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The Index of Holomorphic Vector Fields on Singular Varieties I[1]

Ch. Bonatti and X. Gómez-Mont

Given a complex analytic space V with an isolated singuarity at p, there is a way to associate to a holomorphic vector field X on V an index at p a la Poincaré-Hopf $\mathrm{Ind}(X, V, p)$ (see [Se],[GSV]). The objective of this series of papers is to understand this index. In the present paper we relate it to the V-multiplicity:

$$\mu_V(X,p) = \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^n,p}}{(f_1,\ldots,f_\ell,X^1,\ldots,X^n)}$$

where $f_1,\ldots,f_\ell$ are generators of the ideal defining $V \subset \mathbf{C}^n$, X^j are the coordinate functions of a holomorphic vector field that extends X to a neighbourhood of 0 in $\mathbf{C}^n$ and the denominator denotes the ideal generated by the elements inside the parenthesis in the ring $\mathcal{O}_{\mathbf{C}^n,p}$ of germs of holomorphic functions at p. The main results are:

Theorem 2.2. *Let $(V,0) \subset \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space in the unit ball $\mathbf{B}_1$ which is smooth except for an isolated singularity at 0. Let Θ_r denote the Banach space of holomorphic vector fields on V_r with continuous extensions to ∂V_r, $r < 1$, with its natural structure as an analytic space of infinite dimension. Then:*

a) The function V-multiplicity at 0

$$\mu_V(\ ,0)\colon \Theta_r \to \mathbf{Z}^+ \cup \{\infty\}$$

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is upper semicontinuous and it is locally bounded at those points X where X has an isolated singularity on V at 0.

b) *The subsets of Θ_r defined by $\mu(\ ,0) \geq K$ are analytic subspaces and the minimum value of $\mu_V(\ ,0)$ in Θ_r is attained on an open dense subset $\tilde{\Gamma}_1$ of Θ_r.*

c) *The subset of Θ_r formed by vector fields whose critical set at 0 has positive dimension is an analytic subspace of Θ_r.*

We introduce the Euler characteristic $\chi_V(X,0)$ of $X \in \Theta_r$ at 0 in (2.10) and show:

Theorem 2.5. *For $X \in \Theta_r$ with an isolated singularity at 0, $s \ll r$ and $0 \ll \varepsilon$, we have:*

1) *For any family of vector fields $\{X_t\}_{t\in T}$, parametrized by a finite dimensional analytic space $(T,0) \to (\Theta_r, X)$ such that the V- multiplicity at 0 of the general vector field X_t of the family is minimal μ_V, we have:*

$$\chi_V(X,0) = \chi_0^{\mathrm{tor}}(\mathcal{O}_{z_{T,s}}, \mathcal{O}_{\{X\}})$$

where the right and hand side is the Euler characteristic of higher torsion groups.

2) *For $Z \in U(X,\varepsilon)$ we have*

$$\chi_V(X,0) = \chi_V(Z,0) + \sum_{\substack{Z(p_j)=0 \\ p_j \in V_s - \{0\}}} \mu_V(Z,p_j)$$

3) *For $X \in \Theta_r$ with an isolated critical point at 0, we have:*

$$0 < \chi_V(X,0) \leq \mu_V(X,0)$$

and $\chi_V(X,0) = \mu_V(X,0)$ if and only if the universal critical set $\mathcal{Z}_r$ is π_1-anaflat at $(X,0)$ (in particular this happens in $\tilde{\Gamma}_1$).

Let $X \in \Theta_r$, we say that the *critical set of X does not bifurcate* if there is $\varepsilon > 0$ and $s > 0$ such that for $Y \in \mathbf{U}(X,\varepsilon) \subset \Theta_r$ we have that the only critical point of Y on V_s is 0, (that is, X has an isolated singularity at 0 as well as any sufficiently near vector field in Θ_r and there is no other critical point uniformly in a neighbourhood V_s of 0).

Theorem 2.6. *Let $(V,0) \subset \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space which is smooth except for an isolated singularity at 0, then the set of points in Θ_r whose critical set does not bifurcate contains the connected dense open subset $\tilde{\Gamma}_1 \subset \Theta_r$ consisting of vector fields with minimum V-multiplicity.*

Theorem 3.1. *Let $(V,0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space which is smooth except for an isolated singularity at 0, then there is an integer K such that*

$$Ind_W(X,V,0) = \chi_V(X,0) + K$$

for X in the dense open set Θ' of vector fields in Θ_r with an isolated singularity at 0. For X in the dense open set of Θ' where the universal critical set $\mathcal{Z}_r$ is Θ_r-anaflat we have

$$Ind_W(X,V,0) = \mu_V(X,0) + K$$

Corollary 3.2. *Let $(V,0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space which is smooth except for an isolated singularity at 0, then there is a constant L such that $Ind_W(X,V,0) \geq L$ for every germ of holomorphic vector field X on V with an isolated singularity at 0 on V.*

In the first section we analyse the index on smooth compact manifolds with boundary. We prove:

Proposition 1.1. *Let X and Y be C^1-vector fields defined on the compact manifold with boundary $(W,\partial W)$ and non-vanishing on ∂W and let $[\Gamma_X]$ denote the fundamental class of the graph of $X/\|X\|$ on the sphere bundle $\mathbf{S}$ of unit tangent vectors of W restricted to ∂W (with respect to some Riemannian metric on W). Then*

$$Ind(X,\partial W,W) - Ind(Y,\partial W,W) = [\Gamma_X]\cdot[\Gamma_{-Y}]$$

where we do the intersection in homology of $\mathbf{S}$.

In the second section we develop the properties of the V-multiplicity, and in the third we compare the V-multiplicity with the topological index.

1. The index of vector fields on manifolds with boundary

Let W be a compact oriented manifold of dimension m with boundary, ∂W, oriented in the natural way. Given a never vanishing C^0-vector field X in a neighbourhood of ∂W, the *index of* X *on the boundary of* W, $\mathrm{Ind}(X, \partial W, W)$ may be defined by extending X to a vector field $\tilde{X}$ on W with isolated singularities, and then adding up the indices at the singularities of $\tilde{X}$. The index is independent of the chosen extension $\tilde{X}$ (see [Mi],[Se]).

To understand the dependence of the index on the manifold W, we will prove that the difference of the indices of 2 vector fields may be computed exclusively in terms of boundary data:

Proposition 1.1. *Let* X *and* Y *be* C^1*-vector fields defined on the compact manifold with boundary* $(W, \partial W)$ *and non-vanishing on* ∂W *and let* $[\Gamma_X]$ *denote the fundamental class of the graph of* $X/\|X\|$ *on the sphere bundle* $\mathbf{S}$ *of unit tangent vectors of* W *restricted to* ∂W *(with respect to some Riemannian metric on* W*). Then*

$$Ind(X, \partial W, W) - Ind(Y, \partial W, W) = [\Gamma_X] \cdot [\Gamma_{-Y}]$$

where we do the intersection in homology of $\mathbf{S}$.

Proof. Since the index and the fundamental classes do not change if we make a small perturbation, we will assume that X and Y are in general position. Namely we will assume that if the zeroes $\mathcal{Z} \subset \mathbf{C} \times W$ of the vector fields $\{X_t = (1-t)X + tY\}_{t \in [0,1]}$ intersect ∂W, say at 0_t, then at 0_t: X_t has a zero of multiplicity 1 and the projection of $\mathcal{Z}$ to W is transversal to the boundary ∂W.

The intuitive idea of the proof is very simple. The above family connects X with Y, and the only way the index as a function of $t \in [0,1]$ can change is if a zero leaves W at ∂W, or if a zero arrives at W through ∂W. By the transversality conditions we are assuming, this will happen every time X_t has a zero on ∂W, and it will give a contribution of ± 1, depending whether the index of X_t is ± 1 and whether the point is arriving or leaving W. One has to prove that one obtains the same sign from the contribution of the intersection $[\Gamma_X] \cdot [\Gamma_{-Y}]$ at the above point on ∂W.

