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Perturbations of Critical Fixed Points of Analytic Maps

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This paper has three parts. The first is concerned with an inequality relating the positions of the fixed point, the critical point, and the critical value obtained by a perturbation of an analytic function which has a critical point which is simultaneously a fixed point. The result is local in nature in that the analytic function need only be defined in a neighborhood of the fixed critical point. However, the motivation for this problem comes from a conjecture of Smale about polynomials, [4].

The conjecture states that, given a complex polynomial f of degree d and a non-critical point z, that for some critical point θ

$$(1) \qquad |f(z) - f(\theta)| \,/\, |(z - \theta)f'(z)| < 1.$$

Let $S(f, z, \theta)$ denote the left hand side of (1). For the polynomial $f(z) = z^d + (d/(d-1))z$, and the point $z = 0$, for any choice of critical point θ, $S(f, 0, \theta) = (d-1)/d$. This is possibly the worst case for the conjecture, which would mean that $(d-1)/d$ could replace 1 in (1). In [5], we showed that $(d-1)/d$ is an upper bound for any sufficiently small perturbation of $f(z) = z^d + (d/(d-1))z$ and $z = 0$. We also conjectured that the mean value conjecture might be strengthened to state that for any f of degree d, and any z, for some θ

$$(2) \qquad |S(f, z, \theta) - 1/2| \leq 1/2 - 1/d.$$

In the first part of this paper we will show that (2) is true for any sufficiently small perturbation of $f(z) = z^d + (d/(d-1))z$, $z = 0$.

In the second part of this paper we will consider a topological version of the mean value conjecture. That is, we will consider only critical points θ for which $f(\theta)$ is on the boundary of the largest disk centered at $f(z)$ on which a branch of f^{-1} can be defined. For these critical points, by applying the Koebe 1/4 theorem to f^{-1}, it was shown in [4] that $S(f,z,\theta) \le 4$. The topological version of the mean value conjecture raises the question of whether there is a better version of the Koebe 1/4 theorem for inverse branches of polynomials.

We will consider degrees $d = 3, 4$ where (2) is known to be true, [5]. We will show that the topological version of (2) is true for $d = 3$ and is false for $d = 4$. Whether the topological version of (1) is true is not known.

In the final section we will describe a quasi-conformal model for polynomials which will allow us to verify (2) for roots of polynomials whose critical values have norms that increase fast enough.

§1. Let $f(z)$ be a complex analytic function defined in a neighborhood of the point θ. Suppose that $f(\theta) = \theta$ and that $f'(\theta) = 0$ and $f''(\theta) \neq 0$. Let p be another point different than θ, not necessarily in the domain of f. Since θ is a fixed point of f, $|\theta - p| = |f(\theta) - p|$. Let h denote a complex analytic function which is a peturbation of f in a neighborhood of θ. For any sufficiently small perturbation h of f there will be a fixed point q, $(h(q) = q)$, and a critical point σ, $(h'(\sigma) = 0)$, near θ. The following theorem gives a sufficient condition on perturbations of f so that $|\sigma - p| \ge |h(\sigma) - p|$.

Theorem 1. *Suppose that* $\mathbf{R}((\theta - p)(f''(\theta)) < 0$. *Suppose that* $|\sigma - p| > |q - p|$. *Then if* h *is sufficiently near* f, $|h(\sigma) - p| \le |\sigma - p|$.

Proof. Without loss of generality, for the purposes of this proof, we can assume that $p = 0$. This follows because conjugating f by a translation does not change the second derivative, and translations preserve lengths. Since $p = 0$ and $p \neq \theta$, we can define an analytic function g in a neighborhood of θ by the formula $f(z) = zg(z)$. We are interested in the level curves of g. That is the curves defined by $|g(z)|$ =constant.

Lemma 1. *Let $f(z) = zg(z)$ and suppose $f'(\theta) = 0$ and $g'(\theta) \neq 0$. Then, at θ, the tangent vector $-\theta$ is orthogonal to the level curve of g.*

Proof. Let $w = g(z)$. The level curves of g are the preimages of circles in the w plane centered at 0. The radial vector field $V(w) = w$ pulls back by g to the vector field $g(z)/g'(z)$. Since g is conformal away from critical points we have that the vector field $g(z)/g'(z)$ is orthogonal to the level curves of g. Since $f'(z) = g(z) + zg'(z)$ and θ is a critical point of f, we see that the vector $-\theta$ is orthogonal to the level curve of g at θ. Note that $g'(\theta)$ is not zero because then $g(\theta) = 0$ and then $f(\theta) = 0$ which means that $\theta = 0$ since θ is a fixed point for f, and we have assumed that $\theta \neq p = 0$. Therefore the rays from 0 are transversal to the level curves of g near θ, and $|g(z)|$ increases in the direction of $-\theta$. Let us orient the level curves of g so that at θ the orientation agrees with the direction $-(\sqrt{-1})\theta$. Let k denote the curvature of the level curves of g with the given orientation. A calculation shows that

$$k = |g'(z)/g(z)|\,\mathbf{R}(1 - (g(z)g''(z)/(g'(z)^2)), \text{ see [2,p.359]}.$$

