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INDEX THEOREM FOR ELLIPTIC PAIRS

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Table of Contents

Introduction

In this series of papers, we investigate the relative index theorem in the framework of algebraic analysis.

On a complex manifold X, let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and F an $\mathbb{R}$-constructible sheaf (for the underlying real analytic structure of X). The complex

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om(F, \mathcal{O}_X))$$

is the complex of solutions of the system of PDE represented by $\mathcal{M}$ in the sheaf of generalized holomorphic functions associated to F. For example, if X is the complexification of a real analytic manifold M, and $F = \mathrm{or}_M$, we get the complex of Sato's hyperfunction solutions, or else if F is $\mathbb{C}$-constructible, we find a complex of ramified holomorphic solutions.

A natural problem is to find conditions under which such a complex has finite dimensional global cohomology and then to compute the corresponding Euler-Poincaré characteristic.

In our first paper, we prove the finiteness theorem when $(\mathcal{M}, F)$ has compact support and is "elliptic", i.e.:

$$\mathrm{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X$$

where $\mathrm{char}(\mathcal{M})$ is the characteristic variety of $\mathcal{M}$, $SS(F)$ is the micro-support of F and $T^*_X X$ is the zero section of the cotangent bundle.

In fact, we give a relative version of this finiteness result together with the associated duality theorem and Künneth formula. Our methods rely upon results of functional analysis over a sheaf of Fréchet algebras which are developped in the last paper of this volume.

With finiteness, duality and Künneth formula at hand, we have all the basic tools needed to get an index formula along the line of the Lefschetz fixed point theorem. Such an approach is developped in our second paper. We attach a "microlocal Euler class"

$$\mu\mathrm{eu}(\mathcal{M}, F) \in H^{2\dim X}_{\mathrm{char}\mathcal{M}+SSF}(T^*X; \mathbb{C})$$

to any elliptic pair $(\mathcal{M}, F)$ and prove that, under natural assumptions, this class is compatible with direct images, inverse images and external products. In particular, it is the microlocal product of a class $\mu\mathrm{eu}(\mathcal{M})$ attached to $\mathcal{M}$ and a class $\mu\mathrm{eu}(F)$ attached to F, this last one being nothing but the Kashiwara's Lagrangian cycle of F. We also give the index formula:

$$\chi(\mathrm{R}\Gamma(X; R\mathcal{H}om(\mathcal{M} \otimes F; \mathcal{O}_X))) = \int \mu\mathrm{eu}(\mathcal{M}, F)_{|T^*_X X} = \int_{T^*X} \mu\mathrm{eu}(\mathcal{M}) \cup \mu\mathrm{eu}(F).$$

Note that $(\mathcal{M}, \mathbb{C}_X)$ is always elliptic. Hence our results contain many results of $\mathcal{D}$-module theory. Moreover, choosing $\mathcal{M} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{G}$ for a coherent $\mathcal{O}_X$-module $\mathcal{G}$ allows us to recover classical results of analytic geometry.

When $F = \mathbb{C}_M$, our results for the pair $(\mathcal{M}, F)$ give an index theorem for elliptic systems and we discuss its relations with the Atiyah-Singer theorem.

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Elliptic pairs I. Relative finiteness and duality

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Elliptic Pairs I. Relative Finiteness and Duality

PIERRE SCHAPIRA   JEAN-PIERRE SCHNEIDERS

Contents

1 Introduction

Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds, $\mathcal{M}$ a coherent module over the ring $\mathcal{D}_X$ of differential operators on X, F an $\mathbb{R}$-constructible object on X. In this first paper, we give a criterion insuring that the derived direct images of the $\mathcal{D}_X$-module $F \otimes \mathcal{M}$ are coherent $\mathcal{D}_Y$-modules, and we prove related duality and Künneth formulas. Part of these results were announced in [20, 21].

In [22], making full use of these results, we shall associate to $(\mathcal{M},F)$ a characteristic class and show its compatibility with direct image, thus obtaining an index theorem generalizing (in some sense) the Atiyah-Singer index theorem as well as its relative version [1, 3].

Let us describe our results with more details, beginning with the non-relative case for the sake of simplicity.

An elliptic pair on a complex analytic manifold X is the data of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ and an $\mathbb{R}$-constructible sheaf F on X (more precisely, objects of the derived categories), these data satisfying the transversality condition

$$\operatorname{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X. \tag{1.1}$$

Here $\operatorname{char}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$, $SS(F)$ the micro-support of F (see [12]) and $T^*_X X$ the zero section of the cotangent bundle T^*X.

This notion unifies many classical situations. For example, if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then the pair $(\mathcal{M},\mathbb{C}_X)$ is elliptic. If U is an open subset of X with smooth boundary ∂U, the pair $(\mathcal{M},\mathbb{C}_U)$ is elliptic if and only if ∂U is non characteristic for $\mathcal{M}$. If X is the complexification of a real analytic manifold M, then $(\mathcal{M},\mathbb{C}_M)$ is an elliptic pair if and only if $\mathcal{M}$ is elliptic on M in the classical sense. If F is $\mathbb{R}$-constructible on X, then $(\mathcal{O}_X,F)$ is an elliptic pair. If $\mathcal{G}$ is a coherent $\mathcal{O}_X$-module, we can associate to it the coherent $\mathcal{D}_X$-module $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X$, and the results obtained for the elliptic pair $(\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X,\mathbb{C}_X)$ will give similar results for $\mathcal{G}$. See §8 for a more detailed discussion.

If $f : X \longrightarrow Y$ is a morphism of complex analytic manifolds, we generalize the preceding definition and introduce the notion of an f-elliptic pair, replacing in (1.1) $\operatorname{char}(\mathcal{M})$ by $\operatorname{char}_f(\mathcal{M})$, the f-characteristic variety of $\mathcal{M}$ (this set was already defined in [19] when f is smooth).

The main results of this paper assert that if the pair $(\mathcal{M},F)$ is f-elliptic, f is proper on $\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(F)$ and $\mathcal{M}$ is endowed with a good filtration, then:

1) the direct image (in the sense of $\mathcal{D}$-modules) $\underline{f}_!(\mathcal{M} \otimes F)$ has $\mathcal{D}_Y$-coherent cohomology,

2) the duality morphism

$$\underline{f}_!(D'F \otimes \underline{D}_X\mathcal{M}) \longrightarrow \underline{D}_Y \underline{f}_!(\mathcal{M} \otimes F)$$

is an isomorphism (here, $\underline{D}$ denotes the dualizing functor for $\mathcal{D}$-modules and D' is the simple dual for sheaves),

3) there is a Künneth formula for elliptic pairs,

4) direct image commutes with microlocalization.