Let p be a boundary point, and consider the convex hull $C = \langle X(p), Y(p) \rangle$ in T_pW. If 0 is not contained in C, then the vector fields X_t do not vanish at p for $t \in [0,1]$. If 0 is contained in C, then there is exactly one value of t where X_t vanishes at p. Note that this condition means that $X(p)$ and $Y(p)$ are linearly dependent with distinct orientation, and this is equivalent to the fact that Γ_X and Γ_{-Y} intersect. So the only point left is to show that one obtains the same sign from the intersection $[\Gamma_X] \cdot [\Gamma_{-Y}]$ at p as the difference of the indices $\mathrm{Ind}(X_{t+\varepsilon}, \partial W, W) - \mathrm{Ind}(X_{t-\varepsilon}, \partial W, W)$.

To simplify notation, let $(x_1, \dots, x_n)$ be coordinates around $p = 0$, where W and ∂W are defined by $x_n \geq 0$ and $x_n = 0$ respectively. Let $Z = (Z_1, \dots, Z_n)$ be a C^1-vector field with a critical point at 0 of multiplicity 1, $Y = (Y_1, \dots, Y_n)$ a C^1-vector field with $Y_1(0) > 0$, and we are interested in computing the contribution to the index of the family $Z + tY$, when t passes through 0 in the positive direction.

Let $DZ(0)$ be the derivative of Z at 0. It is an invertible matrix and the sign of $\det[DZ(0)]$ is $\mathrm{Ind}(Z, 0)$, and hence it also $\mathrm{Ind}(Z_t, 0_t)$, where 0_t is the zero of Z_t near to 0. (One may think that everything extends to a neighbourhood of ∂W outside of W, so as to "see" 0_t for all small values of t, and not only for the ones that are in W). A simple calculation shows that the curve 0_t intersects ∂W with velocity vector

$$(d0_t/dt)(0) = -[DZ(0)]^{-1} Y(0)$$

and hence the zero set is entering W if $dx_n[-[DZ(0)]^{-1}Y(0)]$ is positive, and is leaving W if it is negative. By Crammer's rule, we have

$$dx_n[-[DZ(0)]^{-1}Y(0)] = \frac{1}{\det[DZ(0)]}\begin{bmatrix} \partial Z^1/\partial x_1 & \dots & \partial Z^n/\partial x_1 \\ \vdots & & \vdots \\ \partial Z^1/\partial x_1 & \dots & \partial Z^n/\partial x_{n-1} \\ Y^1 & \dots & Y^n \end{bmatrix}(0)$$

$$(1.1) \qquad = -\det[A]/\det[DZ(0)]$$

where the matrix A is defined by the above formula. Hence we obtain for $\varepsilon > 0$:

$$\operatorname{Ind}(Z_\varepsilon, W, \partial W) - \operatorname{Ind}(Z_{-\varepsilon}, W, \partial W) = \operatorname{Ind}(Z, 0)\operatorname{Sign}(-\det[A]/\det[DZ(0)])$$

$$(1.2) \qquad = -\operatorname{Sign}[\det[A]]$$

We will now compute $[\Gamma_{Z+Y}]\cdot[\Gamma_{-(Z-Y)}]$. Dividing by $Z^1 + Y^1$ (respectively by $Y^1 - Z^1$), which is positive, amounts to taking coordinates in the sphere bundle, so Γ_{Z+Y} and $\Gamma_{-(Z-Y)}$ are the graphs of the functions

$$\begin{aligned}
&\gamma^+(x_1, \dots, x_{n-1}) \\
&= ((Z^2+Y^2)/(Z^1+Y^1), \dots, (Z^n+Y^n)/(Z^1+Y^1)(x_1, \dots, x_{n-1}, 0) \\
&\gamma^-(x_1, \dots, x_{n-1}) \\
&= ((Y^2-Z^2)/(Y^1-Z^1), \dots, (Y^n-Z^n)/(Y^1-Z^1)(x_1, \dots, x_{n-1}, 0)
\end{aligned}$$

The intersection number $[\Gamma_{Z+Y}]\cdot[\Gamma_{-(Z-Y)}]$ is equal to the sign of determinant of the matrix obtained by grouping the derivatives of the graphs of γ^+ and γ^-:

$$\det\begin{bmatrix} I_{n-1} & I_{n-1} \\ D\gamma^+ & D\gamma^- \end{bmatrix}(0) = \det[D\gamma^- - D\gamma^+](0)$$

A simple calculation shows that

$$(D\gamma^\pm)_{ij} = \frac{1}{(Y^1(0))^2}\det\begin{bmatrix} Y^1(0) & \frac{\partial}{\partial x_j}[Y^1 \pm Z^1](0) \\ Y^i(0) & \frac{\partial}{\partial x_j}[Y^i \pm Z^i](0) \end{bmatrix}$$

Hence

$$(1.3)\qquad [D\gamma^- - D\gamma^+]_{ij} = -\frac{2}{(Y^1(0))^2}\det\begin{bmatrix} Y^1(0) & \frac{\partial}{\partial x_j}Z^1(0) \\ Y^i(0) & \frac{\partial}{\partial x_j}Z^i(0)\end{bmatrix}$$

We now need to use a formula involving determinants: Let $A = [a_{ij}]$ be an $s \times s$ matrix, and let $B = (b_{ij})$ be the $(s-1)\times(s-1)$ matrix whose general term is

$$(1.4)\qquad b_{ij} = \det\begin{bmatrix} a_{11} & a_{1i} \\ a_{j1} & a_{ji}\end{bmatrix}$$

$i, j = 2, \dots, n$, then we have that

$$(1.5)\qquad (a_{11})^{s-2}\det[A] = \det[B]$$

This formula may be proved first for diagonal matrixes, and then by showing that both sides are left invariant under the elementary operations of rows and columns.

Consider the matrix A in (1.1). Let A' be the matrix obtained by moving the last row to the first. We have $\det[A] = (-1)^{n-1}\det[A']$, and let B be the matrix obtained from A' as in (1.4). Nothing that $[D\gamma^- - D\gamma^+] = -2B/Y^1(0)^2$ we have by (1.5):

$$\begin{aligned}\det[A] &= (-1)^{n-1}\det[A'] = (-1)^{n-1}Y^1(0)^{2-n}\det[B] \\ &= (-1)^{n-1}Y^1(0)^{4-n}\det[D\gamma^- - D\gamma^+]/(-2)^{n-1}\end{aligned}$$

Hence $\det[A]$ and $\det[D\gamma^- - D\gamma^+]$ have the same sign. Using (1.2) we obtain: $\mathrm{Ind}(Z_\varepsilon, W, \partial W) - \mathrm{Ind}(Z_{-\varepsilon}, W, \partial W) = -\mathrm{Sign}[\det[A]] = -\mathrm{Sign}[\det[D\gamma^- - D\gamma^+]] =$

$$= [\Gamma_{-(Z-Y)}]\cdot[\Gamma_{Z+Y}] = [\Gamma_{Z-Y}]\cdot[\Gamma_{-(Z+Y)}]$$

This proves the Proposition. □

If we denote by *Vec*$(\partial W)^+$ the set of C^1-vector fields defined and never zero on a neighbourhood of ∂W, the Index is an integer valued function with *Vec*$(\partial W)^+$ as a domain:

$$\mathrm{Ind}\colon Vec(\partial W)^+ \to \mathbf{Z}$$

Let $(W', \partial W')$ be another compact manifold with boundary and $\phi: \partial W \to \partial W'$ an orientation preserving diffeomorphism of the boundaries. We may extend this diffeomorphism to a diffeomorphism of a neighbourhood of the boundaries $\phi: W_1 \to W_1'$. Given a non-singular C^0-vector field X defined on the neighbourhood W_1 of ∂W, we may transport it via ϕ to a vector field X' defined on W_1'.

$$
\begin{array}{ccc}
\mathrm{Ind}_{\partial W}: Vec(\partial W)^+ & \longrightarrow & \mathbf{Z} \\
\downarrow \phi & & \downarrow \alpha \\
\mathrm{Ind}_{\partial W'}: Vec(\partial W')^+ & \longrightarrow & \mathbf{Z}
\end{array}
\tag{1.6}
$$

It follows from Proposition 1 that for $X, Y \in Vec(\partial W)^+$ we have:

$$
\begin{aligned}
\mathrm{Ind}(X, \partial W, W) - \mathrm{Ind}(Y, \partial W, W) &= [\Gamma_X] \cdot [\Gamma_{-Y}] = [\Gamma_{\partial_* X}] \cdot [\Gamma_{\phi_* - Y}] \\
&= \mathrm{Ind}(\phi_* X, \partial W', W') - \mathrm{Ind}(\phi_* Y, \partial W', W')
\end{aligned}
$$

