AST

Complex analytic methods in dynamical systems - IMPA, January 1992 - Pages préliminaires

Astérisque, tome 222 (1994), p. 1-8

<http://www.numdam.org/item?id=AST_1994__222__1_0>

222 ASTÉRISQUE

1994

COMPLEX ANALYTIC METHODS IN DYNAMICAL SYSTEMS

IMPA, January 1992

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Classification A.M.S. :

Ilyashenko
Morsalani 58F21

Bonatti : 32565
Cerveau : 14C21

Cerveau, Lins-Neto
Brunella
Mattei
Perez-Marco 32L30

Nakai : 58F18
Roche : 57R30
Sibony : 32H50
Ecalle : 58F36
Kostov : 32A20
Tischler : 30D05
Shishikura : 58F23

This volume contains the contributions to the congress Complex Analytic Methods in Dynamical Systems held at Instituto de Matemática Pura e Aplicada, IMPA, in January 1992.

We hope this book will provide an extensive overview on the subject.

This meeting counted with the support of Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Financiadora de Estudos e Projetos (Finep) and especially to IBM.

Rio de Janeiro, September 10, 1993

C. Camacho
A. Lins Neto
R. Moussu
P. Sad,

The Editors

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ABSTRACTS

C. BONATTI and X. GOMEZ-MONT. *The index of holomorphic vector fields on singular varieties I*

Given an analytic space V with an isolated singularity p, a Poincaré-Hopf type of index, $\mathrm{Ind}(X, V, p)$ is associated to every holomorphic vector field X tangent to V for which p is an isolated zero.

In this paper this topological index is related to the algebraic multiplicity $\mu_V(X,p)$. In particular, it is shown that the set of indices $\mathrm{Ind}(X, V, p)$, where X is tangent to V with an isolated zero at p, admits a minimum which is reached for X in the open dense subset of vector fields of smallest V-multiplicity.

M. BRUNELLA. *Vanishing holonomy and monodromy of certain centres and foci*

In this paper we study a particular class of germs of analytic differential equations on the real plane, which present a singularity of the type centre - focus. For these singularities it is defined a monodromy map, which is a germ of analytic diffeomorphism on the real line. A complexification of these germs allows to introduce (following R. Moussu and D. Cerveau) their vanishing holonomy. We study the relation existing between monodromy and vanishing holonomy; corollaries about normal forms are obtained.

D. CERVEAU. *Théorèmes de type fuchs pour les tissus feuilletés*

Après avoir rappelé des résultats pour certains bien anciens - et souvent méconnus - concernant les d-tissus sur un ouvert de $\mathbf{C}^n$ on s'intéresse à la dynamique des 3-tissus feuilletés hexagonaux globaux. Bien souvent - c'est le cas sur les espaces projectifs - un tel objet va présenter des singularités. On se propose, moyennant des hypothèses de type Fuchs, de donner une description des feuilles comme niveaux de fonctions multivaluées de type Liouville ($\Sigma\lambda_i \log f_i + H$, f_i et H holomorphes). Ce travail est motivé par la description de la variété des feuilletages algébriques de codimension un sur des espaces projectifs $\mathbf{CP}(n)$.

D. CERVEAU and A. LINS NETO. *Codimension one foliations in* $\mathbf{C}P^n$, $n \geq 3$, *with Kupka components*

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J. ECALLE. *Compensation of small denominators and ramified linearisation of local objects*

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J.E. FORNAESS and N. SIBONY. *Complex dynamics in higher dimension I*

We study global questions of iteration for holomorphic self maps of $\mathbf{P}^k$. After discussing some basic properties of holomophic and meromorphic maps in $\mathbf{P}^k$, we describe the maps f in $\mathbf{P}^2$ for which there exists a variety V satisfying $f^{-1}(V) = V$. We show that for a Zariski dense set of holomorphic maps in $\mathbf{P}^2$ the complement of the critical orbit is Kobayashi hyperbolic. We then study expansive properties of the maps in the interior of the complement of the critical orbit, under suitable hyperbolicity assumptions. We finally classify maps in $\mathbf{P}^2$ such that the orbit of the critical set is a variety.

Y. ILYASHENKO. *Normal forms for local families and nonlocal bifurcations*

The study of nonlocal bifurcations from the topological point of view requires not only topological, but smooth normal forms of the families of differential equations near singular points. In the first part of the paper a survey of these normal forms is presented. In the second part these normal forms are applied to the study of the bifurcations of planar vector fields. A complete list of polycycles appearing in generic two or three parameter families (Zoo of Kotova) is presented.

The proof of the finite cyclicity of elementary polycycles occuring in typical finite parameter families of planar vector fields is outlined.

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In this paper we prove that the $p+1$ Jordan normal forms of the monodromy operators of a regular linear system on $\mathbf{C}P^1$ with $p+1$ poles and the possible reducibility of the monodromy group define an analytic stratification of $(GL(n,\mathbf{C}))^p$ - the space of monodromy groups of such systems.

J.F. MATTEI and M. NICOLAU. *Equisingular unfoldings of foliations by curves*

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M. EL MORSALANI, A. MOURTADA and R. ROUSSARIE. *Quasi-regularity property for unfoldings of hyperbolic polycycles*

Some years ago Yu. Ilyashenko proved that the return map of any planar analytic hyperbolic polycycle has a quasi-regularity property. This implies that the polycycle is isolated among limit cycles, a key step in the proof that any polynomial planar vector field has just a finite number of limit cycles.

Here one proves a similar property for analytic one-parameter unfoldings of hyperbolic polycycles. As a consequence one deduces that some special unfoldings, with an unbrocken connection and a fixed product of again value ratios, have a finite cyclicity i.e., that the number of created limit cycles is bounded. Such unfoldings arrives for instance in quadratic vector fields, so that the result solves some of the cases in a general program formulated else where about the Hilbert's 16^{th} Problem for quadratic vector fields.

I. NAKAI. *A rigidity theorem for transverse dynamics of real analytic foliations of codimension one*

We prove a topological rigidity theorem for transverse dynamics of real analytic foliations of codimension one (Theorem 1) as well as for pseudogroups of real analytic diffeomorphisms of open neighbourhoods of 0 in the real line $\mathbf{R}$ (Theorem 3). We apply those results to prove the topological rigidity of analytic actions of the surface group Γ^g on the circle S^1 (Corollary 4) and also the topological invariance of the Godbillon-Vey class (Corollary 2).

R. PÉREZ-MARCO et J-C. YOCCOZ. *Germes de feuilletages holomorphes à holonomie prescrite*

Les germes de singularités irreductibles de feuilletages holomorphes de type Siegel sont définis par un champ de vecteurs holomorphe avec valeurs propres λ_1, λ_2 en $0 \in \mathbf{C}$, tels que $\lambda_1 \cdot \lambda_2 \neq 0$ et $\alpha = -\lambda_2/\lambda_1$ soit réel positif. L'holomie d'une des séparatrices determine le feuilletage.

Étant donné un germe holomorphe $f(z) = e^{2\pi i\alpha} + z + \mathcal{O}(z^2)$, on construit un tel feuilletage avec holonomie f. On obtient alors l'équivalence entre la classification analytique de ce type de germes de singularités et la classification analytique de germes de difféomorphismes holomorphes de $(\mathbf{C}, 0)$. La condition optimale arithmétique pour la linéarisation est obtenue.

C. ROCHE. *Densities for certain leaves of real analytic foliations*

Khovanskii's theory applies for non spiralating leaves of real analytic foliations as was shown in a joint work with R. Moussu. This theory proves that these leaves behave much like subanalytic subsets at least for the finiteness properties. Here it is shown that Kurdyka-Raby's technique can be applied to prove the existence of densities in each boundary point of a non spiralating leaf. This result doesn't use regularity assumptions on the boundary set of a very few results are known on these sets.

M. SHISHIKURA. *The boundary of the Mandelbrot set has Hausdorff dimension two*

The boundary of the Mandelbrot set M has Hausdorff dimension two and for a generic $c \in \partial M$, the Julia set of $z \mapsto z^2 + c$ also has Hausdorff dimension two. The proof of these statements is based on the analysis of the bifurcation of parabolic fixed points. This paper is an attempt to explain the main point of the proof, using the notion of geometric limit of rational maps.

D. TISCHLER. *Perturbations of critical fixed points of analytic maps*

We consider perturbations of a locally defined analytic function which has a critical point which is also a fixed point. The second derivative at the fixed point determines an inequality for the positions of the critical point, critical value and fixed point of the perturbed analytic function relative to some reference point. We apply this to the case of critical points of polynomials where the reference point is another fixed point. We also use the topological description of polynomials, all of whose critical points are fixed, to examine some inequalities relating the positions of critical points and critical values for polynomials which depend on the branching of the polynomial.

Astérisque

CH. BONATTI
X. GÓMEZ-MONT

The index of holomorphic vector fields on singular varieties I

Astérisque, tome 222 (1994), p. 9-35
<http://www.numdam.org/item?id=AST_1994__222__9_0>

The Index of Holomorphic Vector Fields on Singular Varieties I[1]

Ch. Bonatti and X. Gómez-Mont

Given a complex analytic space V with an isolated singuarity at p, there is a way to associate to a holomorphic vector field X on V an index at p a la Poincaré-Hopf $\mathrm{Ind}(X, V, p)$ (see [Se],[GSV]). The objective of this series of papers is to understand this index. In the present paper we relate it to the V-multiplicity:

$$\mu_V(X,p) = \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^n,p}}{(f_1,\ldots,f_\ell,X^1,\ldots,X^n)}$$

where $f_1,\ldots,f_\ell$ are generators of the ideal defining $V \subset \mathbf{C}^n$, X^j are the coordinate functions of a holomorphic vector field that extends X to a neighbourhood of 0 in $\mathbf{C}^n$ and the denominator denotes the ideal generated by the elements inside the parenthesis in the ring $\mathcal{O}_{\mathbf{C}^n,p}$ of germs of holomorphic functions at p. The main results are:

Theorem 2.2. *Let $(V,0) \subset \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space in the unit ball $\mathbf{B}_1$ which is smooth except for an isolated singularity at 0. Let Θ_r denote the Banach space of holomorphic vector fields on V_r with continuous extensions to ∂V_r, $r < 1$, with its natural structure as an analytic space of infinite dimension. Then:*

a) The function V-multiplicity at 0

$$\mu_V(\ ,0)\colon \Theta_r \to \mathbf{Z}^+ \cup \{\infty\}$$

[1]Research partially supported by CONACYT-CNRS and CONACYT-CNPq. The second author was a Guggenheim fellow during this research, and he would like to thank Bo Berndtson for useful conversations.

is upper semicontinuous and it is locally bounded at those points X where X has an isolated singularity on V at 0.

b) *The subsets of Θ_r defined by $\mu(\ ,0) \geq K$ are analytic subspaces and the minimum value of $\mu_V(\ ,0)$ in Θ_r is attained on an open dense subset $\tilde{\Gamma}_1$ of Θ_r.*

c) *The subset of Θ_r formed by vector fields whose critical set at 0 has positive dimension is an analytic subspace of Θ_r.*

We introduce the Euler characteristic $\chi_V(X,0)$ of $X \in \Theta_r$ at 0 in (2.10) and show:

Theorem 2.5. *For $X \in \Theta_r$ with an isolated singularity at 0, $s \ll r$ and $0 \ll \varepsilon$, we have:*

1) *For any family of vector fields $\{X_t\}_{t\in T}$, parametrized by a finite dimensional analytic space $(T,0) \to (\Theta_r, X)$ such that the V- multiplicity at 0 of the general vector field X_t of the family is minimal μ_V, we have:*

$$\chi_V(X,0) = \chi_0^{\text{tor}}(\mathcal{O}_{z_{T,s}}, \mathcal{O}_{\{X\}})$$

where the right and hand side is the Euler characteristic of higher torsion groups.

2) *For $Z \in U(X,\varepsilon)$ we have*

$$\chi_V(X,0) = \chi_V(Z,0) + \sum_{\substack{Z(p_j)=0 \\ p_j \in V_s - \{0\}}} \mu_V(Z,p_j)$$

3) *For $X \in \Theta_r$ with an isolated critical point at 0, we have:*

$$0 < \chi_V(X,0) \leq \mu_V(X,0)$$

and $\chi_V(X,0) = \mu_V(X,0)$ if and only if the universal critical set $\mathcal{Z}_r$ is π_1-anaflat at $(X,0)$ (in particular this happens in $\tilde{\Gamma}_1$).

Let $X \in \Theta_r$, we say that the *critical set of X does not bifurcate* if there is $\varepsilon > 0$ and $s > 0$ such that for $Y \in \mathbf{U}(X,\varepsilon) \subset \Theta_r$ we have that the only critical point of Y on V_s is 0, (that is, X has an isolated singularity at 0 as well as any sufficiently near vector field in Θ_r and there is no other critical point uniformly in a neighbourhood V_s of 0).

Theorem 2.6. *Let $(V,0) \subset \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space which is smooth except for an isolated singularity at 0, then the set of points in Θ_r whose critical set does not bifurcate contains the connected dense open subset $\tilde{\Gamma}_1 \subset \Theta_r$ consisting of vector fields with minimum V-multiplicity.*

Theorem 3.1. *Let $(V,0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space which is smooth except for an isolated singularity at 0, then there is an integer K such that*

$$Ind_W(X,V,0) = \chi_V(X,0) + K$$

for X in the dense open set Θ' of vector fields in Θ_r with an isolated singularity at 0. For X in the dense open set of Θ' where the universal critical set $\mathcal{Z}_r$ is Θ_r-anaflat we have

$$Ind_W(X,V,0) = \mu_V(X,0) + K$$

Corollary 3.2. *Let $(V,0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space which is smooth except for an isolated singularity at 0, then there is a constant L such that $Ind_W(X,V,0) \geq L$ for every germ of holomorphic vector field X on V with an isolated singularity at 0 on V.*

In the first section we analyse the index on smooth compact manifolds with boundary. We prove:

Proposition 1.1. *Let X and Y be C^1-vector fields defined on the compact manifold with boundary $(W,\partial W)$ and non-vanishing on ∂W and let $[\Gamma_X]$ denote the fundamental class of the graph of $X/\|X\|$ on the sphere bundle $\mathbf{S}$ of unit tangent vectors of W restricted to ∂W (with respect to some Riemannian metric on W). Then*

$$Ind(X,\partial W,W) - Ind(Y,\partial W,W) = [\Gamma_X] \cdot [\Gamma_{-Y}]$$

where we do the intersection in homology of $\mathbf{S}$.

In the second section we develop the properties of the V-multiplicity, and in the third we compare the V-multiplicity with the topological index.

1. The index of vector fields on manifolds with boundary

Let W be a compact oriented manifold of dimension m with boundary, ∂W, oriented in the natural way. Given a never vanishing C^0-vector field X in a neighbourhood of ∂W, the *index of* X *on the boundary of* W, $\text{Ind}(X, \partial W, W)$ may be defined by extending X to a vector field $\tilde{X}$ on W with isolated singularities, and then adding up the indices at the singularities of $\tilde{X}$. The index is independent of the chosen extension $\tilde{X}$ (see [Mi],[Se]).

To understand the dependence of the index on the manifold W, we will prove that the difference of the indices of 2 vector fields may be computed exclusively in terms of boundary data:

Proposition 1.1. *Let* X *and* Y *be* C^1*-vector fields defined on the compact manifold with boundary* $(W, \partial W)$ *and non-vanishing on* ∂W *and let* $[\Gamma_X]$ *denote the fundamental class of the graph of* $X/\|X\|$ *on the sphere bundle* $\mathbf{S}$ *of unit tangent vectors of* W *restricted to* ∂W *(with respect to some Riemannian metric on* W*). Then*

$$Ind(X, \partial W, W) - Ind(Y, \partial W, W) = [\Gamma_X] \cdot [\Gamma_{-Y}]$$

where we do the intersection in homology of $\mathbf{S}$.

Proof. Since the index and the fundamental classes do not change if we make a small perturbation, we will assume that X and Y are in general position. Namely we will assume that if the zeroes $\mathcal{Z} \subset \mathbf{C} \times W$ of the vector fields $\{X_t = (1-t)X + tY\}_{t\in[0,1]}$ intersect ∂W, say at 0_t, then at 0_t: X_t has a zero of multiplicity 1 and the projection of $\mathcal{Z}$ to W is transversal to the boundary ∂W.

The intuitive idea of the proof is very simple. The above family connects X with Y, and the only way the index as a function of $t \in [0,1]$ can change is if a zero leaves W at ∂W, or if a zero arrives at W through ∂W. By the transversality conditions we are assuming, this will happen every time X_t has a zero on ∂W, and it will give a contribution of ± 1, depending whether the index of X_t is ± 1 and whether the point is arriving or leaving W. One has to prove that one obtains the same sign from the contribution of the intersection $[\Gamma_X] \cdot [\Gamma_{-Y}]$ at the above point on ∂W.

Let p be a boundary point, and consider the convex hull $C = \langle X(p), Y(p) \rangle$ in T_pW. If 0 is not contained in C, then the vector fields X_t do not vanish at p for $t \in [0,1]$. If 0 is contained in C, then there is exactly one value of t where X_t vanishes at p. Note that this condition means that $X(p)$ and $Y(p)$ are linearly dependent with distinct orientation, and this is equivalent to the fact that Γ_X and Γ_{-Y} intersect. So the only point left is to show that one obtains the same sign from the intersection $[\Gamma_X] \cdot [\Gamma_{-Y}]$ at p as the difference of the indices $\mathrm{Ind}(X_{t+\varepsilon}, \partial W, W) - \mathrm{Ind}(X_{t-\varepsilon}, \partial W, W)$.

To simplify notation, let $(x_1, \dots, x_n)$ be coordinates around $p = 0$, where W and ∂W are defined by $x_n \geq 0$ and $x_n = 0$ respectively. Let $Z = (Z_1, \dots, Z_n)$ be a C^1-vector field with a critical point at 0 of multiplicity 1, $Y = (Y_1, \dots, Y_n)$ a C^1-vector field with $Y_1(0) > 0$, and we are interested in computing the contribution to the index of the family $Z + tY$, when t passes through 0 in the positive direction.

Let $DZ(0)$ be the derivative of Z at 0. It is an invertible matrix and the sign of $\det[DZ(0)]$ is $\mathrm{Ind}(Z, 0)$, and hence it also $\mathrm{Ind}(Z_t, 0_t)$, where 0_t is the zero of Z_t near to 0. (One may think that everything extends to a neighbourhood of ∂W outside of W, so as to "see" 0_t for all small values of t, and not only for the ones that are in W). A simple calculation shows that the curve 0_t intersects ∂W with velocity vector

$$(d0_t/dt)(0) = -[DZ(0)]^{-1} Y(0)$$

and hence the zero set is entering W if $dx_n[-[DZ(0)]^{-1}Y(0)]$ is positive, and is leaving W if it is negative. By Crammer's rule, we have

$$dx_n[-[DZ(0)]^{-1}Y(0)] = \frac{1}{\det[DZ(0)]}\begin{bmatrix} \partial Z^1/\partial x_1 & \dots & \partial Z^n/\partial x_1 \\ \vdots & & \vdots \\ \partial Z^1/\partial x_1 & \dots & \partial Z^n/\partial x_{n-1} \\ Y^1 & \dots & Y^n \end{bmatrix}(0)$$

$$= -\det[A]/\det[DZ(0)] \tag{1.1}$$

where the matrix A is defined by the above formula. Hence we obtain for $\varepsilon > 0$:

$$\operatorname{Ind}(Z_\varepsilon, W, \partial W) - \operatorname{Ind}(Z_{-\varepsilon}, W, \partial W) = \operatorname{Ind}(Z, 0)\operatorname{Sign}(-\det[A]/\det[DZ(0)])$$

$$= -\operatorname{Sign}[\det[A]] \tag{1.2}$$

We will now compute $[\Gamma_{Z+Y}]\cdot[\Gamma_{-(Z-Y)}]$. Dividing by $Z^1 + Y^1$ (respectively by $Y^1 - Z^1$), which is positive, amounts to taking coordinates in the sphere bundle, so Γ_{Z+Y} and $\Gamma_{-(Z-Y)}$ are the graphs of the functions

$$\begin{aligned}
&\gamma^+(x_1, \dots, x_{n-1}) \\
&= ((Z^2 + Y^2)/(Z^1 + Y^1), \dots, (Z^n + Y^n)/(Z^1 + Y^1)(x_1, \dots, x_{n-1}, 0) \\
&\gamma^-(x_1, \dots, x_{n-1}) \\
&= ((Y^2 - Z^2)/(Y^1 - Z^1), \dots, (Y^n - Z^n)/(Y^1 - Z^1)(x_1, \dots, x_{n-1}, 0)
\end{aligned}$$

The intersection number $[\Gamma_{Z+Y}]\cdot[\Gamma_{-(Z-Y)}]$ is equal to the sign of determinant of the matrix obtained by grouping the derivatives of the graphs of γ^+ and γ^-:

$$\det\begin{bmatrix} I_{n-1} & I_{n-1} \\ D\gamma^+ & D\gamma^- \end{bmatrix}(0) = \det[D\gamma^- - D\gamma^+](0)$$

A simple calculation shows that

$$(D\gamma^\pm)_{ij} = \frac{1}{(Y^1(0))^2}\det\begin{bmatrix} Y^1(0) & \frac{\partial}{\partial x_j}[Y^1 \pm Z^1](0) \\ Y^i(0) & \frac{\partial}{\partial x_j}[Y^i \pm Z^i](0) \end{bmatrix}$$

Hence

$$[D\gamma^- - D\gamma^+]_{ij} = -\frac{2}{(Y^1(0))^2}\det\begin{bmatrix} Y^1(0) & \frac{\partial}{\partial x_j}Z^1(0) \\ Y^i(0) & \frac{\partial}{\partial x_j}Z^i(0)\end{bmatrix} \tag{1.3}$$

We now need to use a formula involving determinants: Let $A = [a_{ij}]$ be an $s \times s$ matrix, and let $B = (b_{ij})$ be the $(s-1) \times (s-1)$ matrix whose general term is

$$b_{ij} = \det\begin{bmatrix} a_{11} & a_{1i} \\ a_{j1} & a_{ji}\end{bmatrix} \tag{1.4}$$

$i, j = 2, \dots, n$, then we have that

$$(a_{11})^{s-2}\det[A] = \det[B] \tag{1.5}$$

This formula may be proved first for diagonal matrixes, and then by showing that both sides are left invariant under the elementary operations of rows and columns.

Consider the matrix A in (1.1). Let A' be the matrix obtained by moving the last row to the first. We have $\det[A] = (-1)^{n-1}\det[A']$, and let B be the matrix obtained from A' as in (1.4). Nothing that $[D\gamma^- - D\gamma^+] = -2B/Y^1(0)^2$ we have by (1.5):

$$\begin{aligned}\det[A] &= (-1)^{n-1}\det[A'] = (-1)^{n-1}Y^1(0)^{2-n}\det[B] \\ &= (-1)^{n-1}Y^1(0)^{4-n}\det[D\gamma^- - D\gamma^+]/(-2)^{n-1}\end{aligned}$$

Hence $\det[A]$ and $\det[D\gamma^- - D\gamma^+]$ have the same sign. Using (1.2) we obtain: $\mathrm{Ind}(Z_\varepsilon, W, \partial W) - \mathrm{Ind}(Z_{-\varepsilon}, W, \partial W) = -\mathrm{Sign}[\det[A]] = -\mathrm{Sign}[\det[D\gamma^- - D\gamma^+]] =$

$$= [\Gamma_{-(Z-Y)}]\cdot[\Gamma_{Z+Y}] = [\Gamma_{Z-Y}]\cdot[\Gamma_{-(Z+Y)}]$$

This proves the Proposition. □

If we denote by $Vec(\partial W)^+$ the set of C^1-vector fields defined and never zero on a neighbourhood of ∂W, the Index is an integer valued function with $Vec(\partial W)^+$ as a domain:

$$\mathrm{Ind}\colon Vec(\partial W)^+ \to \mathbf{Z}$$

Let $(W', \partial W')$ be another compact manifold with boundary and $\phi\colon \partial W \to \partial W'$ an orientation preserving diffeomorphism of the boundaries. We may extend this diffeomorphism to a diffeomorphism of a neighbourhood of the boundaries $\phi\colon W_1 \to W_1'$. Given a non-singular C^0-vector field X defined on the neighbourhood W_1 of ∂W, we may transport it via ϕ to a vector field X' defined on W_1'.

$$
\begin{array}{ccc}
\mathrm{Ind}_{\partial W}\colon Vec(\partial W)^+ & \longrightarrow & \mathbf{Z} \\
\downarrow \phi & & \downarrow \alpha \\
\mathrm{Ind}_{\partial W'}\colon Vec(\partial W')^+ & \longrightarrow & \mathbf{Z}
\end{array}
\tag{1.6}
$$

It follows from Proposition 1 that for $X, Y \in Vec(\partial W)^+$ we have:

$$
\begin{aligned}
\mathrm{Ind}(X, \partial W, W) - \mathrm{Ind}(Y, \partial W, W) &= [\Gamma_X] \cdot [\Gamma_{-Y}] = [\Gamma_{\partial_* X}] \cdot [\Gamma_{\phi_* - Y}] \\
&= \mathrm{Ind}(\phi_* X, \partial W', W') - \mathrm{Ind}(\phi_* Y, \partial W', W')
\end{aligned}
$$

And hence for variable X, and a fixed Y we obtain:

$$
\mathrm{Ind}_{\partial W'}(\phi_* X) = \mathrm{Ind}_{\partial W}(X) - (\mathrm{Ind}_{\partial W}(Y) - \mathrm{Ind}_{\partial W'}(\phi_* Y))
$$

Hence we may complete (1.6) by a map α which is substraction by an integer. This integer may be computed by taking the difference of the two indices with respect to any pair of vector fields in $Vec(W)^+$. To see who this integer is, let Z be the vector field in $Vec(W)^+$ which is always pointing inward. In this case, by the relative Poincaré-Hopf Index Theorem (see [Pu]), it is $\chi(M) - \chi(\partial M)$, where χ is the Euler Poincaré characteristic. Hence we obtain:

Corollary 1.2. *Let $(W, \partial W)$ and $(W, \partial W')$ be manifolds with diffeomorphic boundaries and ϕ a diffeomorphism of a neighbourhood of the boundaries. Then for any C^0-vector field defined and non-vanishing on a neighbourhood of ∂W we have*

$$
Ind(X, \partial W, W) = Ind(\phi_* X, \partial W', W') + [\chi(W) - \chi(W')]
$$

Remark. The above result may be also obtained from Pugh's Poincaré- Hopf Index Theorem for compact manifolds with boundary ([Pu]) since it expresses the index as the Euler-Poincaré characteristic of the manifold with boundary plus a contribution of the tangency behaviour of the vector field with the boundary. Hence taking the difference of both extensions we obtain that the difference of the indixes will be the difference of the Euler-Poincaré characteristics of the manifolds, since the boundary contributions are equal and hence cancel each other.

We will now give another explanation of the ambiguity of the definition of the index as a number just from its behaviour at the boundary.

Let $(W, \partial W)$ be a compact manifold with boundary, choose a Riemannian metric on W and let T^1W be the unit sphere bundle in the tangent bundle of W, and $\mathbf{S} = T^1W\,|_{\partial W}$ its restriction to the boundary. The natural projection $\rho\colon \mathbf{S} \to \partial W$ has the structure of an $(m-1)$ sphere bundle over ∂W. $\mathbf{S}$ has dimension $2(m-1)$ and its cohomology groups $H^q(\partial W, \mathbf{Z})$ may be calculated using the spectral sequence of the fibration, since the cohomology bundles $R^q\rho_*(\mathbf{Z_S})$ over ∂W are non- vanishing except for dimension 0 and $n-1$ (since it is a sphere bundle) and both of them are trivial bundles with Z as fibers, since the monodromy group is acting trivially (it sends the fundamental class to itself, since everything is oriented). The spectral sequence degenerates since $H^p(\partial W, R^q\rho_*(\mathbf{Z_S}))$ is non-zero only for $q = 0, n-1$. Hence the cohomology of $\mathbf{S}$ consists of 2 copies of the cohomology of ∂W glued together in the middle dimension:

$$H^p(\mathbf{S}, \mathbf{Z}) = H^p(\partial W, \mathbf{Z}) \text{ for } 0 \leq p \leq m-2$$

$$H^p(\mathbf{S}, \mathbf{Z}) = H^{p-(m-1)}(\partial W, \mathbf{Z}) \text{ for } m \leq p \leq 2m-2$$

(1.7)

$$0 \longrightarrow H^{m-1}(\partial W, \mathbf{Z}) \xrightarrow{\rho^*} H^{m-1}(\mathbf{S}, \mathbf{Z}) \xrightarrow{\nu} H^0(\partial W, \mathbf{Z}) \longrightarrow 0$$

We are interested in the middle group $H^{m-1}(\mathbf{S}, \mathbf{Z})$. $H^{m-1}(\partial W, \mathbf{Z_S}) = \oplus_j H^{m-1}(\partial W_j, \mathbf{Z_S})$, where $\{\partial W_j\}$ are the connected components of ∂W, say r of them. Hence $H^{m-1}(\mathbf{S}, \mathbf{Z})$ has a submodule canonically isomorphic to

$\mathbf{Z}^r$, obtained by pulling back the fundamental classes of the boundary components. The quotient group is again canonicaly isomorphic to $\mathbf{Z}^r$, but there is no canonical splitting. $H^{m-1}(\mathbf{S},\mathbf{Z})$ is hence free of rank $2r$.

If X is a C^0 vector field on W, non-vanishing on ∂W, the fundamental class $[\Gamma_X]$ of the graph of $X/\|X\|$ restricted to ∂W is an element of $H_{m-1}(\partial W,\mathbf{Z})$. It is the class $[\Gamma_X]$ which carries the topological information of the index. Since it is a section of ρ, it projects to $(1,\ldots,1)$ in (1.7). The difference of two such fundamental classes will produce integers on each boundary component. If one wants to obtain an integer for a vector field, then one has to choose a splitting of (1.7), which is a non-canonical operation. This is carried out by choosing the bounding manifold W.

2. Holomorphic vector fields on singular spaces

Let $p \in V$ be a point of a complex analytic space of dimension N and let $(V,p) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be a local embedding of V into the unit ball $\mathbf{B}_1$. We will denote $V \cap \mathbf{B}_r$ by V_r, where $\mathbf{B}_r$ is the ball around 0 and radius $r \leq 1$ in $\mathbf{C}^n$. The ring $\mathcal{O}_{V,p}$ of germs of holomoprhic functions at p may be represented by the quotient $\mathcal{O}_{\mathbf{C}^n,0}/\mathcal{J}$, where $\mathcal{J}$ is the ideal of germs of holomorphic functions on $(\mathbf{C}^n,0)$ vanishing on V. A *germ of a holomorphic vector field at p* is a derivation

$$X\colon \mathcal{O}_{V,p} \to \mathcal{O}_{V,p}$$

(see [Ro]). Given a holomorphic vector field on $(V,0)$, it gives rise to a diagram

$$\begin{array}{ccc}
\mathcal{O}_{\mathbf{C}^n,0} & \overset{\tilde{X}}{- \to} & \mathcal{O}_{\mathbf{C}^n,0} \\
\pi \downarrow & & \downarrow \pi \\
\mathcal{O}_{V,p} & \overset{X}{\longrightarrow} & \mathcal{O}_{V,p}
\end{array}$$

We can always lift X to a derivation $\tilde{X}$ on $\mathcal{O}_{\mathbf{C}^n,0}$. To see this let $(z_1,\ldots,z_n)$ be coordinates of $\mathbf{C}^n$, and let A_j be π- liftings to $\mathcal{O}_{\mathbf{C}^n,0}$ of $X(\pi(z_j))$. One easily checks that $\tilde{X} = \sum A_j \frac{\partial}{\partial z_j}$ makes the above diagram commutative on the generators z_j, and applying linearity and Leibnitz's rule, we see that the diagram is commutative. $\tilde{X}$ will send the ideal $\mathcal{J}$ defining V to itself, and conversely, any such derivation will induce a holomorphic vector field on V. A germ of a holomorphic vector field at $(V, 0)$ induces a (usual) holomorphic vector field on the smooth points of V near 0.

If X is a holomorphic vector field defined on the non-singular points of V, then using an embedding of V into $\mathbf{C}^n$, we may express $X = \sum X_j \frac{\partial}{\partial z_j}$, where X_j are holomorphic functions on $V - \mathrm{Sing}(V)$. If V has a normal singularity at p then, by the second Riemann's Removable Singularity Theorem ([Fi], p.120), the functions X_j extend to holomorphic functions on V and the vector field obtained with these extensions gives a holomorphic extension of the vector fied X from $V - \mathrm{Sing}V$ to V. Hence for normal singularities, holomorphic vector fields on V coincide with (usual) holomorphic vector fields on $V - \mathrm{Sing}(V)$. If $(V, p) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n, 0)$ is an analytic space then the sheaf of holomorphic vector fields Θ_V is coherent ([Ro]). We shall denote by Θ_r *the Banach space of continuous vector fields* defined on $\overline{V}_r$ and holomorphic in V_r, with the C^0-norm. We will also denote the ball $\{Y \in \Theta_r / \|X - Y\| < \varepsilon\}$ by $\mathbf{U}(X, \varepsilon)$. The ring of germs of holomorphic vector fields $\Theta_{V,p}$ is endowed with the analytic topology. Recall that a sequence $\{X_n\}$ converges to X in $\Theta_{V,p}$ if they are all defined in a small neighbourhood $\overline{V}_r \subset V$ of p, and they converge in Θ_r (see [G-R]). Note that by the Weierstrass approximation theorems, Θ_r is dense in $\Theta_{V,p}$, so that many properties for germs will follow by considering similar properties in Θ_r.

Proposition 2.1. *Let $(V, 0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n, 0)$ be an analytic space which is smooth except for an isolated singularity at 0, then the subset $\Theta'_r \subset \Theta_r$ consisting of holomorphic vector fields that have at 0 an isolated singularity is a connected dense open subset in Θ_r.*

Proof. Assume that X has an isolated critical point at 0. For $s < r$ small, X restricted to ∂V_s does not vanish. Let 2ε be the minimum value of $\|X\|$

on ∂V_s. If $Y \in \Theta_r$ with $\|Y\| < \varepsilon$, then $X + Y$ cannot vanish on ∂V_s (for then $X(q) = -Y(q)$). This implies that $X + Y$ will have an isolated critical point at 0, since if it vanished on a set of positive dimension passing through 0, this set would have to intersect ∂V_s (otherwise, one would have a compact complex manifold in V_s of positive dimension). This shows that Θ'_r is open in Θ.

Let $X_0 \in \Theta_r$ and let $\varepsilon > 0$ be given. To each vector field X in $\mathbf{U}(X_0, \varepsilon)$ we can associate to it the dimension of its critical set at 0, $\dim_0(\{X = 0\})$. Let Y be a vector field where this minimum is attained. We claim that Y has an isolated singularity at 0. So assume that Y does not have an isolated singularity at 0.

Let $P\mathbf{T}(V - \{0\}) \subset \mathbf{C}^n \times \mathbf{P}^{n-1}_{\mathbf{C}}$ be the (complex) projectivized tangent bundle of $V - \{0\}$, denote by $\mathbf{P}_r$ its closure and $\pi: \mathbf{P}_r \to V_r$ its projection to the first factor. $\mathbf{P}_r$ is an analytic space, π is a proper holomorphic map which is a complex projective bundle outside of 0 and the fibre over 0 is the tangent cone of V at 0 (see [Wh]). Let $A = A_1 \cup \cdots \cup A_m$ be the decomposition in irreducible components of $\{Y = 0\} \subset V_r$ passing through 0. By assumption A does not reduce to 0. Let $\Gamma_Y \subset \mathbf{P}_r$ be the closure of the graph of $\mathrm{Proj}(Y)$ on $V_r - A$. Γ_Y has dimension $N = \dim(V_r)$. The intersection of Γ_Y with $\pi^{-1}(A)$ has dimension at most $n - 1$, since it is contained in the boundary of the graph of Y, which has dimension N. Since $\pi^{-1}(A_j)$ has dimension $N - 1 + \dim(A_j) > N - 1$, we may choose points in $\pi^{-1}(A_j) - \Gamma_Y$. That is, there are points $p_j \in A_j - \{0\}$ arbitrarily close to 0 and vectors v_j tangent to V_r at p_j such that $\mathrm{Proj}(v_j)$ is disjoint from Γ_Y. Since V_r is a Stein space, there is a vector field Z on V_r such that $Z(p_j) = v_j$. We claim that $Y + tZ$, for small values of $t \neq 0$ will have singular set at 0 of dimension smaller than the critical set of Y, contradicting the choice of Y.

To see this, let $s < r$ so that $A \cap V_s = \{Y = 0\} \cap V_s$. Without loss of generality, we may assume that $p_j \in V_s$ (since the set of points that do not satisfy the defining condition of p_j is a proper subvariety of each A_j). Let C be the set of points of $V_s \times \mathbf{C}$ where $Y + tZ$ vanishes and let $\rho: V_s \times \mathbf{C} \to \mathbf{C}$

be the projection to the second factor.

We claim that the A_j's are irreducible components of C. To see this, consider $(Y + tZ)(p) = 0$ for p near to p_j. By the way we chose $Z(p_j)$, one may conclude that $Z(p)$ is linearly independent with $Y(p)$ if $Y(p) \neq 0$. Hence $(Y+tZ)(p) \neq 0$. If $Y(p) = 0$, then for $t \neq 0$ we have $(Y+tZ)(p) = tZ(p) \neq 0$. This implies that the decomposition into irreducible components of C in a neighbourhood of (0,0) is of the form $C = A_1 \cup \cdots \cup A_m \cup C_1 \cup \cdots \cup C_r$. Hence the irreducible components C_k are not contained in $\rho^{-1}(0)$ and its intersection with $\rho^{-1}(0)$ does not contain any A_j. Hence $C_k \cap \rho^{-1}(0)$ has dimension strictly smaller than the dimension of A. By the theorem of upper semicontinuity of the dimension of the fibers of a holomorphic map, we conclude that $(C_1 \cup \cdots \cup C_r) \cap \rho^{-1}(t_0)$ has dimension smaller than the dimension of A, for $t_0 \neq 0$. But this set is exactly the critical set of $Y + t_0 Z$. This contradicts the hypothesis that the minimum dimension of its critical set is attained at Y. Hence Y has isolated singularities. This shows that Θ'_r is dense in Θ_r.

To see that Θ'_r is connected, let X an Y belong to Θ'_r, then consider the family $\{X+tY\}_{t\in\mathbf{C}}$. The critical set C of the family consists of $(t,p) \in \mathbf{C}\times V$ such that $(X + tY)(p) = 0$. C is an analytic subvariety, containing the line $\mathcal{L}_0 = \mathbf{C} \times \{0\}$. By hypothesis (0,0) and (1,0) lie on $\mathcal{L}_0$ and in no other irreducible component of C. Hence $\mathcal{L}_0$ is an irreducible component of C. The other irreducible components of C intersect $\mathcal{L}_0$ on a finite number of points. Hence all points of $\mathcal{L}_0$ except a finite number represent vector fields with isolated singularities. Hence, Θ'_r is connected. $\square$

From now on, we assume that $V \subset \mathbf{B}_1 \subset \mathbf{C}^n$ is a smooth variety of dimension N except for an isolated singularity at 0 (V non-smooth at 0). Let $\mathcal{J} = (f_1(z), \ldots, f_\ell(z))$ be the ideal sheaf defining V_r in $\mathbf{B}_r$. Consider the Banach space Θ_r as an infinite dimensional analytic space (see [Do1]) and let $e\colon \Theta_r \times V_r \to \mathbf{C}^n$ be the valuation map

$$e(X, z_0) = e\left(\sum a_I^j z^I \frac{\partial}{\partial z_j}, z_0\right) = \sum a_I^j z_0^I \frac{\partial}{\partial z_j} = \sum e^j(X, z_0)\frac{\partial}{\partial z_j} = X(z_0)$$

It is an analytic function on the Banach space $\Theta_r \times V_r$, linear in the first variable. The ***universal critical set*** $\mathcal{Z} = \mathcal{Z}_r$ is the analytic subvariety of $\Theta_r \times V_r$ defined by the sheaf of ideals

$$\text{(2.1)} \qquad \mathcal{I}_r = (f_1(z), \dots, f_\ell(z), e^1(X,z), \dots, e^n(X,z)) \subset \mathcal{O}_{\Theta_r \times \mathbf{B}_r}$$

The above generators of the ideal $\mathcal{I}_r$ give a finite presentation of $\mathcal{O}_{\mathcal{Z}}$ as an $\mathcal{O}_{\Theta \times \mathbf{B}}$-module:

$$\text{(2.2)} \qquad \mathcal{O}^{\oplus \ell + n}_{\Theta_r \times \mathbf{B}_r} \xrightarrow{\Phi} \mathcal{O}_{\Theta_r \times \mathbf{B}_r} \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0$$

where the map Φ is matrix multiplication with $(f_1, \dots, f_\ell, e^1, \dots, e^n)$.

Let π_1 and π_2 be the restriction to $\mathcal{Z}$ of the projections to the factors Θ_r and $\mathbf{B}_r$, respectively. We analyse first π_2. Since V has an isolated singularity at 0, all vector fields on V vanish at 0, hence $\Theta_0 \subset \mathcal{Z}$, where $\Theta_0 = \Theta_r \times \{0\}$ is the zero section. This means that $\pi_2^{-1}(0) = \Theta_0$, which is a subvariety of $\Theta_r \times \mathbf{B}_r$ of codimension n. By restricting $\pi_2 : \mathcal{Z} - \Theta_0 \to V_r - \{0\}$ we see that the fiber $\pi_2^{-1}(p)$, with $p \in V_r - \{0\}$, is a vector space of codimension N in Θ_r (since V_r is Stein and $N = \dim V_r$) and hence $\pi_2^{-1}(V_r - \{0\})$ has the structure of a vector bundle over V_r whose fibers have codimension N in Θ_r. Hence $\pi_2^{-1}(V_r - \{0\})$ is smooth of codimension n in $\Theta_r \times \mathbf{B}_r$ (the same codimension as Θ_0). Let $\Theta_{\mathbf{sing}} \subset \mathcal{Z}$ be the closure of $\pi_2^{-1}(V - \{0\})$. Set theoretically $\mathcal{Z} = \Theta_0 \cup \Theta_{\mathbf{sing}}$, but $\mathcal{Z}$ will in general have a non-trivial scheme structure on Θ_0.

We now view $\mathcal{Z}$ as a space over Θ_r via the projection $\pi_1 : \mathcal{Z} \to \Theta_r$. The fibre $\pi_1^{-1}(X)$ over the vector field X is set theoretically the critical set $\{z \in V_r / X(z) = 0\}$ of X. Recall that the process of restricting to a π-fibre $\{X\} \times \mathbf{C}^n$ is carried out by tensoring with $\otimes_{\mathcal{O}_{\Theta_r \times \mathbf{B}_r}} \mathcal{O}_{\{X\} \times \mathbf{B}_r}$. In particular, $\mathcal{O}_{\mathcal{Z}} \otimes_{\mathcal{O}_{\Theta_r \times \mathbf{B}_r}} \mathcal{O}_{\{X\} \times \mathbf{B}_r}$ has support on the critical set of X and for an isolated singularity of X at p, its dimension is the *V-multiplicity of X at $p \in V_r$*:

$$\text{(2.3)} \qquad \mu_V(X,p) = \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^n, p}}{(f_1, \dots, f_\ell, X^1, \dots, X^n)}$$

Note that $\mu_V(X,p)$ depends exclusively on $X\,|_V$, since the contribution from choosing another extension to $\mathbf{C}^n$ is cancelled by the terms $(f_1,\dots,f_\ell)$ and that it is strictly positive exactly at the critical set $\{X=0\}$ of X. Note that (2.3) is the corank of Φ in (2.2) over the point $(X,0)$, or equivalently, (2.2) gives a way to express the V-multiplicity as a corank of a matrix with parameters. We will exploit this expression to describe the dependence of the V-multiplicity on X; but technically it wil be simpler to consider an approximation of Φ on infinitesimal neighbourhoods of Θ_0.

We will now analyse the structure of $\mathcal{Z}$ at the zero section Θ_0. Let $\mathcal{K}=(z_1,\dots,z_n)\subset\mathcal{O}_{\Theta_r\times\mathbf{C}^n}$ be the ideal of definition of Θ_0, and denote by Θ_0^j the j^{th} infinitesimal neighbourhood of Θ_0 defined by the sheaf of ideals $\mathcal{K}^{j+1}\subset\mathcal{O}_{\Theta_r\times\mathbf{C}^n}$ generated by the monomials in z of degree $j+1$. As a space, it consists of Θ_0 but its function theory remembers the Taylor series in the z-variables up to order j. Using the presentation (2.2) of $\mathcal{Z}$, we note that $\Phi((\mathcal{K}^{j+1})^{\oplus\ell+n})\subset\mathcal{K}^{j+1}$, so that it will induce an exact commutative diagram

$$
\begin{array}{cccccccc}
 & \mathcal{O}^{\oplus\ell+n}_{\Theta_r\times\mathbf{B}_r} & \xrightarrow{\Phi} & \mathcal{O}_{\Theta_r\times\mathbf{B}_r} & \longrightarrow & \mathcal{O}_{\mathcal{Z}} & \longrightarrow 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
(2.4) & \mathcal{O}^{\oplus\ell+n}_{\Theta_r\times\mathbf{B}_r}/(\mathcal{K}^{j+1})^{\oplus\ell+n} & \xrightarrow{\Phi^j} & \mathcal{O}_{\Theta_r\times\mathbf{B}_r}/\mathcal{K}^{j+1} & \longrightarrow & \mathcal{O}_{\mathcal{Z}^j} & \longrightarrow 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}
$$

where $\mathcal{Z}^j$ is the analytic intersection of $\mathcal{Z}$ and Θ_0^j, and its defining ideal is spanned by $\mathcal{I}$ and $\mathcal{K}^{j+1}$. From the inclusions

$$\mathcal{J}=(\mathcal{I},\mathcal{K})\supset\cdots\supset(\mathcal{I},\mathcal{K}^{j+1})\supset(\mathcal{I},\mathcal{K}^{j+2})\supset\cdots\supseteq\mathcal{I} \tag{2.5}$$

we obtain the inclusions of analytic spaces

$$\theta_0=\mathcal{Z}^1\subset\cdots\subset\mathcal{Z}^j\subset\mathcal{Z}^{j+1}\subset\cdots\subset\mathcal{Z} \tag{2.6}$$

Φ^j in (2.4) is a sheaf map between free sheaves over Θ_0, so it may be identified with a (finite dimensional) vector bundle map between (trivial) bundles over Θ_0. Hence Φ^j may be represented by a (finite dimensional) matrix with parameters. Denote by $\Phi^j(X)\colon \mathcal{O}_{\{X\},\mathbf{B}_r,0}/m^{j+1} \to \mathcal{O}_{\{X\},\mathbf{B}_r,0}/m^{j+1}$, where m is the maximal ideal in $\mathcal{O}_{\{X\},\mathbf{B}_r,0}$, the restriction of Φ^j to the point $(X,0)$ and define

$$\mu(\mathcal{Z}^j, X) := \operatorname{corank}_{\mathbf{C}}[\Phi^j(X)] = \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^n,p}}{(f_1,\ldots,f_\ell,X^1,\ldots,X^n,z_1^{j+1},\ldots,z_n^{j+1})}$$

We have for $j \le k$:

$$1 \le \mu(\mathcal{Z}^j, X) \le \mu(\mathcal{Z}^k, X) \le \mu_V(X,0)$$

Theorem 2.2. *Let $(V,0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space which is smooth except for an isolated singularity at 0, and let Θ_r denote the Banach space of holomorphic vector fields on V_r with continuous extensions to ∂V_r, $r<1$, and let $\mathcal{Z}$, $\mathcal{Z}^j = \mathcal{Z}_\cap\Theta_0^j \subset \Theta_r \times \mathbf{B}_r$ be the universal critical set and its approximation sets. Then, there is a descending sequence of analytic subvarieties of finite codimension A^k, $A^{k+1} \subset A^k$, and an integer J such that:*

a) $\mathcal{Z}^j \cap (\Theta - A^J) = \mathcal{Z}^k \cap (\Theta - A^J)$ *for* $j,k \ge J$.

b) $\mu_V(X,0) = \mu(\mathcal{Z}^j, X)$ *for* $X \notin A^J$ *and* $j \ge J$.

c) The function V-multiplicity at 0

$$\mu_V(\ ,0)\colon \Theta_r \to \mathbf{Z}^+ \cup \{\infty\}$$

is upper semicontinuous and it is locally bounded at those points X where X has an isolated singularity on V at 0 (Θ_r has for this the topology whose closed sets are the analytic subsets).

d) The subsets of Θ_r defined by $\mu(\ ,0) \ge K$ are analytic subspaces and the minimum value of $\mu_V(\ ,0)$ in Θ_r is attained on an open dense subset $\tilde{\Gamma}_1$ of Θ_r.

e) The subset of Θ_r formed by vector fields whose critical set at 0 has positive dimension is the analytic subspace of Θ_r defined by $\cap A^j$.

Proof. For every j, the inclusion $(\mathcal{I}, \mathcal{K}^{j+2}) \subseteq (\mathcal{I}, \mathcal{K}^{j+1})$ induces an exact sequence of sheaves on $\Theta_r \times \mathbf{B}_r$

$$\begin{array}{lccccccc}
 & & & 0 & & & & \\
 & & & \downarrow & & & & \\
 & & & \mathcal{K}^{j+1}/\mathcal{K}^{j+2} & & & & \\
 & & & \downarrow & & & & \\
(2.7) & \mathcal{O}^{\oplus \ell+n}_{\Theta_r\times\mathbf{B}_r}/(\mathcal{K}^{j+2})^{\oplus \ell+n} & \xrightarrow{\Phi^{j+1}} & \mathcal{O}_{\Theta_r\times\mathbf{B}_r}/\mathcal{K}^{j+2} & \longrightarrow & \mathcal{O}_{\mathcal{Z}^{j+1}} & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
 & \mathcal{O}^{\oplus \ell+n}_{\Theta_r\times\mathbf{B}_r}/(\mathcal{K}^{j+1})^{\oplus \ell+n} & \xrightarrow{\Phi^{j}} & \mathcal{O}_{\Theta_r\times\mathbf{B}_r}/\mathcal{K}^{j+1} & \longrightarrow & \mathcal{O}_{\mathcal{Z}^{j}} & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & & 0 & & 0 & &
\end{array}$$

where the first 2 columns may be interpreted as (finite dimensional) vector bundle maps over Θ_0. The corank of $\Phi^{j+1}(X)$ is equal to the corank of $\Phi^j(X)$ if and only if $\mathcal{K}^{j+1}/\mathcal{K}^{j+2}(X)$ is contained in the image of $\Phi^{j+1}(X)$. The increase in the corank from $\Phi^j(X)$ to $\Phi^{j+1}(X)$ is the codimension of

$$\mathrm{Image}[\Phi^{j+1}(X)] \cap [\mathcal{K}^{j+1}/\mathcal{K}^{j+2}(X)] \subset \mathcal{K}^{j+1}/\mathcal{K}^{j+2}(X).$$

A *stratification of* Θ_0 consist of a disjoint decomposition of Θ_0 by subsets $\Gamma_1, \dots, \Gamma_s$ where each Γ_i is an analytic subvariety minus another analytic subvariety (the ones that will actually appear have finite codimension). Since Θ_0 is irreducible there is one and only one component that is open. We will assume that for any stratification of Θ_0 this open component is the first one Γ_1.

We may first find a stratification of Θ_0 so that the corank of $\Phi^{j+1}(X)$ is constant on each strata. Then one may further stratify according to the dimension of $Im\Phi^{j+1} \cap (\mathcal{K}^{j+1}/\mathcal{K}^{j+2})(X)$. In all, we obtain a stratification $\{\Gamma_1^{j+1}, \dots, \Gamma_c^{j+1}\}$ of Θ_0 such that the codimension of $Im\Phi^{j+1} \cap (\mathcal{K}^{j+1}/\mathcal{K}^{j+2})(X)$ is constant on the stratification, say d_i^{j+1} on Γ_i^{j+1}. Since the numbers d_i^{j+1} are defined as coranks of a matrix with parameters, they behave upper semicontinuosly, in the sense that if Γ_k^{j+1} is in the closure of Γ_h^{j+1}, then $d_k^{j+1} \geq d_h^{j+1}$. Due to this property, we may assume that Γ_1^{j+1} consists of all those points X

of Θ_0 where the minimum is attained (i.e. $d_1^{j+1} < d_k^{j+1}$ for $k \geq 2$). We have

$$\mu(\mathcal{Z}^{j+1}, X) \geq \mu(\mathcal{Z}^j, X) + d_1^{j+1} \qquad X \in \Theta_0$$

d_1^j has to be 0 for j large, due to the fact that the sum of these numbers gives a lower bound to $\mu(\mathcal{Z}^j, X)$, which is finite for X with an isolated singularity at 0. If $d_1^{j+1} = 0$, in Γ_1^{j+1} we have

$$\mathcal{K}^{j+1}/\mathcal{K}^{j+2} \subset \text{Image}[\Phi^{j+1}]$$

or equivalently on the open set Γ_1^{j+1} we have

$$\mathcal{K}^{j+1} \subset (\mathcal{I}.\mathcal{K}^{j+2})$$

This last implies also that on Γ_1^{j+1} we have

$$\mathcal{K}^{j+k} \subset (\mathcal{I}, \mathcal{K}^{j+k+1}), \quad k \geq 2 \tag{2.8}$$

which means that $\{A^k = \Theta_0 - \Gamma_1^k\}_k$ form a descending family of analytic spaces of Θ_0, for $k \geq j$ where $d_1^j = 0$. The intersection of the above family consist of those points where $\mu(\mathcal{Z}^j, X)$ is infinite. This set is exactly the set $\{(X, 0)/0$ is not an isolated critical point of X at $0\}$.

(2.8) also implies that if $d_1^j = 0$, then for $k > j$ we have $\Gamma_1^k \subset \Gamma_1^{k+1}$. Let $\Gamma_1 = \bigcap_k \Gamma_1^k$, which by the previous remark reduces to a finite intersection. Γ_1 *is the open dense subset of* Θ_0 *consisting of vector fields with minimum* V*-multiplicity at 0*, and equal to $d_1^1 + d_1^2 + \cdots + d_1^{j-1}$. Let $\tilde{\Gamma}_1 = \pi_1(\Gamma_1)$. From the above description, the theorem is clear. □

Now we begin to analyse the other component $\mathcal{Z}_{\text{sing}}$ of $\mathcal{Z}$.

Lemma 2.3. *The* V*-multiplicity of the holomorphic vector field* X *on* V *at a smooth point* p *of* V *coincides with the multiplicity (or the index) of the vector field* $X|_V$ *at* p.

Proof. We may find coordinates $(z_1,\dots,z_n)$ around p such that $\mathcal{J} = (z_{N+1},\dots,z_n)$ and the condition that the vector field X is tangent to V is that $X^j \in \mathcal{J}$ for $j = N+1,\dots,n$. Hence

$$\mu_V(X,p) = \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^n,0}}{(z_{N+1},\dots,z_n,X^1,\dots,X^n)}$$

$$= \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^n,0}}{(z_{N+1},\dots,z_n,X^1,\dots,X^N)} =$$

$$= \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^N\times\{0\},0}}{(X^1(\tilde{z},0),\dots,X^N(\tilde{z},0))} = \mu(X\,|_V,p) \quad \square \tag{2.9}$$

A sheaf $\mathcal{F}$ on $\Theta_r \times \mathbf{B}_r$ is Θ_r-*anaflat* ([Do1], 66) if for every point (X,z) there is a finite locally free resolution

$$0 \longrightarrow \mathcal{L}_q \longrightarrow \cdots \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

in a neighbourhood of (X,z) such that its restriction to $\{X\}\times\mathbf{B}_r$ is also an exact sequence.

Proposition 2.4. *If $p \neq 0$ is an isolated critical of $X \in \Theta_r$, then $\mathcal{O}_{\mathcal{Z}_r}$ is Θ_r-anaflat at (X,p).*

Proof. If $(X,p) \in \mathcal{Z}_r$ with $p \neq 0$ an isolated singularity of X, then Lemma 2.3 shows that $\mathcal{Z}_r$ at (X,p) is a complete intersection:

$$\mathcal{I}_{X,p} = (z_{N+1},\dots,z_n,X^1,\dots,X^N).$$

The generators of $\mathcal{I}_{X,p}$ form a regular sequence, so the Koszul complex of the regular sequence ([G-H], p.688) gives a finite locally free resolution of $\mathcal{O}_{\mathcal{Z}_r}$. The restriction of this complex to $\{X\}\times\mathbf{C}^n$ is the Koszul complex of the restricted generators, who also form a regular sequence. Hence the restricted sequence is also exact. So $\mathcal{O}_{\mathcal{Z}_r}$ is Θ_r- anaflat at (X,p). $\square$

Let now $X \in \Theta_r$ with an isolated critical point at 0, let $s < r$ be such that X is non-vanishing on $\overline{V}_s - \{0\}$, and let $2\varepsilon = \min\{\|X(z)\| \,/\, z \in \partial V_s\}$ and consider the ball $\mathbf{U} = \mathbf{U}(X, \varepsilon) \subseteq \Theta_r$. The projection map $\pi_1 : \mathcal{Z}' = \mathcal{Z} \cap (\mathbf{U} \times \mathbf{B}_s) \to \mathbf{U}$ is a finite map by Propositin 2.1 (see [G-R], where a finite map is a closed map with finite fibers). We want to analyse the sheaf $\pi_{1*}\mathcal{O}_{\mathcal{Z}'}$. The points of $\mathcal{Z}' - (\Theta_0 \cap \mathbf{U})$ are π_1-anaflat by Proposition 2.4 and the points of $\Gamma_1 \subset \Theta_0$, consisting of $(W, 0)$ with W of minimal V-multiplicity μ_V at 0, are also π_1-flat (since they have constant multiplicity (see [Do2], p.58)). Hence $\pi_{1*}\mathcal{O}_{\mathcal{Z}'}$ is locally free on $\pi_1(\Gamma_1) = \tilde{\Gamma}_1$ of rank

$$\mu_V + \sum_{\substack{Y(p_j)=0 \\ p_j \in V_s - \{0\}}} \mu_V(Y, p_j) \qquad Y \in \mathbf{U} \cap \tilde{\Gamma}_1 \tag{2.10}$$

where the V-multiplicity of Y at 0 is μ_V. This number is independent of s, for s sufficiently small and of $Y \in \mathbf{U} \cap \tilde{\Gamma}_1$. We will call it the *Euler characteristic of X at 0*, and denote it by $\chi_V^t(X, 0)$ (See [Ser]).

A family of holomophic vector fields parametrized by the irreducible and reduced complex space of finite dimension T is a holomorphic map $\phi : T \to \Theta_r$. The family ϕ induces a map $(\phi, id_s) : T \times V_s \to \Theta_r \times V_s$, and we will denote $(\phi, id_s)^*(\mathcal{Z}) \subset T \times V_s$ by $\mathcal{Z}_{T,s}$. Let $\pi_{1T} : \mathcal{Z}_{T,s} \to T$ be the projection to the first factor. If π_{1T} is a finite map, then $\pi_{1T*}\mathcal{O}_{\mathcal{Z}_{T,s}}$ is a coherent sheaf on T, and hence is locally free on a Zarizki dense set T' of T, say of rank r. For $t \in T'$ we have

$$r = \mu_V(X_t, 0) + \sum_{\substack{X_t(p_j)=0 \\ p_j \in V_s - \{0\}}} \mu_V(X_t, p_j)$$

and for $t \in T$ we have

$$r = \chi_0^{\mathrm{tor}}(\mathcal{O}_{\mathcal{Z}_{T,s}}, \mathcal{O}_{\{t\}}) + \sum_{\substack{X_t(p_j)=0 \\ p_j \in V_s - \{0\}}} \mu_V(X_t, p_j) \tag{2.11}$$

where

$$\chi_0^{\mathrm{tor}}(\mathcal{O}_{\mathcal{Z}_{T,s}}, \mathcal{O}_{\{t\}}) = \sum_q (-1)^q \, Tor^q_{\mathcal{O}_{T \times \mathbf{B}, (t,0)}}(\mathcal{O}_{\mathcal{Z}_{t,s,(t,0)}}, \mathcal{O}_{\{t\}}) \tag{2.12}$$

is the Euler characteristic of torsion groups of $\mathcal{O}_{\mathcal{Z}_{T,s},(t,0)}$ over $\mathcal{O}_{\{t\}'}$, where $Tor^0(\mathcal{O}_{\mathcal{Z}_{T,s}}, \mathcal{O}_{\{t\}}) = \mu_V(X_t, 0)$ (see [Do2]). Recall from Proposition 2.1 that $\Theta'_r \subset \Theta_r$ is the open dense subset consisting of vector fields having an isolated critical point at 0.

Theorem 2.5. *For $X \in \Theta'_r$, $s \ll r$ and $0 \ll \varepsilon$, we have:*

1) *For any family of vector fields $\{X_t\}_{t\in T}$, parametrized by a finite dimensional analytic space $(T, 0) \to (\Theta_r, X)$ such that the V-multiplicity of the general vector field X_t of the family is minimal μ_V we have:*

$$\chi_V(X, 0) = \chi_0^{\mathrm{tor}}(\mathcal{O}_{\mathcal{Z}_{T,s}}, \mathcal{O}_{\{X\}}) \tag{2.13}$$

2) *For $Z \in \mathbf{U}(X, \varepsilon)$ we have*

$$\chi_V(X, 0) = \chi_V(Z, 0) + \sum_{\substack{Z(p_j)=0\\ p_j \in V_s - \{0\}}} \mu_V(Z, p_j) \tag{2.14}$$

3) *For $X \in \Theta'_r$ we have:*

$$0 < \chi_V(X, 0) \leq \mu_V(X, 0)$$

and $\chi_V(X, 0) = \mu_V(X, 0)$ if and only if $\mathcal{Z}_r$ is π_1-anaflat at $(X, 0)$ (in particular in $\tilde{\Gamma}_1$).

Proof. Let $X \in \Theta_r$ with an isolated critical point at 0, let $s < r$ be such that X is non-vanishing on $\overline{V}_s - \{0\}$, $2\varepsilon = \min\{\|X(z)\| \,/\, z \in \partial V_s\}$ and consider the ball $\mathbf{U} = \mathbf{U}(X, \varepsilon) \subseteq \Theta_r$.

1) $\chi_V(X, 0)$ is defined by (2.10), where Y has minimal multiplicity μ_V at 0. If an element X_1 of a family $\{X_t\}$ has minimal V- multiplicity at 0, then the general element will have at 0 minimal multiplicity μ_V. At these points $\mathcal{Z}_{T,s}$ will be T-flat, since they represent Θ_r-anaflat points of $\mathcal{Z}_r$, and so the general rank of $\pi_{1T^*}\mathcal{O}_{\mathcal{Z}_{T,s}}$ is again (2.10). (2.11) applied to X on V_s gives $r = \chi_0^{\mathrm{tor}}(\mathcal{O}_{\mathcal{Z}_{T,s}}, \mathcal{O}_{\{X\}})$, hence we obtain (2.13).

2) Take a 1-parameter family $\{X_t\}_{t\in T=\mathbf{C}}$ in $\mathrm{U}(X,\varepsilon)$ which contains X and Z such that the general element has minimal V-multiplicity at 0. $\chi(X,0)$ is defined by (2.10), where Y has minimal multiplicity μ_V at 0. Since this is the only condition needed to apply part 1 of the theorem, assume that Y is near to Z. Assume that Z vanishes at $0, p_1, \dots, p_c$. Then part of Y is near to each part of the critical set of Z. Since $Z_{T,s}$ is T-flat at $p_1, \dots, p_c$, there are actually as much multiplicity near p_1 for Y as for Z at p_j. The multiplicity of Y near 0 is $\chi_V(Z,0)$ again by definition (2.10) applied to Z, where a new $\varepsilon' < \varepsilon$ is used in the definition in order to get rid of $p_1, \dots, p_c$. Hence we obtain (2.14).

3) Consider a 1-parameter family which contains X with 0 as only critical point in V_s and whose general element has minimal V-multiplicity. Then $\pi_{1*}\mathcal{O}_{\mathcal{Z}_\mathbf{C}}$ is a coherent sheaf on $\mathbf{C}$ whose rank is $\chi_V(X,0)$, by part 1. Hence the dimension of $\pi_{1*}\mathcal{O}_{\mathcal{Z}_\mathbf{C}} \otimes \mathcal{O}_{\{0\}}$ is greater than or equal to the general rank. If the rank is constant, then $\mathcal{Z}$ is π_1- anaflat. □

Let $X \in \Theta_r$, we say that the *zero set of X does not bifurcate* if there is $\varepsilon > 0$ and $s > 0$ such that for $Y \in \mathrm{U}(X,\varepsilon) \subset \Theta_r$ we have that the only critical point of Y on V_s is 0, (that is, X has an isolated singularity at 0 as well as any sufficiently near vector field in Θ_r and there is no other critical point uniformly in a neighbourhood V_s of 0). The critical set of a vector field X on V_r does not bifurcate if and only if the zero section Θ_0 coincides (as sets) with $\mathcal{Z}_r$ in a neighbourhood of $(X,0)$ in $\Theta_r \times V_r$.

Theorem 2.6. *Let $(V,0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space which is smooth except for an isolated singularity at 0, then the set of points in Θ_r whose critical set does not bifurcate contains the connected dense open subset $\tilde{\Gamma}_1 \subset \Theta_r$ consisting of vector fields with minimum V-multiplicity.*

Proof. Using previously introduced notation, what we have to prove is that $\Theta_{\text{sing}} \cap \Gamma_1 = \phi$ or equivalently that if $(X,0) \in \Theta_{\text{sing}}$ then the V-multiplicity at 0 cannot be minimal.

If $(X,0) \in \Theta_{\text{sing}}$, then we may find a 1-parameter linear family $\{X_t = X + tY\}$ in Θ_r such that its critical set

$$C = \{(t,z) \in \mathbf{C} \times \mathbf{B}_r / z \in V_r, X_t(z) = 0\}$$

has at least 2 local irreducible components at $(X,0)$, the zero section $C_0 = \mathbf{C} \times \{0\}$ and the others, say C_1. Formula (2.14) applied to $Z = X + \varepsilon Y$ is

$$\chi_V(X,0) = \chi_V(X+\varepsilon Y, 0) + \sum_{\substack{X+\varepsilon Y(p_j)=0 \\ p_j \in V_s - \{0\}}} \mu_V(X+\varepsilon Y, p_j) \tag{2.15}$$

The points $p_j \in C_1$ have a strictly positive contribution to the right hand side of (2.15), hence $\chi_V(X,0) > \chi_V(X+\varepsilon Y, 0)$. From this inequality we obtain that $\mu_V(X,0)$ cannot be minimal, for in that case $\chi_V(X,0) = \mu_V(X,0)$ would also be minimal. □

Example. Let $X_t = tz_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + (2t-1)z_3\frac{\partial}{\partial z_3}$ be a family of vector fields on $\mathbf{C}^3$ and let V be the surface defined by $f = z_1^2 - z_2z_3$. X_t is tangent to V, since $df(X_t) = 2tf$. As vector fields in $\mathbf{C}^3$, X_t has 0 as only critical point, except if $t = 0$ or $1/2$. X_0 has a line of critical points, but on V it has an isolated critical point. For $t \neq 0, 1/2$ one has that

$$(z_1^2 - z_2z_3, tz_1, z_2, (2t-1)z_3) = (z_1, z_2, z_3)$$

so that the V-multiplicity is 1 for $t \neq 0, 1/2$. For $t = 0$, one has

$$(z_1^2 - z_2z_3, tz_1, z_2, (2t-1)z_3) = (z_1^2, z_2, z_3)$$

so that the V-multiplicity is 2 for $t = 0$. So we see the upper semicontinuity behaviour of the V-multiplicity.

Remark. For a family $\{X+tY\}$ with $X+\varepsilon Y$ of minimal V- multiplicity we have

$$\chi_V(X,0) = \mu_V(X,0) - \dim[Tor^1_{\mathcal{O}_{T\times \mathbf{B},(t,0)}}(\mathcal{O}_{\mathcal{Z}_{T,s,(t,0)}}, \mathcal{O}_{\{t\}})]$$

This second term can be computed as the codimension of

$$(tf_1, \dots, tf_\ell, t(X^1+tY^1), \dots, t(X^n+tY^n))$$

in

$$(t) \cap (f_1, \dots, f_\ell, X^1+tY^1, \dots, X^n+tY^n)$$

(see [Do2]).

3. The index of holomorphic vector fields

Let V be a (reduced complex) analytic space of complex dimension N, with compact singular set and with boundary, ∂V, a smooth manifold of real dimension $2N-1$ oriented in a natural way. Let W be an orientable differentiable manifold of real dimension $2N$ with boundary ∂W diffeomorphic to ∂V (orientation preserving). We may extend this diffeomorphism to a diffeomorphism of a neighbourhood of the boundaries $\phi: V' \to W'$. Given a C^0-vector field X on V', non-singular on ∂V, we may transport it via ϕ to a vector field X' defined on W' and then define the *index of* X *on* V as the index of X' on W', and denote it by $\mathrm{Ind}_W(X, V, \partial V)$. This number depends on the choice of manifold W, but as we have seen in the first section, the choice of a different W' changes the index by an integer uniformly for all vector fields.

Given an analytic space V, one may choose as W a desingularization of V. In case V is a germ of a hypersurface with an isolated singularity defined by the equation $f=0$, then W can be defined by $f=\varepsilon$, for sufficiently small ε (or more generally, if V is a complete intersection, or a smoothable germ with an isolated singularity, then W can be the smoothening (see [Se])).

If p is an isolated singular point of V and X is a holomorphic vector field defined in a neighbourhood of p non-vanishing in a pointed neighbourhood of p, then *the index of* X *at* p $Ind_W(X, V, p)$ is defined as $\mathrm{Ind}_W(X, V', \partial V')$, where V' is a sufficiently small neighbourhood of p in V, and W is a manifold with $\partial W \approx \partial V$. The function $\mathrm{Ind}_W(\_, V, p)$ is well defined up to adding an integer, choice that depends on the election of the bounding manifold W.

The objective of this section is to compare the index with the V-multiplicity of X at 0. We recall that at a smooth point of V, if one uses the model of a ball as bounding a neighbourhood of the boundary of a smooth point, then the index coincides with the multiplicity (Lemma 2.3).

Theorem 3.1. *Let* $(V, 0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n, 0)$ *be an analytic space which is smooth except for an isolated singularity at 0, then there is a constant* K *such that*

$$Ind_W(X, V, 0) = \chi_V(X, 0) + K \tag{3.1}$$

for X *in the dense open set* Θ' *of vector fields in* Θ_r *with an isolated singularity at 0, where* χ_V *denotes the Euler-Poincaré characteristic of* X *at 0. For* X *in the dense open set of* Θ' *where the universal critical set* $\mathcal{Z}_r$ *is* Θ_r*-anaflat we have*

$$Ind_W(X, V, 0) = \mu_V(X, 0) + K \tag{3.2}$$

Proof. If $X \in \Theta_r$ is a vector field on V whose critical set does not bifurcate, then the index is locally constant at X, since the index on the boundary remains constant, and it is equal to the sum of the local indices, but the only critical point is located at 0. Hence the index is constant on the connected set $\mathcal{B}$ of Theorem 2.6. By Theorem 2.2.d the minimum of the V-multiplicity is attained on a dense open subset $\tilde{\Gamma}_1 \subset \mathcal{B}$.

Hence there is an integer K satisfying (3.2) for $X \in \Gamma_1$ (due to the fact that both functions are constant there).

Let now $X \in \Theta_r$ with an isolated critical point at 0, let $s < r$ be such that X is non-vanishing on $\overline{V}_s - \{0\}$, and let $2\varepsilon = \min\{\|X(z)\| / z \in \partial V_s\}$ and consider the ball $\mathbf{U} = \mathbf{U}(X, \varepsilon)$. For $X + tY \in \overline{\Gamma}_1 \cap \mathbf{U}$ we have

$$\mathrm{Ind}_W(X, V, 0) = \mathrm{Ind}_W(X + tY, V, 0) + \sum_{\substack{X+ty(p_j)=0 \\ p_j \in V_s - \{0\}}} \mathrm{Ind}_W(X + tY, V, p_j)$$

And hence

$$\mathrm{Ind}_W(X, V, 0) = [\chi_V(X + tY, V, 0) + K] + \sum_{\substack{X+ty(p_j)=0 \\ p_j \in V_s - \{0\}}} \mu_V(X + tY, p_j)$$

since $X + tY$ has minimal V-multiplicity at 0 and (3.1) and the fact that at the smooth points the V-multiplicity is equal to the index (Lemma 2.3). Using now (2.15) we obtain (3.1). (3.2) follows now from Theorem 2.5.3.

Corollary 3.2. *Let $(V, 0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n, 0)$ be an analytic space wich is smooth except for an isolated singularity at 0, then there is a constant L such that $\mathrm{Ind}_W(X, V, 0) \geq L$ for every germ of holomorphic vector field X on V with an isolated singularity at 0 on V.*

Proof. Let K be as in Theorem 3.2. Since $\chi_V(X, 0) > 0$ for any $X \in \Theta'$, we have

$$\mathrm{Ind}_W(X, V, 0) > K \quad \square$$

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Astérisque

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Vanishing holonomy and monodromy of certain centres and foci

Astérisque, tome 222 (1994), p. 37-48
<http://www.numdam.org/item?id=AST_1994__222__37_0>

VANISHING HOLONOMY AND MONODROMY OF CERTAIN CENTRES AND FOCI

MARCO BRUNELLA

Introduction

Let $\omega(x,y) = A(x,y)dx + B(x,y)dy = 0$ be the germ of an analytic differential equation on $\mathbf{R}^2$, with an algebraically isolated singularity at the origin: $A(0,0) = B(0,0) = 0$, $dim_{\mathbf{R}} \frac{\mathbf{R}\{x,y\}}{(A,B)} < +\infty$.

The singularity $\omega = 0$ is called *monodromic* if there are not separatrices at 0. In this case, given a germ of an analytic embedding $(\mathbf{R}^+,0) \overset{\tau}{\hookrightarrow} (\mathbf{R}^2,0)$ transverse to ω outside 0, it is possible to define a *monodromy map* $P_{\omega,\tau}$: $(\mathbf{R}^+,0) \to (\mathbf{R}^+,0)$, following clockwise the solutions of $\omega = 0$; $P_{\omega,\tau}$ is a germ of homeomorphism of $(\mathbf{R}^+,0)$ analytic outside 0. If $P_{\omega,\tau} = id$ then $\omega = 0$ is called *centre.* Otherwise $P_{\omega,\tau}$ is a contraction or an expansion (by the results of Écalle, Il'yashenko, Martinet, Moussu, Ramis... on "Dulac conjecture") and $\omega = 0$ is called *focus.*

The simplest monodromic singularities are those for which the linear part M_ω of the dual vector field $v(x,y) = B(x,y)\frac{\partial}{\partial x} - A(x,y)\frac{\partial}{\partial y}$ is nondegenerate, i.e. invertible. We distinguish two situations:

i) the eigenvalues λ, μ of M_ω are complex conjugate, non real, with real part different from zero. Then $\omega = 0$ is a focus and it is analytically equivalent to $\omega_{lin} = 0$, where ω_{lin} denotes the linear part of ω (Poincarè's linearization theorem).

ii) the eigenvalues λ, μ of M_ω are complex conjugate, non real, with zero real part. Then if $\omega = 0$ is a centre there exists an analytic first integral (Lyapunov-Poincarè theorem, see [Mou1] and references therein) and $\omega = 0$ is analytically equivalent to $xdx + ydy = 0$. If $\omega = 0$ is a focus the analytic classification is a difficult problem, which requires the theory of Écalle-Martinet-Ramis-Voronin to pass from the formal classification to the analytic one ([M-R]). The monodromy is an analytic diffeomorphism tangent to the identity, and two such equations are analytically equivalent if and only if their monodromies are ([M-R]).

In this paper we shall study the simplest degenerate monodromic singularities, i.e. those with $\lambda = \mu = 0$, $\omega_{lin} \neq 0$, and with "generic" higher order terms. Modulo a change of coordinates ([Mou2]), we may work in the following class.

Definition. Let $\omega = Adx + Bdy = 0$ be the germ of an analytic differential equation on $\mathbf{R}^2$, with an algebraically isolated singularity at 0. This singularity is called *monodromic semidegenerate* if the first nonzero quasihomogeneous jet of type (1,2) of ω is

$$\omega_0(x, y) = x^3dx + (y + ax^2)dy$$

with $a^2 < 2$. Notation: $\omega \in MSD(a)$.

We will denote by P_ω the monodromy map of $\omega \in MSD(a)$ corresponding to the embedding $(\mathbf{R}^+, 0) \hookrightarrow (\mathbf{R}^2, 0)$, $t \mapsto (t, 0)$. P_ω is a germ of analytic diffeomorphism tangent to the identity ([Mou2]), and we may consider P_ω as the restriction to $\mathbf{R}^+$ of a germ of biholomorphism of $(\mathbf{C}, 0)$, tangent to the identity, again denoted by P_ω.

Let $\omega \in MSD(a)$ and let Ω be the germ of holomorphic 1-form on $\mathbf{C}^2$ obtained by complexification of ω. Using a resolution of the singularity we may define as in [Mou3] and [C-M] the *vanishing holonomy* of Ω: it is a subgroup $H(\Omega) \subset Bh(\mathbf{C},0) = \{$ group of germs of biholomorphisms of $(\mathbf{C},0)\}$, generated by $f, g \in Bh(\mathbf{C},0)$ satisfying the relation $(f \circ g)^2 = id$.

Our result is a computation of P_ω in terms of $H(\Omega)$. A similar result was remarked by Moussu in the (simpler) case of nondegenerate monodromic singularities ([Mou1]).

Theorem. *Let $\omega = 0$ be monodromic semidegenerate, then*

$$P_\omega = [f, g]$$

In particular, $H(\Omega)$ is abelian if and only if $\omega = 0$ is a centre. This means, by [C-M], that a nontrivial space of "formal-analytic moduli" can appear only if $\omega = 0$ is a centre (and $a = 0$, see below): for the foci, formal equivalence $\Rightarrow$ analytic equivalence. Hence our situation is very different from the situation of equations of the type $xdx + ydy + \ldots = 0$, where the difficult case is the case of foci whereas all the centres are analytically equivalent (here the vanishing holonomy is always abelian, generated by a single $f \in Bh(\mathbf{C},0)$, and the monodromy is given by f^2, see [Mou1]). On the other hand, it is no more true that the monodromy characterizes the equation: it may happen that $\omega_1, \omega_2 \in MSD(a)$ have the same monodromy without being analytically equivalent.

A consequence of the above relation between monodromy and vanishing holonomy is the following normal form theorem for centres, based again on the results of [C-M]. Let us before remark that $\omega_0(x,y) = x^3dx + (y + ax^2)dy = 0$ is a centre for any $a \in \mathbf{R}$ (but a first integral exists if and only if $a = 0$).

Corollary 1. *Let $\omega \in MSD(a)$ be a centre and let $a \neq 0$, then the germ $\omega = 0$ is analytically equivalent to $\omega_0 = 0$.*

We don't know a similar explicit and "simple" (polynomial?) normal form for foci, even in the case $a \neq 0$; but the triviality of the space of formal-analytic moduli seems here a useful tool. The classification of centres with $a = 0$ requires arguments of the type Écalle - Martinet - Ramis - Voronin (cfr. [C-M]).

As another corollary of the above theorem we give a positive answer to a question posed by Moussu in [Mou2].

Corollary 2. *Let $\omega = 0$ be a monodromic semidegenerate centre, then there exists a nontrivial analytic involution $I : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ which preserves the solutions of $\omega = 0$: $I^*(\omega) \wedge \omega = 0$.*

The above computation may be generalized to the case of germs ω whose first nonzero quasihomogeneous jet of type $(1, n)$ is

$$\omega_0(x, y) = x^{2n-1}dx + (y + ax^n)dy$$

with $a^2 < \frac{1}{4}n$ ([Mou2]). The vanishing holonomy $H(\Omega)$ for these germs is generated by $f, g \in Bh(\mathbf{C}, 0)$ satisfying $(f \circ g)^n = id$ ([C-M]). But now, if $n \geq 3$, the relation between commutativity of $H(\Omega)$ and triviality of P_ω becomes more complicated; in particular, it is no more true that there is equivalence between "$H(\Omega)$ abelian" and "$P_\omega = id$".

The computation of P_ω in terms of $H(\Omega)$ for $n \geq 3$ is straightforward, once one has understood the case $n = 2$. Hence, for sake of simplicity and clarity, we have choose to limit ourselves to the semidegenerate monodromic singularities.

Acknowledgements: I thank R. Moussu and A. Verjovsky who read the manuscript and suggested me some improvements of the exposition.

Resolution of singularities and vanishing holonomy

Let $\omega \in MSD(a)$ and let Ω be its complexification. We recall the desingularization of Ω and the construction of $H(\Omega)$ ([C-M], [Mou3]).

We denote by M the complex manifold of dimension two covered by 3 charts $U_j = \{(x_j, y_j)\} \simeq \mathbf{C}^2$, $j = 1, 2, 3$, glued together by the identifications

$$\begin{cases} x_1 = \dfrac{1}{y_2} \\ y_1 = x_2 y_2^2 \end{cases} \qquad \begin{cases} x_3 = x_2 y_2 \\ y_3 = \dfrac{1}{x_2} \end{cases} \qquad \begin{cases} x_3 = x_1 y_1 \\ y_3 = \dfrac{1}{x_1^2 y_1} \end{cases}$$

Let $h : M \to \mathbf{C}^2$ be the holomorphic map whose expressions h_j in the charts U_j are

$$h_1(x_1, y_1) = (x_1 y_1, y_1), \quad h_2(x_2, y_2) = (x_2 y_2, x_2 y_2^2), \quad h_3(x_3, y_3) = (x_3, x_3^2 y_3)$$

The *divisor* $Z \overset{def}{=} h^{-1}((0,0))$ is a union of two copies of $\mathbf{C}P^1$, which intersect transversally at a point p. If $Z_j = Z \cap U_j$, then

$$Z_1 = \{y_1 = 0\}, \qquad Z_2 = \{x_2 = 0\} \cup \{y_2 = 0\}, \qquad Z_3 = \{x_3 = 0\}$$

The map $h|_{M \setminus Z} : M \setminus Z \to \mathbf{C}^2 \setminus \{(0,0)\}$ is a biholomorphism.

On M there is naturally defined an involution $j : M \to M$ given, in every chart U_j, by $j(x_j, y_j) = (\bar{x}_j, \bar{y}_j)$. The set $M^{\mathbf{R}}$ of fixed points of j is a real analytic manifold, the map h restricts to a real analytic map $h^{\mathbf{R}} : M^{\mathbf{R}} \to \mathbf{R}^2$, $Z^{\mathbf{R}} \overset{def}{=} (h^{\mathbf{R}})^{-1}((0,0)) = Z \cap M^{\mathbf{R}}$ is a union of two copies of $\mathbf{R}P^1$ intersecting transversally at p. The map $h^{\mathbf{R}}|_{M^{\mathbf{R}} \setminus Z^{\mathbf{R}}} : M^{\mathbf{R}} \setminus Z^{\mathbf{R}} \to \mathbf{R}^2 \setminus \{(0,0)\}$ is a real analytic diffeomorphism. The manifold $M^{\mathbf{R}}$ is covered by the charts $U_j^{\mathbf{R}} \overset{def}{=} U_j \cap M^{\mathbf{R}} \simeq \mathbf{R}^2$, $j = 1, 2, 3$, and we will denote again with (x_j, y_j) the corresponding coordinates.

From now on we will consider only the germs of the previous objects (M, h, $M^{\mathbf{R}}$, etc.) along Z or $Z^{\mathbf{R}}$, denoted by the same symbols.

Define $\tilde{\Omega} = h^*(\Omega)$. Its local expressions are

$\tilde{\Omega}_1 = y_1[(1 + \mathcal{O}(|y_1|))dy_1 + (\mathcal{O}(|y_1|))dx_1]$

$\tilde{\Omega}_2 = x_2 y_2^3[(2x_2 + 2ax_2^2 + x_2^3 + \mathcal{O}(|x_2 y_2|))dy_2 + (y_2 + \mathcal{O}(|x_2 y_2|))dx_2]$

$\tilde{\Omega}_3 = x_3^3[(1 + 2ay^3 + 2y_3^2 + \mathcal{O}(|x_3|))dx_3 + (\mathcal{O}(|x_3|))dy_3]$

The only singularities of the foliation $\tilde{\mathcal{F}}$ defined by $\tilde{\Omega}$ are:

– the point p $(= (0,0)$ in $U_2)$, where $\bar{\Omega}_2 \overset{def}{=} \frac{1}{x_2 y_2^3}\tilde{\Omega}_2$ has a singularity of the type "2:1 resonant saddle":

$$\bar{\Omega}_2 = 2x_2 dy_2 + y_2 dx_2 + h.o.t.$$

– the points

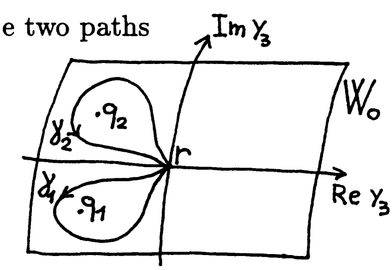
$$q_1 = (x_2 = -a + i\sqrt{2-a^2}, y_2 = 0) = (x_3 = 0, y_3 = \frac{1}{2}(-a - i\sqrt{2-a^2}))$$

$$q_2 = (x_2 = -a - i\sqrt{2-a^2}, y_2 = 0) = (x_3 = 0, y_3 = \frac{1}{2}(-a + i\sqrt{2-a^2})) = j(q_1)$$

where $\bar{\Omega}_2$ has hyperbolic singularities (the ratio of the eigenvalues is not real) if $a \neq 0$, and saddles with $4:1$ resonance if $a = 0$.

We denote by $W_0 \simeq \mathbf{C}P^1$ the component of Z containing q_1 and q_2, and by $W_1 \simeq \mathbf{C}P^1$ the other component; $W_0 \setminus \{p, q_1, q_2\}$ and $W_1 \setminus \{p\}$ are regular leaves of $\tilde{\mathcal{F}}$.

Let $L = cl(h^{-1}(\{y=0\}) \setminus Z) = \{y_3 = 0\}$ and $r = L \cap W_0 = (x_3 = 0, y_3 = 0)$. Let $\gamma_j : [0,1] \to W_0 \setminus \{p, q_1, q_2\}$, $j = 1,2$, be two paths such that $\gamma_j(0) = \gamma_j(1) = r$, $ind_{\gamma_j}(q_i) = \delta_{ij}$. To these paths there correspond two germs of biholomorphisms $f_j : (L, r) \to (L, r)$, given by the holonomy of the foliation $\tilde{\mathcal{F}}$.



We set $f, g \in Bh(\mathbf{C}, 0)$ equal respectively to f_1, f_2 expressed using the coordinate x_3 on L.

Definition ([C-M]). The *vanishing holonomy* $H(\Omega)$ of Ω is the subgroup of $Bh(\mathbf{C}, 0)$ generated by f and g.

An elementary computation shows that

$$f'(0) = -i \cdot exp(-\frac{a}{2\sqrt{2-a^2}}) \qquad g'(0) = -i \cdot exp(+\frac{a}{2\sqrt{2-a^2}})$$

in particular, $H(\Omega)$ is *hyperbolic* (i.e. $|f'(0)| \neq 1$, $|g'(0)| \neq 1$) if and only if $a \neq 0$.

Near the singularity p there exists a first integral, of the form $x_2 y_2^2 + h.o.t.$ ([C-M]). This implies that the holonomy of $\tilde{\mathcal{F}}$ along $\gamma_1 * \gamma_2$ or $\gamma_2 * \gamma_1$ (which are freely homotopic in $W_0 \setminus \{p, q_1, q_2\}$ to small paths around p) is periodic, of period 2. Hence:

$$(f \circ g)^2 = (g \circ f)^2 = id$$

Remark that from the fact that Ω has real coefficients we deduce that

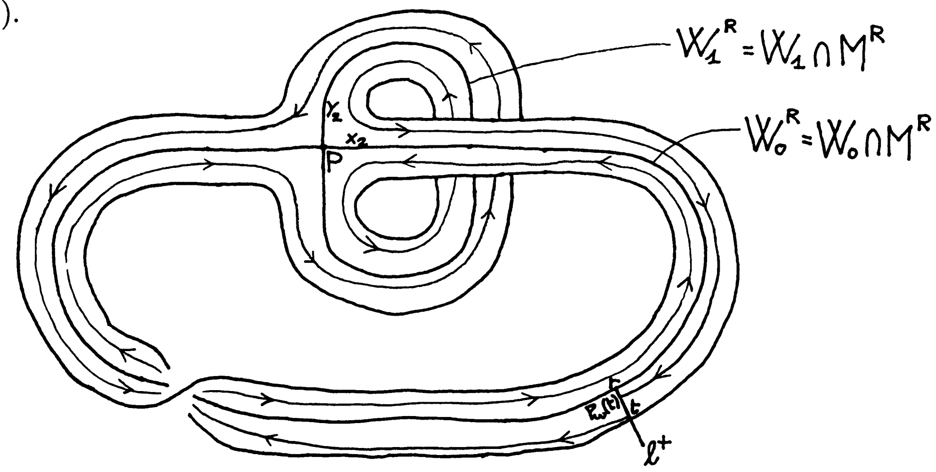
$$g(z) = \overline{f^{-1}(\bar{z})}$$

and, because p is a real point of M, $f \circ g$ and $g \circ f$ are real:

$$(f \circ g)(\bar{z}) = \overline{(f \circ g)(z)} \qquad \text{and} \qquad (g \circ f)(\bar{z}) = \overline{(g \circ f)(z)}$$

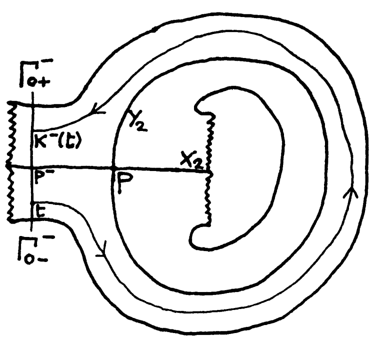
Now we turn to the real 1-form ω. Clearly, $h^{\mathbf{R}} : M^{\mathbf{R}} \to \mathbf{R}^2$ gives a resolution of the singularity. We set $l^+ = cl((h^{\mathbf{R}})^{-1}(\{y = 0, x \geq 0\}) \setminus Z^{\mathbf{R}}) = L \cap M^{\mathbf{R}} \cap \{x_3 \geq 0\}$, then the holonomy of the foliation $\tilde{\mathcal{G}}$ defined by $\tilde{\omega} = (h^{\mathbf{R}})^*(\omega)$ along the "polycicle" $Z^{\mathbf{R}}$ produces a germ of analytic diffeomorphism of (l^+, r). Using the coordinate x_3 on l^+ and complexifying the result we obtain a germ of biholomorphism of $(\mathbf{C}, 0)$ which is nothing else that the (complex) monodromy P_ω of ω (for the appropriate choice of orientation of

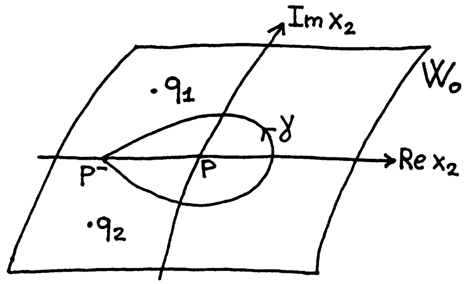
$\tilde{\mathcal{G}}$).



Proof of the theorem

Let $\Gamma^- \subset M$ be a germ of complex line, j-symmetric, transverse to $\tilde{\mathcal{F}}$, passing throught a point p^- of W_0 near p with coordinates (in the chart U_2) $(-\epsilon, 0)$, $\epsilon \in \mathbf{R}^+$ small. Let $\Gamma_0^- = \Gamma^- \cap M^{\mathbf{R}}$ be its real part; it is a germ of real line transverse to $\tilde{\mathcal{G}}$. The holonomy of $\tilde{\mathcal{G}}$ along the polycycle composed by the segment from p^- to p and $W_1^{\mathbf{R}} \overset{def}{=} W_1 \cap M^{\mathbf{R}}$ gives a germ of homeomorphism $k^- : (\Gamma_{0-}^-, p^-) \to (\Gamma_{0+}^-, p^-)$, where $\Gamma_{0-}^- = \Gamma_0^- \cap \{y_2 \le 0\}$, $\Gamma_{0+}^- = \Gamma_0^- \cap \{y_2 \ge 0\}$.





On the other hand, we may consider a path $\gamma : [0,1] \to W_0 \setminus \{p, q_1, q_2\}$ with $\gamma(0) = \gamma(1) = p^-$ and $ind_\gamma(p) = 1$, $ind_\gamma(q_1) = ind_\gamma(q_2) = 0$. The holonomy of $\tilde{\mathcal{F}}$ along γ induces a germ of biholomorphism $K^- : (\Gamma^-, p^-) \to (\Gamma^-, p^-)$, which is an involution because of the first integral of $\tilde{\Omega}$ near p.

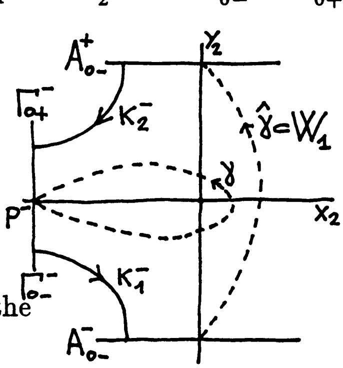
Lemma.

$$k^- = K^-|_{\Gamma^-_{0-}}$$

Proof.

let $A^+, A^- \subset M$ be two germs of complex lines, j-symmetric, transverse to $\tilde{\mathcal{F}}$ and passing throught $(0,\epsilon)$, $(0,-\epsilon)$ (in the chart U_2). Let A_0^+, A_0^- be their real parts, $A_{0-}^+ = A_0^+ \cap \{x_2 \le 0\}$, $A_{0-}^- = A_0^- \cap \{x_2 \le 0\}$. The homeomorphism k^- is the composition of a homeomorphism k_1^- from Γ_{0-}^- to A_{0-}^-, an analytic diffeomorphism $k^* : A_0^- \to A_0^+$, and a homeomorphism k_2^- from A_{0-}^+ to Γ_{0+}^-.

Because $W_1 \setminus \{p\}$ is simply connected, the path which joins $(0,-\epsilon)$ to $(0,\epsilon)$ along $W_1^{\mathbf{R}} \setminus \{p\}$ is contractible in $W_1 \setminus \{p\}$ (endpoints fixed) to a path $\hat{\gamma}$ contained in $\{|y_2| \le \epsilon\}$. Hence k^* is the restriction to A_0^- of a biholomorphism $K^* : A^- \to A^+$, obtained from the holonomy of $\tilde{\mathcal{F}}$ along this path $\hat{\gamma}$.



We choose ϵ so small that $\tilde{\Omega}$ has a first integral $x_2 y_2^2 + \ldots$ defined on $U_\epsilon = \{|x_2| \le \epsilon, |y_2| \le \epsilon\}$. Hence every leaf of $\tilde{\mathcal{F}}|_{U_\epsilon}$ different from a separatrix at p either does not intersect $U_\epsilon^{\mathbf{R}} = U_\epsilon \cap M^{\mathbf{R}}$, or it intersects $U_\epsilon^{\mathbf{R}}$ along two segments "symmetric" w.r. to the x-axis. We deduce that:

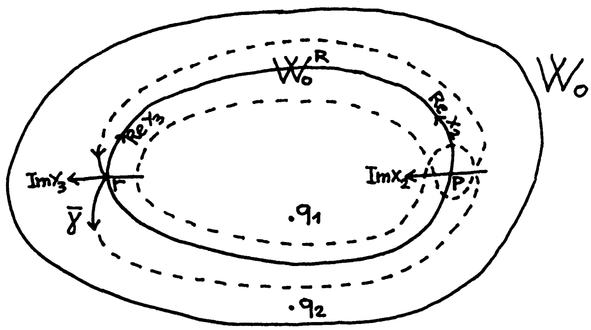
i) if $t \in A_{0-}^-$, then $K^*(t) \in A_{0-}^+$ is the only intersection of A_{0-}^+ with the leaf of $\tilde{\mathcal{F}}|_{U_\epsilon}$ through t;

ii) if $s \in \Gamma_{0-}^-$, then $K^-(s) \in \Gamma_{0+}^-$ is the only intersection of Γ_{0+}^- with the leaf of $\tilde{\mathcal{F}}|_{U_\epsilon}$ through s.

From these two remarks, it is clear that $k_2^- \circ (K^*|_{A_{0-}^-}) \circ k_1^-$ is equal to $K^-|_{\Gamma_{0-}^-}$, i.e. $k^- = K^-|_{\Gamma_{0-}^-}$. Q.E.D.

Obviously, a similar result holds if we start from $\Gamma^+=$ germ of complex line j-symmetric passing throught $(\epsilon, 0) = p^+$, etc..

As a consequence, the complex monodromy $P_\omega : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ may be computed as the holonomy of $\tilde{\mathcal{F}}$ along a path $\bar{\gamma} : [0,1] \to W_0 \setminus \{p, q_1, q_2\}$, $\bar{\gamma}(0) = \bar{\gamma}(1) = r$, as in the following picture:



This path is homotopic to $\gamma_2^{-1} * \gamma_1^{-1} * \gamma_1^{-1} * \gamma_2^{-1}$, hence

$$P_\omega = g^{-1} \circ f^{-1} \circ f^{-1} \circ g^{-1}$$

and from $(g \circ f)^2 = id$ we conclude

$$P_\omega = f \circ g \circ f^{-1} \circ g^{-1} = [f, g]$$

Q.E.D.

Proof of corollary 1

It is sufficient, using the path-lifting argument of [C-M], to show that the vanishing holonomies $H(\Omega)$ and $H(\Omega_0)$ are holomorphically conjugate. The "assertion 1" of [C-M], pag. 478, is here replaced by

Assertion 1_a*:* if $\omega \in MSD(a)$ then there are analytic coordinates (x, y) near $(0,0)$ s.t. ω is, modulo multiplication by a nonvanishing germ:

$$\omega(x,y) = x^3dx + (y + ax^2)dy + f(x,y)(2ydx - xdy), \qquad f \in \mathbf{R}\{x,y\}$$

The proof of this normal form lemma is achieved as in [C-M]: $\Omega = 0$ has a separatrix $X^4 + 2aX^2Y + 2Y^2 = 0$ (in suitable coordinates, preserving the class $MSD(a)$), which is a separatrix also for $2YdX - XdY = 0$. Hence:

$$\Omega \wedge (X^3dX + (Y + aX^2)dY) = (X^4 + 2aX^2Y + 2Y^2) \cdot H_1(X,Y)dX \wedge dY$$

$$\Omega \wedge (2YdX - XdY) = (X^4 + 2aX^2Y + 2Y^2) \cdot H_2(X,Y)dX \wedge dY$$

$$(2YdX - XdY) \wedge (X^3dX + (Y + aX^2)dY) = (X^4 + 2aX^2Y + 2Y^2) \cdot dX \wedge dY$$

and from these formulae the assertion 1_a follows.

Let us denote by f_0, g_0 the generators of $H(\Omega_0)$; $[f_0, g_0] = id$ because ω_0 is a centre, moreover

$$f_0'(0) = f'(0) = \lambda$$
$$g_0'(0) = g'(0) = \frac{1}{\bar{\lambda}} = \frac{-1}{\lambda}$$

and $|\lambda| \neq 1$ because $a \neq 0$.

From the commutativity and the hyperbolicity of $H(\Omega)$ we deduce that $H(\Omega)$ is holomorphically conjugate to the group generated by

$$z \mapsto \lambda z \qquad \text{and} \qquad z \mapsto \frac{1}{\bar{\lambda}} z$$

For the same reasons, $H(\Omega_0)$ also is holomorphically conjugate to that linear group, hence to $H(\Omega)$. Q.E.D.

Proof of corollary 2

Consider the germ $f \circ g \in Bh(\mathbf{C}, 0)$: it is real ($(f \circ g)(\bar{z}) = \overline{(f \circ g)(z)}$), periodic with period 2, and conjugates $H(\Omega)$ with itself thanks to $[f, g] = id$:

$$(f \circ g) \circ g = (g \circ f) \circ g = g \circ (f \circ g)$$
$$(f \circ g) \circ f = f \circ (g \circ f) = f \circ (f \circ g)$$

As in corollary 1, we use the path-lifting technique of [C-M] to suspend $(f \circ g)$ and to obtain a germ of biholomorphism $\tilde{I} : M \to M$, which preserves $\tilde{\mathcal{F}}$:

$\tilde{I}^*(\tilde{\Omega}) \wedge \tilde{\Omega} = 0$. From the fact that $(f \circ g)$ is a real involution, we obtain that $\tilde{I}$ is also a real involution. Taking the projection on $\mathbf{C}^2$ and the restriction to $\mathbf{R}^2$ we obtain the required analytic involution. Q.E.D.

Remark: if $a \neq 0$ the result follows also from corollary 1: ω_0 is invariant by $(x, y) \mapsto (-x, y)$.

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Astérisque

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Théorèmes de type Fuchs pour les tissus feuilletés

Astérisque, tome 222 (1994), p. 49-92
<http://www.numdam.org/item?id=AST_1994__222__49_0>

Théorèmes de type Fuchs pour les tissus feuilletés

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Après avoir rappelé des résultats pour certains bien anciens - et souvent méconnus - concernant les d-tissus sur un ouvert de $\mathbb{C}^n$ on s'intéresse à la dynamique des 3-tissus feuilletés hexagonaux globaux. Bien souvent - c'est le cas sur les espaces projectifs - un tel objet va présenter des singularités. On se propose, moyennant des hypothèses de type Fuchs, de donner une description des feuilles comme niveaux de fonctions multivaluées de type Liouville ($\sum \lambda_i \operatorname{Log} f_i + H$, f_i et H holomorphes). Ce travail est motivé par la description de la variété des feuilletages algébriques de codimension un sur les espaces projectifs $\mathbb{CP}(n)$.

I - Généralités

§ I.1. d-tissus sur un ouvert de $\mathbb{C}^n$.

Dans ce qui suit on use et abuse de la référence [C , G] ainsi que de l'exposé de Beauville au séminaire Bourbaki [B]. Tous les énoncés sont **locaux** et l'on doit entendre "quitte à diminuer l'ouvert de définition U".

Définition 1. Un d-web ou un d-tissu $\mathscr{W}$ sur un ouvert U de $\mathbb{C}^n$ est la donnée de d-feuilletages holomorphes $\mathscr{F}_i$ de codimension un en position générale. On note $\mathscr{W} = [\mathscr{F}_1,\dots,\mathscr{F}_d]$.

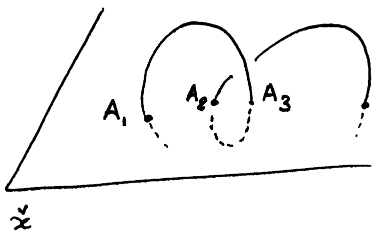
Localement chaque feuilletage $\mathscr{F}_i$ est donné par les niveaux d'une submersion holomorphe u_i ; on note $\mathscr{F}_i = \mathscr{F}(du_i)$. Si $d \leq n$, tout d-web est localement difféomorphe au d-web standard $[\mathscr{F}(dx_1),\dots,\mathscr{F}(dx_d)]$, où $x_1,\dots,x_n$ est un système de coordonnées local de $\mathbb{C}^n$.

Un d-web $\mathscr{W} = [\mathscr{F}_1,\dots,\mathscr{F}_d]$ sur U est **linéaire** si toutes les feuilles des feuilletages $\mathscr{F}_i$ sont les traces sur U d'hyperplans affines.

Un d-web est **linéarisable** s'il est difféomorphe à un d-web linéaire.

Exemple : web associé à une courbe algébrique.

Cet exemple est fameux. Soit $\Gamma \subset \mathbb{CP}(n)$ une courbe algébrique de degré d suffisamment générale pour que ce qui suit ait un sens. Soit $\check{x}_0$ un point général de l'espace dual $\check{\mathbb{CP}}(n)$; si $\check{x}$ est voisin de $\check{x}_0$, l'hyperplan $\check{x}$ (vu dans $\mathbb{CP}(n)$) coupe la courbe Γ en d points distincts, notés $A_1(\check{x}),\ldots,A_d(\check{x})$.



Par dualité ces d-points définissent d-hyperplans passant par $\check{x}$ que l'on décide être les feuilles de notre d-web. On munit ainsi un voisinage U de $\check{x}_0$ d'un d-web que l'on note $\mathcal{W}(\Gamma)$; ce d-web est par construction linéaire. Si z_i, $i = 1,\ldots,d$, sont des coordonnées locales près du point $A_i(\check{x}_0)$, on peut choisir les $u_i = z_i(A_i)$ comme fonctions définissant nos d-feuilletages : $\mathcal{W}(\Gamma) = [\mathcal{F}(du_1),\ldots,\mathcal{F}(du_d)]$. Mais d'autres choix sont évidemment possibles ; ainsi si w est une forme différentielle holomorphe sur Γ, les fonctions :

$$\int_{A_i(\check{x}_0)}^{A_i(\check{x})} w$$

définissent pour $\check{x} \in U(\check{x}_0)$ des fonctions donnant encore nos feuilletages ; notamment :

$$(1) \qquad \int_{A_i(\check{x}_0)}^{A_i(\check{x})} w = \ell_i^{w}(u_i(\check{x}))$$

où $\ell_i^{w}(t)$ est holomorphe au voisinage de $u_i(\check{x}_0)$.

Rappelons ici la fameux théorème d'Abel :

(2) La somme $\sum_{i=1}^{d} \int_{A_i(\check{x}_0)}^{A_i(\check{x})} w$ est indépendante de $\check{x} \in U(\check{x}_0)$.

On traduit alors (1) et (2) par :

(3) $$\sum_{i=1}^{d} \frac{\partial \ell_i^w}{\partial t}(u_i)\, du_i = 0 .$$

On introduit la :

Définition 2 : Soit $\mathscr{W}(\Gamma) = [\mathscr{F}(du_1),\ldots,\mathscr{F}(du_d)]$ un d-web sur l'ouvert U de $\mathbb{C}^n$. On appelle relation abélienne une relation de la forme :

(4) $$\sum_{i=1}^{d} f_i(u_i)\, du_i = 0$$

où les f_i sont holomorphes au voisinage de $u_i(x_0) \in \mathbb{C}$, $x_0 \in U$. L'ensemble des relations abéliennes constitue un espace vectoriel $Ab(\mathscr{W})$ dont la dimension $M(\mathscr{W})$ s'appelle le rang du web $\mathscr{W}$.

Revenons au web $\mathscr{W}(\Gamma)$ associé à une courbe de degré d dans $\mathbb{CP}(n)$. On peut démontrer qu'en fait toutes les relations abéliennes sont de type (3) ; une conséquence du théorème d'Abel est donc la suivante :

$$M(\mathscr{W}(\Gamma)) = g(\Gamma) \leq M(n,d)$$

où $g(\Gamma)$ est le genre de la courbe Γ et $M(n,d)$ le maximum du genre pour une courbe de degré d dans $\mathbb{CP}(n)$ (Castelnuovo).

Dans sa thèse Chern démontre que pour un d-tissu sur un ouvert U de $\mathbb{C}^n$ on a toujours l'inégalité :

$$M(\mathscr{W}) \leq M(n,d) .$$

On voit ainsi que le maximum du rang est précisément réalisé par les d-webs linéaires $\mathscr{W}(\Gamma)$ associés aux courbes extrémales, i.e. celles ayant le genre maximum dans un degré donné.

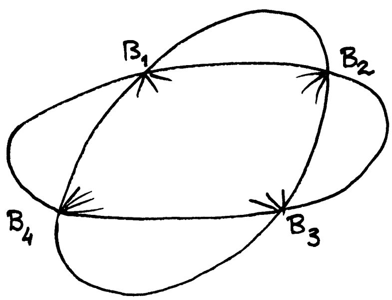
Voici pêle mêle des résultats connus concernant les d-webs linéaires et concernant la linéarisation.

Théorème 1 (Lie-Poincaré-Wirtinger-Griffiths). Soit $\mathscr{W}$ un d-web linéaire sur $U \subset \mathbb{C}^n$ possédant au moins une relation abélienne $\sum f_i(u_i)du_i = 0, f_i \equiv 0$ pour tout i. Alors il existe une courbe algébrique Γ dans $\mathbb{CP}(n)$, $U \subset \mathbb{C}^n \subset \mathbb{CP}(n)$ tel que $\mathscr{W} = \mathscr{W}(\Gamma)_{|U}$.

Concernant la linéarisation on a le :

Théorème 2 (Chern-Griffiths). Soit $\mathcal{W}$ un d-web sur $U \subset \mathbb{C}^n$ de rang maximal $M(n,d)$; si $n \geq 3$ et $d \leq n+1$ ou $d \geq 2n$ alors $\mathcal{W}$ est linéarisable.

Ainsi pour $n = 3$ et $d \neq 5$ tout d-tissus de rang maximal est linéarisable. Pour $n = 2$ le théorème est inopérant ; ainsi on peut construire un 5-tissu du complément dans $\mathbb{CP}(2)$ d'une courbe γ, de rang maximum et qui ne soit pas linéarisable. Cet exemple est dû à Bol : on se donne quatre points B_i en position générale et l'on considère le feuilletage $\mathcal{F}_i$ dont les feuilles sont les droites passant par B_i. On ajoute à nos quatre feuilletages $\mathcal{F}_i$ le feuilletage $\mathcal{F}_5$ dont les feuilles sont les coniques passant par les B_i (fig.).



On a $M(2,5) = \frac{1}{2}(5-1)(5-2) = 6$ et le 5-webs ci-dessus est effectivement de rang 6, ce qui n'est pas si facile à établir.

En dimension deux les d-tissus de rang maximal qui sont linéarisables ont été récemment classifiés par A. Hénaut dans [H].

§ I.2. Les 3-tissus en dimension deux.

Leur rang est inférieur ou égal à 1. Si l'on dispose d'une relation abélienne non triviale pour $\mathcal{W} = [\mathcal{F}(du_1), \mathcal{F}(du_2), \mathcal{F}(du_3)]$:

$$\ell_1(u_1)du_1 + \ell_2(u_2)du_2 + \ell_3(u_3)du_3 = 0 \tag{5}$$

il est clair que chaque ℓ_i est non identiquement nulle. On peut alors choisir des primitives L_i de ℓ_i de sorte que

(6) $$L_1 + L_2 + L_3 = 0 .$$

Visiblement en un point générique (L_1,L_2) est un difféomorphisme local. Dans les coordonnées locales $(L_1(u_1),L_2(u_2))$ le web $\mathscr{W}$ a pour feuilles les lignes horizontales, verticales et les parallèles à la deuxième bissectrice.
Si H_1, H_2, H_3 définissent $\mathscr{W}$, $H_i = H_i(u_i)$, et vérifient (6) un calcul élémentaire assure que :

(7) $$H_i = \lambda . L_i + a_i , \qquad i = 1,2,3$$

où $\lambda \in \mathbb{C}^*$ et $a_i \in \mathbb{C}$.

Un triplet (u_1,u_2,u_3) tel que $\mathscr{W} = [\mathscr{F}(du_1),\mathscr{F}(du_2),\mathscr{F}(du_3)]$ et vérifiant (6), s'il en existe, sera dit basique. Evidemment un triplet basique est caractérisé par deux de ses éléments et deux triplets basiques s'échangent par une transformation affine (d'après 7).

Considérons maintenant une courbe elliptique $E \subset \mathbb{CP}(2)$; le trois web linéaire $\mathscr{W}(E)$ est de rang un (théorème d'Abel) et l'associativité de la loi d'addition sur la cubique se traduit par **l'hexagonalité** du web $\mathscr{W}(E)$.

Définition 3. Un 3-web $\mathscr{W}$ sur $U \subset \mathbb{C}^2$ est hexagonal si en chaque point m les "hexagones" se "referment" (cf. [B] et fig.)

Le fait remarquable est qu'un 3-tissu est hexagonal si et seulement s'il possède une relation abélienne non triviale.

La proposition suivante relie nos différentes notions :

Proposition 1. Soit $\mathscr{W} = [\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3]$ un 3-web sur un ouvert U de $\mathbb{C}^2$; soient ω_i des formes différentielles définissant $\mathscr{F}_i$ telles que $\omega_1+\omega_2+\omega_3= 0$; sont équivalents :

(i) $\mathscr{W}$ est hexagonal

(ii) si $m \in U$, il existe u_1,u_2,u_3 et g holomorphes au voisinage de m,

g unité, tels que :

$$w_i = g\, du_i$$

(iii) il existe une 1-forme holomorphe fermée Θ telle que

$$dw_i = \Theta \wedge w_i\,.$$

Preuve :

1) (i) $\Leftrightarrow$ (ii) d'après la proposition 1.

2) (ii) $\Leftrightarrow$ (iii) : supposons que l'on ait défini Θ_1 et Θ_2 vérifiant

$$dw_i = \Theta_j \wedge w_i\,, \quad j = 1,2, \quad i = 1,2,3$$

sur un petit ouvert $U' \subset U$. Alors

$$0 = (\Theta_1 - \Theta_2) \wedge w_i\,, \qquad i = 1,2,3$$

et par suite $0 = \Theta_1 - \Theta_2$.

Compte tenu de cette remarque il suffit de construire Θ localement le recollement sera automatique ; au voisinage de $m \in U$, on pose :

$\Theta = \dfrac{dg}{g}$, où g est fourni par (ii).

3) (iii) $\Rightarrow$ (ii) : soit $m \in U$; au voisinage de m on a : $\Theta = \dfrac{dg}{g}$ où g est une unité.

Ecrivant $w_i = g.w'_i$ on constate que :

$$dw_i = \frac{dg}{g} \wedge w_i + g\, dw'_i = \frac{dg}{g} \wedge w_i\,.$$

Par suite les w'_i sont fermées et localement $w'_i = du_i$.

La condition $w_1 + w_2 + w_3 = 0$ fournit la relation abélienne :

$du_1 + du_2 + du_3 = 0$.

§ I.3. Webs globaux sur les surfaces. Structures affines.

Soit S une surface holomorphe ; on peut définir un 3-web sur la surface S par la donnée de trois feuilletages holomorphes $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ partout deux à deux transverses. Cette définition est en fait restrictive si ce n'est maladroite. Revenons en effet au web $\mathcal{W}(E)$ associé à une courbe elliptique E. Visiblement $\mathcal{W}(E)$ se laisse définir sur $S = \check{\mathbb{C}\mathbb{P}}(2) - E^*$, où E^* est la courbe duale de E : E^* est une sextique à neuf point cuspidaux. Il est à peu près évident que $\mathcal{W}(E)$ ne provient pas de 3-feuilletages : par un argument géométrique élémentaire on voit que nos 3-feuilletages locaux sur un petit ouvert sont permutés lorsque l'on suit un chemin tournant autour de la sextique E^*.

Pour palier à ce fait on note $\Pi : T^dS \longrightarrow S$ le fibré vectoriel naturel dont la fibre est l'espace $\mathcal{P}(2,d)$ des polynômes homogènes de degré d en deux variables. Pour d=1, T^1S est le fibré cotangent de S ; soit $\mathcal{D}(2,d) \subset \mathcal{P}(2,d)$ l'hyper-surface algébrique constituée des polynômes ayant au moins une droite multiple. Introduisons l'hypersurface $\mathcal{D}is\ \Pi \cap \subset T^dS$ définie dans une carte trivialisante $\psi^{(d)}$ associée à une carte locale $\psi : U \longrightarrow \mathbb{C}^2$ de S :

$$\begin{array}{ccc} \Pi^{-1}(U) & \xrightarrow{\psi^{(d)}} & \mathbb{C}^2 \times \mathcal{P}(2,d) \\ \Pi \big\downarrow & & \big\downarrow pr_1 \\ U & \xrightarrow{\psi} & \mathbb{C}^2 \end{array}$$

par $\mathcal{D}is\ \Pi \cap \Pi^{-1}(U) = \psi^{(d)-1}(\mathbb{C}^2 \times \mathcal{D}(2,d))$.

Définition 4. Un d-web $\mathcal{W}$ sur la surface S est la donnée d'un recouvrement $\mathcal{U} = (U_i)_{i\in I}$ par des ouverts munis de sections

$$\mathcal{W}_i : U_i \longrightarrow T^dS,\ \ \Pi\circ\mathcal{W}_i = id_{U_i}$$

telles que :

1) l'image de $\mathcal{W}_i$ n'est pas contenue dans $\mathcal{D}is\ \Pi$

2) $\mathcal{W}_i / U_i \cap U_j = h_{ij}\ \ \mathcal{W}_j / U_i \cap U_j$

où les $h_{ij} \in \mathcal{O}(U_i \cap U_j)$ sont des unités.

L'ensemble singulier de $\mathcal{W}$ noté Sing $\mathcal{W}$ est par définition l'ensemble analytique défini par :

$$(\text{Sing}\ \mathcal{W}) \cap U_i = \mathcal{W}_i^{-1}(\mathcal{D}is\ \Pi).$$

Le web $\mathcal{W}$ est sans singularité ou ordinaire si Sing $\mathcal{W} = \phi$.

Remarques.

1) Soit $\mathcal{W}$ un d-web sur la surface S ; si S est simplement connexe, $\mathcal{W}$ est donné par d feuilletages $\mathcal{F}_i$ en position générale (aux points non singuliers). L'ensemble singulier Sing $\mathcal{W}$ est constitué des singularités éventuelles des $\mathcal{F}_i$ et des contacts entre les $\mathcal{F}_i$.

2) Du point de vue local il résulte de 1) que l'on retrouve la même notion que précédemment.

3) D'après 2) on peut définir la notion de feuilles locales et par prolongement analytique la notion de feuilles globales. Alors que pour un feuilletage les feuilles ne se recoupent pas c'est tout à fait possible ici... et de façon sauvage comme nous le verrons.

Désignons par S_d le groupe des permutations de l'ensemble $[1,\ldots,d]$; si S est muni d'un d-web on a une action évidente de $\Pi_1(S\text{–Sing}\,\mathcal{W}, m)$ sur S_d définie de la façon suivante :

soient γ un lacet en $m \in S$–Sing $\mathcal{W}$, et U un petit voisinage simplement connexe de m ; U est muni de d-feuilletages $\mathcal{F}_1,\ldots,\mathcal{F}_d$ tels que :

$$\mathcal{W}|_U = [\mathcal{F}_1,\ldots,\mathcal{F}_d]\,.$$

Partant de la feuille locale $\mathcal{L}(\mathcal{F}_i)_{(m)}$ de $\mathcal{F}_i$ passant par m on peut suivre par continuité cette feuille le long de γ ; on revient avec l'une des feuilles $\mathcal{L}(\mathcal{F}_{\sigma_\gamma(i)})_{(m)}$ de $\mathcal{W}|_U$ par m.

On note

$$\sigma : \Pi_1(S\text{–Sing}\,\mathcal{W}, m) \longrightarrow S_d$$

cette action et on l'appelle monodromie **primaire**. Si la monodromie primaire est triviale le d-web est donné par d-feuilletages (éventuellement singuliers) globaux.

Voici quelques exemples :

1) Soit $\mathcal{W}_E$ le 3-web associé à une courbe elliptique ; $\mathcal{W}_E$ est défini sur $\check{\mathbb{C}}\mathbb{P}(2)\text{–}E^*$; la monodromie primaire

$$\sigma_E : \check{\mathbb{C}}\mathbb{P}(2)\text{–}E^* \longrightarrow S_3$$

est surjective.

2) Soit $\mathbb{C}^2$ muni du 3-web $[\mathcal{F}(dx_1), \mathcal{F}(dx_2), \mathcal{F}(dx_1{+}x_2)]$; si $\Lambda = \mathbb{Z}w_1{+}\mathbb{Z}w_2{+}\mathbb{Z}w_3{+}\mathbb{Z}w_4$ est un réseau le quotient $\mathbb{C}^2/\Lambda$ est muni d'un 3-web hexagonal sans singularité. La monodromie primaire est triviale.

3) De même la variété de Hopf $H_\lambda = \mathbb{C}^2\text{–}\{0\} \,/\, \lambda\,\mathrm{id}$, $|\lambda| > 1$, est muni d'un 3-web hexagonal à monodromie primaire triviale.

4) Soit $P(x)$ un polynôme à une variable ; la famille de courbes :

$$(y-c)^3 = P(x)$$

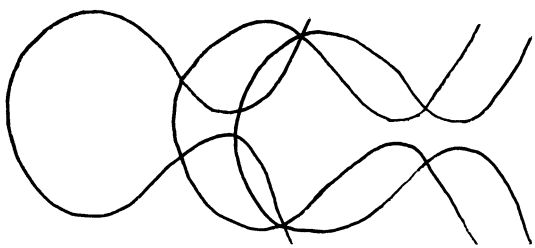
est solution de l'équation différentielle :

$$y'^3 = \frac{1}{9}\,\frac{P'(x)^3}{P(x)^2}$$

et définit un 3-web sur $\mathbb{C}^2$.
Les feuilles s'obtiennent par translation parallèle à l'axe des y de la courbe :

$$y^3 = P(x) .$$

Par un point générique passent 3 feuilles distinctes ; pourtant ce web ne correspond pas à la donnée de 3 feuilletages. En fait si $\tilde{x} : D \longrightarrow X = \mathbb{C}-\{x_1,...x_d\}$ est le revêtement universel de $\mathbb{C}$ moins les racines de P le relèvement par $(\tilde{x},y) : D \times \mathbb{C} \longrightarrow X \times \mathbb{C}$ de notre 3-web - qui est donné par "$P(x)^{2/3}\, dy - \dfrac{1}{9}\, P'(x)dx$" - conduit à un unique feuilletage sur $D \times \mathbb{C}$ donné par $\tilde{P}(\tilde{x})\, dy - \dfrac{1}{9}\, P'(\tilde{x})dx$" où $\tilde{P}$ désigne le relèvement à D de $P(x)^{2/3}$.
On notera que ce trois web est hexagonal



5) En fait toute équation différentielle $P(x,y,y') = 0$ où $P \in \mathbb{C}[x,y,z]$ est un polynôme de degré d en z conduit à un d-web sur $\mathbb{C}^2$ avec singularités, web qui s'étend évidemment à $\mathbb{CP}(2)$.

Considérons maintenant un 3-web hexagonal $\mathscr{W}$ sur la surface S et soit $m \in S$ - Sing $\mathscr{W}$ un point de base - Si U est un voisinage ouvert suffisamment petit de m, $\mathscr{W}|_U$ est donné par les niveaux de $u_1, u_2, u_3 \in \mathcal{O}(U)$ tels que :

$$u_1 + u_2 + u_3 = 0 .$$

D'après (7) on sait que les u_i sont définis à une transformation affine près, ce qui conduit au **principe de prolongement analytique** : si γ est un chemin d'origine m, on peut, en recouvrant γ par des ouverts U_i munis de triplets basiques, effectuer le prolongement analytique du triplet (u_1, u_2, u_3).

En résulte une représentation de monodromie :

$$\text{Mon}\, \mathscr{W} : \Pi_1(S - \text{Sing}\, \mathscr{W}, m) \longrightarrow \mathscr{Aff}(2)$$

du groupe de Poincaré de $S-\mathrm{Sing}\,\mathcal{W}$ dans le groupe des transformations affines de $\mathbb{C}^2$. Si $\gamma(t)$ est un lacet en m, $\gamma(0)=\gamma(1)=m$, et $(\vec{u}_1^{\,t}, \vec{u}_2^{\,t})$ le prolongement analytique de (u_1,u_2) le long de γ au point $\gamma(t)$,$(\vec{u}_1^{\,0}, \vec{u}_2^{\,0})=(u_1,u_2)$, alors Mon $\mathcal{W}(\gamma)$ est défini par :

$$\mathrm{Mon}\,\mathcal{W}(\gamma)\,(u_1,u_2)=(u_1^1,u_2^1)$$

Si le lacet γ est invariant par la monodromie primaire, ce qui est le cas si par exemple $\mathcal{W}$ est donné par trois feuilletages, alors

$$\mathrm{Mon}\,\mathcal{W}(\gamma)\,(u_1,u_2)=\lambda(u_1,u_2)+(a_1,a_2),\ \ \lambda\in\mathbb{C}^*,\ \ a_i\in\mathbb{C}\,.$$

Un dernier point, d'importance, est le fait qu'un web hexagonal sur la surface S muni naturellement $S-\mathrm{Sing}\,\mathcal{W}$ d'une structure affine. Les feuilles de notre web sont des droites pour cette structure. Rappelons qu'une structure affine sur une variété M^n est la donnée d'un atlas de cartes (U_i,φ_i) tel que les $\varphi_i\circ\varphi_j^{-1}$ soient des transformations affines de $\mathbb{C}^n$. Dans notre cas si $m\in S-\mathrm{Sing}\,\mathcal{W}$ et (u_1,u_2,u_3) est un triplet basique, en m on prend par exemple comme carte en m le difféomorphisme local (u_1,u_2,u_3) à valeur dans l'hyperplan de $\mathbb{C}^3$: $\{x+y+z=0\}$. On ne confondra pas cette notion de structure affine avec celle de variété affine en géométrie algébrique.

Ainsi le web hexagonal $\mathcal{W}(E)$ associé à une courbe elliptique muni $\check{\mathbb{P}}\mathbb{C}(2)-E^*$ d'une structure affine. Pour voir les droites pour cette structure affine on paramètre E au moyen de fonctions elliptiques $(P,P')=F$:

$$F:\mathbb{C}/\Lambda\longrightarrow E\subset\mathbb{CP}(2)\,.$$

Se donner un point $\check{x}$ de $\check{\mathbb{P}}\mathbb{C}(2)-E^*$ c'est se donner trois points alignés distincts $A_1(\check{x}), A_2(\check{x}), A_3(\check{x})$ de la cubique.

Si l'on écrit $A_i=F(u_i)$, $u_i\in\mathbb{C}/\Lambda$, finalement se donner un point de $\check{\mathbb{P}}\mathbb{C}(2)-E^*$ c'est se donner un triplet $(u_1,u_2,u_3)\in\mathbb{C}^3 \bmod \Lambda^3$ tel que :

1) $u_1+u_2+u_3=0$ (condition d'alignement)
2) $u_i\neq u_j$
3) les u_i sont définis à permutation près.

Ainsi $\check{\mathbb{P}}\mathbb{C}(2)-E^*$ s'identifie à "l'hyperplan" Σ

$$\Sigma := \{ (u_1, u_2, u_3),\ u_1 + u_2 + u_3 = 0 \}$$

de l'espace $\mathrm{Sym}_3\, \mathbb{C}/\Lambda$:

$$\mathrm{Sym}_3\, \mathbb{C}/\Lambda = \frac{\{(u_1, u_2, u_3) \in (\mathbb{C}/\Lambda)^3, u_i \neq u_j\}}{\{(u_1, u_2, u_3) \sim (u_{\sigma(1)}, u_{\sigma(2)}, u_{\sigma(3)})\ \sigma \in S_3\}}$$

et la structure affine de $\check{\mathbb{P}}\mathbb{C}(2)-E^*$ est celle qu'hérite Σ de $\mathbb{C}^3$.

La structure affine induite sur une surface M par un 3-web hexagonal sans singularités est une structure de similitude, i.e. les changements de cartes $\varphi : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$ vérifient :

$$\| \varphi(x) - \varphi(y) \| = \lambda \| x - y \|$$

où $\lambda \in \mathbb{R}^+$ et $\| \ \|$ dénote la norme euclidienne sur $\mathbb{C}^2 \simeq \mathbb{R}^4$.

D. Fried a classé les structures de similitudes sur les variétés réelles compactes. On déduit de sa classification qu'une surface compacte complexe munie d'un 3-web sans singularité est revêtue par le tore (réel) $\mathbb{T}^4$ ou bien $S^3 \times S^1$ (où l'on retrouve les exemples 2) et 3)) [F].

Voici quelques problèmes :

I.4. Problèmes.

I.4.1 : Soit Γ une courbe algébrique dans $\mathbb{CP}(2)$; à quelle condition peut-on munir $\mathbb{CP}(2)-\Gamma$ d'un 3-web hexagonal sans singularités ? Plus généralement sous quelle condition peut-on munir $\mathbb{CP}(2)-\Gamma$ d'une structure affine. On examinera spécialement la cas où Γ est irréductible.

I.4.2 : Classifier les d-webs (singuliers) sur $\mathbb{CP}(2)$ dont toutes les feuilles sont des courbes algébriques. Il s'agit de montrer que ces d-webs sont donnés par une famille de courbes paramétrée par $c \in \mathbb{CP}(1)$:

$$P(x,y,c) = c^d\, a_0(x,y) + c^{d-1}\, a_1(x,y) + \ldots + a_d(x,y) = 0\ .$$

Pour $d = 3$, classifier parmi ceux-ci, ceux qui sont hexagonaux.

I.4.3 : Soit Γ comme dans I.4.1 ; supposons que Γ soit irréductible et que les singularités de Γ soient des croisements ordinaires ; le π_1 du complément de Γ dans $\mathbb{CP}(2)$ est alors isomorphe à $\mathbb{Z}/p\mathbb{Z}$ où $p =$

degré(Γ).
Si $\mathbb{CP}(2)-\Gamma$ est muni d'un 3-web $\mathcal{W}$ - hexagonal ou non - la monodromie primaire :

$$\sigma : \mathbb{Z}/p\mathbb{Z} \longrightarrow S_3$$

est déterminée par $s = \sigma(1)$, $s^p = \text{id}$.

Comme ou bien $s^2 = \text{id}$ ou bien $s^3 = \text{id}$ suivant le type de la permutation s, pour que la monodromie primaire soit non triviale il est donc nécessaire que l'entier p soit un multiple de 2 ou 3.

Dans le cas où $s = s_0 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, soit

$$\pi : X \longrightarrow \mathbb{CP}(2)$$

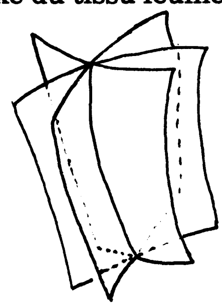
un revêtement ramifié le long de $3.\Gamma$; alors l'image inverse de $\mathcal{W}$ est un feuilletage $\mathcal{F}_{\Pi,\mathcal{W}}$ de X dont toutes les singularités sont dans $\pi^{-1}(\Gamma)$. Inversement si $\mathcal{F}$ est un feuilletage de la surface X dont toutes les singularités sont sur $\pi^{-1}(\Gamma)$ l'image directe de $\mathcal{F}$ est un 3-web dont la monodromie primaire est engendrée par s_0. Ces remarques font penser qu'il est possible de classer les courbes nodales irréductibles dont le complément peut être muni d'un 3-web hexagonal.

II - Tissus feuilletés et variétés de feuilletages sur les espaces projectifs.

§ II.1. Définitions et premières propriétés.

La notion de tissu feuilleté a été introduite par Etienne Ghys qui classifie les variétés réelles compactes de dimension 3 munies de tels objets [G]. Nous ne parlerons ici que de 3-tissus feuilletés, et en fait d'un type particulier de 3-tissus feuilletés : ceux dont la monodromie primaire est triviale.

Le modèle local d'un 3-tissu feuilleté est la donnée d'un triplet de feuilletages $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ obtenu par pull-back par une submersion locale $\sigma : \mathbb{C}^n \longrightarrow \mathbb{C}^2$ d'un 3-web local sur $\mathbb{C}^2$. Localement les feuilletages de codimension deux $(\mathcal{F}_i \cap \mathcal{F}_j)_{i \neq j}$ coïncident avec le feuilletage par les fibres de σ ; Ghys appelle ce feuilletage de codimension deux l'axe du tissu feuilleté



Le trois tissu feuilleté $\mathcal{F}$ est hexagonal s'il l'est en restriction à une section plane transverse à l'axe ; cette notion ne dépend pas bien entendu du choix de la section plane. Du point de vue équations de Pfaff on peut définir $\mathcal{F}$ par trois 1-formes holomorphes w_1, w_2, w_3 telles que :

$$\begin{cases} w_i \wedge dw_i = 0 \quad i = 1,2,3 \ \text{(condition d'intégrabilité)} \\ w_1 + w_2 + w_3 = 0 . \end{cases}$$

L'axe est alors défini par le système de Pfaff $\{w_1, w_2\}$.

Soit $\mathcal{F}$ un 3-tissu feuilleté défini par w_1, w_2, w_3 comme ci-dessus : la droite de formes différentielle $t \longrightarrow w_t = tw_2 + (1-t)w_1$ est bien définie (projectivement) et ne dépend pas du choix des w_i. Comme la trois forme $w_t \wedge dw_t$ dépend quadratiquement du paramètre t et s'annule pour $t = 0,1,\frac{1}{2}$ elle est identique-ment nulle. Est donc attaché au 3-tissu feuilleté une droite de feuilletages pivotant autour de l'axe. Si le 3-tissu feuilleté est hexagonal, alors pour n'importe quel choix de t_1,t_2,t_3, $t_i \neq t_j$, le 3-tissu défini par w_{t_1},w_{t_2},w_{t_3} sera évidemment hexagonal. L'hexagonalité est donc en fait une propriété de la droite de feuilletages.

§ II.2. Variété des feuilletages algébriques sur les espaces projectifs complexes.

Un feuilletage holomorphe singulier $\mathcal{F}$ de codimension un sur l'espace projectif $\mathbb{CP}(n)$ se définit par la donnée d'un ensemble algébrique $Y \subset \mathbb{CP}(n)$, cod $Y \geq 2$, et d'un recouvrement ouvert $\mathcal{U} = (U_i)_{i\in I}$ de $\mathbb{CP}(n)-Y$ chaque U_i étant muni d'un feuilletage holomorphe $\mathcal{F}_i$ de codimension un, $\mathcal{F}_i$ sans singularités, tel que :

$$\mathcal{F}_{i\,|\,U_i\cap U_j} = \mathcal{F}_{j\,|\,U_i\cap U_j} \quad \text{chaque fois que } U_i \cap U_j \neq \phi .$$

Via les théorèmes d'Hartogs et de Chow on démontre qu'un tel feuilletage $\mathcal{F}$ est défini en coordonnées homogènes $x = (x_1 : x_2 : ..... : x_{n+1})$ par une 1 forme homogène :

$$w = \sum a_i(x)\, dx_i\,, \quad a_i \text{ homogènes de degré } d+1$$

avec :

(α) $\sum x_i\, a_i(x) = 0$ (identité d'Euler)

(β) $w \wedge dw = 0$ (condition de Frobenius)

(γ) cod $S(w) \geq 2$ où $S(w) = \{\, x \,/\, a_i(x) = 0 \quad \forall\, i = 1,\dots,n+1 \,\}$.

L'entier d est par définition le degré du feuilletage $\mathcal{F}$; si $D \in \mathbb{CP}(n)$ est une droite générale, d est précisément le nombre de points de D où $\mathcal{F}$ n'est pas transverse à D ; on dit que $\mathcal{F}$ est un feuilletage algébrique de degré d. On note $\mathcal{F}(\mathbb{CP}(n),d)$ l'ensemble des feuilletages algébriques de degré d. Soit $\Lambda^1_{n,d}$

l'espace vectoriel des formes différentielles sur $\mathbb{C}^{n+1}$ homogènes de degré d+1 satisfaisant (α) :

$$\Lambda^1_{n,d} = \left\{ \sum_{i=1}^{n+1} a_i(x)\, dx_i,\ a_i \text{ homogène de degré } d+1, \sum x_i\, a_i(x) = 0 \right\} .$$

On note $\mathcal{I}_{n,d} \subset \Lambda^1_{n,d}$ l'ensemble des formes $w \in \Lambda^1_{n,d}$ satisfaisant la condition d'intégrabilité (β) et par $\mathbb{P}\mathcal{I}_{n,d}$ l'espace $\mathcal{I}_{n,d}/\!\sim$ où l'équivalence

$\sim$ est définie par :
$w \sim w \Leftrightarrow w = \lambda . w', \ \lambda \in \mathbf{C}^*$.

Visiblement $\mathbb{P}\mathcal{I}_{n,d}$ est une intersection de quadriques dans $\mathbb{P}\,\Lambda^1_{n,d}$.

L'espace $\mathcal{F}(\mathbb{CP}(n),d)$ s'identifie donc naturellement à l'ouvert de Zariski $\mathbb{P}\mathcal{I}_{n,d}$ des éléments satisfaisant la condition γ. Un problème naturel si l'on veut s'intéresser à la dynamique des feuilletages holomorphes de $\mathbb{CP}(n)$ est de donner la décomposition en composantes irréductibles de la variété algébrique $\mathbb{P}\mathcal{I}_{n,d}$.

Pour $n \geq 3$ cette décomposition n'est connue que dans les cas $d = 1$ et $d = 2$!

Nous allons essayer de montrer comment la théorie des webs et plus généralement des tissus feuilletés peut aider à donner la décomposition de $\mathcal{F}(\mathbb{CP}(n),d)$ tout du moins pour de "petits" n et d. Par un trois tissu feuilleté sur $\mathbb{CP}(n)$ on entend la donnée de 3 feuilletages $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \mathcal{F}(\mathbb{CP}(n),d)$ tels que au voisinage d'un point générique nos trois feuilletages définissent localement un 3-tissu feuilleté.

§ II .3. Quelques composantes de la variété $\mathbb{P}\mathcal{I}_{n,d}$.