See Theorem 4.2, Theorem 5.15, Theorem 6.7 and Theorem 7.5 below for more details.

In fact, we obtain these results in a relative situation over a smooth complex manifold S, working with the rings of relative differential operators. This relative setting makes notations a little heavy but it gives us the freedom on the base manifold we need in the proofs. Even if we want the final result over a base manifold reduced to a point, in the proofs, we need to use other bases. So, it is better to work in a relative situation everywhere. Moreover, the base change Theorem 6.5 is a natural way to get the Künneth formula for elliptic pairs.

The idea of the proof of the finiteness result goes as follows.

First, using the graph embedding, we are reduced to prove the theorem for a closed embedding (this one does not offer much difficulty) and for a projection. Then, using the same trick as in [8], we reduce to the case $Y = S$. Then it remains to treat the case where $X = Z \times S$, $f : X \longrightarrow S$ is the second projection, $\mathcal{M}$ is a $\mathcal{D}_{X|S}$-module endowed with a good filtration and $F = G \boxtimes \mathbb{C}_S$ where G is an $\mathbb{R}$-constructible sheaf on Z. We call it the projection case and we have to prove that in this case $Rf_!(F \otimes \mathcal{M} \otimes_{\mathcal{D}_{X|S}} \mathcal{O}_X)$ is $\mathcal{O}_S$ coherent and $\mathcal{O}_S$ dual to $Rf_! R\mathcal{H}om_{\mathcal{D}_{X|S}}(F \otimes \mathcal{M}, \Omega_{X|S}[d_X - d_S])$.

For that purpose, we "trivialize" F by replacing it by a bounded complex of sheaves of the form $\oplus_\alpha \mathbb{C}_{U_\alpha}$, the U_α's being relatively compact subanalytic open subsets of X satisfying the regularity condition:

$$D'(\mathbb{C}_{U_\alpha}) = \mathbb{C}_{\overline{U}_\alpha}.$$

This construction is made possible thanks to the triangulation theorem and a result of Kashiwara [11].

Next, we consider the relative realification $\mathcal{M}_{\mathbb{R}|S}$ of $\mathcal{M}$ obtained by adding the relative Cauchy-Riemann system to the $\mathcal{D}_{Z\times S|S}$-module $\mathcal{M}$ and remark that since $\mathcal{M}$ is assumed to be good we may always find a resolution of $\mathcal{M}_{\mathbb{R}|S}$ by finite free $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-modules near subsets of $Z^{\mathbb{R}} \times S$ of the form $K \times \Delta$ where K is a compact subset of Z and Δ is an open polydisc in S. Moreover, since the solutions of the relative Cauchy-Riemann system are the same in the sheaves of analytic functions, differentiable functions or distributions with holomorphic parameters in S, we can compute the holomorphic solutions of $\mathcal{M}$ as the relative analytic, differentiable or distributional solutions of $\mathcal{M}_{\mathbb{R}|S}$.

Now, the elliptic hypothesis insures the regularity theorem, that is, the isomorphism

$$F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X \xrightarrow{\sim} R\mathcal{H}om(D'F, \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X).$$

Applying $Rf_!$ to this isomorphism, we shall compute both sides using the trivialization of F and a finite free resolution of the relative realification of $\mathcal{M}$ using analytic (resp.

differentiable) solutions for the left (resp. right) hand side. This will give us a continuous $\mathcal{O}_S$-linear quasi-isomorphism

$$\mathcal{R}_1^{\cdot} \xrightarrow{\sim} \mathcal{R}_2^{\cdot} \tag{1.2}$$

where the components of the left (resp. right) hand side are DFN-free (resp. FN-free) topological modules over the Fréchet algebra $\mathcal{O}_S$. The coherence then follows from an extension of Houzel's finiteness theorem [7] due to one of the authors [25]. Note that we found no way of applying the original Houzel's theorem in our situation since it is not obvious to find the requested chain of nuclear quasi-isomorphisms for a given elliptic pair.

The duality result is proved along the same lines once we have a clear construction of the general duality morphism which makes it easy to check its compatibility with the various simplifications and transformations used in the proof.

Note that the hypothesis that the $\mathcal{D}$-module $\mathcal{M}$ is endowed with a good filtration could be relaxed by using cohomological descent techniques as in [24]. However, doing so would have cluttered the proof with unessential technical difficulties. This is why we have preferred to stay to a simpler setting, sufficient for all known applications.

Our theorems provide a wide generalization of many classical results as shown in the last section.

In particular, we obtain Grauert's theorem [6] (in the smooth case) on direct images of coherent $\mathcal{O}$-modules and the corresponding duality result of Ramis-Ruget-Verdier [15, 16]. Since we treat $\mathcal{D}$-modules, we are allowed to "realify" the manifolds by adding the Cauchy-Riemann system to the module, and the rigidity of the complex situation disappears, which makes the proofs much simpler and, may be, more natural than the classical ones.

We also obtain Kashiwara's theorem [9] on direct images of coherent $\mathcal{D}$-modules as well as its extension to the non-proper case of [8] (whose detailed proof had never been published) and the corresponding duality result of [23, 24].

In the absolute case, we regain and generalize many well-known theorems concerning regularity, finiteness or duality for $\mathcal{D}$-modules (in particular those of [2, 13, 14]), see §8 for a more detailed discussion.

2 Elliptic pairs and regularity

2.1 Relative $\mathcal{D}$-modules

In this section, we recall some basic facts about relative $\mathcal{D}$-modules.

In the sequel, by an analytic manifold we mean a complex analytic manifold X of finite dimension d_X. Keeping the notations of [12], we denote by

$$\tau : TX \longrightarrow X \qquad \text{and} \qquad \pi : T^*X \longrightarrow X$$

the tangent and cotangent bundles of X.