And hence for variable X, and a fixed Y we obtain:

$$
\mathrm{Ind}_{\partial W'}(\phi_* X) = \mathrm{Ind}_{\partial W}(X) - (\mathrm{Ind}_{\partial W}(Y) - \mathrm{Ind}_{\partial W'}(\phi_* Y))
$$

Hence we may complete (1.6) by a map α which is substraction by an integer. This integer may be computed by taking the difference of the two indices with respect to any pair of vector fields in $Vec(W)^+$. To see who this integer is, let Z be the vector field in $Vec(W)^+$ which is always pointing inward. In this case, by the relative Poincaré-Hopf Index Theorem (see [Pu]), it is $\chi(M) - \chi(\partial M)$, where χ is the Euler Poincaré characteristic. Hence we obtain:

Corollary 1.2. *Let $(W, \partial W)$ and $(W, \partial W')$ be manifolds with diffeomorphic boundaries and ϕ a diffeomorphism of a neighbourhood of the boundaries. Then for any C^0-vector field defined and non-vanishing on a neighbourhood of ∂W we have*

$$
Ind(X, \partial W, W) = Ind(\phi_* X, \partial W', W') + [\chi(W) - \chi(W')]
$$

Remark. The above result may be also obtained from Pugh's Poincaré- Hopf Index Theorem for compact manifolds with boundary ([Pu]) since it expresses the index as the Euler-Poincaré characteristic of the manifold with boundary plus a contribution of the tangency behaviour of the vector field with the boundary. Hence taking the difference of both extensions we obtain that the difference of the indixes will be the difference of the Euler-Poincaré characteristics of the manifolds, since the boundary contributions are equal and hence cancel each other.

We will now give another explanation of the ambiguity of the definition of the index as a number just from its behaviour at the boundary.

Let $(W, \partial W)$ be a compact manifold with boundary, choose a Riemannian metric on W and let T^1W be the unit sphere bundle in the tangent bundle of W, and $\mathbf{S} = T^1W\,|_{\partial W}$ its restriction to the boundary. The natural projection $\rho\colon \mathbf{S} \to \partial W$ has the structure of an $(m-1)$ sphere bundle over ∂W. $\mathbf{S}$ has dimension $2(m-1)$ and its cohomology groups $H^q(\partial W, \mathbf{Z})$ may be calculated using the spectral sequence of the fibration, since the cohomology bundles $R^q\rho_*(\mathbf{Z_S})$ over ∂W are non- vanishing except for dimension 0 and $n-1$ (since it is a sphere bundle) and both of them are trivial bundles with Z as fibers, since the monodromy group is acting trivially (it sends the fundamental class to itself, since everything is oriented). The spectral sequence degenerates since $H^p(\partial W, R^q\rho_*(\mathbf{Z_S}))$ is non-zero only for $q = 0, n-1$. Hence the cohomology of $\mathbf{S}$ consists of 2 copies of the cohomology of ∂W glued together in the middle dimension:

$$H^p(\mathbf{S}, \mathbf{Z}) = H^p(\partial W, \mathbf{Z}) \text{ for } 0 \leq p \leq m-2$$

$$H^p(\mathbf{S}, \mathbf{Z}) = H^{p-(m-1)}(\partial W, \mathbf{Z}) \text{ for } m \leq p \leq 2m-2$$

(1.7)

$$0 \longrightarrow H^{m-1}(\partial W, \mathbf{Z}) \xrightarrow{\rho^*} H^{m-1}(\mathbf{S}, \mathbf{Z}) \xrightarrow{\nu} H^0(\partial W, \mathbf{Z}) \longrightarrow 0$$

We are interested in the middle group $H^{m-1}(\mathbf{S}, \mathbf{Z})$. $H^{m-1}(\partial W, \mathbf{Z_S}) = \oplus_j H^{m-1}(\partial W_j, \mathbf{Z_S})$, where $\{\partial W_j\}$ are the connected components of ∂W, say r of them. Hence $H^{m-1}(\mathbf{S}, \mathbf{Z})$ has a submodule canonically isomorphic to

$\mathbf{Z}^r$, obtained by pulling back the fundamental classes of the boundary components. The quotient group is again canonicaly isomorphic to $\mathbf{Z}^r$, but there is no canonical splitting. $H^{m-1}(\mathbf{S}, \mathbf{Z})$ is hence free of rank $2r$.

If X is a C^0 vector field on W, non-vanishing on ∂W, the fundamental class $[\Gamma_X]$ of the graph of $X/\|X\|$ restricted to ∂W is an element of $H_{m-1}(\partial W, \mathbf{Z})$. It is the class $[\Gamma_X]$ which carries the topological information of the index. Since it is a section of ρ, it projects to $(1, \dots, 1)$ in (1.7). The difference of two such fundamental classes will produce integers on each boundary component. If one wants to obtain an integer for a vector field, then one has to choose a splitting of (1.7), which is a non-canonical operation. This is carried out by choosing the bounding manifold W.

2. Holomorphic vector fields on singular spaces

Let $p \in V$ be a point of a complex analytic space of dimension N and let $(V,p) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n, 0)$ be a local embedding of V into the unit ball $\mathbf{B}_1$. We will denote $V \cap \mathbf{B}_r$ by V_r, where $\mathbf{B}_r$ is the ball around 0 and radius $r \leq 1$ in $\mathbf{C}^n$. The ring $\mathcal{O}_{V,p}$ of germs of holomoprhic functions at p may be represented by the quotient $\mathcal{O}_{\mathbf{C}^n,0}/\mathcal{J}$, where $\mathcal{J}$ is the ideal of germs of holomorphic functions on $(\mathbf{C}^n, 0)$ vanishing on V. A *germ of a holomorphic vector field at* p is a derivation

$$X\colon \mathcal{O}_{V,p} \to \mathcal{O}_{V,p}$$

(see [Ro]). Given a holomorphic vector field on $(V, 0)$, it gives rise to a diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbf{C}^n,0} & \stackrel{\tilde{X}}{\dashrightarrow} & \mathcal{O}_{\mathbf{C}^n,0} \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{O}_{V,p} & \xrightarrow{X} & \mathcal{O}_{V,p} \end{array}$$

We can always lift X to a derivation $\tilde{X}$ on $\mathcal{O}_{\mathbf{C}^n,0}$. To see this let $(z_1, \dots, z_n)$ be coordinates of $\mathbf{C}^n$, and let A_j be π- liftings to $\mathcal{O}_{\mathbf{C}^n,0}$ of $X(\pi(z_j))$. One easily checks that $\tilde{X} = \sum A_j \frac{\partial}{\partial z_j}$ makes the above diagram commutative on the generators z_j, and applying linearity and Leibnitz's rule, we see that the diagram is commutative. $\tilde{X}$ will send the ideal $\mathcal{J}$ defining V to itself, and conversely, any such derivation will induce a holomorphic vector field on V. A germ of a holomorphic vector field at $(V, 0)$ induces a (usual) holomorphic vector field on the smooth points of V near 0.

If X is a holomorphic vector field defined on the non-singular points of V, then using an embedding of V into $\mathbf{C}^n$, we may express $X = \sum X_j \frac{\partial}{\partial z_j}$, where X_j are holomorphic functions on $V - \mathrm{Sing}(V)$. If V has a normal singularity at p then, by the second Riemann's Removable Singularity Theorem ([Fi], p.120), the functions X_j extend to holomorphic functions on V and the vector field obtained with these extensions gives a holomorphic extension of the vector fied X from $V - \mathrm{Sing}V$ to V. Hence for normal singularities, holomorphic vector fields on V coincide with (usual) holomorphic vector fields on $V - \mathrm{Sing}(V)$. If $(V, p) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n, 0)$ is an analytic space then the sheaf of holomorphic vector fields Θ_V is coherent ([Ro]). We shall denote by Θ_r *the Banach space of continuous vector fields* defined on $\overline{V}_r$ and holomorphic in V_r, with the C^0-norm. We will also denote the ball $\{Y \in \Theta_r / \|X - Y\| < \varepsilon\}$ by $\mathbf{U}(X, \varepsilon)$. The ring of germs of holomorphic vector fields $\Theta_{V,p}$ is endowed with the analytic topology. Recall that a sequence $\{X_n\}$ converges to X in $\Theta_{V,p}$ if they are all defined in a small neighbourhood $\overline{V}_r \subset V$ of p, and they converge in Θ_r (see [G-R]). Note that by the Weierstrass approximation theorems, Θ_r is dense in $\Theta_{V,p}$, so that many properties for germs will follow by considering similar properties in Θ_r.

