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CODIMENSION ONE FOLIATIONS IN CP^n, $n \geq 3$, WITH KUPKA COMPONENTS

D. Cerveau and A. Lins Neto

1. Introduction

1.1 – Basic notions:

A codimension one holomorphic foliation in a complex manifold M can be given by an open covering $(U_\alpha)_{\alpha \in A}$ of M and two collections $(w\alpha)_{\alpha \in A}$ and $(g_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \phi}$, such that:

(a) For each $\alpha \in A$, w_α is an integrable ($w_\alpha \wedge dw_\alpha = 0$) holomorphic 1-form in U_α, and $w_\alpha \not\equiv 0$.

(b) If $U_\alpha \cap U_\beta \neq \phi$ then $w_\alpha = g_{\alpha\beta} \cdot w_\beta$, where $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$.

Recall that $\mathcal{O}(V)$ is the set of holomorphic functions in V and $\mathcal{O}^*(V) = \{g \in \mathcal{O}(V) | g(p) \neq 0 \ \forall p \in V\}$.

Let $\mathcal{F} = ((U_\alpha)_{\alpha \in A}, (w_\alpha)_{\alpha \in A,(g_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \phi})}$ be a foliation in M. The singular set of $\mathcal{F}$, $S(\mathcal{F})$, is by definition $S(\mathcal{F}) = \bigcup_{\alpha \in A} S_\alpha$, where $S_\alpha = \{p \in U_\alpha | w_\alpha(p) = 0\}$. It follows from (a) and (b) that $S(\mathcal{F})$ is a proper analytic subset of M. The integrability condition implies that for each $\alpha \in A$ we can define a foliation $\mathcal{F}_\alpha$ (in the usual sense) in $U_\alpha - S_\alpha$, whose leaves are solutions of $w_\alpha = 0$. Condition (b) implies that if $U_\alpha \cap U_\beta \neq \phi$, then $\mathcal{F}_\alpha$ coincides with $\mathcal{F}_\beta$ in $U_\alpha \cap U_\beta - S(\mathcal{F})$. Hence we have a codimension one foliation defined in $M - S(\mathcal{F})$. A leaf of $\mathcal{F}$ is by definition, a leaf of this foliation.

If $S(\mathcal{F})$ has codimension one components, then it is possible to find a new foliation $\mathcal{F}_1 = ((U_\alpha)_{\alpha \in A}, (\tilde{w}_\alpha)_{\alpha \in A}, (\tilde{g}_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \phi})$ such that $S(\mathcal{F}_1)$ has no components of codimension one, $S(\mathcal{F}_1) \subset S(\mathcal{F})$, and the leaves of $\mathcal{F}$ and $\mathcal{F}_1|(M - S(\mathcal{F}))$ are the same (in fact $w_\alpha = f_\alpha \cdot \tilde{w}_\alpha$, $f_\alpha \in \mathcal{O}(U_\alpha)$). From now on all the foliatons that we will consider *will not have codimension 1 singular components.*

1.2 – The Kupka set:

In 1964 I.Kupka proved the following result (see [**K**]);

1.2.1 THEOREM. *Let w be an integrable holomorphic 1-form defined in a neighborhood of $p \in \mathbf{C}^n$, $n \geq 3$. Supose that $w_p = 0$ and $dw_p \neq 0$. Then there exists a holomorphic coordinate system $(x, y, z_3, \ldots, z_n)$ defined in a neighborhood U of p such that $x(p) = y(p) = 0$ and $w = A(x,y)dx + B(x,y)dy$ in this coordinate system, where $A(0,0) = B(0,0) = 0$ and $\frac{\partial B}{\partial x}(0,0) - \frac{\partial A}{\partial y}(0,0) \neq 0$.*

In fact Kupka proved this result in the real context, but his proof adapts very well in the holomorphic case.

1.2.2 Remarks: Let w, A, B and U be as in Theorem 1.2.1.

(i) The set $\{(x, y, z_3, \ldots, z_n) \in U | x = y = 0\} = V$ is containned in U. If the singular set S of w has no codimension 1 components, then V is a smooth codimension 2 piece of S and $(0,0)$ is an isolated solution of $A(x,y) = B(x,y) = 0$. By taking a smaller U if necessary we can suppose that $S \cap U = V$.

(ii) The foliation induced by $w = 0$ in U is equivalent to the product of the singular foliation in $U \cap \{z_3 = c_3, \ldots, z_n = c_n\} \subset \mathbf{C}^2 \times (c_3, \ldots, c_n)$ given by $Adx + Bdy = 0$ (or by the differential equation $\dot{x} = -B, \dot{y} = A$), by the codimension 2 foliation in U given by $x = c_1, y = c_2$. The singular set in this case is $V = \{x = y = 0\}$.

Let $\mathcal{F} = ((U_\alpha)_{\alpha \in A}, (w_\alpha)_{\alpha \in A}, (g_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \phi})$ be a foliation on M. We define the Kupka set of $\mathcal{F}$ by $K(\mathcal{F}) = \bigcup\limits_{\alpha \in A} K_\alpha$, where

$$K_\alpha = \{p \in U_\alpha | w_\alpha(p) = 0 \text{ and } dw_\alpha(p) \neq 0\}$$

Since $w_\alpha = g_{\alpha\beta} w_\beta$ in $U_\alpha \cap U_\beta \neq \phi$, we have $dw_\alpha = dg_{\alpha\beta} \wedge w_\beta + g_{\alpha\beta} dw_\beta$ which implies that $K_\alpha \cap U_\beta = K_\beta \cap U_\alpha$. It follows from (i) that $K(\mathcal{F})$ is a smooth complex codimension 2 submanifold of M. In fact $K(\mathcal{F}) = S(\mathcal{F}) - W(\mathcal{F})$ where $W(\mathcal{F}) = \bigcup_{\alpha \in A} W_\alpha$, $W_\alpha = \{p \in U_\alpha | w_\alpha(p) = 0 \text{ and } dw_\alpha(p) = 0\}$. Observe that $W(\mathcal{F})$ is an analytic subset of M.

1.2.3 Definition: We say that K is a *Kupka component* of $\mathcal{F}$ if K is an irreducible component of $S(\mathcal{F})$ and $K \subset K(\mathcal{F})$. Observe that a Kupka component of $\mathcal{F}$ is in particular a smooth connected codimension 2 analytic subset of M.

Let V be a connected codimension 2 submanifold of $K(\mathcal{F})$. It follows from the local product structure (see 1.2.1 and 1.2.2) that there exists a covering $(B_i)_{i \in I}$ of V by open sets of M, a collection of submersions $(\psi_i)_{i \in I}$, $\psi_i : B_i \to \mathbf{C}^2$, and a 1-form $w = A(x,y)dx + B(x,y)dy$ defined in a neighborhood C of $(0,0) \in \mathbf{C}^2$, such that:

(a) $\psi_i(B_i) \subset C$ for evere $i \in I$.
(b) $(0,0)$ is the unique singularity of w in C and $V \cap B_i = \psi_i^{-1}(0,0)$, for every $i \in I$.
(c) $\mathcal{F}|B_i$ is represented by $w_i^* = \psi_i^*(w)$.

We will say that $\mathcal{F}$ has *transversal type w or X* along V, where X is the vector field $-B\partial/\partial x + A\partial/\partial y$. The *linear transversal type of $\mathcal{F}$ along* V is, by definition, the linear part of X at $(0,0)$ in Jordan's canonical form, modulo multiplication by non-zero constants. Let L be the linear part of X at $(0,0)$ in Jordan's canonical form. We have the following possibilities:

(i) L is diagonal with eigenvalues $\lambda_1 \neq \lambda_2$.
(ii) L is diagonal with eigenvalues $\lambda_1 = \lambda_2 \neq 0$.
(iii) L is not diagonal with eigenvalues $\lambda_1 = \lambda_2 \neq 0$.

Observe that, since $\frac{\partial B}{\partial x}(0,0) - \frac{\partial A}{\partial y}(0,0) \neq 0$, we have $tr(L) \neq 0$ and so the possibilities $\lambda_1 = \lambda_2 = 0$ or $\lambda_1 = -\lambda_2$ cannot occur.

In case (i) the two eigendirections of L induce via the submersions ψ_i, two line subbundles of the normal bundle $\nu(V)$ of V in M. We will call these line bundles L_1 (relative to λ_1) and L_2 (relative to λ_2). It is clear that $\nu(V) = L_1 \oplus L_2$. In case (iii) L has just one eigendirection which induces in the same way a line subbundle L_1 of $\nu(V)$. In the case of Kupka components we have the following (see [**G.M- L.N**]):

1.2.4 - THEOREM. *Let* $\dim(M) \geq 3$ *and* K *be a Kupka compact component of* $\mathcal{F}$. *We have:*

(a) In case (i), if $C(L_i)$ *is the first Chern class of* L_i, $i = 1, 2$, *considered in* $H^2(K, \mathbf{C})$, *then* $\lambda_1 C(L_2) = \lambda_2 C(L_1)$.
(b) In case (iii) we have $C(L_1) = 0$.
(c) In case (i), if $\lambda_2/\lambda_1 = p/q$, *where* $p, q \in \mathbf{Z}_+$ *are relatively primes and* $C(L_1) \neq 0$, *then* X *is linearizable.*

1.3 - Codimension 1 foliations of $\mathbf{C}P^n$, $n \geq 3$:

A holomorphic foliation in $\mathbf{C}P^n$ can be given by an integrable 1-form $w = \sum_{i=0}^{n} w_i dz_i$ ($w \wedge dw = 0$), with the following properties:

(a) $w_0, \ldots, w_n$ are homogeneous polynomials of the same degree ≥ 1.
(b) $i_R(w) = \sum_{i=0}^{n} w_i z_i \equiv 0$ ($R = \sum_{i=0}^{n} z_i \partial/\partial z_i$ is the radial vector field).

This form can be obtained as follows: let $\pi\colon \mathbf{C}^{n+1} - \{0\} \to \mathbf{C}P^n$ be the canonical projection and $\mathcal{F} = ((U_\alpha)_{\alpha \in A}, (w_\alpha)_{\alpha \in A}, (g_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \phi})$ be a foliation in $\mathbf{C}P^n$. Let $\mathcal{F}^* = ((U^*_\alpha)_{\alpha \in A}, (w^*_\alpha)_{\alpha \in A}, (g^*_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \phi})$ be the foliation in $\mathbf{C}^{n+1} - \{0\}$ defined by $U^*_\alpha = \pi^{-1}(U_\alpha), w^*_\alpha = \pi^*(w_\alpha)$ and $g^*_{\alpha\beta} = g_{\alpha\beta} \circ \pi$. Since for $U^*_\alpha \cap U^*_\beta \cap U^*_\gamma \neq \phi$ we have $g^*_{\alpha\beta} \cdot g^*_{\beta\gamma} \cdot g^*_{\gamma_\alpha} = 1$, we can use Cartan's solution of the multiplicative Cousin's problem in $\mathbf{C}^{n+1} - \{0\}$ (see [**G-R**]) to obtain an integrable 1-form η in $\mathbf{C}^{n+1} - \{0\}$ such that for any $\alpha \in A$, we have $\eta | U^*_\alpha = h_\alpha \cdot w^*_\alpha$, where $h_\alpha \in \mathcal{O}^*(U^*_\alpha)$. From Hartog's Theorem (see [**G-R**]), η extends to a holomorphic 1-form μ in $\mathbf{C}^{n+1}$. If $\mu = \mu_k + \mu_{k+1} + \ldots$ is the

Taylor development of μ at 0, where the coefficients of μ_j are homogeneous of degree j and $\mu_k \not\equiv 0$, then it is easy to see that $w = \mu_k$ is integrable. We leave it to the reader the proof of the following facts:

(c) $S(w) = S(\mathcal{F}^*) = \pi^{-1}(S(\mathcal{F})) \cup \{0\}$.

(d) L^* is a leaf of $\mathcal{F}^*$ iff $L^* = \pi^{-1}(L)$, where L is a leaf of $\mathcal{F}$.

(e) If $k =$ degree $(w) \geq 2$, then $K(\mathcal{F}^*) = \pi^{-1}(K(\mathcal{F})) = \{p \in \mathbf{C}^{n+1} | w(p) = 0$ and $dw(p) \neq 0\}$.

When degree $(w) = 1$ we can have $w = z_1 dz_2 - z_2 dz_1$ and in this case $K(\mathcal{F}^*) = \{z_1 = z_2 = 0\} = \pi^{-1}(K(\mathcal{F})) \cup \{0\}$.

Observe that condition (b) is equivalent to conditions (c) and (d) and means that the lines through the origin are tangent to the leaves of $\mathcal{F}^*$.

Observe also that given an integrable 1-form w in $\mathbf{C}^{n+1}$ satisfying (a) and (b) we can induce a foliation $\mathcal{F}(w)$ in $\mathbf{C}P^n$ as follows: let $(U_i)_{i=0}^n$ be the covering of $\mathbf{C}P^n$ by affine coordinate systems, where $U_i = \{[z_0 : \ldots : z_n] \in \mathbf{C}P^n | z_i \neq 0\}$. Let $\psi_i : U_i \to \mathbf{C}^n$, $\psi_i[z_0 : \ldots : z_n] = (z_0/z_i, \ldots, z_{i-1}/z_i, z_{i+1}/z_i, \ldots, z_n/z_2) = (x_0^i, \ldots, x_{i-1}^i, x_{i+1}^i, \ldots, x_n^i)$. Define $\eta_i = \psi_i^*(\eta_i^*)$, where $\eta_i^* = w|(z_i = 1) = \sum_{j \neq i} w_j(z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n) dz_j$. It is not difficult to see that if $dg(w) = k$ then $\eta_i | V_i \cap U_j = (x_j^i)^{k+1} \eta_j | U_i \cap U_j$. Hence $\mathcal{F}(w) = ((U_i)_{i=0}^n, (\eta_i)_{i=0}^n ((x_i^j)^{k+1})_{i \neq j})$ is a foliation on $\mathbf{C}P^n$.

Remark: Two integrable 1-forms w and η in $\mathbf{C}^{n+1}$, with properties (a) and (b) define the same foliation iff $w = \lambda \cdot \eta$ where $\lambda \in \mathbf{C}^*$.

It follows from the above considerations that the space of foliations in $\mathbf{C}P^n$ can be written as $\cup_{m \geq 1} P_m$, where P_m is the projectivization of the following space of polynomial 1-forms: $I_m = \{w | w = \sum_{i=0}^n w_i dz_i, dg(w_i) = m \quad \forall i = 0, \ldots, n, \ w \wedge dw = 0, \sum_{i=0}^n z_j w_j \equiv 0$ and the set $\{w_0 = \cdots = w_n = 0\}$ has all irreducible components of codimension $\geq 2\}$.

Observe that I_m is an open subset of the following algebraic set

$$E_m = \{w | dg(w) = m, w \wedge dw = 0, \sum_{i=0}^{n} z_i w_i \equiv 0\}$$

Problem - Describe in some way the irreducible components of P_m. Let us see some examples.