To every complex analytic map $f : X \longrightarrow Y$, we associate the natural maps

$$\begin{array}{ccccc} TX & \underset{f'}{\longrightarrow} & X \times_Y TY & \underset{f_\tau}{\longrightarrow} & TY \\ T^*X & \underset{{}^tf'}{\longleftarrow} & X \times_Y T^*Y & \underset{f_\pi}{\longrightarrow} & T^*Y. \end{array}$$

Let S be an analytic manifold. A *relative analytic manifold over* S is an analytic manifold X endowed with a surjective analytic submersion $\epsilon_X : X \longrightarrow S$. We often use the notation $X|S$ for such an object when we want to avoid confusion on the basis and set for short $d_{X|S} = d_X - d_S$.

A *morphism* $f : X|S \longrightarrow Y|S$ *of relative analytic manifolds* is the data of a complex analytic map $f : X \longrightarrow Y$ such that $\epsilon_Y \circ f = \epsilon_X$.

Let $X|S$ be a relative analytic manifold over S.

Since $\epsilon : X \longrightarrow S$ is smooth, the map

$$TX \underset{\epsilon'}{\longrightarrow} X \times_S TS$$

is surjective. Its kernel is thus a sub-bundle of TX. We denote it by $TX|S$ and call it the *relative tangent bundle of* $X|S$. Its holomorphic sections form the sheaf $\Theta_{X|S}$ of vertical holomorphic vector fields on $X|S$. Recall that a holomorphic vector field θ is vertical if and only if

$$\theta(h \circ \epsilon_X) = 0$$

for any section h of $\mathcal{O}_S$. The dual map

$$X \times_S T^*S \underset{{}^t\epsilon'}{\longrightarrow} T^*X$$

is injective. Its cokernel is thus a quotient-bundle of T^*X which is isomorphic to the dual of $TX|S$. This is the *relative cotangent bundle of* $X|S$, we denote it by $T^*X|S$ and denote by

$$p_{X|S} : T^*X \longrightarrow T^*X|S$$

the canonical projection. The holomorphic sections of $\bigwedge^p T^*X|S$ form the sheaf $\Omega^p_{X|S}$ of relative holomorphic differential forms of degree p. To shorten the notations, we set

$$\Omega_{X|S} = \Omega^{d_{X|S}}_{X|S}.$$

To every morphism $f : X|S \longrightarrow Y|S$, we associate the natural maps

$$\begin{array}{ccccc} TX|S & \underset{f'}{\longrightarrow} & X \times_Y TY|S & \underset{f_\tau}{\longrightarrow} & TY|S \\ T^*X|S & \underset{{}^tf'}{\longleftarrow} & X \times_Y T^*Y|S & \underset{f_\pi}{\longrightarrow} & T^*Y|S. \end{array}$$

Note that we use the same notations as in the non-relative case since the context will avoid any confusion.

The subring of $\mathcal{H}om_{\mathbb{C}_X}(\mathcal{O}_X, \mathcal{O}_X)$ generated by the derivatives along vertical holomorphic vector fields and multiplication with holomorphic functions is denoted by $\mathcal{D}_{X|S}$. We call it the *ring of relative differential operators on* $X|S$.

The basic algebraic properties of $\mathcal{D}_{X|S}$ are easily obtained using the usual filtration/graduation techniques. We will not review them here and refer the reader to [18, 19].

As usual, we denote by $\mathrm{Mod}(\mathcal{D}_{X|S})$ the abelian category of left $\mathcal{D}_{X|S}$-modules and by $\mathrm{Coh}(\mathcal{D}_{X|S})$ the full subcategory of coherent modules. The category $\mathrm{Coh}(\mathcal{D}_{X|S})$ is a thick subcategory of $\mathrm{Mod}(\mathcal{D}_{X|S})$ (i.e. it is full and stable by kernel, cokernel and extensions).

A coherent $\mathcal{D}_{X|S}$-module $\mathcal{M}$ is *good* if, in a neighborhood of any compact subset of X, $\mathcal{M}$ admits a finite filtration by coherent $\mathcal{D}_{X|S}$-submodules $\mathcal{M}_k$ ($k = 1, \ldots, \ell$) such that each quotient $\mathcal{M}_k/\mathcal{M}_{k-1}$ can be endowed with a good filtration. We denote by $\mathrm{Good}(\mathcal{D}_{X|S})$ the full subcategory of $\mathrm{Coh}(\mathcal{D}_{X|S})$ consisting of good $\mathcal{D}_{X|S}$-modules. This definition ensures that $\mathrm{Good}(\mathcal{D}_{X|S})$ is the smallest thick subcategory of $\mathrm{Mod}(\mathcal{D}_{X|S})$ containing the modules which can be endowed with good filtrations on a neighborhood of any compact subset of X.

We denote by $\mathbf{D}(\mathcal{D}_{X|S})$ the derived category of $\mathrm{Mod}(\mathcal{D}_{X|S})$ and by $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X|S})$ its full triangulated subcategory consisting of objects with bounded amplitude. The full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X|S})$ consisting of objects with coherent (resp. good) cohomology modules is denoted by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{X|S})$ (resp. $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{X|S})$).

We introduce similar notations with the ring $\mathcal{D}_{X|S}$ replaced by the opposite ring $\mathcal{D}^{\mathrm{op}}_{X|S}$ to deal with right $\mathcal{D}_{X|S}$-modules. Since the categories $\mathrm{Mod}(\mathcal{D}_{X|S})$ and $\mathrm{Mod}(\mathcal{D}^{\mathrm{op}}_{X|S})$ are equivalent, we will work only in the most convenient one depending on the problem at hand.

In the sequel, we will often need to work with bimodule structures. Let k be a field. Recall that if A and B are k-algebras, giving a left (A,B)-bimodule structure on an abelian group M is just giving M a left structure of A-module and a left structure of B-module such that

$$\begin{aligned} a \cdot (b \cdot m) &= b \cdot (a \cdot m) \\ (c \cdot a) \cdot (b \cdot m) &= (c \cdot b) \cdot (a \cdot m) \end{aligned}$$

for any $a \in A$, $b \in B$, $c \in k$ and $m \in M$. Hence, it is equivalent to consider that M is endowed with a structure of $A \otimes_k B$-module. Using this point of view it is easy to extend to bimodules the notions and notations defined usually for modules. For example, we will denote by $\mathrm{Mod}(\mathcal{D}_{X|S} \otimes \mathcal{D}_{X|S})$ the category of left $\mathcal{D}_{X|S}$ bimodules and by $\mathbf{D}(\mathcal{D}_{X|S} \otimes \mathcal{D}_{X|S})$ the corresponding derived category.