Proposition 2.1. *Let $(V, 0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n, 0)$ be an analytic space which is smooth except for an isolated singularity at 0, then the subset $\Theta'_r \subset \Theta_r$ consisting of holomorphic vector fields that have at 0 an isolated singularity is a connected dense open subset in Θ_r.*

Proof. Assume that X has an isolated critical point at 0. For $s < r$ small, X restricted to ∂V_s does not vanish. Let 2ε be the minimum value of $\|X\|$

on ∂V_s. If $Y \in \Theta_r$ with $\|Y\| < \varepsilon$, then $X + Y$ cannot vanish on ∂V_s (for then $X(q) = -Y(q)$). This implies that $X + Y$ will have an isolated critical point at 0, since if it vanished on a set of positive dimension passing through 0, this set would have to intersect ∂V_s (otherwise, one would have a compact complex manifold in V_s of positive dimension). This shows that Θ'_r is open in Θ.

Let $X_0 \in \Theta_r$ and let $\varepsilon > 0$ be given. To each vector field X in $\mathbf{U}(X_0, \varepsilon)$ we can associate to it the dimension of its critical set at 0, $\dim_0(\{X = 0\})$. Let Y be a vector field where this minimum is attained. We claim that Y has an isolated singularity at 0. So assume that Y does not have an isolated singularity at 0.

Let $P\mathbf{T}(V - \{0\}) \subset \mathbf{C}^n \times \mathbf{P}^{n-1}_{\mathbf{C}}$ be the (complex) projectivized tangent bundle of $V - \{0\}$, denote by $\mathbf{P}_r$ its closure and $\pi: \mathbf{P}_r \to V_r$ its projection to the first factor. $\mathbf{P}_r$ is an analytic space, π is a proper holomorphic map which is a complex projective bundle outside of 0 and the fibre over 0 is the tangent cone of V at 0 (see [Wh]). Let $A = A_1 \cup \cdots \cup A_m$ be the decomposition in irreducible components of $\{Y = 0\} \subset V_r$ passing through 0. By assumption A does not reduce to 0. Let $\Gamma_Y \subset \mathbf{P}_r$ be the closure of the graph of $\mathrm{Proj}(Y)$ on $V_r - A$. Γ_Y has dimension $N = \dim(V_r)$. The intersection of Γ_Y with $\pi^{-1}(A)$ has dimension at most $n - 1$, since it is contained in the boundary of the graph of Y, which has dimension N. Since $\pi^{-1}(A_j)$ has dimension $N - 1 + \dim(A_j) > N - 1$, we may choose points in $\pi^{-1}(A_j) - \Gamma_Y$. That is, there are points $p_j \in A_j - \{0\}$ arbitrarily close to 0 and vectors v_j tangent to V_r at p_j such that $\mathrm{Proj}(v_j)$ is disjoint from Γ_Y. Since V_r is a Stein space, there is a vector field Z on V_r such that $Z(p_j) = v_j$. We claim that $Y + tZ$, for small values of $t \neq 0$ will have singular set at 0 of dimension smaller than the critical set of Y, contradicting the choice of Y.

To see this, let $s < r$ so that $A \cap V_s = \{Y = 0\} \cap V_s$. Without loss of generality, we may assume that $p_j \in V_s$ (since the set of points that do not satisfy the defining condition of p_j is a proper subvariety of each A_j). Let C be the set of points of $V_s \times \mathbf{C}$ where $Y + tZ$ vanishes and let $\rho: V_s \times \mathbf{C} \to \mathbf{C}$

be the projection to the second factor.

We claim that the A_j's are irreducible components of C. To see this, consider $(Y + tZ)(p) = 0$ for p near to p_j. By the way we chose $Z(p_j)$, one may conclude that $Z(p)$ is linearly independent with $Y(p)$ if $Y(p) \neq 0$. Hence $(Y+tZ)(p) \neq 0$. If $Y(p) = 0$, then for $t \neq 0$ we have $(Y+tZ)(p) = tZ(p) \neq 0$. This implies that the decomposition into irreducible components of C in a neighbourhood of (0,0) is of the form $C = A_1 \cup \cdots \cup A_m \cup C_1 \cup \cdots \cup C_r$. Hence the irreducible components C_k are not contained in $\rho^{-1}(0)$ and its intersection with $\rho^{-1}(0)$ does not contain any A_j. Hence $C_k \cap \rho^{-1}(0)$ has dimension strictly smaller than the dimension of A. By the theorem of upper semicontinuity of the dimension of the fibers of a holomorphic map, we conclude that $(C_1 \cup \cdots \cup C_r) \cap \rho^{-1}(t_0)$ has dimension smaller than the dimension of A, for $t_0 \neq 0$. But this set is exactly the critical set of $Y + t_0 Z$. This contradicts the hypothesis that the minimum dimension of its critical set is attained at Y. Hence Y has isolated singularities. This shows that Θ'_r is dense in Θ_r.

To see that Θ'_r is connected, let X an Y belong to Θ'_r, then consider the family $\{X+tY\}_{t\in\mathbf{C}}$. The critical set C of the family consists of $(t,p) \in \mathbf{C} \times V$ such that $(X + tY)(p) = 0$. C is an analytic subvariety, containing the line $\mathcal{L}_0 = \mathbf{C} \times \{0\}$. By hypothesis (0,0) and (1,0) lie on $\mathcal{L}_0$ and in no other irreducible component of C. Hence $\mathcal{L}_0$ is an irreducible component of C. The other irreducible components of C intersect $\mathcal{L}_0$ on a finite number of points. Hence all points of $\mathcal{L}_0$ except a finite number represent vector fields with isolated singularities. Hence, Θ'_r is connected. □

From now on, we assume that $V \subset \mathbf{B}_1 \subset \mathbf{C}^n$ is a smooth variety of dimension N except for an isolated singularity at 0 (V non-smooth at 0). Let $\mathcal{J} = (f_1(z), \ldots, f_\ell(z))$ be the ideal sheaf defining V_r in $\mathbf{B}_r$. Consider the Banach space Θ_r as an infinite dimensional analytic space (see [Do1]) and let $e\colon \Theta_r \times V_r \to \mathbf{C}^n$ be the valuation map

$$e(X, z_0) = e\left(\sum a_I^j z^I \frac{\partial}{\partial z_j}, z_0\right) = \sum a_I^j z_0^I \frac{\partial}{\partial z_j} = \sum e^j(X, z_0)\frac{\partial}{\partial z_j} = X(z_0)$$

It is an analytic function on the Banach space $\Theta_r \times V_r$, linear in the first variable. The ***universal critical set*** $\mathcal{Z} = \mathcal{Z}_r$ is the analytic subvariety of $\Theta_r \times V_r$ defined by the sheaf of ideals

$$\mathcal{I}_r = (f_1(z), \dots, f_\ell(z), e^1(X,z), \dots, e^n(X,z)) \subset \mathcal{O}_{\Theta_r \times \mathbf{B}_r} \tag{2.1}$$

The above generators of the ideal $\mathcal{I}_r$ give a finite presentation of $\mathcal{O}_{\mathcal{Z}}$ as an $\mathcal{O}_{\Theta \times \mathbf{B}}$-module:

$$\mathcal{O}_{\Theta_r \times \mathbf{B}_r}^{\oplus \ell + n} \xrightarrow{\Phi} \mathcal{O}_{\Theta_r \times \mathbf{B}_r} \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0 \tag{2.2}$$

where the map Φ is matrix multiplication with $(f_1, \dots, f_\ell, e^1, \dots, e^n)$.