Remarquons d'abord que $\mathcal{I}_{2,d} = \Lambda^1_{2,d}$ est un espace vectoriel ; le problème de la décomposition en composantes irréductibles de $\mathbb{P}\mathcal{I}_{2,d}$ est donc trivial. Considérons $w_2 \in \mathcal{I}_{2,d}$ et soit $L : \mathbf{C}^{n+1} \longrightarrow \mathbf{C}^3$ une submersion linéaire ; la forme pull-back L^*w_2 est visiblement un élément de $\mathcal{I}_{n,d}$. Introduisons :

$$\mathcal{I}^2{}_{n,d} = \left\{ \overline{L^*w_2, w_2 \in \mathcal{I}_{2,d}, \ L : \mathbf{C}^{n+1} \longrightarrow \mathbf{C}^3 \text{ submersion linéaire}} \right\}$$

où l'adhérence est prise de façon ordinaire dans $\Lambda^1{}_{n,d}$. On déduit sans peine de [C, L] le :

Théorème. $\mathbb{P}\mathcal{I}^2_{n,d}$ est une composante irréductible de $\mathbb{P}\mathcal{I}_{n,d}$.

La composante $\mathbb{P}\mathcal{I}^2_{n,d}$ s'appelle composante des feuilletages triviaux sur un deux plan. Considérons un élément L^*w_2, $L^*w_2 \in \mathcal{I}^2_{n,d}$; pour L et w_2 génériques $\mathcal{I}_{n,d}$ est lisse en L^*w_2 ; comme $L^*\mathcal{I}_{2,d}$ est un espace vectoriel l'intersection $\mathcal{I}_{n,d} \cap T_{L^*w_2} \mathcal{I}_{n,d}$ de $\mathcal{I}_{n,d}$ avec son espace tangent en L^*w_2 contient un espace vectoriel de dimension dim $\mathcal{I}_{2,d}$. Notamment une droite tracée dans cette intersection nous donne une droite de feuilletages sur $\mathbb{CP}(n)$; trois points sur cette droite conduisent à un 3-tissu feuilleté de $\mathbb{CP}(n)$, tissu feuilleté avec singularités.

Introduisons maintenant ce que l'on appelle les composantes logarithmiques. Désignons par $\mathcal{P}(n+1,\nu)$ l'espace vectoriel des polynômes homogènes de degré ν ; pour $\nu_1,\dots,\nu_p$ entiers positifs tels que $\sum \nu_i = d+2$ on introduit le sous ensemble de $\mathcal{I}_{n,d}$:

$$\Sigma^{p,n,d}_{\nu_1,\dots,\nu_p} = \left\{ \overline{w = P_1 \dots P_p \sum \lambda_i \frac{dP_i}{P_i}, P_i \in \mathcal{P}(n+1,\nu_i), \sum \lambda_i \nu_i = 0, \lambda_i \in \mathbb{C}} \right\}.$$

où cette fois encore l'adhérence est prise au sens ordinaire. On démontre ([C$_e$, M$_a$] dans le cas affine, [0] dans le cas projectif qui nous intéresse ici) le :

Théorème. Les $\mathbb{P}\,\Sigma^{p,n,d}_{\nu_1,\dots,\nu_p}$ sont des composantes irréductibles de $\mathbb{P}\mathcal{I}_{n,d}$.

Malheureusement on n'obtient pas toutes les composantes avec nos deux théorèmes comme on peut le voir dans le cas quadratique $d = 2$ ([C$_e$,M$_a$], [C$_e$,L]). Prenons un point général $w \in \Sigma = \Sigma^{p,n,d}_{\nu_1,\dots,\nu_p}$; on peut encore exhiber des espaces linéaires dans l'intersection $\Sigma \cap T_w \Sigma$ de différentes façons d'ailleurs. Si $w = P_1 \dots P_p \sum_{i=1}^{p} \lambda_i \frac{dP_i}{P_i}$ on peut par exemple considérer la famille $Q P_2 \dots P_p \left(\lambda_1 \frac{dQ}{Q} + \sum_{i=1}^{p} \frac{dP_i}{P_i} \right)$, $Q \in \mathcal{P}(n+1, \nu_1)$ ou bien lorsque $p \geq 3$ la famille $P_1 \dots P_p \sum_{i=1}^{p} t_i \frac{dP_i}{P_i}$ où ici les P_i sont fixés et les t_i parcourent l'hyperplan $\sum t_i \nu_i = 0$. Si l'on prend une droite générale dans l'une de ces deux familles, puis trois points sur cette droite on construit un 3 tissu-feuilleté qui, miracle, s'avère hexagonal : c'est un exercice facile.

Nous allons voir que l'on peut en quelque sorte caractériser les composantes logarithmiques par cette propriété. C'est l'objet du chapitre suivant.

III - Théorème de Fuchs pour les tissus feuilletés et applications

Dans ce chapitre on s'intéresse à des 3-tissus feuilletés sur l'espace projectif $\mathbb{CP}(n)$; comme nous avons en vue des résultats concernant les composantes de la variété des feuilletages de degré d donné on supposera que la monodromie primaire est triviale. Nous préciserons ensuite comment adapter nos méthodes au cas général. Un tel tissu feuilleté $\mathcal{W}$ est donc donné par trois feuilletages algébriques $\mathcal{W}=(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)$, $\mathcal{F}_i \neq \mathcal{F}_j$, définis par les formes homogènes $\omega_1,\omega_2,\omega_3 \in \Lambda^1_{n,d}$ vérifiant l'identité d'Euler et :

$$\begin{cases} w_i \wedge dw_i = 0 \\ \omega_1 + \omega_2 + \omega_3 = 0 \end{cases}$$

L'axe du tissu feuilleté est le feuilletage de codimension deux $\mathcal{F}_1 \cap \mathcal{F}_2$.

Au tissu $\mathcal{W}$ on associe comme précédemment la droite de feuilletages :

$$\omega_t = t\omega_2 + (1\text{-t})\,\omega_1 .$$

Dans toute la suite on suppose ω_1 et ω_2 choisis de façon telle que cod $S(w_i) \geq 2$, $i = 1,2$.

Tous les résultats énoncés sont valables pour $n = 2$, dimension où les deux notions tissu et tissu feuilleté coïncident.

III.1. Points singuliers. Points singuliers réguliers

Soit $\mathcal{W}$ défini par $(\omega_1,\omega_2,\omega_3)$ comme ci-dessus ; l'ensemble singulier Sing $\mathcal{W}$ s'écrit en coordonnées projectives :

$$\text{Sing } \mathcal{W} = \{\, m \in \mathbb{C}^{n+1},\, w_1 \wedge w_2 = 0 \,\}.$$

Notons que si $m \in \mathbb{C}^{n+1} -$ Sing $\mathcal{W}$ alors l'axe du tissu feuilleté trivialise localement $\mathcal{W}$. ■

Lemme : Soit $m \in \mathbb{C}^{n+1} -$ Sing $\mathcal{W}$; il existe une unique 1-forme holomorphe $\Theta_{,m}$ définie au voisinage de m telle que $dw_{i,m} = \Theta_{,m} \wedge w_{i,m}$, $i = 1,2,3$.

Preuve : On peut trouver des coordonnées $x = (x_1,x_2 ...)$ au voisinage de m telles que :

$$w_1 = g_1\, dx_1$$
$$w_2 = g_2\, dx_2$$

où les g_i sont des unités holomorphes au voisinage de m.
Comme w_1+w_2 doit être intégrable on a :
$0 = (g_1 dx_1 + g_2 dx_2) \wedge (dg_1 \wedge dx_1 + dg_2 \wedge dx_2) = (g_1 dg_2 + g_1 dg_1) \wedge dx_2 \wedge dx_2$.

La fonction holomorphe $h = \dfrac{g_1}{g_2}$ ne dépend donc que de x_1, x_2.
On a :

$$dw_1 = \frac{dg_1}{g_1} \wedge w_1 = \left(\frac{dh}{h} + \frac{dg_2}{g_2} \right) \wedge w_1 = \left(\frac{\frac{\partial h}{\partial x_2}}{h}\, dx_2 + \frac{dg_2}{g_2} \right) \wedge w_1$$

$$dw = \frac{dg_1}{g_1} \wedge w_2 = \left(\frac{dg_2}{g_2} \wedge w_2 \right) = \left(\frac{\frac{\partial h}{\partial x_2}}{h}\, dx_2 + \frac{dg_2}{g_2} \right) \wedge w_2 .$$

On pose $\Theta_{,m} = \left(\dfrac{\frac{\partial h}{\partial x_2}}{h}\, dx_2 + \dfrac{dg_2}{g_2} \right)$; si $\Theta'_{,m}$ vérifie encore $dw_i = \Theta'_{,m} \wedge w_{i,m}$, alors : $(\Theta_{,m} - \Theta'_{,m}) \wedge w_{i,m} = 0$ et par suite $\Theta_{,m} = \Theta'_{,m}$. ■

D'après le lemme il existe une 1-forme Θ définie sur $\mathbb{C}^{n+1}$– Sing $\mathcal{W}$ vérifiant $dw_i = \Theta \wedge w_i$; il est innocent de constater que Θ s'étend de façon rationnelle à $\mathbb{C}^{n+1}$. Si $\mathcal{W}$ est hexagonal Θ est fermée, ce qui est la version tissu feuilleté de § I.2 prop. 1.

Corollaire : cod Sing $\mathcal{W} = 1$.

Démonstration : Si cod Sing $\mathcal{W} = 2$, la 1 forme Θ s'étend alors holomorphique-ment à tout $\mathbb{C}^{n+1}$ (Hartogs).
Mais une identité du type

$$dw_i = \Theta \wedge w_i$$

est impossible pour des raisons de degré.

Remarque : En fait la 1-forme rationnelle Θ a ses composantes homogènes de degré -1.

Rappelons qu'une 1-forme α méromorphe sur l'espace projectif est logarithmique si α et $d\alpha$ sont à pôles simples le long d'un diviseur à croisements normaux. Les formes logarithmiques sont fermées : on peut trouver ce résultat dans [D] ou le déduire de calculs élémentaires de [C_e, L].

Introduisons l'hypersurface $\Sigma \subset \mathrm{Sing}\, \mathcal{W}$ constituée de l'adhérence des points où $\mathrm{Sing}\, \mathcal{W}$ est de dimension pure $n-1$.

Proposition 1 : Si Σ est à croisements normaux et Θ et $d\Theta$ à pôles simples alors Θ est fermée et $\mathcal{W}$ est hexagonal.

Preuve : Visiblement Θ a ses pôles le long de Σ ; soit L une forme linéaire suffisamment générale pour que $\Sigma \cup (L=0)$ soit encore à croisements ordinaires. Si $R = \sum x_i \frac{\partial}{\partial x_i}$ désigne le champ radial on a :

$$i_R\, dw_i = (d+2) w_i \,.$$

Par suite :

$$i_R\, dw_i = \Theta(R) w_i$$

implique que $\Theta(R) = d+2$. En résulte que la 1 forme α :

$$\alpha = \Theta - (d+2)\frac{dL}{L}$$

vérifie $L_R\, \alpha = 0 = \alpha(R)$ et définit une 1-forme sur $\mathbb{CP}(n)$ qui est visiblement logarithmique donc fermée ; il en est de même pour Θ. ■

On suppose dans la suite que $\mathcal{W}$ est hexagonal. Soient $m_0 \in \mathbb{C}^{n+1}$ - $\mathrm{Sing}\, \mathcal{W}$ un point de base et (u_1,u_2,u_3) un triplet basique en m_0 ; par prolongement analytique on définit un triplet multiforme basique $(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)$, $\sum \tilde{u}_i = 0$, sur $\mathbb{C}^{n+1}$ - $\mathrm{Sing}\, \mathcal{W}$.

Nous allons décrire les $\tilde{u}_i$ près de $\mathrm{Sing}\, \mathcal{W}$.

Désignons comme précédemment par $\Sigma \subset \mathrm{Sing}\, \mathcal{W}$ l'adhérence des points de $\mathrm{Sing}\, \mathcal{W}$ de dimension pure $n-1$. Soit $m_0 \in (\mathrm{Sing}\, \mathcal{W} - \Sigma)$; comme cod $\mathrm{Sing}\, \mathcal{W}$, $m \geq 2$, si U est une petite boule centrée en m, on a :

$$\Pi_1(U - \mathrm{Sing}\, \mathcal{W}, m) = 0 \,.$$

Par suite chaque détermination v_i de $\tilde{u}_i$ est uniforme sur U et s'étend holomorphiquement à U tout entier par le théorème d'Hartogs. Ainsi le triplet basique $(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)$ s'étend à tout $\mathbb{C}^{n+1} - \Sigma$.

Soit $\Sigma = \bigcup_{i\in I} \Sigma^i$ la décomposition en composantes irréductibles de l'hyper-surface Σ. Si m est un point générique de Σ^j les espaces Ker $w_i(m)$ sont identiques ; ils peuvent être ou bien transverses ou bien égaux à l'espace tangent $T_m\,\Sigma^j$.

Définition 5 : Une composante irréductible Σ^j de $\Sigma \subset$ Sing $\mathcal{W}$ est une composante de contact transverse si en tout point générique $m \in \Sigma^j$, $T_m\,\Sigma^j = \mathrm{Ker}\, w_i(m)$, $i = 1,2$.

Remarque : Soit Σ^j une composante de Σ ; si Σ^j n'est pas de contact transverse alors $(\Sigma^j - (\overline{\Sigma - \Sigma^j}))$ lisse est une feuille commune des $\mathcal{F}_i$.

Proposition 2 : Soit Σ^j une composante de contact transverse ; soit m un point générique de Σ^j. Alors toute détermination v_i de $\tilde{u}_i$ se prolonge holomorphique-ment et uniformément au voisinage de m.

Preuve : Puisque m est générique le feuilletage $\mathcal{F}_i$, i = 1,2, est donné au voisinage de m par les niveaux d'une submersion y_i, $y_i(m) = 0$ et $y_i^{-1}(0)$ est tranverse à Σ^j en m.

Soit $m' \in y_i^{-1}(0) - \Sigma^j$ et v_i une détermination de $\tilde{u}_i$ définie au voisinage de m'. Alors v_i se factorise dans y_i, i.e. il existe ℓ_i holomorphe tel que : $v_i = \ell_i \circ y_i$; comme $\ell_i \circ y_i$ est holomorphe dans un domaine $|y_i| \leq \varepsilon$, on étend v_i de façon uniforme au voisinage de m.

Proposition 3 : Soit Σ^j une composante de contact transverse et m un point générique de $\Sigma^j \cap$ Sing $\mathcal{W}_{\text{lisse}}$. Alors la forme fermée rationnelle Θ est holomorphe en m.

Preuve : Supposons qu'il n'en soit pas ainsi ; alors les pôles de Θ en m coïncident avec $\Sigma^j_{,m}$. Soit x une submersion en m telle que $\Sigma^j_{,m} = (x{=}0)$; visiblement on a :

$\Theta = \lambda \dfrac{dx}{x} + d(\dfrac{h}{x^p})$ où $\lambda \in \mathbb{C}$, h est holomorphe en m et $p \in \mathbb{N}$.
Mais les égalités :

$$dw_i = \Theta \wedge w_i = \left[\lambda \frac{dx}{x} + d(\frac{h}{x^p}) \right] \wedge w_i$$

impliquent que x=0 est feuille des $\mathcal{F}_i$. Ce qui est absurde. ■

On introduit l'ensemble $\Sigma' \subset \Sigma$ constitué de l'union des composantes Σ^j de Σ qui sont feuilles communes des $\mathcal{F}_i$, $i = 1,2$; d'après la proposition 2 le triplet basique $(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)$ multiforme est en fait défini sur $\mathbb{C}^{n+1}-\Sigma'$.

Soit m un point de Σ' ; comme l'ensemble singulier des w_i, $i = 1,2$, est de codimension 2, pour m générique les $\mathcal{F}_i$ sont réguliers en m et $\Sigma'_{,m} = \Sigma_{,m}$ est feuille des $\mathcal{F}_{i,m}$ au sens ordinaire.

Définition 6 : Soit m un point générique de Σ' ; <u>on dit que $\mathcal{W}$ a un contact tangentiel d'ordre 1 en m</u> s'il existe un système de coordonnées locales $y_1,\dots,y_{n+1}$ en m tel que :

1) $y_1^{-1}(0) = \Sigma'_{,m} = \Sigma_{,m} = \operatorname{Sing} \mathcal{W}, m$
2) $w_1 = a_1\, dy_1$, a_1 unité
3) $w_2 = a_2\, d(y_1\alpha)$, a_2 et α unités
4) $dy_1 \wedge d\alpha$ n'est pas identiquement nul le long de $y_1^{-1}(0)$.

Remarque :

1) Les conditions 1, 2, 3 traduisent que le point m est non singulier pour les $\mathcal{F}_i$ et que $\Sigma'_{,m}$ est une feuille. La condition 4 permet de contrôler le contact des feuilletages $\mathcal{F}_1$ et $\mathcal{F}_2$ le long de $\Sigma' \subset \operatorname{Sing} \mathcal{W}$.

2) La condition 4) est intrinsèque i.e. ne dépend pas du choix des coordonnées $y_1,\dots y_{n+1}$ vérifiant 1,2,3.

3) Dans les coordonnées $y_1,y_2,\dots,y_{n+1}$, $w_1 \wedge w_2$ s'écrit :

$$w_1 \wedge w_2 = a_1.a_2\, y_1\, dy_1 \wedge d\alpha$$

et par conséquent $w_1 \wedge w_2$ s'annule le long de $\Sigma'_{,m}$ avec multiplicité

générique un ; c'est visiblement une caractérisation des points de contact tangentiel d'ordre 1.

L'ensemble des points de Σ' qui ont un contact tangentiel d'ordre supérieur à un constituent un sous ensemble analytique fermé de Σ'. On a la :

Proposition 4 : Soit m un point de $\Sigma' \cap \mathrm{Sing}\,\mathcal{W}_{\text{lisse}}$ tel que w_1 et w_2 ne s'annulent pas en m. Alors si $\mathcal{W}$ est à contact tangentiel d'ordre un en m,Θ est au pire à pôle simple le long de $\Sigma'_{,m}$.

Preuve : Supposons $\mathcal{W}$ à contact tangentiel d'ordre un en m et soit $y_1,\ldots,y_{n+1}$ un système de coordonnées comme dans la définition :

$w_1 = a_1\, dy_1$

$w_2 = a_2\, d(y_1\alpha)$, a_i unité.

De $dw_i = \Theta \wedge w_i$ on tire l'existence de fonctions méromorphes α_i telles que :

$$\Theta = \frac{da_1}{a_1} + \alpha_1 w_1 = \frac{da_2}{a_2} + \alpha_2 w_2 .$$

En résulte que :

$$w_1 \wedge \frac{da_1}{a_1} = w_1 \wedge \frac{da_2}{a_2} + \alpha_2 w_1 \wedge w_2 .$$

Comme les $w_1 \wedge \dfrac{da_i}{a_i}$ sont holomorphes, et que $w_1 \wedge w_2$ est à zéro simple le long de $y_1 = 0$, α_2 est au pire à pôle simple le long de $y_1 = 0$; il en est évidemment de même pour Θ. ∎

Définition 7 : Soit m un point de $\Sigma' \cap \mathrm{Sing}\,\mathcal{W}_{\text{lisse}}$; on dit que m est un point singulier régulier de $\mathcal{W}$ si la forme fermée rationnelle Θ est au pire à pôle simple le long de Σ' en m.

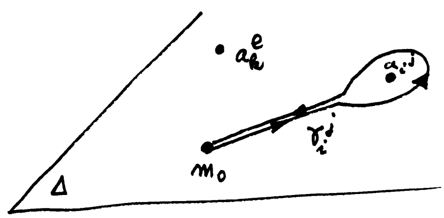
On a donc l'implication :

m point de contact tangentiel d'ordre 1 $\Rightarrow$ m singulier régulier. Les points singuliers réguliers forment un ouvert de Sing $\mathcal{W}$; si m_j appartient à une composante Σ'^j de Σ' et est singulier régulier alors presque tout point de Σ'^j est singulier régulier. Nous dirons que $\mathcal{W}$ est à singularité régulière le long de Σ'^j.

§ III.2. Formes normales près d'un point singulier régulier.

Dans ce paragraphe nous décrivons les $\tilde{u}_i$ près d'un point singulier régulier. Comme nous l'avons vu les $\tilde{u}_i$ se laissent prolonger au complément de l'hypersurface Σ' formée des composantes de Sing $\mathcal{W}$ qui sont feuilles des $\mathcal{F}_i$.

Rappelons comment se décrit le groupe de Poincaré du complément de l'hypersurface Σ'. On se donne un point de base m_0 dans le complément de Σ' et Δ une droite générale passant par m_0. Cette droite coupe les composantes Σ'^j de Σ' aux points a_j^i, $i = 1,\dots,\ell(j)$; on trace alors dans Δ des lacets γ_i^j d'extrémité m_0, d'indice 1 par rapport aux points a_j^i et d'indices 0 par rapport aux autres a_k^ℓ ; suivant Lefsetchz les γ_i^j engendrent $\Pi_1(\mathbb{CP}(n)-\Sigma', m_0)$ [L].



Soient $(\underline{\tilde{u}}_1, \underline{\tilde{u}}_2, \underline{\tilde{u}}_3)$ une détermination fixée en m_0 du triplet basique $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$; la monodromie

$$\text{Mon}\,\mathcal{W} : \Pi_1(\mathbb{CP}(n)-\Sigma', m_0) \longrightarrow \text{Aff}(2)$$

est décrite par les Mon $\mathcal{W}(\gamma_i^j)$, où les γ_i^j sont définis comme ci-dessus.

Comme on l'a vu dans § I.3, si $\gamma \in \Pi_1(\mathbb{CP}(n)-\Sigma', m_0)$ l'application affine Mon $\mathcal{W}(\gamma)$ est du type :

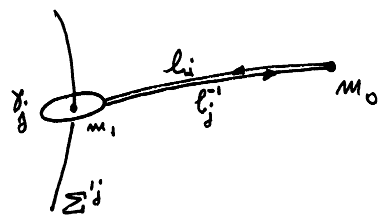
$$(\underline{\tilde{u}}_1, \underline{\tilde{u}}_2) \longmapsto \lambda(\gamma)(\underline{\tilde{u}}_1, \underline{\tilde{u}}_2) + (a_1(\gamma), a_2(\gamma))$$

avec $\lambda(\gamma) \in \mathbb{C}^*$ et $a(\gamma), b(\gamma) \in \mathbb{C}$.

Définition 8 : L'élément γ est dit <u>parabolique</u> si $\lambda(\gamma) = 1$, <u>hyperbolique</u> sinon.

Nous allons travailler avec les objets suivants :

- m_0 un point de base comme ci-dessus
- m_j un point de $\Sigma' \cap \operatorname{Sing} \mathscr{W}_{\text{lisse}}$
- P_j un polynôme homogène irréductible tel que $\Sigma'^j = P_j^{-1}(0)$
- V_j un petit polydisque centré en m_j
- γ_j un lacet obtenu en joignant par un chemin ℓ_j m_0 à un cercle γ'_j tracé autour de Σ'^j dans V_j, γ_j d'indice un autour de Σ'^j (on dira que γ_j est un lacet élémentaire).



- Θ la 1-forme fermée rationnelle telle que $dw_i = \Theta \wedge w_i$.

Proposition 5 : Si γ_j est parabolique, les $d\tilde{\underline{u}}_i$ s'étendent par prolongement analytique le long de γ_j de façon uniforme autour de $V_j \cap \Sigma'^j$; plus précisément les $\tilde{\underline{u}}_i$ sont du type $\varepsilon_i \log P_j + f_i$ où $\varepsilon_i \in \mathbf{C}$ et $f_i \in \mathcal{O}(V_j - \Sigma'^j)$.

L'intégrale $\dfrac{1}{2i\pi} \displaystyle\int_{\gamma_j} \Theta$ est un entier ; si de plus le point m_j est singulier régulier alors f_i s'étend méromorphiquement le long de $\Sigma'^j \cap V_j$.

Preuve : Si γ_j est parabolique, visiblement les prolongements analytiques le long de γ_i, des différentielles $d\tilde{\underline{u}}_i$ sont uniformes et se laissent définir sur $V_j - \Sigma'^j$ au voisinage de m_j. En résulte l'existence de ε_i et f_i comme dans l'énoncé ; on peut préciser la nature des f_i comme suit. Localement Θ s'écrit au voisinage de m_0 :

$$\Theta = \lambda_j \frac{dP_j}{P_j} + d\Big(\frac{h}{P_j^{n_j}}\Big)$$

où $\lambda_j = \frac{1}{2i\pi} \int_{\gamma_j} \Theta$, h est holomorphe sur V_j, $n_j \in \mathbb{N}$ et P_j ne divise pas h.

Au voisinage de m_0 on a (§ I.2 prop. 1) :

$$\begin{cases} w_i = g\, d\tilde{\underline{u}}_i,\ g \text{ holomorphe au voisinage de } m_0, \quad \text{et} \\ \Theta = \dfrac{dg}{g} \end{cases}$$

Comme les $d\tilde{\underline{u}}_i$ sont uniformes, g se laisse prolonger analytiquement de façon uniforme le long de γ_j et par suite sur $V_j - \Sigma'^j$; nécessairement :

$$\lambda_j = \frac{1}{2i\pi} \int_{\gamma_j} \Theta = \frac{1}{2i\pi} \int_{\gamma_j} \frac{dg}{g}$$

est entier.

Si m_j est un point singulier régulier, Θ est à pôle simple, donc l'entier n_j est nul. On en déduit que :

$$\Theta = \frac{dg}{g} = \lambda_j \frac{dP_j}{P_j} + dh\,, \quad \lambda_j \in \mathbb{Z} \text{ et } h \in \theta(V_j).$$

Par suite

$$g = P_j^{\lambda_j}.U, \quad \lambda_j \in \mathbb{Z} \text{ et } U \text{ unité.}$$

De :

$$w_i = g.d\tilde{\underline{u}}_i = P_j^{\lambda_j}.U \left(\varepsilon_j \frac{dP_j}{P_j} + df_i \right)$$

on tire que f_i est méromorphe. ■

Remarque : La façon de "joindre" m_0 à γ'_j pour construire γ_j n'importe pas dans l'énoncé ; en effet si $\tilde{\gamma}_j$ est un autre lacet obtenu en joignant m_0 à γ'_j par un chemin ℓ_j on a visiblement

$$\frac{1}{2i\pi} \int_{\tilde{\gamma}_j} \Theta = \frac{1}{2i\pi} \int_{\gamma_j} \Theta = \lambda_j\,.$$

Le prolongement analytique $\tilde{g}$ de g le long de $\tilde{\gamma}_j$ vérifie :

$$\frac{d\tilde{g}}{\tilde{g}} = \Theta = \lambda_j \frac{dP_j}{P_j} + dh\,, \quad \lambda_j \in \mathbb{Z}\,.$$

Par suite sur V_j on a :

$$\tilde{g}|_{V_j} = c_j g|_{V_j} \text{ où } c_j \in \mathbf{C}^*$$

et $\tilde{g}$ est uniforme au voisinage de $\tilde{\gamma}_j$; il en est de même du prolongement analytique des $d\tilde{u}_i$ le long de $\tilde{\gamma}_j$. Par suite $\tilde{\gamma}_j$ est aussi parabolique.

Ainsi le fait d'être parabolique est une propriété de la composante Σ'^j et non pas du lacet γ_j ; dans toute la suite on parlera de composante parabolique ou de composante hyperbolique.

On a le :

Théorème (de Fuchs pour les webs paraboliques) : Soit $\mathcal{W}$ un web hexagonal sur $\mathbf{CP}(n)$ défini par les forme de Pfaff w_1,w_2,w_3. Si toutes les composantes de $\Sigma' \subset \operatorname{Sing} \mathcal{W}$ sont paraboliques et si $\mathcal{W}$ est à singularités régulières le long de Σ', il existe des entiers $q_j \in \mathbf{Z}$ tels que $d\left(\dfrac{w_i}{P_1^{q_1}\dots P_p^{q_p}}\right) = 0$, $i = 1,2,3$ où $P = P_1 \dots P_p$ est une équation réduite de Σ'. De plus, il existe $\lambda_j(i) \in \mathbf{C}$ et h_i polynômes homogènes tels que :

$$\frac{w_i}{P_1^{q_1}\dots P_p^{q_p}} = \sum_j \lambda_j(i)\,\frac{dP_j}{P_j} + \frac{dh_i}{P_1^{q_1-1}\dots P_p^{q_p-1}}\ , \quad i=1,2,3$$

avec $\displaystyle\sum_{i=1}^{3} \lambda_j(i) = 0 = \sum_{i=1}^{3} h_i$ et degré h_i = degré $P_1^{q_1-1}\dots P_p^{q_p-1}$.

Remarques :

1) Le feuilletage $\mathcal{F}_i = \mathcal{F}(w_i)$ possède alors l'intégrale première

$$I(i) = \sum_j \lambda_j(i) \,.\, \operatorname{Log} P_j + \frac{h_i}{P_1^{q_1-1}\dots P_p^{q_p-1}}$$

ce qui justifie l'appellation théorème de Fuchs.

Il est facile de voir que les formes ω_i sont alors dans l'adhérence d'une composante de la variété des feuilletages décrite dans § II.2.

2) Le feuilletage $\mathcal{F}_t = \mathcal{F}(w_t)$ possède alors l'intégrale première

$$(1-t)I(2) + tI(1) = \Sigma\,[(1-t)\lambda_j(2) + t\lambda_j(1)]\ \mathrm{Log}\, P_j + \left(\frac{(1-t)h_2 + th_1}{P_1^{q_1-1} \dots P_p^{q_p-1}}\right).$$

Preuve du théorème : Soient m_0 un point de base dans le complément de Sing $\mathcal{W}$ et γ_j des lacets comme dans la proposition 5 ; on pose

$$q_j = \frac{1}{2i\pi} \int_{\gamma_j} \Theta .$$

Comme on l'a vu les q_j sont dans $\mathbb{Z}$. Les γ_j engendrant l'homotopie du complément de Σ' la forme différentielle $\Theta - \Sigma q_j \dfrac{dP_j}{P_j}$ est exacte dans $\mathbb{C}^{n+1} - \Sigma'$; en fait [$\mathrm{C_e}$, $\mathrm{M_a}$] :

$$\Theta = \Sigma q_j \frac{dP_j}{P_j} + d\left(\frac{h}{P_1^{n_1} \dots P_p^{n_p}}\right)$$ où h est holomorphe et le quotient $\dfrac{h}{P_1^{n_1} \dots P_p^{n_p}}$ est homogène de degré 0.

De :

$$d\omega_i = \Theta \wedge w_i = \left[\ \Sigma q_i \frac{dP_j}{P_j} + d\left(\frac{h}{P_1^{n_1} \dots P_p^{n_p}}\right)\right] \wedge w_i$$

on déduit que la forme différentielle :

$$\frac{w_i}{P_1^{q_1-1} \dots P_p^{q_p-1} \exp \dfrac{h}{P_1^{n_1-1} \dots P_p^{n_p-1}}}$$

est fermée, résultat qui n'utilise pas d'ailleurs le fait que les singularités soient régulières. Si maintenant $\mathcal{W}$ est à singularité régulière le long de Σ' alors Θ est à pôles simples et donc les n_i et h sont nuls. Dans ce cas on a :

$$d\left(\frac{w_i}{P_1^{q_1} \dots P_p^{q_p}}\right) = 0 .$$

Finalement [$\mathrm{C_e}$, $\mathrm{M_a}$] on trouve $\lambda_j(i)$ et h_i comme dans l'énoncé. ∎

Remarque : Les $\lambda_j(i)$ correspondant à des indices q_j négatifs sont nuls.

Faisons maintenant une étude analogue dans le cas hyperbolique ; nous gardons les mêmes notations que précédemment. Si le lacet γ_j est hyperbolique et $(\tilde{u}_1, \tilde{u}_2)$ une détermination de $(\tilde{u}_1, \tilde{u}_2)$ en m_0, la monodromie le long de γ_j est du type :

$$(\tilde{u}_1, \tilde{u}_2) \xrightarrow{\mathrm{Mon}(\gamma_j)} \lambda(\gamma_j) \, . \, (\tilde{u}_1, \tilde{u}_2) + (a_1(\gamma_j), a_2(\gamma_j))$$

où cette fois $\lambda(\gamma_j) \neq 1$.

On peut voir agir la monodromie sur les différentielles $d\tilde{u}_i$ et l'on a :

$$(d\tilde{u}_1, d\tilde{u}_2) \xrightarrow{\mathrm{Mon}(\gamma_j)} \lambda(\gamma_j) \, . \, (d\tilde{u}_1, d\tilde{u}_2) \, .$$

Par suite, si l'on écrit au voisinage de m_0 :

$$\begin{cases} \omega_i = g \, . \, d\tilde{u}_i, \ i=1,2, \ g \text{ holomorphe au voisinage de } m_0 \\ \Theta = \dfrac{dg}{g} \end{cases}$$

on constate que le prolongement analytique $\tilde{g}$ de g a pour monodromie

$$g \xrightarrow{\mathrm{Mon}(\gamma_j)} \frac{1}{\lambda(\gamma_j)} \cdot g \, .$$

On pose : $\dfrac{1}{\lambda(\gamma_j)} = e^{2i\pi(1-\mu_j)}$ où μ_j est, pour l'instant, défini modulo $\mathbb{Z}$ et l'on écrit le prolongement analytique $\tilde{g}$ dans V_j comme :

$$\tilde{g} = P_j^{1-\mu_j} \, . \, U \quad \text{où } U \text{ est uniforme sur } V_j - (P_j = 0).$$

Notons que $\dfrac{d\tilde{g}}{\tilde{g}}$ est uniforme et :

$$\Theta = \frac{d\tilde{g}}{\tilde{g}} = (1-\mu_j)\frac{dP_j}{P_j} + \frac{dU}{U} = \lambda_j \frac{dP_j}{P_j} + d\left(\frac{h}{P_j^{n_j}}\right)$$

où : $\lambda_j = \dfrac{1}{2i\pi} \displaystyle\int_{\gamma_j} \Theta$, $h \in \theta(V_j)$, $n_j \in \mathbb{N}$ et h est non divisible par P_j.

On peut maintenant choisir μ_j et U de sorte que $1-\mu_j = \lambda_j$. Evidemment si Θ est à pôle simple le long de $(P_j = 0)$, on a $n_j = 0$. Dans ce cas on aura :

$$\tilde{g} = P_j^{1-\mu_j} . U \quad \text{où cette fois} \quad U \in \mathcal{O}^*(V_j), \ U = \exp h.$$

Le prolongement analytique de $\tilde{u}_i$ le long de γ_j dans $V_j - (P_j = 0)$ sera alors de la forme :

$$\tilde{u}_i = P_j^{1-\mu_j} . W_i + c_i \,, \quad W_i \in \mathcal{O}(V_j - (P_j = 0)), \quad c_i \in \mathbf{C}$$

avec :

$$c_k \,(\, 1 - e^{2i\pi\mu_j}\,) = a_k(\gamma_j) \,, \quad k = 1{,}2 \,, \quad e^{2i\pi\,\mu_j} = \lambda(\gamma_j) \neq 1.$$

On a :

$$\omega_i = g \,.\, d\tilde{u}_i \;=\; P_j^{1-\mu_j}\, U\,(\,\mu_j\, P_j^{\mu_j - 1}\, W_i\, dP_j + P_j^{\mu_j}\, d\, W_i\,)$$

$$= \mu_j\, U\, W_i\, dP_j + UP_j\, d\, W_i\,.$$

Afin de préciser la nature de W_i le long de $(P_j = 0)$ choisissons en m_j des coordonnées $y_1,\dots y_{n+1}$ telles que $y_1 = P_1$. Dans ces coordonnées w_i s'écrit :

$w_i = \alpha_i\, dy_1 + y_1 \sum\limits_{k\geq 2} \beta_{i,k}\, dy_k$ où les α_i et $\beta_{i,k}$ sont holomorphes.

Comme :

$$w_i = \mu_j\, U\, W_i\, dy_1 + Uy_1\, d\, W_i$$

les W_i vérifient le système d'équations différentielles linéaires avec second membre :

$$(*) \qquad \left\{ \begin{array}{l} \mu_j W_i + y_1 \dfrac{\partial W_i}{\partial y_1} = \dfrac{\alpha_i}{U} \quad i = 1{,}2 \\[2ex] (k) \quad \dfrac{\partial W_i}{\partial y_k} = \dfrac{\beta_{i,k}}{U}\,, \quad k = 2,\dots,n+1 \quad i = 1{,}2 \end{array} \right.$$

On remarquera que deux solutions uniformes $W_i^{(1)}$ et $W_i^{(2)}$ de ce système coïncident nécessairement ; en effet les équations (k) indiquent que $W_i^{(1)} - W_i^{(2)}$ ne dépendent que de y_1 et sont de plus solution de :

$$\mu_1\, W_i + y_1 \frac{\partial W_i}{\partial y_1} = 0\,.$$

Comme μ_1 est non entier cette équation différentielle a pour seule solution uniforme la solution nulle et donc $W_i^{(1)} = W_i^{(2)}$.

Lorsque $\mathscr{W}$ est à points singuliers réguliers, on sait que U est une unité holomorphe ; en résulte que le second membre de notre système est holomorphe. Puisque le système (*) possède une solution il est complétement intégrable, ce qui se traduit par les équations aux dérivées partielles :

$$\begin{cases} \dfrac{\partial}{\partial y_\ell} \dfrac{\beta_{i,k}}{U} = \dfrac{\partial}{\partial y_k} \dfrac{\beta_{i,\ell}}{U} \quad k=2...n+1 \\ \mu_j \dfrac{\beta_{i,k}}{U} + y_1 \dfrac{\partial}{\partial y_1} \dfrac{\beta_{i,k}}{U} = \dfrac{\partial \alpha_i}{\partial y_k} \end{cases}$$

Le théorème de Poincaré (à paramètre y_1) permet de trouver W_i'' holomorphe en $y_1,...y_{n+1}$ telle que :

$$\frac{\partial}{\partial y_\ell} W_i'' = \frac{\beta_{i,k}}{U}, \quad k = 2,...,n+1 .$$

On cherche W_i' holomorphe solution de (*) sous la forme

$$W_i' = W_i'' + L_i(y_1)$$

soit à résoudre en L_i :

$$\mu_j \; L_i(y_1) + y_1 \frac{\partial L_1}{\partial y_1} = \frac{\alpha_i}{U} - \mu_j W_i'' - y_1 \frac{\partial W_i''}{\partial y_1} = H_i(y_1,...y_{n+1}) .$$

Mais la condition d'intégrabilité implique que $\dfrac{\partial H_i}{\partial y_\ell} = 0, \quad \ell = 2...n+1$. Comme μ_j est non entier, notre équation possède une unique solution holomorphe L_i ; du fait de l'unicité on a $W_i = W_i'$ sur $V_j-(P_j=0)$ et W_i s'étend de façon holomorphe à V_j .

Nous sommes donc en mesure d'énoncer la :

Proposition 6 : Si le lacet γ_j est hyperbolique ($\lambda(\gamma_j) \neq 1$) le prolongement analytique "à V_j" des $\tilde{\mathscr{U}}_i$ le long de γ_j est du type $P_j^{\mu_j}$. $W_i + c_i$ où $W_i \in \mathcal{O}(V_j-(P_j=0))$ et les $c_i, \mu_j \in \mathbb{C}$ vérifient :

(α) $e^{2i\pi\mu_j} = \lambda(\gamma_j), \quad \mu_j \notin \mathbb{Z}$ et $1-\mu_j = \dfrac{1}{2i\pi} \displaystyle\int_{\gamma_j} \Theta$

(β) $c_i = \dfrac{a_i(\gamma_j)}{1-\lambda(\gamma_j)} = \dfrac{a_i(\gamma_j)}{1-e^{2i\pi\mu_j}}$.

De plus si $\mathscr{W}$ est à singularité régulière le long de $(P_j=0)$ alors les W_i sont holomorphes sur V_j.

On a évidemment des remarques analogues à celles suivant la proposition 5.

Remarques : Le choix du lacet élémentaire γ_j n'importe pas ; les μ_j ne dépendent que de la classe d'homologie de γ_j. Toutefois les c_i eux dépendent du choix de γ_j.

§ III.3. Webs hyperboliques abéliens.

Si $\mathscr{W}$ est un 3-tissu feuilleté hexagonal parabolique, i.e. dont toutes les composantes de Σ' sont paraboliques alors l'image de

$$\mathrm{Mon} : \Pi_1(\mathbb{CP}(n) - \Sigma', m_0) \longrightarrow \mathrm{Aff}(2)$$

est un groupe abélien (ici un groupe de translation). Il en est de même lorsque l'intersection de Σ' avec un plan générique $\mathbb{CP}(2)$ est une courbe nodale puisque dans cette éventualité $\Pi_1(\mathbb{CP}(n) - \Sigma', m_0)$ est abélien.

Définition : Le 3-tissu feuilleté hexagonal $\mathscr{W}$ sera dit abélien si la représentation de monodromie $\mathrm{Mon} : \Pi_1(\mathbb{CP}(n) - \Sigma', m_0) \longrightarrow \mathrm{Aff}(2)$ a son image abélienne.

Cette définition ne dépend évidemment pas du choix du point de base m_0. Soit $\mathscr{W}$ un 3-tissu feuilleté abélien hyperbolique ; on peut trouver une composante Σ'^j de Σ' et un lacet élémentaire $\gamma_j \in \Pi_1(\mathbb{CP}(n) - \Sigma', m_0)$ tel que $\lambda(\gamma_j) \neq 1$:

$$(\tilde{\underline{u}}_1, \tilde{\underline{u}}_2) \xrightarrow{\mathrm{Mon}(\gamma_j)} \lambda(\gamma_j) \,.\, (\tilde{\underline{u}}_1, \tilde{\underline{u}}_2) + (a_1(\gamma_j), a_2(\gamma_j)) \,.$$

On peut alors choisir le triplet basique $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$, $\tilde{u}_1 + \tilde{u}_2 + \tilde{u}_3 = 0$, de sorte que $a_1(\gamma_j) = a_2(\gamma_j) = 0$: il suffit pour cela d'ajouter au triplet initial les constantes

$$b_1 = \frac{a_1(\gamma_j)}{\lambda(\gamma_j) - 1}\,, \quad b_2 = \frac{a_2(\gamma_j)}{\lambda(\gamma_j) - 1}\,, \quad b_3 = -b_1 - b_2.$$

Du fait de l'abélianité on aura pour un tel triplet et tout $\gamma \in \Pi_1(\mathbb{CP}(n) - \Sigma', m_0)$:

$$(\tilde{u}_1, \tilde{u}_2) \xrightarrow{\mathrm{Mon}(\gamma_j)} \lambda(\gamma) . (\tilde{u}_1, \tilde{u}_2) .$$

On a le :

Théorème (de Fuchs pour les tissus feuilletés abéliens hyperboliques). Soit $\mathcal{W}$ un tissu feuilleté sur $\mathbb{CP}(n)$ défini par les 1-formes w_1, w_2, w_3. On suppose $\mathcal{W}$ hyperbolique abélien à singularités régulières le long de Σ' et l'on désigne par $P_1 ... P_p = 0$ une équation réduite de Σ' dans $\mathbb{C}^{n+1}$.

Il existe un triplet basique $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ du type :

$$\tilde{u}_i = P_1^{\mu_1} ... P_p^{\mu_p} \ Q_i$$

où les Q_i sont des polynômes homogènes tels que $Q_1 + Q_2 + Q_3 = 0$ et les μ_j sont définis par :

$$1 - \mu_j = \frac{1}{2i\pi} \int_{\gamma_j} \Theta$$

les γ_j étant des lacets élémentaires autour de $P_j = 0$.

Preuve : Supposons $\lambda(\gamma_1) \neq 1$ avec nos notations usuelles.

D'après la proposition 6 le prolongement analytique d'une détermination d'un élément d'un triplet basique suivant γ_1 est du type

$$P_1^{\mu_1} . W_i + c_1$$

avec W_i holomorphe. Comme pour un choix convenable initial du triplet basique l'image du groupe de monodromie sera linéaire, on peut supposer $c_1 = 0$ et écrire $P_1^{\mu_1} W_i$ sous la forme :

$P_1^{\mu_1} ... P_p^{\mu_p} . U_i$ où U_i est holomorphe le long de γ_1 avec $1 - \mu_j = \dfrac{1}{2i\pi} \displaystyle\int_{\gamma_j} \Theta$

Invoquant le fait que les γ_j engendrent le groupe de Poincaré de $\mathbb{CP}(n) - \Sigma'$ et de nouveau la proposition 6 on constate que U_i s'étend de façon uniforme à $\mathbb{C}^{n+1} - \Sigma'$ et se prolonge holomorphiquement le long de la partie lisse de Σ' ; le théorème d'Hartogs permet de prolonger U_i à tout $\mathbb{C}^{n+1}$ en une fonction holomorphe Q_i qui sera visiblement homogène. ∎

Remarques :

1) Le feuilletage $\mathcal{F}(w_t) = \mathcal{F}(tw_2+(1-t)w_1)$ possède l'intégrale première $P_1^{\mu_1}...P_p^{\mu_p}\,.\,[tQ_2+(1-t)Q_1]$; on peut évidemment en prendre le logarithme :

$$\sum \mu_j \operatorname{Log} P_j + \operatorname{Log}(t\,Q_2 + (1-t)\,Q_1)$$

et sous cette forme on voit clairement la différence entre tissus paraboliques et tissus hyperboliques (abéliens) ; dans le cas paraboliques les points de "multi-formités" ($P_i = 0$) sont fixés et les résidus λ_j dépendent linéairement de t alors que dans le contexte hyperbolique les résidus sont fixes et une branche $tQ_2 + (1-t)Q_1$ bouge.

Dans le cas hyperbolique les nombres μ_j apparaissant dans les intégrales sont directement liés aux périodes de la forme Θ alors qu'il n'en est pas ainsi dans le cas parabolique.

2) si $\mathcal{W} = \mathcal{W}(w_1,w_2,w_3)$ est abélien hyperbolique alors la droite $tw_2 + (1-t)w_2$ est située dans une certaine composante $\mathbb{P}\sum_{\nu'_1,\dots,\nu'_p}^{p',n',d'}$ de la variété $\mathbb{P}\mathcal{I}_{n,d}$.

3) Concernant les degrés des P_j et des Q_i on remarque que :

$$i_R\,\Theta = d+2$$

où R est le champ radial et $d+1$ est le degré des composantes des w_i.
Par suite si $\nu_j = \text{degré}(P_j)$ on a :

$$\sum \lambda_j\,\nu_j = d+2$$

soit :

$$\sum_{j=1}^{p} (1-\mu_j)\nu_j = d+2\,.$$

Comme $\tilde{u}_i$ est nécessairement projective on a :

$$\sum \nu_j\,\mu_j \ + \nu(Q_i) = 0$$

soit :

$$\sum \nu_j \ + \nu(Q_i) = d+2\,.$$

Ce qui précise le point 2 : les w_i sont dans $\mathbb{P}\sum_{\nu_1,\dots,\nu_p,\nu(Q_i)}^{p+1,n,d}$.

§ III . 4. Exemples de tissus hyperboliques non abéliens.

Nous allons commencer par des exemples en dimension deux et nous allons cette fois travailler dans une carte affine $\mathbb{C}^2$ de $\mathbb{CP}(2)$.

On désigne par w_ν une 1-forme homogène :

$$w_\nu = a_\nu \, dx + b_\nu \, dy$$

où a_ν et b_ν sont des polynômes homogènes de degré ν ; on note $P_{\nu+1}$ le polynôme homogène :

$$P_{\nu+1} = x a_\nu + y b_\nu$$

et l'on suppose dans toute la suite que $P_{\nu+1}$ est non identiquement nul et réduit. Soit $P_{\nu+1} = \prod_{i=1}^{\nu+1} L_i$ la décomposition en facteurs irréductibles de $P_{\nu+1}$, L_i formes linéaires ; la forme rationnelle $\dfrac{w_\nu}{P_{\nu+1}}$ est fermée et s'écrit :

$$\frac{w_\nu}{P_{\nu+1}} = \sum \mu_j \frac{dL_j}{L_j} \;, \quad \mu_j \in \mathbb{C}, \quad \sum \mu_j = 1 \;,$$

$\mu_j = \dfrac{1}{2i\pi} \displaystyle\int_{\gamma_j} \dfrac{w_\nu}{P_{\nu+1}}$, où les γ_j sont des petits cycles autour de $L_j = 0$.

Proposition 7 : Soient f_ν un polynôme homogène de degré ν et w la forme différentielle :

$$w = w_\nu + f_\nu (xdy - ydx) .$$

On désigne par Θ la forme rationnelle :

$$\Theta = \sum (1+\mu_j) \frac{dL_j}{L_j} .$$

Alors $d\omega = \Theta \wedge w$.

Preuve : Comme $\dfrac{w_\nu}{P_{\nu+1}} = \sum \mu_j \dfrac{dL_j}{L_j}$ est fermée on a :

$$dw_\nu = \frac{dP_{\nu+1}}{P_{\nu+1}} \wedge w_\nu = \left(\frac{dP_{\nu+1}}{P_{\nu+1}} + \sum \mu_j \frac{dL_j}{L_j} \right) \wedge w_\nu = \Theta \wedge w_\nu .$$

D'un autre côté :

$$d(f_\nu (xdy - ydx)) = (\nu+2) f_\nu \, . \, dx \wedge dy$$

et

$\Theta \wedge f_\nu(xdy-ydx) = f_\nu \,.\ \sum_{j=1}^{\nu+1}(1+\mu_j)\ dx \wedge dy = (\nu+2)f_\nu \,.\ dx \wedge dy =$
$d(f_\nu(xdy-ydx))$.
D'où le résultat. ■

Remarque : La forme Θ ne dépend pas du choix de f_ν.

Corollaire : Soient w_ν comme ci-dessus, $f_{\nu,1}, f_{\nu,2}, f_{\nu,3}$ trois polynômes homogènes de degré ν non nuls distincts tels que $f_{\nu,1}+f_{\nu,2}+f_{\nu,3} = 0$ et trois nombres complexes non nuls ε_i tels que $\varepsilon_1+\varepsilon_2+\varepsilon_3 = 0$. Alors les formes différentielles :

$$\underline{w}_i = \varepsilon_i\, w_\nu + f_{\nu,i}(xdy-ydx)$$

engendrent un trois web hexagonal sur $\mathbb{CP}(2)$ à singularités régulières. Ce trois web est hyperbolique dès que l'un des μ_j est non entier. En général un tel tissu est non abélien.

Preuve : D'après la proposition 7 et la remarque seule la non abélianité est à prouver. Supposons qu'un tel web soit abélien et désignons par w_i les homogénéïsées des $\underline{w}_i$ dans $\mathbb{C}^3$. D'après notre théorème de Fuchs les w_i possèderaient une intégrale première de type Liouville. Sa restriction à la carte $\mathbb{C}^2$ procurerait une intégrale Liouville des $\underline{w}_i$.

En résulterait que l'holonomie projective ([C_e, M_a]) de la singularité 0 des $\underline{w}_i$ serait un groupe abélien. Mais un calcul explicite montre que générique-ment sur les μ_j et $f_{\nu,i}$ il n'en est pas ainsi (*). ■

Conjecture : Soit $\mathcal{W}^0 = \mathcal{W}(\underline{w}_1,\underline{w}_2,\underline{w}_3)$ un 3 web sur $\mathbb{CP}(2)$, $\underline{w}_i$ comme dans le corollaire. Soit $\mathcal{W}$ un 3 tissu feuilleté sur $\mathbb{CP}(n)$ tel que dans une certaine section plane $\mathbb{CP}(2) \longrightarrow \mathbb{CP}(n)$, $\mathcal{W}$ coïncident avec $\mathcal{W}^0$, alors $\mathcal{W}$ est trivial au-dessus de $\mathcal{W}^0$.

(*) Un travail sur les webs de ce type est en cours d'élaboration par un de mes élèves Frank Loray.

§ III . 5 . Remarques sur les tissus feuilletés non hexagonaux.

Sur $\mathbb{CP}(2)$ il n'y a pas grand chose à dire puisque la donnée de deux feuilletages algébriques quelconques de même degré $\mathcal{F}(w_1)$ et $\mathcal{F}(w_1)$ conduit à un 3-tissus feuilleté : en effet $w_3 = -w_1 - w_2$ est automatiquement intégrable.

La première dimension où apparaît de façon effective la contrainte d'intégrabilité de la droite de feuilletage $tw_2 + (1-t)w_1$ est la dimension 3. Nous supposons donc $n \geq 3$. Comme d'habitude tout le discours sera fait en coordonnées homogènes sur $\mathbb{C}^{n+1}$. Soit donc $\mathcal{W}$ un 3 tissu feuilleté sur $\mathbb{CP}(n)$ donné par les 1-formes w_i, $i = 1,2,3$ $\sum w_i = 0$, $w_1 \wedge dw_2 + w_2 \wedge dw_1 = 0$. On rappelle que l'ensemble singulier Sing $\mathcal{W} = \{m,\ w_1 \wedge w_2(m) = 0\}$ est nécessaire-ment de codimension m. On note toujours Θ l'unique 1-forme rationnelle, à composantes de degré -1, telle que :

$$dw_i = \Theta \wedge w_i \,.$$

Evidemment dans ce paragraphe on suppose Θ non fermée : $d\Theta \neq 0$; si d est le degré des feuilletages $\mathcal{F}(w_i)$ on a :

$$i_R\, dw_i = (d+2)\, w_i = i_R\, \Theta . w_i$$

où $R = \sum_{i=1}^{n} x_i \dfrac{\partial}{\partial x_i}$ est le champ radial d'Euler.

Voici quelques propriétés de Θ, pour certaines déjà rencontrées :

1) $d\Theta \wedge w_i = 0$ et par suite il existe α rationnelle telle que :

2) $d\Theta = \alpha .\ w_1 \wedge w_2$, degré $\alpha = -2d-4$; notamment $i_R\, d\Theta = 0$

3) la trois forme rationnelle $\Theta \wedge d\Theta$ est non identiquement nulle ; en effet :

$$i_R\, \Theta \wedge d\Theta = i_R\, \Theta . d\Theta + i_R d\Theta \wedge \Theta = (d+2) . d\Theta \neq 0 \,.$$

De plus :

4) $\Theta \wedge d\Theta = \alpha . \Theta \wedge w_1 \wedge w_2 = \alpha . dw_1 \wedge w_2 = -\alpha . dw_2 \wedge w_1$.

Proposition 7 : Si $\mathcal{W} = \mathcal{W}(w_1, w_2, w_3)$, $w_1 + w_2 + w_3 = 0$, est un tissu feuilleté non hexagonal sur $\mathbb{CP}(n)$ il existe une fraction rationnelle R non constante telle que $w_1 \wedge w_2 \wedge dR = 0$; en particulier les feuilles de l'axe sont tracées sur des hypersurfaces de $\mathbb{CP}(n)$.

Preuve : En dérivant 4) on obtient :

$$0 = d.d\Theta = d\alpha \wedge w_1 \wedge w_2 + \alpha[dw_1 \wedge w_2 - w_1 \wedge dw_2]$$

qui conduit après multiplication par Θ à :

$$0 = d\alpha \wedge w_1 \wedge w_2 \wedge \Theta .$$

Faisant agir le champ radial par produit intérieur, il vient :

$$0 = (d^0\alpha).\alpha.w_1 \wedge w_2 \wedge \Theta - i_R\,\Theta\,.\,d\alpha \wedge w_1 \wedge w_2$$

soit, puisque $d^0\alpha = -2(d+2)$ et $i_R\,\Theta = (d+2)$:

$$0 = \Big(\Theta + \frac{1}{2}\,\frac{d\alpha}{\alpha} \Big) \wedge w_1 \wedge w_2 .$$

On peut donc trouver deux fonctions rationnelles K_1 et K_2, degré $K_i = -(d+2)$ telles que :

$$\Theta + \frac{1}{2}\,\frac{d\alpha}{\alpha} = K_1 w_1 + K_2 w_2 .$$

Remarquons que, puisque Θ est supposée non fermée, les K_i ne peuvent être simultanément nuls. Pour des raisons de commodités nous supposerons K_1 et K_2 non identiquement nuls ; ce qui est loisible quitte à faire une combinaison linéaire des w_i.

Par dérivation on obtient :

$$\alpha.w_1 \wedge w_2 = d\Theta = K_1 w_1 + K_2 w_2 + dK_1 \wedge w_1 + dK_2 \wedge w_2 .$$

En multipliant par dw_1 on a :

$$0 = \mathrm{d}K_2 \wedge w_2 \wedge dw_1 = \alpha.dK_2 \wedge \Theta \wedge w_1 \wedge w_2 .$$

De même :

$$0 = dK_1 \wedge \Theta \wedge w_1 \wedge w_2 .$$

En procédant comme précédemment par action du champ radial, on obtient :

$$\Big(\Theta + \frac{dK_1}{K} \Big) \wedge w_1 \wedge w_2 = 0 = \Big(\Theta + \frac{dK_2}{K_2} \Big) \wedge w_1 \wedge w_2 .$$

Finalement en éliminant Θ, on constate que :

$$0 = \Big(\frac{dK_1}{K_1} - \frac{dK_2}{K_2}\Big) \wedge w_1 \wedge w_2 = \Big(2\,\frac{dK_1}{K_1} - \frac{d\alpha}{\alpha}\Big) \wedge w_1 \wedge w_2 = \Big(2\,\frac{dK_2}{K_2} - d\alpha\Big) \wedge w_1 \wedge w_2$$

qui indique que les fonctions rationnelles K_1/K_2, K_1^2/α , K_2^2/α satisfont à l'énoncé. Reste tout de même à vérifier qu'elles sont non constantes !

C'est l'objet du :

Lemme : Les K_1/K_2, K_1^2/α , K_2^2/α ne sont pas toutes constantes.

Preuve : Si tel est le cas $\alpha = c_1K_1^2, K_2 = c_2.K_1$ où $c_i \in \mathbb{C}$ et :

$$\Theta + \frac{1}{2}\frac{d\alpha}{\alpha} = \Theta + \frac{dK_1}{K_1} = K_1(w_1+c_2w_2).$$

On a par suite :

$$d(w_1+c_2w_2) = \Theta \wedge (w_1+c_2w_2) = -\frac{dK_1}{K_1} \wedge (w_1+c_2w_2)$$

qui indique que la forme différentielle $K_1.(w_1+c_2w_2)$ est fermée et donc Θ aussi ; ce qui est absurde. ■

On note $K(\mathcal{A}xe\ \mathcal{W})$ l'ensemble des fractions rationnelles R vérifiant

$$dR \wedge w_1 \wedge w_2 = 0 .$$

On constate que si $R \in K(\mathcal{A}xe\ \mathcal{W})$, R est de degré 0 et constante sur les feuilles de l'axe ; $K(\mathcal{A}xe\ \mathcal{W})$ est un corps non réduit aux constantes d'après la proposition 7.

On introduit l'entier $\tau(\mathcal{W})$:

$\tau(\mathcal{W}) = \sup \{ p \in \mathbb{N}$ tel qu'il existe $R_1,\ldots,R_p \in K(\mathcal{A}xe\ \mathcal{W})$ vérifiant $dR_1\wedge\ldots\wedge dR_p \neq 0 \}$

Puisque les éléments de $K(\mathcal{A}xe\ \mathcal{W})$ sont constants sur les feuilles de l'axe on a visiblement $1 \le \tau(\mathcal{W}) \le 2$. Si $\tau(\mathcal{W}) = 2$ les feuilles de l'axe sont les composantes connexes des fibres d'une application rationnelles.

Proposition 8 : Si $\tau(\mathcal{W}) = 1$, il existe $R_0 \in K(\mathcal{A}xe\ \mathcal{W})$ tel que $K(\mathcal{A}xe\ \mathcal{W}) = \mathbb{C}(R_0)$.

Preuve : Soit $\mathbb{CP}(1) \longrightarrow \mathbb{CP}(n)$ une droite générale ; les restrictions $\bar{R}$ à $\mathbb{CP}(1)$ des éléments R de $K(\mathcal{A}xe\ \mathcal{W})$ forment un sous corps du corps des fractions rationnelles à une variable : d'après le théorème de Luröth il existe $R_0 \in K(\mathcal{A}xe\ \mathcal{W})$ tel que :

$$K(\mathcal{A}xe\ \mathcal{W})\big|_{\mathbf{CP}(1)} = \mathbf{C}(\bar{R}_0),\quad \bar{R}_0 = R_0\big|_{\mathbf{CP}(1)}\ .$$

Si $R \in K(\mathcal{A}xe\ \mathcal{W})$ alors :

$$dR \wedge dR_0 = 0 \ \text{ puisque } \ \tau(\mathcal{W}) = 1$$

et

$$R_0\big|_{\mathbf{CP}(1)} = r(R_0\big|_{\mathbf{CP}(1)}) \ \text{ où } \ r \in \mathbf{C}(t)\,.$$

Par suite $R = r(R_0)$. ■

Remarque : On peut bien sûr définir aussi le corps $K(\mathcal{A}xe\ \mathcal{W})$ dans le cas hexagonal ; génériquement si $\mathcal{W}$ est hexagonal parabolique ou hyperbolique abélien on a $K(\mathcal{A}xe\ \mathcal{W}) = \mathbf{C}$. On peut produire des exemples explicites de $\mathcal{W}$ hexagonaux pour lesquels $\tau(\mathcal{W}) = 1$. Par contre je ne connais pas d'exemples de tissus feuilletés non hexagonaux tels que $\tau(\mathcal{W}) = 1$. En fait la proposition qui suit fait penser qu'il sera bien difficile d'en construire.

Proposition 9 : Soit $w_1 = P_1 \dots P_p \sum\limits_{i=1}^{p} \lambda_i \dfrac{dP_i}{P_i}$ un élément de $\sum_{v_1,\dots,v_p}^{p,n,d}$, $n \geq 3$, $\lambda_i \neq \lambda_j$, $P_i \neq P_j$ pour $i \neq j$ et P_j réduits. Supposons qu'il existe un tissu feuilleté $\mathcal{W}$ tel que $\mathcal{F}(w_1)$ soit l'un des feuilletages de $\mathcal{W}$. Si $\tau(\mathcal{W}) = 1$, alors $\mathcal{W}$ est hexagonal.

Preuve : Elle est un peu lourde. Il existe w_2, w_3, $w_1 + w_2 + w_3 = 0$ tels que
$\mathcal{W} = \mathcal{W}(w_1, w_2, w_3)$; pour vérifier l'hexagonalité on travaille en fait avec la droite $tw_2 + (1-t)w_1$ ce qui nous permet de supposer que w_2 est voisin de w_1 ; on en déduit que w_2 est dans $\sum_{1,\dots,v_p}^{p,n,d}$ et plus précisément s'écrit :

$$w_2 = Q_1 \dots Q_p \sum \mu_i \frac{dQ_i}{Q_i} \quad \text{avec } Q_i \neq Q_j \text{ pour } i \neq j.$$

Si $P = P_1 \dots P_p$ et $Q = Q_1 \dots Q_p$ sont $\mathbf{C}$–colinéaire il est clair que $\mathcal{W}$ sera hexagonal. Si tel n'est pas le cas il se peut toutefois que P et Q aient des composantes communes. On écrit P et Q sous la forme :
$P = F.G$, $Q = G.H$ avec F, G, H réduits sans branches communes, l'un des F, G, H pouvant éventuellement être égal à 1.

Soit R_0 tel que $K(\mathscr{W}) = \mathbb{C}(R_0)$.

Lemme : La fraction rationnelle non constante F/H appartient à $K(\mathscr{W})$; notamment il existe $r \in \mathbb{C}(t)$ tel que $F/H = r(R_0)$.

Preuve : Remarquons que

$$\begin{cases} dw_1 = (\frac{dF}{F} + \frac{dG}{G}) \wedge w_1 \\ dw_2 = (\frac{dG}{G} + \frac{dH}{H}) \wedge w_2 \end{cases}$$

Comme $w_1 \wedge dw_2 + w_2 \wedge dw_1 = 0$ on obtient :
$(\frac{dF}{F} + \frac{dH}{H}) \wedge w_1 \wedge w_2 = 0$ et $\frac{F}{H} \in K(\mathscr{W})$. ■

Remarque : Noter que nécessairement $\frac{F}{H}$ est homogène de degré 1. Par suite seule G peut être éventuellement égal à 1.

Maintenant, puisque $R_0 \in K(\mathscr{W})$ on peut trouver des fractions rationnelles A_1 et A_2, non toutes deux nulles telles que :

$$dR_0 = A_1 . \frac{w_1}{P} + A_1 . \frac{w_2}{Q} .$$

Comme $\frac{w_1}{P}$ et $\frac{w_2}{Q}$ sont fermées, on a par dérivation :

$$0 = dA_1 \wedge \frac{w_1}{P} + dA_2 \wedge \frac{w_2}{Q}$$

et par suite :

$$0 = dA_1 \wedge w_1 \wedge w_2 .$$

Ainsi les A_i sont aussi dans $K(\mathscr{W})$ et il existe $r_i \in \mathbb{C}(t)$ tels que : $A_i = r_i(R_0)$. Supposons A_1 non identiquement nul ; on a :

$$\frac{dR_0}{r_1(R_0)} = \frac{w_1}{P} + \frac{r_2}{r_1}(R_0) . \frac{w_2}{P} .$$

Par suite, si $\frac{r_2}{r_1}$ est non constant on constate par différentiation que R_0 est intégrale première de w_2 ; mais ce discours ne dépendant pas du choix de w_2 sur la droite de feuilletages, tout élément de cette droite a

R_0 comme intégrale première : c'est impossible puisque $w_1 \wedge w_2 \neq 0$.

En résulte que $\dfrac{r_2}{r_1} = c \in \mathbf{C}$ et :

$$\frac{dR_0}{r_1(R_0)} = \frac{w_1}{P} + c \,.\, \frac{w_2}{Q} \,.$$

Notons F_j, G_k, H_ℓ les composantes irréductibles de F, G, H et écrivons :

$$\frac{w_1}{P} = \Sigma\, \alpha_j \frac{dF_j}{F_j} + \Sigma\, \beta_k \frac{dG_k}{G_k}$$

$$\frac{w_2}{Q} = \Sigma\, \gamma_k \frac{dG_k}{G_k} + \Sigma\, \varepsilon_\ell \frac{dH_\ell}{H_\ell}$$

où les α_j, β_k sont pris dans la liste des λ_i et γ_k, ε_ℓ dans la liste des μ_i.
D'après le lemme il existe $r_0 \in \mathbf{C}(t)$ tel que :

$$r_0(R_0)\left(\frac{dF}{F} - \frac{dH}{H}\right) = \frac{w_1}{P} + c\,\frac{w_2}{Q}$$

soit encore :

$$r_0(R_0)\left(\Sigma \frac{dF_j}{F_j} - \Sigma \frac{dH_\ell}{H_\ell}\right) = \Sigma\, \alpha_j \frac{dF_j}{F_j} + \Sigma(\beta_k + c.\gamma_k) \frac{dG_k}{G_k} + \Sigma\, c.\varepsilon_\ell \frac{dH_\ell}{H_\ell} \,.$$

Par intégration sur des cycles élémentaires autour des $G_k = 0$ on constate que :

$$\beta_k + c.\gamma_k = 0 \,.$$

Par suite $r_0(R_0)$ ne peut avoir de pôles que le long des $F_j = 0$ et $H_\ell = 0$. Mais pour des raisons de multiplicité $r_0(R_0)$ ne peut avoir de pôles le long des $F_j = 0$ et $H_\ell = 0$; ainsi $r_0(R_0)$ est constante : $r_0(R_0) = c_0 \in \mathbf{C}$. Visiblement $c_0 \neq 0$.
Finalement :

$$\begin{cases} \alpha_j = c_0 = -c\varepsilon_\ell \quad \text{(qui implique que } c \neq 0) \\ \beta_k + c\,\gamma_k = 0 \end{cases}$$

et :

$$\frac{w_1}{F.G} = c_0 \frac{dF}{F} + \Sigma\, \beta_k \frac{dG_k}{G_k}$$

$$\frac{w_2}{G.H} = -\Sigma\, \frac{\beta_k}{c} \frac{dG_k}{G_k} - \frac{c_0}{c} \frac{dH}{H} \,.$$

La droite de feuilletages $tw_2 + (1-t)w_1$ se laisse écrire :

$$tw_2+(1-t)w_1 = G\left\{ c_0 d\left[(1-t)F-\frac{t}{c}H\right] + \sum \beta_k \frac{dG_k}{G_k}\left[(1-t)F-\frac{t}{c}H\right]\right\}$$
$$= G F_t \left\{ c_0 \frac{dF_t}{F_t} + \sum \beta_k \frac{dG_k}{G_k} \right\}$$

où $F_t = (1-t)F - \frac{t}{c}H$.

Finalement $tw_2 + (1-t)w_1$ possède l'intégrale première $F_t.\Pi.G_k^{\beta_k/c_0}$, ce qui
montre que le tissu feuilleté est hexagonal abélien. ■

Lorsque $\tau(\mathscr{W}) = 2$ il existe suivant Siegel [Si] des fractions rationnelles $Q_1,\dots,Q_s$, $s \geq 2$ telles que :

$$K(\mathscr{W}) = \mathbb{C}(Q_1,\dots,Q_s)\,.$$

Moralement, il existe une surface S dont le corps des fonctions rationnelles est $K(\mathscr{W})$ munie d'un web $\mathscr{W}_s$ et une application $P : \mathbb{CP}(n) \longrightarrow S$ telle que $\mathscr{W} = P^* \mathscr{W}_s$; cette affirmation est correcte sur un ouvert de Zariski de $\mathbb{CP}(n)$. En effet si $K(\mathscr{W}) = 2$, les feuilles de l'axe, tout du moins les feuilles génériques, sont les composantes connexes des fibres d'une application rationnelle (R_1, R_2), $R_i \in K(\mathscr{W})$ algébriquement indépendantes. On récupère une relation d'équivalence qui restreinte à un ouvert de Zariski produit par passage au quotient une variété analytique S munie effectivement d'un 3-web puisque les feuilles de $\mathscr{W}$ sont compatibles avec la relation d'équivalence.

Si $\mathscr{W} = \mathscr{W}(w_1, w_2, w_3)$ les w_i appartiennent vraisemblablement à des composantes de type $\mathscr{I}^2_{n,d}$ où l'on substitue aux submersions linéaires
$L : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^3$ des fractions rationnelles.

IV - Quelques exemples de tissus feuilletés avec monodromie primaire non triviale.

Soient P_j, Q_k et V des polynômes homogènes de degré respectif ν_j, δ_k, ν, irréductibles et sans composantes communes, $j = 1...p$, $k = 1...q$. Soient q' un entier tel que $1 \leq q' < q$ et $\lambda_1,...,\lambda_p$, $\mu_1,...,\mu_q$ des nombres complexes tels que :

$$0 = \sum_{j=1}^{p} \lambda_j \nu_j + \sum_{k=1}^{q} \mu_k \delta_k = \frac{1}{3} \sum_{k=1}^{q'} \delta_k + \frac{2}{3} \sum_{k=q'+1}^{q'} \delta_k + \nu + \sum_{j=1}^{p} \lambda_j \nu_j .$$

Soit X l'hypersurface d'équation $P_1...P_p . Q_1...Q_q = 0$; la formule

$$F = P_1^{\lambda_1}..P_p^{\lambda_p} \left[Q_1^{\mu_1}...Q_q^{\mu_q} + (Q_1...Q_{q'})^{1/3} (Q_{q'+1}...Q_q)^{2/3} V \right] = F_1 + F_2$$

définit une fonction multiforme sur $\mathbb{C}^{n+1} - X$; cette fonction multiforme est homogène de degré 0 du fait des contraintes sur les λ et les μ. On peut calculer la monodromie de F en choisissant des lacets élémentaires α_j, β_k autour de $P_j = 0$ et $Q_k = 0$ respectivement. Si $\underline{F} = \underline{F}_1 + \underline{F}_2$ est une détermination de F en un point base $m_0 \in \mathbb{C}^{n+1} - X$ on a :

$$\mathrm{Mon}(\alpha_j).\underline{F} = e^{2i\pi\lambda_j} F = e^{2i\pi\lambda_j}(\underline{F}_1 + \underline{F}_2) , \quad j = 1,...,p$$
$$\mathrm{Mon}(\beta_k).\underline{F} = \underline{F}_1 + e^{2i\pi/3} \underline{F}_2 \quad \text{pour} \quad k = 1...q'$$
$$\mathrm{Mon}(\beta_k).\underline{F} = \underline{F}_1 + e^{i\pi/3} \underline{F}_2 \quad \text{pour} \quad k = 1...q' .$$

Remarquant qu'il existe une relation abélienne (linéaire) entre $\underline{F}_1 + \underline{F}_2$ $\underline{F}_1 + e^{2i\pi/3} \underline{F}_2$, $\bar{\underline{F}}_1 + e^{i\pi/3} \underline{F}_2$ on munit ainsi $\mathbb{CP}(n)$ d'un trois tissu feuilleté mais avec monodromie primaire non triviale. Je pense savoir démontrer que si les λ et les μ sont choisis suffisamment génériquement alors la feuille générique est dense et se recoupe sur un ensemble dense. Il serait intéressant d'étudier la topologie de tels objets pour des p, q, ν_j, δ_k petits.

Je remercie A. Hénaut qui m'a amené à réfléchir sur les webs ; J.-P. Dufour pour m'avoir invité à participer à Montpellier à un atelier tissu. Et tout particulièrement Françoise Dal'bo qui m'a initié à la géométrie affine.

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Astérisque, tome 222 (1994), p. 93-133

<http://www.numdam.org/item?id=AST_1994__222__93_0>

CODIMENSION ONE FOLIATIONS IN CP^n, $n \geq 3$, WITH KUPKA COMPONENTS

D. Cerveau and A. Lins Neto

1. INTRODUCTION

1.1 – Basic notions:

A codimension one holomorphic foliation in a complex manifold M can be given by an open covering $(U_\alpha)_{\alpha\in A}$ of M and two collections $(w\alpha)_{\alpha\in A}$ and $(g_{\alpha\beta})_{U_\alpha\cap U_\beta\neq\phi}$, such that:

(a) For each $\alpha \in A$, w_α is an integrable ($w_\alpha \wedge dw_\alpha = 0$) holomorphic 1-form in U_α, and $w_\alpha \not\equiv 0$.

(b) If $U_\alpha \cap U_\beta \neq \phi$ then $w_\alpha = g_{\alpha\beta} \cdot w_\beta$, where $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$.

Recall that $\mathcal{O}(V)$ is the set of holomorphic functions in V and $\mathcal{O}^*(V) = \{g \in \mathcal{O}(V) | g(p) \neq 0 \ \forall p \in V\}$.

Let $\mathcal{F} = ((U_\alpha)_{\alpha\in A}, (w_\alpha)_{\alpha\in A,(g_{\alpha\beta})_{U_\alpha\cap U_\beta\neq\phi})}$ be a foliation in M. The singular set of $\mathcal{F}$, $S(\mathcal{F})$, is by definition $S(\mathcal{F}) = \bigcup\limits_{\alpha\in A} S_\alpha$, where $S_\alpha = \{p \in U_\alpha | w_\alpha(p) = 0\}$. It follows from (a) and (b) that $S(\mathcal{F})$ is a proper analytic subset of M. The integrability condition implies that for each $\alpha \in A$ we can define a foliation $\mathcal{F}_\alpha$ (in the usual sense) in $U_\alpha - S_\alpha$, whose leaves are solutions of $w_\alpha = 0$. Condition (b) implies that if $U_\alpha \cap U_\beta \neq \phi$, then $\mathcal{F}_\alpha$ coincides with $\mathcal{F}_\beta$ in $U_\alpha \cap U_\beta - S(\mathcal{F})$. Hence we have a codimension one foliation defined in $M - S(\mathcal{F})$. A leaf of $\mathcal{F}$ is by definition, a leaf of this foliation.

If $S(\mathcal{F})$ has codimension one components, then it is possible to find a new foliation $\mathcal{F}_1 = ((U_\alpha)_{\alpha\in A}, (\tilde{w}_\alpha)_{\alpha\in A}, (\tilde{g}_{\alpha\beta})_{U_\alpha\cap U_\beta\neq\phi})$ such that $S(\mathcal{F}_1)$ has no components of codimension one, $S(\mathcal{F}_1) \subset S(\mathcal{F})$, and the leaves of $\mathcal{F}$ and $\mathcal{F}_1|(M - S(\mathcal{F}))$ are the same (in fact $w_\alpha = f_\alpha \cdot \tilde{w}_\alpha$, $f_\alpha \in \mathcal{O}(U_\alpha)$). From now on all the foliatons that we will consider *will not have codimension 1 singular components.*

1.2 – The Kupka set:

In 1964 I.Kupka proved the following result (see [**K**]);

1.2.1 THEOREM. *Let w be an integrable holomorphic 1-form defined in a neighborhood of $p \in \mathbf{C}^n$, $n \geq 3$. Supose that $w_p = 0$ and $dw_p \neq 0$. Then there exists a holomorphic coordinate system $(x, y, z_3, \ldots, z_n)$ defined in a neighborhood U of p such that $x(p) = y(p) = 0$ and $w = A(x,y)dx + B(x,y)dy$ in this coordinate system, where $A(0,0) = B(0,0) = 0$ and $\frac{\partial B}{\partial x}(0,0) - \frac{\partial A}{\partial y}(0,0) \neq 0$.*

In fact Kupka proved this result in the real context, but his proof adapts very well in the holomorphic case.

1.2.2 Remarks: Let w, A, B and U be as in Theorem 1.2.1.

(i) The set $\{(x, y, z_3, \ldots, z_n) \in U | x = y = 0\} = V$ is containned in U. If the singular set S of w has no codimension 1 components, then V is a smooth codimension 2 piece of S and $(0,0)$ is an isolated solution of $A(x,y) = B(x,y) = 0$. By taking a smaller U if necessary we can suppose that $S \cap U = V$.

(ii) The foliation induced by $w = 0$ in U is equivalent to the product of the singular foliation in $U \cap \{z_3 = c_3, \ldots, z_n = c_n\} \subset \mathbf{C}^2 \times (c_3, \ldots, c_n)$ given by $Adx + Bdy = 0$ (or by the differential equation $\dot{x} = -B, \dot{y} = A$), by the codimension 2 foliation in U given by $x = c_1, y = c_2$. The singular set in this case is $V = \{x = y = 0\}$.

Let $\mathcal{F} = ((U_\alpha)_{\alpha \in A}, (w_\alpha)_{\alpha \in A}, (g_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \phi})$ be a foliation on M. We define the Kupka set of $\mathcal{F}$ by $K(\mathcal{F}) = \bigcup\limits_{\alpha \in A} K_\alpha$, where

$$K_\alpha = \{p \in U_\alpha | w_\alpha(p) = 0 \text{ and } dw_\alpha(p) \neq 0\}$$

Since $w_\alpha = g_{\alpha\beta} w_\beta$ in $U_\alpha \cap U_\beta \neq \phi$, we have $dw_\alpha = dg_{\alpha\beta} \wedge w_\beta + g_{\alpha\beta} dw_\beta$ which implies that $K_\alpha \cap U_\beta = K_\beta \cap U_\alpha$. It follows from (i) that $K(\mathcal{F})$ is a smooth complex codimension 2 submanifold of M. In fact $K(\mathcal{F}) = S(\mathcal{F}) - W(\mathcal{F})$ where $W(\mathcal{F}) = \bigcup_{\alpha \in A} W_\alpha$, $W_\alpha = \{p \in U_\alpha | w_\alpha(p) = 0 \text{ and } dw_\alpha(p) = 0\}$. Observe that $W(\mathcal{F})$ is an analytic subset of M.

1.2.3 Definition: We say that K is a *Kupka component* of $\mathcal{F}$ if K is an irreducible component of $S(\mathcal{F})$ and $K \subset K(\mathcal{F})$. Observe that a Kupka component of $\mathcal{F}$ is in particular a smooth connected codimension 2 analytic subset of M.

Let V be a connected codimension 2 submanifold of $K(\mathcal{F})$. It follows from the local product structure (see 1.2.1 and 1.2.2) that there exists a covering $(B_i)_{i \in I}$ of V by open sets of M, a collection of submersions $(\psi_i)_{i \in I}$, $\psi_i \colon B_i \to \mathbf{C}^2$, and a 1-form $w = A(x,y)dx + B(x,y)dy$ defined in a neighborhood C of $(0,0) \in \mathbf{C}^2$, such that:

(a) $\psi_i(B_i) \subset C$ for evere $i \in I$.
(b) $(0,0)$ is the unique singularity of w in C and $V \cap B_i = \psi_i^{-1}(0,0)$, for every $i \in I$.
(c) $\mathcal{F}|B_i$ is represented by $w_i^* = \psi_i^*(w)$.

We will say that $\mathcal{F}$ has *transversal type w or X* along V, where X is the vector field $-B\partial/\partial x + A\partial/\partial y$. The *linear transversal type of $\mathcal{F}$ along* V is, by definition, the linear part of X at $(0,0)$ in Jordan's canonical form, modulo multiplication by non-zero constants. Let L be the linear part of X at $(0,0)$ in Jordan's canonical form. We have the following possibilities:

(i) L is diagonal with eigenvalues $\lambda_1 \neq \lambda_2$.
(ii) L is diagonal with eigenvalues $\lambda_1 = \lambda_2 \neq 0$.
(iii) L is not diagonal with eigenvalues $\lambda_1 = \lambda_2 \neq 0$.

Observe that, since $\frac{\partial B}{\partial x}(0,0) - \frac{\partial A}{\partial y}(0,0) \neq 0$, we have $tr(L) \neq 0$ and so the possibilities $\lambda_1 = \lambda_2 = 0$ or $\lambda_1 = -\lambda_2$ cannot occur.

In case (i) the two eigendirections of L induce via the submersions ψ_i, two line subbundles of the normal bundle $\nu(V)$ of V in M. We will call these line bundles L_1 (relative to λ_1) and L_2 (relative to λ_2). It is clear that $\nu(V) = L_1 \oplus L_2$. In case (iii) L has just one eigendirection which induces in the same way a line subbundle L_1 of $\nu(V)$. In the case of Kupka components we have the following (see **[G.M- L.N]**):

1.2.4 - THEOREM. *Let* $\dim(M) \geq 3$ *and* K *be a Kupka compact component of* $\mathcal{F}$. *We have:*

(a) In case (i), if $C(L_i)$ *is the first Chern class of* L_i, $i = 1, 2$, *considered in* $H^2(K, \mathbf{C})$, *then* $\lambda_1 C(L_2) = \lambda_2 C(L_1)$.
(b) In case (iii) we have $C(L_1) = 0$.
(c) In case (i), if $\lambda_2/\lambda_1 = p/q$, *where* $p, q \in \mathbf{Z}_+$ *are relatively primes and* $C(L_1) \neq 0$, *then* X *is linearizable.*

1.3 - Codimension 1 foliations of $\mathbf{C}P^n$, $n \geq 3$:

A holomorphic foliation in $\mathbf{C}P^n$ can be given by an integrable 1-form $w = \sum_{i=0}^{n} w_i dz_i$ ($w \wedge dw = 0$), with the following properties:

(a) $w_0, \ldots, w_n$ are homogeneous polynomials of the same degree ≥ 1.
(b) $i_R(w) = \sum_{i=0}^{n} w_i z_i \equiv 0$ ($R = \sum_{i=0}^{n} z_i \partial/\partial z_i$ is the radial vector field).

This form can be obtained as follows: let $\pi\colon \mathbf{C}^{n+1} - \{0\} \to \mathbf{C}P^n$ be the canonical projection and $\mathcal{F} = ((U_\alpha)_{\alpha \in A}, (w_\alpha)_{\alpha \in A}, (g_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \phi})$ be a foliation in $\mathbf{C}P^n$. Let $\mathcal{F}^* = ((U^*_\alpha)_{\alpha \in A}, (w^*_\alpha)_{\alpha \in A}, (g^*_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \phi})$ be the foliation in $\mathbf{C}^{n+1} - \{0\}$ defined by $U^*_\alpha = \pi^{-1}(U_\alpha), w^*_\alpha = \pi^*(w_\alpha)$ and $g^*_{\alpha\beta} = g_{\alpha\beta} \circ \pi$. Since for $U^*_\alpha \cap U^*_\beta \cap U^*_\gamma \neq \phi$ we have $g^*_{\alpha\beta} \cdot g^*_{\beta\gamma} \cdot g^*_{\gamma_\alpha} = 1$, we can use Cartan's solution of the multiplicative Cousin's problem in $\mathbf{C}^{n+1} - \{0\}$ (see **[G-R]**) to obtain an integrable 1-form η in $\mathbf{C}^{n+1} - \{0\}$ such that for any $\alpha \in A$, we have $\eta | U^*_\alpha = h_\alpha \cdot w^*_\alpha$, where $h_\alpha \in \mathcal{O}^*(U^*_\alpha)$. From Hartog's Theorem (see **[G-R]**), η extends to a holomorphic 1-form μ in $\mathbf{C}^{n+1}$. If $\mu = \mu_k + \mu_{k+1} + \ldots$ is the

Taylor development of μ at 0, where the coefficients of μ_j are homogeneous of degree j and $\mu_k \not\equiv 0$, then it is easy to see that $w = \mu_k$ is integrable. We leave it to the reader the proof of the following facts:

(c) $S(w) = S(\mathcal{F}^*) = \pi^{-1}(S(\mathcal{F})) \cup \{0\}$.

(d) L^* is a leaf of $\mathcal{F}^*$ iff $L^* = \pi^{-1}(L)$, where L is a leaf of $\mathcal{F}$.

(e) If $k =$ degree $(w) \geq 2$, then $K(\mathcal{F}^*) = \pi^{-1}(K(\mathcal{F})) = \{p \in \mathbf{C}^{n+1} | w(p) = 0$ and $dw(p) \neq 0\}$.

When degree $(w) = 1$ we can have $w = z_1 dz_2 - z_2 dz_1$ and in this case $K(\mathcal{F}^*) = \{z_1 = z_2 = 0\} = \pi^{-1}(K(\mathcal{F})) \cup \{0\}$.

Observe that condition (b) is equivalent to conditions (c) and (d) and means that the lines through the origin are tangent to the leaves of $\mathcal{F}^*$.

Observe also that given an integrable 1-form w in $\mathbf{C}^{n+1}$ satisfying (a) and (b) we can induce a foliation $\mathcal{F}(w)$ in $\mathbf{C}P^n$ as follows: let $(U_i)_{i=0}^n$ be the covering of $\mathbf{C}P^n$ by affine coordinate systems, where $U_i = \{[z_0 \colon \ldots \colon z_n] \in \mathbf{C}P^n | z_i \neq 0\}$. Let $\psi_i \colon U_i \to \mathbf{C}^n$, $\psi_i[z_0 \colon \ldots \colon z_n] = (z_0/z_i, \ldots, z_{i-1}/z_i, z_{i+1}/z_i, \ldots, z_n/z_2) = (x_0^i, \ldots, x_{i-1}^i, x_{i+1}^i, \ldots, x_n^i)$. Define $\eta_i = \psi_i^*(\eta_i^*)$, where $\eta_i^* = w|(z_i = 1) = \sum\limits_{j \neq i} w_j(z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n) dz_j$. It is not difficult to see that if $dg(w) = k$ then $\eta_i | V_i \cap U_j = (x_j^i)^{k+1} \eta_j | U_i \cap U_j$. Hence $\mathcal{F}(w) = ((U_i)_{i=0}^n, (\eta_i)_{i=0}^n ((x_i^j)^{k+1})_{i \neq j})$ is a foliation on $\mathbf{C}P^n$.

Remark: Two integrable 1-forms w and η in $\mathbf{C}^{n+1}$, with properties (a) and (b) define the same foliation iff $w = \lambda \cdot \eta$ where $\lambda \in \mathbf{C}^*$.

It follows from the above considerations that the space of foliations in $\mathbf{C}P^n$ can be written as $\cup_{m \geq 1} P_m$, where P_m is the projectivization of the following space of polynomial 1-forms: $I_m = \{w | w = \sum_{i=0}^n w_i dz_i, dg(w_i) = m \quad \forall i = 0, \ldots, n,\ \ w \wedge dw = 0, \sum\limits_{i=0}^n z_j w_j \equiv 0$ and the set $\{w_0 = \cdots = w_n = 0\}$ has all irreducible components of codimension $\geq 2\}$.

Observe that I_m is an open subset of the following algebraic set

$$E_m = \{w | dg(w) = m, w \wedge dw = 0, \sum_{i=0}^{n} z_i w_i \equiv 0\}$$

Problem - Describe in some way the irreducible components of P_m. Let us see some examples.

1.3.1 - Example: Let f, g be homogeneous polynomials in $\mathbf{C}^{n+1}$, $n \geq 3$, where $dg(f) = k \geq 1$, $dg(g) = \ell \geq 1$ and $k/\ell = p/q$ where p and q are relatively primes. Assume that:

(*) $\qquad \forall\ z \in \{f = g = 0\} - \{0\}$ we have $df(z) \wedge dg(z) \neq 0$.

We will use the notation $f \pitchfork g$ ($f = 0$ intersects $g = 0$ transversely) in this case. We observe that Noether's lemma implies that if f and g satisfy (*) then $\{f = g = 0\}$ is a complete intersection.

Let $w = qgdf - pfdg$. It follows from Euler's identity that $i_R(w) = 0$. Moreover $w \wedge dw = 0$ because $w = f.g.\eta$, where $\eta = q\frac{df}{f} - p\frac{dg}{g}$ and $d\eta = 0$. Therefore w induces a foliation in $\mathbf{C}P^n$, $\mathcal{F}(w)$, such that:

(i) $S(\mathcal{F}(w)) = \pi\{p \neq 0 \mid w(p) = 0\} = S$ (singular set)
(ii) $K(\mathcal{F}(w)) = \pi\{p \neq 0 \mid f(p) = g(p) = 0\} = K$ (Kupka set)
(iii) f^q/g^p, considered as meromorphic function on $\mathbf{C}P^n$, is a first integral of $\mathcal{F}(w)$. This follows from the fact that $w = g^{p+1}f^{1-q}d(f^q/g^p)$.
(iv) $w \in P_n$ where $n = k + \ell - 1$.

As a consequence of the techniques developed in **[G.M-L.N.]** it is possible to prove the following result:

1.3.2 THEOREM. *Let $\mathcal{F}_0 = \mathcal{F}(w)$, where w is as in example 1.3.1. Then there exists a neighborhood $\mathcal{U}$ of $\mathcal{F}_0$ in P_n such that if $\mathcal{F} \in \mathcal{U}$ then there are polynomials $\tilde{f}$ and $\tilde{g}$ of degrees k and ℓ respectively, and $\mathcal{F} = \mathcal{F}(q\tilde{g}d\tilde{f} - p\tilde{f}d\tilde{g})$.*

As a consequence we have:

1.3.3 COROLLARY. *There are irreducible components in P_m (for all $m \geq 1$) whose foliations are defined by meromorphic functions in $\mathbf{C}P^n$, $n \geq 3$.*

Concerning Kupka components we will prove in §3 the following result:

1.3.4 THEOREM A. *Let $\mathcal{F}$ be a foliation of $\mathbf{C}P^n$, $n \geq 3$, which has a Kupka component K of the form $\{[z] \mid f(z) = g(z) = 0\}$ where f and g are homogeneous polynomials and $f \pitchfork g$. Let $dg(f)/dg(g) = p/q$ where p and q are relatively primes. Then:*

(a) If $dg(f) = dg(g)$, then $\mathcal{F}$ is the foliation induced by the form $fdg - gdf$. In particular f/g is a first integral of $\mathcal{F}$.
(b) If $dg(f) < dg(g)$ then $\mathcal{F}$ is induced by a form of the type $qg_1df - pfdg_1$, where $g_1 = g + h.f$ is homogeneous and $dg(g_1) = dg(g)$. In particular f^q/g_1^p is a first integral of $\mathcal{F}$.

1.3.5 Example: A logarithmic form is one of the type

$$(*) \qquad w = f_1 \dots f_r \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j}$$

where $\lambda_1, \dots, \lambda_r \in \mathbf{C}^*$ and $f_1, \dots, f_r$ are holomorphic functions. When $f_1, \dots, f_r$ are homogeneous polynomials in $\mathbf{C}^{n+1}$, $dg(f_i) \geq 1$ for $i = 1, \dots, r$, and $\sum_{i=1}^{r} \lambda_i dg(f_i) = 0$, then w induces a foliation $\mathcal{F}(w)$ in $\mathbf{C}P^n$, where $\mathcal{F}(w) \in P_m$, $m = \sum_{i=1}^{r} dg(f_i) - 1$. Observe that $\sum_{i=1}^{r} \lambda_i dg(f_i) = 0$ is equivalent to the condition $\sum_{i=0}^{n} z_i w_i \equiv 0$. We will use the notations $F_i = \{[z] \in \mathbf{C}P^n \mid f_i(z) = 0\}$ and $F_{ij} = \{[z] \in \mathbf{C}P^n \mid f_i(z) = f_j(z) = 0\}$ if $i \neq j$. We will assume that $f_1, \dots, f_r$ are irreducibles. The foliation $\mathcal{F}(w)$, induced by w in $\mathbf{C}P^n$ has the following properties:

(i) For every $i = 1, \dots, r$, $F_i^* = F_i - S(\mathcal{F}(w))$ is a leaf of $\mathcal{F}(w)$.
(ii) The holonomy of F_i^* is linearizable and is conjugated to a subgroup of the group of linear transformations of $\mathbf{C}$ generated by the set $\{g_j \mid g_j(z) =$

$\exp(2\pi i\lambda_j/\lambda_i).z, \quad j = 1,\dots,r, \quad j \neq i\}$. The holonomy of a leaf $L \neq F_1^*,\dots,F_r^*$ is trivial.

(iii) For any $i \neq j$, $F_{ij} \subset S(\mathcal{F}(w))$. Moreover, if $\lambda_i \neq \lambda_j$ then $F_{ij} - V$ is contained in the Kupka set of $\mathcal{F}(w)$, where $V = \{[z] \mid df_i(z) \wedge df_j(z) = 0\} \cup V', V' = \bigcup\limits_{k \neq i,j} \{[z] \mid f_k(z) = 0\}$. In particular $K(\mathcal{F}(w)) \subset \bigcup\limits_{i \neq j} F_{ij}$.

(iv) The function $f_1^{\lambda_1} \dots f_r^{\lambda_r}$ (in general multivalued) is a first integral of $\mathcal{F}(w)$. The following result is known:

1.3.6 THEOREM. *(J. Omegar) – Let $\mathcal{F}_0$ be the foliation induced by w in $\mathbf{C}P^n$, $n \geq 3$, where w is like in (*) of 1.3.5. Assume that $f_1,\dots,f_r$ are irreducibles and for some $i \in \{1,\dots,r\}$, say $i = 1$, we have:*

(a) F_1 is smooth.

(b) For any subset $\{j_1,\dots,j_s\} \subset \{2,\dots,r\}$ where $j_1 < \cdots < j_s$ and any $p \in F_1 \cap F_{j_1} \cap \cdots \cap F_{j_s}$ then $F_1, F_{j_1},\dots,F_{j_s}$ intersect multitransversely at p.

(c) For some $j > 1$ we have $\lambda_j/\lambda_1 \notin \mathbf{R}$.

Then there exists a neighborhood $\mathcal{U}$ of $\mathcal{F}_0$ in P_m such that if $\mathcal{F} \in \mathcal{U}$ then $\mathcal{F}$ is induced by a logarithmic form of the same type of w, say $\eta = g_1,\dots,g_r \sum\limits_{j=1}^{r} \eta_j \frac{dg_j}{g_j}$, where $dg(g_j) = dg(f_j)$, $j = 1,\dots,r$.

It follows that:

1.3.7 COROLLARY. *There are irreducible components in P_m (for all $m \geq 1$) whose foliations in an open and dense subset are defined by logarithmic forms.*

1.3.8 Definition: We say that a meromorphic 1-form w, defined in some complex manifold M, has an *integrating factor*, if there exists a meromorphic function f in M, called an integrating factor, such that, $d(\frac{w}{f}) = 0$. Remark that for w as in (*) of 1.3.5, the function $f = f_1 \dots f_r$ is an integrating factor.

In §3 we will prove the following result:

1.3.9 THEOREM B. *Let $\mathcal{F}$ be a foliation in $\mathbf{C}P^n$, $n \geq 3$, such that there is an analytic subset $N \subset S(\mathcal{F})$ with the following properties:*

(a) $\mathrm{cod}(N) = 2$ and $N = \{[z] \in \mathbf{C}P^n \mid f(z) = g(z) = 0\}$, where f and g are homogeneous polynomials on $\mathbf{C}^{n+1}$.

(b) $K(\mathcal{F}) \cap N$ is open and dense in N and moreover for any connected component C of $K(\mathcal{F}) \cap N$ the linear part of the transversal type of $\mathcal{F}$ at C has eigenvalues $\lambda_1(C) \neq 0 \neq \lambda_2(C)$, where $\lambda_2(C)/\lambda_1(C) \notin \mathbf{R}$.

(c) For any $p \in N - K(\mathcal{F})$, $\mathcal{F}$ can be represented in a neighborhood of p by a holomorphic form which has an integrating factor.

Then there exists a closed meromorphic 1-form η in $\mathbf{C}P^n$ which represents $\mathcal{F}$ outside its divisor of poles. In particular $\mathcal{F}$ is induced by a homogeneous 1-form in $\mathbf{C}^{n+1}$ which has a meromorphic integrating factor.

Furthermore $\mathcal{F}$ is of logarithmic type if we assume that:

(d) $K(\mathcal{F})$ is dense in each irreducible component of codimension 2 of $S(\mathcal{F})$.

(e) For any connected component C of $K(\mathcal{F})$ the transversal part of $\mathcal{F}$ at C has linear part non degenerated (i.e. 0 is not an eigenvalue).

Remarks:

1.3.10 – It will follow from the proof that condition (b) can be replaced by;

(b') $\lambda_2(C)/\lambda_1(C) \notin \mathbf{Q}$ and the transversal type is linearizable.

1.3.11 – In **[C-M]** the authors give some sufficient conditions for a holomorphic integrable 1-form have a local integrating factor. One of their results implies that (c) follows from (b) and

(c') For some neighborhood U of $p \in N - K(\mathcal{F})$, $\mathcal{F}|U$ has a finite number of analytic leaves which intersect multitransversely in the points of N.

1.4 An example: Let K be the twisted cubic in $\mathbf{C}P^3$, which is defined in homogeneous coordinates $(x, y, z, w) \in \mathbf{C}^4$ by the equations $f = g = h = 0$, where

$$(*) \qquad f = XW - YZ, \quad g = XZ - Y^2 \text{ and } h = YW - Z^2.$$

We will prove here that there exists no foliation in $\mathbf{C}P^3$ having K as a Kupka component.

Let U_0 and U_4 be the affine coordinate systems in $\mathbf{C}P^3$, whose points are of the form $[1 : u : v : w]$ and $[x : y : z : 1]$, respectively. Then $K \subset U_0 \cup U_4$. Moreover $K \cap U_0$ and $K \cap U_4$ can be parametrized by $\varphi_0(t) = [1 : t : t^2 : t^3]$ and $\varphi_4(s) = [s^3; s^2 : s : 1]$ respectively, where $\varphi_0(t) = \varphi_4(s)$ iff $s = 1/t$. Let $f_0(u, v, w) = f(1, u, v, w) = w - uv$, $g_0(u, v, w) = g(1, u, v, w) = v - u^2$, $f_4(x, y, z) = f(x, y, z, 1) = x - yz$ and $h_4(x, y, z) = h(x, y, z, 1) = y - z^2$. Remark that $K \cap U_0 = \{f_0 = g_0 = 0\}$ and $K \cap U_4 = \{f_4 = h_4 = 0\}$.

Suppose by contradiction that there exists a foliation $\mathcal{F}$ on $\mathbf{C}P^3$ whose Kupka set contains K. Let w be a homogeneous integrable 1-form in $\mathbf{C}^4$ such that $i_R(w) = 0$ and w represents $\mathcal{F}$. Let $w = \sum_{i=0}^{3} \alpha_i dz_i$, where $dg(\alpha_i) = k$, $i = 0, \dots 4$. If $w_0 = w \mid \{z_0 = 1\} = \alpha_1(1, u, v, w)du + \alpha_2(1, u, v, w)dv + \alpha_3(1, u, v, w)dw$ and $w_4 = w \mid \{z_4 = 1\} = \alpha_0(x, y, z, 1)dx + \alpha_1(x, y, z, 1)dy + \alpha_2(x, y, z, 1)dz$, then $\mathcal{F}|U_0$ is represented by w_0 and $\mathcal{F}|U_4$ by w_4. Moreover in $U_0 \cap U_4$ we have $w_0 = x^{-(k+1)} w_4$.

Now, consider the maps $\psi_0, \psi_4 \colon \mathbf{C}^3 \to \mathbf{C}^3$ given by $\psi_0(u, v, w) = (u, g_0(u, v, w), f_0(u, v, w))$ and $\psi_4(x, y, z) = (f_4(x, y, z), h_4(x, y, z), z)$. It is not difficult to see that ψ_0 and ψ_4 are diffeomorphisms, so that we can consider (u, g_0, f_0) and (f_4, h_4, z) as coordinates in U_0 and U_4 respectively. Moreover $\psi_0(K \cap U_0) = \{f_0 = g_0 = 0\}$ and $\psi_4(K \cap U_4) = \{f_4 = h_4 = 0\}$. Observe also that the inverse maps of ψ_0 and ψ_4 are polynomials, so that we can write

$$w_0 = A(u, g_0, f_0)du + B(u, g_0, f_0)dg_0 + C(u, g_0, f_0)df_0$$

$$w_4 = D(f_4, h_4, z)df_4 + E(f_4, h_4, z)dh_4 + F(f_4, h_4, z)dz$$

where A, B, C, D, E and F are polynomials. Let us analyze w_0. Consider the vector field

$$X = \left(\frac{\partial C}{\partial g_0} - \frac{\partial B}{\partial f_0}\right)\frac{\partial}{\partial u} + \left(\frac{\partial A}{\partial f_0} - \frac{\partial C}{\partial u}\right)\frac{\partial}{\partial g_0} + \left(\frac{\partial B}{\partial u} - \frac{\partial A}{\partial g_0}\right)\frac{\partial}{\partial f_0}$$

As the reader can see, the integrability condition is equivalent to $i_X(w_0) = 0$. Moreover, in the proof of Kupka's Theorem (1.2.1) it is proved that the flow of X leaves invariant the Kupka set. Since $\{f_0 = g_0 = 0\} \subset K(\mathcal{F}) \cap U_0$, we must have $X(u,0,0) = \left(\frac{\partial C}{\partial g_0} - \frac{\partial B}{\partial f_0}\right)(u,0,0)\frac{\partial}{\partial u}$, so that,

$$\text{(1)} \qquad \frac{\partial A}{\partial f_0}(u,0,0) = \frac{\partial C}{\partial u}(u,0,0) \text{ and } \frac{\partial B}{\partial u}(u,0,0) = \frac{\partial A}{\partial g_0}(u,0,0).$$

On the other hand, since $K \subset S(\mathcal{F})$, we can write

$$w_0 = (a_1(u)g_0+a_2(u)f_0)du+(b_1(u)g_0+b_2(u)f_0)dg_0+(c_1(u)g_0+c_2(u)f_0)df_0+\dots$$

where $a_1, \dots, c_2$ are polynomials in u and the dots mean terms of order ≥ 2 in (g_0, f_0). This implies that:

$$X = (c_1(u) - b_2(u))\frac{\partial}{\partial u} + a_2(u)\frac{\partial}{\partial g_0} - a_1(u)\frac{\partial}{\partial f_0} + \dots$$

where the dots mean terms of order ≥ 1 in (g_0, f_0). From (1) we get that $a_1 \equiv a_2 \equiv 0$ and $X(u,0,0) = (c_1(u) - b_2(u))\frac{\partial}{\partial u}$. On the other hand, since $K \subset K(\mathcal{F})$, we must have $dw_0(u,0,0) \neq 0$ $\forall u \in \mathbf{C}$, and this implies that $c_1(u) - b_2(u) \neq 0$ $\forall u \in \mathbf{C}$. Hence $c_1 - b_2 \equiv c$, $c \neq 0$ a constant, because $c_1 - b_2$ is a polynomial. From these considerations it is easy to see that:

$$\text{(2)} \qquad dw_0|K \cap U_0 = c\,dg_0 \wedge df_0.$$

With an analogous argument it is possible to conclude that

$$\text{(3)} \qquad dw_4|K \cap U_4 = \hat{c}\,df_4 \wedge dh_4, \text{ where } \hat{c} \neq 0 \text{ is a constant.}$$

Now, recall that $w_0 = x^{-(k+1)}w_4$ in $U_0 \cap U_4$. Since $x = z^3$ along $K \cap U_4$ and $w_4(0,0,z) = 0$, we have:

$$dw_0|K \cap U_0 \cap U_4 = x^{-(k+1)}dw_4(0,0,z) = z^{-3(k+1)}dw_4(0,0,z)$$

which together with (2) and (3) implies that:

$$\text{(4)} \qquad \tilde{c}\,dg_0 \wedge df_0 = z^{-3(k+1)}df_4 \wedge dh_4 \text{ along } K \cap U_0 \cap U_4, \quad \tilde{c} = c/\hat{c}.$$

As the reader can verify easily, we have the following relations

$$\begin{cases} f_0|U_0 \cap U_4 = x^{-2} f_4|U_0 \cap U_4 \\ g_0|U_0 \cap U_4 = x^{-2}[z f_4 - y h_4]|U_0 \cap U_4 \end{cases}$$

This implies that

(5)

$$dg_0 \wedge df_0|K \cap U_0 \cap U_4 = yx^{-4} dh_4 \wedge df_4|K \cap U_0 \cap U_4 = z^{-10} dh_4 \wedge df_4|K \cap U_0 \cap U_4$$

because $yx^{-4} = z^{-10}$ along $K \cap U_4$.

Finally, from (4) and (5) we get that $10 = 3(k+1)$, where $k \in \mathbf{N}$, which is a contradiction.

This example motivates the following:

Problem: Are there foliations on $\mathbf{C}P^n$, $n \geq 3$, which admit a Kupka component which is not a complete intersection?

We think that the answer is no.

2. Basic results

In this section we will state and prove some of the results that we will need in §3.

2.1 Definitions: Let $\varphi\colon \to \mathbf{R}$ be a C^2 function, where $U \subset \mathbf{C}^n$ is an open set. We say that φ is strictly k-subharmonic (briefly $s.k - s.$) or $(n-k+1)$-pseudoconvex, if for any $z \in U$ the $\partial\overline{\partial}$-matrix of φ at z, which is defined by

$$H_\varphi(z) = \left(\frac{\partial^2 \varphi}{\partial \overline{z}_i \partial z_j}(z)\right) 1 \leq i, j \leq n$$

has at least k positive eigenvalues. Observe that $H_\varphi(z)$ is a hermitian matrix, so that all its eigenvalues are real. Moreover, if $f\colon V \to U$ is a biholomorphism then

$$H_{\varphi\circ f}(w) = \overline{P}^t . H_\varphi(f(w)).P$$

where P is the jacobian matrix of f at w. This implies that the concept of $s.k-s.$ can be defined in complex manifolds: if M is a complex manifold of dimension n and $\varphi: M \to \mathbf{R}$ is C^2, we say that φ is $s.k-s.$ if for any $p \in M$ there is a holomorphic local chart $\alpha: U \to V \subset \mathbf{C}^n$, $p \in U$, such that $\varphi \circ \alpha^{-1}$ is $s.k-s.$. It is clear that if $\beta: U_1 \to V_1$, $p \in U_1$, is another holomorphic chart, then $\varphi \circ \beta^{-1}$ is $s.k-s.$.

We say that a connected complex manifold M is *k-complete*, $k \geq 1$, if there exists a $s.k-s.$ function $\varphi: M \to \mathbf{R}$ such that:

$$(*) \qquad \lim_{p \to \infty} \varphi(p) = +\infty,$$

that is, for any sequence $(p_n)_{n \geq 1}$ in M, without accumulation points, we have $\lim_{n \to \infty} \varphi(p_n) = +\infty$. We observe that a $s.k-s.$ function, $k \geq 1$, cannot have a local maximum. This fact follows from the maximum principle for subharmonic functions, as the reader can verify easily. Hence there are no $s.k-s.$ functions on compact manifolds. Remark also that property (*) implies that:

(i) For any $r \in \mathbf{R}$ the sets $\varphi^{-1}(-\infty, r]$ and $\varphi^{-1}(r)$ are compact.
(ii) $\inf \{\varphi(p) \mid p \in M\} = m > -\infty$, and there exists $p_0 \in M$ such that $\varphi(p_0) = m$.

When $k = n = \dim(M)$ a $s.k-s.$ function is also called a strictly subharmonic function.

2.2 Extension of Meromorphic forms:

The main result of this section is the following:

2.2.1 THEOREM. *Let M be a k-complete complex manifold, where $k \geq 2$. Let C be a compact subset of M and w be a meromorphic (resp. holomorphic) ℓ-form defined on $M - C$. Then w extends to a meromorphic (resp. holomorphic) ℓ-form on M.*

Proof: Let $\varphi: M \to \mathbf{R}$ be a $s.k-s.$ function such that $\lim_{p \to \infty} \varphi(p) = +\infty$. Since a $s.k-s$ function is $s.2-s.$ if $k \geq 2$, we can assume that $k = 2$. Let $m =$

$\inf\{\varphi(p) \mid p \in M\}$ and $r = \sup\{\varphi(p) \mid p \in C\}$, so that $M - C \subset \varphi^{-1}(r, +\infty)$ and w is defined on $\varphi^{-1}(r, +\infty)$. The idea is to prove the following:

Assertion 1: If w can be extended to $\varphi^{-1}(s, +\infty)$, where $s \leq r$, then there exists $\varepsilon > 0$ such that w can be extended to $\varphi^{-1}(s - \varepsilon, +\infty)$.

Since $\varphi^{-1}[m, +\infty) = M$, then assertion 1 clearly implies the theorem. On the other hand, assertion 1 is implied by the following:

Assertion 2: Suppose that w has been already extended to $\varphi^{-1}(s, +\infty)$, where $s \leq r$. Given $p \in \varphi^{-1}(s)$, there exist a neighborhood V of p such that w can be extended to $V \cup \varphi^{-1}(s, +\infty)$.

Assertion 1 follows from assertion 2 because $\varphi^{-1}(s)$ is compact. In order to prove assertion 2 we use Levi's Theorem:

Levi's Theorem: (see [S] for the proof). Let $W \subset V \subset \mathbf{C}^{n-1}$ be open sets, where $W \neq \phi$ and V is connected. Let f be a meromorphic (resp. holomorphic) function defined in $(W \times \Delta(r)) \cup (V \times [\Delta(r) - \overline{\Delta(r')}])$, where $\Delta(r) = \{z \in \mathbf{C} \mid |z| < r\}$ and $0 < r' < r$. Then f can be extended to a meromorphic (resp. holomorphic) function on $V \times \Delta(r)$.

An open set A of the form $(W \times \Delta(r)) \cup (V \times [\Delta(r) - \overline{\Delta}(r')])$ is called a Hartog's domain. The set $\hat{A} = V \times \Delta(r)$ is called its envelope of holomorphy. Another fact we will use is the following:

LEMMA 1. *Let $\varphi: U \to \mathbf{R}$ be a s.2 − s. function, where $U \subset \mathbf{C}^n$ is an open set $(n \geq 2)$. Let $p \in U$ be such that $\varphi(p) = s$. Then there exist a biholomorphism $\alpha: V_1 \to U_1$ and a Hartog's domain $A \subset V_1$ such that*

(a) $0 \in V_1$, $\alpha(0) = p \in U_1 \subset U$

(b) $\alpha(A) \subset U_1$ and $p \in \alpha(\hat{A})$, where $\hat{A}$ is the envelope of holomorphy of A.

For the proof see the §8 of [**S-T**].

Assertion 2 follows from Levi's Theorem and Lemma 1. In fact, given $p \in \varphi^{-1}(s)$, by taking a local chart we can assume that $p = 0 \in \mathbf{C}^n$ and $\varphi: U \to \mathbf{R}$, $0 \in U \subset \mathbf{C}^n$. Since $\varphi(0) = s$ and w is defined in $\varphi^{-1}(s, +\infty) \subset U$, we can write $w = \sum_I f_I dz_I$, where $I = (i_1, \ldots, i_\ell)$, $i_1 < \cdots < i_\ell$, $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_\ell}$

and f_I is a meromorphic (resp. holomorphic) function on $\varphi^{-1}(s,+\infty)$. Let α and A be as in lemma 1 and $g_i = f_I \circ \alpha^{-1}$, $I = (i_1 < \cdots < i_\ell)$. From Levi's Theorem g_I can be extended to a meromorphic (resp. holomorphic) function on $\hat{A}$. Hence f_I can be extended to a meromorphic (resp. holomorphic) function on $\alpha(\hat{A})$. Since $\alpha(\hat{A})$ is a neighborhood of $p = 0$, then assertion 2 is proved. ▮

Now we consider the following situation: let $f_1, \ldots, f_k$ be homogeneous (non constant) polynomials in $\mathbf{C}^{n+1}$ and $V(f_1, \ldots, f_k) = \{[p] \in \mathbf{C}P^n \mid f_1(p) = \cdots = f_k(p) = 0\}$.

2.2.2 THEOREM. *$M = \mathbf{C}P^n - V(f_1, \ldots, f_k)$ is ℓ-complete, where $\ell = n-k+1$.*

Proof: Let $dg(f_j) = d_j$, $j = 1, \ldots, k$ and $q_1, \ldots, q_k \in [N]$ be such that $d_1 q_1 = \cdots = d_k q_k = q > 0$. Put $G_j = f_j^{q_j}$, so that $dg(G_j) = q$, $j = 1, \ldots, k$, and $V(f_1, \ldots, f_k) = V(G_1, \ldots, G_k)$. Define $\varphi: M \to \mathbf{R}$ by

$$\varphi([z]) = \ell g \left(\frac{\left(\sum_{j=0}^{n} |z_j|^2\right)^q}{\sum_{j=1}^{k} |G_j(z)|^2} \right)$$

where $[\ ] = \pi: \mathbf{C}^{n+1} - \{0\} \to \mathbf{C}P^n$ is the canonical projection and $z = (z_0, \ldots, z_n)$. It is easy to see that φ is well defined and real analytic on M. Moreover, since $M = \mathbf{C}P^n - V(G_1, \ldots, G_k)$, we have $\lim\limits_{p\to\infty} \varphi(p) = \lim\limits_{p\to V} \varphi(p) = +\infty$, where $V = V(G_1, \ldots, G_k)$. Let us prove that φ is $s.\ell - s.$. Fix $[z^0] = [z_0^0 : \ldots : z_n^0] \in M$. We can suppose that $z_0^0 \neq 0$, so that $[z^0] = [1 : x_1^0, \ldots, x_n^0]$, where $x_j^0 = z_j^0 / z_0^0$. In the affine coordinate system $(x_1, \ldots, x_n) = [1 : x_1 : \ldots : x_n] \in \mathbf{C}^n$, φ can be written as $\varphi = q\varphi_1 - \varphi_2$, where $\varphi_1(x) = \ell g(1 + \sum\limits_{j=1}^{n} |x_j|^2)$ and $\varphi_2(x) = \sum\limits_{j=1}^{k} |g_j(x)|^2$, where $g_j(x) = G_j(1, x_1, \ldots, x_n)$. Therefore we have $H_\varphi = qH_{\varphi_1} - H_{\varphi_2}$. A direct computation shows that $H_{\varphi_1} = (a_{ij})_{1\le i,j\le n}$ and $H_{\varphi_2} = (b_{ij})_{1\le i,j\le n}$, where

$$a_{ii}(x) = \left(1 + \sum_{j\neq i} |x_j|^2\right) \Big/ \left(1 + \sum_{j=1}^{n} |x_j|^2\right)^2$$

$$a_{ij}(x) = -x_i\overline{x}_j \Big/ \left(1 + \sum_{j=1}^{n} |x_j|^2\right)^2$$

$$b_{ij} = \sum_{r<s} \overline{\Delta}^i_{r_s} \Delta^j_{r_s} \Big/ \left(\sum_{j=1}^{k} |g_j|^2\right)^2$$

where $\Delta^i_{r_s} = g_r \frac{\partial g_s}{\partial x_i} - g_s \frac{\partial g_r}{\partial x_i}$. Observe that the quadratic form associated to H_{φ_1} is

$$Q_1(x,w) = \sum_{i,j} \overline{w}_i a_{ij} w_j = \left(1 + \sum_{j=1}^{n} |x_i|^2\right)^{-4} \left[\sum_{i=1}^{n} |w_i|^2 + \sum_{i<j} |w_i x_j - w_j x_i|^2\right].$$

Hence it is positive definite. For some fixed $x \in \mathbf{C}^n - V$, $V = \{g_1 = \cdots = g_k = 0\}$, let $K(x) = \{w \in \mathbf{C}^n \mid \sum_{j=1}^{n} b_{ij}(x).w_j = 0$, for all $i = 1,\ldots,n\}$.

Assertion: $\dim(K(x)) \geq n - k + 1$ for all $x \in \mathbf{C}^n - V$.

Proof: Since $x \in \mathbf{C}^n - V$, let us assume for instance that $g_1(x) \neq 0$. From now on we will omit the point x in the notation. Let S be the space of solutions of the linear system:

$$\sum_{j=1}^{n} \Delta^j_{1s} w_j = 0, \quad s = 2,\ldots,k \tag{6}$$

Since in (6) we have $k-1$ equations, we have $\dim(S) \geq n-k+1$. So it is enough to prove that $S \subset K(x)$. Observe that (6) is equivalent to

$$\sum_{j=1}^{n} \frac{\partial g_s}{\partial x_j} w_j = \frac{g_s}{g_1} \sum_{j=1}^{n} \frac{\partial g_1}{\partial x_j} w_j, \quad s = 2,\ldots,k. \tag{7}$$

On the other hand (7) implies that if $r \neq s$ then

$$\sum_{j=1}^{n} \Delta_{rs}^{j} w_j = g_r \sum_{j=1}^{n} \frac{\partial g_s}{\partial x_j} w_j - g_s \sum_{j=1}^{n} \frac{\partial g_r}{\partial x_j} w_j = 0$$

Therefore, if $w \in S$, then $\sum_{j=1}^{n} \Delta_{rs}^{j} w_j = 0$ for all $r \neq s$. Hence, if $w \in S$, then

$$\begin{aligned}
\sum_{j=1}^{n} b_{ij} w_j &= \left(\sum_{j=1}^{n} |g_j|^2 \right)^{-2} \sum_{j=1}^{n} \sum_{r<s} \overline{\Delta}_{rs}^{i} \Delta_{rs}^{j} w_j \\
&= \left(\sum_{j=1}^{n} |g_j|^2 \right)^{-2} \sum_{r<s} \overline{\Delta}_{rs}^{i} \left(\sum_{j=1}^{n} \Delta_{rs}^{j} w_j \right) = 0
\end{aligned}$$

This proves the assertion.

Now if $w \in K(x) - \{0\}$, we have

$$\overline{w}^t H_\varphi w = qQ_1(x,w) - \sum_{i,j=1}^{n} \overline{w}_i b_{ij} w_j = qQ_1(x,w) > 0$$

This implies that H_φ has at least $n - k + 1$ positive eigenvalues. ∎

2.2.3 COROLLARY. *Let $k \leq n-1$ and $f_1, \ldots, f_k$ be homogeneous non constant polynomials on $\mathbf{C}^{n+1}$. Then any meromorphic ℓ-form defined in a neighborhood of $V(f_1, \ldots, f_k) \subset \mathbf{C}P^n$, can be extended to a meromorphic ℓ-form in $\mathbf{C}P^n$.*

2.3 Noether's lemma of second order:

Let $f, g: U \to \mathbf{C}$ be two analytic functions and $V = \{f = g = 0\}$, where $U \subset \mathbf{C}^n$ is an open set. If $W \subset V$, we say that f intersects g transversely outside W (briefly $f \pitchfork g$ out of W) if for any $z \in V - W$ we have $df(z) \wedge dg(z) \neq 0$. Classical Noether's lemma can be stated as follows:

2.3.1 NOETHER'S LEMMA. *Let f, g, U, V and W be as above. Suppose that:*

(a) W is an analytic subset of U, where $cod(W) \geq 3$.

(b) $\hat{H}^1(U - W, \mathcal{O}) = \{1\}$.

If $h: U \to \mathbf{C}$ is an analytic function such that $h|V \equiv 0$, then there are analytic functions $\alpha, \beta: U \to \mathbf{C}$ such that $h = \alpha.f + \beta.g$.

When $U = \mathbf{C}^n$, $n \geq 3$, f and g are homogeneous polynomials, $W = \{0\}$, and $f \pitchfork g$ out of $\{0\}$, then we have the following: if h is a homogeneous polynomial with $h|V \equiv 0$, then there are homogeneous polynomials α and β such that $h = \alpha.f + \beta.g$, where $dg(\alpha) + dg(f) = dg(\beta) + dg(g) = dg(h)$. In particular if $dg(h) < \min\{dg(f), dg(g)\}$, we must have $h \equiv 0$. This assertion follows from Noether's lemma because $\hat{H}^1(\mathbf{C}^n - \{0\}, \mathcal{O}) = \{1\}$ for $n \geq 3$ (see [C]). Here we are mainly interested in the case where the 1-jet of h is 0 along V.

2.3.2 Definition: Let $U \subset \mathbf{C}^n$ be an open set, $V \subset U$ be a codimension k smooth complex submanifold and $h: U \to \mathbf{C}$ be analytic. We say that the *ℓ-jet of h is zero along V if for any $z^0 = (z_1^0, \ldots, z_n^0) \in V$ there is a holomorphic coordinate system $x = (x_1, \ldots, x_n)$ defined in a neighborhood A of z^0 such that:*

(a) $V \cap A = \{x_1 = \cdots = x_k = 0\}$

(b) $h(x) = \sum_{|\sigma| = \ell+1} a_\sigma(x) x_1^{\sigma_1} \ldots x_k^{\sigma_k}$

where in the above notation $\sigma = (\sigma_1, \ldots, \sigma_k)$, $|\sigma| = \sigma_1 + \cdots + \sigma_k$, and $a_\sigma: A \to \mathbf{C}$ is holomorphic for all σ such that $|\sigma| = \ell+1$. We use the notation $j_V^\ell(h) = 0$ to say that the ℓ-jet of h is zero along V.

2.3.3 THEOREM. *Let $f, g: \mathbf{C}^n \to \mathbf{C}$ be homogeneous polynomials, $n \geq 3$, where $f \pitchfork g$ out of $0 \in \mathbf{C}^n$. Let $V = \{f = g = 0\}$ and $V^* = V - \{0\}$. If h is a homogeneous polynomial with $j_{V^*}^1(h) = 0$, then there are homogeneous polynomials a, b and c such that $h = af^2 + bf.g + cg^2$, where $dg(h) = dg(a) + 2dg(f) = dg(b) + dg(f) + dg(g) = dg(c) + 2dg(g)$.*

Proof: Since $f \pitchfork g$ out of 0, for any $z^0 \in V^*$ there is a holomorphic coordinate system $(x_1, \ldots, x_n)$ defined in a neighborhood U_{z^0} such that:

(a) $f(x) = x_1$, $g(x) = x_2$ $\forall x \in U_{z^0}$, so that $U_{z^0} \cap V = \{x_1 = x_2 = 0\}$.
(b) $h(x) = \alpha(x)x_1^2 + \beta(x)x_1x_2 + \gamma(x)x_2^2$, $\forall x \in U_{z^0}$, where $\alpha, \beta, \gamma \in \mathcal{O}(U_{z^0})$.

It follows that it is possible to find a convering $(U_i)_{i\in I}$ of $\mathbf{C}^n - \{0\}$ by open sets and three collections $\{\alpha_i\}_{i\in I}$, $\{\beta_i\}_{i\in I}$, $\{\gamma_i\}_{i\in I}$, where $\alpha_i, \beta_i, \gamma_i \in \mathcal{O}(U_i)$ and $h|U_i = (\alpha_i f^2 + \beta_i fg + \gamma_i g^2)|U_i$. When $U_i \cap U_j \neq \phi$ we have $\alpha_{ij}f^2 + \beta_{ij}fg + \gamma_{ij}g^2 \equiv 0$, where $\alpha_{ij} = \alpha_j - \alpha_i$, $\beta_{ij} = \beta_j - \beta_i$ and $\gamma_{ij} = \gamma_j - \gamma_i \in \mathcal{O}(U_i \cap U_j)$. Observe that this relation implies that $g|U_i \cap U_j$ divides $\alpha_{ij}f^2|U_i \cap U_j$. This fact together with $f \pitchfork g$ out of 0 implies that $\alpha_{ij} = \delta_{ij}g$, for some $\delta_{ij} \in \mathcal{O}(U_i \cap U_j)$. Now, if $U_i \cap U_j \cap U_k \neq \phi$ we have $(\delta_{ij} + \delta_{jk} + \delta_{ki})g = \alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0$, and so $\delta_{ij} + \delta_{jk} + \delta_{ki} = 0$. It follows from Cartan's solution of Cousin's problem that there exists a collection $(\delta_i)_{i\in I}$, $\delta_i \in \mathcal{O}(U_i)$, such that $\delta_{ij} = \delta_j - \delta_i$. Therefore we can define a function $\alpha \in \mathcal{O}(\mathbf{C}^n - \{0\})$ by

$$\alpha|U_i = \alpha_i - \delta_i g$$

Similarly, there are a collection $(\varepsilon_i)_{i\in I}$, $\varepsilon_i \in \mathcal{O}(U_i)$, and $\gamma \in \mathcal{O}(\mathbf{C}^n - \{0\})$ such that $\gamma|U_i = \gamma_i - \varepsilon_i f$. Let $h_1 = h - \alpha f^2 - \gamma g^2$. It is clear that:

$$h_1|U_i = \beta_i fg + \delta_i g f^2 + \varepsilon_i f g^2 = (\beta_i + \delta_i f + \varepsilon_i g)fg = \varphi_i fg.$$

If $U_i \cap U_j \neq \phi$, we have $(\varphi_j - \varphi_i)fg \equiv 0$, so that $\varphi_i = \varphi_j$. Therefore there exists $\beta \in \mathcal{O}(\mathbf{C}^n - \{0\})$ such that $\beta|U_i = \varphi_i$, $i \in I$, and we have $h = \alpha f^2 + \beta fg + \gamma g^2$. Now, from Hartog's theorem, α, β and γ can be extended to holomorphic functions on $\mathbf{C}^n$, which we call α, β and γ also. Let $\alpha = \sum_{j\geq 0} \alpha_j$, $\beta = \sum_{j\geq 0} \beta_j$ and $\gamma = \sum_{j\geq 0} \gamma_j$, where α_j, β_j and γ_j are homogeneous of degree j. It is clear that if j_1, j_2 and j_3 are such that $j_1 + 2dg(f) = j_2 + dg(f) + dg(g) = j_3 + 2dg(g)$, then we have $h = \alpha_{j_1} f^2 + \alpha_{j_2} fg + \alpha_{j_3} g^2$. So we can take $a = \alpha_{j_1}$, $b = \alpha_{j_2}$ and $c = \alpha_{j_3}$. ∎

Remark: We will need the above result only to prove (a) of theorem A.

3. Proofs

3.1 Proof of Theorem B:

Let $\mathcal{F}$ be a foliation on $\mathbf{C}P^n$ which satisfies hypothesis (a), (b) and (c) of Theorem B. The idea is the following: let w be a homogeneous integrable 1-form in $\mathbf{C}^{n+1}$ which represents $\mathcal{F}$ as in §1.3. We will prove that there exists a homogeneous polynomial f such that $d(w/f) = 0$ and $dg(f) = dg(w) + 1$. After that we will use some results containned in **[C-M]** to conclude the proof.

Let $N = N_1 \cup \cdots \cup N_r$ be the decomposition of N in irreducible components. From the hypothesis we have that for each $i = 1, \ldots, r$, $N_i \cap K(\mathcal{F})$ is open and dense in N_i. Since $N_i - K(\mathcal{F})$ is algebraic we have that $\mathrm{cod}_{N_i}(N_i - K(\mathcal{F})) \geq 1$, and so, $N_i \cap K(\mathcal{F})$ is connected.

Observe now that hypothesis (b) and Poincaré's linearization Theorem imply that $N_i \cap K(\mathcal{F})$ is of linearizable transversal type. This means that there exist λ_1^i and λ_2^i with $\lambda_2^i/\lambda_1^i \notin \mathbf{R}$ with the following property: (1) $\forall p \in N_i \cap K(\mathcal{F})$ there exists a local chart $(x, y, z): U \to \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-2}$, such that $U \cap N_i \cap K(\mathcal{F}) = \{(x, y, z) | x = y = 0\}$ and $\mathcal{F}|U$ is the foliation defined by the 1-form $w_U = \lambda_1^i x dy - \lambda_2^i y dx$. If we divide w_U by $\lambda_1^i xy$ we get the form $\alpha_U = \frac{dy}{y} - a\frac{dx}{x}$, where $a = \lambda_2^i/\lambda_1^i \notin \mathbf{R}$. Observe that α_U is closed, so that w_U has an integrating factor.

Lemma 2. *Let $i \in \{1, \ldots, r\}$ be fixed. There exists a neighborhood A_i of N_i in $\mathbf{C}P^n$, and a meromorphic closed 1-form η_i on A_i such that if P_i is the divisor of poles of η_i, then $\mathcal{F}|(A_i - P_i)$ is represented by $\eta_i|(A_i - P_i)$.*

Proof: It follows from the considerations before Lemma 2 and from hypothesis (c) that it is possible to find a convering of N_i by open sets of $\mathbf{C}P^n$, $(U_j)_{j \in J}$ and a collection $(\alpha_j)_{j \in J}$ such that:

(i) If $j, k, \ell \in J$ are such that $U_j \cap U_k \cap U_\ell \neq \phi$, then $U_j, U_j \cap U_k$ and $U_j \cap U_k \cap U_\ell$ are simply connected. Moreover by using the local structure of analytic sets, we can suppose also that $U_j \cap N_i$, $U_j \cap U_k \cap N_i$ and $U_j \cap U_k \cap U_\ell \cap N_i$ are simply connected.

(ii) $J = J_1 \cup J_2$ where $U_j \cap (N_i - K(\mathcal{F})) = \phi$ if $j \in J_1$, $\bigcup_{j \in J_1} U_j \supset N_i \cap K(\mathcal{F})$, $\bigcup_{j \in J_2} U_j \supset N_i - K(\mathcal{F})$ and if $j, k \in J_2$ is such that $U_j \cap U_k \neq \phi$ then there is $\ell \in J_1$ with $U_\ell \subset U_j \cap U_k$.

(iii) For each $j \in J$, α_j is a *closed* meromorphic 1-form on U_j, such that $\alpha_j = w_j / f_j$, where $f_j \in \mathcal{O}(U_j)$ and w_j is a holomorphic 1-form which represents $\mathcal{F}|U_j$. We can assume also that the singular set of w_j is of codimension ≥ 2.

(iv) For each $j \in J_1$, there is a local chart $(x_j, y_j, z_j): U_j \to \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-2}$ such that $\alpha_j = \frac{dy_j}{y_j} - a\frac{dx_j}{x_j}$, where $a \notin \mathbf{R}$ is as before. In this case $w_j = x_j dy_j - a y_j dx_j$ and $f_j = x_j y_j$.

(v) If $j, k \in J$ is such that $U_j \cap U_k \neq \phi$, then there exists a meromorphic function g_{jk}, defined on $U_j \cap U_k$, with $\alpha_j = g_{jk}\alpha_k$. This function is obtainned as follows: since w_j and w_k define the same foliation on $U_j \cap U_k$ and their singular sets have codimension ≥ 2, we can write $w_j = h_{jk} w_k$, where $h_{jk} \in \mathcal{O}^*(U_j \cap U_k)$. Therefore $g_{jk} = f_k h_{jk} / f_j$ as the reader can verify easily. Moreover, the collection $(g_{jk})_{U_j \cap U_k} \neq \phi$ satisfies the cocycle condition $g_{jk} g_{k\ell} g_{\ell j} \equiv 1$ on $U_j \cap U_k \cap U_\ell$ if this set is non empty.

Assertion: If $j, k \in J$ are such that $U_j \cap U_k \neq \phi$, then g_{jk} is a constant.

Proof: Observe first that $\alpha_j = g_{jk}\alpha$ implies that $dg_{jk} \wedge \alpha_k = 0$ because α_j and α_k are closed.

1^{st} case: $k \in J_1$, so that $\alpha_k = \frac{dy_k}{y_k} - a\frac{dx_k}{x_k}$. Let $z_k = (z^1, \ldots, z^{n-2})$ and $g_{jk} = g$. Relation $dg \wedge \alpha_k \equiv 0$ implies that outside the set of poles of g we have $\frac{\partial g}{\partial z^r} \equiv 0$, $1 \leq r \leq n-2$, and $x_k \frac{\partial g}{\partial x_k} + a y_k \frac{\partial g}{\partial y_k} \equiv 0$. This implies already that g does not depend on z_k. Therefore $g = g(x_k, y_k)$ and we can suppose that g is defined in a neighborhood of $(0,0) \in \mathbf{C}^2$. From now on we will omit the indexes k. Let P be the set of poles of g. Suppose first that $P \supset \{x = 0\}$. In this case, it is not difficult to see that there are a disk $\Delta \subset \{y = 0\}$ and an annulus $A \subset \{x = 0\}$ such that g has no poles on $W = (\Delta - \{0\}) \times A$.

Consider the Laurent development of g in W:

$$g(x,y) = \sum_{m,n=-\infty}^{\infty} b_{mn} x^m y^n, \quad b_{mn} \in \mathbf{C}$$

From $x\frac{\partial g}{\partial x} + ay\frac{\partial g}{\partial y} \equiv 0$ we get $\sum\limits_{m,n=-\infty}^{\infty} b_{mn}(m+na)x^m y^n = 0$, which implies that $b_{mn}(m+na) = 0 \ \forall m,n \in \mathbf{Z}$. Since $a \notin \mathbf{Q}$, this implies that $b_{mn} = 0$ if $(m,n) \neq (0,0)$. Therefore $g|W$ is constant, which implies that g is constant.

If $\{x=0\} \not\subset P$, then it is possible to find a disc $\Delta \subset \{y=0\}$ and an annulus $A \subset \{x=0\}$ such that g is holomorphic on $\Delta \times A$. In this case g admits a Laurent development on $\Delta \times A$ and so g is constant by the same argument as before.

2^{nd} case: $j,k \in J_2$. In this case let $\ell \in J_1$ be such that $U_\ell \subset U_j \cap U_k$. Observe that $g_{jk} \cdot g_{k\ell} \cdot g_{\ell j} \equiv 1$ on $U_\ell = U_\ell \cap U_j \cap U_k$. By the firs case $g_{k\ell}$ and $g_{\ell j}$ are constants. Hence g_{jk} is constant on U_ℓ, and so on $U_j \cap U_k$. This proves the assertion.

Now, if $j,k \in J_1$ and $U_j \cap U_k \neq \phi$ then $g_{jk} = 1$. In fact, on $U_j \cap U_k$ we have:

$$\alpha_j = \frac{dy_j}{y_j} - a\frac{dx_j}{x_j} = g_{jk}\left(\frac{dy_k}{y_k} - a\frac{dx_k}{x_k}\right) = g_{jk}\alpha_k.$$

From (i), there is $p_0 \in U_j \cap U_k \cap N_i$. It is clear that $p_0 = (0,0,z_j^0)$ in the chart (x_j, y_j, z_j) and $p_0 = (0,0,z_k^0)$ in the chart (x_k, y_k, z_k). By analyzing the sets of poles of α_j and α_k we get that either $\{y_j = 0\} \cap U_k = \{y_k = 0\} \cap U_j$ and $\{x_j = 0\} \cap U_k = \{x_k = 0\} \cap U_j$, or $\{y_j = 0\} \cap U_k = \{x_k = 0\} \cap U_j$ and $\{y_k = 0\} \cap U_j = \{x_j = 0\} \cap U_k$.

On the other hand, by comparing the residues of α_j and α_k around $\{x_j = 0\}$, $\{y_j = 0\}$, $\{x_k = 0\}$ and $\{y_k = 0\}$, we obtain in the first case that $g_{jk} = 1$ and in the second case that $1 = -ag_{jk}$ and $-a = g_{jk}$. Well, these last relations imply that $a^2 = 1$, which is not possible. Therefore $g_{jk} = 1$. So we have proved that if $j,k \in J_1$ is such that $U_j \cap U_k \neq \phi$ then $\alpha_j = \alpha_k$ on $U_j \cap U_k$.

It follows that we can define a closed meromorphic 1-form $\tilde{\eta}_i$ on $B_i = \bigcup_{j \in J_1} U_j$ by $\tilde{\eta}_i | U_j = \alpha_j$.

Let us prove that $\tilde{\eta}_i$ can be extended to $A_i = \bigcup_{j \in J} U_j$. Let η_i be defined on A_i by:

(vi) $\eta_i | B_i = \tilde{\eta}_i$

(vii) If $j \in J_2$ then $U_j \cap B_i \neq \phi$. Therefore there is $k \in J_1$ such that $U_k \cap U_j \neq \phi$ (because $B_i \supset N_i \cap K(\mathcal{F})$). We put $\eta_i | U_j = g_{kj} \alpha_j$. Observe that this definition is natural because $\eta_i | U_j \cap U_k = \tilde{\eta}_i | U_j \cap U_k = \alpha_k | U_j \cap U_k = g_{kj} \alpha_j | U_j \cap U_k$.