If we express this formula for k in terms of f and use that θ is both a fixed point and a critical point for f we obtain the formula

$$k = |1/\theta|\,\mathbf{R}(-1 - \theta f''(\theta)).$$

By hypothesis, $\mathbf{R}(\theta f''(\theta)) < 0$. Therefore, $k > -1/|\theta|$, which is the curvature of the circle C_θ with center 0 and which passes through θ. Since $-\theta$ is orthogonal to the g level curve at θ and $-\theta$ is also orthogonal to C_θ at θ, it follows that the g level curve is tangent to C_θ at θ. We conclude that in a small enough neighborhood of θ the level curve of g passing through θ is outside C_θ.

Suppose h is a perturbation of f. Define $j(z) = h(z)/z$. For sufficiently small perturbations h of f the level curves of j are transversal to the rays from 0, the level curve of j through σ is outside the circle C_σ which passes through σ with center 0. Let b denote the intersection of the ray through q and the level curve of j through σ. Let c denote the intersection of the ray through q and the circle C_σ. Note that $|j(z)|$ increases along the rays in the direction towards 0. Therefore $|j(c)| \geq |j(b)|$. By hypothesis, $|\sigma - 0| \geq |q - 0|$, so $|j(c)| \leq |j(q)|$. Since b and σ are on the same level curve of j, we conclude that $|j(\sigma)| \leq |j(q)|$. Since q is a fixed point for h we obtain $1 > |h(\sigma)/\sigma|$ or $|\sigma - 0| > |h(\sigma) - 0|$. Since we have assumed $p = 0$, this completes the proof of Theorem 1.

Remark. Theorem 1 is also true if all three inequality signs are reversed. The proof is analogous to the one given above.

Let us apply Theorem 1, and more particularly, the method of proof to the case of $f(z) = z^d + (d/(d-1))z$ and $p = 0$, which was the case discussed in the introduction. The critical points θ of f satisfy $\theta^{d-1} = -1/(d-1)$. For this polynomial $f(z)$ we find that $g(z) = z^{d-1} + d/(d-1)$ and $f''(\theta) = d(d-1)\theta^{d-2}$ which implies $\mathbf{R}(\theta f''(\theta)) < 0$. Without loss of generality we can restrict our attention to perturbations h of f which are degree d polynomials which satisfy $h(0) = 0$ and $h'(0) = d/(d-1)$. As was shown in [5], for any such perturbation h there is a critical point σ and a fixed point q near one of the critical points θ of f so that $|\sigma - 0| \geq |q - 0|$. From Thm. 1, we conclude that $|\sigma| > |h(\sigma)|$ and $S(h, 0, \sigma) < (d-1)/d$. This was already shown in [5]. Here we want to show in addition that

Theorem 2. *The stronger mean value conjecture (2) is true for any sufficiently small perturbation h of the above f with $z = 0$ and some critical point σ of h.*

Proof. Observe that the image by g of the level curve through θ is a circle centered at the origin. Since θ is fixed by f, $g(\theta) = 1$. Using the same notation as in the proof of Thm. 1, $g(C_{0\theta})$ is the circle of radius $1/(d-1)$ centered at $d/(d-1)$. For a perturbation h close to f, j will be a polynomial approximately the same as g. Therefore, $j(C_\sigma)$ is approximately a circle of radius $1/(d-1)$. So the curvature of $j(C_\sigma)$ is approximately $1/(d-1)$. Furthermore, $j(C_\sigma)$ is tangent at $j(\sigma)$ to the circle centered at 0 passing through σ. Let C_q be the circle centered at 0 passing through q. Like $j(C_\sigma)$, $j(C_q)$ is also approximately a circle with radius $1/(d-1)$. By hypothesis, C_σ is outside C_q and so $j(C_\sigma)$ is outside $j(C_q)$.

Since $h(q) = q$, $j(q) = 1$. As a first estimate, we suppose that $j(C_\sigma)$, is actually a circle of radius $1/(d-1)$ passing through 1, with center at a point e approximately equal to $d/(d-1)$. Denote any such circle by C. Let s denote the point where C is tangent to a circle centered at 0. So $s, 0$, and e are collinear. Let ϕ be the central angle at e between s and 1. Let $u = (1 + 1/(d-1))/2$.

