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THE BOUNDARY OF THE MANDELBROT SET HAS HAUSDORFF DIMENSION TWO

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INTRODUCTION

Let $P_c(z) = z^2 + c$ $(z, c \in \mathbf{C})$. The Julia set J_c of P_c and the Mandelbrot set are defined by

$$J_c = \text{the closure of } \{ \text{ repelling periodic points of } P_c\},$$

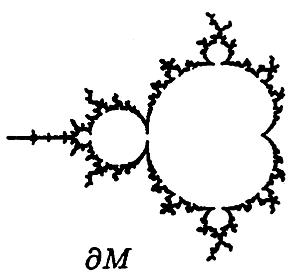
$$M = \{c \in \mathbf{C} \mid J_c \text{ is connected}\} = \{c \in \mathbf{C} \mid \{P_c^n(0)\}_{n=1}^{\infty} \text{ is bounded}\}.$$

In [**Sh**], we proved the following theorems concerning the Hausdorff dimension (denoted by $H - \dim(\,\cdot\,)$) of these sets:

Theorem A. *(conjectured by Mandelbrot [**Ma**] and others)*

$$H - \dim \partial M = 2.$$

Moreover for any open set U which intersects ∂M, we have $H - \dim(\partial M \cap U) = 2$.



∂M

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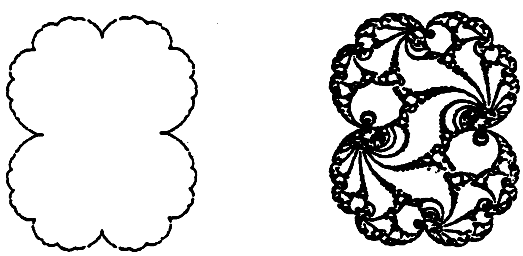
Theorem B. *For a generic* $c \in \partial M$,

$$\text{H-dim}\, J_c = 2.$$

In other words, there exists a residual (hence dense) subset $\mathcal{R}$ *of* ∂M *such that if* $c \in \mathcal{R}$, *then* $\text{H-dim}\, J_c = 2$.



J_c for $c = \frac{1}{4}$ and for $c = 0.25393 + 0.00048i$

In order to explain the idea of the proof, we need to introduce the following.

Definition. A closed subset X of $\overline{\mathbb{C}}$ is called a *hyperbolic subset* for a rational map f, if and only if $f(X) \subset X$ and f is expanding on X (i.e. $\|(f^n)'\| \geq C\kappa^n$ on X $(n \geq 0)$ for some $C > 0$ and $\kappa > 1$, where $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric of $\overline{\mathbb{C}}$). The *hyperbolic dimension* of f is the supremum of $\text{H-dim}(X)$ over all hyperbolic subset X for f.

Theorem A was an immediate consequence of the following claims $(*)$ and $(**)$ (Corollary 3 (i) and (ii) in [Sh]).

$(*)$ If U is an open set containing $c \in \partial M$, then

$$\text{H-dim}(\partial M \cap U) \geq \text{hyp-dim}(P_c).$$

$(**)$ For $c \in \partial M$, there exists a sequence $\{c_n\}$ in ∂M such that $c_n \to c$ and $\text{hyp-dim}(P_{c_n}) \to 2$, as $n \to \infty$.

Theorem B also follows from $(**)$, as we will see in §1.

In this paper, we give a brief explanation on the hyperbolic dimension and on the proof of $(*)$ (in §1), and present a different point of view for the proof of (**) after McMullen [Mc2], although the essential idea came from [Sh]. The claim (**) is reduced to the main theorem in §2, which is reduced to Proposition 5.1 in §5.

The main Theorem has a meaning that the "secondary bifurcation" of the parabolic fixed point leads to high hyperbolic dimension (i.e. close to 2). The proof presented in this paper is based on the notion of geometric limit, which was proposed by McMullen [Mc2]. By a geometric limit, we mean a limit of iterates $f_n^{k_n}$ in some region (usually only in a proper subregion of $\overline{\mathbb{C}}$), where $k_n \in \mathbb{N}$, $k_n \to \infty$.

In §3, we construct a geometric limit of rational maps related to a parabolic fixed point. This case was studied earlier by Lavours [La]. In §4, we construct a second geometric limit, which corresponds to the secondary bifurcation. These limits are used in §5, to construct "good" hyperbolic subsets with high Hausdorff dimension, which leads to the proof of the main theorem. Here the use of second geometric limits is crucial.

I would like to thank specially Curt McMullen for inspiring discussions on this subject. The above pictures are made by using J. Milnor's program.

§1. Hyperbolic subsets and hyperbolic dimension

This section is devoted to the topics related to hyperbolic subsets and hyperbolic dimension. We state some properties the hyperbolic dimension, prove that (**) implies Theorem B, state the outline of the proof of (*), and explain a specific construction of a hyperbolic subset.

Properties of hyperbolic dimension. (see [Sh] §2 for the proof.)

Let X be a hyperbolic subset for a rational map f.

(1.1) *X is a subset of the Julia set $J(f)$ of f. Hence* H-dim $J(f) \geq$ hyp-dim(f).

(1.2) *X is "stable" under a perturbation, i.e. a nearby rational map f_1 has a hyperbolic subset X_1 such that (X_1, f_1) is conjugated to (X, f) by a Hölder homeomorphism with Hölder exponent close to* 1.

(1.3) $f \mapsto$ hyp-dim(f) *is lower semi-continuous, or equivalently, for any number a, the set* $\{f|$ hyp-dim$(f) > a\}$ *is open.*

Now we can prove that (**) implies Theorem B.

Proof of Theorem B. Let $\mathcal{R}_n = \{c \in \partial M |\ \text{hyp-dim}(P_c) > 2 - \frac{1}{n}\}$ $(n = 1, 2 \cdots)$. Then $\mathcal{R} = \bigcap_{n>0} \mathcal{R}_n = \{c \in \partial M |\ \text{hyp-dim}(P_c) = 2\ \} \subset \{c \in \partial M |\ \text{H-dim}\ J(P_c) = 2\ \}$. By Property (1.3), $\mathcal{R}_n$ are open in ∂M. Moreover $\mathcal{R}_n$ is dense in ∂M by (**). Hence $\mathcal{R}$ is residual (i.e. contains the intersection of a countable collection of open dense subsets of ∂M.) Such an $\mathcal{R}$ is dense in ∂M by Baire's theorem. □

Remark 1.4. It follows from (1.3) that if $\text{hyp-dim}(P_{c_0}) = 2$ then both $c \mapsto \text{hyp-dim}(P_c)$ and $c \mapsto \text{H-dim}(J_c)$ are continuous at $c = c_0$. On the other hand, (**) implies that if $c_0 \in \partial M$ and $\text{hyp-dim}(P_{c_0}) \neq 2$ (resp. $\text{H-dim}(J_{c_o}) \neq 2$) then $c \mapsto \text{hyp-dim}(P_c)$ (resp. $c \mapsto \text{H-dim}(J_c)$) is not continuous at $c = c_0$. Note that P_c is J-stable in $\mathbb{C} - \partial M = (\mathbb{C} - M) \cup (\text{interior} M)$, hence $c \mapsto \text{hyp-dim}(P_c)$ and $c \mapsto \text{H-dim}(J_c)$ are continuous.

