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VANISHING HOLONOMY AND MONODROMY OF CERTAIN CENTRES AND FOCI

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Introduction

Let $\omega(x,y) = A(x,y)dx + B(x,y)dy = 0$ be the germ of an analytic differential equation on $\mathbf{R}^2$, with an algebraically isolated singularity at the origin: $A(0,0) = B(0,0) = 0$, $dim_{\mathbf{R}} \frac{\mathbf{R}\{x,y\}}{(A,B)} < +\infty$.

The singularity $\omega = 0$ is called *monodromic* if there are not separatrices at 0. In this case, given a germ of an analytic embedding $(\mathbf{R}^+,0) \overset{\tau}{\hookrightarrow} (\mathbf{R}^2,0)$ transverse to ω outside 0, it is possible to define a *monodromy map* $P_{\omega,\tau}$: $(\mathbf{R}^+,0) \to (\mathbf{R}^+,0)$, following clockwise the solutions of $\omega = 0$; $P_{\omega,\tau}$ is a germ of homeomorphism of $(\mathbf{R}^+,0)$ analytic outside 0. If $P_{\omega,\tau} = id$ then $\omega = 0$ is called *centre.* Otherwise $P_{\omega,\tau}$ is a contraction or an expansion (by the results of Écalle, Il'yashenko, Martinet, Moussu, Ramis... on "Dulac conjecture") and $\omega = 0$ is called *focus.*

The simplest monodromic singularities are those for which the linear part M_ω of the dual vector field $v(x,y) = B(x,y)\frac{\partial}{\partial x} - A(x,y)\frac{\partial}{\partial y}$ is nondegenerate, i.e. invertible. We distinguish two situations:

i) the eigenvalues λ, μ of M_ω are complex conjugate, non real, with real part different from zero. Then $\omega = 0$ is a focus and it is analytically equivalent to $\omega_{lin} = 0$, where ω_{lin} denotes the linear part of ω (Poincarè's linearization theorem).

ii) the eigenvalues λ, μ of M_ω are complex conjugate, non real, with zero real part. Then if $\omega = 0$ is a centre there exists an analytic first integral (Lyapunov-Poincarè theorem, see [Mou1] and references therein) and $\omega = 0$ is analytically equivalent to $xdx + ydy = 0$. If $\omega = 0$ is a focus the analytic classification is a difficult problem, which requires the theory of Écalle-Martinet-Ramis-Voronin to pass from the formal classification to the analytic one ([M-R]). The monodromy is an analytic diffeomorphism tangent to the identity, and two such equations are analytically equivalent if and only if their monodromies are ([M-R]).

In this paper we shall study the simplest degenerate monodromic singularities, i.e. those with $\lambda = \mu = 0$, $\omega_{lin} \neq 0$, and with "generic" higher order terms. Modulo a change of coordinates ([Mou2]), we may work in the following class.

Definition. Let $\omega = Adx + Bdy = 0$ be the germ of an analytic differential equation on $\mathbf{R}^2$, with an algebraically isolated singularity at 0. This singularity is called *monodromic semidegenerate* if the first nonzero quasihomogeneous jet of type (1,2) of ω is

$$\omega_0(x,y) = x^3dx + (y + ax^2)dy$$

with $a^2 < 2$. Notation: $\omega \in MSD(a)$.

We will denote by P_ω the monodromy map of $\omega \in MSD(a)$ corresponding to the embedding $(\mathbf{R}^+,0) \hookrightarrow (\mathbf{R}^2,0)$, $t \mapsto (t,0)$. P_ω is a germ of analytic diffeomorphism tangent to the identity ([Mou2]), and we may consider P_ω as the restriction to $\mathbf{R}^+$ of a germ of biholomorphism of $(\mathbf{C},0)$, tangent to the identity, again denoted by P_ω.

Let $\omega \in MSD(a)$ and let Ω be the germ of holomorphic 1-form on $\mathbf{C}^2$ obtained by complexification of ω. Using a resolution of the singularity we may define as in [Mou3] and [C-M] the *vanishing holonomy* of Ω: it is a subgroup $H(\Omega) \subset Bh(\mathbf{C},0) = \{$ group of germs of biholomorphisms of $(\mathbf{C},0)\}$, generated by $f, g \in Bh(\mathbf{C},0)$ satisfying the relation $(f \circ g)^2 = id$.