1.3.1 - Example: Let f, g be homogeneous polynomials in $\mathbf{C}^{n+1}$, $n \geq 3$, where $dg(f) = k \geq 1$, $dg(g) = \ell \geq 1$ and $k/\ell = p/q$ where p and q are relatively primes. Assume that:

(*) $\quad \forall\ z \in \{f = g = 0\} - \{0\}$ we have $df(z) \wedge dg(z) \neq 0$.

We will use the notation $f \pitchfork g$ ($f = 0$ intersects $g = 0$ transversely) in this case. We observe that Noether's lemma implies that if f and g satisfy (*) then $\{f = g = 0\}$ is a complete intersection.

Let $w = qgdf - pfdg$. It follows from Euler's identity that $i_R(w) = 0$. Moreover $w \wedge dw = 0$ because $w = f.g.\eta$, where $\eta = q\frac{df}{f} - p\frac{dg}{g}$ and $d\eta = 0$. Therefore w induces a foliation in $\mathbf{C}P^n$, $\mathcal{F}(w)$, such that:

(i) $S(\mathcal{F}(w)) = \pi\{p \neq 0 \mid w(p) = 0\} = S$ (singular set)

(ii) $K(\mathcal{F}(w)) = \pi\{p \neq 0 \mid f(p) = g(p) = 0\} = K$ (Kupka set)

(iii) f^q/g^p, considered as meromorphic function on $\mathbf{C}P^n$, is a first integral of $\mathcal{F}(w)$. This follows from the fact that $w = g^{p+1}f^{1-q}d(f^q/g^p)$.

(iv) $w \in P_n$ where $n = k + \ell - 1$.

As a consequence of the techniques developed in **[G.M-L.N.]** it is possible to prove the following result:

1.3.2 THEOREM. *Let $\mathcal{F}_0 = \mathcal{F}(w)$, where w is as in example 1.3.1. Then there exists a neighborhood $\mathcal{U}$ of $\mathcal{F}_0$ in P_n such that if $\mathcal{F} \in \mathcal{U}$ then there are polynomials $\tilde{f}$ and $\tilde{g}$ of degrees k and ℓ respectively, and $\mathcal{F} = \mathcal{F}(q\tilde{g}d\tilde{f} - p\tilde{f}d\tilde{g})$.*

As a consequence we have:

1.3.3 COROLLARY. *There are irreducible components in P_m (for all $m \geq 1$) whose foliations are defined by meromorphic functions in $\mathbf{C}P^n$, $n \geq 3$.*

Concerning Kupka components we will prove in §3 the following result:

1.3.4 THEOREM A. *Let $\mathcal{F}$ be a foliation of $\mathbf{C}P^n$, $n \geq 3$, which has a Kupka component K of the form $\{[z] \mid f(z) = g(z) = 0\}$ where f and g are homogeneous polynomials and $f \pitchfork g$. Let $dg(f)/dg(g) = p/q$ where p and q are relatively primes. Then:*

(a) If $dg(f) = dg(g)$, then $\mathcal{F}$ is the foliation induced by the form $fdg - gdf$. In particular f/g is a first integral of $\mathcal{F}$.
(b) If $dg(f) < dg(g)$ then $\mathcal{F}$ is induced by a form of the type $qg_1df - pfdg_1$, where $g_1 = g + h.f$ is homogeneous and $dg(g_1) = dg(g)$. In particular f^q/g_1^p is a first integral of $\mathcal{F}$.

1.3.5 Example: A logarithmic form is one of the type

$$(*) \qquad w = f_1 \dots f_r \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j}$$

where $\lambda_1, \dots, \lambda_r \in \mathbf{C}^*$ and $f_1, \dots, f_r$ are holomorphic functions. When $f_1, \dots, f_r$ are homogeneous polynomials in $\mathbf{C}^{n+1}$, $dg(f_i) \geq 1$ for $i = 1, \dots, r$, and $\sum_{i=1}^{r} \lambda_i dg(f_i) = 0$, then w induces a foliation $\mathcal{F}(w)$ in $\mathbf{C}P^n$, where $\mathcal{F}(w) \in P_m$, $m = \sum_{i=1}^{r} dg(f_i) - 1$. Observe that $\sum_{i=1}^{r} \lambda_i dg(f_i) = 0$ is equivalent to the condition $\sum_{i=0}^{n} z_i w_i \equiv 0$. We will use the notations $F_i = \{[z] \in \mathbf{C}P^n \mid f_i(z) = 0\}$ and $F_{ij} = \{[z] \in \mathbf{C}P^n \mid f_i(z) = f_j(z) = 0\}$ if $i \neq j$. We will assume that $f_1, \dots, f_r$ are irreducibles. The foliation $\mathcal{F}(w)$, induced by w in $\mathbf{C}P^n$ has the following properties:

(i) For every $i = 1, \dots, r$, $F_i^* = F_i - S(\mathcal{F}(w))$ is a leaf of $\mathcal{F}(w)$.
(ii) The holonomy of F_i^* is linearizable and is conjugated to a subgroup of the group of linear transformations of $\mathbf{C}$ generated by the set $\{g_j \mid g_j(z) =$

$\exp(2\pi i\lambda_j/\lambda_i).z, \quad j = 1,\dots,r, \quad j \neq i\}$. The holonomy of a leaf $L \neq F_1^*,\dots,F_r^*$ is trivial.

(iii) For any $i \neq j$, $F_{ij} \subset S(\mathcal{F}(w))$. Moreover, if $\lambda_i \neq \lambda_j$ then $F_{ij} - V$ is contained in the Kupka set of $\mathcal{F}(w)$, where $V = \{[z] \mid df_i(z) \wedge df_j(z) = 0\} \cup V', V' = \bigcup\limits_{k\neq i,j} \{[z] \mid f_k(z) = 0\}$. In particular $K(\mathcal{F}(w)) \subset \bigcup\limits_{i\neq j} F_{ij}$.

(iv) The function $f_1^{\lambda_1}\dots f_r^{\lambda_r}$ (in general multivalued) is a first integral of $\mathcal{F}(w)$. The following result is known:

1.3.6 THEOREM. *(J. Omegar) – Let $\mathcal{F}_0$ be the foliation induced by w in $\mathbf{C}P^n$, $n \geq 3$, where w is like in (*) of 1.3.5. Assume that $f_1,\dots,f_r$ are irreducibles and for some $i \in \{1,\dots,r\}$, say $i = 1$, we have:*

(a) F_1 is smooth.

(b) For any subset $\{j_1,\dots,j_s\} \subset \{2,\dots,r\}$ where $j_1 < \cdots < j_s$ and any $p \in F_1 \cap F_{j_1} \cap \cdots \cap F_{j_s}$ then $F_1, F_{j_1},\dots,F_{j_s}$ intersect multitransversely at p.

(c) For some $j > 1$ we have $\lambda_j/\lambda_1 \notin \mathbf{R}$.

Then there exists a neighborhood $\mathcal{U}$ of $\mathcal{F}_0$ in P_m such that if $\mathcal{F} \in \mathcal{U}$ then $\mathcal{F}$ is induced by a logarithmic form of the same type of w, say $\eta = g_1,\dots,g_r \sum\limits_{j=1}^{r} \eta_j \frac{dg_j}{g_j}$, where $dg(g_j) = dg(f_j)$, $j = 1,\dots,r$.

It follows that:

1.3.7 COROLLARY. *There are irreducible components in P_m (for all $m \geq 1$) whose foliations in an open and dense subset are defined by logarithmic forms.*

1.3.8 Definition: We say that a meromorphic 1-form w, defined in some complex manifold M, has an *integrating factor*, if there exists a meromorphic function f in M, called an integrating factor, such that, $d(\frac{w}{f}) = 0$. Remark that for w as in (*) of 1.3.5, the function $f = f_1\dots f_r$ is an integrating factor.

In §3 we will prove the following result:

1.3.9 THEOREM B. *Let $\mathcal{F}$ be a foliation in $\mathbf{C}P^n$, $n \geq 3$, such that there is an analytic subset $N \subset S(\mathcal{F})$ with the following properties:*

(a) $\mathrm{cod}(N) = 2$ and $N = \{[z] \in \mathbf{C}P^n \mid f(z) = g(z) = 0\}$, where f and g are homogeneous polynomials on $\mathbf{C}^{n+1}$.

(b) $K(\mathcal{F}) \cap N$ is open and dense in N and moreover for any connected component C of $K(\mathcal{F}) \cap N$ the linear part of the transversal type of $\mathcal{F}$ at C has eigenvalues $\lambda_1(C) \neq 0 \neq \lambda_2(C)$, where $\lambda_2(C)/\lambda_1(C) \notin \mathbf{R}$.

(c) For any $p \in N - K(\mathcal{F})$, $\mathcal{F}$ can be represented in a neighborhood of p by a holomorphic form which has an integrating factor.

Then there exists a closed meromorphic 1-form η in $\mathbf{C}P^n$ which represents $\mathcal{F}$ outside its divisor of poles. In particular $\mathcal{F}$ is induced by a homogeneous 1-form in $\mathbf{C}^{n+1}$ which has a meromorphic integrating factor.

Furthermore $\mathcal{F}$ is of logarithmic type if we assume that:

(d) $K(\mathcal{F})$ is dense in each irreducible component of codimension 2 of $S(\mathcal{F})$.

(e) For any connected component C of $K(\mathcal{F})$ the transversal part of $\mathcal{F}$ at C has linear part non degenerated (i.e. 0 is not an eigenvalue).

Remarks:

1.3.10 – It will follow from the proof that condition (b) can be replaced by;

(b') $\lambda_2(C)/\lambda_1(C) \notin \mathbf{Q}$ and the transversal type is linearizable.

1.3.11 – In **[C-M]** the authors give some sufficient conditions for a holomorphic integrable 1-form have a local integrating factor. One of their results implies that (c) follows from (b) and

(c') For some neighborhood U of $p \in N - K(\mathcal{F})$, $\mathcal{F}|U$ has a finite number of analytic leaves which intersect multitransversely in the points of N.

1.4 An example: Let K be the twisted cubic in $\mathbf{C}P^3$, which is defined in homogeneous coordinates $(x, y, z, w) \in \mathbf{C}^4$ by the equations $f = g = h = 0$, where

$$(*) \qquad f = XW - YZ, \quad g = XZ - Y^2 \text{ and } h = YW - Z^2.$$

We will prove here that there exists no foliation in $\mathbf{C}P^3$ having K as a Kupka component.

Let U_0 and U_4 be the affine coordinate systems in $\mathbf{C}P^3$, whose points are of the form $[1 : u : v : w]$ and $[x : y : z : 1]$, respectively. Then $K \subset U_0 \cup U_4$. Moreover $K \cap U_0$ and $K \cap U_4$ can be parametrized by $\varphi_0(t) = [1 : t : t^2 : t^3]$ and $\varphi_4(s) = [s^3; s^2 : s : 1]$ respectively, where $\varphi_0(t) = \varphi_4(s)$ iff $s = 1/t$. Let $f_0(u, v, w) = f(1, u, v, w) = w - uv$, $g_0(u, v, w) = g(1, u, v, w) = v - u^2$, $f_4(x, y, z) = f(x, y, z, 1) = x - yz$ and $h_4(x, y, z) = h(x, y, z, 1) = y - z^2$. Remark that $K \cap U_0 = \{f_0 = g_0 = 0\}$ and $K \cap U_4 = \{f_4 = h_4 = 0\}$.

Suppose by contradiction that there exists a foliation $\mathcal{F}$ on $\mathbf{C}P^3$ whose Kupka set contains K. Let w be a homogeneous integrable 1-form in $\mathbf{C}^4$ such that $i_R(w) = 0$ and w represents $\mathcal{F}$. Let $w = \sum\limits_{i=0}^{3} \alpha_i dz_i$, where $dg(\alpha_i) = k$, $i = 0, \ldots 4$. If $w_0 = w \mid \{z_0 = 1\} = \alpha_1(1, u, v, w)du + \alpha_2(1, u, v, w)dv + \alpha_3(1, u, v, w)dw$ and $w_4 = w \mid \{z_4 = 1\} = \alpha_0(x, y, z, 1)dx + \alpha_1(x, y, z, 1)dy + \alpha_2(x, y, z, 1)dz$, then $\mathcal{F}|U_0$ is represented by w_0 and $\mathcal{F}|U_4$ by w_4. Moreover in $U_0 \cap U_4$ we have $w_0 = x^{-(k+1)}w_4$.

Now, consider the maps $\psi_0, \psi_4 \colon \mathbf{C}^3 \to \mathbf{C}^3$ given by $\psi_0(u, v, w) = (u, g_0(u, v, w), f_0(u, v, w))$ and $\psi_4(x, y, z) = (f_4(x, y, z), h_4(x, y, z), z)$. It is not difficult to see that ψ_0 and ψ_4 are diffeomorphisms, so that we can consider (u, g_0, f_0) and (f_4, h_4, z) as coordinates in U_0 and U_4 respectively. Moreover $\psi_0(K \cap U_0) = \{f_0 = g_0 = 0\}$ and $\psi_4(K \cap U_4) = \{f_4 = h_4 = 0\}$. Observe also that the inverse maps of ψ_0 and ψ_4 are polynomials, so that we can write

$$w_0 = A(u, g_0, f_0)du + B(u, g_0, f_0)dg_0 + C(u, g_0, f_0)df_0$$

$$w_4 = D(f_4, h_4, z)df_4 + E(f_4, h_4, z)dh_4 + F(f_4, h_4, z)dz$$

where A, B, C, D, E and F are polynomials. Let us analyze w_0. Consider the vector field

$$X = \left(\frac{\partial C}{\partial g_0} - \frac{\partial B}{\partial f_0}\right)\frac{\partial}{\partial u} + \left(\frac{\partial A}{\partial f_0} - \frac{\partial C}{\partial u}\right)\frac{\partial}{\partial g_0} + \left(\frac{\partial B}{\partial u} - \frac{\partial A}{\partial g_0}\right)\frac{\partial}{\partial f_0}$$

As the reader can see, the integrability condition is equivalent to $i_X(w_0) = 0$. Moreover, in the proof of Kupka's Theorem (1.2.1) it is proved that the flow of X leaves invariant the Kupka set. Since $\{f_0 = g_0 = 0\} \subset K(\mathcal{F}) \cap U_0$, we must have $X(u,0,0) = \left(\frac{\partial C}{\partial g_0} - \frac{\partial B}{\partial f_0}\right)(u,0,0)\frac{\partial}{\partial u}$, so that,

$$\text{(1)} \qquad \frac{\partial A}{\partial f_0}(u,0,0) = \frac{\partial C}{\partial u}(u,0,0) \text{ and } \frac{\partial B}{\partial u}(u,0,0) = \frac{\partial A}{\partial g_0}(u,0,0).$$