In fact, if X is a hyperbolic subset for f and a homeomorphism conjugates f to f_1 in a neighbourhood of the Julia sets, then $h(X)$ is a hyperbolic subset for f_1.

Outline of the proof of (*). The main idea is "the similarity between the Mandelbrot set and Julia sets" (cf. [T]). See [Sh] §3 for the complete proof.

Let $c_0 \in \partial M$. For any $\varepsilon > 0$, there exists a hyperbolic subset X for P_{c_0} such that $\text{H-dim}\, X > \text{hyp-dim}(P_{c_0}) - \varepsilon$. Let us take a point $z \in X$ such that X intersected with any neighborhood of z has the same Hausdorff dimension as entire X. By Property (1.2), for c in a neighborhood U of c_0, P_c has a hyperbolic subset X_c ($X_{c_0} = X$), which moves continuously with c.

On the other hand, the family $\{P_c^n(0)\}_{n=1}^{\infty}$ (as functions of c) is not normal in any neighborhood of c_0. In fact, $\{P_{c_0}^n(0)\}$ is bounded, but for arbitarily close $c \in \mathbb{C} - M$, $P_c^n(0) \to \infty$ ($n \to \infty$). Then one can show that for z chosen above, there are a parameter c_1 arbitrarily close to c_0 and a positive integer m such that $P_{c_1}^m(0)$ belongs to X_{c_1} and corresponds to z in X.

Now consider $M_1 = \{c \in U |\ P_c^m(0) \in X_c\}$. One can show that $M_1 \subset \partial M$. For $c \in M_1$, let h(c) be the point in X corresponding to $P_c^m(0)$ in X_c. Consider the case where $P_c^m(0)$ moves "transversally" relative to X_c. Then one can show that h is a homeomorphism from a neighborhood of c_1 in M_1 to a neighborhood of z in X. Moreover it is Hölder with exponent close to 1, for c_1 near c_0. (To see these facts, pretend that X_c did not move at all. Then h would give a local diffeomorphism near c_1.)

Therefore M_1 has Hausdorff dimension close to $\text{H-dim}\, X$ or higher. So we can prove the assertion by letting $c_1 \to c$ and $\varepsilon \to 0$. □

In §5, we will construct a special kind of hyperbolic sets as follows.

Lemma 1.5. *Suppose that U is a simply connected open set in $\mathbb{C}$; $U_1, \ldots, U_N$ are disjoint open subsets of U with $\overline{U}_i \subset U$; $n_1, \ldots, n_N$ are positive integers such that f^{n_i} maps U_i onto U bijectively ($i = 1, \ldots, N$).*

Then there exists a Cantor set X_0 generated by $\tau_i \equiv (f^{n_i}|_{U_i})^{-1} : U \to U_i$ ($i = 1, \ldots, N$), that is, X_0 is the minimal non-empty closed set satisfying

$$X_0 = \tau_1(X_0) \cup \cdots \cup \tau_N(X_0),$$

and the set $X = X_0 \cup f(X_0) \cup \cdots \cup f^{M-1}(X_0)$ *(where* $M = \max n_i$*) is a hyperbolic subset for* f.

Moreover $\delta = \text{H-dim}\, X_0$ *satisfies*

$$1 \geq \sum_{i=1}^{N} \inf_{U} |\tau_i'|^{\delta} \geq N(\max_i \sup_{U_i} |(f^{n_i})'|)^{-\delta}, \quad \textit{hence} \quad \delta \geq \frac{\log N}{\log(\max_i \sup_{U_i} |(f^{n_i})'|)} .$$

See [Sh] Lemmas 2.1 and 2.2 for the proof.

§2. Main theorem

In this section, we state the main theorem of this paper and prove (**) assuming the main theorem.

Definition. A periodic point is called *parabolic* if its multiplier is a root of unity. The *parabolic basin* of a parabolic periodic point ζ of period k is

$$\{z \in \overline{\mathbb{C}} |\ f^{nk} \to \zeta\ (n \to \infty) \ \text{ in a neighborhood of } z\ \},$$

and the *immediate parabolic basin* of ζ is the union of periodic connected components of the parabolic basin.

We assume the following (A1) and (A2).

(A1) f is a rational map of degree d (> 1) such that

$z = 0$ is a parabolic fixed point with multiplier 1 (i.e. $f(0) = 0$, $f'(0) = 1$),

the immediate parabolic basin $\mathcal{B}$ of $z = 0$ contains only one critical point of f, and

the forward orbits of critical points do not contain 0.

(A2) $\{f_\alpha\}$ is a continuous family of rational maps of degree d parametrized by $\alpha \in \mathbb{C}$, $|\alpha| < r$ ($r > 0$), $\operatorname{Re} \alpha \geq 0$, such that $f_0 = f$, $f_\alpha(0) = 0$ and $f_\alpha'(0) = \exp(2\pi i \alpha)$.

Now we can state:

Main Theorem. *Suppose* f *and* $\{f_\alpha\}$ *satisfy* (A1) *and* (A2). *Let* $\{b_n\}$ *be a sequence of integers tending to* ∞ *and* $\{\gamma_n\}$ *a covergent sequence in* $\mathbb{C}$. *Then there exists a sequence* $\{a_n\}$ *of integers tending to* ∞ *such that*

$$\text{hyp-dim}(f_{\alpha_n}) \to 2 \quad \textit{as } n \to \infty,$$

for α_n *defined by*

$$\alpha_n = \cfrac{1}{a_n - \cfrac{1}{b_n - \gamma_n}} .$$

This theorem is a weaker version of Theorem 2 in [Sh]. We give in §5 an alternative proof based on the notion of geometric limits. We can now prove (**) assuming this theorem.

Proof of (**). It follows from [MSS] that for dense parameters c in ∂M, the critical point 0 is preperiodic but not periodic under P_c. Then by [DH2] Theorem 5, such parameters can be approximated by parameters c such that some iterate P_c^k restricted to a neighbourhood of 0 is a polynomial-like mapping which is hybrid conjugate to $P_{1/4}$. For the later parameter c, the iterate $f = P_c^k$ satisfies (A1) after a change of coordinate, since P_c has only one critical point in $\mathbb{C}$. By reparametrization and coordinate change, one can construct a family $\{f_\alpha\}$ satisfying (A2) such that f_α is conjugate to $P_{\lambda(\alpha)}^k$, where and $\lambda(\alpha)$ is a continuous function with $\lambda(0) = c$.