Let $f : X|S \longrightarrow Y|S$ be a morphism of relative analytic manifolds over S.

Recall that

$$\mathcal{D}_{X|S \to Y|S} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_{Y|S}$$

has a natural structure of left $\mathcal{D}_{X|S}$-module compatible with its structure of right $f^{-1}\mathcal{D}_{Y|S}$-module. Using this transfer module, we may define the relative proper direct image of an object $\mathcal{M}$ of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}^{\mathrm{op}}_{X|S})$ by the formula

$$\underline{f}_{|S!}(\mathcal{M}) = Rf_!(\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S \to Y|S})$$

It is an object of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{Y|S}^{\mathrm{op}})$.

Recall also that if $\mathcal{M}$ (resp. $\mathcal{N}$) is a right (resp. left) $\mathcal{D}_{X|S}$-module then there is on $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ a unique structure of right $\mathcal{D}_{X|S}$-module such that

$$\begin{aligned}(m \otimes n) \cdot \theta &= m \cdot \theta \otimes n - m \otimes \theta \cdot n \\ (m \otimes n) \cdot h &= m \cdot h \otimes n = m \otimes h \cdot n\end{aligned}$$

for any sections m, n, θ and h of $\mathcal{M}$, $\mathcal{N}$, $\Theta_{X|S}$ and $\mathcal{O}_X$ respectively.

In the same way, if $\mathcal{N}$, $\mathcal{P}$ are two left $\mathcal{D}_{X|S}$-modules then there is on $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{P}$ a unique structure of left $\mathcal{D}_{X|S}$-module such that

$$\begin{aligned}\theta \cdot (n \otimes p) &= \theta \cdot n \otimes p + n \otimes \theta \cdot p \\ h \cdot (n \otimes p) &= h \cdot n \otimes p = n \otimes h \cdot p\end{aligned}$$

for any sections n, p, θ and h of $\mathcal{N}$, $\mathcal{P}$, $\Theta_{X|S}$ and $\mathcal{O}_X$ respectively.

Finally, recall the following exchange lemma which will be useful in the sequel.

Lemma 2.1 *If $\mathcal{M}$ is a right $\mathcal{D}_{X|S}$-module and $\mathcal{N}$, $\mathcal{P}$ are left $\mathcal{D}_{X|S}$-modules then the map*

$$\begin{aligned}\mathcal{M} \otimes_{\mathcal{D}_{X|S}} (\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{P}) &\longrightarrow (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}) \otimes_{\mathcal{D}_{X|S}} \mathcal{P} \\ m \otimes (n \otimes p) &\mapsto (m \otimes n) \otimes p\end{aligned}$$

is a canonical isomorphism.

Let X be a relative analytic manifold over S. Recall that the characteristic variety of a coherent $\mathcal{D}_{X|S}$-module $\mathcal{M}$ is a conic analytic subset of $T^*X|S$ denoted by $\mathrm{char}_{X|S}(\mathcal{M})$ and that

$$\mathrm{char}(\mathcal{D}_X \otimes_{\mathcal{D}_{X|S}} \mathcal{M}) = p_{X|S}^{-1}\mathrm{char}_{X|S}(\mathcal{M}).$$

Hence theorem 11.3.3 of [12] gives the equality

$$SS(R\mathcal{H}om_{\mathcal{D}_{X|S}}(\mathcal{M}, \mathcal{O}_X)) = p_{X|S}^{-1}\mathrm{char}_{X|S}(\mathcal{M}).$$

The sheaf $\Omega_{X|S}$ of relative holomorphic differential forms of maximal degree is canonically endowed with a structure of right $\mathcal{D}_{X|S}$-module which is compatible with its structure of $\mathcal{O}_X$-module and characterized by the fact that, for every open subset U of X, one has $\omega.\theta = -L_\theta\omega$ if $\omega \in \Omega_{X|S}(U)$ and θ is a vertical vector field defined on U.

Definition 2.2 The *dualizing complex* for right $\mathcal{D}_{X|S}$-modules is the complex of right $\mathcal{D}_{X|S}$-bimodules defined by setting

$$\mathcal{K}_{X|S} = \Omega_{X|S}[d_{X|S}] \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}$$

and using the natural structure of right $\mathcal{D}_{X|S}$-bimodule on the sheaf $\Omega_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}$.

The *dual* of an object $\mathcal{M}$ of $\mathbf{D}^-(\mathcal{D}_{X|S}^{\mathrm{op}})$ is

$$R\mathcal{H}om_{\mathcal{D}_{X|S}}(\mathcal{M}, \mathcal{K}_{X|S})$$

as an object of $\mathbf{D}^+(\mathcal{D}_{X|S}^{\mathrm{op}})$. We denote it by $\underline{D}_{X|S}(\mathcal{M})$.

The functor $\underline{D}_{X|S}$ is the *dualizing functor* for right $\mathcal{D}_{X|S}$-modules.

As in algebraic geometry, the terminology used in the preceding definition is justified by the following biduality result.

Lemma 2.3 *There is a canonical sheaf involution of $\Omega_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}$ interchanging its two right $\mathcal{D}_{X|S}$-module structures.*

Proof: Let us consider the sheaf $\mathcal{D}_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}$ where the tensor product uses the left $\mathcal{O}_X$ module structures of the two copies of $\mathcal{D}_{X|S}$. This sheaf is obviously endowed with one structure of left $\mathcal{D}_{X|S}$-module and two structures of right $\mathcal{D}_{X|S}$-module which are compatible with each other.

The involution

$$\begin{array}{ccc} \mathcal{D}_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S} & \longrightarrow & \mathcal{D}_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S} \\ P \otimes Q & \mapsto & Q \otimes P \end{array}$$

exchanges the two right structures and preserves the left one.