On the other hand, since $K \subset S(\mathcal{F})$, we can write

$$w_0 = (a_1(u)g_0 + a_2(u)f_0)du + (b_1(u)g_0 + b_2(u)f_0)dg_0 + (c_1(u)g_0 + c_2(u)f_0)df_0 + \dots$$

where $a_1, \dots, c_2$ are polynomials in u and the dots mean terms of order ≥ 2 in (g_0, f_0). This implies that:

$$X = (c_1(u) - b_2(u))\frac{\partial}{\partial u} + a_2(u)\frac{\partial}{\partial g_0} - a_1(u)\frac{\partial}{\partial f_0} + \dots$$

where the dots mean terms of order ≥ 1 in (g_0, f_0). From (1) we get that $a_1 \equiv a_2 \equiv 0$ and $X(u,0,0) = (c_1(u) - b_2(u))\frac{\partial}{\partial u}$. On the other hand, since $K \subset K(\mathcal{F})$, we must have $dw_0(u,0,0) \neq 0$ $\forall u \in \mathbf{C}$, and this implies that $c_1(u) - b_2(u) \neq 0$ $\forall u \in \mathbf{C}$. Hence $c_1 - b_2 \equiv c$, $c \neq 0$ a constant, because $c_1 - b_2$ is a polynomial. From these considerations it is easy to see that:

$$\text{(2)} \qquad dw_0 | K \cap U_0 = c\, dg_0 \wedge df_0.$$

With an analogous argument it is possible to conclude that

$$\text{(3)} \qquad dw_4 | K \cap U_4 = \hat{c}\, df_4 \wedge dh_4, \text{ where } \hat{c} \neq 0 \text{ is a constant.}$$

Now, recall that $w_0 = x^{-(k+1)} w_4$ in $U_0 \cap U_4$. Since $x = z^3$ along $K \cap U_4$ and $w_4(0,0,z) = 0$, we have:

$$dw_0 | K \cap U_0 \cap U_4 = x^{-(k+1)} dw_4(0,0,z) = z^{-3(k+1)} dw_4(0,0,z)$$

which together with (2) and (3) implies that:

$$\text{(4)} \qquad \tilde{c}\, dg_0 \wedge df_0 = z^{-3(k+1)} df_4 \wedge dh_4 \text{ along } K \cap U_0 \cap U_4, \quad \tilde{c} = c/\hat{c}.$$

As the reader can verify easily, we have the following relations

$$\begin{cases} f_0|U_0 \cap U_4 = x^{-2} f_4 | U_0 \cap U_4 \\ g_0|U_0 \cap U_4 = x^{-2}[z f_4 - y h_4]|U_0 \cap U_4 \end{cases}$$

This implies that

(5)

$$dg_0 \wedge df_0 | K \cap U_0 \cap U_4 = y x^{-4} dh_4 \wedge df_4 | K \cap U_0 \cap U_4 = z^{-10} dh_4 \wedge df_4 | K \cap U_0 \cap U_4$$

because $yx^{-4} = z^{-10}$ along $K \cap U_4$.

Finally, from (4) and (5) we get that $10 = 3(k+1)$, where $k \in \mathbf{N}$, which is a contradiction.

This example motivates the following:

Problem: Are there foliations on $\mathbf{C}P^n$, $n \geq 3$, which admit a Kupka component which is not a complete intersection?

We think that the answer is no.

2. Basic results

In this section we will state and prove some of the results that we will need in §3.

2.1 Definitions: Let $\varphi\colon \to \mathbf{R}$ be a C^2 function, where $U \subset \mathbf{C}^n$ is an open set. We say that φ is strictly k-subharmonic (briefly $s.k-s.$) or $(n-k+1)$-pseudoconvex, if for any $z \in U$ the $\partial\overline{\partial}$-matrix of φ at z, which is defined by

$$H_\varphi(z) = \left(\frac{\partial^2 \varphi}{\partial \overline{z}_i \partial z_j}(z)\right) 1 \leq i, j \leq n$$

has at least k positive eigenvalues. Observe that $H_\varphi(z)$ is a hermitian matrix, so that all its eigenvalues are real. Moreover, if $f\colon V \to U$ is a biholomorphism then

$$H_{\varphi \circ f}(w) = \overline{P}^t . H_\varphi(f(w)).P$$

where P is the jacobian matrix of f at w. This implies that the concept of $s.k-s.$ can be defined in complex manifolds: if M is a complex manifold of dimension n and $\varphi: M \to \mathbf{R}$ is C^2, we say that φ is $s.k-s.$ if for any $p \in M$ there is a holomorphic local chart $\alpha: U \to V \subset \mathbf{C}^n$, $p \in U$, such that $\varphi \circ \alpha^{-1}$ is $s.k-s.$. It is clear that if $\beta: U_1 \to V_1$, $p \in U_1$, is another holomorphic chart, then $\varphi \circ \beta^{-1}$ is $s.k-s.$.

We say that a connected complex manifold M is *k-complete*, $k \geq 1$, if there exists a $s.k-s.$ function $\varphi: M \to \mathbf{R}$ such that:

$$(*) \qquad \lim_{p \to \infty} \varphi(p) = +\infty,$$

that is, for any sequence $(p_n)_{n \geq 1}$ in M, without accumulation points, we have $\lim_{n \to \infty} \varphi(p_n) = +\infty$. We observe that a $s.k-s.$ function, $k \geq 1$, cannot have a local maximum. This fact follows from the maximum principle for subharmonic functions, as the reader can verify easily. Hence there are no $s.k-s.$ functions on compact manifolds. Remark also that property (*) implies that:

(i) For any $r \in \mathbf{R}$ the sets $\varphi^{-1}(-\infty, r]$ and $\varphi^{-1}(r)$ are compact.
(ii) $\inf \{\varphi(p) \mid p \in M\} = m > -\infty$, and there exists $p_0 \in M$ such that $\varphi(p_0) = m$.

When $k = n = \dim(M)$ a $s.k-s.$ function is also called a strictly subharmonic function.

2.2 Extension of Meromorphic forms:

The main result of this section is the following:

2.2.1 THEOREM. *Let M be a k-complete complex manifold, where $k \geq 2$. Let C be a compact subset of M and w be a meromorphic (resp. holomorphic) ℓ-form defined on $M - C$. Then w extends to a meromorphic (resp. holomorphic) ℓ-form on M.*

Proof: Let $\varphi: M \to \mathbf{R}$ be a $s.k-s.$ function such that $\lim_{p \to \infty} \varphi(p) = +\infty$. Since a $s.k-s$ function is $s.2-s.$ if $k \geq 2$, we can assume that $k = 2$. Let $m =$

$\inf\{\varphi(p) \mid p \in M\}$ and $r = \sup\{\varphi(p) \mid p \in C\}$, so that $M - C \subset \varphi^{-1}(r, +\infty)$ and w is defined on $\varphi^{-1}(r, +\infty)$. The idea is to prove the following:

Assertion 1: If w can be extended to $\varphi^{-1}(s, +\infty)$, where $s \leq r$, then there exists $\varepsilon > 0$ such that w can be extended to $\varphi^{-1}(s - \varepsilon, +\infty)$.

Since $\varphi^{-1}[m, +\infty) = M$, then assertion 1 clearly implies the theorem. On the other hand, assertion 1 is implied by the following:

Assertion 2: Suppose that w has been already extended to $\varphi^{-1}(s, +\infty)$, where $s \leq r$. Given $p \in \varphi^{-1}(s)$, there exist a neighborhood V of p such that w can be extended to $V \cup \varphi^{-1}(s, +\infty)$.

Assertion 1 follows from assertion 2 because $\varphi^{-1}(s)$ is compact. In order to prove assertion 2 we use Levi's Theorem:

Levi's Theorem: (see [S] for the proof). Let $W \subset V \subset \mathbf{C}^{n-1}$ be open sets, where $W \neq \phi$ and V is connected. Let f be a meromorphic (resp. holomorphic) function defined in $(W \times \Delta(r)) \cup (V \times [\Delta(r) - \overline{\Delta(r')}])$, where $\Delta(r) = \{z \in \mathbf{C} \mid |z| < r\}$ and $0 < r' < r$. Then f can be extended to a meromorphic (resp. holomorphic) function on $V \times \Delta(r)$.

An open set A of the form $(W \times \Delta(r)) \cup (V \times [\Delta(r) - \overline{\Delta}(r')])$ is called a Hartog's domain. The set $\hat{A} = V \times \Delta(r)$ is called its envelope of holomorphy. Another fact we will use is the following:

LEMMA 1. *Let $\varphi: U \to \mathbf{R}$ be a s.2 − s. function, where $U \subset \mathbf{C}^n$ is an open set $(n \geq 2)$. Let $p \in U$ be such that $\varphi(p) = s$. Then there exist a biholomorphism $\alpha: V_1 \to U_1$ and a Hartog's domain $A \subset V_1$ such that*

(a) $0 \in V_1$, $\alpha(0) = p \in U_1 \subset U$

(b) $\alpha(A) \subset U_1$ and $p \in \alpha(\hat{A})$, where $\hat{A}$ is the envelope of holomorphy of A.

For the proof see the §8 of [**S-T**].

Assertion 2 follows from Levi's Theorem and Lemma 1. In fact, given $p \in \varphi^{-1}(s)$, by taking a local chart we can assume that $p = 0 \in \mathbf{C}^n$ and $\varphi: U \to \mathbf{R}$, $0 \in U \subset \mathbf{C}^n$. Since $\varphi(0) = s$ and w is defined in $\varphi^{-1}(s, +\infty) \subset U$, we can write $w = \sum\limits_I f_I dz_I$, where $I = (i_1, \ldots, i_\ell)$, $i_1 < \cdots < i_\ell$, $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_\ell}$

and f_I is a meromorphic (resp. holomorphic) function on $\varphi^{-1}(s, +\infty)$. Let α and A be as in lemma 1 and $g_i = f_I \circ \alpha^{-1}$, $I = (i_1 < \cdots < i_\ell)$. From Levi's Theorem g_I can be extended to a meromorphic (resp. holomorphic) function on $\hat{A}$. Hence f_I can be extended to a meromorphic (resp. holomorphic) function on $\alpha(\hat{A})$. Since $\alpha(\hat{A})$ is a neighborhood of $p = 0$, then assertion 2 is proved. ∎

Now we consider the following situation: let $f_1, \ldots, f_k$ be homogeneous (non constant) polynomials in $\mathbf{C}^{n+1}$ and $V(f_1, \ldots, f_k) = \{[p] \in \mathbf{C}P^n \mid f_1(p) = \cdots = f_k(p) = 0\}$.

2.2.2 THEOREM. *$M = \mathbf{C}P^n - V(f_1, \ldots, f_k)$ is ℓ-complete, where $\ell = n-k+1$.*

Proof: Let $dg(f_j) = d_j$, $j = 1, \ldots, k$ and $q_1, \ldots, q_k \in [N]$ be such that $d_1q_1 = \cdots = d_kq_k = q > 0$. Put $G_j = f_j^{q_j}$, so that $dg(G_j) = q$, $j = 1, \ldots, k$, and $V(f_1, \ldots, f_k) = V(G_1, \ldots, G_k)$. Define $\varphi: M \to \mathbf{R}$ by

$$\varphi([z]) = \ell g \left(\frac{\left(\sum_{j=0}^n |z_j|^2\right)^q}{\sum_{j=1}^k |G_j(z)|^2} \right)$$

where $[\ ] = \pi: \mathbf{C}^{n+1} - \{0\} \to \mathbf{C}P^n$ is the canonical projection and $z = (z_0, \ldots, z_n)$. It is easy to see that φ is well defined and real analytic on M. Moreover, since $M = \mathbf{C}P^n - V(G_1, \ldots, G_k)$, we have $\lim\limits_{p \to \infty} \varphi(p) = \lim\limits_{p \to V} \varphi(p) = +\infty$, where $V = V(G_1, \ldots, G_k)$. Let us prove that φ is $s.\ell - s.$. Fix $[z^0] = [z_0^0 : \ldots : z_n^0] \in M$. We can suppose that $z_0^0 \neq 0$, so that $[z^0] = [1 : x_1^0, \ldots, x_n^0]$, where $x_j^0 = z_j^0/z_0^0$. In the affine coordinate system $(x_1, \ldots, x_n) = [1 : x_1 : \ldots : x_n] \in \mathbf{C}^n$, φ can be written as $\varphi = q\varphi_1 - \varphi_2$, where $\varphi_1(x) = \ell g(1 + \sum\limits_{j=1}^n |x_j|^2)$ and $\varphi_2(x) = \sum\limits_{j=1}^k |g_j(x)|^2$, where $g_j(x) = G_j(1, x_1, \ldots, x_n)$. Therefore we have $H_\varphi = qH_{\varphi_1} - H_{\varphi_2}$. A direct computation shows that $H_{\varphi_1} = (a_{ij})_{1\leq i,j\leq n}$ and $H_{\varphi_2} = (b_{ij})_{1\leq i,j\leq n}$, where

$$a_{ii}(x) = \left(1 + \sum_{j\neq i} |x_j|^2\right) \Big/ \left(1 + \sum_{j=1}^{n} |x_j|^2\right)^2$$

$$a_{ij}(x) = -x_i\overline{x}_j \Big/ \left(1 + \sum_{j=1}^{n} |x_j|^2\right)^2$$

$$b_{ij} = \sum_{r<s} \overline{\Delta}^i_{r_s} \Delta^j_{r_s} \Big/ \left(\sum_{j=1}^{k} |g_j|^2\right)^2$$

where $\Delta^i_{r_s} = g_r \frac{\partial g_s}{\partial x_i} - g_s \frac{\partial g_r}{\partial x_i}$. Observe that the quadratic form associated to H_{φ_1} is

$$Q_1(x,w) = \sum_{i,j} \overline{w}_i a_{ij} w_j = \left(1 + \sum_{j=1}^{n} |x_i|^2\right)^{-4} \left[\sum_{i=1}^{n} |w_i|^2 + \sum_{i<j} |w_i x_j - w_j x_i|^2\right].$$

Hence it is positive definite. For some fixed $x \in \mathbf{C}^n - V$, $V = \{g_1 = \cdots = g_k = 0\}$, let $K(x) = \{w \in \mathbf{C}^n \mid \sum_{j=1}^{n} b_{ij}(x).w_j = 0$, for all $i = 1,\dots,n\}$.

Assertion: $\dim(K(x)) \geq n - k + 1$ for all $x \in \mathbf{C}^n - V$.

Proof: Since $x \in \mathbf{C}^n - V$, let us assume for instance that $g_1(x) \neq 0$. From now on we will omit the point x in the notation. Let S be the space of solutions of the linear system:

$$\sum_{j=1}^{n} \Delta^j_{1s} w_j = 0, \quad s = 2,\dots,k \tag{6}$$

Since in (6) we have $k-1$ equations, we have $\dim(S) \geq n - k + 1$. So it is enough to prove that $S \subset K(x)$. Observe that (6) is equivalent to

$$\sum_{j=1}^{n} \frac{\partial g_s}{\partial x_j} w_j = \frac{g_s}{g_1} \sum_{j=1}^{n} \frac{\partial g_1}{\partial x_j} w_j, \quad s = 2,\dots,k. \tag{7}$$

On the other hand (7) implies that if $r \neq s$ then

$$\sum_{j=1}^{n} \Delta_{rs}^{j} w_j = g_r \sum_{j=1}^{n} \frac{\partial g_s}{\partial x_j} w_j - g_s \sum_{j=1}^{n} \frac{\partial g_r}{\partial x_j} w_j = 0$$

Therefore, if $w \in S$, then $\sum_{j=1}^{n} \Delta_{rs}^{j} w_j = 0$ for all $r \neq s$. Hence, if $w \in S$, then

$$\begin{aligned}\sum_{j=1}^{n} b_{ij} w_j &= \left(\sum_{j=1}^{n} |g_j|^2\right)^{-2} \sum_{j=1}^{n} \sum_{r<s} \overline{\Delta}_{rs}^{i} \Delta_{rs}^{j} w_j \\ &= \left(\sum_{j=1}^{n} |g_j|^2\right)^{-2} \sum_{r<s} \overline{\Delta}_{rs}^{i} \left(\sum_{j=1}^{n} \Delta_{rs}^{j} w_j\right) = 0\end{aligned}$$

This proves the assertion.