Let α_n be as in the main theorem and define $c_n = \lambda(\alpha_n)$. Then hyp-dim(P_{c_n}) $\to 2$ $(n \to \infty)$. Moreover if we take $\gamma_n = 0$, then P_{c_n} has a parabolic periodic point, hence $c_n \in \partial M$. Thus we conclude that any parameter in ∂M can be approximated by c in ∂M for which hyp-dim(P_c) is arbitrarily close to 2. $\square$

§3. Geometric limit

In this section and the next section, we state the notion of geometric limit, which provides the basis for the proof of the main theorem in §5. First we state the facts and later, in Appendix, we comment briefly on their proof. Most of the facts stated here can be found in [L] and [Sh].

3.0. Let f be a rational map satisfying (A1) in §2. It implies that $f''(0) \neq 0$. In fact if $f''(0) = 0$, then there would be more than one immediate parabolic basins, each of which must contain a critical point, and this is impossible. By a coordinate change, we may assume that $f''(0) = 1$.

3.1. There exist analytic functions $\varphi : \mathbb{C} \to \overline{\mathbb{C}}$ and $\Phi : \mathcal{B} \to \mathbb{C}$ such that

$$\varphi(w+1) = f \circ \varphi(w) \quad \text{for } w \in \mathbb{C}$$

and

$$\Phi \circ f(z) = \Phi(z) + 1 \quad \text{for } z \in \mathcal{B}.$$

Denote $I(z) = -1/z$. If $R > 0$ is large enough, then $I(\mathcal{B})$ contains $\{w \in \mathbb{C} |\ |w| > R \text{ and } |\arg(w)| < 3\pi/4\ \}$ and $\varphi^{-1}(\mathcal{B})$ contains $\{w \in \mathbb{C} |\ |\operatorname{Im} w| > R\ \}$. The functions satisfy

$$I \circ \varphi(w) = w + c \log w + const + o(1) \quad \text{when } |\arg w - \pi| < \frac{3\pi}{4} \text{ and } |w| \to \infty;$$

$$\Phi \circ I^{-1}(w) = w - c\log w + const + o(1) \quad \text{when} |\arg w| < \frac{3\pi}{4} \text{ and } |w| \to \infty,$$

where $c = 1 - \frac{2f'''(0)}{3(f''(0))^2}$ and in each formula a suitable branch of the logarithm is taken.

The function Φ is uniquely determined up to addition of constant. We adjust it so that $\Phi \circ \varphi(w) = w + o(1)$ when $w \to +i\infty$ (i.e. $w = iy$, $y \in \mathbb{R}$ and $y \to +\infty$). Then taking into account the difference of the branches of the logarithm, we obtain $\Phi \circ \varphi(w) = w + 2\pi ic + o(1)$ when $w \to -i\infty$.

3.2. Fix $\beta \in \mathbb{C}$. Let us denote $\pi(z) = \exp(2\pi iz)$, $T(w) = w + 1$ and $\widetilde{\mathcal{B}} = \varphi^{-1}(\mathcal{B})$. Define $g : \mathcal{B} \to \overline{\mathbb{C}}$ and $\widetilde{g} : \widetilde{\mathcal{B}} \to \mathbb{C}$ by

$$g(z) = \varphi(\Phi(z) + \beta) \quad \text{and} \quad \widetilde{g}(w) = \Phi \circ \varphi(w) + \beta.$$

They satisfy

$$g \circ f = f \circ g \quad \text{on } \mathcal{B}, \quad g \circ \varphi = \varphi \circ \widetilde{g} \quad \text{and} \quad \widetilde{g} \circ T = T \circ \widetilde{g} \quad \text{on } \varphi^{-1}(\mathcal{B}).$$

By the adjustment in 3.1, we have

$$\widetilde{g}(w) = w + \beta + o(1) \text{ as } \operatorname{Im} w \to +\infty \quad \text{and}$$
$$\widetilde{g}(w) = w + \beta + 2\pi ic + o(1) \text{ as } \operatorname{Im} w \to -\infty.$$

Hence $\overline{g} = \pi \circ \widetilde{g} \circ \pi^{-1}$ is well-defined on $\overline{\mathcal{B}} = \pi(\widetilde{\mathcal{B}})$, and extends analytically to 0 and ∞, in such a way that 0 and ∞ are fixed points of $\overline{g}$, with multipliers $e^{2\pi i\beta}$ and $e^{-2\pi i\beta + 4\pi^2 c}$, respectively.

3.3. The map $\overline{g} : \overline{\mathcal{B}} \to \mathbb{C}^*$ is a *branched* covering with only one critical value. That is, there is a critical value $v \in \mathbb{C}^*$ such that $\overline{g} : \overline{\mathcal{B}} - \overline{g}^{-1}(v) \to \mathbb{C}^* - \{v\}$ is a covering map.

There are two distinct connected components $\overline{\mathcal{B}}_0$ and $\overline{\mathcal{B}}_\infty$ of $\overline{\mathcal{B}}$ such that $\overline{\mathcal{B}}_0 \cup \{0\}$ and $\overline{\mathcal{B}}_\infty \cup \{\infty\}$ are neighborhoods of 0 and ∞, respectively.

3.4. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of rational maps such that f_n has a fixed point z_n with multiplier $\exp(2\pi i\alpha_n)$;

$$\alpha_n = \frac{1}{a_n - \beta_n} ;$$

where $a_n \in \mathbb{N}$ and $\beta_n \in \mathbb{C}$; and $f_n \to f$, $z_n \to 0$, $a_n \to \infty$ and $\beta_n \to \beta$, when $n \to \infty$.

Then $f_n^{a_n} \to g$ uniformly on compact sets in $\mathcal{B}$ as $n \to \infty$.

Moreover for any $k, \ell \in \mathbb{Z}$ with $\ell > 0$, there exists $\lim_{n\to\infty} f_n^{\ell a_n + k}$ on $g^{-\ell}(\mathbb{C}^*) = g^{-\ell+1}(\mathcal{B})$ and the limit coincides with $f^k \circ g^\ell = g^\ell \circ f^k$ if $k \geq 0$. For $k < 0$, we also denote the limit by $f^k \circ g^\ell = g^\ell \circ f^k$.

For $\ell > 0$, $f^k g^\ell$ are called *geometric limits* of f. This notion was introduced by McMullen in connection with Kleinian group. See [Mc2]. In §5, we will see that this notion is very useful for the study of the bifurcation of parabolic fixed points.

§4. Second geometric limit

4.0. Let us suppose that $\beta = 0$ in §3. Then we have $\overline{g}'(0) = 1$. Although $\overline{g}$ is defined only in a subregion of $\overline{\mathbb{C}}$, the covering property of $\overline{g} : \overline{\mathcal{B}} \to \mathbb{C}^*$ allows us to treat $\overline{g}$ analogously as f.

One can show that there is an immediate basin $\mathcal{B}'$ of $z = 0$ for $\overline{g}$, such that $\overline{g} : \mathcal{B}' \to \mathcal{B}'$ is a branched covering, $\mathcal{B}'$ contains only one critical point of $\overline{g}$, and $\mathcal{B}'$ is simply connected. As in 3.0, it follows that $\overline{g}''(0) \neq 0$.