Our result is a computation of P_ω in terms of $H(\Omega)$. A similar result was remarked by Moussu in the (simpler) case of nondegenerate monodromic singularities ([Mou1]).

Theorem. *Let $\omega = 0$ be monodromic semidegenerate, then*

$$P_\omega = [f,g]$$

In particular, $H(\Omega)$ is abelian if and only if $\omega = 0$ is a centre. This means, by [C-M], that a nontrivial space of "formal-analytic moduli" can appear only if $\omega = 0$ is a centre (and $a = 0$, see below): for the foci, formal equivalence $\Rightarrow$ analytic equivalence. Hence our situation is very different from the situation of equations of the type $xdx + ydy + ... = 0$, where the difficult case is the case of foci whereas all the centres are analytically equivalent (here the vanishing holonomy is always abelian, generated by a single $f \in Bh(\mathbf{C},0)$, and the monodromy is given by f^2, see [Mou1]). On the other hand, it is no more true that the monodromy characterizes the equation: it may happen that $\omega_1, \omega_2 \in MSD(a)$ have the same monodromy without being analytically equivalent.

A consequence of the above relation between monodromy and vanishing holonomy is the following normal form theorem for centres, based again on the results of [C-M]. Let us before remark that $\omega_0(x,y) = x^3dx + (y + ax^2)dy = 0$ is a centre for any $a \in \mathbf{R}$ (but a first integral exists if and only if $a = 0$).

Corollary 1. *Let $\omega \in MSD(a)$ be a centre and let $a \neq 0$, then the germ $\omega = 0$ is analytically equivalent to $\omega_0 = 0$.*

We don't know a similar explicit and "simple" (polynomial?) normal form for foci, even in the case $a \neq 0$; but the triviality of the space of formal-analytic moduli seems here a useful tool. The classification of centres with $a = 0$ requires arguments of the type Écalle - Martinet - Ramis - Voronin (cfr. [C-M]).

As another corollary of the above theorem we give a positive answer to a question posed by Moussu in [Mou2].

Corollary 2. *Let $\omega = 0$ be a monodromic semidegenerate centre, then there exists a nontrivial analytic involution $I : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ which preserves the solutions of $\omega = 0$: $I^*(\omega) \wedge \omega = 0$.*

The above computation may be generalized to the case of germs ω whose first nonzero quasihomogeneous jet of type $(1, n)$ is

$$\omega_0(x, y) = x^{2n-1}dx + (y + ax^n)dy$$

with $a^2 < \frac{1}{4}n$ ([Mou2]). The vanishing holonomy $H(\Omega)$ for these germs is generated by $f, g \in Bh(\mathbf{C}, 0)$ satisfying $(f \circ g)^n = id$ ([C-M]). But now, if $n \geq 3$, the relation between commutativity of $H(\Omega)$ and triviality of P_ω becomes more complicated; in particular, it is no more true that there is equivalence between "$H(\Omega)$ abelian" and "$P_\omega = id$".

The computation of P_ω in terms of $H(\Omega)$ for $n \geq 3$ is straightforward, once one has understood the case $n = 2$. Hence, for sake of simplicity and clarity, we have choose to limit ourselves to the semidegenerate monodromic singularities.

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Resolution of singularities and vanishing holonomy

Let $\omega \in MSD(a)$ and let Ω be its complexification. We recall the desingularization of Ω and the construction of $H(\Omega)$ ([C-M], [Mou3]).