Now if $w \in K(x) - \{0\}$, we have

$$\overline{w}^t H_\varphi w = qQ_1(x,w) - \sum_{i,j=1}^{n} \overline{w}_i b_{ij} w_j = qQ_1(x,w) > 0$$

This implies that H_φ has at least $n - k + 1$ positive eigenvalues. ∎

2.2.3 COROLLARY. *Let $k \leq n-1$ and $f_1, \ldots, f_k$ be homogeneous non constant polynomials on $\mathbf{C}^{n+1}$. Then any meromorphic ℓ-form defined in a neighborhood of $V(f_1, \ldots, f_k) \subset \mathbf{C}P^n$, can be extended to a meromorphic ℓ-form in $\mathbf{C}P^n$.*

2.3 Noether's lemma of second order:

Let $f, g: U \to \mathbf{C}$ be two analytic functions and $V = \{f = g = 0\}$, where $U \subset \mathbf{C}^n$ is an open set. If $W \subset V$, we say that f intersects g transversely outside W (briefly $f \pitchfork g$ out of W) if for any $z \in V - W$ we have $df(z) \wedge dg(z) \neq 0$. Classical Noether's lemma can be stated as follows:

2.3.1 NOETHER'S LEMMA. *Let f, g, U, V and W be as above. Suppose that:*

(a) W is an analytic subset of U, where $cod(W) \geq 3$.

(b) $\hat{H}^1(U - W, \mathcal{O}) = \{1\}$.

If $h: U \to \mathbf{C}$ is an analytic function such that $h|V \equiv 0$, then there are analytic functions $\alpha, \beta: U \to \mathbf{C}$ such that $h = \alpha.f + \beta.g$.

When $U = \mathbf{C}^n$, $n \geq 3$, f and g are homogeneous polynomials, $W = \{0\}$, and $f \pitchfork g$ out of $\{0\}$, then we have the following: if h is a homogeneous polynomial with $h|V \equiv 0$, then there are homogeneous polynomials α and β such that $h = \alpha.f + \beta.g$, where $dg(\alpha) + dg(f) = dg(\beta) + dg(g) = dg(h)$. In particular if $dg(h) < \min\{dg(f), dg(g)\}$, we must have $h \equiv 0$. This assertion follows from Noether's lemma because $\hat{H}^1(\mathbf{C}^n - \{0\}, \mathcal{O}) = \{1\}$ for $n \geq 3$ (see [C]). Here we are mainly interested in the case where the 1-jet of h is 0 along V.

2.3.2 Definition: Let $U \subset \mathbf{C}^n$ be an open set, $V \subset U$ be a codimension k smooth complex submanifold and $h: U \to \mathbf{C}$ be analytic. We say that the *ℓ-jet of h is zero along V if for any $z^0 = (z_1^0, \ldots, z_n^0) \in V$ there is a holomorphic coordinate system $x = (x_1, \ldots, x_n)$ defined in a neighborhood A of z^0 such that:*

(a) $V \cap A = \{x_1 = \cdots = x_k = 0\}$

(b) $h(x) = \displaystyle\sum_{|\sigma| = \ell + 1} a_\sigma(x) x_1^{\sigma_1} \ldots x_k^{\sigma_k}$

where in the above notation $\sigma = (\sigma_1, \ldots, \sigma_k)$, $|\sigma| = \sigma_1 + \cdots + \sigma_k$, and $a_\sigma: A \to \mathbf{C}$ is holomorphic for all σ such that $|\sigma| = \ell + 1$. We use the notation $j_V^\ell(h) = 0$ to say that the ℓ-jet of h is zero along V.

2.3.3 THEOREM. *Let $f, g: \mathbf{C}^n \to \mathbf{C}$ be homogeneous polynomials, $n \geq 3$, where $f \pitchfork g$ out of $0 \in \mathbf{C}^n$. Let $V = \{f = g = 0\}$ and $V^* = V - \{0\}$. If h is a homogeneous polynomial with $j_{V^*}^1(h) = 0$, then there are homogeneous polynomials a, b and c such that $h = af^2 + bf.g + cg^2$, where $dg(h) = dg(a) + 2dg(f) = dg(b) + dg(f) + dg(g) = dg(c) + 2dg(g)$.*

Proof: Since $f \pitchfork g$ out of 0, for any $z^0 \in V^*$ there is a holomorphic coordinate system $(x_1, \ldots, x_n)$ defined in a neighborhood U_{z^0} such that:

(a) $f(x) = x_1$, $g(x) = x_2$ $\forall x \in U_{z^0}$, so that $U_{z^0} \cap V = \{x_1 = x_2 = 0\}$.
(b) $h(x) = \alpha(x)x_1^2 + \beta(x)x_1x_2 + \gamma(x)x_2^2$, $\forall x \in U_{z^0}$, where $\alpha, \beta, \gamma \in \mathcal{O}(U_{z^0})$.

It follows that it is possible to find a convering $(U_i)_{i\in I}$ of $\mathbf{C}^n - \{0\}$ by open sets and three collections $\{\alpha_i\}_{i\in I}$, $\{\beta_i\}_{i\in I}$, $\{\gamma_i\}_{i\in I}$, where $\alpha_i, \beta_i, \gamma_i \in \mathcal{O}(U_i)$ and $h|U_i = (\alpha_i f^2 + \beta_i fg + \gamma_i g^2)|U_i$. When $U_i \cap U_j \neq \phi$ we have $\alpha_{ij}f^2 + \beta_{ij}fg + \gamma_{ij}g^2 \equiv 0$, where $\alpha_{ij} = \alpha_j - \alpha_i$, $\beta_{ij} = \beta_j - \beta_i$ and $\gamma_{ij} = \gamma_j - \gamma_i \in \mathcal{O}(U_i \cap U_j)$. Observe that this relation implies that $g|U_i \cap U_j$ divides $\alpha_{ij}f^2|U_i \cap U_j$. This fact together with $f \pitchfork g$ out of 0 implies that $\alpha_{ij} = \delta_{ij}g$, for some $\delta_{ij} \in \mathcal{O}(U_i \cap U_j)$. Now, if $U_i \cap U_j \cap U_k \neq \phi$ we have $(\delta_{ij} + \delta_{jk} + \delta_{ki})g = \alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0$, and so $\delta_{ij} + \delta_{jk} + \delta_{ki} = 0$. It follows from Cartan's solution of Cousin's problem that there exists a collection $(\delta_i)_{i\in I}$, $\delta_i \in \mathcal{O}(U_i)$, such that $\delta_{ij} = \delta_j - \delta_i$. Therefore we can define a function $\alpha \in \mathcal{O}(\mathbf{C}^n - \{0\})$ by

$$\alpha|U_i = \alpha_i - \delta_i g$$

Similarly, there are a collection $(\varepsilon_i)_{i\in I}$, $\varepsilon_i \in \mathcal{O}(U_i)$, and $\gamma \in \mathcal{O}(\mathbf{C}^n - \{0\})$ such that $\gamma|U_i = \gamma_i - \varepsilon_i f$. Let $h_1 = h - \alpha f^2 - \gamma g^2$. It is clear that:

$$h_1|U_i = \beta_i fg + \delta_i g f^2 + \varepsilon_i f g^2 = (\beta_i + \delta_i f + \varepsilon_i g)fg = \varphi_i fg.$$

If $U_i \cap U_j \neq \phi$, we have $(\varphi_j - \varphi_i)fg \equiv 0$, so that $\varphi_i = \varphi_j$. Therefore there exists $\beta \in \mathcal{O}(\mathbf{C}^n - \{0\})$ such that $\beta|U_i = \varphi_i$, $i \in I$, and we have $h = \alpha f^2 + \beta fg + \gamma g^2$. Now, from Hartog's theorem, α, β and γ can be extended to holomorphic functions on $\mathbf{C}^n$, which we call α, β and γ also. Let $\alpha = \sum_{j\geq 0} \alpha_j$, $\beta = \sum_{j\geq 0} \beta_j$ and $\gamma = \sum_{j\geq 0} \gamma_j$, where α_j, β_j and γ_j are homogeneous of degree j. It is clear that if j_1, j_2 and j_3 are such that $j_1 + 2dg(f) = j_2 + dg(f) + dg(g) = j_3 + 2dg(g)$, then we have $h = \alpha_{j_1} f^2 + \alpha_{j_2} fg + \alpha_{j_3} g^2$. So we can take $a = \alpha_{j_1}$, $b = \alpha_{j_2}$ and $c = \alpha_{j_3}$. ∎

Remark: We will need the above result only to prove (a) of theorem A.

3. Proofs

3.1 Proof of Theorem B:

Let $\mathcal{F}$ be a foliation on $\mathbf{C}P^n$ which satisfies hypothesis (a), (b) and (c) of Theorem B. The idea is the following: let w be a homogeneous integrable 1-form in $\mathbf{C}^{n+1}$ which represents $\mathcal{F}$ as in §1.3. We will prove that there exists a homogeneous polynomial f such that $d(w/f) = 0$ and $dg(f) = dg(w) + 1$. After that we will use some results containned in **[C-M]** to conclude the proof.

Let $N = N_1 \cup \cdots \cup N_r$ be the decomposition of N in irreducible components. From the hypothesis we have that for each $i = 1, \ldots, r$, $N_i \cap K(\mathcal{F})$ is open and dense in N_i. Since $N_i - K(\mathcal{F})$ is algebraic we have that $\operatorname{cod}_{N_i}(N_i - K(\mathcal{F})) \geq 1$, and so, $N_i \cap K(\mathcal{F})$ is connected.

Observe now that hypothesis (b) and Poincaré's linearization Theorem imply that $N_i \cap K(\mathcal{F})$ is of linearizable transversal type. This means that there exist λ_1^i and λ_2^i with $\lambda_2^i/\lambda_1^i \notin \mathbf{R}$ with the following property: (1) $\forall p \in N_i \cap K(\mathcal{F})$ there exists a local chart $(x, y, z): U \to \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-2}$, such that $U \cap N_i \cap K(\mathcal{F}) = \{(x, y, z) | x = y = 0\}$ and $\mathcal{F}|U$ is the foliation defined by the 1-form $w_U = \lambda_1^i x dy - \lambda_2^i y dx$. If we divide w_U by $\lambda_1^i xy$ we get the form $\alpha_U = \frac{dy}{y} - a\frac{dx}{x}$, where $a = \lambda_2^i/\lambda_1^i \notin \mathbf{R}$. Observe that α_U is closed, so that w_U has an integrating factor.

Lemma 2. *Let $i \in \{1, \ldots, r\}$ be fixed. There exists a neighborhood A_i of N_i in $\mathbf{C}P^n$, and a meromorphic closed 1-form η_i on A_i such that if P_i is the divisor of poles of η_i, then $\mathcal{F}|(A_i - P_i)$ is represented by $\eta_i|(A_i - P_i)$.*

Proof: It follows from the considerations before Lemma 2 and from hypothesis (c) that it is possible to find a convering of N_i by open sets of $\mathbf{C}P^n$, $(U_j)_{j \in J}$ and a collection $(\alpha_j)_{j \in J}$ such that:

(i) If $j, k, \ell \in J$ are such that $U_j \cap U_k \cap U_\ell \neq \phi$, then $U_j, U_j \cap U_k$ and $U_j \cap U_k \cap U_\ell$ are simply connected. Moreover by using the local structure of analytic sets, we can suppose also that $U_j \cap N_i$, $U_j \cap U_k \cap N_i$ and $U_j \cap U_k \cap U_\ell \cap N_i$ are simply connected.

(ii) $J = J_1 \cup J_2$ where $U_j \cap (N_i - K(\mathcal{F})) = \phi$ if $j \in J_1$, $\bigcup_{j \in J_1} U_j \supset N_i \cap K(\mathcal{F})$, $\bigcup_{j \in J_2} U_j \supset N_i - K(\mathcal{F})$ and if $j, k \in J_2$ is such that $U_j \cap U_k \neq \phi$ then there is $\ell \in J_1$ with $U_\ell \subset U_j \cap U_k$.

(iii) For each $j \in J$, α_j is a *closed* meromorphic 1-form on U_j, such that $\alpha_j = w_j/f_j$, where $f_j \in \mathcal{O}(U_j)$ and w_j is a holomorphic 1-form which represents $\mathcal{F}|U_j$. We can assume also that the singular set of w_j is of codimension ≥ 2.

(iv) For each $j \in J_1$, there is a local chart $(x_j, y_j, z_j)\colon U_j \to \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-2}$ such that $\alpha_j = \frac{dy_j}{y_j} - a\frac{dx_j}{x_j}$, where $a \notin \mathbf{R}$ is as before. In this case $w_j = x_j dy_j - a y_j dx_j$ and $f_j = x_j y_j$.

(v) If $j, k \in J$ is such that $U_j \cap U_k \neq \phi$, then there exists a meromorphic function g_{jk}, defined on $U_j \cap U_k$, with $\alpha_j = g_{jk}\alpha_k$. This function is obtainned as follows: since w_j and w_k define the same foliation on $U_j \cap U_k$ and their singular sets have codimension ≥ 2, we can write $w_j = h_{jk} w_k$, where $h_{jk} \in \mathcal{O}^*(U_j \cap U_k)$. Therefore $g_{jk} = f_k h_{jk}/f_j$ as the reader can verify easily. Moreover, the collection $(g_{jk})_{U_j \cap U_k} \neq \phi$ satisfies the cocycle condition $g_{jk} g_{k\ell} g_{\ell j} \equiv 1$ on $U_j \cap U_k \cap U_\ell$ if this set is non empty.

Assertion: If $j, k \in J$ are such that $U_j \cap U_k \neq \phi$, then g_{jk} is a constant.

Proof: Observe first that $\alpha_j = g_{jk}\alpha$ implies that $dg_{jk} \wedge \alpha_k = 0$ because α_j and α_k are closed.

1st case: $k \in J_1$, so that $\alpha_k = \frac{dy_k}{y_k} - a\frac{dx_k}{x_k}$. Let $z_k = (z^1, \ldots, z^{n-2})$ and $g_{jk} = g$. Relation $dg \wedge \alpha_k \equiv 0$ implies that outside the set of poles of g we have $\frac{\partial g}{\partial z^r} \equiv 0$, $1 \leq r \leq n-2$, and $x_k \frac{\partial g}{\partial x_k} + a y_k \frac{\partial g}{\partial y_k} \equiv 0$. This implies already that g does not depend on z_k. Therefore $g = g(x_k, y_k)$ and we can suppose that g is defined in a neighborhood of $(0,0) \in \mathbf{C}^2$. From now on we will omit the indexes k. Let P be the set of poles of g. Suppose first that $P \supset \{x = 0\}$. In this case, it is not difficult to see that there are a disk $\Delta \subset \{y = 0\}$ and an annulus $A \subset \{x = 0\}$ such that g has no poles on $W = (\Delta - \{0\}) \times A$.