We can adjust φ and Φ so that $\bar{g}''(0) = 1$, keeping the normalization $\Phi \circ \varphi(z) = z + o(1)$ $(z \to +i\infty)$. In the following, $\overline{h}$, $\overline{\widetilde{h}}$, $\overline{\overline{h}}$ for $\overline{g}$ are similar to g, $\widetilde{g}$, $\overline{g}$ for f in §3.

4.1. There exist $\psi : \mathbb{C} \to \mathbb{C}^*$ and $\Psi : \mathcal{B}' \to \mathbb{C}$ such that

$$\psi(w+1) = \overline{g} \circ \psi(w) \quad \text{for } w \in \mathbb{C} \quad \text{and} \quad \Psi \circ \overline{g}(z) = \Psi(z) + 1 \quad \text{for } z \in \mathcal{B}.$$

Note here that the range of ψ is $\mathbb{C}^*$, since $\overline{g}(\overline{\mathcal{B}}) = \mathbb{C}^*$. As in 3.1, $\{w|\ |w| > R,\ |\arg(w)| < 3\pi/4\} \subset I(\mathcal{B}')$ and $\{w|\ |\operatorname{Im} w| > R\} \subset \psi^{-1}(\mathcal{B}')$ for $R > 0$ large, and there exists a constant c' such that

$$I \circ \psi(w) = w + c' \log w + const + o(1) \quad \text{when } |\arg w - \pi| < \frac{3\pi}{4} \text{ and } |w| \to \infty;$$

$$\Psi \circ I^{-1}(w) = w - c' \log w + const + o(1) \quad \text{when } |\arg w| < \frac{3\pi}{4} \text{ and } |w| \to \infty.$$

We adjust Ψ so that $\Phi \circ \varphi(w) = w + o(1)$ as $w \to +i\infty$.

4.2. Fix a $\gamma \in \mathbb{C}$. Define $\overline{h} : \mathcal{B}' \to \mathbb{C}^*$ and $\overline{\widetilde{h}} : \widetilde{\mathcal{B}}' \to \mathbb{C}$, where $\widetilde{\mathcal{B}}' = \varphi^{-1}(\mathcal{B}')$, by

$$\overline{h} = \psi(\Psi(z) + \gamma) \quad \text{and} \quad \overline{\widetilde{h}}(w) = \Psi \circ \psi(w) + \gamma.$$

Then

$$\overline{h} \circ \overline{g} = \overline{g} \circ \overline{h} \ \text{ on } \mathcal{B}', \quad \overline{h} \circ \psi = \psi \circ \overline{\widetilde{h}} \ \text{ and } \ \overline{\widetilde{h}} \circ T = T \circ \overline{\widetilde{h}} \ \text{ on } \psi^{-1}(\mathcal{B}').$$

Hence $\overline{\overline{h}} = \pi \circ \overline{\widetilde{h}} \circ \pi^{-1}$ is well-defined on $\overline{\mathcal{B}}' = \pi(\widetilde{\mathcal{B}}')$ and extends analytically to 0 and ∞, which become fixed points.

4.3. Again, $\overline{h} : \overline{\mathcal{B}}' \to \mathbb{C}^*$ is a branched covering map with only one critical value, and there are two distinct connected components $\overline{\mathcal{B}}'_0$ and $\overline{\mathcal{B}}'_\infty$ of $\overline{\mathcal{B}}'$ such that $\overline{\mathcal{B}}'_0 \cup \{0\}$ and $\overline{\mathcal{B}}'_\infty \cup \{\infty\}$ are neighborhoods of 0 and ∞, respectively.

4.4. Let $\widetilde{h} : \pi^{-1}(\mathcal{B}') \to \mathbb{C}$ be a lift of $\overline{h}$ by π such that $\pi \circ \widetilde{h} = \overline{h} \circ \pi$ on $\pi^{-1}(\mathcal{B}')$ and $\widetilde{h}$ commutes with T and $\widetilde{g}$. (Note here that $T(\pi^{-1}(\mathcal{B}')) = \pi^{-1}(\mathcal{B}')$ and $g(\pi^{-1}(\mathcal{B}')) = \pi^{-1}(\mathcal{B}')$). Then there exists $h : \check{\mathcal{B}} \to \overline{\mathbb{C}}$, where $\check{\mathcal{B}} = \varphi(\pi^{-1}(\mathcal{B}'))$, such that $h \circ \varphi = \varphi \circ \widetilde{h}$ and h commutes with f and g.

4.5. Suppose $\{f_\alpha\}$ satisfies (A2) in §2. Let $\{b_n\}$ be a sequence of integers tending to ∞ and $\{\gamma_n\}$ a covergent sequence in $\mathbb{C}$. Then there exist sequences of integers $\{a_n\}$ and $\{p_n\}$ tending to infinity such that

$$f_{\alpha_n}^{p_n} \to h \quad \text{uniformly on compact sets in } \check{\mathcal{B}}' \quad \text{as } n \to \infty,$$

for α_n defined by

$$\alpha_n = \cfrac{1}{a_n - \cfrac{1}{b_n - \gamma_n}}.$$

(In fact, we can take $p_n = a_n b_n + const$, but this is not necessary in this paper.)

Let k, ℓ, m are integers with $m > 0$. Then there exists $\lim_{n\to\infty} f_n^{mp_n + \ell a_n + k}$ on $h^{-m}(\overline{\mathbb{C}}) = h^{-m+1}(\check{\mathcal{B}}')$, and the limit coincides with $f^k g^\ell h^m$ if $\ell \geq 0$ or if $\ell = 0$ and $k > 0$. In other cases, we define $f^k g^\ell h^m$ by the limit. Then they satisfy $(f^{k_1} g^{\ell_1} h^{m_1}) \circ (f^{k_2} g^{\ell_2} h^{m_2}) = f^{k_1+k_2} g^{\ell_1+\ell_2} h^{m_1+m_2}$ for k_i, $\ell_i \in \mathbb{Z}$ and $m_i \in \mathbb{N}$, which justifies this notation. Moreover there are commutation relations:

$$\begin{aligned}
(f^k g^\ell h^m) \circ \varphi &= \varphi \circ (T^k \widetilde{g}^\ell \widetilde{h}^m) && \text{on } \widetilde{h}^{-m}(\mathbb{C}) \\
(\overline{g}^\ell \overline{h}^m) \circ \pi &= \pi \circ (T^k \widetilde{g}^\ell \widetilde{h}^m) && \text{on } \widetilde{h}^{-m}(\mathbb{C}) \\
(\overline{g}^\ell \overline{h}^m) \circ \psi &= \psi \circ (T^\ell \widetilde{\overline{h}}^m) && \text{on } \widetilde{\overline{h}}^{-m}(\mathbb{C}^*),
\end{aligned}$$

where $\widetilde{g}^\ell \widetilde{h}^m$ and $\overline{g}^\ell \overline{h}^m$ are suitably defined for $\ell < 0$ (see Appendix). Incidently, if $m = 0$ and $\ell > 0$ or if $m = \ell = 0$, these formulae are still true, where each formule is considered on the region on which either side of the formula is defined.