Tensoring over $\mathcal{D}_{X|S}$ with $\Omega_{X|S}$ using its right structure and the left structure of $\mathcal{D}_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}$ and applying the exchange lemma 2.1 gives us the requested involution. □

Proposition 2.4 *For any object $\mathcal{M}$ of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{X|S})$, the canonical arrow*

$$\mathcal{M} \longrightarrow R\mathcal{H}om_{\mathcal{D}_{X|S}}(R\mathcal{H}om_{\mathcal{D}_{X|S}}(\mathcal{M}, \mathcal{K}_{X|S}), \mathcal{K}_{X|S})$$

deduced from the involution of the preceding lemma is an isomorphism.

Proof: Since $\mathcal{M}$ is locally isomorphic to a bounded complex of finite free right $\mathcal{D}_{X|S}$-modules it is sufficient to prove the result for $\mathcal{M} = \mathcal{D}_{X|S}$ where it is an easy consequence of the preceding lemma and the fact that $\Omega_{X|S}$ is a locally free $\mathcal{O}_X$ module of rank one. □

It follows that the characteristic variety does not change by duality:

Proposition 2.5 *If $\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{X|S})$ then one has*

$$\mathrm{char}_{X|S}(\mathcal{M}) = \mathrm{char}_{X|S}(\underline{D}_{X|S}\mathcal{M}).$$

2.2 Relative f-characteristic variety

In this subsection, we consider a morphism $f : X|S \longrightarrow Y|S$ of relative analytic manifolds over S and define the relative characteristic variety $\mathrm{char}_f(\mathcal{M})$ of a coherent $\mathcal{D}^{\mathrm{op}}_{X|S}$-module $\mathcal{M}$. First, we consider the case of a relative submersion where such a variety was already defined in [19] for $S = \{\mathrm{pt}\}$. Next, by using the graph embedding, we extend this definition to the general case. Finally, we show how the relative characteristic variety controls the micro-support of $\mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S \to Y|S}$.

Let $f : X|S \longrightarrow Y|S$ be a relative analytic submersion over S. Since f is smooth, we have the following exact sequence of vector bundles on X:

$$0 \longrightarrow X \times_Y T^*Y|S \underset{\psi_f}{\longrightarrow} T^*X|S \underset{\phi_f}{\longrightarrow} T^*X|Y \longrightarrow 0.$$

Working as in paragraph III.1.3 of [19] we get the following lemmas.

Lemma 2.6 *Assume $\mathcal{M}_0$ is a coherent $\mathcal{D}_{X|Y}$-module. Then*

$$\mathrm{char}_{X|S}(\mathcal{D}_{X|S} \otimes_{\mathcal{D}_{X|Y}} \mathcal{M}_0) = \phi_f^{-1}\mathrm{char}_{X|Y}(\mathcal{M}_0).$$

Lemma 2.7 *Assume $\mathcal{M}$ is a coherent $\mathcal{D}_{X|S}$-module and assume $\mathcal{M}_0$, $\mathcal{N}_0$ are two coherent $\mathcal{D}_{X|Y}$-submodules of $\mathcal{M}$ which generates it as a $\mathcal{D}_{X|S}$-module then*

$$\mathrm{char}_{X|Y}(\mathcal{M}_0) = \mathrm{char}_{X|Y}(\mathcal{N}_0).$$

Hence, we may introduce the following definition.

Definition 2.8 Let $\mathcal{M}$ be a coherent $\mathcal{D}_{X|S}$-module. One defines the *relative characteristic variety* $\mathrm{char}_{f|S}(\mathcal{M})$ *of* $\mathcal{M}$ *with respect to* f to be the subset of $T^*X|S$ which coincide on $T^*U|S$ with $\phi_f^{-1}\mathrm{char}_{U|Y}(\mathcal{M}_0)$ for any open subset U and any coherent $\mathcal{D}_{U|Y}$-submodule $\mathcal{M}_0$ of $\mathcal{M}_{|U}$ which generates $\mathcal{M}_{|U}$ as a $\mathcal{D}_{U|S}$-module.

It is clear that $\mathrm{char}_{f|S}(\mathcal{M})$ is a closed conic analytic subvariety of $T^*X|S$ and that

$$\mathrm{char}_{f|S}(\mathcal{M}) = \mathrm{char}_{f|S}(\mathcal{M}) + \psi_f(X \times_Y T^*Y|S).$$

The functor $\mathrm{char}_{f|S}$ is additive:

Proposition 2.9 *If $f : X \longrightarrow Y$ is a relative analytic submersion over S and if*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow 0$$

is an exact sequence of coherent $\mathcal{D}_{X|S}$-modules then

$$\mathrm{char}_{f|S}(\mathcal{M}) = \mathrm{char}_{f|S}(\mathcal{L}) \cup \mathrm{char}_{f|S}(\mathcal{N}).$$

In the sequel we will need the following lemma essentially due to [8].

Lemma 2.10 *Let $f : X|S \longrightarrow Y|S$ be a relative analytic submersion over S and let K be a compact subset of X. Assume $\mathcal{M}$ is a $\mathcal{D}_{X|S}$-module which admits a good filtration in a neighborhood of K. Then, in a neighborhood of K, $\mathcal{M}$ has a left resolution by $\mathcal{D}_{X|S}$-modules of the form*

$$\mathcal{D}_{X|S} \otimes_{\mathcal{D}_{X|Y}} \mathcal{N}$$

where the $\mathcal{D}_{X|Y}$-module $\mathcal{N}$ admits a good filtration and is such that

$$\phi_f^{-1}\mathrm{char}_{X|Y}(\mathcal{N}) \subset \mathrm{char}_{f|S}(\mathcal{M}).$$

Proof: In this proof, we always work in some neighborhood of K.

