1}$$

Let us call the left hand side A; this is a $(Y\times \mathbf{Z})$–graded k–algebra. If Lusztig's conjecture holds, then A is positively graded and its degree zero part is just a product of $|\Omega_1|$ copies of k, with a basis of pairwise orthogonal idempotents $1_\lambda \in A_0$ given as the images of the projections onto $\widetilde{Q}_{S_k}(\lambda)$. This follows from 18.16(2) and the discussion on 17.9. More precisely, we can calculate the Poincaré polynomial $P(A,t)$ of our ring, an $(\Omega_1 \times \Omega_1)$-matrix with entries in $\mathbf{Z}[[t]]$ given as

$$P(A,t)_{\lambda,\mu} = \big(\mathrm{rk}\,\mathrm{Hom}^\sharp_{\widetilde{\mathcal{C}}}(\widetilde{Q}_{S_k}(\lambda), \widetilde{Q}_{S_k}(\mu))\big)(t^{-1}). \tag{2}$$

If Lusztig's conjecture holds, then 18.16($L8$) says

$$P(A,t)_{\lambda,\mu} = \sum_{\nu\in\Omega}\sum_{\tau\in Y} Q_{\nu,w_0\bullet\lambda+\tau}(t^{-1})Q_{\nu,w_0\bullet\mu}(t^{-1})\, t^{\delta(w_0\bullet\lambda+\tau)+\delta(w_0\bullet\mu)-2\delta(\nu)}. \tag{$L3$}$$

Note that indeed $P(A,t)_{\lambda,\mu} \in \mathbf{Z}[t]$.

Proposition: *Suppose that $p > h$. If Lusztig's conjecture holds for k, then A is a Koszul algebra.*

Proof: We want to deduce this from the numerical Koszulity criterion explained in [BGS], 2.11. So we have to know the Poincaré polynomial of $E(A) = \mathrm{Ext}^\bullet_A(A_0, A_0)$. By definition of A the category of finitely generated Y-graded A–modules is equivalent to $\mathcal{C}_k(\Omega)$; under this equivalence $A_0 1_\lambda$ corresponds to $L_k(\lambda)$, for each $\lambda \in \Omega_1$. Hence A_0 corresponds under our equivalence to $L = \bigoplus_{\lambda\in\Omega_1} L_k(\lambda)$ and $E(A) \simeq \bigoplus_{\tau\in Y} \mathrm{Ext}^\bullet_{\mathcal{C}}(L[\tau], L)$ has the Poincaré polynomial $P(E(A), t)$ with entries

$$P(E(A),t)_{\lambda,\mu} = \sum_{\tau\in Y}\sum_{i\in\mathbf{Z}} t^i \dim \mathrm{Ext}^i_{\mathcal{C}}(L_k(\mu), L_k(\lambda+\tau)).$$

The dimensions of Ext–groups between irreducible objects have been deduced from Lusztig's conjecture in [CPS] in the case Ω is regular and $p > h$ in Case 1. Their results extend to Case 2: One simply has to replace the reference to Friedlander and Parshall in [CPS], 3.12.2.3 by a reference to the main result in [GK]. We have by [CPS], 5.8.1

$$P(E(A),t)_{\lambda,\mu} = \sum_{\tau\in Y}\sum_{\nu\in\Omega} t^{\delta(\mu)+\delta(\lambda+\tau)-2\delta(\nu)}\widehat{P}_{\nu,\mu}(t^{-1})\widehat{P}_{\nu,\lambda+\tau}(t^{-1}),$$

where the $\widehat{P}$ are the generic Kazhdan-Lusztig polynomials as in [Kat], 3.3. The $\widehat{P}$ are polynomials in t^2, so $P(E(A),-t)_{\lambda,\mu}$ is equal to

$$\sum_{\tau\in Y}\sum_{\nu\in\Omega}(-1)^{\delta(\lambda+\tau)-\delta(\mu)}t^{\delta(\mu)+\delta(\lambda+\tau)-2\delta(\nu)}\widehat{P}_{\nu,\mu}(t^{-1})\widehat{P}_{\nu,\lambda+\tau}(t^{-1}).$$

So the (λ,μ)-entry of $P(A,t)P(E(A),-t)$ is equal to

$$\sum_{\eta\in\Omega_1}\sum_{\nu,\nu'\in\Omega}\sum_{\tau,\tau'\in Y}(-1)^{\delta(\eta+\tau')-\delta(\mu)}\big(Q_{\nu,w_0\bullet\lambda+\tau}Q_{\nu,w_0\bullet\eta}\widehat{P}_{\nu',\mu}\widehat{P}_{\nu',\eta+\tau'}\big)(t^{-1})\cdot$$
$$\cdot\, t^{\delta(w_0\bullet\lambda+\tau)+\delta(w_0\bullet\eta)-2\delta(\nu)+\delta(\mu)+\delta(\eta+\tau')-2\delta(\nu')}.$$

Substitute $\nu-w_0\tau'$ for ν and $\tau-w_0\tau'$ for τ and apply 18.16(7) to the Q-factors. The sum above turns into

$$\sum_{\eta\in\Omega_1}\sum_{\nu,\nu'\in\Omega}\sum_{\tau,\tau'\in Y}(-1)^{\delta(\eta+\tau')-\delta(\mu)}\big(Q_{\nu,w_0\bullet\lambda+\tau}Q_{\nu,w_0\bullet(\eta+\tau')}\widehat{P}_{\nu',\mu}\widehat{P}_{\nu',\eta+\tau'}\big)(t^{-1})\cdot$$
$$\cdot\, t^{\delta(w_0\bullet\lambda+\tau)+\delta(w_0\bullet(\eta+\tau'))-2\delta(\nu)+\delta(\mu)+\delta(\eta+\tau')-2\delta(\nu')}.$$

(One uses 18.5(1) to rewrite the exponent of t.) If η runs over Ω_1 and τ' over Y, then $\eta+\tau'$ runs over Ω. So we can rewrite the sum as

$$\sum_{\omega\in\Omega}\sum_{\nu,\nu'\in\Omega}\sum_{\tau\in Y}(-1)^{\delta(\omega)-\delta(\mu)}\big(Q_{\nu,w_0\bullet\lambda+\tau}Q_{\nu,w_0\bullet\omega}\widehat{P}_{\nu',\mu}\widehat{P}_{\nu',\omega}\big)(t^{-1})\cdot$$
$$\cdot\, t^{\delta(w_0\bullet\lambda+\tau)+\delta(w_0\bullet\omega)-2\delta(\nu)+\delta(\mu)+\delta(\omega)-2\delta(\nu')}.$$

We have

$$\delta(w_0\bullet\omega)+\delta(\omega)=|R^+|\qquad\text{for all }\omega\in\Omega,\tag{4}$$

e.g., by [Lu2], 1.4.3. On the other hand, [Lu2], 11.10 implies (cf. [Kan], 2.9) that

$$\sum_{\omega\in\Omega}(-1)^{\delta(\omega)-\delta(\nu')}Q_{\nu,\omega}\widehat{P}_{\nu',w_0\bullet\omega}=\delta_{\nu',w_0\bullet\nu}\tag{5}$$

for all $\nu,\nu'\in\Omega$. So the sum simplifies to

$$\sum_{\nu\in\Omega}\sum_{\tau\in Y}(-1)^{\delta(w_0\bullet\nu)-\delta(\mu)}\big(Q_{\nu,w_0\bullet\lambda+\tau}\widehat{P}_{w_0\bullet\nu,\mu}\big)(t^{-1})t^{\delta(w_0\bullet\lambda+\tau)-|R^+|+\delta(\mu)},$$

where we have used in the exponent of t that $-2\delta(\nu)-2\delta(\nu')=-2\delta(\nu)-2\delta(w_0\bullet\nu)=-|R^+|$ by (4). Now apply the counterpart of (5), i.e., the fact that the Q-matrix is also the right inverse to the $(\pm\widehat{P})$-matrix. We get

$$\sum_{\tau\in Y}\delta_{w_0\bullet\lambda+\tau,w_0\bullet\mu}=\delta_{\lambda,\mu}.$$

(For the last step use that λ and μ are from a system of representatives modulo Y.) These results tell us that $P(E(A),-t)$ is inverse to $P(A,t)$. Hence A is Koszul by the numerical Koszulity criterion.

18.18. Recall that Q is a projective Y–generator of $\mathcal{C}(\Gamma, \widehat{S}_k)$. This implies that $Q_k = Q \otimes_{\widehat{S}_k} k$ is a projective Y–generator of $\mathcal{C}_k(\Gamma)$. We have $E_\Gamma \otimes_{S_k} \widehat{S}_k \simeq (\mathrm{End}^\sharp_{\mathcal{C}} P)^{opp}$, hence $E_{\Gamma,k} := E_\Gamma \otimes_{S_k} k \simeq (\mathrm{End}^\sharp_{\mathcal{C}} P_k)^{opp}$. The category of finitely generated Y–graded $E_{\Gamma,k}$–modules is equivalent to $\mathcal{C}_k(\Gamma)$ under $M \mapsto P_k \otimes_{E_{\Gamma,k}} M \simeq P \otimes_{E_\Gamma} M$.

The algebra $E_{\Gamma,k}$ is $(Y \times \mathbf{Z})$–graded; denote by $\widetilde{\mathcal{C}}_k(\Gamma)$ the "graded representation category" of all finitely generated $(Y \times \mathbf{Z})$–graded $E_{\Gamma,k}$–modules. (We can identify $\widetilde{\mathcal{C}}_k(\Gamma)$ with a full subcategory of $\widetilde{\mathcal{C}}(\Gamma, S_k)$: Take all M with $h_\alpha M = 0$ for all α.) We have a functor $\overline{v} : \widetilde{\mathcal{C}}_k(\Gamma) \to \mathcal{C}_k(\Gamma)$ where we first forget the $\mathbf{Z}$–grading and then apply the equivalence of categories from above. If we embed $\widetilde{\mathcal{C}}_k(\Gamma)$ into $\widetilde{\mathcal{C}}(\Gamma, S_k)$ as above and similarly embed $\mathcal{C}_k(\Gamma)$ into $\mathcal{C}(\Gamma, \widehat{S}_k)$ (cf. 18.11), then $\overline{v}$ is just the restriction of the functor v from 18.7.

Objects in $\widetilde{\mathcal{C}}_k(\Gamma)$ are for example the $\widetilde{L}_k(\lambda)$, the $\widetilde{Z}_k(\lambda)$, the $\widetilde{Z}'_k(\lambda) = \widetilde{Z}'_{S_k}(\lambda) \otimes_{S_k} k$ and also the $\widetilde{Q}_k(\lambda) := \widetilde{Q}_{S_k}(\lambda) \otimes_{S_k} k$ (with $\lambda \in \Gamma$). We have $\overline{v}\widetilde{L}_k(\lambda) \simeq L_k(\lambda)$, $\overline{v}\widetilde{Z}_k(\lambda) \simeq Z_k(\lambda)$, $\overline{v}\widetilde{Z}'_k(\lambda) \simeq Z'_k(\lambda)$, $\overline{v}\widetilde{Q}_k(\lambda) \simeq Q_k(\lambda)$ for all $\lambda \in \Gamma$. (Apply Lemma 18.11.) Since $\widetilde{Q}_k(\lambda)$ is projective in $\widetilde{\mathcal{C}}_k(\Gamma)$, the last isomorphism implies that $\widetilde{Q}_k(\lambda)$ is the projective cover of $\widetilde{L}_k(\lambda)$.

The discussion of the functor $\overline{v}$ above shows for any M and N in $\widetilde{\mathcal{C}}_k(\Gamma)$ and any i that

$$\mathrm{Ext}^i_{\mathcal{C}_k}(\overline{v}M, \overline{v}N) \simeq \bigoplus_{r \in \mathbf{Z}} \mathrm{Ext}^i_{\widetilde{\mathcal{C}}_k(\Gamma)}(M\langle r\rangle, N). \tag{1}$$

We can use this to show for all $\lambda, \mu \in \Gamma$ that

$$\mathrm{Ext}^i_{\widetilde{\mathcal{C}}_k(\Gamma)}(\widetilde{Z}_k(\lambda)\langle r\rangle, \widetilde{Z}'_k(\mu)) = \begin{cases} k, & \text{if } i = r = 0,\ \lambda = \mu; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Indeed, (1) and 4.12(1), (2) yield the vanishing for all $i > 0$ and for $i = 0$ when $\lambda \neq \mu$. They also show for $i = 0$ and $\lambda = \mu$ that there is exactly one r where the Hom space is equal to k, whereas it is equal to 0 for all other r. This unique r is readily determined from Theorem 18.8(a). We get similarly

$$\mathrm{Ext}^i_{\widetilde{\mathcal{C}}_k(\Gamma)}(\widetilde{Z}'_k(\lambda)\langle r\rangle, \widetilde{Z}_k(\mu)) = \begin{cases} k, & \text{if } i = 0,\ r = 2(|R^+| - N_\Gamma),\ \lambda = \mu; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

18.19. The first parts of the following result motivated the shifts appearing in the definitions of the graded deformation category and of its objects.

Proposition: *Suppose that $p > h$ and that Lusztig's conjecture holds for k. Let Ω be a regular W_p–orbit in X.*

a) For any $\lambda, \mu \in \Omega$ we have $\mathrm{Ext}^i_{\widetilde{\mathcal{C}}_k(\Omega)}(\widetilde{L}_k(\lambda), \widetilde{L}_k(\mu)\langle n\rangle) = 0$ unless $i = n$.

b) For any $\lambda, \mu \in \Omega$ *we have*

$$\operatorname{Ext}^i_{\widetilde{\mathcal{C}}_k(\Omega)}(\widetilde{Z}_k(\lambda), \widetilde{L}_k(\mu)\langle n\rangle) = 0 = \operatorname{Ext}^i_{\widetilde{\mathcal{C}}_k(\Omega)}(\widetilde{L}_k(\mu), \widetilde{Z}'_k(\lambda)\langle n\rangle)$$

unless $i = n$.

c) Each $\widetilde{Q}_k(\lambda)$ *with* $\lambda \in \Omega$ *admits a* $\widetilde{Z}_k$*-filtration and a* $\widetilde{Z}'_k$*-filtration; we have*

$$\begin{aligned}\sum_r [\widetilde{Q}_k(\lambda) : \widetilde{Z}_k(\mu)\langle r\rangle] t^r &= t^{\delta(\mu)-\delta(\widehat{\lambda})+|R^+|} Q_{\mu,\widehat{\lambda}} \\ &= \sum_r [\widetilde{Q}_k(\lambda) : \widetilde{Z}'_k(\mu)\langle -r\rangle] t^{r+2|R^+|}\end{aligned} \tag{L1}$$

and

$$[\widetilde{Z}_k(\mu) : \widetilde{L}_k(\lambda)\langle r\rangle] = [\widetilde{Q}_k(\lambda)\langle r\rangle : \widetilde{Z}'_k(\mu)\langle 2|R^+|\rangle] = [\widetilde{Q}_k(\lambda) : \widetilde{Z}_k(\mu)\langle r\rangle]. \tag{L2}$$

Proof: a) This follows from general properties of Koszul rings applied to A as in 18.17, cf. [BGS], Proposition 2.1.3.

b) We can apply [CPS], Theorem 3.8, to the category $\widetilde{\mathcal{C}}_k(\Omega)$ with weight poset $\Lambda = \Omega \times \mathbf{Z}$ with the order given by $(\omega, n) < (\omega', m)$ if and only if $\omega < \omega'$, the length $l : \Lambda \to \mathbf{Z}$ given as $l(\omega, n) = \delta(\omega)$, and with $X = \widetilde{L}_k(\lambda)$, $Y = \widetilde{L}_k(\mu)\langle n\rangle$. We obtain in this way

$$\begin{aligned}&\dim \operatorname{Ext}^i_{\widetilde{\mathcal{C}}_k(\Omega)}(\widetilde{L}_k(\lambda), \widetilde{L}_k(\mu)\langle n\rangle) = \\ &\sum_{\nu\in\Omega}\sum_{r\in\mathbf{Z}}\sum_{j\in\mathbf{Z}} \dim \operatorname{Ext}^{i-r}_{\widetilde{\mathcal{C}}_k(\Omega)}(\widetilde{L}_k(\lambda), \widetilde{Z}'_k(\nu)\langle j\rangle) \dim \operatorname{Ext}^r_{\widetilde{\mathcal{C}}_k(\Omega)}(\widetilde{Z}_k(\nu)\langle j\rangle, \widetilde{L}_k(\mu)\langle n\rangle).\end{aligned}$$

From a) we deduce that for $i \neq n$ each summand vanishes, in particular the one for $\nu = \lambda$, $r = i$, $j = 0$ and the one for $\nu = \mu$, $r = 0$ and $j = n$. This establishes b).

c) We start with $(L1)$. As pointed out in 18.16, the permissible $\mathcal{Z}$–filtration of $\mathcal{Q}(\lambda)$ yields a filtration of $\widetilde{Q}_{S_k}(\lambda)$ where the factors have the form $\widetilde{Z}_{S_k}(\mu)\langle n\rangle$. More precisely, each factor $\mathcal{Z}_\mu\langle n\rangle$ leads to a factor $\widetilde{Z}_{S_k}(\mu)\langle n + \delta(\mu)\rangle$, cf. 18.10(1). Tensoring over S_k with k we get a filtration of $\widetilde{Q}_k(\lambda)$ in $\widetilde{\mathcal{C}}_k(\Omega)$, since the $\widetilde{Z}_{S_k}(\mu)$ are free over S_k. Now 17.9($L2$) implies

$$[\widetilde{Q}_k(\lambda) : \widetilde{Z}_k(\mu)\langle r\rangle] = D^{r-\delta(\mu)-|R^+|}_{\mu,\widehat{\lambda}},$$

hence

$$\sum_{r\in\mathbf{Z}} [\widetilde{Q}_k(\lambda) : \widetilde{Z}_k(\mu)\langle r\rangle] t^{r-\delta(\mu)} = t^{|R^+|} D_{\mu,\widehat{\lambda}} = t^{|R^+|-\delta(\widehat{\lambda})} Q_{\mu,\widehat{\lambda}},$$

where we use 18.16(5) for the last step. This yields the first equality. The second one follows similarly from 17.9($L3$).

We now prove ($L2$). The second equation is clear from ($L1$), so we concentrate on the first. Certainly

$$[\widetilde{Z}_k(\mu) : \widetilde{L}_k(\lambda)\langle r\rangle] = \dim \mathrm{Hom}_{\widetilde{\mathcal{C}}_k(\Omega)}(\widetilde{Q}_k(\lambda)\langle r\rangle, \widetilde{Z}_k(\mu)),$$

and this can be written as $[\widetilde{Q}_k(\lambda)\langle r\rangle : \widetilde{Z}'_k(\mu)\langle 2|R^+|\rangle]$ using 18.18(3).

Remark: We expect a) and b) to hold also for $p \geq h$ and for an arbitrary W_p–orbit Ω.

18.20. Suppose for the moment that we are in Case 1. The restricted enveloping algebra $U^{[p]}(\mathfrak{g}_k)$ is $(\mathbf{Z}R)$–graded in a natural way. So we can regard it also as an X–graded algebra and study the category of all X–graded $U^{[p]}(\mathfrak{g}_k)$–modules.

Lemma: *We have a canonical decomposition of the category of all finitely generated X-graded $U^{[p]}(\mathfrak{g}_k)$-modules into $\bigoplus_{\mu\in X/pX}\mathcal{C}_{k[\mu]}$.*

Proof: Any $\lambda \in X/pX \subset \mathfrak{h}_k^*$ determines an algebra homomorphism $\lambda : U^{[p]}(\mathfrak{h}_k) \to k$; together they give an isomorphism

$$U^{[p]}(\mathfrak{h}_k) \xrightarrow{\sim} \prod_{X/pX} k. \tag{1}$$

Consider an X–graded $U^{[p]}(\mathfrak{g}_k)$–module $M = \bigoplus_{\nu\in X} M_\nu$. Since $U^{[p]}(\mathfrak{h}_k)$ is contained in the degree zero part of $U^{[p]}(\mathfrak{g}_k)$, it stabilizes each M_ν and we get a decompososition $M_\nu = \bigoplus_{\lambda\in X/pX} M_\nu^\lambda$ where

$$M_\nu^\lambda = \{m \in M_\nu \mid um = \lambda(u)m \qquad \text{for all } u \in U^{[p]}(\mathfrak{h}_k)\}.$$

It is immediate that for any $\mu \in X/pX$ the subspace

$$M(\mu) = \bigoplus_{\nu\in\lambda+\mu} M_\nu^\lambda \subset M$$

is stable under $U^{[p]}(\mathfrak{g}_k)$ and lies in fact in $\mathcal{C}_{k[\mu]}$. Clearly $M = \bigoplus_{\mu\in X/pX} M(\mu)$ and clearly this decomposition is respected by all morphisms of X–graded $U^{[p]}(\mathfrak{g}_k)$–modules.

Remark: The discussion above generalizes to Case 2, if we replace $U^{[p]}(\mathfrak{g}_k)$ by $\mathbf{u}$ modulo the ideal generated by the (central) elements $K_\beta^p - 1$.

18.21. Suppose again that we are in Case 1. The restricted enveloping algebra $U^{[p]}(\mathfrak{g}_k)$ is finite dimensional. It decomposes uniquely into a direct product of algebras that cannot be decomposed further:

$$U^{[p]}(\mathfrak{g}_k) = \prod_{i=1}^{r} \mathcal{B}_i.$$

The factors $\mathcal{B}_i$ are called the blocks of $U^{[p]}(\mathfrak{g}_k)$. Each block is the annihilator of the product of the other blocks, hence also the annihilator of a suitable direct sum of projective indecomposable modules for $U^{[p]}(\mathfrak{g}_k)$. Since these modules lift to objects in $\mathcal{C}_k$, cf. the proof of 16.17, we can deduce that each block is the direct sum of its weight spaces.

Each $U^{[p]}(\mathfrak{g}_k)$–module M is the direct sum of all $\mathcal{B}_i M$; we say that M belongs to $\mathcal{B}_j$ if $M = \mathcal{B}_j M$. Each indecomposable M belongs to some $\mathcal{B}_i$. In particular, each $L_k(\lambda)$ with $\lambda \in X$ belongs to some block; another $L_k(\mu)$ belongs to the same block if and only if $\mu \in W_\bullet\lambda + pX$. So there is for each W_p–orbit Γ a unique block such that all $L_k(\mu)$ with $\mu \in \Gamma$ belong to this block; denote it by $\mathcal{B}(\Gamma)$. For M in $\mathcal{C}_k(\Gamma)$ all $\mathcal{B}_i \neq \mathcal{B}(\Gamma)$ annihilate M, since they annihilate all composition factors. So we have $M = \mathcal{B}(\Gamma)M$; more precisely, the identity in $\mathcal{B}(\Gamma)$ acts as the identity on M.