Let us prove that η_i is well defined. We can consider $g = (g_{jk})_{U_j \cap U_k \neq \phi}$ as a cocycle in $\check{H}^1(\mathcal{U}, \mathbf{C}^*)$, where $\mathcal{U} = (U_j \cap N_i)_{j \in J}$. It is not difficult to see that if $\mathcal{G}$ is trivial in $\check{H}^1(\mathcal{U}, \mathbf{C}^*)$ then η_i is well defined. Let $\mathcal{U}_1 = (U_j \cap N_i)_{j \in J_1}$ and $\mathcal{G}_1 = \{g_{jk} \in \mathcal{G} \mid j, k \in J_1\}$. Then $g_{jk} = 1$ for any $g_{jk} \in \mathcal{G}_1$ and so $\mathcal{G}_1$ is trivial in $\check{H}^1(\mathcal{U}_1, \mathbf{C}^*)$. On the other hand, since $\mathrm{cod}_{N_i}(N_i - K(\mathcal{F})) \geq 1$ it follows that any closed path $\gamma\colon [0,1] \to N_i$ with end points in $N_i \cap K(\mathcal{F})$ is homotopic, with fixed end points, to a path $\tilde{\gamma}\colon [0,1] \to N_i \cap K(\mathcal{F})$. It follows that the monodromy of a closed path as above (with respect to $\mathcal{G}$) is trivial. This implies that $\mathcal{G}$ is trivial, as the reader can check by himself. This ends the proof of Lemma 2. ∎

From Lemma 2 we get for each N_i , $i = 1, \dots, r$, a neighborhood A_i and a closed meromorphic 1-form η_i on A_i such that, if P_i is the divisor of poles of η_i, then η_i represents $\mathcal{F}$ on $A_i - P_i$. Furthermore if C is a connected component of $A_i \cap A_j$ then $\eta_i | C = \lambda_{ij}(C) \eta_j | C$, where $\lambda_{ij}(C)$ is a constant in $\mathbf{C}^*$. Let $U_0 = \{[1\colon z_1\colon \dots\colon z_n] \mid (z_1, \dots, z_n) \in \mathbf{C}^n\}$ and w_0 be a polynomial integrable 1-form which represents $\mathcal{F}$ on U_0. Then w_0 can be extended to $\mathbf{C}P^n$ as a meromorphic 1-form with poles in $L_0 = \{[z] \in \mathbf{C}^n \mid z_0 = 0\}$. Since $w_0 | A_i$ and η_i represent the same foliation, we have that $w_0 | A_i = f_i \eta_i$, where f_i is a meromorphic function on A_i. On the other hand, if C is a connected component of $A_i \cap A_j$ then we have

$$f_i \lambda_{ij}(C) \eta_j | C = f_i \eta_i | C = w_0 | C = f_j \eta_j | C$$

which implies that $f_j|C = \lambda_{ij}(C) f_i|C$. Since $\lambda_{ij}(C)$ is a constant, it follows that $\frac{df_j}{f_j}|C = \frac{df_i}{f_i}|C$. This implies that we can define a closed 1-form θ on $A = \bigcup_{i=1}^{r} A_i$ by $\theta|A_i = \frac{df_i}{f_i}$. Now, by corollary 2.2.3 of §2.2, θ can be extended to $\mathbf{C}P^n$, because A is a neighborhood of $N = \{f = g = 0\}$ and $2 \leq n-1$. We call θ this extension. Let P be the divisor of poles of θ. Fix $p_0 \in \mathbf{C}P^n - P$ and for each path $\gamma: [0,1] \to \mathbf{C}P^n - P$ with $\gamma(0) = p_0$, put

$$I(\gamma) = \exp\left[\int_\gamma \theta\right]$$

We will prove now that if γ is a closed path then $I(\gamma) = 1$. It is easy to see that this will imply that we can define a holomorphic function $F: \mathbf{C}P^n - P \to \mathbf{C}$ by $F(p) = I(\gamma)$, where $\gamma(0) = p_0$ and $\gamma(1) = p$.

Let γ be a closed path. Since $\mathbf{C}P^n$ is simply connected, γ is homologous in $\mathbf{C}P^n - P$ to $\gamma_1 + \cdots + \gamma_k$ where each γ_j is a small cycle envolving an irreducible component of P. So it is sufficient to prove that if γ is a small cycle envolving an irreducible component of P, say Q, then $\exp[\int_\gamma \theta] = 1$. Now, since $N = \{f = g = 0\}$, it follows from Bézout's Theorem that $Q \cap N_i \neq \phi$ for some i. It follows that we can deform γ keeping it closed along the deformation, to a small cycle $\hat{\gamma}$ envolving Q and containned in A_i. Since $\theta|A_i = \frac{df_i}{f_i}$, this implies that $\int_\gamma \theta = \int_{\hat{\gamma}} \frac{df_i}{f_i} = 2\pi i m$, $m \in \mathbf{Z}$. Hence $I(\gamma) = 1$.

The above argument implies also that $F|A_i = c_i f_i$, where c_i is a constant. Since $\eta_i = w_0/f_i$ is closed, we get that $\eta = w_0/F$ is closed. Hence the first part of the theorem is proved. It follows also that if w is a homogeneous integrable 1-form on $\mathbf{C}^{n+1}$ which induces $\mathcal{F}$ then w has a meromorphic integrating factor say $F = g/h$. Let us prove that g and h are homogeneous polynomials such that $dg(g) - dg(h) = dg(w) + 1$.

In fact, let $dg(w) = k$. Since F is an integrating factor, we have $dF \wedge w = Fdw$. On the other hand, $i_R(w) = 0$ implies that $(k+1)w = L_R(w) = i_R dw + d(i_R w) = i_R dw$, where L_R is the Lie derivative. Therefore we get

$$i_R(dF)w = Fi_R(dw) = (k+1)Fw \Rightarrow i_R(dF) = (k+1)F$$

Integrating $i_R(dF) = (k+1)F$, we get that $F(tz) = t^{k+1}F(z)$, if z is not a pole of F. Hence g and h are homogeneous and $dg(g) - dg(h) = k+1$. We can suppose that g and h do not have common factors. Let $g = g_1^{k_1} \dots g_m^{k_m}$ be the decomposition of g in irreducible factors, where $k_1, \dots, k_m \geq 1$.

LEMMA 3. *There are $\lambda_1, \dots, \lambda_m \in \mathbf{C}$ and a homogeneous polynomial φ such that*

$$\frac{hw}{g} = \sum_{j=1}^{m} \lambda_j \frac{dg_j}{g_j} + d\left(\frac{\varphi}{\psi}\right) \tag{8}$$

where $\psi = g_1^{\ell_1} \dots g_m^{\ell_m}$, $0 \leq \ell_j \leq k_j - 1$, $\sum_{j=1}^{m} \lambda_j dg(g_j) = 0$, φ and ψ have no common factors and $dg(\varphi) = dg(\psi)$.

The proof of the above result can be found in [C-M].

Now let us assume hypothesis (d) and (e) of Theorem B and prove that $\mathcal{F}$ is of logarithmic type. First of all, if we multiply the right hand side of (8) by $g_1^{\ell_1+1} \dots g_m^{\ell_m+1}$ we get the form

$$\eta = g_1 \dots g_m d\varphi - \varphi g_i \dots g_m \sum_{j=1}^{m} \ell_j \frac{dg_j}{g_j} + g_1^{\ell_1+1} \dots g_m^{\ell_m+1} \sum_{j=1}^{m} \lambda_j \frac{dg_j}{g_j} \tag{9}$$

which is holomorphic in $\mathbf{C}^{n+1}$. Observe that if $\ell_i = \lambda_i = 0$ for some $i \in \{1, \dots, m\}$, then g_i is a factor of η and g_i plays no essential role. Hence we can suppose that either $\ell_i \neq 0$ or $\lambda_i \neq 0$ for all $i = 1, \dots, m$. With this condition $G_i = \{[z] \in \mathbf{C}P^n \mid g_i(z) = 0\}$ is invariant under $\mathcal{F}$, which implies that if $i \neq j$ then $G_{ij} = \{[z] \mid g_i(z) = g_j(z) = 0\} \subset S(\mathcal{F})$. Since G_{ij} has codimension 2, hypothesis (d) implies that the set

$$A = K(\mathcal{F}) \cap G_{ij} \cap \{[z] \in \mathbf{C}P^n \mid dg_i(z) \wedge dg_j(z) \neq 0, \quad \varphi(z) \neq 0 \text{ and } g_r(z) \neq 0 \text{ for } r \neq i$$

is open and dense in G_{ij}. In order to simplify the notations we will put $i = 1$ and $j = 2$. Now we will prove that:

(i) If $\ell_1 > 0$ and $\ell_2 = 0$ then the linear part of the transversal type of $\mathcal{F}$ at G_{12} is degenerated.

(ii) If $\ell_1 = \ell_2 = 0$ then the quotient of the eigenvalues of the linear part of the transversal type of $\mathcal{F}$ at G_{12} is $-\lambda_2/\lambda_1$ (or $-\lambda_1/\lambda_2$)

(iii) If $\ell_1 > 0$, $\ell_2 > 0$ then the above quotient is $-\ell_2/\ell_1$ (or $-\ell_1/\ell_2$).

In fact, since A is open and dense in G_{12}, let $p \in \mathbf{C}^{n+1} - \{0\}$ be such that $[p] \in A$. We know that $\varphi(p) \neq 0$ and $g_r(p) \neq 0$ for $r > 2$. This implies that the form $\sum_{j \geq 3} \lambda_j \frac{dg_j}{g_j}$ has a holomorphic primitive, say h_1, defined in a simply connected neighborhood U of p. Therefore we can write:

$$\frac{hw}{g}\bigg|_U = \lambda_1 \frac{dg_1}{g_1} + \lambda_2 \frac{dg_2}{g_2} + dh_1 + d\left(\frac{h_2}{g_1^{\ell_1} g_2^{\ell_2}}\right) = \mu \tag{10}$$

where $h_2 = \varphi / g_3^{\ell_3} \ldots g_m^{\ell_m}$, $h_2(p) \neq 0$.

Proof of (i): Since $\ell_2 = 0$ we have $\lambda_2 \neq 0$. Let α be a branch of h_2^{-1/ℓ_1} and β be a branch of $h_2^{\lambda_1/\ell_1\lambda_2} \exp(h_1/\lambda_2)$ defined in U. These branches can be defined because $h_2(p) \neq 0$. Observe that $d(\alpha g_1)(p) \wedge d(\beta g_2)(p) \neq 0$, so that there is a local coordinate system $(x, y, z) \in \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-1}$ around p, such that $x = \alpha g_1$ and $y = \beta g_2$. It is easy to see that in this coordinate system we have

$$\mu = \lambda_1 \frac{dx}{x} + \lambda_2 \frac{dy}{y} + d(x^{-\ell_1}) = \lambda_1 \frac{dx}{x} + \lambda_2 \frac{dy}{y} - \ell_1 \frac{dx}{x^{\ell_1+1}}$$

If we multiply μ be $x^{\ell_1+1}.y$ we get $(-\ell_1 y + \lambda_1 x^{\ell_1} y)dx + \lambda_2 x^{\ell_1+1} dy$. Therefore the transversal type of $\mathcal{F}$ in G_{12} is given by the vector field $\lambda_2 x^{\ell_1+1} \partial/\partial x + (\ell_1 y - \lambda_1 x^{\ell_1} y)\partial/\partial y$. Since $\ell_1 > 0$, this proves (i).

Proof of (ii): Since $\ell_1 = \ell_2 = 0$ we have $\lambda_1 \neq 0 \neq \lambda_2$ and

$$\mu = \lambda_1 \frac{dg_1}{g_1} + \lambda_2 \frac{dg_2}{g_2} + dh_3$$

where $h_3 = h_1 + h_2$. Let $\alpha = \exp(h_3/\lambda_2)$. We have $dg_1(p) \wedge d(\alpha g_2)(p) \neq 0$, therefore there exists a local coordinate system $(x, y, z) \in \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-1}$ around p, such that $g_1 = x$ and $\alpha g_2 = y$. In this coordinate system we have $\mu = \lambda_1 \frac{dx}{x} + \lambda_2 \frac{dy}{y}$, so that $xy\mu = \lambda_1 y dx + \lambda_2 x dy$, therefore the quotient of the eigenvalues of the normal type is $-\lambda_2/\lambda_1$ (or $-\lambda_1/\lambda_2$).

Proof (iii): In this case we can write (10) as

$$\mu = \lambda_1 \frac{dg_1}{g_1} + \lambda_2 \frac{dg_2}{g_2} + d\left(\frac{h_3}{g_1^{\ell_1} g_2^{\ell_2}}\right)$$

where $h_3 = h_2 + h_1 g_1^{\ell_1} g_2^{\ell_2}$. Since $dg_1(p) \wedge dg_2(p) \neq 0$, there exists a coordinate system $(x, y, z) \in \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-1}$ around p such that $x = g_1$ and $y = g_2$. If we multiply μ by $x^{\ell_1+1}.y^{\ell_2+1}.h_3^{-1}$ we get

$$\begin{aligned} h_3^{-1} x^{\ell_1+1}.y^{\ell_2+1}.\mu = -\ell_1 y dx - \ell_2 x dy + \lambda_1 h_3^{-1} x^{\ell_1} y^{\ell_2+1} dx + \lambda_2 h_3^{-1} x^{\ell_1+1} y^{\ell_2} dy \\ + h_3^{-1} x^{\ell_1+1} y^{\ell_2+1} dh. \end{aligned}$$

It is not difficult to see that this implies (iii).

Now from (i) and hypothesis (e) it follows that either $\ell_1 = \cdots = \ell_m = 0$ or $\ell_1 \ldots \ell_m > 0$. In fact, if this is not the case, then there are $i \neq j$ such that $\ell_i > 0$ and $\ell_j = 0$, which cannot happen by (i). If $\ell_1 = \cdots = \ell_m = 0$ then $\mathcal{F}$ is logarithmic and we are done. Let us suppose that $\ell_1 \ldots \ell_m > 0$. In this case, it follows from (iii) that for any $i \neq j$ the quotient of the eigenvalues of the linear part of the normal type of $\mathcal{F}$ at G_{ij} is rational. Let us prove that this case cannot occur.

In fact, since in $N \cap K(\mathcal{F})$ the quotient of the eigenvalues of the normal type is not real, then in the above situation, there exists $p \in \mathbf{C}^{n+1} - \{0\}$ such that $[p] \in K(\mathcal{F}) \cap N - \bigcup_{i \neq j} G_{ij}$. In this case $p \in K(\pi^*(\mathcal{F}))$ and so there exists a coordinate system $(x, y, z) \in \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-1}$ around p such that $\pi^*(\mathcal{F})$ is defined by the form $\theta = xdy + aydx$, where $a \notin \mathbf{R}$. On the other hand let $\frac{hw}{g}$ be as in (8). Its set of poles is $P = \bigcup_i \{g_i = 0\}$, it is closed and represents $\pi^*(\mathcal{F})$ outside P. Therefore there exists a meromorphic function f_1 on U such that

$$\frac{hw}{g}\Big|_U = f_1(xdy + aydx) \Rightarrow 0 = d(f_1(xdy + aydx)) = d(xyf_1) \wedge \left(\frac{dy}{y} + a\frac{dx}{x}\right)$$

As we have seen in the proof of Lemma 2, the last relation implies $xyf_1 = c$, c a constant. This implies that $P \cap U = \{x = y = 0\}$. Therefore there are $i \neq j$ such that $\{g_i = 0\} \cap U = \{x = 0\}$ and $\{g_j = 0\} \cap U = \{y = 0\}$, and so $[p] \in G_{ij}$, a contradiction. This completes the proof of Theorem B. ∎

3.2 Proof of Theorem A:

We will use Theorem 1.2.4 of §1 (cf. **[G.M.-L.N.]**). We need some preliminary results.

LEMMA 4. *Let $V \xrightarrow{P} M$ be a holomorphic vector bundle with fiber $\mathbf{C}^2$, where M is compact. Assume that $V = E_1 \oplus E_2 = F_1 \oplus F_2$, where E_1, E_2, F_1 and F_2 are holomorphic line bundles, such that $c(E_1) \neq 0$ and $qc(E_1) = pc(E_2)$, where $p, q \in \mathbf{Z}$, $0 \leq q \leq |p|$ (c = first chern class). Then:*

(a) If $p \neq q$ then $F_i = E_j$ for some $i, j \in \{1, 2\}$. Moreover, if we assume $i = j = 1$ then $c(F_2) = c(E_2)$.

(b) If $p = q$ (i.e. $c(E_1) = c(E_2)$) then $c(F_i) = c(E_1)$ for $i = 1, 2$.

Note: As in 1.2.4, we denote by $c(\cdot)$ the first Chern class considered as an element of $H^2(M, \mathbf{C})$.

Proof: Let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be a covering of M by trivializing open sets of the E_i's and F_i's, where $U_\alpha \cap U_\beta$ and $U_\alpha \cap U_\beta \cap U_\gamma$ are simply connected if they are not empty. For each $\alpha \in A$ and $i = 1, 2$ let $e^i_\alpha \colon U_\alpha \to P^{-1}(U_\alpha)$ and $f^i_\alpha \colon U_\alpha \to P^{-1}(U_\alpha)$ be holomorphic local sections such that $e^i_\alpha(p) \in E_i(p) - \{0\}$

and $f^i_\alpha(p) \in F_i(p) - \{0\}$, $p \in U_\alpha$, where $E_i(p)$ and $F_i(p)$ denote the fibers of E_i and F_i over p. It follows that for all $p \in U_\alpha$, $\{e^1_\alpha(p), e^2_\alpha(p)\}$ and $\{f^1_\alpha(p), f^2_\alpha(p)\}$ are basis of $P^{-1}(p)$. Therefore there is a matrix $A_\alpha = (a^{ij}_\alpha)$ such that $a^{ij}_\alpha \in \mathcal{O}(U_\alpha)$, $\Delta_\alpha = \det(A_\alpha) \in \mathcal{O}^*(U_\alpha)$ and $e^i_\alpha = a^{i1}_\alpha f^1_\alpha + a^{i2}_\alpha f^2_\alpha$ on U_α. On the other hand there are collections $\{g^i_{\alpha\beta}\}_{U_\alpha \cap U_\beta \neq \phi}$ and $\{h^i_{\alpha\beta}\}_{U_\alpha \cap U_\beta \neq \phi}$, $i = 1, 2$, where $g^i_{\alpha\beta}$, $h^i_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and $e^i_\alpha = g^i_{\alpha\beta} e^i_\beta$, $g^i_\alpha = h^i_{\alpha\beta} f^i_\beta$ on $U_\alpha \cap U_\beta \neq \phi$ for $i = 1, 2$. These collections are in fact cocycles (i.e. if $U_\alpha \cap U_\beta \cap U_\gamma \neq \phi$ then $g^i_{\alpha\beta} . g^i_{\beta\gamma} . g^i_{\gamma\alpha} = 1$ and $h^i_{\alpha\beta} . h^i_{\beta\gamma} . h^i_{\gamma\alpha} = 1$) and $c(E_i)$ (resp. $c(F_i)$) can be represented in $\check{H}^2(\mathcal{U}, \mathbf{Z})$ by the 2-cocycle

(11)

$$\begin{cases} m^i_{\alpha\beta\gamma} = \frac{1}{2\pi\sqrt{-1}}(\ell_g(g^i_{\alpha\beta}) + \ell_g(g^i_{\beta\gamma}) + \ell_g(g^i_{\gamma\alpha})) \\ \text{resp. } n^i_{\alpha\beta\gamma} = \frac{1}{2\pi\sqrt{-1}}(\ell_g(h^i_{\alpha\beta}) + \ell_g(h^i_{\beta\gamma}) + \ell_g(h^i_{\gamma\alpha})) \end{cases} \qquad U_\alpha \cap U_\beta \cap U_\gamma \neq \phi$$

where the ℓg's are branches of the logarithm arbitrarily chosen.

Now, if we set $G_{\alpha\beta} = \begin{pmatrix} g^1_{\alpha\beta} & 0 \\ 0 & g^2_{\alpha\beta} \end{pmatrix}$ and $H_{\alpha\beta} = \begin{pmatrix} h^1_{\alpha\beta} & 0 \\ 0 & h^2_{\alpha\beta} \end{pmatrix}$, then it is not difficult to see that on $U_\alpha \cap U_\beta \neq \phi$ we have $G_{\alpha\beta} A_\beta = A_\beta H_{\alpha\beta}$ or equivalently

$$(12) \qquad \begin{cases} \text{(i) } g^1_{\alpha\beta} a^{11}_\beta = a^{11}_\alpha h^1_{\alpha\beta} \\ \text{(ii) } g^1_{\alpha\beta} a^{12}_\beta = a^{12}_\alpha h^2_{\alpha\beta} \\ \text{(iii) } g^2_{\alpha\beta} a^{21}_\beta = a^{21}_\alpha h^1_{\alpha\beta} \\ \text{(iv) } g^2_{\alpha\beta} a^{22}_\beta = a^{22}_\alpha h^2_{\alpha\beta}. \end{cases}$$

For fixed $\alpha \in A$ let $f_\alpha = a^{11}_\alpha a^{12}_\alpha a^{21}_\alpha a^{22}_\alpha / \Delta^2_\alpha$, where $\Delta_\alpha = \det(A_\alpha)$. Let us prove that there is $f \in \mathcal{O}(M)$ such that $f|U_\alpha = f_\alpha$. In fact, if we take the product of the relations $(i) \ldots (iv)$ we get

$$(13) \qquad (g^1_{\alpha\beta} g^2_{\alpha\beta})^2 a^{11}_\beta a^{12}_\beta a^{21}_\beta a^{22}_\beta = a^{11}_\alpha a^{12}_\alpha a^{21}_\alpha a^{22}_\alpha (h^1_{\alpha\beta} h^2_{\alpha\beta})^2$$

On the other hand the relation $G_{\alpha\beta} A_\beta = A_\alpha H_{\alpha\beta}$ implies that $g^1_{\alpha\beta} g^2_{\alpha\beta} \Delta_\beta = h^1_{\alpha\beta} h^2_{\alpha\beta} \Delta_\beta$, and so $(g^1_{\alpha\beta} g^2_{\alpha\beta})^2 / (h^1_{\alpha\beta} h^2_{\alpha\beta})^2 = \Delta^2_\alpha / \Delta^2_\beta$. This relation together with (13) implies that $f_\alpha | U_\alpha \cap U_\beta = f_\beta | U_\alpha \cap U_\beta$, which proves the existence of f.

Now, since M is compact, f is a constant. We have two case to consider:

1^{st} case: $f \neq 0$. In this case the $a^{ij}_\alpha \in \mathcal{O}^*(U_\alpha)$ for all α. This together with the relations in (12) imply that all cocycles in (11) are equivalent and so all Chern classes involved are equal. This case, of course, cannot happen in case (a).

2^{nd} case: $f = 0$. Observe that the relations in (12) imply that for all $i, j \in \{1, 2\}$, the collection $\{a^{ij}_\alpha\}_{\alpha \in A}$ defines a divisor in M. Since $f \equiv 0$, one of these divisors is $\equiv 0$. Suppose for instance that $a^{12}_\alpha \equiv 0$ for all $\alpha \in A$. This implies that $E_1 = F_1$ and $a^{11}_\alpha, a^{22}_\alpha \in \mathcal{O}^*(U_\alpha)$ for all $\alpha \in A$, because $\Delta_\alpha = a^{11}_\alpha a^{22}_\alpha \in \mathcal{O}^*(U_\alpha)$ in this case. Analogously, if $a^{11}_\alpha \equiv 0$ for all $\alpha \in A$ we get $E_1 = F_2$ and $c(E_2) = c(F_1)$. The remaining cases are similar and we leave them for the reader. ∎

Now let f and g be homogeneous polynomials on $\mathbf{C}^{n+1}$, $n \geq 3$, such that $f \pitchfork g$. Let $F = \{[z] \in \mathbf{C}P^n \mid f(z) = 0\}$, $G = \{[z] \mid g(z) = 0\}$ and $K = F \cap G$. We denote by $\nu \xrightarrow{P} K$, $\hat{F} \to F$ and $\hat{G} \to G$ the normal bundles of K, F and G in $\mathbf{C}P^n$ respectively. Let $\tilde{F} = \hat{F}|K$, $\tilde{G} = \hat{G}|K$. We will use the notation $c(\cdot)$ to denote the first Chen class of a holomorphic vector bundle in H^2_{DR} of the corresponding base. It is well known that:

(a) $\nu = \tilde{F} \oplus \tilde{G}$ and $c(\nu) = c(\tilde{F}).c(\tilde{G})$ (c f. **[G-A]**)

(b) $c(\tilde{F}) = c(F)|K$ and $c(\tilde{G}) = c(G)|K$. In particular $c(\tilde{F}), c(\tilde{G}) \neq 0$, because $c(F), c(G) \neq 0$ and $\dim(K) > 0$ (cf. **[G-A]**).

(c) $dg(f).c(\tilde{G}) = dg(g).c(\tilde{F})$.

Assertion (c) follows easily from Theorem 1.2.4 and example 1.3.1, as the reader can verify.

Suppose now that $\mathcal{F}$ is a foliation of $\mathbf{C}P^n$ having K as a Kupka component. Let $dg(f)/dg(g) = p/q$ where p, q are relatively primes and $p \leq q$ ($p = q$ iff $p = q = 1$).

Let λ_1 and λ_2 be eigenvalues of the normal type of $\mathcal{F}$ at K. Let us prove first that $\lambda_1 \neq 0 \neq \lambda_2$. In fact, let us suppose by contradiction that $\lambda_2 = 0$. In this case $\lambda_1 \neq 0$, because $K \subset K(\mathcal{F})$. Let E_1 and E_2 be the line subbundles of ν induced by the eigendirections of λ_1 and λ_2 respectively. Then $\nu = \tilde{F} \oplus \tilde{G} = E_1 \oplus E_2$. It follows from Lemma 4 that $c(\tilde{F}) = c(E_1)$ and $c(\tilde{G}) = c(E_2)$ or $c(\tilde{G}) = c(E_1)$ and $c(\tilde{F}) = c(E_2)$. Let us suppose for instance that $c(\tilde{F}) = c(E_1)$ and $c(\tilde{G}) = c(E_2)$. Now (a) of Theorem 1.2.4 implies that $0 = \lambda_2 c(E_1) = \lambda_1 c(E_2)$ and so $c(E_2) = c(\tilde{G}) = 0$, a contradiction.

On the other hand, if $\lambda_1 \neq \lambda_2$ we can define E_1 and E_2 in the same way and get the following relations (assuming $c(\tilde{F}) = c(E_1)$):

$$\begin{cases} \lambda_1 c(E_2) = \lambda_2 c(E_1) \\ c(\tilde{F}) = c(E_1), c(\tilde{G}) = c(E_2) \\ pc(\tilde{G}) = qc(\tilde{F}) \end{cases} \Rightarrow \frac{\lambda_2}{\lambda_1} = \frac{p}{q}$$

We can conclude from the above arguments that:

(i) $\lambda_2 \neq 0 \neq \lambda_1$ and $\lambda_2/\lambda_1 \in \mathbf{Q}_+$.
(ii) If $\lambda_2 \neq \lambda_1$ then $\lambda_2/\lambda_1 = p/q$.

We want to prove that $\lambda_2/\lambda_1 = p/q$ in all cases, but before that we will prove the followingt result.

LEMMA 5. *The transversal type of $\mathcal{F}$ at K is always linearizable and diagonal.*

Proof: Let $\lambda_2/\lambda_1 = r/s$ where $r, s \in \mathbf{Z}_+$, $0 < s \leq r$ and $(r, s) = 1$. Let us suppose by contradiction that the transversal type is either non linearizable or linearizable but not diagonal. In this case, by Poincaré-Dulac Theorem, we must have $1 = s \leq r$ and the transversal type is equivalent to the vector field $X = x\partial/\partial x + (ry + x^r)\partial/\partial y$. The dual form of X is $w = (ry + x^r)dx - xdy$, therefore by Kupka Theorem (1.2.1), there is a covering $(U_\alpha)_{\alpha \in A}$ of K by open sets of $\mathbf{C}P^n$, where each U_α is the domain of a chart $(x_\alpha, y_\alpha, z_\alpha)\colon U_\alpha \to \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-2}$ such that

(i) If $U_\alpha \cap U_\beta \neq \phi$ then it is connected.
(ii) $K \cap U_\alpha = \{x_\alpha = y_\alpha = 0\}$
(iii) $\mathcal{F}|U_\alpha$ is defined by the form $(ry_\alpha + x_\alpha^r)dx_\alpha - x_\alpha dy_\alpha = w_\alpha$.
(iv) There is a multiplicative cocycle $(g_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \phi}$ such that if $U_\alpha \cap U_\beta \neq \phi$ then $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and $w_\alpha = g_{\alpha\beta} w_\beta$ on $U_\alpha \cap U_\beta$.

Now observe that $x_\alpha^{-r-1}w_\alpha$ is closed because

$$\frac{w_\alpha}{x_\alpha^{r+1}} = \frac{dx_\alpha}{x_\alpha} - d\left(\frac{y_\alpha}{x_\alpha^r}\right).$$

Observe also that $\{x_\alpha = 0\}$ is the unique analytic separatrix of $\mathcal{F}$ through $K \cap U_\alpha$. It follows that on $U_\alpha \cap U_\beta \neq \phi$ we must have $x_\alpha = h_{\alpha\beta}x_\beta$ where $h_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. In particular $(x_\alpha)_{\alpha\in A}$ defines an analytic divisor on $U = \bigcup_{\alpha\in A} U_\alpha$. On the other hand we have

$$(14) \qquad \frac{w_\alpha}{x_\alpha^{r+1}} = \frac{g_{\alpha\beta}}{(h_{\alpha\beta})^{r+1}}\frac{w_\beta}{x_\beta^{r+1}} = f_{\alpha\beta}\frac{w_\beta}{x_\beta^{r+1}}, \quad f_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta).$$

This implies that:

$$0 = d(\frac{w_\alpha}{x_\alpha^{r+1}}) = df_{\alpha\beta} \wedge \frac{w_\beta}{x_\beta^{r+1}} \Rightarrow df_{\alpha\beta} \wedge w_\beta = 0.$$

Therefore $f_{\alpha\beta}$ is a first integral of $\mathcal{F}|U_\alpha \cap U_\beta$. As the reader can verify by using the Taylor series of $f_{\alpha\beta} = \sum_{m,n\geq 0} a_{mn}(z_\beta)x_\beta^m y_\beta^n$, the relation $df_{\alpha\beta}\wedge w_\beta = 0$ implies that $f_{\alpha\beta}$ is a constant. In fact this constant is 1 because the residues of $x_\alpha^{-r-1}w_\alpha$ and $x_\beta^{-r-1}w_\beta$ around $\{x_\alpha = 0\}$ are both equal to 1. It follows that $x_\alpha^{-r-1}w_\alpha = x_\beta^{-r-1}w_\beta$ and so there exists a closed meromorphic 1-form $\hat{\eta}$ defined on $U = \bigcup_\alpha U_\alpha$ such that $\hat{\eta}|U_\alpha = x_\alpha^{-r-1}w_\alpha$ and $\hat{\eta}$ defines $\mathcal{F}|U$ outside its divisor of poles. By Corollary 2.2.3 of §2, $\hat{\eta}$ can be extended to a meromorphic closed 1-form η on $\mathbf{C}P^n$ which defines $\mathcal{F}$ outside its divisor of poles. Let $\eta^* = \pi^*(\eta)$.

It follows from Lemma 3 of §3.1 that there exist homogeneous polynomials $g_1, \ldots, g_m$, φ and ψ on $\mathbf{C}^{n+1}$, $a_1, \ldots, a_m \in \mathbf{C}$ and $\ell_1, \ldots, \ell_m$ non negative integers such that:

(v) $\eta^* = \sum_{j=1}^m a_j \frac{dg_j}{g_j} + d(\frac{\varphi}{\psi})$

(vi) $\sum_{j=1}^{m} a_j dg(g_j) = 0$

(vii) $\psi = g_1^{\ell_1} \dots g_m^{\ell_m}$ and $dg(\varphi) = dg(\psi)$.

Let $\alpha \in A$ be fixed. We have

$$\left[\sum_{j=1}^{m} a_j \frac{dg_j}{g_j} + d(\frac{\varphi}{\psi})\right] |U_\alpha = \frac{dx_\alpha}{x_\alpha} - d\left(\frac{y_\alpha}{x_\alpha^r}\right) = \eta|U_\alpha \tag{15}$$

From Bézout's theorem, for every $j = 1, \dots, m$ we know that $G_j \cap K \neq \phi$, where $G_j = \{[z] \mid g_j(z) = 0\}$. If α is such that $U_\alpha \cap G_j \cap K \neq \phi$, we can conclude from (15) that:

(viii) $G_j \cap U_\alpha = \{x_\alpha = 0\} \Rightarrow m = 1$

Then (vi) implies that $a_1 dg(g_1) = 0 \Rightarrow a_1 = 0$. But this implies that $\eta = d(\frac{\varphi}{\psi})$ and its residue around $\{x_\alpha = 0\}$ is zero, a contradiction. ∎

LEMMA 6. *If $\lambda_1 = \lambda_2$ then $p = q = 1$ and $\mathcal{F}$ is induced by the form on $\mathbf{C}^{n+1} fdg - gdf$.*

Proof: Let w be an integrable homogeneous 1-form on $\mathbf{C}^{n+1}$ which induces $\mathcal{F}$. The foliation defined by w on $\mathbf{C}^{n+1}$ is $\mathcal{F}^* = \pi^*\mathcal{F}$. Moreover, if $K^* = \pi^{-1}(K) - \{0\}$, then $K^* \subset K(\mathcal{F}^*)$ and the transversal type of $\mathcal{F}^*$ at K^* is linearizable and diagonal with equal eigenvalues.

Let $p_0 \in K^*$. We assert that there exist a chart $(x, y, z): U \to \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-1}$ around p_0 and a function $\Delta \in \mathcal{O}^*(K^* \cap U)$ such that:

(i) $K^* \cap U = \{x = y = 0\}$

(ii) $f|U = x$, $g|U = y$

(iii) $w|U = \Delta(z)(xdy - ydx) + \theta$, where θ denotes terms of order highter than 1 in (x, y).

(iv) The local expression of the radial vector field in U is

$$R|U = mx\frac{\partial}{\partial x} + nx\frac{\partial}{\partial y} + \frac{\partial}{\partial z_1}, \quad m = dg(f), \quad n = dg(g).$$

Observe that (i) and (ii) follow from the implicit function theorem and the fact that $df(p_0) \wedge dg(p_0) \neq 0$. Let $R|U = A\partial/\partial x + B\partial/\partial y + \sum_{j=1}^{n-1} G_j\partial/\partial u_j$. From Euler's identity we have $R(x) = mx$ and $R(y) = ny$, which implies that $A = mx$ and $B = ny$. The proof of (iv) can be done as follows: let $L_1, \dots, L_{n-1}$ be homogeneous linear polymials such that $L_1(p_0) \neq 0$ and $df(p_0) \wedge dg(p_0) \wedge dL_1(p_0) \wedge \cdots \wedge dL_{n-1}(p_0) \neq 0$. Take $z_j = L_j/L_1$ for $j \geq 2$ and z_1 a branch of $\ell g(L_1)$ defined in a neighborhood of p_0. Then $(x, y, z_1, \dots, z_{n-1})$ is a diffeomorphism in a neighborhood U of p_0 and moreover $R|U = mx\partial/\partial x + ny\partial/\partial y + \partial/\partial z_1$.

Now since the linear part of the normal type of $\mathcal{F}^*$ at K^* has $\lambda_1 = \lambda_2$ and is diagonal, then for each section $\{z = c\}$ the dual form has linear part $xdy - ydx$, which implies that the linear part of $w|\{z = c\}$ is of the form $\Delta(z)(xdy - ydx)$. Therefore the linear part of $w|U$ with respect to (x, y) if of the form

$$w_1 = \Delta(z)(xdy - ydx) + \sum_{j=1}^{n-1}(A_j(z)x + B_j(z)y)dz_j$$

It follows from the integrability condition $w \wedge dw = 0$, that $A_j = B_j \equiv 0$ for $j = 1, \dots, n-1$, as the reader can verify directly by taking the linear part of $w \wedge dw$ with respect to (x, y). This proves (iii).

We can conclude from the above facts, that there exists an open covering $(U_\alpha)_{\alpha \in A}$ of K^* and two collections $(\Delta_\alpha)_{\alpha \in A}$, $((x_\alpha, y_\alpha, z_\alpha))_{\alpha \in A}$, where $\Delta_\alpha \in \mathcal{O}^*(K^* \cap U_\alpha)$ and $(x_\alpha, y_\alpha, z_\alpha)\colon U_\alpha \to \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-1}$ is a local chart such that

(i) $K^* \cap U_\alpha = \{x_\alpha = y_\alpha = 0\}$

(ii) $f|U_\alpha = x_\alpha$ and $g|U_\alpha = y_\alpha$.

(iii) $w|U_\alpha = \Delta_\alpha(z_\alpha)(x_\alpha dy_\alpha - y_\alpha dx_\alpha) + \theta_\alpha$, where θ_α denotes terms of order highter than 1 in (x_α, y_α).

(iv) $R|U_\alpha = mx_\alpha \frac{\partial}{\partial x_\alpha} + ny_\alpha \frac{\partial}{\partial y_\alpha} + \frac{\partial}{\partial z_{\alpha 1}}$.

Now, if $K^* \cap U_\alpha \cap U_\beta \neq \phi$ and $p \in K^* \cap U_\alpha \cap U_\beta$ then

$$dw(p) = 2\Delta_\alpha(p)dx_\alpha(p) \wedge dy_\alpha(p) = 2\Delta_\beta(p)dx_\beta(p) \wedge dy_\beta(p).$$

Since $x_\alpha = x_\beta = f$ and $y_\alpha = y_\beta = g$ on $U_\alpha \cap U_\beta$, then we must have $\Delta_\alpha(p) = \Delta_\beta(p)$. Hence we can define a holomorphic function $\Delta: K^* \to \mathbf{C}^*$ such that $\Delta | U_\alpha \cap K^* = \Delta_\alpha$ for every $\alpha \in A$. Let us prove that Δ is constant.

Let $dg(w) = k \geq 1$ ($dg(w) =$ degree of the coefficients of w) and the two jet of w along $K^* \cap U_\alpha$ be

$$\begin{aligned} j^2_{(x_\alpha, y_\alpha)} w &= \Delta(z_\alpha)(x_\alpha dy_\alpha - y_\alpha dx_\alpha) \\ &+ \sum_{i=1}^{n-1} (A_i(z_\alpha)x_\alpha^2 + B_i(z_\alpha)x_\alpha y_\alpha + c_i(z_\alpha)y_\alpha^2)dz_i \\ &+ P(x_\alpha, y_\alpha, z_\alpha)dx_\alpha + Q(x_\alpha, y_\alpha, z_\alpha)dy_\alpha \end{aligned}$$

where P and Q are homogeneous of degree two in (x_α, y_α). Since $i_R(w) = 0$, get

$$0 = i_R(w) = \Delta(z_\alpha)(n-m)x_\alpha y_\alpha + A_1(z_\alpha)x_\alpha^2 + B_1(z_\alpha)x_\alpha y_\alpha + C_1(z_\alpha)y_\alpha^2 + \theta_3$$

where θ_3 denotes terms of order highter than 2 in (x_α, y_α). This implies that

(v) $A_1 \equiv C_1 \equiv 0$ and $(n-m)\Delta + B_1 \equiv 0$.

On the other hand, we have seen in the proof of Theorem B that $i_R(dw) = (k+1)w$. From this we get

$$\begin{aligned} (k+1)\Delta(x_\alpha dy_\alpha - y_\alpha dx_\alpha) + \theta_\alpha = i_R(d\Delta)&(x_\alpha dy_\alpha - y_\alpha dx_\alpha) \\ &+ 2\Delta(m x_\alpha dy_\alpha - n y_\alpha dx_\alpha) \\ &- B_1(y_\alpha dx_\alpha + x_\alpha dy_\alpha) + r \end{aligned}$$

where r denotes terms either of order highter than 1 in (x_α, y_α) or terms in the $dz_j's$. Comparing these expressions we get:

(vi) $(k+1)\Delta = i_R(d\Delta) + 2m\Delta - B_1 = i_R(d\Delta) + 2n\Delta + B_1$

Therefore,

(vii) $i_R(d\Delta) = (k+1-m-n)\Delta = \ell\Delta$

Equation (vii) implies that $\Delta(tp) = t^\ell \Delta(p)$ for all $t \in \mathbf{C}^*$ and $p \in K^*$. Now, if $\ell = 0$ then we can define a holomorphic function $\varphi: K \to \mathbf{C}$ by

$\varphi([p]) = \Delta(p)$ and this implies that Δ is constant. On the other hand, if $\ell > 0$, let $M: \mathbf{C}^{n+1} \to \mathbf{C}$ be linear. In this case we can define a meromorphic function on K by $\varphi([p]) = \Delta(p)/M^\ell(p)$. Since M is arbitrary, it follows that Δ must vanish in some point $p \in K^*$, a contradiction. Analogously, if $\ell < 0$ the function $\varphi = M^{-\ell}\Delta$ must have some pole, and so Δ also, which is again a contradiction. Therefore Δ is constant and moreover $k + 1 = m + n$.

Let $\mu = w - \Delta(fdg - gdf)$. It is not difficult to see that the 1-jet of μ along K^* is zero and that μ is homogeneous of degree $k = m + n - 1$. From Theorem 2.3.3 we can conclude that $\mu = f^2\mu_1 + fg\mu_2 + g^2\mu_3$, where μ_1, μ_2, μ_3 are 1-forms with homogeneous coefficients and

(viii) $k = 2m + dg(\mu_1) = m + n + dg(\mu_2) = 2n + dg(\mu_3)$

unless some of the μ_j's are $\equiv 0$. Now, if $m = n$, (viii) and $k + 1 = m + n = 2m = 2n$, implies that $\mu_1 \equiv \mu_2 \equiv \mu_3 \equiv 0$, and so $w = \Delta(fdg - gdf)$. Let us suppose by contradiction that $m > n$ for instance. In this case $k = m + n - 1$ implies that $\mu_1 \equiv \mu_2 \equiv 0$, and so

$$w = \Delta(fdg - gdf) + g^2\mu_3.$$

Since $i_R(w) = 0$, we get

$$0 = \Delta(n - m)fg + g^2 i_R(\mu_3) = g(\Delta(n - m)f + g i_R(\mu_3)).$$

This implies that g divides f which is a contradiction. Hence $m = n$ and $w = \Delta(fdg - gdf)$, which proves the lemma. ∎

COROLLARY. $\lambda_2/\lambda_1 = p/q = m/n$.

Now let us suppose that $1 \leq p < q$ (i.e. $dg(f) < dg(g)$). Let w be a homogeneous integrable 1-form on $\mathbf{C}^{n+1}$ which induces $\mathcal{F}$ on $\mathbf{C}P^n$.

LEMMA 7. *In the above situation w has an integrating factor.*

Proof: We know from Lemma 5 that the transversal type of $\mathcal{F}$ at K is linearizable. This implies that there exists a covering of K by open sets

$(U_\alpha)_{\alpha \in A}$ and coordinate systems $((x_\alpha, y_\alpha, z_\alpha): U_\alpha \to \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-2})_{\alpha \in A}$ such that:

(i) $K \cap U_\alpha = \{x_\alpha = y_\alpha = 0\}$

(ii) $\mathcal{F}|U_\alpha$ is defined by $w_\alpha = px_\alpha dy_\alpha - qy_\alpha dx_\alpha$

(iii) If $U_\alpha \cap U_\beta \neq \phi$ then it is connected and there exists $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ such that $w_\alpha = g_{\alpha\beta} w_\beta$ on $U_\alpha \cap U_\beta$.

(iv) If $U_\alpha \cap U_\beta \cap U_\gamma \neq \phi$ then it is connected and $g_{\alpha\beta}.g_{\beta\gamma}.g_{\gamma\alpha} \equiv 1$.

We will consider two cases:

1st case: $1 < p < q$. In this case we will prove that there exists a closed meromorphic 1-form on $U = \sum_\alpha U_\alpha$, which defines $\mathcal{F}$ outside its poles.

Oberve first that $1 < p < q$ implies that $\{x_\alpha = 0\} \cap U_\beta = \{x_\beta = 0\} \cap U_\alpha$ and $\{y_\alpha = 0\} \cap U_\beta = \{y_\alpha = 0\} \cap U_\alpha$. This follows from the fact that the vector field $X = px\partial/\partial x + qy\partial/\partial y$ has only two analytic smooth separatrices through $(0,0)$, which are $\{x = 0\}$ and $\{y = 0\}$, and they correspond to two differet eigenvalues q and p. Let $\eta_\alpha = x_\alpha^{-1} y_\alpha^{-1} w_\alpha = pdy_\alpha/y_\alpha - qdx_\alpha/x_\alpha$. If $U_\alpha \cap U_\beta \neq \phi$ and $h_{\alpha\beta} = x_\beta y_\beta g_{\alpha\beta}/x_\alpha y_\alpha$, then $\eta_\alpha = h_{\alpha\beta}\eta_\beta$ and $h_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ because x_β/x_α and $y_\beta/y_\alpha \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ by the first observation. On the other hand, $0 = d\eta_\alpha = dh_{\alpha\beta} \wedge \eta_\beta \Rightarrow dh_{\alpha\beta} \wedge w_\beta = 0 \Rightarrow h_{\alpha\beta}$ is a first integral of $px_\beta\partial/\partial x_\beta + qy_\beta\partial/\partial y_\beta \Rightarrow h_{\alpha\beta}$ is a constant.

Now, if we compare the residues of η_α and η_β around $\{x_\alpha = 0\} \cap U_\beta$ we get $h_{\alpha\beta} \equiv 1$. This implies that $\eta_\alpha|U_\alpha \cap U_\beta = \eta_\beta|U_\alpha \cap U_\beta$ and so there exists a closed meromorphic 1-form $\tilde{\eta}$ on $U = \bigcup_\alpha U_\alpha$ such that $\tilde{\eta}|U_\alpha = \eta_\alpha$ for all $\alpha \in A$ and $\tilde{\eta}$ represents $\mathcal{F}|U$ outside its poles. It follows from Corollary 2.2.3 that this form can be extended to a closed 1-form η on $\mathbf{C}P^n$ which represents $\mathcal{F}$ outside its poles. This implies the 1st case.

2nd case: $1 = p < q$. In this case we have still $\{x_\alpha = 0\} \cap U_\beta = \{x_\beta = 0\} \cap U_\alpha$ by the same reason as in the 1st case, but $\{y_\alpha = 0\} \cap U_\beta = \{y_\beta - cx_\beta^q = 0\} \cap U_\alpha$, where c is a constant, as the reader can verify easily. For each $\alpha \in A$, let

$\varphi_\alpha = y_\alpha / x_\alpha^q$. Let us prove that if $U_\alpha \cap U_\beta \neq \phi$ then there constants $a_{\alpha\beta} \in \mathbf{C}^*$ and $b_{\alpha\beta} \in \mathbf{C}$ such that $\varphi_\alpha = a_{\alpha\beta}\varphi_\beta + b_{\alpha\beta}$.

In fact, we have

$$d\varphi_\alpha = x_\alpha^{-q-1} w_\alpha = x_\alpha^{-q-1} g_{\alpha\beta} w_\beta = \left(\frac{x_\beta}{x_\alpha}\right)^{q+1} g_{\alpha\beta} d\varphi_\beta = a_{\alpha\beta} d\varphi_\beta$$

where $a_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. The above relation implies:

$$0 = d^2\varphi_\alpha = da_{\alpha\beta} \wedge d\varphi_\beta \Rightarrow da_{\alpha\beta} \wedge w_\beta = 0 \Rightarrow a_{\alpha\beta} \text{ is a constant } \Rightarrow a_{\alpha\beta} \in \mathbf{C}^*.$$

$\Rightarrow \varphi_\alpha = a_{\alpha\beta}\varphi_\beta + b_{\alpha\beta}$.

Now let $\hat{w}$ be a merorphic 1-form on $\mathbf{C}P^n$ which represents $\mathcal{F}$ outside its poles (we can take $\hat{w}$ such that $\pi^*(\hat{w}) = w/M^{k+1}$, where $dg(w) = k$ and M is linear). For each $\alpha \in A$, there exists a meromorphic function f_α on U_α, such that $\hat{w}|U_\alpha = f_\alpha d\varphi_\alpha$. If $U_\alpha \cap U_\beta \neq \phi$ then

$$\hat{w}|U_\alpha = f_\alpha d\varphi_\alpha = f_\alpha a_{\alpha\beta} d\varphi_\beta = f_\beta d\varphi_\beta.$$

Therefore $f_\beta = a_{\alpha\beta} f_\alpha$, and so $\frac{df_\alpha}{d_\alpha} = \frac{df_\beta}{f_\beta}$ on $U_\alpha \cap U_\beta$. This implies that we can define a meromorphic closed 1-form $\hat{\theta}$ on $U = \bigcup_\alpha U_\alpha$ such that $\hat{\theta}|U_\alpha = df_\alpha | f_\alpha$ for each $\alpha \in A$. By Corollary 2.2.3 this form can be extended to a closed 1-form θ on $\mathbf{C}P^n$. With an argument similar to that we have done before in the proof of Theorem B, it can be proved that there exists a meromorphic function f on $\mathbf{C}P^n$ such that $df/f = \theta$. Clearly for each $\alpha \in A$, we have $f|U_\alpha = cf_\alpha$, c a constant. This implies that $d(\hat{w}/f) = 0$ and so $\hat{w}$ has an integrating factor and w also. This proves the lemma. ∎

Let A/B be the integrating factor of w. From Lemma 3 we have

$$\frac{Bw}{A} = \sum_{j=1}^{m} \lambda_j \frac{dg_j}{g_j} + d(\frac{\varphi}{\psi})$$

where $A = g_1^{k_1} \dots g_m^{k_m}$ is the decomposition of A in irreducible factors, $\psi = g_1^{\ell_1} \dots g_m^{\ell_m}$, $0 \le \ell_j \le k_j - 1$, $\sum_{j=1}^{m} \lambda_j dg(g_j) = 0$, φ and ψ have no common factors and $dg(\varphi) = dg(\psi)$. We can suppose also that for any $j \in \{1, \dots, m\}$ we have either $\ell_j \ne 0$ or $\lambda_j \ne 0$.

Let us consider the 1^{st} case in the proof of Lemma 7. In this case we can suppose that for any $\alpha \in A$:

$$\frac{Bw}{A}|\pi^{-1}(U_\alpha) = \pi^* \left(p\frac{dy_\alpha}{y_\alpha} - q\frac{dx_\alpha}{x_\alpha} \right) = p\frac{dy_\alpha^*}{y_\alpha^*} - q\frac{dx_\alpha^*}{x_\alpha^*}$$

On the other hand for $j \in \{1, \dots, m\}$ we have from Bézout's Theorem that $A_j = \{g_j = f = g = 0\} - \{0\} \ne \phi$ and for a point $p \in A_j \cap U_\alpha$ (for some α) we get

$$\sum_{i=1}^{m} \lambda_i \frac{dg_i}{g_i} + d(\frac{\varphi}{\psi}) = p\frac{dy_\alpha^*}{y_\alpha^*} - q\frac{dx_\alpha^*}{x_\alpha^*}.$$

This implies that either $\{g_j = 0\} \cap U_\alpha = \{x_\alpha^* = 0\}$ and $\lambda_i = -q$ or $\{g_j = 0\} \cap U_\alpha = \{y_\alpha^* = 0\}$ and $\lambda_j = p$. Since in the right member the poles are of order 1, we get also $\ell_j = 0$. Moreover $\{g_j = 0\} \cap \pi^{-1}(U)$ coincides with one of the divisors $\{x_\alpha^* = 0\}$ or $\{y_\alpha^* = 0\}$ $(U = \bigcup_\alpha U_\alpha)$. Therefore $m = 2$ and we can suppose that

$$\frac{Bw}{A} = p\frac{dg_1}{g_1} - q\frac{dg_2}{g_2} \text{ and } pdg(g_1) = qdg(g_2).$$

Observe that we have also $\{g_1 = g_2 = 0\} \supset \{f = g = 0\}$. From Noëther's Lemma (2.3.1), we get $g_1 = \alpha_1 f + \beta_1 g$ and $g_2 = \alpha_2 f + \beta_2 g$ where $\alpha_1, \dots, \beta_2$ are homogeneous polynomials. On the other hand, $\{g_1 = g_2 = 0\}$ is connected by Lefschetz's Theorem and $\{g_1 = g_2 = 0\} \subset S(\mathcal{F}^*)$ as we have seen after Lemma 3. This implies that $\{g_1 = g_2 = 0\} = \{f = g = 0\}$. In fact, $\{f = g = 0\}$ is an irreducible component of $\{g_1 = g_2 = 0\}$ and if it has another component, say N, then $N \cap \{f = g = 0\} - \{0\} \ne \phi$. Hence if $z \in N \cap \{f = g = 0\} - \{0\}$, then $dg_1(z) \wedge dg_2(z) = 0$. But this implies that $z \in \{f = g = 0\} - K(\mathcal{F}^*)$ which is not possible. Therefore $\{g_1 = g_2 = 0\} = \{f = g = 0\}$.

By Noëther's lemma the matrix $\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$ is invertible. Since $dg(f) < dg(g)$ this is possible only if β_1 and α_2 are constants and $\beta_2 \equiv 0$. Hence:

$$\frac{Bw}{A} = p\frac{dg_1}{g_1} - q\frac{df}{f} \Rightarrow \mathcal{F} \text{ is induced by } pf dg_1 - qg_1 df.$$

In the 2^{nd} case we can suppose that $\frac{Bw}{A} = \pi^*(\hat{w}/f)$. Therefore if $p \in (\{g_j = f = g = 0\} - \{0\}) \cap \pi^{-1}(U_\alpha)$, we have

$$\sum_{j=1}^{m} \lambda_j \frac{dg_j}{g_j} + d(\frac{\varphi}{\psi}) = a_\alpha d(\varphi_\alpha \circ \pi) = a_\alpha d(y_\alpha / x_\alpha^q)$$

where a_α is a constant. This implies that $\lambda_j = 0$, $j = 1, \ldots, m$. Moreover $\varphi/\psi | U_\alpha = a_\alpha y_\alpha / x_\alpha^q + b_\alpha \Rightarrow m = 1$, $\ell_1 = q$ and $Bw/A = d(\varphi/g_1^q)$. Now, let $\eta = \frac{g_1^q}{\varphi} d(\varphi/g_1^q) = \frac{d\varphi}{\varphi} - q\frac{dg_1}{g_1}$. If we apply the same argument as in the 1^{st} case for η we can conclude that $g_1 = \alpha_1 f$ and $\varphi = \alpha_2 f + \beta_2 g$, where α_1 and β_2 are constants. This proves Theorem A. ∎

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Astérisque

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Astérisque, tome 222 (1994), p. 135-199
<http://www.numdam.org/item?id=AST_1994__222__135_0>

COMPENSATION OF SMALL DENOMINATORS AND RAMIFIED LINEARISATION OF LOCAL OBJECTS

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Contents:

1. REMINDER ABOUT LOCAL OBJECTS. THE THREE MAIN ANALYTIC FACTS.

By *local objects* we will understand either *local analytic vector fields* (or *fields* for short) on $\mathbf{C}^\nu$ at 0 :

$$X = \sum_{i=1}^{\nu} X_i(x)\partial_{x_i} \qquad (X_i(x) \in \mathbf{C}\{x\};\ X_i(0) = 0) \tag{1.1}$$

or *local analytic selfmappings* (or *diffeos*, short for diffeomorphisms) of $\mathbf{C}^\nu$ with 0 as fixed point :

$$f : x_i \mapsto f_i(x) \qquad (i = 1, \ldots, \nu) \qquad (f_i(x) \in \mathbf{C}\{x\};\ f(0) = 0) \tag{1.2}$$

or again, equivalently, the related *substitution operators* :

$$F : \varphi \mapsto F.\varphi \stackrel{\text{def}}{=} \varphi \circ f \qquad (\varphi(x) \text{ and } \varphi \circ f(x) \in \mathbf{C}\{x\}) \tag{1.3}$$

Throughout, we will assume *diagonalisability* of the linear part and work with (analytic) *prepared forms* of the object on hand. That is to say, we will deal with vector fields given by :

$$X = X^{\text{lin}} + \sum \mathbb{B}_n \tag{1.4}$$

$$X^{\text{lin}} = \sum \lambda_i x_i \partial_{x_i} \tag{1.4'}$$

$$\mathbb{B}_n = \text{ homogeneous part of degree } n = (n_1, \ldots, n_\nu) \qquad \text{with } n_i \geq -1 \tag{1.4''}$$

and with diffeos given by :

$$F = \{1 + \sum \mathbb{B}_n\} F^{\text{lin}} \tag{1.5}$$

$$F^{\text{lin}} \varphi(x_1, \ldots, x_\nu) \stackrel{\text{def}}{=} \varphi(\ell_1 x_1, \ldots, \ell_\nu x_\nu) \tag{1.5'}$$

$$\mathbb{B}_n = \text{ homogeneous part of degree } n = (n_1, \ldots, n_\nu) \quad \text{with } n_i \geq -1 \tag{1.5''}$$

Of course, n-homogeneity means that for each monomial $x^m = x_1^{m_1} \ldots x_\nu^{m_\nu}$:

$$(1.6) \qquad \mathbf{B}_n.x^m = \beta_{n,m}x^{n+m} \qquad \text{with } \beta_{n,m} \in \mathbb{C}.$$

Note that, for any given $\mathbf{B}_n$, *at most* one component n_i may assume the value -1.

The scalars λ_i and ℓ_i are the object's *multipliers*. Together, they constitute its spectrum. If the spectrum is "random", the object turns out to be formally and even *analytically linearisable* (see below) and that about ends the matter, as far as the local study is concerned. For interesting problems to arise, at least one of three specific complications C_1, C_2, C_3 (see below) must come into play. Then numerous difficulties, mostly due to divergence, have to be sorted out. Yet the remarkable thing is that three easy, formal statements F_1, F_2, F_3 (F for formal) and three non-trivial, analytic theorems A_1, A_2, A_3 (A for analytic) suffice, between themselves, to give a fairly comprehensive picture of the whole situation. In this paper, we shall be mainly concerned with statement A_3 about the *effective, ramified linearisation of local objects*. Nonetheless, both for completeness and orientation, we shall begin with a brief review of all six statements. But first, we list the three "complications".

C_1. **Resonance.**

For a vector field, this means additive resonance of the λ_i :

$$(1.7) \qquad \sum_{i=1}^{\nu} m_i\,\lambda_i = 0 \qquad \text{or} \qquad (1.7') \qquad \sum_{i=1}^{\nu} m_i\lambda_i = \lambda_j \quad (m_i \in \mathbb{N})$$

and for a diffeo it means multiplicative resonance of the ℓ_i :

$$(1.8) \qquad \prod_{i=1}^{\nu} (\ell_i)^{m_i} = 1 \qquad \text{or} \qquad (1.8') \qquad \prod_{i=1}^{\nu} (\ell_i)^{m_i} = \ell_j \ (m_i \in \mathbb{N})$$

C_2. **Quasiresonance.**

This means that among all the non-vanishing expressions $\alpha(m) = < m, \lambda >$ or $\alpha(m) = \ell^m - 1$ (with all $m_i \geq 0$ except at most one that may be $\equiv -1$) there is a subinfinity that tends to 0 "abnormally" fast, thus violating the two equivalent diophantine conditions :

$$(1.9) \qquad S \stackrel{\text{def}}{=} \sum 2^{-k} \log(1/\varpi(2^k)) \ < +\infty \qquad \text{(A.D. Bruno)}$$

$$(1.9') \qquad S^* \stackrel{\text{def}}{=} \sum k^{-2} \log(1/\varpi(k)) \ < +\infty \qquad \text{(H. Rüssmann)}$$

with $\varpi(k) = \inf|\alpha(m)|$ for $m_1 + \ldots m_\nu \leq k$. (Clearly, $1/2 \leq S^*/S \leq 2$).

C_3. Nihilence.

It amounts to the existence of a "first integral" in the form of a power series H :

$$X.H(x) \equiv 0 \ \text{ or } \ H(f(x)) \equiv H(x) \ \text{ with } \ H(x) \in \mathbb{C}[[x]] = \mathbb{C}[[x_1, \ldots, x_\nu]] \tag{1.10}$$

along with the existence of small denominators :

$$\inf |\alpha(m)| = 0 \ \text{ for } \ \alpha(m) \neq 0. \tag{1.11}$$

Note that conditions (1.10) presupposes *resonance*, but that condition (1.11) differs from (1.9) in that it involves no arithmetical condition.

Resonance is fairly common, if only because it includes all diffeos tangent to the identity map, for which indeed $\ell_1 = \ell_2 = \ldots \ell_\nu = 1$. *Quasiresonance* is decidedly exceptional in single objects, but becomes inescapable when one studies parameter-dependent families of objects. *Nihilence* is common with volume-preserving or symplectic objects, where it may occur, respectively, from dimension 3 and 4 onwards.

All three complications may coexist. They may even occur in layers. Indeed, whenever the ordinary or first-level multipliers of an object X or F are involved in *multiple resonance*, there is a natural notion of reduced object X^{red} or F^{red} acting on the algebra of resonant monomials, and endowed with its own multipliers (second-level multipliers), which may in turn give rise to second-level resonance, quasiresonance or nihilence; and so forth. This daunting multiplicity of cases and subcases makes the existence of universally valid statements like A_1, A_2, A_3 (infra) all the more remarkable. But first let us go through the formal statements F_1, F_2, F_3 which, though fairly trivial, will clear the ground for A_1, A_2, A_3 and settle some useful terminology.

F_1. *In the absence of resonance, a local object is formally linearisable.*

The proof is straightforward. Indeed, inductive coefficient identification yields formal, entire changes of coordinates :

$$h^{\text{ent}} : x \mapsto y \quad \text{with} \quad y_i = h_i^{\text{ent}}(x) = x_i\{1 + \ldots\} \tag{1.12}$$

$$k^{\text{ent}} : y \mapsto x \quad \text{with} \quad x_i = k_i^{\text{ent}}(y) = y_i\{1 + \ldots\} \tag{1.13}$$

which take us from the given analytic chart $x = (x_i)$ to a formal chart $y = (y_i)$ where the object reduces to its linear part X^{lin} or F^{lin}. But to pave the way for the forthcoming analytic study, we require explicit expansions for h^{ent} and k^{ent}, or rather for the corresponding formal substitution operators Θ_{ent} and Θ_{ent}^{-1}. We use the variables x_i throughout :

$$\Theta_{\text{ent}}\ \varphi(x) \stackrel{\text{def}}{\equiv} \varphi \circ h^{\text{ent}}(x) \qquad (\varphi(x), \varphi \circ h^{\text{ent}}(x) \in \mathbb{C}[[x]]) \tag{1.14}$$

$$\Theta_{\mathrm{ent}}^{-1}\,\varphi(x) \overset{\mathrm{def}}{\equiv} \varphi \circ k^{\mathrm{ent}}(x) \qquad (\varphi(x), \varphi \circ k^{\mathrm{ent}}(x) \in \mathbf{C}[[x]]) \tag{1.15}$$

Written down in compact form, those expansions read, for a vector field :

$$\Theta_{\mathrm{ent}} = \sum S^{\bullet}\, \mathbf{B}_{\bullet} \overset{Def}{=} 1 + \sum_{r\geq 1} \sum_{\omega_i} S^{\omega_1,\ldots,\omega_r}\, \mathbf{B}_{\omega_1,\ldots,\omega_r} \tag{1.16}$$

$$\Theta_{\mathrm{ent}}^{-1} = \sum \mathscr{S}^{\bullet}\, \mathbf{B}_{\bullet} = 1 + \ldots \tag{1.17}$$

and for a diffeo :

$$\Theta_{\mathrm{ent}} = \sum \mathcal{S}^{\bullet}\, \mathbf{B}_{\bullet} \overset{Def}{=} 1 + \sum_{r\geq 1} \sum_{\omega_i} \mathcal{S}^{\omega_1,\ldots,\omega_r} \mathbf{B}_{\omega_1,\ldots,\omega_r} \tag{1.18}$$

$$\Theta_{\mathrm{ent}}^{-1} = \sum \mathfrak{S}^{\bullet}\, \mathbf{B}_{\bullet} = 1 + \ldots \tag{1.19}$$

There being no resonance, the cumbersome $n = (n_1, \ldots, n_\nu)$ indexation can be replaced by the handier $\omega = \sum n_i \lambda_i$ indexation (using any determination $\lambda_i = \log \ell_i$ for a diffeo) and each of the above expansions reduces to the contraction of a *mould* $M^{\bullet}$ with a *comould* $\mathbf{B}_{\bullet}$ (see §12). Here, the relevant comould $\mathbf{B}_{\bullet}$ is defined by :

$$\mathbf{B}_{\phi} = 1 \text{ and } \mathbf{B}_{\omega_1,\ldots,\omega_r} = \mathbf{B}_{\omega_r} \ldots \mathbf{B}_{\omega_1} \tag{1.20}$$

from the homogeneous parts $\mathbf{B}_n$ of the local object (see (1.4), (1.15)) after reindexing from n to ω. The relevant moulds involve only the spectrum of the object and are defined as follows for each sequence $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$:

$$S^{\phi} = \mathscr{S}^{\phi} = \mathcal{S}^{\phi} = \mathfrak{S}^{\phi} = 1 \qquad (\phi = \text{ empty sequence}) \tag{1.21}$$

$$S^{\boldsymbol{\omega}} = (-1)^r\,(\,\check{\omega}_1\,\check{\omega}_2 \ldots \check{\omega}_r)^{-1} \qquad \text{with} \quad \check{\omega}_i = \omega_1 + \ldots \omega_i \tag{1.22}$$

$$\mathscr{S}^{\boldsymbol{\omega}} = (\,\hat{\omega}_1\,\hat{\omega}_2 \ldots \hat{\omega}_r)^{-1} \qquad \text{with} \quad \hat{\omega}_i = \omega_i + \ldots \omega_r \tag{1.23}$$

$$\mathcal{S}^{\boldsymbol{\omega}} = (-1)^r e^{-\|\omega\|}\,(1 - e^{-\check{\omega}_1})^{-1} \ldots (1 - e^{-\check{\omega}_r})^{-1} \text{ with } \|\boldsymbol{\omega}\| = \omega_1 + \ldots \omega_r \tag{1.24}$$

$$\mathcal{S}^{\omega} = (e^{\hat{\omega}_1} - 1)^{-1} \dots (e^{\hat{\omega}_r} - 1)^{-1} \tag{1.25}$$

In the case of a field, the moulds $S^{\bullet}, \mathcal{S}^{\bullet}$ are *symmetral* and the comould $\mathbf{B}_{\bullet}$ is *cosymmetral* (see §12). In the case of a diffeo, the moulds $S^{\bullet}, \mathcal{S}^{\bullet}$ are *symmetrel* and the comould $\mathbf{B}_{\bullet}$ is *cosymmetrel* (see §12). Contraction of like-natured moulds and comoulds yields formal automorphisms :

$$\Theta^{\pm 1}_{\text{ent}}(\varphi.\psi) \equiv (\Theta^{\pm 1}_{\text{ent}}\varphi)\ (\Theta^{\pm 1}_{\text{ent}}\psi) \tag{1.26}$$

So the only thing left to check is that the substitution operators $\Theta^{\pm 1}_{\text{ent}}$ just defined do satisfy the linearisation identities :

$$X = \Theta X^{\text{lin}} \Theta^{-1} \tag{1.27}$$

$$F = \Theta X^{\text{lin}} \Theta^{-1} \tag{1.28}$$

This in turn readily follows from the obvious identities :

$$\|\omega\| S^{\omega} = -S^{\omega'} \tag{1.29}$$

$$\|\omega\|\, \mathcal{S}^{\omega} = +\mathcal{S}^{`\omega} \tag{1.30}$$

$$e^{\|\omega\|} S^{\omega} = S^{\omega} - S^{\omega'} + S^{\omega''} - S^{\omega'''} + \dots \tag{1.31}$$

$$e^{\|\omega\|}\, \mathcal{S}^{\omega} = \mathcal{S}^{\omega} + \mathcal{S}^{`\omega} \tag{1.32}$$

with $\omega = (\omega_1, \dots, \omega_r)$; $\|\omega\| = \omega_1 + \dots \omega_r$; $\omega' = (\omega_1, \dots, \omega_{r-1})$; $`\omega = (\omega_2, \dots, \omega_r)$.

$\mathbf{F_2}$. ***In the presence of resonance, there are formal changes of coordinates which bring the local object to a prenormal form (containing only resonant monomials) and even to a normal form (containing a "minimum number" of resonant monomials, with in front of them scalar coefficients which are formal invariants of the object).***

A *resonant monomial* is of couse any $x^m = x_1^{m_1} \dots x_\nu^{m_\nu}$ such that $< m, \lambda >= 0$ for a field or $\ell^m = 1$ for a diffeo. This case being of secondary concern to us here, we refer to [E.2] [E.3] [E.7] and also to §10 *infra*, where we shall investigate the overlap between resonance and ramified linearisation. We may note in passing the rule of thumb : one (resp. several) degree of resonance implies the existence of finitely (resp. infinitely) many independent formal invariants.

F_3. ***Any non-degenerate local object (even if resonant) can be linearised by means of a formal-ramified change of coordinates.***

We now *ramify* the coordinates x_i (relative to a *prepared form* (1.4) or (1.5)), i.e. we regard them as elements of $\underset{\bullet}{\mathbf{C}} = \widetilde{\mathbf{C}\backslash\{0\}}$.

Non-degenerate means $\lambda_i \neq 0$ for a field and $\underset{\bullet}{\ell_i} \neq \underset{\bullet}{0}, \underset{\bullet}{1}$ for a diffeo ($\underset{\bullet}{\ell_i}, \underset{\bullet}{0}, \underset{\bullet}{1} \in \underset{\bullet}{\mathbf{C}}$).

A *formal-ramified change of coordinates* is any formal transformation of the form :

$$y_i = x_i\{1 + \tilde{\varphi}_i(x)\} \tag{1.33}$$

with formal series $\tilde{\varphi}_i(x)$ without constant term and of type :

$$\sum a_{\sigma_1,\ldots,\sigma_\nu} x_1^{\sigma_1} \ldots x_\nu^{\sigma_\nu} \qquad (a_\sigma \in \mathbf{C}) \tag{1.34}$$

or of type

$$\sum a_{\sigma_1,\ldots,\sigma_\nu;n_1,\ldots,n_\nu} x_1^{\sigma_1} \ldots x_\nu^{\sigma_\nu} (\log x_1)^{n_1} \ldots (\log x_\nu)^{n_\nu} \qquad (a_{\sigma,n} \in \mathbf{C}) \tag{1.34'}$$

with real positive powers σ_i ranging over some discrete subset of $(\mathbf{R}^+)^\nu$ and integers n_i constrained by :

$$\limsup\ (n_1 + \ldots n_\nu)/(\sigma_1 + \ldots \sigma_\nu) < +\infty \tag{1.35}$$

or by the stronger conditions :

$$\limsup\ n_i/\sigma_i < +\infty \qquad (\text{for } i = 1, \ldots, \nu) \tag{1.36}$$

Lastly, the transformation of ℓ_i into $\underset{\bullet}{\ell_i}$ means that, for a diffeo, we may replace each multiplier $\ell_i \in \mathbf{C}$ by some $\underset{\bullet}{\ell_i} \in \underset{\bullet}{\mathbf{C}}$ that lies over ℓ_i, subject only to $\underset{\bullet}{\ell_i} \neq \underset{\bullet}{0}, \underset{\bullet}{1}$. For this to be possible, ℓ_i must be $\neq 0$, but it may well be $= 1$. Thus, *non-degeneracy for a diffeo is not the "exponential" of non-degeneracy for a field.*

Strictly speaking, formal-ramified linearisability has to be established for resonant (or nihilent) objects only, because otherwise entire linearisation is available by proposition F_1. For explicit formulae relative to resonant objects, see §10 infra. But we will use ramified linearisation *also*, and even *mostly*, in the absence of resonance, to overcome the divergence caused by quasiresonance.

Now, let us come to the real thing - the three analytic statements A_1, A_2, A_3.

A_1. ***The Siegel-Bruno-Rüssmann theorem about the innocuousness of diophantine small denominators : Any non-resonant and non-quasiresonant local object (i.e. any object free of C_1, C_2, C_3) can be analytically linearised.***

The first result in this direction was obtained by Siegel, but under an unnecessarily strong diophantine condition. The proof of A_1, under the probably optimal diophantine condition (1.9) (see §10) is due to A.D. Bruno [B] for fields and to H. Rüssmann [R] for diffeos. In the introductory paras of [E.8], we gave an essentially similar proof, but in a concise form which easily extends to a host of other situations. We must here give a sketch of this method, because we will require it in the sequel.

It is based on the process of ***arborification-coarborification***, expounded in [E.8], and recalled in §12. The idea is to replace contracted sums of type $\sum M^{\bullet}\mathbf{B}_{\bullet}$ by formally equivalent sums $\sum M^{\overset{<}{\bullet}}\mathbf{B}_{\underset{\bullet}{<}}$, using the dual operations of arborification :

$$M^{\overset{<}{\boldsymbol{\omega}}} = \sum \mathrm{sh}\begin{pmatrix} \overset{<}{\boldsymbol{\omega}} \\ \boldsymbol{\omega} \end{pmatrix} M^{\boldsymbol{\omega}} \text{ (fields) or} \sum \mathrm{ctsh}\begin{pmatrix} \overset{<}{\boldsymbol{\omega}} \\ \boldsymbol{\omega} \end{pmatrix} M^{\boldsymbol{\omega}} \text{ (diffeos)} \tag{1.37}$$

and coarborification :

$$\mathbf{B}_{\boldsymbol{\omega}} = \sum \mathrm{sh}\begin{pmatrix} \overset{<}{\boldsymbol{\omega}} \\ \boldsymbol{\omega} \end{pmatrix} \mathbf{B}_{\overset{<}{\boldsymbol{\omega}}} \text{ (fields) or} \sum \mathrm{ctsh}\begin{pmatrix} \overset{<}{\boldsymbol{\omega}} \\ \boldsymbol{\omega} \end{pmatrix} \mathbf{B}_{\overset{<}{\boldsymbol{\omega}}} \text{ (diffeos)} \tag{1.38}$$

Here, the boldfaced symbols $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ (resp. $\overset{<}{\boldsymbol{\omega}} = (\omega_1, \ldots, \omega_r)^{<}$) denote sequences of ω_i with a ***full*** (resp. ***arborescent***) order on them, and the coefficients $\mathrm{sh}\begin{pmatrix} \overset{<}{\boldsymbol{\omega}} \\ \boldsymbol{\omega} \end{pmatrix}$ (resp. $\mathrm{ctsh}\begin{pmatrix} \overset{<}{\boldsymbol{\omega}} \\ \boldsymbol{\omega} \end{pmatrix}$) are equal to the number of order-preserving ***bijections*** (resp. ***surjections***) of $\overset{<}{\boldsymbol{\omega}}$ into $\boldsymbol{\omega}$. For details, see [E.8] and §12 infra.

Each sum (1.37) or (1.38) contains many terms - close to $r!$ on average. It is predictable therefore, and readily verified, that coarborification $\mathbf{B}_{\boldsymbol{\omega}} \mapsto \mathbf{B}_{\overset{<}{\boldsymbol{\omega}}}$ should entail a drastic ***reduction*** of norms :

$$\|\mathbf{B}_{\boldsymbol{\omega}}\|_{\mathcal{D},\mathcal{D}'} < C^r.r! \qquad (C = C_{\mathcal{D},\mathcal{D}'} = \mathrm{Cste}) \tag{1.39}$$

$$\|\mathbf{B}_{\overset{<}{\boldsymbol{\omega}}}\|_{\mathcal{D},\mathcal{D}'} < C^r \tag{1.40}$$

with norms relative to two bounded neighbourhoods $\mathcal{D}' \subset \mathcal{D}$ of 0 in $\mathbf{C}^\nu$:

$$\|\varphi\|_{\mathcal{D}} = \sup_x \varphi(x) \quad \text{for } x \in \mathcal{D} \qquad (\varphi \in \mathbf{C}\{x\}) \tag{1.41}$$

$$\|\mathbf{B}\|_{\mathcal{D},\mathcal{D}'} = \sup_\varphi \|\mathbf{B}\varphi\|_{\mathcal{D}'} \quad \text{for } \|\varphi\|_{\mathcal{D}} \leq 1 \qquad (\mathbf{B} \in \mathrm{End}\,\mathbf{C}\{x\}) \tag{1.42}$$

Moreover, for any $\epsilon > 0$, one may chose $\mathcal{D}, \mathcal{D}'$ so that $C \leq \epsilon$.

By the same token, one would expect arborification $M^{\boldsymbol{\omega}} \mapsto M^{\overset{<}{\boldsymbol{\omega}}}$ to bring about a corresponding *increase* in norm. Fortunately, however, this is seldom the case, due to massive cancellations in the sums (1.37). In the present case, for instance, we have the easy identities :

$$\mathcal{S}^{\overset{<}{\boldsymbol{\omega}}} = \prod_i (\hat{\omega}_i)^{-1} \tag{1.43}$$

$$\mathscr{S}^{\overset{<}{\boldsymbol{\omega}}} = \prod_i (e^{\hat{\omega}_i} - 1)^{-1} \tag{1.44}$$

the only difference with (1.23) and (1.25) being that now each sum $\hat{\omega}_i = \sum \omega_j$ extends to all j posterior (or equal) to i *relative to the arborescent order* of $\overset{<}{\omega}$. The bounds for $\mathcal{S}^{\overset{<}{\boldsymbol{\omega}}}$ and $\mathscr{S}^{\overset{<}{\boldsymbol{\omega}}}$ are therefore almost as good as those for $\mathcal{S}^{\boldsymbol{\omega}}$ and $\mathscr{S}^{\boldsymbol{\omega}}$. Indeed, under Bruno's diophantine condition (1.9) we have :

$$|\mathcal{S}^{\boldsymbol{\omega}}| \leq C^r \quad ; \quad |\mathscr{S}^{\boldsymbol{\omega}}| \leq C^r \tag{1.45}$$

$$|\mathcal{S}^{\overset{<}{\boldsymbol{\omega}}}| \leq C^r \quad ; \quad |\mathscr{S}^{\overset{<}{\boldsymbol{\omega}}}| \leq C^r \tag{1.46}$$

$\omega = (\omega_1, \ldots, \omega_r)$ and a constant C depending on the spectrum λ or ℓ. The inequalities (1.46) are implicit in Bruno's paper [B]. The inequalities (1.45) are easy and could be had under *slightly weaker* diophantine conditions than (1.9). See also [Y].

Pairing the estimates (1.40) and (1.46) and using (12.22), we immediately deduce the normal convergence of the arborified expansions :

$$\Theta_{\text{ser}}^{-1} = \sum \mathcal{S}^{\overset{<}{\bullet}}\, \mathbf{B}_{\underset{\bullet}{<}} \ \text{(field)} \quad \text{or} \quad \sum \mathscr{S}^{\overset{<}{\bullet}}\, \mathbf{B}_{\underset{\bullet}{<}} \ \text{(diffeo)} \tag{1.47}$$

In contrast, the original, non-arborified expansions (1.17) and (1.19) do not converge in norm, at least in the presence of small denominators.

A$_2$. ***Theorem about the resurgent normalisation of resonant local objects : For resonant local objects, the normalising change of coordinates is usually divergent, but resurgent with respect to one or several "critical times" z. Moreover, the Bridge Equation holds :***

$$\dot{\Delta}_\omega\, x(z,u) = \mathbb{A}_\omega\, x(z,u) \tag{1.48}$$

with on the right-hand side ordinary differential operators $\mathbb{A}_\omega$ which, taken together, constitute a complete system of holomorphic invariants of the object. Lastly, the $x(z,u)$ are usually resummable, but only in sectors of the (z,u) space.

Since this whole topic of resonance-cum-resurgence has been dealt with at lenght in [E.2] and [E.3], and is only of incidental relevance to the present study (see §§9 and 10), we will limit ourselves to a few cursory indications. The ingredients of the Bridge Equation are three. First, a so-called *formal integral* :

$$x(z,u) = \{x_1(z,u_1,\ldots,u_{\nu-1}),\ldots,x_\nu(z,u_1,\ldots,u_{\nu-1})\} \tag{1.49}$$

which is a general (i.e. parameter-saturated) formal solution of the differential system associated with a field X

$$\partial_z\ x_i(z,u) = X_i(x(z,u)) \qquad (\partial_z = \frac{\partial}{\partial z};\ i = 1,\ldots,\nu) \tag{1.50}$$

or of the difference system associated with a diffeo f :

$$x_i(z+1,u) = f_i(x(z,u)) \qquad (i = 1,\ldots,\nu) \tag{1.51}$$

It is thus a formal, non-entire chart $(z,u_1,\ldots,u_{\nu-1})$ in which the object assumes the simplest conceivable form, namely :

$$X = \frac{\partial}{\partial z} \qquad \text{or} \qquad f : z \mapsto z+1 \tag{1.52}$$

The precise shape of $x(z,u)$ varies considerably from case to case. For a simple example, see below in §10. Many more are given in [E.2] [E.3] [E.5] [E.6] [E.7] [E.8].

The symbols $\overset{\bullet}{\Delta}_\omega$ on the left-hand side of (1.48) denote pointed *alien derivations.* (See §12). Their indexes ω range over an enumerable, and usually discrete, subset of $\underset{\bullet}{\mathbb{C}}$.

Lastly, the $\mathbb{A}_\omega$ on the right-hand side of (1.48) are ordinary differential operators in z and u, subject to no other a priori constraints than :

$$[\mathbb{A}_\omega, \partial] = 0 \quad \text{for a field} \qquad (\partial = \frac{\partial}{\partial z}) \tag{1.53}$$

$$[\mathbb{A}_\omega, \exp\partial] = 0 \quad \text{for a diffeo} \qquad (\exp\partial = \ \text{translation of step 1.}) \tag{1.54}$$

The $\mathbb{A}_\omega$ are both *analytic invariants* (an analytic change of coordinates leaves them unchanged) and *holomorphic invariants* (they are holomorphic functions of the object, when the latter ranges over a given formal class). For a full treatment, see [E.3] [E.8] and for the overlap with ramified linearisation, see [E.8] chap. 3 and §10 infra.

A$_3$. *Any non-degenerate local object (whether or not affected by C_1, C_2, C_3) can be linearised by means of a seriable-ramified change of coordinates.*

Non-degeneracy, we recall, is a very mild requirement, meaning simply $\lambda_i \neq 0$ for a field and $\underset{\bullet}{\ell}_i \underset{\bullet}{\neq} \underset{\bullet}{0}, 1$ for a diffeo.

For the precise definition and properties of *seriability*, we will have to wait until §§2,3 but for the moment we may say that a seriable-ramified change of coordinates is a formal change of type (1.34) or (1.34′) with condition (1.35) or its strengthened variant (1.36), which furthermore can be resummed in a *unique way*, by means of a suitable Borel-Laplace procedure, to a sum that is defined, analytic in a ramified, spiralling neighbourhood of the origin, i.e. in a neighbourhood, not of $0 \in \mathbf{C}^\nu$, but of $\underset{\bullet}{0} \in \underset{\bullet}{\mathbf{C}}^\nu$. These linearisation domains are optimally large, and seriable-ramified linearisation, while less simple than analytic linearisation, retains the latter's essential feature, which is "formalisability", namely the quality for a function germ $\varphi(x)$ of being totally and constructively reducible to a formal object $\tilde{\varphi}(x)$.(*)

Before concluding this introductory section, let us introduce the idea of *compensation*, on which seriable-ramified linearisation rests. Assume non-resonance for a start. Then by statement F_1, the local object X or F can be reduced, by means of a formal, entire change of coordinates Θ_{ent}, to its linear part X^{lin} or F^{lin}. But if we allow for ramifications, we find a huge group INV of formal-ramified changes of coordinates Θ_{inv}, which involve only *resonant monomials* $y^\sigma = y_1^{\sigma_1} \dots y_\nu^{\sigma_\nu}$ and leave the linear part of the object invariant :

$$(1.55) \qquad y_i \mapsto y_i^* = y_i\{1 + \sum a_\sigma y^\sigma\} \text{ with } <\sigma, \lambda> = \sum \sigma_i \lambda_i = 0 \quad (\sigma_i \in \mathbf{R}^+)$$

The idea therefore is to look for ramified transformations of the form :

$$(1.56) \qquad \Theta_{\text{ser}} = \Theta_{\text{ent}} \Theta_{\text{inv}}^{-1} \; ; \; \Theta_{\text{ser}}^{-1} = \Theta_{\text{inv}} \Theta_{\text{ent}}^{-1}$$

(with *ser, ent, inv* for *seriable, entire, invariant) where the divergence present in* Θ_{ent} *(due to quasiresonance or nihilence) is offset by a similar divergence in* Θ_{inv}.

For instance, if we are dealing with a local diffeo of $\mathbf{C}^1$ of type :

$$(1.57) \qquad f : x \mapsto x\{\ell + \sum a_n x^n\} \qquad (\ell = e^{2\pi i \lambda^*}; \lambda^* > 0)$$

and if λ^* is strongly liouvillian, we are faced with complication C_1 (quasiresonance) and the entire linearisation Θ_{ent} usually involves an infinity of monomials $c_n x^n$ with nearly integral $n\lambda^*$ and very large coefficients c_n :

$$(1.58) \qquad c_n = a_n(\ell^n - 1)^{-1} + \dots ; \; n\lambda^* = m + \epsilon_n ; \; \epsilon_n \text{ exceptionnally small}$$

which hopefully may be neutralised by a ramified transformation Θ_{inv} involving monomials of type $c_n x^{m/\lambda^*}$.

(*) This is sometimes referred to as *germ quasianalyticity*, but the meaning of this term has been so over-stretched that we prefer to keep it for *function quasianalyticity* of Denjoy-Carleman or related types.

The procedure is not different in the case of ***resonant objects***, with factorisation (1.56) replaced by (10.2) or (10.3).

Now it so happens that $\Theta_{\mathrm{ser}}^{-1}$ is simpler to calculate than Θ_{ser}. So we will look for explicit expansions :

$$\Theta_{\mathrm{ser}}^{-1} = \sum \mathcal{S}_{\mathrm{co}}^{\bullet}(x)\mathbf{B}_{\bullet} \qquad \text{(fields)} \tag{1.59}$$

$$\Theta_{\mathrm{ser}}^{-1} = \sum \mathscr{S}_{\mathrm{co}}^{\bullet}(x)\mathbf{B}_{\bullet} \qquad \text{(diffeos)} \tag{1.60}$$

which differ from (1.17) and (1.19) only in that the moulds $M^{\bullet}(x)$ being contracted with the comould $\mathbf{B}_{\bullet}$ are no longer constant, but functions $\mathcal{S}_{\mathrm{co}}^{\bullet}(x)$ or $\mathscr{S}_{\mathrm{co}}^{\bullet}(x)$ of $x = (x_1, \dots, x_\nu)$ that must meet three main requirements :

First, they must be ***compensators*** in x, i.e. sums of monomials which remain small though their coefficients may be very large.

Second, they must satisfy the following equations :

$$(\|\boldsymbol{\omega}\| + X^{\mathrm{lin}})\, \mathcal{S}_{\mathrm{co}}^{\boldsymbol{\omega}}(x) = \mathcal{S}_{\mathrm{co}}^{\grave{\boldsymbol{\omega}}}(x) \qquad \text{(fields)} \tag{1.61}$$

$$(e^{\|\boldsymbol{\omega}\|} F^{\mathrm{lin}})\ \mathscr{S}_{\mathrm{co}}^{\boldsymbol{\omega}}(x) = \mathscr{S}_{\mathrm{co}}^{\grave{\boldsymbol{\omega}}}(x) + \mathscr{S}_{\mathrm{co}}^{\boldsymbol{\omega}}(x) \qquad \text{(diffeo)} \tag{1.62}$$

(with $\boldsymbol{\omega}$, $\|\boldsymbol{\omega}\|$, $\grave{\boldsymbol{\omega}}$ as in (1.32)) which ensure that the operators $\Theta_{\mathrm{ser}}^{-1}$ defined by (1.59), (1.60) are indeed solutions of the linearisation equations (1.27), (1.28).

Third, they have to be ***symmetral*** (for fields) or ***symmetrel*** (for diffeos) ***functions*** of the sequence $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$, to ensure that the operators $\Theta_{\mathrm{ser}}^{-1}$ be formal automorphisms (like $\Theta_{\mathrm{ent}}^{-1}$ in (1.26)). However, we don't have to worry about this last requirement, since it is an automatic consequence of equations (1.61), (1.62).

In the present paper, we shall prove the theorem about seriable-ramified linearisability, not for all local objects, but only for an important subclass : the ***girators*** (i.e. vector fields with purely imaginary eigenvalues λ_i) and ***girations*** (i.e. diffeos with eigenvalues ℓ_i of modulus 1). The reasons for this restriction are two :

Reason one : Linearisation is specially relevant for girators or girations because, in the presence of analytic, entire linearisation, we have in the y chart continuous or discrete orbits :

$$y_i(t) = y_i(0)\ e^{\lambda_i t} \qquad (t \in \mathbf{R};\ \text{for girators}) \tag{1.63}$$

$$y_i(n) = y_i(0)\ \ell_i^n \qquad (n \in \mathbf{Z};\ \text{for girations}) \tag{1.64}$$

which "girate" indefinitely at a fixed distance from the origin. In the absence of analytic linearisability, the dynamics become much more complicated, but ramified linearisation sheds some light on it.

Reason two : Only for girators and girations may one find simple and nearly "canonical" expansions of type (1.59) or (1.60) with truly explicit moulds $\mathcal{S}^{\omega}_{\text{co}}(x)$ of $\mathcal{S}^{\omega}_{\text{co}}(x)$. For other objects, such expansions still exist, but the construction is more arduous and involves a far higher degree of arbitrariness, so that we must postpone it to a future paper.

The main results are stated and proved in §§5,6,7,8,10. The next three paras (§§2,3,4) introduce the necessary machinery of ***compensation*** and ***seriability*** and §9 investigates the link with resurgence.

2. SYMMETRIC COMPENSATORS.

Compensators are usually finite sums $\sum a_i z^{\sigma_i}$ which, due to internal cancellations, remain small even though the a_i may be large. The most basic ones are the *symmetric compensators* :

2.1 Definition : *symmetric compensator of order r*

(2.2) $$z^{\sigma_0,\sigma_1,\ldots,\sigma_r} \stackrel{\text{def}}{=} z^{\sigma_0} \star z^{\sigma_1} \star \ldots z^{\sigma_r} \qquad (z \in \underset{\bullet}{\mathbb{C}}; \sigma_0, \sigma_1, \ldots, \sigma_r \in \mathbb{R}^+)$$

Here, $\star$ denotes the multiplicative convolution of holomorphic functions on $\underset{\bullet}{\mathbb{C}}$, defined as follows :

(2.3) $$\varphi_1 \star \varphi_2(z) = \int_{\underset{\bullet}{1}}^{z} \varphi_1(z_1)\varphi_2(z/z_1)\frac{dz_1}{z_1}$$

for z close to $\underset{\bullet}{1}$ and, in the large, by analytic continuation. Convolution being commutative, $z^{\sigma_0,\ldots,\sigma_r}$ is a symmetric function of the σ_i, uniform for z ranging over $\underset{\bullet}{\mathbb{C}}$. For distinct σ_i we have :

(2.4) $$z^{\sigma_0,\ldots,\sigma_r} = \sum_{0\leq i\leq r} z^{\sigma_i} \prod_{j\neq i} (\sigma_i - \sigma_j)^{-1}$$

and in case of repetitions :

(2.5) $$z^{\sigma_0^{(1+n_0)},\ldots,\sigma_r^{(1+n_r)}} = (n_0!\ldots n_r!)^{-1}(\partial_{\sigma_0})^{n_0}\ldots(\partial_{\sigma_r})^{n_r} z^{\sigma_0,\ldots,\sigma_r}$$

with $\sigma_i \neq \sigma_j$ for $i \neq j$; $\partial_{\sigma_i} = \partial/\partial\sigma_i$; and with $\sigma_i^{(1+n_i)}$ denoting a subsequence of $1+n_i$ identical σ_i. We have in particular :

$$(2.5') \qquad z^{\sigma_0,\ldots,\sigma_r} = (1/r!)z^{\sigma}(\log z)^r \;\text{ if }\; \sigma_0 = \sigma_1 = \ldots \sigma_r = \sigma$$

Compensators multiply according to the rule :

$$(2.6) \qquad z^{\sigma_0,\ldots,\sigma_r}.z^{\tau_0,\ldots,\tau_s} = \sum z^{\sigma_{i_0}+\tau_{j_0},\ldots,\sigma_{i_{r+s}}+\tau_{j_{r+s}}}$$

with a sum extending to all (i_n, j_n) such that :

$$(2.6') \qquad i_n + j_n = n \;\text{ for }\; n = 0, 1, \ldots, r+s$$

$$(2.6'') \qquad 0 = i_0 \leq i_1 \leq \ldots \leq i_{r+s} = r$$

$$(2.6''') \qquad 0 = j_0 \leq j_1 \leq \ldots \leq j_{r+s} = s$$

The quickest derivation of (2.6) is by formula (4.10) which relates *symmetric* and *symmetral* compensators, the latter obeying the universal multiplication rule (12.4). That same formula (4.10), combined with the integral representation (4.8) of symmetral compensators, also yields the following useful bounds :

$$(2.7) \qquad |z^{\sigma_0,\ldots,\sigma_r}| \leq (1/r!)\; |\log z|^r |z|^{\sigma_*}$$

$$(2.8) \qquad \left|\frac{z^{\sigma_0,\ldots,\sigma_r}}{x_0^{\sigma_1,\ldots,\sigma_r}}\right| \leq \left|\frac{\log z}{\log x_0}\right|^r \left|\frac{z}{x_0}\right|^{\sigma_*}$$

valid for any $z \in \underset{\bullet}{\mathbf{C}}$ with $|z| < x_0 < 1$, and $\sigma_i > 0$, $\sigma_* = \inf \sigma_i$. Moreover, for $0 < x < 1$:

$$(2.9) \qquad (-1)^r x^{\sigma_0,\ldots,\sigma_r} > 0$$

3. COMPENSATION AND SERIATION.

We are going to introduce germs of functions which, though defined on ramified neighbourhoods of $\underset{\bullet}{0} \in \underset{\bullet}{\mathbf{C}}$ and having generically divergent asymptotic series, yet retain most of the regularity proper to holomorphic germs at $0 \in \mathbf{C}$.

Fundamental spiralling domains.

For each real pair x_0, κ_0 such that :

$$(3.1) \qquad 0 < x_0 < e^{-\kappa_0} < 1$$

let $\mathcal{D} = \mathcal{D}_{x_0,\kappa_0}$ denote the connected part of $\underset{\bullet}{\mathbb{C}}$ defined by :

$$\mathcal{D}_{x_0,\kappa_0} = \{z; |z||\log z|^{\kappa_0} \leq x_0|\log x_0|^{\kappa_0}\} \tag{3.2}$$

x_0 is the "radius" of $\mathcal{D}$ and κ_0 measures the speed at which its boundary $\partial\mathcal{D}$ coils in towards $\underset{\bullet}{0}$. The smaller κ_0, the more $\mathcal{D}$ resembles the covering space of a punctured disc.

The norms $\|\bullet\|^{\mathcal{D}}$ and $\|\bullet\|^{\mathcal{D}}_{\text{comp}}$ on ramified polynomials.

A ramified polynomial is a finite sum of the form :

$$\varphi(z) = \sum a_{\sigma,r}\, z^{\sigma}(\log z)^{r} \qquad (\sigma \text{ real } > 0;\ r \text{ integral } \geq 0) \tag{3.3}$$

On the algebra spanned by such polynomials we introduce the usual *uniform norms* :

$$\|\varphi\|^{\mathcal{D}} \overset{\text{def}}{=} \sup_{z\in\mathcal{D}} |\varphi(z)| \quad (\|\varphi\|^{\mathcal{D}} < +\infty \text{ iff } \sup(r/\sigma) \leq \kappa_0) \tag{3.4}$$

as well as the *compensation norms* defined by :

$$\|\varphi\|^{\mathcal{D}}_{\text{comp}} \overset{\text{def}}{=} \inf\left\{\sum |A_{\boldsymbol{\sigma}}|.\|z^{\boldsymbol{\sigma}}\|^{\mathcal{D}}\right\} \leq \|\varphi\|^{\mathcal{D}} \tag{3.5}$$

with an inf relative to all possible finite decompositions of $\varphi(z)$:

$$\varphi(z) \equiv \sum A_{\boldsymbol{\sigma}} z^{\boldsymbol{\sigma}} = \sum A_{\sigma_0,\ldots,\sigma_r} z^{\sigma_0,\ldots,\sigma_r} \tag{3.5$'$}$$

into sums of compensators of arbitrary order r, but with $r \leq \kappa_0\sigma_i$.

The norms $\|\bullet\|^{\mathcal{D}}$ are obviously multiplicative, but so are the norms $\|\bullet\|^{\mathcal{D}}_{\text{comp}}$:

$$\|\varphi_1\varphi_2\|^{\mathcal{D}}_{\text{comp}} \leq \|\varphi_1\|^{\mathcal{D}}_{\text{comp}}\|\varphi_2\|^{\mathcal{D}}_{\text{comp}} \tag{3.6}$$

because for any fundamental domain $\mathcal{D} = \mathcal{D}_{x_0,\kappa_0}$:

$$\|z^{\boldsymbol{\sigma}}\|^{\mathcal{D}} = |x_0^{\boldsymbol{\sigma}}| \qquad \text{(by (2.8)) if } r \leq \kappa_0\sigma_i. \tag{3.7}$$

$$|x_0^{\boldsymbol{\sigma}}.x_0^{\boldsymbol{\tau}}| = |x_0^{\boldsymbol{\sigma}}|\ |x_0^{\boldsymbol{\tau}}| \qquad \text{(by (2.6) and (2.9))} \tag{3.8}$$

The latter identity stems from the fact that for $0 < z < 1$ all terms on the right-hand side of (2.6) have the same sign, namely $(-1)^{r+s}$.

The algebras of seriable or compensable functions.

For any discrete, additive semigroup $T \in \mathbf{R}^+$ and any fundamental spiralling domain $\mathcal{D}$, we denote by $\mathrm{Ser}^{T,\mathcal{D}}(z)$ and $\mathrm{Comp}^{T,\mathcal{D}}(z)$ the closure under $\| \bullet \|^{\mathcal{D}}$ and $\| \bullet \|^{\mathcal{D}}_{\mathrm{comp}}$ of the algebra of ramified polynomials (3.3) with coefficients σ in T, and we put :

(3.9) $$\mathrm{Ser}(z) \stackrel{\mathrm{def}}{=} \bigcup_{T,\mathcal{D}} \mathrm{Ser}^{T,\mathcal{D}}(z)$$

(3.10) $$\mathrm{Comp}(z) \stackrel{\mathrm{def}}{=} \bigcup_{T,\mathcal{D}} \mathrm{Comp}^{T,\mathcal{D}}(z)$$

Clearly $\mathrm{Ser}(z)$ and $\mathrm{Comp}(z)$ are both seminormed algebras. Their elements are known as *seriable* and *compensable* (germs of) functions respectively. $\mathrm{Ser}(z)$ contains $\mathrm{Comp}(z)$ but it is unclear to us whether the inclusion is strict.

3.11 Proposition : Asymptotic expansions

Each seriable or compensable (germ of) function $\varphi(z)$ admits a unique asymptotic expansion of the form :

(3.12) $$\tilde{\varphi}(z) = \sum a_{\sigma,r}\, z^{\sigma}(\log z)^r \quad \text{with } r/\sigma \leq \kappa$$

for some finite κ and z going radially to $\underset{\bullet}{0}$. Although the coefficients admit universal bounds in terms of T and $\mathcal{D}$:

(3.13) $$|a_{\sigma,r}| < C^{T,\mathcal{D}}_{\sigma,r}.\|\varphi\|^{\mathcal{D}}$$

(3.14) $$|a_{\sigma,r}| < D^{T,\mathcal{D}}_{\sigma,r}.\|\varphi\|^{\mathcal{D}}_{\mathrm{comp}}$$

the series $\tilde{\varphi}(z)$ is usually divergent.

3.15 Proposition : Resummation of $\tilde{\varphi}$.

Each $\tilde{\varphi}(z)$ is resummable to $\varphi(z)$ by an appropriate Borel-Laplace procedure :

(3.16) $$\tilde{\varphi}(z) \stackrel{1}{\mapsto} \tilde{\varphi}(e^{-t}t^{-\rho}) = \tilde{\psi}(t) \stackrel{2}{\mapsto} \hat{\psi}(\tau) \stackrel{3}{\mapsto} \psi(t) \stackrel{4}{\mapsto} \varphi(z)$$

for $\rho > 0$ large enough.

Proof of Propositions 3.11 and (3.15).

The arrows 1 and 4 in (3.16) simply denote the change of variable $z = e^{-t}t^{-\rho}$. Arrow 2 denotes the *formal* or *term-wise* Borel transform (12.29) and arrow 3 the Laplace transform (12.28), which reverses it. Under arrow 1 the monomials of (3.12) become :

(3.17) $$P_{\sigma,r}(z) = z^{\sigma}(\log z)^r \stackrel{1}{\mapsto} Q_{\sigma,r}(t) = (-1)^r e^{-\sigma t} t^{r-\sigma\rho}(1 + \rho t^{-1}\log t)^r$$

and under arrow 2 they become functions on $\mathbf{R}^+$:

$$\hat{Q}_{\sigma,r}(\tau) = \begin{cases} 0 \text{ if } \tau \leq \sigma \\ (-1)^r \frac{(\tau-\sigma)^{\sigma\rho-r-1}}{\Gamma(\sigma\rho-r)}. \{1+\ldots\} \text{ if } \tau > \sigma \end{cases} \tag{3.18}$$

which can be rendered as smooth as one wishes (at the singularity $\tau = \sigma$) by choosing ρ large enough. Finally, arrows 3 and 4 take $\hat{Q}_{\sigma,r}(\tau)$ back to $Q_{\sigma,r}(t)$ and $P_{\sigma,r}(z)$ Now, for t_0 and ρ large enough, any given $\mathcal{D} = \mathcal{D}_{x_0,\kappa_0}$ contains the image of the half-plane $\mathrm{Re}(t) \geq t_0$ under $t \mapsto z = e^{-t}t^{-\rho}$. Therefore, the summation procedure (3.16) clearly applies to all ramified polynomials from a given algebra $\mathrm{Ser}^{T,\mathcal{D}}(z)$ and yields the universal bounds (3.13) which remain valid under *closure*, thus ensuring the existence of an asymptotic expansion $\tilde{\varphi}(z)$ of type (3.12) for all elements $\varphi(z)$, polynomial or not, of $S^{T,\mathcal{D}}(z)$. It also ensures that for any such pair $\varphi, \tilde{\varphi}$ the function :

$$\hat{\psi}(\tau) \stackrel{\text{def}}{=} \sum a_{\sigma,r}\hat{Q}_{\sigma,r}(\tau) \tag{3.19}$$

obtained after step 2, and which for each finite τ contains only a finite number of non-vanishing terms, has at most exponential growth

$$|\hat{\psi}(\tau)| \leq \mathrm{Const}\,.e^{\kappa\tau} \qquad (\tau \in \mathbf{R}^+) \tag{3.20}$$

although the coefficients $a_{\sigma,r}$ may themselves *fail to possess exponential bounds* :

$$\limsup |a_{\sigma,r}|^{1/\sigma} \leq +\infty \tag{3.21}$$

so that, for too small a choice of ρ, we might have ended up with a $\hat{\psi}(\tau)$ of faster-than-exponential growth.

The same results hold a fortiori for *compensable functions*.

Remark 1 : Stability under composition.

The algebra $\mathrm{Ser}(z)$ is clearly stable under composition :

$$\varphi, \psi \mapsto \varphi \circ \psi \equiv \varphi(\psi) \qquad (\text{if } \psi(0) = 0) \tag{3.22}$$

The same holds, though less obviously so, for $\mathrm{Comp}(z)$.

Remark 2 : Choice of a critical "slow time".

The auxiliary variable t used in (3.16) is known in general resummation theory as a *critical slow time*. Here, it depends on ρ only. If $\kappa \stackrel{\text{def}}{=} \limsup(r/\sigma) = 0$, then any positive ρ goes. If $\kappa > 0$, then any $\rho > \kappa$ goes. But κ may be unknown, in which case one may choose a still slower time, valid for all κ. For instance :

$$\log(1/z) = t + (\log t)(\log\log t) \tag{3.23}$$

$$\log(1/z) = t + (\log t)^2. \tag{3.24}$$

Remark 3 : Justification of the term "seriable".

Most of the time, the Borel-Laplace resummation of divergent series $\tilde{\varphi}(z) = \sum a_n \epsilon_n(z)$ involves *critical times* $t = h(z)$ relative to which the monomials $e_n(t) \equiv \epsilon_n(z)$ *decrease subexponentially* as t goes to ∞. Their Borel transforms $\hat{e}_n(\tau)$ are therefore analytic on $\underset{\bullet}{\mathbb{C}}$ and, for each value of τ, calculating $\sum a_n \hat{e}_n(\tau)$ involves summing an infinite series, usually for small values of τ, and then resorting to analytic or quasianalytic continuation (See for ex. [E.5], chap 2). In the present instance, however, each monomial $e_n(t)$ has a strictly exponential rate of decrease, leading to $\hat{e}_n(\tau)$ which vanish for small $\tau > 0$ and to finite sums $\sum a_n \hat{e}_n(\tau)$ (for each *given* τ). We have therefore a much simpler procedure, under which the contribution of each single monomial e_n remains clearly individualised and can be dealt with "serially".

In the sequel, we will apply the terms *seriable* and *compensable* not only to (germs of) functions $\varphi(z)$ but also to the corresponding asymptotic series $\tilde{\varphi}(z)$.

Examples of seriable functions $\varphi(x)$ with divergent series $\tilde{\varphi}(x)$.

Example 1 :

$$\tilde{\varphi}(x) \stackrel{\text{def}}{=} \sum_{n \geq 0} n!\; x^{\boldsymbol{\sigma}^n} = \sum_{n \geq 0} n!\; x^{\sigma_0^n, \sigma_1^n, \ldots, \sigma_n^n} \tag{3.25}$$

with fixed α, β $(0 < \alpha < \beta)$ and

$$\alpha + n = \sigma_0^n \leq \sigma_1^n \leq \ldots \leq \sigma_n^n = \beta + n. \tag{3.25'}$$

The *seriability* of $\tilde{\varphi}(x)$ follows from the estimates (2.7), while its *divergence* follows from direct coefficient calculation : see the next examples.

Example 2 :

We take $\tilde{\varphi}(x)$ as above, but with $\alpha = 0,\ \beta = 1$ and with coefficients σ_i^n equally spaced in $\boldsymbol{\sigma}^n$. An immediate calculation yields the obviously divergent, "decompensated " series :

$$\tilde{\varphi}(x) = \sum_{p,q \geq 0} x^{p+q+p(p+q)^{-1}} (-1)^p (p+q)^{p+q} (p+q)!(p!)^{-1}(q!)^{-1} \tag{3.26}$$

Example 3 :

Again, we take $\tilde{\varphi}(x)$ as in (3.25) but with :

(3.27) $$\alpha = 0, \beta = 1;\ \sigma_0^n = n;\ \sigma_1^n = \sigma_2^n = \ldots \sigma_n^n = 1 + n$$

Using formula (2.5) we get :

(3.28) $$\tilde{\varphi}(x) = \sum (-1)^{n-1} \left[\partial_{\sigma_1}^{n-1} \left(\frac{x^{\sigma_0} - x^{\sigma_1}}{\sigma_0 - \sigma_1}\right)\right]_{\substack{\sigma_0 = n \\ \sigma_1 = n+1}}$$

and after some regrouping of terms this becomes :

(3.29) $$\tilde{\varphi}(x) = \tilde{v}(x) - x\tilde{v}(x(1 + x\log x)^{-1}) \in \mathbf{C}[[x, x\log x]]$$

with

(3.29′) $$\tilde{v}(x) = -\sum n!\ x^{n+1}$$

The divergent series $\tilde{v}(x)$ is ***resurgent*** in $z = x^{-1}$. It has ***sectorial sums***, which are regular only in sectorial neighbourhoods of $x = \dot{0}$, with finite apertures. However, the two terms on the right-hand side of (3.29) balance each other in such a way that, although $\tilde{\varphi}(x)$ is left to diverge, the resurgence in it is "compensated away", so that the sum $\varphi(x)$ is now regular in a fundamental neighbourhood $\mathcal{D}$ of $\dot{0}$, with infinite aperture.

This phenomenon is relevant for all ***resonant local objects*** and will be investigated in §§9,10. Indeed, putting $x = z^{-1}$ and anticipating on the notations of §9, we find :

(3.30) $$\tilde{v}(x) \equiv \tilde{\mathcal{V}}^{\eta}(z) \qquad \text{with } \eta = \binom{\omega}{\sigma} = \binom{+1}{-1} \text{ with } \rho = -1$$

(3.31) $$\tilde{\varphi}(x) \equiv \tilde{\mathcal{V}}^{\eta}_{\mathrm{co}}(z) \text{ with } \rho = -1$$

where $\tilde{\mathcal{V}}^{\eta}(z)$ is the "simplest" ***resurgent monomial***, and $\tilde{\mathcal{V}}^{\eta}_{\mathrm{co}}(z)$ the "simplest" ***compensated-resurgent monomial.***

Example 4 : The Riemann zeta function $\zeta(s)$:

Denote by $\mu(n)$ the arithmetical Möbius function and put :

(3.32) $$\varphi(z) \stackrel{\text{def}}{=} \zeta^{-1}(2^{-1}\log(z^{-1})) = \sum_{n \geq 1} \mu(n) z^{(\log n)/2} \qquad (z \in \dot{\mathbf{C}})$$

(3.33) $$\psi(t) \stackrel{\text{def}}{=} \varphi(e^{-t-\rho\log t}) = \zeta^{-1}(2^{-1}(t + \rho\log t)) = \sum \mu(n) n^{-t/2-\rho(\log t)/2}$$

Clearly, the Riemann hypothesis is true ***iff*** the series $\tilde{\psi}(t)$ is Borel-Laplace summable to the ***function*** $\psi(t)$, for each given $\rho > 0$, with summation abscissa $t = 1$. The question, of course, is whether, underneath this "overconvergence", there lies hidden an expansion of $\varphi(z)$ into a series of compensators :

(3.34) $$\sum_{n \geq 1} \mu(n) z^{(\log n)/2} = \sum a_{\sigma_0, \ldots, \sigma_r}\ z^{\sigma_0, \ldots, \sigma_r}$$

with the desired domain of absolute convergence, namely $|z| < e^{-1}$.

Seriable functions of several variables. Rigid algebras.

A function $\varphi(z_1,\ldots,z_\nu)$ defined in a neighbourhood of $\underset{\bullet}{0} \in \underset{\bullet}{\mathbb{C}}^\nu$ respectively of the form :

$$(3.35) \qquad (\sup_i |z_i|)(\sup_i |\log z_i|)^{\kappa_0} < x_0 \qquad (0 < x_0; 0 < \kappa_0)$$

or of the form :

$$(3.36) \qquad \sup_i (|z_i| \, |\log z_i|)^{\kappa_i} < x_0 \qquad (0 < x_0; 0 < \kappa_i)$$

is said to be *weakly* (resp. *strongly*) seriable if for each *positive* (resp. *non-negative*) vector $\alpha = (\alpha_1,\ldots,\alpha_\nu)$ with $\alpha_i > 0$ (resp $\alpha_i \geq 0$) and for any fixed $z = (z_1,\ldots,z_\nu)$ in $\underset{\bullet}{\mathbb{C}}^\nu$, the function :

$$(3.37) \qquad \varphi_{\alpha,z}(t) \overset{\text{def}}{=} \varphi(z_1 t^{\alpha_1},\ldots,z_\nu t^{\alpha_\nu}) \qquad (t \in \underset{\bullet}{\mathbb{C}})$$

is a seriable function of its one variable t.

A subalgebra RIG of seriable functions $\varphi(z)$ with asymptotic expansions $\tilde{\varphi}(z)$ is said to be *rigid* if all $\tilde{\varphi}(z)$ are convergent. For instance, a formal entire power series $\tilde{\varphi}(z)$ of one or several variables is automatically *convergent* if *seriable.*

4. SYMMETRAL AND SYMMETREL COMPENSATORS.

The *symmetric compensators* of the preceding sections are simplest from the theoretical point of view, but for applications we shall require *six new compensators* which, when viewed as *moulds* (see §12), turn out to be either *symmetral* or *symmetrel* (see §12) ***functions*** of the sequence $\omega = (\omega_1,\ldots,\omega_r)$. Those six compensators go in pairs of mutually inverse moulds :

$$(4.1) \qquad 1^\bullet = S^\bullet_{\text{co}}(z) \times \mathcal{S}^\bullet_{\text{co}}(z) = \mathcal{S}^\bullet_{\text{core}}(z) \times \mathcal{S}^\bullet_{\text{core}}(z) = \mathcal{S}^\bullet_{\text{coim}}(z) \times \mathcal{S}^\bullet_{\text{coim}}(z)$$

relative to mould multiplication (see §12). Here *co* stands for *compensation, re* for *real, im* for *imaginary.*

The compensators from the first pair are *symmetral*; easily deducible from the symmetric compensators; and of use in the study of differential equations or vector fields.

The compensators from the second and third pairs are *symmetrel*; not reducible to finite combinations of symmetric compensators; and of use in the study of difference equations or diffeomorphisms.

A. The symmetral compensators $S^\bullet_{\text{co}}(z)$ and $\mathcal{S}^\bullet_{\text{co}}(z)$.

Their simplest definition is by factorisation :

$$S^\bullet_{\text{co}}(z) = \mathcal{S}^\bullet(z) \times S^\bullet \qquad \text{with } \mathcal{S}^{\omega_1,\ldots,\omega_r}(z) = z^{\omega_1+\ldots\omega_r}\,\mathcal{S}^{\omega_1,\ldots,\omega_r} \tag{4.2}$$

$$\mathcal{S}^\bullet_{\text{co}}(z) = \mathcal{S}^\bullet \times S^\bullet(z) \qquad \text{with } S^{\omega_1,\ldots,\omega_r}(z) = z^{\omega_1+\ldots\omega_r}\,S^{\omega_1,\ldots,\omega_r} \tag{4.3}$$

from the symmetral moulds $S^\bullet$ and $\mathcal{S}^\bullet$ introduced in (1.22), (1.23).

We also have the inductive characterisation by :

$$\begin{cases} (\|\bullet\| - z\partial_z)S^\bullet_{\text{co}}(z) = -S^\bullet_{\text{co}}(z) \times I^\bullet & \|\bullet\| = \|\bullet\| = \omega_1 + \ldots\omega_r \qquad (4.4)\\ (\|\bullet\| - z\partial_z)\,\mathcal{S}^\bullet_{\text{co}}(z) = +I^\bullet \times \mathcal{S}^\bullet_{\text{co}}(z) & I^\bullet \text{ as in } (12.9) \qquad (4.5)\end{cases}$$

with the initial conditions :

$$S^\bullet_{\text{co}}(1) = \mathcal{S}^\bullet_{\text{co}}(1) = 1^\bullet \qquad (1^\bullet \text{ as in } (12.8)) \tag{4.6}$$

or alternatively :

$$S^\bullet_{\text{co}}(0) = S^\bullet; \mathcal{S}^\bullet_{\text{co}}(0) = \mathcal{S}^\bullet \qquad (\text{if } \operatorname{Re}(\omega_i) > 0) \tag{4.7}$$

The symmetral compensators also admit useful integral representations :

$$S^{\omega_1,\ldots,\omega_r}_{\text{co}}(z) = (-1)^r \int z_1^{\omega_1-1}\ldots z_r^{\omega_r-1}dz_1\ldots dz_r \tag{4.8}$$

with integration along the multipath $\{z < z_1 < \ldots < z_r < 1\}$, and

$$\mathcal{S}^{\omega_1,\ldots,\omega_r}_{\text{co}}(z) = \int z_1^{\omega_1-1}\ldots z_r^{\omega_r-1}dz_1\ldots dz_r \tag{4.9}$$

with integration along the multipath $\{z < z_r < \ldots < z_1 < 1\}$. Lastly, they are linked to the symmetric compensator of §2 by the following formulae :

$$S^{\omega_1,\ldots,\omega_r}_{\text{co}}(z) = z^{0,\,\check{\omega}_1,\,\check{\omega}_2,\ldots,\,\check{\omega}_r} = z^{0,\omega_1,\omega_1+\omega_2,\ldots,\omega_1+\ldots\omega_r} \tag{4.10}$$

$$\mathcal{S}^{\omega_1,\ldots,\omega_r}_{\text{co}}(z) = (-1)^r z^{\hat{\omega}_1,\,\hat{\omega}_2,\ldots,\,\hat{\omega}_r,0} = z^{\omega_1+\ldots\omega_r,\omega_2+\ldots\omega_r,\ldots,\omega_r,0} \tag{4.11}$$

B. The symmetrel compensators $\mathcal{S}^{\bullet}_{\text{core}}(z)$ and $\mathscr{S}^{\bullet}_{\text{core}}(z)$.

Their simplest definition is by factorisation :

$$\mathcal{S}^{\bullet}_{\text{core}}(z) = \mathscr{S}^{\bullet}_{\text{re}}(z) \times \mathcal{S}^{\bullet} \qquad \text{with } \mathscr{S}^{\omega_1,\dots,\omega_r}_{\text{re}}(z) = z^{\omega_1+\dots\omega_r}\,\mathscr{S}^{\omega_1,\dots,\omega_r} \tag{4.12}$$

$$\mathscr{S}^{\bullet}_{\text{core}}(z) = \mathscr{S}^{\bullet} \times \mathcal{S}^{\bullet}_{\text{re}}(z) \qquad \text{with } \mathcal{S}^{\omega_1,\dots,\omega_r}_{\text{re}}(z) = z^{\omega_1+\dots\omega_r}\,\mathcal{S}^{\omega_1,\dots,\omega_r} \tag{4.13}$$

from the symmetrel moulds $\mathcal{S}^{\bullet}$ and $\mathscr{S}^{\bullet}$ introduced in (1.24) and (1.25).

We also have the inductive characterisation by :

$$e^{\|\bullet\|}\mathcal{S}^{\bullet}_{\text{core}}(z/e) = \mathcal{S}^{\bullet}_{\text{core}}(z) \times (1^{\bullet} + I^{\bullet})^{-1} \tag{4.14}$$

$$e^{\|\bullet\|}\,\mathscr{S}^{\bullet}_{\text{core}}(z/e) = (1^{\bullet} + I^{\bullet}) \times \mathscr{S}^{\bullet}_{\text{core}}(z) \tag{4.15}$$

with the initial conditions :

$$\mathcal{S}^{\bullet}_{\text{core}}(1) = \mathscr{S}^{\bullet}_{\text{core}}(1) = 1^{\bullet} \tag{4.16}$$

or alternatively

$$\mathcal{S}^{\bullet}_{\text{core}}(0) = \mathcal{S}^{\bullet} ; \mathscr{S}^{\bullet}_{\text{core}}(0) = \mathscr{S}^{\bullet} \qquad (\text{if } \operatorname{Re}(\omega_i) > 0) \tag{4.17}$$

The preceding definitions give rise to factors $(1-\exp\ \check{\omega}_i)^{-1}$ and $(1-\exp\ \hat{\omega}_i)^{-1}$. Yet, due to internal cancellations, for any given z on $\dot{\mathbb{C}}$ the scalars $\mathcal{S}^{\omega_1,\dots,\omega_r}_{\text{core}}(z)$ and $\mathscr{S}^{\omega_1,\dots,\omega_r}_{\text{core}}(z)$, as functions of the sequence $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$, are free of singularities on $\mathbf{R}^r$ (though not on $\mathbf{C}^r$) and therefore deserve the label of "compensators". That fact is easiest to derive from the following "integral" representations (4.19) (4.20), which involve the *compensation measures* $\delta_{\boldsymbol{\sigma}}$:

$$\delta_{\boldsymbol{\sigma}}(t) = \delta_{\sigma_0,\dots,\sigma_r}(t) = \sum_{i=0}^{r} \delta_{\sigma_i}(t) \prod_{j\neq i}(\sigma_i - \sigma_j)^{-1} \qquad (\delta_{\sigma_i} = \text{ Dirac}) \tag{4.18}$$

and mirror the integral formulae (4.8) (4.9) for the symmetral compensators.

$$\mathcal{S}^{\omega_1,\dots,\omega_r}_{\text{core}}(z) = \sigma_0\sigma_1\dots\sigma_{r-1} \int t^{\log z}\delta_{\sigma_0,\dots,\sigma_r}(t) \tag{4.19}$$

with $\sigma_0 = 1$ and $\sigma_i = \exp\ \check{\omega}_i = \exp(\omega_1 + \dots\omega_i)$

$$\mathcal{S}_{\text{core}}^{\omega_1,\ldots,\omega_r}(z) = (-1)^r \int t^{r-1+\log z}\delta_{\sigma_0,\ldots,\sigma_r}(t) \tag{4.20}$$

with $\sigma_0 = 1$ and $\sigma_i = \exp\ \overset{\wedge}{\omega}_i = \exp(\omega_i + \ldots \omega_r)$.

C. The symmetrel compensators $S^\bullet_{\text{coim}}(z)$ and $\mathcal{S}^\bullet_{\text{coim}}(z)$.

After the symmetrel real-compensators (*core*) indexed by real sequences $\omega = (\omega_1, \ldots, \omega_r)$, we shall require symmetrel imaginary-compensators (*coim*) indexed by pure imaginary sequences :

$$\omega = (\omega_1, \ldots, \omega_r) = (2\pi i\omega_1^*, \ldots, 2\pi i\omega_r^*) \text{ with } \omega_j^* \in \mathbf{R}^+ \tag{4.21}$$

The new compensators still admit remarkable factorisations :

$$S^\bullet_{\text{coim}}(z) = \mathcal{S}^\bullet_{\text{im}}(z) \times S^\bullet \tag{4.22}$$

$$\mathcal{S}^\bullet_{\text{coim}}(z) = \mathcal{S}^\bullet \times S^\bullet_{\text{im}}(z) \tag{4.23}$$

with the elementary, scalar-valued moulds $S^\bullet$ and $\mathcal{S}^\bullet$ as in (1.24), (1.25) and with non-elementary moulds $S^\bullet_{\text{im}}(z)$ and $\mathcal{S}^\bullet_{\text{im}}(z)$ which, up to a trivial factor, are convergent power series of z :

$$S_{\text{im}}^{\omega_1,\ldots,\omega_r}(z) \text{ and } \mathcal{S}_{\text{im}}^{\omega_1,\ldots,\omega_r}(z) \in z^{-(\omega_1^*+\ldots\omega_r^*)}\mathbf{C}\{z\} \tag{4.24}$$

The moulds $S^\bullet_{\text{coim}}(z)$ and $\mathcal{S}^\bullet_{\text{coim}}(z)$ are therefore *mixed* power series, with components drawn from several spaces $z^{-\alpha}\mathbf{C}\{z\}$. However, despite this composite character, they admit an easy, direct definition (see (4.29) below) so that equations (4.22) (4.23) should be viewed as characterising $S^\bullet_{\text{im}}(z)$ and $\mathcal{S}^\bullet_{\text{im}}(z)$ rather than $S^\bullet_{\text{coim}}(z)$ or $\mathcal{S}^\bullet_{\text{coim}}(z)$. To define the latter moulds, the quickest way is to introduce a linear operator θ acting on all positive powers of z according to :

$$\theta.z^\sigma \overset{\text{def}}{=} \frac{z^\sigma}{e^{2\pi i\sigma}-1} + \frac{1}{2\pi i}\sum_{n\geq 0}\frac{z^n}{n-\sigma} \quad \text{if } \sigma \in \mathbf{R}^+ - \mathbf{N} \tag{4.25}$$

$$\theta.z^\sigma \overset{\text{def}}{=} \frac{1}{2\pi i}\, z^\sigma \log z + \frac{1}{2\pi i}\sum_{n\geq 0; n\neq\sigma}\frac{z^n}{n-\sigma} \quad \text{if } \sigma \in \mathbf{N} \tag{4.26}$$

For any z on $\underset{\bullet}{\mathbf{C}}$ and $|z| < 1$, the series (4.25) and (4.26) converge towards a ramified function $J(z)$ equal in both cases to the integral

$$J(z) = -\frac{1}{2\pi i}\int_0^1 \frac{z_1^\sigma}{z_1 - z}\, dz_1 \qquad (\text{when } 0 < \arg z < 2\pi) \tag{4.27}$$

so that, for any z on $\underset{\bullet}{\mathbb{C}}$ and close to $\underset{\bullet}{0}$, we have :

$$J(e^{2\pi i}z) - J(z) \equiv z^{\sigma}. \tag{4.28}$$

Next, for imaginary ω_j and real ω_j^* as in (4.21), we put :

$$\mathcal{S}_{\text{coim}}^{\omega_1,\ldots,\omega_r}(z) \stackrel{\text{def}}{\equiv} z^{-(\omega_1^*+\ldots\omega_r^*)}\theta z^{\omega_1^*}\theta z^{\omega_2^*}\ldots\theta z^{\omega_r^*} \tag{4.29}$$

where each θ acts on all terms to its right, and not just on its immediate neighbour. Thus $\theta z^a \theta z^b$ should really read $\theta(z^a\theta(z^b))$.

This defines the mould $\mathcal{S}^{\bullet}_{\text{coim}}(z)$ and its multiplicative inverse $S^{\bullet}_{\text{coim}}(z)$, according to (4.1).

For the coming applications, we require the induction formulae :

$$e^{\|\bullet\|}S^{\bullet}_{\text{coim}}(e^{2\pi i}z) = S^{\bullet}_{\text{coim}}(z) \times (1^{\bullet} + I^{\bullet})^{-1} \tag{4.30}$$

$$e^{\|\bullet\|}\,\mathcal{S}^{\bullet}_{\text{coim}}(e^{2\pi i}z) = (1^{\bullet} + I^{\bullet}) \times \mathcal{S}^{\bullet}_{\text{coim}}(z) \tag{4.31}$$

with $\| \bullet \| = \|\omega\| = \omega_1 + \ldots \omega_r$ and $1^{\bullet} + I^{\bullet}$ as in §12. Unlike in the earlier examples, however, the present induction falls far short of characterizing the mould $S^{\bullet}_{\text{coim}}(z)$ and $\mathcal{S}^{\bullet}_{\text{coim}}(z)$, even in combination with initial conditions involving the values of those moulds for $z = 0$ or $z = 1$.

Lastly, from (4.27) and (4.29) we deduce the crucially important integral representations :

$$S_{\text{coim}}^{\omega_1,\ldots,\omega_r}(z) = \left(\frac{1}{2\pi i}\right)^r \int_0^1 \frac{(z_1/z)^{\omega_1^*}(z_2/z)^{\omega_2^*}\ldots(z_r/z)^{\omega_r^*}}{(z_1-z_2)(z_2-z_3)\ldots(z_r-z)}\,dz_1\ldots dz_r \tag{4.32}$$

$$\mathcal{S}_{\text{coim}}^{\omega_1,\ldots,\omega_r}(z) = \left(\frac{1}{2\pi i}\right)^r \int_0^1 \frac{(z_1/z)^{\omega_1^*}(z_2/z)^{\omega_2^*}\ldots(z_r/z)^{\omega_r^*}}{(z-z_1)(z_1-z_2)\ldots(z_{r-1}-z_r)}\,dz_1\ldots dz_r \tag{4.33}$$

In (4.32) we integrate successively in $z_1, z_2, \ldots, z_r$ and let each z_i bypass the *following* variables $z_{i+1}, z_{i+2}, \ldots, z_r, z$ *to the right.* In (4.33) we integrate successively in $z_r, z_{r-1}, \ldots, z_1$ and let each z_i bypass the *preceding* variables $z_{i-1}, z_{i-2}, \ldots, z_1, z$ also *to the right.*We may take the variable z first on the segment $[\underset{\bullet}{0}, \underset{\bullet}{1}]$ and then let it range freely over the universal covering of $\underset{\bullet}{\mathbb{C}} \setminus \{\underset{\bullet}{0}, \underset{\bullet}{1}, \underset{\bullet}{\infty}\}$.

We now have at our disposal all the *compensators* which we shall require in the sequel, but we still lack the main information about them, namely their *bounds*, both *before* and *after* arborification. These bounds will be established singly, when each will first be needed, and then a general synopsis will be appended to the concluding section §11.

5. RAMIFIED LINEARISATION OF GIRATORS (SEMI-MIXED SPECTRUM).

Recall that *girators* are non-degenerate vector fields $2\pi i X$ with purely imaginary spectrum. Their study is obviously equivalent to that of vector fields X with real spectrum $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ and with the non-degeneracy condition, meaning :

(*) diagonalisable linear part

(**) no vanishing eigenvalue $(\lambda_i \neq 0)$

Three cases may occur :

(i) all λ_i have the same sign.

(ii) there is one positive (or negative) λ_i and all others have the opposite sign.

(iii) there are at least two positive and two negative λ_i.

Case (i) is entirely trivial, since it ensures analytic linearisability. Indeed, it rules out small denominators, even diophantine ones, so that the expansions (1.16) and (1.17) of the linearisators $\Theta_{\text{ent}}^{\pm 1}$ become *normally* convergent, even prior to arborification, due to the obvious bounds :

$$|S^{\boldsymbol{\omega}}| < c/r! \ ; \ |\mathcal{S}^{\boldsymbol{\omega}}| < c/r! \qquad (C = \text{Const}; \ \boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)) \tag{5.1}$$

which in case (i) improve upon (1.45), (1.46) and match with the bounds (1.39) for $\mathbb{B}_\omega$.

That leaves only case (ii), which will be dealt with in the present section, and the significantly more complex case (iii), to be studied in the next section.

So, let us for the moment consider a girator $2\pi i X$, with X of semi-mixed real spectrum $(\lambda_1, \ldots, \lambda_\nu)$. We may assume :

$$\lambda_1 < 0; \ \lambda_2 > 0, \ \lambda_3 > 0, \ldots, \ \lambda_r > 0 \tag{5.2}$$

and we easily check that there always exist analytic prepared forms (1.14) with homogeneous components $\mathbb{B}_n$ such that :

$$n_1 \geq 1/\kappa_0 > 0 \tag{5.3}$$

$$n_1 - \omega/\lambda_1 = -(n_2\lambda_2 + \ldots n_\nu\lambda_\nu)/\lambda_1 \geq 1/\kappa_0 > 0 \tag{5.4}$$

for some suitable finite constant κ_0. As usual, $n = (n_1, \ldots, n_\nu)$ and $\omega = < n, \lambda >= \Sigma n_i \lambda_i$.

Proposition 5.1. (Seriable linearisation of semi-mixed girators.)

The following expansion

$$\Theta_{\text{ser}}^{-1} = \sum_{\bullet} \mathcal{S}_{\text{co}}^{\bullet}(z_1)\mathbb{B}_{\bullet} \qquad (\text{with } z_1 = x_1^{-1/\lambda_1}) \tag{5.5}$$

relative to a prepared form of type (5.3), (5.4), defines an inverse linearisator Θ_{ser}^{-1} *and is normally convergent on a spiral-like domain* $\underset{\bullet}{\mathbb{C}} \times \mathbb{C}^{\nu-1}$ *of the form*

$$\{|x_1|\ |\log x_1|^{\kappa_0} \le x_0; \| x_2| \le x_0; \ldots, |x_\nu| \le x_0\} \tag{5.6}$$

Proof. In agreement with the general contraction rule, the sum (5.5) stands for :

$$\sum \mathcal{S}_{\text{co}}^{\omega_1,\ldots,\omega_r}(z_1)\mathbf{B}_{n_r}\ldots\mathbf{B}_{n_1} \tag{5.7}$$

which in the non-resonant case may be re-indexed to :

$$\sum \mathcal{S}_{\text{co}}^{\omega_1,\ldots,\omega_r}(z_1)\mathbf{B}_{\omega_r}\ldots\mathbf{B}_{\omega_1} \tag{5.8}$$

with $\omega_i =< n_i, \lambda >= n_{i,1}\lambda_1 + \ldots + n_{i,\nu}\lambda_\nu$.

Using the derivation rule (4.5) of symmetral compensators, one checks (1.61), so that the series (5.5) formally satisfies the conjugacy equation (1.27) and therefore defines a formal linearisator. Next, we must establish the convergence of the expansion (5.5). Reverting to the definition (4.3) of $\mathcal{S}_{\text{co}}^{\bullet}(z)$ (or its counterpart in the case of vanishing indexes ω_i) we may count the powers of x_1 in each term :

$$\mathcal{S}_{\text{co}}^{\omega_1,\ldots,\omega_r}(z_1)\ x_1^{n_{1,1}+n_{2,1}+\ldots n_{r,1}} \qquad (\text{with } n_i = (n_{i,1},\ldots,n_{i,\nu})) \tag{5.9}$$

Using $z_1 = x_1^{-1/\lambda_1}$ and discounting possible logarithmic factors $(\log x_1)^{r'}$ (with $r' \le r$) we find a finite number of terms of the form :

$$x_1^{\sigma_i} = x_1^{\sigma_i' + \sigma_i''} \qquad (i = 0, 1, \ldots, \nu) \tag{5.10}$$

with

$$\sigma_i' = n_{1,1} + n_{2,1} + \ldots n_{i-1,1} \tag{5.11}$$

$$\sigma_i'' = (n_{i,1} - \omega_i/\lambda_1) + (n_{i+1,1} - \omega_{i+1}/\lambda_1) + \ldots (n_{r,1} - \omega_r/\lambda_1) \tag{5.12}$$

Using (5.3), (5.4) we find :

$$\sigma_i' \ge i/\kappa_0;\ \sigma_i'' \ge (r-i)/\kappa_0;\ \sigma_i \ge r/\kappa_0 \tag{5.13}$$

Each expression (5.9) is therefore, due to (4.11), of the form :

$$x_1^{\sigma_0,\ldots,\sigma_r} \text{ with } \sigma_i \ge r/\kappa_0 \text{ for } i = 0, 1, \ldots, \nu. \tag{5.14}$$

Using the estimates (2.7), (2.8) for symmetric compensators, we find for (5.9) bounds of the form :

$$\frac{1}{r!}\left(|x_1|\ \left|\log\frac{1}{x_1}\right|^{\kappa_0}\right)^{r/\kappa_0} \tag{5.15}$$

which, paired with the obvious estimates

$$\left\| x_1^{-(n_{1,1}+n_{2,1}+\dots n_{r,1})} \mathbf{B}_{n_r} \dots \mathbf{B}_{n_2} \mathbf{B}_{n_1} \right\|_{\mathcal{D},\mathcal{D}'} \leq r!\,\mathrm{Const} \tag{5.16}$$

(for suitable polydiscs $\mathcal{D}, \mathcal{D}'$) ensure the normal convergence of the expansion (5.5) on a domain of the form (5.6).

Remark 1. If $\mathbf{B}_n.x_1 \equiv 0$ for each n (and one may always achieve this by premultiplying X by an analytic unit), the direct linearisator admits the normal convergent expansion :

$$\Theta_{\mathrm{ser}} = \sum_{\bullet} S^{\bullet}_{\mathrm{co}}(z_1)\mathbf{B}_{\bullet} \qquad (\text{with } z_1 = x_1^{-1/\lambda_1}) \tag{5.17}$$

Even without the assumption $\mathbf{B}_n.x_1 \equiv 0$, a similar expansion holds, with a slightly modified comoulds $\mathbf{B}_{\bullet}$.

Remark 2. For a non-resonant spectrum, the constant κ_0 in (5.3), (5.4) may be chosen as small as we wish. This is a favourable circumstance, since κ_0 turns out to measure the "spiralling speed" of the ramified domain (3.2) where seriable linearisation holds.

Remark 3. For a resonant spectrum, with one or several degrees of resonance, there is a limit to the smallness of the acceptable constants κ_0, due to the generic presence of *unremovable resonant monomials* x^m (unremovable, that is, under formal, entire changes of coordinates). In that case, there is usually *one* optimal domain (3.2), which cannot be improved upon, and where seriable linearisation holds.

Remark 4. For a canonical factorisation of the seriable linearisator Θ_{ser}, see §8 in the non-resonant case and §10 in the resonant case.

6. RAMIFIED LINEARISATION OF GIRATORS (GENERAL SPECTRUM).

Consider as before a girator $2\pi i X$, but assume now that X has more than one positive λ_i and more than one negative λ_i. Formula (5.5) for $\Theta^{-1}_{\mathrm{ser}}$ continues to make formal sense, but becomes unsatisfactory from the point of view of analysis, because it entails *negative* powers of x_1, irrespective of whether λ_1 is positive or negative. In order to end up with positive powers only, we must "compensate" or "seriate" not in one, but in at least two variables.

Let λ_1 be one of the *negative* λ_i and λ_2 one of the *positive* λ_i and put :

$$z_1 = x_1^{-1/\lambda_1} \; z_2 = x_2^{-1/\lambda_2} \qquad (\lambda_1 < 0 < \lambda_2) \tag{6.1}$$

Instead of considering compensators $\mathcal{S}^{\bullet}_{\mathrm{co}}(z_1)$ of z_1 only, we shall use *mixed compensators* $\mathcal{S}^{\bullet}_{\mathrm{co}}(z_1, z_2)$ which will involve only positive powers of z_1 and negative powers of z_2, that is to say only ***positive powers of both*** x_1 ***and*** x_2, but which will otherwise retain all the useful properties of $\mathcal{S}^\bullet_{\text{co}}(z_1)$ and $\mathcal{S}^\bullet_{\text{co}}(z_2)$. The passage from $\mathcal{S}^\bullet_{\text{co}}(z_1)$ and $\mathcal{S}^\bullet_{\text{co}}(z_2)$ to $\mathcal{S}^\bullet_{\text{co}}(z_1,z_2)$ is achieved by an important mould-theoretical operation, known as *mixing*.