Let π_1 and π_2 be the restriction to $\mathcal{Z}$ of the projections to the factors Θ_r and $\mathbf{B}_r$, respectively. We analyse first π_2. Since V has an isolated singularity at 0, all vector fields on V vanish at 0, hence $\Theta_0 \subset \mathcal{Z}$, where $\Theta_0 = \Theta_r \times \{0\}$ is the zero section. This means that $\pi_2^{-1}(0) = \Theta_0$, which is a subvariety of $\Theta_r \times \mathbf{B}_r$ of codimension n. By restricting $\pi_2 \colon \mathcal{Z} - \Theta_0 \to V_r - \{0\}$ we see that the fiber $\pi_2^{-1}(p)$, with $p \in V_r - \{0\}$, is a vector space of codimension N in Θ_r (since V_r is Stein and $N = \dim V_r$) and hence $\pi_2^{-1}(V_r - \{0\})$ has the structure of a vector bundle over V_r whose fibers have codimension N in Θ_r. Hence $\pi_2^{-1}(V_r - \{0\})$ is smooth of codimension n in $\Theta_r \times \mathbf{B}_r$ (the same codimension as Θ_0). Let $\Theta_{\mathbf{sing}} \subset \mathcal{Z}$ be the closure of $\pi_2^{-1}(V - \{0\})$. Set theoretically $\mathcal{Z} = \Theta_0 \cup \Theta_{\mathbf{sing}}$, but $\mathcal{Z}$ will in general have a non-trivial scheme structure on Θ_0.

We now view $\mathcal{Z}$ as a space over Θ_r via the projection $\pi_1 \colon \mathcal{Z} \to \Theta_r$. The fibre $\pi_1^{-1}(X)$ over the vector field X is set theoretically the critical set $\{z \in V_r / X(z) = 0\}$ of X. Recall that the process of restricting to a π-fibre $\{X\} \times \mathbf{C}^n$ is carried out by tensoring with $\otimes_{\mathcal{O}_{\Theta_r \times \mathbf{B}_r}} \mathcal{O}_{\{X\} \times \mathbf{B}_r}$. In particular, $\mathcal{O}_{\mathcal{Z}} \otimes_{\mathcal{O}_{\Theta_r \times \mathbf{B}_r}} \mathcal{O}_{\{X\} \times \mathbf{B}_r}$ has support on the critical set of X and for an isolated singularity of X at p, its dimension is the *V-multiplicity of X at $p \in V_r$*:

$$\mu_V(X,p) = \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^n, p}}{(f_1, \dots, f_\ell, X^1, \dots, X^n)} \tag{2.3}$$

Note that $\mu_V(X,p)$ depends exclusively on $X\mid_V$, since the contribution from choosing another extension to $\mathbf{C}^n$ is cancelled by the terms $(f_1,\dots,f_\ell)$ and that it is strictly positive exactly at the critical set $\{X=0\}$ of X. Note that (2.3) is the corank of Φ in (2.2) over the point $(X,0)$, or equivalently, (2.2) gives a way to express the V-multiplicity as a corank of a matrix with parameters. We will exploit this expression to describe the dependence of the V-multiplicity on X; but technically it wil be simpler to consider an approximation of Φ on infinitesimal neighbourhoods of Θ_0.

We will now analyse the structure of $\mathcal{Z}$ at the zero section Θ_0. Let $\mathcal{K}=(z_1,\dots,z_n)\subset\mathcal{O}_{\Theta_r\times\mathbf{C}^n}$ be the ideal of definition of Θ_0, and denote by Θ_0^j the j^{th} infinitesimal neighbourhood of Θ_0 defined by the sheaf of ideals $\mathcal{K}^{j+1}\subset\mathcal{O}_{\Theta_r\times\mathbf{C}^n}$ generated by the monomials in z of degree $j+1$. As a space, it consists of Θ_0 but its function theory remembers the Taylor series in the z-variables up to order j. Using the presentation (2.2) of $\mathcal{Z}$, we note that $\Phi((\mathcal{K}^{j+1})^{\oplus\ell+n})\subset\mathcal{K}^{j+1}$, so that it will induce an exact commutative diagram

$$
\begin{array}{ccccccc}
\mathcal{O}^{\oplus\ell+n}_{\Theta_r\times\mathbf{B}_r} & \xrightarrow{\Phi} & \mathcal{O}_{\Theta_r\times\mathbf{B}_r} & \longrightarrow & \mathcal{O}_{\mathcal{Z}} & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\mathcal{O}^{\oplus\ell+n}_{\Theta_r\times\mathbf{B}_r}/(\mathcal{K}^{j+1})^{\oplus\ell+n} & \xrightarrow{\Phi^j} & \mathcal{O}_{\Theta_r\times\mathbf{B}_r}/\mathcal{K}^{j+1} & \longrightarrow & \mathcal{O}_{\mathcal{Z}^j} & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 &
\end{array}
\tag{2.4}
$$

where $\mathcal{Z}^j$ is the analytic intersection of $\mathcal{Z}$ and Θ_0^j, and its defining ideal is spanned by $\mathcal{I}$ and $\mathcal{K}^{j+1}$. From the inclusions

$$
\mathcal{J}=(\mathcal{I},\mathcal{K})\supset\cdots\supset(\mathcal{I},\mathcal{K}^{j+1})\supset(\mathcal{I},\mathcal{K}^{j+2})\supset\cdots\supseteq\mathcal{I}
\tag{2.5}
$$

we obtain the inclusions of analytic spaces

$$
\theta_0=\mathcal{Z}^1\subset\cdots\subset\mathcal{Z}^j\subset\mathcal{Z}^{j+1}\subset\cdots\subset\mathcal{Z}
\tag{2.6}
$$

Φ^j in (2.4) is a sheaf map between free sheaves over Θ_0, so it may be identified with a (finite dimensional) vector bundle map between (trivial) bundles over Θ_0. Hence Φ^j may be represented by a (finite dimensional) matrix with parameters. Denote by $\Phi^j(X)\colon \mathcal{O}_{\{X\},\mathbf{B}_r,0}/m^{j+1} \to \mathcal{O}_{\{X\},\mathbf{B}_r,0}/m^{j+1}$, where m is the maximal ideal in $\mathcal{O}_{\{X\},\mathbf{B}_r,0}$, the restriction of Φ^j to the point $(X,0)$ and define

$$\mu(\mathcal{Z}^j, X) := \operatorname{corank}_{\mathbf{C}}[\Phi^j(X)] = \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^n,p}}{(f_1,\dots,f_\ell,X^1,\dots,X^n,z_1^{j+1},\dots,z_n^{j+1})}$$

We have for $j \le k$:

$$1 \le \mu(\mathcal{Z}^j, X) \le \mu(\mathcal{Z}^k, X) \le \mu_V(X,0)$$

Theorem 2.2. *Let $(V,0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space which is smooth except for an isolated singularity at 0, and let Θ_r denote the Banach space of holomorphic vector fields on V_r with continuous extensions to ∂V_r, $r < 1$, and let $\mathcal{Z}$, $\mathcal{Z}^j = \mathcal{Z} \cap \Theta_0^j \subset \Theta_r \times \mathbf{B}_r$ be the universal critical set and its approximation sets. Then, there is a descending sequence of analytic subvarieties of finite codimension A^k, $A^{k+1} \subset A^k$, and an integer J such that:*

a) $\mathcal{Z}^j \cap (\Theta - A^J) = \mathcal{Z}^k \cap (\Theta - A^J)$ for $j,k \ge J$.

b) $\mu_V(X,0) = \mu(\mathcal{Z}^j, X)$ for $X \notin A^J$ and $j \ge J$.

c) The function V-multiplicity at 0

$$\mu_V(\ ,0)\colon \Theta_r \to \mathbf{Z}^+ \cup \{\infty\}$$

is upper semicontinuous and it is locally bounded at those points X where X has an isolated singularity on V at 0 (Θ_r has for this the topology whose closed sets are the analytic subsets).

d) The subsets of Θ_r defined by $\mu(\ ,0) \ge K$ are analytic subspaces and the minimum value of $\mu_V(\ ,0)$ in Θ_r is attained on an open dense subset $\tilde{\Gamma}_1$ of Θ_r.

e) The subset of Θ_r formed by vector fields whose critical set at 0 has positive dimension is the analytic subspace of Θ_r defined by $\cap A^j$.