If Γ and Γ' are two W_p–orbits, then we have $\mathcal{B}(\Gamma) = \mathcal{B}(\Gamma')$ if and only if there is $\mu \in X$ with $\Gamma' = \Gamma + p\mu$. Under a decomposition as in Lemma 18.20, the category of finitely generated X–graded $\mathcal{B}(\Gamma)$–modules corresponds to the product of all $\mathcal{C}_{k[\nu]}(\Gamma')$ with ν running over representatives for X/pX and Γ' running over all orbits of the form $\Gamma + p\mu$ with $\mu \in X$. Then $M \mapsto M[p\mu][\nu]$ is an equivalence of categories between $\mathcal{C}_k(\Gamma)$ and $\mathcal{C}_{k[\nu]}(\Gamma + p\mu)$. If P is a projective Y–generator P of $\mathcal{C}_k(\Gamma)$, then P is also a projective X–generator of the category of finitely generated X–graded $\mathcal{B}(\Gamma)$–modules. So $\mathcal{B}(\Gamma)$ is X–Morita equivalent (in the sense of F.6) to $(\mathrm{End}^\sharp_{\mathcal{C}} P)^{opp}$. (This endomorphism ring is in fact Y–graded, but we can also consider it as an X–graded ring.)

Proposition: *Suppose that $p > h$ and that we are in Case 1. Let Ω be a regular W_p–orbit in X. If Lusztig's conjecture holds for k, then $\mathcal{B}(\Omega)$ admits a Koszul grading compatible with its X–grading.*

Proof: This is now an immediate consequence of Proposition 18.17 and Lemma F.7.

Remarks: 1) If a) and b) in Proposition 18.19 extend to all W_p–orbits, then one could show that the restricted enveloping algebra $U^{[p]}(\mathfrak{g}_k)$ admits a T_k–stable Koszul grading. This would allow to determine the dimensions of the Ext groups $\mathrm{Ext}^i_{\mathcal{C}_k(\Gamma)}(L_k(\lambda), L_k(\mu))$ and $\mathrm{Ext}^i_{\mathcal{C}_k(\Gamma)}(Z_k(\lambda), L_k(\mu))$ following for example the line of reasoning given in the proof of [BGS], Theorem 3.11.4.

2) Everything carries over to Case 2; we simply have to replace $U^{[p]}(\mathfrak{g}_k)$ everywhere by the quotient of $\mathbf{u}$ mentioned in 18.20.

19. Examples of Endomorphism Algebras

We return to the situation of Section 14 with the same restriction on p. Assume that R is indecomposable. We choose $A = A(k)$ and the h_β as in 14.1. So we can regard A as an S–algebra with S as in 14.4. (In Case 1 we could also work with $A = B(k)$. However, then A would not be local and that would make several proofs more complicated.)

19.1. Let Ω be a W_p–orbit. We want to use our theory to determine explicitly $\mathrm{End}_{\mathcal{C}} Q$ for a projective module Q in $\mathcal{D}_A(\Omega)$ in a very special situation. We shall assume that

$$Q^\emptyset \simeq \bigoplus_{i \in I} Z^\emptyset(\mu_i) \tag{1}$$

for some finite index set I *where the* $\mu_i \in \Omega$ *are distinct.* Furthermore we assume for all $\beta \in R^+$ that there is a subset $I(\beta)$ of I such that

$$Q^\beta \simeq \bigoplus_{i \in I(\beta)} Q^\beta(\mu_i) \tag{2}$$

where $Q^\beta(\mu) = Z^\beta(\mu)$ if $\beta \uparrow \mu = \mu$, and where $Q^\beta(\mu)$ is the middle term in a representative of $e^\beta(\mu)$ if $\beta \uparrow \mu \neq \mu$. (This is the same convention as in the proof of 9.4. Note that the isomorphism class of $Q^\beta(\mu)$ does not depend on the choice of $e^\beta(\mu)$.) The existence of a decomposition as in (2) would follow from Theorem 4.19.b, if A^β were a local ring. Since the characters of distinct $Q^\beta(\mu)$ are linearly independent, the set $I(\beta)$ is determined by the character of Q, i.e., by the family $(\mu_i)_{i \in I}$.

19.2. Since A is local, each $\mathrm{Hom}_{\mathcal{C}}(Z_A(\lambda), Q)$ is free over A, cf. Lemma 4.14. Choose for all $i \in I$ a homomorphism $f_i : Z_A(\mu_i) \to Q$ such that

$$\mathrm{Hom}_{\mathcal{C}}(Z_A(\mu_i), Q) = A f_i, \tag{1}$$

i.e., f_i is a basis of this A–module. So

$$\mathcal{V}Q(\mu) = \begin{cases} A^\emptyset f_i, & \text{if } \mu = \mu_i; \\ 0, & \text{if } \mu \notin \{\mu_i \mid i \in I\}. \end{cases} \tag{2}$$

Fix a positive root β. Let $\mu \in X$. If $\beta \uparrow \mu = \mu$, then the definition 9.3(2) implies that

$$\mathcal{V}Q(\mu,\beta) = \begin{cases} A^\beta f_i, & \text{if } \mu = \mu_i; \\ 0, & \text{if } \mu \notin \{\mu_i \mid i \in I\}. \end{cases} \tag{3}$$

If $\beta \uparrow \mu \neq \mu$, then choose an exact sequence

$$0 \to Z^\beta(\beta\uparrow\mu) \xrightarrow{f} Q^\beta(\mu) \xrightarrow{g} Z^\beta(\mu) \to 0 \tag{4}$$

representing $e^\beta(\mu)$. If neither μ nor $\beta \uparrow \mu$ belongs to the set of the μ_j, then obviously $\mathcal{V}Q(\mu,\beta) = 0$. If $\mu = \mu_i$ for some $i \in I$ and if $\beta\uparrow\mu$ is not in the set of the μ_j, then $h \mapsto h \circ g$ is an isomorphism

$$\mathrm{Hom}_\mathcal{C}(Z^\beta(\mu), Q^\beta) \xrightarrow{\sim} \mathrm{Hom}_\mathcal{C}(Q^\beta(\mu), Q^\beta).$$

This yields

$$\mathcal{V}Q(\mu,\beta) = A^\beta(f_i, 0). \tag{5}$$

If $\beta\uparrow\mu = \mu_i$ for some $i \in I$ and if μ is not in the set of the μ_j, then $h \mapsto h \circ f$ is an isomorphism

$$\mathrm{Hom}_\mathcal{C}(Q^\beta(\mu), Q^\beta) \xrightarrow{\sim} \mathrm{Hom}_\mathcal{C}(Z^\beta(\beta\uparrow\mu), Q^\beta).$$

(Here we use Lemma 4.14.) This yields

$$\mathcal{V}Q(\mu,\beta) = A^\beta(0, f_i). \tag{6}$$

Suppose now that $\mu = \mu_i$ and $\beta \uparrow \mu = \mu_j$ for some $i, j \in I$, $i \neq j$. There are two possible cases: Either $Q^\beta(\mu)$ is isomorphic to a direct summand of Q^β, i.e., $i \in I(\beta)$, or it isn't. Consider first the case where $i \in I(\beta)$. Identify $Q^\beta(\mu)$ with a direct summand of Q^β. Under such an identification f is a basis of $\mathrm{Hom}_\mathcal{C}(Z^\beta(\mu_j), Q^\beta)$, hence differs from f_j by a unit in B^β. We can now multiply our identification by that unit and shall assume that $f = f_j$. On the other side, under the identification we can regard g as a basis of $\mathrm{Hom}_\mathcal{C}(Q^\beta, Z^\beta(\mu_i))$, so there is (by Lemma 8.10.b) a unit d_i in A^β with $h_\beta \mathrm{id} = d_i\, g \circ f_i$. Working with the basis $(f_i \circ g, \mathrm{id})$ of $\mathrm{End}_\mathcal{C} Q^\beta(\mu)$ we get

$$\mathcal{V}Q(\mu,\beta) = A^\beta(f_i, 0) \oplus A^\beta(d_i f_i h_\beta^{-1}, f_j). \tag{7}$$

Consider now the case where $i \notin I(\beta)$. Then Q^β has a direct summand of the form $Q^\beta(\beta\uparrow\mu) \oplus Q^\beta(\mu')$ where μ' is the unique weight with $\beta\uparrow\mu' = \mu$. Choose a representative

$$0 \to Z^\beta(\mu) \xrightarrow{f'} Q^\beta(\mu') \xrightarrow{g'} Z^\beta(\mu') \to 0$$

of $e^\beta(\mu')$. We have isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(Q^\beta(\mu), Q^\beta) \simeq \mathrm{Hom}_{\mathcal{C}}(Q^\beta(\mu), Q^\beta(\beta\uparrow\mu)) \oplus \mathrm{Hom}_{\mathcal{C}}(Q^\beta(\mu), Q^\beta(\mu')),$$

and

$$\mathrm{Hom}_{\mathcal{C}}(Q^\beta(\mu), Q^\beta(\beta\uparrow\mu)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z^\beta(\mu_j), Q^\beta(\beta\uparrow\mu))$$

given by $h \mapsto h \circ f$, and

$$\mathrm{Hom}_{\mathcal{C}}(Q^\beta(\mu), Z^\beta(\mu_i)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Q^\beta(\mu), Q^\beta(\mu'))$$

given by $h \mapsto f' \circ h$. So $\mathrm{Hom}_{\mathcal{C}}(Q^\beta(\mu), Q^\beta)$ has a basis (h_1, h_2) such that h_2 factors through $Q^\beta(\beta \uparrow \mu)$ satisfying $h_2 \circ f = f_j$, and such that h_1 factors through $Q^\beta(\mu')$ satisfying $h_1 = f_i \circ g$. Composing with f and g^{-1} we get $(0, f_j)$ and $(f_i, 0)$, i.e.,

$$\mathcal{V}Q(\mu,\beta) = A^\beta(f_i, 0) \oplus A^\beta(0, f_j). \tag{8}$$

19.3. Keep the assumptions and notations from the last two subsections.

Proposition: *The algebra* $\mathrm{End}_{\mathcal{C}}Q$ *is isomorphic to the subalgebra of* A^I *consisting of those families* $(a_i)_{i\in I}$ *that satisfy the following condition for each* $\beta \in R^+$*: If* Q^β *has a direct summand that is a nonsplit extension of* $Z^\beta(\mu_i)$ *by* $Z^\beta(\mu_j)$*, then* $a_i \equiv a_j \pmod{h_\beta}$.

Proof: An endomorphism φ of Q is given over $A^\emptyset$ by a family $(a_i)_{i\in I}$ of elements in $A^\emptyset$ such that $\varphi \circ f_i = a_i f_i$ for all i. In order to map Q into itself all a_i have to be in A. Furthermore all $\mathcal{V}Q(\mu,\beta)$ have to be stabilized. This is automatical in all cases except for those of type (7) in 19.2. There we need

$$(d_i a_i f_i h_\beta^{-1}, a_j f_j) = a_j(d_i f_i h_\beta^{-1}, f_j) + (a_i - a_j) d_i h_\beta^{-1}(f_i, 0) \in \mathcal{V}Q(\mu,\beta),$$

i.e.,

$$a_i - a_j \in A \cap A^\beta h_\beta = A h_\beta.$$

The claim follows.

19.4. The example considered in 10.14 is obviously a special case of our discussion above. We shall generalize it somewhat. Let $\mu \in X$ be in the closure of the first dominant alcove, i.e., satisfying

$$0 \le \langle \mu + \rho, \alpha^\vee \rangle \le p \qquad \text{for all } \alpha \in R^+.$$

We want to apply 19.1–3 to $\Omega = W_p{\bullet}\mu$ and $Q = TZ_A(-\rho)$, where T is translation from $W_p{\bullet}(-\rho)$ to Ω. If we regard k as an A–module via augmentation, then $Q \otimes_A k \simeq TZ_k(-\rho)$ is isomorphic to the projective indecomposable module $Q_k(w_0{\bullet}\mu)$, cf. [Ja6], II.11.10. So Theorem 4.19.b implies that $Q \simeq Q_A(w_0{\bullet}\mu)$.

We get from 7.5 that Q has a Z–filtration with factors $Z_A(w{\bullet}\mu)$ with $w \in W$ such that each $w{\bullet}\mu$ occurs exactly once. This implies that 19.1(1) is satisfied; we can take for I a set of representatives of W modulo the stabiliser of μ. Let β in R^+. We can decompose T as in 7.10 into a direct sum of translation functors for $W_{\beta,p} = \ < s_{\beta,rp} \mid r \in \mathbf{Z} >$. This leads to a decomposition of Q^β as in 19.1(2). So all assumptions in 19.1 are satisfied, and we can apply Proposition 19.3.

Theorem: *a) If $\langle \mu + \rho, \alpha_0^\vee \rangle < p$, then $\mathrm{End}_{\mathcal{C}} Q_A(w_0{\bullet}\mu)$ is isomorphic to the algebra of all functions $\xi : W \to A$ such that for all $w \in W$*

$$\xi(ws_\alpha) = \xi(w) \qquad \text{for all } \alpha \in R \text{ with } s_\alpha{\bullet}\mu = \mu \tag{1}$$

and

$$\xi(s_\beta w) \equiv \xi(w) \pmod{h_\beta} \qquad \text{for all } \beta \in R^+. \tag{2}$$

b) If $\langle \mu + \rho, \alpha_0^\vee \rangle = p$ and $0 < \langle \mu + \rho, \alpha^\vee \rangle < p$ for all $\alpha \in R^+$, $\alpha \neq \alpha_0$, then $\mathrm{End}_{\mathcal{C}} Q_A(w_0{\bullet}\mu)$ is isomorphic to the algebra of all functions $\xi : W \to A$ such that for all $\beta \in R^+$ and $w \in W$

$$\xi(s_\beta w) \equiv \xi(w) \pmod{h_\beta} \qquad \text{if} \quad \beta \neq \pm w(\alpha_0). \tag{3}$$

Proof: We can apply 19.1–3 with $I = W{\bullet}\mu$. We get thus an algebra of functions from $W{\bullet}\mu$ to A satisfying certain congruences. We can regard these functions as maps $\xi : W \to A$ with $\xi(w) = \xi(w')$ whenever $w{\bullet}\mu = w'{\bullet}\mu$. This last condition is equivalent to (1) since the stabilizer of μ is generated by reflections. In b) this condition is empty, since there the stabilizer of μ in W is trivial.

In order to see which congruence conditions occur we have to understand the decomposition 19.1(2) better. If $\langle \mu + \rho, w^{-1}\beta^\vee \rangle \in \{0, \pm p\}$, then $\beta \uparrow (w{\bullet}\mu) = w{\bullet}\mu$ and $Q^\beta(w{\bullet}\mu) = Z^\beta(w{\bullet}\mu)$ occurs in 19.1(2), but does not lead to a congruence condition. If $0 < \langle \mu + \rho, w^{-1}\beta^\vee \rangle < p$, then $Q^\beta(w{\bullet}\mu)$ is an extension of $Z^\beta(w{\bullet}\mu)$ by $Z^\beta(s_\beta w{\bullet}\mu)$. Then $Q^\beta(w{\bullet}\mu)$ occurs in Q^β, whereas $Q^\beta(s_\beta w{\bullet}\mu)$ does not occur.

This shows that $\xi(w)$ is congruent to $\xi(s_\beta w)$ unless $\langle \mu + \rho, w^{-1}\beta^\vee \rangle \in \{0, \pm p\}$. If we get here 0, then $s_\beta w{\bullet}\mu = w{\bullet}\mu$, hence $\xi(s_\beta w) = \xi(w)$ by (1). So we may as well add the corresponding congruence. (This case cannot occur in b).) If $\langle \mu + \rho, w^{-1}\beta^\vee \rangle = \pm p$, then we have to be in the situation of b) and we have to have $w^{-1}\beta = \pm\alpha_0$. This leads to the exceptions in (3).

Remarks: 1) Note that it is enough to take in (1) all simple roots α with $s_\alpha{\bullet}\mu = \mu$, since these already generate the stabilizer of μ.

2) Note that μ as in b) is a weight as in 13.10.

19.5. Let S be as in 15.2 and in Appendix D. Set

$$E_S = \{\xi : W \to S \mid \xi(s_\beta w) \equiv \xi(w) \pmod{h_\beta} \text{ for all } \beta \in R^+,\ w \in W\} \quad (1)$$

and

$$E_A = \{\xi : W \to A \mid \xi(s_\beta w) \equiv \xi(w) \pmod{h_\beta} \text{ for all } \beta \in R^+,\ w \in W\}. \quad (2)$$

It is shown in D.5 that E_S is a free module over S of rank $|W|$. More precisely, there is a basis $(\eta^w)_{w\in W}$ of E_S such that $\eta^w(w')$ is homogeneous of degree $l(w)$ for all $w' \in W$, such that

$$\eta^w(w') \neq 0 \implies w' \geq w, \qquad \text{and} \qquad \eta^w(w) = \prod_{\alpha\in R(w)} h_\alpha \quad (3)$$

where $R(w) = \{\alpha \in R^+ \mid w^{-1}\alpha < 0\}$ as in 5.11. The group W acts on E_S and E_A via

$$(w\xi)(w') = \xi(w'w) \qquad \text{for all } w, w' \in W. \quad (4)$$

If α is a simple root, then

$$w < ws_\alpha \implies s_\alpha \eta^w = \eta^w \quad (5)$$

by D.2(4).

Let I be a subset of the set of simple roots. Set W_I equal to the subgroup of W generated by the s_α with $\alpha \in I$. Set

$$W^I = \{w \in W \mid w < ws_\alpha \text{ for all } \alpha \in I\}.$$

This is a set of coset representatives for W/W_I. Set

$$E_S(I) = \{\xi \in E_S \mid \xi(w) = \xi(ww') \text{ for all } w \in W,\ w' \in W_I\} = (E_S)^{W_I} \quad (6)$$

and

$$E_A(I) = \{\xi \in E_A \mid \xi(w) = \xi(ww') \text{ for all } w \in W,\ w' \in W_I\} = (E_A)^{W_I}. \quad (7)$$

The η^w with $w \in W^I$ are in $E_S(I)$ by (5).

19.6. Recall that we can regard A as an S–algebra. We get thus a homomorphism $E_S \otimes_S A \to E_A$. We denote the image of $\eta^w \otimes 1$ in E_A again by η^w. The new η^w satisfy the analogue of 19.5(3) over A as well as the same homogeneity condition as the old η^w.

Lemma: *a) The η^w with $w \in W^I$ are a basis of $E_S(I)$ as an S–module.*

b) The η^w with $w \in W^I$ are a basis of $E_A(I)$ as an A–module.

c) The η^w with $w \in W$ are a basis of E_A as an A–module.

Proof: a) The η^w are linearly independent as part of a basis of E_S. So it is enough to show that they generate $E_S(I)$. Let $\xi \in E_S(I)$, $\xi \neq 0$. Let $w \in W$ minimal with $\xi(w) \neq 0$. If $\alpha \in I$, then $\xi(w) = \xi(ws_\alpha)$, hence $w < ws_\alpha$ by the minimality of w. Thus $w \in W^I$. We have for all $\beta \in R(w)$

$$\xi(w) \equiv \xi(s_\beta w) = 0 \pmod{h_\beta},$$

so $\xi(w)$ is divisible by $\prod_{\beta\in R(w)} h_\beta = \eta^w(w)$ in S. Let $a \in S$ with $\xi(w) = a\eta^w(w)$. Then $\xi' = \xi - a\eta^w \in E_S(I)$ with $\xi'(w) = 0$ and $\xi'(w') = \xi(w')$ for all $w' \not\geq w$. We can now apply induction to ξ'.

b) Since A is integral, the formula 19.5(2) implies immediately that the η^w are linearly independent over A. Now the proof of a) generalizes.

c) This is a special case of b).

Remark: We have to use the fact that the h_β are non-proportional primes in A to get the divisibility of $\xi(w)$ by $\eta^w(w)$. This is where we need that $p \neq 2$ if R has two root lengths, and that $p \neq 3$, if R is of type G_2, cf. 9.1.

19.7. Lemma 19.6 implies that the map $S \to A$ induces isomorphisms

$$E_S \otimes_A A \simeq E_A \qquad \text{and} \qquad E_S(I) \otimes_A A \simeq E_A(I). \tag{1}$$

If μ is a weight as in Theorem 19.4.a, then we have in our new notation

$$E_A(I) \simeq \mathrm{End}_{\mathcal{C}} Q_A(w_0 \bullet \mu) \qquad \text{where } I = \{\alpha \in \Sigma \mid \langle \mu + \rho, \alpha^\vee \rangle = 0\}. \tag{2}$$

We get especially for $\mu = \lambda_0$ with λ_0 as in 14.4 that

$$E_A \simeq \mathrm{End}_{\mathcal{C}}(Q). \tag{3}$$

19.8. We can generalize the definition of E_S and $E_S(I)$ to any S–algebra. We shall consider especially the S–algebra $S_k = S \otimes_{\mathbf{Z}} k$. It is easy to see that Lemma 19.6 extends to S_k. (Use the same arguments as in 19.6.)

Suppose in Case 1 that $p > h$. Proposition D.10 shows that the map

$$\Phi : S_k \otimes_{S_k^W} S_k \to E_{S_k} \tag{1}$$

with

$$\Phi(a \otimes b)(w) = w(a) \cdot b \qquad \text{for all } w \in W \tag{2}$$

is an isomorphism of k–algebras. It is W–equivariant, if we let W act on the left hand side via $w(a \otimes b) = w(a) \otimes b$. Since S_k is a free module over S_k^W (under our assumption on p in Case 1, cf. [Bo2], chap. V, §5, n° 5, Thm. 4) the map in (1) induces an isomorphism

$$S_k^{W'} \otimes_{S_k^W} S_k \xrightarrow{\sim} (E_{S_k})^{W'} \tag{3}$$

for all subgroups W' of W, especially for all subsets I of Σ:

$$S_k^{W_I} \otimes_{S_k^W} S_k \xrightarrow{\sim} E_{S_k}(I). \tag{4}$$

Tensoring over S_k with A yields isomorphisms

$$S_k \otimes_{S_k^W} A \xrightarrow{\sim} E_A \qquad \text{and} \qquad S_k^{W_I} \otimes_{S_k^W} A \xrightarrow{\sim} E_A(I). \tag{5}$$

The algebras on the right hand side are by 19.7(2) endomorphism algebras of certain modules $Q_A(w_0{\scriptstyle\bullet}\mu)$. If we tensor over A with k, then we get by 3.3 the endomorphism algebra of $Q_k(w_0{\scriptstyle\bullet}\mu)$. We get thus isomorphisms

$$S_k^{W_I} \otimes_{S_k^W} k \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}_k} Q_A(w_0{\scriptstyle\bullet}\mu) \tag{6}$$

with I as in 19.7(2). Here k is regarded as an algebra over S_k^W via the augmentation map $S_k \to k$. This shows:

Proposition: *Let μ be in the closure of the first dominant alcove with $\langle \mu + \rho, \alpha_0^\vee \rangle < p$. If μ is p-regular, then $\mathrm{End}_{\mathcal{C}_k} Q_k(w_0{\scriptstyle\bullet}\mu)$ is isomorphic to S_k modulo the ideal generated by all homogeneous W-invariants of positive degree, i.e., to the covariant algebra. In general, the endomorphism algebra is isomorphic to the ring of W_I-invariants in the covariant algebra (with I as in 19.7(2)).*

19.9. Set E'_S equal to the set of all maps $\xi : W \to S$ such that for all $\beta \in R^+$ and $w \in W$:

$$\xi(s_\beta w) \equiv \xi(w) \pmod{h_\beta} \qquad \text{if} \quad \beta \neq \pm w(\alpha_0). \tag{1}$$

Define similarly E'_A by replacing S by A. We have for all μ as in Theorem 19.4.b:

$$E'_A \simeq \mathrm{End}_{\mathcal{C}} Q_A(w_0{\scriptstyle\bullet}\mu). \tag{2}$$

Before we describe the algebras E'_S and E'_A more explicitly, we state an elementary property of the Chevalley order:

Observation: Let $w, w' \in W$ with $w' \geq w$. If $w\alpha_0 < 0$, then $w'\alpha_0 < 0$.