Lemma 2. *For $d \geq 4$, there is a positive constant M, independent of ϕ, such that $|u - s| < 1 - u - M\phi^2$.*

Proof. Let A be the angle between the rays $0s$ and 01. Let r be the point in the interval [0,1] which is equidistant from 1 and s. So $|s - r| = 1 - r$. In triangle $0rs$, the interior angle at s is $A + \phi$. By the law of sines, $(1-r)/r = \sin(A)/\sin(A+\phi)$. In triangle $0, 1, e$, the interior angle at 1 is $\pi - (A+\phi)$. By the law of sines in this triangle $\sin(A)/\sin(\pi - (A + \phi)) = (1/(d-1))/|e| < 1/(d-1)$. Since $\sin(A+\phi) = \sin(\pi - (A+\phi))$ we conclude that $r > (d-1)/d$. For $d \geq 4$, $u < (d-1)/d$ so that $u < r$. In triangle u, r, s the interior angle at r is $\pi - (2A + \phi)$. Note that $A = 0(\phi)$ and that ϕ is near 0 since $\phi = 0$ for the uperturbed function f. Using the law of cosines in triangle u, r, s and the fact that $|1 - u| = |1 - r| + |r - u|$ we find that $|u - s| < |1 - u|$ and $|u - s|^2 - |1 - u|^2 = 0(\phi^2)$. Therefore, $|u - s| < |1 - u| - M\phi^2$ for some positive number M. This proves the lemma for $d \geq 4$.

Remark. The cases of $d = 3, 4$ of Thm. 2 are treated in [5].

Suppose that the curvature k of $j(C_\sigma)$, and that of $j(C_q)$, satisfy $|k - 1/(d-1)| < \varepsilon$. Let A_σ be the angle between the rays 01 and $0j(\sigma)$ and let A_q be the angle between 01 and $0t$, where t is an element of $j(C_q)$, with

$0, t, e$ collinear. Then $|A - A_\sigma|$ and $|A - A_q|$ are both less than $\pi\varepsilon$. Furthermore, the distance from 1 to t along $j(C_q)$ is $0(\phi)$. Therefore, $|s - t| < L\varepsilon\phi^2$, for some constant L, independent of ϕ and ε. Therefore, for small enough ϕ and ε, $|u - t| \leq |1 - u| - M\phi^2 + L\varepsilon\phi^2 < |1 - u|$. Since $j(C_\sigma)$ is outside of $j(C_q)$ we can make a similar estimate to show $|u - j(\sigma)| < |u - t| + N\varepsilon\phi^2$ for some N independent of ϕ and ε. This shows that for a sufficiently small perturbation h of f that $|j(\sigma) - u| < |1 - u|$.

In order to complete the proof of Theorem 2 we recall the normalization discussed just before the statement of Thm. 2 to obtain

$$S(h, 0, \sigma) = h(\sigma)/(\sigma h'(0)) = j(\sigma)(d - 1)/d.$$

Therefore, $((d-1)/d)\,|j(\sigma) - u| = |S(h, 0, \sigma) - 1/2|$. Also, $((d-1)/d)\,|1 - u| = 1/2 - 1/d$. Therefore $|S(h, 0, \sigma) - 1/2| < 1/2 - 1/d$ and the proof of Theorem 2 is complete.

§**2.** If A and B are first degree polynomials, then it is easy to check that

$$S(A \circ f \circ B, z, \theta) = S(f, B(z), B(\theta)).$$

Suppose f is a degree three polynomial that has two distinct critical points. Choose B so that the critical points are at $z = 1/2$ and $z = -1/2$, and choose A so that $f(1/2) = 1/2$ and $f(-1/2) = -1/2$. Then $f(z) = -2z^3 + (3/2)z$. In [5], we gave a topological description of polynomials, all of whose critical points are also fixed points. We will use that description to show that f satisfies the topological version of (2).

The topological description of $f(z) = -2z^3 + (3/2)z$ is as follows. There are two closed topological disks, one around $z = -1/2$ and one around $z = 1/2$ on each of which f is topologically conjugate to $z \to z^2$. In the interior of each disk the conjugation is analytic. The two disks have the point $z = 0$ as their intersection, and it is a repelling fixed point for f. The imaginary axis is invariant by f. Denote the positive imaginary axis by J_1 and the negative imaginary axis by J_4. The points $z = +(\sqrt{3})/2$ and $-(\sqrt{3})/2$ are the two other preimages of $z = 0$. See Fig. 1 as reference for the following part of the description of f. The curves J_2, J_6 are preimages of J_4 whereas, J_3, J_5 are preimages of J_1. The region between J_1 and J_2 and outside of the disk around $z = 1/2$ is mapped by f onto the part of the right half plane which lies outside the disk around $z = 1/2$. The region between J_2 and J_3 is mapped onto the half plane $\mathbf{R}(z) < 0$, sending $[(\sqrt{3})/2, \infty)$ onto $(-\infty, 0]$. Each of these maps is a homeomorphism. There are similar homeomorphisms of the regions outside the union of the two disks and between the curves J_i and J_{i+1}, $i = 1$ to 6, (note that J_7 means J_1).