Proof. For every j, the inclusion $(\mathcal{I}, \mathcal{K}^{j+2}) \subseteq (\mathcal{I}, \mathcal{K}^{j+1})$ induces an exact sequence of sheaves on $\Theta_r \times \mathbf{B}_r$

$$
\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \mathcal{K}^{j+1}/\mathcal{K}^{j+2} & & & & \\
 & & \downarrow & & & & \\
\mathcal{O}^{\oplus \ell+n}_{\Theta_r\times\mathbf{B}_r}/(\mathcal{K}^{j+2})^{\oplus \ell+n} & \xrightarrow{\Phi^{j+1}} & \mathcal{O}_{\Theta_r\times\mathbf{B}_r}/\mathcal{K}^{j+2} & \longrightarrow & \mathcal{O}_{\mathcal{Z}^{j+1}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{O}^{\oplus \ell+n}_{\Theta_r\times\mathbf{B}_r}/(\mathcal{K}^{j+1})^{\oplus \ell+n} & \xrightarrow{\Phi^{j}} & \mathcal{O}_{\Theta_r\times\mathbf{B}_r}/\mathcal{K}^{j+1} & \longrightarrow & \mathcal{O}_{\mathcal{Z}^{j}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
\tag{2.7}
$$

where the first 2 columns may be interpreted as (finite dimensional) vector bundle maps over Θ_0. The corank of $\Phi^{j+1}(X)$ is equal to the corank of $\Phi^j(X)$ if and only if $\mathcal{K}^{j+1}/\mathcal{K}^{j+2}(X)$ is contained in the image of $\Phi^{j+1}(X)$. The increase in the corank from $\Phi^j(X)$ to $\Phi^{j+1}(X)$ is the codimension of

$$\mathrm{Image}[\Phi^{j+1}(X)] \cap [\mathcal{K}^{j+1}/\mathcal{K}^{j+2}(X)] \subset \mathcal{K}^{j+1}/\mathcal{K}^{j+2}(X).$$

A *stratification of* Θ_0 consist of a disjoint decomposition of Θ_0 by subsets $\Gamma_1, \dots, \Gamma_s$ where each Γ_i is an analytic subvariety minus another analytic subvariety (the ones that will actually appear have finite codimension). Since Θ_0 is irreducible there is one and only one component that is open. We will assume that for any stratification of Θ_0 this open component is the first one Γ_1.

We may first find a stratification of Θ_0 so that the corank of $\Phi^{j+1}(X)$ is constant on each strata. Then one may further stratify according to the dimension of $Im\Phi^{j+1} \cap (\mathcal{K}^{j+1}/\mathcal{K}^{j+2})(X)$. In all, we obtain a stratification $\{\Gamma_1^{j+1}, \dots, \Gamma_c^{j+1}\}$ of Θ_0 such that the codimension of $Im\Phi^{j+1} \cap (\mathcal{K}^{j+1}/\mathcal{K}^{j+2})(X)$ is constant on the stratification, say d_i^{j+1} on Γ_i^{j+1}. Since the numbers d_i^{j+1} are defined as coranks of a matrix with parameters, they behave upper semicontinuosly, in the sense that if Γ_k^{j+1} is in the closure of Γ_h^{j+1}, then $d_k^{j+1} \geq d_h^{j+1}$. Due to this property, we may assume that Γ_1^{j+1} consists of all those points X

of Θ_0 where the minimum is attained (i.e. $d_1^{j+1} < d_k^{j+1}$ for $k \geq 2$). We have

$$\mu(\mathcal{Z}^{j+1}, X) \geq \mu(\mathcal{Z}^j, X) + d_1^{j+1} \qquad X \in \Theta_0$$

d_1^j has to be 0 for j large, due to the fact that the sum of these numbers gives a lower bound to $\mu(\mathcal{Z}^j, X)$, which is finite for X with an isolated singularity at 0. If $d_1^{j+1} = 0$, in Γ_1^{j+1} we have

$$\mathcal{K}^{j+1}/\mathcal{K}^{j+2} \subset \mathrm{Image}[\Phi^{j+1}]$$

or equivalently on the open set Γ_1^{j+1} we have

$$\mathcal{K}^{j+1} \subset (\mathcal{I}.\mathcal{K}^{j+2})$$

This last implies also that on Γ_1^{j+1} we have

$$\mathcal{K}^{j+k} \subset (\mathcal{I}, \mathcal{K}^{j+k+1}), \quad k \geq 2 \tag{2.8}$$

which means that $\{A^k = \Theta_0 - \Gamma_1^k\}_k$ form a descending family of analytic spaces of Θ_0, for $k \geq j$ where $d_1^j = 0$. The intersection of the above family consist of those points where $\mu(\mathcal{Z}^j, X)$ is infinite. This set is exactly the set $\{(X, 0)/0$ is not an isolated critical point of X at $0\}$.

(2.8) also implies that if $d_1^j = 0$, then for $k > j$ we have $\Gamma_1^k \subset \Gamma_1^{k+1}$. Let $\Gamma_1 = \bigcap_k \Gamma_1^k$, which by the previous remark reduces to a finite intersection. Γ_1 *is the open dense subset of* Θ_0 *consisting of vector fields with minimum* V*-multiplicity at 0*, and equal to $d_1^1 + d_1^2 + \cdots + d_1^{j-1}$. Let $\tilde{\Gamma}_1 = \pi_1(\Gamma_1)$. From the above description, the theorem is clear. $\square$

Now we begin to analyse the other component $\mathcal{Z}_{\mathrm{sing}}$ of $\mathcal{Z}$.

Lemma 2.3. *The* V*-multiplicity of the holomorphic vector field* X *on* V *at a smooth point* p *of* V *coincides with the multiplicity (or the index) of the vector field* $X|_V$ *at* p.

Proof. We may find coordinates $(z_1, \dots, z_n)$ around p such that $\mathcal{J} = (z_{N+1}, \dots, z_n)$ and the condition that the vector field X is tangent to V is that $X^j \in \mathcal{J}$ for $j = N+1, \dots, n$. Hence

$$\mu_V(X,p) = \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^n,0}}{(z_{N+1}, \dots, z_n, X^1, \dots, X^n)}$$

$$= \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^n,0}}{(z_{N+1}, \dots, z_n, X^1, \dots, X^N)} =$$

$$(2.9) \qquad = \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^N \times \{0\},0}}{(X^1(\tilde{z},0), \dots, X^N(\tilde{z},0))} = \mu(X\,|_V, p) \quad \square$$

A sheaf $\mathcal{F}$ on $\Theta_r \times \mathbf{B}_r$ is Θ_r-*anaflat* ([Do1], 66) if for every point (X, z) there is a finite locally free resolution

$$0 \longrightarrow \mathcal{L}_q \longrightarrow \cdots \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

in a neighbourhood of (X, z) such that its restriction to $\{X\} \times \mathbf{B}_r$ is also an exact sequence.

Proposition 2.4. *If $p \neq 0$ is an isolated critical of $X \in \Theta_r$, then $\mathcal{O}_{\mathcal{Z}_r}$ is Θ_r-anaflat at (X,p).*

Proof. If $(X,p) \in \mathcal{Z}_r$ with $p \neq 0$ an isolated singularity of X, then Lemma 2.3 shows that $\mathcal{Z}_r$ at (X,p) is a complete intersection:

$$\mathcal{I}_{X,p} = (z_{N+1}, \dots, z_n, X^1, \dots, X^N).$$

The generators of $\mathcal{I}_{X,p}$ form a regular sequence, so the Koszul complex of the regular sequence ([G-H], p.688) gives a finite locally free resolution of $\mathcal{O}_{\mathcal{Z}_r}$. The restriction of this complex to $\{X\} \times \mathbf{C}^n$ is the Koszul complex of the restricted generators, who also form a regular sequence. Hence the restricted sequence is also exact. So $\mathcal{O}_{\mathcal{Z}_r}$ is Θ_r- anaflat at (X,p). $\square$