Consider the Laurent development of g in W:

$$g(x,y) = \sum_{m,n=-\infty}^{\infty} b_{mn}x^m y^n, \quad b_{mn} \in \mathbf{C}$$

From $x\frac{\partial g}{\partial x} + ay\frac{\partial g}{\partial y} \equiv 0$ we get $\sum\limits_{m,n=-\infty}^{\infty} b_{mn}(m+na)x^m y^n = 0$, which implies that $b_{mn}(m+na) = 0\ \forall m,n \in \mathbf{Z}$. Since $a \notin \mathbf{Q}$, this implies that $b_{mn} = 0$ if $(m,n) \neq (0,0)$. Therefore $g|W$ is constant, which implies that g is constant.

If $\{x = 0\} \not\subset P$, then it is possible to find a disc $\Delta \subset \{y = 0\}$ and an annulus $A \subset \{x = 0\}$ such that g is holomorphic on $\Delta \times A$. In this case g admits a Laurent development on $\Delta \times A$ and so g is constant by the same argument as before.

2^{nd} case: $j,k \in J_2$. In this case let $\ell \in J_1$ be such that $U_\ell \subset U_j \cap U_k$. Observe that $g_{jk} \cdot g_{k\ell} \cdot g_{\ell j} \equiv 1$ on $U_\ell = U_\ell \cap U_j \cap U_k$. By the firs case $g_{k\ell}$ and $g_{\ell j}$ are constants. Hence g_{jk} is constant on U_ℓ, and so on $U_j \cap U_k$. This proves the assertion.

Now, if $j,k \in J_1$ and $U_j \cap U_k \neq \phi$ then $g_{jk} = 1$. In fact, on $U_j \cap U_k$ we have:

$$\alpha_j = \frac{dy_j}{y_j} - a\frac{dx_j}{x_j} = g_{jk}\left(\frac{dy_k}{y_k} - a\frac{dx_k}{x_k}\right) = g_{jk}\alpha_k.$$

From (i), there is $p_0 \in U_j \cap U_k \cap N_i$. It is clear that $p_0 = (0,0,z_j^0)$ in the chart (x_j,y_j,z_j) and $p_0 = (0,0,z_k^0)$ in the chart (x_k,y_k,z_k). By analyzing the sets of poles of α_j and α_k we get that either $\{y_j = 0\} \cap U_k = \{y_k = 0\} \cap U_j$ and $\{x_j = 0\} \cap U_k = \{x_k = 0\} \cap U_j$, or $\{y_j = 0\} \cap U_k = \{x_k = 0\} \cap U_j$ and $\{y_k = 0\} \cap U_j = \{x_j = 0\} \cap U_k$.

On the other hand, by comparing the residues of α_j and α_k around $\{x_j = 0\}$, $\{y_j = 0\}$, $\{x_k = 0\}$ and $\{y_k = 0\}$, we obtain in the first case that $g_{jk} = 1$ and in the second case that $1 = -ag_{jk}$ and $-a = g_{jk}$. Well, these last relations imply that $a^2 = 1$, which is not possible. Therefore $g_{jk} = 1$. So we have proved that if $j,k \in J_1$ is such that $U_j \cap U_k \neq \phi$ then $\alpha_j = \alpha_k$ on $U_j \cap U_k$.

It follows that we can define a closed meromorphic 1-form $\tilde{\eta}_i$ on $B_i = \bigcup_{j\in J_1} U_j$ by $\tilde{\eta}_i|U_j = \alpha_j$.

Let us prove that $\tilde{\eta}_i$ can be extended to $A_i = \bigcup_{j\in J} U_j$. Let η_i be defined on A_i by:

(vi) $\eta_i|B_i = \tilde{\eta}_i$

(vii) If $j \in J_2$ then $U_j \cap B_i \neq \phi$. Therefore there is $k \in J_1$ such that $U_k \cap U_j \neq \phi$ (because $B_i \supset N_i \cap K(\mathcal{F})$). We put $\eta_i|U_j = g_{kj}\alpha_j$. Observe that this definition is natural because $\eta_i|U_j \cap U_k = \tilde{\eta}_i|U_j \cap U_k = \alpha_k|U_j \cap U_k = g_{kj}\alpha_j|U_j \cap U_k$.

Let us prove that η_i is well defined. We can consider $g = (g_{jk})_{U_j\cap U_k\neq\phi}$ as a cocycle in $\check{H}^1(\mathcal{U}, \mathbf{C}^*)$, where $\mathcal{U} = (U_j \cap N_i)_{j\in J}$. It is not difficult to see that if $\mathcal{G}$ is trivial in $\check{H}^1(\mathcal{U}, \mathbf{C}^*)$ then η_i is well defined. Let $\mathcal{U}_1 = (U_j \cap N_i)_{j\in J_1}$ and $\mathcal{G}_1 = \{g_{jk} \in \mathcal{G} \mid j,k \in J_1\}$. Then $g_{jk} = 1$ for any $g_{jk} \in \mathcal{G}_1$ and so $\mathcal{G}_1$ is trivial in $\check{H}^1(\mathcal{U}_1, \mathbf{C}^*)$. On the other hand, since $\mathrm{cod}_{N_i}(N_i - K(\mathcal{F})) \geq 1$ it follows that any closed path $\gamma: [0,1] \to N_i$ with end points in $N_i \cap K(\mathcal{F})$ is homotopic, with fixed end points, to a path $\tilde{\gamma}: [0,1] \to N_i \cap K(\mathcal{F})$. It follows that the monodromy of a closed path as above (with respect to $\mathcal{G}$) is trivial. This implies that $\mathcal{G}$ is trivial, as the reader can check by himself. This ends the proof of Lemma 2. ∎

From Lemma 2 we get for each N_i , $i = 1,\ldots,r$, a neighborhood A_i and a closed meromorphic 1-form η_i on A_i such that, if P_i is the divisor of poles of η_i, then η_i represents $\mathcal{F}$ on $A_i - P_i$. Furthermore if C is a connected component of $A_i \cap A_j$ then $\eta_i|C = \lambda_{ij}(C)\eta_j|C$, where $\lambda_{ij}(C)$ is a constant in $\mathbf{C}^*$. Let $U_0 = \{[1{:}z_1{:}\ldots{:}z_n] \mid (z_1,\ldots,z_n) \in \mathbf{C}^n\}$ and w_0 be a polynomial integrable 1-form which represents $\mathcal{F}$ on U_0. Then w_0 can be extended to $\mathbf{C}P^n$ as a meromorphic 1-form with poles in $L_0 = \{[z] \in \mathbf{C}^n \mid z_0 = 0\}$. Since $w_0|A_i$ and η_i represent the same foliation, we have that $w_0|A_i = f_i\eta_i$, where f_i is a meromorphic function on A_i. On the other hand, if C is a connected component of $A_i \cap A_j$ then we have

$$f_i \lambda_{ij}(C) \eta_j | C = f_i \eta_i | C = w_0 | C = f_j \eta_j | C$$

which implies that $f_j|C = \lambda_{ij}(C) f_i|C$. Since $\lambda_{ij}(C)$ is a constant, it follows that $\frac{df_j}{f_j}|C = \frac{df_i}{f_i}|C$. This implies that we can define a closed 1-form θ on $A = \bigcup_{i=1}^{r} A_i$ by $\theta|A_i = \frac{df_i}{f_i}$. Now, by corollary 2.2.3 of §2.2, θ can be extended to $\mathbf{C}P^n$, because A is a neighborhood of $N = \{f = g = 0\}$ and $2 \leq n-1$. We call θ this extension. Let P be the divisor of poles of θ. Fix $p_0 \in \mathbf{C}P^n - P$ and for each path $\gamma: [0,1] \to \mathbf{C}P^n - P$ with $\gamma(0) = p_0$, put

$$I(\gamma) = \exp\left[\int_\gamma \theta\right]$$

We will prove now that if γ is a closed path then $I(\gamma) = 1$. It is easy to see that this will imply that we can define a holomorphic function $F: \mathbf{C}P^n - P \to \mathbf{C}$ by $F(p) = I(\gamma)$, where $\gamma(0) = p_0$ and $\gamma(1) = p$.

Let γ be a closed path. Since $\mathbf{C}P^n$ is simply connected, γ is homologous in $\mathbf{C}P^n - P$ to $\gamma_1 + \cdots + \gamma_k$ where each γ_j is a small cycle envolving an irreducible component of P. So it is sufficient to prove that if γ is a small cycle envolving an irreducible component of P, say Q, then $\exp[\int_\gamma \theta] = 1$. Now, since $N = \{f = g = 0\}$, it follows from Bézout's Theorem that $Q \cap N_i \neq \phi$ for some i. It follows that we can deform γ keeping it closed along the deformation, to a small cycle $\hat{\gamma}$ envolving Q and containned in A_i. Since $\theta|A_i = \frac{df_i}{f_i}$, this implies that $\int_\gamma \theta = \int_{\hat{\gamma}} \frac{df_i}{f_i} = 2\pi i m$, $m \in \mathbf{Z}$. Hence $I(\gamma) = 1$.

The above argument implies also that $F|A_i = c_i f_i$, where c_i is a constant. Since $\eta_i = w_0/f_i$ is closed, we get that $\eta = w_0/F$ is closed. Hence the first part of the theorem is proved. It follows also that if w is a homogeneous integrable 1-form on $\mathbf{C}^{n+1}$ which induces $\mathcal{F}$ then w has a meromorphic integrating factor say $F = g/h$. Let us prove that g and h are homogeneous polynomials such that $dg(g) - dg(h) = dg(w) + 1$.

In fact, let $dg(w) = k$. Since F is an integrating factor, we have $dF \wedge w = Fdw$. On the other hand, $i_R(w) = 0$ implies that $(k+1)w = L_R(w) = i_R dw + d(i_R w) = i_R dw$, where L_R is the Lie derivative. Therefore we get

$$i_R(dF)w = Fi_R(dw) = (k+1)Fw \Rightarrow i_R(dF) = (k+1)F$$

Integrating $i_R(dF) = (k+1)F$, we get that $F(tz) = t^{k+1}F(z)$, if z is not a pole of F. Hence g and h are homogeneous and $dg(g) - dg(h) = k+1$. We can suppose that g and h do not have common factors. Let $g = g_1^{k_1} \dots g_m^{k_m}$ be the decomposition of g in irreducible factors, where $k_1, \dots, k_m \geq 1$.

LEMMA 3. *There are $\lambda_1, \dots, \lambda_m \in \mathbf{C}$ and a homogeneous polynomial φ such that*

$$\frac{hw}{g} = \sum_{j=1}^{m} \lambda_j \frac{dg_j}{g_j} + d\left(\frac{\varphi}{\psi}\right) \tag{8}$$

where $\psi = g_1^{\ell_1} \dots g_m^{\ell_m}$, $0 \leq \ell_j \leq k_j - 1$, $\sum_{j=1}^{m} \lambda_j dg(g_j) = 0$, φ and ψ have no common factors and $dg(\varphi) = dg(\psi)$.

The proof of the above result can be found in [C-M].

Now let us assume hypothesis (d) and (e) of Theorem B and prove that $\mathcal{F}$ is of logarithmic type. First of all, if we multiply the right hand side of (8) by $g_1^{\ell_1+1} \dots g_m^{\ell_m+1}$ we get the form

$$\eta = g_1 \dots g_m d\varphi - \varphi g_i \dots g_m \sum_{j=1}^{m} \ell_j \frac{dg_j}{g_j} + g_1^{\ell_1+1} \dots g_m^{\ell_m+1} \sum_{j=1}^{m} \lambda_j \frac{dg_j}{g_j} \tag{9}$$

which is holomorphic in $\mathbf{C}^{n+1}$. Observe that if $\ell_i = \lambda_i = 0$ for some $i \in \{1, \dots, m\}$, then g_i is a factor of η and g_i plays no essential role. Hence we can suppose that either $\ell_i \neq 0$ or $\lambda_i \neq 0$ for all $i = 1, \dots, m$. With this condition $G_i = \{[z] \in \mathbf{C}P^n \mid g_i(z) = 0\}$ is invariant under $\mathcal{F}$, which implies that if $i \neq j$ then $G_{ij} = \{[z] \mid g_i(z) = g_j(z) = 0\} \subset S(\mathcal{F})$. Since G_{ij} has codimension 2, hypothesis (d) implies that the set

$$A = K(\mathcal{F}) \cap G_{ij} \cap \{[z] \in \mathbf{C}P^n \mid dg_i(z) \wedge dg_j(z) \neq 0, \quad \varphi(z) \neq 0 \text{ and } g_r(z) \neq 0 \text{ for } r \neq \imath$$

is open and dense in G_{ij}. In order to simplify the notations we will put $i = 1$ and $j = 2$. Now we will prove that:

(i) If $\ell_1 > 0$ and $\ell_2 = 0$ then the linear part of the transversal type of $\mathcal{F}$ at G_{12} is degenerated.

(ii) If $\ell_1 = \ell_2 = 0$ then the quotient of the eigenvalues of the linear part of the transversal type of $\mathcal{F}$ at G_{12} is $-\lambda_2/\lambda_1$ (or $-\lambda_1/\lambda_2$)

(iii) If $\ell_1 > 0$, $\ell_2 > 0$ then the above quotient is $-\ell_2/\ell_1$ (or $-\ell_1/\ell_2$).

In fact, since A is open and dense in G_{12}, let $p \in \mathbf{C}^{n+1} - \{0\}$ be such that $[p] \in A$. We know that $\varphi(p) \neq 0$ and $g_r(p) \neq 0$ for $r > 2$. This implies that the form $\sum\limits_{j \geq 3} \lambda_j \frac{dg_j}{g_j}$ has a holomorphic primitive, say h_1, defined in a simply connected neighborhood U of p. Therefore we can write:

$$\frac{hw}{g}\bigg|_U = \lambda_1 \frac{dg_1}{g_1} + \lambda_2 \frac{dg_2}{g_2} + dh_1 + d\left(\frac{h_2}{g_1^{\ell_1} g_2^{\ell_2}}\right) = \mu \tag{10}$$

where $h_2 = \varphi / g_3^{\ell_3} \dots g_m^{\ell_m}$, $h_2(p) \neq 0$.