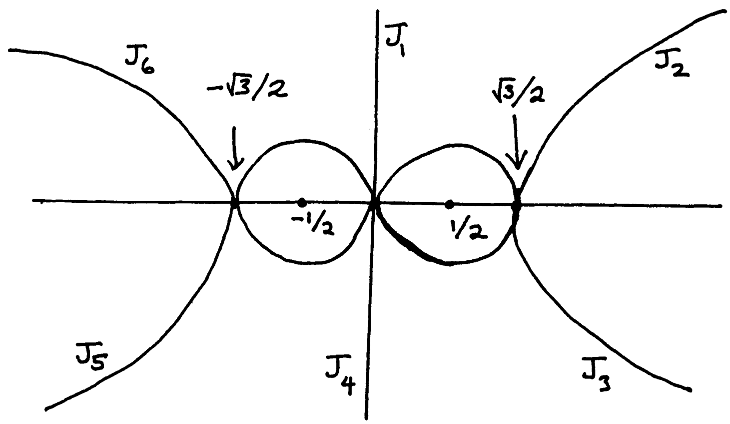

Figure 1

In studying the topological mean value conjecture, we make a choice of a non- critical point p. There are three points p_i, $i = 1, 2, 3$ with the same output $f(p)$. We draw a straight segment c from $f(p)$ to whichever critical value is closest to $f(p)$. Let us assume that $\mathbf{R}(f(p)) < 0$ so that $z = -1/2$ is the closest critical value. The preimage of c consists of three arcs I_i, $i = 1, 2, 3$. I_1 and I_2 have an endpoint at $-1/2$ and I_3 has an endpoint at the other preimage of $z = -1/2$ which $z = 1$. Since the segment c does not intersect J_1 and J_4 neither does any I_i intersect any J_j. The other endpoints of the I_i are the p_i. Then $\theta_- = -1/2$ is the critical point topologically related to p_1 and p_2, whereas $\theta_+ = 1/2$ is topologically related to p_3.

We can describe this correspondence in terms of the level curves $|f(z) - f(p)| =$ constant, which are preimages by f of circles centered at $f(p)$. $\{z: |f(z) - f(p)| = |f(p) + 1/2|\}$ is the preimage of the circle passing through θ_- and it has a component which is a topological figure-eight which contains both I_1 and I_2, one arc inside each loop of the figure-eight. The other component is a loop around p_3 which contains I_3.

The preimage of the circle through θ_+ is also a figure-eight. Each loop of this figure-eight contains one of the components of the preimage of the circle through θ_-, see Fig. 2.

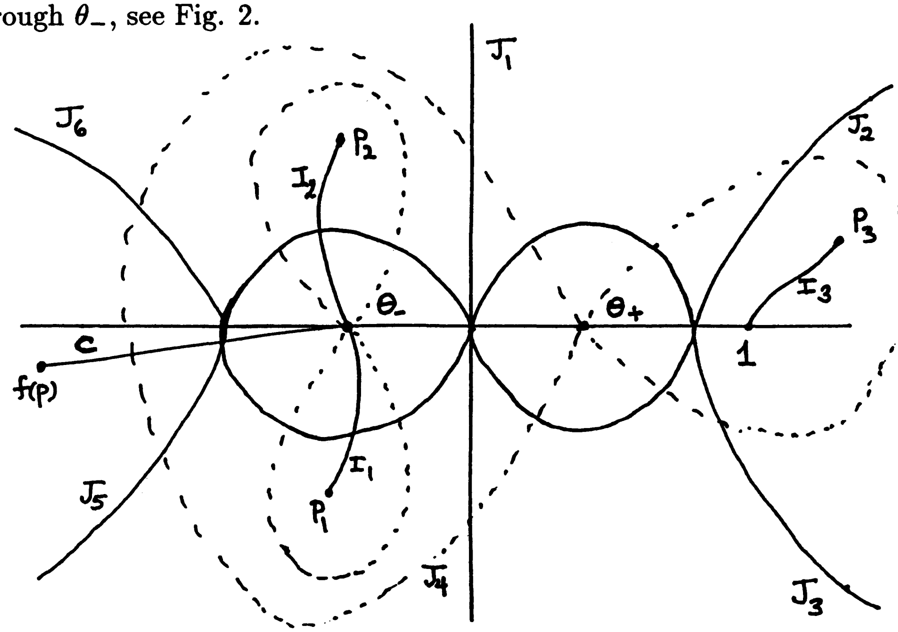

Figure 2

From the Fig. 2, we see that a branch of f^{-1} which sends $f(p)$ to p_3 can be defined on the disk centered at $f(p)$ and which has θ_+ on its boundary. Similarly, for p_1 and p_2, with θ_-. So we see that θ_j is topologically related to p_i, if $|\theta_j - p_i| \leq |\theta_{-j} - p_i|$.

From [5], we have that $S(f, p_i, \theta_j) - 1/2 = -(1/6)(\theta_j - p_i)/(\theta_{-j} - p_i)$, for $i = 1, 2, 3$, and for $j = +, -$. Therefore, if $|\theta_j - p_i| \leq |\theta_{-j} - p_i)|$, $|S(f, p_i, \theta_j) - 1/2| \leq 1/6 = 1/2 - 1/3$. This proves the topological version of (2) when $d = 3$.