Let now $X \in \Theta_r$ with an isolated critical point at 0, let $s < r$ be such that X is non-vanishing on $\overline{V}_s - \{0\}$, and let $2\varepsilon = \min\{\|X(z)\| \,/\, z \in \partial V_s\}$ and consider the ball $\mathbf{U} = \mathbf{U}(X, \varepsilon) \subseteq \Theta_r$. The projection map $\pi_1 \colon \mathcal{Z}' = \mathcal{Z} \cap (\mathbf{U} \times \mathbf{B}_s) \to \mathbf{U}$ is a finite map by Propositin 2.1 (see [G-R], where a finite map is a closed map with finite fibers). We want to analyse the sheaf $\pi_{1*}\mathcal{O}_{\mathcal{Z}'}$. The points of $\mathcal{Z}' - (\Theta_0 \cap \mathbf{U})$ are π_1-anaflat by Proposition 2.4 and the points of $\Gamma_1 \subset \Theta_0$, consisting of $(W, 0)$ with W of minimal V-multiplicity μ_V at 0, are also π_1-flat (since they have constant multiplicity (see [Do2], p.58)). Hence $\pi_{1*}\mathcal{O}_{\mathcal{Z}'}$ is locally free on $\pi_1(\Gamma_1) = \tilde{\Gamma}_1$ of rank

$$(2.10) \qquad \mu_V + \sum_{\substack{Y(p_j)=0 \\ p_j \in V_s - \{0\}}} \mu_V(Y, p_j) \qquad Y \in \mathbf{U} \cap \tilde{\Gamma}_1$$

where the V-multiplicity of Y at 0 is μ_V. This number is independent of s, for s sufficiently small and of $Y \in \mathbf{U} \cap \tilde{\Gamma}_1$. We will call it the *Euler characteristic of X at 0*, and denote it by $\chi_V^t(X, 0)$ (See [Ser]).

A family of holomophic vector fields parametrized by the irreducible and reduced complex space of finite dimension T is a holomorphic map $\phi \colon T \to \Theta_r$. The family ϕ induces a map $(\phi, id_s) \colon T \times V_s \to \Theta_r \times V_s$, and we will denote $(\phi, id_s)^*(\mathcal{Z}) \subset T \times V_s$ by $\mathcal{Z}_{T,s}$. Let $\pi_{1T} \colon \mathcal{Z}_{T,s} \to T$ be the projection to the first factor. If π_{1T} is a finite map, then $\pi_{1T*}\mathcal{O}_{\mathcal{Z}_{T,s}}$ is a coherent sheaf on T, and hence is locally free on a Zarizki dense set T' of T, say of rank r. For $t \in T'$ we have

$$r = \mu_V(X_t, 0) + \sum_{\substack{X_t(p_j)=0 \\ p_j \in V_s - \{0\}}} \mu_V(X_t, p_j)$$

and for $t \in T$ we have

$$(2.11) \qquad r = \chi_0^{\mathrm{tor}}(\mathcal{O}_{\mathcal{Z}_{T,s}}, \mathcal{O}_{\{t\}}) + \sum_{\substack{X_t(p_j)=0 \\ p_j \in V_s - \{0\}}} \mu_V(X_t, p_j)$$

where

$$(2.12) \qquad \chi_0^{\mathrm{tor}}(\mathcal{O}_{\mathcal{Z}_{T,s}}, \mathcal{O}_{\{t\}}) = \sum_q (-1)^q \, Tor^q_{\mathcal{O}_{T \times \mathbf{B}, (t,0)}}(\mathcal{O}_{\mathcal{Z}_{t,s,(t,0)}}, \mathcal{O}_{\{t\}})$$

is the Euler characteristic of torsion groups of $\mathcal{O}_{\mathcal{Z}_{T,s,(t,0)}}$ over $\mathcal{O}_{\{t\}'}$, where $Tor^0(\mathcal{O}_{\mathcal{Z}_{T,s}}, \mathcal{O}_{\{t\}}) = \mu_V(X_t, 0)$ (see [Do2]). Recall from Proposition 2.1 that $\Theta'_r \subset \Theta_r$ is the open dense subset consisting of vector fields having an isolated critical point at 0.

Theorem 2.5. *For $X \in \Theta'_r$, $s \ll r$ and $0 \ll \varepsilon$, we have:*

1) *For any family of vector fields $\{X_t\}_{t\in T}$, parametrized by a finite dimensional analytic space $(T,0) \to (\Theta_r, X)$ such that the V-multiplicity of the general vector field X_t of the family is minimal μ_V we have:*

$$\chi_V(X,0) = \chi_0^{\text{tor}}(\mathcal{O}_{\mathcal{Z}_{T,s}}, \mathcal{O}_{\{X\}}) \tag{2.13}$$

2) *For $Z \in \mathbf{U}(X,\varepsilon)$ we have*

$$\chi_V(X,0) = \chi_V(Z,0) + \sum_{\substack{Z(p_j)=0 \\ p_j \in V_s - \{0\}}} \mu_V(Z, p_j) \tag{2.14}$$

3) *For $X \in \Theta'_r$ we have:*

$$0 < \chi_V(X,0) \leq \mu_V(X,0)$$

and $\chi_V(X,0) = \mu_V(X,0)$ if and only if $\mathcal{Z}_r$ is π_1-anaflat at $(X,0)$ (in particular in $\tilde{\Gamma}_1$).

Proof. Let $X \in \Theta_r$ with an isolated critical point at 0, let $s < r$ be such that X is non-vanishing on $\overline{V}_s - \{0\}$, $2\varepsilon = \min\{\|X(z)\| / z \in \partial V_s\}$ and consider the ball $\mathbf{U} = \mathbf{U}(X,\varepsilon) \subseteq \Theta_r$.

1) $\chi_V(X,0)$ is defined by (2.10), where Y has minimal multiplicity μ_V at 0. If an element X_1 of a family $\{X_t\}$ has minimal V- multiplicity at 0, then the general element will have at 0 minimal multiplicity μ_V. At these points $\mathcal{Z}_{T,s}$ will be T-flat, since they represent Θ_r-anaflat points of $\mathcal{Z}_r$, and so the general rank of $\pi_{1T^*}\mathcal{O}_{\mathcal{Z}_{T,s}}$ is again (2.10). (2.11) applied to X on V_s gives $r = \chi_0^{\text{tor}}(\mathcal{O}_{\mathcal{Z}_{T,s}}, \mathcal{O}_{\{X\}})$, hence we obtain (2.13).

2) Take a 1-parameter family $\{X_t\}_{t\in T=\mathbf{C}}$ in $\mathbf{U}(X,\varepsilon)$ which contains X and Z such that the general element has minimal V-multiplicity at 0. $\chi(X,0)$ is defined by (2.10), where Y has minimal multiplicity μ_V at 0. Since this is the only condition needed to apply part 1 of the theorem, assume that Y is near to Z. Assume that Z vanishes at $0, p_1, \dots, p_c$. Then part of Y is near to each part of the critical set of Z. Since $Z_{T,s}$ is T-flat at $p_1, \dots, p_c$, there are actually as much multiplicity near p_1 for Y as for Z at p_j. The multiplicity of Y near 0 is $\chi_V(Z,0)$ again by definition (2.10) applied to Z, where a new $\varepsilon' < \varepsilon$ is used in the definition in order to get rid of $p_1, \dots, p_c$. Hence we obtain (2.14).

3) Consider a 1-parameter family which contains X with 0 as only critical point in V_s and whose general element has minimal V-multiplicity. Then $\pi_{1*}\mathcal{O}_{\mathcal{Z}_\mathbf{C}}$ is a coherent sheaf on $\mathbf{C}$ whose rank is $\chi_V(X,0)$, by part 1. Hence the dimension of $\pi_{1*}\mathcal{O}_{\mathcal{Z}_\mathbf{C}} \otimes \mathcal{O}_{\{0\}}$ is greater than or equal to the general rank. If the rank is constant, then $\mathcal{Z}$ is π_1- anaflat. □

Let $X \in \Theta_r$, we say that the *zero set of X does not bifurcate* if there is $\varepsilon > 0$ and $s > 0$ such that for $Y \in \mathbf{U}(X,\varepsilon) \subset \Theta_r$ we have that the only critical point of Y on V_s is 0, (that is, X has an isolated singularity at 0 as well as any sufficiently near vector field in Θ_r and there is no other critical point uniformly in a neighbourhood V_s of 0). The critical set of a vector field X on V_r does not bifurcate if and only if the zero section Θ_0 coincides (as sets) with $\mathcal{Z}_r$ in a neighbourhood of $(X,0)$ in $\Theta_r \times V_r$.