We call $f^k g^\ell h^m$ *second geometric limits* of f.

The following diagram shows the relationship between the maps, where the arrows indicates semi-conjugacies between the maps.

$$\begin{tikzcd}
I \circ f^k g^\ell h^m \circ I^{-1} & T^k \widetilde{g}^\ell \widetilde{h}^m \arrow[l, "I\circ\varphi"'] \arrow[d, "\pi"] & \\
f^k g^\ell h^m \arrow[u, "I"] & \overline{g}^\ell \overline{h}^m \arrow[d, "I"] & \\
 & I \circ \overline{g}^\ell \overline{h}^m \circ I^{-1} & T^\ell \widetilde{\overline{h}}^m \arrow[l, "I\circ\psi"'] \arrow[d, "\pi"] \\
 & & \overline{\overline{h}}^m
\end{tikzcd}$$

§5. The proof of the main theorem

In this section, we prove the following:

Proposition 5.1. *Let f, g, h be as in the previous section. Then there exist an open set $U \subset \overline{\mathbb{C}}$ containing 0 and constants $a, C, C' > 0$ such that for large $\eta > 0$, there exist an integer N, open sets U_i and integers k_i, ℓ_i, m_i $(i = 1, \ldots, N)$ such that*

$N > a\eta^2$; $m_i > 0$; $\overline{U}_1, \ldots, \overline{U}_N$ are disjoint and contained in U;

$f^{k_i} g^{\ell_i} h^{m_i}$ maps $\overline{U}_i$ bijectively onto $\overline{U}$, and

$$C\eta(\log\eta)^2 < |(f^{k_i} g^{\ell_i} h^{m_i})'| < C'\eta(\log\eta)^2 \quad \text{on } U_i .$$

Let us first show:

Claim. *Proposition 5.1 implies the main theorem.*

Proof. Fix a large $\eta > 0$ and take U, N, etc. in Proposition 5.1. Denote $f_j = f_{\alpha_j}$, where $\{\alpha_j\}$ is the sequence chosen in 4.5. It follows from 4.5 that for each $i = 1, \ldots, N$, there exists a sequence of integers $\{n(i,j)\}_{j=1}^{\infty}$ such that $f_j^{n(i,j)} \to f^{k_i} g^{\ell_i} h^{m_i}$ uniformly on a neighborhood on $\overline{U}_i$ when $j \to \infty$. If j is large enough, there exist $U_{i,j}$ (near U_i) such that $\overline{U}_{1,j}, \ldots, \overline{U}_{N,j}$ are disjoint and contained in U; $f_j^{n(i,j)}$ maps $\overline{U}_{i,j}$ bijectively onto $\overline{U}$, and

$$\frac{1}{2}C\eta(\log\eta)^2 < |(f^{k_i} g^{\ell_i} h^{m_i})'| < 2C'\eta(\log\eta)^2 \quad \text{on } U_{i,j}$$

$(i = 1, \ldots, N)$. Then by Lemma 1.5, f_j has a hyperbolic subset X_j such that

$$\text{H-dim}(X_j) \geq \frac{\log(a\eta^2)}{\log(2C'\eta(\log\eta)^2)} = \frac{2\log\eta + \log a}{\log\eta + 2\log\log\eta + \log(2C')} .$$

Therefore hyp-dim$(f_j) \to 2$ when $j \to \infty$. $\square$

Proof of Proposition 5.1.

Step 1. There exist w_0 and ζ_0 such that

$$\varphi(w_0) = 0,\ \varphi'(w_0) \neq 0,\ \pi(w_0) = \psi(\zeta_0) \text{ and } \psi'(\zeta_0) \neq 0.$$

Proof. Recal that a point z is called an *exceptional point* of f if $f^{-2}(z) = \{z\}$. For example, ∞ is an exceptional point for a polynomial. There are at most two exceptional points and if exist, they belong to $\overline{\mathbb{C}} - J_f$. Let us prove:

Lemma 5.2.

$$\varphi(\mathbb{C}) = \overline{\mathbb{C}} - \{\textit{exceptional points of } f\} \supset J_f,$$

$$\psi(\mathbb{C}) = \mathbb{C}^*.$$

Proof. Suppose that $z \in \overline{\mathbb{C}}$ is not an exceptional point of f. It is known that the closure of $\bigcup_{n\geq 1} f^{-n}(z)$ contains the Julia set. Since the image of φ must intersect Julia set, there exist z' and $n \geq 0$ such that $f^n(z') = z$ and z' is in the image of φ. Then z is also in the image of φ, since if $z' = \varphi(w)$ then $\varphi(w+n) = f^n(\varphi(w)) = f^n(z') = z$.

As to ψ, first pick a sequence $\{z_j\}_{j=0}^{\infty}$ in $\mathbb{C}^*$ such that $\overline{g}(w_j) = w_{j-1}$ and $w_j \to 0$ as $j \to \infty$. Let w be any point in $\mathbb{C}^*$. Since the unique critical value must be in the basin $\mathcal{B}'$ of $z = 0$ by 4.0, one can take a simply connected open set D in $\mathbb{C}^*$ containing w and w_0 but not the forward orbit of the critical value of $\overline{g}$, if necessary, after replacing w by its inverse image and w_0 by some w_j. In D, there is a branch of $\overline{g}^{-j}$ which sends w_0 to w_j. Let w'_j be the image of w by this branch. Since these branches have three omitted values 0, ∞ and the critical value of g, they form a normal family by Montel's theorem. So the fact $w_j \to 0$ implies that $w'_j \to 0$ as $j \to \infty$. Therefore w'_j belong to the image of ψ, for large j. So we conclude $\psi(\mathbb{C}) = \mathbb{C}^*$ as in the case of φ. □

It follows from the proof of Lemma 5.2 and the assumption (A1) that $\varphi'(w_0) \neq 0$, by using $\varphi(w) = f^n(\varphi(w-n))$. The forward orbit of the critical value for $\overline{g}$ is defined and converges to 0. On the other hand, $\pi(w_0)$ is not in $\overline{\mathcal{B}}$, since $\varphi(w_0) = 0 \notin \mathcal{B}$. Therefore the forward orbit of the critical value does not contain $\pi(w_0)$. So one can show $\psi'(\zeta_0) = 0$, as before. □(Step 1)