For the case of $d = 4$, we will describe a case where the topological version of (2) is false. Consider $f(z) = -3z^4 + 4z^3$. Note that $z = 0$ and $z = 1$ are the only critical points and they are each fixed points for f. Since all the critical points are fixed we can describe a topological model for f. There are topological disks around $z = 0$ and $z = 1$ on which f is topologically conjugate to $z \to z^3$ and $z \to z^2$ respectively. The two disks have one point q in common, that is, $q = (-1 + \sqrt{13})/6$. This is a repelling fixed point as is $(-1 - \sqrt{13})/6$. There are three invariant expanding curves J_i, $i = 1, 2, 3$, see Fig. 3. Extend J_3 to q by adding on the invariant segment $[p, q]$. These curves bound three regions of the plane denoted by R_i, $i = 1, 2, 3$, where R_i is the region not bordered by J_i. The preimages of the region R_i will be denoted by R_i^{-1}. Let z be of the form $1 + t\sqrt{(-1)}$. With t a sufficiently large negative real number, one can check that $z \in R_1^{-1}$ since as $t \to -\infty$, $|f(z)| \to \infty$, and $\arg(f(z)) \to \pi^-$. In particular, the component of R_1^{-1} which contains z, also contains 0 in its boundary. Take a straight segment c joining the critical point 0 to $f(z)$. The segment c will have four preimage arcs I_i, three of which, say for $i = 1, 2, 3$, emanate from 0 and lie inside the three components of R_1^{-1}, respectively which have 0 in their boundary. We denote the other endpoint of I_i by z_i, for $i = 1, 2, 3$. The other component I_4, lies in the other component of R_1^{-1} which has as one boundary component a part of the positive real axis. One endpoint of I_4 is $z = 4/3$ and we will denote the other endpoint by z_4. We see that the critical point $z = 1$ is topologically related to z_4, whereas the critical point $z = 0$ is topologically related to the other three preimages of $f(z)$.

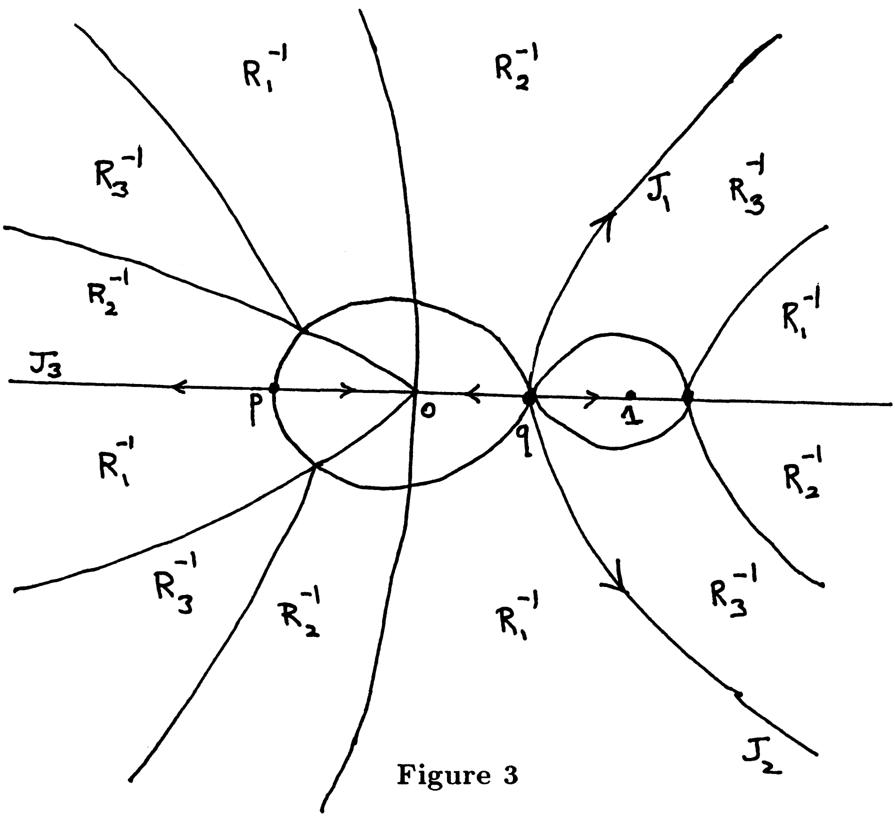

Figure 3

In [5], it is shown that

$$S(f, z, \theta_j) - 1/2 = (-1/4)\{(\theta_j - z)(\theta_j' - z)\}/\{(\theta_{j+1} - z)(\theta_{j+2} - z)\},$$

where $\theta_j' = 2G - \theta_j$, $G = (1/3)\sum_1^3(\theta_k)$. We can use this formula to calculate which critical points satisfy (2) for a given z. In our case, $\theta_1 = 0$, $\theta_2 = 0$, $\theta_3 = 1$, $G = 1/3$, $\theta_1' = \theta_2' = 2/3$ and $\theta_3' = -1/3$.