Theorem 2.6. *Let $(V,0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space which is smooth except for an isolated singularity at 0, then the set of points in Θ_r whose critical set does not bifurcate contains the connected dense open subset $\tilde{\Gamma}_1 \subset \Theta_r$ consisting of vector fields with minimum V-multiplicity.*

Proof. Using previously introduced notation, what we have to prove is that $\Theta_{\text{sing}} \cap \Gamma_1 = \phi$ or equivalently that if $(X,0) \in \Theta_{\text{sing}}$ then the V-multiplicity at 0 cannot be minimal.

If $(X,0) \in \Theta_{\text{sing}}$, then we may find a 1-parameter linear family $\{X_t = X + tY\}$ in Θ_r such that its critical set

$$C = \{(t,z) \in \mathbf{C} \times \mathbf{B}_r / z \in V_r, X_t(z) = 0\}$$

has at least 2 local irreducible components at $(X,0)$, the zero section $C_0 = \mathbf{C} \times \{0\}$ and the others, say C_1. Formula (2.14) applied to $Z = X + \varepsilon Y$ is

$$\chi_V(X,0) = \chi_V(X + \varepsilon Y, 0) + \sum_{\substack{X+\varepsilon Y(p_j)=0 \\ p_j \in V_s - \{0\}}} \mu_V(X + \varepsilon Y, p_j) \tag{2.15}$$

The points $p_j \in C_1$ have a strictly positive contribution to the right hand side of (2.15), hence $\chi_V(X,0) > \chi_V(X + \varepsilon Y, 0)$. From this inequality we obtain that $\mu_V(X,0)$ cannot be minimal, for in that case $\chi_V(X,0) = \mu_V(X,0)$ would also be minimal. □

Example. Let $X_t = tz_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + (2t-1)z_3\frac{\partial}{\partial z_3}$ be a family of vector fields on $\mathbf{C}^3$ and let V be the surface defined by $f = z_1^2 - z_2 z_3$. X_t is tangent to V, since $df(X_t) = 2tf$. As vector fields in $\mathbf{C}^3$, X_t has 0 as only critical point, except if $t = 0$ or $1/2$. X_0 has a line of critical points, but on V it has an isolated critical point. For $t \neq 0, 1/2$ one has that

$$(z_1^2 - z_2 z_3, tz_1, z_2, (2t-1)z_3) = (z_1, z_2, z_3)$$

so that the V-multiplicity is 1 for $t \neq 0, 1/2$. For $t = 0$, one has

$$(z_1^2 - z_2 z_3, tz_1, z_2, (2t-1)z_3) = (z_1^2, z_2, z_3)$$

so that the V-multiplicity is 2 for $t = 0$. So we see the upper semicontinuity behaviour of the V-multiplicity.

Remark. For a family $\{X+tY\}$ with $X+\varepsilon Y$ of minimal V- multiplicity we have

$$\chi_V(X,0)=\mu_V(X,0)-\dim[Tor^1_{\mathcal{O}_{T\times \mathbf{B},(t,0)}}(\mathcal{O}_{\mathcal{Z}_{T,s,(t,0)}},\mathcal{O}_{\{t\}})]$$

This second term can be computed as the codimension of

$$(tf_1,\dots,tf_\ell,t(X^1+tY^1),\dots,t(X^n+tY^n))$$

in

$$(t)\cap(f_1,\dots,f_\ell,X^1+tY^1,\dots,X^n+tY^n)$$

(see [Do2]).

3. The index of holomorphic vector fields

Let V be a (reduced complex) analytic space of complex dimension N, with compact singular set and with boundary, ∂V, a smooth manifold of real dimension $2N-1$ oriented in a natural way. Let W be an orientable differentiable manifold of real dimension $2N$ with boundary ∂W diffeomorphic to ∂V (orientation preserving). We may extend this diffeomorphism to a diffeomorphism of a neighbourhood of the boundaries $\phi: V'\to W'$. Given a C^0-vector field X on V', non-singular on ∂V, we may transport it via ϕ to a vector field X' defined on W' and then define the *index of* X *on* V as the index of X' on W', and denote it by $\mathrm{Ind}_W(X,V,\partial V)$. This number depends on the choice of manifold W, but as we have seen in the first section, the choice of a different W' changes the index by an integer uniformly for all vector fields.

Given an analytic space V, one may choose as W a desingularization of V. In case V is a germ of a hypersurface with an isolated singularity defined by the equation $f=0$, then W can be defined by $f=\varepsilon$, for sufficiently small ε (or more generally, if V is a complete intersection, or a smoothable germ with an isolated singularity, then W can be the smoothening (see [Se])).

If p is an isolated singular point of V and X is a holomorphic vector field defined in a neighbourhood of p non-vanishing in a pointed neighbourhood of p, then *the index of X at p $Ind_W(X, V, p)$* is defined as $\mathrm{Ind}_W(X, V', \partial V')$, where V' is a sufficiently small neighbourhood of p in V, and W is a manifold with $\partial W \approx \partial V$. The function $\mathrm{Ind}_W(\_, V, p)$ is well defined up to adding an integer, choice that depends on the election of the bounding manifold W.

The objective of this section is to compare the index with the V-multiplicity of X at 0. We recall that at a smooth point of V, if one uses the model of a ball as bounding a neighbourhood of the boundary of a smooth point, then the index coincides with the multiplicity (Lemma 2.3).

Theorem 3.1. *Let $(V,0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n, 0)$ be an analytic space which is smooth except for an isolated singularity at 0, then there is a constant K such that*

$$Ind_W(X, V, 0) = \chi_V(X, 0) + K \tag{3.1}$$

for X in the dense open set Θ' of vector fields in Θ_r with an isolated singularity at 0, where χ_V denotes the Euler-Poincaré characteristic of X at 0. For X in the dense open set of Θ' where the universal critical set $\mathcal{Z}_r$ is Θ_r-anaflat we have

$$Ind_W(X, V, 0) = \mu_V(X, 0) + K \tag{3.2}$$

Proof. If $X \in \Theta_r$ is a vector field on V whose critical set does not bifurcate, then the index is locally constant at X, since the index on the boundary remains constant, and it is equal to the sum of the local indices, but the only critical point is located at 0. Hence the index is constant on the connected set $\mathcal{B}$ of Theorem 2.6. By Theorem 2.2.d the minimum of the V-multiplicity is attained on a dense open subset $\tilde{\Gamma}_1 \subset \mathcal{B}$.

Hence there is an integer K satisfying (3.2) for $X \in \Gamma_1$ (due to the fact that both functions are constant there).

Let now $X \in \Theta_r$ with an isolated critical point at 0, let $s < r$ be such that X is non-vanishing on $\overline{V}_s - \{0\}$, and let $2\varepsilon = \min\{\|X(z)\| / z \in \partial V_s\}$ and consider the ball $\mathbf{U} = \mathbf{U}(X, \varepsilon)$. For $X + tY \in \overline{\Gamma}_1 \cap \mathbf{U}$ we have

$$\mathrm{Ind}_W(X, V, 0) = \mathrm{Ind}_W(X + tY, V, 0) + \sum_{\substack{X+ty(p_j)=0 \\ p_j \in V_s - \{0\}}} \mathrm{Ind}_W(X + tY, V, p_j)$$

And hence

$$\mathrm{Ind}_W(X, V, 0) = [\chi_V(X + tY, V, 0) + K] + \sum_{\substack{X+ty(p_j)=0 \\ p_j \in V_s - \{0\}}} \mu_V(X + tY, p_j)$$

since $X + tY$ has minimal V-multiplicity at 0 and (3.1) and the fact that at the smooth points the V-multiplicity is equal to the index (Lemma 2.3). Using now (2.15) we obtain (3.1). (3.2) follows now from Theorem 2.5.3.

Corollary 3.2. *Let $(V, 0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n, 0)$ be an analytic space wich is smooth except for an isolated singularity at 0, then there is a constant L such that $\mathrm{Ind}_W(X, V, 0) \geq L$ for every germ of holomorphic vector field X on V with an isolated singularity at 0 on V.*

Proof. Let K be as in Theorem 3.2. Since $\chi_V(X, 0) > 0$ for any $X \in \Theta'$, we have

$$\mathrm{Ind}_W(X, V, 0) > K \quad \square$$

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