We denote by M the complex manifold of dimension two covered by 3 charts $U_j = \{(x_j, y_j)\} \simeq \mathbf{C}^2$, $j = 1,2,3$, glued together by the identifications

$$\begin{cases} x_1 = \dfrac{1}{y_2} \\ y_1 = x_2 y_2^2 \end{cases} \qquad \begin{cases} x_3 = x_2 y_2 \\ y_3 = \dfrac{1}{x_2} \end{cases} \qquad \begin{cases} x_3 = x_1 y_1 \\ y_3 = \dfrac{1}{x_1^2 y_1} \end{cases}$$

Let $h : M \to \mathbf{C}^2$ be the holomorphic map whose expressions h_j in the charts U_j are

$$h_1(x_1, y_1) = (x_1 y_1, y_1), \quad h_2(x_2, y_2) = (x_2 y_2, x_2 y_2^2), \quad h_3(x_3, y_3) = (x_3, x_3^2 y_3)$$

The *divisor* $Z \overset{def}{=} h^{-1}((0,0))$ is a union of two copies of $\mathbf{C}P^1$, which intersect transversally at a point p. If $Z_j = Z \cap U_j$, then

$$Z_1 = \{y_1 = 0\}, \qquad Z_2 = \{x_2 = 0\} \cup \{y_2 = 0\}, \qquad Z_3 = \{x_3 = 0\}$$

The map $h|_{M \setminus Z} : M \setminus Z \to \mathbf{C}^2 \setminus \{(0,0)\}$ is a biholomorphism.

On M there is naturally defined an involution $j : M \to M$ given, in every chart U_j, by $j(x_j, y_j) = (\bar{x}_j, \bar{y}_j)$. The set $M^{\mathbf{R}}$ of fixed points of j is a real analytic manifold, the map h restricts to a real analytic map $h^{\mathbf{R}} : M^{\mathbf{R}} \to \mathbf{R}^2$, $Z^{\mathbf{R}} \overset{def}{=} (h^{\mathbf{R}})^{-1}((0,0)) = Z \cap M^{\mathbf{R}}$ is a union of two copies of $\mathbf{R}P^1$ intersecting transversally at p. The map $h^{\mathbf{R}}|_{M^{\mathbf{R}} \setminus Z^{\mathbf{R}}} : M^{\mathbf{R}} \setminus Z^{\mathbf{R}} \to \mathbf{R}^2 \setminus \{(0,0)\}$ is a real analytic diffeomorphism. The manifold $M^{\mathbf{R}}$ is covered by the charts $U_j^{\mathbf{R}} \overset{def}{=} U_j \cap M^{\mathbf{R}} \simeq \mathbf{R}^2$, $j = 1,2,3$, and we will denote again with (x_j, y_j) the corresponding coordinates.

From now on we will consider only the germs of the previous objects (M, h, $M^{\mathbf{R}}$, etc.) along Z or $Z^{\mathbf{R}}$, denoted by the same symbols.

Define $\tilde{\Omega} = h^*(\Omega)$. Its local expressions are

$\tilde{\Omega}_1 = y_1[(1 + \mathcal{O}(|y_1|))dy_1 + (\mathcal{O}(|y_1|))dx_1]$

$\tilde{\Omega}_2 = x_2 y_2^3[(2x_2 + 2ax_2^2 + x_2^3 + \mathcal{O}(|x_2 y_2|))dy_2 + (y_2 + \mathcal{O}(|x_2 y_2|))dx_2]$

$\tilde{\Omega}_3 = x_3^3[(1 + 2ay^3 + 2y_3^2 + \mathcal{O}(|x_3|))dx_3 + (\mathcal{O}(|x_3|))dy_3]$

The only singularities of the foliation $\tilde{\mathcal{F}}$ defined by $\tilde{\Omega}$ are:
– the point p (= $(0,0)$ in U_2), where $\bar{\Omega}_2 \overset{def}{=} \frac{1}{x_2 y_2^3}\tilde{\Omega}_2$ has a singularity of the type "2:1 resonant saddle":

$$\bar{\Omega}_2 = 2x_2 dy_2 + y_2 dx_2 + h.o.t.$$

– the points

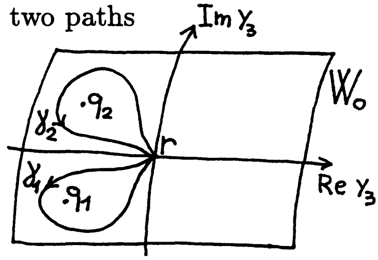
$$q_1 = (x_2 = -a + i\sqrt{2-a^2}, y_2 = 0) = (x_3 = 0, y_3 = \frac{1}{2}(-a - i\sqrt{2-a^2}))$$

$$q_2 = (x_2 = -a - i\sqrt{2-a^2}, y_2 = 0) = (x_3 = 0, y_3 = \frac{1}{2}(-a + i\sqrt{2-a^2})) = j(q_1)$$

where $\bar{\Omega}_2$ has hyperbolic singularities (the ratio of the eigenvalues is not real) if $a \neq 0$, and saddles with $4:1$ resonance if $a = 0$.