Step 2. Let ζ_0 be as in Step 1.
(i) There exist a sequence $\{\zeta_m\}_{m=1}^{\infty}$ in $\mathbb{C}$ and constants C_0, C_1, $C'_1 > 0$ such that $\widetilde{\overline{h}}(\zeta_m) = \zeta_{m-1}$, $\operatorname{Im}\zeta_m \to \pm\infty$ $(m \to \infty)$, $|\zeta_m - \zeta_{m-1}| < C_0$ and $C_1 < |(\widetilde{\overline{h}}^{m})'(\zeta_m)| < C'_1$.
(ii) Moreover if U is a sufficiently small neighborhood of $z = 0$ in $\overline{\mathbb{C}}$, then there exist a neighborhood V of w_0 and neighborhoods W_m of ζ_m in $\mathbb{C}$ such that

$diam\ W_m < 1$;
$T^\ell W_m$ $(m = 0, 1, 2, \ldots;\ \ell \in \mathbb{Z})$ are disjoint;
$\varphi : \overline{V} \to \overline{U}$, $\pi : \overline{V} \to \pi(\overline{V})$, $\psi : \overline{W}_0 \to \pi(\overline{V})$ and $\tilde{\overline{\overline{h}}}^m : \overline{W}_m \to \overline{W}_0$ are bijections with non-zero derivative;
$\frac{1}{2}C_1 < |(\tilde{\overline{\overline{h}}}^m)'| < 2C_1'$ on W_m .

Proof. (i) Let $\overline{\mathcal{B}}_0'$ and $\overline{\mathcal{B}}_\infty'$ be as in 4.3. At least one of them does not contain the critical value of $\overline{\overline{h}}$. So suppose $\overline{\mathcal{B}}_0'$ does not contain it.

There is a well-defined analytic branch of $\overline{\overline{h}}^{-1}$, $H : \overline{\mathcal{B}}_0' \cup \{0\} \to \overline{\mathcal{B}}_0' \cup \{0\}$, such that $\overline{\overline{h}} \circ H = id$, $H(0) = 0$ and $|H'(0)| < 1$ by Schwarz's lemma. Then it is well-known (see [Mi]) that there exists a linearizing coordinate $L : \overline{\mathcal{B}}_0' \cup \{0\} \to \mathbb{C}$ such that L is conformal near 0, $L(0) = 0$, $L'(0) = 1$ and $L \circ H(z) = H'(0) \cdot L(z)$ in $\overline{\mathcal{B}}_0'$. By $\pi : \mathbb{C} \to \mathbb{C}^*$, we can lift H and L to $\tilde{H} : \tilde{\mathcal{B}}_0' \to \tilde{\mathcal{B}}_0'$ and $\tilde{L} : \tilde{\mathcal{B}}_0' \to \mathbb{C}$ so that $\tilde{H}$ is a branch of $\tilde{\overline{\overline{h}}}^{-1}$, $\tilde{L} \circ \tilde{H}(z) = \tilde{L}(z) + const$ and $\tilde{L}'(z) \to 1$ as $\operatorname{Im} z \to +\infty$.

Now choose $\zeta_1 \in \tilde{\overline{\overline{h}}}^{-1}(\zeta_0) \cap \tilde{\mathcal{B}}_0'$, and define $\zeta_m = \tilde{H}^{m-1}(\zeta_1)$ for $m \geq 2$. Then $\operatorname{Im} \zeta_m \to +\infty$ $(m \to \infty)$, since $H^m \to 0$. The sequence $\{|\zeta_m - \zeta_{m-1}|\}$ is bounded, since $H(z) - z \to const$ as $\operatorname{Im} z \to +\infty$. The derivative can be estimated by the fact that when $m \to \infty$,

$$(\tilde{\overline{\overline{h}}}^m)'(\zeta_m) = \tilde{\overline{\overline{h}}}'(\zeta_1)/(\tilde{H}^{m-1})'(\zeta_1) = \tilde{\overline{\overline{h}}}'(\zeta_1)\tilde{L}'(\zeta_m)/\tilde{L}'(\zeta_1) \ \to\ \tilde{\overline{\overline{h}}}'(\zeta_1)/\tilde{L}'(\zeta_1)\ .$$

If $\overline{\mathcal{B}}_0'$ contains the critical value, we can use $\overline{\mathcal{B}}_\infty'$ and $I \circ \overline{\overline{h}} \circ I^{-1}$ instead of $\overline{\mathcal{B}}_0'$ and $\overline{\overline{h}}$ to prove the assertion.
(ii) This follows from Step 1 and the proof of (i). □

Step 3. If $\eta > 0$ is sufficiently large, then there exist an integer $N > \frac{1}{C_0}\eta^2$ and N distinct pairs of integers $(\ell_i,\ m_i)$ $(i = 1, \ldots, N)$ such that $m_i > 0$ and $T^{-\ell_i} W_{m_i} \subset S_\eta = \{z \in \mathbb{C} \mid |\operatorname{Re} z| < \eta,\ \eta < |\operatorname{Im} z| < 2\eta\}$ for $i = 1, \ldots, N$.

Proof. By Step 2, there is an m^* such that $\eta + 1 < |\operatorname{Im} \zeta_m| < 2\eta - 1$ if $m^* \leq m < m^* + [(\eta - 2)/C_0]$. For such an m, at least $[2\eta - 2]$ of $\{T^\ell W_m \mid \ell \in \mathbb{Z}\}$ are contained in S_η. The assertion follows, since $[(\eta - 2)/C_0] \cdot [2\eta - 2] > \frac{1}{C_0}\eta^2$ for large η. □

Step 4. It follows from §4.1 that if η is large, then $I \circ \psi$ is injective on S_η and the image $I \circ \psi(S_\eta)$ is contained in $\{z \in \mathbb{C} \mid \eta/2 < |z| < 3\eta\} - \mathbb{R}$.

For each $i (= 1, \ldots, N)$, we can choose a component V_i of $\pi^{-1}(\psi(T^{-\ell_i} W_{m_i}))$ $= (I \circ \pi)^{-1}(I \circ \psi(T^{-\ell_i} W_{m_i}))$ such that $V_i \subset S_\eta' = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z <$

$1,\ \frac{1}{2\pi}(\log\eta - \log 2) < |\operatorname{Im} z| < \frac{1}{2\pi}(\log\eta + \log 3)\}$. By the diagram in 4.5 and the definition of W_m, there are integers k_i ($i = 1, \ldots, N$) such that $T^{k_i}\widetilde{g}^{\ell_i}\widetilde{h}^{m_i}$ maps bijectively $\overline{V}_i$ onto $\overline{V}$.

If η is large, then by 3.1, $I \circ \varphi$ is injective on S'_η and the image $I \circ \varphi(S'_\eta)$ is contained in $\{z \in \mathbb{C} |\ \frac{\log\eta}{4\pi} < |z| < \frac{\log\eta}{\pi}\}$. Then define $U_i = \varphi(V_i)$ ($i = 1, \ldots, N$). Their closures $\overline{U}_1, \ldots, \overline{U}_N$ are disjoint, and contained in U, for η large.

By the construction of U_i, k_i, etc. and the diagram in 4.5, we see that $f^{k_i}g^{\ell_i}h^{m_i}$ maps $\overline{U}_i$ bijectively onto $\overline{U}$, for each i.