Proof: Since α_0 is dominant, $w' \geq w$ implies $w'\alpha_0 \leq w\alpha_0 < 0$.

Proposition: *The S-module E'_S is free of rank equal to $|W|$. There is a basis $(\xi^w)_{w \in W}$ of E'_S over S such that $\xi^w(w') = 0$ whenever $w' \not\geq w$ and such that*

$$\xi^w(w) = \begin{cases} \prod_{\alpha \in R(w)} h_\alpha, & \textit{if } w\alpha_0 > 0; \\ \prod_{\alpha \in R(w), \alpha \neq -w\alpha_0} h_\alpha, & \textit{otherwise.} \end{cases} \tag{3}$$

Proof: It will be enough to prove the existence of elements $\xi^w \in E'_S$ with these properties. The lower triangular form of the matrix of all $\xi^w(w')$ yields the linear independence. In order to show that the ξ^w generate E'_S we argue as in 19.6. Consider an arbitrary $\xi \in E'_S$. Let $w \in W$ be minimal for $\xi(w) \neq 0$. We have for all $\alpha \in R(w)$ with $\alpha \neq -w(\alpha_0)$ that $\xi(w) \equiv \xi(s_\alpha w) = 0 \pmod{h_\alpha}$. So $\xi(w)$ is divisible by $\xi^w(w)$ and we can form $\xi' = \xi - (\xi(w)\xi^w(w)^{-1})\xi^w$. We then apply induction on w.

So we just have to construct the ξ^w. We can obviously take $\xi^w = \eta^w$ if $w\alpha_0 > 0$. So fix from now on w with $w\alpha_0 < 0$. Set for all $w' \in W$:

$$\xi^w(w') = \eta^w(w')(-w'h_{\alpha_0})^{-1}. \tag{4}$$

Let us check first that $\xi^w(w')$ is in S. We can assume that $w' \geq w$ since we get 0 otherwise. Then the observation implies that $w'\alpha_0 < 0$ and $w's_{\alpha_0} \not\geq w$. So

$$0 = \eta^w(w's_{\alpha_0}) = \eta^w(s_{-w'\alpha_0}w') \equiv \eta^w(w') \pmod{h_{-w'\alpha_0}}.$$

Therefore $\eta^w(w')$ is divisible by $h_{-w'\alpha_0} = -w'h_{\alpha_0}$ and $\xi^w(w')$ is in S.

We now have to check for all $w' \in W$ and $\alpha \in R^+$ that

$$\xi^w(w') \equiv \xi^w(s_\alpha w') \pmod{h_\alpha} \qquad \text{if } \alpha \neq \pm w'\alpha_0.$$

We can assume $w' \geq w$, hence (as above) $\gamma = -w'\alpha_0 > 0$. If $s_\alpha w' \not\geq w$, then $0 = \eta^w(s_\alpha w') = \xi^w(s_\alpha w')$, and $\eta^w(w') \equiv 0 \pmod{h_\alpha}$. So h_α divides $\eta^w(w')$, hence also $\xi^w(w') = \eta^w(w')/h_\gamma$, since h_α and h_γ are coprime. On the other hand, if $s_\alpha w' \geq w$, then

$$\begin{aligned}\xi^w(w') - \xi^w(s_\alpha w') &= \big(\eta^w(w') - \eta^w(s_\alpha w')\big)h_\gamma^{-1} + \eta^w(s_\alpha w')\big(h_\gamma^{-1} - (s_\alpha h_\gamma)^{-1}\big) \\ &= \big(\eta^w(w') - \eta^w(s_\alpha w')\big)h_\gamma^{-1} + \eta^w(s_\alpha w')\alpha(h_\gamma)h_\alpha h_\gamma^{-1}(s_\alpha h_\gamma)^{-1}.\end{aligned}$$

This expression is divisible by h_α in $S[h_\gamma^{-1}, (s_\alpha h_\gamma)^{-1}]$, hence in S, since h_α is coprime with h_γ and $(s_\alpha h_\gamma)$.

Remark: We have a natural map $E'_S \otimes_S A \to E'_A$. The first paragraph of the proof above shows that the images of the $\xi^w \otimes 1$ are a basis of E'_A. This implies that the map is an isomorphism

$$E'_S \otimes_S A \xrightarrow{\sim} E'_A. \tag{5}$$

19.10. Let λ be regular in the first dominant alcove (as in 13.2), and let μ be semiregular in the closure of that alcove with $\langle \mu + \rho, \alpha_0^\vee \rangle = p$ (as in 13.10 or Theorem 19.4.b). So the stabilizer of μ in W_p is equal to $\{1, s_0\}$ where $s_0 = s_{\alpha_0,p}$. Set $\Omega = W_p{\bullet}\lambda$ and $\Gamma = W_p{\bullet}\mu$, set $T = T_\Omega^\Gamma$ and $T' = T_\Gamma^\Omega$.

Let Q be the module we get from translating $Z_A(-\rho)$ to Ω. So we have $Q = Q_0(A)$ in the notation from 14.4 (for $\lambda_0 = \lambda$) and $Q \simeq Q_A(w_0{\bullet}\lambda)$ as observed in 19.4. We have also seen that

$$\mathrm{End}_{\mathcal{C}} Q \simeq E_A, \tag{1}$$

cf. 19.7(3). Set

$$Q_1 = TQ \qquad \text{and} \qquad Q_2 = T'Q_1.$$

Since A is local, Theorem 4.19.b implies that Q_1 is isomorphic to $Q_A(w_0{\scriptstyle\bullet}\mu)$, so we have

$$\mathrm{End}_{\mathcal{C}} Q_1 \simeq E'_A \tag{2}$$

by 19.9(2). We have $Q_2 = Q_{(s_0)}(A)$ in the notation from 14.4. We get from [Ja6], II.11.10(1), that $Q_2 \otimes_A k$ is isomorphic to $Q_k(w_0 s_0{\scriptstyle\bullet}\lambda)$, hence that $Q_2 \simeq Q_A(w_0 s_0{\scriptstyle\bullet}\lambda)$.

We claim that also Q_2 satisfies the assumptions of 19.1–3. Since $Q_1^{\emptyset}$ is the direct sum of the $Z^{\emptyset}(w{\scriptstyle\bullet}\mu)$ with $w \in W$, we get from 7.5 that

$$Q_2^{\emptyset} = \bigoplus_{w \in W \cup W s_0} Z^{\emptyset}(w{\scriptstyle\bullet}\lambda). \tag{3}$$

Let $\beta \in R^+$. In order to describe Q_2^{β} we use the decomposition of T' over A^{β} according to orbits for $W_{\beta,p}$. Let $w \in W$. If $\beta \neq \pm w\alpha_0$, then Q_1^{β} has a block component that is an extension of $Z^{\beta}(w{\scriptstyle\bullet}\mu)$ and $Z^{\beta}(s_\beta w{\scriptstyle\bullet}\mu)$. If $w^{-1}\beta < 0$, then the component is isomorphic to $Q^{\beta}(w{\scriptstyle\bullet}\mu)$, otherwise to $Q^{\beta}(s_\beta w{\scriptstyle\bullet}\mu)$. Applying T' yields two block components for Q_2^{β} isomorphic to two different $Q^{\beta}(\lambda')$. One of them is an extension of $Z^{\beta}(w{\scriptstyle\bullet}\lambda)$ and $Z^{\beta}(s_\beta w{\scriptstyle\bullet}\lambda)$, the other one of $Z^{\beta}(w_0{\scriptstyle\bullet}\lambda)$ and $Z^{\beta}(s_\beta w s_0{\scriptstyle\bullet}\lambda)$. On the other hand, if $\beta = \pm w\alpha_0$, then Q_1^{β} has a block component isomorphic to $Z^{\beta}(w{\scriptstyle\bullet}\mu)$. Applying T' yields a block component for Q_2^{β} that is an extension of $Z^{\beta}(w{\scriptstyle\bullet}\lambda)$ and $Z^{\beta}(w s_0{\scriptstyle\bullet}\lambda)$ and isomorphic to $Q^{\beta}(\lambda')$, where λ' is the smaller of the two weights. So we get a decomposition as in 19.1(2).

Proposition: *The algebra* $\mathrm{End}_{\mathcal{C}} Q_2$ *is isomorphic to the algebra of all functions* $\zeta : W \cup W s_0 \to A$ *satisfying the following congruences: We have for all* $\beta \in R^+$ *and* $w \in W$

$$\left.\begin{aligned} \zeta(s_\beta w) &\equiv \zeta(w) \pmod{h_\beta} \\ \zeta(s_\beta w s_0) &\equiv \zeta(w s_0) \pmod{h_\beta} \end{aligned}\right\} \qquad \text{if } \beta \neq \pm w\alpha_0, \tag{4}$$

and

$$\zeta(w) \equiv \zeta(w s_0) \pmod{h_\beta} \qquad \text{if } \beta = \pm w\alpha_0. \tag{5}$$

Proof: This is an easy consequence of Proposition 19.3 and of the explicit decomposition of Q_2^{β} found above.

19.11. Set E''_S equal to the algebra of functions $\zeta : W \cup W s_0 \to S$ satisfying all congruences 19.10(4), (5). Define similarly E''_A by replacing S by A. So Proposition 19.10 says

$$E''_A \simeq \mathrm{End}_{\mathcal{C}} Q_2.$$

We can embed the algebra E'_S into E''_S as follows: For any $\xi \in E'_S$ let $\widehat{\xi}$ be the function on $W \cup W s_0$ with

$$\widehat{\xi}(w) = \widehat{\xi}(w s_0) = \xi(w) \qquad \text{for all } w \in W.$$

It is then obvious that $\widehat{\xi}$ satisfies 19.10(5) and that 19.10(4) is inherited from the similar property 19.9(1) for ξ. So $\widehat{\xi}$ is in E''_S, and $\xi \mapsto \widehat{\xi}$ is easily checked to be an isomorphism from E'_S onto the subalgebra

$$\widehat{E'_S} = \{\zeta \in E''_S \mid \zeta(w) = \zeta(w s_0) \qquad \text{for all } w \in W\}. \tag{1}$$

The elements $\zeta^w = \widehat{\xi^w}$ with $w \in W$ will turn out to be part of a basis of E''_S over S.

Set

$$E''_S(0) = \{\zeta \in E''_S \mid \zeta(w) = 0 \qquad \text{for all } w \in W\}. \tag{2}$$

This is obviously an ideal in E''_S with $E''_S(0) \cap \widehat{E'_S} = 0$. We claim that

$$E''_S = \widehat{E'_S} \oplus E''_S(0). \tag{3}$$

Indeed, for any $\zeta \in E''_S$ the function ζ_1 on $W \cup W s_0$ with $\zeta_1(w) = \zeta_1(w s_0) = \zeta(w)$ for all $w \in W$ inherits 19.10(4) from ζ. So ζ_1 is in $\widehat{E'_S}$ and $\zeta - \zeta_1$ is in $E''_S(0)$.

It is easy to check that the function ζ^{s_0} with

$$\zeta^{s_0}(w) = 0 \quad \text{and} \quad \zeta^{s_0}(w s_0) = w h_{\alpha_0} \qquad \text{for all } w \in W \tag{4}$$

is in E''_S, hence in $E''_S(0)$. We can therefore define for each $w \in W$:

$$\zeta^{w s_0} = \zeta^w \zeta^{s_0}. \tag{5}$$

(For $w = 1$ this is compatible with (4) since $\zeta^1 = 1$.) Our usual arguments yield now the first two parts of:

Proposition: *a) The S-module E''_S is free of rank $2|W|$. The ζ^w with $w \in W \cup W s_0$ are a basis of E''_S over S.*

b) The S-module $E''_S(0)$ is free of rank $|W|$. The $\zeta^{w s_0}$ with $w \in W$ are a basis of $E''_S(0)$ over S.

c) We have a natural isomorphism $E''_S \otimes_S A \xrightarrow{\sim} E''_A$.

Proof: As far as c) is concerned, note that we can carry out the constructions above also inside E''_A. The claim follows now from 19.7(1) and 19.9(5).

19.12. The description of the basis in 19.11 implies especially that the multiplication with ζ^{s_0} is a bijection from $\widehat{E'_S}$ onto $E''_S(0)$. This can be seen directly as follows. The injectivity is clear since all $\zeta^{s_0}(ws_0) = wh_{\alpha_0}$ are nonzero and since S is an integral domain. To get the surjectivity, define for a given $\zeta \in E''_S(0)$ a function ξ on W via

$$\xi(w) = \zeta(ws_0)(wh_{\alpha_0})^{-1} \qquad \text{for all } w \in W.$$

As soon as we know that $\xi \in E'_S$ we see that $\zeta = \zeta^{s_0}\widehat{\xi}$. Well, the congruences 19.10(5) imply that $\xi(w) \in S$ for all w. The congruences 19.9(1) follow from 19.10(4) using the same calculation as used for the ξ^w following 19.9(4).

If we apply this construction to $(\zeta^{s_0})^2$ we get (Exercise!)

$$(\zeta^{s_0})^2 = \zeta^{s_0}\big(h_{\alpha_0}\zeta^1 - \sum_{\alpha\in\Sigma}\langle\alpha_0, \alpha^\vee\rangle\zeta^{s_\alpha}\big). \tag{1}$$

Since $E''_S = \widehat{E'_S}[\zeta^{s_0}]$ the structure of E''_S is determined by that of E'_S and by this formula.

19.13. Let us now give another description of the algebra E''_A. Keep the notations from 19.11–12. We can embed E_S into E''_S as follows. For any $\eta \in E_S$ define a function $\widetilde{\eta}$ on $W \cup Ws_0$ by

$$\widetilde{\eta}(w) = \eta(w) \quad \text{and} \quad \widetilde{\eta}(ws_0) = \eta(ws_{\alpha_0}) \qquad \text{for all } w \in W. \tag{1}$$

It is clear that $\widetilde{\eta}$ satisfies the congruences 19.10(4). Since

$$\widetilde{\eta}(ws_0) = \eta(ws_{\alpha_0}) = \eta(s_{w\alpha_0}w) \equiv \eta(w) = \widetilde{\eta}(w) \pmod{h_{w\alpha_0}}$$

for all $w \in W$, also 19.10(5) holds, so indeed $\widetilde{\eta} \in E''_S$. In order to simplify notation set

$$s' = s_{\alpha_0}.$$

The map $\eta \mapsto \widetilde{\eta}$ is obviously injective and identifies E_S with

$$\widetilde{E}_S = \{\zeta \in E''_S \mid \zeta(ws_0) = \zeta(ws') \qquad \text{for all } w \in W\}. \tag{2}$$

Write $\alpha_0^\vee = \sum_{\beta\in\Sigma} m_\beta\beta^\vee$. We claim: If R is not of type A_1, then

$$\widetilde{\eta^\alpha} = \zeta^{s_\alpha} + m_\alpha\zeta^{s_0} \qquad \text{for all } \alpha \in \Sigma. \tag{3}$$

(We use here the abbreviation $\eta^\alpha = \eta^{s_\alpha}$ for any $\alpha \in \Sigma$.) The assumption on R makes sure that $\xi^{s_\alpha} = \eta^{s_\alpha}$, so that both ζ^{s_α} and $\widetilde{\eta^\alpha}$ coincide on W. Since ζ^{s_0} vanishes on W, both sides in (3) agree on W. We have $\eta^\alpha(w) = \omega_\alpha - w\omega_\alpha$ for all $w \in W$ where ω_α is the fundamental weight corresponding to α, cf. D.1.e. This implies:

$$\begin{aligned}\widetilde{\eta^\alpha}(ws_0) &= \eta^\alpha(ws') = \omega_\alpha - ws'\omega_\alpha \\ &= \omega_\alpha - w\omega_\alpha + \langle \omega_\alpha, \alpha_0^\vee \rangle w h_{\alpha_0} \\ &= \eta^\alpha(w) + m_\alpha w h_{\alpha_0} = \zeta^{s_\alpha}(ws_0) + m_\alpha \zeta^{s_0}(ws_0).\end{aligned}$$

So both sides in (3) agree also on Ws_0. For R of type A_1 one has $\xi^{s_\alpha} = h_\alpha^{-1}\eta^\alpha$ and gets

$$\widetilde{\eta^\alpha} = h_\alpha \zeta^{s_\alpha} + \zeta^{s_0}.$$

The m_β have greatest common divisor 1, so (3) implies that ζ^{s_0} is in $\widetilde{E}_S + \widehat{E'_S}$. Therefore E''_S is generated as an algebra by $\widetilde{E}_S$ and $\widehat{E'_S}$, and the multiplication is a surjective homomorphism $\widetilde{E}_S \otimes_S \widehat{E'_S} \to E''_S$.

We have

$$\{\eta \in E_S \mid \widetilde{\eta} \in \widehat{E'_S}\} = \{\eta \in E_S \mid \eta(w) = \eta(ws') \quad \text{for all } w \in W\} = (E_S)^{s'}.$$

We can therefore refine the statement above and get a surjective algebra homomorphism

$$E_S \otimes_{E_S^{s'}} E'_S \to E''_S \qquad \text{with} \qquad \eta \otimes \xi \mapsto \widetilde{\eta}\widehat{\xi}. \tag{4}$$

We want to show that it is an isomorphism.

Suppose that E_S is free of rank 2 over $E_S^{s'}$. Now E'_S is free of rank $|W|$ over S, therefore $E_S \otimes_{E_S^{s'}} E'_S$ is free of rank $2|W|$ over S. On the other hand E''_S is free over S of the same rank. Any surjective homomorphism between free modules of the same finite rank is an isomorphism (in the commutative case). So the map in (4) is an isomorphism.

If $\operatorname{char}(k) \neq 2$, then S_k is free over $(S_k)^{s'}$ with basis $(1, h_{\alpha_0})$. Suppose that we are in Case 1 that $p > h$ or in Case 2. Then 19.8(1)–(3) imply now that E_{S_k} is free over $E_{S_k}^{s'}$ of rank 2. We can develop the theory above also over S_k (and over A). Now our remarks imply that (4) induces isomorphisms

$$E_{S_k} \otimes_{(E_{S_k})^{s'}} E'_{S_k} \xrightarrow{\sim} E''_{S_k} \tag{5}$$

and

$$E_A \otimes_{(E_A)^{s'}} E'_A \xrightarrow{\sim} E''_A \tag{6}$$

(for k as above).

19.14. We want to compute $\mathrm{Hom}_{\mathcal{C}}(Q, Q_2)$ and $\mathrm{Hom}_{\mathcal{C}}(Q_2, Q)$ with Q and Q_2 as in 19.10. We cannot proceed as in the proof of Proposition 19.3, because the unknown units d_i from 19.2(7) will no longer cancel. Instead, we shall use the detailed description of $\mathcal{Q} = \mathcal{V}_\Omega Q$ and $\mathcal{Q}_2 = \mathcal{V}_\Omega Q_2$ that is provided by the theory from Sections 10 and 13. Recall that we have described $\mathcal{Q}$ in 10.14. The precise values of the $a^{\beta}_{w\bullet\lambda}$ from 10.14(4) are given by 13.23(1); however, they will turn out to be irrelevant for our present purpose.

Note that $\mathcal{Q}_2 = \mathcal{Q}_{(s_0),0}(A)$ in the notation from 16.5. We get from 10.11 and 13.4 the following description: We have

$$\mathcal{Q}_2(w\bullet\lambda) = \mathcal{Q}(w\bullet\lambda) = A^{\emptyset} = \mathcal{Q}_2(ws_0\bullet\lambda)$$

for all $w \in W$; all other $\mathcal{Q}_2(\lambda')$ are zero. For $\beta \in R^+$ and $w \in W$ with $w^{-1}\beta \neq \pm\alpha_0$ we get

$$\mathcal{Q}_2(w\bullet\lambda, \beta) = \mathcal{Q}(w\bullet\lambda, \beta)$$

and (in case $w^{-1}\beta < 0$)

$$\mathcal{Q}_2(w\bullet\lambda - p\beta, \beta) = \mathcal{Q}(w\bullet\lambda - p\beta, \beta).$$

We shall not have to know $\mathcal{Q}_2(ws_0\bullet\lambda, \beta)$ and $\mathcal{Q}_2(ws_0\bullet\lambda - p\beta, \beta)$ for our calculations. For β and w with $w^{-1}\beta = \alpha_0$ we get

$$\begin{aligned}
\mathcal{Q}_2(w\bullet\lambda, \beta) &= A^{\beta}(h_\beta, 0) \oplus A^{\beta}(1, h_\beta),\\
\mathcal{Q}_2(ws_0\bullet\lambda, \beta) &= A^{\beta}(h_\beta, 0),\\
\mathcal{Q}_2(s_\beta w\bullet\lambda, \beta) &= A^{\beta}(1, 0) \oplus A^{\beta}(0, h_\beta),\\
\mathcal{Q}_2(s_\beta ws_0\bullet\lambda, \beta) &= A^{\beta}(1, 0) \oplus A^{\beta}(h_\beta^{-1}, 1).
\end{aligned}$$

It is now easy to check that $\mathrm{Hom}_{\mathcal{C}}(Q_2, Q)$ is identified with the set of all maps $\xi : W \to A$ such that for all $\beta \in R^+$ and $w \in W$:

$$\xi(s_\beta w) \equiv \xi(w) \pmod{h_\beta} \qquad \text{if} \quad \beta \neq \pm w(\alpha_0),$$

i.e.,

$$\mathrm{Hom}_{\mathcal{C}}(Q_2, Q) \simeq E'_A. \tag{1}$$

Similarly, one can identify $\mathrm{Hom}_{\mathcal{C}}(Q, Q_2)$ with the set of all maps $\xi : W \to A$ such that for all $\beta \in R^+$ and $w \in W$:

$$\xi(s_\beta w) \equiv \xi(w) \pmod{h_\beta} \qquad \text{if} \quad \beta \neq \pm w(\alpha_0)$$

and

$$\xi(w) \in A h_{w\alpha_0}.$$

One checks easily that one gets an isomorphism

$$E'_A \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Q, Q_2), \qquad \xi \mapsto \check{\xi} \tag{2}$$

where (for all $w \in W$)

$$\check{\xi}(w) = \xi(w) h_{w\alpha_0}.$$

It should not come as a surprise that both Hom spaces are isomorphic to $E'_A \simeq \mathrm{End}_{\mathcal{C}} Q_1$, since we have (in the notations from 10.1) isomorphisms

$$\mathrm{adj}_2 : \mathrm{End}_{\mathcal{C}} Q_1 \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Q_2, Q)$$

and

$$\mathrm{adj}_1 : \mathrm{Hom}_{\mathcal{C}}(Q, Q_2) \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}} Q_1.$$

The remarks in 10.13 imply that we take in (1) the identification arising from adj_2 whereas in (2) we modify the identification from adj_1 and make it more compatible with the grading. Note also that the discussion in 10.12 implies that the map

$$\mathrm{End}_{\mathcal{C}} Q \to \mathrm{End}_{\mathcal{C}} Q_1, \qquad h \mapsto Th$$

corresponds to the inclusion of E_A into E'_A, and that the map

$$\mathrm{End}_{\mathcal{C}} Q_1 \to \mathrm{End}_{\mathcal{C}} Q_2, \qquad h \mapsto T'h$$

corresponds to the embedding of E'_A into E''_A given by $\xi \mapsto \widehat{\xi}$ as in 19.11.