Definition 6.1. (Mould mixing)

To any pair $A^\bullet, B^\bullet$ of moulds with real indices ω_i, the mixing associates a mould $C^\bullet$ defined by :

$$C^{\omega_1,\ldots,\omega_r} \stackrel{\text{def}}{=} \sum_{\left\{\substack{0\le m\le r\\ \pi\in\Sigma_r}\right.} H^{\omega_1,\ldots,\omega_r}_{\pi,m}\,\tilde{B}^{\omega_{\pi(1)},\ldots,\pi_{\pi(m)}}A^{\omega_{\pi(m+1)},\ldots,\pi_{\pi(r)}} \tag{6.2}$$

with a sum extending to all permutations π of the set $\{1,\ldots,r\}$ and involving the conjugate mould $\tilde{B}$:

$$\tilde{B}^{\omega_1,\ldots,\omega_r} \stackrel{\text{def}}{=} (-1)^r B^{\omega_r,\ldots,\omega_1} \tag{6.3}$$

as well as universal ***disorder coefficients*** *:*

$$H^{\omega_1,\ldots,\omega_r}_{\pi,m} = s_{\epsilon_1}(\hat{\omega}_1)s_{\epsilon_2}(\hat{\omega}_2)\ldots s_{\epsilon_r}(\hat{\omega}_r) \tag{6.4}$$

which assume only the values $0,+1,-1$ and are made up of the following ingredients :

$$\hat{\omega}_i \stackrel{\text{def}}{=} \omega_i+\omega_{i+1}+\ldots\omega_r \tag{6.5}$$

$$s_+(\omega) \stackrel{\text{def}}{=} \begin{cases} 1 & if\ \omega\ge 0\\ 0 & if\ \omega<0\end{cases} \qquad s_-(t) \stackrel{\text{def}}{=} \begin{cases} 0 & if\ \omega> 0\\ -1 & if\ \ \omega\le 0\end{cases} \tag{6.6}$$

$$\epsilon_1 \stackrel{\text{def}}{=} \begin{cases} + & if\ \ m<\pi^{-1}(1)\\ - & if\ \ \pi^{-1}(1)\le m\end{cases} \tag{6.7}$$

$$\epsilon_i \stackrel{\text{def}}{=} \begin{cases} + & if\ \ \pi^{-1}(i-1)<\pi^{-1}(i)\\ - & if\ \ \pi^{-1}(i)<\pi^{-1}(i-1)\end{cases} \qquad (for\ i>1) \tag{6.8}$$

$C^\bullet$ is called the "mixture" of $A^\bullet$ and $B^\bullet$, and denoted by ($A^\bullet$ mix $B^\bullet$).

Lemma 6.1 (Main properties of mould mixing)

$$\{A^\bullet \text{ and } B^\bullet \text{ symmetral}\} \Rightarrow \{A^\bullet \text{ mix } B^\bullet \text{ symmetral}\} \tag{6.9}$$

$$\{A^\bullet \text{ symmetral}\} \Rightarrow \{A^\bullet \text{ mix} A^\bullet\ = A^\bullet\} \tag{6.10}$$

$$(C^\bullet \times A^\bullet) \text{ mix } (C^\bullet \times B^\bullet) = C^\bullet \times (A^\bullet \text{ mix } B^\bullet) \tag{6.11}$$

Moreover, the mixing "separates signs" in the sense that the definition (6.2) of $C^{\omega_1,\ldots,\omega_r}$ involves only terms $A^{\alpha_1,\ldots,\alpha_{r_1}}$ and $B^{\beta_1,\ldots,\beta_{r_2}}$ such that :

$$\overset{\wedge}{\alpha}_i \overset{\text{def}}{=} \alpha_i + \ldots \alpha_{r_1} \geq 0 \quad (\forall i \leq r_1) \text{ and } \overset{\wedge}{\beta}_i \overset{\text{def}}{=} \beta_i + \ldots \beta_{r_2} < 0 \quad (\forall i \leq r_2) \tag{6.12}$$

The shortest way to prove the above lemma is by introducing on the free commutative algebra generated by the symmetral symbols A^ω and B^ω a ***derivation operator D*** and an infinity of ***"integration" operators $I_\pm^{\omega_0}$*** acting as follows :

$$DA^\bullet \overset{\text{def}}{=} I^\bullet \times A^\bullet;\ DB^\bullet \overset{\text{def}}{=} I^\bullet \times B^\bullet;\ D\tilde{B}^\bullet \overset{\text{def}}{=} -\tilde{B}^\bullet \times I^\bullet$$

$$D(\tilde{B}^{\beta_1,\ldots,\beta_r} A^{\alpha_1,\ldots,\alpha_s}) \overset{\text{def}}{=} \tilde{B}^{\beta_1,\ldots,\beta_r} A^{\alpha_2,\ldots,\alpha_s} - \tilde{B}^{\beta_1,\ldots,\beta_{r-1}} A^{\alpha_1,\ldots,\alpha_s}$$

$$I_+^{\omega_0}(\tilde{B}^{\beta_1,\ldots,\beta_r} A^{\alpha_1,\ldots,\alpha_s}) \overset{\text{def}}{=} \sum_{0\leq i\leq r} \tilde{B}^{\beta_1,\ldots,\beta_i} A^{\beta_{i+1},\ldots,\beta_r,\omega_0,\alpha_1,\ldots,\alpha_s}$$

$$I_-^{\omega_0}(\tilde{B}^{\beta_1,\ldots,\beta_r} A^{\alpha_1,\ldots,\alpha_s}) \overset{\text{def}}{=} \sum_{0\leq i\leq s} \tilde{B}^{\beta_1,\ldots,\beta_r,\omega_0,\alpha_1,\ldots,\alpha_i} A^{\alpha_{i+1},\ldots,\alpha_s}$$

and then to observe that $I_+^{\omega_0}$ and $I_-^{\omega_0}$ reverse D :

$$DI_+^{\omega_0} = DI_-^{\omega_0} = 1 \qquad (\forall \omega_0)$$

and that (6.2) may be written :

$$C^{\omega_1,\ldots,\omega_r} = \sum_{\epsilon_i=\pm} s_{\epsilon_1}(\overset{\wedge}{\omega_1})\ldots s_{\epsilon_r}(\overset{\wedge}{\omega_r}) C^{\omega_1,\ldots,\omega_r}_{\epsilon_1,\ldots,\epsilon_r}$$

with the following induction :

$$DC^{\omega_1,\ldots,\omega_r}_{\epsilon_1,\ldots,\epsilon_r} = \epsilon_1 C^{\omega_2,\ldots,\omega_r}_{\epsilon_2,\ldots,\epsilon_r}$$

$$C^{\omega_1,\ldots,\omega_r}_{\epsilon_1,\ldots,\epsilon_r} = \epsilon_1 I^{\omega_1}_{\epsilon_1} C^{\omega_2,\ldots,\omega_r}_{\epsilon_2,\ldots,\epsilon_r}$$

The details are formal and left to the reader.

Lemma 6.2. (Definition and properties of mixed compensators)

$$S^\bullet_{\text{co}}(z_1, z_2) \times \mathcal{S}^\bullet_{\text{co}}(z_1, z_2) = 1^\bullet \tag{6.13}$$

$$\mathcal{S}^\bullet_{\text{co}}(z_1, z_2) \overset{\text{def}}{=} \mathcal{S}^\bullet_{\text{co}}(z_1) \text{ mix } \mathcal{S}^\bullet_{\text{co}}(z_2) = \mathcal{S}^\bullet \times (S^\bullet(z_1) \text{ mix } S^\bullet(z_2)) \tag{6.14}$$

$$(6.15) \qquad (\omega_1 + \dots \omega_r - z_1\partial_{z_1} - z_2\partial_{z_2}) S_{\text{co}}^{\omega_1,\dots,\omega_r}(z_1,z_2) = -S_{\text{co}}^{\omega_1,\dots,\omega_{r-1}}(z_1,z_2)$$

$$(6.16) \qquad (\omega_1 + \dots \omega_r - z_1\partial_{z_1} - z_2\partial_{z_2}) \mathcal{S}_{\text{co}}^{\omega_1,\dots,\omega_r}(z_1,z_2) = \mathcal{S}_{\text{co}}^{\omega_2,\dots,\omega_r}(z_1,z_2)$$

For any sequence $\boldsymbol{\omega} = (\omega_1,\dots,\omega_r)$, of whatever signs, the mixed compensators $S_{\text{co}}^{\boldsymbol{\omega}}(z_1,z_2)$ and $\mathcal{S}_{\text{co}}^{\boldsymbol{\omega}}(z_1,z_2)$ involve only positive powers of z_1 and negative powers of z_2.

Proof : (6.13) defines $S_{\text{co}}^{\bullet}(z_1,z_2)$ indirectly in terms of $\mathcal{S}_{\text{co}}^{\bullet}(z_1,z_2)$ and (6.14) defines $\mathcal{S}_{\text{co}}^{\bullet}(z_1,z_2)$ directly in terms of the one-variable compensators $\mathcal{S}_{\text{co}}^{\bullet}(z_i)$ or the even more elementary moulds $S^{\bullet}(z_i)$, defined as in (4.3). The equivalence between the last two terms of (6.14) follows from (6.11) and (4.3). The derivation rules (6.15) and (6.16) follow from the derivation rules (4.4) and (4.5) applied to the identity (6.2) with the symmetral moulds :

$$(6.17) \qquad A^{\bullet}, B^{\bullet}, \tilde{B}^{\bullet}$$

replaced by

$$(6.18). \qquad \mathcal{S}_{\text{co}}^{\bullet}(z_1),\ \mathcal{S}_{\text{co}}^{\bullet}(z_2),\ S_{\text{co}}^{\bullet}(z_2)$$

Lastly, the concluding remark in Lemma 6.2 about the "separation of signs" in mixed compensators follows from (6.12) with (6.18) in place of (6.17).

Now, let us return to a general *girator* $2\pi i X$ with truly mixed spectrum. As in the preceding section, one establishes the existence of prepared analytic forms (1.14) with homogeneous components $\mathbf{B}_n$ such that :

$$(6.19) \qquad \sum_{\lambda_i<0} n_i \geq -\lambda_1/\kappa_1 > 0 \qquad (\lambda_1 < 0)$$

$$(6.20) \qquad \sum_{\lambda_i>0} n_i \geq +\lambda_2/\kappa_2 > 0 \qquad (\lambda_2 > 0)$$

for some suitable finite constants κ_1, κ_2. The reason for this is that one may go from any analytic form (1.14) to one satisfying (6.19), (6.20) by means of a formal change of variables which involves no small denominators (not even diophantine) and is therefore not merely formal, but analytic.

Proposition 6.1. (Seriable linearisation of general girators)

The following expansion

$$(6.21) \qquad \Theta_{\text{ser}}^{-1} = \sum \mathcal{S}_{\text{co}}^{\bullet}(z_1,z_2)\mathbf{B}_{\bullet}$$

with z_1, z_2 as in (6.1), with a mixed compensator $\mathcal{S}^{\bullet}_{\mathrm{co}}(z_1, z_2)$ as in (6.14) and with a comould $\mathbf{B}_{\bullet}$ relative to a prepared form of type (6.19)(6.20), defines an inverse linearisator $\Theta^{-1}_{\mathrm{ser}}$ which involves only positive powers of all variables x_i (and of $\log x_1, \log x_2$ in case of resonance). Such as it stands, the expansion (6.21) generally fails to converge in norm, but its arborified counterpart :

$$\Theta^{-1}_{\mathrm{ser}} = \sum \mathcal{S}^{\overset{<}{\bullet}}_{\mathrm{co}}(z_1, z_2)\mathbf{B}_{\overset{<}{\bullet}} \tag{6.22}$$

is always normally convergent on a spiral-like domain of $\underset{\bullet}{\mathbf{C}}^2 \times \mathbf{C}^{\nu-2}$ of the form

$$\left\{|\log x_1|^{\kappa_1} \sup_{\lambda_i<0} |x_i| \leq x_0\right\} \times \left\{|\log x_2|^{\kappa_2} \sup_{\lambda_i>0} |x_i| \leq x_0\right\} \tag{6.23}$$

Proof of Proposition 6.1 :

As usual, the contracted sum (6.21) stands for :

$$\sum \mathcal{S}^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(z_1, z_2)\mathbf{B}_{n_r} \ldots \mathbf{B}_{n_1} \tag{6.24}$$

with $\omega_i =< n_i, \lambda >$. Recalling the definition (6.1) of z_1, z_2, we see that :

$$X^{\mathrm{lin}}\, \mathcal{S}^{\bullet}_{\mathrm{co}}(z_1, z_2) = -(z_1\partial_{z_1} + z_2\partial_{z_2})\, \mathcal{S}^{\bullet}_{\mathrm{co}}(z_1, z_2) \tag{6.25}$$

Using the derivation rule (6.16), we check (1.61), so that the series (6.24) formally satisfies the conjugacy equation (1.27) and therefore defines a formal linearisator.

But when we turn to the question of convergence, we find that mixed compensators no longer admit good estimates of types (5.15) with $r!$ as a denominator. Indeed, for $|x_1|$ and $|x_2|$ small enough, we have :

$$\mathcal{S}^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(z_1, z_2) = \sum_{\pi, m} H^{\omega_1,\ldots,\omega_r}_{\pi,m}\, \mathcal{S}^{\omega_{\pi(1)},\ldots,\omega_{\pi(m)}}_{\mathrm{co}}(z_2)\, \mathcal{S}^{\omega_{\pi(m+1)},\ldots,\omega_{\pi(r)}}_{\mathrm{co}}(z_1) \tag{6.26}$$

$$|\mathcal{S}^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(z_1, z_2)| \leq \sum_{\pi, m} \left|\mathcal{S}^{\omega_{\pi(1)},\ldots,\omega_{\pi(m)}}_{\mathrm{co}}(z_2)\right| \cdot \left|\mathcal{S}^{\omega_{\pi(m+1)},\ldots,\omega_{\pi(r)}}_{\mathrm{co}}(z_1)\right| \tag{6.27}$$

$$|\mathcal{S}^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(z_1, z_2)| \leq \sum_{\pi, m} (|\log x_2|^m/m!)\ \ (|\log x_1|^{r-m}/(r-m)!)) \tag{6.27bis}$$

For badly mixed sequence ω_i (the worst being the alternate sequence $(-1)^i\omega_i > 0$) the sum (6.27) may contain more than $r!2^{-r}$ terms, but of course less than $(r+1)!$, so that we get bounds of type :

$$|\mathcal{S}_{\mathrm{co}}^{\omega_1,\ldots,\omega_r}(z_1,z_2)| \leq C_0 C_1^r(|\log x_1| + |\log x_2|)^r \tag{6.28}$$

Using the special case when the indices ω_i are very small, plus some combinatorics, one verifies that the estimates (6.28) are essentially sharp. In view of the equally sharp bounds (1.39) for $\mathbf{B}_\bullet$, this means that the expansion (6.21) is, generally speaking, divergent in norm. So we must take recourse to arborification and use the following lemma :

Lemma 6.3 (Mixing and arborification)

Take $A^\bullet, B^\bullet$ and their mixture $C^\bullet$ as defined in (6.2). Let $\overset{<}{C}{}^\bullet$ denote the arborification of $C^\bullet$ as defined in (12.17). Then, for any arborescent sequence $\overset{<}{\omega} = (\omega_1,\ldots,\omega_r)^<$ we have

$$C^{\overset{<}{\omega}} = \sum_{\left\{\substack{0\leq m\leq r\\ \pi\in\Sigma_r}\right.} H_{\pi,m}^{\overset{<}{\omega}} \tilde{B}^{\omega_{\pi(1)},\ldots,\omega_{\pi(m)}}\ A^{\omega_{\pi(m+1)},\ldots,\omega_{\pi(r)}} \tag{6.29}$$

with coefficients $H_{\pi,m}^{\overset{<}{\omega}}$ still defined by :

$$H_{\pi,m}^{\overset{<}{\omega}} = s_{\epsilon_1}(\,\hat{\omega}_1)s_{\epsilon_2}(\,\hat{\omega}_2)\ldots s_{\epsilon_r}(\,\hat{\omega}_r) \tag{6.30}$$

but with partial sums $\hat{\omega}_i$ now relative to the arborescent order of $\overset{<}{\omega}$:

$$\hat{\omega}_i \overset{\text{def}}{=} \sum \omega_j \ \ for\ all\ j\ equal\ or\ posterior\ to\ i\ in\ \overset{<}{\omega}. \tag{6.31}$$

and with signs ϵ_i reflecting the compatibility, or otherwise, of the substitution π with the arborescent order $\overset{<}{\omega}$

$$\epsilon_i \overset{\text{def}}{=} \begin{cases} + & if\ \ m < \pi^{-1}(i) \\ - & if\ \ \ \pi^{-1}(i) \leq m \end{cases} \qquad (i\ root\ of\ \overset{<}{\omega}\ ) \tag{6.32}$$

$$\epsilon_i \overset{\text{def}}{=} \begin{cases} + & if\ \pi^{-1}(i_-) < \pi^{-1}(i) \\ - & if\ \pi^{-1}(i) < \pi^{-1}(i_-) \end{cases} \qquad (i_-\ antecedent\ of\ i) \tag{6.33}$$

(6.33) holds when i has a (necessarily unique) antecedent i_- in $\overset{<}{\omega}$ and (6.32) holds when i is a ***least element****, or* ***root****, of $\overset{<}{\omega}$.*

The verification of the above lemma is straightforward, by induction on the length r of $\overset{<}{\omega}$, but its significance is startling. Indeed, the right-hand sides of (6.2) and (6.29) contain exactly the same number of terms, namely $(r+1)!$, and both sums involve coefficients $H^{\omega}_{\pi,m}$ or $H^{\overset{<}{\omega}}_{\pi,m}$ which assume only the values $0,+1,-1$. However, the arborification construction (12.17) implies :

$$C^{\overset{<}{\omega}} = \sum \mathrm{sh}\begin{pmatrix}\overset{<}{\omega}\\ \omega\end{pmatrix} C^{\omega} \tag{6.34}$$

$$H^{\overset{<}{\omega}}_{\pi,m} = \sum \mathrm{sh}\begin{pmatrix}\overset{<}{\omega}\\ \omega\end{pmatrix} H^{\omega}_{\pi,m} \tag{6.35}$$

with sums Σ involving on average $r!2^{-r}$ terms. Now, what the lemma 6.3 says, in essence, is that in spite of this the $C^{\overset{<}{\omega}}$ and $H^{\overset{<}{\omega}}_{\pi,m}$ are not significantly larger than the unarborified C^{ω} and $H^{\omega}_{\pi,m}$

If we now apply (6.29) to the triplet (6.18) in place of (6.17), we get for the mixed compensators the following estimates :

Lemma 6.4. *For $|z_1| \ll 1 \ll |z_2|$ and real indices ω_i :*

$$|S^{\overset{<}{\omega}}_{co}(z_1,z_2)| \leq C_0 C_1^r(|\log z_1| + |\log z_2|)^r \tag{6.36}$$

$$|\mathcal{S}^{\overset{<}{\omega}}_{co}(z_1,z_2)| \leq C_0 C_1^r(|\log z_1| + |\log z_2|)^r \tag{6.37}$$

which in view of (6.1) are *no worse* than the estimates (6.28). But these have to be paired with the estimates for $\mathbf{B}_{\overset{<}{\bullet}}$:

$$\|x^{-(n_1+\dots n_r)}\mathbf{B}_{(n_1,\dots,n_r)^<}\|_{\mathcal{D}_1,\mathcal{D}_2} < C^r \tag{6.38}$$

which are much better than the corresponding estimates for $\mathbf{B}_\bullet$. Taking the "preparation" (6.19), (6.20) into account, we immediately obtain the normal convergence of the arborified expansion (6.22), and therefore of the seriable linearisators $\Theta^{\pm 1}_{\mathrm{ser}}$, on domains of the form (6.23), which ends the proof of Proposition 6.1.

Remark 1. By regrouping in (6.21) all terms $\mathcal{S}^{\bullet}_{\mathrm{co}}\,\mathbf{B}_\bullet$ relative to sequences $\omega = (\omega_1,\dots,\omega_r)$ that differ only by the order of the ω_i, we can restore *normal convergence* in a seemingly simpler way than through arborification. Such regroupings are indeed useful when it comes to actually *calculating* the seriable linearisators, but for the purpose of *proving* normal convergence, they are "too large" for an easy identification of cancellations, and there is no convenient alternative to arborification.

Remark 2. If there exists an analytic ***prepared form*** such that :

$$\mathbf{B}_n.x_1 = \mathbf{B}_n.x_2 = 0 \qquad (\forall x) \tag{6.39}$$

(which is exceptional, even after premultiplication of X by an analytic unit), then the direct linearisator admits the expansion :

$$\Theta_{\text{ser}} = \sum S^{\bullet}_{\text{co}}(z_1, z_2)\mathbf{B}_{\bullet} = \sum \quad S^{\overset{<}{\bullet}}_{\text{co}}(z_1, z_2)\mathbf{B}_{\underset{\bullet}{<}} \tag{6.40}$$

with normal convergence of the ***arborified*** sum (to the right). Even without assuming (6.39), there exists a variant of (6.40), with a slightly modified comould $\mathbf{B}_{\bullet}$.

Remark 3. For a non-resonant spectrum, seriable linearisation involves positive real powers of x_1, x_2 and positive integral powers of the remaining x_i. The constants κ_1 and κ_2 in (6.19) (6.20) may be chosen as small as we wish, leading to excellent domains of seriable linearisation.

Remark 4. For a resonant spectrum, seriable linearisation also involves logarithms of x_1 and x_2. The constants κ_1 and κ_2 cannot, generally speaking, be chosen smaller than a certain optimal size, which depends on the resonance relations.

Remark 5. For a canonical factorisation of the seriable linearisator Θ_{ser}, see §8 in the non-resonant case and §10 in the resonant case.

Remark 6. There exist other explicit seriable linearisations, with expansions of type (6.21), but with better domains of convergence, namely domains of type (3.36) instead of type (3.35) as in (6.23). However, these fine-tuned linearisations involve more than two ramified variables $z_i = x_i^{-1/\lambda_i}$ and rely on higher-order compensators $S^{\bullet}_{\text{co}}(z_1, z_2, z_3, \ldots)$ obtained by repeated use of "mould mixing" (see lemma 6.1).

7. RAMIFIED LINEARISATION OF GIRATIONS.

Unlike *girators*, whose study reduces to that of vector fields with real spectrum, *girations* (that is to say diffeomorphisms with multipliers of modulus 1) differ markedly from diffeomorphisms with real eigenvalues.

Let us first consider a one-dimensional giration :

$$f : x \mapsto \ell x + \sum_{n \geq 1} a_n x^{n+1}; \qquad (\mathbb{C}, 0) \to (\mathbb{C}, 0) \tag{7.1}$$

and let us lift f to a giration $\underset{\bullet}{f}$ of $(\underset{\bullet}{\mathbb{C}}, \underset{\bullet}{0})$:

$$\underset{\bullet}{f} \colon x \mapsto \underset{\bullet}{\ell}\, x + \sum_{n \geq 1} a_n x^{n+1}; \qquad (\underset{\bullet}{\mathbb{C}}, \underset{\bullet}{0}) \to (\underset{\bullet}{\mathbb{C}}, \underset{\bullet}{0}) \tag{7.2}$$

by choosing for $\ell \in \mathbb{C}$ a determination $\underset{\bullet}{\ell} \in \underset{\bullet}{\mathbb{C}}$:

$$\underset{\bullet}{\ell} = e^{\lambda} = e^{2\pi i \lambda^*} \qquad (\lambda^* \in \mathbf{R}^+) \tag{7.3}$$

It is essential for our purpose that λ^* should be $\neq 0$. Let us assume for definiteness that it is positive. With f and $\underset{\bullet}{f}$ we associate the substitution operators F and $\underset{\bullet}{F}$ with their (common) homogeneous parts $\mathbf{B}_n$ as in (1.5).

Of course, linearisation (resp. normalisation) difficulties arise for f only if ℓ is liouvillian (resp. a root of unity), but for the lifted $\underset{\bullet}{f}$ seriable linearisation becomes possible as soon as $\underset{\bullet}{\ell} \neq \underset{\bullet}{1}$, which can always be achieved, even if $\ell = 1$.

Proposition 7.1 (Seriable linearisation of one-dimensional girations)
The following expansions :

$$\Theta_{\text{ser}}^{-1} = \sum \mathcal{S}^{\bullet}_{\text{coim}}(z/u)\, \mathbf{B}_{\bullet} = \sum \mathcal{S}^{\overset{<}{\bullet}}_{\text{coim}}(z/u) \mathbf{B}_{\underset{\bullet}{<}} \tag{7.4}$$

with the symmetrel compensators introduced in (4.23) and :

$$z = x^{1/\lambda^*}; \ \lambda_* > 0; \ u \ \textit{fixed on } \underset{\bullet}{\mathbb{C}} \textit{ but close enough to } \underset{\bullet}{0} \tag{7.4bis}$$

$$\mathcal{S}^{\bullet}_{\text{coim}} = \mathcal{S}^{\omega_1,\ldots,\omega_r}_{\text{coim}}; \ \mathbf{B}_{\bullet} = \mathbf{B}_{n_r} \ldots \mathbf{B}_{n_1}; \ \omega_j = \lambda n_j = 2\pi i \lambda^* n_j \tag{7.4ter}$$

define an inverse linearisator of the giration $\underset{\bullet}{f}$ *. The unarborified expansion (in the middle) is almost never normally convergent, but its arborified counterpart (to the right) always is, on spiral-like domains of the form :*

$$\{|x| \cdot |\log x|^{\kappa_0} \leq x_0 |\log x_0|^{\kappa_0}\} \qquad (x \in \underset{\bullet}{\mathbb{C}}, \ 0 < x_0 < e^{-\kappa_0} < 1) \tag{7.5}$$

Proof of Proposition 7.1.

That Θ_{ser}^{-1} as defined by (7.4) satisfies the linearisation equation (1.28) directly follows from (1.62) and (4.15). So the real issue is normal convergence. For this, we need :

Lemma 7.1. *Consider indices $\omega_j^* > 0$ and $\omega_j = 2\pi i \omega_j^*$, and define a mould $Q^\bullet$ by*

$$Q^{\omega_1,\dots,\omega_r}(z) = z^{\omega_1^*+\dots\omega_r^*}\,\mathcal{S}_{\mathrm{coim}}^{\omega_1,\dots,\omega_r}(z) = \theta z^{\omega_1^*}\dots\theta z^{\omega_r^*} \tag{7.6}$$

with θ as in (4.27) and $\mathcal{S}^\bullet_{\mathrm{coim}}$ as in (4.29). The mould $Q^\bullet$ and its symmetrel arborification $Q^{\overset{<}{\bullet}}$ (see (12.19)) admit essentially the same bounds, namely :

$$|Q^{\boldsymbol{\omega}}(z)| < (C(\epsilon))^r(|z\log z| + |\log(1-\overset{\bullet}{z})|)^r \tag{7.7}$$

$$|Q^{\overset{<}{\boldsymbol{\omega}}}(z)| < (C(\epsilon))^r(|z\log z| + |\log(1-\overset{\bullet}{z})|)^r \tag{7.8}$$

for $|z| < 1$ and $0 < \epsilon < \omega_j^*$ $\quad(z \in \underset{\bullet}{\mathbb{C}};\ \overset{\bullet}{z} \in \mathbb{C})$.

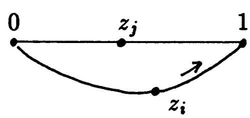
This is unexpected, since in view of (12.19), the $Q^{\overset{<}{\boldsymbol{\omega}}}$ are sums of many terms $Q^{\boldsymbol{\omega}}$, sometimes as many as $r!$ or even more, due to possible sequence-contractions (12.7). However, as usual with mould arborification, there happen to exist handy identities which explain why $Q^{\overset{<}{\boldsymbol{\omega}}}$ is no larger than $Q^{\boldsymbol{\omega}}$. They read :

$$Q^{\boldsymbol{\omega}}(z) = \int_0^1 \prod_{i=1}^{r} (z_i^{\omega_i^*}(z_{i-1}-z_i)^{-1}dz_i) \qquad (z_0 \overset{\mathrm{def}}{=} z) \tag{7.9}$$

$$Q^{\overset{<}{\boldsymbol{\omega}}}(z) = \int_0^1 \prod_{i=1}^{r} (z_i^{\omega_i^*}(z_{i_-}-z_i)^{-1}dz_i) \qquad (z_0 \overset{\mathrm{def}}{=} z) \tag{7.10}$$

In (7.10) i_- denotes the unique antecedent of i within the arborescent order $\overset{<}{\boldsymbol{\omega}}$ or, if i has no antecedent (if it is a *root* or *least element*), we put $i_- \overset{\mathrm{def}}{=} 0$ and $z_0 \overset{\mathrm{def}}{=} z$. In both (7.9) and (7.10) we integrate first in those variables z_i whose index i is largest and we systematically circumvent the still unused variables z_j (j prior to i within $\boldsymbol{\omega}$ or $\overset{<}{\boldsymbol{\omega}}$) to the right, like this :

(7.11)



Identity (7.9) is merely a rewriting of identity (4.33) in terms of $Q^{\boldsymbol{\omega}}$. Identity (7.10) follows from (7.9) by putting :

$$M^{z_1,\dots,z_r}(z) \overset{\mathrm{def}}{=} (z-z_1)^{-1}(z_1-z_2)^{-1}\dots(z_{r-1}-z_r)^{-1} = M^{z_1-z,\dots,z_r-z}(0) \tag{7.12}$$

and by observing that the coefficients $M^\bullet$, regarded as *functions*, define a ***symmetral mould*** (see §12) but when regarded as *functionals* on the space of analytic functions on $[0,1]^r$:

$$M^\bullet : \varphi \mapsto \int_0^1 \varphi(z_1,\dots,z_r) M^{z_1,\dots,z_r}(z)\, dz_1 \dots dz_r \tag{7.13}$$

with integration rule (7.11)), those same coefficients define a ***symmetrel mould*** (see §12). In other words, the *function* $M^\bullet$ satisfies identities of type (12.6) whereas the *functional* $M^\bullet$ satisfies identities of type (12.7). This difference stems from the fact that the integration rule (7.11) is not preserved under reshufflings of the sequence $(z_1,\dots,z_r)$, which calls for corrective terms corresponding precisely to sequence-contractions (12.7). Using (7.9) and (7.10) we now find the induction formulae :

$$Q^{\omega_1 \bullet \boldsymbol{\omega}}(z) = \int_0^1 z_1^{\omega_1^*} \cdot (z-z_1)^{-1} Q^{\boldsymbol{\omega}}(z_1)\, dz_1 \tag{7.14}$$

$$Q^{\omega_1 \bullet \overset{<}{\boldsymbol{\omega}}}(z) = \int_0^1 z_1^{\omega_1^*} \cdot (z-z_1)^{-1} Q^{\overset{<}{\boldsymbol{\omega}}}(z_1)\, dz_1 \tag{7.15}$$

$$Q^{\overset{<}{\boldsymbol{\omega}}' \oplus \overset{<}{\boldsymbol{\omega}}'' \oplus \dots}(z) = Q^{\overset{<}{\boldsymbol{\omega}}'}(z) Q^{\overset{<}{\boldsymbol{\omega}}''}(z) \dots \tag{7.16}$$

with right-side circumvention of z by z_1 (for z on $[\dot{0},\dot{1}]$). Here, $\omega_1 \bullet \boldsymbol{\omega}$ (resp. $\omega_1 \bullet \overset{<}{\boldsymbol{\omega}}$) denotes the fully ordered sequence (resp. the arborescent sequence) consisting of ω_1 followed by the fully ordered sequence $\boldsymbol{\omega}$ (resp. the arborescent sequence $\overset{<}{\boldsymbol{\omega}}$); and $\overset{<}{\boldsymbol{\omega}}' \oplus \overset{<}{\boldsymbol{\omega}}'' \dots$ denotes the arborescent sequence obtained by juxtaposition of $\overset{<}{\boldsymbol{\omega}}', \overset{<}{\boldsymbol{\omega}}'' \dots$. From (7.14) (7.15) (7.16) we infer (7.7) and (7.8) by induction on r, which proves lemma 7.1.

We may now revert to the expansions (7.4) and rewrite them as :

$$\sum (u^{\|\boldsymbol{\omega}^*\|} Q^{\boldsymbol{\omega}}(z/u))(x^{-\|n\|}\mathbf{B}_n) = \sum (u^{\|\boldsymbol{\omega}^*\|} Q^{\overset{<}{\boldsymbol{\omega}}}(z/u))(x^{-\|n\|}\mathbf{B}_{\overset{<}{\mathbf{n}}}) \tag{7.17}$$

with the usual bounds on $\mathbf{B}_\bullet$ and $\mathbf{B}_{\overset{<}{\bullet}}$ which, combined with (7.7) and (7.8), yield the *normal convergence* of the arborified expansion (7.4), along with the generic normal divergence of the unarborified expansion.

Remark 1 : There exist for Θ_{ser} expansions analogous to (7.4), but with a slightly redefined comould $\mathbf{B}_\bullet$.

Remark 2 : In the non-resonant case (i.e. for an irrational λ^*) we have a seriable linearisation of the form :

$$x = k^{\text{ser}}(y) \in y\, \mathbf{C}[[y, y^{1/\lambda^*}]] \tag{7.18}$$

and the spiralling index κ_0 may be chosen as small as one wishes. The series $k^{\rm ser}(y)$ is convergent if λ^* is diophantine, and generically divergent if λ^* is liouvillian. Yet $k^{\rm ser}(y)$ is always seriable and therefore summable by the methods of §3.

Remark 3 : In the resonant case (i.e. for a rational λ^*) we have :

$$x = k^{\rm ser}(y) \in y\, \mathbb{C}[[y, y^{1/m}(\log y)^{\kappa/m}]] \qquad (m \in \mathbb{N};\ \kappa \in Q^+) \tag{7.19}$$

which makes it plain that the spiralling constant κ_0 in (4.5) cannot be taken smaller than κ.

Remark 4 : For the canonical factorisation of $\Theta_{\rm ser}$ and $k^{\rm ser}$, see §8 in the non-resonant case and §10 in the resonant case.

Remark 5. Seriability and conformal maps. Connection with the Douady-Ghys lemma.

We shall be pairing local diffeos f, g of $\mathbb{C}$ with "opposite" multipliers ℓ, ℓ', or rather lifted diffeos $\underset{\bullet}{f}, \underset{\bullet}{g}$ of $\underset{\bullet}{\mathbb{C}}$ with multipliers $\underset{\bullet}{\ell}, \underset{\bullet}{\ell'}$ such that :

$$\log \ell + \log \ell' = 0 \ (\mathrm{mod}\, 2\pi i);\ \log \underset{\bullet}{\ell} + \log \underset{\bullet}{\ell'} = 0 \ \ (\text{exactly}) \tag{7.20}$$

$$\underset{\bullet}{f}\colon x \mapsto \underset{\bullet}{\ell}\, x + \ldots \ \text{ with } \underset{\bullet}{\ell} = e^{2\pi i \lambda^*},\ \lambda^* > 0,\ x \in \underset{\bullet}{\mathbb{C}} \tag{7.21}$$

$$\underset{\bullet}{g}\colon x \mapsto \underset{\bullet}{\ell'}\, x + \ldots \ \text{ with } \underset{\bullet}{\ell'} = e^{2\pi i/\lambda^*},\ \lambda^* > 0,\ x \in \underset{\bullet}{\mathbb{C}} \tag{7.22}$$

For the moment we forget about $\underset{\bullet}{g}$ and start with a given $\underset{\bullet}{f}$. We fix $\underset{\bullet}{\epsilon}$ on $\underset{\bullet}{\mathbb{C}}$ small enough ($\underset{\bullet}{\epsilon}$ close to $\underset{\bullet}{0}$) and real ($\arg \underset{\bullet}{\epsilon} = 0$) and introduce on $\underset{\bullet}{\mathbb{C}}$ three sectorial neighbourhoods D^1, D^2, D^3 of $\underset{\bullet}{0}$, with apertures respectively $2\pi, 2\pi\lambda^*, 2\pi\lambda^*$:

$$D^1 \overset{\text{def}}{=} \{x \in \underset{\bullet}{\mathbb{C}};\ 0 < \arg x < 2\pi,\ |x| < \epsilon\} \tag{7.23}$$

$$D^2 \overset{\text{def}}{=} \{x \in \underset{\bullet}{\mathbb{C}};\ 0 < \arg x < 2\pi\lambda^*,\ |x| < \epsilon^{\lambda^*}\} \tag{7.24}$$

(7.25) $D^3 \overset{\text{def}}{=}$ domain of $\underset{\bullet}{\mathbb{C}}$ enclosed by the ray I from $\underset{\bullet}{0}$ to $\underset{\bullet}{\epsilon}^{\lambda^*}$, by its image $\underset{\bullet}{f}\,(I)$, and by any given curve Γ joining $\underset{\bullet}{\epsilon}^{\lambda^*}$ to $\underset{\bullet}{f}\,(\underset{\bullet}{\epsilon}^{\lambda^*})$ without crossing itself, nor I, nor $\underset{\bullet}{f}\,(I)$.

Lastly, we denote by $k^{j,i}$ *the* conformal mapping from D^i to D^j that preserves $\underset{\bullet}{0}$ $(i, j = 1, 2, 3)$. Obviously, $k^{2,1}(x) \equiv x^{\lambda^*}$ and $k^{1,2}(x) \equiv x^{1/\lambda^*}$.

The Douady-Ghys lemma

$k^{1,3}$ conjugates $\underset{\bullet}{f}$ with the one-turn rotation $\underset{\bullet}{r}$:

$$\underset{\bullet}{f} = k^{3,1} \circ \underset{\bullet}{r} \circ k^{1,3} \qquad (\textit{with } \underset{\bullet}{r}(x) \stackrel{\text{def}}{=} e^{2\pi i}x) \tag{7.26}$$

and the correspondence :

$$\underset{\bullet}{f} \mapsto \underset{\bullet}{g} \stackrel{\text{def}}{=} k^{1,3} \circ \underset{\bullet}{r} \circ k^{3,1} \qquad (\textit{with same } \underset{\bullet}{r}) \tag{7.27}$$

though dependent on the choice of ϵ and Γ, induces an intrinsic, one-to-one correspondence between analytic conjugacy classes of diffeos $\underset{\bullet}{f}$ and $\underset{\bullet}{g}$ of type (7.21) and (7.22).

Let us now reinterpret this result in the light of *seriability*. We replace the domain D^1 by D^2, which has the same aperture $2\pi\lambda^*$ as D^3, and we observe that :

$$k^{2,3} \circ \underset{\bullet}{f} \circ k^{3,2}(x) \equiv e^{2\pi i\lambda^*}x \qquad (\forall x \in D^2) \tag{7.28}$$

since $k^{2,3} \circ \underset{\bullet}{f} \circ k^{3,2}$ leaves invariant $\underset{\bullet}{0}$ and the circle $|x| = \epsilon$. If we now introduce the seriable linearisation k^{ser} and put :

$$k^{3,2} = k^{\text{ser}} \circ k \qquad (k^{\text{ser}}(x) \stackrel{\text{def}}{=} \Theta_{\text{ser}}^{-1}.x) \tag{7.29}$$

that *defines* a map k which leaves $\underset{\bullet}{0}$ invariant as well as the radii $\{\arg x = 0\}$ and $\{\arg x = 2\pi\lambda^*\}$, so that $k(x)$ must be the sum of a real, convergent power series $\tilde{k}(x)$ of the form :

$$\tilde{k}(x) = x\{1 + \sum d_n x^{n/\lambda^*}\} \in x.\mathbf{R}\{x^{1/\lambda^*}\} \tag{7.30}$$

But Proposition 7.1 shows that k^{ser} is a *seriable* function with asymptotic series $\tilde{k}^{\text{ser}}$. Therefore the map $k^{3,2}$ in (7.29), as indeed all conformal maps $k^{i,j}$ of this section, are also *seriable* at $\underset{\bullet}{0}$.

Moreover, in the non-resonant case (λ^* irrational) we have the obvious factorisations (see also §8) between asymptotic series :

$$\tilde{k}^{3,1} = \tilde{a}^{-1} \circ P_{\lambda^*} \circ \tilde{b} \qquad (\tilde{k}^{3,1}(x) = x^{\lambda^*} + \ldots) \tag{7.31}$$

$$\tilde{k}^{1,3} = \tilde{b}^{-1} \circ P_{1/\lambda^*} \circ \tilde{a} \qquad (\tilde{k}^{1,3}(x) = x^{1/\lambda^*} + \ldots) \tag{7.32}$$

$$\underset{\bullet}{\tilde{f}} = \tilde{a}^{-1} \circ \underset{\bullet}{r}^{\lambda^*} \circ \tilde{a} \qquad (\underset{\bullet}{\tilde{f}}(x) = e^{2\pi i\lambda^*} x + \ldots) \tag{7.33}$$

$$\underset{\bullet}{\tilde{g}} = \tilde{b}^{-1} \circ \underset{\bullet}{r}^{1/\lambda^*} \circ \tilde{b} \qquad (\underset{\bullet}{\tilde{g}}(x) = e^{2\pi i/\lambda^*} x + \ldots) \tag{7.34}$$

with

$$P_{\lambda^*}(x) \overset{\text{def}}{=} x^{\lambda^*};\ P_{1/\lambda^*}(x) \overset{\text{def}}{=} x^{1/\lambda^*} \tag{7.35}$$

$$\underset{\bullet}{r}^{\lambda^*}(x) \overset{\text{def}}{=} e^{2\pi i\lambda^*} x;\ \underset{\bullet}{r}^{1/\lambda^*}(x) \overset{\text{def}}{=} e^{2\pi i/\lambda^*} x \tag{7.36}$$

and with entire, but usually divergent power series $\tilde{a}, \tilde{b}$ of the form :

$$\tilde{a}(x) = x + \sum a_n x^{n+1};\ \tilde{b}(x) = x + \sum b_n x^{n+1} \tag{7.37}$$

Indeed, due to the small denominators present in their coefficients a_n, b_n, the series $\tilde{a}$ and $\tilde{b}$ are not only generically divergent, but hopelessly so : when divergent, there is no canonical way of summing them *separately*, even in sectorial neighbourhoods of $\underset{\bullet}{0}$. When *combined*, however, $\tilde{a}$ and $\tilde{b}$ undergo *compensation* and yield either *seriable series* $\tilde{k}^{3,1}, \tilde{k}^{1,3}$ as in (7.31), (7.32) or plain *convergent series* $\underset{\bullet}{\tilde{f}}, \underset{\bullet}{\tilde{g}}$ as in (7.33), (7.34).

Remark 6. Extension to Carleman classes $\mathcal{C}(M)$:

The alternative approach outlined in Remark 5, being based on conformal mappings, doesn't extend smoothly to higher dimensions, and even in dimension 1, like all geometrical methods, it fails in the face of classes $\mathcal{C}(M)$ of power series larger than the convergent or "analytic" class $\mathcal{A} = \mathcal{C}(1)$. Indeed, let $M_\sigma = (m_\sigma)^\sigma$ be defined for $\sigma > 0$, with $\log m_\sigma$ positive, convex and growing to $+\infty$ as σ goes to $+\infty$. Then the linear operator :

$$M : x^\sigma \mapsto M.x^\sigma \overset{\text{def}}{=} M_\sigma.x^\sigma \qquad (\forall \sigma \in \mathbf{R}^+) \tag{7.38}$$

turns the convergent class $\mathcal{A}$ into the so-called *Carleman class* $\mathcal{C}(M) \overset{\text{def}}{=} M.\mathcal{A}$ which, just like $\mathcal{A}$, enjoys stability under all useful operations, including postcomposition by elements of the form $\ell x + o(x)$. However, for every Carleman class $\mathcal{C}(M)$, no matter how large, the conjugacy equation :

$$(7.39) \qquad \tilde{h} \circ \tilde{f}_1 = \tilde{f}_2 \circ \tilde{h} \qquad (\tilde{f}_1(x) = e^{2\pi i \lambda^*} x + \ldots, \tilde{f}_2(x) = e^{2\pi i \lambda^*} x + \ldots)$$

for a given pair $\tilde{f}_1, \tilde{f}_2$ in $\mathcal{C}(M)$, may fail to have solutions $\tilde{h}$ in $\mathcal{C}(M)$, due to small denominators which, for strongly liouvillian multipliers $\ell = \exp(2\pi i \lambda^*)$, can be as small as one wishes. Within a class $\mathcal{C}(M)$, therefore, neither the conjugacy problem, nor the linearisation problem, nor the correspondence between conjugacy classes $\mathrm{cl}(\underset{\bullet}{\tilde{f}})$ and $\mathrm{cl}(\underset{\bullet}{\tilde{g}})$ as in (7.21), (7.22), can be tackled by conformal mapping, but the methods based on seriability still apply with only slight changes. Actually, all it takes is to redefine the symmetrel compensators $\mathcal{S}^{\bullet}_{\mathrm{coim}}$, still using formula (4.29) but with the operator $M^{-1}.\theta.M$ in place of θ, and to define M-seriability of a series $\tilde{\varphi}(z) = \Sigma a_\sigma z^\sigma$ as meaning that, after changing variables :

$$(7.40) \qquad \tilde{\varphi}(z) \equiv \tilde{\psi}(t) \quad \text{with } z = e^{-t}.t^{-\rho} \text{ and } \rho \text{ positive large}$$

and Borel-transforming $\tilde{\psi}(t)$ into $\hat{\psi}(\tau)$, we have the bounds :

$$(7.41) \qquad |\hat{\psi}(\tau)| < \mathrm{Const}\,.\, M_\tau e^{\kappa\tau} \qquad (\tau \in \mathbf{R}^+)$$

instead of (3.20). If we do this, expansion (7.4) still gives us a M-seriable linearisator $\Theta_{\mathrm{ser}}^{-1}$ of $\underset{\bullet}{\tilde{f}}$, while formula (7.27) becomes :

$$(7.41) \qquad \underset{\bullet}{\tilde{f}} \mapsto \underset{\bullet}{\tilde{g}} \stackrel{\mathrm{def}}{=} \tilde{k}^{\mathrm{ser}} \circ \underset{\bullet}{r} \circ \tilde{h}^{\mathrm{ser}}$$

with

$$(7.41bis) \qquad \tilde{h}^{\mathrm{ser}}(x) = \Theta_{\mathrm{ser}}.x, \tilde{k}^{\mathrm{ser}}(x) = \Theta_{\mathrm{ser}}^{-1}.x;\ \underset{\bullet}{r}\,(x) = e^{2\pi i}.x$$

and still yields an intrinsic, one-to-one correspondence $\mathrm{cl}(\underset{\bullet}{\tilde{f}}) \to \mathrm{cl}(\underset{\bullet}{\tilde{g}})$ between conjugacy classes of diffeos of types (7.21) and (7.22) but in the class $\mathcal{C}(M)$.

Remark 7.Seriable linearisation of girators in higher dimension.

In dimension larger than one, the study of *girations* doesn't run exactly parallel to that of *girators*. Given a collection of ν multipliers ℓ_j on the unit circle of $\mathbf{C}$, we face the choice of lifting them into multipliers $\underset{\bullet}{\ell_j} = \exp(2\pi i \lambda_j^*)$ on $\underset{\bullet}{\mathbf{C}}$ with real numbers λ_j^* of one sign only, or of mixed signs. The first choice leads to linearisators that are close in form to (7.4), yet involve an infinity of independent ramified monomials $x_i^{\sigma_i}$. The second choice avoids this infinity, but at the cost of less explicit formulae. See the concluding section §11.

8. GENERALISED HOLONOMY. CORRESPONDENCES BETWEEN CONJUGACY CLASSES OF LOCAL OBJECTS.

In this section, we shall have two concerns. *First*, factorising (canonically and explicitly) the seriable linearisators of local objects. *Second*, generalising the notion of holonomy; that is to say, associating with a local object *Ob* (field or diffeo) a local diffeo $\underset{\bullet}{f}^{\text{inv}}$ of $\underset{\bullet}{\mathbf{C}}^{\nu}$ in such a way that the analytic conjugacy class of Ob should determine that of $\underset{\bullet}{f}^{\text{inv}}$.

For our first purpose, we will provisionally assume *Ob* to be non-resonant (that restriction will be lifted in §10), but our second construction will apply right away to all cases, resonant and non-resonant.

We recall the definition of the invariant algebra INV introduced in §1. For any given local object X or $\underset{\bullet}{F}$ with diagonal linear part X^{lin} or $\underset{\bullet}{F}^{\text{lin}}$, INV is the algebra of formal power series spanned by all monomials :

$$x^{\sigma} = x_1^{\sigma_1} \dots x_{\nu}^{\sigma_{\nu}} \qquad (\sigma_i \in \mathbf{R}^+,\ x_i \in \underset{\bullet}{\mathbf{C}}) \tag{8.1}$$

invariant under X^{lin} or $\underset{\bullet}{F}^{\text{lin}}$:

$$X^{\text{lin}}.x^{\sigma} = 0; \qquad \underset{\bullet}{F}^{\text{lin}}.x^{\sigma} = x^{\sigma} \tag{8.1bis}$$

Proposition 8.1. Factorisation of seriable linearisators.

Assuming non-resonance, all seriable linearisators hitherto defined factor as follows :

$$\Theta_{\text{ser}} = \Theta_{\text{ent}} \cdot \Theta_{\text{inv}}^{-1}; \qquad \Theta_{\text{ser}}^{-1} = \Theta_{\text{inv}} \cdot \Theta_{\text{ent}}^{-1} \tag{8.2}$$

or scalarly (mark the reversals) :

$$h^{\text{ser}} = k^{\text{inv}} \circ h^{\text{ent}}; \qquad k^{\text{ser}} = k^{\text{ent}} \circ h^{\text{inv}} \tag{8.2bis}$$

*into a usually divergent "**integral part**" Θ_{ent}, which preserves entire series, and a usually divergent "**invariant part**" Θ_{inv}, which preserves INV and commutes with the linear part of the girator or giration. The substitution operators Θ_{ent} and Θ_{inv} (we also recall Θ_{ser} for completeness) possess the following expansions in the case of **girators*** :

(8.3)* $\Theta_{\text{ser}} = \sum S^{\bullet}_{\text{co}}(z)\mathbf{B}_{\bullet}$ (8.3′) $\Theta_{\text{ser}}^{-1} = \sum \mathcal{S}^{\bullet}_{\text{co}}(z)\mathbf{B}_{\bullet}$

(8.4) $\Theta_{\text{ent}} = \sum S^{\bullet}\mathbf{B}_{\bullet}$ (8.4′) $\Theta_{\text{ent}}^{-1} = \sum \mathcal{S}^{\bullet}\,\mathbf{B}_{\bullet}$

(8.5) $\Theta_{\text{inv}} = \sum S^{\bullet}(z)\mathbf{B}_{\bullet}$ (8.5′)* $\Theta_{\text{inv}}^{-1} = \sum \mathcal{S}^{\bullet}(z)\mathbf{B}_{\bullet}$

and in the case of girations :

(8.6)* $\Theta_{\mathrm{ser}} = \sum \mathcal{S}^{\bullet}_{\mathrm{coim}}(z)\mathbf{B}_{\bullet}$ (8.6′) $\Theta^{-1}_{\mathrm{ser}} = \sum \mathcal{S}^{\bullet}_{\mathrm{coim}}(z)\mathbf{B}_{\bullet}$

(8.7) $\Theta_{\mathrm{ent}} = \sum \mathcal{S}^{\bullet}\mathbf{B}_{\bullet}$ (8.7′) $\Theta^{-1}_{\mathrm{ent}} = \sum \mathcal{S}^{\bullet}\,\mathbf{B}_{\bullet}$

(8.8) $\Theta_{\mathrm{inv}} = \sum \mathcal{S}^{\bullet}_{\mathrm{im}}(z)\mathbf{B}_{\bullet}$ (8.8′)* $\Theta^{-1}_{\mathrm{inv}} = \sum \mathcal{S}^{\bullet}_{\mathrm{im}}(z)\mathbf{B}_{\bullet}$

Proof and comments :

The 12 above expansions involve standard mould-comould contractions of type $\Sigma A^{\bullet}\mathbf{B}_{\bullet}$ and the way to deduce the expansions of Θ_{ent} and Θ_{inv} from those of Θ_{ser} is by using the canonical factorisations $A^{\bullet} = A^{\bullet}_1 \times A^{\bullet}_2$ of compensators along with the identity

(8.9) $$\sum (A^{\bullet}_1 \times A^{\bullet}_2)\mathbf{B}_{\bullet} = (\sum A^{\bullet}_2\mathbf{B}_{\bullet})(\sum A^{\bullet}_1\mathbf{B}_{\bullet})$$

which is valid as soon as the differential operators $\mathbf{B}_n$ which go into the making of the comould $\mathbf{B}_{\bullet}$ commute with $A^{\bullet}_1$, that is to say, do not act on the variable or variables (if any) present in the mould $A^{\bullet}_1$. Note that there is no restriction here on the mould $A^{\bullet}_2$. This explains why the four star-marked formulae $(\ldots)^*$ in Proposition 8.1 hold only when the $\mathbf{B}_n$ don't act on the variable(s) carried by the corresponding mould. But even when this is not the case, as with diffeos, the formulae $(\ldots)^*$ retain their validity after a slight redefinition of the comould $\mathbf{B}_{\bullet}$. Observe that the scalar moulds $S^{\bullet}, \mathcal{S}^{\bullet}, \mathcal{S}^{\bullet}, \mathcal{S}^{\bullet}$ appearing in the formulae, are defined as in §1. The one or two variable compensators $\mathcal{S}^{\bullet}_{\mathrm{co}}(z), \mathcal{S}^{\bullet}_{\mathrm{co}}(z), \mathcal{S}^{\bullet}_{\mathrm{coim}}(z), \mathcal{S}^{\bullet}_{\mathrm{coim}}(z)$ are the same as in sections 5,6,7. Lastly, the moulds $S^{\bullet}(z), \mathcal{S}^{\bullet}(z), \mathcal{S}^{\bullet}_{\mathrm{im}}(z), \mathcal{S}^{\bullet}_{\mathrm{im}}(z)$ are defined as in (4.2) (4.3) (4.22) (4.23) in the case of *one* ramified variable, and by using mould mixing as in (6.14) in the case of *two* ramified variables.

Now, let us address the construction of the "generalised holonomy" operator $\underset{\bullet}{F}_{\mathrm{inv}}$. Denote by $\underset{\bullet}{R}$ any rotation of r_i turns in each component $\underset{\bullet}{\mathbf{C}}$ of $\underset{\bullet}{\mathbf{C}}^{\nu}$:

(8.10) $$\underset{\bullet}{R}\,\varphi(x_1,\ldots,x_\nu) \overset{\mathrm{def}}{=} \varphi(e^{2\pi i r_1}x_1,\ldots,e^{2\pi i r_\nu}x_\nu) \quad (r_i \in \mathbf{Z})$$

Then, for any girator X or any lifted giration $\underset{\bullet}{F}$ with seriable linearisator Θ_{ser}, put

(8.11) $$\underset{\bullet}{F}_{\mathrm{inv}} \overset{\mathrm{def}}{=} \Theta^{-1}_{\mathrm{ser}}\,\underset{\bullet}{R}\,\Theta_{\mathrm{ser}} = \Theta_{\mathrm{inv}}\,\underset{\bullet}{R}\,\Theta^{-1}_{\mathrm{inv}}$$

or scalarly :

(8.11*bis*) $$\underset{\bullet}{f}^{\mathrm{inv}} = h^{\mathrm{ser}} \circ \underset{\bullet}{r} \circ k^{\mathrm{ser}} = k^{\mathrm{inv}} \circ \underset{\bullet}{r} \circ h^{\mathrm{inv}}$$

The middle and right terms in (8.11) and (8.11bis) are indeed equal,because the factors $\Theta^{\pm 1}_{\mathrm{ent}}$ in $\Theta^{\pm 1}_{\mathrm{ser}}$ (see (8.2)) commute with $\underset{\bullet}{R}$ and cancel out. Moreover, as a product of two seriable factors $\Theta^{\pm 1}_{\mathrm{ser}}$ with an elementary $\underset{\bullet}{R}$ in between, $\underset{\bullet}{F}_{\mathrm{inv}}$ is automatically seriable.

Now, the above construction of $\underset{\bullet}{F}_{\mathrm{inv}}$ is not intrinsic. Suppose indeed that we calculate, according to the formulae of sections 5,6,7, different seriable linearisators Θ^i_{ser} relative to different charts $i = 1, 2, \ldots$. These linearisators will factor into :

$$\Theta^i_{\mathrm{ser}} = (\Theta_{\mathrm{ent}}) \cdot (\Theta^i_{\mathrm{inv}})^{-1} \qquad (i = 1, 2, \ldots) \tag{8.12}$$

with identical Θ_{ent} but chart-dependent factors Θ^i_{inv} connected by :

$$\Theta^i_{\mathrm{inv}} = \Theta^{ij}_{\mathrm{inv}} \Theta^j_{\mathrm{inv}} \qquad (i, j = 1, 2, 3 \ldots) \tag{8.13}$$

Consequently, we also get chart-dependent holonomies F^i_{inv} mutually connected by :

$$\underset{\bullet}{F}^i_{\mathrm{inv}} = \Theta^{ij}_{\mathrm{inv}} \underset{\bullet}{F}^j_{\mathrm{inv}} \Theta^{ji}_{\mathrm{inv}} \qquad (i, j = 1, 2 \ldots) \tag{8.14}$$

with

$$\Theta^{ij}_{\mathrm{inv}} = (\Theta^i_{\mathrm{inv}})(\Theta^j_{\mathrm{inv}})^{-1} = (\Theta^i_{\mathrm{ser}})^{-1}(\Theta^j_{\mathrm{ser}}) \tag{8.15}$$

The last term shows that $\Theta^{i,j}_{\mathrm{inv}}$ is seriable and the middle term shows that it preserves INV. Therefore, if the factors Θ^i_{inv} implicit in the Θ^i_{ser} *preserve a rigid subalgebra* RIG of INV (see end of §3), so do the connection factors $\Theta^{i,j}_{\mathrm{inv}}$, which means that they correspond to power series $\tilde{k}^{ij}_{\mathrm{inv}} \in$ RIG. *But being also seriable elements of a rigid algebra, the series $\tilde{k}^{ij}_{\mathrm{inv}}$ are in fact convergent.* Therefore we have proved :

Proposition 8.2. Invariance of the holonomy class $\underset{\bullet}{F}_{\mathrm{inv}}$

Whenever the seriable linearisators Θ_{ser} of a local object Ob (girator or giration) draw only on a rigid subalgebra RIG of the invariant algebra INV (and such is the case for all linearisators constructed in sections 5,6,7) the various (generalised) holonomies $\underset{\bullet}{F}_{\mathrm{inv}}$ relative to a given multirotation $\underset{\bullet}{R}$ but to different charts, are all conjugate according to (8.14) with **convergent** *connection factors $\Theta^{ij}_{\mathrm{inv}}$. The "convergent conjugacy class" of $\underset{\bullet}{F}_{\mathrm{inv}}$ is therefore well-defined and dependent only on the analytic conjugacy class of Ob.*

Now, let us look for the mould expansion of the holonomy $\underset{\bullet}{F}_{\mathrm{inv}}$ of a girator $2\pi i X$, assuming for simplicity that the condition for the validity of the star-marked formulae are met. Using definition (8.11) along with expansions (8.3)* and (8.3′) we find :

$$\left\{\begin{aligned} \underset{\bullet}{F}_{\mathrm{inv}} &= \left(\sum \mathcal{S}^{\bullet}_{\mathrm{co}}(z)\mathbf{B}_{\bullet}\right) \underset{\bullet}{R} \left(\sum S^{\bullet}_{\mathrm{co}}(z)\mathbf{B}_{\bullet}\right) \\ &= \underset{\bullet}{R}\left(\sum\left(\underset{\bullet}{R}^{-1} \mathcal{S}^{\bullet}_{\mathrm{co}}(z)\right)\mathbf{B}_{\bullet}\right)\left(\sum S^{\bullet}_{\mathrm{co}}(z)\mathbf{B}_{\bullet}\right) \end{aligned}\right. \tag{8.16}$$

and finally :

$$\underset{\bullet}{F}_{\mathrm{inv}} = \underset{\bullet}{R} \left(\sum D^{\bullet}(z)\mathbf{B}_{\bullet}\right) \tag{8.17}$$

with the customary order reversal :

(8.18) $$D^\bullet(z) = (S^\bullet_{\text{co}}(z)) \times (\underset{\bullet}{R}^{-1}\, \mathcal{S}^\bullet_{\text{co}}(z))$$

Due to (4.2) (4.3) or (6.13)(6.14) this becomes :

(8.19) $$D^\bullet(z) = (\mathcal{S}^\bullet(z)) \times (\underset{\bullet}{R}^{-1} S^\bullet(z))$$

For girators with semi-mixed spectrum (see §5) we have only one ramified variable :

(8.20) $$z = z_1 = x_1^{-1/\lambda_1} \qquad (\lambda_1 < 0)$$

and, due to (4.2) (4.3), the mould $D^\bullet(z)$ simplifies to :

(8.21) $$D^\bullet(z) = z^{\|\bullet\|}\, \mathcal{S}^\bullet_{\text{co}}(e_1)$$

with $e_1 = \exp(2\pi i r_1/\lambda_1)$ and $z^{\|\omega\|} = x_1^{-\|\omega\|/\lambda_1}$. In other words, the ramified variable x_1 factors away and a mere ***constant*** e_1 becomes responsible for the compensation effect. For girators with truly mixed spectrum, on the other hand, we have (see §6) two ramified variables :

(8.21) $$z = (z_1, z_2) = (x_1^{-1/\lambda_1}, x_2^{-1/\lambda_2}) \qquad (\text{with } \lambda_1 < 0 < \lambda_2)$$

which do not factor out and remain essentially involved in the compensation, which is now more directly apparent in (8.18) than (8.19).

To conclude, let us observe that the correspondence $\underset{\bullet}{f} \to \underset{\bullet}{g}$ studied in the preceding section (for girations) is essentially the same (up to ramification) as the correspondence $\underset{\bullet}{f} \to \underset{\bullet}{f}^{\text{inv}}$ of the present section, since $\underset{\bullet}{g}$ is analytically conjugate to $P_{1/\lambda^*} \circ \underset{\bullet}{f}^{\text{inv}} \circ P_{\lambda^*}$. Moreover, like in Remarks 5 and 6 of §7, the "holonomy" correspondence $\text{Ob} \to \underset{\bullet}{F}_{\text{inv}}$ extends, thanks to seriability, to non-analytic classes $\mathcal{C}(M)$ while retaining its basic invariance property.

9. LINK BETWEEN COMPENSATION AND RESURGENCE.

In the case of resonant objects, the seriable linearisators Θ_{ser} still exist, with unchanged expansions into series of compensators, but instead of their usual factorisation $\Theta_{\text{ent}}\Theta_{\text{inv}}^{-1}$ into two hopelessly divergent factors, they admit (see §10) an alternative factorisation $\Theta_{\text{nor}}\Theta_{\text{inv}}^{-1}$ into two *resurgent* (and usually resummable) factors. But since *effective resurgence* prevents regular summation in a *full* neighbourhood of $\underset{\bullet}{0}$ and since on the other hand Θ_{ser}, on account of its seriability, does have a regular sum in such a full neighbourhood, it follows that the resurgence present in both factors Θ_{nor} and $\Theta_{\mathrm{inv}}^{-1}$ must somehow cancel out. This is indeed the case, as we shall see in the next section. But as a preparation, we must first study the interplay of resurgence, seriability and compensation in the simplest possible context, namely that of resurgent monomials. To that end we will have to study the following moulds :

$$
(9.1)\qquad \left\{ \begin{array}{ccc}
\{S^\bullet, \not{S}^\bullet\} & \xrightarrow{1} & \{S^\bullet_{\mathrm{co}}(z), \not{S}^\bullet_{\mathrm{co}}(z)\} \\
\downarrow 3 & & \downarrow 4 \\
\{\mathcal{V}^\bullet(z), \not{\mathcal{V}}^\bullet(z)\} & \xrightarrow{2} & \{\mathcal{V}^\bullet_{\mathrm{co}}(z), \not{\mathcal{V}}^\bullet_{\mathrm{co}}(z)\}
\end{array} \right.
$$

and pay full attention to their intricate symmetries and interconnections.

All eight moulds listed in (9.1) are ***symmetral*** (see §12) and those bracketted together are ***mutually inverse*** (for mould multipliation, see §12). The first pair are scalar-valued moulds; all others are divergent series of decreasing powers of a complex variable z, but for simplicity we drop in this section the usual twiddle ($\sim$) indicative of formalness.

The moulds of the first line are indexed by sequences $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ with real or complex ω_i. They were already defined in (1.22) (1.23) for the first pair and in (4.2) (4.3) for the second pair.

The moulds of the second line are indexed by sequences $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_r)$ with $\eta_i = \binom{\omega_i}{\sigma_i}$ and real or complex ω_i and σ_i.

The $\mathcal{V}^{\boldsymbol{\eta}}, \not{\mathcal{V}}^{\boldsymbol{\eta}}$ are so-called ***resurgence monomials*** and their definition is given by the following induction :

$$
(9.2)\qquad \mathcal{V}^{\emptyset}(z) \stackrel{\mathrm{def}}{\equiv} \not{\mathcal{V}}^{\emptyset}(z) \stackrel{\mathrm{def}}{\equiv} 1
$$

$$
(9.3)\qquad (\omega_1 + \ldots \omega_r + \partial_z)\mathcal{V}^{\eta_1,\ldots,\eta_r}(z) \stackrel{\mathrm{def}}{=} -\mathcal{V}^{\eta_1,\ldots,\eta_{r-1}}(z) z^{\sigma_r}
$$

$$
(9.4)\qquad (\omega_1 + \ldots \omega_r + \partial_z)\,\not{\mathcal{V}}^{\eta_1,\ldots,\eta_r}(z) \stackrel{\mathrm{def}}{=} +z^{\sigma_1}\,\not{\mathcal{V}}^{\eta_2,\ldots,\eta_r}(z)
$$

The $\mathcal{V}^{\boldsymbol{\eta}}_{\mathrm{co}}, \not{\mathcal{V}}^{\boldsymbol{\eta}}_{\mathrm{co}}$ depend on a complex parameter ρ, which is left unwritten for the sake of brevity. They are called ***compensated resurgence monomials***, for reasons that will become plain in a moment (Proposition 9.1). Their shortest definition is by means of the mould factorisation :

$$\mathcal{V}^{\bullet}_{\mathrm{co}}(z) \overset{\mathrm{def}}{=} (z^{\|\bullet\|\rho}\, \mathcal{V\!\!\!/}^{\bullet}(z+\rho\log z)) \times (\mathcal{V}^{\bullet}(z)) \tag{9.5}$$

$$\mathcal{V\!\!\!/}^{\bullet}_{\mathrm{co}}(z) \overset{\mathrm{def}}{=} (\mathcal{V\!\!\!/}^{\bullet}(z)) \times (z^{\|\bullet\|\rho}\mathcal{V}^{\bullet}(z+\rho\log z)) \tag{9.6}$$

(with $\|\bullet\| = \omega_1 + \ldots \omega_r$ as usual) which mirrors the definition of $S^{\bullet}_{co}, \mathcal{S\!\!\!/}^{\bullet}_{co}$ in terms $S^{\bullet}, \mathcal{S\!\!\!/}^{\bullet}$.

Proposition 9.1. (Compensation of resurgence in $\mathcal{V}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$ and $\mathcal{V\!\!\!/}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$)

Each monomial $\mathcal{V}^{\eta}_{\mathrm{co}}(z)$ and $\mathcal{V\!\!\!/}^{\eta}_{\mathrm{co}}(z)$ is a resurgence constant, meaning that all their alien derivatives vanish :

$$\Delta_{\omega_0}\mathcal{V}^{\boldsymbol{\eta}}_{\mathrm{co}}(z) \equiv \Delta_{\omega_0}\mathcal{V\!\!\!/}^{\boldsymbol{\eta}}_{\mathrm{co}}(z) \equiv 0 \qquad (\forall \omega_0 \in \underset{\bullet}{\mathbb{C}};\ \forall \boldsymbol{\eta} = (\eta_1,\ldots,\eta_r)) \tag{9.7}$$

Proof : We start from the alien derivation rules for resurgent monomials (see e.g. [E.1] or [E.3]) :

$$\Delta_{\omega_0}\mathcal{V}^{\boldsymbol{\eta}}(z) = \sum_{\boldsymbol{\eta}=\boldsymbol{\eta}^1\boldsymbol{\eta}^2} V^{\boldsymbol{\eta}^1}_{\omega_0}\mathcal{V}^{\boldsymbol{\eta}^2}(z) \tag{9.8}$$

$$\Delta_{\omega_0}\mathcal{V\!\!\!/}^{\boldsymbol{\eta}}(z) = -\sum_{\boldsymbol{\eta}=\boldsymbol{\eta}^1\boldsymbol{\eta}^2} \mathcal{V\!\!\!/}^{\boldsymbol{\eta}^1}(z) V^{\boldsymbol{\eta}^2}_{\omega_0} \tag{9.9}$$

Applying this to the factorisations (9.5) and (9.6) in conjunction with the rule (12.39), which in the present instance yields :

$$\Delta_{\omega}(z^{\|\bullet\|\rho}\mathcal{A}^{\boldsymbol{\eta}}(z+\rho\log z)) \equiv (\Delta_{\omega}\mathcal{A}^{\boldsymbol{\eta}})(z+\rho\log z) \tag{9.10}$$

(with $\mathcal{A}^{\bullet} = \mathcal{V}^{\bullet}$ or $\mathcal{V\!\!\!/}^{\bullet}$) we immediately arrive at (9.7).

Proposition 9.2. (Seriability of $\mathcal{V}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$ and $\mathcal{V\!\!\!/}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$)

Each monomial $\mathcal{V}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$ and $\mathcal{V\!\!\!/}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$, belongs to $z^{\|\boldsymbol{\sigma}\|}\mathbb{C}[[z^{-1}, z^{-1}\log z]]$ (with $\|\boldsymbol{\sigma}\| = \sigma_1 + \ldots \sigma_r$) as well as to the algebra $\mathrm{Ser}(z^{-1})$ *of* ***seriable series*** *($z \sim \infty$) and can therefore be resummed by the procedure of §3 in a full neighbourhood of $\underset{\bullet}{0} \in \underset{\bullet}{\mathbb{C}}$. Moreover, each $\mathcal{V}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$ and $\mathcal{V\!\!\!/}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$ is also* ***compensable****, with the explicit expansions (9.29) and (9.30) into convergent series of compensators.*

Proof : The factorisations (4.2) (4.3) and (9.5) (9.6) make explicit the horizontal arrows 1 and 2 in diagram (9.1). The way to prove Proposition 9.2 is by studying the vertical arrows 3 and 4. The following two lemmas will do this for us :

Lemma 9.1. *($\{\mathcal{V}^\bullet(z), \not{\mathcal{V}}^\bullet(z)\}$ in terms of $\{\mathcal{S}^\bullet, \not{\mathcal{S}}^\bullet\}$).* .

$$\mathcal{V}^{\boldsymbol{\eta}}(z) = (z+\partial_{\omega_1})^{\sigma_1}\dots(z+\partial_{\omega_r})^{\sigma_r}\mathcal{S}^{\boldsymbol{\omega}} \tag{9.11}$$

$$\not{\mathcal{V}}^{\boldsymbol{\eta}}(z) = (z+\partial_{\omega_1})^{\sigma_1}\dots(z+\partial_{\omega_r})^{\sigma_r}\not{\mathcal{S}}^{\boldsymbol{\omega}} \tag{9.12}$$

Lemma 9.2. *($\{\mathcal{V}^\bullet_{\text{co}}(z), \not{\mathcal{V}}^\bullet_{\text{co}}(z)\}$ in terms of $\{\mathcal{S}^\bullet_{\text{co}}(z), \not{\mathcal{S}}^\bullet_{\text{co}}(z)\}$)* .

$$\mathcal{V}^{\boldsymbol{\eta}}_{\text{co}}(z) = (z+\partial_{\omega_1})^{\sigma_1}\dots(z+\partial_{\omega_r})^{\sigma_r}\mathcal{S}^{\boldsymbol{\omega}}_{\text{co}}(z) \tag{9.13}$$

$$\not{\mathcal{V}}^{\boldsymbol{\eta}}_{\text{co}}(z) = (z+\partial_{\omega_1})^{\sigma_1}\dots(z+\partial_{\omega_r})^{\sigma_r}\not{\mathcal{S}}^{\boldsymbol{\omega}}_{\text{co}}(z) \tag{9.14}$$

Here $(z+\partial_\omega)^\sigma$ is short-hand for :

$$(z+\partial_\omega)^\sigma = z^\sigma(1+z^{-1}\partial_\omega)^\sigma = z^\sigma + \sum_{n\geq 1}\frac{\sigma(\sigma-1)\dots(\sigma-n+1)}{n!}\,z^{\sigma-n}(\partial_\omega)^n \tag{9.15}$$

with $\partial_\omega = \partial/\partial\omega$. To prove the above lemmas, we now subdot all $\boldsymbol{\omega}$-indexed or $\boldsymbol{\eta}$-indexed moulds to denote multiplication by the same factor $e^{\|\boldsymbol{\omega}\|z}$ in both cases :

$$A^{\boldsymbol{\omega}} \mapsto \underset{\bullet}{A}^{\boldsymbol{\omega}} \overset{\text{def}}{=} e^{\|\boldsymbol{\omega}\|z}A^{\boldsymbol{\omega}} \tag{9.16}$$

$$A^{\boldsymbol{\eta}} \mapsto \underset{\bullet}{A}^{\boldsymbol{\eta}} \overset{\text{def}}{=} e^{\|\boldsymbol{\omega}\|z}A^{\eta} \tag{9.17}$$

and we observe that the identites (9.11), (9.12), (9.13) (9.14) simplify to :

$$\underset{\bullet}{\mathcal{V}}^{\boldsymbol{\eta}}(z) = (\partial_{\omega_1})^{\sigma_1}\dots(\partial_{\omega_r})^{\sigma_r}\underset{\bullet}{\mathcal{S}}^{\boldsymbol{\omega}} \tag{9.18}$$

$$\underset{\bullet}{\not{\mathcal{V}}}^{\boldsymbol{\eta}}(z) = (\partial_{\omega_1})^{\sigma_1}\dots(\partial_{\omega_r})^{\sigma_r}\underset{\bullet}{\not{\mathcal{S}}}^{\boldsymbol{\omega}} \tag{9.19}$$

$$\underset{\bullet\,\text{co}}{\mathcal{V}}^{\boldsymbol{\eta}}(z) = (\partial_{\omega_1})^{\sigma_1}\dots(\partial_{\omega_r})^{\sigma_r}\underset{\bullet\,\text{co}}{\mathcal{S}}^{\boldsymbol{\omega}}(z) \tag{9.20}$$

$$\underset{\bullet\,\mathrm{co}}{\mathcal{V}\!\!\!/}^{\boldsymbol{\eta}}(z) = (\partial_{\omega_1})^{\sigma_1} \dots (\partial_{\omega_r})^{\sigma_r} \underset{\bullet\,\mathrm{co}}{\mathcal{S}\!\!\!/}^{\boldsymbol{\omega}}(z) \tag{9.21}$$

although their interpretation is still according to (9.15) after elimination of the exponential factors $e^{\|\boldsymbol{\omega}\| z}$. We readily check that the series $\underset{\bullet}{\mathcal{V}}^{\boldsymbol{\eta}}(z)$ and $\underset{\bullet}{\mathcal{V}\!\!\!/}^{\boldsymbol{\eta}}(z)$ *as defined* by (9.18) (9.19) satisfy the induction :

$$\partial_z \underset{\bullet}{\mathcal{V}}^{\eta_1,\dots,\eta_r}(z) = -\underset{\bullet}{\mathcal{V}}^{\eta_1,\dots,\eta_{r-1}}(z).(e^{\omega_r z} z^{\sigma_r}) \tag{9.22}$$

$$\partial_z \underset{\bullet}{\mathcal{V}\!\!\!/}^{\eta_1,\dots,\eta_r}(z) = (e^{\omega_1 z} z^{\sigma_1}).\underset{\bullet}{\mathcal{V}\!\!\!/}^{\eta_2,\dots,\eta_r}(z) \tag{9.23}$$

which is equivalent to the relations (9.3) (9.4) which characterize $\mathcal{V}^{\eta}(z)$ and $\mathcal{V}\!\!\!/^{\eta}(z)$. This proves lemma 9.1.

The derivation of (9.13) (9.14) is not so straightforward. Putting $z_* = z + \rho log z$ we observe that (9.5) (9.6) translate into the relations :

$$\partial_z \underset{\bullet\,\mathrm{co}}{\mathcal{V}}^{\eta_1,\dots,\eta_r}(z) = \begin{cases} +\left(\frac{dz_*}{dz}\right)(e^{\omega_1 z_*} z_*^{\sigma_1}) \left(\underset{\bullet\,\mathrm{co}}{\mathcal{V}}^{\eta_2,\dots,\eta_r}(z)\right) \\ -\left(\underset{\bullet\,\mathrm{co}}{\mathcal{V}}^{\eta_1,\dots,\eta_{r-1}}(z)\right)(e^{\omega_r z} z^{\sigma_r}) \end{cases} \tag{9.24}$$

$$\partial_z \underset{\bullet\,\mathrm{co}}{\mathcal{V}\!\!\!/}^{\eta_1,\dots,\eta_r}(z) = \begin{cases} +\left(e^{\omega_r 1 z} z^{\sigma_1}\right) \left(\underset{\bullet\,\mathrm{co}}{\mathcal{V}\!\!\!/}^{\eta_2,\dots,\eta_r}(z)\right) \\ -\left(\underset{\bullet\,\mathrm{co}}{\mathcal{V}\!\!\!/}^{\eta_1,\dots,\eta_{r-1}}(z)\right)(e^{\omega_r z_*} z_*^{\sigma_r})\left(\frac{dz_*}{dz}\right) \end{cases} \tag{9.25}$$

which provides an inductive definition of $\underset{\bullet\,\mathrm{co}}{\mathcal{V}}^{\boldsymbol{\eta}}(z)$ and $\underset{\bullet\,\mathrm{co}}{\mathcal{V}\!\!\!/}^{\boldsymbol{\eta}}(z)$. Next, using the differentiation rules (4.4) (4.5) for the compensators $\underset{\bullet\,\mathrm{co}}{\mathcal{S}}^{\boldsymbol{\omega}}(z)$ and $\underset{\bullet\,\mathrm{co}}{\mathcal{S}\!\!\!/}^{\boldsymbol{\omega}}(z)$, we check that, *if we now regard* $\underset{\bullet\,\mathrm{co}}{\mathcal{V}}^{\boldsymbol{\eta}}(z)$ *and* $\underset{\bullet\,\mathrm{co}}{\mathcal{V}\!\!\!/}^{\boldsymbol{\eta}}(z)$ *as being defined by* (9.20) (9.21), they still satisfy the characteristic induction (9.24) (9.25). This shows the equivalence of both definitions and proves lemma 9.2.

Now, we require complex coefficient $c^{\mathbf{n}}_{\boldsymbol{\sigma}}$ and $d^{\mathbf{n}}_{\boldsymbol{\sigma}}$ defined by the generating function (9.27) :

$$d^{n_1,\dots,n_r}_{\sigma_1,\dots,\sigma_r} \stackrel{\text{def}}{=} (-1)^{n_1+\dots n_r}\, c^{n_r,\dots,n_1}_{\sigma_r,\dots,\sigma_1} \tag{9.26}$$

$$\prod_{i=1}^{r} (z + \partial_{\omega_i})^{\sigma_i} \stackrel{\text{def}}{=} \sum c^{n_1,\dots,n_r}_{\sigma_1,\dots,\sigma_r}\, z^{(\sigma_1+\dots\sigma_r)-(n_1+\dots n_r)}\, \frac{(\partial_{\tau_1})^{n_1}}{n_1!} \dots \frac{(\partial_{\tau_r})^{n_r}}{n_r!} \tag{9.27}$$

$$\partial_{\omega_i} \stackrel{\text{def}}{=} \partial_{\tau_i} + \partial_{\tau_{i+1}} + \dots \partial_{\tau_r} \qquad (i = 1, \dots, r) \tag{9.28}$$

Using these coefficients, we find that equations (9.20) (9.21) immediately translate into the following expansions of $\mathcal{V}_{co}^{\boldsymbol{\eta}}(z)$ and $\mathcal{V}_{co}^{\boldsymbol{\eta}}(z)$ into ***convergent series of compensators*** :

$$\mathcal{V}_{co}^{\eta_1,\dots,\eta_r}(z) = z^{\sigma_1+\dots\sigma_r} \sum_{n_i} c_{\sigma_1,\dots,\sigma_r}^{n_1,\dots,n_r} (\rho/z)^{n_1+\dots n_r} \mathcal{S}^{\omega_1,0^{(n_1)},\dots,\omega_r,0^{(n_r)}}(z^\rho) \tag{9.28bis}$$

$$\mathcal{V}_{co}^{\eta_1,\dots,\eta_r}(z) = z^{\sigma_1+\dots\sigma_r} \sum_{n_i} d_{\sigma_1,\dots,\sigma_r}^{n_1,\dots,n_r} (\rho/z)^{n_1+\dots n_r} \mathcal{S}^{0^{(n_1)},\omega_1,\dots,0^{(n_r)},\omega_r}(z^\rho) \tag{9.28ter}$$

where $0^{(n_i)}$ denotes a subsequence of n_i consecutive zeros. This vindicates the claim about the *compensability* (in the sense of §3) of $\mathcal{V}_{co}^{\boldsymbol{\eta}}(z)$ and $\mathcal{V}_{co}^{\boldsymbol{\eta}}(z)$ and terminates the proof of Proposition 9.2.

Remark : For the simplest conceivable instance of *compensated resurgence*, corresponding to $r = 1$ and $\eta_1 = \binom{\omega_1}{\sigma_1} = \binom{+1}{-1}$, see (3.28) (3.29).

10. RAMIFIED LINEARISATION OF RESONANT OBJECTS.

Resonant local objects present us with a rich situation, as they fall within the purview of two of the formal statements F_i and two of the analytic statements A_i of §1. Indeed, on the one hand, resonant local objects can be brought to a *normal form* (containing only resonant monomials with formally invariant coefficients) by a change of coordinates that is not merely *formal* (statement F_2) but also *resurgent* (statement A_2) with regular summability in *sectorial neighbourhoods* of $\underset{\bullet}{0}$. On the other hand, resonant, non-degenerate local objects admit *ramified linearisations* which are not merely *formal* (statement F_3) but may be chosen to be *seriable* (statement A_3) with regular summability in full spiral-like neighbourhoods of $\underset{\bullet}{0}$.

This dual nature, and in particular the passage from sectorial to full neighbourhoods of $\underset{\bullet}{0}$, calls for a close study of the interplay between, first F_2 and F_3, then A_2 and A_3.

Interplay of F_2 and F_3 : from formal-entire normalisation to formal-ramified linearisation.

Statement F_2 is classic and statement F_3 follows from statement A_3 with its explicit expansions of the linearisation map into convergent sums of compensators. Nonetheless, it is interesting to study directly the passage from *normal* to *linear* forms, if only because it helps understand the compensation of resurgence. (See (10.25) and (10.26)).

We will restrict ourselves to vector fields, which we will write as dynamical systems (with respect to a complex time z) first in a ***normal, entire*** (but generally ***non-analytic)*** ***chart*** $x = (x_i)$, then in a ***linear, ramified chart*** $y = (y_i)$:

$$\partial_z \; x_i = x_i.\{\lambda_i + \sum c_{i,m} x^m\} \qquad (1 \le i \le \nu; \; <\lambda, m>= 0) \tag{10.1}$$

$$\partial_z \; y_i = y_i \lambda_i \qquad (1 \le i \le \nu) \tag{10.2}$$

and we will look for ramified changes of coordinates :

$$x_i = k_i^{\text{lin}}(y) \quad and \quad y_i = h_i^{\text{lin}}(x) \qquad (1 \le i \le \nu) \tag{10.3}$$

of the form (1.35) or the still better form (1.36)

Example 1. Vector fields with one degree of resonance : Simplest type.

This is the type $(p, \rho) = (1, 0)$. See e.g. [E.3]. In this case, the normal form is as follows :

$$\partial_z \; x_i = x_i\{\lambda_i + \tau_i x^m\} \qquad (1 \le i \le \nu) \tag{10.4}$$

with $x^m = x_1^{m_1} \ldots x_\nu^{m_\nu}$ $(m_i \in \mathbb{N})$ and :

$$<m, \lambda>= 0 \; ; \quad <m, \tau>= -1 \tag{10.5}$$

For any real vector $\alpha = (\alpha_1, \ldots, \alpha_\nu)$ normalised to λ :

$$<\alpha, \lambda> \stackrel{\text{def}}{=} \sum \alpha_i \lambda_i = 1 \qquad (\alpha_i \in \mathbb{R}) \tag{10.6}$$

we have the ramified linearisation :

$$x_i = k_i^{\text{lin}}(y) = y_i\{1 + y^m \log y^\alpha\}^{\tau_i} \qquad (1 \le i \le \nu) \tag{10.7}$$

with an explicit reciprocal linearisation :

$$y_i = h_i^{\text{lin}}(x) = x_i\{1 + x^m \log x^\alpha\}^{\tau_i} \qquad (1 \le i \le \nu) \tag{10.8}$$

which, unlike (10.7), is valid only if α is also orthogonal to τ (i.e. $<\alpha, \tau>= 0$). Both linearisations have the required form (1.35) and when :

$$\{m_i = 0\} \Rightarrow \{\alpha_i = 0\} \qquad (\forall i) \tag{10.9}$$

they even have the better form (1.36).

The quickest way to check (10.7) (10.8) is to compare the formal integrals of (10.4) and (10.2), which read :

(10.10) $$x_i = u_i e^{\lambda_i z} z^{\tau_i} \qquad \text{with} \quad u^m \equiv 1$$

(10.11) $$y_i = v_i e^{\lambda_i z} \qquad \text{with} \quad v^m \log v^\alpha \equiv -1$$

with the identities :

(10.12) $$z \equiv x^{-m} \equiv y^{-m} + \log y^\alpha$$

(10.13) $$u_i \equiv v_i (v^m)^{\tau_i} \qquad (\forall i)$$

Example 2. Vector fields with one degree of resonance : General type.

This is the type with general "level" $p \in N^*$ and general "residue" $\rho \in \mathbb{C}$. The normal form is now as follows :

(10.14) $$\partial_z x_i = x_i (1 + \rho x^{mp})^{-1} . (\lambda_i + \sum_{q=1}^{p-1} \lambda_i^q x^{mq} + \tau_i x^{mp})$$

with (10.5) still in force and

(10.15) $$< m, \lambda^q > \stackrel{\text{def}}{=} \sum m_i \lambda_i^q = 0 \qquad (1 \leq q \leq p-1)$$

From (10.14) we calculate the formal integral :

(10.16) $$x_i = u_i z_*^{\tau_i} \exp \left\{ \lambda_i z_* + \sum_{q=1}^{p-1} \Lambda_i^q z_*^{1-q/p} \right\}$$

with

(10.17) $$z \equiv z_* + \rho \log z_*; \ u^m \equiv 1; \ (1 - q/p) \Lambda_i^q = \lambda_i^q \ \ (\forall i)$$

This can be rewritten as :

(10.18) $$x_i = u_i \ \exp \Lambda_i(z) \qquad (\text{with } \Lambda_i(z) = \lambda_i z + o(z))$$

and yields an explicit ramified linearisation :

(10.19) $$x_i = k_i^{\text{lin}}(y) = y_i \exp\{\Lambda_i(y^{-pm} + \log y^\alpha) - \Lambda_i(y^{-pm}) - \lambda_i \log y^\alpha\}$$

which is indeed of the form :

$$k_i^{\text{lin}}(y) = y_i\{1 + \varphi_i(y^m, y^m \log y^\alpha)\} \tag{10.20}$$

$$\varphi_i(a,b) \in \mathbb{C}\{a,b\} \tag{10.20bis}$$

and hence of the form (1.35) or even (1.36) if α is chosen so as to meet condition (10.9) (on top of (10.6)).

Example 3. Vector fields with one degree of resonance : Nihilent type.

This is the case $p = +\infty$, but although fields with finite level p have $(\nu - 1)(p+1)+1$ formal invariants in their normal form, the fields of infinite level still have finitely many invariants. We may however start from any normal or prenormal (see statement F_2 in §1) form :

$$\partial_z x_i = x_i\{\lambda_i + \sum_{1\le q} \lambda_i^q x^{mq}\} \tag{10.21}$$

and arrive at the ramified linearisation :

$$x_i = k_i^{\text{lin}}(y) = y_i \exp\left\{(\log y^\alpha)(\sum_{1\le q} \lambda_i^q y^{mq})\right\} \tag{10.22}$$

with (10.6) and (10.20) as usual.

Example 4. Vector fields with several degrees of resonance.

Whenever the resonance degree μ is ≥ 2, there exist infinitely many formal invariants and the normal form (10.1) is generally divergent. One may still construct formal integrals (1.49) and deduce from them ramified linearisations, as was done in the previous examples, but one may also take recourse to explicit expansions into sums of compensators, such as (5.5) or (6.21). Note that these expansions, when applied to normal (or prenormal) forms, undergo a drastic simplification, because the compensators in them carry only vanishing indexes ($\omega_i \equiv 0$), so that they become polynomials of $\log x_i$, for one or several variables x_i. In the case of vector fields with a general complex spectrum (irreducible to girators), it is advisable to introduce *one ramified variable per degree of resonance* and to make repeated use of *mould mixing* (see §6) in order to construct compensators of several variables.

Example 5. One-dimensional, resonant diffeos.

They are local diffeos $x \mapsto f(x)$ with multipliers $\ell = f'(0)$ equal to roots of unity. If nilpotent (i.e. if some of their iterates reduce to the identity map) such diffeos possess analytic linearisations. Otherwise, they have a definite level $p \in \mathbb{N}^*$ and can be linearised by a formal ramified change of coordinates $y = \tilde{h}(x)$ with $\tilde{h}(x)$ in $\mathbb{C}[[x, x^p \log x]]$.

But let us leave these formal aspects for the real thing, namely the analytic study, with the phenomenon of resurgence compensation leading to seriability.

Interplay of F_2 and F_3 : from resurgent normalisation to seriable linearisation.

As we saw in §8, for non-resonant local objects, seriable linearisators admit the canonical factorisation

$$\Theta_{\mathrm{ser}} = \Theta_{\mathrm{ent}}\ \Theta_{\mathrm{inv}}^{-1} \tag{10.23}$$

For resonant local objects this becomes :

$$\Theta_{\mathrm{ser}} = \Theta_{\mathrm{nor}}\ \Theta_{\mathrm{inv}}^{-1} \tag{10.24}$$

or alternatively :

$$\Theta_{\mathrm{ser}} = (\Theta_{\mathrm{nor}}\ \Theta_{\mathrm{lin}}) \cdot (\Theta_{\mathrm{inv}}\ \Theta_{\mathrm{lin}})^{-1} \tag{10.24bis}$$

with an *entire change of coordinates* Θ_{nor} taking the object into a normal form, and a simple *ramified change of coordinates* Θ_{lin} taking that normal form into the linear form (such as in the above examples). Like (10.23), the factorisation (10.24) is essentially unique, at least for a given normal form, but the factorisation (10.24bis) has a larger element of arbitrariness in it.

Now, *for any seriable linearisation* Θ_{ser}, *both factors in* (10.24) *are resurgent and satisfy the same bridge equation* :

$$[\ \overset{\bullet}{\Delta}_\omega, \Theta_{\mathrm{nor}}] = -\Theta_{\mathrm{nor}}\mathbb{A}_\omega \qquad (\forall\omega\in\Omega) \tag{10.25}$$

$$[\ \overset{\bullet}{\Delta}_\omega, \Theta_{\mathrm{inv}}] = -\Theta_{\mathrm{inv}}\mathbb{A}_\omega \qquad (\forall\omega\in\Omega) \tag{10.26}$$

with respect to the same "resurgence lattice" Ω *and the same holomorphic invariants* $\mathbb{A}_\omega$. Here, the $\overset{\bullet}{\Delta}_\omega$ are the usual *pointed alien derivations* (see §12) acting on $z = x^{-m}$ (for level one) or $z = x^{-mp}$ (for level p) or some suitable resonance monomial in the multiresonant case (see [E.3] or [E.7]) and the $\mathbb{A}_\omega$ are ordinary differential operators intrinsically attached to the object (see §1).

Equation (10.25) is of course nothing but equation (1.48) in operatorial form (and with a minus sign), but the remarkable thing is that Θ_{inv} should satisfy the same equation. The immediate consequence is that :

$$[\ \overset{\bullet}{\Delta}_\omega, \Theta_{\mathrm{inv}}^{-1}] = +\mathbb{A}_\omega\Theta_{\mathrm{inv}}^{-1} \tag{10.26bis}$$

so that for each ω :

$$[\,\overset{\bullet}{\Delta}_\omega, \Theta_{\mathrm{ser}}] = [\,\overset{\bullet}{\Delta}_\omega, \Theta_{\mathrm{nor}}]\Theta_{\mathrm{inv}}^{-1} + \Theta_{\mathrm{nor}}[\,\overset{\bullet}{\Delta}_\omega, \Theta_{\mathrm{inv}}^{-1}] = 0 \tag{10.27}$$

which means that Θ_{ser} is a *resurgence constant* : the resurgence in both factors of (10.24) cancels out. This in turn explains how Θ_{ser} can have a regular sum in a full neighbourhood of $\underset{\bullet}{0}$, while the resurgent factors Θ_{nor} and $\Theta_{\mathrm{inv}}^{-1}$ have regular sums in sectorial neighbourhoods only.

We may also observe (reasoning on a girator for definiteness) that while the factors $\Theta_{\mathrm{nor}}^{\pm 1}$ and $\Theta_{\mathrm{inv}}^{\pm 1}$ admit convergent mould expansions involving the moulds $\mathcal{V}^\bullet(z)$ and $\mathcal{V}\!\!\!/^\bullet(z)$ of §8, the product $\Theta_{\mathrm{ser}}^{\pm 1}$ admits simultaneously two types of expansions : one (already encountered in §5 and §6) involving the *plain compensators* $\mathcal{S}_{\mathrm{co}}^\bullet(z)$ and $\mathcal{S}\!\!\!/_{\mathrm{co}}^\bullet(z)$; and another involving the *compensated resurgence monomials* $\mathcal{V}_{\mathrm{co}}^\bullet(z)$ and $\mathcal{V}\!\!\!/_{\mathrm{co}}^\bullet(z)$ studied in §9.

We cannot enter into details for lack of space, but the following two examples will clarify the preceding points.

Example 6. Vector fields with one degree of resonance. Compensation of resurgence.

This is example 1 studied from the analytic angle. Here, the general bridge equation (1.48) involves holomorphic invariants of the form (see [E.3] or [E.7]) :

$$\mathbb{A}_\omega = u^{n(\omega)}.\left\{A_\omega^0 \partial_z + \sum_{i=1}^{\nu-1} A_\omega^i \; u_i \partial_{u_i}\right\} \tag{10.28}$$

with constant coefficients A_ω^i. In the normal chart this becomes :

$$\mathbb{A}_\omega = I^\omega \; {}^*\mathbb{A}_\omega \tag{10.29}$$

with :

$${}^*\mathbb{A}_\omega = \left\{{}^*A_\omega^0 X^{\mathrm{nor}} + \sum_{i=1}^{\nu} {}^*A_\omega^i x_i \partial_{x_i}\right\} \tag{10.30}$$

$$\sum_{i=1}^{\nu} m_i \; {}^*A_\omega^i = 0 \tag{10.31}$$

$$I^\omega = x^{\sigma(\omega)} e^{-\omega x^{-m}} \tag{10.32}$$

$$\sigma_i(\omega) = n_i(\omega) + \left(\sum_{i=1}^{\nu} \tau_i n_i(\omega)\right) \; m_i \quad \text{if} \quad \omega = \sum_{i=1}^{\nu} n_i(\omega)\lambda_i \tag{10.33}$$

Due to the resonance, the $n_i(\omega)$ are not determined by ω, but the $\sigma_i(\omega)$ are. If we now revert to the notations of example 1 and choose :

$$\alpha_1 = 1/\lambda_1;\ \alpha_2 = \alpha_3 = \ldots \alpha_\nu = 0 \tag{10.34}$$

the normal variable x_i and the linear variables y_i are connected by :

$$x^{-m} \equiv y^{-m} + (1/\lambda_1)\log y_1 \tag{10.35}$$

Using the *unpointed* alien derivations Δ_ω and Δ'_ω relative to $z = x^{-m}$ and $z' = y^{-m}$, we obtain for the factors (10.24) of Θ_{ser} two parallel bridge equations :

$$[\Delta_\omega, \Theta_{\mathrm{nor}}] = -\Theta_{\mathrm{nor}}.(x^{\sigma(\omega)}\ {}^*\mathbb{A}_\omega) \tag{10.36}$$

$$[\Delta'_\omega, \Theta_{\mathrm{inv}}] = -\Theta_{\mathrm{inv}}.(y^{\sigma(\omega)}\ y_1^{-\omega/\lambda_1}{}^*\mathbb{A}_\omega) \tag{10.37}$$

which make perfect sense if we interpret them component-wise, isolating on both sides the power series in x^m and y^m and regarding the other variables (minus one) as mere parameters, inert under alien differentiation. In this identification process, the factor $y_1^{-\omega/\lambda_1}$ must of course be interpreted as an entire series of $\log y_1$, via the obvious formula.

Example 7. Resonant local diffeos of C. Compensation of resurgence.

This is the *analytic counterpart* of example 5, as well as the *resonant counterpart* of the example studied under remark 5 of section 7.

Let us consider two local diffeos f_j with multipliers $\ell_j = 1$ and levels $p_j = 1$ (to simplify) but with general residues ρ_j :

$$f_j(x) = x + a_j x^2 + (a_j)^2(1+\rho_j)x^3 + \ldots \qquad (j = 1,2;\ a_j \neq 0) \tag{10.38}$$

Let us also consider the usual formal iterators $\tilde{f}_j^*$ of the form :

$$\tilde{f}_j^*(x) = x^{-1} - \rho_j \log x + \sum_{n\geq 1} c_{j,n} x^n \tag{10.39}$$

and characterised by :

$$\tilde{f}_j^* \circ f_j(x) \equiv 1 + \tilde{f}_j^*(x) \tag{10.40}$$

so that for any $r \in \mathbb{Z}$:

$$\tilde{f}_j^*(e^{2\pi i r}x) \equiv (e^{-2\pi i r}\tilde{f}_j^*(x) - 2\pi i r \rho_j \tag{10.41}$$

Next, consider the formal series $\tilde{f}_{ij}$ of the form :

$$\tilde{f}_{ij}(x) = x + x\tilde{\varphi}_{ij}(x) \ \text{ with } \ \tilde{\varphi}_{ij}(x) \in \mathbf{C}[[x, x\log x]] \tag{10.42}$$

and characterised by :

$$\tilde{f}_i^* = \tilde{f}_{ij} \circ \tilde{f}_j^* \qquad \text{with } (i,j) = (1,2) \text{ or } (2,1) \tag{10.43}$$

If we now denote by $\underset{\bullet}{r}$ any rotation of r turns :

$$\underset{\bullet}{r}\,(x) \overset{\text{def}}{=} e^{2\pi i r}x \qquad (x \in \underset{\bullet}{\mathbf{C}},\ r \in \mathbf{Z}^*) \tag{10.44}$$

and if we assume :

$$a_1 = a_2 = 1; \qquad 2\pi i r(\rho_1 - \rho_2) = 1 \tag{10.45}$$

then, using (10.41), we readily check the following formal linearisation equations :

$$\underset{\bullet}{f_1} = \tilde{f}_{12} \circ \underset{\bullet}{r} \circ \tilde{f}_{21} \ \text{ with } \ \underset{\bullet}{f_1} \overset{\text{def}}{=} \underset{\bullet}{r} \circ f_1 \tag{10.46}$$

$$\underset{\bullet}{f_2}^{-1} = \tilde{f}_{21} \circ \underset{\bullet}{r} \circ \tilde{f}_{12} \ \text{ with } \ \underset{\bullet}{f_2} \overset{\text{def}}{=} \underset{\bullet}{r}^{-1} \circ f_2 \tag{10.47}$$

But iterators $\tilde{f}_j^*(x)$ are resurgent in $z = x^{-1}$ and satisfy the bridge equation :

$$\dot{\Delta}_\omega\, \tilde{f}_j^* = -A_\omega \exp(-\omega \tilde{f}_j^*) \qquad (\forall \omega \in 2\pi i \mathbf{Z}^*) \tag{10.48}$$

Therefore if we assume that f_1 and f_2 have the same holomorphic invariants, that is to say, if the coefficient A_ω in (10.48) do not depend on j, we immediately check that the series $\tilde{f}_{ij}$ are *resurgence constants* :

$$\dot{\Delta}_\omega\, \tilde{f}_{ij} \equiv 0 \qquad (\forall \omega) \tag{10.49}$$

with *seriable sums* f_{ij} which, unlike the sectorial sums f_i^* of the iterators, are defined and regular in a full, spiral-like neighbourhood of $\underset{\bullet}{0}$, so that the linearisation equations (10.46) (10.47), from formal, become effective. This is possibly (after the examples of §9) one of the simplest instances of resurgence compensation.

11. CONCLUSION AND SUMMARY.

We have shown in this paper that ***girators and girations admit seriable linearisations valid in ramified neighbourhoods*** of $\underset{\bullet}{0} \in \underset{\bullet}{\mathbb{C}}^{\nu}$. Actually, as we propose to show in a follow-up investigation, all local analytic objects (whether vector fields or diffeos) under the sole assumption of non-degeneracy (no vanishing multipliers) also admit seriable linearisations, although for them the relevant constructions are somewhat less explicit than for girators and girations, and the geometric-dynamical interpretation somewhat less direct than (1.63) (1.64).

To conclude, let us review the main merits of seriable linearisation.

(M1) Like analytic-entire linearisation, seriable-ramified linearisation $x_i = k_i^{\mathrm{ser}}(y)$ is "quasianalytic" in the sense that the formal series $\tilde{k}_i^{\mathrm{ser}}$ constructively determine k_i^{ser}.

(M2) But unlike analytic linearisation, whose existence is guaranteed only in the absence of the three complications C_1, C_2, C_3 (resonance, quasiresonance, nihilence), seriable linearisation holds ***in all cases***, subject only to non-degeneracy.

(M3) Even when analytic linearisation is available (according to statement A_2 of section 1), the size of the Siegel linearisation domain is no continuous function of the multipliers of the object. It is highly sensitive to their arithmetical properties : when the sum S in (1.19) is large, the Siegel domain tends to be very small. The spiral-like domain of seriable linearisation, on the other hand, depends continuously on the multipliers and is always fairly large.

(M4) The optimal spiralling speed of the domain of seriable linearisation can be explicitly specified in function of the resonance or quasiresonance of the multipliers.

(M5) For resonant objects, there is a remarkable interplay between ***seriability*** and ***compensated resurgence***, which gets reflected in the symmetry between ***plain compensators*** and ***compensated resurgence monomials*** (see §9 ans §10).

(M6) Seriable linearisation leads to a generalisation of the notion of ***holonomy*** (see §8) and provides a systematic, unified method for constructing intrinsic correspondences between analytic conjugacy classes of local objects (see §8).

(M7) Seriable linearisation, suitably reinterpreted, ***extends to non-analytic local objects*** (defined for instance by power series in Gevrey or more general Carleman classes), which are ***ex hypothesi*** beyond the pale of geometry, and yet give rise to non-trivial conjugacy problems (see for ex. §7, remark 6, for the extension of the Douady-Ghys lemma to non-analytic classes).

12. REMINDER ABOUT MOULDS; ARBORIFICATION; RESURGENT FUNCTIONS; ALIEN DERIVATIONS.

The following ultraconcise reminders are no substitute for full definitions, but references are appended to each subsection.

A. Moulds and comoulds.

A *mould* $A^\bullet$ is a family $\{A^{\boldsymbol{\omega}} = A^{\omega_1,\dots,\omega_r}\}$ of elements of a commutative algebra $\mathcal{A}$, indexed by sequences $\boldsymbol{\omega}$ of ω_i ranging through a commutative semigroup Ω. ***Mould multiplication*** is defined by :

$$A_3^\bullet = A_1^\bullet \times A_2^\bullet \Leftrightarrow A_3^{\omega_1,\dots,\omega_r} = \sum_{i=0}^{r} A_1^{\omega_1,\dots,\omega_i}\, A_2^{\omega_{i+1},\dots,\omega_r} \tag{12.1}$$

For any three sequences $\boldsymbol{\omega}, \boldsymbol{\omega}', \boldsymbol{\omega}''$ we put :

$$\mathrm{sh}\begin{pmatrix} \boldsymbol{\omega}', \boldsymbol{\omega}'' \\ \boldsymbol{\omega} \end{pmatrix} = \text{number of order} - \text{preserving bijections of } \boldsymbol{\omega}' \oplus \boldsymbol{\omega}'' \text{ into } \boldsymbol{\omega} \tag{12.2}$$

$$\mathrm{ctsh}\begin{pmatrix} \boldsymbol{\omega}', \boldsymbol{\omega}'' \\ \boldsymbol{\omega} \end{pmatrix} = \text{number of order} - \text{preserving surjections of } \boldsymbol{\omega}' \oplus \boldsymbol{\omega}'' \text{ into } \boldsymbol{\omega} \tag{12.3}$$

A mould $A^\bullet$ is said to be *symmetral* or *symmetrel* if $A^\emptyset = 1$ and if for any pair $\boldsymbol{\omega}', \boldsymbol{\omega}''$:

$$A^{\boldsymbol{\omega}'} A^{\boldsymbol{\omega}''} = \sum \mathrm{sh}\begin{pmatrix} \boldsymbol{\omega}', \boldsymbol{\omega}'' \\ \boldsymbol{\omega} \end{pmatrix} A^{\boldsymbol{\omega}} \qquad (\textit{symmetral}) \tag{12.4}$$

$$A^{\boldsymbol{\omega}'} A^{\boldsymbol{\omega}''} = \sum \mathrm{ctsh}\begin{pmatrix} \boldsymbol{\omega}', \boldsymbol{\omega}'' \\ \boldsymbol{\omega} \end{pmatrix} A^{\boldsymbol{\omega}} \qquad (\textit{symmetrel}) \tag{12.5}$$

For instance :

$$A^{\omega_1} A^{\omega_2,\omega_3} = A^{\omega_1,\omega_2,\omega_3} + A^{\omega_2,\omega_1,\omega_3} + A^{\omega_2,\omega_3,\omega_1} \qquad (\textit{symmetral}) \tag{12.6}$$

$$A^{\omega_1} A^{\omega_2,\omega_3} = \text{idem } + A^{\omega_1+\omega_2,\omega_3} + A^{\omega_2,\omega_1+\omega_3} \qquad (\textit{symmetrel}) \tag{12.7}$$

The simplest moulds are $1^\bullet$ and $I^\bullet$:

$$1^\emptyset = 1;\ 1^{\omega_1,\dots,\omega_r} = 0 \qquad \text{if } r \geq 1 \tag{12.8}$$

$$I^{\omega_1} \equiv 1\ (\forall \omega_1);\ I^{\omega_1,\dots,\omega_r} = 0 \qquad \text{if } r = 0 \text{ or } r \geq 2. \tag{12.9}$$

$1^\bullet$ is neutral respective to mould multiplication X and $I^\bullet$ is neutral respective to mould composition 0 (which we don't require here).

Comoulds $\mathbf{B}_\bullet$ have sequences $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$ as lower indexes. They usually assume values in bialgebras $\mathcal{B}$ of differential operators, endowed with a non-commutative, associative product and a commutative co-product σ :

$$\sigma : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B} \tag{12.10}$$

Most comoulds are either cosymmetral or cosymmetrel :

$$\sigma(\mathbf{B}_{\boldsymbol{\omega}}) = \sum \mathrm{sh}\begin{pmatrix} \boldsymbol{\omega}', \boldsymbol{\omega}'' \\ \boldsymbol{\omega} \end{pmatrix} \mathbf{B}_{\boldsymbol{\omega}'} \otimes \mathbf{B}_{\boldsymbol{\omega}''} \ (cosymmetral) \tag{12.11}$$

$$\sigma(\mathbf{B}_{\boldsymbol{\omega}}) = \sum \mathrm{ctsh}\begin{pmatrix} \boldsymbol{\omega}', \boldsymbol{\omega}'' \\ \boldsymbol{\omega} \end{pmatrix} \mathbf{B}_{\boldsymbol{\omega}'} \otimes \mathbf{B}_{\boldsymbol{\omega}''} \ (cosymmetrel) \tag{12.12}$$

Thus if the $\mathbf{B}_{\omega_i}$ are ordinary derivations, we have :

$$\sigma(\mathbf{B}_{\omega_i}) = \mathbf{B}_{\omega_i} \otimes 1 + 1 \otimes \mathbf{B}_{\omega_i} \tag{12.13}$$

and the comould $\mathbf{B}_\bullet$ defined by:

$$\mathbf{B}_{\omega_1,\ldots,\omega_r} \stackrel{\text{def}}{=} \mathbf{B}_{\omega_r} \ldots \mathbf{B}_{\omega_1} \tag{12.14}$$

is cosymmetral.

The *contraction* of a mould $A^\bullet$ with a comould $\mathbf{B}_\bullet$ is a sum of the form :

$$\Theta = \sum A^\bullet B_\bullet = \sum A^{\boldsymbol{\omega}} \mathbf{B}_{\boldsymbol{\omega}} \tag{12.15}$$

extending to *all* sequences $\boldsymbol{\omega}$, including $\boldsymbol{\omega} = \emptyset$. If $A^\bullet$ and $\mathbf{B}_\bullet$ are well-matched (i.e. symmetral and cosymmetral, or symmetrel and cosymmetrel), their contraction Θ is a formal automorphism :

$$\sigma(\Theta) = \Theta \otimes \Theta \tag{12.16.}$$

For details and examples, see [E.1],[E.3],[E.8].

B. Arborification and coarborification.

Arborification means replacing fully ordered sequences $\boldsymbol{\omega}$ by sequences $\overset{<}{\boldsymbol{\omega}}$ with an *arborescent* partial order on them (i.e. each element ω_i has at most one immediate antecedent ω_{i_-}).

For any pair $(\boldsymbol{\omega}, \overset{<}{\boldsymbol{\omega}})$ we define sh $\begin{pmatrix} \overset{<}{\boldsymbol{\omega}} \\ \boldsymbol{\omega} \end{pmatrix}$ and ctsh $\begin{pmatrix} \overset{<}{\boldsymbol{\omega}} \\ \boldsymbol{\omega} \end{pmatrix}$ as the number of order-preserving bijections (resp. surjections) of $\overset{<}{\boldsymbol{\omega}}$ into $\boldsymbol{\omega}$.

Symmetral arborification-coarborification obeys the formulae :

$$A^{\overset{<}{\boldsymbol{\omega}}} = \sum \mathrm{sh}\begin{pmatrix} \overset{<}{\boldsymbol{\omega}} \\ \boldsymbol{\omega} \end{pmatrix} A^{\boldsymbol{\omega}} \tag{12.17}$$

(12.18) $$\mathbf{B}_{\omega} = \sum \mathrm{sh}\begin{pmatrix} \overset{<}{\omega} \\ \omega \end{pmatrix} \mathbf{B}_{\overset{<}{\omega}}$$

Symmetrel arborification-coarborification obeys the formulae :

(12.19) $$A^{\overset{<}{\omega}} = \sum \mathrm{ctsh}\begin{pmatrix} \overset{<}{\omega} \\ \omega \end{pmatrix} A^{\omega}$$

(12.20) $$\mathbf{B}_{\omega} = \sum \mathrm{ctsh}\begin{pmatrix} \overset{<}{\omega} \\ \omega \end{pmatrix} \mathbf{B}_{\overset{<}{\omega}}$$

Whereas (12.17) or (12.19) define $A^{\overset{<}{\omega}}$, neither (12.18) nor (12.20) suffice to determine $\mathbf{B}_{\overset{<}{\omega}}$. So we add other natural conditions. For instance, for differential operators $\mathbf{B}_{\omega_i}$ as in (12.13) and a comould $\mathbf{B}_{\bullet}$ as in (12.14), we define $\mathbf{B}_{\overset{<}{\bullet}}$ by :

(12.21) $$\mathbf{B}_{\overset{<}{\omega}}.\varphi \overset{\text{def}}{=} \{\mathbf{B}_{\omega_r}, \ldots, \mathbf{B}_{\omega_1}\}\, \varphi$$

by letting each $\mathbf{B}_{\omega_i}$ in $\{\ldots\}$ act on φ alone if ω_i has no antecedent in $\overset{<}{\omega}$, or on the coefficients of $\mathbf{B}_{\omega_{i_-}}$ if ω_{i_-} is the (unique) antecedent of ω_i in $\overset{<}{\omega}$.

Since arborification and coarborification (whether symmetral or symmetrel) are dual operations, we have :

(12.22) $$\Theta = \sum A^{\bullet}\mathbf{B}_{\bullet} = \sum A^{\overset{<}{\bullet}}\,\mathbf{B}_{\overset{<}{\bullet}}$$

but in very numerous instances the seemingly harmless passage from $\bullet$ to $\overset{<}{\bullet}$ restores *normal convergence.*

For details and examples, see [E.8].

C. The algebras RES of resurgent functions.

There are three models : *formal, convolutive, geometric.* The *formal model* $\widetilde{\mathrm{R}}\mathrm{ES}$ consists of formal series like :

(12.23) $$\tilde{\varphi}(z) = \sum a_n z^{-n} \qquad (n \in \mathbf{N}^* \text{ or } \mathbf{R}^+)$$

or of a more general type :

$$\tilde{\varphi}(z) = \sum a_n \epsilon_n(z) \qquad (\epsilon_n(z) \gg \epsilon_{n+1}(z)) \tag{12.24}$$

with each "monomial" $\epsilon_n(z)$ decreasing subexponentially as $z \to \underset{\bullet}{\infty} \in \underset{\bullet}{\mathbb{C}}$.

The product here is the (formal) multiplication of (formal) series.

The *convolutive model* $\overset{\nabla}{\mathrm{RES}}$ consists of pairs :

$$\overset{\nabla}{\varphi} = (\overset{\wedge}{\varphi}, \overset{\vee}{\varphi}) = \text{(minor, major)} \tag{12.25}$$

of germs $\overset{\wedge}{\varphi}(\zeta)$ and $\overset{\vee}{\varphi}(\zeta)$ at $\underset{\bullet}{0} \in \underset{\bullet}{\mathbb{C}}$ in the ζ-plane conjugate to the z-plane.

The *minors* $\overset{\wedge}{\varphi}(\zeta)$ are assumed to be *endlessly continuable* in the ζ-plane, along any (discretely punctured) broken line. The *majors* $\overset{\vee}{\varphi}(\zeta)$ are defined up to regular germs at $\underset{\bullet}{0}$ and relate to the minors as follows (for ζ close to $\underset{\bullet}{0}$) :

$$\overset{\wedge}{\varphi}(\zeta) = \overset{\vee}{\varphi}(\zeta) - \overset{\vee}{\varphi}(e^{-2\pi i}\zeta) \tag{12.26}$$

The product here is *convolution* ($*$), which for integrable functions (at $\underset{\bullet}{0}$) reduces to *minor convolution* :

$$\overset{\wedge}{\varphi}_1 * \overset{\wedge}{\varphi}_2(\zeta) = \int_0^\zeta \overset{\wedge}{\varphi}_1(\zeta_1)\, \overset{\wedge}{\varphi}_2(\zeta - \zeta_1) d\zeta_1 \qquad (\text{for } \zeta \sim \underset{\bullet}{0}) \tag{12.27}$$

The *geometric model* ${}^\theta\mathrm{RES}$ of direction θ consists of holomorphic germs $\varphi(z)$ which are defined and have (at most) subexponential growth in half-planes $\mathrm{Re}(ze^{i\theta}) \geq x(\varphi) > 0$. The product here is ordinary, point-wise multiplication.

The passage from the convolutive model to the geometric model of direction θ is via the *Laplace transform* $\mathcal{L}$, which reduces to :

$$\overset{\nabla}{\varphi}(\zeta) \to \varphi(z) = \int_{\arg \zeta = \theta} \overset{\wedge}{\varphi}(\zeta) e^{-\zeta z} d\zeta \tag{12.28}$$

when $\overset{\nabla}{\varphi}(\zeta)$ is integrable at $\underset{\bullet}{0}$.

The passage from the geometric (resp. formal) models to the convolutive model is via the effective (resp. formal or term-wise) *Borel transform* $\mathcal{B}$:

$$\varphi(z) \mapsto \overset{\wedge}{\varphi}(\zeta) = \frac{1}{2\pi i} \int_{\infty_1}^{\infty_2} \varphi(z) e^{+\zeta z} dz \tag{12.29}$$

$$\varphi(z) \mapsto \overset{\vee}{\varphi}(\zeta) = \frac{1}{2\pi i} \int_u^{\infty_3} \varphi(z) e^{+\zeta z} dz \tag{12.30}$$

for $\infty_1, \infty_2, \infty_3$ suitably located at infinity.

For details and examples, see [E.1], [E.3], [E.5], [E.8].

D. Alien derivations.

These are linear operators Δ_ω, with indexes ω in $\underset{\bullet}{\mathbb{C}}$, which are useful on three accounts.

First, they are easy to handle, being derivations :

$$\Delta_\omega(\overset{\nabla}{\varphi}_1 * \overset{\nabla}{\varphi}_2) = (\Delta_\omega \overset{\nabla}{\varphi}_1) * \overset{\nabla}{\varphi}_2 + \overset{\nabla}{\varphi}_1 * (\Delta_\omega \overset{\nabla}{\varphi}_2) \tag{12.31}$$

Second, they measure the singularities of minors $\overset{\wedge}{\varphi}(\zeta)$ over the point ω, which is essential, since those singularities are responsible for the divergence of the corresponding series $\tilde{\varphi}(z)$ in the formal model.

Third, they enable us to describe, by means of so-called *resurgence equations* :

$$E_\omega((\overset{\nabla}{\varphi}, \Delta_\omega \overset{\nabla}{\varphi}) \equiv 0 \tag{12.32}$$

the close connection which usually exists between the behaviour of $\overset{\wedge}{\varphi}(\zeta)$ near $\underset{\bullet}{0}$ and near its other singularities ω. This self-reproduction property is an outstanding feature of all *resurgent functions* of natural origin.

Alien derivations act as follows on the *convolutive model* :

$$\Delta_\omega : \qquad \overset{\nabla}{\varphi} = (\overset{\wedge}{\varphi}, \overset{\vee}{\varphi}) \mapsto \overset{\nabla}{\varphi}_\omega = (\overset{\wedge}{\varphi}_\omega, \overset{\vee}{\varphi}_\omega) \tag{12.33}$$

with :

$$\overset{\wedge}{\varphi}_\omega(\zeta) \overset{\text{def}}{=} \sum_{\epsilon_i = \pm} \epsilon_r \delta^{\epsilon_1, \ldots, \epsilon_{r-1}} \overset{\wedge}{\varphi}{}^{\epsilon_1, \ldots, \epsilon_r}_{\omega_1, \ldots, \omega_r} (\zeta + \omega) \tag{12.34}$$

$$\overset{\vee}{\varphi}_\omega(\zeta) \overset{\text{def}}{=} \sum_{\epsilon_i = \pm} \delta^{\epsilon_1, \ldots, \epsilon_{r-1}} \overset{\wedge}{\varphi}{}^{\epsilon_1, \ldots, \epsilon_{r-1}, +}_{\omega_1, \ldots, \omega_{r-1}, \omega_r} (\zeta + \omega) \tag{12.35}$$

first for ζ close to $\overset{\bullet}{0}$ on the radius $\arg\zeta = \arg\omega$, and then in the large by analytic continuation. Here, the ω_i (with $\omega_r = \omega$) denote the successive singularities of $\hat{\varphi}\,(\zeta)$ on the segment $[\underset{\bullet}{0},\omega]$ and $\hat{\varphi}^{\epsilon_1,\dots,\epsilon_r}_{\omega_1,\dots,\omega_r}(\zeta+\omega)$ denotes the determination of $\hat{\varphi}\,(\zeta+\omega)$ that corresponds to the right (resp. left) circumvention of ω_i if $\epsilon_i = +$ (resp. $\epsilon_i = -$). Lastly, the *weights* $\delta^\bullet$ are given by :

$$\delta^{\epsilon_1,\dots,\epsilon_{r-1}} = \delta_{p,q} = \frac{p!q!}{(p+q+1)!} = \frac{p!q!}{r!} \tag{12.36}$$

where p and q are the number of $+$ and $-$ signs in $(\epsilon_1,\dots,\epsilon_{r-1})$.

The alien derivations Δ_ω *generate a free Lie algebra.* There being no danger of confusion, we retain the same symbols Δ_ω to denote their (pulled-back) action in the *multiplicative models* (formal or geometric). We also introduce the *pointed alien derivations* :

$$\overset{\bullet}{\Delta}_\omega \overset{\text{def}}{=} e^{-\overset{\bullet}{\omega} z}\,\Delta_\omega \tag{12.37}$$

which act in the *multiplicative models tensored by exponentials* $\exp(-\overset{\bullet}{\omega}\,z)$, and obey the rules :

$$[\,\overset{\bullet}{\Delta}_\omega,\partial] \equiv 0 \qquad (\text{with } \partial = \partial/\partial z) \tag{12.38}$$

$$\overset{\bullet}{\Delta}_\omega\,(\varphi\circ f) \equiv (\,\overset{\bullet}{\Delta}_\omega\,\varphi)\circ f + (\partial\varphi)\circ f\;.\;\overset{\bullet}{\Delta}_\omega\,f \tag{12.39}$$

if $f(z) = z + o(z)$.

For details and examples, see [E.1], [E.3], [E.7], [E.8].

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The notion of compensation, as used in this article, along with its applications to vector fields and diffeomorphisms, was introduced by us in 1986 and expounded in an unpublished manuscript. We mentioned special applications in [E.4] and [E.5]. It was also brought to our attention that Eliason had already made use of the word "compensation" (see [Eℓ]) but with another meaning and in a different context (with entire series and without ramified variables).

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Complex dynamics in higher dimension. I

Astérisque, tome 222 (1994), p. 201-231
<http://www.numdam.org/item?id=AST_1994__222__201_0>

COMPLEX DYNAMICS IN HIGHER DIMENSION. I.

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1. Introduction

Given a polynomial equation $P(x) = a_n x^n + \cdots + a_0 = 0$, in one variable, x, one asks what are the solutions. The main advantage of the complex number system is that if x is allowed to be complex then the solutions always exist. However, to find the actual values of the solutions is impossible. One can only find approximate solutions.

A traditional method is Newton's method. One starts with a value x_0 and finds inductively a sequence $\{x_n\}$, $x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)}$. If x_0 is near a simple root, this sequence converges to this root.

Shröder [Sc] was the first to study Newton's method for complex numbers. He was led to the study of iteration of the rational function $R(z) := z - \frac{P(z)}{P'(z)}$. Mainly he studied the local behavior of rational functions near attractive fixed points, $R(z_0) = z_0$, $| R'(z_0) |< 1$. He actually studied general rational functions rather than the special ones from Newton's method, because he discovered that Newton's could be replaced by infinitely many rational functions.

If instead one considers polynomial equation in two (or more) variables, $P(x,y) = Q(x,y) = 0$, where $P(x,y) = \sum a_{n,m} x^n y^n$, one is likewise led to study iteration of rational fonctions in two or more variables. In this case Newton's method takes the inductive form

$$(x_{n+1}, y_{n+1}) = R(x_n, y_n)$$

where the rational map R is given by

$$R(x,y) = (x,y) - \frac{1}{P_x Q_y - P_y Q_x}(PQ_y - QP_y, QP_x - PQ_x).$$

As in one variable there is an infinite family of other rational maps that could be used as well. The simplest one is $R(x,y) = (x,y) - A(P,Q)$ where A is a constant matrix equal to the inverse of the Jacobian matrix of $(P, Q)'$ at some point close to a fixed point.

More precisely consider the mapping in $\mathbf{C}^2$ given by

$$(P, Q) = \left(\frac{1}{2}x - (x - 2y)^2, \frac{1}{2}y - x^2\right).$$

Obviously $(0, 0)$ is a root of the system $P = 0$, $Q = 0$. If we apply the Schroder method to this system with $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ we get in homogeneous coordinate the mapping $f[x : y : t] = [2(z - 2w)^2 : 2z^2 : t^2]$ which is a holomorphic map in $\mathbf{P}^2$. For any invertible matrix B the map $g = I - B(P, Q)$ in homogeneous coordinates is a holomorphic map of $\mathbf{P}^2$.

The analogue of Schröder's study indicated above is the local study of R around attractive fixed points. This was studied extensively in dimension 2, starting by Leau [Le] in the end of the last century and carried through by Lattes [La] and Fatou [Fa].

As far as the global study of iteration is concerned, that is, if we start with a value x_0, perhaps far from the roots of the polynomial, does Newton's method still converge? Schröder was able to decide this only for quadratic polynomials. In this case he found that there is a circle in the sphere, $\mathbf{C} \cup \{\infty\} = \mathbf{P}^1$, dividing it into two open sets. Each of these open sets contains one of the two roots and each starting point x_0 in these open sets give a sequence $\{x_n\}$ by Newton's method converging to the root in the same open set.

The global study of iteration in one variable only became possible in the second decade of this century after the introduction by Montel of normal families, in particular the normality of the family of holomorphic maps from the unit disc to the sphere $\mathbf{P}^1$ minus three points is crucial.

The analogue of this in higher dimensions was unavailable at Fatou's time, so essentially all the study of iteration of rational maps was local.

In this paper we will discuss mainly global questions of iteration of rational maps in higher dimension. The analogue of Montel's Theorem comes from the Kobayashi hyperbolicity of the complement of certain complex hypersurfaces in $\mathbf{P}^k$, the complex projective space of dimension k.

We start, here, with some basic facts on holomorphic endomorphisms of $\mathbf{P}^k$ (i.e. holomorphic maps). For simplicity we sometimes restrict our attention to $\mathbf{P}^2$. In a forthcoming paper we will study the structure of Julia's and Fatou components.

In section 2 we discuss some basic properties of holomorphic and meromorphic maps on $\mathbf{P}^k$.

Section 3 is an estimate of the number of periodic points, counted without multiplicity.

Then in section 4 we give a description of the family of exceptional maps. This family generalizes the map $z \to z^d$ on $\mathbf{P}^1$ which is characterized by the property that the points $\{0, \infty\}$ are totally invariant.

In section 5 we discuss the Kobayashi hyperbolicity of the complement of part of the critical orbit. We show that this holds for a Zariski dense set of maps. See Theorem 5.3 for a precise statement.

In section 6 we consider expansive properties of the map in the complement of the closure of the critical orbit under suitable hyperbolicity assumption and finally, in section 7, we classify critically finite maps in $\mathbf{P}^2$.

2. Holomorphic maps, Fatou and Julia sets.

We first describe the holomorphic maps from $\mathbf{P}^k$ to $\mathbf{P}^k$.

THEOREM 2.1. *Let f be a non constant holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$. Then f is given in homogeneous coordinates by $[f_0 : f_1 : \cdots : f_k]$ where each f_j is a homogeneous polynomial of degree d and the f_j have no common zero except the origin.*

Proof. Let $[z_0 : z_1 : \cdots : z_k]$ be homogeneous coordinates in $\mathbf{P}^k$. We can assume that the image of f is not contained in any $(z_j = 0)$ (otherwise rotate coordinates). By the Weierstrass-Hurwitz Theorem [Gu] it follows that each of the meromorphic functions $\frac{z_i}{z_0} \circ f$ is a quotient of two homogeneous polynomials $\frac{F_j}{G_j}$ of the same degree.

Let $\tilde{F}$ denote the map $[\tilde{F}_0 : \cdots : \tilde{F}_k]$ where the $\tilde{F}_j$'s are homogeneous polynomials of the same degree obtained by dividing out common factors from the polynomials $\frac{F_j}{G_j} \cdot \Pi G_\ell$. We will show that $\tilde{F}$ is a lifting of f to $\mathbf{C}^{k+1}$. For this we only need to show that the $\tilde{F}_j$ have no common zeros except the origin. Suppose to the contrary that $p \in \mathbf{C}^{k+1} \mid (0)$ is a common zero. Choose a local lifting $\tilde{f} = [f_0 : \cdots : f_k]$ of f in a neighborhood of p. We may assume that one of the $f_j \equiv 1$. Say $f_0 \equiv 1$. Then it follows that $\tilde{F}_j = \tilde{F}_0 f_j$ and that $\tilde{F}_0(p) = 0$. But this implies that the common zero set of the $\tilde{F}_j$ is a complex hypersurface, which implies that they have a common factor, contradicting that we have already divided out all common factors.

Let $\mathcal{H}$ denote the space of non constant holomorphic maps on $\mathbf{P}^k$ and $\mathcal{H}_d$ the holomorphic maps given by homogeneous polynomials of degree d. Observe that $\mathcal{H}$ is stable under composition.

On the other hand there are the (not necessarily everywhere well defined) maps of degree d from $\mathbf{P}^k$ to $\mathbf{P}^k$, which are given in homogeneous coordinates by $[f_0 : f_1 : \cdots : f_k]$, but now the degree d homogeneous polynomials f_j are allowed to have common zeros. This later space is easily identified with $\mathbf{P}^N$ where $N = (k+1)\frac{(d+k)!}{d!k!} - 1$.

We will also consider the space $\mathcal{M}_d$ of meromorphic maps, consisting of those $[f_0 : \cdots : f_k]$ in P^N which have maximal rank on some nonempty open set.

It follows from Bezout's theorem that for f in $\mathcal{H}_d$ the number of points in $f^{-1}(a)$ is d^k counting multiplicity. Consequently f is of maximal rank and hence $\mathcal{H}_d \subset \mathcal{M}_d \subset \mathbf{P}^N$.

In analogy with one complex variable we define the Fatou set and Julia sets of a holomorphic map f in $\mathcal{H}_d$. More precisely we have the following definition.

DEFINITION 2.2. Given $f : \mathbf{P}^k \to \mathbf{P}^k$ in $\mathcal{H}_d$, $0 \leq \ell \leq k-1$, a point $p \in \mathbf{P}^k$ belongs to the Fatou set $\mathcal{F}_\ell$ if there exists a neighborhood $U(p)$ such that for every $q \in U(p)$ there exists a complex variety X_q through q of codimension ℓ and $\{f^n \mid_{X_q}\}$ is equicontinuous.

Observe that $\mathcal{F}_0$ is the largest open set where $\{f^n\}$ is equicontinuous. We call $\mathcal{F}_0$ the Fatou set. Also observe that each $\mathcal{F}_\ell$ is open and $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{k-1}$.

Correspondingly, let $\mathcal{J}_\ell = \mathbf{P}^k \backslash \mathcal{F}_\ell$. We call $\mathcal{J}_0$ the Julia set.

THEOREM 2.3. *The Julia set of a holomorphic map in*$\mathcal{H}_d$, $d \geq 2$, *is always non empty.*

Proof. Assume $\mathcal{F}_0 = \mathbf{P}^k$. Let h be the limit of a subsequence $\{f^{n_k}\}$. Then h is a non constant holomorphic map of finite degree. As in one variable this contradicts that the degrees of f^{n_k} are unbounded, see [Mi].

THEOREM 2.4. *The sets*$\mathcal{H}_d$ *and*$\mathcal{M}_d$ *are Zariski open sets of*$\mathbf{P}^N$. *In particular*$\mathcal{H}_d$ *and*$\mathcal{M}_d$ *are connected. If* $f \in \mathcal{H}_d$, *then the critical set of* f *is an algebraic variety of degree*$(k+1)(d-1)$.

Proof. Consider $\sum$, the analytic set in $\mathbf{P}^N \times \mathbf{P}^k$ defined by the equation $f(z) = 0$. Let $\sum_d$ be the projection of $\sum$ in $\mathbf{P}^N$. Then $\sum_d$ is equal to $\mathbf{P}^N \backslash \mathcal{H}_d$. Since the projection is proper, by Tarski Theorem, we get that $\sum_d$ is an analytic set. The fact that $\mathcal{M}_d$ is Zariski open follows from the equation $\mathbf{P}^N \backslash \mathcal{M}_d = \bigcap_{z \in \mathbf{P}^k} \{f; J(f, z) = 0\}$ where $J(f, z)$ is the Jacobian of the lifted map on $\mathbf{C}^{k+1}$.

Let $f = [f_0 : f_1 : \cdots : f_k] \in \mathcal{H}_d$. Then the critical set of f is the projection under the canonical map $\Pi : \mathbf{C}^{k+1} \to \mathbf{P}^k$ of the critical set of $(f_0, f_1, \cdots, f_k)$. The degree of this set is clearly $\leq (k+1)(d-1)$ so the degree of the critical set of f in $\mathbf{P}^k$ is $\leq (k+1)(d-1)$. On the other hand for the map $[z_0^d : z_1^d : \cdots : z_k^d]$ the critical set has degree $(k+1)(d-1)$ and therefore since $\mathcal{H}_d$ is connected we get that for any $f \in \mathcal{H}_d$ the critical set has degree exactly $(k+1)(d-1)$.

3. Periodic points.

We show that the fixed point set of $f \in \mathcal{H}_d$ is discrete. More precisely we have :

THEOREM 3.1. *Let$f$: $\mathbf{P}^k \to \mathbf{P}^k$ be a holomorphic map of degree$d \geq 2$, and$g$ be a meromorphic map of degree$d' < d$. LetΩ be the Zariski dense open subset where$g$ is holomorphic. There can be no compact algebraic curve$Z$ such that$f = g$ on$Z \cap \Omega$ and$Z \cap \Omega \neq \emptyset$. If$g$ is holomorphic, the number of points where$f = g$ equals$(d^{k+1} - d'^{k+1})/(d-d')$ counted with multiplicity if$g$ is holomorphic.*

The proof is to apply the Bezout theorem. We include it for the convenience of the reader.

*Proof.*Suppose that $\{f = g\}$ contains an open set of a compact complex subvariety Z of dimension one. We will arrive at a contradiction. First we write $f = [f_0 : f_1 : \cdots : f_k]$ and $g = [g_0 : g_1 : \cdots : g_k]$, where the f_j's are homogeneous holomorphic polynomials of degree $d > 1$ and the g_j's are homogeneous holomorphic polynomials of degree $d' \geq 1$, $d' < d$. Hence we can lift f, g to map on $\mathbf{C}^{k+1}$, $F = (f_0, f_1, \cdots, f_k)$, $G = (g_0, \cdots, g_k)$. Also the variety Z lifts to conic two dimensional surface X in $\mathbf{C}^{k+1}$. Introduce one more complex variable t and consider the $k+1$ equations $f_j - t^{d-d'} g_j = 0$. These are homogeneous equations of degree d in $\mathbf{C}^{k+2}$. Hence the common zero set is a conic complex variety Y. Consider at first the intersection with the hyperplane $t = 0$. Then the equations reduce to $f_0 = f_1 = \cdots = f_k = 0$. Since f is a well defined holomorphic map this zero set consists only of the origin. The natural projection of Y to P^{k+1} is therefore a compact complex space which does not intersect the hyperplane $t = 0$ at infinity. Hence the image is a compact subvariety in $\mathbf{C}^{k+1}$ and hence must be finite. This means that$Y$ consists of a finite number of complex lines in $\mathbf{C}^{k+2}$ through the origin. Suppose next that p is in $Z \cap \Omega$, so $f(p) = g(p)$. Then there exists a complex value $t \neq 0$ and $(z_0, \cdots, z_k) \neq 0$ such that $p = [z_0 : \cdots : z_k]$ and $f_j(z_0, \cdots, z_k) = t^{d-d'} g_j(z_0, \cdots, z_k)$. Hence the point $(z_0, \cdots, z_k, t)$ belongs to Y. But this implies that Y is two dimensional, a contradiction. Hence we have shown that there is no such Z. In the case g is holomorphic this implies that $\{f = g\}$ is finite. Next we need to count the number of points. First we count the number of solutions using Bezout's theorem on the equations $f_j - t^{d-d'}g_j = 0$. There are d^{k+1} of these. However d'^{k+1} of these occur at the point $[0 : 0 : \cdots : 1]$, so this gives $d^{k+1} - d'^{k+1}$ solutions, but rotation of t by a $d - d'$ root of unity produces an equivalent solution, so the total number of solutions to $f = g$ is $(d^{k+1} - d'^{k+1})/(d - d')$. This complete the proof of the Theorem.

Using the above Theorem in the case $g = Id$ we obtain the number of periodic points.

COROLLARY 3.2. *Let* $f : \mathbf{P}^k \to \mathbf{P}^k$, $f \in \mathcal{H}_d$, $d \geq 2$. *The number of periodic points of order* n *counted with multiplicity is* $(d^{n(k+1)} - 1)/(d^n - 1)$.

We show that any holomorphic map of degree $d \geq 2$ has infinitely many disjoint periodic orbits.

THEOREM 3.3. *Let* $f : \mathbf{P}^2 \to \mathbf{P}^2$ *be a holomorphic map of degree* $d \geq 2$. *Then there exists infinitely many distinct periodic orbits.*

Proof. Recall that we have shown in Corollary 3.2 that the iterate f^n has $d^{2n} + d^n + 1$ fixed points counted with multiplicity. So to prove the theorem we only need to control the multiplicity. The control is trivial in dimension one but less obvious in higher dimension.

LEMMA 3.4. *Let 0 be a fixed point for a germ at zero of a local holomorphic map* $f : \mathbf{C}^2 \to \mathbf{C}^2$. *Assume that 0 is an isolated point of* $\{f^n(z,w) = (z,w)\}$ *for all integers* $n \geq 1$. *Then there exists an integer* N *such that for all iterates* f^n, $n \geq 1$, *the inequality* $\| f^n(z,w) - (z,w) \| \geq c_n \| (z,w) \|^N, c_n > 0$ *holds in some neighborhood (depending on* n*) of zero.*

We prove at first the lemma before we continue with the proof of the theorem.