Proof of (i): Since $\ell_2 = 0$ we have $\lambda_2 \neq 0$. Let α be a branch of h_2^{-1/ℓ_1} and β be a branch of $h_2^{\lambda_1/\ell_1\lambda_2} \exp(h_1/\lambda_2)$ defined in U. These branches can be defined because $h_2(p) \neq 0$. Observe that $d(\alpha g_1)(p) \wedge d(\beta g_2)(p) \neq 0$, so that there is a local coordinate system $(x, y, z) \in \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-1}$ around p, such that $x = \alpha g_1$ and $y = \beta g_2$. It is easy to see that in this coordinate system we have

$$\mu = \lambda_1 \frac{dx}{x} + \lambda_2 \frac{dy}{y} + d(x^{-\ell_1}) = \lambda_1 \frac{dx}{x} + \lambda_2 \frac{dy}{y} - \ell_1 \frac{dx}{x^{\ell_1+1}}$$

If we multiply μ be $x^{\ell_1+1}.y$ we get $(-\ell_1 y + \lambda_1 x^{\ell_1} y)dx + \lambda_2 x^{\ell_1+1} dy$. Therefore the transversal type of $\mathcal{F}$ in G_{12} is given by the vector field $\lambda_2 x^{\ell_1+1} \partial/\partial x + (\ell_1 y - \lambda_1 x^{\ell_1} y)\partial/\partial y$. Since $\ell_1 > 0$, this proves (i).

Proof of (ii): Since $\ell_1 = \ell_2 = 0$ we have $\lambda_1 \neq 0 \neq \lambda_2$ and

$$\mu = \lambda_1 \frac{dg_1}{g_1} + \lambda_2 \frac{dg_2}{g_2} + dh_3$$

where $h_3 = h_1 + h_2$. Let $\alpha = \exp(h_3/\lambda_2)$. We have $dg_1(p) \wedge d(\alpha g_2)(p) \neq 0$, therefore there exists a local coordinate system $(x, y, z) \in \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-1}$ around p, such that $g_1 = x$ and $\alpha g_2 = y$. In this coordinate system we have $\mu = \lambda_1 \frac{dx}{x} + \lambda_2 \frac{dy}{y}$, so that $xy\mu = \lambda_1 y dx + \lambda_2 x dy$, therefore the quotient of the eigenvalues of the normal type is $-\lambda_2/\lambda_1$ (or $-\lambda_1/\lambda_2$).

Proof (iii): In this case we can write (10) as

$$\mu = \lambda_1 \frac{dg_1}{g_1} + \lambda_2 \frac{dg_2}{g_2} + d\left(\frac{h_3}{g_1^{\ell_1} g_2^{\ell_2}}\right)$$

where $h_3 = h_2 + h_1 g_1^{\ell_1} g_2^{\ell_2}$. Since $dg_1(p) \wedge dg_2(p) \neq 0$, there exists a coordinate system $(x, y, z) \in \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-1}$ around p such that $x = g_1$ and $y = g_2$. If we multiply μ by $x^{\ell_1+1}.y^{\ell_2+1}.h_3^{-1}$ we get

$$h_3^{-1} x^{\ell_1+1}.y^{\ell_2+1}.\mu = -\ell_1 y dx - \ell_2 x dy + \lambda_1 h_3^{-1} x^{\ell_1} y^{\ell_2+1} dx + \lambda_2 h_3^{-1} x^{\ell_1+1} y^{\ell_2} dy$$
$$+ h_3^{-1} x^{\ell_1+1} y^{\ell_2+1} dh.$$

It is not difficult to see that this implies (iii).

Now from (i) and hypothesis (e) it follows that either $\ell_1 = \cdots = \ell_m = 0$ or $\ell_1 \ldots \ell_m > 0$. In fact, if this is not the case, then there are $i \neq j$ such that $\ell_i > 0$ and $\ell_j = 0$, which cannot happen by (i). If $\ell_1 = \cdots = \ell_m = 0$ then $\mathcal{F}$ is logarithmic and we are done. Let us suppose that $\ell_1 \ldots \ell_m > 0$. In this case, it follows from (iii) that for any $i \neq j$ the quotient of the eigenvalues of the linear part of the normal type of $\mathcal{F}$ at G_{ij} is rational. Let us prove that this case cannot occur.

In fact, since in $N \cap K(\mathcal{F})$ the quotient of the eigenvalues of the normal type is not real, then in the above situation, there exists $p \in \mathbf{C}^{n+1} - \{0\}$ such that $[p] \in K(\mathcal{F}) \cap N - \bigcup_{i \neq j} G_{ij}$. In this case $p \in K(\pi^*(\mathcal{F}))$ and so there exists a coordinate system $(x, y, z) \in \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-1}$ around p such that $\pi^*(\mathcal{F})$ is defined by the form $\theta = xdy + aydx$, where $a \notin \mathbf{R}$. On the other hand let $\frac{hw}{g}$ be as in (8). Its set of poles is $P = \bigcup\limits_i \{g_i = 0\}$, it is closed and represents $\pi^*(\mathcal{F})$ outside P. Therefore there exists a meromorphic function f_1 on U such that

$$\frac{hw}{g}\bigg|_U = f_1(xdy + aydx) \Rightarrow 0 = d(f_1(xdy + aydx)) = d(xyf_1) \wedge \left(\frac{dy}{y} + a\frac{dx}{x}\right)$$

As we have seen in the proof of Lemma 2, the last relation implies $xyf_1 = c$, c a constant. This implies that $P \cap U = \{x = y = 0\}$. Therefore there are $i \neq j$ such that $\{g_i = 0\} \cap U = \{x = 0\}$ and $\{g_j = 0\} \cap U = \{y = 0\}$, and so $[p] \in G_{ij}$, a contradiction. This completes the proof of Theorem B. ∎

3.2 Proof of Theorem A:

We will use Theorem 1.2.4 of §1 (cf. **[G.M.-L.N.]**). We need some preliminary results.

LEMMA 4. *Let $V \xrightarrow{P} M$ be a holomorphic vector bundle with fiber $\mathbf{C}^2$, where M is compact. Assume that $V = E_1 \oplus E_2 = F_1 \oplus F_2$, where E_1, E_2, F_1 and F_2 are holomorphic line bundles, such that $c(E_1) \neq 0$ and $qc(E_1) = pc(E_2)$, where $p, q \in \mathbf{Z}$, $0 \leq q \leq |p|$ (c = first chern class). Then:*

(a) If $p \neq q$ then $F_i = E_j$ for some $i, j \in \{1, 2\}$. Moreover, if we assume $i = j = 1$ then $c(F_2) = c(E_2)$.

(b) If $p = q$ (i.e. $c(E_1) = c(E_2)$) then $c(F_i) = c(E_1)$ for $i = 1, 2$.

Note: As in 1.2.4, we denote by $c(\cdot)$ the first Chern class considered as an element of $H^2(M, \mathbf{C})$.

Proof: Let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be a covering of M by trivializing open sets of the E_i's and F_i's, where $U_\alpha \cap U_\beta$ and $U_\alpha \cap U_\beta \cap U_\gamma$ are simply connected if they are not empty. For each $\alpha \in A$ and $i = 1, 2$ let $e^i_\alpha: U_\alpha \to P^{-1}(U_\alpha)$ and $f^i_\alpha: U_\alpha \to P^{-1}(U_\alpha)$ be holomorphic local sections such that $e^i_\alpha(p) \in E_i(p) - \{0\}$

and $f^i_\alpha(p) \in F_i(p)-\{0\}, p \in U_\alpha$, where $E_i(p)$ and $F_i(p)$ denote the fibers of E_i and F_i over p. It follows that for all $p \in U_\alpha$, $\{e^1_\alpha(p), e^2_\alpha(p)\}$ and $\{f^1_\alpha(p), f^2_\alpha(p)\}$ are basis of $P^{-1}(p)$. Therefore there is a matrix $A_\alpha = (a^{ij}_\alpha)$ such that $a^{ij}_\alpha \in \mathcal{O}(U_\alpha)$, $\Delta_\alpha = \det(A_\alpha) \in \mathcal{O}^*(U_\alpha)$ and $e^i_\alpha = a^{i1}_\alpha f^1_\alpha + a^{i2}_\alpha f^2_\alpha$ on U_α. On the other hand there are collections $\{g^i_{\alpha\beta}\}_{U_\alpha \cap U_\beta \neq \phi}$ and $\{h^i_{\alpha\beta}\}_{U_\alpha \cap U_\beta \neq \phi}$, $i = 1,2$, where $g^i_{\alpha\beta}$, $h^i_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and $e^i_\alpha = g^i_{\alpha\beta} e^i_\beta$, $g^i_\alpha = h^i_{\alpha\beta} f^i_\beta$ on $U_\alpha \cap U_\beta \neq \phi$ for $i = 1,2$. These collections are in fact cocycles (i.e. if $U_\alpha \cap U_\beta \cap U_\gamma \neq \phi$ then $g^i_{\alpha\beta}.g^i_{\beta\gamma}.g^i_{\gamma\alpha} = 1$ and $h^i_{\alpha\beta}.h^i_{\beta\gamma}.h^i_{\gamma\alpha} = 1$) and $c(E_i)$ (resp. $c(F_i)$) can be represented in $\check{H}^2(\mathcal{U}, \mathbf{Z})$ by the 2-cocycle

(11)
$$\begin{cases} m^i_{\alpha\beta\gamma} = \frac{1}{2\pi\sqrt{-1}}(\ell_g(g^i_{\alpha\beta}) + \ell_g(g^i_{\beta\gamma}) + \ell_g(g^i_{\gamma\alpha})) & \\ & U_\alpha \cap U_\beta \cap U_\gamma \neq \phi \\ \text{resp. } n^i_{\alpha\beta\gamma} = \frac{1}{2\pi\sqrt{-1}}(\ell_g(h^i_{\alpha\beta}) + \ell_g(h^i_{\beta\gamma}) + \ell_g(h^i_{\gamma\alpha})) & \end{cases}$$

where the ℓg's are branches of the logarithm arbitrarily chosen.

Now, if we set $G_{\alpha\beta} = \begin{pmatrix} g^1_{\alpha\beta} & 0 \\ 0 & g^2_{\alpha\beta} \end{pmatrix}$ and $H_{\alpha\beta} = \begin{pmatrix} h^1_{\alpha\beta} & 0 \\ 0 & h^2_{\alpha\beta} \end{pmatrix}$, then it is not difficult to see that on $U_\alpha \cap U_\beta \neq \phi$ we have $G_{\alpha\beta} A_\beta = A_\beta H_{\alpha\beta}$ or equivalently

$$(12) \qquad \begin{cases} \text{(i) } g^1_{\alpha\beta} a^{11}_\beta = a^{11}_\alpha h^1_{\alpha\beta} \\ \text{(ii) } g^1_{\alpha\beta} a^{12}_\beta = a^{12}_\alpha h^2_{\alpha\beta} \\ \text{(iii) } g^2_{\alpha\beta} a^{21}_\beta = a^{21}_\alpha h^1_{\alpha\beta} \\ \text{(iv) } g^2_{\alpha\beta} a^{22}_\beta = a^{22}_\alpha h^2_{\alpha\beta}. \end{cases}$$

For fixed $\alpha \in A$ let $f_\alpha = a^{11}_\alpha a^{12}_\alpha a^{21}_\alpha a^{22}_\alpha / \Delta^2_\alpha$, where $\Delta_\alpha = \det(A_\alpha)$. Let us prove that there is $f \in \mathcal{O}(M)$ such that $f|U_\alpha = f_\alpha$. In fact, if we take the product of the relations $(i) \ldots (iv)$ we get

$$(13) \qquad (g^1_{\alpha\beta} g^2_{\alpha\beta})^2 a^{11}_\beta a^{12}_\beta a^{21}_\beta a^{22}_\beta = a^{11}_\alpha a^{12}_\alpha a^{21}_\alpha a^{22}_\alpha (h^1_{\alpha\beta} h^2_{\alpha\beta})^2$$

On the other hand the relation $G_{\alpha\beta} A_\beta = A_\alpha H_{\alpha\beta}$ implies that $g^1_{\alpha\beta} g^2_{\alpha\beta} \Delta_\beta = h^1_{\alpha\beta} h^2_{\alpha\beta} \Delta_\beta$, and so $(g^1_{\alpha\beta} g^2_{\alpha\beta})^2 / (h^1_{\alpha\beta} h^2_{\alpha\beta})^2 = \Delta^2_\alpha / \Delta^2_\beta$. This relation together with (13) implies that $f_\alpha | U_\alpha \cap U_\beta = f_\beta | U_\alpha \cap U_\beta$, which proves the existence of f.

Now, since M is compact, f is a constant. We have two case to consider:

1^{st} case: $f \neq 0$. In this case the $a_\alpha^{ij} \in \mathcal{O}^*(U_\alpha)$ for all α. This together with the relations in (12) imply that all cocycles in (11) are equivalent and so all Chern classes involved are equal. This case, of course, cannot happen in case (a).

2^{nd} case: $f = 0$. Observe that the relations in (12) imply that for all $i, j \in \{1,2\}$, the collection $\{a_\alpha^{ij}\}_{\alpha \in A}$ defines a divisor in M. Since $f \equiv 0$, one of these divisors is $\equiv 0$. Suppose for instance that $a_\alpha^{12} \equiv 0$ for all $\alpha \in A$. This implies that $E_1 = F_1$ and $a_\alpha^{11}, a_\alpha^{22} \in \mathcal{O}^*(U_\alpha)$ for all $\alpha \in A$, because $\Delta_\alpha = a_\alpha^{11} a_\alpha^{22} \in \mathcal{O}^*(U_\alpha)$ in this case. Analogously, if $a_\alpha^{11} \equiv 0$ for all $\alpha \in A$ we get $E_1 = F_2$ and $c(E_2) = c(F_1)$. The remaining cases are similar and we leave them for the reader. ∎

Now let f and g be homogeneous polynomials on $\mathbf{C}^{n+1}$, $n \geq 3$, such that $f \pitchfork g$. Let $F = \{[z] \in \mathbf{C}P^n \mid f(z) = 0\}$, $G = \{[z] \mid g(z) = 0\}$ and $K = F \cap G$. We denote by $\nu \xrightarrow{P} K$, $\hat{F} \to F$ and $\hat{G} \to G$ the normal bundles of K, F and G in $\mathbf{C}P^n$ respectively. Let $\tilde{F} = \hat{F}|K$, $\tilde{G} = \hat{G}|K$. We will use the notation $c(\cdot)$ to denote the first Chen class of a holomorphic vector bundle in H^2_{DR} of the corresponding base. It is well known that:

(a) $\nu = \tilde{F} \oplus \tilde{G}$ and $c(\nu) = c(\tilde{F}).c(\tilde{G})$ (c f. **[G-A]**)

(b) $c(\tilde{F}) = c(F)|K$ and $c(\tilde{G}) = c(G)|K$. In particular $c(\tilde{F}), c(\tilde{G}) \neq 0$, because $c(F), c(G) \neq 0$ and $\dim(K) > 0$ (cf. **[G-A]**).

(c) $dg(f).c(\tilde{G}) = dg(g).c(\tilde{F})$.

Assertion (c) follows easily from Theorem 1.2.4 and example 1.3.1, as the reader can verify.

Suppose now that $\mathcal{F}$ is a foliation of $\mathbf{C}P^n$ having K as a Kupka component. Let $dg(f)/dg(g) = p/q$ where p, q are relatively primes and $p \leq q$ ($p = q$ iff $p = q = 1$).