Appendix A

The computations in Section 11 lead to certain functions of the weights involved. In this appendix we introduce notations for these functions and prove some of their properties. In A.2–11 all terms are to be regarded as elements of the fraction field of U^0, in A.12–14 as elements of the ground field.

A.1. For all $\alpha \in R$ and $\lambda \in X$ set

$$\overline{\alpha}(\lambda) = \begin{cases} 1, & \text{if } \langle \lambda, \alpha^\vee \rangle < 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

We have obviously

$$\overline{(w\alpha)}(w\lambda) = \overline{\alpha}(\lambda) \tag{2}$$

for all $w \in W$, and

$$\overline{(-\alpha)}(-\lambda) = \overline{\alpha}(\lambda). \tag{3}$$

Furthermore

$$\overline{\alpha}(-\lambda) = \begin{cases} 1 - \overline{\alpha}(\lambda), & \text{if } \langle \lambda, \alpha^\vee \rangle \neq 0; \\ 0 = \overline{\alpha}(\lambda), & \text{if } \langle \lambda, \alpha^\vee \rangle = 0, \end{cases} \tag{4}$$

hence

$$\langle \lambda, \alpha^\vee \rangle \overline{\alpha}(-\lambda) = \langle \lambda, \alpha^\vee \rangle (1 - \overline{\alpha}(\lambda)). \tag{5}$$

A.2. For all $\lambda, \mu \in X$ and all $\alpha \in R$ set in Case 1:

$$d(\mu, \lambda, \alpha) = \left(\frac{H_\alpha + \langle \mu + \rho, \alpha^\vee \rangle}{H_\alpha + \langle \lambda + \rho, \alpha^\vee \rangle} \right)^{\overline{\alpha}(\mu - \lambda)} \tag{1}$$

and in Case 2:

$$d(\mu, \lambda, \alpha) = \left(\frac{[K_\alpha; \langle \mu + \rho, \alpha^\vee \rangle]}{[K_\alpha; \langle \lambda + \rho, \alpha^\vee \rangle]} \right)^{\overline{\alpha}(\mu - \lambda)} \tag{2}$$

Using A.1(4) one checks easily that

$$d(\mu,\lambda,\alpha)=\frac{H_\alpha+\langle\mu+\rho,\alpha^\vee\rangle}{H_\alpha+\langle\lambda+\rho,\alpha^\vee\rangle}\,d(\lambda,\mu,\alpha) \tag{3}$$

resp.

$$d(\mu,\lambda,\alpha)=\frac{[K_\alpha;\langle\mu+\rho,\alpha^\vee\rangle]}{[K_\alpha;\langle\lambda+\rho,\alpha^\vee\rangle]}\,d(\lambda,\mu,\alpha). \tag{4}$$

A.3. We have

$$[K_{-\alpha};-m]=-[K_\alpha;m] \tag{1}$$

for all $\alpha\in R$ and $m\in\mathbf{Z}$, hence

$$[K_{-\alpha};\langle\lambda+\rho,(-\alpha)^\vee\rangle]=-[K_\alpha;\langle\lambda+\rho,\alpha^\vee\rangle]$$

for all $\lambda\in X$. In the other case, we have obviously

$$H_{-\alpha}+\langle\lambda+\rho,(-\alpha)^\vee\rangle=-(H_\alpha+\langle\lambda+\rho,\alpha^\vee\rangle).$$

Therefore A.1(3) yields easily

$$d(\mu,\lambda,-\alpha)=d(\lambda,\mu,\alpha)^{-1} \tag{2}$$

for all $\lambda,\mu\in X$ and $\alpha\in R$.

A.4. One sees easily for all $w\in W$ and $\alpha\in R$ that

$$w(H_\alpha+m)=H_{w\alpha}+m \qquad \text{resp.} \qquad w[K_\alpha;m]=[K_{w\alpha};m] \tag{1}$$

for all $m\in\mathbf{Z}$. This together with A.1(2) implies

$$d(w\bullet\mu,w\bullet\lambda,w\alpha)=wd(\mu,\lambda,\alpha) \tag{2}$$

for all $\lambda,\mu\in X$.

A.5. In Case 1 we work in characteristic p, so we have $H_\alpha+m+rp=H_\alpha+m$ for all $r\in\mathbf{Z}$. In Case 2 we have similarly $[K_\alpha;m+rp]=[K_\alpha;m]$ since ζ is a p-th root of unity. This implies

$$d(\mu+p\omega,\lambda+p\omega,\alpha)=d(\mu,\lambda,\alpha) \tag{1}$$

for all $\lambda,\mu,\omega\in X$, and $\alpha\in R$. We have obviously

$$\lambda+\rho,\mu+\rho\in pX \implies d(\mu,\lambda,\alpha)=1 \quad \text{for all } \alpha\in R. \tag{2}$$

A.6. For all $\lambda, \mu, \nu \in X$ and $\alpha \in R$ set

$$c_\alpha(\nu, \mu, \lambda) = d(\mu, \nu, \alpha)^{-1} d(\lambda, \mu, \alpha)^{-1} d(\lambda, \nu, \alpha). \tag{1}$$

Using A.2(3) resp. (4) one checks easily that always

$$c_\alpha(\nu, \mu, \lambda) = d(\nu, \mu, \alpha)^{-1} d(\mu, \lambda, \alpha)^{-1} d(\nu, \lambda, \alpha). \tag{2}$$

The formulas for the d terms in A.3–5 yield now formulas for the c_α terms. We get (for all $\lambda, \mu, \nu \in X$ and $\alpha \in R$)

$$c_{-\alpha}(\nu, \mu, \lambda) = c_\alpha(\nu, \mu, \lambda)^{-1} \tag{3}$$

— the proof here requires (2) — and

$$c_{w\alpha}(w \bullet \nu, w \bullet \mu, w \bullet \lambda) = w c_\alpha(\nu, \mu, \lambda) \tag{4}$$

for all $w \in W$, and (for all $\omega \in X$)

$$c_\alpha(\nu + p\omega, \mu + p\omega, \lambda + p\omega) = c_\alpha(\nu, \mu, \lambda). \tag{5}$$

A.7. For all $w \in W$ and $\lambda, \mu, \nu \in X$ set

$$d(\mu, \lambda, w) = \prod_{\alpha > 0, w^{-1}\alpha < 0} d(\mu, \lambda, \alpha) \tag{1}$$

and

$$c_w(\nu, \mu, \lambda) = \prod_{\alpha > 0, w^{-1}\alpha < 0} c_\alpha(\nu, \mu, \lambda). \tag{2}$$

Comparing these definitions to A.6(1), (2), we see

$$\begin{aligned} c_w(\nu, \mu, \lambda) &= d(\mu, \nu, w)^{-1} d(\lambda, \mu, w)^{-1} d(\lambda, \nu, w) \\ &= d(\nu, \mu, w)^{-1} d(\mu, \lambda, w)^{-1} d(\nu, \lambda, w). \end{aligned} \tag{3}$$

A.8. Lemma: *We have for all $w \in W$ and $\lambda, \mu \in X$:*

$$w d(\mu, \lambda, w^{-1}) = d(w \bullet \lambda, w \bullet \mu, w)^{-1}.$$

Proof: We use A.7(1), A.4(2), A.3(2) to get

$$\begin{aligned} wd(\mu,\lambda,w^{-1}) &= \prod_{\alpha>0,w\alpha<0} wd(\mu,\lambda,\alpha) = \prod_{\alpha>0,w\alpha<0} d(w_\bullet\mu, w_\bullet\lambda, w\alpha) \\ &= \prod_{\alpha>0,w^{-1}\alpha<0} d(w_\bullet\lambda, w_\bullet\mu, \alpha)^{-1} = d(w_\bullet\lambda, w_\bullet\mu, w)^{-1}. \end{aligned}$$

Remark: Using A.7(3) the lemma implies easily for all $w \in W$ and $\lambda, \mu, \nu \in X$:

$$wc_{w^{-1}}(\nu,\mu,\lambda) = c_w(w_\bullet\nu, w_\bullet\mu, w_\bullet\lambda)^{-1}. \tag{1}$$

A.9. Lemma: *We have for all $\beta \in R, w \in W$ and all $\lambda, \mu, \nu \in X$:*

$$c_{s_\beta}(w_\bullet\nu, w_\bullet\mu, w_\bullet\lambda) = w\left(c_{w^{-1}s_\beta}(\nu,\mu,\lambda)c_{w^{-1}}(\nu,\mu,\lambda)^{-1}\right).$$

Proof: Fix λ, μ, ν and write $c_\alpha = c_\alpha(\nu,\mu,\lambda)$ for all $\alpha \in R$. Then A.6(4) and A.7(2) imply

$$\begin{aligned} w^{-1}c_{s_\beta}(w_\bullet\nu, w_\bullet\mu, w_\bullet\lambda) &= \prod_{\alpha>0,s_\beta\alpha<0} c_{w^{-1}\alpha} = \prod_{w\alpha>0,s_\beta w\alpha<0} c_\alpha \\ &= \prod_{\alpha>0,w\alpha>0,s_\beta w\alpha<0} c_\alpha \prod_{\alpha<0,w\alpha>0,s_\beta w\alpha<0} c_\alpha \\ &= \prod_{\alpha>0,w\alpha>0,s_\beta w\alpha<0} c_\alpha \prod_{\alpha>0,w\alpha<0,s_\beta w\alpha>0} c_\alpha^{-1} \\ &= \prod_{\alpha>0,s_\beta w\alpha<0} c_\alpha \prod_{\alpha>0,w\alpha<0} c_\alpha^{-1} \\ &= c_{w^{-1}s_\beta}(\nu,\mu,\lambda)c_{w^{-1}}(\nu,\mu,\lambda)^{-1}, \end{aligned}$$

using A.6(3) to get from line 2 to line 3.

A.10. Lemma: *Suppose that $\beta \in R^+$ and $w \in W$ with $w^{-1}\beta$ simple. Then we have for all $\lambda, \mu, \nu \in X$:*

$$s_\beta c_w(s_{\beta\bullet}\nu, s_{\beta\bullet}\mu, s_{\beta\bullet}\lambda) = c_w(\nu,\mu,\lambda)c_{s_\beta}(\nu,\mu,\lambda)^{-1}c_\beta(\nu,\mu,\lambda).$$

Proof: Abbreviate again $c_\alpha = c_\alpha(\nu, \mu, \lambda)$. By A.6(4) the left hand side is equal to

$$\prod_{\alpha>0, w^{-1}\alpha<0} c_{s_\beta \alpha} = \prod_{s_\beta\alpha>0, w^{-1}s_\beta\alpha<0} c_\alpha = \prod_{\substack{\alpha>0, s_\beta\alpha>0,\\ w^{-1}s_\beta\alpha<0}} c_\alpha \prod_{\substack{\alpha<0, s_\beta\alpha>0,\\ w^{-1}s_\beta\alpha<0}} c_\alpha.$$

A root α with $\alpha > 0$ satisfies $w^{-1}\alpha < 0$ if and only if it satisfies $w^{-1}s_\beta\alpha < 0$ and $\alpha \neq \beta$. (Recall that $w^{-1}\beta$ is simple.) So we can rewrite the product as

$$\prod_{\substack{\alpha>0, s_\beta\alpha>0,\\ w^{-1}\alpha<0}} c_\alpha \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}s_\beta\alpha>0}} c_{-\alpha} = \prod_{\alpha>0, w^{-1}\alpha<0} c_\alpha \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha<0}} c_\alpha^{-1} \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha>0, \alpha\neq\beta}} c_\alpha^{-1}.$$

(We have used A.6(3) to replace the $c_{-\alpha}$ by c_α^{-1}.) In the last expression the first product yields the c_w term in our claim, the second and third product yield the $c_{s_\beta}^{-1}$ term except for a missing factor c_β^{-1}. The claim follows.

A.11. If $\lambda + \rho$ is dominant, then $\overline{\alpha}(\lambda + \rho) = 0$ for all $\alpha \in R^+$, hence

$$d(\lambda, -\rho, \alpha) = 1 = d(\lambda, -\rho, w) \qquad \text{for all } \alpha \in R^+ \text{ and } w \in W. \tag{1}$$

If $\lambda + \rho$ and $\mu + \rho$ are dominant, then

$$c_w(\lambda, \mu, -\rho) = d(\lambda, \mu, w)^{-1} \qquad \text{for all } w \in W \tag{2}$$

by A.7(3), and

$$c_{s_\gamma}(w{\scriptstyle\bullet}\lambda, w{\scriptstyle\bullet}\mu, w{\scriptstyle\bullet}(-\rho)) = w\Big(d(\lambda, \mu, w^{-1})d(\lambda, \mu, w^{-1}s_\gamma)^{-1}\Big) \tag{3}$$

for all $\gamma \in R$ and $w \in W$ by Lemma A.9. (Of course, $w{\scriptstyle\bullet}(-\rho) = -\rho$.)

A.12. Recall that ζ is a primitive p–th root of unity in Case 2. In order to avoid too many separate formulas for our two cases, we set $\zeta = 1$ in Case 1. In that case all expressions ζ_α and ζ_w to be defined will be equal to 1 and all formulas involving these terms will be trivial.

For all $\lambda \in X$ and $\alpha \in R$ set

$$\zeta_\alpha(\lambda) = \zeta^{d_\alpha\langle\lambda,\alpha^\vee\rangle\overline{\alpha}(\lambda)} \qquad \text{and} \qquad z_\alpha(\lambda) = (-1)^{\langle\lambda,\alpha^\vee\rangle\overline{\alpha}(\lambda)}. \tag{1}$$

We have then by A.1(5)

$$\zeta_\alpha(-\lambda) = \zeta_\alpha(\lambda)\zeta^{-d_\alpha\langle\lambda,\alpha^\vee\rangle} \qquad \text{and} \qquad z_\alpha(-\lambda) = z_\alpha(\lambda)(-1)^{\langle\lambda,\alpha^\vee\rangle}. \tag{2}$$

For all $\lambda, \mu, \nu \in X$ set

$$\zeta_\alpha(\nu, \mu, \lambda) = \zeta_\alpha(\mu - \nu)\zeta_\alpha(\lambda - \mu)\zeta_\alpha(\lambda - \nu)^{-1} \tag{3}$$

and

$$z_\alpha(\nu, \mu, \lambda) = z_\alpha(\mu - \nu)z_\alpha(\lambda - \mu)z_\alpha(\lambda - \nu). \tag{4}$$

Then (2) implies

$$\zeta_\alpha(\nu, \mu, \lambda) = \zeta_\alpha(\nu - \mu)\zeta_\alpha(\mu - \lambda)\zeta_\alpha(\nu - \lambda)^{-1} \tag{5}$$

and

$$z_\alpha(\nu, \mu, \lambda) = z_\alpha(\nu - \mu)z_\alpha(\mu - \lambda)z_\alpha(\nu - \lambda). \tag{6}$$

A.13. The formulas in A.1 imply

$$\zeta_{w\alpha}(w\lambda) = \zeta_\alpha(\lambda) = \zeta_{-\alpha}(-\lambda) \qquad \text{and} \qquad z_{w\alpha}(w\lambda) = z_\alpha(\lambda) = z_{-\alpha}(-\lambda)$$

for all $\lambda \in X$, $\alpha \in R$, $w \in W$. This implies for all $\lambda, \mu, \nu \in X$ and $\alpha \in R$ that

$$\zeta_{w\alpha}(w{\bullet}\nu, w{\bullet}\mu, w{\bullet}\lambda) = \zeta_\alpha(\nu, \mu, \lambda) \quad \text{and} \quad z_{w\alpha}(w{\bullet}\nu, w{\bullet}\mu, w{\bullet}\lambda) = z_\alpha(\nu, \mu, \lambda) \tag{1}$$

for all $w \in W$, and — using A.12(5), (6) —

$$\zeta_{-\alpha}(\nu, \mu, \lambda) = \zeta_\alpha(\nu, \mu, \lambda) \qquad \text{and} \qquad z_{-\alpha}(\nu, \mu, \lambda) = z_\alpha(\nu, \mu, \lambda). \tag{2}$$

We have obviously for all $\omega \in X$:

$$\zeta_\alpha(\nu+\omega, \mu+\omega, \lambda+\omega) = \zeta_\alpha(\nu, \mu, \lambda) \quad \text{and} \quad z_\alpha(\nu+\omega, \mu+\omega, \lambda+\omega) = z_\alpha(\nu, \mu, \lambda). \tag{3}$$

A.14. For all $\lambda, \mu, \nu \in X$ and $w \in W$ set

$$\zeta_w(\nu, \mu, \lambda) = \prod_{\alpha>0, w^{-1}\alpha<0} \zeta_\alpha(\nu, \mu, \lambda) \quad \text{and} \quad z_w(\nu, \mu, \lambda) = \prod_{\alpha>0, w^{-1}\alpha<0} z_\alpha(\nu, \mu, \lambda). \tag{1}$$

A calculation similar to the one in A.10 shows for all $w \in W$ and $\beta \in R^+$ with $w^{-1}\beta$ simple

$$\zeta_w(\nu, \mu, \lambda)^{-1}\zeta_w(s_\beta{\bullet}\nu, s_\beta{\bullet}\mu, s_\beta{\bullet}\lambda) = \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ w^{-1}\alpha<0}} \zeta_\alpha^{-1} \prod_{\substack{\alpha>0, s_\beta\alpha<0,\\ \alpha\neq\beta, w^{-1}\alpha>0}} \zeta_\alpha \tag{2}$$

where we use the abbreviation $\zeta_\alpha = \zeta_\alpha(\nu, \mu, \lambda)$.

If we imitate the calculations in A.9, we get

$$z_{s_\beta}(w{\bullet}\nu, w{\bullet}\mu, w{\bullet}\lambda) = z_{(s_\beta w)^{-1}}(\nu, \mu, \lambda)z_{w^{-1}}(\nu, \mu, \lambda). \tag{3}$$

Appendix B

In this appendix we shall use some results from Lusztig's new book [Lu10] to choose extremal weight vectors in the finite dimensional simple modules for quantum groups and to give explicit formulae for the action of the braid group on these vectors. (In the simply laced case these formulae can also be obtained from [Lu9], Section 5.)

B.1. As in 1.3 we let U_1 denote the quantized enveloping algebra corresponding to our root system R. Recall that we agreed in 1.3 to choose the braid group operator T_{s_α} on U_1 in accordance with the convention in [Lu7]. (In [Lu10], Chapter 37 this operator is denoted $T''_{i,1}$ where i corresponds to α.)

Let M be an integrable U_1–module. For each simple root α Lusztig introduces (in [Lu10], 5.2.1) an operator $P(s_\alpha) : M \to M$ by

$$P(s_\alpha)x = \sum_{a,b,c;a-b+c=\langle\lambda,\alpha^\vee\rangle} (-1)^b v^{d_\alpha(ac-b)} F_\alpha^{(a)} E_\alpha^{(b)} F_\alpha^{(c)}$$

for all $x \in M_\lambda$. (In [Lu10] this operator is denoted by $T'_{i,-1}$.) We then have the following formula, see [Lu10], Proposition 37.1.2

$$P(s_\alpha)(T_{s_\alpha}(u)x) = uP(s_\alpha)(x) \qquad \text{for all } u \in U_1,\ x \in M. \tag{1}$$

Moreover, we have by [Lu10], Proposition 5.2.2: If $x \in M_\nu$ satisfies $E_\alpha x = 0$, then $n = \langle \nu, \alpha^\vee\rangle \geq 0$ and

$$P(s_\alpha)x = F_\alpha^{(n)}x, \tag{2}$$

and

$$P(s_\alpha)F_\alpha^{(n)}x = (-1)^n v^{-d_\alpha n}x. \tag{3}$$

B.2. According to [Lu10], Theorem 39.4.3, the $P(s_\alpha)$ satisfy the braid relations. So we can define for each $w \in W$ with reduced decomposition $w = s_1 s_2 \cdots s_n$ the operator

$$P(w) = P(s_1)P(s_2)\cdots P(s_n)$$

that is independent of the choice of decomposition. We have of course $P(1) =$ id, and B.1(1) implies

$$P(w)(T_{w^{-1}}(u)x) = u\, P(w)x \qquad \text{for all } x \in M \text{ and } u \in U_1. \tag{1}$$

Suppose from now on that M is simple with highest weight ω. Choose a highest weight vector x_1 and set

$$x_w = P(w)x_1 \tag{2}$$

for each $w \in W$.

Lemma: *Each x_w has weight $w\omega$. One has for all $w \in W$ and all simple roots α:*

$$P(s_\alpha)x_w = \begin{cases} x_{s_\alpha w}, & \text{if } w^{-1}\alpha > 0; \\ (-1)^{\langle w\omega,\alpha^\vee\rangle} v^{d_\alpha \langle w\omega,\alpha^\vee\rangle} x_{s_\alpha w}, & \text{if } w^{-1}\alpha < 0, \end{cases} \tag{3}$$

and

$$F_\alpha^{(\langle w\omega,\alpha^\vee\rangle)} x_w = x_{s_\alpha w} \qquad \text{if } w^{-1}\alpha > 0, \tag{4}$$

and

$$E_\alpha^{(-\langle w\omega,\alpha^\vee\rangle)} x_w = x_{s_\alpha w} \qquad \text{if } w^{-1}\alpha < 0. \tag{5}$$

Proof: We get the first part by induction on $l(w)$. If the claim holds for a given w and if α is simple with $w^{-1}\alpha < 0$, then $n = \langle w\omega, \alpha^\vee\rangle \geq 0$ and

$$x_{s_\alpha w} = P(s_\alpha w)x_1 = P(s_\alpha)P(w)x_1 = P(s_\alpha)x_w = F_\alpha^{(n)} x_w$$

using the induction hypothesis, and B.1(2) for the last step. So $x_{s_\alpha w}$ has indeed weight $s_\alpha w\omega$ and we have also established (4) and the first case in (3). On the other hand, if $w^{-1}\alpha < 0$, then $-n = \langle w\omega, \alpha^\vee\rangle \leq 0$ and

$$P(s_\alpha)x_w = P(s_\alpha)F_\alpha^{(n)} x_{s_\alpha w} = (-1)^n v^{-nd_\alpha} x_{s_\alpha w}$$

by B.1(3). This yields the second part of (3). Since $E_\alpha^{(n)} F_\alpha^{(n)} x_{s_\alpha w} = x_{s_\alpha w}$, also (5) follows.