Proof of the Lemma. Let the eigenvalues of f' be α, γ. Then we can assume that the map f has the form $f(z,w) = \alpha z + \beta w + P(z,w), \gamma w + Q(z,w)$ where P, Q vanish to at least second order and $\beta = 0$ if $\alpha \neq \gamma$. Then $f^n = (\alpha^n z + \beta_n w + P_n(z,w), \gamma^n w + Q_n(z,w)))$ for some constants β_n and functions P_n, Q_n vanishing to at least second order. It follows that the estimate holds immediately if $|\alpha|, |\gamma| \neq 1$ with $N = 1$. Next consider the situation where say $1\alpha| \neq 1$, $|\gamma| = 1$. Then $\beta = \beta_n = 0$ so the situation is equivalent to the case with α, γ reversed. We write $\gamma = e^{2\pi i\theta}$ where θ belongs to the unit interval. If θ is irrational it follows again that the estimate in the lemma holds for $N = 1$. Suppose then that $\theta = p/q$ for relatively prime integers p, q where $p = 0$, $q = 1$ or $q \geq 2$ and $p \in \{1, 2, \cdots, q-1\}$. In that case the estimate holds with $N = 1$ for all powers f^n as soon as n is not a multiple of q. To handle the missing case, we can replace f by the iterate f^q. This then is the case when $\gamma = 1$. So we can write $f = (\alpha z + P(z, w), w + Q(z, w))$.

We consider for this the more general case $\alpha \neq 1$ because we will anyhow need this below. The equation $\alpha z + P(z, w) = z$ can be solved implicitly in a small neighborhood of zero. We obtain $z = h(w)$ for some holomorphic function h vanishing at least to second order at the origin. We can now rewrite the map f in terms of holomorphic functions $P'(z, w), Q'(z, w)$ vanishing to at least first order at the origin and $Q''(w)$ vanishing to some finite order $k > 1$ at the origin as:

$$f(z, w) = (\alpha(z - h(w)) + h(w) + (z - h(w))P'(z, w),$$

$$w + Q''(w) + (z - h(w))(Q'(z, w)).$$

For any point (z_0, w_0) close to zero, let $(z_n, w_n) := f^n(z_0, w_0)$. We will write $(z_n, w_n) = (h(w_0) + \Delta_n, w_0 + \delta_n)$ and inductively estimate the error terms (Δ_n, δ_n). So at first, $(\Delta_0, \delta_0) = (z_0 - h(w_0), 0)$. We get $(z_{n+1}, w_{n+1}) = f(z_n, w_n) = (\alpha(h(w_0) + \Delta_n - h(w_0 + \delta_n)) + (h(w_0 + \delta_n) + (h(w_0) + \Delta_n - h(w_0 + \delta_n))P'(h(w_0) + \Delta_n, w_0 + \delta_n), w_0 + \delta_n + Q'(w_0 + \delta_n) + (h(w_0) + \Delta_n - h(w_0 + \delta_n))Q'(h(w_0) + \Delta_n, w_0 + \delta_n)) = o(\Delta_n + h(w_0), w_0 + \delta_n + Q'(w_0))$. Hence it follows that $(\Delta_{n+1}, \delta_{n+1}) = (\alpha\Delta_n + o(\mid \Delta_n \mid, \mid \delta_n \mid), \delta_n + Q'(w_0) + o(\mid \Delta_n \mid, \mid \delta_n \mid))$. From this we inductively prove the estimates $(\Delta_n, \delta_n) = (\alpha^n \Delta_0 + o(\mid \Delta_0, Q'(w_0) \mid))), nQ'(w_0) + o(\mid \Delta_0, Q'(w_0) \mid))$.

With these error estimates we estimate $f^n(z_0, w_0) - (z_0, w_0)$. We get $\mid (z_n - z_0, w_n - w_0) \mid = \mid (\Delta_n - \Delta_0, \delta_n) \mid \geq \mid ((\alpha^n - 1) \mid \Delta_0, nQ'(w_0)) \mid - o(\mid \Delta_0, Q'(w_0) \mid)$. Hence for all n such that $\alpha^n \neq 1$, we have that $\mid f^n(z, w) - (z, w) \mid \geq c_n(\mid z - h(w) \mid + \mid Q'(w) \mid)$ close enough to the origin for some $c_n > 0$. But if $\mid z \mid \leq \mid w \mid$ the second factor is of order of $\mid w \mid^k$ while if $\mid z \mid \geq \mid w \mid$ the first factor is at least $\mid z \mid /2$, so the estimate of the lemma follows with $N = k$ in all these cases. In particular we are done when $\mid \alpha \mid \neq 1$, $\mid \gamma \mid = 1$ (or vice versa).

We continue with the case when both α and γ have modulus one. Suppose at first that $\alpha = \gamma$ and they are irrational rotations. Then $f^n = (\alpha^n z + \beta_n w + P_n(z, w), \gamma^n w + Q_n(z, w))$ as above and clearly the estimate of the Lemma holds with $N = 1$. Suppose next that $\alpha = \gamma = e^{2\pi i p/q}$ where $(p, q) = (0, 1)$

or $q \geq 2$ and $p = 1, \cdots, q-1$ relatively prime to q. Then the estimate of the Lemma holds for all iterates with n not a multiple of q and $N = 1$. It remains to consider iterates which are multiples of q. But this reduces to the case $\alpha = \gamma = 1$ and β arbitrary.

First consider the case when $\beta \neq 0$. We may then assume that $\beta = 1$. So the map has the form $f = (z + w + P(z,w), w + Q(z,w))$ where P, Q vanish to at least second order at the origin. There are several cases to consider. Assume at first that P, Q vanish on the z-axis. Then $f(z,0) = (z,0)$, contradicting that 0 is an isolated point of $\{f(z,w) = (z,w)\}$. Next assume that $P = wP'(z,w)$ while $Q(z,0) = az^\ell + \cdots$ for some integer $\ell \geq 2$ and $a \neq 0$. Inductively we see that $f^n = (z + nw(1 + O(\mid (z,w) \mid)) + O(\mid z^\ell \mid), w + N_n az^\ell + o(\mid (w, z^\ell) \mid))$. If $\mid w \mid \leq \mid z \mid^{\ell - 1/2}$ then the second component of $f^n - Id$ is at least of the order of $\| (z,w) \|^\ell$ while if $\mid w \mid \geq \mid z \mid^{\ell-1/2}$, the first component is at least of the order of $\| (z,w) \|^{\ell - 1/2}$. Hence the estimate in the lemma follows. Now consider the case when $P(z,0) = az^k + \cdots$ for some integer $k \geq 2$ and some constant $a \neq 0$, while $Q(z,w) = wQ'(z,w)$. Then note that $w + P(z,w)$ vanishes on a complex manifold $w = h(z)$ for some holomorphic function h vanishing to finite order at least 2 at the origin.

Make the change of coordinates $w' = w - h(z)$, $z' = z$. In this coordinate system f has the same linear terms. But $f(z', 0) = (z', h(z')Q'(z', h(z')))$ and the second component can only vanish to finite order at the origin. (Otherwise in the (z,w) coordinate system $f(z, h(z)) = (z, h(z))$ and there is a whole curve of fixed points.) But then we are back in the previously considered case.

The next case is when $P(z,0) = az^k + \cdots$, $Q(z,0) = bz^\ell + \cdots$ for integers $k, \ell \geq 2$ and numbers $a, b \neq 0$. In this case we can again use the same coordinate change as above, $w' = w - h(z)$, $z' = z$ with $h(z) + P(z, h(z)) = 0$ to reduce to the case $P(z,0) = 0$.

Next consider the case when $\beta = 0$. Then the map has the form $f(z,w) = (z + p(z,w), w + Q(z,w))$ where P, Q vanish to at least second order. Inductively we can then prove the estimate $f^n = (z + nP + o(\| (P,Q) \|), w + nQ + o(\| (P,Q) \|))$. Hence $\| f^n - Id \| \geq \| (P,Q) \| \geq \| (z,w) \|^N$ for fixed N, close enough to the origin. The last inequality is just the Lojasiewicz inequality since (0,0) is an isolated fixed point of f.

Now we investigate the case when $\alpha \neq \gamma$ but they both have modulus one. In both and θ are irrational, then $f^n(z,w) = \alpha^n z + \cdots, \gamma^n w + \cdots$ and it is clear that $f - Id$ vanishes to first order for any n. Next assusme that φ is irrational and that $\gamma = p/q$ where $(p,q) = (0,1)$ or $q > 1$ and $0 < p < q$ is relatively prime to q. But now this case follows as above when $\mid \alpha \mid \neq 1$ and θ is rational. So assume that both φ and θ are rational. Then for some iterates, the map is of this form with both powers equal one. In that case the above applies. If one of the eigenvalues is one and the other is different from one, then this case is also handled above, and if both are different from one we are also done. Hence we have covered all cases.

Having thus finished the proof of the lemma we can continue with the proof of the theorem. Suppose that there are only finitely many periodic orbits. Then for some point p the multiplicity of $f^n - Id$ at p can be chosen arbitrarily large. Taking local coordinates we can assume $p = 0$. From the lemma we have $| f^n(z,w) - (z,w) | \geq c_n | (z,w) |^N$. Let P_N denote the Taylor polynomial of $f^n - Id$ of order N. Then for r sufficiently small $| f^n - Id - P_N | < | f^n - Id |$ on the sphere around zero of radius r. Hence by Rouché's Theorem [AY] the multiplicity of $f^n - Id$ at zero is at most N^2, a contradiction.

4. Exceptional varieties.

For a rational map on $\mathbf{P}^1$, a finite set E is exceptional if $f^{-1}(E) = E$. Similarly we introduce the following notion.

DEFINITION 4.1. *Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be in $\mathcal{H}_d$ and V a compact subvariety in $\mathbf{P}^k$. Then V is exceptional if $f^{-1}(V) = V$.*

If V is exceptional, necessarily $f(V) = V$ also. Let $f : \mathbf{P}^k \to \mathbf{P}^k$ denote a holomorphic map of degree $d \geq 2$. Assume f has an exceptional hypersurface V. Note that replacing f with an iterate we may assume that each irreducible branch of V is mapped to itself. Hence any collection of irreducible branches of V is also exceptional.

PROPOSITION 4.2. *Let $V_1, \cdots, V_\ell$ denote irreducible branches of the exceptional variety of f. Assume V_i is the zero set of the irreducible polynomial h_i. Then $\sum$ degree$(h_i) \leq k+1$. In particular there are at most $k+1$ irreducible branches of the exceptional set, and if there are $k+1$ they are all linear.*

Proof. Let $\pi : \mathbf{C}^{k+1} \to \mathbf{P}^k$ be the natural projection. We denote by $\tilde{V}_i$ the pull back of V_i to $\mathbf{C}^{k+1}$. Denote also by f the lifted map $f = (f_0, f_1, \cdots, f_k)$ to $\mathbf{C}^{k+1}$. Then $\tilde{V}_i$ is an irreducible homogeneous complex hypersurface, so we can write it as $\tilde{V}_i = \{h_i(z_0, z_1, \cdots, z_k) = 0\}$ for an irreducible homogeneous polynomial h_i.

LEMMA 4.3. *Let X be an exceptional hypersurface for $f \in \mathcal{H}_d$. Assume $X =$*

$\{h = 0\}$ whereh is a homogeneous polynomial. Then there exists a non zero constantc such thath, f satisfy the Böttcher functional equation$h \circ f = ch^d$.

Proof. Since X is totally invariant, the polynomial $h \circ f$ only vanishes on $\tilde{X}$ the pull back of X to $\mathbf{C}^{k+1}$. However the degree of this composition is $(\deg h) \cdot d$. Hence the equality follows.

We continue the proof of the proposition. An exceptional variety given by $h = 0$ as above is part of the critical set of f. Moreover the Jacobian determinant of f has $(\Pi h_i)^{d-1}$ as a factor. From Theorem 2.4 it follows that $\sum \deg h_i \leq k + 1$.

PROPOSITION 4.4. *The set of holomorphic maps without exceptional hypersurfaces is a nonempty Zariski open set in$\mathcal{H}_d$.*

Proof. We identify homogeneous polynomials h of degree $\ell \leq k + 1$ with their space of coefficients $\mathbf{P}^{N_\ell}$ and we define $\sum_\ell := \{(f, h) \in \mathcal{H}_d \times \mathbf{P}^{N_\ell}$; $h \circ f = ch^d$ for some constant $c\}$. If f has an exceptional variety then (f, h) is in $\sum_\ell$ for some h and some $\ell \leq k + 1$. The projection of $\sum_\ell$ on $\mathcal{H}_\ell$ is again an analytic variety. Since there exists one map in each $\mathcal{H}_d$ which is not exceptional, the proposition follows.

We will next discuss the various possibilities in $\mathbf{P}^2$. At first suppose that there is an exceptional variety V_1 which is a line. By a linear change of coordinates we can suppose that this line is given by $\{[z : w : t]\}; t = 0$, i.e. V_1 is the hyperplane at infinity. Since V_1 is totally invariant this means that f is a polynomial map of degree d on $\mathbf{C}^2(z, w)$. Moreover the condition that the map is well defined on $\mathbf{P}^2$ simply means that the highest degree terms of the two components of the polynomial are of degree d and have no common zeros except at the origin. This means that if the hyperplane at infinity is exceptional, then the map has the form $[z : w : t] \to [f_0 : f_1 : t^d]$ where the functions $f_0(z, w, 0)$, $f_1(z, w, 0)$ have nondegenerate degree d-terms. Next assume that the exceptional variety contains two complex lines. We may then assume that they are given by $t = 0$, $w = 0$ respectively. It follows that f_1 has the form w^d. Hence in this case the map has the form $[z : w : t] \to [f_0 : w^d : t^d]$ where f_0 is a homogeneous polynomial of degree d with a nonzero coefficients in front of the z^d term.

Then consider the case when the exceptional variety contains three lines. We may assume that two of them are given by $t = 0$, $w = 0$ so that the map has the above mentioned form. We first consider the possibility that all three lines intersect at the same point, i.e. $[1 : 0 : 0]$. Hence the third line must have the form $w = \alpha$ in the $t = 1$ coordinate system. But this contradicts that the line is exceptional because all roots of $w^d = \alpha^d$ define lines in the preimage.

So the three lines only intersect in pairs. Hence we can assume that the lines are $z = 0$, $w = 0$, $t = 0$. But then we see that the map f_0 must be of the form $f_0 = z^d$.

In conclusion we have proved :

THEOREM 4.5. **1.** *If the map* $f \in \mathcal{H}_d$ *has an exceptional variety containing one complex line, then we can assume that* f *has the form* $[f_0([z : w : t]) : f_1([z : w : t]) : t^d]$ *where the functions* $f_0(z, w, 0)$, $f_1(z, w, 0)$ *have nondegenerate degreed-terms.*

2. *If the map* $f \in \mathcal{H}_d$ *has an exceptional variety containing two complex lines, then we can assume that* f *has the form* $[f_0([z : w : t]) : w^d : t^d]$ *where the function* $f_0(z, 0, 0) = z^d$.

3. *If the map* $f \in \mathcal{H}_d$ *has an exceptional variety containing three complex lines, then we can assume that* f *has the form* $[z^d : w^d : t^d]$.

Assume next that the exceptional variety $\{h = 0\}$ is cubic, but not a product of three lines. There are three possibilities, first $\{h = 0\}$ is the union of an irreducible quadric and a complex line, second $\{h = 0\}$ is a smooth torus and finally $\{h = 0\}$ could be a singular cubic variety.

We will show that none of these cases can occur.

If $\{h = 0\}$ is the union of an irreducible quadric Q, and a complex line L, notice that $f\mid_Q$ and $f\mid_L$ are both exceptional maps on $\mathbf{P}^2$. Hence they must have two critical points. Since these critical points must be intersection points of Q and L, it follows that $Q \cap L$ consists of two points. We can assume that $Q = (zw = t^2)$ and $L = (t = 0)$ after a linear change of coordinates.

By Theorem 4.5 we can also assume that f has the form $[f_0 : f_1 : t^d]$. Hence we have the identity

$$f_0 f_1 - t^{2d} = c(zw - t^2)^d$$

for some $c \neq 0$, see Lemma 4.3.

It follows that $f_0 f_1 = \prod_{i=1}^d (t^2 - c_i(zw - t^2))$ for distinct constants c_i. Hence some irreducible factor g_0 of f_0 will divide some term $t^2 - c_i(zw - t^2)$ while some irreducible factor g_1 of f_1 will divide another term $t^2 - c_j(zw - t^2)$, $c_i \neq c_j$. Let p be a common zero of g_0 and g_1. It follows that $f_0(p) = f_1(p) = t(p) = 0$ so f is undefined at p.

This controdiction proves that an exceptional variety cannot consist of an irreducible quadric and a line.

Our next case is when the exceptional set $\{h = 0\} := V$ is a smooth cubic.

We are grateful to R. Narasimhan for proving that this is impossible. His argument goes as follows.

Let $\Omega := \mathbf{P}^2 \backslash V$. Then f is a covering map of Ω, $f : \Omega \to \Omega$. It is known that $\pi_1(\Omega)$ is finite ([D]), in fact $\pi_1(\Omega) = \mathbf{Z}/3\mathbf{Z}$. Moreover $f* : \pi_1(\Omega) \to \pi_1(\Omega)$

is injective and hence bijective. But this is impossible since f is nontrivial. Hence the exceptional set cannot be a torus.

We turn to the remaining cubic case, when $\{h = 0\}$ is irreducible and singular.

If $\{h = 0\}$ has a normal crossing singularity then $\pi_1(\Omega)$ is still $\mathbf{Z}/3\mathbf{Z}$ ([D]) so Narasimhan's proof above still applies. So suppose that $\{h = 0\}$ has a cusp singularity. Then the normalization of the exceptional set is a $\mathbf{P}^1$ and the map restricted to it has only one critical point. Since f is d to 1 on $\{h = 0\}$ this is impossible.

There is only one remaining possibility for the exceptional set, namely a nonsingular quadratic curve. In this case we can assume h has the form $zw - t^2$ and that $f_0 f_1 - f_2^2 = (zw - t^2)^d$.

Then d must be an odd integer : if $d = 2k$ is even, we can write

$$f_0 f_1 = \left(f_2 - (zw - t^2)^k\right)\left(f_2 + (zw - t^2)^k\right)$$

and we can show as above that f_0, f_1 and f_2 must have a common zero contradicting the assumption that f is well defined.

We discuss now finite exceptional sets.

THEOREM 4.6. *For fixed $d \geq 2$ the set $\tilde{\mathcal{H}}_d$ of holomorphic maps f from $\mathbf{P}^k \to \mathbf{P}^k$ that have no exceptional finite set is a nonempty Zariski open set of $\mathcal{H}_d$.*

Proof. Given $f \in \mathcal{H}_d$, and $a \in \mathbf{P}^k$ let $\Phi_i(a, f)$ denote the solutions of $f(z) = a$. If E is an exceptional finite set, then f induces a bijection of E. If a is in E then $f^{-1}(a)$ is one point, i.e. all the $\Phi_i(a, f)$ coincide. Hence $\tilde{\mathcal{H}}_d$ is a Zariski open set in $\mathcal{H}_d$. Since there are maps without exceptional points, $\tilde{\mathcal{H}}_{\mathbf{D}}$ is nonempty.

Next we give an example of a holomorphic map on $\mathbf{P}^2$ with an exceptional point belonging to the Julia set, contradicting the situation in $\mathbf{P}^1$ where all exceptional points are superattractive

$$f([z : w : t]) = [w^d + \lambda z t^{d-1} : z^d : t^d + P(z, w)]$$

where P is any homogeneous polynomial of degree d. Then $p = [0 : 0 : 1]$ is an exceptional point : $f^{-1}(p) = p$. If $| \lambda |> 1$ then the point is in the Julia set, since the two eigenvalues of $f'(p)$ are λ and 0. However, it follows from the stable manifold theorem, see [Ru] or [Sh], that $p \in \mathcal{F}_1$, since one of the eigenvalue of f at p is zero.

THEOREM 4.7. *There exist constants $c(d)$ so that for any $f : \mathbf{P}^2 \to \mathbf{P}^2$, $f \in \mathcal{H}_d$, any finite exceptional set has at most $c(d)$ points.*

Proof. We can assume that $d \geq 3$ since $c(2) \leq c(4)$. Observe that the degree of the map f is d^2, counting multiplicity. Notice that an exceptional point (wether it is fixed or on a periodic orbit) can have only one preimage. Hence, necessarily, all exceptional points lie in the critical set. Let p be an exceptional point. Assume at first that p is a regular point of the critical set C and $f(p)$ is a regular point for $f(C)$. Then we can choose local coordinates near p and $f(p)$ such that the map has the form $(z, w) \to (z, w^\ell)$ for some integer ℓ. So the map is locally ℓ to 1. But by Theorem 2.4, $\ell \leq 3(d-1)+1$, and this last number is $< d^2$ if $d \geq 3$. Hence p cannot be exceptional. It follows that p is a singular point of C or $f(p)$ is a singular point of $f(C)$. Since $f(C)$ has degree at most $3d(d-1)$ and the number of singular points of C or $f(C)$ is bounded by Bezout's Theorem, by a constant, the Theorem follows.

5. Generic hyperbolicity in $\mathbf{P}^2$.

One of the main tools in holomorphic dynamics in one variable is Montel's Theorem, more precisely, a family of holomorphic maps from the unit disc to $\mathbf{P}^1 \backslash \{0, 1, \infty\}$ is locally equicontinuous.

We want to prove here an analogue for maps on $\mathbf{P}^2$. We recall some properties of the Kobayashi-Royden infinitesimal distance, and we refer to Lang's book [La] for background. Let $\Delta(0, r)$ denote the disc of radius r centered at the origin in $\mathbf{C}$.

DEFINITION 5.1. Let M be a complex manifold and (p, ξ) be in the unit tangent bundle. Define

$$K_M(p, \xi) = \inf\{\frac{1}{r}; \exists \varphi : \Delta(0, r) \to M \ \text{ holomorphic}, \ \varphi(0) = p, \ \varphi'(0) = \xi\}.$$

In case M is the unit disc $\Delta(0, 1)$, K_M is the Poincaré metric, more precisely $K_M(z, \xi) = \frac{|\xi|}{1-|z|^2}$.

If $f : M \to N$ is a holomorphic map, then f is distance decreasing, i.e.

$$K_N(f(p), f'(p)\xi) \leq K_M(p, \xi).$$

If f is a covering map, then f is a local isometry, i.e.

$$K_N(f(p), f'(p)\xi) = K_N(p, \xi).$$

DEFINITION 5.2. The complex manifold M is Kobayashi hyperbolic if for every $p \in M$ there exists a neighborhood $U(p)$ and a constant $c > 0$ such that for all $q \in U$, $K_M(q,\xi) \geq c \mid \xi \mid$.

The integrated form of K_M is then a distance d_M, the Kobayashi metric, on M. If M is hyperbolic and complete with respect to d_M, we say that M is complete hyperbolic. If $M \subset N$ is an open subset of a complex manifold N with a hermitian metric ds, then M is said to be hyperbolically embedded, if M is Kobayashi hyperbolic and for every $p \in \partial M$ there is an open neighborhood $U(p)$ in N and a constant $c > 0$ such that $K_M(q,\xi) \geq c\, ds(q,\xi)$ for every $q \in U(p) \cap M$. This definition is independent of the choice of Hermitian metric on N.

THEOREM 5.3. *Fix an integer $d \geq 2$. Then there exists a Zariski dense open set $\mathcal{H}' \subset \mathcal{H}_d$ with the following properties. If $f \in \mathcal{H}'$ and C denotes it's critical set. Then*
i) No point of $\mathbf{P}^2$ lies in $f^n(C)$ for three different n, $0 \leq n \leq 4$.
ii) $\mathbf{P}^2 \setminus \left(\bigcup_{n=0}^4 f^n(C)\right)$ is Kobayashi complete hyperbolic and hyperbolically embedded in $\mathbf{P}^2$.

We prove first a lemma.

LEMMA 5.4. *Let $f = [z^d : w^d : t^d]$. There exists an arbitrarily small perturbation g of f such that the five (reducible) varieties $g^n(C)$, $n = 0, \cdots, 4$ have no triple intersections.*

Proof. We will find such a g by choosing a suitable $(3,3)$ matrix A arbitrarily close to the identity and letting $g = g_A := A(f)$. This ensures that the critical set of g is the same as the critical set of f. Let us denote by C_1, C_2, C_3 the components $(z = 0)$, $(w = 0)$, $(t = 0)$ respectively. For each choice of integers $0 \leq n_1 < n_2 < n_3 \leq 4$ and $1 \leq m_1, m_2, m_3 \leq 3$ let $A_{n,m}$ denote those matrices A for which $g^{n_i}(C_{m_i})$, $i = 1,2,3$ have a triple intersection. Then $A_{n,m}$ is necessarily a closed subvariety. Namely we consider the complex manifold $\mathbf{P}^2 \times G$ where G is the space of invertible matrices. In there consider the complex varieties $V_n := \{g_A^n(C), A\}$. By Tarski's proper mapping theorem these are varieties and the projection to G is proper with one dimensional compact fibers. Hence the projection of the intersection of any three of these is a complex subvariety of G, again by Tarski's proper mapping theorem.

Hence, if the lemma is false, then some $A_{n,m} = G$. Notice that if the m_s's are all distinct, then $A_{n,m}$ must be empty (near the Identity). Hence, by the symmetry in the situation we may assume that $m_1 = 1$, $1 \leq m_2, m_3 \leq 2$. Consider next the special case $g = [z^d + \epsilon t^d : w^d + \epsilon t^d : t^d]$. Observe that if $h(z) = z^d + \epsilon$ then $g^n(w = 0) = \{[z : h^n(0) : 1]\}$ and $g^n((z = 0)) = \{[h^n(0) : w : 1]\}$. It is clear that if some $m_i = 2$, there are no triple intersections if $\epsilon \neq 0$ is small enough. It remains to consider the case when all the $m_i = 1$, $i = 1, 2, 3$.

First let us prove that $n_1 > 0$. Consider at first the family $g_\epsilon = [z^d + \epsilon w^d : w^d : t^d]$. Then $g_\epsilon^n(C_1)$ are lines of the form $z = \eta_n w$ (in $(t = 1)$) where the η_n are distinct for suitable small ϵ while $\eta_0 = 0$. Next consider for fixed ϵ the maps $g_{\epsilon,\delta} = [z^d + \epsilon w^d + \delta t^d : w^d : t^d]$. The image of $g(C_1)$ is parametrized by $[\epsilon w^d + \delta : w^d : 1]$. In general

$$g^n(C_1) = [\eta_n w^{d^n} + O(\delta)(w, \cdots, w^{2^d - 1}) + \delta + O(\delta^2) : w^{2^d} : 1].$$

It follows that the intersection points with C_1 are estimated by

$$w = (-\delta/\eta_n)^{1/d^n}(1 + O(\delta^{1/\delta^n}).$$

Hence their w-coordinate is $-\delta/\eta_n \cdot (1 + O(\delta^{1/d^n}))$ which rules out triple intersections with C_1.

Hence we are reduced to the case $1 \leq n_1 < n_2 < n_3 \leq 4$. Observe that in the previous case we may assume that the intersections of the various images of C_1 with C_1 have different modulus, otherwise modify ϵ. This will enable us to deal with the next case when $n_1 = 1$. Assume that there always is a triple intersection when $n_1 = 1$ for some n_2, n_3 as above. Let p be such a triple intersection. Notice that there are at most d preimages of p, all of which are in C_1 and all with the same $| w |$ value. But necessarily also, one of these points must be in $g^{n_2-1}(C_1) \cap C_1$ and one of these points must be in $g^{n_3-1}(C_1) \cap C_1$, which contradicts that such intersection points have different modulus.

Hence we have only one more case to consider, $n_1 = 2$, $n_2 = 3$, $n_3 = 4$. For this consider a map far from the Identity composed with A, namely $g = [t^d : z^d : w^d]$. Then the orbit of C_1 is $(z = 0) \to (w = 0) \to (t = 0) \to (z = 0) \to (w = 0)$. In particular the images $g^2(C_1)$, $g^3(C_1)$, $g^4(C_1)$ have no point on common. Hence this is true as well for some small perturbations of f. This completes the proof of the lemma.

We will use the following two theorems by M. Greene.

THEOREM 5.5 ([Gr1]). *If V is a compact complex manifold and $D_1, \cdots, D_m$ are hypersurfaces (possibly singular), then $V \backslash D$ is complete hyperbolic and hyperbolically embedded provided*

1) There is no non constant holomorphic map $\mathbf{C} \to V \backslash D$.

2) There is no non constant holomorphic map $\mathbf{C} \to D_{i_1} \cap \cdots \cap D_{i_k} \setminus (D_{j_1} \cup \cdots \cup D_{j_\ell})$ *for any choice of indices* $\{i_1, \cdots, i_k, j_1, \cdots, j_\ell\} = \{1, \cdots, m\}$.

THEOREM 5.6 ([Gr2]). *Suppose f is a holomorphic map from $\mathbf{C}$ to $\mathbf{P}^k$ omitting $k+2$ distinct irreducible compact hypersurfaces. Then $f(\mathbf{C})$ is contained in a compact complex hypersurface.*

We now prove Theorem 5.3.

Proof. By the Lemma and Tarski's Theorem there exists a Zariski dense open set $\mathcal{H}' \subset \mathcal{H}_d$ satisfying condition 1 of Theorem 5.3. We prove 2. Let $f \in \mathcal{H}'$ we will show that $\Omega := \mathbf{P}^2 \setminus \bigcup_{n=0}^4 f^n(C)$ satisfies the conditions of Greene's Theorem. Let φ be a holomorphic map from $\mathbf{C}$ to Ω. By Theorem 5.6 $\varphi(\mathbf{C})$ is contained in an algebraic subvariety $X \setminus \bigcup_{n=0}^4 f^n(C)$. Then $\varphi(\mathbf{C})$ omits at least three points in X. Hence φ is constant. Similarly we check condition 2. This completes the proof of Theorem 5.3.

If three varieties have a common point, then their image under a map must also have a common point. Hence we get the following immediate corollary.

COROLLARY 5.7. *Fix an integer $d \geq 2$. Let $f \in \mathcal{H}'$, defined in Theorem 5.3, and let C denote it's critical set. Then*
1. *No point lies in $f^n(C)$ for three different n, $k-4 \leq n \leq k$ if $k \leq 3$.*
2. $\mathbf{P}^2 \setminus \left(\bigcup_{n=k-4}^{n=k} f^n(C)\right)$, $k \leq 0$ *is Kobayashi complete hyperbolic and hyperbolically embedded.*

We show next that generically components of $f^n(C)$ have genus larger then 1.

PROPOSITION 5.8. *For every $d \geq 2$, $n \in \mathbf{Z}$, there exists an open set $\Omega \subset \mathcal{H}_d$ such that for every $g \in \mathcal{H}_d$ there is an open neighborhood $U(g)$ of g such that $U(g) \setminus \Omega$ is a countable union of compact subsets of varieties. Moreover, for every $f \in \Omega$, all the irreducible components of $f^n(C)$ have genus at least one.*

Remark. We define the genus of a singular curve to be the genus of the normalization.

We first prove a lemma.

LEMMA 5.9. *For each $n \in \mathbf{Z}$, there exists a nonempty open subset V of $\mathcal{H}_d$ so that for each $f \in V$, with critical set C, each irreducible component of $f^n(C)$ is a compact curve of genus at least 1.*

Proof. We study the maps

$$f_r := [z^d + rwt^{d-1} + rtw^{d-1} : w^d + rzt^{d-1} + rtz^{d-1} : t^d + rzw^{d-1} + rwz^{d-1}]$$

for complex numbers r close to zero. Note that these maps are symmetric in the coordinates. Hence it suffices to study the map in the $(t = 1)$ coordinates in the set $| z |, | w | \leq 1$. There the map has the form

$$(z, w) \to \left(\frac{z^d + rw + rw^{d-1}}{1 + rzw^{d-1} + rwz^{d-1}}, \frac{w^d + rz + rz^{d-1}}{1 + rzw^{d-1} + rwz^{d-1}} \right).$$

We compute the Jacobian determinant J. We obtain

$$J = d^2(zw)^{d-1} - r^2(1 + (d-1)z^{d-2})(1 + (d-1)w^{d-2}) + rR(z, w) + r^2 Q(z, w)$$

where Q vanishes to order at least d and R vanishes to order at least $2d - 1$ and each term contains a factor $(zw)^{d-1}$. Consider the set $| z |, | w | \leq s_d$ for some small $s_d > 0$ independant of r. Suppose that $J(z, w) = 0$. It follows that $| zw | > k_d | r |^{2/(d-1)}$ for some fixed constant $k_d > 0$. This implies that the gradient of J is nonzero. It follows that inside this disc of radius s_d, the critical set consists of $d - 1$ branches each of which is a small perturbation of $zw = cr^{2/d-1}$. Since $\{J = 0\}$ is close to $\{zwt = 0\}$ and since there is a closed noncontractible curve in $\{zwt = 0\}$ joining $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$, it follows that each irreducible branch of C has genus at least one. For any fixed $n > 0$ and for r small enough the n^{th} forward image of the noncontractible curves in C are still noncontractible in $f_r^n(C)$. Since there is no nonconstant holomorphic map from $\mathbf{P}^1$ to a compact Riemann surface of genus larger or equal to one, it follows also that each irreducible branch of $f_r^{-n}(C)$ has genus ≥ 1. The same argument holds for an open neighborhood of f_r in $\mathcal{H}_d$.

Proof of the Proposition. There is a proper subvariety $\sum$ of $\mathcal{H}_d$ such that for $h \in \sum$ there exists a regular point in $h^n(C_h)$ of sheet number strictly less than the local maximum.

For each $g \in \mathcal{H}_d \backslash \sum$ there is an open neighborhood U_g of g and a proper subvariety X of U_g such that for $h \in X$ there exists a point $(z_1, z_2) \in S := h^n(C_h)$ and a polydisc $\Delta_1 \times \Delta_2$ around (z_1, z_2) such that S is a ramified cover over Δ_1 and either the number of irreducible branches of the germ $S_{(z_1, z_2)}$ is less than maximal or one of them has sheet number larger than minimal.

For maps $h \in U_g \backslash X$ the irreducible branches have constant topology.

The lemma now implies the proposition.

6. Expansion in the presence of hyperbolicity.

We show that periodic orbits of holomorphic self maps of P^k are non attractive in the complement of the critical orbits under the hypothesis of Kobayashi hyperbolicity.

THEOREM 6.1. *Let* $f : \mathbf{P}^k \to \mathbf{P}^k$ *be a holomorphic map with critical set* C. *Let* $\mathcal{C}$ *be the closure of* $\bigcup_{j=0}^{\infty} f^j(C)$. *Assume that* $P^k \backslash \mathcal{C}$ *is Kobayashi hyperbolic and hyperbolically embedded. If* p *is a periodic point for* f, $f^\ell(p) = p$, *with eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_k$ *and* $p \notin C$, *then* $|\lambda_i| \geq 1$, $1 \leq i \leq k$. *Also* $|\lambda_1 \cdots \lambda_k| > 1$ *or* f *is an automorphism of the component of* $P^k \backslash \mathcal{C}$ *containing* p.

Proof. We show at first that the eigenvalues of the derivative of the ℓ^{th} iterate at the periodic point are all at least one. Let $U := P^k \backslash C$ and let $U_1 := U \backslash f^{-1}(C)$. Observe that $U_1 \subset U$. As $f : U_1 \to U$ is a covering map we obtain that the infinitesimal Kobayashi metric at a point x and tangent vector ξ satisfy

$$K_U(f(x), f'(x)\xi) = K_{U_1}(x, \xi) \geq K_U(x, \xi).$$

So if $x \notin C$, and $f^\ell(x) = x$, then all eigenvalues λ_i of $(f^\ell)'(x)$ have modulus at least one.

Next we consider the Jacobian determinant of the ℓ^{th} iterate. First, let Ω be a component of U, $p \in \Omega$, and let $\Omega_\ell \subset \Omega$ be the connected component of $f^{-\ell}(\Omega)$ containing p. Let M be the universal covering of Ω_ℓ and $\pi : M \to \Omega_\ell$ the projection. Observe that M is hyperbolic and that for the Kobayashi metric biholomorphic mappings are isometries. Also observe that $(M, f^\ell \circ \pi =: \pi')$ is the universal cover of Ω. Pick any nonvanishing holomorphic k-form α at p. Fix a Hermitian metric on TM. Let $\| \ \|$ be a volume form on the space of $(0, k)$ forms, such that holomorphic automorphisms preserve the volume. Fix a point $q \in M$ with $\pi(q) = p$. Define

$$E_\Omega^M(p, q, \alpha) := \inf\{\| \gamma \|_q^2 ; g(q) = p, g_*(\gamma) = \alpha\}$$

where g runs through all holomorphic maps with non vanishing Jacobian from M to Ω with $g(q) = p$. Similarly define

$$E_{\Omega_1}^M(p, q, \alpha) := \inf\{\| \gamma \|_q^2 ; g(q) = p, \pi(q) = p, g_*(\gamma) = \alpha\}$$

where g runs through all holomorphic maps with non vanishing Jacobian and $\pi(q) = p$ from M to Ω_ℓ, $g(q) = p$. To proceed we need a lemma.

LEMMA 6.2. *The extremal maps exist and are surjective.*

We prove the lemma for E_{Ω}^{M}. The proof for $E_{\Omega_\ell}^{M}$ is the same.

Proof of lemma. Let g_n be a minimizing sequence. Consider the g_n's as maps from M to $\mathbf{P}^k \backslash \mathcal{C}$ which is hyperbolically embedded. Then g_n is equicontinuous with respect to a metric on $\mathbf{P}^k$. Hence by Ascoli theorem there exists a subsequence $g_{n_k} \to g$ and $g(p) = p$ and $\det g' \neq 0$ and hence g has values in Ω. Let $\tilde{g}$ be such that $\pi' \circ \tilde{g} = g$ and $\tilde{g}(q) = q$. If $|\det \tilde{g}'(q)| < 1$ then by the chain rule, this will contradict that g is extremal. Since M is hyperbolic we must have that $|\det \tilde{g}'(q)| \leq 1$, ([K, Thm 3. 3]). Hence, it follows that $\tilde{g}$ is an automorphism, ([K, Thm 3. 3]), and hence g is surjective.

We next continue with the proof of the last assertion of the theorem. Since f^ℓ is a covering map from Ω_ℓ to Ω, since $f^\ell(p) = p$

$$(*) \qquad E_{\Omega_\ell}^{M}(p, q, \alpha) = E_{\Omega}^{M}(f^\ell(p), q, (f^\ell)_*(p)(\alpha)).$$

If $\Omega_\ell = \Omega$, then (*) implies that $|\det(f^\ell)'(p)| = 1$. Hence f^ℓ is an automorphism of Ω ([Ko]).

If Ω_ℓ is a proper subset of Ω, then the Lemma implies that $E_{\Omega_\ell}^{M}(z, \alpha) > E_{\Omega}^{M}(z, \alpha)$. Hence $|\det(f^\ell)'(p)| > 1$.

Observe that $E_{\Omega}^{M}(p, q, \alpha) = \| \gamma \|_q$ where $\pi'_*(q)\gamma = \alpha$. Since $\| \ \|$ is invariant under biholomorphisms, this is independent of q. We will denote by $E_{\Omega}^{M}(p, \alpha)$ this volume form.

Remark 6.3. *If $k = 1$, then Theorem 6.1 says that attractive, rationally indifferent and Cremer points are in $\mathcal{C}$.*

The following result generalizes a classical result of Fatou and Julia on rational maps on P^1. Let $f : \mathbf{P}^k \to \mathbf{P}^k$ and let $C = C_1 \cup \cdots \cup C_\ell$ denote the irreducible components of the critical set.

DEFINITION 6.4. *We say that a Fatou component $\Omega \subset P^k$ is a Siegel domain if there exists a subsequence f^{n_i} converging to the identity map on Ω.*

PROPOSITION 6.5. *Let C denote the critical set of a holomorphic map $f : \mathbf{P}^k \to \mathbf{P}^k$ of degree at least 2. Assume that the complement of the closure of $\bigcup_{n=0}^{\infty} f^{-n}(C)$ is hyperbolically embedded. Then*

$$J \subset \bigcup_{N>0} \overline{\cup_{n \geq N} f^{-n}(C)} =: J(C).$$

Hence all periodic points with one eigenvalue of modulus strictly larger than 1 are in $J(C)$.

Proof. Assume that $p \notin J(C)$. Then $\exists N$ such that $B(p,r)$ does not intersect $\overline{\bigcup_{n\geq N} f^{-n}(C)}$. This implies that for $n \geq N$, $f^n(B(p,r)) \cap \overline{\bigcup_{k=0}^{\infty} f^{-k}(C)} = \emptyset$. Hence f^n is a normal family on $B(p,r)$, which proves the statement.

THEOREM 6.6. *Under the assumptions of Theorem 3.3 we have : If there is a component* U *of the Fatou set of* f *such that* $f^n(U)$ *does not converge u. c.c to* $\mathcal{C}$, *then* U *is preperiodic to a Siegel domain* Ω *with* $\partial\Omega \subset \mathcal{C}$.

Note : This does not exclude that part of the Siegel domain is in $\mathcal{C}$.

Proof. Let U be a component of the Fatou set satisfying the hypothesis of the theorem. Assume that f^n does not converge to $\mathcal{C}$ on compact subsets of U. Then there exist a nonempty open subset $U' \subset U$ and a neighborhood N of $\mathcal{C}$ and a subsequence f^{n_i} such that $f^{n_i}(U') \cap N = \emptyset$. Moreover we may assume that $f^{n_i} \to h$ on U'. Since $K_{\mathbf{P}^k \setminus \mathcal{C}}(f(x), f'(x)(\xi)) \geq K_{\mathbf{P}^k \setminus \mathcal{C}}(x,\xi)$, and the Kobayashi metric is upper semicontinuous, it follows that h is nondegenerate. Let Ω be the Fatou component containing $h(U')$. On Ω there is a subsequence such that $f^{n_{k+1}-n_k} \to I$, and Ω is periodic under f^{ℓ}. Let E_X denote the Kobayashi volume form with respect to holomorphic maps of maximal rank at every point on the complex manifold X. Denote $E_{\mathbf{P}^k \setminus f^{-m\ell}(\mathcal{C})}$ by E_m. We have $E_0(x,\alpha) \leq E_m(x,\alpha) \leq E_{m'}(x,\alpha) = E_0(f^{m'\ell}(x), Df^{m'\ell}(x)(\alpha))$ for any $0 < m < m'$. Taking limits and using the upper semicontinuity of the Kobayashi volume form gives $E_{\mathbf{P}^k \setminus \mathcal{C}}(x,\alpha) = E_m(x,\alpha)$ for $x \in \Omega \setminus \mathcal{C}$. Let $\mathcal{C}' = \overline{\bigcup f^{-m}(\mathcal{C})}$. Passing to the limit we obtain that $E_{\mathbf{P}^k \setminus \mathcal{C}}(x,\alpha) = E_{\mathbf{P}^k \setminus \mathcal{C}'}(x,\alpha)$ for $x \in \Omega \setminus \mathcal{C}$ nonempty.

If a point of $\partial\Omega$ is not in $\mathcal{C}$, then $E_{\mathbf{P}^k \setminus \mathcal{C}'}(x,\alpha)$ must blow up, see ([FS], Theorem 8.4) which is a contradiction.

The proof of Theorem 1.8 shows that if U is a Fatou component which is not preperiodic to a Siegel domain, then all limit functions are in $\mathcal{C}$.

We are going to prove the dual statement of Theorem 1.7 for non attractive fixed points.

Remark 6.7. *The same statement holds if instead of* C *we consider an analytic variety* A *such that the complement of the closure of* $\bigcup_{n=0}^{\infty} f^n(A)$ *is hyperbolic, in which case* $J \subset \bigcap_{N>0} \overline{\bigcup_{n>N} f^{-n}(A)} := J(A)$. *In general we*

don't have $J = J(A)$:
Example : $[z:w:t] \to [(z-2w)^2 : z^2 : t^2]$ $(z = \alpha w) \to (Z = ((\alpha-2)/\alpha)^2 W)$ and preimages of lines are lines. The union of the closure of $\cup f^{-n}(C)$ equals $P^2 \cdot J \neq P^2$.

COROLLARY 6.8. *Assume that the complement of the closure of*

$$\bigcup_{n=0}^{\infty} f^{-n}(C)$$

is hyperbolic and hyperbolically embedded. Then any Siegel domain is a Fatou component and a domain of holomorphy which is hyperbolic.

Proof. Let Ω be a Siegel domain. It is clear that $\Omega \cap \bigcup_{n=0}^{\infty} f^{-n}(C) = \emptyset$. Hence Ω is a component of $P^k \backslash J(C)$ which is clearly locally Stein. The domain Ω is hyperbolic since $\Omega \subset P^k \backslash \overline{\bigcup_{n=0}^{\infty} f^{-n}(C)}$.

We describe more precisely the behavior of f at a fix point when some of the eigenvalues are of modulus 1.

PROPOSITION 6.7. *Let* $f : \mathbf{P}^k \to \mathbf{P}^k$ *be a holomorphic map in* $\mathcal{H}_d$, $d \geq 2$. *Assume* $\Omega := \mathbf{P}^k \backslash \mathcal{C}$ *is Kobayashi hyperbolic.*

Suppose p is a fix point for f, $f(p) = p$, and $p \in J \backslash \mathcal{C}$. Let $\lambda_1, \cdots, \lambda_k$ be the eigenvalues of $f'(p)$. Assume $\mid \lambda_i \mid = 1$, $1 \leq i \leq s$, $\mid \lambda_j \mid > 1$ for $j > s$. Then there exists a subvariety $\sum$ through p such that $f_{|\sum}$ is linearizable.

We first prove a Lemma.

LEMMA 6.8. *Let* g *be a holomorphic map from* $\mathbf{C}^s$ *to* $\mathbf{C}^s$. *Assume* $g(0) = 0$ *and* $g'(0) = A$ *is unitary. If* (g^n) *is a normal family in a neighborhood of 0, then* g *is linearizable.*

Proof. Observe first that (A^n) and (A^{-n}) are bounded. Define, as in one variable,

$$\varphi(z) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} A^{-j} \circ g^j.$$

The mapping φ is well defined in a neighborhood of 0 and $\varphi(g(z)) = A\varphi(z)$.

We proceed to prove the proposition.

Proof. We know that f is a local isometry from $\Omega_1 = \mathbf{P}^k \backslash f^{-1}(\mathcal{C})$ to $\Omega = \mathbf{P}^k \backslash \mathcal{C}$. Let d_Ω denote the Kobayashi distance of Ω, and let $B(p,r)$ be the ball of radius r and center p for d_Ω. Since $\Omega_1 \subset \Omega$, $d_\Omega(f(z), f(p)) \geq d_\Omega(z, p)$. The Kobayashi distance induces the topology of Ω, hence we can assume that f is invertible in a neighborhood of $\bar{B}(p, r)$ and let $g = f^{-1}$. We have $g(B(p, r)) \subset B(p, r)$, and the family g^n is normal. Let $h = \lim g^{n_i}$ and define $\sum' := h(B(p, r))$. Clearly we have $g(\sum) \subset \sum$. If we consider the sequence $g^{n_{i+1}-n_i}$, we can assume $h_{|\sum}$ is identity. It follows that $\sum$ is contained in the complex manifold $S := \{q \in B(p, r) \;\; h(q) = q\}$. Let $\sum$ be the connected component of S through p. $\sum$ is a complex manifold of dimension s, $g^n_{|\sum}$ is normal hence by the Lemma $g_{|\sum}$ is linearizable around p, and the same result holds for f.

Example. Let $f[z : w : t] = [\lambda zt + z^2 : w^2 + ct^2 : t^2]$ where $| \lambda |= 1$ is such that $\lambda z + z^2$ is linearizable in a neighborhood of 0 in $\mathbf{C}$. Let $p = [0 : w_0 : 1]$ where w_0 is such that $f(p) = p$. One easily check that if $| c |>> 1$ then $p \notin \mathcal{C}$ and that f is linearizable in a disc through p.

In one dimension, it is well known that if the Julia set J of $R : \mathbf{P}^1 \to \mathbf{P}^1$ is disjoint from the closure of the orbit of the critical set then J is hyperbolic in the dynamical sense, i.e. there exists an infinitesimal metric in a neighborhood of J such that R is expanding for that metric.

We prove here a similar result in higher dimension.

THEOREM 6.9. *Let$f \in \mathcal{H}_d$, $f : \mathbf{P}^k \to \mathbf{P}^k$, $d \geq 2$. Let$\mathcal{C} = \overline{\bigcup_{n\geq 0} f^n(C)}$. Assume$\mathbf{P}^k \backslash \mathcal{C}$ is Kobayashi hyperbolic and hyperbolically embedded. Let$X$ be a compact set in$J$, the Julia set of$f$. Assume$f(X) \subset X$ and$X \cap \mathcal{C} = \emptyset$, then there exists a continuous volume form$E$ in a neighborhood of$X$, and a constant$k > 1$ such that*

$$E(f^n(x), (f^n)'(x)\alpha) \geq k^n E(x, \alpha).$$

Hereα denotes a nonvanishing holomorphic$k$-form.

Proof. Let (Ω_j), $1 \leq j \leq \ell$ denote the components of $U := \mathbf{P}^2 \backslash \mathcal{C}$ intersecting X. Define $\Omega = \bigcup_{j=1}^{\ell} \Omega_j$. Observe that for every j, the components of $f^{-1}(\Omega_j)$ are contained in one of the Ω_s, $1 \leq s \leq \ell$. We claim that there is an N such no component of $f^{-N}(\Omega)$ coincide with one of the Ω_j's. Otherwise, since the number of components is finite, there are two integers n_1, n_2 such that say a component of $f^{-n_1}(\Omega_1)$ is equal to Ω_2 and a component of $f^{-n_2}(\Omega_2)$ is equal to Ω_1 and hence $f^{n_1+n_2}$ is an isometry of the Kobayashi hyperbolic domain Ω_1. As a consequence Ω_1 is a Siegel domain, contradicting that X is in the Julia set and intersects Ω_1.

Let $\Omega' := f^{-N}(\Omega)$, f^N is a covering map from Ω' to Ω. Denote by $E(p,\alpha)$ the volume form $E^M_\Omega(p,\alpha)$ constructed in the proof of Theorem 6.1. Observe that $E(p,\alpha)$ is continuous and that for all p in Ω'

$$E^M_\Omega(f(p), f'(p)\alpha) = E^M_{\Omega'}(p,\alpha) > E^M_\Omega(p,\alpha)$$

the inequality holds since no component of Ω' coincide with a component of Ω. The continuity of E implies the theorem.

7. Classification of critically finite maps in $\mathbf{P}^2$.

In the study of iteration of rational maps $R : P^1 \to P^1$ the case where every critical point is preperiodic or periodic is quite central, see ([Th]). Here we consider the corresponding problem in higher dimension. The question was raised by McMullen [B].

It is quite difficult to construct non trivial examples of critically finite maps in dimension 2. Some examples where studied in [FS]. It is proved in [FS] that the map $g : \mathbf{P}^2 \to \mathbf{P}^2$ defined by

$$g[z : w : t] = [(z-2w)^2 : (z-2t)^2 : z^2]$$

has $\mathbf{P}^2$ as Julia's set. It is probably an interesting question to study maps in $\mathcal{H}_d$ close to g, as was done in one variable by M. Rees [Re].

DEFINITION. *Let$F : M^2 \to M^2$ be a finite holomorphic self mapping of a compact complex manifold of dimension 2. Denote by$C$ the critical set. LetC_i be the irreducible components of$C$. We assume that eachC_i is periodic or preperiodic,*

$$C_i \to F(C_i) \to \cdots \to F^{\ell_i}(C_i) \to \cdots \to F^{\ell_i+n_i}(C_i) = F^{\ell_i}(C_i)$$

where$\ell_i \geq 0$, $n_i \geq 1$ are (minimally chosen) integers. We then say that$F$ is critically finite. We say that$F$ is strictly critically finite if all the mapsF^{n_i} : $F^{\ell_i+j} \to F^{\ell_i+j}$ are critically finite self maps (on possibly singular Riemann surfaces).

Observe that F is critically finite if $V := \bigcup_{n=0}^{\infty} F^n(C)$ is a closed complex hypersurface of M, in this case we define $W = F^{-1}(V)$ which is a complex hypersurface containing V.

We recall the following result, see ([Su],[Th]).

THEOREM 7.2. *Let* $R : \mathbf{P}^1 \to \mathbf{P}^1$ *be a rational map of degree* $d \geq 2$. *Assume that* R *is critically finite. Then the only Fatou components of* R *are preperiodic to superattractive components.*

Recall that Ω is a superattractive component if $\exists p \in \Omega$, $\exists \ell$, $R^\ell(p) = p$ and $(R^\ell)'(p) = 0$. The theorem shows in particular that if R is critically finite and if every critical point is preperiodic that the Julia set of R is $\mathbf{P}^1$.

We give here a proof of Theorem 7.2 that does not use Sullivan's non wandering theorem nor the construction of an expanding metric.

Proof. Let Ω be a Fatou component for R. It follows from Theorem 6.6 that Ω is not preperiodic to a Siegel domain and that all the limits are in $\bigcup_{j=0}^{\infty} R^j(C)$ which is finite. Replacing R by some iterate we can assume that all the possible limits $a_1, a_2, \cdots, a_r$ are fix points for R. Let $U_1, U_2, \cdots, U_r$ be neighborhoods of $a_1, a_2, \cdots, a_r$, 2 by 2 disjoint and such that $R(U_j)$ are also 2 by 2 disjoint. Let $\omega \subset\subset \Omega$. For $n \geq N_0$, $R^n(\omega) \subset \bigcup_{j=1}^{r} U_j$ but since $R(U_j)$ are disjoint there exists j_0 such that for $n \geq N_0$, $R^n(\omega) \subset U_{j_0}$. Hence $\{R^n\}$ converges to $a_{j_0} =: a$. Let $\lambda := f'(a)$, clearly $| \lambda | \leq 1$.

Assume that $| \lambda | = 1$, $\lambda = e^{2\pi i \alpha}$. The map R cannot be linearizable near a since the corresponding Fatou component requires an infinite orbit of critical points. Similarly if α is rational or a Cremer point, see ([Mi]). So $| \lambda | < 1$ but then if $\lambda \neq 0$ we still need an infinite orbit of critical points. So finally $\lambda = 0$ and Ω is preperiodic to a superattractive Fatou component.

We will need to apply the above theorem to Riemann surfaces with singularities whose normalizations are biholomorphic to $\mathbf{P}^1$. More precisely, let X be an irreducible analytic set of dimension 1, let $\hat{X}$ denote it's normalization with covering map $\pi : \hat{X} \to X$. If f is a holomorphic map from X to X, we will denote by $\hat{f}$ the lifted map from $\hat{X}$ to $\hat{X}$. We will say that a point p in X is critical for f if $\pi^{-1}(p)$ is critical for $\hat{f}$. It is straightforward to verify that p is critical if and only if f restricted to a neighborhood of p is not injective. The terminology of Julia sets and Fatou components extends to this context.

DEFINITION 7.3. *Let* X *be a compact analytic set. If* $f : X \to X$ *is holomorphic, we will say that* f *is critically finite if* $\hat{X} = \mathbf{P}^1$ *and the orbit of the critical set is finite.*

Observe that this implies that $\hat{f} : \mathbf{P}^1 \to \mathbf{P}^1$ is critically finite. If a is a fixed point in $\hat{X}$, we will say that it has eigenvalue λ if $\hat{f}'(\hat{a}) = \lambda$ where $\pi(\hat{a}) = a$.

The following version of the above theorem is then clear.

THEOREM 7.4. *Let* $f : X \to X$ *be a* d *to 1 holomorphic map from an irreducible compact analytic set to itself. Assume that* $\hat{X} = \mathbf{P}^1$. *If* f *is critically finite, then the only Fatou components for* f *are preperiodic to superattractive basins.*

Proof. We just apply the above theorem to $\hat{f} : \hat{X} \to \hat{X}$.

We will need the following result which shows that if an algebraic curve X is invariant under f then f is not injective on X.

PROPOSITION 7.5. *Let* $X \subset \mathbf{P}^2$ *be an algebraic curve of degree* r. *Let* $f : \mathbf{P}^2 \to \mathbf{P}^2$ *be a holomorphic map which is* d^2 *to 1. Assume that* $f(X) \subset X$. *If* f *is an* ℓ *to 1 map on* X, *then* $\ell \geq d$.

Proof. Let H be a generic complex plane such that the number of points in $H \cap X$ is r. Suppose that $H = \{z_2 = 0\}$. Let $f^n = [f_0^n : f_1^n : f_2^n]$ in homogeneous coordinates. The number of points of $X \cap \{f_2^n = 0\}$ is $r\ell^n$. So there exists a homogeneous polynomial h_n of degree $r\ell^n$ vanishing on $X \cap \{f_2^n = 0\}$ but not identically. Let $\pi : \mathbf{C}^3 \to \mathbf{P}^2$ be the canonical map, and denote $\tilde{X} = \pi^{-1}(X)$. Define $\Phi_n = (h_n f_0^n / f_2^n, h_n f_1^n / f_2^n, h_n)$ from $\tilde{X}$ with values in $\mathbf{C}^3$. The map is weakly holomorphic in $\tilde{X}$ so there exists p such that for every n, $z_0^p \Phi_n$ extend as a holomorphic map. Since $z_0^p \Phi_n$ is homogeneous of degree $p + r\ell^n$ we can assume that the extension is also homogeneous of degree $p + r\ell^n$. Let g_n be this extension, $g_n = f^n$ on X. Then by Theorem 1.5 we should have $p + r\ell^n \geq d^n$ for every n, which implies $\ell \geq d$.

COROLLARY 7.6. *If* $f(X) = X$, *and if* f *is holomorphic of degree* $d \geq 2$ *then* f *does not induce an automorphism on* X.

THEOREM 7.7. *Let* $f : \mathbf{P}^2 \to \mathbf{P}^2$ *be a holomorphic map with critical set* C. *Assume that* f *is strictly critically finite and that* $\mathbf{P}^2 \setminus C$ *is hyperbolic. Then the only Fatou components of* f *are (preperiodic or equal to) superattractive components. In particular if no critical point is periodic, then the Julia set of* f *is* $\mathbf{P}^2$.

Proof. Recall that $V := \bigcup_{j=0}^{\infty} f^j(C)$ is a closed hypersurface of $\mathbf{P}^2$ and that every irreducible component of V is preperiodic or periodic. Without lack of generality (replacing f by an iterate if necessary), we can assume that every component is preperiodic to some irreducible V_j which is invariant under f. Let $\hat{V}_j$ denote the normalization of V_j. Observe first that $\hat{V}_j$ cannot be a hyperbolic Riemann surface since one of the iterates of f will be the identity on V_j and we have seen in Theorem 3.1 that this is not possible. So V_j can be $\mathbf{P}^1$ or a torus T. Let A denote the singular set of V. Then A consists of intersections of different branches as well as self intersections and cusps. Note that A is a finite set, $\{a_1, a_2, \cdots, a_r\}$. We can assume all superattractive periodic points with respect to some V_j are fixed.

Fix a Fatou component Ω. Assume at first that the iterates $f^n \mid \Omega$ converges u. c. c. to A. The argument is the same as in Theorem 5.2. Replacing f by an iterate we may assume that $f^n \mid \Omega$ converges u. c. c. to a_1. We will show that this implies that a_1 must be superattractive, completing the proof in this first case. For this, let λ_1, λ_2 be the eigenvalues of f' at a_1. Since $\{f^n\}$ is normal on Ω it follows that necessarily $\mid \lambda_1 \mid, \mid \lambda_2 \mid \leq 1$. Suppose that $a_1 \in V_1$ say. Then we exclude at first the possibility that $\hat{V}_1$ is a torus. If $\hat{V}_1$ is a torus, then if $f \mid V_1$ is not an automorphism, then this contradicts that a_1 is a nonrepelling fixed point. So necessarily $f \mid V_1$ is an automorphism. This is impossible by Theorem 3.1. Hence we may assume that $\hat{V}_1 = \mathbf{P}^1$. Since $f \mid V_1$ is critically finite, the eigenvalue corresponding to $f \mid V_1$ at a_1 is zero.

Assume at first that V_1 has a cusp singularity at a_1 which is mapped to itself. In local coordinates we can parametrize this singularity as $t \to (t^p + t^q + \cdots)$ for some integers $1 < p < q$, where q is not a multiple of p. If f' has a nonzero eigenvalue at 0, then we may assume that $f(z,w) = (O(\mid z,w \mid^2), \alpha w + O(\mid z,w \mid^2))$ for some nonzero α. Clearly this contradicts that f maps this singularity to itself.

Hence a_1 can be assumed to be a reducible singular point of V and all branches of V there, which are mapped to themselves by some iterate are nonsingular. If these branches are not all tangent to each other then both eigenvalues of f' must be zero, so we are done. Assume next that there are two such irreducible branches that are tangent. We may assume these have the form $w = 0$ and $w = \alpha z^k + O(\mid z \mid^{k+1})$ for some integer $k > 1$. If there is a nonzero eigenvalue at a_1 we may again assume that f has the form: $(z,w) \to (O(\mid z,w \mid^2), \beta w + O(\mid z,w \mid^2))$ for some nonzero β. This is again impossible. It remains to consider the case when there is one nonsingular branch, $w = 0$ which is mapped to itself and at least one more branch which is mapped to $w = 0$. So again we may assume that the other eigenvalue is nonzero and then that the map has the form $f(z,w) = (O(w* \mid z,w \mid^2), \alpha w + w * O(\mid z,w \mid)$ for some nonzero α. Hence no other branches can be mapped into the z-axis.

The second case to consider is when for some nonempty open subset $\omega \subset\subset \Omega$ there exists some subsequence $\{f^{n_j}\}$ converging to h with values in $V_1 \setminus$ (neighborhood of order ϵ of $A =: A_\epsilon$). We first assume that Ω intersects some preimage of V. Then some forward orbit of Ω intersects some V_j. Clearly this intersection must be with the Fatou set of the restriction map to V_j. But since this restriction is critically finite this implies that any such Fatou component is superattractive as a Fatou component of V_j for a superattractive fixed point p. It remains to show that this is a superattractive fixed point for f on $\mathbf{P}^2$ as well. If $p \in A$ this is as in the first part of the proof. So we may assume that p is a nonsingular point of V. So we may assume that $V_j = (w = 0)$ near the origin and that f has the form $f(z, w) = (z^k + O(\mid z \mid^{k+1}, w* \mid z, w \mid),$ $\alpha w + w * O(\mid z, w \mid)$ for some $k > 1$. Computing the Jacobian we see hat this forces them to have a branch of the critical set through the axis, implying that $p \in A$, a contradiction. Note that if we replace f by a high iterate, the set A may increase, but the argument in the first part still applies. So if some forward orbit of Ω intersects V_j, then this is a superattractive basin, and we are done.

Hence we can assume that Ω does not intersect any preimage of V. Since the Kobayashi infinitesimal metric of $\mathbf{P}^2 \backslash V$ does not blow up in directions parallell to $V_1 \backslash A_\epsilon$, when we approach V_1, we find that h is nonconstant.

We can assume that $\hat{V}_1 = \mathbf{P}^1$ or a torus. Then the image of h cannot contain a repelling periodic point. Say if $f^\ell(p) = p, \mid f^{\ell'}(p) \mid > 1$. Then $\{f^{n_j + \ell r_j}\}$ will have some derivative blowing up in ω. So $\hat{V}_1 = \mathbf{P}^1$ and the image of h will be in a Fatou component of $f \mid V_1$. Hence Ω is preperiodic to a superattractive component. Indeed $f \mid V_1$ has a superattractive component since $f \mid V_1$ is critically finite, the same proof as above shows that the superattractive points with respect to $f \mid V_1$ is superattractive.

The only remaining case is when $f^n \mid \Omega$ does not converge to V. But this cannot happen, by Theorem 6.6, because $\mathbf{P}^2 \backslash V$ is hyperbolic.

We now consider some cases when we do not assume that for some N, $\mathbf{P}^2 \backslash \bigcup_{n=0}^N f^n(C)$ is hyperbolic.

THEOREM 7.8. *Assume that* $f : \mathbf{P}^2 \to \mathbf{P}^2$ *is strictly critically finite. Suppose that for some irreducible component* C_0 *of the critical set we have* $f(C_0) = C_0$. *Then the only Fatou components are preperiodic to superattractive basins.*

We start at first with two lemmas of independent interest.

LEMMA 7.9. *Suppose that* $g : \mathbf{P}^2 \to \mathbf{P}^2$ *is a holomorphic map of degree* d *and that* g *maps a compact complex hypersurface* Z *to itself and that* Z *is contained in the critical set of* g. *Then we have the estimate* $\text{dist}(f(z), Z) = o(\text{dist}(z, Z))$.

Proof of the Lemma. We reduce this to a local statement in $\mathbf{C}^2$. Let X, Y

be one dimensional closed analytic subsets of the unit ball B containing zero. Suppose that $f : B \to B$, $f(0) = 0$, is a holomorphic map with $f(X) \subset Y$ such that the Jacobian of f vanishes on X. Moreover assume that
Moreover assume that f is a finite map. Then for every constant $c > 0$ there exists a neighborhood V of 0 such that $\text{dist}(f(p), Y) \geq c\text{dist}(p, X)$ for all $p \in V$. Note that we may assume that X, Y are irreducible at zero. After changes of coordinates we may assume that X can be parametrized by $t \to (t^p, t^q(g(t))$ where $g(0) = 1$ and $1 \leq p < q$. Similarly Y can be parametrized by $\tau \to (\tau^{p'}, \tau^{q'}(g'(\tau))$ where $g'(0) = 1$ and $1 \leq p' < q'$. Next observe that in a small enough neighborhood of 0, we can measure the distance to X parallell to the w-axis, i.e. $d_w := \text{dist}((z_0, w_0), X \cap \{(z_0, w) \in X\} \geq \text{dist}((z_0, w_0), X) \geq \frac{1}{2} d_w$. To prove that, the left inequality is obvious. For the right inequality, observe at first that if we consider any smooth curve in X, then the total variation in w is less than ϵ times the total variation in z. It follows that the bidisc $\Delta((z_0, w_0); d_w/2)$ does not intersect X. Hence the right inequality follows. If the derivative $\frac{\partial f}{\partial w}$ vanishes, it follows from considering the image of lines parallell to the w-axis that $\text{dist}(f(p), Y) \leq c\,\text{dist}(p, X)$ for all p close enough to zero. Hence we are left with the case that $\frac{\partial f}{\partial w} \neq 0$ at the origin. The image of lines parallell with the origin are therefore nonsingular curves (near 0). Note that since the Jacobian vanishes on X, necessarily these lines must hit Y tangentially except possibly at the origin. However this implies by continuity that $\frac{\partial f_1}{\partial w} \neq 0$ and $\frac{\partial f_2}{\partial w} = 0$ at the origin, where we have written $f = (f_1, f_2)$. Next consider a line L parallell to the w-axis through (z_0, w_0). Inside L consider the straight line γ of lenght r from (z_0, w_0) to the nearest point q in $\{(z_0, w) \cap X\}$. Consider the image $f(\gamma)$. Then the horizontal length of this image is of order of magnitude r, while the vertical lenght is $o(r)$ where the little $o(r)$ refers to an expression bounded by an arbitrarily small multiple of r as the neighborhood shrinks. Also there must be a point on Y with the same z-coordinate as $f(z_0, w_0)$ with a w-coordinate differing by $o(r)$. But this shows that $\text{dist}(f(z_0, w_0), Y) \leq c\,\text{dist}((z_0, w_0), X)$ as desired.

LEMMA 7.10. *Let* $f \in \mathcal{H}$, $f : \mathbf{P}^2 \to \mathbf{P}^2$. *If* C_0 *is a critical invariant component, then except for Siegel domains all limit functions for* $\{f^n\}$ *in a Fatou component are in the closure of* $\bigcup_{n=0}^{\infty} f^n(C)$.

Proof of the Lemma. By the previous lemma there exists a neighborhood U of C_0 such that $f(U) \subset\subset U$. It is easy to construct enough algebraic hypersurfaces contained in U and to show that $\mathbf{P}^2 \backslash \bar{U}$ is hyperbolic.

Let Ω be a Fatou component for f, which is not a Siegel domain. If $f^{n_0}(\Omega)$ intersects U for some n_0, then all possible limits are in C_0. Otherwise $f^n(\Omega)$ stays in the hyperbolic set $\mathbf{P}^2 \backslash \bar{U}$ and the argument about Siegel domains of Theorem 1.8 applies to show that Ω is preperiodic to a Siegel domain, a contradiction.

The proof of the theorem then follows the same lines as the previous theorem.

We give here an example of a critically finite map on $\mathbf{P}^2$ where the Julia set has nonempty interior, but the Julia set is not all of $\mathbf{P}^2$. This contrasts with the one dimensional case where the Julia set is either everything or has empty interior.

Define $f : \mathbf{P}^2 \to \mathbf{P}^2$, $[z : w : t] \to [(z - 2w)^2 : z^2 : t^2]$. Then the critical orbits are

$$(z = 2w) \to (z = 0) \to (w = 0) \to (z = w) \leftarrow$$

$$(t = 0) \leftarrow .$$

First, observe that f is strictly critically finite.

Note that inside $(t = 0)$ the map is of the form $z \to \frac{(z-2)^2}{z^2}$ which is strictly preperiodic. Hence the Julia set contains all of $(t = 0)$. Since $(t = 0)$ is critical, there is, (Lemma 7.9) an open neighborhood U of $(t = 0)$ such that $f^n(U) \to (t = 0)$. Hence if we restrict to any nonempty open subset of U, the iterates cannot be a normal family there, since there is no superattractive point on $(t = 0)$. However the point $[0 : 0 : 1]$ is an attractive fixed point, so has a nonempty open basin of attraction. Hence the Fatou set is nonempty as well.

One can also give a direct proof. Note that f maps the space of lines to itself. $(w = \alpha z) \to (w = (1 - 2\alpha)^{-2} z)$ and high iterates of an open set of lines cover the space of all lines. One shows that the Fatou set is just the basin of attraction of $[0 : 0 : 1]$. It is Kobayashi hyperbolic since if $\| (z, w) \|$ is large enough, then $f^k[z : w : 1] \to (t = 0)$. And on the dense set of periodic lines, the map is exceptional so has only two Fatou components. As in the end of the proof of Theorem 5.7 the derivatives of the iterates of f must blow up on any open set disjoint from the basin of attraction of $[0 : 0 : 1]$. Hence the Julia set has nonempty interior. However int J is not Kobayashi hyperbolic since it contains $(t = 0)$.

Note that if $[P(z, w) : Q(z, w)]$ is any rational map on $\mathbf{P}^1$ which is strictly preperiodic, critically finite, the same argument works to show that the map $[P(z, w) : Q(z, w) : t^d]$ has nonempty Fatou set and Julia set with interior, $d = \deg P, Q$. This is a class of examples where the complement of the closure of $\bigcup_{n=0}^{\infty} f^n(C)$ is not Kobayashi hyperbolic for any rational map $[P : Q]$. Indeed $[0 : 0 : 1]$ is superattractive. The action of f on $(t = 0)$ has Julia set $\mathbf{P}^1$ and f maps lines to lines. Hence the argument is as above.

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Astérisque

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Normal forms for local families and nonlocal bifurcations

Astérisque, tome 222 (1994), p. 233-258
<http://www.numdam.org/item?id=AST_1994__222__233_0>

Normal forms for local families and nonlocal bifurcations

Yu. S. Ilyashenko

Introduction

This paper deals with two closely related topics:

1. Finitely smooth normal forms for local families.
2. Bifurcations of polycycles of few- and many- parameter families. Here "few" is "no greater than 3"

The exposition is the summary of two large paper [I,Y3] and [K,S] which are to be published in the forthcoming book [I]. Therefore all the proofs are brief in this text; there detailed exposition would be found in the book, quoted above.

It appears, that for the study of nonlocal behavior of the orbits of vector field from the *topological* point of view, the *smooth* normal forms of vector field near singular points are necessary. For instance, consider a separatrix loop of a hyperbolic saddle (Figure 1).

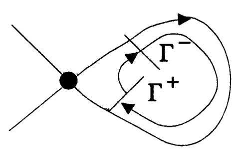


FIGURE 1

We want to know, wether the positive semiorbits winging inside the separatrix loop come *to* or *off* this loop. The topological normal form of the field near the saddle is one and the same for all the fields and give no information on the subject; it is

$$\dot{x} = x, \quad \dot{y} = -y$$

Meanwhile, the smooth normal form in the nonresonant case is

$$\dot{x} = \lambda_1, \quad \dot{y} = -\lambda_2 y$$

$\lambda_1 > 0, -\lambda_2 < 0$ are the eigenvalues of the singular point. The *correspondence map* of the entrance semitransversal Γ^+ onto the exit one Γ^- is equal to

$$\Delta(x) = x^\lambda, \lambda = \lambda_2/\lambda_1$$

Suppose $\lambda \neq 1$. Then the correspondence map in the small neighborhood of O on Γ^+ has a large Lipshitz constant; in the case $\lambda > 1$ this constant tends to zero as the neighborhood contracts to a point. The smooth map from Γ^+ to Γ^- along the orbits cannot neutralize this contraction; therefor in the case $\lambda > 1$, the separatrix loop is orbitally stable from inside. In the same way, it is unstable if $\lambda < 1$.

The example motivates the study of smooth normal forms of local families.

On the other hand, the bifurcations of polycycles are closely related with to Hilbert 16^{th} problem, as is discussed below.

§1. Numbers related to the Hilbert 16^{th} problem.

Consider a family of differential equations

$$\frac{dy}{dx} = \frac{P_n(x,y)}{Q_n(x,y)} \tag{1}$$

where P_n and Q_n are polynomials of degree no larger than the fixed constant n. The following definition is popular in the survey literature.

Definition 1. The Hilbert number $H(n)$ is the maximal possible number of limit cycles of the equation of the family (1).

It is obvious, that $H(1) = 0$. Indeed, a linear vector field has no limit cycles at all.

Nothing is known about the numbers $H(2)$; its mere existence is an open problem.

One can figure out, why Hubert has chosen the family (1) for the study of limit cycles. In the end of the last century polynomial families gave probably the only natural example of finite parameter families of vector fields.

Now, when the mode and viewpoints have reasonably changed, generic finite parameter families became respectful. Therefore a smooth version of the Hilbert 16^{th} problem may be stated; it is written between the lines of some text due to Arnold [AAIS].

Hilbert-Arnold conjecture. *The number of limit cycles of the equation of the typical finite parameter family* (here and below "family" means " $\mathbb{C}^{\infty}$ family of vector fields in S^2 ") *with the compact base is uniformly bounded with respect to the parameter.*

This conjecture is closely related to some nonlocal bifurcation problem. We will first state it and then recall necessary natural definitions.

Conjecture. *Cyclicity of any polycycle appearing in the typical finite parameter family is finite.*

Definition 2. A polycycle is a finite union of singular point and continual phase curves of the field which is connected and cannot be contracted along itself to any proper subset.

A limit cycle is generated by a polycycle γ in the family

$$\dot{x} = v(x, \epsilon), \quad x \in S^2, \epsilon \in B \subset \mathbb{R}^k$$

if the path $\epsilon(t)$ in the parameter space exits such that for any $t \in (0, 1]$ the equation corresponding to $\epsilon(t)$ has a limit cycle $l(t)$, continuously depending on the parameter t, $l(1) = l$, and

$$l(t) \to \gamma \text{ as } t \to 0$$

in sense of the Hausdorff distance.

Cyclicity of the polycycle in the family is the maximal number of limit cycles generated by this polycycle and corresponding to the parameter value, close to the critical one; the last corresponds to the equation with the polycycle.

Theorem (Roussarie). *The equations of the family with the compact base and the polycycles having finite cyclicity only have a uniformly bounded number of limit cycles.*

Therefore the last Conjecture implies the Hilbert-Arnold one. Some *bifurcation numbers* related to these Conjectures, are naturally defined.

Recall that a singular point of a planar vector field is called *elementary* if it has at least one nonzero eigenvalue. A polycycle is called elementary if all its vertexes are elementary.

Definition 3. $B(n)$ is the maximal number of limit cycles which can be generated by a polycycle met in a typical n-parameter family.

$E(n)$ is the maximal numbers of limit cycles which can be generated by an *elementary* polycycle in a typical n-parameter family.

$C(n)$ is the maximal number of limit cycles which can bifurcate in a typical n-parameter family from *all* the polycycles of the field, corresponding to the "critical" value of the parameter.

Conjecture. *$B(n)$ exists and is finite for any n.*

This Conjecture is stronger then Hilbert-Arnold one.

§2. Statements of results.

Theorem 1 (Ilyashenko & Yakovenko). *For any n the number $E(N)$ exists.*

Theorem 2 (Kotova). $C(3) = \infty$.

This means that for any N one can find a generic 3-parameter family, in which some differential equation generates more than N limit cycles.

Moreover, a complete list of polycycles which can generate limit cycles and appear in generic 2 and 3- families is given; this is so called "Zoo of Kotova", Table 1 below.

Theorem 3. $B(2) = 2$.

Theorem 4. $C(2) = 3$.

Last two theorems are due to Grosowskii, Druzkova, Chelubeev and Seregin.

Theorem 5 (Stanzo). *For generic three parameter families there is a countable number of topologicaly nonequivalent germs of bifurcation diagrams.*

In this form the Theorem 5 is an easy consequence of the Theorem 2. In fact Stanzo describes the topological and even the smooth structure of bifurcation diagrams for unfoldings of the phase portrait called "lips"(Figure 2), and constructs the invariants of the topological structure of these diagrams.

As a by product of this study a *generalized Legendre duality* is found.

Comments to the Table 1. In the Table 1 all the polycycles which can appear in the generic 2 and 3 parameter families are presented. For sure, "all" means "all equivalence clases"; the equivalence relation is a following. Two polycycles are equivalent, if they have diffeomorphic neighborhoods in $\mathbb{R}^2$ and a diffeomorphism of one of them to another exists which transforms one

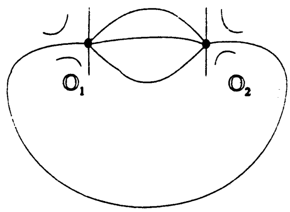


FIGURE 2. Lips

polycycle to another in such a way, that for a correspondent singular points the multiplicity and other characteristics shown in the table coincide. We do not claim, that the equivalent polycycles have equivalent unfoldings; on the contrary, for the most part of cases already investigated this is not the fact. The presence or absence of a punctured heteroclynic curve distinguishes two nonequivalent polycycles in the cell 3.9. Abbreviations in the table mean the following:

res resonant sadoble,

k-degen degeneration in the nonlinear terms of codimension k,

$\int = 0$ $\int_\gamma \operatorname{div} v \, dt = 0$, where v is the correspondent vector field, t is time and γ is a separatrix loop.

The sign $\oplus$ means that the correspondent case is investigated, but not published; the capital letter stands for the name of the author. In case, when the result is already published, the abbreviations mean

B Bogdanov, T Takens, L Lukianov, L-R Leontovich, Roussarie, R Reun, L.W Li Weign, DRS Dumortier, Roussarie, Sotomayor. Most part of the references may be found in [AAIS].

The authors of papers in preparation are

K Kotova, Ch Chelubeev, G Grosovski, S Seregin, St Stanzo.

They are young Moscow mathematicians.

§3. NORMAL FORMS FOR LOCAL FAMILIES

The detailed exposition is published in [IY1], therefore we give only a brief summary here.

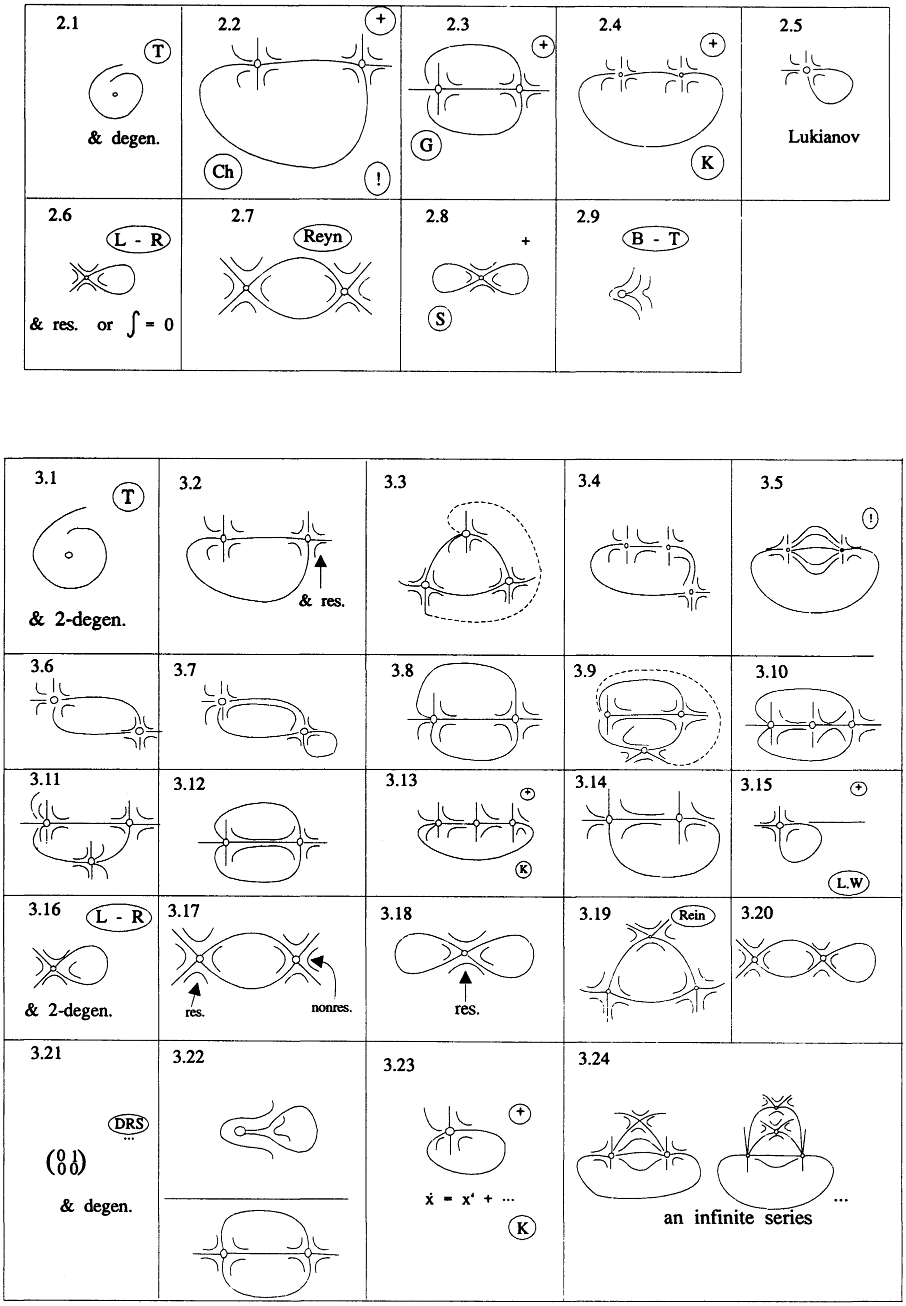

2.1
T
& degen.
2.2
+
Ch
!
2.3
+
G
2.4
+
K
2.5
Lukianov
2.6
L - R
& res. or $\int = 0$
2.7
Reyn
2.8
+
S
2.9
B - T
3.1
T
& 2-degen.
3.2
& res.
3.3
3.4
3.5
!
3.6
3.7
3.8
3.9
3.10
3.11
3.12
3.13
+
K
3.14
3.15
+
L.W
3.16
L - R
& 2-degen.
3.17
res.
nonres.
3.18
res.
3.19
Rein
3.20
3.21
DRS
...
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
& degen.
3.22
3.23
+
$\dot{x} = x^4 + \dots$
K
3.24
an infinite series
...

TABLE 1

Theorem 6. *1. The deformations of the hyperbolic germs of vector fields in a fix point which is nonresonant or oneresonant (all the resonance relations are the consequences of a single one $(\lambda, r) = 0$, $r \in \mathbb{Z}_+^n$, λ is a tuple of eigenvalues of a singular point) have polynomial integrable normal forms with respect to C^k-equivalence for any $k < \infty$.*

2. Analogous statement holds for germs of diffeomorphisms with the only change: $(\lambda, r) = 0$ must be replaced by $\lambda^r = 0$, where λ is a tuple of multoplicators of the fix point.

3. Deformations of saddlenodes of vector field in $\mathbb{R}^n$ (one eigenvalue is zero) having finite multiplicity and no supplementary resonances are C^k equivalent for any k to the linear suspension over one dimensional polynomial integrable family.

The explicit formulae are listed in [IY1], and we shall not repeat it here. Note that the elementary singular points of the planar vector fields fall under conditions of the previous theorems. The list of finitely smooth normal forms of their unfoldings will be given in §6 and used below.

The above theorem exhausts the positive results of this kind. Unfoldings in the other cases corresponding to the codimension one degenerations has functional module of smooth classification, or have no reasonable classification at all.

Theorem 7. *[IY2] 1. Typical one parameter deformation of germs of one dimensional diffeos with multiplicator $\lambda = 1$ or $\lambda = -1$ has the functional modules of C^1-classification.*

2. The same is true for the Andronov-Hopf families: deformations of planar vector fields with $\lambda_{1,2} = \pm i\omega, w \neq 0$.

The modules in the above theorem are explicitly described.

The deformations of saddle suspensions over the above families are finitely smooth equivalent to linear suspensions over these families.

The result form the end of the long chain built by Belitski, Bogdanov, Brjuno, Dumortier, Kostov, Roussarie, Samovol, Takens. See [B], [Bo], [Br], [D], [K], [R], [S], [T].

§4. Lips or Why $C(3) = \infty$?

Consider a vector field in $\mathbb{R}^2$ having two saddlenodes O_1, O_2 of multiplicity two and a saddle connection, like it is shown in Figure 2. These three requirements produce a vector field with degeneration of codimension 3.

Theorem 8 (Kotova). *For any N in a typical 3 parameter family such a vector field with "lips" may be met that its unfolding will have the equations with more than N limit cycles.*

Remark. This theorem immediately implies Theorem 2.

Sketch of the proof. The finitely smooth orbital versal deformation of a saddlenode of multiplicity two has the form [IY1]:

$$\dot{x} = (x^2 + \varepsilon)(1 + ax)^{-1}$$
$$\dot{y} = \pm y$$

Consider the unfolding of "lips". After the suitable reparametrization and coordinate change near saddlenodes one obtain the following local systems near O_1 and O_2 respectively (see Figure 2):

$$\dot{x} = (x^2 + \varepsilon)(1 + a(\bar{\varepsilon})x)^{-1}, \ \dot{y} = -y$$
$$\dot{x} = (x^2 + \delta)(1 + b(\bar{\varepsilon})x)^{-1}, \ \dot{y} = y \tag{4.1}$$

where $\bar{\varepsilon} = (\varepsilon, \delta, \lambda) \in (\mathbb{R}^3, 0)$ and $\lambda = 0$ corresponds to the saddle connection.

Now consider the *limit cycle equations.* This will be an equation for the fix points of the Poincaré map¡ written in the appropriate form. For this sake decompose the Poincaré map for the unfolding of the polycycle, in the domain, where it is defined, into the composition of the four maps, shown in the figure 3.

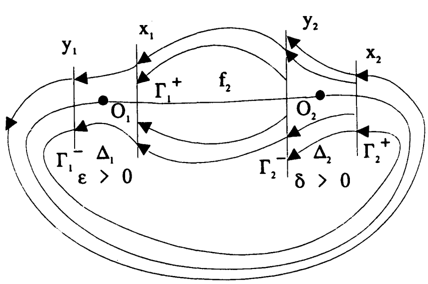


FIGURE 3

The transversals Γ_1^+, Γ_1^- are taken in the neighborhood of the point O_1, where the normalizing chart for the local family is defined. Let Γ_1^+ be the entrance and Γ_1^- the exit transversal through which the phase curves enter in and come off the mentioned neighborhood of O_1. Let x_1 and y_1 be restrictions of the y-function of the normalizing chart to Γ_1^+ and Γ_1^- respectively. Then the *correspondence map*

$$\Delta_1 : \Gamma_1^+ \to \Gamma_1^-$$

along the phase curves will take the form

$$\Delta_1(x_1) = y_1, \; y_1 = C_1(\varepsilon)x_1$$
$$C_1(\varepsilon) \to 0 \text{ as } \varepsilon \to 0$$

This map is called the *funnel*. In fact

$$C_1(\varepsilon) = \phi(\varepsilon)\exp(-\frac{\pi}{\sqrt{\varepsilon}})$$
$$\Phi(\varepsilon) = \exp\frac{2}{\sqrt{\varepsilon}}(-\frac{\pi}{2} + \arctan\frac{1}{\sqrt{\varepsilon}}) = O(1).$$

An analogous construction near the point O_2 gives the transversals Γ_2^+, Γ_2^- with the charts x_2, y_2 and the correspondence map

$$\Delta_2(x_2) = y_2, y_2 = C_2(\delta)x_2. C_2(\delta) \to \infty \text{ as } \sigma \to 0,$$
$$C_2(\delta) = (\phi(\delta))^{-1}\exp\frac{\pi}{\sqrt{\delta}}$$

The map Δ_2 is called is *shower*.

There are also two regular maps along the phase curves depending on $\bar{\varepsilon}$ as a parameter:

$$f_{\bar{\varepsilon}} : \Gamma_1^- \to \Gamma_2^+ \text{ and } g_{\bar{\varepsilon}} : \Gamma_2^- \to \Gamma_1^+.$$

The Poincaré map $\Delta : \Gamma_1^+ \to \mathbb{R} \supset \Gamma_1^+$ for any fixed $\bar{\varepsilon}$ is a composition

$$\Delta = g_{\bar{\varepsilon}} \circ \Delta_2 \circ f_{\bar{\varepsilon}} \circ \Delta_1$$

The limit cycle equation has the form

$$\Delta(x) = x,$$

or equally

$$\Delta_2 \circ f_{\bar{\varepsilon}} \circ \Delta_1 = g_{\bar{\varepsilon}}^{-1} \tag{4.2}$$

The last equation will be studied below.

Note that the function $g_{\bar{\varepsilon}}$ is not a germ but an actual function on a segment. The situation completely loses locality and, therefore, the following construction goes.

Chose the curve

$$\gamma = \{\bar{\varepsilon}(\varepsilon)\}, \ \bar{\varepsilon}(\varepsilon) = (\varepsilon, \delta(\varepsilon), \lambda(\varepsilon)), \ \lambda(\varepsilon) \equiv 0,$$

with the endpoint zero in the parameter space such that

$$\begin{aligned} &f_{\bar{\varepsilon}}(x_1) = 0 \text{ for } x_1 = 0, \ \bar{\varepsilon} = \gamma(\varepsilon), \\ &C_1(\varepsilon)C_2(\delta(\varepsilon)) \equiv 1 \end{aligned}$$

The left hand side of (4.2) for small ε will be the rescaling $\tilde{f}_\varepsilon$ of the smooth function $f_{\gamma(\varepsilon)}$ with $f_{\gamma(\varepsilon)}(0) = 0$:

$$\tilde{f}_\varepsilon(x_1) = C^{-1}(\varepsilon) \circ f_{\gamma(\varepsilon)} \circ C(\varepsilon)(x_1)$$

The limit of the rescaled smooth function with the zero value in zero is linear. The Figure 4 shows that for a function $g_{\bar{\varepsilon}}$ properly chosen the limit cycle equation (4.2) for $\bar{\varepsilon} = 0$ may have a prescribed number of solutions, and a situation is structurally stable.

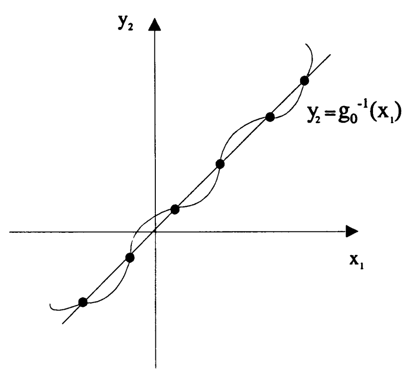


FIGURE 4

This proves the Theorem 8.

§5. Bifurcation diagram for the "lips" and generalized Legendre duality.

In the first half of this section the bifurcation diagram for "lips" will be described. This will give the sketch of the proof of the Theorem 5.

5.1 Bifurcation diagram and Legendre transformation.

Recall that the point in the parameter space of the family of vector fields belong to the bifurcation diagram if the corresponding vector field is not structurally stable in its domain.

We will describe the intersection of the bifurcation diagram (BD) for "lips" with the narrow funnel U centered on the curve

$$\varepsilon = \delta, \ \lambda = 0$$

Let

$$\delta_1 = \frac{\delta - \varepsilon}{\varepsilon^{\frac{3}{2}}}, \ \lambda_1 = \frac{\lambda}{\exp(-\sqrt{\varepsilon})}$$

$$U = \{(\varepsilon, \delta, \lambda) | \delta_1 \in \sigma, \ \lambda_1 \in \sigma\} \tag{5.1}$$

Here $\sigma = [-A, A]$ is such a segment, that

$$A \gg \max |g_{\bar{\varepsilon}}^{-1}|, \ A \gg \max(g_{\bar{\varepsilon}}^{-1})'$$

Let $(\varepsilon, \delta_1, \lambda_1)$ be the new coordinates in U. Consider only those point on the BD which correspond to semistable limit cycle. Fix a small value of ε. The limit cycle equation (4.2) has the form

$$C_2(\delta) f_{\bar{\varepsilon}}(C_1(\varepsilon)x) = g_{\bar{\varepsilon}}^{-1}(x) \tag{5.2}$$

Suppose that ε runs the curve in U having a definite limit point $(0, \delta_1, \lambda_1)$ in the chart $(\varepsilon, \delta_1, \lambda_1)$, see (5.1). Then the solution of (5.2) $g_{\bar{\varepsilon}}^{-1}$ is given by the intersection of the graph of $f_{\bar{\varepsilon}}$ and the rescaled graph of $f_{\bar{\varepsilon}}$ which is almost a straight line. When the parameters δ_1, λ_1 change, these "almost straight lines" changes also; the bifurcation diagram contains the points corresponding to the tangency of these graphs (Figure 5).

In the case, when $\varepsilon = 0$, the set of parameters (δ_1, λ_1), corresponding to the tangencies (2) on the Figure 5, will form the part of the "blown up vertex"

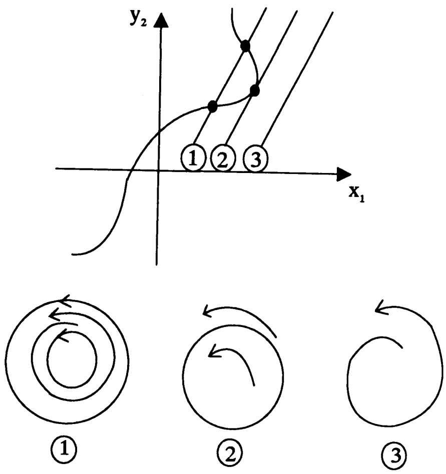


FIGURE 5

of BD in the funnel U. This set will be the Legendre transformation of the graph of g_0^{-1} on the parameter plane (a, b) of the straight lines

$$y_2 = ax_1 - b,$$
$$b = -\lambda_1, \; a = f_0'(0)\exp(-\frac{\pi}{2}\delta_1)$$

The last formulae may be easily obtained analyzing the rescaling in the left hand side of (5.2).

Therefore, the intersection $BD \cap U$ contains a surface, which becomes diffeomorphic to a cylinder over the Legendre transformation of a graph of g_0^{-1} after the blowing up (5.1). This proves the Theorem 5, §2.

Now discuss the intersection

$$\Sigma = BD \cap U \cap \{\varepsilon = \varepsilon_0\}, \; \varepsilon_0 > 0$$

Then the lines would be replaced by the curves, still forming the two parameter family; the parameter values corresponding to tangency with the graph of g_0^{-1} belong to Σ. This intersection is equal, up to some details, to *the generalized Legendre transformation* of the graph of Σ. First recall some classical definitions [A].

5.2. Dual second order differential equations..

Definition 4. A two parameter family of curves in the plane is the divergent diagram of maps:

$$\mathbb{R}^2 \xleftarrow{\phi} \Omega \xrightarrow{\psi} \mathbb{R}^2,\ \Omega \subset \mathbb{R}^3 \tag{5.3}$$

Remark. In the generic case near a typical point $\mathcal{O}$ of Ω both maps φ and ψ are regular (have rank 2). Consider the set $\psi(\Omega) = \{(a,b)\} \subset \mathbb{R}^2$ as a space of parameters and the set $\varphi(\Omega) = \{(p,q)\} \subset \mathbb{R}^2$ as a phase space. The level curves of the germ $\psi : (\Omega, \mathcal{O}) \to (\mathbb{R}^2, \mathcal{O}_1)$, $\mathcal{O}_1 = \psi(\mathcal{O})$ form the germ of a one dimensional foliation in $\mathbb{R}^3$. The image of this foliation under the map ψ is the two parameter family of plane curves (in the naive sense), see Figure 6a.

Remark. Let (t,a,b) be the local chart near $\mathcal{O}$, and let $\varphi = (\varphi_1, \varphi_2)$. Denote by dot the derivation with respect to t along the level curves of ψ : $a = \text{const}$, $b = \text{const}$ in $(\Omega, \mathcal{O})$. Then the function

$$p = \frac{\dot{\varphi}_2}{\dot{\varphi}_1}$$

is the derivative of a function $y = f(x)$ with the graph $\varphi(\psi_{-1}(a,b))$ for suitable a and b. The function $\frac{d^2 f}{dx^2}$ may be also expressed through a, b, t, φ_1, and φ_2 as a function q on $(\Omega, \mathcal{O})$. But *in the generic case functions x, y, p form a chart in* $(\Omega, \mathcal{O})$. Therefore one can write

$$q = \Phi(x, y, p)$$

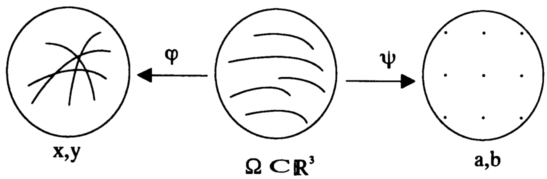
The curves $\{\varphi(\psi^{-1}(a,b) \mid (a,b) \in \psi(\Omega, \mathcal{O})\}$ are the graphs of the solutions of the differential equations

$$y'' = \Phi(x, y, y') \tag{5.4}$$

Therefore, in a genetic case, the diagram (5.3) near a genetic points, gives *a germ of a second order differential equation.*

Definition 5. The local second order differential equation is the genetic diagram (5.3) near a regular point of a map ψ, with the phase space Im φ and a parameter space Im ψ.

Now define the differential equation *dual* to the previous one. For this sake we should nearly change the roles of the maps φ and ψ.



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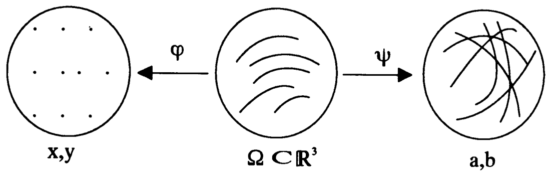

FIGURE 6

Definition 6. The second order differential equation dual to the one described in definition 5, is the same diagram (5.3) considered near a regular point of a map φ with the phase space Im ψ and a parameter space Im φ (see Figure 6 b)

5.3. Generalized Legendre duality.

The definition of the generalized Legendre duality is naturally realized with the constructions of the Figure 5.

Definition 7. The generalized Legendre transformation of the planer curve γ with respect to the family (5.3) with the parameter space Im ψ is the set $\hat{\gamma}$ of the parameters $\alpha = (a, b)$ such that the curves

$$\gamma \text{ and } \varphi(\psi^{-1}(\alpha))$$

are tangent.

Remark. For the family

$$y = ax - b$$

(the maps φ and ψ have the form

$$(x, ax - b) \xleftarrow{\varphi} (a, b, x) \xrightarrow{\psi} (a, b) \ )$$

the definition 7 give the classical Legendre transformation. The following theorem is a classical.

Theorem [A]. *The Legendre transformation is an involution for the genetic germ of a curve γ. Then means that the curve $\hat{\gamma}$ obtain from γ gives the same curve γ after a Legendre transformation.*

The equivalent statement: the Legendre transformation is the inverse to itself on the set of genetic germs of curves.

Theorem 9. *(Stanzo). The generalized Legendre transformations with respect to dual two parameter families, in sense of Definition 6, are inverse to each other on the set of genetic germ of plane curves.*

Remark. The first proof of this theorem was given by Stanzo and will be published in [K,S]. Here I reproduce the proof, proposed by Cromov *without looking to the explicit statement of the theorem.* I allow myself to reproduce here approximately a fragment of our conservation. I ask Gromov, does he know the fact called the "Generalized Legendre duality". He says " I don't, but the proof must be similar to the classical one. The crucial point is, that the curves of the first family passing through one and the same point correspond to the curve of the dual family on the parameter plane. Let me find the proof in some classical book, for the traditional Legendre transformation". We try to find it in some books of Klein and fail. "Well, – says Gromov, – in this case of lines it looks like what follows" – and he gave a sketch of the proof of the theorem [A], which will be extended below to the general context.

This proof is based on incidence reasons only. We will give a sketch of it using the consequences of some genericity assumptions without formulating them explicitly.

The principal fact used below is that the tangent line is the limit of chords. Similarly, the curve of the two parameter planar family tending to some curve γ in a point α, is a limit of the "chordal" curves of the family passing through the points α and β, where β is the point of the same curve γ tending to α.

Consider two dual families of planar curves given by diagram (5.3). Define by $A, B, \ldots$ points on Im ψ. Let Φ and Ψ_A be the curves of the dual families, correspondent to the parameter values A and α respectively:

$$\Phi_\alpha = \varphi(\psi^{-1}(\alpha)), \ \Psi_A = \psi(\varphi^{-1}(A)).$$

Let

$$\begin{gathered}\widetilde{\Phi}_\alpha = \psi^{-1}(\alpha), \ \widetilde{\Psi}_A = \varphi^{-1}(A), \\ \Phi = \{\Phi_\alpha \mid \alpha \in \text{Im } \psi\}, \ \psi = \{\Psi_A \mid A \in \text{Im } \varphi\}\end{gathered}$$

Take a curve $\gamma \subset$ Im φ, a point $A \in \gamma$ and a curve $\Phi_\alpha \in \Phi$ tangent to γ at a point A. Let γ^* be the generalized Legendre transformation of γ with respect to the family Φ. We want to prove that the curve of the family Ψ tangent to the curve γ^* in the point α is ψ_A, that is to say, corresponds to the point A. This will give the desired duality.

Instead of this study the "chordal" curve; the curve of the family Ψ passing through α and the nearby point $\beta \in \gamma^*$. Let C be corresponding point of Im φ:

$$\Psi_C \ni \alpha,\ \beta;\ \Psi_C = \psi(\varphi^{-1}(C)).$$

The point β corresponds, by definition of γ^*, to the curve Φ_β of the family Φ, tangent to γ in the point B close to A. We state that the point C corresponding to the "chordal" curve Ψ_C is to the point of intersection of Φ_α and Φ_β (see Figure 7). Indeed, the curve

$$\widetilde{\Psi}_C = \varphi^{-1}(C)$$

has a non empty intersections with the curves

$$\widetilde{\Psi}_\alpha = \psi^{-1}(\alpha) \text{ and } \widetilde{\Psi}_\beta = \psi^{-1}(\beta)$$

because these curves form the total inverse image of α and β with respect to the map ψ, and the curve $\psi(\widetilde{\Psi}_C)$ contains α and β. Therefore the image

$$C = \varphi(\widetilde{\psi}_C) \text{ belongs to } \Phi_\alpha = \varphi(\widetilde{\psi}_\alpha) \text{ and } \Phi_\beta = \varphi(\widetilde{\psi}_\beta).$$

This means that the point C is the intersection point of Φ_α and Φ_β.

On the other hand C tends to A, as B tends to A. This proves the theorem.

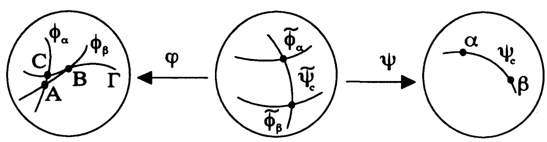

FIGURE 7

Remarks. 1. The generalized Legendre duality may be extended to higher dimensions.

2. After my talk in the conference on Dynamical systems in Triest, June 92, Zakalukin communicated me that this generalized Legendre duality may be derived from some of his recent results (though the statement was unknown to him before).

§6. Bifurcations of elementary polycycles.

In this section the weakened version of the Theorem 1 is discussed.

6.1 Statement of result and four steps of the proof..

Theorem 10 (Ilyashenko & Yakovenko). *An elementary polycycle met in a typical finite parameter family generates only finitely many limit cycles in this family.*

We will give the brief sketch of the proof here; the detailed exposition is given in [IY3]. This proof splits into four steps.

Step I. Replace the Poincaré equation for limit cycles (the equation for fix points of the monodromy map) by the functional-Pfaffian system with the polynomial Pfaffian equations. This step uses the normal polynomial forms for the unfoldings of elementary singular points mentioned in §3 and summarized in the Table 2 below. The Poincaré equation is singular: its right hand side is not defined in the full neighborhood of zero point in the space of phase variables and parameters. The functional-Pfaffian system is regular in a likely neighborhood.

Step II. Replace the functional-Pfaffian system by purely functional system which is regular in the entire neighborhood of zero. This is done using the Khovanskii procedure [Kh].

Step III. Generalize Gabrielov finiteness theorem from real analytic to finitely smooth case. This means, find a sufficient property for finitely smooth maps of real manifolds with boundary to have a uniformly bounded number of universe images of the regular value of the map. This leads to the definition of the so called *nice maps.*

Step IV. The Khovanski procedure reduces the estimate of the cyclicity of elementary polycycle of the upper estimate of the regular solutions of the so called "special chain map". The simplest, not exactly the necessary one, example of the special chain map is the following:

$$g = P \circ f,\ f : B \to \mathbb{R}^N,\ P : \mathbb{R}^N \to \mathbb{R}^n, \tag{6.1}$$

$B \subset \mathbb{R}^n$ is a ball; f is generic, P is polynomial.

Theorem 11 [IY3]. *For any fixed polynomial P and for generic f, the map (6.1) is nice.*

We will now explain these four steps in more details.

6.2 Step I. Reduction to functional-Pfaffian system.

Theorem 12 [IY3]. *The generic unfoldings of elementary singular points of the vector fields in the plane are finitely smooth equivalent to ones listed in the Table below. The correspondence maps for these unfoldings satisfy the Pfaffian equations listed in the column 3 of the same table.*

Type	Normal form	Pfaffian equation for the correspondence map
Nonresonant saddle	$\dot{x} = x$, $\dot{y} = -\lambda(\varepsilon)y$, $\lambda > 0$	$xdy = \lambda(\varepsilon)ydx$ $x > 0$, $y > 0$
Resonant saddle	$\dot{x} = x(\frac{m}{n} + f_\mu(u, \varepsilon))$, $\dot{y} = -y$, $u = x^m y^n$ is the resonant monial, $f_\mu = P_{\mu-1} \pm u^\mu(1 + a(\varepsilon u^\mu)$ $P_{\mu-1}(u, \varepsilon) = \varepsilon_1 + \varepsilon_2 u + \dots + \varepsilon_\mu u^{\mu-1}$	$(f_\mu(x^m, \varepsilon) - mx^m)f_\mu(y^n, \varepsilon)dx$ $+nxy^{n-1}f_\mu(x^m, \varepsilon)dy = 0$ $x > 0$, $y > 0$
Degenerated elementary singular point (saddlenode)	$\dot{x} = g_\mu(x, \varepsilon)$, $\dot{y} = -y$ $g_\mu = P_{\mu-1}(x, \varepsilon) \pm x^{\mu+1}\cdot$ $\cdot(1 + a(\varepsilon)x^\mu)$, $P_{\mu-1}$ is the same as before	a. $xdy = ydx$ b. $g_\mu(x, \varepsilon)dy - ydx = 0$, $y > 0$

TABLE 2

Commentary. The case **a** in the third row of the Table corresponds to the map of the transversals crossing the central manifold.

The case **b** corresponds to the map of the segment transversal to the stable manifold onto segment transversal to the central one.

Let now γ be an elementary polycycle met in a typical k-parameter family. Denote by

$$\Delta : (x, \varepsilon) \mapsto \Delta(x, \varepsilon)$$

the Poincaré map of this polycycle defined for some domain in the space of phase variable x on transversal and parameter ε. This domain contains in its closure zero point corresponding to the polycycle. The limit cycle equation has the form

$$\Delta(x, \varepsilon) = x \tag{6.2}$$

Our goal is to prove the existence of the upper estimate for number of solutions of this equation. The solution of (6.2) is intersection point of the cycle with transversal Γ, see Figure 8. Replace this equation by the system corresponding to the intersection points of the limit cycles generated by γ with transversals separating the singular points $\mathcal{O}_1, \ldots, \mathcal{O}_n$ from the other part of the polycycle, Figure 8.

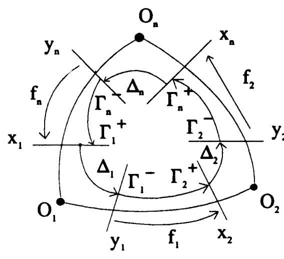

FIGURE 8

Let Γ_j^+ and Γ_j^- be the entrance and exit transversals in the neighborhood of the point $\mathcal{O}_j$. Let Δ_j be the correspondence map of Γ_j^+ to Γ_j^-. Consider the normalizing charts near the singular points, see Table 2. Let x_j, y_j be the restrictions of the appropriate coordinate functions of these charts to Γ_j^+ and Γ_j^- respectively. Then the equation (6.2) may be replaced by the system

$$
\begin{aligned}
y_1 &= \Delta_1(x_1, \varepsilon) \\
x_2 &= f_1(y_1, \varepsilon) \\
&\cdots \\
y_n &= \Delta_n(x_n, \varepsilon) \\
x_1 &= f_n(y_n, \varepsilon)
\end{aligned}
\tag{6.3}
$$

Traditionally the correspondence equations bring the main difficulties. After the Table 2 is written down they become standard. The explicit formulae for them may be derived from this table. What about the functions f_j, we only know that they are regular and generic.

The correspondence maps Δ_j are solution of the Pfaffian equations from the Table 2. Replace these maps in the system (6.3) by the corespondent equations; we will obtain the system

$$
\begin{aligned}
&\omega_1(x_1, y_1, \varepsilon) \\
x_2 &= f_1(y_1, \varepsilon) \\
&\cdots
\end{aligned}
\tag{6.4}
$$

The 1-forms ω_j have the polynomial coefficients with respect to x and ε. This is the end of the step I.

Step II. Khovanskii procedure..

This procedure allows to replace Pfaffian equations once more by the "functional" ones, but the system constructed this time appears to be regular. The algorithm is described in [Kh], its realization may be found in [IY3]. The following system is obtained as a result

$$\mathcal{F}(j_c^n f, \varepsilon) = a \tag{6.5}$$

Here

$$f(y, \varepsilon) = (f_1(y_1, \varepsilon), \ldots, f_n(y_n, \varepsilon)), \tag{6.6}$$

$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in B$ is a parameter, B is a ball in $\mathbb{R}^k$. The notation on the left hand side of (6.5) is explained by the following

Definition 8. Let C (C of Cartesian) denote the space of all maps

$$f : (\mathbb{R}^{n+k}, 0) \to (\mathbb{R}^n, 0)$$

of the form (6.6). The (n, C) jet of the map (6.6) in a point (y, ε) is the set of maps (6.6) (maps from the space C) which difference with f is n-flat in the point (y, ε). The space of all jets $j_C^n f$, $f \in C$ is denoted by J_C^n.

The map $\mathcal{F}$ is polynomial:

$$\mathcal{F} : \mathbb{R}^N \to \mathbb{R}^{n+k}$$

for appropriate N.Next two steps allow to prove that the map (6.5) has a uniformly bounded number of regular inverse images of any of its regular values.

6.4. Step III. Finitely smooth maps with Gabrielov property..

The following theorem of Gabrielov is well known.

Theorem [G]. *Let M be a compact analytic set in the real space and $g : M \to \mathbb{R}^m$ an analytic map. Than for any $a \in \mathbb{R}^m$ the number of connected components of the inverse image $g^{-1}(a)$ is uniformly bounded with respect to a.*

If the map (6.5) would be analytic, then the system (6.5) would have a uniformly bounded number of isolated solutions by the previous theorem. Unfortunately, the map (6.6) is only finitely smooth. In the general case the

Gabrielov theorem for finitely smooth maps is obviously wrong. We must find the sufficient local conditions for the finitely smooth map to have a uniform bound for the number of the inverse images of any of its regular values. These sufficient conditions were obtained analyzing the original proof of the Gabrielov theorem.

Analytic sets form the so called stratified manifolds [W], [M]. Therefor we will consider the smooth ones, referring to the necessary definitions in [M].

Let M be a stratified manifold. Denote by sk M, the ***skeleton*** of M, the union of its strata of all the dimensions lower than the maximal one; the last is called ***the dimension of*** M. Consider a complete flag in $\mathbb{R}^m$ with the orthogonal projections π_j:

$$\mathbb{R}^m \xrightarrow{\pi_m} \mathbb{R}^{m-1} \to \cdots \to \mathbb{R}^1 \xrightarrow{\pi_1} \mathbb{R}^0 = \{0\} \tag{6.7}$$

Definition 9. The map $g : M \to \mathbb{R}^m$ of m-dimensional compact stratified manifold in $\mathbb{R}^m$ is called nice if a flag (6.7) and a commutative diagram

$$\begin{array}{ccccccccc} M^m & \xleftarrow{i_m} & M^{m-1} & \xleftarrow{i_{m-1}} & \dots & \xleftarrow{i_2} & M^1 & \xleftarrow{i_1} & M^0 \\ \downarrow{\scriptstyle g_m} & & \downarrow{\scriptstyle g_{m-1}} & & & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} \\ \mathbb{R}^m & \xrightarrow{\pi_m} & \mathbb{R}^{m-1} & \xrightarrow{\pi_{m-1}} & \dots & \xrightarrow{\pi_2} & \mathbb{R}^1 & \xrightarrow{\pi_1} & \mathbb{R}^0 \end{array} \tag{6.8}$$

exist having the following properties:

$$M^m = M, \ M^{j-1} = \text{sk } M^j, \ g_m = g, \tag{6.9}$$

M^j is a j-dimensional stratified manifold, i_j is a natural embedding. On all the strata of M^j of the higher dimension the following dichotomy holds: either g_j is regular in any point of the stratum. or rank $df_j < \dim M^j$ on entire stratum.

Remark. The maps g_j are well defined by map g, the flag (6.7) and the diagram (6.8).

Definition 10.. The contiguity number for the stratified n-dimensional manifold is the maximal number of n-strata adjacent to the $n-1$ strata in the small neighborhood of the points of these last strata, see Figure 9.

Denote the contiguity number for m_j in (6.8) by ν_j.

Theorem 13. . *If the map g_n is nice and (6.8) is the correspondent diagram with the contiguity number ν_j, then the number of inverse images of any regular values g admits the following estimate*

$$\#\{g^{-1}(a)\} \le \frac{1}{2^m}\nu_1 \cdot \ldots \cdot \nu_m \cdot \#\{M^0\}$$

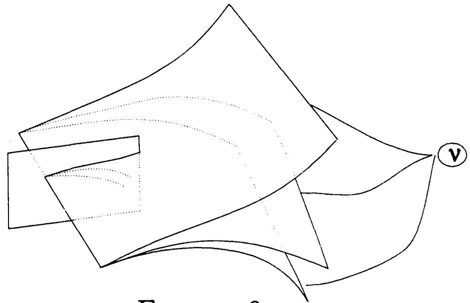

FIGURE 9

◁ The proof goes by induction by j. Suppose the likely estimate for $g_{m-1} : M^{m-1} \to \mathbb{R}^{m-1}$ is obtained. Eliminate from M^m all the higher strata where the rank of dg drops. This does not change the number of regular preimages and the map g with the domain narrowed in this way still remains to be nice. Take a regular value b of g_{m-1} and consider a function

$$\varphi(a) = \#g^{-1}(a) \text{ for } a \in \mathbb{R}^1 = \pi_m^{-1}b.$$

This function is piecewise constant and jumps in the points of the image of the visible contour of M^m only. The magnitude of each jump is, roughly speaking, no larger than ν_m, and the number of jumps is no larger than

$$J = \frac{1}{2^{m-1}}\nu_1 \cdot \ldots \cdot \nu_{m-1} \cdot \#\{M^0\}$$

by the induction assumption, see Figure 10. For the values of a close to $-\infty$ and ∞, $\varphi(a) = 0$. Therefore its maximal value is no greater than one half of its oscillation, which may be in turn estimated;

$$\text{osc } \varphi \leq J \cdot \nu_m.$$

This proves the theorem. ▷

The theorem shows that the nice map has the Gabrielov property. Now we have to prove that the map

$$g : (y, \varepsilon) \mapsto (\mathcal{F}(j_C^n f, \varepsilon), \varepsilon) \in \mathbb{R}^{n+k} \tag{6.10}$$

with the generic $f \in C$ is nice.

6.5. Step IV. Thom–Boardmann like classes..

We replace the family of equations (6.5), (6.6) by a single equation

$$g = (a, \varepsilon)$$

g is the map (6.10), because we want to have a uniform estimate of the number of regular inverse images with respect to ε. In order to investigate the map g, introduce some notations used for the nice maps.

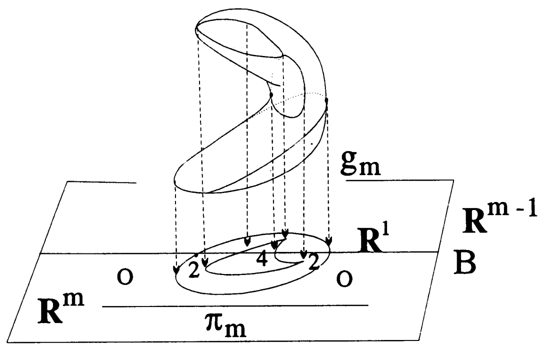

FIGURE 10

Definition 11. . The criminant set K_g of the map $g : M^m \to \mathbb{R}^m$ of the stratified manifold is the union of sk M^n and the set of critical points of g on the n-strata of $M : \Sigma = \{x \in$ higher strata | rank $dg < n\}$

We say that the map g agrees with the stratification $\mathcal{M}$ of M^n, or $\mathcal{M}$ stratifies g, if the following dichotomy holds: each n-stratum $M_\alpha \in M$ entirely consist either or regular, or of critical points of g. The skeleton of the stratification which stratifies g will be called the essential criminant set of g (it is defined up to a choice of stratification which agrees with g).

Remark. The sufficient condition for the map g to be nice is the existence of the commutative diagram (6.8) with the properties (6.9) and the following one:

each M^{j-1} is the essential criminant set for the map $g_j : M^j \to \mathbb{R}^j$.

Fix a polynomial map $\mathcal{F}$ in (6.10) and a flag (6.7).

Theorem 14. . *There exist algebraic sets $\mathcal{K}_l$ in the jet space J_C^{n+l} such that:*

codim $\mathcal{K}_l = l$;

the set $\mathcal{K}_l$ is the "universal criminant set" in the following sense:

if the $n+l$-jet extension of $f \in C$ is transversal to $\mathcal{K}_l$ for any l, then the map g (6.10) is nice.

The constructions goes by induction with respect to $l = m - \dim M^j$ in (6.8). It is like the classical one, due to Thom– Boardmann.

Let $M^m \subset \mathbb{R}^{n+k}$, $m = n+k$ be the ball in the space of y, ε, phase variables and parameters in (6.6). Let $f \in C$ be a Cartesian map (6.6)

$$M^m \to \mathbb{R}^n$$

Consider the first universal criminant set

$$\mathcal{K}_1 \subset J_C^{n+1}$$
$$\mathcal{K}_1 = \{j_C^{n+1} f \mid \text{rank } dg_x > m, \ \bar{x} \in \text{int } B\} \cup \{j_C^{n+1} f \mid \bar{x} \in \partial B\}$$

where $\bar{x}$ is the source of the jet $j_C^{n+1} f$, g is the map(6.10).

The set $\mathcal{K}_1$ is obviously an algebraic variety in J_C^{n+1}.

The following construction makes use of two important remarks.

1. Consider a map

$$\mathbb{R}^r \oplus \mathbb{R}^t \to \mathbb{R}^{r+t},$$
$$x \mapsto (g_1, ..., g_r, h_1, ..., h_t)(x)$$

and suppose that $g = (g_1, ..., g_r)$, $h = (h_1, ..., h_t)$ and rank $dg|_{g=0} = r$.

Then the set of critical points of the restriction of h onto the set $g = 0$ is given by the equations

$$g = 0, \ dg_1 \wedge ... \wedge dg_r \wedge dh_1 \wedge ... \wedge dh_s = 0$$

This fact lies in the foundation of the classical Thom–Boardmann construction.

2. Consider a semialgebraic set $\mathcal{K}$ in the jet space J_C^L for some L given by the polynomial system

$$G = 0, \ G = (G_1, ..., G_r)$$

with the additional requirement rank $dg|\mathcal{K} = r$.

Suppose that the L-jet extension of a map $f \in C$ is transversal to $\mathcal{K}$. Then the map

$$g = G \cdot j^L f$$

has rank r on the set $\{g = 0\}$.

These remarks allow to proceed the construction of the universal criminant sets $\mathcal{K}_l$, using the following theorem of Whitney.

Theorem. . *For any algebraic variety $\overline{\mathcal{K}}$ in the affine space there is a semi-algebraic set $\mathcal{K}$, which is in fact a manifold such that:*

1. For any point $x \in \mathcal{K}$ a system G of polynomials exist such that in some neighborhood U of x

$$\mathcal{K} \cap U = \{G = 0\} \cap U;$$
$$\text{rank } dG = \text{codim } \mathcal{K}$$

on $\mathcal{K} \cap U$;

2. $\dim(\overline{\mathcal{K}} \setminus \mathcal{K}) < \dim \mathcal{K}$.

This concludes the proof of Theorem 14.

6.6 Cartesian transversality theorem..

To conclude the proof of Theorem 10 we need a Cartesian analogue of the classical Thom's transversality theorem.

Theorem 15. . *Let C, as before, be a space of maps of the form (6.6), and let $\mathcal{K}$ be an algebraic variety in the space J^L_C. Then a generic map $f \in C$ has L-jet extension which is transversal to all the strata of $\mathcal{K}$.*

This theorem is proved by Shelkovnikov. The proof is analogous to that of the classical transversality theorem.

This concludes the proof of the Theorem 10.

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Astérisque

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Astérisque, tome 222 (1994), p. 259-283
<http://www.numdam.org/item?id=AST_1994__222__259_0>

REGULAR LINEAR SYSTEMS ON CP^1 AND THEIR MONODROMY GROUPS

V.P. KOSTOV

1. INTRODUCTION

1.1

A meromorphic linear system of differential equations on $\mathbf{C}P^1$ can be presented in the form

$$\dot{X} = A(t)X \tag{1}$$

where $A(t)$ is a meromorphic on $\mathbf{C}P^1$ $n \times n$ matrix function, " $\cdot$ " $\equiv d/dt$. Denote its poles $a_1, \dots, a_{p+1}$, $p \geq 1$. We consider the dependent variable X to be also $n \times n$-matrix.

Definition. System (1) is called *fuchsian* if all the poles of the matrix-function $A(t)$ are of first order.

Definition. System (1) is called *regular* at the pole a_j if in its neighbourhood the solutions of the system are of moderate growth rate, i.e.

$$||X(t - a_j)|| = 0\left(|t - a_j|^{N_j}\right), \quad N_j \in \mathbf{R}, \ \ j = 1, \dots, p+1$$

Here $|| \cdot ||$ denotes an arbitrary norm in $gl(n, \mathbf{C})$ and we consider a restriction of the solution to a sector with vertex at a_j and of a sufficiently small radius, i.e. not containing other poles of $A(t)$. Every fuchsian system is regular, see [1]. The restriction to a sector is essential, if we approach the pole along a spiral encircling it sufficiently fast, then we can obtain an exponential growth rate for $||X||$.

Two systems (1) with the same set of poles are called *equivalent* if there exists a meromorphic transformation (equivalency) on $\mathbf{C}P^1$

$$X \mapsto W(t)X \tag{2}$$

with $W \in \mathcal{O}(\mathbf{C}P^1\backslash\{a_1,...,a_{p+1}\})$ and $\det W(t) \neq 0$ for $t \in \mathbf{C}P^1\backslash\{a_1,...,a_{p+1}\}$ which brings the first system to the second one. A transformation (2) changes system (1) according to the rule

$$A(t) \to -W^{-1}(t)\dot{W}(t) + W^{-1}(t)A(t)W(t) \tag{3}$$

1.2

The monodromy group of system (1) is defined as follows: fix a point $a \neq a_j$ for $j = 1,...,p+1$, fix a matrix $B \in GL(n,\mathbf{C})$ and fix p closed contours on $\mathbf{C}P^1$ beginning at the point a each of which contains exactly one of the poles a_j of system (1), see Fig. 1. The *monodromy operator* corresponding to such a contour is the linear operator mapping the matrix B onto the value of the analytic continuation of the solution of system (1) which equals B for $t = a$ along the contour encircling a_j; we assume that all the contours are positively orientated. Monodromy operators act on the right, i.e. we have $B \mapsto BM_j$. The monodromy operators $M_1,...,M_p$ corresponding to $a_1,...,a_p$ generate the *monodromy group* of system (1) which is a presentation of the fundamental group $\pi_1(\mathbf{C}P^1\backslash(a_1,\ldots,a_{p+1})$ into GL$(n,\mathbf{C})$; we have

$$M_{p+1} = (M_1....M_p)^{-1} \tag{4}$$

for a suitable ordering of the points a_j and the contours, see Fig. 1.

It is clear that

1. the monodromy group is defined up to conjugacy due to the freedom in choosing the point a and the matrix B.

2. the monodromy groups of equivalent systems are the same.

The monodromy group of a regular system is its only invariant under meromorphic equivalence.

Capital Latin letters (in most cases) denote matrices or their blocks; by I we denote diag$(1,\ldots,1)$.

1.3

It is natural to consider $GL(n,\mathbf{C})^p$ as the space of monodromy groups of regular systems on $\mathbf{C}P^1$ with $p+1$ prescribed poles (because the operators $M_1,\ldots,M_p$ define the monodromy group of system (1)). Condition (4) allows one to consider M_{p+1} as an analytic matrix-function defined on $GL(n,\mathbf{C})^p$. Of course, in a certain sense, $M_1,\ldots,M_{p+1}$ are 'equal', i.e. anyone of them can play the role of

M_{p+1}. We define *an analytic stratification* of $(GL(n,\mathbf{C}))^p$ by the Jordan normal forms of the operators $M_1,\ldots,M_{p+1}$ and the possible reducibility of the group $\{M_1,\ldots,M_p\}$. Fixing the Jordan normal form of $M_1,\ldots,M_p$ is equivalent to restricting the matrix-function $M_{p+1}=(M_1\ldots M_p)^{-1}$ to a smooth analytic subvariety of $GL(n,\mathbf{C})^p$, but if we want to fix the one of M_{p+1} as well, then we a priori can say nothing about the smoothness of the subset of $GL(n,\mathbf{C})^p$ (called *superstratum*) obtained in this way. The basic aim of this paper is to begin the study of the stratification of $GL(n,\mathbf{C})^p$ and the smoothness of the strata and superstrata.

Throughout the paper 'to fix the Jordan normal form' means 'to define the multiplicities of the eigenvalues and the sizes and numbers of Jordan blocks corresponding to each of them', but *not* to fix the eigenvalues as well; this is called 'to fix the orbit'.

2 The stratification of the space of monodromy groups

Definition. Let the group $\{M_1,\ldots,M_p\}\subset GL(n,\mathbf{C})$ be conjugate to one in block-diagonal form, the diagonal blocks (called *big blocks*) being themselves block upper-triangular; their block structure is defined by their diagonal blocks (called *small blocks*). The restriction of the group to everyone of the small blocks is assumed to be an irreducible matrix group of the corresponding size. The sizes of the big and small blocks are correctly defined modulo permutation of the big blocks (if we require that the sizes of the big blocks are the minimal possible) and define the *reducibility type* of the group.

Example : The reducibility type $\begin{pmatrix} A & B & 0\\ 0 & C & 0\\ 0 & 0 & Q\end{pmatrix}$ has two big ($\begin{pmatrix} A & B\\ 0 & C\end{pmatrix}$ and Q) and three small blocks (A, C and Q).

Definition. A *stratum* of $GL(n,\mathbf{C})$ is its subset of matrices with one and the same Jordan normal form. A group $\{M_1,\ldots,M_p\}\subset GL(n,\mathbf{C})$ defines a *stratum* of $GL(n,\mathbf{C})^p$: the stratum is defined by

1) the reducibility type of the group;

2) the Jordan normal forms of the small and big blocks of the matrices $M_1,\ldots,M_{p+1}$ and the ones of the matrices M_j themselves;

3) two groups whose matrices $M_1,\ldots,M_p$ are blocked as their reducibility type belong to the same stratum if and only if the corresponding M_j are conjugate to each other by matrices (in general, different for the different j) blocked as the reducibility type.

A stratum is called *irreducible* if its reducibility type is one big and at the same time small block.

A reducible stratum is called *special* if there exists a pair of small blocks of the same size, belonging to one and the same big block, such that the restrictions of the matrices M_j to them have the same Jordan normal form for all $j = 1, \ldots, p+1$.

Remark: Suppose that the definition of a stratum doesn't contain 3). Then some of the reducible strata defined in this way will turn out to be reducible analytic varieties (see the example below; note the double sense of 'reducible'). The good definition of a stratum is obtained when the strata defined above are decomposed into irreducible components if this is possible. After such a decomposition we obtain again a finite number of strata.

Example: Let the reducibility type be $\begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}$, P, Q and R being 3×3.

Let $M_2, \ldots, M_{p+1}$ have distinct eigenvalues. Let $M_1 = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & a \\ 0 & 0 & \lambda & b & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$.

For $a = 0, b \neq 0$ and for $a \neq 0, b = 0$ the Jordan normal forms of the $P-$ and $Q-$block of M_1 and of M_1 itself are the same (M_1 has one eigenvalue - λ - and three Jordan blocks, of sizes 3, 2 and 1 respectively). In the first case the dimension of the intersection of the subspace invariant for M_1 upon which M_1 acts as one Jordan block of size 3 with the subspace invariant for all operators M_j is equal to 1, in the second case it is equal to 2. It can be checked directly that the two matrices (corresponding to $(a, b) = (*, 0)$ and $(a, b) = (0, *), * \neq 0$) aren't conjugate to each other by a matrix blocked in the same way.

Remark: The following example shows that the definition of a stratum of $GL(n, \mathbf{C})^p$ is still not good - there exist several connected components for irreducible strata in which every operator M_j, $j = 1, \ldots, p$ has one eigenvalue only. On the other hand-side, let there exist M_j with at least two different eigenvalues. Consider two systems belonging to the same stratum. One can deform continuously the sets of their eigenvalues, i.e. perform a homotopy from the first into the second set, keeping their product equal to 1 and their multiplicities unchanged, i.e. different (equal) eigenvalues remain such for every value of the homotopy parameter. Whether for any such homotopy there exists a homotopy of the monodromy group, irreducible for every value of the homotopy parameter - this is an open question.

Example: The monodromy groups of the following three systems are irreducible. Every monodromy operator M_1, M_2, M_3 is conjugate to one 3×3-Jordan block. The eigenvalues of M_1 and M_2 are equal to 1, the ones of M_3 in the first case are equal to 1, in the second case – to $e^{4\pi i/3}$, in the third case – to $e^{2\pi i/3}$. By t_j, $j = 1,2,3$ we denote $1/(t-a_j)$.

$$\dot{X} = \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t_1 + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} t_2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} t_3 \right] X$$

$$\dot{X} = \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t_1 + \begin{pmatrix} 0 & -26/27 & 0 \\ 0 & 0 & 0 \\ 1 & -1/3 & 1 \end{pmatrix} t_2 + \begin{pmatrix} 0 & -1/27 & 0 \\ 0 & 0 & -1 \\ -1 & 1/3 & -1 \end{pmatrix} t_3 \right] X$$

$$\dot{X} = \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t_1 + \begin{pmatrix} 0 & -19/27 & 0 \\ 0 & 0 & 0 \\ 1 & -4/3 & 2 \end{pmatrix} t_2 + \begin{pmatrix} 0 & -8/27 & 0 \\ 0 & 0 & -1 \\ -1 & 4/3 & -2 \end{pmatrix} t_3 \right] X$$

Definition. Consider a subset Σ of $GL(n, \mathbf{C})$ consisting of matrices blocked as a given reducibility type. A *stratification* of this set is defined by

1) the Jordan normal forms of the small blocks, taking into account whether two small blocks have common eigenvalues or not

2) two matrices with the same reducibility type and orbits of the small blocks belong to the same stratum if and only if they can be conjugated with one another by a matrix blocked as the reducibility type.

Lemma 2.1. *Any stratum from this stratification is a connected smooth algebraic variety.*

The lemma is proved at the end of Section 3.

Definition. A *superstratum* of $GL(n, \mathbf{C})^p$ is defined by the Jordan normal forms of the matrices $M_1, \ldots, M_{p+1}$. Hence, every superstratum consists of a finite number of strata.

Evidently, every stratum and superstratum is locally an analytic subvariety of $(GL(n, \mathbf{C}))^p$.

Theorem 2.2.

1) All irreducible strata are locally smooth analytic subvarieties of $GL(n, \mathbf{C})^p$.

2) All strata and superstrata in which at least one of the matrices M_j, $j = 1, \ldots ,p+1$ *has n different eigenvalues are globally smooth irreducible semi-analytic subvarieties of* $GL(n, \mathbf{C})^p$ *('semi-analytic' means 'defined by a finite number of equalities and by inequalities of the kind* $P \neq 0$*').*

3) A reducible group $\{M_1, \ldots, M_p\}$ *(in block upper-triangular form, same as the reducibility type) is a singular point of its superstratum only in case that* *if $Q_{11j}, \ldots, Q_{ssj}$ are the restrictions of M_j to the small blocks of the reducibility type, then there exist linear relations of the kind*

$$\alpha_1 \text{var tr}Q_{11j} + \ldots + \alpha_s \text{var tr}Q_{ssj} = 0 \quad , \quad j = 1, \ldots, p+1 \quad , \quad \alpha_k \in \mathrm{Z}$$

Here 'var tr' denotes the possible variation of the trace when every eigenvalue varies independently (equal eigenvalues have equal variations) and the stratum to which $\{M_1, \ldots, M_p\}$ belongs is fixed. Hence, the singular points of a superstratum are contained in one or more of its reducible strata – last equalities mean that for a fixed reducibility type the multiplicities of the eigenvalues of the small blocks $Q_{kkj}, j = 1, \ldots p+1$ for which $\alpha_k \neq 0$ remain the same.

4) Every reducible stratum is locally a smooth analytic variety.

5) An upper-triangular group with M_j having one eigenvalue only , $j = 1, \ldots, p+1$, is a singular point of its superstratum.

3 Proof of Theorem 2.2.

0^0. We prove 1) in $1^0 - 3^0$, 2) and 3) in $4^0 - 7^0$, 4) in $8^0 - 13^0$ and 5) in 14^0. The proofs of the lemmas involved are given after the proof of the theorem.

1^0. Prove 1) (see $1^0 - 3^0$). Fix the Jordan normal forms of $M_1, \ldots, M_p$. This defines a smooth subvariety $\mathcal{T}$ of $(GL(n, \mathrm{C}))^p$. If we fix the Jordan normal form of M_{p+1} (or, equivalently, of M_{p+1}^{-1}), then this defines a smooth analytic subvariety $\mathcal{S}$ of $GL(n, \mathrm{C})$. Let the group $\mathrm{M} = \{M_1, \ldots, M_p\}$ be irreducible. We prove that the differential of the mapping

$$(M_1, \ldots, M_p) \mapsto M_{p+1} = (M_1 \ldots M_p)^{-1}$$

is non-degenerate at M; in fact, we prove (what is equivalent) that the differential of the mapping

$$(M_1, \ldots, M_p) \mapsto M_{p+1}^{-1} = M_1 \ldots M_p \qquad (*)$$

is such, see $2^0 - 3^0$. Hence, the graph of the mapping (*) is a smooth analytic subvariety of $\mathcal{T} \times GL(n, \mathrm{C})$, transversal at M to the smooth analytic subvariety $\mathcal{U} = \mathcal{T} \times \mathcal{S}$; therefore their intersection is locally a smooth analytic subvariety.

2^0. The differential of (*) is the sum of two terms – the first (denoted by Φ_p) is obtained when M_j are conjugated by matrices of the kind $G_j = I + \varepsilon Y_j$, i.e. we move infinitesimally along the orbit without changing the eigenvalues. Hence, Φ_p is the coefficient before ε in the product $G_1^{-1} M_1 G_1 \ldots G_p^{-1} M_p G_p$. Note that for small values of ε the group $\{G_1^{-1} M_1 G_1, \ldots, G_p^{-1} M_p G_p\}$ is irreducible. The second term (denoted by Δ_p) is obtained when we change infinitesimally the eigenvalues (every eigenvalue changes independently, for every $M_j, j = 1, \ldots, p$).

3^0. **Lemma 3.1.**

$$\Phi_p(\mathrm{M}; \mathrm{Y}) \equiv \Phi_p(M_1, \ldots, M_p; Y_1, \ldots, Y_p) =$$

$$= [M_1, Y_1]M_2 \ldots M_p + M_1[M_2, Y_2]M_3 \ldots M_p + \ldots + M_1 \ldots M_{p-1}[M_p, Y_p] =$$
$$= M_1 \ldots M_p \Psi_p(N_1, \ldots, N_p; Z_1, \ldots, Z_p)$$

where $\Psi_p = [N_1, Z_1] + \ldots + [N_p, Z_p]$, $Z_j = S_j^{-1}M_j^{-1}Y_jS_j$, $N_j = S_j^{-1}M_jS_j$, $S_j = M_{j+1} \ldots M_p$, $j = 1, \ldots, p-1$, $S_p = I$. *Hence, the groups* $\{M_1, \ldots, M_p\}$ *and* $\{N_1, \ldots, N_p\}$ *coincide.*

The lemma is checked directly.