Since $\mathcal{M}$ admits a good filtration, we can find a coherent $\mathcal{O}_X$-submodule $\mathcal{M}_0$ of $\mathcal{M}$ which generates it as a $\mathcal{D}_{X|S}$-module. Set $\mathcal{N}_0 = \mathcal{D}_{X|Y}\mathcal{M}_0$. By construction, $\mathcal{N}_0$ is a $\mathcal{D}_{X|Y}$-submodule of $\mathcal{M}$ which generates it as a $\mathcal{D}_{X|S}$-module. Obviously, $\mathcal{N}_0$ admits a good filtration. Moreover, by definition,

$$\phi_f^{-1}\mathrm{char}_{X|Y}(\mathcal{N}_0) = \mathrm{char}_{f|S}(\mathcal{M}).$$

The kernel $\mathcal{K}$ of the canonical $\mathcal{D}_{X|S}$-linear epimorphism

$$\mathcal{D}_{X|S} \otimes_{\mathcal{D}_{X|Y}} \mathcal{N}_0 \longrightarrow \mathcal{M} \longrightarrow 0$$

is a $\mathcal{D}_{X|S}$-module which admits a good filtration and we have

$$\mathrm{char}_{f|S}(\mathcal{K}) \subset \mathrm{char}_{f|S}(\mathcal{D}_{X|S} \otimes_{\mathcal{D}_{X|Y}} \mathcal{N}_0) = \mathrm{char}_{f|S}(\mathcal{M}).$$

We may thus start over the same construction with $\mathcal{M}$ replaced by $\mathcal{K}$ and build the requested resolution by induction. □

Now, by using the graph factorization, we will define the notion of relative characteristic variety for a map which is not necessarily a relative submersion.

Let $f : X|S \longrightarrow Y|S$ be any morphism of relative analytic manifolds.

Denote by

$$X \underset{i}{\longrightarrow} X \times_S Y \underset{q}{\longrightarrow} Y$$

the relative graph factorization of f.

First, we notice:

Lemma 2.11 *Assume f is a relative submersion and $\mathcal{M}$ is a coherent $\mathcal{D}_{X|S}$-module. Then*

$$\mathrm{char}_{f|S}(\mathcal{M}) = {}^t i' i_\pi^{-1}\mathrm{char}_{q|S}(\underline{i}_{|S!}(\mathcal{M})).$$

Hence, for a general f, the following definition is a natural extension of our previous one.

Definition 2.12 For any coherent $\mathcal{D}_{X|S}$-module $\mathcal{M}$, we set

$$\mathrm{char}_{f|S}(\mathcal{M}) = {}^t i' i_\pi^{-1}\mathrm{char}_{q|S}(\underline{i}_{|S!}(\mathcal{M})).$$

As usual, for an object $\mathcal{M}$ of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{X|S})$, we also set

$$\mathrm{char}_{f|S}(\mathcal{M}) = \bigcup_{j\in\mathbb{Z}} \mathrm{char}_{f|S}(\mathcal{H}^j(\mathcal{M})).$$

We also introduce similar definitions for right $\mathcal{D}_{X|S}$-modules.

Note that $\mathrm{char}_{f|S}(\mathcal{M})$ is a closed conic analytic subset of $T^*X|S$ and that

$$\mathrm{char}_{f|S}(\mathcal{M}) = \mathrm{char}_{f|S}(\mathcal{M}) + \psi_f(X \times_Y T^*Y|S).$$

A link between the relative characteristic variety and the micro-local theory of sheaves is given in the following theorem:

Theorem 2.13 *Let $f : X|S \longrightarrow Y|S$ be a morphism of relative analytic manifolds over S and assume $\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{X|S})$. Then*

$$SS(\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \subset p^{-1}_{X|S}\mathrm{char}_{f|S}(\mathcal{M}).$$

Proof: Consider the graph factorization of f:

$$X \underset{i}{\longrightarrow} X \times_S Y \underset{q}{\longrightarrow} Y.$$

By Proposition 5.4.4 of [12], if F is a sheaf on X, then $SS(i_*F)$ is the natural image of $SS(F)$. Hence, in view of the definition of $\mathrm{char}_{f|S}$, it is enough to prove the inclusion:

$$SS(i_!(\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S})) \subset p^{-1}\mathrm{char}_{q|S}(\underline{i}_{|S!}(\mathcal{M}))$$

where we write p instead of $p_{X\times_S Y|S}$. Since

$$i_!(\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \simeq \underline{i}_{|S!}(\mathcal{M}) \otimes^L_{\mathcal{D}_{X\times_S Y|S}} \mathcal{D}_{X\times_S Y|S\to Y|S},$$

we have reduced the proof to the case where f is a relative submersion, what we shall assume now.

Since the problem is local on X and

$$\mathrm{char}_{f|S}(\mathcal{M}) = \bigcup_{j\in\mathbb{Z}} \mathrm{char}_{f|S}(\mathcal{H}^j(\mathcal{M})),$$

we may assume $\mathcal{M}$ is a coherent right $\mathcal{D}_{X|S}$-module. Using Lemma 2.10, we are then reduced to consider the case where $\mathcal{M} = \mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}$, for a coherent right $\mathcal{D}_{X|Y}$-module $\mathcal{M}_0$. Now,

$$\begin{aligned}
\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S} &\simeq \mathcal{M}_0 \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S\to Y|S} \\
&\simeq (\mathcal{M}_0 \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{O}_X) \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_{Y|S}.
\end{aligned}$$

This last sheaf is locally on X a direct sum of an infinite number of copies of $\mathcal{M}_0 \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{O}_X$. Applying [12] Exercise V.5(i) (which is an easy consequence of Proposition 5.1.1(3) [loc. cit.]) we get successively

$$\begin{aligned}
SS(\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) &= SS(\mathcal{M}_0 \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{O}_X) \\
&= SS(\mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_X \otimes^L_{\mathcal{D}_X} \mathcal{O}_X) \\
&= \mathrm{char}(\mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_X) \\
&= p^{-1}\phi_f^{-1}\mathrm{char}_{X|Y}(\mathcal{M}_0) \\
&= p^{-1}\mathrm{char}_{f|S}(\mathcal{M})
\end{aligned}$$

where the third equality comes from theorem 11.3.3 of [12]. □

2.3 Relative elliptic pairs

We shall now define the main object of study of this paper.

Let $\mathbf{D}(X)$ denote the derived category of the category of sheaves of $\mathbb{C}$-vector spaces on X and let $\mathbf{D}^{\mathrm{b}}(X)$ denote the full triangulated subcategory of complexes with bounded amplitude.