B.3. Proposition: *We have for all $w, y \in W$:*

$$P(y)x_w = c(y,w)x_{yw}$$

where

$$c(y,w) = \prod_{\substack{\gamma>0, y\gamma<0, \\ w^{-1}\gamma<0}} (-v^{d_\gamma})^{\langle w\omega,\gamma^\vee\rangle}. \tag{1}$$

Proof: We use induction on the length of y. We have obviously $c(1,w) = 1$ for all $w \in W$, and B.2(3) implies for any simple root α that

$$c(s_\alpha, w) = \begin{cases} (-v^{d_\alpha})^{\langle w\omega, \alpha^\vee\rangle}, & \text{if } w^{-1}\alpha < 0; \\ 1, & \text{if } w^{-1}\alpha > 0. \end{cases} \tag{2}$$

For $y \neq 1$ there is a simple root α such that $s_\alpha y$ is shorter that y. Since obviously

$$c(y,w) = c(s_\alpha, s_\alpha y w) c(s_\alpha y, w),$$

induction and (2) yield the claim.

Remark: Using the notation from A.1, we can rewrite (1) as

$$c(y,w) = \prod_{\gamma>0, y\gamma<0} (-v^{d_\gamma})^{\langle w\omega, \gamma^\vee\rangle \overline{\gamma}(w\omega)}. \tag{3}$$

B.4. Lemma: *Let $y, w \in W$ and α simple with $m = \langle -w\omega, y(\alpha)^\vee\rangle \geq 0$. Set $\beta = y\alpha$. Then*

$$T_y(E_\alpha^{(m)}) x_w = c(y^{-1}, w) c(y^{-1}, s_\beta w)^{-1} x_{s_\beta w}.$$

Proof: We have

$$\begin{aligned} T_y(E_\alpha^{(m)}) x_w &= P(y^{-1})^{-1}(E_\alpha^{(m)} P(y^{-1}) x_w) \qquad \text{— by B.2(1) —} \\ &= c(y^{-1}, w) P(y^{-1})^{-1}(E_\alpha^{(m)} x_{y^{-1}w}) \\ &= c(y^{-1}, w) P(y^{-1})^{-1} x_{s_\alpha y^{-1} w} \qquad \text{— by B.2(5) —} \\ &= c(y^{-1}, w) c(y^{-1}, y s_\alpha y^{-1} w)^{-1} x_{y s_\alpha y^{-1} w} \end{aligned}$$

using Proposition B.3 twice.

B.5. For all $\nu \in X$, $\beta \in R$ and $y \in W$ set

$$\varepsilon(\nu, \beta, y) = \prod_{\gamma>0, y^{-1}\gamma<0} (-1)^{\langle \nu, \gamma^\vee\rangle \overline{\gamma}(\nu) - \langle s_\beta \nu, \gamma^\vee\rangle \overline{\gamma}(s_\beta \nu)} \tag{1}$$

and

$$v(\nu, \beta, y) = \prod_{\gamma>0, y^{-1}\gamma<0} v^{d_\gamma\left(\langle \nu, \gamma^\vee\rangle \overline{\gamma}(\nu) - \langle s_\beta \nu, \gamma^\vee\rangle \overline{\gamma}(s_\beta \nu)\right)}. \tag{2}$$

We have obviously:

$$\langle \nu, \beta^\vee\rangle = 0 \implies \varepsilon(\nu, \beta, y) = v(\nu, \beta, y) = 1. \tag{3}$$

We have also

$$\varepsilon(s_\beta \nu, \beta, y) = \varepsilon(\nu, \beta, y) \tag{4}$$

and

$$v(s_\beta \nu, \beta, y) = v(\nu, \beta, y)^{-1}. \tag{5}$$

Indeed, if we plug $s_\beta \nu$ into the definitions (1), (2), the exponents get multiplied by -1.

B.6. Proposition: *Let $\beta \in R^+$. Suppose that we have $y \in W$ with $\alpha = y^{-1}\beta$ simple and that we set $E_\beta = T_y(E_\alpha)$ and $E_{-\beta} = T_y(F_\alpha)$. Let $w \in W$. If $w^{-1}\beta < 0$, then*

$$E_\beta^{(\langle -w\omega,\beta^\vee\rangle)} x_w = \varepsilon(w\omega,\beta,y)v(w\omega,\beta,y)x_{s_\beta w}, \tag{1}$$

If $w^{-1}\beta > 0$, then

$$E_{-\beta}^{(\langle w\omega,\beta^\vee\rangle)} x_w = \varepsilon(w\omega,\beta,y)v(w\omega,\beta,y)x_{s_\beta w}. \tag{2}$$

Proof: (1) is an immediate consequence of Lemma B.4 and B.3(3). We get (2) from (1) using B.5(4), (5) and the fact that $E_\beta^{(m)}E_{-\beta}^{(m)}x_w = x_w$ where $m = \langle w\omega, \beta^\vee\rangle$.

Remark: If $\langle w\omega,\beta^\vee\rangle = 0$, then we get $x_{s_\beta w} = x_w$. More generally, if $w\omega = v\omega$, then $x_w = x_v$.

B.7. Let $\beta \in R^+$ and $y \in W$ with $\alpha = y^{-1}\beta$ simple. Consider the definition B.5(2); we get for all $\nu \in X$

$$v(\nu,\beta,y) = \prod_{\gamma>0,y^{-1}\gamma<0} v^{d_\gamma\langle\nu,\gamma^\vee\rangle\overline{\gamma}(\nu)} \prod_{s_\beta\gamma>0,y^{-1}s_\beta\gamma<0} v^{-d_\gamma\langle\nu,\gamma^\vee\rangle\overline{\gamma}(\nu)}$$

by substituting $s_\beta\gamma$ for γ in the second product. Since $y^{-1}s_\beta = s_\alpha y^{-1}$ with α simple, we see that $y^{-1}s_\beta\gamma < 0$ if and only if $y^{-1}\gamma < 0$ as long as $\gamma \neq \pm\beta$. We can therefore replace the condition in the last product by

$$s_\beta\gamma > 0,\ y^{-1}\gamma < 0,\ \gamma \neq -\beta.$$

Now the terms with $s_\beta\gamma > 0$ in the first product and the terms with $\gamma > 0$ in the second product cancel. We get

$$v(\nu,\beta,y) = \prod_{\substack{\gamma>0,s_\beta\gamma<0,\\ y^{-1}\gamma<0}} v^{d_\gamma\langle\nu,\gamma^\vee\rangle\overline{\gamma}(\nu)} \prod_{\substack{\gamma<0,s_\beta\gamma>0,\\ y^{-1}\gamma<0,\gamma\neq-\beta}} v^{-d_\gamma\langle\nu,\gamma^\vee\rangle\overline{\gamma}(\nu)}.$$

We substitute $-\gamma$ for γ in the second product and get

$$v(\nu,\beta,y) = \prod_{\substack{\gamma>0,s_\beta\gamma<0,\\ y^{-1}\gamma<0}} v^{d_\gamma\langle\nu,\gamma^\vee\rangle\overline{\gamma}(\nu)} \prod_{\substack{\gamma>0,s_\beta\gamma<0,\\ \gamma\neq\beta,y^{-1}\gamma>0}} v^{d_\gamma\langle\nu,\gamma^\vee\rangle\overline{\gamma}(-\nu)}. \tag{1}$$

Similarly, one gets

$$\varepsilon(\nu,\beta,y) = \prod_{\substack{\gamma>0, s_\beta\gamma<0,\\ y^{-1}\gamma<0}} (-1)^{\langle\nu,\gamma^\vee\rangle\overline{\gamma}(\nu)} \prod_{\substack{\gamma>0, s_\beta\gamma<0,\\ \gamma\neq\beta, y^{-1}\gamma>0}} (-1)^{\langle\nu,\gamma^\vee\rangle\overline{\gamma}(-\nu)}. \tag{2}$$

Using A.1(5) one can rewrite this as

$$\varepsilon(\nu,\beta,y) = \prod_{\substack{\gamma>0, s_\beta\gamma<0,\\ \gamma\neq\beta}} (-1)^{\langle\nu,\gamma^\vee\rangle\overline{\gamma}(\nu)} \prod_{\substack{\gamma>0, s_\beta\gamma<0,\\ \gamma\neq\beta, y^{-1}\gamma>0}} (-1)^{\langle\nu,\gamma^\vee\rangle}. \tag{3}$$

B.8. We can now move away from $\mathbf{Q}(v)$. Since both $P(w)$ and its inverse involve only divided powers of the E_i and F_i as well as ordinary powers of the K_i and v, they preserve Lusztig's lattice over $\mathbf{Z}[v, v^{-1}]$. This implies that the x_w are also bases of their weight spaces in this lattice. If we now specialize v, say to a root of unity (such as 1), we get the action of the appropriate divided power of $T_y(E_\alpha)$ on the extremal vectors in the specialized module.

Appendix C

In this appendix we want to prove a certain property of coproducts $\Delta(E_\beta)$ with $\beta \in R$. For $\pm\beta$ simple this coproduct is part of the definition of Δ, cf. 7.1. For arbitrary β the formulas get more complicated and there is no closed formula in general. We shall use the approach from [LS] to these coproducts, cf. [LS], Proposition 2.4.1. However, they introduce the Hecke operators T_i in a different way from Lusztig (in [Lu3], 5.1 and [Lu5], 1.2). Therefore we include for the benefit of the reader a self contained proof of the crucial step (Proposition C.4). This proposition can also be found in [Lu10], see Proposition 37.3.2. (Note that our T_α corresponds to Lusztig's $T''_{i,1}$ and that our R_α is obtained from his $\mathbf{L}''_i$ by applying $T_\alpha \otimes T_\alpha$.) One may also compare the approach in [KR], Section 7, that is similar to that in [LS].

C.1. Set $\mathcal{A} = \mathbf{Q}[v, v^{-1}]$ and let $U_\mathcal{A}$ be Lusztig's $\mathcal{A}$–form of the quantized enveloping algebra U_1 over $\mathbf{Q}(v)$ (generated by all $E_\alpha^{(n)}$, $F_\alpha^{(n)}$ and $K_\alpha^{\pm n}$ with $\alpha \in \Sigma$ and $n \in \mathbf{Z}$, $n \geq 0$). It is a free module over $\mathcal{A}$ with a PBW-type basis. All T_w induce automorphisms of $U_\mathcal{A}$, and Δ induces a comultiplication $U_\mathcal{A} \to U_\mathcal{A} \otimes_\mathcal{A} U_\mathcal{A}$.

Let $\mathcal{A}'$ be the local ring of $\mathcal{A}$ at the maximal ideal generated by $v-1$. Set $U_{\mathcal{A}'} = U_\mathcal{A} \otimes_\mathcal{A} \mathcal{A}'$; the T_w and Δ extend to $U_{\mathcal{A}'}$.

In the subsections C.2 – C.6 we shall work with $U_{\mathcal{A}'}$. In C.7 we shall describe how to make the transition to the algebra U that we consider in Case 2.

C.2. Fix (until C.4) a simple root α. We shall write T_α for T_{s_α}. Set

$$R_\alpha = \sum_{n=0}^{\infty} a_n K_\alpha^{-n} E_\alpha^{(n)} \otimes F_\alpha^{(n)} K_\alpha^n \tag{1}$$

where

$$a_n = a_n(\alpha) = (-1)^n v^{-d_\alpha n(n-1)/2} (v^{d_\alpha} - v^{-d_\alpha})^n [n]^!_{d_\alpha}. \tag{2}$$

Here R_α is an element of the completion of the $\mathcal{A}'$–algebra $U_{\mathcal{A}'} \otimes U_{\mathcal{A}'}$ with respect to the maximal ideal $\mathcal{A}'(v-1)$ of $\mathcal{A}'$. Note that a_n is contained in the

n–th power of that ideal. We have the following iteration formula for the a_n:

$$a_{n+1} = -v^{-d_\alpha n}(v^{d_\alpha} - v^{-d_\alpha})[n+1]_{d_\alpha} a_n; \tag{3}$$

obviously $a_0 = 1$. Note that we can rewrite (1) as

$$R_\alpha = \sum_{n=0}^{\infty} a_n T_\alpha(F_\alpha^{(n)}) \otimes T_\alpha(E_\alpha^{(n)}) \tag{4}$$

since $T_\alpha(E_\alpha) = -F_\alpha K_\alpha$ and $T_\alpha(F_\alpha) = -K_\alpha^{-1} E_\alpha$ and

$$(F_\alpha K_\alpha)^n = v^{-n(n-1)d_\alpha} F_\alpha^n K_\alpha^n \qquad \text{and} \qquad (K_\alpha^{-1} E_\alpha)^n = v^{n(n-1)d_\alpha} K_\alpha^{-n} E_\alpha^n.$$

C.3. An elementary calculation shows that R_α has an inverse R_α^{-1} in the completion of $U_{\mathcal{A}'} \otimes U_{\mathcal{A}'}$ given by

$$R_\alpha^{-1} = \sum_{n=0}^{\infty} \overline{a_n} K_\alpha^{-n} E_\alpha^{(n)} \otimes F_\alpha^{(n)} K_\alpha^n \tag{1}$$

where $\overline{a_n}$ is the image of a_n under the automorphism of A over $\mathbf{Q}$ that maps v to v^{-1}, i.e., we have

$$\overline{a_n} = \overline{a_n(\alpha)} = v^{d_\alpha n(n-1)/2}(v^{d_\alpha} - v^{-d_\alpha})^n [n]_{d_\alpha}^! . \tag{2}$$

There is an involutory antiautomorphism Ω of $U_{\mathcal{A}'}$ that maps E_β to F_β and K_β to K_β^{-1} (for all simple roots β) and v to v^{-1}, cf. [Lu7], 1.1. It satisfies

$$\Delta \circ \Omega = (\Omega \otimes \Omega) \circ \sigma \circ \Delta \tag{3}$$

where σ is the automorphism with $\sigma(a \otimes b) = b \otimes a$. Comparing (1) to C.2(4) one checks easily that

$$R_\alpha^{-1} = (\Omega \otimes \Omega) \circ \sigma(R_\alpha). \tag{4}$$

C.4. Proposition: *We have* $\Delta(T_\alpha(x)) = R_\alpha^{-1}(T_\alpha \otimes T_\alpha)\Delta(x) R_\alpha$ *for all* $x \in U_{\mathcal{A}'}$.

Proof: Let us first show: If the claim holds for x, then it holds for $\Omega(x)$. By [Lu7], 3.1, Ω commutes with T_α. So we get using C.3(3), (4)

$$\begin{aligned}
\Delta\, T_\alpha\, \Omega(x) &= \Delta\, \Omega\, T_\alpha(x) = (\Omega \otimes \Omega)\, \sigma\, \Delta\, T_\alpha(x) \\
&= (\Omega \otimes \Omega)\, \sigma(R_\alpha^{-1} \cdot (T_\alpha \otimes T_\alpha)\Delta(x) \cdot R_\alpha) \\
&= (\Omega \otimes \Omega)\big(\sigma(R_\alpha^{-1}) \cdot (T_\alpha \otimes T_\alpha)\sigma\, \Delta(x) \cdot \sigma(R_\alpha)\big) \\
&= (\Omega \otimes \Omega)\sigma(R_\alpha) \cdot (\Omega \otimes \Omega)(T_\alpha \otimes T_\alpha)\sigma\, \Delta(x) \cdot (\Omega \otimes \Omega)\sigma(R_\alpha^{-1}) \\
&\qquad\qquad (\text{since } \Omega \otimes \Omega \text{ is an antiautomorphism})
\end{aligned}$$

$$
\begin{aligned}
&= R_\alpha^{-1} \cdot (T_\alpha \otimes T_\alpha)(\Omega \otimes \Omega)\sigma\, \Delta(x) \cdot R_\alpha \\
&= R_\alpha^{-1} \cdot (T_\alpha \otimes T_\alpha)\Delta\, \Omega(x) \cdot R_\alpha
\end{aligned}
$$

It is obviously enough to prove the claim for generators of $U_{\mathcal{A}'}$. For $x = K_\beta$ (with $\beta \in R$) we have $\Delta(x) = x \otimes x$. Since clearly R_α commutes with $x \otimes x$ the claim follows easily for $x = K_\beta$. It is therefore enough to look at $x = E_\beta$ with $\pm\beta$ a simple root. Since $\Omega(E_\beta) = E_{-\beta}$ the argument above shows that we can assume that β is a simple root. We have now to check that

$$
R_\alpha \cdot \Delta T_\alpha(E_\beta) = (T_\alpha \otimes T_\alpha)(E_\beta \otimes 1 + K_\beta \otimes E_\beta) \cdot R_\alpha. \tag{1}
$$

Case 1: $\langle \beta, \alpha^\vee \rangle = 0$. In this case $T_\alpha(E_\beta) = E_\beta$ and also $T_\alpha(K_\beta) = K_\beta$ so that the claim comes down to checking that R_α commutes with $E_\beta \otimes 1 + K_\beta \otimes E_\beta$. This is clear since $\langle \beta, \alpha^\vee \rangle = 0$.

Case 2: $\langle \beta, \alpha^\vee \rangle = 2$, i.e., $\alpha = \beta$. Let us drop the subscripts α and also let us pretend that $d = d_\alpha = 1$ (i.e. we replace v^d by v). We have $T(E) = -FK$ and $\Delta(T(E)) = -(FK \otimes 1 + K \otimes FK)$. Also, $(T \otimes T)\Delta(E) = -(FK \otimes 1 + K^{-1} \otimes FK)$, so that the relation we have to check is

$$
R(FK \otimes 1 + K \otimes FK) = (FK \otimes 1 + K^{-1} \otimes FK)R. \tag{2}
$$

We compute

$$
\begin{aligned}
K^{-n}E^{(n)}FK &= K^{-n}(FE^{(n)} + [K; 1-n]E^{(n-1)})K \\
&= FK^{-n+1}E^{(n)} + \frac{K^{-n+2}v^{3(1-n)} - K^{-n}v^{1-n}}{v - v^{-1}} E^{(n-1)},
\end{aligned}
$$

and

$$
(K^{-n}E^{(n)} \otimes F^{(n)}K^n)(K \otimes FK) = v^{-4n}[n+1]K^{-n+1}E^{(n)} \otimes F^{(n+1)}K^{n+1}.
$$

So for the left hand side in (2) we get the following contributions

$$
a_n FK^{1-n}E^{(n)} \otimes F^{(n)}K^n,
$$

$$
\big(a_{n+1}\frac{v^{-3n}}{v - v^{-1}} + a_n v^{-4n}[n+1]\big)K^{1-n}E^{(n)} \otimes F^{(n+1)}K^{n+1},
$$

$$
-a_{n+1}\frac{v^{-n}}{v - v^{-1}}K^{-n-1}E^{(n)} \otimes F^{(n+1)}K^{n+1}.
$$

Note that by C.2(3) the middle term is zero. For the right hand side in (2) we compute

$$
(FK \otimes 1)(K^{-n}E^{(n)} \otimes F^{(n)}K^n) = FK^{-n+1}E^{(n)} \otimes F^{(n)}K^n
$$

and

$$(K^{-1} \otimes FK)(K^{-n}E^{(n)} \otimes F^{(n)}K^n) = v^{-2n}[n+1]K^{-n-1}E^{(n)} \otimes F^{(n+1)}K^{n+1}.$$

We now compare the contributions from this with the above; using C.2(3) we discover that the terms do indeed match up nicely.

Case 3: $\langle \beta, \alpha^\vee \rangle = -1$. Again we drop the subscript α and we shall write $E' = E_\beta$ and similarly for F' and K'. We have $d = d_\alpha = -\langle \alpha, \beta^\vee \rangle$ and $d_\beta = 1$. Let us also set $E_0 = T(E')$ and $K_0 = T(K') = K'K$. Then $E_0 = -EE' + v^{-d}E'E$ and

$$\Delta(E_0) = E_0 \otimes 1 + K_0 \otimes E_0 - (v^d - v^{-d})K'E \otimes E'.$$

Moreover,

$$(T \otimes T)\Delta(E') = E_0 \otimes 1 + K_0 \otimes E_0.$$

It is easy to check that R commutes with $E_0 \otimes 1$. (Use C.2(4) and observe that $T(F^{(n)})$ clearly commutes with $T(E')$.) So the claim is equivalent to the relation

$$R\left(K_0 \otimes E_0 - (v^d - v^{-d})K'E \otimes E'\right) = (K_0 \otimes E_0)R. \tag{3}$$

For the left hand side we observe

$$(K^{-n}E^{(n)} \otimes F^{(n)}K^n)(K_0 \otimes E_0) = v^{-dn}K^{1-n}K'E^{(n)} \otimes F^{(n)}K^nE_0$$

and

$$(K^{-n}E^{(n)} \otimes F^{(n)}K^n)(K'E \otimes E') = v^{dn}[n+1]_dK^{-n}K'E^{(n+1)} \otimes F^{(n)}K^nE'.$$

For the right hand side we need the following commutation formula between F and E_0:

$$E_0F^{(n)} = F^{(n)}E_0 + v^{d(n-2)}F^{(n-1)}K^{-1}E'. \tag{4}$$

This gives

$$\begin{aligned}(K_0 \otimes E_0)(K^{-n}E^{(n)} \otimes F^{(n)}K^n) = v^{-dn}&K^{1-n}K'E^{(n)} \otimes F^{(n)}K^nE_0\\ &+ v^{2d(n-1)}K^{1-n}K'E^{(n)} \otimes F^{(n-1)}K^{n-1}E'\end{aligned}$$

and comparing these contributions we again find that they match. (Recall C.2(3).)