We denote by $W_0 \simeq \mathbf{C}P^1$ the component of Z containing q_1 and q_2, and by $W_1 \simeq \mathbf{C}P^1$ the other component; $W_0 \setminus \{p, q_1, q_2\}$ and $W_1 \setminus \{p\}$ are regular leaves of $\tilde{\mathcal{F}}$.

Let $L = cl(h^{-1}(\{y=0\}) \setminus Z) = \{y_3 = 0\}$ and $r = L \cap W_0 = (x_3 = 0, y_3 = 0)$. Let $\gamma_j : [0,1] \to W_0 \setminus \{p, q_1, q_2\}$, $j = 1, 2$, be two paths such that $\gamma_j(0) = \gamma_j(1) = r$, $ind_{\gamma_j}(q_i) = \delta_{ij}$. To these paths there correspond two germs of biholomorphisms $f_j : (L, r) \to (L, r)$, given by the holonomy of the foliation $\tilde{\mathcal{F}}$.



We set $f, g \in Bh(\mathbf{C}, 0)$ equal respectively to f_1, f_2 expressed using the coordinate x_3 on L.

Definition ([C-M]). The *vanishing holonomy* $H(\Omega)$ of Ω is the subgroup of $Bh(\mathbf{C}, 0)$ generated by f and g.

An elementary computation shows that

$$f'(0) = -i \cdot exp(-\frac{a}{2\sqrt{2-a^2}}) \qquad g'(0) = -i \cdot exp(+\frac{a}{2\sqrt{2-a^2}})$$

in particular, $H(\Omega)$ is *hyperbolic* (i.e. $|f'(0)| \neq 1$, $|g'(0)| \neq 1$) if and only if $a \neq 0$.

Near the singularity p there exists a first integral, of the form $x_2 y_2^2 + h.o.t.$ ([C-M]). This implies that the holonomy of $\tilde{\mathcal{F}}$ along $\gamma_1 * \gamma_2$ or $\gamma_2 * \gamma_1$ (which are freely homotopic in $W_0 \setminus \{p, q_1, q_2\}$ to small paths around p) is periodic, of period 2. Hence:

$$(f \circ g)^2 = (g \circ f)^2 = id$$

Remark that from the fact that Ω has real coefficients we deduce that

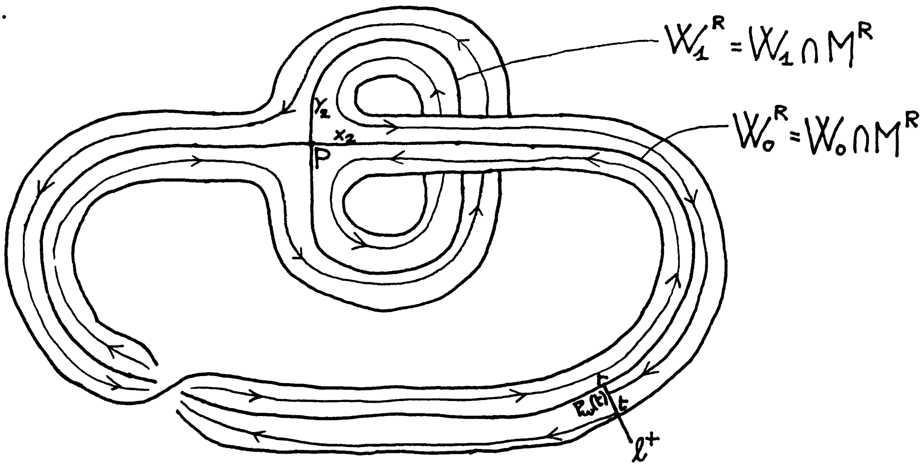
$$g(z) = \overline{f^{-1}(\bar{z})}$$

and, because p is a real point of M, $f \circ g$ and $g \circ f$ are real:

$$(f \circ g)(\bar{z}) = \overline{(f \circ g)(z)} \qquad \text{and} \qquad (g \circ f)(\bar{z}) = \overline{(g \circ f)(z)}$$