If $j = 1$ then $\{(\theta_1 - z)(\theta_1' - z)\}/\{(\theta_2 - z)(\theta_3 - z)\} = ((2/3) - z))/(1 - z)$. Therefore, if $\mathbf{R}(z) > 5/6$, $S(f, z, \theta_1)$ does not satisfy (2).

So, for $z = 1 + t\sqrt{(-1)}$ for t sufficiently negative, $|S(f, z, \theta_1) - 1/2| > 1/2 - 1/4$. However, θ_1 is the only critical point topologically related to z, and so the topological version of (2) is false for $d = 4$.

§**3.** Given a polynomial f of degree d, let $\theta_1, \dots, \theta_{d-1}$, denote the critical points of f. Assume that $|f(\theta_1)| < |f(\theta_2)| < \cdots < |f(\theta_{d-1})|$. We will decompose the domain of f into $d-1$ pieces and describe a simple model for each piece. The $d-1$ pieces are defined as the connected components of the complement of a set of $d-2$ loops C_i, $i = 1, \dots, d-2$.

The loops C_i are defined as follows. Choose u_i, $i = 1, \dots, d-2$, so that $|f(\theta_i)| < u_i < |f(\theta_{i+1})|$. Denote the connected component of the level curve of f passing through each θ_i by ℓ_1. For each i, $1 \leq i \leq d-1$, ℓ_i is a figure-eight. For each u_i, the level curve at level u_i has a connected component C_i which is a loop which together with the figure-eight through θ_i bounds an annulus.

The complement of $C = UC_i$ consists of $d-1$ connected components K_i. Each K_i contains exactly one critical point θ_i. There are only a small number of topological types for the K_i, which we will now catalog. Let D_i denote the open disk bounded by C_i.

The region $\overline{K}_1$ is a closed disk since it is the closure of the disk bounded by C_1. For $1 < i < d-1$, there are three possible topological types for $\overline{K}_i$, namely, a disk, a disk minus one disk, or a disk minus two disks. One boundary component of $\overline{K}_i$ is C_i. We will call it the outer boundary since $|f(z)|$ restricted to $\overline{K}_i$ is maximal along C_i. The level curve ℓ_i passing through θ_i consists of two loops. Inside each loop there may or may not be any critical points of f. If there are none in either loop then $\overline{K}_i$ is a closed disk. If there are critical points inside only one loop, then $\overline{K}_i$ is the closed annulus bounded by C_i and C_j, where θ_j is the critical point of largest critical value inside one loop of ℓ_i. If ℓ_i has critical points inside each of its loops then $\overline{K}_i = \overline{D}_i - \{D_j \cup D_k\}$ where θ_j, θ_k are the critical points inside each loop of largest critical value inside their respective loops.

Finally, the region $\overline{K}_{d-1}$ differs from the last cases, only in that, there is no outer boundary C_{d-1}. That is, $\overline{K}_{d-1} = \mathbf{C}$, for $d = 2$, or $\overline{K}_{d-1} = \mathbf{C} - D_j$ or $\overline{K}_{d-1} = \mathbf{C} - D_j - D_k$.

Suppose that for some i, $\overline{K}_i$ has three boundary loops C_i, C_j, and C_k. We will describe a set $K_i^* \subset \mathbf{C}$ and a polynomial function f_i defined on K_i^* so that there is a conformal map $h_i: K_i^* \to \overline{K}_i$ satisfying $f_i = f \circ h_i$. Let m_r =degree of f restricted to C_r, for $r = 1, \ldots, d-1$. The degree m_r is the number of roots of f which lie inside C_r. Define $f_i(z) = \lambda_i z^{m_j}(z-1)^{m_k}$, where λ_i is a constant chosen so that, at the critical point $\sigma_i = m_j/(m_j + m_k)$ of f_i, the critical value $f_i(\sigma_i) = f_i(\theta_i)$.

The level curve of f_i corresponding to u_j, i.e., $\{z: |f_i(z)| = u_j\}$ consists of two loops, one around $z = 0$ and one around $z = 1$.

Recall, $u_j < |f_i(\theta_i)|$ and that all the roots of f_i lie at $z = 0$ and $z = 1$. The loop around $z = 0$, denoted by C_j^*, will be one boundary component of K_i^*. Similarly, for the u_k level of f_i, there is a loop around $z = 1$, denoted by C_k^*, which will be another boundary component of K_i^*.

The outside boundary of K_i^* is defined to be the level curve of f_i at the level u_i, denoted by C_i^*, which is a single loop inside of which is the critical point σ_i.