Case 4: $\langle \beta, \alpha^\vee \rangle = -2$. We shall use the same abbreviations as in Case 3 except that have now $d = d_\alpha = 1$ and $d_\beta = 2$.
In this case we have

$$E_0 = E^{(2)}E' - v^{-1}EE'E + v^{-2}E'E^{(2)} \quad \text{and} \quad K_0 = K^2K'.$$

Hence

$$\begin{aligned}\Delta(E_0) = (&E^{(2)} \otimes 1 + vEK \otimes E + K^2 \otimes E^{(2)})(E' \otimes 1 + K' \otimes E') \\ &- v^{-1}(E \otimes 1 + K \otimes E)(E' \otimes 1 + K' \otimes E')(E \otimes 1 + K \otimes E) \\ &+ v^{-2}(E' \otimes 1 + K' \otimes E')(E^{(2)} \otimes 1 + vEK \otimes E + K^2 \otimes E^{(2)}).\end{aligned}$$

When we combine the product of the first two terms in the first product with the product of the three first terms in the second product and with the product of the two first terms in the third product, we get $E_0 \otimes 1$. Likewise the combination of the last terms in the 3 products give $K_0 \otimes E_0$. The following terms cancel out: the combination $(3\times 1, 2\times 1\times 2, 1\times 3)$ (this means the product of the third and first term in the first product together with the product of the second, the first and the second terms in the second product etc.), and the combination $(2 \times 1, 1 \times 1 \times 2 + 2 \times 1 \times 1, 1 \times 2)$. From $(-, 1 \times 2 \times 2, 2 \times 2)$ we get the contribution

$$v^{-3}(1 - v^2)K'KE \otimes E'E$$

and from $(2 \times 2, 2 \times 2 \times 1, -)$

$$v^{-1}(v^2 - 1)KK'E \otimes EE'.$$

So this combines to

$$v^{-1}(v^2 - 1)KK'E \otimes (EE' - v^{-2}E'E).$$

Finally from $(1 \times 2, 1 \times 2 \times 1, 2 \times 1)$ we get

$$(v^2 - 1)(v^2 - v^{-2})K'E^{(2)} \otimes E' = v(v - v^{-1})^2[2]K'E^{(2)} \otimes E',$$

so altogether we find that $\Delta(E_0)$ is equal to

$$E_0 \otimes 1 + K_0 \otimes E_0 + (v - v^{-1})KK'E \otimes (EE' - v^{-2}E'E) + v(v - v^{-1})^2[2]K'E^{(2)} \otimes E'.$$

We also need to see how E_0 and $F^{(n)}$ commute:

$$E_0F^{(n)} = E^{(2)}F^{(n)}E' - v^{-1}EE'EF^{(n)} + v^{-2}E'E^{(2)}F^{(n)} = a + b + c,$$

where

$$a = F^{(n)}E^{(2)}E' + F^{(n-1)}[K; -n]EE' + F^{(n-2)}\begin{bmatrix} K; 2-n \\ 2 \end{bmatrix} E',$$

$$\begin{aligned} b &= -v^{-1}EE'(F^{(n)}E + F^{(n-1)}[K;1-n]) \\ &= -v^{-1}(F^{(n)}EE'E + F^{(n-1)}[K;1-n]E'E + F^{(n-1)}[K;1-n]EE' \\ &\qquad + F^{(n-2)}[K;2-n][K;3-n]E') \end{aligned}$$

and

$$c = v^{-2}(F^{(n)}E'E^{(2)} + F^{(n-1)}[K;2-n]E'E + F^{(n-2)}\begin{bmatrix} K;4-n \\ 2 \end{bmatrix} E').$$

So we see that $E_0F^{(n)}$ is equal to

$$\begin{aligned} &F^{(n)}E_0 + F^{(n-1)}([K;-n] - v^{-1}[K;1-n])EE' \\ &\quad - F^{(n-1)}(v^{-1}[K;1-n] - v^{-2}[K;2-n])E'E \\ &\quad + F^{(n-2)}(\begin{bmatrix} K;2-n \\ 2 \end{bmatrix} - v^{-1}[K;2-n][K;3-n] + v^{-2}\begin{bmatrix} K;4-n \\ 2 \end{bmatrix})E' \\ &\quad = F^{(n)}E_0 - v^{n-1}F^{(n-1)}K^{-1}(EE' - v^{-2}E'E) + v^{2n-6}F^{(n-2)}K^{-2}E'. \end{aligned}$$

Again it is easy to check that R commutes with $E_0 \otimes 1$. The above formula for $\Delta(E_0)$ shows that our claim reduces to show that

$$R(K_0 \otimes E_0 + (v-v^{-1})KK'E \otimes (EE' - v^{-2}E'E) + v(v-v^{-1})^2[2]K'E^{(2)} \otimes E') \quad (5)$$

is equal to $(K_0 \otimes E_0)R$. We use the above formula for $E_0F^{(n)}$ to compute $(K_0 \otimes E_0)R$ and get the following contributions

$$\begin{aligned} a_nK^{2-n}K'E^{(n)} \otimes (v^{-2n}F^{(n)}K^nE_0 - v^{n-1}F^{(n-1)}K^{n-1}(EE' - v^{-2}E'E) \\ + v^{4n-6}F^{(n-2)}K^{n-2}). \end{aligned}$$

In (5) we get

$$\begin{aligned} a_n(&v^{-2n}K^{2-n}K'E^{(n)} \otimes F^{(n)}K^nE_0 \\ &+ (v-v^{-1})[n+1]K^{-n+1}K'E^{(n+1)} \otimes F^{(n)}K^n(EE' - v^{-2}E'E) \\ &+ v(v-v^{-1})^2[2]v^{2n}\frac{[n+1][n+2]}{[2]}K^{-n}K'E^{(n+2)} \otimes F^{(n)}K^nE'). \end{aligned}$$

So the terms match if and only if a_n satisfies

$$-a_nv^{n-1} = a_{n-1}(v-v^{-1})[n]$$

and

$$a_nv^{4n-6} = a_{n-2}(v-v^{-1})^2v^{2n-3}[n-1][n].$$

But these relations are indeed satisfied by our a_n, cf. C.2(3).

Case 5: $\langle\beta,\alpha^\vee\rangle = -3$. In this case $d_\alpha = 1$ and $d_\beta = 3$. The proof proceeds along the same lines as in the last case. We shall write down only the crucial formulas and leave the rest to the reader. We shall use the same abbreviations as before. We have

$$E_0 = T(E') = v^{-3}E'E^{(3)} - v^{-2}EE'E^{(2)} + v^{-1}E^{(2)}E'E - E^{(3)}E'$$

and $K_0 = K^3K'$ and

$$\begin{aligned}\Delta(E_0) = E_0 \otimes 1 + K_0 \otimes E_0 \\ &- (v-v^{-1})K^2K'E \otimes (E^{(2)}E' - v^{-2}EE'E + v^{-4}E'E^{(2)}) \\ &- v(v-v^{-1})^2[2]KK'E^{(2)} \otimes (EE' - v^{-3}E'E) \\ &- v^3(v-v^{-1})^3[2][3]K'E^{(3)} \otimes E'\end{aligned}$$

and

$$\begin{aligned}E_0F^{(n)} = F^{(n)}E_0 &+ v^nF^{(n-1)}K^{-1}(E^{(2)}E' - v^{-2}EE'E + v^{-4}E'E^{(2)}) \\ &- v^{2n-4}F^{(n-2)}K^{-2}(EE' - v^{-3}E'E) \\ &+ v^{3n-12}F^{(n-3)}K^{-3}E'.\end{aligned}$$

C.5. We shall say that $x \in U_{\mathcal{A}'} \otimes U_{\mathcal{A}'}$ has weight (μ,ν) if it is contained in $(U_{\mathcal{A}'})_\mu \otimes (U_{\mathcal{A}'})_\nu$.

Proposition: *Let α be a simple root and $w \in W$ with $\beta = w\alpha > 0$. Set $E_\beta = T_wE_\alpha$. There are weights γ_j with $0 < \gamma_j < \beta$ and $w^{-1}\gamma_j < 0$ and $w^{-1}(\beta-\gamma_j) > 0$ for all j such that*

$$\Delta(E_\beta) = E_\beta \otimes 1 + K_\beta \otimes E_\beta + \sum_j x_j \tag{1}$$

with $x_j \in U_{\mathcal{A}'} \otimes U_{\mathcal{A}'}$ of weight $(\gamma_j, \beta-\gamma_j)$.

Proof: Let $w = s_1s_2\cdots s_r$ be a reduced decomposition of w with $s_i = s_{\alpha_i}$ where α_i is a simple root (for $1 \le i \le r$). Set $S_i = T_{\alpha_i} \otimes T_{\alpha_i}$ and $R_i = R_{\alpha_i}$ for all i. Proposition C.4 applied r times yields

$$\Delta(T_wE_\alpha) = R^{-1}\cdot(T_w \otimes T_w)\Delta(E_\alpha)\cdot R = R^{-1}\cdot(E_\beta \otimes 1 + K_\beta \otimes E_\beta)\cdot R \tag{2}$$

where

$$R = (S_1S_2\cdots S_{r-1})(R_r)\cdots(S_1S_2)(R_3)\cdot S_1(R_2)\cdot R_1. \tag{3}$$

Set $\beta_i = s_1s_2\cdots s_{i-1}\alpha_i$ for $1 \le i \le r$. Then the β_i are exactly the positive roots γ with $w^{-1}\gamma < 0$. The definition of the R_α implies that each

$(S_1 S_2 \cdots S_{i-1})(R_i)$ and each $(S_1 S_2 \cdots S_{i-1})(R_i)^{-1}$ has the form $1\otimes 1+\sum_{n=1}^{\infty} y_n$ with y_n of weight $(n\beta_i, -n\beta_i)$. If we now evaluate (2) we get $E_\beta \otimes 1 + K_\beta \otimes E_\beta$ when we multiply all the $1\otimes 1$ terms. Besides that we get from $E_\beta \otimes 1$ elements of weight $(\beta + \sum_{i=1}^r n_i\beta_i, -\sum_{i=1}^r n_i\beta_i)$ with all $n_i \geq 0$ and at least one $n_i > 0$. Similarly, we get from $K_\beta \otimes E_\beta$ elements of weight $(\sum_{i=1}^r n_i\beta_i, \beta - \sum_{i=1}^r n_i\beta_i)$ with all $n_i \geq 0$ and at least one $n_i > 0$. However, $\Delta(E_\beta)$ is contained in $U^0_{\mathcal{A}'} U^+_{\mathcal{A}'} \otimes U^+_{\mathcal{A}'}$, so after cancellations only terms of weight $(\gamma, \beta - \gamma)$ with $0 \leq \gamma$ and $0 \leq \beta - \gamma$ can survive. This shows that all contributions from $E_\beta \otimes 1$ will have to cancel and that only contributions from $K_\beta \otimes E_\beta$ with $0 < \gamma = \sum_{i=1}^r n_i\beta_i \leq \beta$ can survive. We have then $w^{-1}\gamma = \sum_{i=1}^r n_i w^{-1}\beta_i < 0$, whereas $w^{-1}\beta = \alpha > 0$. This shows also that $\gamma < \beta$. Finally we have $w^{-1}(\beta - \gamma) = \alpha - w^{-1}\gamma > 0$.

C.6. Corollary: *Let α be a simple root and $w \in W$ with $\beta = w\alpha > 0$. Set $E_{-\beta} = T_w E_{-\alpha}$. Then there are weights γ'_j with $-\beta < \gamma'_j < 0$ and $w^{-1}\gamma'_j < 0$ and $w^{-1}(-\beta - \gamma'_j) > 0$ for all j such that*

$$\Delta(E_{-\beta}) = E_{-\beta} \otimes K_\beta^{-1} + 1 \otimes E_{-\beta} + \sum_j y_j \tag{1}$$

where $y_j \in U_{\mathcal{A}'} \otimes U_{\mathcal{A}'}$ has weight $(\gamma'_j, -\beta - \gamma'_j)$.

Proof: We get this from C.5 using the involutory antiautomorphism Ω as in C.3. Besides C.3(3) we have to use that Ω maps a weight vector of weight γ to a weight vector of weight $-\gamma$ and that it commutes with T_w.

C.7. It is clear that C.5(1) and C.6(1) extend to $U_{\mathcal{A}} \subset U_{\mathcal{A}'}$ since $U_{\mathcal{A}} \otimes U_{\mathcal{A}} \subset U_{\mathcal{A}'} \otimes U_{\mathcal{A}'}$ and since Δ and the T_w on $U_{\mathcal{A}}$ are the restrictions of the analogous maps over $U_{\mathcal{A}'}$. It then follows that C.5(1) and C.6(1) hold also in $U_3 = U_{\mathcal{A}} \otimes_{\mathcal{A}} k$ (where we regard k as an $\mathcal{A}$-algebra via $v \mapsto \zeta$) and in the subalgebra $\mathbf{u}$ of U_3, cf. 1.3.

Let us now look at the algebra U from 1.3. The formulas in 7.1 for Δ imply for all $\mu \geq 0$ that

$$\Delta((U^+)_\mu) \subset \bigoplus_{0 \leq \nu \leq \mu} K_\nu (U^+)_{\mu-\nu} \otimes (U^+)_\nu \tag{1}$$

and

$$\Delta((U^-)_{-\mu}) \subset \bigoplus_{0 \leq \nu \leq \mu} (U^-)_{-\nu} \otimes (U^-)_{\nu-\mu} K_\nu^{-1}. \tag{2}$$

We have a canonical map $\overline{f} : U \to \mathbf{u}$, cf. 1.3. It is compatible with Δ and each T_w. It induces isomorphisms $U^+ \xrightarrow{\sim} \mathbf{u}^+$ and $U^- \xrightarrow{\sim} \mathbf{u}^-$. Since weight space decompositions in U and $\mathbf{u}$ are direct and since the K_ν with $\nu \in X$ are units in both rings, it follows that $\overline{f}$ maps the right hand sides in (1) and (2) bijectively to the analogous objects in $\mathbf{u}$. This implies that C.5(1) and C.6(1) hold also in U.

Appendix D

D.1. In [KK], 4.20, the authors define for each $w \in W$ a function

$$\xi^w : W \to S_{\mathbf{C}} = S \otimes_{\mathbf{Z}} \mathbf{C}, \tag{1}$$

where S is the symmetric algebra of the $\mathbf{Z}$–module $\mathbf{Z}R$. Kostant and Kumar work in that paper with Kac–Moody algebras over $\mathbf{C}$. We can apply their results to the special case of a complex semi–simple Lie algebra with root system R. We can then identify $S_{\mathbf{C}}$ as above with the symmetric algebra of the dual of a Cartan subalgebra of that Lie algebra (and that is what Kostant and Kumar use).

For our purposes a minor change in notation will be useful: We define for all $w \in W$:

$$\eta^w : W \to S_{\mathbf{C}}, \qquad \text{with} \qquad \eta^w(x) = \xi^{w^{-1}}(x^{-1}) \tag{2}$$

for all $x \in W$. The following properties (for all $w \in W$) of η^w are immediate translations of [KK], Prop. 4.24(a), (c), (d):

a) If $x \in W$ with $\eta^w(x) \neq 0$, then $w \leq x$.

b) $\eta^w(w) = \prod_{\alpha>0, w^{-1}\alpha<0} \alpha$.

c) Each $\eta^w(x)$ with $x \in W$ is homogeneous of degree $l(w)$.

d) $\eta^1(x) = 1$ for all $x \in W$.

e) If α is a simple root, then

$$\eta^{s_\alpha}(x) = \omega_\alpha - x\omega_\alpha \qquad \textit{for all } x \in W,$$

where ω_α is the fundamental weight corresponding to α.

Though d) is not stated explicitly in [KK], 4.24, it is an easy consequence. (We can also use D.2(4) below.)

We use in a) the Chevalley ordering (often called "Bruhat ordering") of W. For more details on this ordering, we refer to [Hu3], 5.9.

D.2. We define for each simple root α an operator A_α on all functions f from W to the fraction field of $S_{\mathbf{C}}$ by

$$(A_\alpha f)(w) = \frac{f(ws_\alpha) - f(w)}{w\alpha} \qquad \text{for all } w \in W. \tag{1}$$

Now [KK], 4.24(b) implies for all $w \in W$ and all α simple

$$A_\alpha \eta^w = \begin{cases} 0, & \text{if } ws_\alpha > w; \\ \eta^{ws_\alpha}, & \text{if } ws_\alpha < w. \end{cases} \tag{2}$$

More explicitly, we have for all $w, x \in W$ and all simple roots α

$$\eta^w(xs_\alpha) = \begin{cases} \eta^w(x), & \text{if } l(ws_\alpha) > l(w); \\ \eta^w(x) + (x\alpha)\eta^{ws_\alpha}(x), & \text{if } l(ws_\alpha) < l(w). \end{cases} \tag{3}$$

We can express the first equation as

$$s_\alpha \eta^w = \eta^w \qquad \text{if } \alpha \text{ simple with } w < ws_\alpha. \tag{4}$$

We use here the operation of W on functions f on W given by

$$(wf)(x) = f(xw) \qquad \text{for all } w, x \in W. \tag{5}$$

D.3. We can use D.2(3) to compute each $\eta^w(x)$ inductively. We use first induction on w starting with $w = 1$ given by D.1(d). Then we use for each w induction on x starting with $x = 1$, i.e., with $\eta^w(1) = 0$ for $w \neq 1$ by D.1(a). If $x \neq 1$, then we choose a simple root α with $xs_\alpha < x$; then apply D.2(3) to xs_α instead of x and get $\eta^w(x)$ in terms of $\eta^w(xs_\alpha)$ and — if $ws_\alpha < w$ — of $\eta^{ws_\alpha}(xs_\alpha)$. This formula shows especially:

Lemma: *Each η^w takes values in S.*

Remark: We leave it to the reader to show more precisely the following (by induction on r): Suppose that $x = s_1 s_2 \cdots s_r$ with $s_i = s_{\alpha_i}$ for some simple root α_i. Then

$$\eta^w(x) = \sum_{\mathbf{i}} \prod_{\nu=1}^{l} s_1 s_2 \cdots s_{i_\nu - 1}(\alpha_{i_\nu}), \tag{1}$$

where the sum is over all sequences $\mathbf{i} = (i_1 < i_2 < \ldots < i_l)$ with $i_1 \geq 1$ and $i_l \leq r$ such that $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ is a reduced expression of w. (Here $l = l(w)$.)

One can, alternatively, use (1) to define the η^w. The main work is to show the independence of the sequence chosen for x. Once that is done, the proof of the other properties (mentioned above or below) is easy.

D.4. Let w_0 be the longest element in W. We have by D.1(a), (b) for all $x \in W$

$$\eta^{w_0}(x) = \begin{cases} \prod_{\beta>0} \beta, & \text{if } x = w_0; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Lemma: *Let $w, x \in W$ and $\alpha \in R^+$. Then*

$$\eta^w(s_\alpha x) \equiv \eta^w(x) \pmod{S\alpha}.$$

Proof: We use induction on $l(w)$ from above. The case $w = w_0$ is obvious by (1). Suppose that $w \neq w_0$ and choose a simple root β with $ws_\beta > w$. Let $x \in W$. If $\alpha = \pm x\beta$, then $s_\alpha x = xs_\beta$ and $A_\beta \eta^w = 0$ implies $\eta^w(x) = \eta^w(xs_\beta) = \eta^w(s_\alpha x)$. Suppose now that $\alpha \neq \pm x\beta$. It is enough to prove

$$\eta^w(s_\alpha x) \equiv \eta^w(x) \pmod{S[(x\beta)^{-1}, (s_\alpha x\beta)^{-1}]\alpha}.$$

Well, we have $\eta^w = A_\beta \eta^v$ where $v = ws_\beta$, hence (using the induction hypothesis for the second step)

$$\begin{aligned} \eta^w(s_\alpha x) - \eta^w(x) &= \frac{\eta^v(s_\alpha x s_\beta) - \eta^v(s_\alpha x)}{s_\alpha x\beta} - \frac{\eta^v(xs_\beta) - \eta^v(x)}{x\beta} \\ &\equiv \frac{\eta^v(xs_\beta) - \eta^v(x)}{s_\alpha x\beta} - \frac{\eta^v(xs_\beta) - \eta^v(x)}{x\beta} \\ &= \big(\eta^v(xs_\beta) - \eta^v(x)\big)\big(s_\alpha \frac{1}{x\beta} - \frac{1}{x\beta}\big) \equiv 0 \pmod{\alpha}. \end{aligned}$$

D.5. Set

$$E_S = \{f : W \to S \mid f(s_\alpha x) \equiv f(x) \pmod{S\alpha} \quad \text{for all } x \in W \text{ and } \alpha \in R.\} \tag{1}$$

Lemma D.4 says that $\eta^w \in E_S$ for all $w \in W$.

Lemma: *The η^w with $w \in W$ are a basis of E_S as a module over S.*

Proof: It is clear by D.1(a), (b) that the η^w are linearly independent over S. We have to show that they generate E_S. Choose a total order on W compatible with the Chevalley order. Let $f \in E_S$ with $f \neq 0$; let $w \in W$ be the smallest element with $f(w) \neq 0$. We want to use induction (from above) on w in the total order to show that f is in the span of the η^x. We have $s_\alpha w < w$ for all $\alpha \in R^+$ with $w^{-1}\alpha < 0$, hence $f(s_\alpha w) = 0$ and $f(w) \equiv 0 \pmod{\alpha}$. Since the $\alpha \in R^+$ are prime in S and pairwise not multiples of each other, we see that $f(w)$ is divisible in S by the product of all $\alpha > 0$ with $w^{-1}\alpha < 0$, i.e., by $\eta^w(w)$, cf. D.1(b). Let $a \in S$ with $f(w) = a\eta^w(w)$; then $g = f - a\eta^w \in E_S$ and we can apply induction.

D.6. It is clear that E_S is closed under multiplication. So any product $\eta^w \eta^y$ is a linear combination of the η^w. Obviously

$$\eta^1 \eta^w = \eta^w \qquad \text{for all } w \in W. \tag{1}$$

Besides that, let us mention only a special case that is the translation of Proposition 4.30 in [KK]. One has for all $w \in W$ and all simple roots α:

$$\eta^{s_\alpha} \eta^w = \eta^{s_\alpha}(w) \eta^w + \sum_{w \xrightarrow{\beta} w'} \langle w\omega_\alpha, \beta^\vee \rangle \eta^{w'} \tag{2}$$

where we use the notation $w \xrightarrow{\beta} w'$ from [BGG2] to indicate that β is a positive root with $w' = s_\beta w$ and $l(w') = l(w) + 1$. (For the general case, see [KK], 4.31.)

D.7. We can define E_A for any S–algebra A by replacing S in D.5(1) by A. If A is an unique factorization domain, and if the greatest common divisor in A of two distinct positive roots is equal to 1, then we can generalize Lemma D.5 to E_A. Strictly speaking, we have to replace any η^w by its composition with the structural map $S \to A$, but by abuse of notation we call this composition again η^w.

Let k be any field, but suppose that $\operatorname{char}(k) \neq 2$, if R has two root lengths, and that $\operatorname{char}(k) \neq 3$, if R has a component of type G_2. We can then apply the discussion above to $A = S_k = S \otimes_{\mathbf{Z}} k$, cf. the proof of Lemma 9.1. Note that we get especially

$$E_{S_k} \simeq E_S \otimes_{\mathbf{Z}} k. \tag{1}$$

We can regard S_k as the symmetric algebra of the vector space $\mathbf{Z}R \otimes_{\mathbf{Z}} k$. We denote the homogeneous components of this graded algebra by S_k^r $(r \geq 0)$, and set $S_k^+ = \bigoplus_{r>0} S_k^r$. We extend the notation $\langle \mu, \alpha^\vee \rangle$ to all $\mu \in S_k^1 = \mathbf{Z}R \otimes_{\mathbf{Z}} k$ such that $s_\alpha(\mu) = \mu - \langle \mu, \alpha^\vee \rangle \alpha$ (for all $\alpha \in R$).

Suppose that G is a connected semisimple algebraic group defined over k with root system R, and suppose that T is a maximal torus in G, defined and split over k. If $\operatorname{char}(k) = 0$ or if $\operatorname{char}(k) = p > 0$ with p prime to the index of connection of R, then we can identify S_k with the symmetric algebra of $\operatorname{Lie}(T)^*$. If G is of adjoint type, then this holds without restriction on k. (The identification is supposed to identify the roots and to be W–equivariant.)