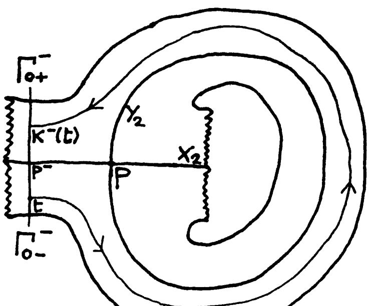
Now we turn to the real 1-form ω. Clearly, $h^{\mathbf{R}} : M^{\mathbf{R}} \to \mathbf{R}^2$ gives a resolution of the singularity. We set $l^+ = cl((h^{\mathbf{R}})^{-1}(\{y = 0, x \geq 0\}) \setminus Z^{\mathbf{R}}) = L \cap M^{\mathbf{R}} \cap \{x_3 \geq 0\}$, then the holonomy of the foliation $\tilde{\mathcal{G}}$ defined by $\tilde{\omega} = (h^{\mathbf{R}})^*(\omega)$ along the "polycicle" $Z^{\mathbf{R}}$ produces a germ of analytic diffeomorphism of (l^+, r). Using the coordinate x_3 on l^+ and complexifying the result we obtain a germ of biholomorphism of $(\mathbf{C}, 0)$ which is nothing else that the (complex) monodromy P_ω of ω (for the appropriate choice of orientation of

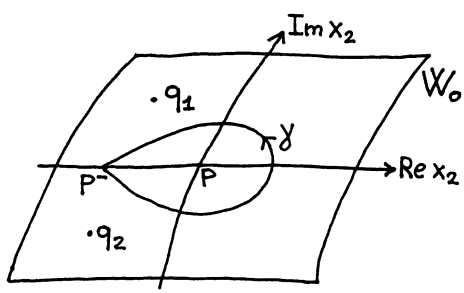
$\tilde{\mathcal{G}}$).



Proof of the theorem

Let $\Gamma^- \subset M$ be a germ of complex line, j-symmetric, transverse to $\tilde{\mathcal{F}}$, passing throught a point p^- of W_0 near p with coordinates (in the chart U_2) $(-\epsilon, 0)$, $\epsilon \in \mathbf{R}^+$ small. Let $\Gamma_0^- = \Gamma^- \cap M^{\mathbf{R}}$ be its real part; it is a germ of real line transverse to $\tilde{\mathcal{G}}$. The holonomy of $\tilde{\mathcal{G}}$ along the polycycle composed by the segment from p^- to p and $W_1^{\mathbf{R}} \stackrel{def}{=} W_1 \cap M^{\mathbf{R}}$ gives a germ of homeomorphism $k^- : (\Gamma_{0-}^-, p^-) \to (\Gamma_{0+}^-, p^-)$, where $\Gamma_{0-}^- = \Gamma_0^- \cap \{y_2 \le 0\}$, $\Gamma_{0+}^- = \Gamma_0^- \cap \{y_2 \ge 0\}$.





On the other hand, we may consider a path $\gamma : [0,1] \to W_0 \setminus \{p, q_1, q_2\}$ with $\gamma(0) = \gamma(1) = p^-$ and $ind_\gamma(p) = 1$, $ind_\gamma(q_1) = ind_\gamma(q_2) = 0$. The holonomy of $\tilde{\mathcal{F}}$ along γ induces a germ of biholomorphism $K^- : (\Gamma^-, p^-) \to (\Gamma^-, p^-)$, which is an involution because of the first integral of $\tilde{\Omega}$ near p.