Lemma 3.2. *Let* $M_j = Q_j^{-1}J_jQ_j$ *where* $Q_j \in GL(n, \mathrm{C})$ *and* J_j *is the Jordan normal form of* M_j. *Then*

$$\Delta_p(\mathrm{M}, \mathrm{V}) = V_1M_2 \ldots M_p + M_1V_2M_3 \ldots M_p + \ldots + M_1M_2 \ldots M_{p-1}V_p$$

where $V_j = Q_j^{-1}D_jQ_j$, D_j *being a diagonal matrix whose diagonal entries are the variations of the eigenvalues of* M_j, *i.e. of the diagonal entries of* J_j *(equal eigenvalues have equal variations). We have*

$$\Delta_p = M_1 \ldots M_p\kappa_p \quad , \quad \kappa_p = \sum_{j=1}^{p} S_j^{-1}M_j^{-1}V_jS_j \quad , \quad S_j = M_{j+1} \ldots M_p \quad , \quad S_p = I$$

The lemma is checked directly.

Lemma 3.3. *Let the group* $\{M_1, \ldots, M_p\}$ *be irreducible. Then for every matrix* $L \in gl(n, \mathrm{C})$, $tr(M_{p+1}L) = 0$ *there exist matrices* $Y_1, \ldots, Y_p$ *such that* $L = \Phi_p(\mathrm{M}; \mathrm{Y})$, *see Lemma 3.1.*

Lemma 3.4. *For every* $d \in \mathrm{C}$ *there exist matrices* $V_1, \ldots, V_p$, *see Lemma 3.2., such that* $tr(M_{p+1}\Delta_p(\mathrm{M}; \mathrm{V})) = d$.

The first statement of the theorem follows from Lemmas 3.3. and 3.4. Really, for $L \in gl(n, \mathrm{C})$ choose $V_1, \ldots, V_p$ such that $\mathrm{tr}(M_{p+1}\Delta_p(\mathrm{M}; \mathrm{V})) = \mathrm{tr}(M_{p+1}L)$. Hence, $\mathrm{tr}(M_{p+1}^{-1}(L - \Delta_p)) = 0$ and we can choose $Y_1, \ldots, Y_p$ such that $L - \Delta_p = \Phi_p(\mathrm{M}; \mathrm{Y})$.

4^0. Prove 2) and 3). To this end we use a similar idea to the one of the proof of 1). We show for what groups M the tangent spaces to the graph of (*) (denoted by $T(*)$) and to the variety $\mathcal{U}$, see 1^0, are transversal. The space $T\mathcal{U}$ contains the tangent space to $\mathcal{T}$, therefore it suffices to find the cases when the sum of the projections of the spaces $T(*)$ and $T\mathcal{U}$ into $GL(n,\mathbf{C})$, the space of M_{p+1}^{-1}, is the whole space $gl(n,\mathbf{C})$. Similarly to Lemmas 3.1. and 3.2. we find that the projection of $T\mathcal{U}$ into $GL(n,\mathbf{C})$ is equal to

$$\Lambda_{p+1}(M_{p+1},Y_{p+1},V_{p+1}) = [M_{p+1}^{-1},Y_{p+1}] + V_{p+1}$$

where $Y_{p+1} \in gl(n,\mathbf{C})$ and if $M_{p+1}^{-1} = Q_{p+1}^{-1}J_{p+1}Q_{p+1}$, J_{p+1} being the Jordan normal form of M_{p+1}^{-1}, $Q_{p+1} \in GL(n,\mathbf{C})$, then $V_{p+1} = Q_{p+1}^{-1}D_{p+1}Q_{p+1}$, D_{p+1} being diagonal, whose diagonal entries are the variations of the eigenvalues of M_{p+1}^{-1}. Hence, the two varieties $\mathcal{U}$ and the graph of $(*)$ are transversal if and only if every matrix $L \in gl(n,\mathbf{C})$ can be presented as

$$L = \mathcal{L}_p(\mathbf{M};\mathbf{Y},\mathbf{V},Y_{p+1},V_{p+1}) =$$

$$\Phi_p(\mathbf{M};\mathbf{Y}) + \Delta_p(\mathbf{M};\mathbf{V}) + \Lambda_{p+1}(M_{p+1},Y_{p+1},V_{p+1}) \qquad (**)$$

Present (**) with $Y_{p+1} = V_{p+1} = 0$, i.e. with $\Lambda_{p+1} = 0$, in the form

$$L' \equiv (M_1 \dots M_p)^{-1}L = \mathcal{L}'_p(\mathbf{M};\mathbf{Y};\mathbf{V}) \equiv \Psi_p(\mathbf{N};\mathbf{Z}) + \kappa_p(\mathbf{M};\mathbf{V}) \qquad (***)$$

From now on we most often consider equation (***) instead of equation (**) (if we can solve (***), then we can solve (**)).

5^0. Let the reducible group $\mathbf{M}$ be in block upper-triangular form (same as the reducibility type). Decompose any matrix $A \in \mathrm{gl}(n,\mathbf{C})$ in blocks, the decomposition being induced by the sizes of the small blocks of the reducibility type. Then for the following ordering of the blocks operator $\mathcal{L}'_p$, see (***), is block upper-triangular: if the blocks are denoted by Q_{ks}, k (s) being the number of row (of column) of blocks, then $Q_{k_1s_1}$ precedes $Q_{k_2s_2}$ if and only if $k_1 - s_1 < k_2 - s_2$ or $k_1 - s_1 = k_2 - s_2$ and $k_1 < k_2$. Hence, it suffices to consider the action of $\mathcal{L}'_p$ upon matrices $Y_j, V_j, j = 1,\dots,p+1$ whose elements outside a fixed block are equal to 0.

Lemma 3.5. *Denote by Q_{ijk} the restriction of M_k for $k = 1,\dots,p$ or M_{p+1}^{-1} for $k = p+1$ to the block Q_{ij}. Then equation*

$$\sum_{k=1}^{p+1} Q_{iik}Z_k - Z_kQ_{jjk} = A$$

has a solution for any matrix $A \in gl(n,\mathbf{C})$ if and only if the two following conditions don't hold simultaneously:

1) Q_{ii} and Q_{jj} are of the same size (denoted by l);

2) there exists a matrix $B \in GL(l,\mathbf{C})$ such that $B^{-1}Q_{iik}B = Q_{jjk}$ for $k = 1,\dots,p$ (hence, for $k = p+1$ as well).

Note that for $i > j$ the left hand-side of the equation gives the restriction of the image of operator $\mathcal{L}'_p$ to the block Q_{ij} (κ_p has no influence upon blocks under the diagonal). Conditions 1) and 2) together are a private case of the non-smoothnes condition from 3) of the theorem.

The block upper-triangular form of operator $\mathcal{L}'_p$ implies that one could try to solve equation (***) successively for each block, in the opposite order of the blocks. If we fail at one of them, then, probably, we can't solve equation (**) ('probably' means that solving equation (***) is not equivalent to solving equation (**)). If conditions 1) and 2) from the lemma hold simultaneously, then it doesn't follow from Lemma 3.5. that we can solve (***). If they don't, but the non-smoothness condition from 3) of the theorem holds, then we can solve equation (***) restricted to all blocks Q_{ij} under the diagonal, i.e. with $i > j$. For $i = j$ operator Ψ_p can give a solution only for matrices L' with $\mathrm{tr}L'|_{Q_{ii}} = 0$. Hence, operator κ_p must be used to make the trace of $L'|_{Q_{ii}}$ equal to 0 and he'll fail to do it for all small blocks simultaneously exactly if the variations of the traces of the small blocks are linearly dependent.

7^0. We proved in 6^0 that for strata verifying the non-smoothness condition from 3) of the theorem equation (**) possibly can't be solved and, hence, the variety $\mathcal{U}$, see 1^0, might not be transversal to the graph of (*). We prove now that non-transversality would imply local non-smoothness of their intersection. Introduce in $GL(n, \mathrm{C})$ local coordinates $q = (q_1, \ldots, q_{n^2})$ such that at the intersection point of $\mathcal{U}$ and the graph of (*) the projection of $\mathcal{U}$ into $\mathrm{GL}(n, \mathrm{C})$, i.e. $\mathcal{S}$, should be given by equations $q_1 = \ldots = q_s = 0$, $s = \dim\mathcal{S}$. Hence, the points of non-transversal intersection of the graph of (*) and $\mathcal{U}$ are the ones where the differential of (*) degenerates and we have $q_1 = \ldots = q_s = 0$. These are the singular points of the intersection of the image of (*) with $\{q_1 = \ldots = q_s = 0\}$, i.e. with $\mathcal{U}$.

If a superstratum of $(\mathrm{GL}(n, \mathrm{C}))^p$ contains among the Jordan normal forms of M_j, $j = 1, \ldots, p+1$ one with distinct eigenvalues, then their variations are independent and we never have conditions 1) and 2) of Lemma 3.5. fulfilled together. On the contrary, if there is at least a pair of equal eigenvalues in every M_j, $j = 1, \ldots, p+1$, then it is always possible to find a reducibility type such that conditions 1) and 2) of Lemma 3.5. will be fulfilled together.

Irreducible strata and superstrata in which at least one of the operators M_j is with distinct eigenvalues are connected. Really, let this be M_{p+1}. The image of (*) is a semi-algebraic subset of $GL(n, \mathrm{C})$ for every fixed set of Jordan normal forms of $M_1, \ldots, M_p$. The differential of (*) is non-degenerate, hence, the image is an open subset of $GL(n, \mathrm{C})$. Hence, if the Jordan normal forms of $M_1, \ldots, M_p$ are fixed, then the points of the graph of (*) for which M_{p+1} is not with distinct eigenvalues is locally a proper subvariety of the graph. This completes the proof of 2) and 3) of the theorem.

8^0. **Lemma 3.6.** *Let the group* M *be reducible. Then in its neighbourhood* $U \in (GL(n,\mathbf{C}))^p$ *there exists a holomorphic and holomorphically invertible matrix* C *such that it conjugates every group of the intersection of* U *with the stratum to which* M *belongs to one blocked as the reducibility type.*

The proof of 4) is similar to the one of 1). We consider the mapping (*) defined for $M_1, \ldots, M_p$ blocked as the reducibility type, making use of the lemma. Denote the set of these matrices by Σ. Consider the following subset of $(\Sigma)^p$: the multiplicities of the eigenvalues of every operator $M_1, \ldots, M_p$ and their distribution among the small blocks are fixed and the Jordan normal forms of the small blocks as well. Denote this set by $\mathcal{T}'$ and consider (*) as a mapping $(*):\mathcal{T}' \mapsto \Sigma$. For $M_{p+1}^{-1} \in \Sigma$ fix the multiplicities of its eigenvalues, their distribution among the small blocks and the Jordan normal forms of the small blocks. This defines a subset $\mathcal{S}' \subset \Sigma$.

Lemma 3.7. *$\mathcal{T}'$ and $\mathcal{S}'$ are smooth analytic subvarieties.*

The intersection of the graph of (*) with $\mathcal{S}' \times \mathcal{T}'$ (denote it by $\mathcal{R}$) consists of a finite number of strata of $(GL(n,\mathbf{C}))^p$ (not necessarily of a single one). Really, though the eigenvalues of M_j and the Jordan normal forms of their small blocks are fixed, the Jordan structures of M_j depend on the elements in the blocks above the diagonal as well.

Example : Consider the matrix $\begin{pmatrix} \lambda & 1 & a & b \\ 0 & \lambda & c & d \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$. For $c \neq 0$ it is conjugate to one 4×4 Jordan block. For $c = 0$, $a + d \neq 0$ it is conjugate to a matrix with one 3×3- and one 1×1-block, for $c = a + d = 0$ it is cojugate to a matrix with two 2×2-blocks.

9^0. In the case of non-special strata we have

Lemma 3.8. *Every reducible non-special stratum consists of a finite number of smooth analytic varieties.*

10^0. We prove 4) of the theorem for reducibility types with one big block only; for such with several big blocks the proof is similar. The proof is carried out by induction with respect of the number k of small blocks. Let $k = 2$. Set $M_j = \begin{pmatrix} P_j & Q_j \\ 0 & R_j \end{pmatrix}$. Let the stratum be special, i.e. the Jordan normal forms of P_j and R_j be the same for $j = 1, \ldots, p+1$ (for non-special strata the answer is given by Lemma 3.8.). By Lemma 3.5., equation (***) in which all matrices are block upper triangular (as M_j) can't be solved only if the sizes of P_j and R_j are equal and we have $B^{-1}P_jB = R_j$, $j = 1, \ldots, p+1$ for some matrix B.

Really, we first solve equation (***) for the diagonal blocks as in the irreducible case and then for the block B.

Let $P_j = R_j$, $j = 1, \dots, p+1$. Denote by $\mathcal{C}$ the space of matrices blocked as $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$. Then we have

Lemma 3.9. *The image of operator $\mathcal{L}'_p$ restricted to $\mathcal{C}$ is either $\mathcal{C}$ or $\mathcal{C} \cap \{N \in \mathcal{C} | \mathrm{tr}(N|_B) = 0\}$. The second case occurs only if $Q_j = [D, P_j]$ for some matrix D, $j = 1, \dots, p+1$.*

Lemma 3.10. *Let $M_j = \begin{pmatrix} P_j & Q_j \\ 0 & P_j \end{pmatrix}$ be conjugate to $\begin{pmatrix} P_j & 0 \\ 0 & P_j \end{pmatrix}$. Then $Q_j = [D_j, P_j]$ for some square matrices D_j. The opposite implication is also true.*

Hence, for $k = 2$ the only case in which equation (***) can't be solved is the one when M_j can be simultaneously conjugated to the form $\begin{pmatrix} P_j & 0 \\ 0 & P_j \end{pmatrix}$. But in this case the reducibility type has two big blocks.

11^0. Let the reducibility type contain $k \geq 3$ small blocks. Let in the block decomposition induced by the sizes of the small blocks $C \cup D$ be the set of blocks in the first row, $B \cup D$ – the one of blocks in the last column and A – the set of all other blocks on and above the diagonal, see Fig. 2. Item 3) from the definition of the stratification of $(GL(n, \mathbf{C}))^p$ implies that once the stratification of the restriction of the group $\mathbf{M}$ to A is defined, its definition for $\mathbf{M}|_{A\cup B}$, $\mathbf{M}|_{A\cup C}$, $\mathbf{M}|_{A\cup B\cup C\cup D}$ does not change the one of $\mathbf{M}|_A$.

Fix a stratum of $\mathbf{M}|_{A\cup B}$ and a stratum of $\mathbf{M}|_{A\cup C}$ such that their restrictions to A coincide. The stratification of $\mathbf{M}|_{A\cup B\cup C\cup D}$ imposes (for every stratum) analytic conditions on the block D. The restrictions $M_j|_D$, $j = 1, \dots, p$, consist of a finite number of smooth analytic varieties for every stratum of Σ, see Lemma 2.1.. In (*) $M_{p+1}|_D$ is presented as an analytic function of $M_1, \dots, M_p$ restricted to the corresponding strata. To prove the local smoothness of $\mathbf{M}|_D$ it suffices to prove that the graph of $M_{p+1}|_D$ is transversal to the level sets $M_{p+1}|_D$ where M_{p+1} is restricted to some stratum of Σ, as in the proof of 1).

12^0. Introduce the following notation for the blocks of M_j:

$$\begin{pmatrix} P & Q_1 & Q_2 & \dots & Q_{k-2} & D \\ 0 & T & U_2 & \dots & U_{k-2} & S_1 \\ 0 & 0 & V & \dots & H_{k-2} & S_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & W & S_{k-2} \\ 0 & 0 & 0 & \dots & 0 & R \end{pmatrix}$$

The tangent space to the graph of $M_{p+1}|_D$ contains the space

$$\mathcal{Y} = \{Y|Y = (M_1 \dots M_p)|_P \sum_{j=1}^{p}((N_j|_P)X_j^0 - X_j^0(N_j|_R)\}$$

X_j^0 is of the size of D; see the definition of $\mathcal{L}_p$. Really, to see this it suffices to restrict in Ψ_p and $\mathcal{L}_p$ the matrices Z_j to D to obtain X_j^0. The space $\mathcal{Y}$ is not the whole space $\mathcal{D}$ (the space of matrices of the size of D) if and only if P and R are of the same size and we have $N_j|_P = B^{-1}(N_j|_R)B$, $j = 1, \dots, p$, see Lemma 3.5.. This implies that $M_j|_P = B^{-1}(M_j|_R)B$. Without loss of generality assume that $M_j|_P = M_j|_R$, $j = 1 \dots, p+1$.

13^0. The tangent space to the graph of $(*)|_D$ contains as well the space

$$\mathcal{Y}^1 = \{Y|Y = (M_1 \dots M_p)|_P \sum_{j=1}^{p}((N_j|_P)X_j^0 - X_j^0(M_j|_R))+$$

$$(M_1 \dots M_p)|_{Q_1} \sum_{j=1}^{p}(N_j|_{Q_1})X_j^1\}, \mathcal{Y} \subset \mathcal{Y}^1$$

where Y, (X_j^0, X_j^1) is of the size of D, (of D, of S_1) and X_j^1 belong to the subspace

$$\mathcal{Y}' = \{\sum_{j=1}^{p}((Mj|_T)X_j^1 - X_j^1(M_j|_R)) = 0\}$$

As $M_j|_P = M_j|_R$ and $\mathrm{M}|_P(\equiv \mathrm{M}|_R)$ is irreducible, then we have either $\mathcal{Y}^1 = \mathcal{D}$ or $\mathcal{Y}^1 = \mathcal{D} \cap \{\mathrm{tr}Y = 0\}$ ($\mathcal{D}$ is the space of matrices of the size of D); the second case occurs only if

$$\mathrm{tr} \sum_{j=1}^{p}(M_j|_{Q_1})X_j^1 = 0 \quad \forall (X_1^1, \dots, X_p^1) \in \mathcal{Y}' \qquad (****)$$

(the proof of this fact is similar to the one of Lemma 3.9.). Then condition (****) must be a corollary of equation

$$\sum_{j=1}^{p}((M_j|_T)X_j^1 - X_j^1(M_j|_R)) = 0$$

As in the proof of Proposition 3.17., we prove that the group M is conjugate to one blocked as M, with $M_j|_{Q_1} = 0$, $j = 1, \dots, p$. In this case note that the stratifications of the sets of monodromy groups blocked as

$$\begin{pmatrix} P & 0 & Q_2 & \dots & Q_{k-2} & D \\ 0 & T & U_2 & \dots & U_{k-2} & S_1 \\ 0 & 0 & V & \dots & H_{k-2} & S_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & W & S_{k-2} \\ 0 & 0 & 0 & \dots & 0 & R \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} T & 0 & U_2 & \dots & U_{k-2} & S_1 \\ 0 & P & Q_2 & \dots & Q_{k-2} & D \\ 0 & 0 & V & \dots & H_{k-2} & S_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & W & S_{k-2} \\ 0 & 0 & 0 & \dots & 0 & R \end{pmatrix}$$

are isomorphic, the isomorphism being generated by the conjugation with the permutation matrix which permutes the lines of blocks in which P and T are. Hence, the smoothness of the stratification of the block D for fixed strata of the other blocks follows from the inductive assumption applied to the set of monodromy groups blocked as

$$\begin{pmatrix} P & Q_2 & \dots & Q_{k-2} & D \\ 0 & V & \dots & H_{k-2} & S_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & W & S_{k-2} \\ 0 & 0 & \dots & 0 & W \end{pmatrix}$$

This proves 4) of the theorem.

14^0. Prove 5). If in (**) all matrices except L are upper-triangular, the diagonal entries of each M_j being equal, then the image of $\mathcal{L}_p$ belongs to the subspace of matrices whose left lowest element is equal to 0. As in 7^0, this implies non-smoothness of the corresponding superstratum at $\mathbf{M}$.

Proof of Lemma 3.3.: 1^0. Making use of Lemma 3.1., we prove that every matrix $L \in gl(n, \mathbf{C})$, $\mathrm{tr}L = 0$ can be presented as $L = \Psi_p(\mathbf{M}; \mathbf{X})$ (it would be, of course, more precise to write $\Psi_p(\mathbf{N}, \mathbf{Z})$; we hope that the reader will not ge mixed up).

Proposition 3.11. *Let J be a Jordan matrix. Then any matrix $A \in gl(n, \mathbf{C})$ can be presented in a unique way as $A = [J, X] + Y$, where $[Y\ ,\ {}^tJ] = 0$; ${}^tJ$ denotes the transposed of J.*

Proof: Let the matrix J have one eigenvalue only. Then the matrices commuting with ${}^tJ$ are shown on Fig. 3. The numbers on one and the same interval are the same, all other numbers are equal to zero, see [2]. The intervals are parallel to the diagonal and they begin and end at the bords of the blocks. The block decomposition is in accordance with the Jordan structure. The number of intervals of a diagonal (of an off-diagonal) block is equal to the size (to the least of the sizes) of the block.

If on Fig. 3. we assume that the sum of all the elements lying on one and the same interval is equal to 0 (for every interval) and that no other conditions are imposed on the matrix, then we obtain the definition of an arbitrary matrix presentable as $[J, X]$ for some $X \in gl(n, \mathrm{C})$ (the reader will check this easily, because one needs to consider the action of the operator $[J, .]$ on every block on Fig. 3. separately).

If J has several eigenvalues, then one must take a direct sum of figures like Fig. 3. corresponding to the different eigenvalues. The proposition is proved.

Corollary 3.12. *Let $M_j = Q_j^{-1} J_j Q_j$ where J_j is the Jordan normal form of M_j. Then for a fixed Q_j (Q_j is not unique) any matrix $A \in gl(n, \mathrm{C})$ can be presented as $A = [M_j, X] + Y$, $[Y, Q_j^{-1}({}^t J_j) Q_j] = 0$.*

2^0. **Proposition 3.13. (Schur's Lemma)** *If there exists a non-scalar matrix $S \in gl(n, \mathrm{C})$ such that $[S, M_j] = 0$, $j = 1, \ldots, p$, then the group $\{M_1, \ldots, M_p\}$ is reducible.*

Proof: Without loss of generality one can assume that S is in Jordan normal form. If it has at least two different eigenvalues, then M_j must be all block diagonal and the proposition is proved. If not, then Fig. 4. (it is the transposed of Fig. 3.) shows one possible way how to choose the invariant subspace in the case when there are Jordan blocks of different sizes; for the case of Jordan blocks of the same size see Fig. 5. The vectors to the right describe the invariant subspaces. Asteriscs denote elements which can be arbitrary.

3^0. **Proposition 3.14.** *For every group $\mathrm{M} = \{M_1, \ldots, M_p\}$ there exists a group $\mathrm{K} = \{K_1, \ldots, K_p\}$ such that*

i) M and K are simultaneously (ir)reducible;

ii) the images of the mappings $\Psi_p : (X_1, \ldots, X_p) \mapsto [N_1, X_1] + \ldots + [N_p, X_p]$ for $N_j = M_j$ and for $N_j = K_j$ are the same;

iii) the matrices K_j have one and the same Jordan normal form.

For the rest of the proof of the lemma we consider the Jordan normal forms of $M_1, \ldots, M_p$ to be the same, making use of the proposition.

Proof: We have $[M_1, X_1] + [M_2, X_2] = [M_1 + \alpha M_2, X_1] + [M_2, X_2 - \alpha X_1]$. Hence, the change $(M_1, M_2) \mapsto (M_1 + \alpha M_2, M_2)$ preserves the image of Ψ_p and the (ir)reductibility of the group. If $M_1 + \alpha M_2$ fails to be non-degenerate, then we replace it by $M_1 + \alpha M_2 + \beta I$ for a suitable β; the image and the (ir)reductibility are preserved again. If M_1 and M_2 have different Jordan normal forms, then either

i) one of them (say, M_1) belongs to a stratum S_1 of $GL(n, \mathrm{C})$ from the closure of the stratum S_2 to which belongs the other or

ii) this is not the case.

In case *i)* either

i1) the whole line $M_1 + \alpha M_2$, $\alpha \in \mathbf{C}$ belongs to the closure of S_2 or

i2) for almost all α the matrix $M_1 + \alpha M_2$ belongs to a stratum S_3 of a higher dimension.

In case *i1)* we choose $\alpha \in \mathbf{C}$ such that $M_1+\alpha M_2 \in S_2$ (or $M_1+\alpha M_2+\beta I \in S_2$). In case *i2)* we choose $\alpha \in \mathbf{C}$ such that $M_1+\alpha M_2 \in S_3$ (or $M_1+\alpha M_2+\beta I \in S_3$). Case *ii)*, in fact, coincides with *i2)*. Hence, after a finite number of transformations all conditions of the proposition will be fulfilled, due to the finite number of strata of $GL(n, \mathbf{C})$.

Proposition 3.15. *Let all the matrices $M_1, \dots, M_p$ have the same Jordan normal form, i.e. let $M_j = Q_j^{-1} J_j Q_j$, where $Q_j \in GL(n, \mathbf{C})$ and the Jordan matrices J_j belong to the same stratum of $GL(n, \mathbf{C})$, i.e. they have the same Jordan normal form. Set $J_1 = J$. Then the (ir)reducibility of the group and the image of the mapping Ψ_p (with $N_j = M_j$, see Proposition 3.14.) are preserved if the group $\{M_1, \dots, M_p\}$ is replaced by the group $\{Q_j^{-1} J Q_j\}, j = 1, \dots, p$.*

In accordance with this proposition, during the rest of the proof of the lemma we consider $M_1, \dots, M_p$ to be from one and the same orbit. To prove the proposition it suffices to notice that the image of Ψ_p does not depend on the eigenvalues of the matrices J_j if the eigenvalues vary so that the Jordan normal forms of M_j are preserved and the matrices Q_j are fixed. The set of invariant subspaces is also preserved under such a change of the eigenvalues. The details are left for the reader.

4^0. **Proposition 3.16.** *Let $M_j = Q_j^{-1} J Q_j$, see Proposition 3.15., and let the image of the mapping Ψ_p with $N_j = M_j$, see Proposition 3.14., be not the whole of $sl(n, \mathbf{C})$. Then there exists a non-scalar matrix V such that $[{}^tV, Q_j^{-1} J Q_j] = 0, j = 1, \dots, p$.*

Proof: Regard $\mathrm{gl}(n, \mathbf{C})$ as a vector space of dimension n^2. Denote its coordinates by x_{ks}, $1 \le k, s \le n$. Let $\mathcal{S}_j$ be the set of elements on one interval on Fig. 3., see the proof of Proposition 3.11. Define the linear forms φ_j on $\mathrm{gl}(n, \mathbf{C})$ as $\varphi_j = \sum_{\mathcal{S}_j} x_{ks}$. Set $d = \sharp\varphi_j$. Suppose (which is not restrictive) that $Q_1 = I$, i.e. $M_1 = J$ is in Jordan normal form. Then the image of the mapping $X_1 \mapsto [M_1, X_1]$ is given by $\varphi_j = 0, j = 1, \dots, d$. Hence, the image of the mapping Ψ_p with $N_j = M_j$ is a subspace of $\mathrm{sl}(n, \mathbf{C})$ described by a system of equations of the kind $\varphi \equiv \sum_{j=1}^{d} s_j \varphi_j = 0$, $s_j \in \mathbf{C}$. Fix one such equation. It denotes the set of zeros of a linear form (of x_{ks}) on $\mathrm{gl}(n, \mathbf{C})$. The coefficients of the form are the coordinates of an $n \times n$-matrix V commuting with ${}^tJ$, see Proposition 3.11.

The conjugation $X \mapsto Q_j^{-1} J Q_j$ induces an automorphism of $\mathrm{gl}(n, \mathbf{C})$: $Y = AX$ where A is $n^2 \times n^2$ and X is considered as a vector column (the $(k+1)$-st column of the matrix X follows the k-th one). The matrix A is explicitely described below. The form φ can be presented as $\varphi = \tilde{\varphi}X$ where $\tilde{\varphi}$ is a vector-line of size n^2; if V is considered as a vector-column, then we have ${}^t\tilde{\varphi} = V$. After the conjugation the form $\tilde{\varphi}$ changes to $\tilde{\varphi}A^{-1}$. Really, if $Y = AX$ are the new coordinates, then $\tilde{\varphi}X = \tilde{\varphi}A^{-1}Y$.

We have $\tilde{\varphi}A^{-1}Y ={}^t Y^t(A^{-1})^t\tilde{\varphi} \equiv^t (\tilde{\varphi}A^{-1}Y)$. The n^2-vector-column ${}^t(A^{-1})^t\tilde{\varphi}$ is (in the sense above) equal to an $n\times n$-matrix. This is the matrix $({}^tQ_j)V({}^tQ_j)^{-1}$. Really, for $n = 3$ let $Q_j = \begin{pmatrix} m & n & p \\ q & r & s \\ t & u & v \end{pmatrix}$. Then we have $A = LR = RL$ where

$L = \begin{pmatrix} Q_j^{-1} & 0 & 0 \\ 0 & Q_j^{-1} & 0 \\ 0 & 0 & Q_j^{-1} \end{pmatrix}$ corresponds to multiplying by Q_j^{-1} to the left and

$R = \begin{pmatrix} mI & qI & tI \\ nI & rI & uI \\ pI & sI & vI \end{pmatrix}$ corresponds to multiplying by Q_j to the right. It is clear how to construct L and R for arbitrary n. Hence, ${}^tA^{-1} ={}^t (L^{-1})^tR^{-1}$, i.e. ${}^tA^{-1}$ is the matrix of the transformation $X \mapsto ({}^tQ_j)X({}^tQ_j)^{-1}$. We have $[({}^tQ_j)V({}^tQ_j)^{-1}, ({}^tQ_j)({}^tJ)({}^tQ_j)^{-1}] = 0$.

The linear form φ describes a subspace of $\mathrm{sl}(n, \mathbf{C})$ to which the image of every mapping $X \mapsto [Q_j^{-1}JQ_j, X]$ belongs. Hence, the matrix V corresponding to φ plays the role of the matrix ${}^tQ_jV^tQ_j^{-1}$ above as well, i.e. it commutes with ${}^tJ$ and ${}^t(Q_j)^tJ^t(Q_j^{-1}), j = 2, \ldots, p$, i.e. $[{}^tV, J] = [{}^tV, Q_j^{-1}JQ_j] = 0$. The proposition is proved.

5^0. Sum up the proof of the lemma. Suppose that the image of Ψ_p is not the whole of $\mathrm{sl}(n, \mathbf{C})$. Then one can replace the group $\mathbf{M}$ by another group which satisfies the conclusion of Proposition 3.15. (namely, that all generators belong to one and the same orbit), without changing the image of Ψ_p. This leads to the existence of the nonscalar matrix ${}^tV$ commuting with the generators of the new (irreducible!) group which is a contradiction with Proposition 3.13. The lemma is proved.

Proof of Lemma 3.4.: It suffices to vary any eigenvalue of any of the matrices $M_1, \ldots, M_p$.

Proof of Lemma 3.5.: The proof resembles the one of Lemma 3.3. Consider the matrices $P_k = \begin{pmatrix} Q_{iik} & 0 \\ 0 & Q_{jjk} \end{pmatrix}$. The left hand-side of the equation in Lemma 3.5. is the restriction of the action of Ψ_p with $N_j = P_j$ to the left lower block; Ψ_p is defined in Lemma 3.1.

Like in Proposition 3.16. we show that if the equation from Lemma 3.5. has no solution for some choice of the right hand-side, then there exists a nonscalar matrix V whose right upper and diagonal blocks are equal to 0 such that $[{}^tV, P_k] = 0, k = 1, \ldots, p$. Conjugating V and $P_1, \ldots, P_p$ by one and the same block-diagonal matrix, we can achieve the following form of ${}^tV$: zeros in the left lower and in the diagonal blocks, the right upper block being of the form $V' = \begin{pmatrix} I & 0 \\ S & T \end{pmatrix}$, where the blocks S and T are equal to 0; we assume that Q_{iik} is of size $n_1 \leq n_2$ where n_2 is the size of Q_{jjk}. The size r of I is equal to the rank of ${}^tV$, S and/or T can be empty.

The condition $[{}^tV, P_k] = 0$ implies that for $r < n_1 \leq n_2$ last $n_1 - r$ columns of $Q_{iik}V'$ and last $n_2 - r$ rows of $V'Q_{jjk}$ are equal to 0. Hence, we must have that the elements in last $n_1 - r$ rows and first $n_1 - r$ columns of Q_{iik} must be zeros for $k = 1, \ldots, p$, i.e. the group $\{Q_{iik}\}, k = 1, \ldots, p$ is reducible which is a contradiction. If $r = n_1 < n_2$, then the elements of last $n_2 - r$ columns and first $n_2 - r$ rows of Q_{jjk} must be zeros (for $k = 1, \ldots, p$), i.e. the group $\{Q_{jjk}\}, k = 1, \ldots, p$ is reducible which again is a contradiction. Finally, if $r = n_1 = n_2$, we have $Q_{iik} = Q_{jjk}, k = 1, \ldots, p$ which gives the result claimed by the lemma.

Proof of Lemma 3.6.: We prove the lemma in the case of one big and two small blocks. In the general case the proof consists in repeating the same construction the necessary number of times. There exists a holomorphic and holomorphically invertible matrix $C'(\varepsilon)$ conjugating $M_1(\varepsilon)$ with its Jordan normal form; ε denotes the local coordinates in the neighbourhood of M. The invariant subspaces of $M_1|_{\varepsilon=0}$ are described on Fig. 4 and Fig. 5. One of them must be invariant for $M_2|_{\varepsilon=0}, \ldots, M_{p+1}|_{\varepsilon=0}$. But then at least one such subspace must be invariant for $M_2, \ldots, M_{p+1}$ for all $\varepsilon \in U$ as well – the number of invariant subspaces is finite, the set of values of ε for which an invariant subspace of $C'^{-1}M_1C'$ is such for $C'^{-1}M_jC'$, $j = 2, \ldots, p+1$ as well is closed. Hence, there exists a cojugation with a permutation matrix such that all the matrices M_j will be blocked as follows: $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. The superposition of the two conjugations gives the necessary matrix C.

Proof of Lemma 3.7.: We have $\mathcal{T}' = \mathcal{T}^* \times \mathcal{D}$ where $\mathcal{D}$ is the subspace of $(\mathrm{gl}(n, \mathbf{C}))^p$ consisting of the p-tuples of matrices with the same reducibility type as the ones of Σ, the elements of whose small blocks and of the blocks outside the big blocks are 0 and the elements of whose superdiagonal blocks in the big blocks are arbitrary. $\mathcal{T}^*$ is the set of matrices the elements of whose off-diagonal blocks are 0 and whose small blocks are same as the ones of $\mathcal{T}'$.

Consider a fixed small block. A matrix of the size of the small block – $T = D^{-1}JD$, J being its Jordan normal form which is fixed – can be locally parametrised by the matrix D and by the eigenvalues of J; for D one can fix a subspace of minimal dimension (D is not defined uniquely but modulo multiplication by matrices commuting with J). Hence, for every small block one can introduce local coordinates (d, g) where d are the coordinates of D and g are the ones of the eigenvalues of J. The matrices with a fixed Jordan normal form depend analytically on (d, g) and their locally smooth analytic variety is an analytic fibration over the base g. One can locally parametrise $\mathcal{T}^*$ as follows: for every M_j parametrise its every small block independently as above; denote the coordinates by $(d_1, g_1), \ldots, (d_s, g_s)$. If the second small block has eigenvalues equal to such of the first one, then we set $g_{2j} = g_{1k}$ for the corresponding eigenvalues and obtaining a subset g' of g. In the analytic fibration $(d_1, g_1) \times (d_2, g_2) \mapsto (d_1, g_1)$ consider the subset $\{g_{2j} = g_{1k}\}$ – this is an analytic subvariety of the initial one. We obtain in the same way the subspaces $g'_3, \ldots, g'_s$ (considering the fibrations $(d_1, g_1) \times (d_3, g_3) \mapsto (d_1, g_1)$ etc.). The variety obtained in this way (setting $g_{3j} = g_{1k}$) is a smooth analytic subvariety of the initial one. Replacing in the reasoning above g_1 and g_2 by $g_1 \cup g'_2$ and g_3, we obtain the subspaces $g''_3, \ldots, g''_s$ etc. Finally, we obtain the subspace $g_1 \cup g'_2 \cup g''_3 \cup \ldots \cup g_s^{(s-1)} \subset g_1 \cup \ldots \cup g_s$ and the necessary smooth analytic subvariety, i.e. $\mathcal{T}^*$ which has local coordinates $(d_1, \ldots, d_s, g_1, g'_2, \ldots, g_s^{(s-1)})$ (constructed for every M_j separately).

For $\mathcal{S}'$ the proof is similar to the one for $\mathcal{T}'$.

Proof of Lemma 3.8.: 1^0. Consider the case when the reducibility type is $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, i.e. it consists of one big and two small blocks. Consider M_1. Suppose that its blocks A and C are in Jordan normal form for all values of the coordinates upon which the elements of $\mathcal{R}$ (see 8^0 of the proof of the theorem) depend. A holomorphic conjugation to such a form exists (locally). Let λ_1 be an eigenvalue of M_1. Then condition $\text{rk}(M_1 - \lambda_1 I) \leq a_1$, $a_1 \in \mathbf{N}$ ('rk'='rank') defines a finite number of smooth subvarieties $\mathcal{T}_k^{(1)}$ of $\mathcal{T}'$. The same is true for $M_2, \ldots, M_p$. If we consider conditions $\text{rk}(M_s - \lambda_s I) \leq a_s$, $s = 1, \ldots, p$ simultaneously, then this defines a finite number of smooth analytic subvarieties $\mathcal{T}''_k$. Applied to M_{p+1}, (i.e. for $s = p+1$), these conditions define a finite number of smooth analytic subvarieties $\mathcal{S}''_j$ of $\mathcal{S}'$. The graph of (*) restricted to $\mathcal{T}''_k$ intersects $\mathcal{S}''_j \times \mathcal{T}''_k$ and the intersection (for each (k, j)) is a smooth analytic subvariety. This is proved in the same way as of 1) of the theorem, considering the restriction of equation (***) to each block on or above the diagonal (in the big blocks), in the opposite to the order of the blocks as described in 5^0 of the proof of the theorem. This is possible because the strata are non-special and we don't have problems with Lemma 3.5.

On each $(\mathcal{S}''_j, \mathcal{T}''_k)$ consider conditions $\text{rk}(M_s - \lambda_s I)^2 \leq b_s$, $s = 1, \ldots, p+1$. They define smooth analytic subvarieties; the graph of (*) restricted to $\mathcal{T}''_k$ intersects $\mathcal{S}''_i \times \mathcal{T}''_k$ and the intersection is a smooth analytic variety. Then we consider conditions $\mathrm{rk}(M_s-\lambda_s I)^3 \leq c_s$ etc. These conditions define the closures of the strata; they are connected with the finding of the Jordan normal form of M_s (more precisely – how many blocks and of what size correspond to the given eigenvalue). When $\mathrm{rk}(M_s - \lambda_s I)$ is maximal possible, then this doesn't define a subvariety, but the compliment to the analytic varieties from which the other strata are composed. In the case of one big and two small blocks the proof of the smoothness is easy because the B-blocks of the matrices $(M_s - \lambda_s I)^k$ are equal to $\sum_{j=0}^k P_s^j Q_s R_s^{k-j}$ where $P_s = M_s|_A$, $Q_s = M_s|_B$, $R_s = M_s|_C$, i.e. they depend linearly on Q_s.

2^0. Let the reducibility type consist of one big and r small blocks:

$$\begin{pmatrix} Q_{11} & Q_{12} & \dots & Q_{1,r-1} & Q_{1r} \\ 0 & Q_{22} & \dots & Q_{2,r-1} & Q_{2r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & Q_{r-1,r-1} & Q_{r-1,r} \\ 0 & 0 & \dots & 0 & Q_{rr} \end{pmatrix}$$

In this case the proof is carried out by induction with respect to r, in the same way as in 1^0; the roles of A, B and C are played respectively by $Q' = \begin{pmatrix} Q_{11} & Q_{12} & \dots & Q_{1,r-1} \\ 0 & Q_{22} & \dots & Q_{2,r-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_{r-1,r-1} \end{pmatrix}$, $\begin{pmatrix} Q_{1r} \\ \vdots \\ Q_{r-1,r} \end{pmatrix}$ and Q_{rr}. It is assumed that the Q'- and Q_{rr}-blocks of $M_1, \dots, M_{p+1}$ are restricted to irreducible components of given strata. Smoothness of strata is proved as it is explained in 1^0. For the case of many big blocks the lemma is proved in the same way.

Proof of Lemma 3.9.: Set in operator $\mathcal{L}'_p$, see (***), $Y_j = \begin{pmatrix} Y'_j & Y''_j \\ 0 & Y'''_j \end{pmatrix}$. Find first Y'_j , Y'''_j and V_j , $j = 1, \dots, p+1$ as in the irreducible case; they solve the restriction of equation (***) to the A- and C-blocks. After this the equation can be solved for the B-block as well if its trace after the fixing of Y'_j , Y'''_j is 0. If not, then we can try to make the substitution $Y'_j \mapsto Y'_j + U_j$ (or $Y'''_j \mapsto Y'''_j + U_j$) where U_j are matrices of the size of P_j such that $\sum_{j=1}^p [U_j, P_j] = 0$. This will not change the A- and C-blocks. It will fail to change the trace of the B- block if and only if $\mathrm{tr}\sum_{j=1}^p U_j Q_j (\equiv \mathrm{tr}\sum_{j=1}^p Q_j U_j) = 0$ for every set of U_j such that $\sum_{j=1}^p [U_j, P_j] = 0$.

Proposition 3.17. *Let* $tr\sum_{j=1}^p U_j Q_j (\equiv tr\sum_{j=1}^p Q_j U_j) = 0$ *for every set of* U_j *such that* $\sum_{j=1}^p [U_j, P_j] = 0$. *Then* $Q_j = [P_j, D^0]$ *for some matrix* D^0 *of the size of* P_j, $j = 1, \dots, p+1$.

The lemma follows from the proposition, setting $D = -D^0$.

Proof : Condition $\mathrm{tr}\sum_{j=1}^{p} U_jQ_j = 0$ must be a corollary from condition $\sum_{j=1}^{p}[U_j, P_j] = 0$. Every such corollary is of the form $\mathrm{tr}\sum_{j=1}^{p}[U_j, P_j]D^0 = 0$. We have $\mathrm{tr}\sum_{j=1}^{p}[U_j, P_j]D^0 = \mathrm{tr}\sum_{j=1}^{p}(P_jD^0U_j - D^0P_jU_j)$. Hence, we must have $Q_j = [P_j, D^0]$.

Proof of Lemma 3.10.: For every $\varepsilon \in \mathbf{C}$ the matrix $\begin{pmatrix} P_j & \varepsilon Q_j \\ 0 & P_j \end{pmatrix}$ is conjugate to $M_j^0 = \begin{pmatrix} P_j & 0 \\ 0 & P_j \end{pmatrix}$. Hence, the matrix $\begin{pmatrix} 0 & Q_j \\ 0 & 0 \end{pmatrix}$ belongs to the tangent space to the orbit of M_j^0 for every ε, i.e. $Q_j = [P_j, D_j]$ for some D_j. The opposite implication follows from $M_j = S^{-1}M_j^0S$ with $S = \begin{pmatrix} I & -D_j \\ 0 & I \end{pmatrix}$.

Proof of Lemma 2.1.: We combine the ideas used in the proofs of Lemmas 3.7. and 3.8. For every stratum of Σ the Jordan normal forms of its small blocks define smooth analytic varieties in Σ. This is proved as the smoothness of $\mathcal{T}^*$, see the proof of Lemma 3.7. Let λ be an eigenvalue of $M \in \Sigma$. Then conditions $\mathrm{rk}(M - \lambda I)^i \leq a_i$, $i = 1, 2, \ldots$, $a_i \in \mathbf{N}$ define a finite number of smooth analytic varieties, see the proof of Lemma 3.8., which are the closures of a finite number of strata. The lemma is proved.

Acknowledgement

The author is grateful to A.A. Bolibrukh, A.H.M. Levelt, D.Siersma, B.L.J. Braaksma and C. Simpson for discussing the problem, to V.I. Arnold and Yu. S. Il'yashenko for having introduced him into the analytic theory of fuchsian systems.

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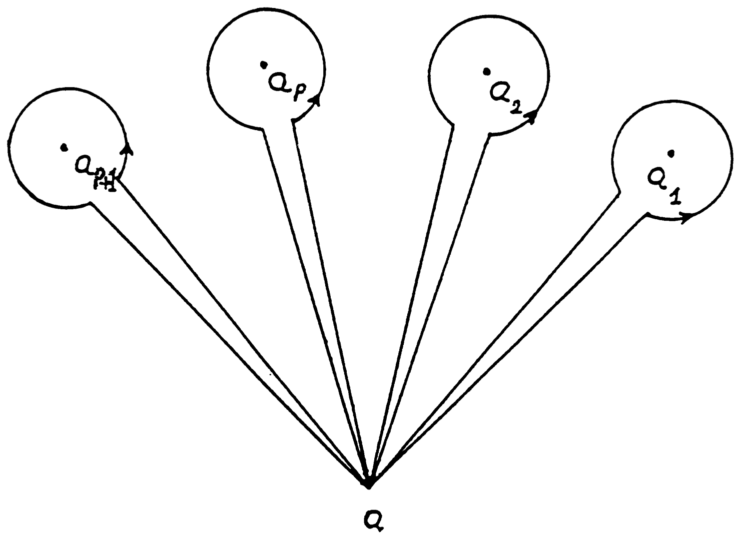

Fig. 1

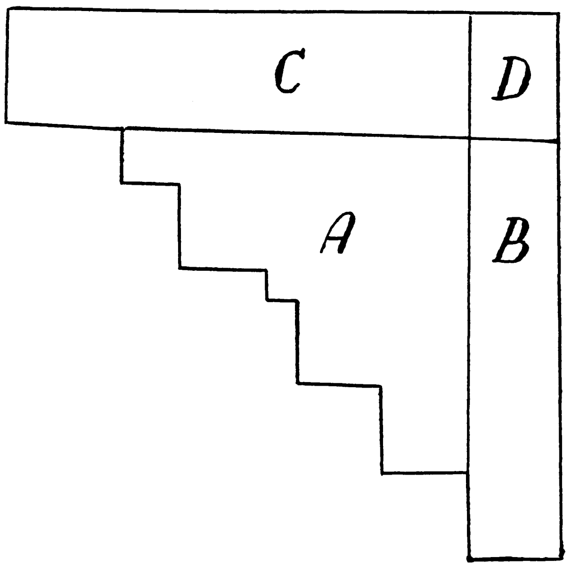

Fig. 2.

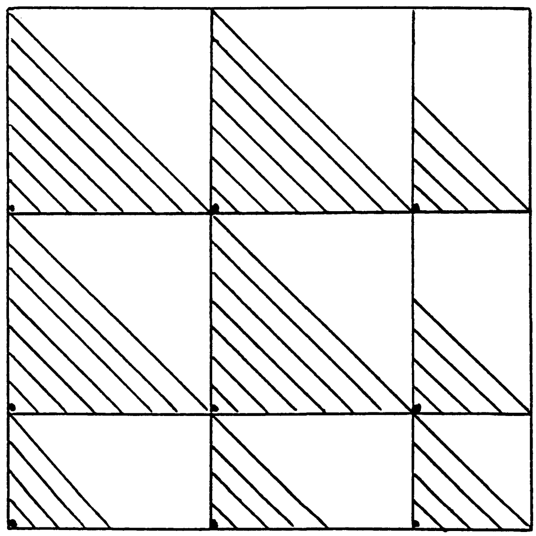

Fig. 3.

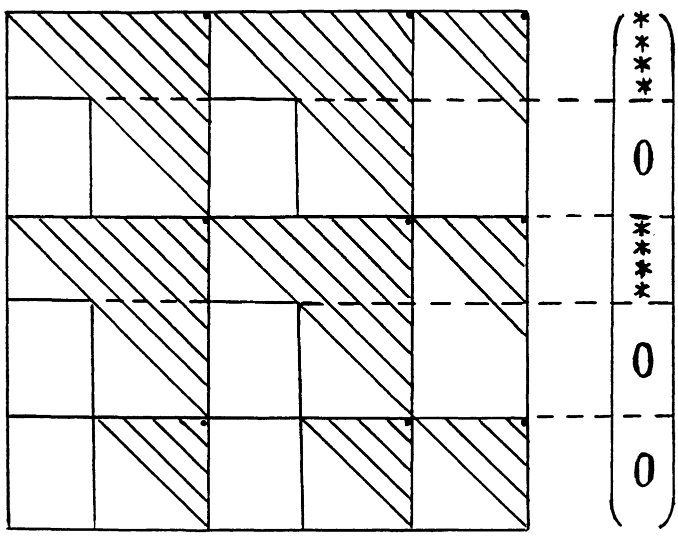

Fig. 4.

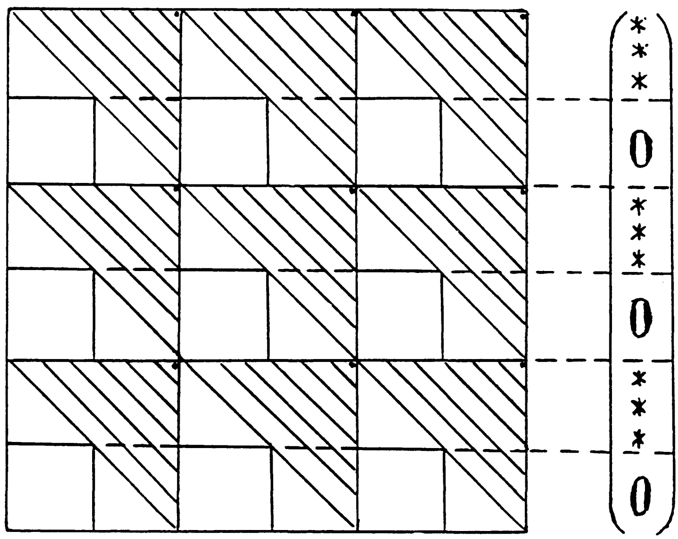

Fig. 5.

Astérisque

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Equisingular unfoldings of foliations by curves

Astérisque, tome 222 (1994), p. 285-302
<http://www.numdam.org/item?id=AST_1994__222__285_0>

EQUISINGULAR UNFOLDINGS OF FOLIATIONS BY CURVES

Jean François MATTEI and Marcel NICOLAU*

0. Introduction. Let F denote a holomorphic foliation by curves with isolated singularities on a complex surface M. The first author constructed in [M] a versal equisingular unfolding, parametrized by a smooth space of parameters K_i^{loc}, of the germ of the foliation F at one of its singular points q_i. The aim of this paper is to show the existence of a versal equisingular unfolding of the global foliation F when M is compact. In this case the parameter space K_e of the versal unfolding can be singular. The problem of finding conditions on F assuring the triviality of any unfolding has been considered by X. Gomez-Mont in [G-M].

An equisingular unfolding of F is an unfolding admiting a reduction of the singularities "with parameters". It is claimed in [M] that there is a one-to-one correspondence between equisingular unfoldings of F and locally trivial unfoldings (cf. Definition 1.6) of the reduction $\widetilde{F}$ of F preserving the divisor which comes from the singular points of F. So we are led to construct a versal locally trivial unfolding of a (possibly non saturated) foliation by curves. The construction of the versal space is carried out in the first two sections. The key point is the identification of locally trivial unfoldings with a certain type of deformations of the complex structure of the underlying manifold. Then we consider the relationship between the global versal space K_e and the local versal spaces K_i^{loc}. We show that under some cohomological assumptions K_e is smooth and naturally identified with the product $\prod K_i^{\text{loc}}$. Finally we apply the above results to show that any equisingular unfolding of a germ of algebraic foliation is still algebraic.

1. Locally trivial unfoldings of foliations by curves.

Let M be a n-dimensional compact complex manifold and let TM be its holomorphic tangent bundle. Given a holomorphic vector bundle E over M we denote by $\mathcal{O}(E)$ the sheaf of germs of holomorphic sections of E. In

*Partially supported by grant PB90-0686 from DGICYT.

particular $\mathcal{O}_M = \mathcal{O}(\mathbb{C})$ and $\Theta_M = \mathcal{O}(TM)$ are respectively the sheaves of germs of holomorphic functions and holomorphic vector fields on M.

By a (singular) holomorphic foliation on M we mean a locally free $\mathcal{O}_M$-submodule F of Θ_M which is closed under the Lie bracket of vector fields. The ***singular locus*** $S(F)$ of F is the analytic subset whose complementary $M - S(F)$ is the maximal open set on which F defines a foliation in the usual sense, i.e. without singularities. The foliation F is called ***saturated*** if one has

$$\Gamma(U, \Theta_M) \cap \Gamma(U - S(F), F) = \Gamma(U, F)$$

for any open subset $U \subset M$. It is well known that a foliation F is saturated if and only if $\operatorname{codim} S(F) \geq 2$. In fact it follows from Hartogs' extension theorem that if there is given an analytic subset $\Sigma \subset M$ of codimension greater than one and a non singular foliation F' on $M - \Sigma$ then there is a uniquely defined saturated foliation F on M which coincides with F' on $M - \Sigma$. In particular $S(F) \subset \Sigma$.

A ***foliation by curves*** is a locally free subsheaf F of Θ_M of rank one. Therefore a foliation by curves is determined by a pair (L, κ) where L is a line bundle over M and $\kappa : L \to TM$ is a non identically zero bundle morphism. The bundle morphism κ induces an injective morphism of sheaves $\mathcal{O}(L) \to \Theta_M$ and we identify $\mathcal{O}(L)$ with its image in Θ_M. Then $S(F)$ is the set of those points $z \in M$ for which $\kappa : L_z \to T_z M$ is the zero map.

In an equivalent way a holomorphic foliation by curves can be defined by a collection of local holomorphic vector fields $\xi_i \in \Gamma(U_i, \Theta_M)$ such that $\mathcal{U} = \{U_i\}$ is an open cover of M and $\xi_j = u_{ji} \cdot \xi_i$ on $U_i \cap U_j$ for suitable non vanishing holomorphic functions u_{ji}. Then L is the line bundle associated to the 1-cocycle $\{u_{ji}\}$. Moreover, if $\sigma_i : U_i \to L$ are non vanishing sections with $\sigma_j = u_{ji} \cdot \sigma_i$ then $\kappa : L \to TM$ is the bundle morphism determined by the condition $\kappa \circ \sigma_i = \xi_i$. The singular locus $S(F)$ is just the union $\bigcup \operatorname{Sing}(\xi_i)$ where $\operatorname{Sing}(\xi_i)$ is the subset of U_i where ξ_i vanishes.

To any foliation F there is naturally associated a saturated foliation ${}^sF$ which coincides with F outside $S(F)$. In the case of a foliation by curves ${}^sF$ can be described as follows. Assume that F is defined by local vector fields $\xi_i \in \Gamma(U_i, \Theta_M)$ where each U_i is holomorphically equivalent to an open polidisc Δ of $\mathbb{C}^n$. Let $z^1, \ldots, z^n$ be the coordinates on U_i induced by the identification $U_i \equiv \Delta$ and set $\xi_i = \sum \xi_i^\alpha \partial/\partial z^\alpha$. Let v_i be a m.c.d. of the functions $\xi_i^1, \ldots, \xi_i^n$ and define $\hat{\xi}_i = \xi_i / v_i$. Then $\operatorname{codim}(\operatorname{Sing}(\hat{\xi}_i)) \geq 2$. Since the foliation on $U_i \cap U_j$ defined by $\hat{\xi}_j$ is saturated there is a holomorphic function $\hat{u}_{ij}$ such that $\hat{\xi}_i = \hat{u}_{ij} \cdot \hat{\xi}_j$ on $U_i \cap U_j$. Furthermore the functions $\hat{u}_{ij}$ do not vanish. If not $\operatorname{Sing}(\hat{\xi}_j)$ would be of codimension one. Hence the local vector fields $\hat{\xi}_i$ define a saturated foliation ${}^sF$ called the ***saturation*** of F.

1.1. Example. Let a saturated foliation F and an analytic hypersurface D on M be given. One can find an open cover $\{U_i\}$ of M with the property that there exist $\xi_i \in \Gamma(U_i, \Theta_M)$ and $h_i \in \Gamma(U_i, \mathcal{O}_M)$ such that the collection $\{\xi_i\}$ defines F and $h_i = 0$ defines D on U_i (i.e. if $f \in \Gamma(W, \mathcal{O}_M)$ vanishes on D then $f = \lambda \cdot h_i$ on $W \cap U_i$). Set $\eta_i = h_i \cdot \xi_i$. The collection of vector fields $\{\eta_i\}$ defines a non saturated foliation by curves F^D with singular locus $S(F^D) = D \cup S(F)$ and whose saturation is just F. With more generality and for any given positive integer $k \in \mathbb{N}^*$ one can define the foliation $F^{k\cdot D}$ as the foliation by curves defined by the local vector fields $\eta_i^{(k)} = (h_i)^k \cdot \xi_i$. In section 3 we will consider locally trivial unfoldings of foliations by curves obtained in this way.

From now on F will be a fixed foliation by curves on a compact manifold M defined by a pair (L, κ). Let Ω be an open neighbourhood of 0 in $\mathbb{C}^m$. The product $\Omega \times M$ is endowed with a holomorphic foliation F_Ω of the same codimension as F obtained as the product of F by the foliation on Ω consisting of a single leaf; i.e. F_Ω is the foliation defined by the subsheaf $\mathrm{pr}_1^*\Theta_\Omega \oplus \mathrm{pr}_2^*F$ of $\Theta_{\Omega\times M}$. Here pr_1 and pr_2 denote respectively the natural projections from $\Omega \times M$ onto the first and second factors. We call F_Ω the *trivial unfolding* of F parametrized by Ω.

1.2. Definition. *Let W be an open subset of $\Omega \times M$ and let $\Psi : W \to \Psi(W) \subset \Omega \times M$ be an $\mathbb{C}$-analytic diffeomorphism over the identity of Ω such that the restriction of Ψ to $W \cap (\{0\} \times M)$ is the identity. We say that Ψ is a relative automorphism of the trivial unfolding F_Ω if there is a bundle morphism $\Psi^\sharp : (T\Omega \times L)|W \to (T\Omega \times L)|\Psi(W)$ over Ψ such that the diagram*

$$\begin{array}{ccc}
(T\Omega \times L)|W & \xrightarrow{id\times\kappa} & (T\Omega \times TM)|W \\
{\scriptstyle\Psi^\sharp}\big\downarrow & & \big\downarrow{\scriptstyle\Psi_*} \\
(T\Omega \times L)|\Psi(W) & \xrightarrow{id\times\kappa} & (T\Omega \times TM)|\Psi(W)
\end{array}$$

is commutative, where Ψ_* denotes the tangent map of Ψ.

1.3. Remark. Let Ψ be a local biholomorphism over the identity of Ω and inducing the identity on $\{0\} \times M$. Suppose that Ψ maps the singular locus $S(F_\Omega)$ of F_Ω identically into itself and preserves the foliation F_Ω outside $S(F_\Omega)$. If the foliation F is saturated then Ψ_* induces a bundle morphism $\Psi^\sharp$ fulfiling the conditions required in the above definition. This is no longer true in general if the foliation is not saturated and in this case Ψ need not to be an automorphism of the trivial unfolding. Nevertheless there is a particular type of non saturated foliations for which this property still holds. It is considered in Proposition 1.5.

Assume $\Psi : W \to \Psi(W)$ is a holomorphic diffeomorphism such that W, $\Psi(W)$ are subsets of $\Omega \times U$ where U is a coordinate open subset of M, with coordinates $z = (z^1, \ldots, z^n)$, on which F is defined by a holomorphic vector field ξ. Let $s = (s^1, \ldots, s^m)$ denote the linear coordinates on Ω. Then Ψ is a relative automorphism of F_Ω if and only if it is of the form $\Psi(s, z) = (s, \psi(s, z))$ where ψ is a holomorphic map such that $\psi(0, z) = z$ and fulfiling

$$\psi_* \xi = (u \circ \Psi^{-1}) \cdot \xi, \tag{1}$$

$$\psi_* \frac{\partial}{\partial s^\mu} = (v_\mu \circ \Psi^{-1}) \cdot \xi \qquad \text{for } \mu = 1, \ldots, m, \tag{2}$$

for suitable functions u, v_μ on W. For example, if $\varphi = \varphi(t, z)$ denotes the local flow of ξ and $\sigma = \sigma(s, z)$ is a holomorphic function with $\sigma(0, z) = 0$ then $\Psi(s, z) = (s, \varphi(\sigma(s, z), z))$ is a relative automorphism of the trivial unfolding. The following proposition states that any relative automorphism of F_Ω is locally of this form.

1.4. Proposition. *Let $\Psi(s, z) = (s, \psi(s, z))$ be a relative automorphism of F_Ω with domain $W \subset \Omega \times M$. Assume $U = W \cap (\{0\} \times M)$ is an open subset of M holomorphically equivalent to an open polidisc on which F is defined by a holomorphic vector field ξ. Let $\varphi = \varphi(t, z)$ be the local flow associated to ξ. Then there is a holomorphic function $\sigma = \sigma(s, z)$ defined in a neighbourhood W' of U in W with $\sigma(0, z) = 0$ and such that $\psi(s, z) = \varphi(\sigma(s, z), z)$ on W'.*

Proof. Because of (2) a function $\sigma = \sigma(s, z)$ fulfils the required conditions if and only if σ is a solution of the total differential equation (with parameter $z \in U$)

$$\begin{cases} \dfrac{\partial \sigma}{\partial s^\mu}(s, z) = v_\mu(s, z) \\ \sigma(0, z) = 0. \end{cases} \tag{3}$$

The existence of such a function σ outside the singular locus $S(F_\Omega)$ of F_Ω can be easily seen by using local coordinates $w^1, \ldots, w^n$ such that $\xi = \partial/\partial w^n$. In particular the integrability condition of the equation (3), i.e.

$$\frac{\partial v_\mu}{\partial s^\nu} = \frac{\partial v_\nu}{\partial s^\mu} \qquad \text{for } \mu, \nu = 1, \ldots, m, \tag{4}$$

is fulfiled outside $S(F_\Omega)$. But equalities (4) must then also be verified on the singular set by continuity. So the complex Frobenius theorem with holomorphic parameters applies showing that (3) has a unique solution defined in a neighbourhood of $\{0\} \times U$ in W. ∎

Let $F^{k \cdot D}$ denote a foliation of the type defined in Example 1.1. A relative automorphism of $(F^{k \cdot D})_\Omega$ is also a relative automorphism of F_Ω. The converse is not true in general. In the following proposition we consider a particular situation in which the converse still holds. It will be used in section 3.

1.5. Proposition. *Assume F is saturated and let D be a hypersurface of M. Let Ψ be a given relative automorphism of F_Ω. If Ψ is the identity on $\Omega \times D$ then it is also a relative automorphism of the trivial unfolding $(F^D)_\Omega$.*

Proof. Using the notation in the above proposition we can write Ψ locally as $\Psi = \varphi(\sigma(s,z),z)$ for a given function σ. Suppose D is defined by the equation $h(z) = 0$, set $\hat{\xi} = h \cdot \xi$ and denote by $\hat{\varphi} = \hat{\varphi}(t,z)$ the local flow of $\hat{\xi}$. Then $\hat{\varphi}(t,z) = \varphi(a(t,z),z)$ where $a = a(t,z)$ is the holomorphic function determined by the equation

$$\begin{cases} \dfrac{\partial a}{\partial t}(t,z) = h(\hat{\varphi}(t,z)) \\ a(0,z) = 0. \end{cases}$$

Since $z = \hat{\varphi}(t,z)$ for any $z \in D$ we have $a(t,z) = 0$ for $z \in D - S(F)$. The foliation F being saturated $D - S(F)$ is dense in D. So $a(t,\cdot)$ must vanish on D. From this fact we deduce that $a(t,z) = h(z)\cdot b(t,z)$ with $b(t,z) = t + t^2 b_1(t,z)$.

The maps $\Psi(s,\dot{})$ are also the identity on D by hypothesis. The above argument also shows that $\sigma(s,z) = h(z)\cdot\tau(s,z)$. The implicit function theorem implies the existence of a function $\hat{\sigma} = \hat{\sigma}(s,z)$ defined for small s such that $\tau(s,z) = b(\hat{\sigma}(s,z),z)$. So we can write $\psi(s,z) = \hat{\varphi}(\hat{\sigma}(s,z),z)$ showing that Ψ is a relative automorphism of $(F^D)_\Omega$. ∎

Let $U \subset M$ be open. Two relative automorphisms Ψ and Ψ' of F_Ω whose domains contain $\{0\} \times U$ are identified if they coincide in a neighbourhood of $\{0\} \times U$ in $\Omega \times M$. The set $\mathcal{G}_\Omega(U)$ of these equivalence classes is a group under the composition of automorphisms. When U runs over the open subsets of M the family $\{\mathcal{G}_\Omega(U)\}$ defines a sheaf $\mathcal{G}_\Omega$ of non abelian groups over M.

In order to obtain a versality theorem for locally trivial unfoldings we are led to consider (germs of) analytic spaces as spaces of parameters. So the general space of parameters we will be the germ $(S,0)$ at $0 \in \mathbb{C}^m$ of a (possibly non reduced) analytic space S defined by $S = \operatorname{supp}(\mathcal{O}_\Omega/\mathcal{I})$ where $\mathcal{I}$ is a coherent sheaf of ideals of $\mathcal{O}_\Omega$. The restriction F_S of F_Ω to $S \times M$ will be called the *trivial unfolding of F parametrized by S*. The restrictions to $S \times M$ of the elements of $\mathcal{G}_\Omega$ form a sheaf of non abelian groups over M denoted by $\mathcal{G}_S$.

1.6. Definition. *A (germ of) locally trivial unfolding $(\mathcal{F}, \mathcal{M}, \pi, S, \iota)$ (sometimes simply denoted by $\mathcal{F}$) of F parametrized by $(S,0)$ is given by an analytic space $\mathcal{M}$, a proper $\mathbb{C}$-analytic morphism $\pi : \mathcal{M} \to S$ and a holomorphic isomorphism $\iota : M \to M_0 := \pi^{-1}(0)$ in such a way that there exists an atlas $\{(W_i, \phi_i)\}$ of $\mathcal{M}$, where $\phi_i : W_i \to \phi(W_i) \subset S \times M$ are $\mathbb{C}$-analytic diffeomorphisms, fulfiling*

(i) $\mathrm{pr}_1 \circ \phi_i = \pi$,

(ii) $\phi_i|M_0 = \iota^{-1}$,
(iii) $\Psi_{ij} = \phi_i \circ \phi_j^{-1}$ *is a section of* $\mathcal{G}_S$.

For s close to zero $M_s := \pi^{-1}(s)$ is a compact complex manifold and we can think of $\pi : \mathcal{M} \to S$ as a family of deformations of the complex manifold $M \cong M_0$. The restriction F_s of $\mathcal{F}$ to M_s is a foliation by curves locally isomorphic to $F = \iota^* F_0$.

1.7. Remark. Let $\mathcal{M}$ be a complex manifold, $\pi : \mathcal{M} \to \Omega$ a proper holomorphic map and $\iota : M \to M_0 := \pi^{-1}(0)$ a biholomorphism. Let a saturated foliation $\mathcal{F}$ on $\mathcal{M}$ which is transverse to the fibres of π (outside $S(\mathcal{F})$) be given. Then ι induces a well defined saturated foliation $\iota^*\mathcal{F}$ on M. Therefore in the case of saturated foliations one can define the (general) notion of *unfolding* of F parametrized by the smooth space Ω as a 5-tuple $(\mathcal{F}, \mathcal{M}, \pi, \Omega, \iota)$ where the foliation $\mathcal{F}$ is saturated and transverse to the fibres of π and such that $\iota^*\mathcal{F} = F$.

Two locally trivial unfoldings $(\mathcal{F}, \mathcal{M}, \pi, S, \iota)$ and $(\mathcal{F}', \mathcal{M}', \pi', S, \iota')$ of F parametrized by $(S, 0)$ and defined respectively by the atlas $\{(W_i, \phi_i)\}$ and $\{(W'_k, \phi'_k)\}$ are said to be isomorphic if there is a $\mathbb{C}$-analytic isomorphism over the identity of S, $\Phi : \mathcal{M} \to \mathcal{M}'$, such that: (i) $\Phi \circ \iota = \iota'$ and (ii) $\phi'_k \circ \Phi \circ \phi_i^{-1}$ are sections of $\mathcal{G}_S$.

The underlying analytic space $\mathcal{M}$ of a locally trivial unfolding $\mathcal{F}$ is obtained by glueing together open subsets of $S \times M$ by means of relative automorphisms $\Psi_{ij} \in \Gamma(U_i \cap U_j, \mathcal{G}_S)$. Assume that ξ_i are local vector fields defining F on U_i and let $\varphi_i = \varphi_i(t, z)$ denote the corresponding local flows. By virtue of Proposition 1.4, $\Psi_{ij} = (\mathrm{pr}_1, \psi_{ij})$ where ψ_{ij} is of the form

$$\psi_{ij}(s, z) = \varphi_i(\sigma_{ij}(s, z), z)$$

for suitable functions σ_{ij}. Given $\partial/\partial s \in T_0 S$, let $\varrho_{\mathcal{F}}(\partial/\partial s)$ denote the cohomology class of the cocycle associating to $U_i \cap U_j$ the section of $\mathcal{O}(L)$ given by

$$\theta_{ij} = \left.\frac{\partial \psi_{ij}}{\partial s}\right|_{s=0} = \left(\left.\frac{\partial \sigma_{ij}}{\partial s}\right|_{s=0}\right) \cdot \xi_i.$$

The $\mathbb{C}$-linear map $\varrho_{\mathcal{F}} : T_0 S \to H^1(M, \mathcal{O}(L))$ defined in this way only depends on the isomorphism class of the unfolding and is called the *Kodaira-Spencer map* of $\mathcal{F}$.

Given a morphism of germs of analytic spaces $f : (T, 0) \to (S, 0)$ the fibered product $T \times_S \mathcal{M}$ is constructed by means of sections $\Psi^f_{ij} = (\mathrm{pr}_1, \psi^f_{ij})$ of $\mathcal{G}_T$ where $\psi^f_{ij} = \psi_{ij} \circ (f \times 1)$. In this way $T \times_S \mathcal{M}$ is endowed with a locally trivial unfolding $f^*\mathcal{F}$ of F parametrized by $(T, 0)$ called the pull-back of $\mathcal{F}$

by f. A locally trivial unfolding $(\mathcal{F}, \mathcal{M}, \pi, S, \iota)$ of F is called ***versal*** if for any other locally trivial unfolding $(\mathcal{F}', \mathcal{M}', \pi', S', \iota')$ of F there is a morphism of germs of analytic spaces $f : (S', 0) \to (S, 0)$ such that: (i) $\mathcal{F}'$ and $f^*\mathcal{F}$ are isomorphic and (ii) $d_0 f$ is unique. In this case the Kodaira-Spencer map of $\mathcal{F}$ is an isomorphism. If the morphism f itself and not only its linear part is unique then the unfolding $\mathcal{F}$ is called *universal.*

We end this paragraph by showing that any locally trivial unfolding is globally differentiably trivial. This is a consequence of the existence of the "exponential map" stated in the proposition below. This exponential map is also used in the proof of the versality theorem 2.5. Let $(\mathcal{F}, \mathcal{M}, \pi, S, \iota)$ be a locally trivial unfolding of F defined by an atlas $\{(W_i, \phi_i)\}$ of $\mathcal{M}$ as in Definition 1.6. Assume that F is defined on $U_i = \iota^{-1}(W_i \cap M_0)$ by a local vector field ξ_i and let $\varphi_i = \varphi_i(t, z)$ be the associated flow. Let us consider $S \times L$ as a vector bundle over $S \times M$ and let $\gamma : S \times M \to S \times L$ denote the zero section. In this situation we have

1.8. Proposition. *By shrinking S if necessary we can find an open neighbourhood $\mathcal{V}$ of $\gamma(S \times M)$ in $S \times L$ and a C^∞ map $\tilde{g}_{\mathcal{F}} : \mathcal{V} \to \mathcal{M}$ over the identity of S and holomorphic with respect to S such that the restriction of $\phi_i \circ \tilde{g}_{\mathcal{F}}$ to $\mathcal{V} \cap p^{-1}(U_i)$ can be written in the form*

$$\tilde{g}_{\mathcal{F}}(s, z, t) = (s, \varphi_i(a_i(s, z, t), z)) \tag{5}$$

where t denotes the linear coordinate on $L|U_i$ induced by the choice of ξ_i and $a_i = a_i(s, z, t)$ is a C^∞ function depending holomorphically on s and t and fulfiling

$$a_i(0, z, 0) = 0 \tag{6}$$

$$\frac{\partial a_i}{\partial t}(0, z, 0) = 1. \tag{7}$$

Proof. A straightforward computation shows that the condition for the mapping $\tilde{g}_{\mathcal{F}}$ to admit a local writing like (5) does not depend on the choice of ξ_i nor on the choice of ϕ_i. Thus there is a globally defined sheaf $\mathcal{S}$ over $S \times L$ whose elements are the germs of such mappings. The restriction $\mathcal{S}_0$ of $\mathcal{S}$ to $M \equiv \gamma(\{0\} \times M)$ is locally isomorphic to the sheaf $\mathcal{S}_0'$ over $\{0\} \times \mathbb{C}^n \times \{0\}$ which is the restriction of the sheaf over $S \times \mathbb{C}^n \times \mathbb{C}$ whose elements are the germs of C^∞ functions $a = a(s, z, t)$ depending holomorphically on s and t and fulfuling conditions (6) and (7).

By means of partitions of unity one sees that $\mathcal{S}_0'$ is locally soft ("mou"). This implies that $\mathcal{S}_0$ is globally soft (cf. [G, th. 3.4.1]) and it admits a section on $\gamma(\{0\} \times M)$. Since $\gamma(\{0\} \times M)$ has a fundamental system of paracompact neighbourhoods in $S \times L$ the above section can be extended to a section of $\mathcal{S}$ on a neighbourhood $\mathcal{V}$ of $\gamma(\{0\} \times M)$ (cf. [G, th. 3.3.1]). Because of the compacity of M the domain $\mathcal{V}$ can be taken as a neighbourhood of $\gamma(S' \times M)$ in $S' \times L$ for a suitable neighbourhood S' of 0 in S. ∎

1.9. Corollary. *By shrinking S if necessary we can find a C^∞ diffeomorphism $g_\mathcal{F} : S \times M \to \mathcal{M}$ over the identity of S and holomorphic with respect to S which takes the leaves of the foliation F in $\{s\} \times M$ onto the leaves of F_s in M_s.*

Proof. Set $g_\mathcal{F} = \tilde{g}_\mathcal{F} \circ \gamma$. ∎

2. Locally trivial unfoldings of F as families of complex structures on M.

For a given holomorphic vector bundle E over M let $A^{0,k}(E)$ denote the space of differential forms on M of type $(0,k)$ with values in E, i.e. $A^{0,k}(E) = C^\infty(M, \Lambda^k \overline{TM}^* \otimes E)$. Set $\mathcal{T} = TM \oplus \overline{TM}$. We recall that a complex structure τ on M close to the original one is given by an involutive subbundle $T^{0,1}$ of $\mathcal{T}$ such that $\mathcal{T} = T^{1,0} \oplus T^{0,1}$ and $T^{0,1} \cap TM = 0$. Here $T^{1,0} = \overline{T^{0,1}}$. In this situation there is a unique element $\omega \in A^{0,1}(TM)$ such that $T^{0,1} = T^{0,1}_\omega := (\mathrm{id} - \omega)(\overline{TM})$. Involutiveness of $T^{0,1}_\omega$ is equivalent to the integrability of ω, i.e. to the condition $\bar\partial\omega - 1/2[\omega,\omega] = 0$. Thus complex structures on M close to the original one are parametrized by a neighbourhood of zero in the space J defined as

$$J = \{\omega \in A^{0,1}(TM) | \bar\partial\omega - \frac{1}{2}[\omega,\omega] = 0\}.$$

Given a family of deformations $\pi : \mathcal{M} \to S$ of the complex manifold $M \equiv M_0 = \pi^{-1}(0)$ and a differentiable trivialization $g : S \times M \to \mathcal{M}$ (i.e. a diffeomorphism over the identity of S, depending holomorphically on S and fulfiling $g|\{0\} \times M = \mathrm{id}$) there is a family of complex structures on M for which g is holomorphic. This family of complex structures is parametrized by a family $\{\omega_s\}$ of elements of J depending holomorphically on S and such that $\omega_0 = 0$. We will say that $\{\omega_s\}$ is the family of elements of J associated to $\mathcal{M}$ and the differentiable trivialization g. Conversely, given a holomorphic family $\{\omega_s\}$ of *real analytic* elements of J with $\omega_0 = 0$ there is a (uniquely determined) family of complex structures on M which is parametrized by $\{\omega_s\}$ (cf. [W, Prop. II.3.2] and [D]).

Given $\omega \in J$ close to zero and a diffeomorphism h of M which is C^1-close to the identity we will denote by $\psi = \omega \circ h$ the unique element of J such that $h : M_\psi \to M_\omega$ is holomorphic.

Let us consider now the pair (L, κ) defining the foliation by curves F on M. The bundle morphism κ induces injective $\mathbb{C}$-linear maps $\kappa : A^{0,k}(L) \to A^{0,k}(TM)$. We will identify $A^{0,k}(L)$ with its image in $A^{0,k}(TM)$. In this way the elements of $J_F = J \cap A^{0,1}(L)$ close to zero can be thought as a certain type of complex structures on M. In this paragraph we will see that locally trivial unfoldings of F are naturally parametrized by families of elements of J_F.

A diffeomorphism h of M which is C^1-close to the identity will be called *tangent to* F if, in any local chart $(U, z = (z^1, \dots, z^n))$ of M on which F is defined by a vector field ξ, it can be written

$$h(z) = \varphi(b(z), z) \tag{8}$$

where $b = b(z)$ is a certain differentiable function close to zero and $\varphi = \varphi(t, z)$ is the local flow associated to ξ.

2.1. Proposition. *Let $\omega \in J_F$ be close to zero and let h be a diffeomorphism of M tangent to F. Then $\psi = \omega \circ h$ belongs to J_F.*

Proof. The statement being purely local we can assume that M is just the domain of the local chart $(U, z = (z^1, \dots, z^n))$ in which h is given by (8). Clearly $\psi = \omega \circ h$ is integrable so we only have to see that ψ belongs to $A^{0,1}(L)$. In order to prove this we need the following identities

$$\bar\partial h^\beta = \xi^\beta \cdot \bar\partial b \qquad \text{for } \beta = 1, \dots, n, \tag{9}$$

$$h_* \xi = (1 + \xi(b)) \cdot \xi + \xi(\bar b) \cdot \xi. \tag{10}$$

The first one follows directely from the chain rule. The second equality can be easily checked on $U - \mathrm{Sing}(\xi)$ by taking local coordinates $\{w^1, \dots, w^n\}$ on which $\xi = \partial/\partial w^n$. Then (10) must also be verified on $\mathrm{Sing}(\xi)$ by continuity. From (10) one gets the identities

$$\xi(h^\beta) = (1 + \xi(b)) \cdot \xi^\beta, \tag{11}$$

$$\xi(\bar h^\beta) = \xi(\bar b) \cdot \bar\xi^\beta. \tag{12}$$

Let us write $\omega = \sum_\lambda \omega_\lambda d\bar z^\lambda \otimes \xi = \sum_{\beta,\lambda} \omega_\lambda \xi^\beta d\bar z^\lambda \otimes \partial/\partial z^\beta$ and let $\zeta^1, \dots, \zeta^n$ be ω-holomorphic local coordinates. They must verify $\bar\partial \zeta^\alpha = \omega(\zeta^\alpha)$, that is

$$\frac{\partial \zeta^\alpha}{\partial \bar z^\lambda} = \sum_\beta \omega_\lambda \xi^\beta \frac{\partial \zeta^\alpha}{\partial z^\beta}. \tag{13}$$

The vector 1-form ψ is determined by the condition that the functions $\zeta^\alpha \circ h$ are ψ-holomorphic, i.e.

$$\bar\partial(\zeta^\alpha \circ h) = \psi(\zeta^\alpha \circ h). \tag{14}$$

We want to prove that ψ can be written $\psi = \sum_\delta \psi_\delta d\bar{z}^\delta \otimes \xi$ where ψ_δ are C^∞ functions. If this were the case then equality (14) could be written, using (13),

$$(15)\quad \sum_\beta \frac{\partial \zeta^\alpha}{\partial h^\beta}(\bar{\partial} h^\beta + \sum_\lambda \omega_\lambda \xi^\beta \bar{\partial} \bar{h}^\lambda)$$
$$= \sum_\beta \frac{\partial \zeta^\alpha}{\partial h^\beta}(\sum_\gamma \xi^\gamma \frac{\partial h^\beta}{\partial z^\gamma} + \sum_{\gamma,\lambda} \xi^\gamma \omega_\lambda \xi^\beta \frac{\partial \bar{h}^\lambda}{\partial z^\gamma}) \cdot \psi^o$$

where $\psi^o = \sum_\delta \psi_\delta d\bar{z}^\delta$. Since $\left(\frac{\partial \zeta^\alpha}{\partial h^\beta}\right)$ is an invertible matrix we see, making use of (9) and (11), that (15) is equivalent to

$$\xi^\beta(\bar{\partial} b + \sum_\lambda \omega_\lambda \bar{\partial} \bar{h}^\lambda) = \xi^\beta(1 + \xi(b) + \sum_\lambda \omega_\lambda \xi(\bar{h}^\lambda)) \cdot \psi^o.$$

But the expression

$$\bar{\partial} b + \sum_\lambda \omega_\lambda \bar{\partial} \bar{h}^\lambda = (1 + \xi(b) + \sum_\lambda \omega_\lambda \xi(\bar{h}^\lambda)) \cdot \psi^o$$

defines ψ^o as a differential form of type $(0,1)$ because the function $(1 + \xi(b) + \sum_\lambda \omega_\lambda \xi(\bar{h}^\lambda))$ does not vanish by hipothesis. Thus $\psi = \psi^o \otimes \xi$ is an element of J_F fulfiling (14). This concludes the proof. ∎

2.2. Remark. Let $\{\omega_s\}$ be a family of elements of J_F with $\omega_0 = 0$ and depending holomorphically on $s \in S$ and let h_s be a holomorphic family of diffeomorphisms of M tangent to F such that $h_0 = \mathrm{id}$. Then the same argument shows that $\psi_s = \omega_s \circ h_s$ is a holomorphic family of elements of J_F.

Let $\tilde{g}_{\mathcal{F}}$ be the exponential map for the unfolding $\mathcal{F}$ constructed in Proposition 1.8. The restriction $g_{\mathcal{F}}$ of $\tilde{g}_{\mathcal{F}}$ to the zero section is a differentiable trivialization of $\mathcal{M}$. In this situation we have

2.3. Proposition. *The family $\{\omega_s\}$ of elements of J associated to $\mathcal{M}$ and $g_{\mathcal{F}}$ is in fact a family of elements of J_F.*

Proof. By construction the composition $\phi_i \circ g_{\mathcal{F}}$, where $\phi_i : W_i \to S \times M$ is a local chart of $\mathcal{M}$ defining the locally trivial unfolding $\mathcal{F}$, is a diffeomorphism tangent to F. The restriction of $\{\omega_s\}$ to $g_{\mathcal{F}}^{-1}(W_i)$ coincides with $0_s \circ (\phi_i \circ g_{\mathcal{F}})$ which, by virtue of Remark 2.2, is a family of integrable local sections of $\overline{TM}^* \otimes L$. Here 0_s denotes the family of J_F which is constantly equal to zero. ∎

2.4. Proposition. *Let $\{\omega_s\}$ be a given holomorphic family of real analytic elements of J_F with $\omega_0 = 0$. Then there exist a locally trivial unfolding $(\mathcal{F}, \mathcal{M}, \pi, S, \iota)$ of F and a real analytic trivialization $g : S \times M \to \mathcal{M}$ such that $\{\omega_s\}$ is the family of vector 1-forms induced by $\mathcal{M}$ and g.*

Proof. For a given point $x \in S \times M$ let $W'_x \subset S \times M$ be a coordinate neighbourhood of x in which F_S is defined by a vector field ξ with associated local flow $\varphi = \varphi(t, z)$. We will construct a structure of $\mathbb{C}$-analytic space on the topological space $S \times M$ by giving, for any point $x \in S \times M$, a local chart $\phi_x : W_x \to S \times M$, where $W_x \subset W'_x$ is a neighbourhood of x, with the properties: (i) ϕ_x is holomorphic when we consider on W_x the complex structure defined by $\{\omega_s\}$, and (ii) ϕ_x can be written in the form

$$\phi_x(s, z) = (s, \varphi(b(s, z), z))$$

where $b = b(s, z)$ is a real analytic function depending holomorphically on $s \in S$ such that $b(0, z) = 0$. Then the compositions $\phi_x \circ \phi_{x'}^{-1}$ will be sections of $\mathcal{G}_S$ and we will obtain an analytic space $\mathcal{M}$ endowed with a trivial unfolding $\mathcal{F}$ of F. The differentiable trivialization will then be given by the map which identifies $S \times M$ with the topological space which underlies $\mathcal{M}$.

Assume $x \in S \times M$ does not belong to the singular locus $S(F_S)$ of F_S. In a neighbourhood of x we can find local coordinates $(s, w^1, \ldots, w^n)$ with $\xi = \partial/\partial w^n$. Then $\varphi(t, w^1, \ldots, w^n) = (w^1, \ldots, w^n + t)$. Since the elements of $\{\omega_s\}$ are real analytic one can prove as in [W, Prop. II.3.2] that there is a real analytic function $\tilde{w}^n = \tilde{w}^n(s, w^1, \ldots, w^n)$ depending holomorphically on S with $\tilde{w}^n(0, w^1, \ldots, w^n) = w^n$ such that $(s, w^1, \ldots, w^{n-1}, \tilde{w}^n)$ is a system of local coordinates holomorphic with respect to $\{\omega_s\}$ on a certain neighbourhood of x. In this case we set

$$b(s, w^1, \ldots, w^n) = \tilde{w}^n(s, w^1, \ldots, w^n) - w^n$$

getting $\phi_x(s, w^1, \ldots, w^n) = (s, w^1, \ldots, w^{n-1}, \tilde{w}^n)$.

Let now $x \in S(F_S)$. On W'_x set $\xi = \sum_\beta \xi^\beta \partial/\partial z^\beta$ and $\omega_s = \sum_\lambda \omega_\lambda d\bar{z}^\lambda \otimes \xi$ where $\omega_\lambda = \omega_\lambda(s, z)$. Let us consider on $W'_x \times \mathbb{C}$ the non-singular vector field $\tilde{\xi} = \xi + \partial/\partial t$. The map $\tilde{\varphi} : W'_x \times \mathbb{C} \to W'_x \times \mathbb{C}$ given by $\tilde{\varphi}(s, z, t) = (s, \varphi(t, z), t)$ is a holomorphic diffeomorphism fulfiling $\tilde{\varphi}_*(\partial/\partial t) = \tilde{\xi}$. The family $\{\tilde{\omega}_s\}$ of real analytic vector 1-forms on $W'_x \times \mathbb{C}$ defined as

$$\tilde{\omega}_s = \sum_\lambda \omega_\lambda d\bar{z}^\lambda \otimes \tilde{\xi}$$

is holomorphic with respect to S and integrable. Let $\widetilde{\mathcal{W}'_x}$ and $\mathcal{W}'_x$ stand for $W'_x \times \mathbb{C}$ and W'_x endowed with the complex structure defined by $\{\tilde{\omega}_s\}$ and $\{\omega_s\}$

respectively. The canonical projection $\pi : \widetilde{\mathcal{W}'_x} \to \widetilde{\mathcal{W}'_x}$ is holomorphic. As in the non-singular case one can see that, if we consider on $W'_x \times \mathbb{C}$ the only complex structure for which $\tilde{\varphi} : W'_x \times \mathbb{C} \to \widetilde{\mathcal{W}'_x}$ is holomorphic, then there is a real analytic function $z^{n+1} = z^{n+1}(s, z^1, \dots, z^n, t)$ holomorphic with respect to S such that $(s, z^1, \dots, z^n, z^{n+1})$ is a system of holomorphic local coordinates. Let $b = b(s, z^1, \dots, z^n)$ be the function defined by the condition

$$z^{n+1}(s, z^1, \dots, z^n, b(s, z^1, \dots, z^n)) = 0$$

and set $\psi(s, z^1, \dots, z^n) = (s, z^1, \dots, z^n, b(s, z^1, \dots, z^n))$. Then the composition $\phi_x = \pi \circ \tilde{\varphi} \circ \psi$ fulfils the required conditions. ∎

The above propositions illustrate the way in what locally trivial unfoldings of F can be thought as a certain type of deformations of the complex structure of M. Namely those parametrized by holomorphic families of elements of J_F. Kuranishi's theorem (cf. [K]) states the existence of a versal space for the deformations of a complex structure on a compact manifold. The proof of this theorem given by Douady in [D] can be adapted here to obtain the following

2.5. Theorem. *Let F be a foliation by curves on a complex compact manifold M. There is a germ of analytic space $(K_{tr}, 0)$ parametrizing a versal locally trivial unfolding $\mathcal{F}$ of F.*

2.6. Remarks. (i) As in the proof of Kuranishi's theorem one can see that there is an open neighbourhood V of 0 in $H^1(M, \mathcal{O}(L))$ and an analytic map $\Phi : V \to H^2(M, \mathcal{O}(L))$ such that $(K_{tr}, 0)$ is identified with the germ at 0 of $\Phi^{-1}(0)$. Moreover the jet of order two of Φ at 0 is the quadratic map $v \to [v, v]$. This implies in particular that K_{tr} is smooth in the case $H^2(M, \mathcal{O}(L)) = 0$. Here the braket $[\ ,\ ]$ refers to the structure of graded Lie algebra on $H^*(M, \mathcal{O}(L))$ induced by the inclusion of sheaves $\mathcal{O}(L) \hookrightarrow \Theta_M$.

(ii) Through the proof of the theorem one sees that if $H^0(M, \mathcal{O}(L)) = 0$ then the unfolding $\mathcal{F}$ is universal.

Sketch of the proof. Let us fix a real analytic Hermitian metric on L and a real analytic Riemannian metric on M. From these metrics one constructs an Hermitian product on $A^{0,*}(L)$ in a standard way. The differential complex

$$A^{0,0}(L) \xrightarrow{\bar{\partial}} A^{0,1}(L) \xrightarrow{\bar{\partial}} A^{0,2}(L) \longrightarrow \dots \longrightarrow A^{0,n}(L)$$

is elliptic and if we denote by ϑ the adjoint operator of $\bar{\partial}$ with respect to the above Hermitian product then the Laplacian $\Delta = \bar{\partial}\vartheta + \vartheta\bar{\partial}$ is a real analytic elliptic operator. The space

$$N = \{\omega \in A^{0,1}(L) \mid \vartheta(\bar{\partial}\omega - \frac{1}{2}[\omega, \omega]) + \bar{\partial}\vartheta\omega = 0\}$$

is a finite dimensional submanifold whose tangent space at 0 is the space $\mathbb{H}^1 \cong H^1(M, \mathcal{O}(L))$ of harmonic elements of $A^{0,1}(L)$. The elements of N are real analytic because they are solutions of an elliptic equation with real analytic coefficients.

Let $(K_{tr}, 0)$ be the germ at 0 of the intersection $K_{tr} = N \cap J_F$. The analytic space K_{tr} can also be defined as (cf. [K, p. 83])

$$K_{tr} = \{\omega \in N \mid H[\omega, \omega] = 0\}$$

where H denotes the orthogonal projection of $A^{0,2}(L)$ onto the space $\mathbb{H}^2$ of harmonic elements. A neighbourhood of 0 in K_{tr} can be thought as a holomorphic family of real analytic elements of J_F. Proposition 2.4 implies that this family defines a locally trivial unfolding $\mathcal{F}$ of F parametrized by the space K_{tr} itself.

Let $\tilde{g}_{\mathcal{F}}$ denote the exponential map for the unfolding $\mathcal{F}$ constructed in Proposition 1.8. Given $\eta \in A^{0,0}(L)$ close to zero then $h_\eta = \tilde{g}_{\mathcal{F}} \circ \eta$ is a diffeomorphism of M tangent to F. By virtue of Proposition 2.1 there is a map

$$\rho : A^{0,0}(L) \times K_{tr} \longrightarrow J_F$$

defined in a neighbourhood of $(0,0)$ such that $\rho(\eta, \omega) = \omega \circ h_\eta$. One can adapt here Douady's proof of Kuranishi's theorem to see that ρ is an isomorphism when restricted to $E \times K_{tr}$ where E is a subspace of $A^{0,0}(L)$ complementary to $\ker\{\bar{\partial} : A^{0,0}(L) \to A^{0,1}(L)\} \cong H^0(M, \mathcal{O}(L))$. From this fact and Proposition 2.3 it follows the versality of the unfolding $\mathcal{F}$. ∎

3. Equisingular unfoldings.

In this paragraph we assume that M has dimension 2 and that the foliation by curves F is saturated. In this case the singular locus $S(F)$ is a finite set $\{q_1, \dots, q_k\} \subset M$. We recall that F is called *reduced* at a singular point q_i if there are local coordinates w^1, w^2 centered at q_i such that F is defined in a neighbourhood of q_i by a vector field with a 1-jet of the form

$$w^1 \frac{\partial}{\partial w^1} + \lambda w^2 \frac{\partial}{\partial w^2}$$

where $\lambda \in \mathbb{C}$ is not a strictly positive rational number.

There is a procedure of reduction of the singular foliation F similar to the reduction of singular plane curves (cf. [S]). More precisely there exist a compact manifold $\widetilde{M}$, a divisor $D \subset \widetilde{M}$, a saturated foliation $\widetilde{F}$ on $\widetilde{M}$ and a holomorphic map $\varpi : \widetilde{M} \to M$ with the following properties: (i) ϖ maps $\widetilde{M} - D$ biholomorphically onto $M - \{q_1, \dots, q_k\}$ identifying $\widetilde{F}|\widetilde{M} - D$ with $F|M - \{q_1, \dots, q_k\}$, (ii) the singularities of $\widetilde{F}$ are reduced and (iii) if

$x \in D - S(\widetilde{F})$ then the leaf of $\widetilde{F}$ through x is transverse to D or contained in D if x is a regular point of D and is a local irreducible component of D if x is a singular point of D. Furtheremore the 4-tuple $(\widetilde{F}, \widetilde{M}, D, \varpi)$ which is called the ***reduction*** of F is unique up to isomorphism. We denote by D_0 the union of all the irreducible components of D which are *dicritical*, i.e. those components which are generically transverse to $\widetilde{F}$.

An unfolding $(\mathcal{F}, \mathcal{M}, \pi, \Omega, \iota)$ of F parametrized by an open neighbourhood Ω of 0 in $\mathbb{C}^m$ (cf. Remark 1.7) is said to be *equisingular* if there exist a complex manifold $\widetilde{\mathcal{M}}$, a hypersurface $\mathcal{D} \subset \widetilde{\mathcal{M}}$, a saturated foliation $\widetilde{\mathcal{F}}$ on $\widetilde{\mathcal{M}}$ and a holomorphic map $\Pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ such that: (i) Π maps $\widetilde{\mathcal{M}} - \mathcal{D}$ biholomorphically onto $\mathcal{M} - S(\mathcal{F})$ identifying $\widetilde{\mathcal{F}}|\widetilde{\mathcal{M}} - \mathcal{D}$ with $\mathcal{F}|\mathcal{M} - S(\mathcal{F})$, (ii) $\widetilde{\mathcal{F}}$ is transverse to the fibres of $\pi \circ \Pi$ outside its singular locus and (iii) the restriction of $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{M}}, \mathcal{D}, \Pi)$ to $\pi^{-1}(0) = \iota(M)$ is the reduction of F_0. Then the 4-tuple $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{M}}, \mathcal{D}, \Pi)$ is unique up to isomorphism and is called the reduction of $\mathcal{F}$. Because of the unicity of the reduction of a foliation there is a biholomorphism Υ from $\widetilde{M}$ onto $(\pi \circ \Pi)^{-1}(0)$ making commutative the diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\Upsilon} & \widetilde{\mathcal{M}} \\ \varpi \downarrow & & \downarrow \Pi \\ M & \xrightarrow{\iota} & \mathcal{M} \end{array}$$

and such that $\Upsilon^* \widetilde{\mathcal{F}} = \widetilde{F}$. We can thus think of the 5-tuple $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{M}}, \widetilde{\pi}, \Omega, \Upsilon)$ as an unfolding of the reduction $\widetilde{F}$ of F. Here $\widetilde{\pi} = \pi \circ \Pi$. As above $\mathcal{D}_0$ will denote the union of all the dicritical irreducible components of $\mathcal{D}$.

The first author proved in [M] that any unfolding of a germ of foliation at a reduced singularity is trivial. From this fact he deduces that for any equisingular unfolding $\mathcal{F}$ of F the corresponding reduction $\widetilde{\mathcal{F}}$ is a locally trivial unfolding of $\widetilde{F}$. Furthermore, if $\widetilde{\phi}_i : \widetilde{\mathcal{M}} \to \Omega \times \widetilde{M}$ are the local charts trivializing $\widetilde{\mathcal{F}}$ then the compositions $\widetilde{\phi}_j \circ \widetilde{\phi}_i^{-1}$ are the identity on $\Omega \times D_0$. This implies using Proposition 1.5 that $\widetilde{\mathcal{F}}^{\mathcal{D}_0}$ is in fact a locally trivial unfolding of $\widetilde{F}^{D_0}$. It is also shown in [M, Lemme 1.3.1] that, conversely, given a locally trivial unfolding $\widetilde{\mathcal{F}}'$ of $\widetilde{F}^{D_0}$ parametrized by Ω there is an equisingular unfolding $\mathcal{F}$ of F also parametrized by Ω such that its reduction $\widetilde{\mathcal{F}}$ is the saturation ${}^s(\widetilde{\mathcal{F}}')$ of $\widetilde{\mathcal{F}}'$. These facts suggest the following

3.1. Definition. *An equisingular unfolding of F parametrized by a germ of analytic space $(S, 0)$ is a locally trivial unfolding $(\widetilde{\mathcal{F}}^{\mathcal{D}_0}, \widetilde{\mathcal{M}}, \widetilde{\pi}, S, \Upsilon)$ of $\widetilde{F}^{D_0}$ where $(\widetilde{F}, \widetilde{M}, D, \varpi)$ is the reduction of F and $\widetilde{F}^{D_0}$ has the meaning of example 1.1.*

Because of the above remarks in the case S is smooth the above definition of equisingular unfolding is equivalent to the first one. The notion of versal equisingular unfolding has an evident meaning and its existence is a corollary of Theorem 2.5.

3.2. Theorem. *Let F be a foliation by curves with isolated singularities on a compact surface M. There is a germ of analytic space $(K_e, 0)$ which parametrizes a versal equisingular unfolding of F.*

3.3. Remarks. (i) The space K_e is nothing but the parameter space $\widetilde{K}_{tr}$ of the versal locally trivial unfolding $(\widetilde{\mathcal{F}}^{\mathcal{D}_0}, \widetilde{\mathcal{M}}, \widetilde{\pi}, \widetilde{K}_{tr}, \Upsilon)$ of $\widetilde{F}^{D_0}$. So the tangent space of K_e at 0 is naturally identified with $H^1(\widetilde{M}, \mathcal{O}(\widetilde{L}))$ where $\widetilde{L}$ is the line bundle associated to the foliation by curves $\widetilde{F}^{D_0}$. Furthermore if $H^2(\widetilde{M}, \mathcal{O}(\widetilde{L})) = 0$ then K_e is smooth.

(ii) In the case K_e is smooth there is an equisingular unfolding in the first sense $(\mathcal{F}, \mathcal{M}, \pi, K_e, \iota)$ of F whose reduction is just ${}^s(\widetilde{\mathcal{F}}^{\mathcal{D}_0})$. Then any other equisingular unfolding of F parametrized by a smooth space is a pull-back of $\mathcal{F}$.

For a given germ of singular foliation by curves on $(\mathbb{C}^2, 0)$ there is a versal equisingular unfolding of it parametrized by a (germ of) smooth space. This local versal space is constructed by the first author in [M]. In our situation and using the above notation the parameter space K_i^{loc} of the versal equisingular unfolding of the germ of F at a singular point q_i turns out to be isomorphic to $H^1(D_i, \mathcal{O}(\widetilde{L}))$ where $D_i = \varpi^{-1}(q_i)$. The versality property of the local versal spaces K_i^{loc} induces a "localization" map

$$\chi : K_e \longrightarrow K_1^{\text{loc}} \times \ldots \times K_k^{\text{loc}}. \tag{16}$$

In certain cases the map χ is in fact a biholomorphism. More precisely

3.4. Theorem. *Let F be a foliation by curves with isolated singularities on a compact surface M defined by a pair (L, κ). Assume that $H^1(M, \mathcal{O}(L)) = 0$. Then the differential map $d_0\chi$ of χ at 0 is an isomorphism. Moreover if $H^2(M, \mathcal{O}(L))$ also vanishes then K_e is smooth and χ is an isomorphism.*

Proof. For each $i = 1, \ldots, k$ let B_i and B_i' be open neighbourhoods of q_i identified by a suitable local chart to the polydiscs $\Delta(1)$ and $\Delta(1/2)$ of $\mathbb{C}^2$ of polyradius 1 and 1/2 respectively. Assume that the open sets B_i are disjoint. Set $U = \cup B_i$, $V = M - \cup \overline{B}_i'$ and define $\widetilde{B}_i = \varpi^{-1}(B_i)$, $\widetilde{U} = \varpi^{-1}(U)$ and $\widetilde{V} = \varpi^{-1}(V)$. Notice that any section of $\widetilde{L}$ on $\widetilde{U} \cap \widetilde{V}$ can be extended because of Hartogs' theorem to a section on $\widetilde{U}$. Using this fact and the identifications induced by ϖ the Mayer-Vietoris sequence gives

$$0 \to H^1(\widetilde{M}, \mathcal{O}(\widetilde{L})) \xrightarrow{\alpha^1} H^1(\widetilde{U}, \mathcal{O}(\widetilde{L})) \oplus H^1(V, \mathcal{O}(L)) \xrightarrow{\beta^1} H^1(U \cap V, \mathcal{O}(L)) \to$$
$$\to H^2(\widetilde{M}, \mathcal{O}(\widetilde{L})) \xrightarrow{\alpha^2} H^2(\widetilde{U}, \mathcal{O}(\widetilde{L})) \oplus H^2(V, \mathcal{O}(L)) \xrightarrow{\beta^2} H^2(U \cap V, \mathcal{O}(L)) \to 0.$$

The intersection $U \cap V$ is the union of two Stein open subsets. Thus $H^2(U \cap V, \mathcal{O}(L)) = 0$. A theorem by Andreotti and Grauert [A,G] states that $H^*(\widetilde{U}, \mathcal{O}(\widetilde{L})) = \bigoplus_{i=1}^k H^*(\widetilde{B}_i, \mathcal{O}(\widetilde{L}))$ is isomorphic to $\bigoplus_{i=1}^k H^*(D_i, \mathcal{O}(\widetilde{L}))$ and also implies that the restriction of β^* to $H^*(\widetilde{U}, \mathcal{O}(\widetilde{L}))$ is the zero map. It is also proved in [M] that $H^2(D_i, \mathcal{O}(\widetilde{L})) = 0$. Using now the Mayer-Vietoris sequence which computes $H^*(M, \mathcal{O}(L))$ by means of the decomposition $M = U \cup V$ one can see that $H^1(M, \mathcal{O}(L)) = 0$ implies that the restriction of β^1 to $H^1(V, \mathcal{O}(L))$ is injective and therefore α^1 maps $H^1(\widetilde{M}, \mathcal{O}(\widetilde{L}))$ isomorphically onto $\bigoplus_{i=1}^k H^1(D_i, \mathcal{O}(\widetilde{L}))$ but this map is just the differential $d_0\chi$ of χ at 0. The same exact sequence also shows that in the case $H^2(M, \mathcal{O}(L)) = 0$ then the restriction of β^1 to $H^1(V, \mathcal{O}(L))$ is surjective and $H^2(V, \mathcal{O}(L)) = 0$. This implies $H^2(\widetilde{M}, \mathcal{O}(\widetilde{L})) = 0$ concluding the proof. ∎

Let us apply the above result to the case of a foliation by curves on $\mathbb{P}^2$. Recall that line bundles L_d on $\mathbb{P}^2$ are classified by its Chern class $d \in \mathbb{Z} \cong H^2(\mathbb{P}^2, \mathbb{Z})$ and that there are non identically zero morphisms $L_d \to T\mathbb{P}^2$ if and only if $d \leq 1$ (cf. [G-M,O-B]). It is also known from Serre's computations that $H^1(\mathbb{P}^2, \mathcal{O}(L)) = 0$ for any $d \in \mathbb{Z}$ and that $H^2(\mathbb{P}^2, \mathcal{O}(L)) = 0$ for $d > -3$. So we obtain

3.5. Corollary. *Let $F = (L_d, \kappa)$ be a foliation by curves on $\mathbb{P}^2$ with isolated singularities $q_1, \dots, q_k$. Let K_e (resp. K_i^{loc}) denote the parameter space of the versal equisingular unfolding of F (resp. of the germ of F at q_i). Then the tangent map at 0 of the localization morphism $\chi : K_e \to K_1^{\mathrm{loc}} \times \cdots \times K_k^{\mathrm{loc}}$ is an isomorphism. The map χ itself is an isomorphism if $d = 1, 0, -1, -2$.*

3.6. Remarks. (i) As it is pointed out in [G-M,O-B] the foliations by curves on $\mathbb{P}^2$ having an associated line bundle L_d with Chern class $d = 1, 0, -1, -2$ are those obtained respectively by projectivization of the vector fields

$$X = X_1 \frac{\partial}{\partial z^1} + X_2 \frac{\partial}{\partial z^2} + X_3 \frac{\partial}{\partial z^3}$$

on $\mathbb{C}^3$ such that X_1, X_2 and X_3 are homogeneous polinomials of degree $0, 1, 2$ or 3.

(ii) The space K_e is not smooth in general as it is shown by the examples studied by I. Luengo in [L].

Finally we come back from global to local foliations and using the above methods we obtain

3.7. Theorem. *Let ξ be a polynomial vector field on $\mathbb{C}^2$ having an isolated singularity at the origin. Let F_ξ denote the germ of foliation by curves at 0 defined by ξ. Then any equisingular unfolding of F_ξ is algebraic. More precisely the versal equisingular unfolding of F_ξ is given by an integrable holomorphic differential form*

$$\omega = a(z,t)dz_1 + b(z,t)dz_2 + \sum_{i=1}^{p} c_i(z,t)dt_i$$

on an open subset $\mathbb{C}^2 \times U$ of $\mathbb{C}^2 \times \mathbb{C}^p$ where p is the dimension of the versal space and a, b, c_i are functions which are polynomials with respect to the coordinates $z = (z_1, z_2)$ of $\mathbb{C}^2$.

Proof. The foliation F_ξ extends to a foliation by curves F on $\mathbb{P}^2$ having only isolated singularities at $q_1 = 0, q_2, \ldots, q_k$. Let $(\widetilde{F}, \widetilde{M}, D, \varpi)$ denote the reduction of F. Let us chose a straight line C in $\mathbb{P}^2$ which do not meet any singular point of F. There is a positive number $m \in \mathbb{N}^*$ such that the line bundle associated to the foliation $F^{m \cdot C}$ (cf. Example 1.1) is L_0, i.e. its Chern class is zero. Set $\widetilde{C} = \varpi^{-1}(C)$. Then the argunent used to prove the above theorem can be repeated to show that the parameter space K' of the versal locally trivial unfolding $(\widetilde{\mathcal{F}}^{m \cdot \widetilde{C}}, \widetilde{\mathcal{M}}, \widetilde{\pi}, K', \Upsilon)$ of $\widetilde{\mathcal{F}}^{m \cdot \widetilde{C}}$ is smooth and can be identified with the product $K_1^{\mathrm{loc}} \times \cdots \times K_k^{\mathrm{loc}}$. Moreover one can prove as in [M, Lemme 1.3.1] that there is an equisingular unfolding (in the first sense) $(\mathcal{F}, \mathcal{M}, \pi, K', \iota)$ of F whose reduction is just the saturation of $\widetilde{\mathcal{F}}^{m \cdot \widetilde{C}}$. Then $\mathcal{F}$ is algebraic and the germ at zero of the restriction of $\mathcal{F}$ to the subspace $K_1^{\mathrm{loc}} \times \{0\} \times \cdots \times \{0\}$ of K' is the versal equisingular unfolding of F_ξ. ∎

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Astérisque, tome 222 (1994), p. 303-326
<http://www.numdam.org/item?id=AST_1994__222__303_0>

QUASI-REGULARITY PROPERTY FOR UNFOLDINGS OF HYPERBOLIC POLYCYCLES

M. El Morsalani, A. Mourtada and R. Roussarie

1. INTRODUCTION.

Let X be a real analytical vector field on $\mathbf{R}^2$. A polycycle Γ of X is an immersion of the circle, union of trajectories (Singular points and separatrices whose α and ω limits are contained in this set of singular points). Moreover one supposes that Γ is oriented by the flow of X and that a return map $P(x)$ along Γ is defined on some interval σ with one end point on Γ : σ is parametrized by analytical variable $x \in [0, x_1]$, $\{x = 0\} = \sigma \cap \Gamma = \{q\}$ and $P(x) : [0, x_0] \longrightarrow [0, x_1]$ for some $x_0 \in]0, x_1[$

We say that Γ is an hyperbolic polycycle if all the singular points in γ are hyperbolic saddle points. Let $\{p_1, \dots , p_k\}$ the set of these singular points listing in the way they are encountered when we describe Γ starting at q. We define the hyperbolicity ratio of p_i, $i = 1, \dots , k$ to be $r_i = \dfrac{\mu_i'}{\mu_i''}$ where $-\mu_i'$, μ_i'' are the eigenvalues at p_i $(\mu_i', \mu_i'' > 0)$.

The Poincaré map $P(x)$ is analytic for $x > 0$, and extends continuously at 0 by $P(0) = 0$.

In 1985, Yu. Ilyashenko [I1] introduced a notion (the almost-regularity) similar to the following one up to a composition by the logarithm :

Definition. *Let $g(x) : [0, x_0] \longrightarrow \mathbf{R}$ a function, analytic for $x > 0$, and continuous at $x = 0$. One says that g is quasi-regular if:*

$\mathbf{Q}R_1$) *$g(x)$ has a formal expansion of Dulac type. This means that there exists a formal series:*

$$\hat{g}(x) = \sum_{i=0}^{\infty} x^{\lambda_i} P_i(lnx)$$

where λ_i is a strictly increasing sequence of positive real numbers $0 < \lambda_0 < \lambda_1 < \dots$ tending to infinity and for each i, P_i is a polynomial, and $\hat{g}$ is a formal expansion of $g(x)$ in the following sense:

$$\forall n \geq 0 \qquad g(x) - \sum_{i=0}^{n} x^{\lambda_i} P_i(lnx) = o(x^{\lambda_n}).$$

QR_2*) Let $G(\xi) = g(e^{-\xi})$ for $\xi \in [\xi_0 = -logx_0, \infty[$.*
*Then G has a bounded holomorphic extension in a domain $\Omega(C) \subset$ **C** where $\Omega(C) = \{\zeta = \xi + i\eta \mid \xi^4 \geq C(1+\eta^2)\}$ for some $C > 0$.*

In the same paper [I1], Ilyashenko proved that the shift map $\delta(x) = P(x) - x$ is quasi-regular. The property **Q**R_1 was already established by Dulac in [D].

As a consequence of the Phragmen-Lindelöf theorem (see [C]) a flat quasi-regular function ($g(x) = o(x^n), \forall n$) is necessarily equal to zero, and it follows from this that Γ cannot be accumulated by limit cycles of X (a limit cycle of X is an isolated periodic orbit).

This result was a first step in the solution of the "Dulac problem", for which one needs to look not only at hyperbolic polycycles but more generally at all elementary polycycles. As it is well known, this general solution ([EMMR], [E1], [E2], [I2], [I3]) involved more elaborated technics, and we limit ourselves to the hyperbolic polycycles in this paper.

Here we want to consider the unfoldings (X_λ, Γ) of a hyperbolic polycycle Γ, germs of finite parameter family (X_λ), with $X_0 = X$ defined by a representative family on $V \times W$ where V is a neighborhood of Γ and W neighborhood of 0 in the parameter space.

As it was shown in [R], it is useful to obtain quasi-regularity property for 1-parameter unfoldings, in order to study finite cyclicity for general unfoldings of hyperbolic polycycles. In the present paper, we extend to any 1-parameter unfoldings a result of [R], proved there for hyperbolic loops (singular cycles with just 1 singular point):

Theorem 1. *Let (X_ϵ, Γ) a 1-parameter analytic unfolding of an hyperbolic polycycle Γ for X_0 with k vertices. Let $P(x, \epsilon)$ the unfolding of the return map where x is some analytic parameter defined as above for X_0. Let $\delta(x, \epsilon) =$*

$P(x,\epsilon)-x$. Let $\widehat{\delta}(x,\epsilon)=\sum_{i=0}^{\infty}\delta_i(x)\epsilon^i$ (i.e: $\delta_i(x)=\frac{1}{i!}\frac{\partial^i\delta(x,0)}{\partial\epsilon^i}$) the formal expansion of δ in ϵ.

Then, there exists some $R>0$ (depending on $r_1(0),\dots,r_k(0)$) such that for $\forall i\in\mathbf{N}$, $x^{iR}\delta_i(x)$ is quasi-regular.

Remarks.

1) Given an unfolding X_ϵ and a transversal $\sigma\simeq[0,x_1]$ chosen as above for X_0, the return map $P(x,\lambda)$ is defined in a domain $D=\cup_{\epsilon\in W}[\alpha(\epsilon),x_1]$ where $\alpha(\epsilon)$ is a continuous function, such that $\alpha(0)=0$. So, given any $x\in]0,x_1]$, the return map $P(x,\lambda)$ is defined for x if $|\epsilon|$ is small enough. From this it follows that the functions $\delta_i(x)$ in the above theorem, are defined for $\forall x\in]0,x_1]$.

2) Theorem 1 extends Ilyashenko's one which corresponds to the quasi-regularity of $\delta_0(x)$.

The generalization brought by theorem 1 is useful to study unfolding of identical polycycles, i.e polycycles such that $\delta(x)=P(x)-x\equiv 0$. Suppose for instance that $\lambda=\epsilon\in\mathbf{R}$. Then, if (Γ,X_0) is an identical polycycle, one can write:

$$\delta(x,\epsilon)=\epsilon^n\bar{\delta}(x,\epsilon)$$

for some $n\geq 1$, with a function $\bar{\delta}(x,\epsilon)$ such that $\bar{\delta}(x,0)\not\equiv 0$. Then from theorem 1, we have that $\bar{\delta}(x,0)$ has a non-trivial Dulac expansion.

So, the equation for limit cycles $\{\delta(x,\epsilon)=0\}$, which is equivalent to $\{\bar{\delta}(x,\epsilon)=0\}$, has the same properties that in the non-identical case ($\delta(x,0)\not\equiv 0$).

This allows us to develop for some identical unfoldings a proof similar to the one for unfolding of non-identical polycycles. In [R] these ideas were applied to prove the finite cyclicity of any analytic unfolding of loops (Singular cycles with just one singular hyperbolic point). Here we extend it to some polycycles with 2 singular points:

Theorem 2. *Let (X_λ,Γ) an analytic unfolding of an hyperbolic 2-polycycle Γ (a polycycle with 2 singular points p_1,p_2). Let $r_1(\lambda)$, $r_2(\lambda)$ the λ-depending hyperbolicity ratio at p_1,p_2. Suppose that:*

1) For all λ, $r_1(\lambda)r_2(\lambda) \equiv 1$
2) at least one of the two saddle connexions remains unbroken (for all λ).
Then (X_λ, Γ) has a finite cyclicity.

Remark.

A part the conditions 1,2, no other conditions are imposed on (X_λ, Γ) and the polycycle Γ may be identical. The non-identical case was already worked out in a previous paper [El.M]. Moreover, if $r_1(0) = r_2(0)^{-1} \notin \mathbf{Q}$, a result of finite cyclicity was obtained in [M], without the conditions 1,2.

The conditions 1,2 in the theorem 2 may seem very restrictive. Nethertheless the theorem has the following natural application to polynomial vector fields. Let P_{2p} be the family of all polynomial vector fields of some even degree $2p$, $p \geq 1$. It is easy to extend P_{2p} in an analytic family of vector fields on the sphere (X_λ). This family (X_λ) is equivalent to P_{2p} on the interior of a 2-disk D^2, whose boundary ∂D^2 corresponds to the "circle at infinity γ_∞". Singular points of (X_λ) appears at infinity in pairs of opposite points (p, q) and a consequence of the even degree is that the tangential eigenvalues at p, q are opposite and the same for the two radial eigenvalues. It follows that the product of the ratios of hyperbolicity at p and q is one. Then if for some value λ_0 (that we can suppose equal to 0), $X_{\lambda_0} = X_0$ has just a pair of singular points p, q on γ_∞ and if there exists a connection Γ_1 of p and q in int(D^2), one can apply theorem 2 to the unfolding (X_λ, Γ) where Γ is one of the 2 polycycles containing Γ_1 and an arc Γ_2 of γ_∞ joigning p and q; we have $r_1(\lambda)r_2(\lambda) \equiv 1$ as noted above and the connection Γ_2 at infinity remains unbroken. This applies to the quadratic family $\mathcal{P}_2$ and allows to prove the finite cyclicity of some of the 121 possible cases of periodic limit sets listed for this family in [DRR] (cases labelled: H_1^1, H_2^1 in this article).

In the first paragraph, we prove the theorem 1. Of course, we hope that the quasi-regularity property proved here will have a more general application that the one given in theorem 2 and proved below in the second paragraph. In fact the proof uses the existence of a well ordered expansion for $\delta(x, \lambda)$ at any order of differentiability. This expansion was established for unfoldings like in theorem 2 in [El.M] and we recall it bellow. In this paper it was used to prove the finite cyclicity in the non-identical case. Here, we use it to reduce in some sense the general case to the non-identical case, by the method already described in the loop case in [R]. This is made in the second paragraph.

Firstly a natural ideal in the space of parameter functions germs, the coefficient Ideal $\mathcal{J}$ is associated to any unfolding of identical graphic . If $\{\phi_1, \dots, \phi_l\}$ is a system of generators for $\mathcal{J}$, and, in our case, when it exists a well- ordered expansion for $\delta(x, \lambda)$, one can divide $\delta(x, \lambda)$ in the ideal:

$$\delta(x, \lambda) = \sum \phi_i(\lambda)\delta_i(x, \lambda)$$

with functions $\delta_i(x, \lambda)$ having also a well-ordered expansion. The theorem 1 is then used to prove that "for most of indices i", $\delta_i(x, 0)$ is quasi-regular and then has a non-trivial Dulac expansion; and we can apply, as in [R] for the loop case, a derivation-division algorithm similar to the one used for the non-identical case. Here we will use the precise procedure developed in [El.M] for the non identical case.

2. QUASI-REGULARITY PROPERTY

2.1 Reduction to the quasi-regularity property for saddle transitions.— We recall here a definition used in [I1]:

DEFINITION 1 [I1].— *A domain L of $\mathbf{C}$ is said to be of class $\mathcal{I}$ if it contains a domain $\Omega(C)$ of the form*

$$\Omega(C) = \{\zeta = \xi + \eta; \xi^4 \geq C(1 + \eta^2)\}.$$

for some $C > 0$.

In the neighbourhood of each saddle point p_i, choose as in [I1] a chart analytical in (x_i, y_i, ϵ) in which the field X_ϵ takes the form

$$\begin{cases} \dot{x}_i &= x_i \\ \dot{y}_i &= -y_i[r_i(\epsilon) + x_i^{n_i} y_i f_i(x_i, y_i, \epsilon)] \end{cases}$$

where $n_i \in \mathbf{N}$ and $n_i \geq r_i(0)$ and the functions f_i are analytical on $\Delta_i = \{|x_i| \leq 1\} \times \{|y_i| \leq 1\} \times] - \epsilon_0, \epsilon_0[$ and satisfy

$$\sup_{\Delta_i} \mid f_i \mid \leq \inf_{|\epsilon| < |\epsilon_0|} (1, r_i(\epsilon)/2).$$

Denote by $\sigma_i = \{(x_i, y_i); y_i = 1\}$, $\sigma_i^+ = \sigma_i \cap [0, 1] \times \{1\}$, $\tau_i = \{(x_i, y_i); x_i = 1\}$, $\tau_i^+ = \tau_i \cap \{1\} \times [0, 1]$, $D_i(., \epsilon)$ the Dulac map which send σ_i^+ on τ_i^+

$$y_i = D_i(x_i, \epsilon) = D_{i,\epsilon}(x_i)$$

and $G_i(.,\epsilon)$ the analytical map which send τ_i on σ_{i+1} (with cyclic notation)

$$x_{i+1} = G_i(y_i, \epsilon) = G_{i,\epsilon}(y_i).$$

The return map on σ_1^+ is given by

$$P(x_1, \epsilon) = G_{k,\epsilon} \circ D_{k,\epsilon} \circ G_{k-1,\epsilon} \circ D_{k-1,\epsilon} \circ \cdots \circ G_{1,\epsilon} \circ D_{1,\epsilon}.$$

Put $H_1 = D_1$, $H_2 = G_1, \ldots$, $H_{2k-1} = D_k$, $H_{2k} = G_k$ and agree that the composition is made with respect to the first variable

$$P(x_1, \epsilon) = H_{2k} \circ H_{2k-1} \circ \cdots \circ H_1(x_1, \epsilon),$$

for all $n \in \mathbf{N}$, we can write

$$\begin{aligned}\frac{\partial^n P}{\partial \epsilon^n}(x_1, \epsilon) = & \sum_{p_{2k}+q_{2k}=l_{2k}=1}^{n} \ \sum_{p_{2k-1}+q_{2k-1}=l_{2k-1}=1}^{n} \cdots \sum_{l_1=1}^{n} A_{l_1, p_2, q_2, \ldots, p_{2k}, q_{2k}} \times \\ & \times \left(\frac{\partial^{l_{2k}} H_{2k}}{\partial x_1^{p_{2k}} \epsilon^{q_{2k}}} \circ H_{2k-1} \circ \ldots \circ H_1(x_1, \epsilon)\right) \times \\ & \times \left(\frac{\partial^{l_{2k-1}} H_{2k-1}}{\partial x_1^{p_{2k-1}} \epsilon^{q_{2k-1}}} \circ H_{2k-2} \circ \ldots \circ H_1(x_1, \epsilon)\right) \times \ldots \times \left(\frac{\partial^{l_1} H_1}{\partial \epsilon^{l_1}}(x_1, \epsilon)\right)\end{aligned}$$

the coefficients $A_{l_1, p_2, q_2, \ldots, p_{2k}, q_{2k}} \in \mathbf{Z}$.

Using [I1], we see that the maps $x_1 \mapsto H_i \circ H_{i-1} \circ \cdots \circ H_1(x_1, 0)$ are quasi-regular and their continuation to the complex plane, after conjugacy by the map $e^{-\xi}$, send a domain of class $\mathcal{I}$ on a domain of class $\mathcal{I}$. Furthermore, the analytical maps G_i can be naturally continued to complex disks in biholomorphic maps and their partial derivatives of all order $\partial^{l_i} G_i / \partial x_1^{p_i} \epsilon^{q_i}(x_1, 0)$ are quasi-regular. So the Theorem 1 is a consequence of the Lemma 1 below.

2.2 Quasi-regularity for unfolding of hyperbolic saddle transition.— This section is devoted to the proof of the following Lemma

LEMMA 1. *Let P be a saddle point of an analytical planar field X_0 and X_λ an analytical unfolding of X_0 near P (the parameter λ belonging to some neighbourhood $\mathcal{V}^m$ of 0 in $\mathbf{R}^m$). Let $D(.,\lambda)$ the Dulac map defined as above and $r(\lambda)$ the hyperbolicity ratio of the saddle point $P(\lambda)$. Let $R = Max(1, r(0))$, then for all $n \in \mathbf{N}$ and $p + q_1 + q_2 + \cdots + q_m = n$, the map*

$$x \mapsto x^{nR} \frac{\partial^n D}{\partial x^p \lambda_1^{q_1} \cdots \lambda_m^{q_m}}(x, 0)$$

is quasi-regular.

Remark—.

The Lemma for $n = 0$ is proved in [I1]; the proof in the general case is based on this one. The multiplying function x^{nR} is not the best one for some values of $(p, q_1, \dots, q_m)$. But this choice allows an easy estimate of the constant R in Theorem 1: If we put $R_i = \mathrm{Max}(1, r_i(0))$ then in Theorem 1, take $R = R_1 \cdots R_k$. The result of this Lemma and the remark above show that Theorem 1 may be extended to any unfolding X_λ with $m > 1$: the maps $x_1 \mapsto x_1^{nR} \partial^n P / \partial \lambda_1^{q_1} \dots \lambda_m^{q_m}(x_1, 0)$ are quasi-regular.

Proof of Lemma 1.— We use the notations of [I1]. Choose an analytical chart (x, y, λ) so that the field X_λ takes the form

$$\left\{ \begin{array}{ll} \dot{x} & = x \\ \dot{y} & = -y[r(\lambda) + x^n y f(x, y, \lambda)] \end{array} \right. \tag{2.2.1}$$

with $n \geq r_0 = r(0)$ and f analytical on $\Delta = \{|x| \leq 1\} \times \{|y| \leq 1\} \times \mathcal{V}^m$ and satisfying

$$\sup_{\Delta} |f| \leq \inf_{\lambda \in \mathcal{V}} (1, r(\lambda)/2) \tag{2.2.2}$$

the family (X_λ) is induced by the family (X_μ) given by

$$\left\{ \begin{array}{ll} \dot{x} & = x \\ \dot{y} & = -y[r(\mu) + x^n y f(x, y, \lambda)] \end{array} \right. \tag{2.2.3}$$

where $\mu = (\mu_0, \lambda) \in \mathcal{V}^{m+1} \subset \mathbf{R}^{m+1}$ and $r(\mu) = r_0 + \mu_0$. Denote by $\sigma^+ = \{(x, y); y = 1 \text{ and } x \in [0, 1]\}$, $\tau^+ = \{(x, y); x = 1 \text{ and } y \in [0, 1]\}$ and $D(., \mu)$ the Dulac map which sends σ^+ on τ^+

$$y = D(x, \mu).$$

Extend the real field X_μ to a field $\widehat{X}_\mu$ defined on $\mathbf{C}^2$ with local variables (z, w) and complexify the parameter λ to $\widehat{\lambda} \in \mathcal{V}_{\mathbf{C}}^m \subset \mathbf{C}^2$

$$\left\{ \begin{array}{ll} \dot{z} & = z \\ \dot{w} & = -w[r + z^n w f(z, w, \widehat{\lambda})] \end{array} \right. \tag{2.2.4}$$

we keep the parameter μ_0 real for reason given below.

We will say that the complex family (2.2.4) as above, with $\mu_0 \in \mathbf{R}$ belongs to the Siegel domain because the ratio of hyperbolicity remains real. Denote by d_0 and d_1 the punctured disks of coordinate z

$$d_0 = \{(z, w); 0 < |z| \le 1, w = 0\},$$

$$d_1 = \{(z, w); 0 < |z| \le 1, w = 1\},$$

and by $\widehat{d_0}$, $\widehat{d_1}$ their universal covering with base point respectively on (1,0), (1,1) and with coordinate $\zeta = -\mathrm{Ln}z$. Denote also by d_2 the punctured disk of coordinate w

$$d_2 = \{(z, w); 0 < |w| \le 1, z = 1\}$$

and by $\widehat{d_2}$ his universal covering with base point on (1,1) and coordinate $\nu = -\mathrm{Ln}w$. Let us show that for $\widehat{\lambda} \in \mathcal{V}^m_{\mathbf{C}}$, ϵ small enough and $C > 0$ big enough, there exists a map $\widehat{D}$ holomorphic in $(\zeta, \widehat{\lambda}) \in \Omega(C) \times \mathcal{V}^m_{\mathbf{C}}$, analytical in $\mu_0 \in]-\epsilon, \epsilon[$ and with values in $\widehat{d_2}$; furthermore, for $\widehat{\lambda} \in \mathcal{V}^m = \mathcal{V}^m_{\mathbf{C}} \cap \mathbf{R}^m$, the map $\widehat{D}$ is the complex continuation of the map D defined on $\sigma^+ \subset d_1$.

Let $\zeta = \xi + i\eta \in \Omega(C)$ and α^ζ the union of the two segments $[0, \xi], [\xi, \zeta]$ parametrized by the arc-length s (see fig. 2a): $s(0) = 0$, $s(\xi) = \xi$, $s(\zeta) = \xi + |\eta| = S$. Let γ_0 and γ_1 the curves on fig. 2b defined by

$$\gamma_1 = \gamma_{\zeta,1} : \quad [0, S] \mapsto \mathbf{C} \times \{1\}, \quad s \to (\mathrm{Exp}(-\alpha^\zeta(s)), 1),$$

$$\gamma_0 = \gamma_{\zeta,0} : \quad [0, S] \mapsto \mathbf{C} \times \{0\}, \quad s \to (\mathrm{Exp}(\alpha^\zeta(s) - \zeta), 0),$$

and $u = r(\mu)\mathrm{Ln}z + \mathrm{Ln}w$ the first integral of the linear field associated to the field $\widehat{X}_\mu$. The formula

$$\frac{d}{dt}(|w|^2) = -2|w|^2[r(\mu) + \mathcal{R}e(z^n w f(z, w, \widehat{\lambda}))]$$

and the hypothesis (2.2.2) show that through each point $p = (z, w)$ with $z \neq 0$ and $|w| \le 1$ passes a curve, solution of the system (2.2.4), which cover the segment $[z, z/|z|]$ of the curve γ_0 under the projection $\pi_z : (z, w) \to z$ and which is entirely contained in the polydisk $P = \{|z| \le 1, |w| \le 1\}$. Let us show now that if $p = (z, 1) = (\mathrm{Exp}(-\zeta), 1)$, then this curve can be extented to a curve $\widehat{\gamma}_\zeta$ which cover the curve $\gamma_{\zeta,0}$ under the projection π_z and is contained in the intersection of P with the surface φ_p (the complex solution of the system (2.2.4) passing through p). To prove this point, an estimate of u along this curve is usefull. Parametrize the arc of $\widehat{\gamma}_\zeta$ defined above by $s \in [0, \xi]$; on the end of this arc, we have $|w| < 1$. Hence, suppose that the curve $\widehat{\gamma}_\zeta$ exists for $s \in [0, T]$ with $T > \xi$ and differentiate u along this curve

$$\dot{u} = z^n w f(z, w, \widehat{\lambda});$$

one can compute $|z^n w| \leq |z^{r_0} w| = |z^{-\mu_0}|.|z^r w| = |z^{-\mu_0}|.|e^u|$; along the curve $\widehat{\gamma}_\zeta$, we have $e^{-\xi} \leq |z| \leq 1$, and $|z^{-\mu_0}| \leq A = \text{Sup}(e^{\mu_0 \zeta}, 1)$; therefore, we get $|\dot{u}| \leq A.|e^u|$. Put $v(s) = |u(\widehat{\gamma}_\zeta(s)) - u(p)|$; as along the curve $\widehat{\gamma}_\zeta$ we have $|dt/ds| = 1$, one can easily verify that

$$|dv/ds| \leq A.e^{-r\xi + v(s)} \leq e^{-\frac{r_0}{2}\xi + v(s)}.$$

From now on, the same arguments as in [I1] can be used; so we conclude that for $C > 0$ big enough, we get $v(s) \leq 1$ along the curve $\widehat{\gamma}_\zeta$ for all $\zeta \in \Omega(C)$ and for all $\mu = (\mu_0, \widehat{\lambda}) \in]-\epsilon, \epsilon[\times \mathcal{V}^m_{\mathbf{C}}$. The extension of th curve $\widehat{\gamma}_\zeta$ for $s \in [0, S]$ is done as in [I1] and we put

$$\widehat{D}(\zeta, \mu) = \nu(\widehat{\gamma}_\zeta(S)) = u(\widehat{\gamma}_\zeta(S)) = r(\mu)\zeta + h(\zeta, \mu)$$

with $|h(\zeta, \mu)| = v(S) \leq 1$; the same estimate as above shows that $h(\zeta, \mu) \to 0$ as $\zeta \to \infty$ and $\zeta \in \Omega(C)$ uniformly on $\mu \in]-\epsilon, \epsilon[\times \mathcal{V}^m_{\mathbf{C}}$.

Remark that for $\mu_0 \in \mathbf{C}$, the results above are false in domains of class $\mathcal{I}$, but still valid in domains of the form $\omega(C, C') = \{(\xi, \eta);\quad \xi \geq C(1 + C'\eta^2)^{1/2}\}$. Unfortunately, the Phragmen-Lindelöf theorem does not apply on such domains.

The extension map $\widehat{D}$ is holomorphic in $(\zeta, \widehat{\lambda}) \in \Omega(C) \times \mathcal{V}^m_{\mathbf{C}}$ and analytic in $\mu_0 \in]-\epsilon, \epsilon[$ and we have $D(x, \mu) = e^{-\widehat{D}(-\text{Ln}x, \mu_0, \lambda)}$ for $x \in]0, x_0]$ and $\mu = (\mu_0, \lambda) \in \mathcal{V}^{m+1}_{\mathbf{R}}$. Denote by $F(\zeta, \mu_0, \widehat{\lambda}) = e^{-\widehat{D}(\zeta, \mu_0, \widehat{\lambda})}$; the analytic extension of the partial derivatives $\partial^n D/\partial x^p \mu_0^q \lambda_1^{q_1} \dots \lambda_m^{q_m}$ to the domain $\Omega(C) \times]-\epsilon, \epsilon[\times \mathcal{V}^m_{\mathbf{C}}$ is a function of the form

$$e^{p\zeta} \sum_{l=1}^{p} a_l \frac{\partial^{n-p+l} F}{\partial \zeta^l \mu_0^q \widehat{\lambda}_1^{q_1} \dots \widehat{\lambda}_m^{q_m}}$$

with $a_l \in \mathbf{Z}$. As the function F is bounded on $\Omega(C) \times]-\epsilon, \epsilon[\times \mathcal{V}^m_{\mathbf{C}}$ and holomorphic in $(\zeta, \widehat{\lambda})$, we begin by studying the functions $\partial^n F/\partial \mu_0^n$; but we have $\widehat{D}(\zeta, \mu) = -\text{Ln}w(t(S), p, \mu)$. Then if we put $w_n(t(S), p, \mu) = (\partial^n w/\partial \mu_0^n)(t(S), p, \mu)$, we see that it suffices to study the functions $u_n = w_n/w$. Let us begin by the function u_1: using the fact that $r(\mu) = r_0 + \mu_0$ and the second line of the system (2.2.4), we get

$$\dot{w}_1 = -w_1[r + z^n w f_{11}] - w$$

and then $\dot{u}_1 = u_1 z^n w f_1 - 1$, where f_{11} and f_1 are holomorphic functions in $(z, w, \widehat{\lambda})$ bounded on $\mathcal{L} = \{|z| \leq 1\} \times \{|w| \leq 1\} \times \mathcal{V}^m_{\mathbf{C}}$. Put $v_1 = |u_1|$, then one can easily show that $|dv_1/ds| \leq A_{11} v_1 |e^u| + 1$ where $A_{11} = A.S_1$ and $S_1 = \mathrm{Sup}\{|f_1(z, w, \widehat{\lambda})|, \quad (z, w, \widehat{\lambda}) \in \mathcal{L}\}$; so we get

$$|\frac{dv_1}{ds}| \leq A_1 v_1 e^{-\frac{r_0}{2}\xi} + 1$$

for some constant A_1; this yield after integration between $s = 0$ and $s = \xi + |\eta|$

$$v_1 \leq A_1^{-1} e^{\frac{r_0}{2}\xi} (e^{A_1(\xi+|\eta|)e^{-\frac{r_0}{2}\xi}} - 1)$$

and this show that there exists $B_1 > 0$ such that $|e^{-r_0\zeta} u_1(\zeta, \mu)| \leq B_1$ for all $(\zeta, \mu) \in \Omega(C) \times] - \epsilon, \epsilon[\times \mathcal{V}^m_{\mathbf{C}}$.

Remark that the multiplying function can be replaced by the function ζ^{-1} and this is optimal for the linear part of the field. The same procedure and an induction on n permit us to show that there exist $B_n > 0$ such that $|e^{-nr_0\zeta} u_n(\zeta, \mu)| \leq B_n$ for all $(\zeta, \mu) \in \Omega(C) \times] - \epsilon, \epsilon[\times \mathcal{V}^m_{\mathbf{C}}$.

Now, let $\mathcal{V}^m_{\mathbf{C}} = d_1(0, a_1) \times \cdots \times d_m(0, a_m)$ where $a_i > 0$, $(C_l)_{l \in \mathbf{N}}$ some strictly increasing sequences with $C_0 = C$ and tending to some $C' < \infty$ and $(a_{i,l})_{l \in \mathbf{N}}$ some strictly decreasing sequences with $a_{i,0} = a_i$, tending to some $a'_i > 0$ for all $i = 1, \ldots, m$. The theorem of derivation under the integral sign and the Cauchy's integral formulas show that for all $n \in \mathbf{N}$ and $p + q + q_1 + \cdots + q_m = n$, there exist $B_{p,q,q_1,\ldots,q_m} > 0$ such that

$$|e^{-(p+qr_0)\zeta} \frac{\partial^n F}{\partial \zeta^p \mu_0^q \widehat{\lambda}_1^{q_1} \ldots \widehat{\lambda}_m^{q_m}} (\zeta, \mu)| < B_{p,q,q_1,\ldots,q_m}$$

for all $(\zeta, \mu) \in \Omega(C') \times] - \epsilon, \epsilon[\times d_1(0, a'_1) \times \cdots \times d_m(0, a'_m)$ and this finish the proof of Lemma 1.