K_i^* is the union of three annuli bounded by four curves: the level curve through σ_i, C_i^*, C_j^*, and C_k^*. Similarly $\overline{K}_i$ is the union of three annuli bounded by ℓ_i, C_i, C_j, and C_k. Both $f_i \,|\, K_i^*$ and $f_i \,|\, \overline{K}_i$ are branched mappings with the same critical value. On each pair of corresponding annuli, f_i and f are covering maps with the same image. Note that the degrees of f_i and f are the same on each pair of corresponding annuli. Define $h_i: K_i^* \to \overline{K}_i$, on each of the three closed annuli, as a mapping between covering spaces which satisfies $h_i(\sigma_i) = \theta_i$.

Therefore, $f_i \,|\, K_i^* = f \circ h_i$ is conformal since it is conformal on each of the three annuli and is well defined on the level curve passing through σ_i.

There is an oriented tree T associated to the pieces $\overline{K}_i$ by associating a vertex v_i to each $\overline{K}_i$ and an edge e_{im}, oriented from v_i to v_m provided that the outer boundary C_i of $\overline{K}_i$ is an inner boundary of the piece $\overline{K}_m$.

We can choose complex affine mappings A_i, $1 \le i \le d-1$, of $\mathbf{C}$ with the following property. The subsets $A_i(K_i^*)$ are deployed in such a way that, for each edge e_{im} of T, $A_i(K_i^*)$ is inside the loop $A_m((h_m)^{-1}(C_i))$, see Fig. 4.

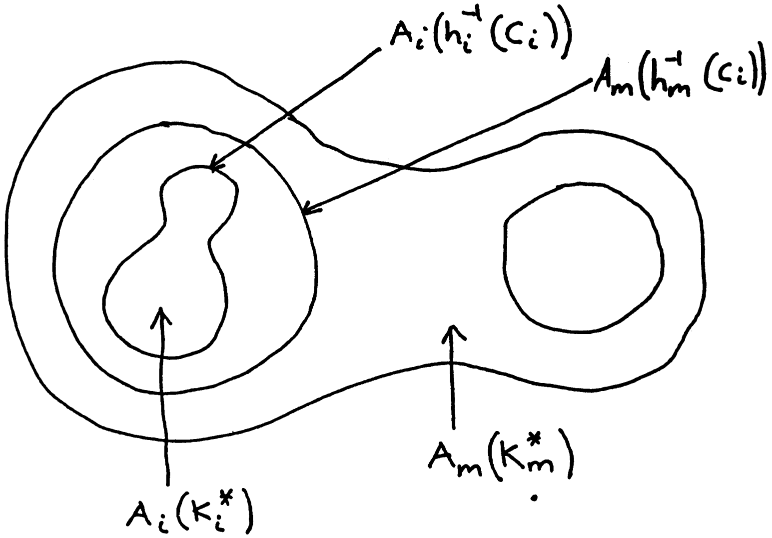

Figure 4

In general $A_i((h_i^{-1}(C_i))$ is not the same affine shape as $A_m((h_m^{-1}(C_i))$. However there is an analytic homeomorphism $\varepsilon_{im}: A_i((h_i)^{-1}(C_i)) \rightarrow A_m((h_m)^{-1}(C_i))$ defined by $\varepsilon_{im} = A_m \circ (h_m)^{-1} \circ h_i \circ (A_i)^{-1}$.

Suppose C is a boundary component of some K_i^*, corresponding to an f level curve at level u. For $\varepsilon > 0$, if $\varepsilon < ||u| - |f(\sigma_i)||$ then there is an ε-collar neighborhood of C in K_i^* defined by all level curves in K_i^* whose level is within ε of u. There is a product structure on the collar given by the level curves and the orthogonal trajectories to the level curves. For small enough ε, define $J_i^* = K_i^*$ minus an ε-collar around each boundary component of K_i^*. The region between the outer boundary of $A_i(J_i^*)$ and the inner boundary of $A_m(J_m^*)$ is topologically an annulus denoted by E_{im}. On one of the boundary components of E_{im}, $f_m \circ (A_m)^{-1}$ is defined and is a covering map of degree m_i onto the circle of radius $u_i + \varepsilon$, centered at 0. On the other boundary component of E_{im}, $f_i \circ (A_i)^{-1}$ is defined and it is a covering map of degree m_i onto the circle of radius $u_i - \varepsilon$, centered at 0. Therefore, there is a covering map f_{im} of degree m_i from E_{im} onto the annulus between the circles of radii $u_i - \varepsilon$ and $u_i + \varepsilon$ which restricts to $f_i \circ (A_i)^{-1}$ and $f_m \circ (A_m)^{-1}$ on the two boundary components respectively. This covering map can be chosen in many ways.

Suppose that for each E_{im} a covering map f_{im} has been chosen. Then we can define a branched covering f^* from $\mathbf{C}$ to $\mathbf{C}$ as follows. If z is in J_r^*, let $f^*(z) = f_r \circ (A_r)^{-1}$. If z is in E_{im} let $f^*(z) = f_{im}$. There is a complex structure on the domain of f^* which makes f^* holomorphic as follows. On each J_r^* one has the usual complex structure of $\mathbf{C}$ and on each E_{im} define the complex structure as the pullback of the usual one by f_{im}.