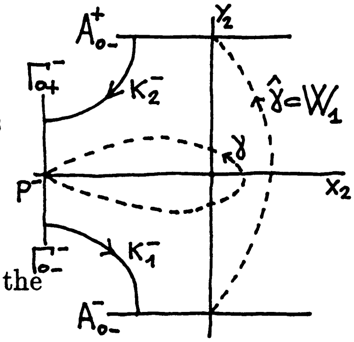
Lemma.

$$k^- = K^-|_{\Gamma^-_{0-}}$$

Proof.

let $A^+, A^- \subset M$ be two germs of complex lines, j-symmetric, transverse to $\tilde{\mathcal{F}}$ and passing throught $(0,\epsilon)$, $(0,-\epsilon)$ (in the chart U_2). Let A_0^+, A_0^- be their real parts, $A_{0-}^+ = A_0^+ \cap \{x_2 \le 0\}$, $A_{0-}^- = A_0^- \cap \{x_2 \le 0\}$. The homeomorphism k^- is the composition of a homeomorphism k_1^- from Γ_{0-}^- to A_{0-}^-, an analytic diffeomorphism $k^* : A_0^- \to A_0^+$, and a homeomorphism k_2^- from A_{0-}^+ to Γ_{0+}^-.

Because $W_1 \setminus \{p\}$ is simply connected, the path which joins $(0,-\epsilon)$ to $(0,\epsilon)$ along $W_1^{\mathbf{R}} \setminus \{p\}$ is contractible in $W_1 \setminus \{p\}$ (endpoints fixed) to a path $\hat{\gamma}$ contained in $\{|y_2| \le \epsilon\}$. Hence k^* is the restriction to A_0^- of a biholomorphism $K^* : A^- \to A^+$, obtained from the holonomy of $\tilde{\mathcal{F}}$ along this path $\hat{\gamma}$.



We choose ϵ so small that $\tilde{\Omega}$ has a first integral $x_2 y_2^2 + \ldots$ defined on $U_\epsilon = \{|x_2| \le \epsilon, |y_2| \le \epsilon\}$. Hence every leaf of $\tilde{\mathcal{F}}|_{U_\epsilon}$ different from a separatrix at p either does not intersect $U_\epsilon^{\mathbf{R}} = U_\epsilon \cap M^{\mathbf{R}}$, or it intersects $U_\epsilon^{\mathbf{R}}$ along two segments "symmetric" w.r. to the x-axis. We deduce that:

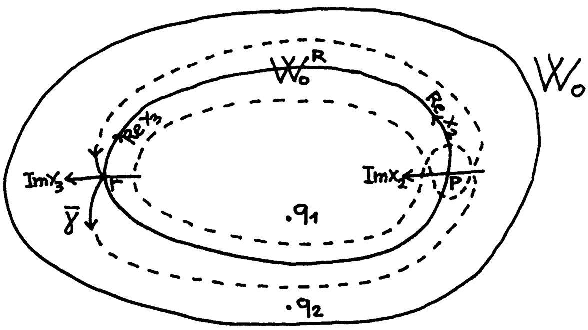
i) if $t \in A_{0-}^-$, then $K^*(t) \in A_{0-}^+$ is the only intersection of A_{0-}^+ with the leaf of $\tilde{\mathcal{F}}|_{U_\epsilon}$ through t;

ii) if $s \in \Gamma_{0-}^-$, then $K^-(s) \in \Gamma_{0+}^-$ is the only intersection of Γ_{0+}^- with the leaf of $\tilde{\mathcal{F}}|_{U_\epsilon}$ through s.

From these two remarks, it is clear that $k_2^- \circ (K^*|_{A_{0-}^-}) \circ k_1^-$ is equal to $K^-|_{\Gamma_{0-}^-}$, i.e. $k^- = K^-|_{\Gamma_{0-}^-}$. Q.E.D.

Obviously, a similar result holds if we start from Γ^+= germ of complex line j-symmetric passing throught $(\epsilon, 0) = p^+$, etc..