3. FINITE CYCLICITY RESULT.

3. .1 The well ordered expansion for the shift map

We consider a real analytic family of vector fields X_λ on the plane. This family depends on a parameter $\lambda \in \mathbf{R}^\Lambda$, for some $\Lambda \in \mathbf{N}$. Suppose that for $\lambda = 0$, X_λ has an identical hyperbolic polycycle Γ_0 with two vertices P_1 and P_2. In order to study the cyclicity of Γ_0 in the family X_λ, we restrict ourselves to a fixed neighbourhood U of Γ_0 in the plane. We choose U as union of three sets A_1, A_2 and A_3 i.e $U = A_1 \cup A_2 \cup A_3$, and we denote W a neighbourhood of $\lambda = 0$ in $\mathbf{R}^\Lambda$. Now X_λ will be represented in $A_1 \times W, A_2 \times W$ and $A_3 \times W$ respectively by X_λ^1, X_λ^2 and X_λ^3 three analytic vector fields depending analytically in $(m, \lambda) \in \mathbf{R}^2 \times R^\Lambda$. The three charts verify the following properties :

i) In A_1 with local coordinates (x_1, y_1), $A_1 = \{(x_1, y_1); |x_1| \leq 2 \text{ and } |y_1| \leq 2\}$ the family X_λ^1 has a unique singular point $P_1(\lambda)$, which is an hyperbolic saddle point situated in the origin of A_1 i.e $P_1(\lambda) \equiv 0$. Also the stable separatrix and the unstable one are respectively the axis oy_1 and ox_1. Finally, the 1– jet of X_λ^1 in 0 is equal to :

$$j^1 X_\lambda^1(0) = x_1 \frac{\partial}{\partial x_1} - r_1(\lambda) y_1 \frac{\partial}{\partial y_1} \tag{3.1.1}$$

this formula defines on W an analytic function $r_1(\lambda)$: the hyperbolicity ratio of the saddle point P_1.

ii) In A_2 with local coordinates (x_2, y_2), $A_2 = \{(x_2, y_2); |x_2| \leq 2 \text{ and } |y_2| \leq 2\}$, the family X_λ^2 has a unique singular point $P_2(\lambda)$ which is an hyperbolic saddle point, situated in the origin i.e $P_2(\lambda) \equiv 0$.The 1– jet of X_λ^2 in 0 is given by :

$$j^1(-X_\lambda^2)(0) = y_2 \frac{\partial}{\partial y_2} - r_2(\lambda) x_2 \frac{\partial}{\partial x_2} \tag{3.1.2}$$

the stable and unstable separatrices of $(-X_\lambda^2)$ at 0 are respectively the axis ox_2 and oy_2; and the hyperbolicity ratio of $(-X_\lambda^2)$ at P_2 is $r_2(\lambda)$.

iii) In A_3 the vector field X_λ^3 has no singularities. Furthermore the points $Q_i(1, 0)$, $s_i(0, 1)$ in the two charts A_i $i = 1, 2$ and the regular segments of Γ_0 joining them are contained in A_3 (figure 3).

The family X_λ verifies the two following conditions :
a) for all λ in W $\quad r_1(\lambda) = r_2(\lambda)$.

b) at least one of the saddle connections remains unbroken for all λ.

Remark.

The condition a) is equivalent to the first condition in Theorem 2.

Now let us define the maps that will permit us to study the cyclicity of Γ_0. Firstly consider:

$$\sigma_i = \{(x_i, y_i) \in A_i; y_i = 1\} \quad \text{and} \quad \tau_i = \{(x_i, y_i) \in A_i; x_i = 1\} \quad i = 1, 2$$

$$\sigma_i^+ = \{(x_i, y_i) \in \sigma_i ; x_i \geq 0\} \quad \text{and} \quad \tau_i^+ = \{(x_i, y_i) \in \tau_i ; y_i \geq 0\} \quad i = 1, 2$$

the segments σ_i, τ_i are parametrized respectively by x_i, y_i $i = 1, 2$ and are transversal to the vector field X_λ for all λ in W.
The flow of X_λ^3, $\lambda \in W$ in A_3 defines two analytic diffeomorphisms, the regular transition maps, $R_{1,\lambda}$ and $R_{2,\lambda}$

$$R_{1,\lambda} : \tau_1^+ \longrightarrow \sigma_2, \quad R_{2,\lambda} : \tau_2^+ \longrightarrow \sigma_1$$

The flow of X_λ^1 in A_1 (resp of $(-X_\lambda^2)$ in A_2) defines the transition map $D_{1,\lambda}$ (resp $D_{2,\lambda}$) called the Dulac map.

$$D_{1,\lambda} : \sigma_1^+ \longrightarrow \tau_1, \quad D_{2,\lambda} : \tau_2^+ \longrightarrow \sigma_2$$

the map $D_{1,\lambda}$ (resp $D_{2,\lambda}$) is analytic for $x_1 > 0$ (resp $y_2 > 0$), but it's extended by continuity in 0: $D_{1,\lambda}(0) = 0$ and $D_{2,\lambda}(0) = 0$ for all λ in W.

Remark.

To define the above maps, we have perhaps to reduce the neighbourhood W to a some smaller one.
Finally the shift map will be defined by :

$$\delta(x, \lambda) = R_{1,\lambda} \circ D_{1,\lambda} \circ R_{2,\lambda}(x) - D_{2,\lambda}(x)$$

where $x = y_2$ is the parametrization of the transversal τ_2.

Proposition 1. *Given K arbitrary integer, there exists a neighbourhood $W_K \subset W$ of 0 in $\mathbf{R}^\Lambda$, analytic functions $\gamma_{ij}^K : W_K \longrightarrow \mathbf{R}$ such that on $[0, x_0] \times W_K$ the map $\delta(x, \lambda)$ has the form :*

$$1)\delta(x, \lambda) = \sum_{ir(0)+j \leq K+1} \gamma_{ij}^K x^{ir(\lambda)+j} + \phi_K(x, \lambda) \qquad \textit{if} \quad r(0) \notin \mathbf{Q}$$

$$2)\delta(x, \lambda) = \sum_{0 \leq j \leq i \leq K+1} \gamma_{ij}^K x^{ir(0)} \omega^j + \phi_K(x, \lambda) \qquad \textit{if} \quad r(0) = \frac{p}{q} \in \mathbf{Q}, p \wedge q = 1$$

where $r(\lambda)$ is the common hyperbolicity's ratio of X_λ^1 at P_1 and $(-X_\lambda^2)$ at P_2. The function ω is defined by: let $\alpha_1(\lambda) = r(0) - r(\lambda)$,

$$\omega(x, \lambda) = \begin{cases} \dfrac{x^{-\alpha_1(\lambda)} - 1}{\alpha_1(\lambda)}, & \textit{for } \alpha_1(\lambda) \neq 0 \\ -\ln x, & \textit{for } \alpha_1(\lambda) = 0. \end{cases}$$

$\phi_K(x, \lambda)$ is a C^k function, k-flat at $x = 0$ for any λ.

In order to prove this proposition we have to extend the field X_λ in the complex domain; besides we'll restrict ourselves to the case $r(0) = 1$ because the other cases are resolved in the same way.

3. .2 The complex continuation of δ

The family $X_\lambda, \lambda \in \mathbf{R}^\Lambda$ has a natural holomorphic extension $\mathbf{X}_{\widehat{\lambda}}$. This extension is obtained by extending the vector fields X^i_λ $i = 1, 2, 3$ to holomorphic ones in domains extending the different charts $A_i \times W$.

In the following we will denote this holomorphic extension by $\mathbf{X}_{\widehat{\lambda}}$, $\widehat{\lambda} \in \mathbf{C}^\Lambda$, we will work with the same notations as in the real domains with caps symbols to subline that we are in the complex ones.

We can suppose, up to a holomorphic conjugacy, that the vector fields $\mathbf{X}_{\widehat{\lambda}}{}^i$ are defined in the charts $\mathbf{A}_i$: polydisks $|\mathbf{x}_i|^2 + |\mathbf{y}_i|^2 \leq 2; (\mathbf{x}_i, \mathbf{y}_i) \in \mathbf{C}^2, i = 1, 2$. The origin in each chart $\mathbf{A}_i$ is the only singular point with hyperbolic 1-jet:

$$j^1\mathbf{X}_{\widehat{\lambda}}{}^1(0) = \mathbf{x}_1\frac{\partial}{\partial \mathbf{x}_1} - (1 - \widehat{\alpha}_1)\mathbf{y}_1\frac{\partial}{\partial \mathbf{y}_1}$$

$$j^1(-\mathbf{X}_\lambda{}^2)(0) = \mathbf{y}_2\frac{\partial}{\partial \mathbf{y}_2} - (1 - \widehat{\alpha}_1)\mathbf{x}_2\frac{\partial}{\partial \mathbf{y}_2}$$

where $1 - \widehat{\alpha}_1(\widehat{\lambda}) = \mathbf{r}(\widehat{\lambda})$ is the complex continuation of the hyperbolicity ratio.

We define :

$$\sigma_i = \{(\mathbf{x}_i, \mathbf{y}_i) \in \mathbf{A}_i; \mathbf{y}_i = 1\} \quad \text{and} \quad \tau_i = \{(\mathbf{x}_i, \mathbf{y}_i) \in \mathbf{A}_i; \mathbf{x}_i = 1\}$$

σ_i^+ is (resp τ_i^+) a sector in σ_i (resp in τ_i) defined by :

$$\sigma_i^+ = \{(\mathbf{x}_i, \mathbf{y}_i) \in \sigma_i; |Arg(\mathbf{x}_i)| \leq \theta_0\} \qquad 0 < \theta_0 \leq \frac{\pi}{2}$$

respectively

$$\tau_i^+ = \{(\mathbf{x}_i, \mathbf{y}_i) \in \tau_i; |Arg(\mathbf{y}_i)| \leq \theta_0\}$$

the disks σ_i, τ_i are transversal to the local invariant manifolds of $\mathbf{X}^i_{\widehat{\lambda}}$.

The Dulac and the regular transition maps defined above have unique holomorphic extensions. So the shift map has a unique holomorphic extension noted by $\widehat{\delta}(x, \widehat{\lambda})$ in $(\tau_2^+ \setminus \{0\} \times \mathbf{W})$ and prolonged in 0 by 0.
Let the function $\widehat{\omega}(\mathbf{x}, \widehat{\lambda})$ be the continuation of the real one defined above.

Theorem 3. *For any arbitrary integer K, there exists a neighbourhood $\mathbf{W}_K$ of 0 in $\mathbf{W} \subset \mathbf{C}^\Lambda$ and holomorphic functions $\widehat{\gamma}_{ij}^K : \mathbf{W}_K \longrightarrow \mathbf{C}$ continuation of the real functions γ_{ij}^K such that on $\tau_2^+ \times \mathbf{W}_K$ the function $\widehat{\delta}(\mathbf{x}, \widehat{\lambda})$ has the form :*

$$\widehat{\delta}(\mathbf{x}, \widehat{\lambda}) = \sum_{0 \leq j \leq i \leq K} \widehat{\gamma}_{ij}^K(\widehat{\lambda})\mathbf{x}^i\widehat{\omega}^j + \psi^K(\mathbf{x}, \widehat{\lambda})$$

where ψ^K is a function of class C^K, in the real sense, K–flat in $\mathbf{x} = 0$.

Remark.

This development is what we call the well ordered expansion of order K. The monomials $\mathbf{x}^i\widehat{\omega}^j$ are totally ordered by the lexicographic order : $\mathbf{x}^i\widehat{\omega}^j \preceq \mathbf{x}^m\widehat{\omega}^n$ if and only if $i < m$ or $i = m$ and $j > n$.

The proposition 1 is an immediat consequence of theorem 3, it suffices to restrict all the different neighbourhoods, charts and functions in $\mathbf{C}^\Lambda$ and $\mathbf{C}^2$ respectively to $\mathbf{R}^\Lambda$ and $\mathbf{R}^2$.

Proof of the theorem 3.

Given an integer $K \neq 0$ we may apply the results of [R]. There exists a neighbourhood $\mathbf{W}_K$ of 0 in $\mathbf{C}^\Lambda$, some transversals depending on the parameter $\widehat{\lambda} \in \mathbf{C}^\Lambda$, $\widetilde{\sigma}_i$ (resp $\widetilde{\tau}_i$) tangent to $\sigma_i$in 0 (resp to σ_i) such that the Dulac maps : $\widetilde{\mathbf{D}}_{1,\widehat{\lambda}} : \widetilde{\sigma}_1^+ \longrightarrow \widetilde{\tau}_1$ and $\widetilde{\mathbf{D}}_{2,\widehat{\lambda}} : \widetilde{\tau}_2^+ \longrightarrow \widetilde{\sigma}_2$ are written under the form :

$$\widetilde{\mathbf{D}}_{1,\widehat{\lambda}}(\widetilde{\mathbf{x}}_1) = \widetilde{\mathbf{x}}_1 + \sum_{1 \leq j \leq i \leq K} \widetilde{\alpha}_{ij}(\widehat{\lambda})\widetilde{\mathbf{x}}_1^i\widetilde{\omega}^j + \cdots + \widetilde{\alpha}_{K+1,1}\widetilde{\mathbf{x}}_1^{K+1}\widetilde{\omega} + \widetilde{\psi}_K^1(\widetilde{\mathbf{x}}_1, \widehat{\lambda})$$

$$\widetilde{\mathbf{D}}_{2,\widehat{\lambda}}(\widetilde{\mathbf{y}}_2) = \widetilde{\mathbf{y}}_2 + \sum_{1 \leq j \leq i \leq K+1} \widetilde{\beta}_{ij}\widetilde{\mathbf{y}}_2^i\widehat{\omega}^j + \cdots + \widetilde{\beta}_{K+1,1}\widetilde{\mathbf{y}}_2^{K+1}\widehat{\omega} + \widetilde{\psi}_K^2(\widetilde{\mathbf{y}}_2, \widehat{\lambda})$$

where $\widetilde{\alpha}_{ij}$,$\widetilde{\beta}_{ij}$ are holomorphic functions on $\mathbf{W}_K$ $\widetilde{\psi}_K^1, \widetilde{\psi}_K^2$ are C^K in the real sense, K–flat resp. to $\widetilde{\mathbf{x}}_1 = 0$, $\widetilde{\mathbf{y}}_2 = 0$.

The same arguments as in [R] work here because $\mathbf{X}_{\widehat{\lambda}}^1$ and $(-\mathbf{X}_{\widehat{\lambda}}^2)$ have respectively the same hyperbolicity ratio $\mathbf{r}(\widehat{\lambda})$ in P_1 and P_2.
Now there exist $\varphi_{i,\widehat{\lambda}}$ $i = 1, 2$ (resp $\phi_{i,\widehat{\lambda}}$ $i = 1, 2$) holomorphic diffeomorphisms defined by the flow of $\mathbf{X}_{\widehat{\lambda}}^i$ between σ_i and $\widetilde{\sigma}_i$ (resp τ_i and $\widetilde{\tau}_i$.)

$$\varphi_{i,\widehat{\lambda}} : \sigma_i \longrightarrow \widetilde{\sigma}_i \qquad \phi_{i,\widehat{\lambda}} : \tau_i \longrightarrow \widetilde{\tau}_i$$

with $\forall \widehat{\lambda} \in \mathbf{W}_K$, $\phi_{i,\widehat{\lambda}}(0) = \varphi_{i,\widehat{\lambda}}(0) = 0 \qquad i = 1, 2$.

So the Dulac maps :

$$\mathbf{D}_{1,\widehat{\lambda}} : \sigma_1^+ \longrightarrow \tau_1 \quad \text{and} \quad \mathbf{D}_{2,\widehat{\lambda}} : \tau_2^+ \longrightarrow \sigma_2$$

are written :

$$\mathbf{D}_{1,\widehat{\lambda}}(\mathbf{x}_1) = \left(\phi_{1,\widehat{\lambda}}^{-1} \circ \widetilde{\mathbf{D}}_{1,\widehat{\lambda}} \circ \varphi_{1,\widehat{\lambda}}\right)(\widetilde{\mathbf{x}}_1)$$
$$\mathbf{D}_{2,\widehat{\lambda}}(\mathbf{y}_2) = \left(\varphi_{2,\widehat{\lambda}}^{-1} \circ \widetilde{\mathbf{D}}_{2,\widehat{\lambda}} \circ \phi_{2,\widehat{\lambda}}\right)(\widetilde{\mathbf{y}}_2)$$

by using the lemma 2 below we obtain that the Dulac maps have the following development for a choosen K :

$$\mathbf{D}_1(\mathbf{x}_1) = \mathbf{x}_1 + \sum_{1 \le j \le i \le K} \alpha_{ij}(\widehat{\lambda})\mathbf{x}_1^i \widehat{\omega}^j + \cdots + \alpha_{K+1,1}\mathbf{x}_1^{K+1}\widehat{\omega} + \psi_K^1(\mathbf{x}_1, \widehat{\lambda})$$

$$\mathbf{D}_2(\mathbf{y}_2) = \mathbf{y}_2 + \sum_{1 \le j \le i \le K} \beta_{ij}(\widehat{\lambda})\mathbf{y}_2^i \widehat{\omega}^j + \cdots + \beta_{K+1,1}\mathbf{y}_2^{K+1}\widehat{\omega} + \psi_K^2(\mathbf{y}_2, \widehat{\lambda})$$

the functions $\alpha_{ij}, \beta_{ij}, \psi_K^l$ have the same properties as $\widetilde{\alpha}_{ij}, \widetilde{\beta}_{ij}, \widetilde{\psi}_K^l$.

Lemma 2. *Let* $\mathbf{f}$ *be a holomorphic function of* $\sigma^+ \times \mathbf{W}_K$ *where* σ^+ *is a sector as the ones defined below. If* $\mathbf{f}(0, \widehat{\lambda}) = 0$ *then there exists a holomorphic function* $\mathbf{g}$ *such that :*

$$\widehat{\omega}(\mathbf{x}(1+\mathbf{f}), \widehat{\lambda}) = (1 + \widehat{\alpha}_1(\widehat{\lambda})\mathbf{g})\widehat{\omega}(\mathbf{x}, \widehat{\lambda}) + g$$

$$\textit{if} \quad \mathbf{a} \neq 0 : \widehat{\omega}(\mathbf{a}\mathbf{x}, \widehat{\lambda}) = (1 + o(\widehat{\alpha}_1))\widehat{\omega}(\mathbf{x}, \widehat{\lambda}) - \ln \mathbf{a}(1 + o(\widehat{\alpha}_1)$$

$$\widehat{\omega}(\mathbf{a}\mathbf{x}(1+\mathbf{f}), \widehat{\lambda}) = (1 + o(\widehat{\alpha}_1))\widehat{\omega}(\mathbf{x}, \widehat{\lambda}) - \ln \mathbf{a}(1 + o(\widehat{\alpha}_1)) + \mathbf{g}$$

To finish the proof, we have to develop :

$$\widehat{\delta}(\mathbf{x}, \widehat{\lambda}) = \left(\mathbf{R}_{1,\widehat{\lambda}} \circ \mathbf{D}_{1,\widehat{\lambda}} \circ \mathbf{R}_{2,\widehat{\lambda}}\right)(\mathbf{x}) - \mathbf{D}_{2,\widehat{\lambda}}(\mathbf{x})$$

where $\mathbf{R}_{1,\widehat{\lambda}}$ and $\mathbf{R}_{2,\widehat{\lambda}}$ are the regular transition maps. We can write them as :

$$\mathbf{R}_{1,\widehat{\lambda}}(\mathbf{x}) = \mathbf{b}_0(\widehat{\lambda}) + \mathbf{b}_1(\widehat{\lambda})\mathbf{x} + \mathbf{b}_2(\widehat{\lambda})\mathbf{x}^2 + \cdots + \mathbf{b}_K(\widehat{\lambda})\mathbf{x}^K + o(\mathbf{x}^K)$$

$$\mathbf{R}_{2,\widehat{\lambda}}(\mathbf{x}) = \mathbf{a}_1(\widehat{\lambda})\mathbf{x} + \mathbf{a}_2(\widehat{\lambda})\mathbf{x}^2 + \mathbf{a}_3(\widehat{\lambda})\mathbf{x}^3 + \cdots + \mathbf{a}_K(\widehat{\lambda})\mathbf{x}^K + o(\mathbf{x}^K)$$

Using again the lemma 2, we find the expansion of $\widehat{\delta}$.

3. .3 Division in the ideal of coefficients

Let's recall some definitions and results from $[R]$:
For all $x_0 \in]0, \varepsilon_0]$, the domain of the function $\delta_\lambda(x) = \delta(x, \lambda)$ and for all $\lambda \in W$, δ is an analytic function in (x_0, λ). Then we can write it as follows :

$$\delta(x, \lambda) = \sum_{i=0}^{\infty} a_i(\lambda, x_0)(x - x_0)^i \tag{3.3.1}$$

for x close to x_0.

Consider the ideal J_{x_0} generated by the germs of the functions a_i in $\lambda = 0$. We will note them by $\widetilde{a}_i$. In $[R]$, it is proved that the ideal J_{x_0} does not depend on the point $x_0 \neq 0$. J is called the ideal of coefficients associated to δ. $J \subset \mathcal{O}$ the ring of the germs of analytic functions in $\lambda = 0$.

In the following we will suppose that $J \neq \mathcal{O}$ ie $\delta(x, 0) \equiv 0$. This corresponds to the case: Γ_0 is identical. The other case $J = \mathcal{O}$, ie $\delta(x, 0) \not\equiv 0$, was studied in $[El.M]$ and corresponds Γ_0 non identical. The definitions introduced here are available also in the complex domain.
So let $\mathbf{J}$ the complexified ideal of J. It's easy to see that $\mathbf{J} = \mathbf{J}_{\mathbf{x_0}}$ for any $\mathbf{x_0} \in \widehat{\tau}_2^+$ where $\mathbf{J}_{\mathbf{x_0}}$ is the ideal of coefficients of $\widehat{\delta}(\mathbf{x}, \widehat{\lambda})$, extension of $\delta(x, \lambda)$ defined above.

Proposition 1. *Let $\widehat{\gamma}_{ij}^K, K \geq 2$ the coefficients of the expansion of $\widehat{\delta}$ to an order K. Given any k such that $1 \leq k < K$ then the germ $\widetilde{\widehat{\gamma}}_{ij}^K \in \mathbf{J}$ for $0 \leq j \leq i \leq k$.*

Proof.

We will apply the same algorithm as in $[El.M]$
Consider the well ordered development of $\widehat{\delta}$ up to order K :

$$\widehat{\delta}(\mathbf{x}, \widehat{\lambda}) = \sum_{0 \leq j \leq i \leq k} \widehat{\gamma}_{ij}^K \mathbf{x}^i \widehat{\omega}^j + \cdots + \widehat{\gamma}_{k+1.1}^K \mathbf{x}^{k+1} \widehat{\omega} + \psi^K(\mathbf{x}, \widehat{\lambda}) \tag{3.3.2}$$

For $\mathbf{x} \neq 0$, the germ in $\widehat{\lambda} = 0$ of the function $\widehat{\lambda} \longmapsto \widehat{\delta}_{\mathbf{x}}(\widehat{\lambda}) = \widehat{\delta}(\mathbf{x}, \widehat{\lambda})$ is in $\mathbf{J}$. Moreover each monomial in (3.3.2), apart the first one which is equal to 1, corresponds to a nonzero power of $\mathbf{x}$. It follows that :

$$\widehat{\delta}(\mathbf{x}, \widehat{\lambda}) = \widehat{\gamma}_{00} + \psi_0(\mathbf{x}, \widehat{\lambda}) \tag{3.3.3}$$

where $\psi_0(\mathbf{x},\widehat{\lambda}) \longmapsto 0$ when $\mathbf{x} \longmapsto 0$, uniformly in $\widehat{\lambda}$. Because the ideal <u>J is closed</u> we conclude that $\widehat{\gamma}^K_{00}$ has its germ in $\mathbf{J}$.

Suppose now that we have proved that the germs of $\widehat{\gamma}^K_{ij}$ are in $\mathbf{J}$ for all the monomials $\mathbf{x}^i\widehat{\omega}^j \preceq \mathbf{x}^m\widehat{\omega}^n$ where $\preceq$ is the order introduced above. We use also the lexicographic order between the couples $(i,j) < (m,n)$. Let :

$$\begin{aligned}\widehat{\delta}_{mn} &= \widehat{\delta} - \sum_{(i,j)<(m,n)} \widehat{\gamma}^K_{ij}\mathbf{x}^i\widehat{\omega}^j \\ &= \widehat{\gamma}^K_{mn}(\widehat{\lambda})\mathbf{x}^m\widehat{\omega}^n + \cdots + \psi^K(\mathbf{x},\widehat{\lambda})\end{aligned} \tag{3.3.4}$$

For each $\mathbf{x} \neq 0$, the germ in $\widehat{\lambda} = 0$ of the function $\widehat{\lambda} \longmapsto \widehat{\delta}_{mn}(\mathbf{x},\widehat{\lambda})$ is in $\mathbf{J}$. But we have to remark that the sequence of monomials $\mathbf{x}^i\widehat{\omega}^j$ does not form a scale of infinitisimals in $\mathbf{x}$ (in uniform way in $\widehat{\lambda}$), because the ratio of $\mathbf{x}^i\widehat{\omega}^j$ and $\mathbf{x}^i\widehat{\omega}^l$, for $j < l$, is equal to $\widehat{\omega}^{-j+l}$ and does not tend to zero, uniformly in $\widehat{\lambda}$, if $\mathbf{x} \longmapsto 0$. So we cannot apply directly to $\widehat{\gamma}^K_{mn}$, the same argument we have applied to $\widehat{\gamma}^K_{00}$. We will apply it after a first step where we will transform $\widehat{\delta}_{mn}$ by <u>division</u> and <u>derivation</u>. This is based on the following observation : if a function $\varphi(\mathbf{x},\widehat{\lambda})$ has a well ordered development up to some order K, like the function $\widehat{\delta}$, then for $\forall s,l \in \mathbf{R}$ and any order of derivation $r < K$, the function $\widehat{\lambda} \longmapsto \mathbf{x}^l\widehat{\omega}^s\dfrac{\partial^r\varphi}{\partial\mathbf{x}^r}(\mathbf{x},\widehat{\lambda})$ has a germ in $\widehat{\lambda} = 0$ in $\mathbf{J}$, for $\forall\mathbf{x} \neq 0$.

Starting with the monomials $\mathbf{x}^i\widehat{\omega}^j$, $i,j \in \mathbf{N}$, the derivation with respect to $\mathbf{x}$ produces more general monomials $\mathbf{x}^{i+s\widehat{\alpha}_1}\widehat{\omega}^j$. So, firstly we extend the total order introduced above in a *partial* one between these new monomials. For $i,j \in \mathbf{N}$ and $s,l \in \mathbf{Z}$ we take :

$$\mathbf{x}^{i+l\widehat{\alpha}_1}\widehat{\omega}^r \preceq \mathbf{x}^{j+k\widehat{\alpha}_1}\widehat{\omega}^s \iff \begin{cases} i < j & \text{or} \\ i = j, l = k & \text{and} \quad r > s. \end{cases}$$

the notation "$f + \cdots$ " will be for a sum of f and a combination of monomials with larger order.

Now let us explain our first step. Starting with $\Delta_0 = \widehat{\delta}_{mn}$, we divide it by $\mathbf{x}^m$:

$$\Delta_1 = \mathbf{x}^{-m}\Delta_0 = \widehat{\gamma}^K_{mn}\widehat{\omega}^n + \widehat{\gamma}^K_{mn-1}\widehat{\omega}^{n-1} + \cdots + R_1 \tag{3.3.5}$$

with $R_1 = \psi^K\mathbf{x}^{-m}$ of order $(K-m)$, is more differentiable and flatter than the last term in $+\cdots$

If $n = 0$, our first step is achieved: $\Delta_1 = \widehat{\gamma}_{mn} + \varphi_1(\mathbf{x},\widehat{\lambda})$ where $\varphi_1(\mathbf{x},\widehat{\lambda}) \mapsto 0$ when $\mathbf{x} \longmapsto 0$, uniformly in $\widehat{\lambda}$, and we can repeat now, the argument used above for $\widehat{\gamma}^K_{00}$.

If $n > 0$, after noticing that $\dfrac{\partial \widehat{\omega}}{\partial \mathbf{x}} = -\mathbf{x}^{-1-\widehat{\alpha}_1}$, we have :

$$\Delta_2 = \mathbf{x}^{1+\widehat{\alpha}_1} \frac{\partial \Delta_1}{\partial \mathbf{x}} = n\widehat{\gamma}^K_{mn}\widehat{\omega}^{n-1} + \cdots + R_2 \tag{3.3.6}$$

where R_2 is a convenient remaining term as R_1 above. Repeating $n-1$ times again the same procedure, we obtain finally :

$$\begin{aligned} \Delta_{n+1} &= n!\widehat{\gamma}^K_{mn} + \cdots + R_{n+1} \\ &= n!\widehat{\gamma}^K_{mn} + \varphi_{n+1}(\mathbf{x}, \widehat{\lambda}) \end{aligned} \tag{3.3.7}$$

with a convenient remaining term R_{n+1}. Now φ_{n+1} has an expansion whose first monomial has a positive power in $\mathbf{x}$, so that $\varphi_{n+1}(\mathbf{x}, \widehat{\lambda}) \longmapsto 0$ when $\mathbf{x} \longmapsto 0$, uniformly in $\widehat{\lambda}$. As above this implies that the germ of $\widehat{\gamma}^K_{mn}$ is in $\mathbf{J}$.

3. .4 The proof of theorem 2

As $\mathcal{O}$ is noetherian, $\mathbf{J}$ has a finite system of generators $\widetilde{\phi}_1, \widetilde{\phi}_2, \cdots, \widetilde{\phi}_l$; where $(\phi_1, \phi_2, \cdots, \phi_l)$ are holomorphic in $\mathbf{W}$.
Using the proposition 1 and the same arguments as in theorem 7 of [R],we can write $\widehat{\delta}(\mathbf{x}, \widehat{\lambda})$ under the form :

$$\widehat{\delta}(\mathbf{x}, \widehat{\lambda}) = \sum_{i=1}^{l} \phi_i(\widehat{\lambda})\mathbf{h}_i^K(\mathbf{x}, \widehat{\lambda}) \tag{3.4.1}$$

where K is an arbitrary integer,the functions $\mathbf{h}_i^K(\mathbf{x}, \widehat{\lambda})$ are holomorphic for $\mathbf{x} \neq 0$ and have the well ordered expansions of order K. We deduce the following proposition in the real domain :

Proposition 2 [R]. *Let $(\phi_1, \phi_2, \cdots, \phi_l)$ analytic functions in W whose germs in $\lambda = 0$ generate the ideal of coefficients J. Let K an arbitrary integer, then there exists a neighbourhood $W_K \subset W$ in $\mathbf{R}^\Lambda$ and functions $h_i^K(x, \lambda)$, with $1 \leq i \leq l$ having well ordered expansions of order K :*

$$h_i^K(x, \lambda) = D^K h_i^K(x, \lambda) + \psi_i^K(x, \lambda) \qquad in \qquad [0, x_0] \times W_K$$

$$D^K h_i^K(x, \lambda) = \sum_{0 \leq n \leq m \leq K} \gamma^{iK}_{mn} x^m \omega^n + \cdots + \gamma^{iK}_{K+1,1} x^{K+1} \omega$$

the h_i^K are analytic for $x \neq 0$. They permit us to write $\delta(x,\lambda)$ as follows :

$$\delta(x,\lambda) = \sum_{i=1}^{l} \phi_i(\lambda) h_i^K(x,\lambda) \tag{3.4.2}$$

we can choose a system of generators $(\phi_1, \phi_2, \cdots, \phi_l)$ verifying some properties as in [*R*] :

i) $(\phi_1, \phi_2, \cdots, \phi_l)$ is minimal in the sense that it is a basis of the vector space $J/\mathcal{M}J$ where $\mathcal{M}$ is the maximal ideal of $\mathcal{O}$.

ii) For $\lambda = 0$ the values of $h_i^K(x,0)$ of the expression (3.4.2) don't depend on K. So we can define the functions $h_i(x) = h_i^K(x,0)$ for any K. In the neighbourhood of $x = 0$, we can associate to them a formal power serie called the Dulac's development, we note by :

$$D^\infty h_i(x) = \sum_{0 \le n \le m}^{\infty} \gamma_{mn}^i(o) x^m (-\ln x)^n \tag{3.4.3}$$

where $\gamma_{mn}^i(0) = \gamma_{mn}^{iK}(0)$ for any $K \geq Sup\{m,n\}$. This development is unique. We obtain it from (3.4.2) by remarking that for $\lambda = 0$: $\quad x^m \omega^n = x^m(-\ln x)^n$. The functions h_i are analytic for $x \neq 0$ and $h_i \not\equiv 0$. But, we cannot assert that $D^\infty h_i \not\equiv 0$, this would be true if h_i was quasi–regular. If $D^\infty h_i \not\equiv 0$ then it will be equivalent to x^m or $x^m \ln x$. This equivalency allows us to define an order of flatness between the h_i such that $D^\infty h_i \not\equiv 0$ by : $order(h_i) < order(h_j)$ if and only if $h_j/h_i \longmapsto 0$ when $x \longmapsto 0$. We say that $order(h_i) = \infty$ if $D^\infty h_i \equiv 0$.

iii) There exists an index $s, 0 \leq s \leq l$ such that :

$$order(h_1) < order(h_2) < \cdots < order(h_s) < \infty$$

and

$$order(h_j) = \infty \qquad \text{for} \quad j \geq s+1$$

we say that h_i are ordered.

The properties **i,ii,iii** of the system $(\phi_1, \phi_2, \cdots, \phi_l)$ are not sufficient to conclude the finite cyclicity of Γ_0. That is why, we consider the map of desingularization of the set $\{\lambda \setminus \phi_1\phi_2 \cdots \phi_l = 0\}$.

There exists $\varphi : \widetilde{W} \longmapsto W$ a proper analytic map of a compact domain $\widetilde{W}$ onto W neighbourhood of $0 \in R^\Lambda$. φ is the map of Hironaka's desingularization.

We consider the family $X_{\widehat{\lambda}} = X_{\varphi(\lambda)}$ for $\widehat{\lambda} \in \widetilde{W}$. Take $D = \widehat{\varphi}^{-1}\{0\}$ we associate to $X_{\widehat{\lambda}}$ at every point $\widehat{\lambda}_0 \in D$, an ideal of coefficients noted $\widetilde{J}_{\widehat{\lambda}} \subset \widehat{\mathcal{O}}_{\widehat{\lambda}}$, ring of analytic germs at $\widehat{\lambda} = 0$. As D is compact, the cyclicity of Γ_0 in X_λ will be finite if this is true for the $X_{\widehat{\lambda}}$-germ at every $\widehat{\lambda}_0 \in D$.

Let $\widehat{\delta}(x, \widehat{\lambda}) = \delta(x, \varphi(\widehat{\lambda}))$ the shift map of $X_{\widehat{\lambda}}$, $\widehat{\lambda}$ in a neighbourhood of $\widehat{\lambda}_0 \in D$. Then it's easy to see that $\widetilde{J}_{\widehat{\lambda}_0}$ will be generated by $\widehat{\phi}_i(\widehat{\lambda}) = \phi_i \circ \varphi(\widehat{\lambda})$. Furthermore, there exists $\widetilde{W}_{\widehat{\lambda}_0}$ a neighbourhood of $\widehat{\lambda}_0$ with coordinates $z_1, z_2, \cdots, z_\Lambda$(where $\widehat{\lambda}_0 = (0, 0, 0, \cdots, 0)$) such that :

$$\widehat{\phi}_i(\widehat{\lambda}) = u_i(\widehat{\lambda}) \prod_{i=1}^{\Lambda} z_j^{p_j^i} \tag{3.4.4}$$

the functions $u_i(\widehat{\lambda})$ are analytic and nonzero for all $\widehat{\lambda} \in \widetilde{W}_{\widehat{\lambda}_0}$, p_j^i are integers. Let's note $\prod_{j=1}^{i} z^{p_j^i} = \psi_i(\widehat{\lambda})$, then $\widetilde{\phi}_i(\widehat{\lambda}) = u_i(\widehat{\lambda})\psi_i(\widehat{\lambda})$.

Proposition 3 [*R*]. *From the system $(\widehat{\phi}_1, \widehat{\phi}_2, \cdots, \widehat{\phi}_l)$ we can extract a system $(\widehat{\phi}_{i_1}, \widehat{\phi}_{i_2}, \cdots, \widehat{\phi}_{i_L})$ possessing properties* **i,ii,iii** *as $(\phi_1, \phi_2, \cdots, \phi_l)$. i.e there exists $s, 0 \leq s \leq L$ such that if $\widehat{\delta}(x, \widehat{\lambda}) = \sum_{j=1}^{L} \widehat{\phi}_{i_j} H_j^K(x, \widehat{\lambda})$ then $order(H_1) < order(H_2) < \cdots < order(H_s) < \infty$ and $order(H_j) = \infty$, where H_i is defined as below.*

Remark.

The division of $\widehat{\delta}$ in $\widetilde{J}_{\widehat{\lambda}_0}$ is not degenerated in a sense we will explain below. Until the end we are going to work with the family $X_{\widehat{\lambda}}, \widehat{\delta}(x, \widehat{\lambda}), \widehat{\phi}_{i_j}, H_i^K, \ldots$ and we will show the finite cyclicity of Γ_0 for this family.

From now on, we discard the caps in the notation $\widehat{\lambda}, \widehat{\phi}, \ldots$ So that, we suppose we have a family X_λ with :

$$\delta(x, \lambda) = \sum_{i=1}^{L} \phi_i(\lambda) h_i^K(x, \lambda)$$

where

$$order(h_1) < order(h_2) < \cdots < order(h_s) < \infty$$

and

$$order(h_i) = \infty \quad i > s$$

and $\phi_i(\lambda) = u_i(\lambda)\psi_(\lambda)$ with $\psi_i(\lambda) = \prod_{j=1}^{\Lambda} z_j^{p_j^i}$ and $(z_1, z_2, \cdots, z_\Lambda)$ local coordinates in W_{λ_0}.

Lemma 3 [*R*]. *Consider* $W_i = \{\lambda \in W_{\lambda_0} \setminus |\psi_i(\lambda)| \geq |\psi_j(\lambda)| \quad \text{for } j \neq i\}$. *Let* $I = \{i \setminus \lambda_0 \in W_i\}$. *Then we have the following results :*
i) Let $i \in I$ *then there exists an analytic arc* $\lambda(\varepsilon) : [0, \varepsilon_0] \longmapsto W_{\lambda_0}$, *with* $\lambda(0) = \lambda_0$ *such that :*

$$order(\psi_i \circ \lambda)_{\varepsilon=0} < order(\psi_j \circ \lambda)_{\varepsilon=0} \qquad \text{for} \qquad j \neq i$$

ii) $\cup_{i \in I} W_i$ *is a neighbourhood of* λ_0.

Proposition 3. *If* $i \in I$ *then* $D^\infty h_i \not\equiv 0$. *This means that* $I \subset \{1, 2, \cdots, s\}$.

Proof.
Let $i \in I$ and $\lambda(\varepsilon)$ the analytic arc in W_{λ_0}. Let us consider the subfamily depending on 1–parameter :

$$X_\varepsilon = X_{\lambda(\varepsilon)} \qquad \varepsilon \in [0, \varepsilon_0]$$

let $\widetilde{\delta}(x, \varepsilon)$ the map δ associated to this family. obviously, $\widetilde{\delta}(x, \varepsilon) = \delta(x, \lambda(\varepsilon))$ where $\delta(x, \lambda)$ is the shift map of X_λ. So, for a given integer K, we can write :

$$\widetilde{\delta}(x, \varepsilon) = \sum_{j=1}^{L} \phi_j \circ \lambda(\varepsilon) h_j^K(x, \lambda(\varepsilon)) \tag{3.4.5}$$

for all indices j :

$$h_j^K(x, \lambda_\varepsilon)) = h_j(x) + O(\varepsilon)$$

and

$$\phi_j(\lambda(\varepsilon)) = a_j \varepsilon^{n_j} + O(\varepsilon^{n_j}) \qquad a_i \neq 0$$

We replace in equality (3.4.5) to obtain :

$$\widetilde{\delta}(x, \varepsilon) = a_i h_i(x) \varepsilon^{n_i} + O(\varepsilon^{n_i}) \tag{3.4.6}$$

the formula (3.4.6) indicates that up a multiplication by nonzero coefficient, $h_i(x)$ is the principal part of the development of $\widetilde{\delta}$ in serie of ε. Furthermore, by theorem 1 of this article, there exist quasi regular functions $I_j(x)$, up to some factor x^{jR}, such that :

$$\widetilde{\delta}(x,\varepsilon) = \varepsilon I_1(x) + \varepsilon^2 I_2(x) + \cdots + \varepsilon^j I_j(x) + O(\varepsilon^j) \tag{3.4.7}$$

for any integer j.

If we equalize the two expressions (3.4.6) and (3.4.7), we find that :

$$a_i h_i(x) = I_{n_i}(x)$$

as $h_i(x)$ is not identically zero, its Dulac's development that coincides with the one of $I_i(x)$ is not identically null. (Here I_{n_i} is eventually quasi- regular because it is the first non zero term in the expansion (3.4.7).)

Remember that $\phi_i(x,\lambda) = u_i(\lambda)\psi_i(\lambda)$ with $u_i(\lambda) \neq 0$ for every $\lambda \in W_{\lambda_0}$, so we may find a real $r : 0 < r \leq 1$ such that : if $V_i^r = \{\lambda; |\phi_i(\lambda)| \geq r|\phi_j(\lambda)|$ for $i \neq j\}$ then $\cup_{i=1}^s V_i^r$ is a neighbourhood of λ_0.

To end the proof, we remark that we have the same situation as in paragraph 8 of [R], therefore we can conclude that for all $i : 1 \leq i \leq s$, there exists $N_i \in \mathbf{N}$, a neighbourhood W_i of λ_o and a real $x_i : 0 < x_i \leq x_0$ such that $\delta(x,\lambda)$ has less than N_i zeros in $[0, x_i]$ for all $\lambda \in V_i^r \cap W_i$.

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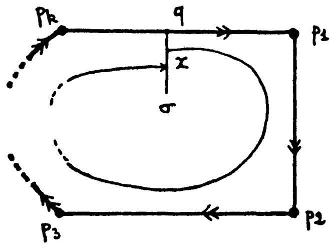

Figure 1.

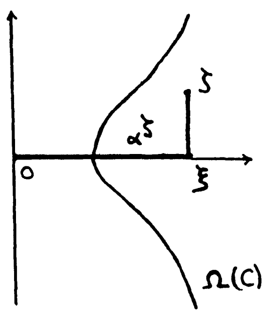

Figure 2.a

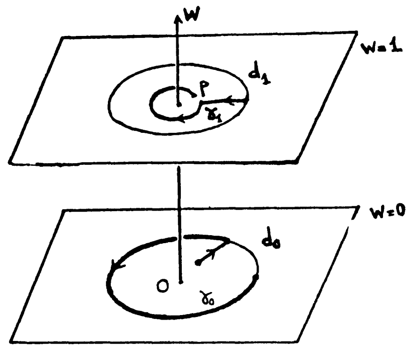

Figure 2.b

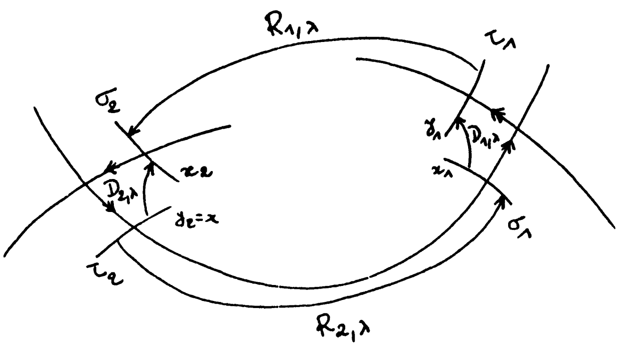

Figure 3.

Astérisque

ISAO NAKAI

A rigidity theorem for transverse dynamics of real analytic foliations of codimension one

Astérisque, tome 222 (1994), p. 327-343
<http://www.numdam.org/item?id=AST_1994__222__327_0>

A rigidity theorem for transverse dynamics of real analytic foliations of codimension one

Isao Nakai

The purpose of this paper is to prove

Theorem 1. *Let $(M_i^n, \mathcal{F}_i), i = 1, 2$, be real analytic and orientable foliations of n-manifolds of codimension 1 and $h : (M_1^n, \mathcal{F}_1) \to (M_2^n, \mathcal{F}_2)$ a foliation preserving homeomorphism. Assume that all leaves of $\mathcal{F}_1$ are dense and there exists a leaf of $\mathcal{F}_1$ with holonomy group $\neq 1, \mathbb{Z}$. Then h is transversely real analytic.*

This applies to prove the following topological rigidity of the Godbillon-Vey class of real analytic foliations of codimension one.

Corollary 2. *Let $(M_i, \mathcal{F}_i), h$ be as in Theorem 1. Then $h^*(\mathrm{GV}(\mathcal{F}_2) = \mathrm{GV}(\mathcal{F}_1)$ holds.*

Here $\mathrm{GV}(\mathcal{F}_i) \in H^3(M, \mathbb{R})$ denotes the Godbillon-Vey class of $\mathcal{F}_i$, which is represented by the 3-form $\alpha \wedge d\alpha$ with a C^∞ -1-form α on M such that $d\theta = \theta \wedge \alpha$ holds with a C^∞ -1-form θ defining $\mathcal{F}$. It is easy to see that the Godbillon-Vey class is invariant under C^2-diffeomorphisms. Ghys, Tsuboi [9] and Raby [18] proved the invariance under C^1-diffeomorphisms, while the invariance is known to fail in some C^0-cases (see [5,9,11]). (Corollary 2 seems to admit the various generalisations allowing the existence of compact leaves. But we will not touch on those generalisations. See also the papers [5,7].)

The proof of the C^1-invariance due to Ghys and Tsuboi is based on a certain rigidity for C^1-conjugacies of transverse dynamics of foliations along compact leaves as well as minimal exceptional leaves cutting Cantor sets on transverse sections. The proof of Theorem 1 is based on the topological rigidity theorem for pseudogroups of diffeomorphisms of $\mathbb{R}$ (Theorem 3(1)).

To state Theorem 3 we prepare some notions. Let Γ_+^ω be the pseudogroup of real analytic and orientation preserving diffeomorphisms of open neighbourhoods of the line $\mathbb{R}$ respecting 0. We call a mapping $\phi : G \to \Gamma_+^\omega$ of a group G to the pseudogroup Γ_+^ω a *morphism* if the set $\phi(G)_0$ of germs of $\phi(f), f \in G$ form a group and ϕ induces a group homomorphism of G to $\phi(G)_0$. Therefore $\phi(f) : U_{\phi(f)}, 0 \to \phi(f)(U_{\phi(f)}), 0$ is a real analytic diffeomorphism of open neighbourhoods of $0 \in \mathbb{R}$ for $f \in G$ representing the germ of $\phi(f)$. We call $\phi(G)_0$ the germ of $\phi(G)$ and say ϕ is *solvable* (respectively *commutative, etc*) if $\phi(G)_0$ is so. The *orbit* $\mathcal{O}(x)$ of an $x \in \mathbb{R}$ is the set of those x_l joined by a sequence $(x_0, x_1, ..., x_l)$ with $x = x_0, x_{i+1} = \phi(f_i)(x_i), x_i \in U_{\phi(f_i)}, i = 0, ..., l-1$ for arbitrary $l \geq 0$. The *basin* $B_{\phi(G)}$ of 0 is the set of those x for which the closure of the orbit $\mathcal{O}(x)$ contains 0. If $\phi(G)$ is non trivial, i.e. $\phi(f) \neq \mathrm{id}$ for an $f \in G$, $B_{\phi(G)}$ is an open neighbourhood of 0 [17]. Morphisms $\phi, \psi : G \to \Gamma_+^\omega$ are *topologically* (resp. C^r-) *conjugate* if there exists a homeomorphism (resp. C^r-diffeomorphism) $h : U, 0 \to h(U), 0$ of open neighbourhoods of 0 such that $U_{\phi(f)}, \phi(f)(U_{\phi(f)}) \subset U, U_{\psi(f)}, \psi(f)(U_{\psi(f)}) \subset h(U)$ and $h \circ \phi(f) = \psi(f) \circ h$ holds on $U_{\phi(f)}$ for all $f \in G$. We call h a *linking homeomorphism* (resp. *linking diffeomorphism*) and we denote $h : \phi \to \psi$.

Theorem 3 (The rigidity theorem for pseudogroups). *Let $\phi, \psi : G \to \Gamma_+^\omega$ be morphisms which are topologically conjugate with each other and $h : \phi \to \psi$ a linking homeomorphism.*

(1) If $\phi(G)_0, \psi(G)_0$ are not isomorphic to $\mathbb{Z}$ and non trivial, the restriction $h : B_{\phi(G)} - 0 \to B_{\psi(G)} - 0$ is a real analytic diffeomorphism.

(2) If $\phi(G)_0, \psi(G)_0$ are non commutative, h is unique and there exist even positive integers i, j such that $|h(\epsilon x^i)|^{1/j} : \tilde{B}^\epsilon_{\phi(G)} \to \tilde{B}^\epsilon_{\psi(G)}$ is a real analytic diffeomorphism for $\epsilon = \pm 1$. Here $\tilde{B}^\epsilon_{\phi(G)}$ is the set of those x such that $\epsilon x^i \in B_{\phi(G)}$ and $\tilde{B}^\epsilon_{\psi(G)}$ is the set of those x such that $x^j (resp. -x^j) \in B_{\psi(G)}$ if h maps $\mathbb{R}^\epsilon$ to $\mathbb{R}^+$ (resp. $\mathbb{R}^-$).

Now we apply the above rigidity theorem to the analytic action of the surface group on the circle S^1. Let Σ_g be the oriented closed surface of genus g and $\Gamma^g = \pi_1(\Sigma_g)$. For $r = 1, \ldots, \infty$ and ω, $\mathrm{Diff}^r_+(S^1)$ denotes the group of orientation preserving C^r-diffeomorphisms of the circle. The ***suspension*** M of a homomorphism $\phi : \Gamma^g \longrightarrow \mathrm{Diff}^r_+(S^1)$ is the quotient of $S^1 \times D^2$ by the product $\phi \times \Gamma$ with a discrete cocompact subgroup $\Gamma^g \simeq \Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ acting freely on the interior of the Poincaré disc D^2. The second projection of $S^1 \times D^2$ induces the submersion of M onto $\Sigma_g = D^2/\Gamma$ with the fiber S^1. Since the action $\phi \times \Gamma$ respects the foliation of $S^1 \times D^2$ by the discs $x \times D^2, x \in S^1$, the suspension M is a foliated S^1-bundle of which the fibres are the quotients of the discs. In this way the topology of foliated S^1-bundles interchanges with that of the actions of Γ^g on S^1. The Euler number $\mathrm{eu}(\phi)$ of a homomorphism $\phi : \Gamma^g \longrightarrow \mathrm{Diff}^r_+(S^1)$ is defined to be that of the S^1-bundle associated to ϕ. The Milnor-Wood inequality [15,22] asserts

$$|eu(\phi)| \leq |\chi(\Sigma_g)| = 2g - 2.$$

The Euler number enjoys the following relations with the orbit structure:

(1) $\mathrm{eu}(\phi) = 0$ if there exists a finite orbit,

(2) If $\mathrm{eu}(\phi) \neq 0$, there exist a minimal set $\mathcal{M} \subset S^1$ of ϕ , an $x \in \mathcal{M}$ and an $f \in \mathrm{stab}(x)$ such that $\phi(f)|_{\mathcal{M}} \neq id$ [13], and if $r = \omega$ all orbits are dense [6] (see also [16]),

(3) If $|\mathrm{eu}(\phi)| = |\chi(\Sigma_g)|$ and $r \geq 2$, all orbits are dense [6],

where $\mathrm{stab}(x)$ denotes the stabiliser of x consisting of $f \in \Gamma^g$ with $\phi(f)(x) = x$. Homomorphisms $\phi, \psi : \Gamma^g \longrightarrow \mathrm{Diff}^r_+(S^1)$ are ***C^s-conjugate*** if there exists a C^s-diffeomorphism h of S^1 such that $\psi(f) \circ h = h \circ \phi(f)$ holds for $f \in \Gamma^g$. We say ϕ, ψ are ***topologically conjugate*** if $s = 0$, ***semi conjugate*** if h is monotone map of degree one (possibly discontinuous). We call h a ***linking homeomorphism*** and denote $h : \phi \longrightarrow \psi$. It is known that the Euler number (and the bounded Euler class) concentrate the homotopic property of the action, namely

Theorem(Ghys [3]). *ϕ, ψ are semi conjugate if and only if* $\phi^*(\chi_{\mathbb{Z}}) = \psi^*(\chi_{\mathbb{Z}})$

in the bounded cohomology group $H_b^2(\Gamma^g : \mathbb{Z})$*, where* $\chi_{\mathbb{Z}} \in H_b^2(\mathrm{Diff}_+^0(S^1) : \mathbb{Z}) = \mathbb{Z}$ *is the generator, the bounded Euler class.*

Theorem (Matsumoto [13]). *If* $\mathrm{eu}(\phi) = \mathrm{eu}(\psi) = \pm\chi(\Sigma_g)$, ϕ, ψ *are semi conjugate, and if* $2 \le r$*, they are topologically conjugate with each other, and in particular, conjugate with a discrete cocompact subgroup of* $\mathrm{PSL}(2, \mathbb{R})$ *naturally acting on* S^1 *the boundary of the Poincaré disc.*

Theorem Ghys [8]. *If a homomorphism* $\phi : \Gamma^g \to \mathrm{Diff}_+^r(S^1)$ *attains the maximum of* $|\mathrm{eu}(\phi)|$ *and* $3 \le r$, ϕ *is* C^r*-smoothly conjugate with a discrete cocompact subgroup of* $\mathrm{PSL}(2, \mathbb{R})$.

In contrast to the above results, the properties of homomorphisms with $|\mathrm{eu}(\phi)| \lneqq |\chi(\Sigma_g)|$ are less known (see [16]). Applying Theorem 3 to the action of the stabiliser subgroup $\mathrm{stab}(x)$ on (S^1, x) for an $x \in S^1$, we obtain

Corollary 4. *Let* $\phi, \psi : \Gamma_g \to \mathrm{Diff}_+^\omega(S^1)$ *be homomorphisms with* $|\mathrm{eu}(\phi)|$, $|\mathrm{eu}(\psi)| \neq 0, |\chi(\Sigma_g)|$*, which are topologically conjugate, and* $h : \phi \to \psi$ *a linking homeomorphism. Assume that for an* $x \in S^1$*, the stabiliser subgroup* $\mathrm{stab}(x) \subset \Gamma_g$ *of* x *is not isomorphic to* $\mathbb{Z}$ *and non trivial. Then* h *is a real analytic diffeomorphism and orientation preserving or reversing respectively whether* $\mathrm{eu}(\phi) = \mathrm{eu}(\psi)$ *or* $\mathrm{eu}(\phi) = -\mathrm{eu}(\psi)$.

The statement remains valid for morphisms of groups G into $\mathrm{Diff}_+^\omega(S^1)$ replacing the condition on the Euler number by the existence of a dense orbit.

The author would like to thank Matsumoto, Minakawa, Nishimori, Tsuboi and Moriyama for their helpful comments.

2. Sequence geometry

In this paper $f^{(n)}$ denotes the n-fold iteration $f \circ \cdots \circ f$ of $f : U_f \to f(U_f)$ in Γ_+^ω. Let $\mathcal{X} = \{x_i\}, \mathcal{Y} = \{y_i\}, i = 1.2.\dots$ be monotone sequences of positive numbers decreasing to 0. Define the *address function* $\mathrm{add}_{\mathcal{Y}}(x)$ of an $x > 0$ relative to $\mathcal{Y}$ to be the smallest integer i such that $y_i \le x$. It is easy to see that $\mathrm{add}_{\mathcal{Y}}(x)$ is a decreasing function of x and $y_{\mathrm{add}_{\mathcal{Y}}(x)-1} > x \ge y_{\mathrm{add}_{\mathcal{Y}}(x)}$.

Define the *address function* $\mathrm{add}_{\mathcal{X},\mathcal{Y}}$ by

$$\mathrm{add}_{\mathcal{X},\mathcal{Y}}(i) = \mathrm{add}_{\mathcal{Y}}(x_i)$$

for $i = 1, 2, \ldots$. The address function enjoys the following inequality for a triple of sequences $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z} = \{z_i\}$.

Proposition 6. *Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z} = \{z_i\}$ be sequences of positive numbers decreasing to 0. Then*

$$\mathrm{add}_{\mathcal{Y},\mathcal{Z}}(\mathrm{add}_{\mathcal{X},\mathcal{Y}}(i) - 1) \le \mathrm{add}_{\mathcal{X},\mathcal{Z}}(i) \le \mathrm{add}_{\mathcal{Y},\mathcal{Z}}(\mathrm{add}_{\mathcal{X},\mathcal{Y}}(i))$$

for $x_i - 1 < y_0$.

We say two functions $P, Q : \mathrm{N} \cup 0 \to \mathrm{N} \cup 0$ are *equivalent* if there exist integers $c_1, ..., c_4$ such that

$$Q(i + c_1) + c_2 \le P(i) \le Q(i + c_3) + c_4$$

holds for all sufficiently large i.

Now let $\phi : G \to \Gamma_+^\omega$ be a morphism, and let $x_0 \in U_{\phi(g)}, y_0 \in U_{\phi(f)}$ be positive and sufficiently small and assume that $x_i = \phi(g)^{(i)}(x_0), y_i = \phi(f)^{(i)}(y_0)$ are decreasing to 0 as $i \to \infty$, replacing f, g by their inverses if necessary, and denote $\mathcal{X} = \{x_i\}, \mathcal{Y} = \{y_i\}$.

Proposition 7. *The equivalence class of the address function $\mathrm{add}_{\mathcal{X},\mathcal{Y}}$ is independent of the choice of the initial values x_0, y_0.*

proof. To prove the statement let $x_0 \neq x_0' > 0, y_0 \neq y_0' > 0$ and define the sequences $\mathcal{X}', \mathcal{Y}'$ similarly with x_0', y_0'. It is easy to see

$$\mathrm{add}_{\mathcal{X}',\mathcal{X}}(i) = i + c$$

for sufficiently large i, where

$$c = \begin{cases} \mathrm{add}_{\mathcal{X}}(x_0'), & \text{if } \; x_0 \ge x_0' \\ 1 - \mathrm{add}_{\mathcal{X}'}(x_0) & \text{if } \; x_0' > x_0, \; x_0 \neq x_j', \; j = 0, 1, \ldots \\ -\mathrm{add}_{\mathcal{X}'}(x_0) & \text{if } \; x_0' > x_0, \; x_0 \in \mathcal{X}' \end{cases}$$

From Proposition 6 we obtain

$$\text{add}_{\mathcal{X},\mathcal{Y}}(i+c-1) \le \text{add}_{\mathcal{X}',\mathcal{Y}}(i) \le \text{add}_{\mathcal{X},\mathcal{Y}}(i+c) \tag{1}$$

for sufficiently large i. Similarly we obtain

$$\begin{aligned}\text{add}_{\mathcal{X}',\mathcal{Y}} + c' - 1 &= (\text{add}_{\mathcal{Y},\mathcal{Y}'}(\text{add}_{\mathcal{X}',\mathcal{Y}} - 1) \\ &\le \text{add}_{\mathcal{X}',\mathcal{Y}'} \\ &\le \text{add}_{\mathcal{Y},\mathcal{Y}'}(\text{add}_{\mathcal{X}',\mathcal{Y}}) \\ &= \text{add}_{\mathcal{X}',\mathcal{Y}} + c'\end{aligned}$$

with

$$c' = \begin{cases} \text{add}_{\mathcal{Y}'}(y_0), & \text{if } \; y_0' \ge y_0 \\ 1 - \text{add}_{\mathcal{Y}}(y_0'), & \text{if } \; y_0 > y_0',\ y_0' \ne y_j,\ j = 0, 1, \ldots \\ -\text{add}_{\mathcal{Y}}(y_0'), & \text{if } \; y_0 > y_0',\ y_0' \in \mathcal{Z} \end{cases}$$

and by (1),

$$\text{add}_{\mathcal{X},\mathcal{Y}}(i+c-1)c' - 1 \le \text{add}_{\mathcal{X}',\mathcal{Y}'}(i) \le \text{add}_{\mathcal{X},\mathcal{Y}}(i+c) + c'$$

for sufficiently large i. This completes the proof.

3. Formal invariants for non solvable pseudogroups

It is shown in the paper [17] that the non solvable group $\phi(G)$ contains diffeomorphisms $\phi(f), f \in G$ with Taylor expansion at $x = 0$

$$\phi(f)(x) = x - \frac{K}{i}(x^{i+1} + \cdots),$$

$K \ne 0$ with i greater than an arbitrary large integer. So let

$$\phi(g)(x) = x - \frac{L}{j}(x^{j+1} + \cdots),$$

$L \ne 0, i < j$ for a $g \in G$. We call the i, j the *orders of the flatness* for $\phi(f), \phi(g)$ respectively. By Proposition 6 the equivalence class of the address function

$\mathrm{add}_{\mathcal{X},\mathcal{Y}}$ is independent of the choice of x_0, y_0. We denote the equivalence class by $\mathrm{add}_{\phi(g),\phi(f)}$.

First we consider the orbit $\mathcal{Y}$ of y_0 under $\phi(f)$. It is known ([20]) that with a suitable analytic coordinate we may assume $\phi(f)$ has the Taylor expantion

$$\phi(f)(x) = x - \frac{K}{i}(x^{i+1} + (-A + \frac{i+1}{2})x^{2i+1} + \cdots),$$

which is formally conjugate with

$$\phi'(f)(x) = \exp - \frac{K}{i}(\frac{x^{i+1}}{1 + Ax^i})\partial/\partial x.$$

The $-iA/K$ is known as the *residue* of f. By a result due to Takens [20] there exists a C^∞ diffeomorphism $\lambda : \mathbb{R}, 0 \to \mathbb{R}, 0$ i-flat at 0 such that $\lambda \circ \phi(f) = \phi'(f) \circ \lambda$ holds on $U_{\phi(f)}$ shrinking $U_{\phi(f)}$. Introducing the coordinate $\tilde{x} = \xi_{i,A}(x) = x^{-i} + A \log x^{-i}$ for $x > 0$, $\phi'(f)$ induces the translation $\tilde{\phi}(f) = \exp K\partial/\partial\tilde{x}$ on the $\tilde{x}$-line at ∞. Let $y'_n = \lambda(y_n)$ and $\tilde{y}_n = \xi_{i,A}(y'_n)$ for $n = 0, 1, \ldots$. Then

$$\text{(a)} \qquad \tilde{y}_n = \tilde{\phi}(f)^{(n)}(\tilde{y}_0) = \tilde{y}_0 + nK.$$

(The existence of the coordinate $\tilde{x}$ with Property (a) is proved by the sectorial normalisation theorem [12,21] as well as the existence of the solution of Abel's equation by Szekeres [19]. Those results imply the existence of the nomalising diffeomorphism λ real analyticity off 0. But the differentiability at 0 is not an obvious consequence. The analyticity of the conjugacy h off 0 in Theorem 3(1) follows from that of λ. In this paper the smoothness of h (Proposition 9) is first proved and analyticity is proved by the uniqueness (Proposition 10) and the convergence of the formal conjugacy due to Cerveau and Moussu [2].)

We apply the same argument to the slow dynamics $\phi(g)$. Let $\mu : \mathbb{R}, 0 \to \mathbb{R}, 0$ be a C^∞ diffeomorphism j-flat at 0 such that $\mu \circ \phi(g) = \phi'(g) \circ \mu$ holds on $U_{\phi(g)}$, where $\phi'(g)(x) = \exp\ - \frac{L}{j}(\frac{x^{j+1}}{1+Bx^j})\partial/\partial x$ with a constant B. Let

$\tilde{\tilde{x}} = \xi_{j,B}(x) = x^{-j} + B\log x^{-j}$ for $x > 0$. On the $\tilde{\tilde{x}}$-line, $\phi'(g)$ lifts to the translation $\tilde{\tilde{\phi}}(g) = \exp L\ \partial/\partial\tilde{\tilde{x}}$ at ∞.

Let $x'_n = \mu(x_n)$ and $\tilde{\tilde{x}}_n = \xi_{j,B}(x'_n)$ for $n = 0, 1, \ldots$. Then $\tilde{\tilde{x}}_n = \tilde{\tilde{x}}_0 + nL$, from which we obtain the estimate for the $\phi(g)$-orbit $\mathcal{X}$, $x_n = (nL)^{-1/j} + o(n^{-1/j})$ for $n = 0, 1, \ldots$. To compare $\mathcal{X}$ to $\mathcal{Y}$, let

(b) $$\tilde{x}_n = x_n^{-i} + A\ \log x_n^{-i} = (nL)^{i/j} + o(n^{i/j}).$$

From (a) and (b) we obtain

(c) $$\mathrm{add}_{\phi(g),\phi(f)}(n) = \frac{L^{i/j}}{K}\ n^{\frac{i}{j}} + o(n^{\frac{i}{j}}).$$

Proposition 8. *$L^{\frac{i}{j}}/K$ and $\frac{i}{j}$ are topological invariants for the pseudogroup generated by $\phi(f)$ and $\phi(g)$.*

Proof. Assume h is orientation preserving. The linking homeomorphism h sends the pairs of the orbits of x_0 under $\phi(f), \phi(g)$ to that of $h(x_0)$ under $\psi(f), \psi(g)$, and those pairs have the same topological structure and define the same address function up to the equivalence relation. By (c) the i/j is the exponent of the address function and $L^{\frac{i}{j}}/K$ is its coefficient, which are clearly invariant under the equivalence relation. If h is orientation reversing, an alternative argument goes through.

4. Proof of The Theorem 3 for non solvable pseudogroups

First we prove Theorem 3(1) for non solvable pseudogroups. If the linking homeomorphism h is orientation reversing, the homeomorphism $-h$ is orientation preserving and links ϕ to the reversed pseudogroup ψ' consinting of the orientation preserving diffeomorphisms $\psi'(f) : -U_f \to -f(U_f), f \in G$ defined by $\psi'(f)(x) = -\psi(f)(-x)$. So we assume that h is orientation preserving throughout this section. Let $\psi(f)(x) = x - \frac{K'}{i'}(x^{i'+1} + \ldots)$ and $\psi(g)(x) = x - \frac{L'}{j'}(x^{j'+1} + \ldots)$. First assume $(i, j) = (i', j')$ and h is orientation preserving for simplicity. By a linear coordinate transformation we may assume $K = K'$ and then it follows $L = L'$ from Proposition 8. By an analytic coordinate transformation we may assume

$$\psi(f)(x) = x - \frac{K}{i}(x^{i+1} + (-A' + \frac{i+1}{2})x^{2i+1} + \cdots).$$

Let $\lambda' : \mathbb{R}, 0 \to \mathbb{R}, 0$ be a C^∞-diffeomorphism j-flat at 0 such that $\lambda' \circ \psi(f) = \psi'(f) \circ \lambda'$ holds on $U_{\psi(f)}$, where

$$\psi'(f) = \exp\ - \frac{K}{i}\frac{x^{i+1}}{1 + A'x^i}\ \partial/\partial x.$$

Let $\tilde{y} = \xi_{i,A'}(x) = x^{-i} + A' \log x^{-i}$. Since $\phi(f)^{(n)}(x_0) \to 0$, we see $K > 0$.

On the $\tilde{x}$-line the diffeomorphism $\phi(g)$ induces the "non-linear translation"

$$\tilde{\phi}(g)(\tilde{x}) = \tilde{x} + \frac{i}{j}L\ \tilde{x}^{\frac{i-j}{i}} + o(\tilde{x}^{\frac{i-j}{i}})$$

from which

$$\tilde{\phi}(f)^{(-n)} \circ \tilde{\phi}(g) \circ \tilde{\phi}(f)^{(n)}(\tilde{x}) = \tilde{x} + \frac{i}{j}L(nK)^{\frac{i-j}{i}} + o(n^{\frac{i-j}{i}})$$

from which

$$\lim_{n\to\infty} n^{\frac{j-i}{i}}(\tilde{\phi}(f)^{(-n)} \circ \tilde{\phi}(g) \circ \tilde{\phi}(f)^{(n)}\ -\ \mathrm{id})\ \partial/\partial\tilde{x} = \frac{iL}{j}K^{\frac{i-j}{i}}\partial/\partial\tilde{x}$$

holds at the end of the $\tilde{x}$-line. The flow of the above limit vector field is approximated arbitrarily closely by the discrete dynamical system of type

$$\tilde{\phi}(f)^{(-n)} \circ \tilde{\phi}(g)^{(m)} \circ \tilde{\phi}(f)^{(n)}, \qquad m = 0, 1, \ldots$$

with a sufficiently large $n > 0$ ([17]).

Similarly the $\tilde{\psi}(f), \tilde{\psi}(g)$ define the vector field $\frac{iL}{j}K^{\frac{i-j}{i}}\partial/\partial\tilde{y}$ on the $\tilde{y}$-line. The lift $\tilde{h}_+ : \tilde{x} - \text{line}, \infty \to \tilde{y} - \text{line}, \infty$ of the restiction h_+ of h to $\mathbb{R}^+$ sends the orbit of

$$\tilde{\phi}(f)^{(-n)} \circ \tilde{\phi}(g)^{(m)} \circ \tilde{\phi}(f)^{(n)}$$

to that of

$$\tilde{\psi}(f)^{(-n)} \circ \tilde{\psi}(g)^{(m)} \circ \tilde{\psi}(f)^{(n)}.$$

Therefore $\tilde{h}_+$ is compatible with the above flows respecting time hence it is a translation by a constant α_+ (see [17] for a detailed argument) and

$$h_+(x) = \lambda'^{(-1)} \circ \xi_{i,A'}^{-1}(\xi_{i,A} \circ \lambda(x) + \alpha_+),$$

which is i-flat at 0. Similarly we can show that the restriction h_- of h to $\mathbb{R}^-$ is of the form

$$h_-(x) = \lambda'^{(-1)} \circ \xi_{i,A'}^{-1}(\xi_{i,A} \circ \lambda(x) + \alpha_-),$$

with a constant α_-, which is i-flat at 0. With both the above smoothness of h_+ and h_-, we see that the linking homeomorphism h is a C^i-smooth diffeomorphism on a neighbourhood of 0 and i-flat at 0.

Proposition 9. *The linking homeomorphism h is C^∞-smooth on a neighbourhood of 0.*

Proof. Since $\phi(G)_0$ is non solvable, the i can be chosen arbitrary large. Therefore h is C^∞-smooth at 0. The smoothness off 0 is clear by the form of $h_\pm$ above presented.

By the proposition $\phi(f)$ and $\psi(f)$ are C^∞-conjugate. Since the residues A, A' are invariant under formal conjugacy relation of germs of analytic diffeomorphisms, we obtain$A = A'$ hence $\tilde{\phi}(f) = \tilde{\psi}(f)$ and

$$\begin{cases} \lambda' \circ h_+ \circ \lambda^{(-1)} = \exp \frac{-\alpha_+}{i}\chi & \text{on} \quad \mathbb{R}^+ \\ \lambda' \circ h_- \circ \lambda^{(-1)} = \exp\frac{-\alpha_-}{i}\chi & \text{on} \quad \mathbb{R}^-, \end{cases}$$

where χ denotes $\frac{x^{i+1}}{1+Ax^i}\partial/\partial x$.

Proposition 10. *$\alpha_+ = \alpha_-$ and the germ of h at 0 is unique.*

Proof. Since $h_+^{(-1)} \circ \phi(g) \circ h_+ = \psi(g)$ and $h_-^{(-1)} \circ \phi(g) \circ h_- = \psi(g)$ hold on $\mathbb{R}^+$ and $\mathbb{R}^-$ respectively at 0, we obtain the formal equalities

$$\lambda^{(-1)} \circ \exp \frac{\alpha_+}{i}\chi \circ \lambda' \circ \phi(g) \circ \lambda'^{(-1)} \circ \exp \frac{-\alpha_+}{i}\chi \circ \lambda = \phi(f)$$

and

$$\lambda^{(-1)} \circ \exp \frac{\alpha_-}{i}\chi \circ \lambda' \circ \phi(g) \circ \lambda'^{(-1)} \circ \exp \frac{-\alpha_-}{i}\chi \circ \lambda = \phi(f).$$

This shows that $\lambda'^{(-1)} \circ \exp \frac{\alpha_+ - \alpha_+}{i}\chi \circ \lambda$ commutes with $\phi(g)$, and by formal calculation, it follows $\alpha_+ = \alpha_- = \alpha$ (since $i \neq j$). Therefore $h = \lambda^{(-1)} \circ \exp \frac{\alpha}{i}\chi \circ \lambda'$.

Next assume $h' = \lambda^{(-1)} \circ \exp\ -\frac{\beta}{i}\chi \circ \lambda'$ satisfies $h'^{(-1)} \circ \phi(g) \circ h' = \psi(g)$. Then it follows $\alpha = \beta$ from a similar argument. This shows the uiqueness of h.

By a result due to Cerveau and Moussu [2], a formal conjugacy is convergent to give a real analytic conjugacy for non solvable groups of germs of diffeomorphisms. Therefore the Taylor series of h at 0 is convergent to an analytic diffeomorphism $\tilde{h}$ linking $\phi(G)_0$ to $\psi(G)_0$. Then the uniqueness of the linking homeomorphism (Proposition 10) asserts that the germ of h is nothing but the $\tilde{h}$ real analytic on a neighbourhood of 0. The analyticity propagates to whole $B_{\phi(G)}$ by the same argument in the proof of Theorem 1 in §6. This completes the proof of Theorem 3 for the case $(i,j) = (i',j')$ and h is orientation preserving.

Now we prove the theorem for general non solvable pseudogroups. Assume that $\phi(f), \phi(g)$ and $\psi(f), \psi(g)$ have the orders of flatness i,j and i',j' respectively. By Proposition 7, we may write $i'/i = j'/j = p/q$ with even positive integers p,q. Define the lift $\phi_p^\epsilon : G \to \Gamma_+^\omega$ by $\phi_p^\epsilon(f) : U_{\phi_p^\epsilon(f)} \to \phi_p^\epsilon(f)(U_{\phi_p^\epsilon(f)})$, $\phi_p^\epsilon(f)(x) = (\epsilon\phi(f)(\epsilon x^p))^{1/p}$ for $\epsilon = \pm 1$, where $U_{\phi\epsilon_p(f)}$ is the preimage of $U_{\phi(f)}$ by $x :\longrightarrow \epsilon x^p$. Define the lift $\psi_q^\epsilon : G \to \Gamma_+^\omega$ similarly. Then $\phi_p^\epsilon(f), \phi_p^\epsilon(g)$ have the orders of flatness pi, pj respectively. The linking homeomorphism h lifts to the orientation preserving homeomorphism $K^\epsilon = (\epsilon\ h(\epsilon x^p))^{1/q}$ of $U_p^\epsilon = \{x |\ \epsilon x^p \in U\}$ to $U_q^\epsilon = \{y |\ \epsilon y^q \in h(U)\}$, which is linking ϕ_p^ϵ to ψ_q^ϵ for $\epsilon = \pm 1$.

Proposition 11. *(1) ϕ is solvable if and only if ϕ_p^1 is solvable if and only if ϕ_p^{-1} is solvable.*

(2) $B_{\phi_p^\epsilon} = \{x|\ \epsilon x^p \in B_\phi\}$ *for* $\epsilon = \pm 1$.

Proof. The homomorphism of pseudogroups which asigns $\phi_p^\epsilon(f)$ to $\phi(f)$ for $f \in G$ induces a group isomorphism of the germs $\phi(G)_0$ to $\phi_p^\epsilon(G)_0$ for $\epsilon = \pm 1$. So Statement (1) is clear. Statement (2) for the basin follows from the definition.

By the result obtained previously in this section, the lift K^ϵ is a unique real analytic diffeomorphism. In particular h is unique and the restriction $h : B_\phi(G) - 0 \to B_\psi(G) - 0$ is a real analytic diffeomorphism. This completes the proof of Theorem 3 for non solvable pseudogroups.

5. Proof of Theorem 3 for solvable pseudogroups

Theorem 12 ([17]). *A solvable subgroup H of the group of germs of analytic diffeomorphisms of $\mathbb{R}$ respecting 0 is C^ω-conjugate with one of the following:*

(1) H consists of linear functions ax with the coefficients a in a subgroup L of $\mathbb{R}^*$.

(2) H consists of $f^{(\alpha)} = x + \alpha K x^{i+1} + \cdots, \alpha \neq 0$ with α in a subgroup $\Lambda \subset \mathbb{R}$, $1 \in \Lambda$. Here $f \in H, f(x) = x + Kx^{i+1} + \cdots$ and $f^{(\alpha)}$ is the unique real analytic diffeomorphism with the Taylor expantion $f^{(\alpha)}(x) = x + \alpha K x^{i+1} + \cdots$ such that $f^{(\alpha)} \circ f = f \circ f^{(\alpha)} = f^{(\alpha+1)}$. If Λ is dense in $\mathbb{R}$, those $f^{(\alpha)}$ are written as exp $\alpha\chi$ with an i-flat real analytic vector field χ on $\mathbb{R}$. (for the definition of the α-times iteration $f^{(\alpha)}$ see the papers [17.19].)

(3) H consists of those $f^{(\alpha)}$ and $-f^{(\alpha+\beta)}$ with $\alpha \in \Lambda \subset \mathbb{R}$ and a $\beta, 2\beta \in \Lambda$ and f satisfies the relation $f(-x) = -f(x)$.

(4) H consists of those f^α in (2) and $af^{(\alpha+\beta(a))}$ with a in a subgroup $L \subset \mathbb{R}^*, a^i \neq 1$. Here f satisfies the relation $a^{-1}f(ax) = f^{(a^i)}$ for $a \in L$ and $\beta : L \to \mathbb{R}$ is a function and $\mathrm{res}(f) = 0$*. i.e. f is formally and C^∞-conjugate with exp $Kx^{i+1}\partial/\partial x, K \neq 0$.*

In Cases (1),(2) and (3), the H is commutative. and in Case (4), H is non commutative but solvable.

Since the members of our pseudogroups $\phi(G), \psi(G)$ are all orientation preserving, the germs $\phi(G)_0, \psi(G)_0$ are C^ω-conjugate to one of the H in Cases (1),(2) and (4). In the following we assume the germs are of the form in those cases and prove the the analyticity of the restrictions h_+, h_- of the linking homeomorphism h to $\mathbb{R}^+, \mathbb{R}^-$ on a neighbourhood of 0. The differentiability propagates to whole $B_{\phi(G)} - 0$ by the same argument as in the proof of theorem 1 in §6.

Case (1). Assume $\phi(G)_0 \neq \mathbb{Z}$. This assumption is equivalent to that the linear term group L_ϕ of $\phi(G)_0$ is a dense subgroup of $\mathbb{R}^*$, in other words, all orbits are dense nearby 0. Let $\log L_\phi$ denote the subgroup of $\mathbb{R}$ consisting of the logarithms of the linear terms of $\phi(f), f \in G$. Since h sends the $\phi(G)$-orbit of an x to the $\psi(G)$-orbit of $h(x)$, h induces a homomorphism $\tilde{h}$ of the subgroups $\log L_\phi$ to $\log L_\psi$, which extends to a linear function kx. By this form we see $\log \circ h \circ \exp(x)$ is an affine transformation $kx + l$, from which $h(x) = (\exp l)x^k$ for $x > 0$. A similar argument shows the analyticity of h_-.

Case (2). In this case the germs of $\phi(f)^{(\alpha)}$ are of the form $\exp \alpha\chi$ with a flat analytic vector field χ and α in a subgroup $\Lambda \subset \mathbb{R}$. The hypothesis that $\phi(G)_0$ is not isomorphic to $\mathbb{Z}$ implies that Λ is a dense subgroup. Let $\Lambda' \subset \mathbb{R}$ be the group associated to $\psi(G)$. The correspondence of $\phi(G)$-orbits and $\psi(G)$-orbits in $\mathbb{R}^+$ by h induces a linear transformation of Λ to Λ', which describes the h conversely. Therefore the h_+ is real analytic off 0, and similarly it is shown that h_- is analytic off 0.

Case (4). Let $\phi(G)_0^0 \subset \phi(G)_0$ denote the subgroup consisting of the i-flat germs of diffeomorphisms $\phi(f)^{(\alpha)}, \alpha \in \Lambda \subset \mathbb{R}$ of $\phi(G)$, and $\psi(G)_0^0 \subset \psi(G)_0$ the subgroup consisting of j-flat germs of diffeomorphisms $\psi(f)^{(\alpha)}, \alpha \in \Lambda \subset \mathbb{R}$. It suffices here to prove the analyticity of h for the case $i = j$.

Lemma 13. *Let $\phi(f), \psi(f) : \mathbb{R}, 0 \to \mathbb{R}, 0$ be germs of analytic diffeomorphisms with the linear term x and the order of flatness $i \geq 1$, and let $h : \mathbb{R}, 0 \to \mathbb{R}, 0$ be a germ of homeomorphism such that $h \circ \phi(f) = \psi(f) \circ h$. Then h is differentiable at 0.*

Proof. By C^∞- coordinate change we may assume $\phi(f) = \exp - \frac{K}{i}\frac{x^{i+1}}{1+Ax^i}\partial/\partial x$ and $\psi(f) = \exp - \frac{L}{i}\frac{x^{i+1}}{1+Bx^i}\partial/\partial x$, and by a linear coordinate transformation, $K = L > 0$. These diffeomorphisms lift to the translations by K respectively on the $\tilde{x}$-line, $\tilde{x} = \xi_{i,A}(x) = x^{-i} + A\log x^{-i} (x > 0)$, and the $\tilde{y}$-line, $\tilde{y} = \xi_{i,B}(y)$. And these translations are conjugate by the lift $\tilde{h} : \tilde{x} - line \to \tilde{y} - line$ of h . So we obtain an extimate $|\tilde{h}(\tilde{x}) - \tilde{x} - T| \leq K$, with a constant T, from which

$$\xi_{i,B}^{-1}(\xi_{i,A}(x) + T + K) \leq h(x) \leq \xi_{i,B}^{-1}(\xi_{i,A}(x) + T - K)$$

This implies the differentiability of h at 0.

Next let $\phi(g)(x) = ax + \cdots$, $a \neq 0, 1$ be a diffeomorphism non commutative with $\phi(f)$ and $\psi(g)(x) = a'x + \cdots$ $a' \neq 0, 1$. By assumption $\psi(g) \circ h = h \circ \phi(g)$ holds, and by the differentiability of h at 0, we obtain $a = a'$.

Lemma 14. *Let $h : \mathbb{R}, 0 \to \mathbb{R}, 0$ be the germ of a mapping commutating with a linear function ax. If h is differentiable at 0, h is linear.*

Proof. By the commutativity, $h(a^i x)/a^i x = h(x)/x$ for all x and $i = 0, 1, \ldots$. By the differentiability, $h(x)/x$ is a constant independent of x.

By the Poincaré linearization theorem $\phi(g), \psi(g)$ are analytically conjugate with ax. Here Lemma 14 applies to say that the germ of h at 0 is linear. In this situation the relation $h \circ \phi(f) = \psi(f) \circ h$ admits the unique linear map h. This completes the proof of Theorem 3.

6. Proof of Theorem 1and Corollaries 2,4

Proof of Theorem 1. Let L be a leaf of $\mathcal{F}_1$ with holonomy group $\neq 0, \mathbb{Z}$. Then the image $h(L)$ has holonomy isomorphic to that of L and, by Theorem 4, h is transversely analytic on a deleted neighbourhood $U - L$ of an $x \in L$. Let $x' \in M_1$ be an arbitrary point. The leaf $L_{x'}$ of $\mathcal{F}_1$ containing x' is dense by assumption, hence a point $x'' \in L_{x'}$ is contained in $U - L$. Clearly the translation $T_{x',x''}$ along a path in $L_{x'}$ sending the transverse section at x' to that of x'' is analytic, and the germs of h at x', x'' link the $T_{x',x''}$ to the transverse dynamics $T_{h(x'),h(x'')}$ along $h(L_{x'}) = L_{h(x')}$. Therefore the transverse analyticity of h at x'' induces the transverse analyticity on a neighbourhood of x'. This completes the proof of Theorem 1.

Proof of Corollary 2. The Godbillon-Vey class $\mathrm{GV}(\mathcal{F})$ of $\mathcal{F}$ may be defined by the pull back $\rho(\mathcal{F})^*c$ of a cocycle $c \in H^3(B\Gamma_{\mathbb{R}}^{\infty}, \mathbb{R})$ of the classifying space $B\Gamma_{\mathbb{R}}^{\infty}$ of the pseudogroup $\Gamma_{\mathbb{R}}^{\infty}$ of orientation preserving C^{∞}-diffeomrphisms of open subsets of $\mathbb{R}$ by the classifying map $\rho(\mathcal{F}) : M \longrightarrow B\Gamma_{\mathbb{R}}^{\infty}$ ([1]). Since $h(\mathcal{F}) = \mathcal{F}'$ and h is transversely real analytic, if follows $\rho(\mathcal{F}') \circ h = \rho(\mathcal{F}')$, from which $\mathrm{GV}(\mathcal{F}) = h^*\mathrm{GV}(\mathcal{F}')$. This completes the proof of Corollary 2.

Proof of Corollary 4. Let $\phi, \psi : \Gamma^g \longrightarrow \mathrm{Diff}_+^{\omega}(S^1)$ be homomorphisms and $h : \phi \longrightarrow \psi$ a linking homeomorhism. Let $\mathrm{stab}(x_0) \subset \Gamma^g$ be the stabiliser of an $x_0 \in S^1$. Then h links the restriction of ϕ to $\mathrm{stab}(x_0)$ to that of ψ. Assume that $\phi(\mathrm{stab}(x_0))$ is not isomorphic to $\mathbb{Z}$ and non trivial. Then by the rigidity theorem (Theorem 3), h is a real analytic diffeomorphism on a deleted neighbourhood $U - x_0$ of x_0 in S^1. By a result due to Ghys [6], if $|\mathrm{eu}(\phi)| \neq 0$, all orbits are dense in S^1. So, for any $y \in S^1$, there is a $g \in G$ such that $\phi(g)(y) \in U - x_0$. Then the equality $h \circ \phi(g) = \psi(g) \circ h$ implies that h is a real analytic diffeomorphism at y. This completes the proof of Corollary 4.

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Astérisque

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Astérisque, tome 222 (1994), p. 345-371
<http://www.numdam.org/item?id=AST_1994__222__345_0>

GERMES DE FEUILLETAGES HOLOMORPHES À HOLONOMIE PRESCRITE

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I. INTRODUCTION

Considérons au voisinage de l'origine dans $\mathbf{C}^2$ un germe de champ de vecteurs holomorphe, $X = X_1(x,y)\frac{\partial}{\partial x} + X_2(x,y)\frac{\partial}{\partial y}$ s'annulant en 0. On veut étudier le feuilletage holomorphe singulier $\mathcal{F}$ défini au voisinage de 0 par l'équation différentielle:

$$\begin{cases} \dot{x} = & X_1(x,y) \\ \dot{y} = & X_2(x,y) \end{cases}$$

Son caractère dépend essentiellement des valeurs propres, notées λ_1 et λ_2, de la partie linéaire de X en 0.

Par le théorème de A. Seidenberg ([**Se**]), après un nombre fini d'éclatements on aboutit à des singularités primitives (ou irréductibles) où $\lambda_2 \neq 0$ et λ_1/λ_2 n'est pas un rationnel strictement positif. Pour ces singularités deux situations se présentent:

• Domaine de Poincaré: C'est le cas où $\lambda_1/\lambda_2 \in \mathbf{C}-\mathbf{R}_-$. H. Poincaré ([**Po**]) a montré que l'équation différentielle est *linéarisable* au voisinage de 0, i.e. par

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un changement de variables holomorphe le champ X s'écrit $\lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y}$. La structure du feuilletage local $\mathcal{F}$ est alors topologiquement et analytiquement déterminée.

• Domaine de Siegel: Ce cas correspond à $\lambda_1/\lambda_2 \in \mathbf{R}_-$. On a encore deux cas à distinguer: $\lambda_1/\lambda_2 \in \mathbf{Q}_-$-c'est le cas *résonant*-et $\lambda_1/\lambda_2 \in \mathbf{R}_- - \mathbf{Q}_-$.

L'étude du cas résonant a été complètement effectuée par J. Martinet et J.-P. Ramis (**[M-R1]** et **[M-R2]**) qui ont déterminé tous les invariants holomorphes d'un tel germe de feuilletage.

Dans le cas non résonant, on n'a pas de *résonances* (d'où la terminologie) du type:

$$\lambda_i = q_1\lambda_1 + q_2\lambda_2,$$

avec $i = 1, 2$ et $q_1, q_2 \in \mathbf{N}^*$.

Ceci permet de montrer facilement que le champ X est formellement linéarisable; cependant ce changement de variables formel peut diverger à cause des problèmes de petits diviseurs. Un théorème remarquable de C.- L. Siegel (**[Si1]**,**[Si2]**) montre que la linéarisante converge lorsque $\alpha = -\lambda_2/\lambda_1 \in \mathbf{R} - \mathbf{Q}$ satisfait à une condition diophantinne, i.e. il existe $\gamma, \tau > 0$ tels que pour $p/q \in \mathbf{Q}$,

$$|\alpha - p/q| \geq \frac{\gamma}{q^\tau}.$$

Cette condition sur α est de mesure de Lebesgue totale. A. Bruno a amélioré la démonstration de Siegel et obtient le résultant sous une condition arithmétique plus faible: Si $(p_n/q_n)_{n\geq 0}$ est la suite des réduites de α cette conditon s'écrit

$$(\mathcal{B}) \qquad \sum_{n\geq 0} \frac{\log q_{n+1}}{q_n} < +\infty.$$

Par ailleurs il est connu qu'il existe des valeurs de $\alpha \in \mathbf{R} - \mathbf{Q}$ pour lesquelles la linéarisante peut diverger (Bruno, Ilyashenko, Pyartli). La topologie du feuilletage dans le cas non linéarisable n'est pas comprise, on va voir qu'elle peut être très complexe.

Dans la suite on se placera dans le domaine de Siegel avec $\alpha = -\lambda_2/\lambda_1 > 0$ rationnel ou irrationnel. Dans ce cadre il existe deux variétés analytiques invariantes par le champ de vecteurs X qui passent par l'origine (Briot-Bouquet [Bri], H. Dulac [Du]). Plus précisement, après un changement de variables holomorphe, le champ X s'écrit

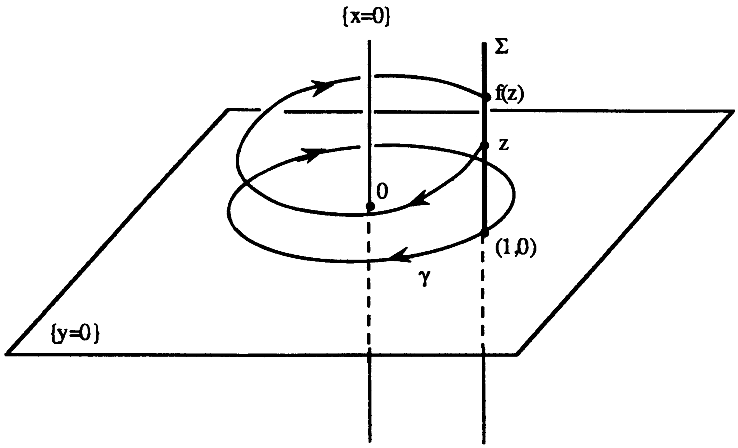
$$X = \lambda_1 x(1 + \ldots)\frac{\partial}{\partial x} + \lambda_2 y(1 + \ldots)\frac{\partial}{\partial y},$$

donc les axes $\{x = 0\}$ et $\{y = 0\}$ sont des variétés invariantes de X (une démonstration se trouve dans [M-M] appendice II). On peut alors considérer l'*holonomie* de la variété invariante $\mathcal{F}_0 = \{(x, 0); x \neq 0\}$ suivant le lacet $\gamma : [0,1] \to \mathbf{C}^2$, $\gamma(t) = (e^{-2\pi i t}, 0)$ (quitte à conjuguer par une homothétie). Pour cela on choisit une transversale holomorphe Σ en $(1, 0)$ à $\mathcal{F}_0$ (par exemple $\{x = 1\}$), et on relève γ aux feuilles voisines de $\mathcal{F}_0$. L'application de retour sur Σ définit (en considérant une carte sur Σ) un germe de difféomorphisme holomorphe de $(\mathbf{C}, 0)$, $f(z) = \lambda z + \mathcal{O}(z^2)$, et on montre facilement que $\lambda = e^{2\pi i\alpha}$ ([M-M] p. 480 pour plus de détails).

Les choix du lacet γ dans sa classe d'homotopie dans $\mathcal{F}_0$, de la transversale Σ et de la carte sur Σ n'affectent pas la classe de conjugaison de f. De même, si on transforme le feuilletage par un germe de difféomorphisme holomorphe de $(\mathbf{C}^2, 0)$, l'holonomie obtenue pour la variété invariante correspondante est encore dans la même classe que f.

En résumé, on obtient ainsi une application, notée Hol_α, définie sur l'ensemble des classes de conjugaison de germes de feuilletages holomorphes singuliers de $(\mathbf{C}^2, 0)$ dans le domaine de Siegel avec $\alpha = -\lambda_2/\lambda_1 > 0$ à valeurs dans l'ensemble des classes de conjugaison des germes de difféomorphismes holomorphes de $(\mathbf{C}, 0)$ de partie linéaire $z \mapsto e^{2\pi i\alpha} z$.

On démontrera ici que cette application est bijective. En d'autres termes, la classification analytique de tels germes de feuilletages singuliers est la même que celle des germes de difféomorphismes holomorphes de $(\mathbf{C}, 0)$.



J.-F. Mattei et R. Moussu ([M-M]) ont montré que $\mathcal{F}$ est linéarisable si et seulement si f est ***linéarisable*** (i.e. conjugué par un difféomorphisme de $(\mathbf{C}, 0)$ à sa partie linéaire), et, plus généralement, que les holomomies de deux feuilletages sont conjuguées si et seulement s'ils sont eux-mêmes conjugués. La démonstration consiste à prolonger la conjugaison, définie sur les transversales, le long des feuilles. Ceci démontre l'injectivité de Hol_α pour toute valeur $\alpha > 0$.

La bijectivité est immédiate lorsque $\alpha \in \mathcal{B}$. En effet, les techniques de Siegel s'appliquent aussi pour les germes de difféomorphismes holomorphes de $(\mathbf{C}, 0)$ pour démontrer la linéarisabilité lorsque $\alpha \in \mathcal{B}$ ([Si1], [Br]). Dans ce cas, le deux ensembles de classes de conjugaison sont réduits à un seul élément. Le cas $\alpha \in \mathbf{Q}_+^*$ a été résolu par J. Martinet et J.-P. Ramis ([M-R]) qui, à partir de la classification des germes obtenue par J. Ecalle et S.-M. Voronin (pour $\alpha \in \mathbf{Q}_+^*$, [Ec], [Vo]) ont montré qu'on pouvait réaliser tout germe holomorphe $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$ comme holonomie des feuilletages considérés. Ceci prouve la surjectivité de Hol_α lorsque $\alpha \in \mathbf{Q}_+^*$.

On se propose de démontrer la surjectivité dans le cas général $\alpha > 0$:

THÉORÈME. *Pour $\alpha > 0$, tout germe holomorphe $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$ est réalisé comme holonomie d'un feuilletage défini par*

$$\begin{cases} \dot{x} = -x(1+\ldots) \\ \dot{y} = \alpha y(1+\ldots) \end{cases}$$

i.e. l'application Hol_α est bijective.

La démonstration n'utilise aucun résultat relatif à la structure des classes de conjugaison qui, comme on sait (voir ci-dessous), est compliquée dans le cas non linéarisable. Elle ne distingue pas le cas rationnel du cas irrationnel. Ceci donne une démonstration du théorème de Martinet et Ramis qui n'utilise pas la structure connue de la dynamique rationnelle.

Le théorème montre l'équivalence de deux problèmes. Il se trouve que celui relatif aux germes de difféomorphismes de $(\mathbf{C}, 0)$ est mieux étudié. On a déjà indiqué que pour $\alpha \in \mathcal{B}$ tout germe holomorphe $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$ est linéarisable. Le deuxième auteur a montré que cette condition arithmétique est optimale : Si $\alpha \notin \mathcal{B}$ il existe un germe holomorphe $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$ non linéarisable. D'où l'optimalité pour les feuilletages:

Corollaire 1. *Si $\alpha \notin \mathcal{B}$ il existe un feuilletage singulier défini par un germe de champ de vecteurs holomorphe $X = -x(1+\ldots)\frac{\partial}{\partial x} + \alpha y(1+\ldots)\frac{\partial}{\partial y}$ non linéarisable.*

Les applications f construites dans [Yo] ont des orbites périodiques qui tendent vers l'origine et on peut fixer avec liberté la classe de conjugaison de l'application de retour au voisinage de ces orbites périodiques. Ceci fourni une grande variété de singularités de feuilletages holomorphes de topologie très complexe.

D'autres exemples sont obtenus à partir des applications $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$ sans autre orbite périodique que 0 construites par le premier auteur ([PM]) lorsque α satisfait $\sum_{n\geq 1} q_n^{-1} \log\log q_{n+1} = +\infty$. On a

Corollaire 2. *Sous la condition arithmétique précédente, il existe un feuilletage singulier défini dans un voisinage U de 0 par un germe de champ de vecteurs holomorphe $X = -x(1+\ldots)\frac{\partial}{\partial x} + \alpha y(1+\ldots)\frac{\partial}{\partial y}$ non linéarisable et dont toute feuille dans $U - \{xy = 0\}$ est simplement connexe.*

Un autre résultat de [PM] montre qu'ici encore la condition arithmétique précédente est optimale pour avoir ce phénomène.

Il se pose naturellement la question de savoir quand est-ce qu'on peut réaliser globalement, par exemple dans $\mathbf{P}^2\mathbf{C}$, ces feuilletages locaux. Les considérations sur les germes de difféomorphismes peuvent laisser penser que c'est le cas pour tout $\alpha \notin \mathcal{B}$ pour les feuilletages non linéarisables du corollaire 1 : Si $\alpha \notin \mathcal{B}$, $f(z) = e^{2\pi i\alpha}z + z^2$ est non linéarisable ([Yo]). Mais, on peut être plus sceptique quand à ceux du corollaire 2 : Pour $\alpha \notin \mathcal{B}$, le deuxième auteur a montré que $f(z) = e^{2\pi i\alpha}z + z^2$ a une suite d'orbites périodiques qui tendent vers 0.

On décrit ci-dessous l'idée de la démonstration du théorème faite en II.

Une facon naturelle de considérer le feuilletage singulier $\mathcal{F}$.

On a vu que dans le domaine de Siegel avec $\alpha = -\lambda_2/\lambda_1 > 0$, le feuilletage $\mathcal{F}$ est défini dans des coordonnées convenables, par :

$$\begin{cases} \dot{x} = -x(1+\ldots) \\ \dot{y} = \alpha y(1+\ldots) \end{cases}$$

La géométrie intéressante de $\mathcal{F}$ se trouve en dehors des axes invariants, i.e. dans un voisinage de 0 privé de $\{xy=0\}$. Il est donc naturel et commode de se placer dans le revêtement universel $E : \mathbf{C}^2 \to \mathbf{C}^2 - \{xy=0\}$ défini par

$$E(z_1, z_2) = (e^{2\pi i z_1}, e^{2\pi i z_2}) = (x, y).$$

L'ouvert $\{0 < |x| < \varepsilon, 0 < |y| < \varepsilon\}$ se relève en $\{\operatorname{Im} z_1 > C, \operatorname{Im} z_2 > C\}$ (avec $\exp(-2\pi C) = \varepsilon$), et le champ en un champ

$$\begin{cases} 2\pi i \dot{z}_1 = -1 + \mathcal{O}(\exp[-2\pi \operatorname{Max}(\operatorname{Im} z_1, \operatorname{Im} z_2)]) \\ 2\pi i \dot{z}_2 = \alpha + \mathcal{O}(\exp[-2\pi \operatorname{Max}(\operatorname{Im} z_1, \operatorname{Im} z_2)]) \end{cases}.$$

Les feuilles du feuilletage associé sont proches de plans $\{z_2 + \alpha z_1 = \text{cte}\}$. On récupère le feuilletage $\mathcal{F}$ initial en quotientant $\mathbf{C}^2$ par T_1 et T_2,

$$T_1(z_1, z_2) = (z_1 + 1, z_2),$$
$$T_2(z_1, z_2) = (z_1, z_2 + 1),$$

et en transportant par E. Il existe un changement de variables holomorphe, proche de l'identité, $(z_1, z_2) \mapsto (w_1, w_2)$ qui rend le feuilletage linéaire et perturbe T_1 et T_2 en des applications F_1 et F_2 qui commutent et sont proches de T_1 et T_2 respectivement.

Il est donc naturel de penser qu'on peut récupérer tout feuilletage à partir du feuilletage linéaire de $\mathbf{C}^2$, $z_2 + \alpha z_1 = \text{cte}$, en quotientant un domaine approprié par deux applications F_1 et F_2 telles que ci-dessus. C'est à dire, faire passer la non linéarité dans les applications de recollement plutôt que dans le feuilletage. C'est cette idée qui est mise en oeuvre.

Comment reconstruire un feuilletage à partir de son holonomie.

Soit $\alpha > 0$. Le cas d'une holonomie linéaire est très simple. Considérons le feuilletage de $\mathbf{C}^2$ en droites complexes $z_2 + \alpha z_1 = \text{cte}$, i.e. défini par la $(1,0)$-forme $dz_2 + \alpha dz_1$. On quotiente $\mathbf{C}^2$ par les translations T_1 et T_2 définies auparavant. La $(1,0)$-forme $dz_2 + \alpha dz_1$ est invariante par T_1 et T_2, passe au quotient, et définit un feuilletage holomorphe de $\mathbf{C}^2/\mathbf{Z}^2$. Or l'application $E : \mathbf{C}^2/\mathbf{Z}^2 \to \mathbf{C}^2 - \{xy = 0\}$, est un difféomorphisme holomorphe. La forme $dz_2 + \alpha dz_1$ devient $\dfrac{dy}{y} + \alpha \dfrac{dx}{x}$. On a construit ainsi le feuilletage linéaire de $\mathbf{C}^2$ définit par $xdy + \alpha ydx$, d'holonomie linéaire.

On veut réaliser de la même facon l'holonomie d'un représentant $f(z) = e^{2\pi i\alpha} z + \mathcal{O}(z^2)$ d'une classe quelconque de conjugaison. Quitte à conjuguer par une homothétie, on peut supposer f défini sur le disque unité $\mathbf{D}$ et on peut relever à $\mathbf{H} = \{z \in \mathbf{C}; \operatorname{Im} z > 0\}$ vu comme revêtement de $\mathbf{D} - \{0\}$ via l'application $z \mapsto \exp(2\pi i z)$. On obtient $F : \mathbf{H} \to \mathbf{C}$, $F(z) = z + \alpha + \Phi(z)$, avec Φ $\mathbf{Z}$-périodique et $\lim\limits_{\operatorname{Im} z \to +\infty} \Phi(z) = 0$. On peut supposer $||\Phi||_{C^0(\mathbf{H})}$ petit.

On considère à nouveau le feuilletage de $\mathbf{C}^2$ par les droites complexes $\{z_2 + \alpha z_1 = \text{cte}\}$. Cette fois on voudrait quotienter un domaine de $\mathbf{C}^2$ par l'action de

$$F_1(z_1, z_2) = (z_1 + 1, z_2 + \Phi(z_2 + \alpha z_1)),$$
$$F_2(z_1, z_2) = (z_1, z_2 + 1).$$

On ne considère F_1 et F_2 que dans un domaine U où $\operatorname{Im}(z_2 + \alpha z_1) > 0$ et Φ est définie et petite en valeur absolue. Donc on a $F_1 \circ F_2 = F_2 \circ F_1$ et F_1 et F_2 sont proches respectivement de T_1 et T_2. La feuille passant par le point

$(0, z_2)$ sera identifiée avec celle contenant $(1, z_2 + \Phi(z_2))$ qui passe elle-même par le point $(0, z_2 + \alpha + \Phi(z_2)) = (0, F(z_2))$. Donc l'application de retour du lacet $\gamma(t) = (1 - t, z_2 + \Phi(z_2) + \alpha t)$, $0 \leq t \leq 1$, sur la transversale $\{z_1 = 0\}$ est précisément F.

On peut quotienter U par l'action de F_1 et F_2, et obtenir une variété complexe M. Pour poursuivre la construction il faut montrer que M est biholomorphe à un voisinage de 0 dans $\mathbf{C}^2$ privé de $\{xy = 0\}$. Ceci est vrai pour la structure C^∞ : En effet, on peut conjuguer de facon C^∞ F_1 et F_2 à T_1 et T_2 respectivement (II.3), ce qui revient à construire un difféomorphisme v de classe C^∞ de M dans $W \subset \mathbf{C}^2/\mathbf{Z}^2$. On peut aussi multiplier la forme $dz_2 + \alpha dz_1$, par une fonction de classe C^∞ pour qu'elle passe au quotient (II.3). Puis le difféomorphisme $E : \mathbf{C}^2/\mathbf{Z}^2 \to \mathbf{C}^2 - \{xy = 0\}$ envoi W sur un voisinage de 0 dans $\mathbf{C}^2$ privé de $\{xy = 0\}$, où l'on obtient un feuilletage de classe C^∞. Ainsi la construction en classe C^∞ est triviale.

La clé de la construction est que l'on peut perturber le difféomorphisme v en un difféomorphisme holomorphe de M dans $\mathbf{C}^2/\mathbf{Z}^2$. Le défaut d'holomorphie de v est localisé dans une de ces coordonnées (II.3). On est amené à résoudre une équation $\bar{\partial}$ avec des estimées (II.6 et II.7). Pour cela l'outil essentiel est une variante d'un résultat de L. Hörmander ([Ho1], [Ho2]), repris dans l'appendice (III). Il faut d'abord se placer dans une variété de Stein. Or, en choisissant convenablement le bord de l'ouvert U où est réalisé le quotient, on peut construire dans M une fonction strictement plurisousharmonique, minorée et propre (II.5); donc M sera une variété de Stein (II.5). Ensuite, en utilisant que le difféomorphisme C^∞, v, de M dans W est C^4 proche de l'identité dans les coordonnées (z_1, z_2) sur U (II.4), on résout dans M une équation $\bar{\partial}$ dont la solution est petite (le défaut d'holomorphie l'est aussi, cf. estimées II.4 et II.7); ceci permet de perturber v en un difféomorphisme holomorphe $\tilde{v} : M \to \tilde{W} \subset \mathbf{C}^2/\mathbf{Z}^2$. Finalement $E \circ \tilde{v}$ est l'uniformisation cherchée. Il faut encore résoudre une équation $\bar{\partial}$ pour le défaut d'holomorphie de la forme C^∞ qui définit le feuilletage sur M et obtenir ainsi une $(1,0)$-forme holomorphe pour le feuilletage final (I.8).

II. Démonstration du théorème.

1. Nous rappelons les notations déjà introduites. On notera dans la suite $\mathbf{D} = \{z \in \mathbf{C}; |z| < 1\}$ et $\mathbf{H} = \{z \in \mathbf{C}; \operatorname{Im} z > 0\}$.

2. Fixons une fois pour toutes un nombre réel $\alpha > 0$, un nombre réel $\varepsilon \in]0,1[$, ainsi qu'une fonction $\eta : \mathbf{R} \to \mathbf{R}$ monotone de classe C^∞ vérifiant $0 \leq \eta \leq 1$, $\eta(t) = 0$ pour $t \geq 2/3$ et $\eta(t) = 1$ pour $t \leq 1/3$.

Nous posons $\alpha_1 = (1+\varepsilon)\alpha$, $\alpha_2 = (1-\varepsilon)\alpha$ et noterons C_1, C_2, ... des constantes ne dépendant que de ε, η, α. Donnons-nous un germe de difféomorphisme holomorphe $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$ de $(\mathbf{C}, 0)$. Quitte à conjuguer f par une homothétie, il existe un relèvement

$$z \mapsto F(z) = z + \alpha + \Phi(z)$$

de f par le revêtement $z \mapsto e^{2\pi iz}$ de $\mathbf{D}^*$ par $\mathbf{H}$, tel que F soit holomorphe et injectif dans $\mathbf{H}$, et que la fonction $\mathbf{Z}$-périodique Φ vérifie :

$$\lim_{\operatorname{Im} z \to \infty} \Phi(z) = 0,$$

et pour $0 \leq i \leq 4$, $z \in \mathbf{H}$,

$$|D^i\Phi(z)| \leq \frac{1}{2}e^{-2\pi \operatorname{Im} z}.$$

3. Posons :

$$U = \{(z_1, z_2) \in \mathbf{C}^2; \operatorname{Im}(z_2 + \alpha z_1) > 0\}.$$

Définissons dans U les applications :

$$F_1(z_1, z_2) = (z_1 + 1, z_2 + \Phi(z_2 + \alpha z_1)),$$
$$F_2(z_1, z_2) = (z_1, z_2 + 1).$$

On a $F_1 \circ F_2 = F_2 \circ F_1$. On va quotienter une partie de U par l'action de F_1 et F_2.

On définit

$$v_0(z_1, z_2) = \eta(\operatorname{Re} z_1)\Phi(z_2 + \alpha z_1),$$
$$v_1(z_1, z_2) = \eta(\operatorname{Re} z_1)\log(1 + D\Phi(z_2 + \alpha z_1)).$$

On considère l'application

$$v(z_1, z_2) = (z_1, z_2 + v_0(z_1, z_2)) = (w_1, w_2),$$

puis la $(1,0)$-forme de classe C^∞ :

$$\Omega_0 = e^{v_1(z_1,z_2)}(dz_2 + \alpha dz_1).$$

D'après les estimations sur Φ dans $\mathbf{H}$, nous avons :

Lemme 1. *Soit ∂^l une dérivée partielle d'ordre inférieur ou égal à trois. Pour $i = 0, 1$ et $(z_1, z_2) \in U$, on a*

$$\left|\partial^l v_i(z_1, z_2)\right| \leq C_1 \exp[-2\pi \operatorname{Im}(z_2 + \alpha z_1)].$$

Notons $T_1 : (w_1, w_2) \mapsto (w_1 + 1, w_2)$ et $T_2 : (w_1, w_2) \mapsto (w_1, w_2 + 1)$ les translations indiquées de $\mathbf{C}^2$. Nous avons clairement :

$$v \circ F_2 = T_2 \circ v,$$
$$F_2^* \Omega_0 = \Omega_0.$$

D'après le choix de η, nous avons aussi, pour $(z_1, z_2) \in U \cap F_1^{-1}(U)$, $|\operatorname{Re} z_1| \leq 1/3$:

$$v \circ F_1 = T_1 \circ v,$$
$$F_1^* \Omega_0 = \Omega_0.$$

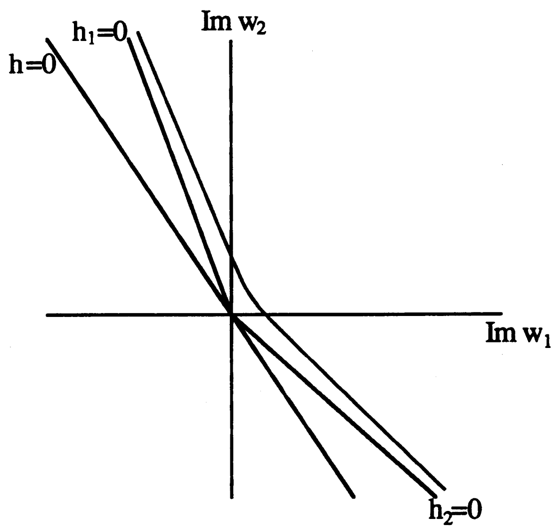
Posons alors, pour $R > 0$ (figure) :

$$\begin{aligned} V_R = \{(w_1, w_2) \in \mathbf{C}^2;\ -1/4 < \operatorname{Re} w_i < 5/4,\ \operatorname{Im}(w_2 + \alpha w_1) > 0, \\ \operatorname{Im}(w_2 + \alpha_1 w_1) \operatorname{Im}(w_2 + \alpha_2 w_1) > R\}, \\ U_R = v^{-1}(V_R) \subset U. \end{aligned}$$

La partie,

$$\{(w_1, w_2) \in \mathbf{C}^2;\ \operatorname{Im}(w_2 + \alpha w_1) > 0,\ \ \operatorname{Im}(w_2 + \alpha_1 w_1) \operatorname{Im}(w_2 + \alpha_2 w_1) = R\},$$

est invariante par T_1 et T_2. D'après les relations ci-dessus et le lemme 1, on obtient :



Lemme 2. *Supposons $R > C_2$.*

a) La restriction de v à U_R est un difféomorphisme de classe C^∞ sur V_R.

b) En quotientant U_R par F_1 et F_2, on obtient une variété complexe M_R, et v induit un difféomorphisme de classe C^∞, noté $\bar{v}$, de M_R sur $V_R/\mathbf{Z}^2$ (V_R quotienté par T_1 et T_2).

c) La forme Ω_0 définit une $(1,0)$-forme de classe C^∞ sur M_R, qu'on notera Ω_1.

4. On notera π l'application canonique de U_R dans M_R. On définit des applications de classe C^∞ dans M_R (ou U_R) par :

$$h_1(z_1, z_2) = \operatorname{Im}(z_2 + v_0(z_1, z_2) + \alpha_1 z_1) = \operatorname{Im}(w_2 + \alpha_1 w_1)$$
$$h_2(z_1, z_2) = \operatorname{Im}(z_2 + v_0(z_1, z_2) + \alpha_2 z_1) = \operatorname{Im}(w_2 + \alpha_2 w_1)$$
$$h(z_1, z_2) = \frac{1}{2}(h_1 + h_2) = \operatorname{Im}(z_2 + v_0(z_1, z_2) + \alpha z_1) = \operatorname{Im}(w_2 + \alpha w_1).$$

Pour $j = 1, 2$, posons $\omega_j = \partial h_j$. D'après le lemme 1, on obtient :

Lemme 3. *Supposons $R > C_3 > C_2$.*

a) Les $(1,0)$*-formes* ω_1, ω_2 *constituent en tout point de* M_R *une base de l'espace cotangent holomorphe à* M_R.

b) Posons $dV = (i/2)^2 \omega_1 \wedge \bar{\omega}_1 \wedge \omega_2 \wedge \bar{\omega}_2$, *et notons* dV_0 *l'élément de volume canonique de* $\mathbf{C}^2$. *Dans* U_R *on a :*

$$C_4^{-1} dV_0 \leq \pi^* dV \leq C_4 dV_0.$$

c) On a

$$\begin{aligned} \partial \omega_1 &= \partial \omega_2 = 0, \\ \bar{\partial} \omega_1 &= \bar{\partial} \omega_2 = \bar{\partial} \partial \operatorname{Im} v_0 = \sum_{j,k} c_{j,k} \, \bar{\omega}_j \wedge \omega_k, \end{aligned}$$

où les $c_{j,k} \in C^\infty(M_R)$ *vérifient*

$$\begin{aligned} |c_{j,k}| &\leq C_5 e^{-2\pi h}, \\ |\partial_l c_{j,k}| &\leq C_5 e^{-2\pi h}, \\ |\bar{\partial}_l c_{j,k}| &\leq C_5 e^{-2\pi h}, \end{aligned}$$

avec $dc_{j,k} = \sum_l \partial_l c_{j,k} \, \omega_l + \sum_l \bar{\partial}_l c_{j,k} \, \bar{\omega}_l$.

5. Soit $\tilde{\psi}$ une fonction de classe C^2, à valeurs réelles, définie dans

$$\{(h_1, h_2) \in \mathbf{R}^2; h_1 > 0, \quad h_2 > 0, \quad h_1 h_2 > R\}.$$

On suppose $R > C_3$, et on associe à $\tilde{\psi}$ la fonction

$$z \mapsto \psi(z) = \tilde{\psi}(h_1(z), h_2(z)),$$

définie dans M_R (ou U_R). D'après le lemme 3.a, on peut écrire

$$\partial \bar{\partial} \psi = \sum_{j,k} \psi_{j,k} \, \omega_j \wedge \bar{\omega}_k,$$

avec, d'après le lemme 3.c,

$$\psi_{j,k} = \left[\frac{\partial^2 \tilde{\psi}}{\partial h_j \partial h_k} + \overline{c_{j,k}} \left(\frac{\partial \tilde{\psi}}{\partial h_1} + \frac{\partial \tilde{\psi}}{\partial h_2} \right) \right] \circ (h_1, h_2).$$

Considérons d'abord la fonction :

$$\tilde{\psi}^{(0)}(h_1, h_2) = -\log(h_1h_2 - R) = -\log\rho.$$

Nous avons :

$$\frac{\partial\tilde{\psi}^{(0)}}{\partial h_1} + \frac{\partial\tilde{\psi}^{(0)}}{\partial h_2} = -2h\rho^{-1},$$
$$\frac{\partial^2\tilde{\psi}^{(0)}}{\partial h_1^2} = h_2^2\rho^{-2},$$
$$\frac{\partial^2\tilde{\psi}^{(0)}}{\partial h_2^2} = h_1^2\rho^{-2},$$
$$\frac{\partial^2\tilde{\psi}^{(0)}}{\partial h_1\partial h_2} = R\rho^{-2}.$$

Observons qu'on a, pour $(\zeta_1, \zeta_2) \in \mathbf{C}^2$:

$$\begin{aligned} h_2^2|\zeta_1|^2 + h_1^2|\zeta_2|^2 + R(\zeta_1\bar{\zeta_2} + \zeta_2\bar{\zeta_1}) &\geq \frac{h_1^2h_2^2 - R^2}{h_1^2 + h_2^2}(|\zeta_1|^2 + |\zeta_2|^2), \\ &\geq (2h^2)^{-1}\rho R(|\zeta_1|^2 + |\zeta_2|^2). \end{aligned}$$

D'après le lemme 3.c, on en déduit :

$$(1) \qquad \sum \psi_{j,k}^{(0)}\,\zeta_j\bar{\zeta_k} \geq \rho^{-1}\left((2h^2)^{-1}R - C_6he^{-2\pi h}\right)(|\zeta_1|^2 + |\zeta_2|^2).$$

De même, avec $\tilde{\psi}^{(1)}(h_1, h_2) = h_1^2 + h_2^2$ et $\tilde{\psi}^{(2)}(h_1, h_2) = -\pi(h_1 + h_2)$, nous obtenons :

$$(2) \qquad \sum \psi_{j,k}^{(1)}\,\zeta_j\bar{\zeta_k} \geq \left(2 - C_6he^{-2\pi h}\right)(|\zeta_1|^2 + |\zeta_2|^2),$$

$$(3) \qquad \sum \psi_{j,k}^{(2)}\,\zeta_j\bar{\zeta_k} \geq -C_6e^{-2\pi h}(|\zeta_1|^2 + |\zeta_2|^2).$$

Lemme 4. *Pour $R > C_7 > C_3$, M_R est une variété de Stein.*

Démonstration. Soit $\psi = \psi^{(0)} + \psi^{(1)} \in C^\infty(M_R)$. Comme on a $h > R^{1/2}$ dans M_R, la fonction ψ est, d'après les inégalités (1) et (2), strictement plurisousharmonique dans M_R si R est assez grand. Par ailleurs, pour tout $a \in \mathbf{R}$, $\psi^{-1}(]-\infty, a])$ est clairement compact.◇

6. Nous reprenons les notations de l'appendice, ω_1, ω_2 étant définis comme ci-dessus.

D'après le lemme 3.c, nous avons dans l'appendice (A_1) :

$$a^i_{j,k} \equiv 0,$$

$$c^1_{j,k} = c^2_{j,k} = c_{j,k},$$

et nous pouvons choisir

$$\theta_0 = \theta_1 = C_5 e^{-2\pi h}.$$

Nous allons appliquer le théorème de l'appendice, en choisissant pour ϕ la fonction associée à $\tilde{\phi} = \tilde{\psi}^{(0)} + \tilde{\psi}^{(2)}$, et en prenant :

$$\theta = (3h^2\rho)^{-1}R,$$
$$\rho = h_1 h_2 - R.$$

D'après les inégalités (1), (3) ci-dessus, nous avons effectivement, si $R > C_8 > C_7$:

$$\sum \phi_{j,k}\, \zeta_j \bar{\zeta}_k \geq (\theta + A(\theta_0^2 + \theta_1))(|\zeta_1|^2 + |\zeta_2|^2).$$

D'après la définition de η, pour $i = 0, 1$, $-\bar{\partial} v_i$ définit une $(0,1)$-forme f_i de classe C^∞ sur M_R, évidement $\bar{\partial}$-fermée. D'après le lemme 1 et le lemme 3.b, on a, pour $R > C_9 > C_8$:

$$\int_{M_R} \theta^{-1} |f_i|^2 e^{-\phi} dV \leq 1.$$

Il existe donc, d'après le théorème de l'appendice, une fonction $u_i \in C^\infty(M_R)$ vérifiant:

$$\bar{\partial} u_i = f_i, \tag{4}$$

$$\int_{M_R} |u_i|^2 \rho e^{2\pi h} dV \leq 1. \tag{5}$$

7. Posons

$$\hat{V}_R = \{(w_1, w_2) \in \mathbf{C}^2; 0 \leq \operatorname{Re} w_i \leq 1,\ \operatorname{Im}(w_2 + \alpha w_1) > 0,$$
$$\operatorname{Im}(w_2 + \alpha_i w_1) > (1 + \alpha_i)/2 + R^{1/2}\},$$

$$\tilde{V}_R = \{(w_1, w_2) \in \mathbf{C}^2; 0 \leq \operatorname{Re} w_i \leq 1,\ \operatorname{Im}(w_2 + \alpha w_1) > 0,$$
$$\operatorname{Im}(w_2 + \alpha_i w_1) > (1 + \alpha_i) + R^{1/2}\},$$

$$\hat{U}_R = v^{-1}(\hat{V}_R),$$

et pour $i = 0, 1$,

$$\tilde{v}_i = v_i + u_i \circ \pi.$$

Les applications $\tilde{v}_0$, $\tilde{v}_1$ sont holomorphes dans U_R.

Lemme 5. *Supposons $R > C_{10} > C_9$. Dans $\hat{U}_R$, on a, pour $i = 0, 1$, $j = 1, 2$:*

$$|\tilde{v}_i| \leq e^{-\pi h},$$
$$\left|\frac{\partial \tilde{v}_i}{\partial z_j}\right| \leq e^{-\pi h}.$$

Démonstration. Soit $z^0 \in \hat{U}_R$, et Δ le polydisque:

$$\Delta = \{(z_1, z_2) \in \mathbf{C}^2; |z_1 - z_1^0| < \frac{1}{10}, \quad |z_2 - z_2^0| < \frac{1}{10}\}.$$

D'après le lemme 1, on a $\Delta \subset U_R$ si R est assez grand. Pour $z \in \Delta$, et R assez grand, on a aussi

$$|h(z) - h(z_0)| \leq C_{11},$$

$$h_i(z) > \frac{1}{4}(1 + \alpha_i) + R^{1/2},$$

d'où $\rho(z) = h_1(z)h_2(z) - R > \frac{1}{2}R^{1/2}$.

Le lemme résulte donc de l'inégalité (5), du lemme 1 et des estimations de Cauchy (compte tenu du lemme 3.b).$\diamondsuit$

8. Posons :

$$\tilde{v}(z_1, z_2) = (z_1, z_2 + \tilde{v}_0(z_1, z_2)),$$
$$\tilde{\Omega}_0 = e^{\tilde{v}_1(z_1, z_2)}(dz_2 + \alpha dz_1),$$
$$\tilde{U}_R = \tilde{v}^{-1}(V_R) \cap \hat{U}_R,$$
$$\tilde{M}_R = \pi(\tilde{U}_R).$$

A partir des estimations du lemme 5, on obtient immédiatement

Lemme 6. *Supposons $R > C_{12} > C_{10}$.*

a) La restriction de $\tilde{v}$ à $\tilde{U}_R$ est un difféomorphisme biholomorphe sur $\tilde{V}_R$ et induit un difféomorphisme biholomorphe (qu'on note encore $\tilde{v}$) de $\tilde{M}_R$ sur $\tilde{V}_R/\mathbf{Z}^2 = \tilde{W}_R$.

b) *La 1-forme $\tilde{\Omega}_0$ induit une 1-forme holomorphe (encore notée $\tilde{\Omega}_0$) sur $\tilde{M}_R$. La 1-forme $\Omega_2 = (\tilde{v}^{-1})^*\tilde{\Omega}_0$ sur $\tilde{W}_R$ s'écrit:*

$$\Omega_2 = A_1(\omega_1, \omega_2)\alpha d\omega_1 + A_2(\omega_1, \omega_2)d\omega_2,$$

où les A_j vérifient:

$$|A_j(\omega_1, \omega_2) - 1| \leq C_{13}e^{-\pi Im(\omega_2+\alpha\omega_1)}.$$

9. Les fonctions A_j étant $\mathbf{Z}^2$-périodiques, on peut écrire, d'après le lemme 6:

$$A_j(\omega_1, \omega_2) = 1 + e^{2\pi i(\omega_1+\omega_2)}B_j(\omega_1, \omega_2),$$

les fonctions B_j étant $\mathbf{Z}^2$-périodiques et bornées. Posons:

$$\begin{aligned} D_R =& \{(x,y) \in \mathbf{C}^2; |y|\,|x|^{\alpha_i} < e^{-2\pi(1+\alpha_i+R^{1/2})}\}, \\ D_R^* =& \{(x,y) \in D_R; x \neq 0, y \neq 0\}. \end{aligned}$$

et notons

$$E: (\omega_1, \omega_2) \mapsto (x,y) = (e^{2\pi i\omega_1}e^{2\pi i\omega_2}).$$

C'est un difféomorphisme biholomorphe de $\tilde{W}_R$ sur D_R^*. Avec $\tilde{B}_i = B_i \circ E$, la 1-forme $\Omega = (E^{-1})^*\Omega_2$ s'écrit:

$$\Omega = (y^{-1} + x\tilde{B}_2(x,y))dy + (x^{-1} + y\tilde{B}_1(x,y))\alpha dx.$$

Les fonctions $\tilde{B}_i$, holomorphes et bornées dans D_R^*, s'étendent en des fonctions holomorphes dans le domaine de Reinhart D_R. La 1-forme Ω est donc bien du type recherché.

10. Notons $\mathcal{F}_0$ le feuilletage holomorphe défini par $\tilde{\Omega}_0$ dans $\tilde{U}_R$, et $\mathcal{F}$ le feuilletage holomorphe défini par Ω dans D_R. Les feuilles de $\mathcal{F}_0$ sont les surfaces de niveau de la fonction $z_2 + \alpha z_1$, et $\mathcal{F}$ est l'image directe de $\mathcal{F}_0$ par $E \circ \tilde{v}$. Posons

$$\begin{aligned} \Sigma &= \{(x,y) \in D_R; x = 1\}, \\ \Sigma^* &= \Sigma - \{(1,0)\}, \\ \Sigma_0 &= \{(z_1,z_2) \in \tilde{U}_R; z_1 = 0\}, \\ \Sigma_1 &= \{(z_1,z_2) \in \tilde{U}_R; z_1 = 1\} = F_1(\Sigma_0). \end{aligned}$$

L'application $E \circ \tilde{v}/\Sigma_0$ est un revêtement universel de Σ^* par Σ_0, et $z = e^{2\pi i z_2}$ est une coordonnée sur Σ. Les points $(0, z_2)$ et $F_1(0, z_2) = (1, z_2 + \Phi(z_2))$ ont même image par $E \circ \tilde{v}$, et le point $(1, z_2 + \Phi(z_2))$ appartient à la même feuille de $\mathcal{F}_0$ que $(0, z_2 + \alpha + \Phi(z_2))$. Par conséquent, l'holonomie sur Σ de $\mathcal{F}$, correspondant au lacet $t \in [0,1] \mapsto (e^{-2\pi i t}, 0)$ de la feuille $\{y = 0\}$, est égale à F dans la coordonnée z sur Σ.

Ceci achève la démonstration du théorème.

III. Appendice.

$\mathbf{A_0}$. Nous reproduisons ci-dessous, dans la situation particulière qui nous intéresse, des estimations L^2 dues à Hörmander ([Ho1], [Ho2]) sur l'opérateur $\bar\partial$ dans les variétés de Stein. Nous ne traitons, pour simplifier les notations, que le cas des fonctions, qui est le seul dont nous avons besoin.

$\mathbf{A_1}$. Soit M une variété de Stein, de dimension n, munie d'une métrique hermitienne.

On suppose qu'il existe des formes $\omega_1, \dots, \omega_n$ de classe C^∞ sur M, de type $(1,0)$, formant en tout point de M une base orthonormée de l'espace cotangent holomorphe à M.

On note $dV = (i/2)^n \omega_1 \wedge \bar\omega_1 \wedge \dots \wedge \omega_n \wedge \bar\omega_n$ l'élément de volume associé à la métrique hermitienne.

Pour une $(0,1)$-forme $f = \sum f_i \bar\omega_i$ (resp. une $(0,2)$-forme $f = \sum_{i<j} f_{i,j} \bar\omega_i \wedge \bar\omega_j$), on pose :

$$|f|^2 = \sum_i |f_i|^2 \quad (\text{resp. } |f|^2 = \sum_{i<j} |f_{i,j}|^2).$$

Pour une fonction $w \in C^\infty(M)$, on pose :

$$dw = \partial w + \bar\partial w = \sum_i \partial_i w \; \omega_i + \sum_i \bar\partial_i w \; \bar\omega_i,$$

$$\partial\bar\partial w = \sum_{j,k} w_{j,k} \; \omega_j \wedge \bar\omega_k.$$

On introduit des fonctions $a^i_{j,k}$, $c^i_{j,k} \in C^\infty(M)$ par les formules,

$$\partial\omega_i = \frac{1}{2} \sum_{j,k} a^i_{j,k} \; \omega_j \wedge \omega_k, \quad a^i_{j,k} = -a^i_{k,j},$$

$$\bar\partial\omega_i = \sum_{j,k} c^i_{j,k} \; \bar\omega_j \wedge \omega_k.$$

Finalement, on se donne deux fonctions réelles θ_0, $\theta_1 \in C^0(M)$ vérifiant:

$$\begin{array}{lll} |c^i_{j,k}| \le \theta_0 & , & |a^i_{j,k}| \le \theta_0, \\ |\partial_l c^i_{j,k}| \le \theta_1 & , & |\partial_l a^i_{j,k}| \le \theta_1, \\ |\bar\partial_l c^i_{j,k}| \le \theta_1 & , & |\bar\partial_l a^i_{j,k}| \le \theta_1. \end{array}$$

A_2. Le résultat suivant est essentiellement du à Hörmander (cf [Ho1] p.92).

Théorème. *Il existe une constante A, ne dépendant que de la dimension n de M, possédant la propriété suivante : étant données une fonction $\theta \in C^0(M)$, strictement positive sur M, une fonction $\phi \in C^2(M)$ strictement plurisousharmonique sur M, et une $(0,1)$-forme f de classe C^∞ sur M vérifiant*

$$\sum_{j,k} \phi_{j,k}\ z_j \bar{z}_k \geq \left(\theta + A(\theta_0^2 + \theta_1)\right) \left(\sum_j |z_j|^2\right),$$

$$\int_M \theta^{-1} |f|^2 e^{-\phi} dV < +\infty, \quad \bar{\partial} f = 0,$$

il existe $u \in C^\infty(M)$ vérifiant :

$$\bar{\partial} u = f,$$

$$\int_M |u|^2 e^{-\phi} dV \leq \int_M \theta^{-1} |f|^2 e^{-\phi} dV.$$

A_3. On fixe jusqu'en A_7 une fonction $\phi \in C^2(M)$. Pour une forme ω de classe C^1, on pose,

$$\delta\omega = e^{\phi} \partial(e^{-\phi}\omega),$$

$$\bar{\delta}\omega = e^{\phi} \bar{\partial}(e^{-\phi}\omega).$$

Pour une fonction $w \in C^1(M)$, on pose aussi,

$$\begin{array}{llll} \delta w & = \sum \delta_i w\ \omega_i & , \quad \delta_i w & = e^{\phi} \partial_i (e^{-\phi} w), \\ \bar{\delta} w & = \sum \bar{\delta}_i w\ \bar{\omega}_i & , \quad \bar{\delta}_i w & = e^{\phi} \bar{\partial}_i (e^{-\phi} w). \end{array}$$

Lemme 1. *Pour une forme ω de classe C^2, on a :*

$$\left(\bar{\partial}\delta + \delta\bar{\partial}\right)(\omega) = \partial\bar{\partial}\phi \wedge \omega.$$

Démonstration. En effet, on a

$$\delta\omega = \partial\omega - \partial\phi \wedge \omega,$$

$$\bar{\delta}\omega = \bar{\partial}\omega - \bar{\partial}\phi \wedge \omega,$$

$$\delta\bar{\partial}\omega = \partial\bar{\partial}\omega - \partial\phi \wedge \bar{\partial}\omega,$$

$$\bar{\partial}\delta\omega = \bar{\partial}(\partial\omega - \partial\phi \wedge \omega) = \bar{\partial}\partial\omega - \bar{\partial}\partial\phi \wedge \omega + \partial\phi \wedge \bar{\partial}\omega,$$

d'où le lemme.◇

Lemme 2. *Pour $w \in C^2(M)$, on a :*

$$\phi_{j,k}\ w = (\delta_j \bar{\partial}_k - \bar{\partial}_k \delta_j)(w) + \sum_i \overline{c^i_{j,k}} \bar{\partial}_i w - \sum_i c^i_{k,j} \delta_i w.$$

Démonstration. En effet, on a :

$$\bar{\partial}\delta w = \bar{\partial}\left(\sum_i \delta_i w\ \omega_i\right) = \sum_{j,k}\left(\bar{\partial}_k \delta_j w + \sum_i c^i_{j,k}\ \delta_i w\right)\ \bar{\omega}_k \wedge \omega_j,$$

$$\delta\bar{\partial} w = \delta\left(\sum_i \bar{\partial}_i w\ \bar{\omega}_i\right) = \sum_{j,k}\left(\delta_j \bar{\partial}_k w + \sum_i \overline{c^i_{j,k}}\ \bar{\partial}_i w\right)\ \omega_j \wedge \bar{\omega}_k,$$

$$w\ \partial\bar{\partial}\phi = \sum_{j,k} \phi_{j,k} w\ \omega_j \wedge \bar{\omega}_k,$$

et on conclut d'après le lemme 1.◇

A₄. Pour $0 \le i \le 2$, on note D_i l'espace des $(0,i)$ formes de classe C^∞ à support compact sur M. On note $T : D_o \to D_1$ et $S : D_1 \to D_2$ l'opérateur $\bar{\partial}$. Pour $f \in D_i$, on pose :

$$||f||^2 = \int_M |f|^2 e^{-\phi} dV.$$

Pour $1 \le j \le n$, posons :

$$b_j = \sum_k (c^k_{j,k} + \overline{a^k_{j,k}}).$$

Lemme 3 (Intégration par parties). *Pour $v, w \in D_0$, on a :*

$$\int_M (\bar{\partial}_j v\ \bar{w} + v\ \overline{\delta_j w} + b_j v \bar{w}) e^{-\phi} dV = 0.$$

Démonstration. Posons

$$\omega^{(j)} = (i/2)^n \omega_1 \wedge \bar{\omega}_1 \wedge \ldots \wedge \omega_{j-1} \wedge \bar{\omega}_{j-1} \wedge \omega_j \wedge \omega_{j+1} \wedge \bar{\omega}_{j+1} \wedge \ldots \wedge \bar{\omega}_n.$$

On vérifie qu'on a :

$$d\omega^{(j)} = \bar{\partial}\omega^{(j)} = -b_j dV.$$

Le lemme résulte donc de la formule de Stokes appliquée à la forme $v\bar{w}e^{-\phi}\omega^{(j)}$.◇

A$_5$. Un calcul immédiat donne le :

Lemme 4. *Soit $f = \sum_i f_i\bar{\omega}_i \in D_1$. On a*

$$Sf = \sum_{j<k}\left(\bar{\partial}_j f_k - \bar{\partial}_k f_j + \sum_i \overline{a^i_{j,k}} f_i\right)\ \bar{\omega}_j \wedge \bar{\omega}_k.$$

Soit $T^* : D_1 \to D_0$ l'opérateur défini, pour $f = \sum_i f_i\bar{\omega}_i \in D_1$, par

$$T^* f = -\sum_i (\delta_i f_i + \bar{b}_i f_i).$$

Lemme 5. *Pour $v \in D_0$, $f \in D_1$, on a,*

$$< Tv, f >_{D_1} = < v, T^* f >_{D_0},$$

les produits scalaires hermitiens étant ceux associés aux normes préhilbertiennes introduites en A_4.

Cela résulte immédiatement du lemme 3 appliqué à $w = f_j$, par sommation sur j.