The Beltrami coefficient determined by this map is given by $\mu = 0$ on each J_r^* and $\mu = (\partial f_{im}/\partial \overline{z})/(\partial f_{im}/\partial z)$ otherwise. By the measurable riemann mapping theorem, there is a quasi-conformal homeomorphism $\phi\colon \mathbf{C} \to \mathbf{C}$ so that $f^* \circ \phi$ is holomorphic in the usual complex strucutre on $\mathbf{C}$. Since $f^* \circ \phi$ has the branching structure of a polynomial, it is a polynomial. Furthermore $f^* \circ \phi$ has the same critical values as f and the same critical level curve combinatorics as recorded by T. Therefore, there is a homeomorphism $A\colon \mathbf{C} \to \mathbf{C}$ so that $f = f^* \circ \phi \circ A$. Since $f^* \circ \phi$ and f are polynomials of degree d, A is a degree one polynomial. By the Ahlfors- Bers theorem, ϕ will be close to the identity if μ is sufficiently close to 0.

Theorem 3. *Given f, if ϕ is sufficiently C^0 close to the identity then the topological version of (2) is true for $S(f, z, \theta)$ where z is a root of f.*

Proof. We can assume that A =identity since $S(f \circ A, z, \theta) = S(f, A(z), A(\theta))$. Note that ϕ is automatically conformal in the interior of the pieces J_r^*, so that if ϕ is uniformly near the identity then $\phi'(z)$ is near one. Consider a piece K_i^* where $f_i(z) = \lambda_i z^{m_j}(z-1)^{m_k}$ and where $m_j = 1$. For f_i, the critical point $\sigma_i = 1/(1+m_k)$ is topologically related to $z = 0$ which is a root of f_i. For ease of notation denote m_k simply by m. Then $S(f_i, 0, \sigma_i) = (m/(1+m))^m$. For all m, $(m/(1+m))^m$ is contained in the interval $(1/e, 1/2]$, so $|S(f_i, 0, \sigma_i) - 1/2| < 1/2 - 1/d$, for $d \geq 3$. On the piece K_i, $f = f_i \circ \phi$.

Let $z = \phi^{-1}(0)$ and $\theta_i = \phi^{-1}(\sigma_i)$. Then z is a root of f and θ_i is a critical point of f topologically related to z. A calculation shows that $S(f, z, \theta_i) = \{(0 - \sigma_i)/((z - \theta_i)\phi'(z))\}S(f_i, 0, \sigma_i)$. For ϕ sufficiently near the identity $\phi'(z)$ is near 1 and we conclude that $|S(f_i, 0, \sigma_i) - 1/2| < 1/2 - 1/d$, for $d \geq 3$.

Proposition 1. *For a polynomial f of degree d, if $|f(\theta_{i+1})/f(\theta_i)|$ is sufficiently large for all i then the topological version of (2) is true for $S(f, z, \theta)$ where z is a root of f.*

Proof. This is a corollary of Thm. 3 if we show that in the construction above that we can make the Beltrami coefficient sufficiently close to zero.

Let $f(z) = \lambda z^a(z-1)^b$. For $\delta > 0$ there is a degree one polynomial A so that $f \circ A(z) = z^a(\varphi(z))$, with $|\varphi(z) - 1| < \delta$ for all z in some neighborhood of $z = 0$. Furthermore there is a degree one polynomial B so that $f \circ B(z) = z^{a+b}(\psi(z))$, with $|\psi(z) - 1| < \delta$, for all z in some neighborhood of $z = \infty$. Suppose that for $|z| < R$, $f_m \circ A(z) = z^n(\varphi(z))$ and for $|z| > r$, $f_i \circ B(z) = z^n(\psi(z))$. One shows that φ and ψ can be made C^1 close to 1 near $|z| = R$, r respectively by the Cauchy inequalities. If $|f(\theta_{i+1})/f(\theta_i)|$ is sufficiently large then we can choose u_i, so that $|f(\theta_i)| < u_i < |f(\theta_{i+1})| < |f(\theta_m)|$, and so that $A_m(C_i) \subset \{z: |z| < R\}$ and $A_i(C_i) \subset \{z: |z| > r\}$. With these choices the covering map f_{im} must agree on the respective components of the boundary of E_{im} with $z^n(\varphi(z))$ and $z^n(\psi(z))$. Since both are C^1 close to the constant function 1, f_{im} can be chosen so that $f_{im} = z^n(H(z))$ where $H(z)$ is C^1 close to 1 and hence the Beltrami coefficient μ on E_{im} is close to zero. For a fixed degree d, there are only a finite number of polynomials $z^a(z-1)^b$ with $a + b \leq d$. Therefore, there is a constant L so that if $|f(\theta_{i+1})/f(\theta_i)| > L$ for all i, the proposition is true.

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