As a consequence, the complex monodromy $P_\omega : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ may be computed as the holonomy of $\tilde{\mathcal{F}}$ along a path $\bar\gamma : [0,1] \to W_0 \setminus \{p, q_1, q_2\}$, $\bar\gamma(0) = \bar\gamma(1) = r$, as in the following picture:



This path is homotopic to $\gamma_2^{-1} * \gamma_1^{-1} * \gamma_1^{-1} * \gamma_2^{-1}$, hence

$$P_\omega = g^{-1} \circ f^{-1} \circ f^{-1} \circ g^{-1}$$

and from $(g \circ f)^2 = id$ we conclude

$$P_\omega = f \circ g \circ f^{-1} \circ g^{-1} = [f, g]$$

Q.E.D.

Proof of corollary 1

It is sufficient, using the path-lifting argument of [C-M], to show that the vanishing holonomies $H(\Omega)$ and $H(\Omega_0)$ are holomorphically conjugate. The "assertion 1" of [C-M], pag. 478, is here replaced by

Assertion 1_a*:* if $\omega \in MSD(a)$ then there are analytic coordinates (x, y) near $(0,0)$ s.t. ω is, modulo multiplication by a nonvanishing germ:

$$\omega(x,y) = x^3 dx + (y + ax^2)dy + f(x,y)(2ydx - xdy), \qquad f \in \mathbf{R}\{x,y\}$$

The proof of this normal form lemma is achieved as in [C-M]: $\Omega = 0$ has a separatrix $X^4 + 2aX^2Y + 2Y^2 = 0$ (in suitable coordinates, preserving the class $MSD(a)$), which is a separatrix also for $2YdX - XdY = 0$. Hence:

$$\Omega \wedge (X^3 dX + (Y + aX^2)dY) = (X^4 + 2aX^2Y + 2Y^2) \cdot H_1(X,Y)dX \wedge dY$$

$$\Omega \wedge (2YdX - XdY) = (X^4 + 2aX^2Y + 2Y^2) \cdot H_2(X,Y)dX \wedge dY$$

$$(2YdX - XdY) \wedge (X^3 dX + (Y + aX^2)dY) = (X^4 + 2aX^2Y + 2Y^2) \cdot dX \wedge dY$$

and from these formulae the assertion 1_a follows.

Let us denote by f_0, g_0 the generators of $H(\Omega_0)$; $[f_0, g_0] = id$ because ω_0 is a centre, moreover

$$f_0'(0) = f'(0) = \lambda$$
$$g_0'(0) = g'(0) = \frac{1}{\bar{\lambda}} = \frac{-1}{\lambda}$$

and $|\lambda| \neq 1$ because $a \neq 0$.

From the commutativity and the hyperbolicity of $H(\Omega)$ we deduce that $H(\Omega)$ is holomorphically conjugate to the group generated by

$$z \mapsto \lambda z \qquad \text{and} \qquad z \mapsto \frac{1}{\bar{\lambda}} z$$

For the same reasons, $H(\Omega_0)$ also is holomorphically conjugate to that linear group, hence to $H(\Omega)$. Q.E.D.

<u>Proof of corollary 2</u>

Consider the germ $f \circ g \in Bh(\mathbf{C}, 0)$: it is real $((f \circ g)(\bar{z}) = \overline{(f \circ g)(z)})$, periodic with period 2, and conjugates $H(\Omega)$ with itself thanks to $[f, g] = id$:

$$(f \circ g) \circ g = (g \circ f) \circ g = g \circ (f \circ g)$$
$$(f \circ g) \circ f = f \circ (g \circ f) = f \circ (f \circ g)$$

As in corollary 1, we use the path-lifting technique of [C-M] to suspend $(f \circ g)$ and to obtain a germ of biholomorphism $\tilde{I} : M \to M$, which preserves $\tilde{\mathcal{F}}$:

$\tilde{I}^*(\tilde{\Omega}) \wedge \tilde{\Omega} = 0$. From the fact that $(f \circ g)$ is a real involution, we obtain that $\tilde{I}$ is also a real involution. Taking the projection on $\mathbf{C}^2$ and the restriction to $\mathbf{R}^2$ we obtain the required analytic involution. Q.E.D.

Remark: if $a \neq 0$ the result follows also from corollary 1: ω_0 is invariant by $(x, y) \mapsto (-x, y)$.

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