Let λ_1 and λ_2 be eigenvalues of the normal type of $\mathcal{F}$ at K. Let us prove first that $\lambda_1 \neq 0 \neq \lambda_2$. In fact, let us suppose by contradiction that $\lambda_2 = 0$. In this case $\lambda_1 \neq 0$, because $K \subset K(\mathcal{F})$. Let E_1 and E_2 be the line subbundles of ν induced by the eigendirections of λ_1 and λ_2 respectively. Then $\nu = \tilde{F} \oplus \tilde{G} = E_1 \oplus E_2$. It follows from Lemma 4 that $c(\tilde{F}) = c(E_1)$ and $c(\tilde{G}) = c(E_2)$ or $c(\tilde{G}) = c(E_1)$ and $c(\tilde{F}) = c(E_2)$. Let us suppose for instance that $c(\tilde{F}) = c(E_1)$ and $c(\tilde{G}) = c(E_2)$. Now (a) of Theorem 1.2.4 implies that $0 = \lambda_2 c(E_1) = \lambda_1 c(E_2)$ and so $c(E_2) = c(\tilde{G}) = 0$, a contradiction.

On the other hand, if $\lambda_1 \neq \lambda_2$ we can define E_1 and E_2 in the same way and get the following relations (assuming $c(\tilde{F}) = c(E_1)$):

$$\begin{cases} \lambda_1 c(E_2) = \lambda_2 c(E_1) \\ c(\tilde{F}) = c(E_1), c(\tilde{G}) = c(E_2) \\ pc(\tilde{G}) = qc(\tilde{F}) \end{cases} \Rightarrow \frac{\lambda_2}{\lambda_1} = \frac{p}{q}$$

We can conclude from the above arguments that:

(i) $\lambda_2 \neq 0 \neq \lambda_1$ and $\lambda_2/\lambda_1 \in \mathbf{Q}_+$.
(ii) If $\lambda_2 \neq \lambda_1$ then $\lambda_2/\lambda_1 = p/q$.

We want to prove that $\lambda_2/\lambda_1 = p/q$ in all cases, but before that we will prove the followingt result.

LEMMA 5. *The transversal type of $\mathcal{F}$ at K is always linearizable and diagonal.*

Proof: Let $\lambda_2/\lambda_1 = r/s$ where $r, s \in \mathbf{Z}_+$, $0 < s \leq r$ and $(r, s) = 1$. Let us suppose by contradiction that the transversal type is either non linearizable or linearizable but not diagonal. In this case, by Poincaré-Dulac Theorem, we must have $1 = s \leq r$ and the transversal type is equivalent to the vector field $X = x\partial/\partial x + (ry + x^r)\partial/\partial y$. The dual form of X is $w = (ry + x^r)dx - xdy$, therefore by Kupka Theorem (1.2.1), there is a covering $(U_\alpha)_{\alpha \in A}$ of K by open sets of $\mathbf{C}P^n$, where each U_α is the domain of a chart $(x_\alpha, y_\alpha, z_\alpha): U_\alpha \to \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-2}$ such that

(i) If $U_\alpha \cap U_\beta \neq \phi$ then it is connected.
(ii) $K \cap U_\alpha = \{x_\alpha = y_\alpha = 0\}$
(iii) $\mathcal{F}|U_\alpha$ is defined by the form $(ry_\alpha + x_\alpha^r)dx_\alpha - x_\alpha dy_\alpha = w_\alpha$.
(iv) There is a multiplicative cocycle $(g_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \phi}$ such that if $U_\alpha \cap U_\beta \neq \phi$ then $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and $w_\alpha = g_{\alpha\beta} w_\beta$ on $U_\alpha \cap U_\beta$.

Now observe that $x_\alpha^{-r-1}w_\alpha$ is closed because

$$\frac{w_\alpha}{x_\alpha^{r+1}} = \frac{dx_\alpha}{x_\alpha} - d\left(\frac{y_\alpha}{x_\alpha^r}\right).$$

Observe also that $\{x_\alpha = 0\}$ is the unique analytic separatrix of $\mathcal{F}$ through $K \cap U_\alpha$. It follows that on $U_\alpha \cap U_\beta \neq \phi$ we must have $x_\alpha = h_{\alpha\beta}x_\beta$ where $h_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. In particular $(x_\alpha)_{\alpha\in A}$ defines an analytic divisor on $U = \bigcup\limits_{\alpha\in A} U_\alpha$. On the other hand we have

$$\text{(14)} \qquad \frac{w_\alpha}{x_\alpha^{r+1}} = \frac{g_{\alpha\beta}}{(h_{\alpha\beta})^{r+1}}\frac{w_\beta}{x_\beta^{r+1}} = f_{\alpha\beta}\frac{w_\beta}{x_\beta^{r+1}}, \quad f_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta).$$

This implies that:

$$0 = d(\frac{w_\alpha}{x_\alpha^{r+1}}) = df_{\alpha\beta} \wedge \frac{w_\beta}{x_\beta^{r+1}} \Rightarrow df_{\alpha\beta} \wedge w_\beta = 0.$$

Therefore $f_{\alpha\beta}$ is a first integral of $\mathcal{F}|U_\alpha \cap U_\beta$. As the reader can verify by using the Taylor series of $f_{\alpha\beta} = \sum\limits_{m,n\geq 0} a_{mn}(z_\beta)x_\beta^m y_\beta^n$, the relation $df_{\alpha\beta}\wedge w_\beta = 0$ implies that $f_{\alpha\beta}$ is a constant. In fact this constant is 1 because the residues of $x_\alpha^{-r-1}w_\alpha$ and $x_\beta^{-r-1}w_\beta$ around $\{x_\alpha = 0\}$ are both equal to 1. It follows that $x_\alpha^{-r-1}w_\alpha = x_\beta^{-r-1}w_\beta$ and so there exists a closed meromorphic 1-form $\hat{\eta}$ defined on $U = \bigcup\limits_\alpha U_\alpha$ such that $\hat{\eta}|U_\alpha = x_\alpha^{-r-1}w_\alpha$ and $\hat{\eta}$ defines $\mathcal{F}|U$ outside its divisor of poles. By Corollary 2.2.3 of §2, $\hat{\eta}$ can be extended to a meromorphic closed 1-form η on $\mathbf{C}P^n$ which defines $\mathcal{F}$ outside its divisor of poles. Let $\eta^* = \pi^*(\eta)$.

It follows from Lemma 3 of §3.1 that there exist homogeneous polynomials $g_1, \ldots, g_m$, φ and ψ on $\mathbf{C}^{n+1}$, $a_1, \ldots, a_m \in \mathbf{C}$ and $\ell_1, \ldots, \ell_m$ non negative integers such that:

(v) $\eta^* = \sum\limits_{j=1}^m a_j \dfrac{dg_j}{g_j} + d(\dfrac{\varphi}{\psi})$

(vi) $\sum_{j=1}^{m} a_j dg(g_j) = 0$

(vii) $\psi = g_1^{\ell_1} \dots g_m^{\ell_m}$ and $dg(\varphi) = dg(\psi)$.

Let $\alpha \in A$ be fixed. We have

$$\left[\sum_{j=1}^{m} a_j \frac{dg_j}{g_j} + d(\frac{\varphi}{\psi})\right] |U_\alpha = \frac{dx_\alpha}{x_\alpha} - d\left(\frac{y_\alpha}{x_\alpha^r}\right) = \eta|U_\alpha \tag{15}$$

From Bézout's theorem, for every $j = 1, \dots, m$ we know that $G_j \cap K \neq \phi$, where $G_j = \{[z] \mid g_j(z) = 0\}$. If α is such that $U_\alpha \cap G_j \cap K \neq \phi$, we can conclude from (15) that:

(viii) $G_j \cap U_\alpha = \{x_\alpha = 0\} \Rightarrow m = 1$

Then (vi) implies that $a_1 dg(g_1) = 0 \Rightarrow a_1 = 0$. But this implies that $\eta = d(\frac{\varphi}{\psi})$ and its residue around $\{x_\alpha = 0\}$ is zero, a contradiction. ∎

LEMMA 6. *If $\lambda_1 = \lambda_2$ then $p = q = 1$ and $\mathcal{F}$ is induced by the form on $\mathbf{C}^{n+1} fdg - gdf$.*

Proof: Let w be an integrable homogeneous 1-form on $\mathbf{C}^{n+1}$ which induces $\mathcal{F}$. The foliation defined by w on $\mathbf{C}^{n+1}$ is $\mathcal{F}^* = \pi^*\mathcal{F}$. Moreover, if $K^* = \pi^{-1}(K) - \{0\}$, then $K^* \subset K(\mathcal{F}^*)$ and the transversal type of $\mathcal{F}^*$ at K^* is linearizable and diagonal with equal eigenvalues.

Let $p_0 \in K^*$. We assert that there exist a chart $(x, y, z): U \to \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-1}$ around p_0 and a function $\Delta \in \mathcal{O}^*(K^* \cap U)$ such that:

(i) $K^* \cap U = \{x = y = 0\}$

(ii) $f|U = x$, $g|U = y$

(iii) $w|U = \Delta(z)(xdy - ydx) + \theta$, where θ denotes terms of order highter than 1 in (x, y).

(iv) The local expression of the radial vector field in U is

$$R|U = mx\frac{\partial}{\partial x} + nx\frac{\partial}{\partial y} + \frac{\partial}{\partial z_1}, \quad m = dg(f), \quad n = dg(g).$$

Observe that (i) and (ii) follow from the implicit function theorem and the fact that $df(p_0) \wedge dg(p_0) \neq 0$. Let $R|U = A\partial/\partial x + B\partial/\partial y + \sum_{j=1}^{n-1} G_j \partial/\partial u_j$. From Euler's identity we have $R(x) = mx$ and $R(y) = ny$, which implies that $A = mx$ and $B = ny$. The proof of (iv) can be done as follows: let $L_1, \ldots, L_{n-1}$ be homogeneous linear polymials such that $L_1(p_0) \neq 0$ and $df(p_0) \wedge dg(p_0) \wedge dL_1(p_0) \wedge \cdots \wedge dL_{n-1}(p_0) \neq 0$. Take $z_j = L_j/L_1$ for $j \geq 2$ and z_1 a branch of $\ell g(L_1)$ defined in a neighborhood of p_0. Then $(x, y, z_1, \ldots, z_{n-1})$ is a diffeomorphism in a neighborhood U of p_0 and moreover $R|U = mx\partial/\partial x + ny\partial/\partial y + \partial/\partial z_1$.

Now since the linear part of the normal type of $\mathcal{F}^*$ at K^* has $\lambda_1 = \lambda_2$ and is diagonal, then for each section $\{z = c\}$ the dual form has linear part $xdy - ydx$, which implies that the linear part of $w|\{z = c\}$ is of the form $\Delta(z)(xdy - ydx)$. Therefore the linear part of $w|U$ with respect to (x, y) if of the form

$$w_1 = \Delta(z)(xdy - ydx) + \sum_{j=1}^{n-1} (A_j(z)x + B_j(z)y)dz_j$$

It follows from the integrability condition $w \wedge dw = 0$, that $A_j = B_j \equiv 0$ for $j = 1, \ldots, n-1$, as the reader can verify directly by taking the linear part of $w \wedge dw$ with respect to (x, y). This proves (iii).

We can conclude from the above facts, that there exists an open covering $(U_\alpha)_{\alpha \in A}$ of K^* and two collections $(\Delta_\alpha)_{\alpha \in A}$, $((x_\alpha, y_\alpha, z_\alpha))_{\alpha \in A}$, where $\Delta_\alpha \in \mathcal{O}^*(K^* \cap U_\alpha)$ and $(x_\alpha, y_\alpha, z_\alpha) \colon U_\alpha \to \mathbf{C} \times \mathbf{C} \times \mathbf{C}^{n-1}$ is a local chart such that

(i) $K^* \cap U_\alpha = \{x_\alpha = y_\alpha = 0\}$

(ii) $f|U_\alpha = x_\alpha$ and $g|U_\alpha = y_\alpha$.

(iii) $w|U_\alpha = \Delta_\alpha(z_\alpha)(x_\alpha dy_\alpha - y_\alpha dx_\alpha) + \theta_\alpha$, where θ_α denotes terms of order highter than 1 in (x_α, y_α).

(iv) $R|U_\alpha = mx_\alpha \frac{\partial}{\partial x_\alpha} + ny_\alpha \frac{\partial}{\partial y_\alpha} + \frac{\partial}{\partial z_{\alpha 1}}$.

Now, if $K^* \cap U_\alpha \cap U_\beta \neq \phi$ and $p \in K^* \cap U_\alpha \cap U_\beta$ then

$$dw(p) = 2\Delta_\alpha(p) dx_\alpha(p) \wedge dy_\alpha(p) = 2\Delta_\beta(p) dx_\beta(p) \wedge dy_\beta(p).$$

Since $x_\alpha = x_\beta = f$ and $y_\alpha = y_\beta = g$ on $U_\alpha \cap U_\beta$, then we must have $\Delta_\alpha(p) = \Delta_\beta(p)$. Hence we can define a holomorphic function $\Delta: K^* \to \mathbf{C}^*$ such that $\Delta | U_\alpha \cap K^* = \Delta_\alpha$ for every $\alpha \in A$. Let us prove that Δ is constant.

Let $dg(w) = k \geq 1$ ($dg(w) =$ degree of the coefficients of w) and the two jet of w along $K^* \cap U_\alpha$ be

$$\begin{aligned} j^2_{(x_\alpha, y_\alpha)} w = {} & \Delta(z_\alpha)(x_\alpha dy_\alpha - y_\alpha dx_\alpha) \\ & + \sum_{i=1}^{n-1} (A_i(z_\alpha)x_\alpha^2 + B_i(z_\alpha)x_\alpha y_\alpha + c_i(z_\alpha)y_\alpha^2)dz_i \\ & + P(x_\alpha, y_\alpha, z_\alpha)dx_\alpha + Q(x_\alpha, y_\alpha, z_\alpha)dy_\alpha \end{aligned}$$

where P and Q are homogeneous of degree two in (x_α, y_α). Since $i_R(w) = 0$, get

$$0 = i_R(w) = \Delta(z_\alpha)(n-m)x_\alpha y_\alpha + A_1(z_\alpha)x_\alpha^2 + B_1(z_\alpha)x_\alpha y_\alpha + C_1(z_\alpha)y_\alpha^2 + \theta_3$$

where θ_3 denotes terms of order highter than 2 in (x_α, y_α). This implies that

(v) $A_1 \equiv C_1 \equiv 0$ and $(n-m)\Delta + B_1 \equiv 0$.