D.8. We keep our assumptions on k and the notations introduced in D.7 until the end of this appendix. For all $a \in S_k$ the map $F(a) : W \to S_k$ with

$$F(a)(w) = wa \qquad \text{for all } w \in W \tag{1}$$

takes values in E_{S_k} since

$$s_\alpha wa - wa \equiv 0 \pmod{S_k \alpha} \qquad \text{for all } \alpha \in R \text{ and } w \in W.$$

So there are $c_w(a) \in S_k$ such that (for all $a \in S_k$):

$$F(a) = \sum_{w \in W} c_w(a)\eta^w. \tag{2}$$

For example, one has for all $\mu \in S_k^1$

$$F(\mu) = \mu\eta^1 - \sum_{\alpha \in \Sigma} \langle \mu, \alpha^\vee \rangle \eta^{s_\alpha}. \tag{3}$$

If we look at $F(a)(1) = a$ and $F(a)(s_\alpha) = s_\alpha a$, we see that

$$c_1(a) = a \qquad \text{and} \qquad c_{s_\alpha}(a) = \frac{s_\alpha a - a}{\alpha} \quad \text{for all } \alpha \in \Sigma. \tag{4}$$

(Use that $\eta^w(1) = 0$ for $w \neq 1$, that $\eta^w(s_\alpha) = 0$ for $w \neq 1, s_\alpha$, that $\eta^1(1) = \eta^1(s_\alpha) = 1$, and that $\eta^{s_\alpha}(s_\alpha) = \alpha$, cf. D.1.) For all $a \in S_k^r$ all $F(a)(w)$ are in S_k^r. So D.1(c) implies

$$a \in S_k^r \implies c_w(a) \in S_k^{r-l(w)} \quad \text{for all } w \in W, \tag{5}$$

where $S_k^i = 0$ for $i < 0$.

We have for all $\mu \in \mathbf{Z}R$ and $a \in S_k$

$$F(a\mu) = F(\mu)F(a) = \Big(\mu\eta^1 - \sum_{\alpha \in \Sigma} \langle \mu, \alpha^\vee \rangle \eta^{s_\alpha}\Big) \sum_{w \in W} c_w(a)\eta^w$$

by (2) and (3). We can evaluate the last product using D.6(1), (2). If we compare coefficients with $F(a\mu) = \sum_{w \in W} c_w(a\mu)\eta^w$, we get

$$c_w(a\mu) = w(\mu)c_w(a) - \sum_{w' \xrightarrow{\beta} w} \langle w'\mu, \beta^\vee \rangle c_{w'}(a). \tag{6}$$

D.9. Define the Demazure operator $\Lambda_\alpha : S_k \to S_k$ for $\alpha \in \Sigma$ by

$$\Lambda_\alpha(a) = \frac{s_\alpha a - a}{\alpha} \qquad \text{for all } a \in S_k. \tag{1}$$

(This differs from [BGG2] or [Dem] by a minus sign.) Define for all $w \in W$ the operator Λ_w via

$$\Lambda_w = \Lambda_{\alpha_1} \circ \Lambda_{\alpha_2} \circ \cdots \circ \Lambda_{\alpha_m} \tag{2}$$

for any reduced decomposition $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_m}$ of w, cf. [BGG2] or [Dem]. Now [BGG2], 3.7 implies (taking into account that our convention differs by $(-1)^{l(w)}$ from theirs)

$$\Lambda_w(a\mu) = w(\mu)\Lambda_w(a) - \sum_{w' \xrightarrow{\beta} w} \langle w'\mu, \beta^\vee \rangle \Lambda_{w'}(a) \tag{3}$$

for all $a \in S_k$ and $\mu \in \mathbf{Z}R$. The c_w satisfy by D.8(6) the same formula. We get now by induction on the degree of a

Lemma: *One has $c_w = \Lambda_w$ for all $w \in W$.*

(Note that $\Lambda_w(\mu) = 0$ for $\mu \in S_k^1$ and w with $l(w) > 1$.)

Remark: The definition of the Λ_w makes it clear that there are elements $b_w(w')$ in the fraction field of S_k, independent of a, such that

$$\Lambda_w(a) = \sum_{w' \in W} b_w(w')w'a \tag{4}$$

for all $a \in S_k$. A priori, these $b_w(w')$ might depend on a reduced decomposition of w. But we can regard the $w \in W$ as characters on the multiplicative monoid of S_k with values in the fraction field. This yields the necessary linear independence that says that (4) determines the $b_w(w')$ uniquely. If we plug (4) into D.8(2) using the lemma we get for all $w_1 \in W$ and $a \in S_k$:

$$w_1 a = F(a)(w_1) = \sum_w \Lambda_w(a)\eta^w(w_1) = \sum_{w' \in W} \Big(\sum_{w \in W} b_w(w')\eta^w(w_1)\Big)w'a.$$

Using again the linear independence we see that the matrix with entries $\eta^w(w')$ is the inverse of the matrix with entries $b_{w'}(w)$.

D.10. Denote the augmentation ideal of S_k by $\mathfrak{m}$. Set

$$E_k = E_{S_k}/\mathfrak{m}E_{S_k} \simeq E_{S_k} \otimes_{S_k} k. \tag{1}$$

Denote the canonical map $E_{S_k} \to E_k$ by $f \mapsto \overline{f}$, denote by $\overline{F} : S_k \to E_k$ the composition of F with this canonical map. Then $\overline{F}$ is given by

$$\overline{F}(a) = \sum_{l(w)=r} \Lambda_w(a)\overline{\eta^w} \qquad \text{for all } a \in S_k^r. \tag{2}$$

(and for all r). The $\overline{\eta^w}$ are a basis of E_k. In [Dem], Cor. 4 de la Prop. 3 the author considers a map from S_k to an algebra H with a basis $(z_w)_{w \in W}$ given by the same formula (2) with $\overline{\eta^w}$ replaced by z_w. Provided that char(k) is 0 or prime to $|W|$, Demazure shows (Thm. 2) that his map is onto and has as kernel the ideal generated by $(S_k^+)^W$. Assume that this assumption on k is satisfied. Then $\overline{F} : S_k \to E_k$ is onto and induces an isomorphism of E_k with the covariant algebra. The graded Nakayama lemma implies now that

$$\widetilde{F} : S_k \otimes S_k \to E_{S_k} \qquad \text{with} \qquad \widetilde{F}(a \otimes b)(w) = w(a)b \tag{3}$$

(for all $a, b \in S_k$ and $w \in W$) is surjective. It factors clearly over a (surjective) homomorphism

$$\Phi : S_k \otimes_{S_k^W} S_k \to E_{S_k}. \tag{4}$$

Now Φ is S_k–linear, if we let S_k act on the tensor product via multiplication on the second factor. Since S_k is free of rank $|W|$ over S_k^W — for our k, cf. [Bou2], chap. V, §5, n° 5, Thm. 4 — both sides in (4) are free of rank $|W|$ over S_k, and we see:

Proposition: *If char(k) $= 0$ or if char(k) $= p > 0$ with p prime to $|W|$, then Φ is an isomorphism.*

Appendix E

Throughout this appendix let Y be an abelian group. From E.6 on let k be a field.

E.1. We shall generally denote graded pieces of a (Y–)graded abelian group M by M_ν, i.e., write $M = \bigoplus_{\nu \in Y} M_\nu$. A subgroup N of M is called homogeneous, if it is the (direct) sum of all $N \cap M_\nu$; it is then naturally graded. For any Y–graded abelian group M as above and any $\mu \in Y$ we define $M\langle\mu\rangle$ as M with the grading shifted by μ, i.e., with

$$(M\langle\mu\rangle)_\nu = M_{\nu-\mu} \qquad \text{for all } \nu \in Y. \tag{1}$$

Suppose that A is a Y–graded ring. An A–module will always be a left A–module, unless explicitly stated otherwise. If M is a Y–graded A–module, then each $M\langle\nu\rangle$ with $\nu \in Y$ is again a Y–graded A–module. If M, N are Y–graded A–modules, then we denote by $\mathrm{Hom}_A(M,N)$ the group of all A–linear maps from M to N, and by $\mathrm{Hom}_{A,Y}(M,N)$ the group of all $\varphi \in \mathrm{Hom}_A(M,N)$ that preserve the grading, i.e., with $\varphi(M_\nu) \subset N_\nu$ for all ν. More generally, set for all $\mu \in Y$

$$\mathrm{Hom}_A(M,N)_\mu = \{\varphi \in \mathrm{Hom}_A(M,N) \mid \varphi M_\nu \subset N_{\nu+\mu} \quad \text{for all } \nu\}.$$

We have obviously

$$\mathrm{Hom}_A(M,N)_\mu = \mathrm{Hom}_{A,Y}(M\langle\mu\rangle, N) = \mathrm{Hom}_{A,Y}(M, N\langle-\mu\rangle). \tag{2}$$

Let us denote by $\mathrm{Hom}^\sharp_{A,Y}(M,N)$ the Y–graded abelian group with graded pieces $\mathrm{Hom}^\sharp_{A,Y}(M,N)_\mu = \mathrm{Hom}_A(M,N)_\mu$. If A is commutative, this is even a Y–graded A–module. We have for all $\nu \in Y$

$$\mathrm{Hom}^\sharp_{A,Y}(M\langle-\nu\rangle, N) = \mathrm{Hom}^\sharp_{A,Y}(M,N)\langle\nu\rangle = \mathrm{Hom}^\sharp_{A,Y}(M, N\langle\nu\rangle). \tag{3}$$

There is a natural embedding of $\mathrm{Hom}_A(M,N)$ into the direct product of all $\mathrm{Hom}_A(M,N)_\mu$. If M is finitely generated over A, then this induces an isomorphism of $\mathrm{Hom}_A(M,N)$ with $\mathrm{Hom}^\sharp_{A,Y}(M,N)$, so $\mathrm{Hom}_A(M,N)$ admits a Y–grading.

We denote the Ext groups in the graded category by $\mathrm{Ext}^i_{A,Y}$. If A is left noetherian, then we can extend the statements above to the higher Ext^i. If M and N are finitely generated Y–graded A–modules, then each group $\mathrm{Ext}^i_A(M,N)$ has natural Y–grading given by

$$\mathrm{Ext}^i_A(M,N)_\mu = \mathrm{Ext}^i_{A,Y}(M\langle\mu\rangle, N)$$

and we have for all $\nu \in Y$ canonical isomorphisms of Y–graded groups

$$\mathrm{Ext}^i_A(M,N\langle\nu\rangle) = \mathrm{Ext}^i_A(M,N)\langle\nu\rangle = \mathrm{Ext}^i_A(M\langle-\nu\rangle, N). \tag{4}$$

We shall need even a more general version of the above. Namely, if Y,Z are two abelian groups and A is a $(Y\times Z)$–graded left noetherian ring, then for any finitely generated $(Y\times Z)$–graded A–modules M,N and any i the group $\mathrm{Ext}^i_{A,Y}(M,N)$ has a natural Z–grading given by $\mathrm{Ext}^i_{A,Y}(M,N)_\mu = \mathrm{Ext}^i_{A,Y\times Z}(M\langle\mu\rangle,N)$ for each $\mu\in Z$, and again we have for all $\nu\in Z$ canonical isomorphisms of Z–graded spaces

$$\mathrm{Ext}^i_{A,Y}(M,N\langle\nu\rangle) = \mathrm{Ext}^i_{A,Y}(M,N)\langle\nu\rangle = \mathrm{Ext}^i_{A,Y}(M\langle-\nu\rangle, N). \tag{5}$$

Here we have abused notation to write $\langle\nu\rangle$ instead of $\langle(0,\nu)\rangle$. We shall continue to do so in the future.

E.2. Let A be a Y–graded ring and let M resp. N be a Y–graded right resp. left A–module. Then $M\otimes_A N$ admits a natural Y–grading. Indeed, it is clear how to give $M\otimes_{\mathbf{Z}} N$ a Y–grading, and we only have to show that the kernel of the surjection $M\otimes_{\mathbf{Z}} N \twoheadrightarrow M\otimes_A N$ contains with an element all its homogeneous components. But by definition this kernel is generated by all expressions $mr\otimes n - m\otimes rn$ for $m\in M$, $r\in A$ and $n\in N$, and it is clear that it is generated as well by all such expressions with m,r,n homogeneous. (We could have used this argument in 2.10. However, it would not have given the more precise character information contained in 2.10(4).)

Let A, E be Y–graded rings and T a Y–graded (A,E)–bimodule. Then the functor

$$N \mapsto T\otimes_E N \tag{1}$$

from Y–graded E–modules to Y–graded A–modules is left adjoint to the functor

$$M \mapsto \mathrm{Hom}^\sharp_{A,Y}(T,M) \tag{2}$$

in the opposite direction. Indeed one checks that such an adjointness is induced by the classical isomorphism

$$\mathrm{Hom}_A(T\otimes_E M, N) \xrightarrow{\sim} \mathrm{Hom}_E(M,\mathrm{Hom}_A(T,N)).$$

Observe that if T is finitely generated as a left A–module, then

$$\mathrm{Hom}^{\sharp}_{A,Y}(T,N) = \mathrm{Hom}_A(T,N).$$

E.3. By a Y–category we mean an additive category $\mathcal{C}$ along with a collection of "shift"-functors $M \mapsto M\langle\nu\rangle$ for $\nu \in Y$ such that $(M\langle\nu\rangle)\langle\mu\rangle = M\langle\nu+\mu\rangle$ for all $\nu,\mu \in Y$ and $M\langle 0\rangle = M$ for all M in $\mathcal{C}$. More precisely, a Y-category is a system $(\mathcal{C}, (\langle\nu\rangle)_{\nu\in Y}, (\phi_{\nu,\mu})_{\nu,\mu\in Y})$ where $\mathcal{C}$ is an additive category, each $\langle\nu\rangle : \mathcal{C} \to \mathcal{C}$, $M \mapsto M\langle\nu\rangle$ an additive functor, and each $\phi_{\nu,\mu}$ a natural equivalence $\phi_{\nu,\mu} : \langle\nu\rangle \circ \langle\mu\rangle \to \langle\nu+\mu\rangle$, such that $\langle 0\rangle$ is the identity functor and the natural transformations $\phi_{\nu,\mu+\tau} \circ (\langle\nu\rangle\phi_{\mu,\tau})$ and $\phi_{\nu+\mu,\tau} \circ (\phi_{\nu,\mu}\langle\tau\rangle)$ from $\langle\nu\rangle \circ \langle\mu\rangle \circ \langle\tau\rangle$ to $\langle\nu+\mu+\tau\rangle$ coincide. In particular $M \mapsto M\langle\nu\rangle$ and $M \mapsto M\langle-\nu\rangle$ are inverse equivalences of categories (for each $\nu \in Y$).

For example, the category of Y–graded A–modules as in E.1 is a Y–category.

If $\mathcal{A}$ is another Y–category, a functor $\mathcal{T} : \mathcal{C} \to \mathcal{A}$ is called a Y–functor if $\mathcal{T} \circ \langle\nu\rangle = \langle\nu\rangle \circ \mathcal{T}$ for all $\nu \in Y$. More precisely, a Y-functor is a system $(\mathcal{T}, (\psi_\nu)_{\nu\in Y})$ where $\mathcal{T} : \mathcal{C} \to \mathcal{A}$ is an additive functor and the ψ_ν are natural equivalences $\psi_\nu : \mathcal{T} \circ \langle\nu\rangle \xrightarrow{\sim} \langle\nu\rangle \circ \mathcal{T}$ such that ψ_0 is the identity and for all $\nu,\mu \in Y$ the obvious pentagon of natural transformations commutes, i.e., the two natural transformations $\psi_{\nu+\mu} \circ (\mathcal{T}\phi_{\nu,\mu})$ and $(\phi_{\nu,\mu}\mathcal{T}) \circ (\langle\nu\rangle\psi_\mu) \circ (\psi_\nu\langle\mu\rangle)$ from $\mathcal{T} \circ \langle\nu\rangle \circ \langle\mu\rangle$ to $\langle\nu+\mu\rangle \circ \mathcal{T}$ coincide.

For $M, N \in \mathcal{C}$ we form the Y–graded group

$$\mathrm{Hom}^{\sharp}_{\mathcal{C}}(M,N) = \bigoplus_{\mu\in Y} \mathrm{Hom}_{\mathcal{C}}(M\langle\mu\rangle, N). \tag{1}$$

(Note that this is compatible with the convention from E.1.) Given M_1, M_2, and M_3 in $\mathcal{C}$ we define a bilinear map

$$\mathrm{Hom}^{\sharp}_{\mathcal{C}}(M_1,M_2) \times \mathrm{Hom}^{\sharp}_{\mathcal{C}}(M_2,M_3) \longrightarrow \mathrm{Hom}^{\sharp}_{\mathcal{C}}(M_1,M_3) \tag{2}$$

as follows: Let ν, $\nu' \in Y$ and $\varphi \in \mathrm{Hom}_{\mathcal{C}}(M_1\langle\nu\rangle, M_2)$, $\psi \in \mathrm{Hom}_{\mathcal{C}}(M_2\langle\nu'\rangle, M_3)$. Then we map (φ,ψ) in (2) to $\psi\circ\langle\nu'\rangle\varphi$ where $\langle\nu'\rangle\varphi \in \mathrm{Hom}_{\mathcal{C}}(M_1\langle\nu+\nu'\rangle, M_2\langle\nu'\rangle)$ is the image of φ under the shift functor. More precisely, we map (φ,ψ) to $\psi \circ \langle\nu'\rangle\varphi \circ \phi_{\nu',\nu}(M_1)^{-1}$.

It is easy to check that (2) has the usual associativity properties and that $\mathcal{C}$ together with $\mathrm{Hom}^{\sharp}$ is an additive category. In particular, each $\mathrm{End}^{\sharp}_{\mathcal{C}}(M)$ with M in $\mathcal{C}$ has a ring structure.

For $M \in \mathcal{C}$ let M_Y be the full subcategory of $\mathcal{C}$ of all objects that are finite direct sums of some $M\langle\nu\rangle$ with $\nu \in Y$. If $\mathcal{C}$ is abelian and $M \in \mathcal{C}$ is such that every object of $\mathcal{C}$ is a quotient of some object in M_Y, then we say that M is a Y–generator of $\mathcal{C}$.

E.4. Proposition: *Let $\mathcal{C}$ be an abelian Y–category and P a projective Y–generator of $\mathcal{C}$. Then $M \mapsto \mathrm{Hom}^\sharp_{\mathcal{C}}(P, M)$ is an equivalence of categories from $\mathcal{C}$ to the category of finitely presented Y–graded $(\mathrm{End}^\sharp_{\mathcal{C}} P)^{opp}$–modules.*

Proof: Let us abbreviate the notation and put $E = (\mathrm{End}^\sharp_{\mathcal{C}} P)^{opp}$ and $\mathcal{H} = \mathrm{Hom}^\sharp_{\mathcal{C}}(P, \ )$. At first $\mathcal{H}$ is just a functor with values in the Y–graded E–modules. It is exact since P is projective. By inspection it induces an equivalence of categories $\mathcal{H} : P_Y \to E_Y$. By our assumptions every $M \in \mathcal{C}$ admits a two-step resolution $K_2 \to K_1 \twoheadrightarrow M$ with $K_1, K_2 \in P_Y$, and if we apply $\mathcal{H}$ we see that $\mathcal{H}M$ is finitely presented. This proves that our functor lands indeed in the finitely presented modules.

Next we prove that $\mathcal{H}$ is fully faithful. We show first that for any $K_3 \in P_Y$ and $M \in \mathcal{C}$ our functor induces a bijection

$$\mathrm{Hom}_{\mathcal{C}}(K_3, M) \longrightarrow \mathrm{Hom}_{E,Y}(\mathcal{H}K_3, \mathcal{H}M).$$

Indeed this follows by the five-lemma from the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(K_3, K_2) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(K_3, K_1) \arrow[r, two heads] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(K_3, M) \arrow[d] \\
\mathrm{Hom}_{E,Y}(\mathcal{H}K_3, \mathcal{H}K_2) \arrow[r] & \mathrm{Hom}_{E,Y}(\mathcal{H}K_3, \mathcal{H}K_1) \arrow[r, two heads] & \mathrm{Hom}_{E,Y}(\mathcal{H}K_3, \mathcal{H}M)
\end{tikzcd}$$

where we took a resolution of M in P_Y as above and use that by what we know already the first two verticals are isomorphisms. Then consider for any $N \in \mathcal{C}$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(M, N) \arrow[r, hook] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(K_1, N) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(K_2, N) \arrow[d] \\
\mathrm{Hom}_{E,Y}(\mathcal{H}M, \mathcal{H}N) \arrow[r, hook] & \mathrm{Hom}_{E,Y}(\mathcal{H}K_1, \mathcal{H}N) \arrow[r] & \mathrm{Hom}_{E,Y}(\mathcal{H}K_2, \mathcal{H}N).
\end{tikzcd}$$

By what we know already the last two verticals are isomorphisms, hence so is the first one by the five-lemma and $\mathcal{H}$ is fully faithful.

It is left to show that every finitely presented Y–graded E–module N is isomorphic to $\mathcal{H}M$ for some $M \in \mathcal{C}$. By assumption N admits a two-step resolution $F_2 \to F_1 \twoheadrightarrow N$ with $F_1, F_2 \in E_Y$. Since $\mathcal{H}$ induces an equivalence $P_Y \to E_Y$, there is a morphism $K_2 \to K_1$ in P_Y and a commutative diagram

$$\begin{tikzcd}
\mathcal{H}K_2 \arrow[r] \arrow[d] & \mathcal{H}K_1 \arrow[d] \\
F_2 \arrow[r] & F_1
\end{tikzcd}$$

whose verticals are isomorphisms. Now $\mathcal{H}$ is exact, so $N \xrightarrow{\sim} \mathcal{H}M$ where $M = \mathrm{coker}(K_2 \to K_1)$.