Recall that a sheaf F of $\mathbb{C}$-vector spaces is $\mathbb{R}$-constructible if there is a subanalytic stratification of X along the strata of which $H^j(F)$ is a locally constant sheaf of finite rank for any $j \in \mathbb{Z}$. Following [12], we denote by $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(X)$ consisting of complexes with $\mathbb{R}$-constructible cohomology sheaves. We say for short that an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$ is an $\mathbb{R}$-constructible complex. For such an object, SS(F) is a closed subanalytic Lagrangian subset of $T^*X^{\mathbb{R}}$ where $X^{\mathbb{R}}$ denotes X considered with its underlying real analytic manifold structure. We shall identify $T^*X^{\mathbb{R}}$ with $(T^*X)^{\mathbb{R}}$ as for example in [12] and simply denote it by T^*X. In this paper, we will have to consider most of the time the simple dual $D'F$ of F and not its Poincaré-Verdier dual DF. Recall that since X is an oriented topological manifold of dimension $2d_X$:

$$D'F = R\mathcal{H}om(F, \mathbb{C}_X) \qquad \text{and} \qquad DF = R\mathcal{H}om(F, \omega_X) \simeq R\mathcal{H}om(F, \mathbb{C}_X[2d_X])$$

so the two duals coincide up to shift. Since F is constructible, we have the local biduality isomorphism $F \xrightarrow{\sim} D'D'F$.

Definition 2.14 Let $f : X|S \longrightarrow Y|S$ be a morphism of relative analytic manifolds over S. A pair $(\mathcal{M},F)$ is a *relative f-elliptic pair* if:

- $\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{X|S})$,
- F is an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$,
- $p^{-1}\mathrm{char}_{f|S}(\mathcal{M}) \cap SS(F) \subset T^*_X X$.

Such a pair is *good* if moreover $\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{X|S})$. Its *support* is the set $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$. When f is the canonical map $\epsilon_X : X|S \longrightarrow S|S$ we will say for short that $(\mathcal{M},F)$ is a *(good) relative elliptic pair on* $X|S$. When $S = \{\mathrm{pt}\}$, we drop the word "relative" in the preceding definitions.

Since $\mathrm{char}_{f|S}(\mathcal{M})$ contains $\mathrm{char}_{X|S}(\mathcal{M})$, a relative f-elliptic pair is a relative elliptic pair. Moreover, on a neighborhood of $\mathrm{supp}\,\mathcal{M}$,

$$SS(F) \cap X \times_S T^*S \subset T^*_X X.$$

In particular, an elliptic pair $(\mathcal{M},F)$ on X is the data of a complex of coherent right $\mathcal{D}_X$-modules $\mathcal{M}$ and an $\mathbb{R}$-constructible complex F such that

$$\mathrm{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X.$$

We shall see in §8 below why this notion is a natural generalization of that of an elliptic system on a real manifold. There, we will also explain why Theorem 2.15 below may be considered as a generalization of the classical regularity theorem for elliptic systems.

Theorem 2.15 *Let $(\mathcal{M},F)$ be an f-elliptic pair. Then the canonical morphism*

$$F \otimes (\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \longrightarrow R\mathcal{H}om(D'F, \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}),$$

induced by the morphism $F \longrightarrow D'D'F$, is an isomorphism.

Proof: By [12] Proposition 5.4.14, we know that if G belongs to $D^b(X)$ (and F is $\mathbb{R}$-constructible as above), the natural morphism

$$F \otimes G \longrightarrow R\mathcal{H}om(D'F, G)$$

is an isomorphism as soon as

$$SS(F)^a \cap SS(G) \subset T^*_X X.$$

Hence, the conclusion follows from Theorem 2.13 by applying the preceding result to

$$G = \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}.$$

□

When $Y = S$, we get:

Corollary 2.16 *Let $\mathcal{M}$ and F be objects of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{X|S})$ and $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$ respectively. Assume the transversality condition*

$$p^{-1}\mathrm{char}_{X|S}(\mathcal{M}) \cap SS(F) \subset T^*_X X$$

where $p : T^*X \longrightarrow T^*X|S$ is the canonical projection. Then the canonical morphism

$$F \otimes (\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X) \longrightarrow R\mathcal{H}om_{\mathcal{D}_{X|S}}(D'F, \mathcal{M} \otimes_{\mathcal{D}_{X|S}} \mathcal{O}_X)$$

is an isomorphism.

Definition 2.17 The *dual* of a relative pair $(\mathcal{M},F)$ is the pair $(\underline{D}_{X|S}(\mathcal{M}),D'F)$.

It follows from this definition that a relative pair is f-elliptic if and only if so is its dual pair.

3 Tools

If X is a complex analytic manifold, we have already encountered $X^{\mathbb{R}}$, the real underlying analytic manifold to X. Here, we shall also make use of $\overline{X}$, the complex manifold with $X^{\mathbb{R}}$ as underlying space for which the holomorphic functions are the anti-holomorphic functions on X. Recall that $X \times \overline{X}$ is a natural complexification of $X^{\mathbb{R}}$ via the diagonal embedding.

3.1 Dolbeault complexes with parameters

Let Z and S be complex analytic manifolds and let $q_Z : Z \times S \longrightarrow Z$ be the second projection.

We will denote by $\mathcal{A}_{Z\times S|S}$ (resp. $\mathcal{F}_{Z\times S|S}$, $\mathcal{D}b_{Z\times S|S}$) the sheaf of real analytic functions (resp. infinitely differentiable functions, distributions) on $Z \times S$ which are holomorphic in S.

We will also set

$$\mathcal{D}_{Z^{\mathbb{R}}\times S|S} = (\mathcal{D}_{Z\times\overline{Z}\times S|S})_{|Z^{\mathbb{R}}\times S}.$$

using the diagonal embedding of $Z^{\mathbb{R}}$ in $Z \times \overline{Z}$. Locally, operators in $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$ are of the form

$$\sum_{\alpha,\beta} a_{\alpha,\beta}(z,\overline{z},s) D_z^{\alpha} D_{\overline{z}}^{\beta}$$

where $a_{\alpha,\beta}(z,\overline{z},s)$ is a section of $\mathcal{A}_{Z\times S|S}$; $(z : U \longrightarrow \mathbb{C}^{d_Z})$ and $(s : V \longrightarrow \mathbb{C}^{d_S})$ being holomorphic local coordinate systems on Z and S respectively.