A$_6$. On a,

Proposition. *Il existe une constante $A > 0$, ne dépendant que de la dimension n de M, telle qu'on ait, pour $f = \sum_i f_i\bar{\omega}_i \in D_1$:*

$$\int_M e^{-\phi}\sum_{j,k}(\phi_{j,k} f_j \bar{f}_k + \frac{1}{2}|\bar{\partial}_j f_k|^2)dV \leq ||Sf||^2 + ||T^* f||^2 + A\int_M (\theta_0^2 + \theta_1)|f|^2 e^{-\phi} dV.$$

Démonstration. On désigne par $A_1, A_2, \ldots$ des constantes ne dépendant que de n. On pose :

$$B_0 = \int_M \sum_{j,k} |\bar{\partial}_j f_k|^2 \, e^{-\phi} dV,$$

$$B_1 = \int_M \theta_1 |f|^2 e^{-\phi} dV,$$

$$B_2 = \int_M \theta_0^2 |f|^2 e^{-\phi} dV.$$

D'après le lemme 4 et l'inégalité de Cauchy-Schwarz, on a

$$(1) \qquad \int_M \sum_{j,k} |\bar{\partial}_j f_k - \bar{\partial}_k f_j|^2 \, e^{-\phi} dV \leq ||Sf||^2 + A_1 B_2 + A_2 (B_0 B_2)^{1/2}.$$

D'après le lemme 3, on a,

$$(2) \qquad \int_M \bar{b}_j f_j \overline{\delta_i f_i} e^{-\phi} dV = -\int_M (\bar{\partial}_i \bar{b}_j f_j + \bar{b}_j \bar{\partial}_i f_j + b_i \bar{b}_j f_j) \bar{f}_i e^{-\phi} dV,$$

donc on obtient, par l'inégalité de Cauchy-Schwarz :

$$(3) \qquad \int_M \sum_{j,k} \delta_j f_j \overline{\delta_k f_k} \, e^{-\phi} dV \leq ||T^* f||^2 + A_3 B_2 + A_4 B_1 + A_5 (B_0 B_2)^{1/2}.$$

D'après le lemme 3 on a :

$$(4) \qquad \int_M \delta_j f_j \overline{\delta_k f_k} e^{-\phi} dV = -\int_M (\bar{\partial}_k \delta_j f_j + b_k \delta_j f_j) \bar{f}_k e^{-\phi} dV.$$

En procédant comme pour (2) et (3), on obtient :

$$(5) \qquad \left| \int_M b_k \bar{f}_k \delta_j f_j e^{-\phi} dV \right| \leq A_3 B_2 + A_4 B_1 + A_5 (B_0 B_2)^{1/2}.$$

D'autre part, d'après le lemme 2, on a :

$$(6) \qquad -\bar{f}_k \bar{\partial}_k \delta_j f_j = \phi_{j,k} f_j \bar{f}_k - \bar{f}_k \delta_j \bar{\partial}_k f_j + \bar{f}_k \sum_i (c^i_{k,j} \delta_i f_j - \overline{c^i_{j,k}} \bar{\partial}_i f_j).$$

D'après le lemme 3, on a :

$$(7) \qquad -\int_M \bar{f}_k \delta_j \bar{\partial}_k f_j e^{-\phi} dV = \int_M (\bar{\partial}_k f_j \overline{\bar{\partial}_j f_k} + \bar{b}_j \bar{f}_k \bar{\partial}_k f_j) e^{-\phi} dV.$$

En rapprochant (4),(5),(6) et (7) on obtient, en procédant comme précédemment

(8)
$$\left|\int_M (\delta_j f_j \overline{\delta_k f_k} - \phi_{j,k} f_j \bar{f}_k - \bar{\partial}_k f_j \overline{\bar{\partial}_j f_k}) e^{-\phi} dV\right| \leq A_6 B_2 + A_7 B_1 + A_8 (B_0 B_2)^{1/2}.$$

Finalement, on observe que

$$\sum_{j<k} \left|\bar{\partial}_j f_k - \bar{\partial}_k f_j\right|^2 = \sum_{j,k} \left(|\bar{\partial}_j f_k|^2 - \bar{\partial}_k f_j \overline{\bar{\partial}_j f_k}\right),$$

donc en joignant (1), (2) et (8), on obtient :

$$B_0 + \int_M \left(\sum_{j,k} \phi_{j,k} f_j \bar{f}_k\right) e^{-\phi} dV \leq ||Sf||^2 + ||T^* f||^2 + A_9 B_2 + A_{10} B_1 + A_{11} (B_0 B_2)^{1/2}.$$

Il suffit alors d'observer qu'on a $A_{11}(B_0 B_2)^{1/2} \leq \frac{1}{2}(B_0 + A_{11}^2 B_2)$.$\diamond$

A₇. Pour $0 \leq i \leq 2$, notons H_i l'espace de Hilbert complété de D_i pour la norme préhilbertienne introduite en A_4; H_i est l'espace des $(0,1)$ formes f à coefficients dans L^2_{loc} telles que:

$$||f||^2 = \int_M |f|^2 e^{-\phi} dV < +\infty.$$

Nous notons encore T, T^*, S les extensions naturelles de ces opérateurs et $D(T)$, $D(T^*)$, $D(S)$ leurs domaines respectifs. D'après le lemme 5, T^* est l'adjoint de T. Le lemme suivant est repris de [Hol], p. 109.

Lemme 6. *Supposons qu'il existe une suite $(\eta_k)_{k\geq 0}$ dans D_0 satisfaisant:*
(i) Pour tout $k \geq 0$, $0 \leq \eta_k \leq 1$, $|\bar{\partial}\eta_k| \leq 1$.
(ii) Pour tout compact $K \subset M$, $\eta_k = 1$ sur K si k est assez grand.
Alors, D_1 est dense dans $D(S) \cap D(T^*)$ pour la norme :

$$|||f|||^2 = ||f||^2 + ||Sf||^2 + ||T^* f||^2.$$

A₈. On se donne maintenant θ, ϕ, f comme dans l'énoncé du théorème , ainsi qu'une fonction $s \in C^\infty(M)$, strictement plurisousharmonique sur M, telle que $M_a = s^{-1}(]-\infty, a[)$ soit relativement compact dans M pour tout réel a. On peut supposer qu'on a

$$\int_M \theta^{-1}|f|^2 e^{-\phi} dV = 1. \tag{10}$$

A_9. Donnons-nous un réel a, fixé jusqu'en A_{12}. Choisissons alors une suite $(\eta_k)_{k\geq 0}$ dans D_0 vérifiant:

(i)' Pour tout $k \geq 0$, $0 \leq \eta_k \leq 1$.

(ii)' Pour tout $k \geq 0$, $\eta_k = 1$ sur $\overline{M_{a+2}}$.

(iii)' Pour tout compact $K \subset M$, $\eta_k = 1$ sur K si k est assez grand.

Soit $h \in C^\infty(M)$ une fonction satisfaisant

(iv)' $h \geq 1$ sur M, $h = 1$ sur $\overline{M_{a+1}}$.

(v)' Pour tout $k \geq 0$, $|\bar{\partial}\eta_k| \leq h$.

Une telle fonction existe d'après (iii)'.

Posons alors $\omega'_i = h\omega_i$; introduisons sur M une nouvelle métrique hermitienne, faisant de $\omega'_1, \ldots, \omega'_n$ une base orthonormée de l'espace cotangent holomorphe à M en tout point de M.

Les hypothèses du lemme 6 sont satisfaites par la suite $(\eta_k)_{k\geq 0}$, relativement à cette nouvelle métrique. Nous notons $dV' = h^{2n}dV$ l'élément de volume relatif à cette nouvelle métrique, et θ'_0, θ'_1 des fonctions continues sur M, coincidant avec θ_0, θ_1 sur $\overline{M_{a+1}}$, qui vérifient les estimations de A_1 relatives aux formes ω'_i et à la nouvelle métrique hermitienne.

A_{10}. Soit $\theta' \in C^0(M)$ une fonction coïncidant avec θ sur $\overline{M_{a+1}}$ et vérifiant

$$0 < \operatorname*{Inf}_M \theta' < \operatorname*{Sup}_M \theta' < +\infty.$$

Posons

$$\phi' = \phi + \chi \circ s,$$

où $\chi : \mathbf{R} \to \mathbf{R}$ est une fonction de classe C^2, nulle sur $]-\infty, a]$, croissante et convexe. Nous choisissons χ croissant suffisamment rapidement pour avoir:

$$\int_M \theta'^{-1}|f|^2 e^{-\phi'} dV' \leq 1 \quad (\text{cf } (10)); \tag{11}$$

$$\sum_{j,k} \phi'_{j,k} z_j \bar{z}_k \geq \left[\theta' + A(\theta_0'^2 + \theta_1')\right] \sum_j |z_j|^2, \tag{12}$$

en tout point de M, avec

$$\partial\bar{\partial}\phi' = \sum_{j,k} \phi'_{j,k}\ \omega'_j \wedge \omega'_k.$$

Notons alors H'_i (pour $0 \leq i \leq 2$) l'espace de Hilbert défini en A_7, relatif à la fonction ϕ' et à la nouvelle métrique hermitienne.

$\mathbf{A_{11}}$. D'après la proposition et la propriété (12), nous avons, pour $g \in D_1$

$$\int_M \theta' |g|^2 e^{-\phi'} dV' \leq ||T^* g||^2_{H'_0} + ||Sg||^2_{H'_2}. \tag{13}$$

D'après le lemme 6, cette inégalité est encore valable pout tout $g \in D(T^*) \cap D(S)$. Comme $\mathrm{Inf}_M \theta' > 0$, ceci implique (cf [Hol], p. 78) que l'image F de T est égale au noyau de S, et aussi au sous-espace (fermé) de H'_1 orthogonal au noyau de T^* (L'opérateur T^* est, bien entendu, celui relatif à la nouvelle métrique et à la fonction ϕ').

$\mathbf{A_{12}}$. D'après (12) on a $f \in H'_1$, d'où $f \in F$ puisque $\bar{\partial} f = 0$. Pour $g \in D_1$ écrivons:

$$g = g_1 + g_2, \quad g_1 \in F, \quad g_2 \in F^{\perp}.$$

Nous avons alors

$$\begin{aligned} < f, g >_{H'_1} &= < f, g_1 >_{H'_1}, \\ T^* g &= T^* g_1, \\ S g_1 &= 0. \end{aligned}$$

D'après l'inégalité de Cauchy-Schwarz et l'inégalité (11), nous obtenons:

$$< f, g_1 >^2_{H'_1} \leq \int_M \theta' |g_1|^2 e^{-\phi'} dV',$$

d'où nous déduisons, grâce à (13) :

$$< f, g >^2_{H'_1} \leq ||T^* g||^2_{H'_0}.$$

On conclut qu'il existe $u \in H_0'$ vérifiant

$$\begin{aligned} \|u\|_{H_0'} &\leq 1, \\ Tu &= f. \end{aligned}$$

A_{13}. On a donc obtenu, pour tout $a \in \mathbf{R}$, une fonction $u_a \in L^2_{\text{loc}}$ vérifiant $\bar{\partial} u_a = f$ au sens des distributions et

$$\int_{M_a} |u_a|^2 e^{-\phi} dV \leq 1,$$

(car $dV' = dV$ et $\phi' = \phi$ sur M_a).

Un procédé diagonal fournit alors une suite $(a_j)_{j\geq 0}$ tendant vers l'infini et une fonction $u \in L^2_{\text{loc}}$ telle que $(1_{M_a} u_{a_j})_{j\geq 0}$ converge faiblement vers $1_{M_a} u$ dans H_0, pour tout réel a. On a alors

$$\begin{aligned} \bar{\partial} u &= f, \\ \int_M |u|^2 e^{-\phi} dV &\leq 1. \end{aligned}$$

Finalement, la théorie générale (ellipticité) nous garantit que $u \in C^\infty(M)$ si f est C^∞.

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Densities for certain leaves of real analytic foliations

Astérisque, tome 222 (1994), p. 373-387
<http://www.numdam.org/item?id=AST_1994__222__373_0>

DENSITIES FOR CERTAIN LEAVES OF REAL ANALYTIC FOLIATIONS

C. ROCHE[1]

I. Introduction.

Let suppose an n dimensional real analytic manifold M be given. We will supose M to be paracompact connected and oriented. A real analytic $n-1$ *foliation with singularities* $\mathcal{F}$ on M is determined by giving an open covering (U_i) of M together with real analytic integrable 1-forms $\omega_i \in \Omega^1(U_i)$ such that on the overlapping charts, $U_i \cap U_j \neq \emptyset$, there exists a non vanishing function $g_{i,j} : U_i \cap U_j \to \mathbf{R}^*$ such that $\omega_i = g_{i,j}\omega_j$. Leaves of $\mathcal{F}$ on U_i are unions of the integral manifolds of the pfaffian equation $\omega_i = 0$.

The singular set of the foliation $\mathrm{Sing}(\mathcal{F})$ is the analytic subspace of M defined by the annulation of the forms ω_i. In local coordinates of M, each ω_i can be written as

$$\omega_i(x) = \sum_{l=1}^{n} a_l^i(x)dx^l$$

and locally $\mathrm{Sing}(\mathcal{F})$ is determined by the equations

$$a_1^i(x) = 0, \ldots, a_n^i(x) = 0 \qquad x \in U_i.$$

The hypothesis that the $g_{i,j}$ be non vanishing allowes to suppose that the singular set is of codimension at least 2. Such $\mathcal{F}$ defines on $M \setminus \mathrm{Sing}(\mathcal{F})$ an $n-1$ dimensional analytic foliation: $\mathcal{F}_{reg}$. Leaves of $\mathcal{F}_{reg}$ are called regular leaves of $\mathcal{F}$.

Morover if we suppose $\mathcal{F}$ to be transversally orientable, as will be done in this paper, Theorem A and B of Cartan in the real case [3] show that we can glue the 1-forms in order to suppose that the foliation $\mathcal{F}$ is given by a globally defined real analytic differential form ω, that is $\omega_i = \omega_{|U_i}$.

Consider now a union Γ of regular leaves of such a foliation $\mathcal{F}$, Γ is an immersed $n-1$ real analytic submanifold of M. Γ is called a separating solution

[1] This research was partially supported by Brazilian CNPq

by Khovanskii if there are two disjoint open sets, L_1 ans L_2 of M such that $M\setminus$ Sing$(\mathcal{F})$ $\setminus\Gamma = L_1 \cup L_2$, $\Gamma = \bar{L_1} \setminus L_1\setminus$Sing$(\mathcal{F})$ and finally ω points inside L_1 all along Γ.

In [17] we generalize this notion introducing Rollian pfaffian hypersurfaces. A regular leaf V of $\mathcal{F}$ is so called if for each analytic path $\gamma : [0,1] \to M$ intersecting the set V twice, say $\gamma(0) \in V$ and $\gamma(1) \in V$ there is an intermediate point, say $\gamma(t)$, $t \in [0,1]$ where the path is tangent to $\mathcal{F}$. At this point, if $\mathcal{F}$ is determined by the pfaff equation $\omega = 0$

$$\omega(\gamma(t)).\gamma'(t) = 0.$$

Such a Rollian pfaffian hypersurface (Rollian leaf or Rollian ph for short) will be denoted $\{V, \mathcal{F}, M\}$ to emphasize the pfaffian equation verified by V.

Khovanskii's Rolle theorem asserts that every separating solution of $\omega = 0$ is a union of Rollian ph. Separating solutions are not easy to find but, as it was shown in [17], an argument of Haefliger proves that if $M\setminus$Sing$(\mathcal{F})$ is simply connected, each regular leaf of $\mathcal{F}$ is a Rollian ph.

In [17] we used this generalisation to prove the following general finiteness theorem.

Theorem on uniform finiteness. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_q$ be transversally oriented singular foliations on M. If X is a semianalytic subset of M for each compact set K of M there is a constant $b \in \mathbf{R}$ such that for any set of Rollian pfaffian hypersurfaces $\{V_i, \mathcal{F}_i, M\}, i = 1, \ldots, q$ the number of connected components of $X \cap V_1 \cap \cdots \cap V_q$ meeting K is bounded by b.*

A carefull reading of the proof of this theorem in [17] shows that a separating manifold is in fact a locally finite union of Rollian ph as was shown by Khovanskii [5].

As an easy consequence of this result we can mention that a Rollian ph $\{V, \mathcal{F}, M\}$ is a real analytic submanifold of M closed in $M\setminus$Sing$(\mathcal{F})$.

In developping the ideas sketched in Khovanskii's work [5] [6] in joint work with R. Moussu, J.-M. Lion and J.-Ph. Rolin (started in [16]) we tried to consider Rollian ph just as building blocks for a theory similar to that of semianalytic sets. By different methods the same goal is pursued by Tougeron [19]. This idea leads to the problem of the behaviour of the boundary of a Rollian ph. At present time it is not known if the closure of a Rollian ph $\{V, \mathcal{F}, M\}$, $\bar{V}$ can be stratified with some regularity condition. In a forthcomming paper of F. Cano, J.-M. Lion and R. Moussu an important result on the regularity of the boundary $\bar{V} \setminus V$ of such a Rollian ph will be described. [2].

The study of the boundary of a sole Rollian ph $\{V, \mathcal{F}, M\}$ is most usefull for further research if we describe the structure of the boundary of an intersection $X \cap V$ where X is a semianalytic subset of M. If X is open connected and relatively compact in M, $X \cap V$ is a finite union of leaves of the restricted foliation $\mathcal{F}_{|X}$ each of them is a Rollian ph in X. The behabiour of V at the ends of M is so permited in the case the foliation can be regularly continued.

Let's define a pfaffian subset of M as a finite intersection $W = X \cap V_1 \cap \cdots \cap V_q$ where X is any semianalytic subset of M and the $V_i's$ are Rollian ph of foliations $\mathcal{F}_i$.

The following properties are known for the set $\partial W = \bar{W} \setminus W$. See [8][10].

Theorem on finiteness of the boundary. *The set ∂W is locally arc connected. Morover if $B_a(\rho)$ is the euclidean open ball of center a and radius ρ for $a \in \bar{W}$ the number of connected components of $\partial W \cap B_a(\rho)$ can be bounded by a constant depending only on the foliations $\mathcal{F}_i$ but not on the particular Rollian ph chosen.*

Let $C_y(A)$ be the tangent cone of $A \subset M$ at $y \in M$.

Curve selection lemma. *Let $a \in \partial W$, $u \in C_a(W)$, with $||u|| = 1$ be given, there is a semianalytic subset Y of M such that $W \cap Y$ is a union of paths $\gamma_i((0,1))$ one of them, say γ_0, can be extended in a C^1 way at 0 by $\gamma_0(0) = a$ and $\gamma_0'(0) = u$.*

These curves are pfaffian curves.

In this paper we show that Rollian ph have local volume properties similar to those of semianalytic and subanalytic sets.

A subset Y of $\mathbf{R}^n$ has a k-dimensional density at $y \in \mathbf{R}^n$ if the k-dimensional volume of $B_y(\epsilon) \cap Y$, $vol_k(B_y(\epsilon) \cap Y)$ is finite for small enough $\epsilon > 0$ and the following limit exists

$$\Theta_k(Y, y) = \lim_{\epsilon \to 0+} \frac{vol_k(B_y(\epsilon) \cap Y)}{\epsilon^k}.$$

This quantity is called density of Y at y. If these conditions are not fullfilled we can always consider the corresponding superior limit and inferior limit, which are denoted by $\bar{\Theta}_k(Y, y)$ and $\underline{\Theta}_k(Y, y) \in \bar{\mathbf{R}}_+$ respectively.

In a recent paper [7] Kurdyka and Raby show that subanalytic subsets have a density at every point. Our result is similar, but restricted to the case of Rollian ph as we cannot, at present time, obtain a general decomposition into graphs theorem for pfaffian sets.

Precisely, let M be an open semianalytic subset of $\mathbf{R}^n$

Theorem 1. *Let $\{V, \mathcal{F}, M\}$ be a Rollian pfaffian hypersurface then V has a density at each point of $\bar{V}$.*

The proof of this result uses the same idea of Kurdyka ans Raby and needs a new result on decomposition of Rollian ph into graphs. This decomposition gives a precision to a similar result of Lion [8], [9] and is obtained in a more elementary way. Namely

Proposition 1. *Let ω be an integrable real analytic 1-form, in a neighborhood of $0 \in \mathbf{R}^n$ and a small enough $\epsilon > 0$ be given. Then there is a finite number of hyperplans (H_i) and a subanalytic stratification $\mathcal{N}$ of a ball $B_0(\rho)$ such that: if $\{V, \omega, B_0(\rho)\}$ is a Rollian ph and $N \in \mathcal{N}$ then either*
$V \cap N$ is included in a smooth submanifold of dimension less than $n-1$,
or $V \cap N \subset H_i \oplus H_i^{\perp} \subset \mathbf{R}^n$ is the graph of a locally ϵ-lipschitzian analytic function on an open subset of H_i.

That is, up to a smaller dimensional set, each Rollian ph is a graph of an analytic funtion. This function can be supposed to have a very small derivative.

It is known that strong regularity conditions for stratified objects doesn't imply the existence of densities. Theorem 1 gives an interesting information on the good behaviour of the boundary of a Rollian ph even in case a theorem of regular stratification happens to be obtained.
The generalisation of theorem 1 to all pfaffian sets would be not difficult provided a result similar to Proposition 1 for several pfaffian equations can be proved.

II. Tangents to semianalytic sets and pfaffian equations.

Here we discuss a general stratification procedure preparing a graph decomposition of Rollian ph. In the first two paragraphs the discusion is fairly general and we restrict to the case of a single pfaffian equation in the third paragraph in order to get the proof of Proposition 1. We will use freely the theory of semianalytic sets [1] and stratifications [15]. A stratification is said to be adapted to a set if this set is a union of strata.
The proofs being local we will suppose from now on that M is an open semianalytic subset of $\mathbf{R}^n$.

1.Strongly analytic submanifolds. A subset X of M is a strongly analytic submanifold of M if it is semianalytic in M and a submanifold of M. That is locally at each point *of* X , X is given by the level set of an analytic submersion and at each point *of* M, X is determined locally by a finite number of analytic equalities and inequalities.

Let U be an open semianalytic subset of M, a semianalytic subset X of M is *normal* in U if $X \cap U$ can be described by a finite number of functionally independent analytic functions defined on U. That is, there exist $n-k+1$ analytic functions on U, $f_0, f_1, \ldots, f_{n-k}$ such that

$$X \cap U = \{x \in U / f_0(x) > 0, f_1(x) = 0, \cdots, f_{n-k}(x) = 0 \text{ and}$$
$$df_1(x) \wedge df_2(x) \wedge \cdots \wedge df_{n-k}(x) \neq 0\}.$$

In this case, $X \cap U$ is a strongly analytic submanifold of dimension k.

We recall a fundamental result of the local theory of semianalytic sets.

Lojasiewicz's stratification theorem. *Let X be a semianalytic subset of M, there is a strongly analytic stratification of M adapted to X. Morover for each $x \in M$ there is a semianalytic open neighborhood U of x in M and a strongly analytic stratification of U adapted to $X \cap U$ such that each stratum is normal in U.*

The modern proof of this result is obtained as in Corollary 2.11 of [1] if we observe that a stratum described by all the elements of a separating familly in an open set U is normal in U.

This local decomposition in normal strata will be used to obtain properties of the map $x \mapsto T_x X$ for a strongly analytic submanifold X.

Let U be an open set of M and N a natural number, a subset X of $U \times \mathbf{R}^N$ is called relatively semialgebraic over U if it is of the form

$$X = \bigcup_{i=1}^{r} \bigcap_{j=1}^{s} X_{i,j}$$

where each $X_{i,j}$ is either $\{(x,T) \in U \times \mathbf{R}^N / f_{i,j}(x,T) = 0\}$ or $\{(x,T) \in U \times \mathbf{R}^N / f_{i,j}(x,T) > 0\}$ for $f_{i,j}$ polynomial in T with coefficients analytic in U.

A recent result of Lojasiewicz asserts that if $X \subset U \times \mathbf{R}^N$ is relatively semialgebraic over U the closure of X in $U \times \mathbf{R}^N$ is also relatively semialgebraic over U [12]. We have also

Tarski-Seidenberg-Lojasiewicz Theorem. *If $X \subset U \times \mathbf{R}^N$ is relatively semialgebraic over U and $\pi : (x,T) \mapsto x$ is the natural projection, $\pi(X)$ is a semianalytic subset of U.*

Consider now the Grassmann manifold $G_p, p < n$ of p-dimensional vector subspaces of $\mathbf{R}^n$ naturally embedded as a smooth semialgebraic subset of some $\mathbf{R}^N$.

We will need later the following remark. If T is a p-plane, $T \in G_p$ and $0 \leq s \leq p$ the set of all the s-planes contained in T is an algebraic set in G_s. We can denote this set by $G_s(T)$. So if $X \subset U \times G_p \subset U \times \mathbf{R}^N$ is relatively semialgebraic over U the subset $\{(x, L) \in U \times G_s / \exists T \in G_p, L \subset T, (x, T) \in X\}$ denoted $G_s(X)$ is also relatively semialgebraic over U.

For each p-dimensional strongly analytic submanifold X of M, the map $\gamma_X : X \to M \times G_p : x \mapsto (x, T_x X)$ has been considered by Lojasiewicz[13] and Verdier [20] in the subanalytic setting.

Proposition 2. *For each $x \in \bar{X}$ there is a neighborhood U of x in M such that the image of $\gamma_{X \cap U}$ is a relatively semialgebraic set over U.*

Proof. Let be given a strongly analytic stratification $\mathcal{N}$ of a neighborhood U of x adapted to X so that each stratum is normal in U according to Lojasiewicz's stratification theorem. Let's suppose that S is a stratum of $\mathcal{N}$ of dimension $p = \dim X$ included in X. We have that if $y \in S$ $T_y S = T_y X$. It's easy to show that the map γ_S has a relatively semialgebraic image, as if the normal semianalytic subset S is described by the functionnally independent equations $f_1 = 0, \ldots, f_{n-p} = 0$ and $f_0 > 0$ the image of γ_S is

$$\{(y, T) \in U \times \mathbf{R}^N / df_1(y) \wedge T = 0, \ldots, df_{n-p} \wedge T = 0, f_0(y) > 0\}$$

considering naturally each element of G_p as an $n-p$-linear form.

As finite unions, intersections and closure of relatively semialgebraic sets over U are relatively semialgebraic over U. It's enough now to prove that the image of $\gamma_{X \cap U}$ is the union of the closures of the images of the γ_S for all the strata S of $\mathcal{N}$ of dimension p included in X. That is elementary as at each point of X one can take analytic local coordinates because X is a submanifold of M.

For later reference for each $\epsilon > 0$ let's fix in each dimension $p = 1, 2, \ldots, n$ a finite familly $\mathcal{H}_{p,\epsilon} = \{H_i^p(\epsilon), i = 1, \ldots l(p.\epsilon)\}$, $H_i^p(\epsilon) \in G_p$ such that the balls in $\mathbf{R}^N$ of center $H_i^p(\epsilon)$ and radius ϵ give an open covering of G_p. Denote finally $\mathcal{H}(\epsilon)$ the union of these families for every dimension. We will denote d_G the euclidean distance of $\mathbf{R}^N$ restricted to the Grassmann manifold.

2.Stratification adapted to a system of pfaffian equations. Let $\omega_1, \ldots, \omega_q$ be integrable analytic differential 1-forms on M. For each strongly analytic submanifold X of M we will obtain information on the solutions of the pfaffian system

$$\omega_1 = 0, \cdots, \omega_q = 0$$

restricted to X. This will be done by local stratification of M such that the kernel of the restricted pfaffian system is nearly parallel to a given vectorial space along each stratum.

Denote Ω the familly $\{\omega_i/i = 1, 2, \ldots, q\}$ and if

$$J : \{1, 2, \ldots, s\} \to \{1, 2, \ldots, q\}$$

is a map put $|J| = s$ and $\Omega_J = \{\omega_{J(j)}/j = 1, 2, \ldots, s\}$.

Let N be a strongly analytic submanifold of M, we will say that the familly Ω_J is transverse to N if

$$\dim(T_xN \cap \bigcap_{j=1}^{s} \ker\, \omega_{J(j)}(x)) = \dim N - s.$$

at every point $x \in N$. This is denoted by $\Omega_J \pitchfork N$.

The familly Ω_J is a basis of Ω along N if $\Omega_J \pitchfork N$ and

$$(*) \qquad T_xN \cap \bigcap_{j=1}^{s} \ker\, \omega_{J(j)}(x) = T_xN \cap \bigcap_{i=1}^{q} \ker\, \omega_i(x)$$

at every point $x \in N$.

In [16] we obtained for each semianalytic subset of M a local stratification adapted to it, such that a basis of Ω can be chosen along each stratum.

The following statement says that we can obtain a better precision. Such a stratification can be obtained so that along each stratum N the tangent space of the restricted foliation, $(*)$, is ϵ–parallel to a fixed vector space.

Proposition 3. *Let $\epsilon > 0$ be fixed. If X is a p-dimensional strongly analytic submanifold of M, $\Omega \pitchfork X$ and $x \in \bar{X}$ then there is a semianalytic subset Y of dimension at most $p-1$ an $\rho > 0$ such that if c is a connected component of $B_x(\rho) \cap (X \setminus Y)$ we can choose a $p-q$-plane $H_c = H^{p-q}_{i(c)}(\epsilon)$ so that the distance in the grasmaniann G_{p-q} from H_c to $\cap_{\omega\in\Omega} \ker\, \omega_{|X}(y)$ is less than ϵ for every $y \in c$.*

Corollary. *Let Ω and $\epsilon > 0$ be given. For each semianalytic subset X of M and $x \in \bar{X}$ there is a strongly analytic stratification of a neighborhood of x, $\mathcal{N}$ adapted to X such that we can choose for each stratum N of $\mathcal{N}$ a map J_N from a natural interval $\{1, 2, \ldots, s(N)\}$ to $\{1, 2, \ldots, q\}$ and a vector space $H_N = H^{\dim N - s(N)}_{i(N)}(\epsilon)$ of the familly $\mathcal{H}(\epsilon)$ verifying:*

1)Ω_{J_N} is a basis of Ω along N;

2)$d_G(H_N, \cap_{j=1}^{s(N)} \ker\, \omega_{J(j)|_N}) < \epsilon$.

The proof of the corollary is a straightforward consequence of the proof of proposition 1 in [16], applying Proposition 3 at each step of the induction and will not be reproduced here.

Proof of Proposition 3. Let $\rho > 0$ be such that if $U = B_0(\rho)$ the image of $\gamma_{X\cap U}$ is relatively semialgebraic over U.

It's clear that the assumption that $\Omega \not\pitchfork X$ implies that if for $x \in X$ we denote by $\Omega(x) = \cap_{\omega\in\Omega} \ker\, \omega(x)$ the set $\{(x, L)/x \in X, L = \Omega(x)\}$ is a relatively semialgebraic subset of $U \times G_{n-q}$.

Next observe that if Q is any relatively semialgebraic set over U, $Q \subset U \times G_{p-q}$ then

$$\{x \in X \cap U \quad / \quad T_x X \cap \bigcap_{\omega\in\Omega} \ker\, \omega(x) \in Q\}$$

is a semianalytic subset of M. We will denote this set by $Q(X, \Omega)$. It suffices to write this set as the natural projection of the set $G_{p-q}(\gamma_{X\cap U}(X \cap U)) \cap G_{p-q}(\Omega(X \cap U)) \cap Q$. This later set being relatively semialgebraic according to Proposition 2. The assertion follows from the Tarski-Seidenberg-Lojasiewicz Theorem.

Let's construct Y and choose the $p-q$-planes by induction. Define $Y_0 = \emptyset$, $X_0 = \emptyset$ and $Q_0 = \emptyset$. Consider the first element H_1 of the familly $\mathcal{H}_{p-q,\epsilon}$. If Q'_1 is the open ball in G_{p-q} with center H_1 and radius ϵ put $Q_1 = Q'_1 \setminus Q_0$, the set $Q_1(X, \Omega)$ is a semianalytic subset of X. If X'_1 is the interior of this set *in* X put $X_1 = X'_1 \cup X_0$. For each connected component c of X_1 we know by the definition of $Q_1(X, \Omega)$ that a proper choice for H_c is clearly H_1. That is at each point of X_1 the intersections of the kernels of the forms in Ω restricted to X are ϵ-nearly parallel to the plane H_1.

To proceed with the next element of the familly $\mathcal{H}$ we can put if Y'_1 is the complementary to X of the closure *in* X of $Q_1(X, \Omega)$, $Y_1 = Y'_1 \cup Y_0$. So Y_1 is a semianalytic subset of X of codimension at least 1.

Suppose that the sets X_l and Y_l have been constructed in such a way to have: X_l is an open semianalytic subset of X, for each connected component c of which a $p-q$-plane H_c in the familly $\{H_j^{p-q}(\epsilon), j < l+1\}$ can be chosen verifying the claim of proposition 3; Y_l is a semianalytic subset of X of codimension at least 1, $Y_l \cap X_l = \emptyset$. It's clear that if we define Q'_{l+1} as the open ball in G_{p-q} with center H_{l+1} and radius ϵ and we put $Q_{l+1} = Q'_{l+1} \setminus Q_l$, the set $Q_{l+1}(X, \Omega)$ is a semianalytic subset of X. If X'_{l+1} is the interior of this set *in* X put $X_{l+1} = X'_{l+1} \cup X_l$. For each connected component c of X_{l+1} we know by the definition of $Q_{l+1}(X, \Omega)$ that a proper choice for H_c is clearly H_{l+1}.

It is now clear that $X_{l(p-q,\epsilon)}$ is a open and dense semianalytic subset of X and Y can be chosen to be it's complement to X. On each connected component of $X \setminus Y$ the choice of $p-q$-plane is done.

3.Graph decomposition of Rollian pfaffian hypersurfaces. In this paragraph we will state the consequences of Proposition 3 for pfaffian sets and with the help of a Theorem of Hardt we will deduce the decomposition into graphs of Rollian ph.

Proposition 4. *Let Ω and $\epsilon > 0$ be given as before. For each semianalytic subset X of M and $x \in \bar{X}$ there is a strongly analytic stratification of a neighborhood of x, $\mathcal{N}$ adapted to X such that we can choose for each stratum N of $\mathcal{N}$ a vector space in $\mathcal{H}(\epsilon)$, H_N such that: for any familly $\{V_i, \omega_i, M\}, i = 1, \dots, q$ of Rollian pfaffian hypersurfaces the intersection $N \cap X \cap V_1 \cap \dots \cap V_q$ if non void is a union of a finite number of analytic submanifolds of M : W_m. Morover the restriction to each of the W_m of the orthogonal projection onto H_N, π_N, is a local isomorphism with finite fiber and the norm of the derivative of a sufficiently small local section can be bounded by ϵ.*

This proposition is a direct consequence of the corollary of proposition 3 using the theorem on uniform finiteness.

Let W be an analytic submanifold of M. Suppose that there is an open set U of some vectorial subspace E of $\mathbf{R}^n$ such that W is the graph in $E \oplus E^\perp$ of a map φ from U to the orthogonal space of E, $E^\perp$. Then we say [7] that W is an ϵ-analytic piece if the norm of the derivative of φ is everywhere bounded by ϵ.

An interesting goal is to decompose pfaffian sets in ϵ-analytic pieces in a finite way. Proposition 4 approches this goal, but we cannot, at present time achieve a further decomposition of M to get pieces where the restriction of the orthogonal projection becomes one to one. The case of codimension 1 pfaffian sets can be treated by an specific argument which seems not to be general enough. We will need for this argument the following Theorem of Hardt as it's proved in [14]. Compare also [4], [18].

Let O be a real analytic manifold.

Hardt's theorem on stratification of maps. *Let $f : E \to O$ be a proper continuous subanalytic map from the subanalytic closed set E of M, and let $\mathcal{G}$ and $\mathcal{T}$ be locally finite families of subanalytic subsets of M and O respectively. There exist a subanalytic stratification $\mathcal{A}$ of M adapted to each element of $\mathcal{G}$ and E and a subanalytic stratification $\mathcal{B}$ of O adapted to each element of $\mathcal{T}$ such that if $\Gamma \in \mathcal{A}$ and $\Gamma \subset E$, we have $f(\Gamma) \in \mathcal{B}$ and the restriction*

$f_{|\Gamma} : \Gamma \to f(\Gamma)$ *is analytically equivalent to the projection* $f(\Gamma) \times \Delta \to f(\Gamma)$ *where* Δ *is a simplex.*

We are now ready to prove the decomposition in ϵ-analytic pieces announced in Proposition 1.

Let ω and ϵ be as in the statement of proposition 1. Let $\mathcal{N}$ be the stratification obtained in Proposition 4. We are going to use Hardt's theorem for each orthogonal projection $\pi_N, N \in \mathcal{N}$ if $\dim H_N = n-1$ and $\dim N = n$ forgetting the frontier condition for a stratification at strata where the intersection with any regular solution of $\omega = 0$ is either the whole stratum of dimension $n-1$ and so a semianalytic subset of M or is of dimension less than $n-1$.

Denote $\mathcal{G}$ the set of elements N of $\mathcal{N}$ such that $\dim H_N = n-1$ and $\dim N = n$. For each $N \in \mathcal{G}$ consider the stratification $\mathcal{A}$ trivialising π_N. Let's show that in each stratum Γ of this new stratification restricted to N any Rollian ph associated to ω is an ϵ-analytic piece. That is, if $\{V, \omega, M\}$ is a Rollian ph meeting Γ, $\pi_{\Gamma|_{V\cap\Gamma}}$ is necessarily one to one.

Suppose there exist two points x_1, x_2 in $V \cap \Gamma$ such that $\pi_N(x_1) = \pi_N(x_2)$. The fiber of π_N restricted to Γ being isomorphic to a simplex, is connected. It's an analytic path meeting the Rollian ph in two points. The fundamental hypothesis proves that this fiber is tangent somewhere inbetween, to the distribution $\omega = 0$. This is not possible as Γ is included in N and the distribution $\omega = 0$ is ϵ-parallel to the orthogonal to this fiber everywhere in N..

The number of elements of $\mathcal{G}$ being finite the proof of proposition 1 is complete.

III. The density of a Rollian leaf at a boundary point

In this short chapter we will give the proof of theorem 1. The whole plan of the proof and the arguments are taken from the beautiful paper [7] of C. Kurdyka and G. Raby. We need only to verify that Rollian ph don't behave more wildly than subanalytic sets. This proof uses the decomposition into ϵ-analytic pieces of Proposition 1 and seems not to work with the result obtained in Proposition 4 alone.

1. Limit values of an ϵ-analytic pfaffian piece. Consider a pfaffian set W verifying the conditions of pieces found in Proposition 4.

That is,

–W is an analytic submanifold of M;

–we can associate to W a vector subspace $H \subset \mathbf{R}^n$ and $\epsilon > 0$ such that the orthogonal projection $\pi_W : H \oplus H^\perp \to H$ is a local isomorphism when

restricted to W;
–for every point $w \in W, d_G(T_w W, H) < \epsilon$.

We will call such a pfaffian set, an ϵ-horizontal piece.

Let's denote by U_W the open set $\pi_W(W)$.

We can think of W as being the graph of some kind of multivalued analytic mapping φ_W. The first problem is to understand the behaviour of W when its argument goes to the boundary of U_W.

Following [7] we state

Lemma 1. *Let W be a pfaffian set which is an ϵ-horizontal piece. If $w \in \bar{W}$ then the tangent cone*

$$C_w(W) \subset \{(x, y) \in H \times H^{\perp} / ||y|| \leq \epsilon ||x||\}.$$

The proof is the same of that of [7] page 757. You need only to know that by the Curve selection lemma you can reach the point w from inside W following a C^1 curve γ_u chosen to have a given u of the tangent cone as limit direction at w.

As this curve is a pfaffian set it can be chosen short enough so that it has no double points. If γ_1 is the projection $\pi_W \circ \gamma_u$, γ_1 behaves as a curve in an ϵ-analytic piece over γ_1 and the argument in [7] applies.

Corollary. *If W is a pfaffian set which is an ϵ-horizontal relatively compact piece, for small enough ϵ. Then the fiber of the projection π_W restricted to $\bar{W}$ is a finite set.*

The uniform finiteness theorem shows that $\bar{W} \cap (\{x\} \times H^{\perp})$ has a finite number of connected components for each x in H. If one of those fibers had an accumulation point z we could not have at that point

$$C_z(\bar{W} \cap (\{x\} \times H^{\perp})) \subset C_z(W) \cap H^{\perp} \subset \{z\}.$$

As Lemma 1 proves.

For pfaffian ϵ-analytic pieces this shows that the underlying analytic map has a finite number of limit values at each point in the closure of its domain.

2.Density of an open subpfaffian set. The subpfaffian sets are defined here in a naïve way. The whole theory of this class is not, at present time, clear enough. Some further properties of these sets can be found in [2]. Precisely let be given a vector subspace H of $\mathbf{R}^n$ a subset Z of H is called subpfaffian in H if there is a relatively compact pfaffian set W in $\mathbf{R}^n$ such that if π is the orthogonal projection onto H we have $Z = \pi(W)$. If Y is a relatively compact subanalytic subset of $\mathbf{R}^n$, Y is a finite union of subanalytic sets Y_i linear projections of relatively compact semianalytic sets L_i of some $\mathbf{R}^{n_i}$: $\pi_i : \mathbf{R}^{n_i} \to \mathbf{R}^n$, $\quad Y_i = \pi_i(L_i)$. As $\pi_i^{-1}(W)$ is a a pfaffian set in $\mathbf{R}^{n_i}$, $W \cap Y_i = \pi_i(\pi_i^{-1}(W) \cap L_i)$ is a subpfaffian set. So $W \cap Y$ is a finite union of subpfaffian sets. It is so, also, for $\pi(W \cap Y)$.

The following property of our subpfaffian sets is not difficult.

Lemma 2. *If Z is a subpfaffian set in some $\mathbf{R}^k$ and if X is a semianalytic subset of $\mathbf{R}^k$ then $Z \cap X$ has a finite number of connected components.*

We can suppose that there is an l and a relatively compact pfaffian set $W \in \mathbf{R}^l$ such that if π is the first k coordinates projection $\mathbf{R}^l \to \mathbf{R}^k$, $\pi(W) = Z$. As $\pi^{-1}(X)$ is semianalytic in $\mathbf{R}^l$ the set $W \cap \pi^{-1}(X)$ has a finite number of connected components. In this particular case we have $\pi(W \cap \pi^{-1}(X)) = Z \cap X$. So $Z \cap X$ has a finite number of connected components.

This result implies, in particular, that if Z is a finite union of subpfaffian sets and X is a line segment the set $Z \cap X$ is a finite union of segments.

We will say that a set Z is *line-finite* if it has this property for any line segment X. The set $Z \cap X$ is so, necessarilly, semianalytic.

We state next a minor generalisation of an argument of [7] in a form most usefull.

Proposition 5. *An open line-finite subset Z of $\mathbf{R}^k$ has a density at every point $z \in \mathbf{R}^k$. Morover*

$$\Theta_k(Z, z) \leq vol_k(B_0(1) \cap C_z(Z)).$$

So open subpfaffian sets have densities everywhere.

We will not reproduce here Kurdyka-Raby's proof as it's one page long. We can remark that in their case the density of Z subanalytic is exactly the upper bound given here in proposition 5. In the subpfaffian case this fact is not known.

All the arguments are ready for the proof of Theorem 1.

3. Densities for Rollian leaves. We will give some details of the final argument of [7] in order to stress the difficulty in the general pfaffian set case. The following lemma is the point where the use of analytic pieces cannot be avoided.

lemma 3. *Let $\varphi : U \to \mathbf{R}^{n-k}$ be a locally ϵ-lipschitzian map defined on an open set $U \subset \mathbf{R}^k$. Let $a \in \bar{U}$ such that φ has a limit b at a. Then for any*

$r > 0$

$$vol_k(U' \cap B_a(\frac{r}{1+\epsilon})) \leq vol_k(\Gamma \cap B_{(a,b)}(r)) \leq (1+\epsilon)^k vol_k(U \cap B_a(r))$$

where Γ is the graph of φ and $U' = \{x \in U / ||\varphi(x) - b|| \leq \epsilon ||x - a||\}$.

The proof is not difficult if we consider the locally $1+\epsilon$-lipschitzian map $q : x \mapsto (x, \varphi(x))$ and the 1-lipschitzian projection $p : \mathbf{R}^k \times \mathbf{R}^{n-k} \to \mathbf{R}^k$. Just observe that $\Gamma \cap B_{(a,b)}(r) \subset q(U \cap B_a(r))$ and

$$U' \cap B_a(\frac{r}{1+\epsilon}) \subset p(\Gamma \cap B_{(a,b)}(r)).$$

This is a good estimate, as it is shown in [7] that the set U' has the same density as U at point a.

You cannot get such an estimate if you dont have a map φ at hand. In the case of ϵ-horizontal pieces W we can obtain a similar estimate but the upper term must be multiplied by the maximun number of points in the fiber of the projection π_W restricted to W. All the information is lost in that way.

The proof can be concluded. Let $\{V, \mathcal{F}, M\}$ be like in Theorem 1 and $x \in \bar{V}$. Fix $\epsilon > 0$ small enough and apply Proposition 1. Let X be an open semianalytic neighborhood of x such that there is a finite collection of submanifolds $\{\Sigma_i\}$ of X of dimensions less or equal to $n-2$ such that $V \cap X \setminus \cup\Sigma_i$ is a finite union of $N(\epsilon)$ ϵ-analytic pieces : W_j^ϵ =graph(φ_j^ϵ) $j = 1, 2, \ldots, N(\epsilon)$. Denote U_j^ϵ the domain of φ_j^ϵ. It's a finite union of subpfaffian open sets in an $n-1$ plane of $\mathbf{R}^n$.

If $r > 0$ we have

$$vol_{n-1}(V \cap B_x(r)) = \sum_{j=1}^{N(\epsilon)} vol_{n-1}(W_j^\epsilon).$$

We can suppose, for short that $x = 0$, and for r small enough that only one limit value for each function φ_j^ϵ remains in the ball, namelly 0.

Applying the majorations of the last lemma, and the remark about the density of the domain $U_j'^\epsilon$ of φ_j^ϵ we get: if we denote $\lambda(\epsilon) = \sum_{j=1}^{N(\epsilon)} \Theta_{n-1}(U_j^\epsilon, 0)$,

$$\frac{\lambda(\epsilon)}{(1+\epsilon)^{n-1}} \leq \liminf_{r - 0} \frac{vol_{n-1}(V \cap B_0(r))}{r^{n-1}}$$

and

$$\limsup_{r \to 0} \frac{vol_{n-1}(V \cap B_0(r))}{r^{n-1}} \leq (1+\epsilon')^{n-1}\lambda(\epsilon').$$

For any two real positive ϵ and ϵ'.

That is $\lambda(\epsilon)(1+\epsilon)^{1-n} \leq \underline{\Theta}(V,0)$ and $\bar{\Theta}(V,0) \leq (1+\epsilon')^{n-1}\lambda(\epsilon')$.

Kurdyka and Raby conclude by showing that these two lateral densities coincide. That is so because λ is bounded as $\epsilon \to 0$ (just use above inequalities to write $(1+\epsilon)^{1-n}\lambda(\epsilon) \leq \lambda(1)(1+1)^{n-1} \leq \lambda(1)$.)

It's clear from this proof that the point at stake in controlling densities of pfaffian sets is that of decomposing them into graphs.

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Astérisque, tome 222 (1994), p. 389-405

<http://www.numdam.org/item?id=AST_1994__222__389_0>

THE BOUNDARY OF THE MANDELBROT SET HAS HAUSDORFF DIMENSION TWO

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INTRODUCTION

Let $P_c(z) = z^2 + c$ ($z, c \in \mathbf{C}$). The Julia set J_c of P_c and the Mandelbrot set are defined by

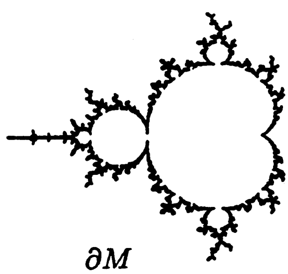
$$J_c = \text{the closure of } \{ \text{ repelling periodic points of } P_c\},$$
$$M = \{c \in \mathbf{C} \mid J_c \text{ is connected}\} = \{c \in \mathbf{C} \mid \{P_c^n(0)\}_{n=1}^{\infty} \text{ is bounded}\}.$$

In [**Sh**], we proved the following theorems concerning the Hausdorff dimension (denoted by $H - \dim(\,\cdot\,)$) of these sets:

Theorem A. *(conjectured by Mandelbrot [**Ma**] and others)*

$$H - \dim \partial M = 2.$$

Moreover for any open set U which intersects ∂M, we have $H - \dim(\partial M \cap U) = 2$.



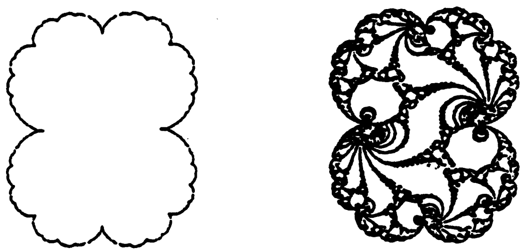
∂M

[1]1991 *Mathematics Subject Classification.* Primary 58F23; Secondary 30D05.

Theorem B. *For a generic $c \in \partial M$,*

$$\text{H-dim}\, J_c = 2.$$

In other words, there exists a residual (hence dense) subset $\mathcal{R}$ of ∂M such that if $c \in \mathcal{R}$, then H-dim $J_c = 2$.



J_c for $c = \frac{1}{4}$ and for $c = 0.25393 + 0.00048i$

In order to explain the idea of the proof, we need to introduce the following.

Definition. A closed subset X of $\overline{\mathbb{C}}$ is called a *hyperbolic subset* for a rational map f, if and only if $f(X) \subset X$ and f is expanding on X (i.e. $\|(f^n)'\| \geq C\kappa^n$ on X $(n \geq 0)$ for some $C > 0$ and $\kappa > 1$, where $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric of $\overline{\mathbb{C}}$). The *hyperbolic dimension* of f is the supremum of H-dim(X) over all hyperbolic subset X for f.

Theorem A was an immediate consequence of the following claims $(*)$ and $(**)$ (Corollary 3 (i) and (ii) in [Sh]).

$(*)$ If U is an open set containing $c \in \partial M$, then

$$\text{H-dim}(\partial M \cap U) \geq \text{hyp-dim}(P_c).$$

$(**)$ For $c \in \partial M$, there exists a sequence $\{c_n\}$ in ∂M such that $c_n \to c$ and hyp-dim$(P_{c_n}) \to 2$, as $n \to \infty$.

Theorem B also follows from $(**)$, as we will see in §1.

In this paper, we give a brief explanation on the hyperbolic dimension and on the proof of $(*)$ (in §1), and present a different point of view for the proof of (**) after McMullen [Mc2], although the essential idea came from [Sh]. The claim (**) is reduced to the main theorem in §2, which is reduced to Proposition 5.1 in §5.

The main Theorem has a meaning that the "secondary bifurcation" of the parabolic fixed point leads to high hyperbolic dimension (i.e. close to 2). The proof presented in this paper is based on the notion of geometric limit, which was proposed by McMullen [Mc2]. By a geometric limit, we mean a limit of iterates $f_n^{k_n}$ in some region (usually only in a proper subregion of $\overline{\mathbb{C}}$), where $k_n \in \mathbb{N}$, $k_n \to \infty$.

In §3, we construct a geometric limit of rational maps related to a parabolic fixed point. This case was studied earlier by Lavours [La]. In §4, we construct a second geometric limit, which corresponds to the secondary bifurcation. These limits are used in §5, to construct "good" hyperbolic subsets with high Hausdorff dimension, which leads to the proof of the main theorem. Here the use of second geometric limits is crucial.

I would like to thank specially Curt McMullen for inspiring discussions on this subject. The above pictures are made by using J. Milnor's program.

§1. Hyperbolic subsets and hyperbolic dimension

This section is devoted to the topics related to hyperbolic subsets and hyperbolic dimension. We state some properties the hyperbolic dimension, prove that (**) implies Theorem B, state the outline of the proof of (*), and explain a specific construction of a hyperbolic subset.

Properties of hyperbolic dimension. (see [Sh] §2 for the proof.)

Let X be a hyperbolic subset for a rational map f.

(1.1) *X is a subset of the Julia set $J(f)$ of f. Hence* H-dim $J(f) \geq$ hyp-dim(f).

(1.2) *X is "stable" under a perturbation, i.e. a nearby rational map f_1 has a hyperbolic subset X_1 such that (X_1, f_1) is conjugated to (X, f) by a Hölder homeomorphism with Hölder exponent close to* 1.

(1.3) $f \mapsto$ hyp-dim(f) *is lower semi-continuous, or equivalently, for any number a, the set* $\{f \mid$ hyp-dim$(f) > a\}$ *is open.*

Now we can prove that (**) implies Theorem B.

Proof of Theorem B. Let $\mathcal{R}_n = \{c \in \partial M \mid \text{hyp-dim}(P_c) > 2 - \frac{1}{n}\}$ $(n = 1, 2 \cdots)$. Then $\mathcal{R} = \bigcap_{n \geq 0} \mathcal{R}_n = \{c \in \partial M \mid \text{hyp-dim}(P_c) = 2\} \subset \{c \in \partial M \mid \text{H-dim } J(P_c) = 2\}$. By Property (1.3), $\mathcal{R}_n$ are open in ∂M. Moreover $\mathcal{R}_n$ is dense in ∂M by (**). Hence $\mathcal{R}$ is residual (i.e. contains the intersection of a countable collection of open dense subsets of ∂M.) Such an $\mathcal{R}$ is dense in ∂M by Baire's theorem. $\square$

Remark 1.4. It follows from (1.3) that if hyp-dim$(P_{c_0}) = 2$ then both $c \mapsto$ hyp-dim(P_c) and $c \mapsto$ H-dim(J_c) are continuous at $c = c_0$. On the other hand, $(**)$ implies that if $c_0 \in \partial M$ and hyp-dim$(P_{c_0}) \neq 2$ (resp. H-dim$(J_{c_o}) \neq 2$) then $c \mapsto$ hyp-dim(P_c) (resp. $c \mapsto$ H-dim(J_c)) is not continuous at $c = c_0$. Note that P_c is J-stable in $\mathbb{C} - \partial M = (\mathbb{C} - M) \cup (\text{interior} M)$, hence $c \mapsto$ hyp-dim(P_c) and $c \mapsto$ H-dim(J_c) are continuous.

In fact, if X is a hyperbolic subset for f and a homeomorphism conjugates f to f_1 in a neighbourhood of the Julia sets, then $h(X)$ is a hyperbolic subset for f_1.

Outline of the proof of $(*)$. The main idea is "the similarity between the Mandelbrot set and Julia sets" (cf. [T]). See [Sh] §3 for the complete proof.

Let $c_0 \in \partial M$. For any $\varepsilon > 0$, there exists a hyperbolic subset X for P_{c_0} such that H-dim $X >$ hyp-dim$(P_{c_0}) - \varepsilon$. Let us take a point $z \in X$ such that X intersected with any neighborhood of z has the same Hausdorff dimension as entire X. By Property (1.2), for c in a neighborhood U of c_0, P_c has a hyperbolic subset X_c ($X_{c_0} = X$), which moves continuously with c.

On the other hand, the family $\{P_c^n(0)\}_{n=1}^{\infty}$ (as functions of c) is not normal in any neighborhood of c_0. In fact, $\{P_{c_0}^n(0)\}$ is bounded, but for arbitarily close $c \in \mathbb{C} - M$, $P_c^n(0) \to \infty$ $(n \to \infty)$. Then one can show that for z chosen above, there are a parameter c_1 arbitrarily close to c_0 and a positive integer m such that $P_{c_1}^m(0)$ belongs to X_{c_1} and corresponds to z in X.

Now consider $M_1 = \{c \in U |\ P_c^m(0) \in X_c\}$. One can show that $M_1 \subset \partial M$. For $c \in M_1$, let h(c) be the point in X corresponding to $P_c^m(0)$ in X_c. Consider the case where $P_c^m(0)$ moves "transversally" relative to X_c. Then one can show that h is a homeomorphism from a neighborhood of c_1 in M_1 to a neighborhood of z in X. Moreover it is Hölder with exponent close to 1, for c_1 near c_0. (To see these facts, pretend that X_c did not move at all. Then h would give a local diffeomorphism near c_1.)

Therefore M_1 has Hausdorff dimension close to H-dim X or higher. So we can prove the assertion by letting $c_1 \to c$ and $\varepsilon \to 0$. $\square$

In §5, we will construct a special kind of hyperbolic sets as follows.

Lemma 1.5. *Suppose that U is a simply connected open set in $\mathbb{C}$; $U_1, \ldots, U_N$ are disjoint open subsets of U with $\overline{U}_i \subset U$; $n_1, \ldots, n_N$ are positive integers such that f^{n_i} maps U_i onto U bijectively $(i = 1, \ldots, N)$.*

Then there exists a Cantor set X_0 generated by $\tau_i \equiv (f^{n_i}|_{U_i})^{-1} : U \to U_i$ $(i = 1, \ldots, N)$, that is, X_0 is the minimal non-empty closed set satisfying

$$X_0 = \tau_1(X_0) \cup \cdots \cup \tau_N(X_0),$$

and the set $X = X_0 \cup f(X_0) \cup \cdots \cup f^{M-1}(X_0)$ *(where* $M = \max n_i$*) is a hyperbolic subset for* f.

Moreover $\delta = \text{H-dim}\, X_0$ *satisfies*

$$1 \geq \sum_{i=1}^{N} \inf_U |\tau_i'|^\delta \geq N(\max_i \sup_{U_i} |(f^{n_i})'|)^{-\delta}, \quad \textit{hence} \quad \delta \geq \frac{\log N}{\log(\max_i \sup_{U_i} |(f^{n_i})'|)} .$$

See [Sh] Lemmas 2.1 and 2.2 for the proof.

§2. Main theorem

In this section, we state the main theorem of this paper and prove (**) assuming the main theorem.

Definition. A periodic point is called *parabolic* if its multiplier is a root of unity. The *parabolic basin* of a parabolic periodic point ζ of period k is

$$\{z \in \overline{\mathbb{C}} |\ f^{nk} \to \zeta\ (n \to \infty) \text{ in a neighborhood of } z\ \},$$

and the *immediate parabolic basin* of ζ is the union of periodic connected components of the parabolic basin.

We assume the following (A1) and (A2).

(A1) f is a rational map of degree d (> 1) such that

$z = 0$ is a parabolic fixed point with multiplier 1 (i.e. $f(0) = 0$, $f'(0) = 1$),

the immediate parabolic basin $\mathcal{B}$ of $z = 0$ contains only one critical point of f, and

the forward orbits of critical points do not contain 0.

(A2) $\{f_\alpha\}$ is a continuous family of rational maps of degree d parametrized by $\alpha \in \mathbb{C}$, $|\alpha| < r$ ($r > 0$), $\operatorname{Re} \alpha \geq 0$, such that $f_0 = f$, $f_\alpha(0) = 0$ and $f_\alpha'(0) = \exp(2\pi i \alpha)$.

Now we can state:

Main Theorem. *Suppose* f *and* $\{f_\alpha\}$ *satisfy* (A1) *and* (A2). *Let* $\{b_n\}$ *be a sequence of integers tending to* ∞ *and* $\{\gamma_n\}$ *a covergent sequence in* $\mathbb{C}$. *Then there exists a sequence* $\{a_n\}$ *of integers tending to* ∞ *such that*

$$\text{hyp-dim}(f_{\alpha_n}) \to 2 \quad \textit{as } n \to \infty,$$

for α_n *defined by*

$$\alpha_n = \cfrac{1}{a_n - \cfrac{1}{b_n - \gamma_n}} .$$

This theorem is a weaker version of Theorem 2 in [Sh]. We give in §5 an alternative proof based on the notion of geometric limits. We can now prove (**) assuming this theorem.

Proof of (**). It follows from [MSS] that for dense parameters c in ∂M, the critical point 0 is preperiodic but not periodic under P_c. Then by [DH2] Theorem 5, such parameters can be approximated by parameters c such that some iterate P_c^k restricted to a neighbourhood of 0 is a polynomial-like mapping which is hybrid conjugate to $P_{1/4}$. For the later parameter c, the iterate $f = P_c^k$ satisfies (A1) after a change of coordinate, since P_c has only one critical point in $\mathbb{C}$. By reparametrization and coordinate change, one can construct a family $\{f_\alpha\}$ satisfying (A2) such that f_α is conjugate to $P^k_{\lambda(\alpha)}$, where and $\lambda(\alpha)$ is a continuous function with $\lambda(0) = c$.

Let α_n be as in the main theorem and define $c_n = \lambda(\alpha_n)$. Then hyp-dim(P_{c_n}) $\to 2$ $(n \to \infty)$. Moreover if we take $\gamma_n = 0$, then P_{c_n} has a parabolic periodic point, hence $c_n \in \partial M$. Thus we conclude that any parameter in ∂M can be approximated by c in ∂M for which hyp-dim(P_c) is arbitrarily close to 2. $\square$

§3. Geometric limit

In this section and the next section, we state the notion of geometric limit, which provides the basis for the proof of the main theorem in §5. First we state the facts and later, in Appendix, we comment briefly on their proof. Most of the facts stated here can be found in [L] and [Sh].

3.0. Let f be a rational map satisfying (A1) in §2. It implies that $f''(0) \neq 0$. In fact if $f''(0) = 0$, then there would be more than one immediate parabolic basins, each of which must contain a critical point, and this is impossible. By a coordinate change, we may assume that $f''(0) = 1$.

3.1. There exist analytic functions $\varphi : \mathbb{C} \to \overline{\mathbb{C}}$ and $\Phi : \mathcal{B} \to \mathbb{C}$ such that

$$\varphi(w+1) = f \circ \varphi(w) \quad \text{for } w \in \mathbb{C}$$

and

$$\Phi \circ f(z) = \Phi(z) + 1 \quad \text{for } z \in \mathcal{B}.$$

Denote $I(z) = -1/z$. If $R > 0$ is large enough, then $I(\mathcal{B})$ contains $\{w \in \mathbb{C} |\ |w| > R \text{ and } |\arg(w)| < 3\pi/4\ \}$ and $\varphi^{-1}(\mathcal{B})$ contains $\{w \in \mathbb{C} |\ |\operatorname{Im} w| > R\ \}$. The functions satisfy

$$I \circ \varphi(w) = w + c \log w + const + o(1) \quad \text{when } |\arg w - \pi| < \frac{3\pi}{4} \text{ and } |w| \to \infty;$$

$$\Phi \circ I^{-1}(w) = w - c\log w + const + o(1) \quad \text{when} |\arg w| < \frac{3\pi}{4} \text{ and } |w| \to \infty,$$

where $c = 1 - \frac{2f'''(0)}{3(f''(0))^2}$ and in each formula a suitable branch of the logarithm is taken.

The function Φ is uniquely determined up to addition of constant. We adjust it so that $\Phi \circ \varphi(w) = w + o(1)$ when $w \to +i\infty$ (i.e. $w = iy$, $y \in \mathbb{R}$ and $y \to +\infty$). Then taking into account the difference of the branches of the logarithm, we obtain $\Phi \circ \varphi(w) = w + 2\pi ic + o(1)$ when $w \to -i\infty$.

3.2. Fix $\beta \in \mathbb{C}$. Let us denote $\pi(z) = \exp(2\pi iz)$, $T(w) = w + 1$ and $\widetilde{\mathcal{B}} = \varphi^{-1}(\mathcal{B})$. Define $g : \mathcal{B} \to \overline{\mathbb{C}}$ and $\widetilde{g} : \widetilde{\mathcal{B}} \to \mathbb{C}$ by

$$g(z) = \varphi(\Phi(z) + \beta) \text{ and } \widetilde{g}(w) = \Phi \circ \varphi(w) + \beta.$$

They satisfy

$$g \circ f = f \circ g \text{ on } \mathcal{B}, \quad g \circ \varphi = \varphi \circ \widetilde{g} \text{ and } \widetilde{g} \circ T = T \circ \widetilde{g} \text{ on } \varphi^{-1}(\mathcal{B}).$$

By the adjustment in 3.1, we have

$$\widetilde{g}(w) = w + \beta + o(1) \text{ as } \operatorname{Im} w \to +\infty \text{ and}$$
$$\widetilde{g}(w) = w + \beta + 2\pi ic + o(1) \text{ as } \operatorname{Im} w \to -\infty.$$

Hence $\overline{g} = \pi \circ \widetilde{g} \circ \pi^{-1}$ is well-defined on $\overline{\mathcal{B}} = \pi(\widetilde{\mathcal{B}})$, and extends analytically to 0 and ∞, in such a way that 0 and ∞ are fixed points of $\overline{g}$, with multipliers $e^{2\pi i\beta}$ and $e^{-2\pi i\beta + 4\pi^2 c}$, respectively.

3.3. The map $\overline{g} : \overline{\mathcal{B}} \to \mathbb{C}^*$ is a *branched* covering with only one critical value. That is, there is a critical value $v \in \mathbb{C}^*$ such that $\overline{g} : \overline{\mathcal{B}} - \overline{g}^{-1}(v) \to \mathbb{C}^* - \{v\}$ is a covering map.

There are two distinct connected components $\overline{\mathcal{B}}_0$ and $\overline{\mathcal{B}}_\infty$ of $\overline{\mathcal{B}}$ such that $\overline{\mathcal{B}}_0 \cup \{0\}$ and $\overline{\mathcal{B}}_\infty \cup \{\infty\}$ are neighborhoods of 0 and ∞, respectively.

3.4. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of rational maps such that f_n has a fixed point z_n with multiplier $\exp(2\pi i\alpha_n)$;

$$\alpha_n = \frac{1}{a_n - \beta_n} ;$$

where $a_n \in \mathbb{N}$ and $\beta_n \in \mathbb{C}$; and $f_n \to f$, $z_n \to 0$, $a_n \to \infty$ and $\beta_n \to \beta$, when $n \to \infty$.
Then $f_n^{a_n} \to g$ uniformly on compact sets in $\mathcal{B}$ as $n \to \infty$.

Moreover for any $k, \ell \in \mathbb{Z}$ with $\ell > 0$, there exists $\lim_{n\to\infty} f_n^{\ell a_n + k}$ on $g^{-\ell}(\mathbb{C}^*) = g^{-\ell+1}(\mathcal{B})$ and the limit coincides with $f^k \circ g^\ell = g^\ell \circ f^k$ if $k \geq 0$. For $k < 0$, we also denote the limit by $f^k \circ g^\ell = g^\ell \circ f^k$.

For $\ell > 0$, $f^k g^\ell$ are called *geometric limits* of f. This notion was introduced by McMullen in connection with Kleinian group. See [Mc2]. In §5, we will see that this notion is very useful for the study of the bifurcation of parabolic fixed points.

§4. Second geometric limit

4.0. Let us suppose that $\beta = 0$ in §3. Then we have $\overline{g}'(0) = 1$. Although $\overline{g}$ is defined only in a subregion of $\overline{\mathbb{C}}$, the covering property of $\overline{g} : \overline{\mathcal{B}} \to \mathbb{C}^*$ allows us to treat $\overline{g}$ analogously as f.

One can show that there is an immediate basin $\mathcal{B}'$ of $z = 0$ for $\overline{g}$, such that $\overline{g} : \mathcal{B}' \to \mathcal{B}'$ is a branched covering, $\mathcal{B}'$ contains only one critical point of $\overline{g}$, and $\mathcal{B}'$ is simply connected. As in 3.0, it follows that $\overline{g}''(0) \neq 0$.

We can adjust φ and Φ so that $\bar{g}''(0) = 1$, keeping the normalization $\Phi \circ \varphi(z) = z + o(1)$ $(z \to +i\infty)$. In the following, $\overline{h}$, $\widetilde{\overline{h}}$, $\overline{\overline{h}}$ for $\overline{g}$ are similar to g, $\widetilde{g}$, $\overline{g}$ for f in §3.

4.1. There exist $\psi : \mathbb{C} \to \mathbb{C}^*$ and $\Psi : \mathcal{B}' \to \mathbb{C}$ such that

$$\psi(w+1) = \overline{g} \circ \psi(w) \quad \text{for } w \in \mathbb{C} \quad \text{and} \quad \Psi \circ \overline{g}(z) = \Psi(z) + 1 \ \text{ for } z \in \mathcal{B}.$$

Note here that the range of ψ is $\mathbb{C}^*$, since $\overline{g}(\overline{\mathcal{B}}) = \mathbb{C}^*$. As in 3.1, $\{w \mid |w| > R,\ |\arg(w)| < 3\pi/4\} \subset I(\mathcal{B}')$ and $\{w \mid |\operatorname{Im} w| > R\} \subset \psi^{-1}(\mathcal{B}')$ for $R > 0$ large, and there exists a constant c' such that

$$I \circ \psi(w) = w + c' \log w + const + o(1) \ \text{ when } |\arg w - \pi| < \frac{3\pi}{4} \text{ and } |w| \to \infty;$$

$$\Psi \circ I^{-1}(w) = w - c' \log w + const + o(1) \ \text{ when } |\arg w| < \frac{3\pi}{4} \text{ and } |w| \to \infty.$$

We adjust Ψ so that $\Phi \circ \varphi(w) = w + o(1)$ as $w \to +i\infty$.

4.2. Fix a $\gamma \in \mathbb{C}$. Define $\overline{h} : \mathcal{B}' \to \mathbb{C}^*$ and $\widetilde{\overline{h}} : \widetilde{\mathcal{B}}' \to \mathbb{C}$, where $\widetilde{\mathcal{B}}' = \varphi^{-1}(\mathcal{B}')$, by

$$\overline{h} = \psi(\Psi(z) + \gamma) \ \text{ and } \ \widetilde{\overline{h}}(w) = \Psi \circ \psi(w) + \gamma.$$

Then

$$\overline{h} \circ \overline{g} = \overline{g} \circ \overline{h} \ \text{ on } \mathcal{B}', \quad \overline{h} \circ \psi = \psi \circ \widetilde{\overline{h}} \ \text{ and } \ \widetilde{\overline{h}} \circ T = T \circ \widetilde{\overline{h}} \ \text{ on } \psi^{-1}(\mathcal{B}').$$

Hence $\overline{\overline{h}} = \pi \circ \widetilde{\overline{h}} \circ \pi^{-1}$ is well-defined on $\overline{\mathcal{B}}' = \pi(\widetilde{\mathcal{B}}')$ and extends analytically to 0 and ∞, which become fixed points.

4.3. Again, $\overline{h} : \overline{\mathcal{B}}' \to \mathbb{C}^*$ is a branched covering map with only one critical value, and there are two distinct connected components $\overline{\mathcal{B}}'_0$ and $\overline{\mathcal{B}}'_\infty$ of $\overline{\mathcal{B}}'$ such that $\overline{\mathcal{B}}'_0 \cup \{0\}$ and $\overline{\mathcal{B}}'_\infty \cup \{\infty\}$ are neighborhoods of 0 and ∞, respectively.

4.4. Let $\widetilde{h} : \pi^{-1}(\mathcal{B}') \to \mathbb{C}$ be a lift of $\overline{h}$ by π such that $\pi \circ \widetilde{h} = \overline{h} \circ \pi$ on $\pi^{-1}(\mathcal{B}')$ and $\widetilde{h}$ commutes with T and $\widetilde{g}$. (Note here that $T(\pi^{-1}(\mathcal{B}')) = \pi^{-1}(\mathcal{B}')$ and $g(\pi^{-1}(\mathcal{B}')) = \pi^{-1}(\mathcal{B}')$). Then there exists $h : \check{\mathcal{B}} \to \overline{\mathbb{C}}$, where $\check{\mathcal{B}} = \varphi(\pi^{-1}(\mathcal{B}'))$, such that $h \circ \varphi = \varphi \circ \widetilde{h}$ and h commutes with f and g.

4.5. Suppose $\{f_\alpha\}$ satisfies (A2) in §2. Let $\{b_n\}$ be a sequence of integers tending to ∞ and $\{\gamma_n\}$ a covergent sequence in $\mathbb{C}$. Then there exist sequences of integers $\{a_n\}$ and $\{p_n\}$ tending to infinity such that

$$f_{\alpha_n}^{p_n} \to h \quad \text{uniformly on compact sets in } \check{\mathcal{B}}' \quad \text{as } n \to \infty,$$

for α_n defined by

$$\alpha_n = \cfrac{1}{a_n - \cfrac{1}{b_n - \gamma_n}}.$$

(In fact, we can take $p_n = a_n b_n + \mathit{const}$, but this is not necessary in this paper.)

Let k, ℓ, m are integers with $m > 0$. Then there exists $\lim_{n\to\infty} f_n^{mp_n+\ell a_n+k}$ on $h^{-m}(\overline{\mathbb{C}}) = h^{-m+1}(\check{\mathcal{B}}')$, and the limit coincides with $f^k g^\ell h^m$ if $\ell \geq 0$ or if $\ell = 0$ and $k > 0$. In other cases, we define $f^k g^\ell h^m$ by the limit. Then they satisfy $(f^{k_1} g^{\ell_1} h^{m_1}) \circ (f^{k_2} g^{\ell_2} h^{m_2}) = f^{k_1+k_2} g^{\ell_1+\ell_2} h^{m_1+m_2}$ for $k_i,\ \ell_i \in \mathbb{Z}$ and $m_i \in \mathbb{N}$, which justifies this notation. Moreover there are commutation relations:

$$\begin{aligned}
(f^k g^\ell h^m) \circ \varphi &= \varphi \circ (T^k \widetilde{g}^\ell \widetilde{h}^m) && \text{on } \widetilde{h}^{-m}(\mathbb{C}) \\
(\overline{g}^\ell \overline{h}^m) \circ \pi &= \pi \circ (T^k \widetilde{g}^\ell \widetilde{h}^m) && \text{on } \widetilde{h}^{-m}(\mathbb{C}) \\
(\overline{g}^\ell \overline{h}^m) \circ \psi &= \psi \circ (T^\ell \widetilde{\overline{h}}^m) && \text{on } \widetilde{\overline{h}}^{-m}(\mathbb{C}^*),
\end{aligned}$$

where $\widetilde{g}^\ell \widetilde{h}^m$ and $\overline{g}^\ell \overline{h}^m$ are suitably defined for $\ell < 0$ (see Appendix). Incidently, if $m = 0$ and $\ell > 0$ or if $m = \ell = 0$, these formulae are still true, where each formule is considered on the region on which either side of the formula is defined.

We call $f^k g^\ell h^m$ *second geometric limits* of f.

The following diagram shows the relationship between the maps, where the arrows indicates semi-conjugacies between the maps.

$$\begin{tikzcd}
I \circ f^k g^\ell h^m \circ I^{-1} & T^k \widetilde{g}^\ell \widetilde{h}^m \arrow[l, "I\circ\varphi"'] \arrow[d, "\pi"] & \\
f^k g^\ell h^m \arrow[u, "I"] & \overline{g}^\ell \overline{h}^m \arrow[d, "I"] & \\
& I \circ \overline{g}^\ell \overline{h}^m \circ I^{-1} & T^\ell \widetilde{\overline{h}}^m \arrow[l, "I\circ\psi"'] \arrow[d, "\pi"] \\
& & \overline{\overline{h}}^m
\end{tikzcd}$$

§5. The proof of the main theorem

In this section, we prove the following:

Proposition 5.1. *Let f, g, h be as in the previous section. Then there exist an open set $U \subset \overline{\mathbb{C}}$ containing 0 and constants $a, C, C' > 0$ such that for large $\eta > 0$, there exist an integer N, open sets U_i and integers k_i, ℓ_i, m_i $(i = 1, \dots, N)$ such that*
$N > a\eta^2$; $m_i > 0$; $\overline{U}_1, \dots, \overline{U}_N$ are disjoint and contained in U;
$f^{k_i} g^{\ell_i} h^{m_i}$ maps $\overline{U}_i$ bijectively onto $\overline{U}$, and

$$C\eta(\log\eta)^2 < |(f^{k_i} g^{\ell_i} h^{m_i})'| < C'\eta(\log\eta)^2 \quad \text{on } U_i\ .$$

Let us first show:

Claim. *Proposition 5.1 implies the main theorem.*

Proof. Fix a large $\eta > 0$ and take U, N, etc. in Proposition 5.1. Denote $f_j = f_{\alpha_j}$, where $\{\alpha_j\}$ is the sequence chosen in 4.5. It follows from 4.5 that for each $i = 1, \dots, N$, there exists a sequence of integers $\{n(i,j)\}_{j=1}^{\infty}$ such that $f_j^{n(i,j)} \to f^{k_i} g^{\ell_i} h^{m_i}$ uniformly on a neighborhood on $\overline{U}_i$ when $j \to \infty$. If j is large enough, there exist $U_{i,j}$ (near U_i) such that $\overline{U}_{1,j}, \dots, \overline{U}_{N,j}$ are disjoint and contained in U; $f_j^{n(i,j)}$ maps $\overline{U}_{i,j}$ bijectively onto $\overline{U}$, and

$$\frac{1}{2}C\eta(\log\eta)^2 < |(f^{k_i} g^{\ell_i} h^{m_i})'| < 2C'\eta(\log\eta)^2 \quad \text{on } U_{i,j}$$

$(i = 1, \dots, N)$. Then by Lemma 1.5, f_j has a hyperbolic subset X_j such that

$$\text{H-dim}(X_j) \geq \frac{\log(a\eta^2)}{\log(2C'\eta(\log\eta)^2)} = \frac{2\log\eta + \log a}{\log\eta + 2\log\log\eta + \log(2C')}\ .$$

Therefore hyp-dim$(f_j) \to 2$ when $j \to \infty$. $\square$

Proof of Proposition 5.1.

Step 1. There exist w_0 and ζ_0 such that

$$\varphi(w_0) = 0,\ \varphi'(w_0) \neq 0,\ \pi(w_0) = \psi(\zeta_0) \text{ and } \psi'(\zeta_0) \neq 0.$$

Proof. Recal that a point z is called an *exceptional point* of f if $f^{-2}(z) = \{z\}$. For example, ∞ is an exceptional point for a polynomial. There are at most two exceptional points and if exist, they belong to $\overline{\mathbb{C}} - J_f$. Let us prove:

Lemma 5.2.

$$\varphi(\mathbb{C}) = \overline{\mathbb{C}} - \{\textit{exceptional points of } f\} \supset J_f,$$

$$\psi(\mathbb{C}) = \mathbb{C}^*.$$

Proof. Suppose that $z \in \overline{\mathbb{C}}$ is not an exceptional point of f. It is known that the closure of $\bigcup_{n\geq 1} f^{-n}(z)$ contains the Julia set. Since the image of φ must intersect Julia set, there exist z' and $n \geq 0$ such that $f^n(z') = z$ and z' is in the image of φ. Then z is also in the image of φ, since if $z' = \varphi(w)$ then $\varphi(w+n) = f^n(\varphi(w)) = f^n(z') = z$.

As to ψ, first pick a sequence $\{z_j\}_{j=0}^{\infty}$ in $\mathbb{C}^*$ such that $\overline{g}(w_j) = w_{j-1}$ and $w_j \to 0$ as $j \to \infty$. Let w be any point in $\mathbb{C}^*$. Since the unique critical value must be in the basin $\mathcal{B}'$ of $z = 0$ by 4.0, one can take a simply connected open set D in $\mathbb{C}^*$ containing w and w_0 but not the forward orbit of the critical value of $\overline{g}$, if necessary, after replacing w by its inverse image and w_0 by some w_j. In D, there is a branch of $\overline{g}^{-j}$ which sends w_0 to w_j. Let w_j' be the image of w by this branch. Since these branches have three omitted values 0, ∞ and the critical value of g, they form a normal family by Montel's theorem. So the fact $w_j \to 0$ implies that $w_j' \to 0$ as $j \to \infty$. Therefore w_j' belong to the image of ψ, for large j. So we conclude $\psi(\mathbb{C}) = \mathbb{C}^*$ as in the case of φ. $\square$

It follows from the proof of Lemma 5.2 and the assumption (A1) that $\varphi'(w_0) \neq 0$, by using $\varphi(w) = f^n(\varphi(w-n))$. The forward orbit of the critical value for $\overline{g}$ is defined and converges to 0. On the other hand, $\pi(w_0)$ is not in $\overline{\mathcal{B}}$, since $\varphi(w_0) = 0 \notin \mathcal{B}$. Therefore the forward orbit of the critical value does not contain $\pi(w_0)$. So one can show $\psi'(\zeta_0) = 0$, as before. $\square$(Step 1)

Step 2. Let ζ_0 be as in Step 1.
(i) There exist a sequence $\{\zeta_m\}_{m=1}^{\infty}$ in $\mathbb{C}$ and constants C_0, C_1, $C_1' > 0$ such that $\tilde{\overline{h}}(\zeta_m) = \zeta_{m-1}$, $\operatorname{Im}\zeta_m \to \pm\infty$ $(m \to \infty)$, $|\zeta_m - \zeta_{m-1}| < C_0$ and $C_1 < |(\tilde{\overline{h}}^m)'(\zeta_m)| < C_1'$.
(ii) Moreover if U is a sufficiently small neighborhood of $z = 0$ in $\overline{\mathbb{C}}$, then there exist a neighborhood V of w_0 and neighborhoods W_m of ζ_m in $\mathbb{C}$ such that

$diam\ W_m < 1$;

$T^\ell W_m$ $(m = 0, 1, 2, \ldots;\ \ell \in \mathbb{Z})$ are disjoint;

$\varphi : \overline{V} \to \overline{U}$, $\pi : \overline{V} \to \pi(\overline{V})$, $\psi : \overline{W}_0 \to \pi(\overline{V})$ and $\tilde{\tilde{h}}^m : \overline{W}_m \to \overline{W}_0$ are bijections with non-zero derivative;

$\frac{1}{2}C_1 < |(\tilde{\tilde{h}}^m)'| < 2C_1'$ on W_m .

Proof. (i) Let $\overline{\mathcal{B}}_0'$ and $\overline{\mathcal{B}}_\infty'$ be as in 4.3. At least one of them does not contain the critical value of $\overline{\overline{h}}$. So suppose $\overline{\mathcal{B}}_0'$ does not contain it.

There is a well-defined analytic branch of $\overline{\overline{h}}^{-1}$, $H : \overline{\mathcal{B}}_0' \cup \{0\} \to \overline{\mathcal{B}}_0' \cup \{0\}$, such that $\overline{\overline{h}} \circ H = id$, $H(0) = 0$ and $|H'(0)| < 1$ by Schwarz's lemma. Then it is well-known (see [Mi]) that there exists a linearizing coordinate $L : \overline{\mathcal{B}}_0' \cup \{0\} \to \mathbb{C}$ such that L is conformal near 0, $L(0) = 0$, $L'(0) = 1$ and $L \circ H(z) = H'(0) \cdot L(z)$ in $\overline{\mathcal{B}}_0'$. By $\pi : \mathbb{C} \to \mathbb{C}^*$, we can lift H and L to $\tilde{H} : \widetilde{\mathcal{B}}_0' \to \widetilde{\mathcal{B}}_0'$ and $\tilde{L} : \widetilde{\mathcal{B}}_0' \to \mathbb{C}$ so that $\tilde{H}$ is a branch of $\tilde{\tilde{h}}^{-1}$, $\tilde{L} \circ \tilde{H}(z) = \tilde{L}(z) + const$ and $\tilde{L}'(z) \to 1$ as $\operatorname{Im} z \to +\infty$.

Now choose $\zeta_1 \in \tilde{\tilde{h}}^{-1}(\zeta_0) \cap \widetilde{\mathcal{B}}_0'$, and define $\zeta_m = \tilde{H}^{m-1}(\zeta_1)$ for $m \geq 2$. Then $\operatorname{Im} \zeta_m \to +\infty$ $(m \to \infty)$, since $H^m \to 0$. The sequence $\{|\zeta_m - \zeta_{m-1}|\}$ is bounded, since $H(z) - z \to const$ as $\operatorname{Im} z \to +\infty$. The derivative can be estimated by the fact that when $m \to \infty$,

$$(\tilde{\tilde{h}}^m)'(\zeta_m) = \tilde{\tilde{h}}'(\zeta_1)/(\tilde{H}^{m-1})'(\zeta_1) = \tilde{\tilde{h}}'(\zeta_1)\tilde{L}'(\zeta_m)/\tilde{L}'(\zeta_1) \ \to\ \tilde{\tilde{h}}'(\zeta_1)/\tilde{L}'(\zeta_1)\ .$$

If $\overline{\mathcal{B}}_0'$ contains the critical value, we can use $\overline{\mathcal{B}}_\infty'$ and $I \circ \overline{\overline{h}} \circ I^{-1}$ instead of $\overline{\mathcal{B}}_0'$ and $\overline{\overline{h}}$ to prove the assertion.

(ii) This follows from Step 1 and the proof of (i). □

Step 3. If $\eta > 0$ is sufficiently large, then there exist an integer $N > \frac{1}{C_0}\eta^2$ and N distinct pairs of integers $(\ell_i,\ m_i)$ $(i = 1, \ldots, N)$ such that $m_i > 0$ and $T^{-\ell_i} W_{m_i} \subset S_\eta = \{z \in \mathbb{C} \mid |\operatorname{Re} z| < \eta,\ \eta < |\operatorname{Im} z| < 2\eta\}$ for $i = 1, \ldots, N$.

Proof. By Step 2, there is an m^* such that $\eta + 1 < |\operatorname{Im} \zeta_m| < 2\eta - 1$ if $m^* \leq m < m^* + [(\eta - 2)/C_0]$. For such an m, at least $[2\eta - 2]$ of $\{T^\ell W_m \mid \ell \in \mathbb{Z}\}$ are contained in S_η. The assertion follows, since $[(\eta - 2)/C_0] \cdot [2\eta - 2] > \frac{1}{C_0}\eta^2$ for large η. □

Step 4. It follows from §4.1 that if η is large, then $I \circ \psi$ is injective on S_η and the image $I \circ \psi(S_\eta)$ is contained in $\{z \in \mathbb{C} \mid \eta/2 < |z| < 3\eta\} - \mathbb{R}$.

For each $i(= 1, \ldots, N)$, we can choose a component V_i of $\pi^{-1}(\psi(T^{-\ell_i} W_{m_i}))$ $= (I \circ \pi)^{-1}(I \circ \psi(T^{-\ell_i} W_{m_i}))$ such that $V_i \subset S_\eta' = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z <$

$1,\ \frac{1}{2\pi}(\log\eta - \log 2) < |\operatorname{Im} z| < \frac{1}{2\pi}(\log\eta + \log 3)\}$. By the diagram in 4.5 and the definition of W_m, there are integers k_i ($i = 1, \dots, N$) such that $T^{k_i}\widetilde{g}^{\ell_i}\widetilde{h}^{m_i}$ maps bijectively $\overline{V}_i$ onto $\overline{V}$.

If η is large, then by 3.1, $I \circ \varphi$ is injective on S'_η and the image $I \circ \varphi(S'_\eta)$ is contained in $\{z \in \mathbb{C} |\ \frac{\log\eta}{4\pi} < |z| < \frac{\log\eta}{\pi}\}$. Then define $U_i = \varphi(V_i)$ ($i = 1, \dots, N$). Their closures $\overline{U}_1, \dots, \overline{U}_N$ are disjoint, and contained in U, for η large.

By the construction of U_i, k_i, etc. and the diagram in 4.5, we see that $f^{k_i}g^{\ell_i}h^{m_i}$ maps $\overline{U}_i$ bijectively onto $\overline{U}$, for each i.

Step 5. Now we give the estimate of $(f^{k_i}g^{\ell_i}h^{m_i})'$. By the above diagram , we have the following factorization:

$$f^{k_i}g^{\ell_i}h^{m_i}|_{U_i} = (\varphi|_V) \circ (T^{k_i}\widetilde{g}^{\ell_i}\widetilde{h}^{m_i}|_{V_i}) \circ (I \circ \varphi|_{V_i})^{-1} \circ (I|_{U_i})$$

$$T^{k_i}\widetilde{g}^{\ell_i}\widetilde{h}^{m_i}|_{V_i} = (\pi|_V)^{-1} \circ (\overline{g}^{\ell_i}\overline{h}^{m_i} \circ I^{-1})|_{I\circ\pi(V_i)} \circ (I \circ \pi|_{V_i})$$

$$(\overline{g}^{\ell_i}\overline{h}^{m_i} \circ I^{-1})|_{I\circ\pi(V_i)} = (\psi|_{W_0}) \circ (\widetilde{\widetilde{h}}^{m_i}|_{W_{m_i}}) \circ (T^{\ell_i}|_{T^{-\ell_i}W_{m_i}}) \circ (I \circ \psi|_{T^{-\ell_i}W_{m_i}})^{-1}\ .$$

Note here that $I \circ \pi(V_i) = I \circ \psi(T^{-\ell_i}W_{m_i})$.

Let us estimate on the factors in the above. First of all, $\varphi|_V$, $\pi|_V$, $\psi|_{W_0}$ are independent of η and their derivatives have estimates from above and below, i.e.

$$C_2 < |(\varphi|_V)'| < C'_2,\ C_2 < |(\pi|_V)'| < C'_2,\ C_2 < |(\psi|_{W_0})'| < C'_2\ ,$$

where $C_2, C'_2 > 0$ are constants independent of η. By 3.1 and 4.1, if η is large, then we have

$$\frac{1}{2} < |(I \circ \varphi|_{V_i})'| < 2,\ \frac{1}{2} < |(I \circ \psi|_{W_{m_i}})'| < 2\ ,$$

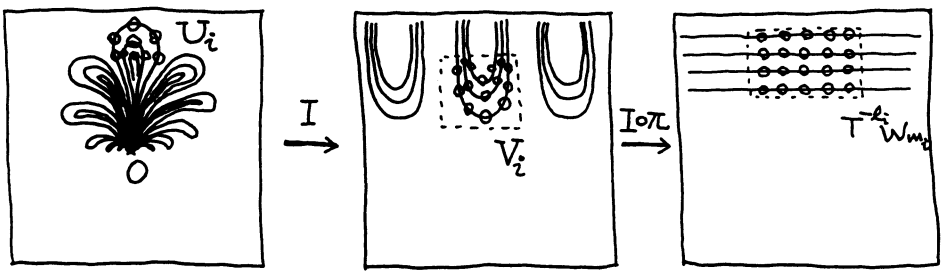
since $V_i \subset S'_\eta$ and $W_{m_i} \subset S_\eta$. The functions $I(z) = -1/z$, $I \circ \pi(z) = -e^{-2\pi i z}$, $T(z) = z + 1$ are estimated explicitly:

$$\left(\frac{\log\eta}{4\pi}\right)^2 < |(I|_{U_i})'| < \left(\frac{\log\eta}{\pi}\right)^2,\quad \pi\eta < |(I \circ \pi|_{V_i})'| < 6\pi\eta,\quad (T^{\ell_i})' = 1.$$

(Note that $U_i \subset \{z |\ \frac{\pi}{\log\eta} < |z| < \frac{4\pi}{\log\eta}\}$ and $V_i \subset S'_\eta$). Finally by Step 2, $\frac{1}{2}C_1 < |(\widetilde{\widetilde{h}}^m|_{W_m})'| < 2C'_1$.

Combining all the estimates, we conclude $C\eta(\log\eta)^2 < |(f^{k_i}g^{\ell_i}h^{m_i}|_{U_i})'| < C'\eta(\log\eta)^2$, where $C, C' > 0$ are constants independent of η. We have thus proved Proposition 5.1. $\square$

Remark. In order to understand the structure in the proof, let us make a "caricature" which was proposed by McMullen [Mc2]. First note that $I\circ\varphi|_{V_i}$, $I\circ\psi|_{T^{-\ell_i}W_{m_i}}$, $\tilde{\tilde{h}}^{m_i}|_{W_{m_i}}$ and $(\varphi|_V)\circ(\pi|_V)^{-1}\circ(\pi|_V)^{-1}$ had bounded effects in the estimate of the derivative. So let us pretend that they are all affine maps, and moreover $I\circ\varphi = id$, $I\circ\psi = id$ and $\tilde{\tilde{h}}^{m_i}(z) = z+\mu$, where μ is a constant with $\operatorname{Im}\mu < 0$. Then $T^{-\ell_i}W_{m_i} = \psi(T^{-\ell_i}W_{m_i})$ are distributed along the lattice generated by T and $\tilde{\tilde{h}}$. They are transformed by $(I\circ\pi)^{-1}$ and I^{-1} as in the figure.



Placement of U_i, V_i and $T^{-\ell_i}W_{m_i}$

The unbouded effects come from $I|_{U_i}$ and $I\circ\pi|_{V_i}$, and the latter is dominant. This observation, together with the figure, suggests that the phenomenon which we see here is very similar to McMullen's result [Mc1] that $\text{H-dim}(J(\lambda e^z)) = 2$ for $\lambda\in\mathbb{C}^*$.

Appendix. Note on the proof of §3 and §4

We give a brief sketch of the proof. In order to complete the proof, see [Sh] (§4, Proposition 5.3 and Appendix) or [La].

3.1. The functional equations in 3.1 mean that φ and Φ conjugate f to $T(w) = w+1$ in some regions. It is classically known (see [Mi], for example) that such φ and Φ exist at least in regions $\{w|\operatorname{Re}w < -R\}$ and $I^{-1}\{w|\operatorname{Re}w > R\}$, respectively, and that they satisfy the estimates. (Note here that $I\circ f\circ I^{-1}(w) = w+1+\frac{c}{w}+O(w^2)$ $(w\to\infty)$.)

Then it is possible to extend them to $\mathbb{C}$ or $\mathcal{B}$. The estimates in the regions stated in 3.1 can be obtained either by the classical estimate or by Koebe's distortion theorem. (See [Sh] Lemma A.2.4.)

3.2. The assertions in 3.2 are easy consequences of 3.1.

3.3. This follows from the facts that φ and Φ are diffeomorphic in $\{\mathrm{Re}\, w < -R\}$ and $I^{-1}\{\mathrm{Re}\, w > R\}$, respectively, and that f has only one critical point in $\mathcal{B}$. (The latter is used only here.)

3.4. In [Sh] (§4 and Appendix), we proved that there exist analytic mappings $\varphi_n = \varphi_{f_n} : \mathcal{D}om(\varphi_n) \to \mathbb{C}$ and $\widetilde{\mathcal{R}}_n = \widetilde{\mathcal{R}}_{f_n} : \mathcal{D}om(\widetilde{\mathcal{R}}_n) \to \mathbb{C}$ such that

$\mathcal{D}om(\varphi_n) = \{w \in \mathbb{C} |\ |\arg(w + \frac{1}{\alpha_n} - \xi_0)| < 2\pi/3\ \}$, where $\xi_0 > 0$ is a constant;

$\varphi_n(w+1) = f_n \circ \varphi_n(w)$ for $w \in \mathcal{D}om(\varphi_n)$;

$\varphi_n \to \varphi$ $(n \to \infty)$ uniformly on compact sets in $\mathbb{C}$;

$\mathcal{D}om(\widetilde{\mathcal{R}}_n) \nearrow \widetilde{\mathcal{B}}$ $(n \to \infty)$;

$\widetilde{\mathcal{R}}_n \circ T = T \circ \widetilde{\mathcal{R}}_n$;

if $w \in \mathcal{D}om(\varphi_n)$, $k, \ell \in \mathbb{N}$ and $\widetilde{\mathcal{R}}_n^{\ell}(w) + k = w' \in \mathcal{D}om(\varphi_n)$, then $f_n^k(\varphi_n(w)) = \varphi_n(w')$;

$\widetilde{\mathcal{R}}_n + \alpha_n \to \Phi \circ \varphi$ $(= \mathcal{E}_{f_0}$ in [Sh]) $(n \to \infty)$ uniformly on compact sets in $\widetilde{\mathcal{B}}$.

Let us prove 3.4 assuming these facts. If $w \in \widetilde{\mathcal{B}}$, Then w belongs both to $\mathcal{D}om(\widetilde{\mathcal{R}}_n)$ and to $\mathcal{D}om(\varphi_n)$ for large n and when $n \to \infty$,

$$\widetilde{\mathcal{R}}_n(w) + a_n = (\widetilde{\mathcal{R}}_n(w) + \frac{1}{\alpha_n}) + (a_n - \frac{1}{\alpha_n}) \to \Phi \circ \varphi(w) + \beta = \widetilde{g}(w)$$

and

$$f_n^{a_n}(\varphi_n(w)) = \varphi_n(\widetilde{\mathcal{R}}_n(w) + a_n) \to \varphi \circ \widetilde{g}(w) = g \circ \varphi(w).$$

For any compact set $K \subset \mathcal{B}$, there is a compact set $K' \subset \widetilde{\mathcal{B}}_0$ such that $\varphi(K')$ contains K in its interior. The above argument applies uniformly on K'. Hence, it follows that $f_n^{a_n} \to g$ uniformly on K, when $n \to \infty$.

Similarly one can prove that $\lim_{n\to\infty} f_n^{a_n+k}(z)$ exists for $z \in \mathcal{B}$ and $k \in \mathbb{Z}$, and coincides with $\varphi(\Phi(z) + \beta + k)$ (which is equal to $f^k \circ g$ if $k \geq 0$). Then the assertion for $f_n^{\ell a_n + k} = (f_n^{a_n+k}) \circ f_n^{(\ell-1)a_n+k}$ follows.

4.0. There is a "petal" D for $z = 0$ such that D is a simply connected open set in $\mathbb{C}^*$, $\overline{g}(D) \subset D$ and the parabolic basin of $z = 0$ is $\bigcup_{n\geq 0} \overline{g}^{-n}(D)$. Since $\overline{g} : \overline{\mathcal{B}} \to \mathbb{C}^*$ is a branched covering with one critical value, every component of $\overline{g}^{-1}(D)$ is simply connected and contains at most one critical point, and so are the components of $\overline{g}^{-n}(D)$. Then the assertions follow, since $\{f^{-n}(D)\}$ is increasing.

4.1-4.3. These are similar to 3.1-3.3.

4.4 and 4.5. Let $f_{n,k} = f_{\alpha(n,k)}$, where $\alpha(n,k) = 1/(k - 1/(b_n - \gamma_n))$, and define g_n, $\widetilde{g}_n$, $\overline{g}_n$ to be g, $\widetilde{g}$, $\overline{g}$ in §3 with $\beta = 1/(b_n - \gamma_n) \equiv \beta_n$. Then by §3, we have

$$f_{n,k}^k \to g_n \quad \text{when } k \to \infty$$

and $\overline{g}_n = e^{2\pi i \beta_n}\overline{g}$. Let $\widetilde{\mathcal{R}}_{n,k}$ be $\widetilde{\mathcal{R}}_n$ in the proof of 3.4 corresponding to $f_{n,k}$. Then $T^k\widetilde{\mathcal{R}}_{n,k} = \widetilde{\mathcal{R}}_{n,k} + k \to \widetilde{g}_n$ on $\widetilde{\mathcal{B}}$ as $k \to \infty$.

On the other hand, when $n \to \infty$, $\overline{g}_n \to \overline{g}$, since $\beta_n \to 0$. Hence as in 3.4, we have $\overline{g}_n^{b_n} \to \overline{h}$ on $\mathcal{B}'$ as $n \to \infty$. Since $\widetilde{g}_n$ and $\widetilde{h}$ are the lifts of $\overline{g}_n$ and $\overline{h}$ by π and commute with T, there exists a sequence of integers $\{q_n\}$ such that $T^{q_n}\widetilde{g}_n^{b_n} \to \widetilde{h}$ on $\pi^{-1}(\mathcal{B}')$ $(n \to \infty)$. If we choose a sequence of integers $\{k_n\}$ tending to infinity sufficiently fast, then on $\pi^{-1}(\mathcal{B}')$

$$\widetilde{\mathcal{R}}_{n,k_n}^{b_n} + (k_n b_n + q_n) = T^{q_n}(T^{k_n}\widetilde{\mathcal{R}}_{n,k_n})^{b_n} \to \widetilde{h} \ \ (n \to \infty).$$

Then it can be shown, as in the proof of 3.4, that for any $w \in \pi^{-1}(\mathcal{B}')$,

$$f_n^{p_n}(\varphi(w)) \to \varphi(\widetilde{h}(w)) \ \ (n \to \infty),$$

where $p_n = k_n b_n + q_n$. So there exists $\lim_{n\to\infty} f_n^{p_n}$ on $\breve{\mathcal{B}}'$, which is defined to be h. Then it satisfies $h \circ \varphi = \varphi \circ \widetilde{h}$ and commutes with f and g.

The assertions for $f^k g^\ell h^m$ can be proved similarly. For the commutation relations, note that $\overline{g}^\ell\overline{h}^m = \lim_{n\to\infty} \overline{g}_n^{mb_n+\ell}$ and $\widetilde{g}^\ell\widetilde{h}^m = \lim_{n\to\infty} T^{mq_n}\widetilde{g}_n^{mb_n+\ell}$, which are definitions if $\ell < 0$.

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Astérisque

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Perturbations of critical fixed points of analytic maps

Astérisque, tome 222 (1994), p. 407-422

<http://www.numdam.org/item?id=AST_1994__222__407_0>

Perturbations of Critical Fixed Points of Analytic Maps

David Tischler

This paper has three parts. The first is concerned with an inequality relating the positions of the fixed point, the critical point, and the critical value obtained by a perturbation of an analytic function which has a critical point which is simultaneously a fixed point. The result is local in nature in that the analytic function need only be defined in a neighborhood of the fixed critical point. However, the motivation for this problem comes from a conjecture of Smale about polynomials, [4].

The conjecture states that, given a complex polynomial f of degree d and a non-critical point z, that for some critical point θ

$$|f(z) - f(\theta)| \,/\, |(z - \theta)f'(z)| < 1. \tag{1}$$

Let $S(f, z, \theta)$ denote the left hand side of (1). For the polynomial $f(z) = z^d + (d/(d-1))z$, and the point $z = 0$, for any choice of critical point θ, $S(f, 0, \theta) = (d-1)/d$. This is possibly the worst case for the conjecture, which would mean that $(d-1)/d$ could replace 1 in (1). In [5], we showed that $(d-1)/d$ is an upper bound for any sufficiently small perturbation of $f(z) = z^d + (d/(d-1))z$ and $z = 0$. We also conjectured that the mean value conjecture might be strengthened to state that for any f of degree d, and any z, for some θ

$$|S(f, z, \theta) - 1/2| \leq 1/2 - 1/d. \tag{2}$$

In the first part of this paper we will show that (2) is true for any sufficiently small perturbation of $f(z) = z^d + (d/(d-1))z$, $z = 0$.

In the second part of this paper we will consider a topological version of the mean value conjecture. That is, we will consider only critical points θ for which $f(\theta)$ is on the boundary of the largest disk centered at $f(z)$ on which a branch of f^{-1} can be defined. For these critical points, by applying the Koebe 1/4 theorem to f^{-1}, it was shown in [4] that $S(f,z,\theta) \leq 4$. The topological version of the mean value conjecture raises the question of whether there is a better version of the Koebe 1/4 theorem for inverse branches of polynomials.

We will consider degrees $d = 3, 4$ where (2) is known to be true, [5]. We will show that the topological version of (2) is true for $d = 3$ and is false for $d = 4$. Whether the topological version of (1) is true is not known.

In the final section we will describe a quasi-conformal model for polynomials which will allow us to verify (2) for roots of polynomials whose critical values have norms that increase fast enough.

§1. Let $f(z)$ be a complex analytic function defined in a neighborhood of the point θ. Suppose that $f(\theta) = \theta$ and that $f'(\theta) = 0$ and $f''(\theta) \neq 0$. Let p be another point different than θ, not necessarily in the domain of f. Since θ is a fixed point of f, $|\theta - p| = |f(\theta) - p|$. Let h denote a complex analytic function which is a peturbation of f in a neighborhood of θ. For any sufficiently small perturbation h of f there will be a fixed point q, $(h(q) = q)$, and a critical point σ, $(h'(\sigma) = 0)$, near θ. The following theorem gives a sufficient condition on perturbations of f so that $|\sigma - p| \geq |h(\sigma) - p|$.

Theorem 1. *Suppose that* $\mathbf{R}((\theta - p)(f''(\theta)) < 0$. *Suppose that* $|\sigma - p| > |q - p|$. *Then if* h *is sufficiently near* f, $|h(\sigma) - p| \leq |\sigma - p|$.

Proof. Without loss of generality, for the purposes of this proof, we can assume that $p = 0$. This follows because conjugating f by a translation does not change the second derivative, and translations preserve lengths. Since $p = 0$ and $p \neq \theta$, we can define an analytic function g in a neighborhood of θ by the formula $f(z) = zg(z)$. We are interested in the level curves of g. That is the curves defined by $|g(z)|$ =constant.

Lemma 1. *Let $f(z) = zg(z)$ and suppose $f'(\theta) = 0$ and $g'(\theta) \neq 0$. Then, at θ, the tangent vector $-\theta$ is orthogonal to the level curve of g.*

Proof. Let $w = g(z)$. The level curves of g are the preimages of circles in the w plane centered at 0. The radial vector field $V(w) = w$ pulls back by g to the vector field $g(z)/g'(z)$. Since g is conformal away from critical points we have that the vector field $g(z)/g'(z)$ is orthogonal to the level curves of g. Since $f'(z) = g(z) + zg'(z)$ and θ is a critical point of f, we see that the vector $-\theta$ is orthogonal to the level curve of g at θ. Note that $g'(\theta)$ is not zero because then $g(\theta) = 0$ and then $f(\theta) = 0$ which means that $\theta = 0$ since θ is a fixed point for f, and we have assumed that $\theta \neq p = 0$. Therefore the rays from 0 are transversal to the level curves of g near θ, and $|g(z)|$ increases in the direction of $-\theta$. Let us orient the level curves of g so that at θ the orientation agrees with the direction $-(\sqrt{-1})\theta$. Let k denote the curvature of the level curves of g with the given orientation. A calculation shows that

$$k = |g'(z)/g(z)|\,\mathbf{R}(1 - (g(z)g''(z)/(g'(z)^2)), \text{ see [2,p.359]}.$$

If we express this formula for k in terms of f and use that θ is both a fixed point and a critical point for f we obtain the formula

$$k = |1/\theta|\,\mathbf{R}(-1 - \theta f''(\theta)).$$

By hypothesis, $\mathbf{R}(\theta f''(\theta)) < 0$. Therefore, $k > -1/|\theta|$, which is the curvature of the circle C_θ with center 0 and which passes through θ. Since $-\theta$ is orthogonal to the g level curve at θ and $-\theta$ is also orthogonal to C_θ at θ, it follows that the g level curve is tangent to C_θ at θ. We conclude that in a small enough neighborhood of θ the level curve of g passing through θ is outside C_θ.

Suppose h is a perturbation of f. Define $j(z) = h(z)/z$. For sufficiently small perturbations h of f the level curves of j are transversal to the rays from 0, the level curve of j through σ is outside the circle C_σ which passes through σ with center 0. Let b denote the intersection of the ray through q and the level curve of j through σ. Let c denote the intersection of the ray through q and the circle C_σ. Note that $|j(z)|$ increases along the rays in the direction towards 0. Therefore $|j(c)| \geq |j(b)|$. By hypothesis, $|\sigma - 0| \geq |q - 0|$, so $|j(c)| \leq |j(q)|$. Since b and σ are on the same level curve of j, we conclude that $|j(\sigma)| \leq |j(q)|$. Since q is a fixed point for h we obtain $1 > |h(\sigma)/\sigma|$ or $|\sigma - 0| > |h(\sigma) - 0|$. Since we have assumed $p = 0$, this completes the proof of Theorem 1.

Remark. Theorem 1 is also true if all three inequality signs are reversed. The proof is analogous to the one given above.

Let us apply Theorem 1, and more particularly, the method of proof to the case of $f(z) = z^d + (d/(d-1))z$ and $p = 0$, which was the case discussed in the introduction. The critical points θ of f satisfy $\theta^{d-1} = -1/(d-1)$. For this polynomial $f(z)$ we find that $g(z) = z^{d-1} + d/(d-1)$ and $f''(\theta) = d(d-1)\theta^{d-2}$ which implies $\mathbf{R}(\theta f''(\theta)) < 0$. Without loss of generality we can restrict our attention to perturbations h of f which are degree d polynomials which satisfy $h(0) = 0$ and $h'(0) = d/(d-1)$. As was shown in [5], for any such perturbation h there is a critical point σ and a fixed point q near one of the critical points θ of f so that $|\sigma - 0| \geq |q - 0|$. From Thm. 1, we conclude that $|\sigma| > |h(\sigma)|$ and $S(h, 0, \sigma) < (d-1)/d$. This was already shown in [5]. Here we want to show in addition that

Theorem 2. *The stronger mean value conjecture (2) is true for any sufficiently small perturbation h of the above f with $z = 0$ and some critical point σ of h.*

Proof. Observe that the image by g of the level curve through θ is a circle centered at the origin. Since θ is fixed by f, $g(\theta) = 1$. Using the same notation as in the proof of Thm. 1, $g(C_{0\theta})$ is the circle of radius $1/(d-1)$ centered at $d/(d-1)$. For a perturbation h close to f, j will be a polynomial approximately the same as g. Therefore, $j(C_\sigma)$ is approximately a circle of radius $1/(d-1)$. So the curvature of $j(C_\sigma)$ is approximately $1/(d-1)$. Furthermore, $j(C_\sigma)$ is tangent at $j(\sigma)$ to the circle centered at 0 passing through σ. Let C_q be the circle centered at 0 passing through q. Like $j(C_\sigma)$, $j(C_q)$ is also approximately a circle with radius $1/(d-1)$. By hypothesis, C_σ is outside C_q and so $j(C_\sigma)$ is outside $j(C_q)$.

Since $h(q) = q$, $j(q) = 1$. As a first estimate, we suppose that $j(C_\sigma)$, is actually a circle of radius $1/(d-1)$ passing through 1, with center at a point e approximately equal to $d/(d-1)$. Denote any such circle by C. Let s denote the point where C is tangent to a circle centered at 0. So $s, 0$, and e are collinear. Let ϕ be the central angle at e between s and 1. Let $u = (1 + 1/(d-1))/2$.

Lemma 2. *For $d \geq 4$, there is a positive constant M, independent of ϕ, such that $|u - s| < 1 - u - M\phi^2$.*

Proof. Let A be the angle between the rays $0s$ and 01. Let r be the point in the interval [0,1] which is equidistant from 1 and s. So $|s - r| = 1 - r$. In triangle $0rs$, the interior angle at s is $A + \phi$. By the law of sines, $(1-r)/r = \sin(A)/\sin(A+\phi)$. In triangle $0, 1, e$, the interior angle at 1 is $\pi - (A+\phi)$. By the law of sines in this triangle $\sin(A)/\sin(\pi - (A+\phi)) = (1/(d-1))/|e| < 1/(d-1)$. Since $\sin(A+\phi) = \sin(\pi - (A+\phi))$ we conclude that $r > (d-1)/d$. For $d \geq 4$, $u < (d-1)/d$ so that $u < r$. In triangle u, r, s the interior angle at r is $\pi - (2A + \phi)$. Note that $A = 0(\phi)$ and that ϕ is near 0 since $\phi = 0$ for the uperturbed function f. Using the law of cosines in triangle u, r, s and the fact that $|1 - u| = |1 - r| + |r - u|$ we find that $|u - s| < |1 - u|$ and $|u - s|^2 - |1 - u|^2 = 0(\phi^2)$. Therefore, $|u - s| < |1 - u| - M\phi^2$ for some positive number M. This proves the lemma for $d \geq 4$.

Remark. The cases of $d = 3, 4$ of Thm. 2 are treated in [5].

Suppose that the curvature k of $j(C_\sigma)$, and that of $j(C_q)$, satisfy $|k - 1/(d-1)| < \varepsilon$. Let A_σ be the angle between the rays 01 and $0j(\sigma)$ and let A_q be the angle between 01 and $0t$, where t is an element of $j(C_q)$, with

$0, t, e$ collinear. Then $|A - A_\sigma|$ and $|A - A_q|$ are both less than $\pi\varepsilon$. Furthermore, the distance from 1 to t along $j(C_q)$ is $0(\phi)$. Therefore, $|s - t| < L\varepsilon\phi^2$, for some constant L, independent of ϕ and ε. Therefore, for small enough ϕ and ε, $|u - t| \leq |1 - u| - M\phi^2 + L\varepsilon\phi^2 < |1 - u|$. Since $j(C_\sigma)$ is outside of $j(C_q)$ we can make a similar estimate to show $|u - j(\sigma)| < |u - t| + N\varepsilon\phi^2$ for some N independent of ϕ and ε. This shows that for a sufficiently small perturbation h of f that $|j(\sigma) - u| < |1 - u|$.

In order to complete the proof of Theorem 2 we recall the normalization discussed just before the statement of Thm. 2 to obtain

$$S(h, 0, \sigma) = h(\sigma)/(\sigma h'(0)) = j(\sigma)(d - 1)/d.$$

Therefore, $((d-1)/d)\,|j(\sigma) - u| = |S(h, 0, \sigma) - 1/2|$. Also, $((d-1)/d)\,|1 - u| = 1/2 - 1/d$. Therefore $|S(h, 0, \sigma) - 1/2| < 1/2 - 1/d$ and the proof of Theorem 2 is complete.

§2. If A and B are first degree polynomials, then it is easy to check that

$$S(A \circ f \circ B, z, \theta) = S(f, B(z), B(\theta)).$$

Suppose f is a degree three polynomial that has two distinct critical points. Choose B so that the critical points are at $z = 1/2$ and $z = -1/2$, and choose A so that $f(1/2) = 1/2$ and $f(-1/2) = -1/2$. Then $f(z) = -2z^3 + (3/2)z$. In [5], we gave a topological description of polynomials, all of whose critical points are also fixed points. We will use that description to show that f satisfies the topological version of (2).

The topological description of $f(z) = -2z^3 + (3/2)z$ is as follows. There are two closed topological disks, one around $z = -1/2$ and one around $z = 1/2$ on each of which f is topologically conjugate to $z \to z^2$. In the interior of each disk the conjugation is analytic. The two disks have the point $z = 0$ as their intersection, and it is a repelling fixed point for f. The imaginary axis is invariant by f. Denote the positive imaginary axis by J_1 and the negative imaginary axis by J_4. The points $z = +(\sqrt{3})/2$ and $-(\sqrt{3})/2$ are the two other preimages of $z = 0$. See Fig. 1 as reference for the following part of the description of f. The curves J_2, J_6 are preimages of J_4 whereas, J_3, J_5 are preimages of J_1. The region between J_1 and J_2 and outside of the disk around $z = 1/2$ is mapped by f onto the part of the right half plane which lies outside the disk around $z = 1/2$. The region between J_2 and J_3 is mapped onto the half plane $\mathbf{R}(z) < 0$, sending $[(\sqrt{3})/2, \infty)$ onto $(-\infty, 0]$. Each of these maps is a homeomorphism. There are similar homeomorphisms of the regions outside the union of the two disks and between the curves J_i and J_{i+1}, $i = 1$ to 6, (note that J_7 means J_1).

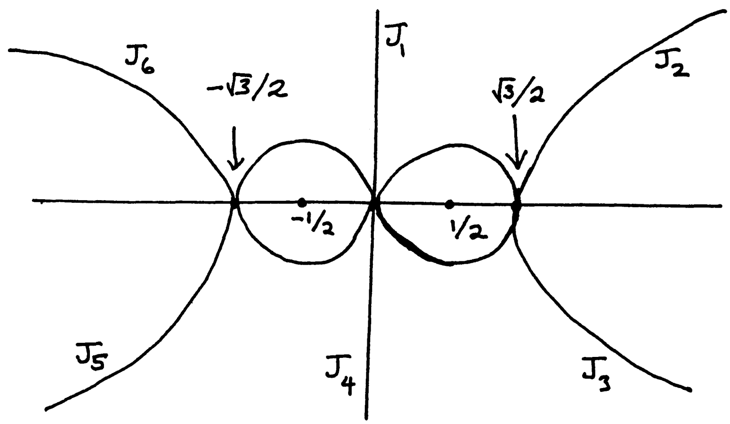

Figure 1

In studying the topological mean value conjecture, we make a choice of a non- critical point p. There are three points p_i, $i = 1, 2, 3$ with the same output $f(p)$. We draw a straight segment c from $f(p)$ to whichever critical value is closest to $f(p)$. Let us assume that $\mathbf{R}(f(p)) < 0$ so that $z = -1/2$ is the closest critical value. The preimage of c consists of three arcs I_i, $i = 1, 2, 3$. I_1 and I_2 have an endpoint at $-1/2$ and I_3 has an endpoint at the other preimage of $z = -1/2$ which $z = 1$. Since the segment c does not intersect J_1 and J_4 neither does any I_i intersect any J_j. The other endpoints of the I_i are the p_i. Then $\theta_- = -1/2$ is the critical point topologically related to p_1 and p_2, whereas $\theta_+ = 1/2$ is topologically related to p_3.

We can describe this correspondence in terms of the level curves $|f(z) - f(p)| =$ constant, which are preimages by f of circles centered at $f(p)$. $\{z: |f(z) - f(p)| = |f(p) + 1/2|\}$ is the preimage of the circle passing through θ_- and it has a component which is a topological figure-eight which contains both I_1 and I_2, one arc inside each loop of the figure-eight. The other component is a loop around p_3 which contains I_3.

The preimage of the circle through θ_+ is also a figure-eight. Each loop of this figure-eight contains one of the components of the preimage of the circle through θ_-, see Fig. 2.

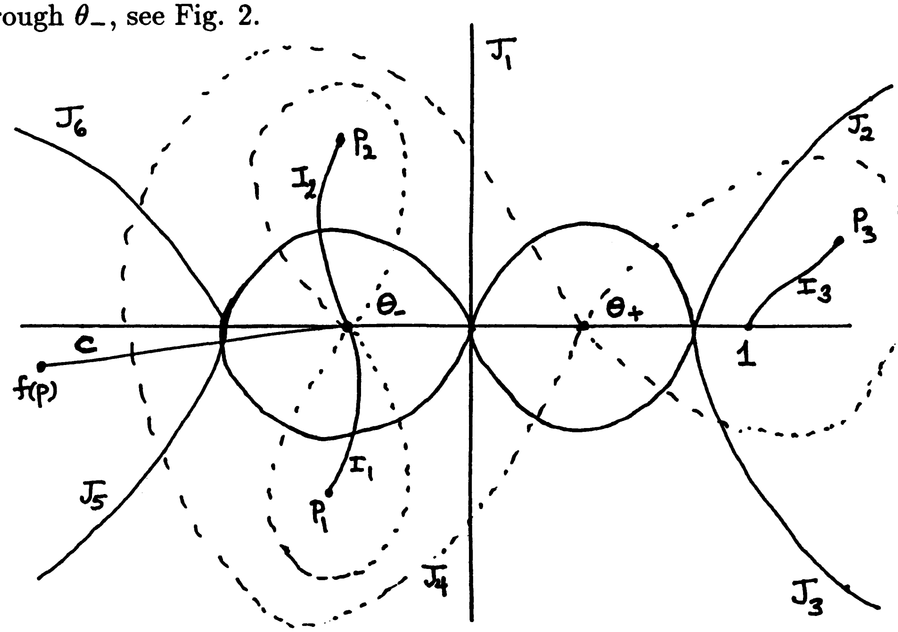

Figure 2

From the Fig. 2, we see that a branch of f^{-1} which sends $f(p)$ to p_3 can be defined on the disk centered at $f(p)$ and which has θ_+ on its boundary. Similarly, for p_1 and p_2, with θ_-. So we see that θ_j is topologically related to p_i, if $|\theta_j - p_i| \leq |\theta_{-j} - p_i|$.

From [5], we have that $S(f, p_i, \theta_j) - 1/2 = -(1/6)(\theta_j - p_i)/(\theta_{-j} - p_i)$, for $i = 1,2,3$, and for $j = +,-$. Therefore, if $|\theta_j - p_i| \leq |\theta_{-j} - p_i)|$, $|S(f, p_i, \theta_j) - 1/2| \leq 1/6 = 1/2 - 1/3$. This proves the topological version of (2) when $d = 3$.

For the case of $d = 4$, we will describe a case where the topological version of (2) is false. Consider $f(z) = -3z^4 + 4z^3$. Note that $z = 0$ and $z = 1$ are the only critical points and they are each fixed points for f. Since all the critical points are fixed we can describe a topological model for f. There are topological disks around $z = 0$ and $z = 1$ on which f is topologically conjugate to $z \to z^3$ and $z \to z^2$ respectively. The two disks have one point q in common, that is, $q = (-1 + \sqrt{13})/6$. This is a repelling fixed point as is $(-1 - \sqrt{13})/6$. There are three invariant expanding curves J_i, $i = 1, 2, 3$, see Fig. 3. Extend J_3 to q by adding on the invariant segment $[p, q]$. These curves bound three regions of the plane denoted by R_i, $i = 1, 2, 3$, where R_i is the region not bordered by J_i. The preimages of the region R_i will be denoted by R_i^{-1}. Let z be of the form $1 + t\sqrt{(-1)}$. With t a sufficiently large negative real number, one can check that $z \in R_1^{-1}$ since as $t \to -\infty$, $|f(z)| \to \infty$, and $\arg(f(z)) \to \pi^-$. In particular, the component of R_1^{-1} which contains z, also contains 0 in its boundary. Take a straight segment c joining the critical point 0 to $f(z)$. The segment c will have four preimage arcs I_i, three of which, say for $i = 1, 2, 3$, emanate from 0 and lie inside the three components of R_1^{-1}, respectively which have 0 in their boundary. We denote the other endpoint of I_i by z_i, for $i = 1, 2, 3$. The other component I_4, lies in the other component of R_1^{-1} which has as one boundary component a part of the positive real axis. One endpoint of I_4 is $z = 4/3$ and we will denote the other endpoint by z_4. We see that the critical point $z = 1$ is topologically related to z_4, whereas the critical point $z = 0$ is topologically related to the other three preimages of $f(z)$.

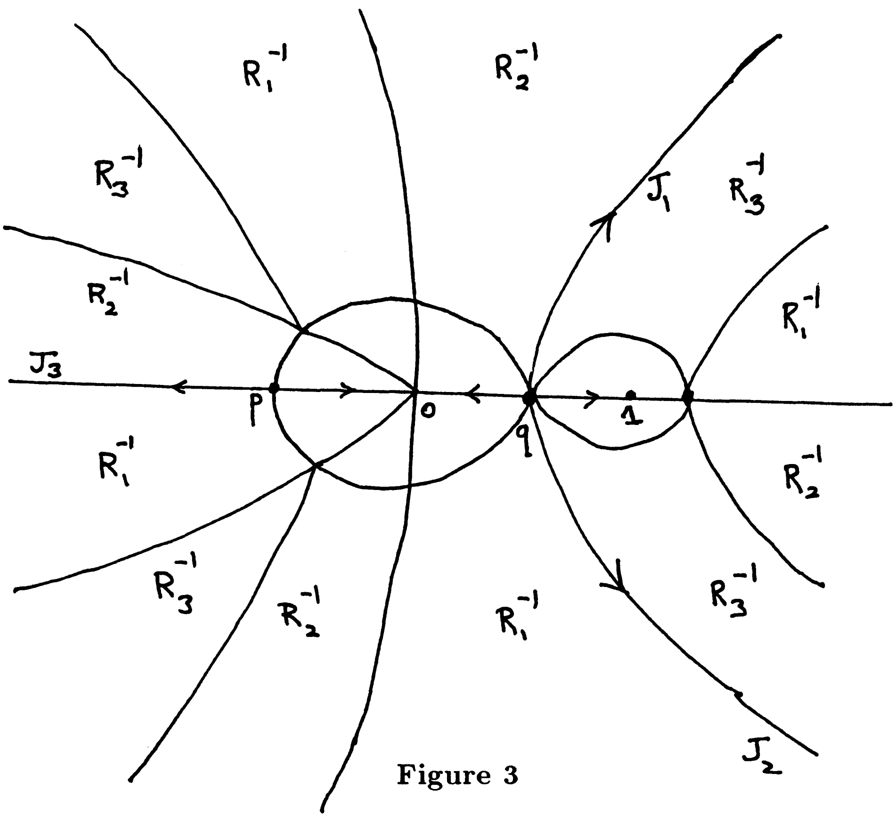

Figure 3

In [5], it is shown that

$$S(f, z, \theta_j) - 1/2 = (-1/4)\{(\theta_j - z)(\theta_j' - z)\}/\{(\theta_{j+1} - z)(\theta_{j+2} - z)\},$$

where $\theta_j' = 2G - \theta_j$, $G = (1/3)\sum_1^3(\theta_k)$. We can use this formula to calculate which critical points satisfy (2) for a given z. In our case, $\theta_1 = 0$, $\theta_2 = 0$, $\theta_3 = 1$, $G = 1/3$, $\theta_1' = \theta_2' = 2/3$ and $\theta_3' = -1/3$.

If $j = 1$ then $\{(\theta_1 - z)(\theta_1' - z)\}/\{(\theta_2 - z)(\theta_3 - z)\} = ((2/3) - z))/(1 - z)$. Therefore, if $\mathbf{R}(z) > 5/6$, $S(f, z, \theta_1)$ does not satisfy (2).

So, for $z = 1 + t\sqrt{(-1)}$ for t sufficiently negative, $|S(f, z, \theta_1) - 1/2| > 1/2 - 1/4$. However, θ_1 is the only critical point topologically related to z, and so the topological version of (2) is false for $d = 4$.

§**3.** Given a polynomial f of degree d, let $\theta_1, \dots, \theta_{d-1}$, denote the critical points of f. Assume that $|f(\theta_1)| < |f(\theta_2)| < \cdots < |f(\theta_{d-1})|$. We will decompose the domain of f into $d-1$ pieces and describe a simple model for each piece. The $d-1$ pieces are defined as the connected components of the complement of a set of $d-2$ loops C_i, $i = 1, \dots, d-2$.

The loops C_i are defined as follows. Choose u_i, $i = 1, \dots, d-2$, so that $|f(\theta_i)| < u_i < |f(\theta_{i+1})|$. Denote the connected component of the level curve of f passing through each θ_i by ℓ_1. For each i, $1 \leq i \leq d-1$, ℓ_i is a figure-eight. For each u_i, the level curve at level u_i has a connected component C_i which is a loop which together with the figure-eight through θ_i bounds an annulus.

The complement of $C = UC_i$ consists of $d-1$ connected components K_i. Each K_i contains exactly one critical point θ_i. There are only a small number of topological types for the K_i, which we will now catalog. Let D_i denote the open disk bounded by C_i.

The region $\overline{K}_1$ is a closed disk since it is the closure of the disk bounded by C_1. For $1 < i < d-1$, there are three possible topological types for $\overline{K}_i$, namely, a disk, a disk minus one disk, or a disk minus two disks. One boundary component of $\overline{K}_i$ is C_i. We will call it the outer boundary since $|f(z)|$ restricted to $\overline{K}_i$ is maximal along C_i. The level curve ℓ_i passing through θ_i consists of two loops. Inside each loop there may or may not be any critical points of f. If there are none in either loop then $\overline{K}_i$ is a closed disk. If there are critical points inside only one loop, then $\overline{K}_i$ is the closed annulus bounded by C_i and C_j, where θ_j is the critical point of largest critical value inside one loop of ℓ_i. If ℓ_i has critical points inside each of its loops then $\overline{K}_i = \overline{D}_i - \{D_j \cup D_k\}$ where θ_j, θ_k are the critical points inside each loop of largest critical value inside their respective loops.

Finally, the region $\overline{K}_{d-1}$ differs from the last cases, only in that, there is no outer boundary C_{d-1}. That is, $\overline{K}_{d-1} = \mathbf{C}$, for $d = 2$, or $\overline{K}_{d-1} = \mathbf{C} - D_j$ or $\overline{K}_{d-1} = \mathbf{C} - D_j - D_k$.

Suppose that for some i, $\overline{K}_i$ has three boundary loops C_i, C_j, and C_k. We will describe a set $K_i^* \subset \mathbf{C}$ and a polynomial function f_i defined on K_i^* so that there is a conformal map $h_i: K_i^* \to \overline{K}_i$ satisfying $f_i = f \circ h_i$. Let m_r =degree of f restricted to C_r, for $r = 1, \ldots, d-1$. The degree m_r is the number of roots of f which lie inside C_r. Define $f_i(z) = \lambda_i z^{m_j}(z-1)^{m_k}$, where λ_i is a constant chosen so that, at the critical point $\sigma_i = m_j/(m_j + m_k)$ of f_i, the critical value $f_i(\sigma_i) = f_i(\theta_i)$.

The level curve of f_i corresponding to u_j, i.e., $\{z: |f_i(z)| = u_j\}$ consists of two loops, one around $z = 0$ and one around $z = 1$.

Recall, $u_j < |f_i(\theta_i)|$ and that all the roots of f_i lie at $z = 0$ and $z = 1$. The loop around $z = 0$, denoted by C_j^*, will be one boundary component of K_i^*. Similarly, for the u_k level of f_i, there is a loop around $z = 1$, denoted by C_k^*, which will be another boundary component of K_i^*.

The outside boundary of K_i^* is defined to be the level curve of f_i at the level u_i, denoted by C_i^*, which is a single loop inside of which is the critical point σ_i.

K_i^* is the union of three annuli bounded by four curves: the level curve through σ_i, C_i^*, C_j^*, and C_k^*. Similarly $\overline{K}_i$ is the union of three annuli bounded by ℓ_i, C_i, C_j, and C_k. Both $f_i \mid K_i^*$ and $f_i \mid \overline{K}_i$ are branched mappings with the same critical value. On each pair of corresponding annuli, f_i and f are covering maps with the same image. Note that the degrees of f_i and f are the same on each pair of corresponding annuli. Define $h_i: K_i^* \to \overline{K}_i$, on each of the three closed annuli, as a mapping between covering spaces which satisfies $h_i(\sigma_i) = \theta_i$.

Therefore, $f_i \mid K_i^* = f \circ h_i$ is conformal since it is conformal on each of the three annuli and is well defined on the level curve passing through σ_i.

There is an oriented tree T associated to the pieces $\overline{K}_i$ by associating a vertex v_i to each $\overline{K}_i$ and an edge e_{im}, oriented from v_i to v_m provided that the outer boundary C_i of $\overline{K}_i$ is an inner boundary of the piece $\overline{K}_m$.

We can choose complex affine mappings A_i, $1 \leq i \leq d-1$, of $\mathbf{C}$ with the following property. The subsets $A_i(K_i^*)$ are deployed in such a way that, for each edge e_{im} of T, $A_i(K_i^*)$ is inside the loop $A_m((h_m)^{-1}(C_i))$, see Fig. 4.

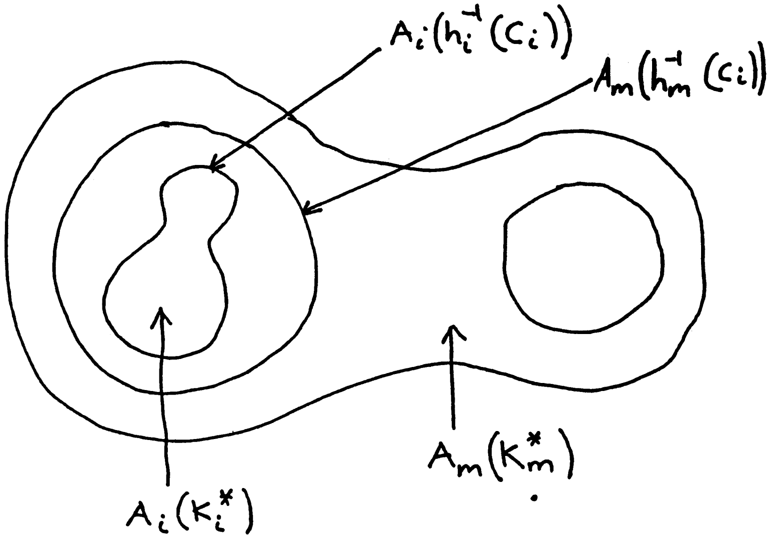

Figure 4

In general $A_i((h_i^{-1}(C_i))$ is not the same affine shape as $A_m((h_m^{-1}(C_i))$. However there is an analytic homeomorphism $\varepsilon_{im}: A_i((h_i)^{-1}(C_i)) \to A_m((h_m)^{-1}(C_i))$ defined by $\varepsilon_{im} = A_m \circ (h_m)^{-1} \circ h_i \circ (A_i)^{-1}$.

Suppose C is a boundary component of some K_i^*, corresponding to an f level curve at level u. For $\varepsilon > 0$, if $\varepsilon < ||u| - |f(\sigma_i)||$ then there is an ε-collar neighborhood of C in K_i^* defined by all level curves in K_i^* whose level is within ε of u. There is a product structure on the collar given by the level curves and the orthogonal trajectories to the level curves. For small enough ε, define $J_i^* = K_i^*$ minus an ε-collar around each boundary component of K_i^*. The region between the outer boundary of $A_i(J_i^*)$ and the inner boundary of $A_m(J_m^*)$ is topologically an annulus denoted by E_{im}. On one of the boundary components of E_{im}, $f_m \circ (A_m)^{-1}$ is defined and is a covering map of degree m_i onto the circle of radius $u_i + \varepsilon$, centered at 0. On the other boundary component of E_{im}, $f_i \circ (A_i)^{-1}$ is defined and it is a covering map of degree m_i onto the circle of radius $u_i - \varepsilon$, centered at 0. Therefore, there is a covering map f_{im} of degree m_i from E_{im} onto the annulus between the circles of radii $u_i - \varepsilon$ and $u_i + \varepsilon$ which restricts to $f_i \circ (A_i)^{-1}$ and $f_m \circ (A_m)^{-1}$ on the two boundary components respectively. This covering map can be chosen in many ways.

Suppose that for each E_{im} a covering map f_{im} has been chosen. Then we can define a branched covering f^* from $\mathbf{C}$ to $\mathbf{C}$ as follows. If z is in J_r^*, let $f^*(z) = f_r \circ (A_r)^{-1}$. If z is in E_{im} let $f^*(z) = f_{im}$. There is a complex structure on the domain of f^* which makes f^* holomorphic as follows. On each J_r^* one has the usual complex structure of $\mathbf{C}$ and on each E_{im} define the complex structure as the pullback of the usual one by f_{im}.

The Beltrami coefficient determined by this map is given by $\mu = 0$ on each J_r^* and $\mu = (\partial f_{im}/\partial \overline{z})/(\partial f_{im}/\partial z)$ otherwise. By the measurable riemann mapping theorem, there is a quasi-conformal homeomorphism $\phi\colon \mathbf{C} \to \mathbf{C}$ so that $f^* \circ \phi$ is holomorphic in the usual complex strucutre on $\mathbf{C}$. Since $f^* \circ \phi$ has the branching structure of a polynomial, it is a polynomial. Furthermore $f^* \circ \phi$ has the same critical values as f and the same critical level curve combinatorics as recorded by T. Therefore, there is a homeomorphism $A\colon \mathbf{C} \to \mathbf{C}$ so that $f = f^* \circ \phi \circ A$. Since $f^* \circ \phi$ and f are polynomials of degree d, A is a degree one polynomial. By the Ahlfors- Bers theorem, ϕ will be close to the identity if μ is sufficiently close to 0.

Theorem 3. *Given f, if ϕ is sufficiently C^0 close to the identity then the topological version of (2) is true for $S(f, z, \theta)$ where z is a root of f.*

Proof. We can assume that A =identity since $S(f \circ A, z, \theta) = S(f, A(z), A(\theta))$. Note that ϕ is automatically conformal in the interior of the pieces J_r^*, so that if ϕ is uniformly near the identity then $\phi'(z)$ is near one. Consider a piece K_i^* where $f_i(z) = \lambda_i z^{m_j}(z-1)^{m_k}$ and where $m_j = 1$. For f_i, the critical point $\sigma_i = 1/(1 + m_k)$ is topologically related to $z = 0$ which is a root of f_i. For ease of notation denote m_k simply by m. Then $S(f_i, 0, \sigma_i) = (m/(1+m))^m$. For all m, $(m/(1+m))^m$ is contained in the interval $(1/e, 1/2]$, so $|S(f_i, 0, \sigma_i) - 1/2| < 1/2 - 1/d$, for $d \geq 3$. On the piece K_i, $f = f_i \circ \phi$.

Let $z = \phi^{-1}(0)$ and $\theta_i = \phi^{-1}(\sigma_i)$. Then z is a root of f and θ_i is a critical point of f topologically related to z. A calculation shows that $S(f, z, \theta_i) = \{(0 - \sigma_i)/((z - \theta_i)\phi'(z))\}S(f_i, 0, \sigma_i)$. For ϕ sufficiently near the identity $\phi'(z)$ is near 1 and we conclude that $|S(f_i, 0, \sigma_i) - 1/2| < 1/2 - 1/d$, for $d \geq 3$.

Proposition 1. *For a polynomial f of degree d, if $|f(\theta_{i+1})/f(\theta_i)|$ is sufficiently large for all i then the topological version of (2) is true for $S(f, z, \theta)$ where z is a root of f.*

Proof. This is a corollary of Thm. 3 if we show that in the construction above that we can make the Beltrami coefficient sufficiently close to zero.

Let $f(z) = \lambda z^a(z-1)^b$. For $\delta > 0$ there is a degree one polynomial A so that $f \circ A(z) = z^a(\varphi(z))$, with $|\varphi(z) - 1| < \delta$ for all z in some neighborhood of $z = 0$. Furthermore there is a degree one polynomial B so that $f \circ B(z) = z^{a+b}(\psi(z))$, with $|\psi(z) - 1| < \delta$, for all z in some neighborhood of $z = \infty$. Suppose that for $|z| < R$, $f_m \circ A(z) = z^n(\varphi(z))$ and for $|z| > r$, $f_i \circ B(z) = z^n(\psi(z))$. One shows that φ and ψ can be made C^1 close to 1 near $|z| = R, r$ respectively by the Cauchy inequalities. If $|f(\theta_{i+1})/f(\theta_i)|$ is sufficiently large then we can choose u_i, so that $|f(\theta_i)| < u_i < |f(\theta_{i+1})| < |f(\theta_m)|$, and so that $A_m(C_i) \subset \{z: |z| < R\}$ and $A_i(C_i) \subset \{z: |z| > r\}$. With these choices the covering map f_{im} must agree on the respective components of the boundary of E_{im} with $z^n(\varphi(z))$ and $z^n(\psi(z))$. Since both are C^1 close to the constant function 1, f_{im} can be chosen so that $f_{im} = z^n(H(z))$ where $H(z)$ is C^1 close to 1 and hence the Beltrami coefficient μ on E_{im} is close to zero. For a fixed degree d, there are only a finite number of polynomials $z^a(z-1)^b$ with $a + b \leq d$. Therefore, there is a constant L so that if $|f(\theta_{i+1})/f(\theta_i)| > L$ for all i, the proposition is true.

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