On the other hand, we have seen in the proof of Theorem B that $i_R(dw) = (k+1)w$. From this we get

$$\begin{aligned} (k+1)\Delta(x_\alpha dy_\alpha - y_\alpha dx_\alpha) + \theta_\alpha = {} & i_R(d\Delta)(x_\alpha dy_\alpha - y_\alpha dx_\alpha) \\ & + 2\Delta(m x_\alpha dy_\alpha - n y_\alpha dx_\alpha) \\ & - B_1(y_\alpha dx_\alpha + x_\alpha dy_\alpha) + r \end{aligned}$$

where r denotes terms either of order highter than 1 in (x_α, y_α) or terms in the $dz_j' s$. Comparing these expressions we get:

(vi) $(k+1)\Delta = i_R(d\Delta) + 2m\Delta - B_1 = i_R(d\Delta) + 2n\Delta + B_1$

Therefore,

(vii) $i_R(d\Delta) = (k+1-m-n)\Delta = \ell\Delta$

Equation (vii) implies that $\Delta(tp) = t^\ell \Delta(p)$ for all $t \in \mathbf{C}^*$ and $p \in K^*$. Now, if $\ell = 0$ then we can define a holomorphic function $\varphi: K \to \mathbf{C}$ by

$\varphi([p]) = \Delta(p)$ and this implies that Δ is constant. On the other hand, if $\ell > 0$, let $M: \mathbf{C}^{n+1} \to \mathbf{C}$ be linear. In this case we can define a meromorphic function on K by $\varphi([p]) = \Delta(p)/M^{\ell}(p)$. Since M is arbitrary, it follows that Δ must vanish in some point $p \in K^*$, a contradiction. Analogously, if $\ell < 0$ the function $\varphi = M^{-\ell}\Delta$ must have some pole, and so Δ also, which is again a contradiction. Therefore Δ is constant and moreover $k+1 = m+n$.

Let $\mu = w - \Delta(fdg - gdf)$. It is not difficult to see that the 1-jet of μ along K^* is zero and that μ is homogeneous of degree $k = m+n-1$. From Theorem 2.3.3 we can conclude that $\mu = f^2\mu_1 + fg\mu_2 + g^2\mu_3$, where μ_1, μ_2, μ_3 are 1-forms with homogeneous coefficients and

(viii) $k = 2m + dg(\mu_1) = m + n + dg(\mu_2) = 2n + dg(\mu_3)$

unless some of the μ_j's are $\equiv 0$. Now, if $m = n$, (viii) and $k+1 = m+n = 2m = 2n$, implies that $\mu_1 \equiv \mu_2 \equiv \mu_3 \equiv 0$, and so $w = \Delta(fdg - gdf)$. Let us suppose by contradiction that $m > n$ for instance. In this case $k = m+n-1$ implies that $\mu_1 \equiv \mu_2 \equiv 0$, and so

$$w = \Delta(fdg - gdf) + g^2\mu_3.$$

Since $i_R(w) = 0$, we get

$$0 = \Delta(n-m)fg + g^2 i_R(\mu_3) = g(\Delta(n-m)f + g i_R(\mu_3)).$$

This implies that g divides f which is a contradiction. Hence $m = n$ and $w = \Delta(fdg - gdf)$, which proves the lemma. ∎

COROLLARY. $\lambda_2/\lambda_1 = p/q = m/n$.

Now let us suppose that $1 \leq p < q$ (i.e. $dg(f) < dg(g)$). Let w be a homogeneous integrable 1-form on $\mathbf{C}^{n+1}$ which induces $\mathcal{F}$ on $\mathbf{C}P^n$.

LEMMA 7. *In the above situation w has an integrating factor.*

Proof: We know from Lemma 5 that the transversal type of $\mathcal{F}$ at K is linearizable. This implies that there exists a covering of K by open sets

$(U_\alpha)_{\alpha\in A}$ and coordinate systems $((x_\alpha, y_\alpha, z_\alpha): U_\alpha \to \mathbf{C}\times\mathbf{C}\times\mathbf{C}^{n-2})_{\alpha\in A}$ such that:

(i) $K\cap U_\alpha = \{x_\alpha = y_\alpha = 0\}$

(ii) $\mathcal{F}|U_\alpha$ is defined by $w_\alpha = px_\alpha dy_\alpha - qy_\alpha dx_\alpha$

(iii) If $U_\alpha\cap U_\beta \neq \phi$ then it is connected and there exists $g_{\alpha\beta}\in \mathcal{O}^*(U_\alpha\cap U_\beta)$ such that $w_\alpha = g_{\alpha\beta}w_\beta$ on $U_\alpha\cap U_\beta$.

(iv) If $U_\alpha\cap U_\beta\cap U_\gamma \neq \phi$ then it is connected and $g_{\alpha\beta}.g_{\beta\gamma}.g_{\gamma\alpha}\equiv 1$.

We will consider two cases:

1^{st} case: $1<p<q$. In this case we will prove that there exists a closed meromorphic 1-form on $U=\sum_\alpha U_\alpha$, which defines $\mathcal{F}$ outside its poles.

Oberve first that $1<p<q$ implies that $\{x_\alpha = 0\}\cap U_\beta = \{x_\beta = 0\}\cap U_\alpha$ and $\{y_\alpha = 0\}\cap U_\beta = \{y_\alpha = 0\}\cap U_\alpha$. This follows from the fact that the vector field $X = px\partial/\partial x + qy\partial/\partial y$ has only two analytic smooth separatrices through $(0,0)$, which are $\{x=0\}$ and $\{y=0\}$, and they correspond to two differet eigenvalues q and p. Let $\eta_\alpha = x_\alpha^{-1}y_\alpha^{-1}w_\alpha = pdy_\alpha/y_\alpha - qdx_\alpha/x_\alpha$. If $U_\alpha\cap U_\beta\neq\phi$ and $h_{\alpha\beta} = x_\beta y_\beta g_{\alpha\beta}/x_\alpha y_\alpha$, then $\eta_\alpha = h_{\alpha\beta}\eta_\beta$ and $h_{\alpha\beta}\in\mathcal{O}^*(U_\alpha\cap U_\beta)$ because x_β/x_α and $y_\beta/y_\alpha\in\mathcal{O}^*(U_\alpha\cap U_\beta)$ by the first observation. On the other hand, $0 = d\eta_\alpha = dh_{\alpha\beta}\wedge\eta_\beta \Rightarrow dh_{\alpha\beta}\wedge w_\beta = 0 \Rightarrow h_{\alpha\beta}$ is a first integral of $px_\beta\partial/\partial x_\beta + qy_\beta\partial/\partial y_\beta \Rightarrow h_{\alpha\beta}$ is a constant.

Now, if we compare the residues of η_α and η_β around $\{x_\alpha = 0\}\cap U_\beta$ we get $h_{\alpha\beta}\equiv 1$. This implies that $\eta_\alpha|U_\alpha\cap U_\beta = \eta_\beta|U_\alpha\cap U_\beta$ and so there exists a closed meromorphic 1-form $\tilde\eta$ on $U=\bigcup_\alpha U_\alpha$ such that $\tilde\eta|U_\alpha = \eta_\alpha$ for all $\alpha\in A$ and $\tilde\eta$ represents $\mathcal{F}|U$ outside its poles. It follows from Corollary 2.2.3 that this form can be extended to a closed 1-form η on $\mathbf{C}P^n$ which represents $\mathcal{F}$ outside its poles. This implies the 1^{st} case.

2^{nd} case: $1=p<q$. In this case we have still $\{x_\alpha=0\}\cap U_\beta = \{x_\beta = 0\}\cap U_\alpha$ by the same reason as in the 1^{st} case, but $\{y_\alpha = 0\}\cap U_\beta = \{y_\beta - cx_\beta^q = 0\}\cap U_\alpha$, where c is a constant, as the reader can verify easily. For each $\alpha\in A$, let

$\varphi_\alpha = y_\alpha / x_\alpha^q$. Let us prove that if $U_\alpha \cap U_\beta \neq \phi$ then there constants $a_{\alpha\beta} \in \mathbf{C}^*$ and $b_{\alpha\beta} \in \mathbf{C}$ such that $\varphi_\alpha = a_{\alpha\beta}\varphi_\beta + b_{\alpha\beta}$.

In fact, we have

$$d\varphi_\alpha = x_\alpha^{-q-1} w_\alpha = x_\alpha^{-q-1} g_{\alpha\beta} w_\beta = \left(\frac{x_\beta}{x_\alpha}\right)^{q+1} g_{\alpha\beta} d\varphi_\beta = a_{\alpha\beta} d\varphi_\beta$$

where $a_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. The above relation implies:

$$0 = d^2\varphi_\alpha = da_{\alpha\beta} \wedge d\varphi_\beta \Rightarrow da_{\alpha\beta} \wedge w_\beta = 0 \Rightarrow a_{\alpha\beta} \text{ is a constant } \Rightarrow a_{\alpha\beta} \in \mathbf{C}^*.$$

$\Rightarrow \varphi_\alpha = a_{\alpha\beta}\varphi_\beta + b_{\alpha\beta}$.

Now let $\hat{w}$ be a merorphic 1-form on $\mathbf{C}P^n$ which represents $\mathcal{F}$ outside its poles (we can take $\hat{w}$ such that $\pi^*(\hat{w}) = w/M^{k+1}$, where $dg(w) = k$ and M is linear). For each $\alpha \in A$, there exists a meromorphic function f_α on U_α, such that $\hat{w}|U_\alpha = f_\alpha d\varphi_\alpha$. If $U_\alpha \cap U_\beta \neq \phi$ then

$$\hat{w}|U_\alpha = f_\alpha d\varphi_\alpha = f_\alpha a_{\alpha\beta} d\varphi_\beta = f_\beta d\varphi_\beta.$$

Therefore $f_\beta = a_{\alpha\beta} f_\alpha$, and so $\frac{df_\alpha}{d_\alpha} = \frac{df_\beta}{f_\beta}$ on $U_\alpha \cap U_\beta$. This implies that we can define a meromorphic closed 1-form $\hat{\theta}$ on $U = \bigcup\limits_\alpha U_\alpha$ such that $\hat{\theta}|U_\alpha = df_\alpha | f_\alpha$ for each $\alpha \in A$. By Corollary 2.2.3 this form can be extended to a closed 1-form θ on $\mathbf{C}P^n$. With an argument similar to that we have done before in the proof of Theorem B, it can be proved that there exists a meromorphic function f on $\mathbf{C}P^n$ such that $df/f = \theta$. Clearly for each $\alpha \in A$, we have $f|U_\alpha = cf_\alpha$, c a constant. This implies that $d(\hat{w}/f) = 0$ and so $\hat{w}$ has an integrating factor and w also. This proves the lemma. ∎

Let A/B be the integrating factor of w. From Lemma 3 we have

$$\frac{Bw}{A} = \sum_{j=1}^{m} \lambda_j \frac{dg_j}{g_j} + d\left(\frac{\varphi}{\psi}\right)$$

where $A = g_1^{k_1} \dots g_m^{k_m}$ is the decomposition of A in irreducible factors, $\psi = g_1^{\ell_1} \dots g_m^{\ell_m}$, $0 \leq \ell_j \leq k_j - 1$, $\sum_{j=1}^{m} \lambda_j dg(g_j) = 0$, φ and ψ have no common factors and $dg(\varphi) = dg(\psi)$. We can suppose also that for any $j \in \{1, \dots, m\}$ we have either $\ell_j \neq 0$ or $\lambda_j \neq 0$.

Let us consider the 1^{st} case in the proof of Lemma 7. In this case we can suppose that for any $\alpha \in A$:

$$\frac{Bw}{A}|\pi^{-1}(U_\alpha) = \pi^* \left(p\frac{dy_\alpha}{y_\alpha} - q\frac{dx_\alpha}{x_\alpha} \right) = p\frac{dy_\alpha^*}{y_\alpha^*} - q\frac{dx_\alpha^*}{x_\alpha^*}$$

On the other hand for $j \in \{1, \dots, m\}$ we have from Bézout's Theorem that $A_j = \{g_j = f = g = 0\} - \{0\} \neq \phi$ and for a point $p \in A_j \cap U_\alpha$ (for some α) we get

$$\sum_{i=1}^{m} \lambda_i \frac{dg_i}{g_i} + d(\frac{\varphi}{\psi}) = p\frac{dy_\alpha^*}{y_\alpha^*} - q\frac{dx_\alpha^*}{x_\alpha^*}.$$

This implies that either $\{g_j = 0\} \cap U_\alpha = \{x_\alpha^* = 0\}$ and $\lambda_i = -q$ or $\{g_j = 0\} \cap U_\alpha = \{y_\alpha^* = 0\}$ and $\lambda_j = p$. Since in the right member the poles are of order 1, we get also $\ell_j = 0$. Moreover $\{g_j = 0\} \cap \pi^{-1}(U)$ coincides with one of the divisors $\{x_\alpha^* = 0\}$ or $\{y_\alpha^* = 0\}$ ($U = \bigcup_\alpha U_\alpha$). Therefore $m = 2$ and we can suppose that

$$\frac{Bw}{A} = p\frac{dg_1}{g_1} - q\frac{dg_2}{g_2} \text{ and } pdg(g_1) = qdg(g_2).$$

Observe that we have also $\{g_1 = g_2 = 0\} \supset \{f = g = 0\}$. From Noëther's Lemma (2.3.1), we get $g_1 = \alpha_1 f + \beta_1 g$ and $g_2 = \alpha_2 f + \beta_2 g$ where $\alpha_1, \dots, \beta_2$ are homogeneous polynomials. On the other hand, $\{g_1 = g_2 = 0\}$ is connected by Lefschetz's Theorem and $\{g_1 = g_2 = 0\} \subset S(\mathcal{F}^*)$ as we have seen after Lemma 3. This implies that $\{g_1 = g_2 = 0\} = \{f = g = 0\}$. In fact, $\{f = g = 0\}$ is an irreducible component of $\{g_1 = g_2 = 0\}$ and if it has another component, say N, then $N \cap \{f = g = 0\} - \{0\} \neq \phi$. Hence if $z \in N \cap \{f = g = 0\} - \{0\}$, then $dg_1(z) \wedge dg_2(z) = 0$. But this implies that $z \in \{f = g = 0\} - K(\mathcal{F}^*)$ which is not possible. Therefore $\{g_1 = g_2 = 0\} = \{f = g = 0\}$.

By Noëther's lemma the matrix $\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$ is invertible. Since $dg(f) < dg(g)$ this is possible only if β_1 and α_2 are constants and $\beta_2 \equiv 0$. Hence:

$$\frac{Bw}{A} = p\frac{dg_1}{g_1} - q\frac{df}{f} \Rightarrow \mathcal{F} \text{ is induced by } pfdg_1 - qg_1df.$$

In the 2^{nd} case we can suppose that $\frac{Bw}{A} = \pi^*(\hat{w}/f)$. Therefore if $p \in (\{g_j = f = g = 0\} - \{0\}) \cap \pi^{-1}(U_\alpha)$, we have

$$\sum_{j=1}^{m} \lambda_j \frac{dg_j}{g_j} + d(\frac{\varphi}{\psi}) = a_\alpha d(\varphi_\alpha \circ \pi) = a_\alpha d(y_\alpha/x_\alpha^q)$$

where a_α is a constant. This implies that $\lambda_j = 0$, $j = 1, \ldots, m$. Moreover $\varphi/\psi|U_\alpha = a_\alpha y_\alpha/x_\alpha^q + b_\alpha \Rightarrow m = 1$, $\ell_1 = q$ and $Bw/A = d(\varphi/g_1^q)$. Now, let $\eta = \frac{g_1^q}{\varphi}d(\varphi/g_1^q) = \frac{d\varphi}{\varphi} - q\frac{dg_1}{g_1}$. If we apply the same argument as in the 1^{st} case for η we can conclude that $g_1 = \alpha_1 f$ and $\varphi = \alpha_2 f + \beta_2 g$, where α_1 and β_2 are constants. This proves Theorem A. ∎

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