Step 5. Now we give the estimate of $(f^{k_i}g^{\ell_i}h^{m_i})'$. By the above diagram , we have the following factorization:

$$f^{k_i}g^{\ell_i}h^{m_i}|_{U_i} = (\varphi|_V) \circ (T^{k_i}\widetilde{g}^{\ell_i}\widetilde{h}^{m_i}|_{V_i}) \circ (I \circ \varphi|_{V_i})^{-1} \circ (I|_{U_i})$$

$$T^{k_i}\widetilde{g}^{\ell_i}\widetilde{h}^{m_i}|_{V_i} = (\pi|_V)^{-1} \circ (\overline{g}^{\ell_i}\overline{h}^{m_i} \circ I^{-1})|_{I\circ\pi(V_i)} \circ (I \circ \pi|_{V_i})$$

$$(\overline{g}^{\ell_i}\overline{h}^{m_i} \circ I^{-1})|_{I\circ\pi(V_i)} = (\psi|_{W_0}) \circ (\widetilde{\widetilde{h}}^{m_i}|_{W_{m_i}}) \circ (T^{\ell_i}|_{T^{-\ell_i}W_{m_i}}) \circ (I \circ \psi|_{T^{-\ell_i}W_{m_i}})^{-1} .$$

Note here that $I \circ \pi(V_i) = I \circ \psi(T^{-\ell_i}W_{m_i})$.

Let us estimate on the factors in the above. First of all, $\varphi|_V$, $\pi|_V$, $\psi|_{W_0}$ are independent of η and their derivatives have estimates from above and below, i.e.

$$C_2 < |(\varphi|_V)'| < C'_2,\ C_2 < |(\pi|_V)'| < C'_2,\ C_2 < |(\psi|_{W_0})'| < C'_2 ,$$

where $C_2, C'_2 > 0$ are constants independent of η. By 3.1 and 4.1, if η is large, then we have

$$\frac{1}{2} < |(I \circ \varphi|_{V_i})'| < 2,\ \frac{1}{2} < |(I \circ \psi|_{W_{m_i}})'| < 2 ,$$

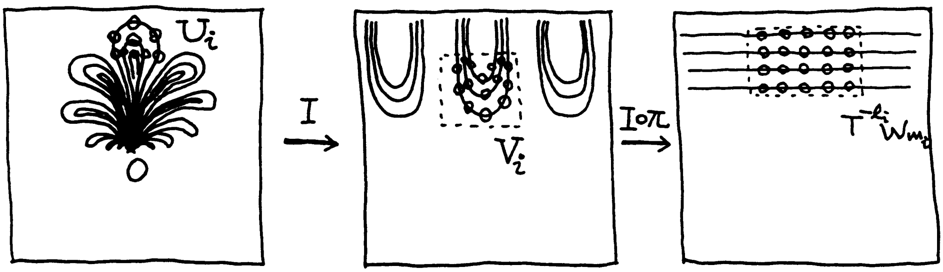
since $V_i \subset S'_\eta$ and $W_{m_i} \subset S_\eta$. The functions $I(z) = -1/z$, $I \circ \pi(z) = -e^{-2\pi i z}$, $T(z) = z + 1$ are estimated explicitly:

$$\left(\frac{\log\eta}{4\pi}\right)^2 < |(I|_{U_i})'| < \left(\frac{\log\eta}{\pi}\right)^2,\quad \pi\eta < |(I \circ \pi|_{V_i})'| < 6\pi\eta,\quad (T^{\ell_i})' = 1.$$

(Note that $U_i \subset \{z |\ \frac{\pi}{\log\eta} < |z| < \frac{4\pi}{\log\eta}\}$ and $V_i \subset S'_\eta$). Finally by Step 2, $\frac{1}{2}C_1 < |(\widetilde{\widetilde{h}}^{m}|_{W_m})'| < 2C'_1$.

Combining all the estimates, we conclude $C\eta(\log\eta)^2 < |(f^{k_i}g^{\ell_i}h^{m_i}|_{U_i})'| < C'\eta(\log\eta)^2$, where $C, C' > 0$ are constants independent of η. We have thus proved Proposition 5.1. $\square$

Remark. In order to understand the structure in the proof, let us make a "caricature" which was proposed by McMullen [Mc2]. First note that $I\circ\varphi|_{V_i}$, $I\circ\psi|_{T^{-\ell_i}W_{m_i}}$, $\overset{\approx}{h}^{m_i}|_{W_{m_i}}$ and $(\varphi|_V)\circ(\pi|_V)^{-1}\circ(\pi|_V)^{-1}$ had bounded effects in the estimate of the derivative. So let us pretend that they are all affine maps, and moreover $I\circ\varphi = id$, $I\circ\psi = id$ and $\overset{\approx}{h}^{m_i}(z) = z+\mu$, where μ is a constant with $\operatorname{Im}\mu < 0$. Then $T^{-\ell_i}W_{m_i} = \psi(T^{-\ell_i}W_{m_i})$ are distributed along the lattice generated by T and $\overset{\approx}{h}$. They are transformed by $(I\circ\pi)^{-1}$ and I^{-1} as in the figure.



Placement of U_i, V_i and $T^{-\ell_i}W_{m_i}$

The unbouded effects come from $I|_{U_i}$ and $I\circ\pi|_{V_i}$, and the latter is dominant. This observation, together with the figure, suggests that the phenomenon which we see here is very similar to McMullen's result [Mc1] that $\text{H-dim}(J(\lambda e^z)) = 2$ for $\lambda\in\mathbb{C}^*$.

Appendix. Note on the proof of §3 and §4

We give a brief sketch of the proof. In order to complete the proof, see [Sh] (§4, Proposition 5.3 and Appendix) or [La].

3.1. The functional equations in 3.1 mean that φ and Φ conjugate f to $T(w) = w+1$ in some regions. It is classically known (see [Mi], for example) that such φ and Φ exist at least in regions $\{w|\operatorname{Re} w < -R\}$ and $I^{-1}\{w|\operatorname{Re} w > R\}$, respectively, and that they satisfy the estimates. (Note here that $I\circ f\circ I^{-1}(w) = w+1+\frac{c}{w}+O(w^2)$ $(w\to\infty)$.)

Then it is possible to extend them to $\mathbb{C}$ or $\mathcal{B}$. The estimates in the regions stated in 3.1 can be obtained either by the classical estimate or by Koebe's distortion theorem. (See [Sh] Lemma A.2.4.)

3.2. The assertions in 3.2 are easy consequences of 3.1.

3.3. This follows from the facts that φ and Φ are diffeomorphic in $\{\mathrm{Re}\, w < -R\}$ and $I^{-1}\{\mathrm{Re}\, w > R\}$, respectively, and that f has only one critical point in $\mathcal{B}$. (The latter is used only here.)