E.5. Proposition: *Let $\mathcal{T} : \mathcal{C} \to \mathcal{A}$ be an additive right exact Y-functor between abelian Y-categories. Let $P \in \mathcal{C}$, $Q \in \mathcal{A}$ be projective objects, put $E_P = (\mathrm{End}^\sharp_{\mathcal{C}} P)^{opp}$, $E_Q = (\mathrm{End}^\sharp_{\mathcal{A}} Q)^{opp}$, and form the Y-graded (E_Q, E_P)-bimodule ${}_Q T_P = \mathrm{Hom}^\sharp_{\mathcal{A}}(Q, \mathcal{T}P)$. Suppose in addition that P is a Y-generator of $\mathcal{C}$. Then the following functorial diagram commutes up to natural equivalence:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathrm{Hom}^\sharp_{\mathcal{C}}(P,\ )} & \{Y\text{-graded } E_P\text{-modules}\} \\ \downarrow{\scriptstyle \mathcal{T}} & & \downarrow{\scriptstyle {}_Q T_P \otimes_{E_P}} \\ \mathcal{A} & \xrightarrow{\mathrm{Hom}^\sharp_{\mathcal{A}}(Q,\ )} & \{Y\text{-graded } E_Q\text{-modules}\}. \end{array}$$

Proof: For every $M \in \mathcal{C}$ we have a natural map

$$\begin{aligned} \tau_M : {}_Q T_P \otimes_{E_P} \mathrm{Hom}^\sharp_{\mathcal{C}}(P, M) &\longrightarrow \mathrm{Hom}^\sharp_{\mathcal{A}}(Q, \mathcal{T}P) \otimes_{E_P} \mathrm{Hom}^\sharp_{\mathcal{A}}(\mathcal{T}P, \mathcal{T}M) \\ &\longrightarrow \mathrm{Hom}^\sharp_{\mathcal{A}}(Q, \mathcal{T}M) \end{aligned}$$

where the first arrow comes from the functor $\mathcal{T}$, the second one from the composition of morphisms. These maps form a natural transformation τ from the composition a of the two functors above the diagonal of our functorial diagram to the composition b of the two functors below, $\tau : a \to b$. We just have to show that all τ_M are isomorphisms. This is clear for $M = P$, hence for all $M \in P_Y$. For $M \in \mathcal{C}$ arbitrary choose a two-step resolution $G \to F \twoheadrightarrow M$ with $F, G \in P_Y$. Since both a and b are right exact, we obtain a commutative diagram

$$\begin{array}{ccccc} aG & \longrightarrow & aF & \twoheadrightarrow & aM \\ \downarrow & & \downarrow & & \downarrow \\ bG & \longrightarrow & bF & \twoheadrightarrow & bM \end{array}$$

with right exact rows, where the vertical arrows are τ_G, τ_F, and τ_M respectively. Since F and G lie in P_Y, both τ_F and τ_G are isomorphisms. But then also τ_M has to be an isomorphism, by the five-lemma.

E.6. Fix a field k. Let S_k be a polynomial ring over k in finitely many variables h_α, with a $\mathbf{Z}$-grading given by some prescription $\deg h_\alpha = n_\alpha$ for certain $n_\alpha > 0$. Regard S_k as a $(Y \times \mathbf{Z})$-graded ring via the trivial Y-grading, i.e., with $(S_k)_{0,i} = (S_k)_i$ and $(S_k)_{\nu,i} = 0$ for all $\nu \in Y$, $\nu \neq 0$ and all $i \in \mathbf{Z}$.

Let A be a finite $(Y \times \mathbf{Z})$-graded S_k-algebra. Thus A is a $(Y \times \mathbf{Z})$-graded ring along with a homomorphism from S_k to the center of A that is compatible with the $(Y \times \mathbf{Z})$-grading; furthermore A is finitely generated as an S_k-module.

Lemma: *a) If M is an indecomposable finitely generated $(Y \times \mathbf{Z})$-graded A-module, then its endomorphism ring (in the graded category) is local.*

b) The Krull-Schmidt theorem holds in the category of all finitely generated $(Y \times \mathbf{Z})$-graded A-modules.

Proof: It is enough to prove a). Let $\varphi \in \mathrm{End}_{A,Y\times\mathbf{Z}}M$ be an endomorphism. Since every $\mathbf{Z}$-homogeneous component M_i of M has finite dimension, there is some $n(i)$ such that $\varphi^m(M_i)$ does not depend on m for $m \geq n(i)$. Since M is generated in finitely many degrees, there is even an n such that $\varphi^m(M)$ does not depend on m for $m \geq n$. It then follows from counting dimensions in each degree that the surjection $\varphi^n : \varphi^n(M) \twoheadrightarrow \varphi^{2n}(M) = \varphi^n(M)$ is an isomorphism, hence that the surjection $\varphi^n : M \twoheadrightarrow \varphi^n(M)$ is split. So if M is indecomposable, φ is either nilpotent or an isomorphism, hence $\mathrm{End}_{A,Y\times\mathbf{Z}}M$ is local.

E.7. For a $\mathbf{Z}$-graded abelian group $M = \bigoplus_{i\in\mathbf{Z}} M_i$ we have the associated filtration, i.e., the filtration by all $\bigoplus_{i\geq j} M_i$ with $j \in \mathbf{Z}$. This filtration defines a topology on M. We denote the completion of M with respect to this topology by $\widehat{M}$. It is equal to the union of the $\prod_{i\geq j} M_i$ over all $j \in \mathbf{Z}$ (taken inside $\prod_{i\in\mathbf{Z}} M_i$). If there is an integer r with $M = \bigoplus_{i\geq r} M_i$, then $\widehat{M} = \prod_{i\geq r} M_i$. Usually a family $m = (m_i)_{i\geq r} \in \widehat{M}$ (where each $m_i \in M_i$) will be denoted by $m = \sum_{i=r}^{\infty} m_i$.

The topology on S_k defined by the grading is the same topology as defined by the maximal ideal generated by all h_α. So the completion of S_k with respect to the grading is equal to the completion studied in 14.4, and the notation $\widehat{S}_k$ is unambiguous.

For each $\mathbf{Z}$-graded S_k-module M the completion $\widehat{M}$ is an $\widehat{S}_k$-module in a natural way. If M is finitely generated over S_k, then we have a natural isomorphism

$$M \otimes_{S_k} \widehat{S}_k \xrightarrow{\sim} \widehat{M}, \tag{1}$$

cf. [AM], Prop. 10.13.

If A is finite $(Y \times \mathbf{Z})$-graded S_k-algebra as in E.6, then its completion $\widehat{A}$ identifies with $A \otimes_{S_k} \widehat{S}_k$; it is a Y-graded $\widehat{S}_k$-algebra. The completion functor takes a finitely generated $(Y \times \mathbf{Z})$-graded A-module to a finitely generated Y-graded $\widehat{A}$-module. If M and N are two such modules, then canonically for all i

$$\mathrm{Ext}^i_{A,Y}(M,N) \otimes_{S_k} \widehat{S}_k \simeq \mathrm{Ext}^i_{\widehat{A},Y}(M \otimes_{S_k} \widehat{S}_k, N \otimes_{S_k} \widehat{S}_k). \tag{2}$$

E.8. Lemma: *a) If M is a $\mathbf{Z}$-graded S_k-module, then*

$$M = 0 \iff M \otimes_{S_k} \widehat{S}_k = 0.$$

b) If $f : M \to N$ is a graded homomorphism of $\mathbf{Z}$-graded S_k-modules, then f is injective (resp. surjective, resp. bijective), if and only if $f \otimes 1 : M \otimes_{S_k} \widehat{S}_k \to N \otimes_{S_k} \widehat{S}_k$ is injective (resp. surjective, resp. bijective).

c) If M is a $\mathbf{Z}$-graded S_k-module, then M is finitely generated over S_k if and only if $M \otimes_{S_k} \widehat{S}_k$ is finitely generated over $\widehat{S}_k$.

d) If M is a $\mathbf{Z}$-graded S_k-module such that $M \otimes_{S_k} \widehat{S_k}$ is free of finite rank over $\widehat{S}_k$, then M is isomorphic to a finite direct sum of $\mathbf{Z}$-graded S_k-modules of the form $S_k\langle n_i \rangle$.

Proof: a) Suppose that $M \neq 0$. If M is finitely generated over S_k, then $M \otimes_{S_k} \widehat{S}_k \simeq \widehat{M} \neq 0$. In general, consider a finitely generated homogeneous submodule M_1 of M and use that $M_1 \otimes_{S_k} \widehat{S}_k \subset M \otimes_{S_k} \widehat{S}_k$.
b) Apply a) to the kernel and the cokernel of f; use that $\widehat{S}_k$ is flat over S_k.
c), d) Set $M^\wedge = M \otimes_{S_k} \widehat{S}_k$ and $\overline{M} = M \otimes_{S_k} k = M^\wedge \otimes_{\widehat{S}_k} k$. There is a $\mathbf{Z}$-grading on the k-vector space $\overline{M}$ such that the canonical surjection $\pi : M \to \overline{M}$ preserves the grading. If $M^\wedge$ is finitely generated over $\widehat{S}_k$, then $\overline{M}$ if finite dimensional over k. (If $M^\wedge$ is free of finite rank r over $\widehat{S}_k$, then M has dimension r over k.) Choose a basis v_i, $1 \leq i \leq r$ of $\overline{M}$ over k consisting of homogeneous elements, say $v_i \in \overline{M}_{n(i)}$. We can then find $m_i \in M_{n(i)}$ with $\pi(m_i) = v_i$. Now the Nakayama lemma implies that the $m_i \otimes 1$ generate $M^\wedge$ over $\widehat{S}_k$. (If $M^\wedge$ is free over $\widehat{S}_k$, then these elements have to be a basis of $M^\wedge$.) We have now a homomorphism f of $\mathbf{Z}$-graded S_k-modules from the (finite) direct sum of all $S_k\langle n(i) \rangle$ to M taking a family $(a_i)_i$ to $\sum a_i m_i$. Then $f \otimes 1$ is surjective (resp. bijective), hence so is f by b). We get thus one direction in c) resp. d). The other direction in c) is obvious.

E.9. Consider again a finite $(Y \times \mathbf{Z})$-graded S_k-algebra A as in E.6. If N is a finitely generated Y-graded $\widehat{A}$-module, then a "graded form" of N is by definition a finitely generated $(Y \times \mathbf{Z})$-graded A-module $\widetilde{N}$ along with an isomorphism $\widetilde{N} \otimes_{S_k} \widehat{S}_k \xrightarrow{\sim} N$.

Lemma: *Let M be a finitely generated $(Y \times \mathbf{Z})$-graded A-module.*

a) M is projective if and only if $M \otimes_{S_k} \widehat{S}_k$ is projective.

b) If $M \simeq M\langle i \rangle$ for some $i \in \mathbf{Z}$, $i \neq 0$, then $M = 0$.

c) Suppose that $M \otimes_{S_k} \widehat{S}_k$ is indecomposable. If M' is a graded form of $M \otimes_{S_k} \widehat{S}_k$, then there is $i \in \mathbf{Z}$ with $M' \simeq M\langle i \rangle$.

Proof: a) If M is projective, then it is a summand of an object from $\mathcal{A}_{Y \times \mathbf{Z}}$, hence $M \otimes_{S_k} \widehat{S}_k$ is a summand of an object from $\widehat{\mathcal{A}}_Y$, hence is projective.

On the other hand, if $M \otimes_{S_k} \widehat{S}_k$ is projective, then $\mathrm{Ext}^1_{\widehat{A},Y}(M \otimes_{S_k} \widehat{S}_k, N \otimes_{S_k} \widehat{S}_k) = 0$ for all finitely generated $(Y \times \mathbf{Z})$-graded A-modules N. Then E.7(2) and Lemma E.8.a imply $\mathrm{Ext}^1_{A,Y}(M,N) = 0$ for all these N, and M is projective.

The proof of b) resp. c) is very similar to that of 15.6.a resp. 15.6.c. One uses that for all finitely generated $(Y \times \mathbf{Z})$-graded A-modules M and N the $\mathbf{Z}$-graded S_k-module $\mathrm{Hom}_{A,Y}(M,N)$ is finitely generated; so there is an integer r with $\mathrm{Hom}_{A,Y}(M,N)_i = 0$ for all $i < r$. Applied to $N = M$ we get easily b). For c) one uses in addition Lemma E.6 and E.7(2) [for Hom $= \mathrm{Ext}^0$]. Details are left to the reader.

E.10. Consider the following property of Y and k:

(E) *For each finite subset $Y_1 \subset Y$ the restrictions to Y_1 of the characters from Y to the multiplicative group $k^\times$ of k generate the full ring of functions $Y_1 \to k$.*

For example, (E) is satisfied for $Y = \mathbf{Z}$, if k is infinite. Indeed, for any $n+1$ pairwise distinct elements λ_i in $k^\times$ (or even in k) the square matrix $(\lambda_i^j)_{i,j=0}^n$ is nonsingular (because a polynomial of degree n has at most n roots). By the same argument the condition (E) is satisfied if $Y = \mathbf{Z}/n\mathbf{Z}$, the characteristic of k does not divide n and k contains all n-th roots of unity. Furthermore, if (E) holds for two groups, it holds for their direct product. It follows that (E) holds for all Y if $\mathrm{char}(k) = 0$ and k contains all roots of unity, which then form an injective $\mathbf{Z}$-module isomorphic to $\mathbf{Q}/\mathbf{Z}$.

Let A be a Y-graded k-algebra. If M is an A-module, we write $\mathrm{rad}_A M$ for the Jacobson radical of M, i.e., we set $\mathrm{rad}_A M = \bigcap N$ where N runs over all proper maximal A-submodules of M.

Proposition: *If* (E) *is satisfied, then the radical* $\mathrm{rad}_A M$ *is homogeneous for each Y-graded A-module M.*

Proof: For every character $\chi : Y \to k^\times$ there is a unique k-linear automorphism $\Phi_\chi \in \mathrm{End}_k M$ such that $\Phi_\chi(m) = \chi(\nu)m$ for all $m \in M_\nu$, $\nu \in Y$. Although the Φ_χ are not always A-linear they always map submodules to submodules, since $\Phi_\chi(am) = \chi(\nu)a\Phi_\chi(m)$ for $a \in A_\nu$, $\nu \in Y$. In particular $\Phi_\chi(\mathrm{rad}_A M) = \mathrm{rad}_A M$ for all χ, and in view of our conditions on k and Y this means that $\mathrm{rad}_A M$ is homogeneous.

E.11. We need in 18.12 a slightly more general result. Let k and Y be as above. Suppose Z is another abelian group and A is a $(Z \times Y)$-graded ring. If M is a Z-graded A-module consider $\mathrm{rad}_{A,Z} M$, the intersection over all maximal Z-graded submodules of M.

Proposition: *If* (E) *is satisfied, then the Z-radical* $\mathrm{rad}_{A,Z} M$ *is homogeneous for each $(Z \times Y)$-graded A-module M.*

Proof: Almost identical to the above proof and omitted.

Appendix F

F.1. We recall the notion of a Koszul ring.

Definition 1: A *Koszul ring* is a positively $\mathbf{Z}$–graded ring $A = \bigoplus_{i \geq 0} A_i$ such that (1) A_0 is semisimple and (2) the graded left A–module $A_0 = A/A_{>0}$ admits a graded projective resolution

$$\cdots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow P^0 \longrightarrow A_0 \to 0$$

such that P^i is generated by its degree i part, $P^i = AP^i_i$ for all i.

Definition 2: Let A be a ring. A *Koszul grading* on A is a $\mathbf{Z}$–grading such that the corresponding $\mathbf{Z}$–graded ring is Koszul.

Here come some generalities concerning Koszul gradings.

F.2. Lemma: *Let A be an Artinian ring and $A = \bigoplus_{i \geq 0} A_i$ a Koszul grading on A. Then* $\mathrm{rad}^i_A A = \bigoplus_{j \geq i} A_j$ *and hence the obvious morphism*

$$A = \bigoplus_{i \geq 0} A_i \longrightarrow \bigoplus_{i \geq 0} \mathrm{rad}^i_A A / \mathrm{rad}^{i+1}_A A$$

is an isomorphism of graded rings.

Proof: The lemma holds in fact for every grading on A such that (1) A is generated by A_0, A_1 and that (2) A_0 is semisimple as is established in [BGS], proof of 2.4.1. Every Koszul grading has these properties, see, e.g., [BGS], 2.3.

F.3. Lemma: *Suppose A and B are Morita-equivalent Artinian rings. Then A admits a Koszul grading if and only if B does.*

Proof: Suppose $A = \bigoplus_{i \geq 0} A_i$ is a Koszul grading. Then every simple A–module has the form $A_0 p$ for a suitable idempotent $p \in A_0$. Its projective cover is Ap. Hence any finitely generated projective A–module P admits a grading compatible with the given grading on A, and this grading can even be chosen such that P is generated by its part of degree zero, $P = AP_0$.

Now B has the form $B = (\mathrm{End}_A P)^{opp}$ for some projective generator P of the category of finitely generated A–modules, and if we choose a grading on P as above we get a positive $\mathbf{Z}$–grading on B. We want to prove that this is a Koszul grading. Note first that by our assumptions the functor $\mathrm{Hom}_A(P,\ )$ is an equivalence between the categories of all finitely generated $\mathbf{Z}$–graded modules over A and over B. It transforms P_0 into B_0. In particular $B_0^{opp} = \mathrm{End}_B B_0 = \mathrm{End}_A P_0$ is semisimple, and so is B_0. In addition

$$\mathrm{Ext}^i_{B,\mathbf{Z}}(B_0, B_0\langle n\rangle) = \mathrm{Ext}^i_{A,\mathbf{Z}}(P_0, P_0\langle n\rangle) = 0$$

unless $i = n$, the latter since A is Koszul, see [BGS], Proposition 2.1.3. But then also B is Koszul by [BGS], Proposition 2.1.3.

F.4. Lemma: *An Artinian ring A admits a Koszul grading if and only if all its blocks admit Koszul gradings.*

Proof: Observe first that a finite direct product of $\mathbf{Z}$–graded rings is Koszul if and only if all of its factors are: This is immediate from the definition. In particular, if all blocks of A admit Koszul gradings, so does A. On the other hand, consider the block decomposition $A = \prod B$ where B runs over the blocks of A. If A is given a Koszul grading, then by Lemma F.2 we have an isomorphism of $\mathbf{Z}$–graded rings $A \simeq \bigoplus_{i\geq 0} \mathrm{rad}^i A/\mathrm{rad}^{i+1} A$. Since $\mathrm{rad}^i A = \prod \mathrm{rad}^i B$, we deduce an isomorphism of $\mathbf{Z}$–graded rings

$$A \simeq \prod \Big(\bigoplus_{i\geq 0} \mathrm{rad}^i B/\mathrm{rad}^{i+1} B\Big).$$

By our first remark, all the factors on the right hand side are Koszul. But since a block decomposition is unique, we deduce ring isomorphisms

$$B' \simeq \bigoplus_{i\geq 0} \mathrm{rad}^i B/\mathrm{rad}^{i+1} B$$

for a suitable permutation $B \mapsto B'$ of the blocks (the identity, but why bother), and thus all blocks B' of A admit a Koszul grading.

F.5. Analogues of the preceding lemmas hold in a graded context. More precisely, let Y be an abelian group and A an Artinian Y–graded ring. We say that a $\mathbf{Z}$–grading on A is compatible with the Y–grading if and only if they fit together to a $(Y \times \mathbf{Z})$–grading.

Lemma: *If A is given a Koszul grading compatible with its Y–grading, then the $\mathrm{rad}^i A = \bigoplus_{j\geq i} A_i$ are $(Y \times \mathbf{Z})$–homogeneous and the obvious morphism*

$$A = \bigoplus_{i\geq 0} A_i \longrightarrow \bigoplus_{i\geq 0} \mathrm{rad}^i A/\mathrm{rad}^{i+1} A$$

is an isomorphism of $(Y \times \mathbf{Z})$–graded rings.

Proof: This is an obvious consequence of Lemma F.2.

F.6. To generalize the next lemma we have to treat graded Morita equivalence. Let Y be an abelian group and A, B two Artinian Y–graded rings.

Definition: *We say B is Y–Morita equivalent to A if and only if the category of all finitely generated Y–graded A–modules has a projective Y–generator P such that there is an isomorphism of Y–graded rings $B \simeq (\mathrm{End}_A P)^{opp}$.*

Remark: It follows then from Proposition E.4 that $\mathrm{Hom}_A(P, \ )$ is an equivalence between the Y–categories of all finitely generated Y–graded modules over A and over B. Since this equivalence is right exact, we see from Proposition E.5 that it can also be written as a tensor functor $Q \otimes_A$ where Q is the Y–graded (B, A)–bimodule $Q = \mathrm{Hom}_A(P, A)$. We find $A = (\mathrm{End}_B Q)^{opp}$ and the inverse equivalence of categories is given by the adjoint $\mathrm{Hom}_B(Q, \ )$, which can also be written $P \otimes_B$. We deduce that B is Y–Morita equivalent to A if and only if A is Y–Morita equivalent to B.

One might even prove that A is Y–Morita equivalent to B if and only if the Y–categories of all finitely generated Y–graded modules over A and over B are equivalent. We won't dive into the details.

F.7. Lemma: *Suppose A and B are Y–Morita equivalent Y–graded Artinian rings. Then A admits a Koszul grading compatible with its Y–grading if and only if B does.*

Proof: Suppose $A = \bigoplus_{i \geq 0} A_i$ is a Koszul grading of A compatible with its Y–grading. Then every simple Y–graded A–module has the form $A_0 p\langle \nu \rangle$ for a suitable Y–homogeneous idempotent $p \in A_0$ and $\nu \in Y$. Its projective cover is $Ap\langle \nu \rangle$. Hence any projective finitely generated Y–graded A–module P admits a $(Y \times \mathbf{Z})$–grading compatible with the given $(Y \times \mathbf{Z})$–grading on A, and this grading can even be chosen such that P is generated by its part of $\mathbf{Z}$–degree zero, $P = AP_0$.

Choose a projective Y–graded A–module P such that $B \simeq (\mathrm{End}_A P)^{opp}$ as Y–graded rings. A $(Y \times \mathbf{Z})$–grading on P as above induces a $(Y \times \mathbf{Z})$–grading on B extending its Y–grading. As in the proof of Lemma F.3 we show that the underlying $\mathbf{Z}$–grading makes B into a Koszul ring.

F.8. To formulate the next lemma, we have to first discuss the graded analogue of block decomposition. If A is an Y–graded Artinian ring, then its block decomposition is not necessarily compatible with the Y–grading. However, it is clear that A decomposes in a unique way into a direct product of Y–graded subrings that are indecomposable as Y–graded rings and will be called the Y–blocks.

Lemma: *Suppose A is an Artinian Y–graded ring. Then A admits a Koszul grading compatible with its Y–grading if and only if all its Y–blocks do.*

Proof: This is just a direct transposition of the proof of Lemma F.4 into the graded setting. Note first that a finite direct product of $(Y \times \mathbf{Z})$–graded rings is Koszul for the underlying $\mathbf{Z}$–grading if and only if all of its factors are: This is immediate from the definition.

To prove the other direction, consider the Y–block decomposition $A = \prod B$ where B runs over the Y–blocks of A. If A is given a Koszul grading compatible with its Y–grading, then by Lemma F.5 we have an isomorphism of $(Y \times \mathbf{Z})$–graded rings $A \simeq \bigoplus_{i\geq 0} \operatorname{rad}^i A/\operatorname{rad}^{i+1} A$. Since $\operatorname{rad}^i A = \prod \operatorname{rad}^i B$, we deduce an isomorphism of $(Y \times \mathbf{Z})$–graded rings

$$A \simeq \prod \Big(\bigoplus_{i\geq 0} \operatorname{rad}^i B/\operatorname{rad}^{i+1} B\Big).$$

By our first remark, all the factors on the right hand side are Koszul. Since a Y–block decomposition is unique, we deduce isomorphisms of Y–graded rings

$$B' \simeq \bigoplus_{i\geq 0} \operatorname{rad}^i B/\operatorname{rad}^{i+1} B$$

for a suitable permutation $B \mapsto B'$ of the Y–blocks (the identity, but why bother), and thus all Y-blocks B' admit a Koszul grading compatible with their Y–grading.

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