3.4. In [Sh] (§4 and Appendix), we proved that there exist analytic mappings $\varphi_n = \varphi_{f_n} : \mathcal{D}om(\varphi_n) \to \mathbb{C}$ and $\widetilde{\mathcal{R}}_n = \widetilde{\mathcal{R}}_{f_n} : \mathcal{D}om(\widetilde{\mathcal{R}}_n) \to \mathbb{C}$ such that

$\mathcal{D}om(\varphi_n) = \{w \in \mathbb{C} \mid |\arg(w + \frac{1}{\alpha_n} - \xi_0)| < 2\pi/3\ \}$, where $\xi_0 > 0$ is a constant;

$\varphi_n(w+1) = f_n \circ \varphi_n(w)$ for $w \in \mathcal{D}om(\varphi_n)$;

$\varphi_n \to \varphi$ $(n \to \infty)$ uniformly on compact sets in $\mathbb{C}$;

$\mathcal{D}om(\widetilde{\mathcal{R}}_n) \nearrow \widetilde{\mathcal{B}}$ $(n \to \infty)$;

$\widetilde{\mathcal{R}}_n \circ T = T \circ \widetilde{\mathcal{R}}_n$;

if $w \in \mathcal{D}om(\varphi_n)$, $k, \ell \in \mathbb{N}$ and $\widetilde{\mathcal{R}}_n^{\ell}(w) + k = w' \in \mathcal{D}om(\varphi_n)$, then $f_n^k(\varphi_n(w)) = \varphi_n(w')$;

$\widetilde{\mathcal{R}}_n + \alpha_n \to \Phi \circ \varphi$ $(= \mathcal{E}_{f_0}$ in [Sh]) $(n \to \infty)$ uniformly on compact sets in $\widetilde{\mathcal{B}}$.

Let us prove 3.4 assuming these facts. If $w \in \widetilde{\mathcal{B}}$, Then w belongs both to $\mathcal{D}om(\widetilde{\mathcal{R}}_n)$ and to $\mathcal{D}om(\varphi_n)$ for large n and when $n \to \infty$,

$$\widetilde{\mathcal{R}}_n(w) + a_n = (\widetilde{\mathcal{R}}_n(w) + \frac{1}{\alpha_n}) + (a_n - \frac{1}{\alpha_n}) \to \Phi \circ \varphi(w) + \beta = \widetilde{g}(w)$$

and

$$f_n^{a_n}(\varphi_n(w)) = \varphi_n(\widetilde{\mathcal{R}}_n(w) + a_n) \to \varphi \circ \widetilde{g}(w) = g \circ \varphi(w).$$

For any compact set $K \subset \mathcal{B}$, there is a compact set $K' \subset \widetilde{\mathcal{B}}_0$ such that $\varphi(K')$ contains K in its interior. The above argument applies uniformly on K'. Hence, it follows that $f_n^{a_n} \to g$ uniformly on K, when $n \to \infty$.

Similarly one can prove that $\lim_{n\to\infty} f_n^{a_n+k}(z)$ exists for $z \in \mathcal{B}$ and $k \in \mathbb{Z}$, and coincides with $\varphi(\Phi(z) + \beta + k)$ (which is equal to $f^k \circ g$ if $k \geq 0$). Then the assertion for $f_n^{\ell a_n + k} = (f_n^{a_n+k}) \circ f_n^{(\ell-1)a_n+k}$ follows.

4.0. There is a "petal" D for $z = 0$ such that D is a simply connected open set in $\mathbb{C}^*$, $\overline{g}(D) \subset D$ and the parabolic basin of $z = 0$ is $\bigcup_{n\geq 0} \overline{g}^{-n}(D)$. Since $\overline{g} : \overline{\mathcal{B}} \to \mathbb{C}^*$ is a branched covering with one critical value, every component of $\overline{g}^{-1}(D)$ is simply connected and contains at most one critical point, and so are the components of $\overline{g}^{-n}(D)$. Then the assertions follow, since $\{f^{-n}(D)\}$ is increasing.

4.1-4.3. These are similar to 3.1-3.3.

4.4 and 4.5. Let $f_{n,k} = f_{\alpha(n,k)}$, where $\alpha(n,k) = 1/(k - 1/(b_n - \gamma_n))$, and define g_n, $\widetilde{g}_n$, $\overline{g}_n$ to be g, $\widetilde{g}$, $\overline{g}$ in §3 with $\beta = 1/(b_n - \gamma_n) \equiv \beta_n$. Then by §3, we have

$$f_{n,k}^k \to g_n \quad \text{when } k \to \infty$$

and $\overline{g}_n = e^{2\pi i\beta_n}\overline{g}$. Let $\widetilde{\mathcal{R}}_{n,k}$ be $\widetilde{\mathcal{R}}_n$ in the proof of 3.4 corresponding to $f_{n,k}$. Then $T^k\widetilde{\mathcal{R}}_{n,k} = \widetilde{\mathcal{R}}_{n,k} + k \to \widetilde{g}_n$ on $\widetilde{\mathcal{B}}$ as $k \to \infty$.

On the other hand, when $n \to \infty$, $\overline{g}_n \to \overline{g}$, since $\beta_n \to 0$. Hence as in 3.4, we have $\overline{g}_n^{b_n} \to \overline{h}$ on $\mathcal{B}'$ as $n \to \infty$. Since $\widetilde{g}_n$ and $\widetilde{h}$ are the lifts of $\overline{g}_n$ and $\overline{h}$ by π and commute with T, there exists a sequence of integers $\{q_n\}$ such that $T^{q_n}\widetilde{g}_n^{b_n} \to \widetilde{h}$ on $\pi^{-1}(\mathcal{B}')$ $(n \to \infty)$. If we choose a sequence of integers $\{k_n\}$ tending to infinity sufficiently fast, then on $\pi^{-1}(\mathcal{B}')$

$$\widetilde{\mathcal{R}}_{n,k_n}^{b_n} + (k_n b_n + q_n) = T^{q_n}(T^{k_n}\widetilde{\mathcal{R}}_{n,k_n})^{b_n} \to \widetilde{h} \ \ (n \to \infty).$$

Then it can be shown, as in the proof of 3.4, that for any $w \in \pi^{-1}(\mathcal{B}')$,

$$f_n^{p_n}(\varphi(w)) \to \varphi(\widetilde{h}(w)) \ \ (n \to \infty),$$

where $p_n = k_n b_n + q_n$. So there exists $\lim_{n\to\infty} f_n^{p_n}$ on $\breve{\mathcal{B}}'$, which is defined to be h. Then it satisfies $h \circ \varphi = \varphi \circ \widetilde{h}$ and commutes with f and g.

The assertions for $f^k g^\ell h^m$ can be proved similarly. For the commutation relations, note that $\overline{g}^\ell\overline{h}^m = \lim_{n\to\infty} \overline{g}_n^{mb_n+\ell}$ and $\widetilde{g}^\ell\widetilde{h}^m = \lim_{n\to\infty} T^{mq_n}\widetilde{g}_n^{mb_n+\ell}$, which are definitions if $\ell < 0$.

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