For any $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-module $\mathcal{M}$ we will consider the parametric Dolbeault complex

$$\mathcal{A}^{\cdot,\cdot}_{Z\times S|S}(\mathcal{M})$$

defined by setting

$$\mathcal{A}^{p,q}_{Z\times S|S}(\mathcal{M}) = q_Z^{-1}\mathcal{A}^{p,q}_Z \otimes_{q_Z^{-1}\mathcal{A}_Z} \mathcal{M}$$

the formulas for the differentials being given locally by

$$\begin{aligned} \partial : \mathcal{A}^{p,q}_{U\times S|S}(\mathcal{M}) &\longrightarrow \mathcal{A}^{p+1,q}_{U\times S|S}(\mathcal{M}) \\ a^{p,q}\otimes m &\longmapsto \partial a^{p,q}\otimes m + \sum_{i=1}^{d_Z} dz^i \wedge a^{p,q} \otimes D_{z^i} m \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \overline{\partial} : \mathcal{A}^{p,q}_{U\times S|S}(\mathcal{M}) &\longrightarrow \mathcal{A}^{p,q+1}_{U\times S|S}(\mathcal{M}) \\ a^{p,q}\otimes m &\longmapsto \overline{\partial} a^{p,q}\otimes m + \sum_{i=1}^{d_Z} d\overline{z}^i \wedge a^{p,q} \otimes D_{\overline{z}^i} m \end{aligned} \tag{3.2}$$

where $(z : U \longrightarrow \mathbb{C}^{d_Z})$ is a holomorphic local coordinate system on Z. Obviously, this definition is independent on the chosen local coordinate system.

When $\mathcal{M}$ is equal to $\mathcal{A}_{Z\times S|S}$ (resp. $\mathcal{F}_{Z\times S|S}$, $\mathcal{D}b_{Z\times S|S}$) we will denote the corresponding parametric Dolbeault complex simply by $\mathcal{A}^{\cdot,\cdot}_{Z\times S|S}$ (resp. $\mathcal{F}^{\cdot,\cdot}_{Z\times S|S}$, $\mathcal{D}b^{\cdot,\cdot}_{Z\times S|S}$). Of course, the natural maps

$$\Omega^p_{Z\times S|S} \longrightarrow \mathcal{A}^{p,\cdot}_{Z\times S|S} \longrightarrow \mathcal{F}^{p,\cdot}_{Z\times S|S} \longrightarrow \mathcal{D}b^{p,\cdot}_{Z\times S|S}$$

are quasi-isomorphisms.

Let $\mathcal{N}$ be another $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-module. Using the natural structure of $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-module on the sheaf $\mathcal{N}\otimes_{\mathcal{A}_{Z\times S|S}}\mathcal{M}$ we define the parametric Dolbeault complex of $\mathcal{M}$ with coefficients in $\mathcal{N}$ by the formula

$$\mathcal{N}^{\cdot,\cdot}(\mathcal{M}) = \mathcal{A}^{\cdot,\cdot}_{Z\times S|S}(\mathcal{N}\otimes_{\mathcal{A}_{Z\times S|S}}\mathcal{M}).$$

The associated simple complex is the parametric de Rham complex of $\mathcal{M}$ with coefficients in $\mathcal{N}$. We denote it by $\mathcal{N}^{\cdot}(\mathcal{M})$.

In this paper, we will only use the preceding notions when $\mathcal{N}$ is $\mathcal{F}_{Z\times S|S}$ or $\mathcal{D}b_{Z\times S|S}$. In this case, we have of course

$$\begin{aligned}\mathcal{F}^{p,q}_{Z\times S|S}(\mathcal{M}) &= \mathcal{F}^{p,q}_{Z\times S|S}\otimes_{\mathcal{A}_{Z\times S|S}}\mathcal{M}\\ \mathcal{D}b^{p,q}_{Z\times S|S}(\mathcal{M}) &= \mathcal{D}b^{p,q}_{Z\times S|S}\otimes_{\mathcal{A}_{Z\times S|S}}\mathcal{M}\end{aligned}$$

and the differentials ∂ and $\overline{\partial}$ are given locally by formulas similar to (3.1) and (3.2).

3.2 Realification with parameters

Let Z and S be complex analytic manifolds and set $n = d_Z$. Consider $Z\times S$ as a relative manifold over S through the second projection ϵ.

The *parametric realification* of a left $\mathcal{D}_{Z\times S|S}$-module $\mathcal{M}$ is the sheaf

$$\mathcal{M}_{\mathbb{R}|S} = \mathcal{A}_{Z\times S|S}\otimes_{\mathcal{O}_{Z\times S}}\mathcal{M}.$$

In this formula, the $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-module structure is described locally by the formulas

$$\begin{aligned}D_{z_j}(a\otimes m) &= D_{z_j}a\otimes m + a\otimes D_{z_j}m\\ D_{\overline{z}_j}(a\otimes m) &= D_{\overline{z}_j}a\otimes m\\ f(a\otimes m) &= fa\otimes m\end{aligned}$$

where a,f and m are sections of $\mathcal{A}_{Z\times S|S}$ and $\mathcal{M}$ respectively; $(z : U \longrightarrow \mathbb{C}^n)$ being a local holomorphic coordinate system on Z.

Since $\mathcal{A}_{Z\times S|S}$ is flat over $\mathcal{O}_{Z\times S}$, parametric realification is an exact functor.

Let us consider the map

$$\begin{aligned}\delta : Z\times S &\longrightarrow Z\times\overline{Z}\times S\\ (z,s) &\mapsto (z,z,s).\end{aligned}$$

It is clear that

$$\delta^{-1}(\mathcal{D}_{Z\times\overline{Z}\times S|S}) = \mathcal{D}_{Z^{\mathbb{R}}\times S|S}.$$

Hence the sheaf inverse image by δ of a $\mathcal{D}_{Z\times\overline{Z}\times S|S}$-module is naturally a $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-module. Moreover, one checks easily that

$$\mathcal{M}_{\mathbb{R}|S} = \delta^{-1}(\mathcal{O}_{Z\times\overline{Z}\times S}\otimes_{q^{-1}\mathcal{O}_{Z\times S}} q^{-1}\mathcal{M}) = \delta^{-1}(\mathcal{M}\boxtimes\mathcal{O}_{\overline{Z}}).$$

where $q : Z \times \overline{Z} \times S \longrightarrow Z \times S$ is the natural projection and $\boxtimes$ denotes the external product of $\mathcal{D}$-modules.

As usual, using "side changing" functors, we may also define the parametric realification of a right $\mathcal{D}_{Z\times S|S}$-module $\mathcal{M}$. We still denote it by $\mathcal{M}_{\mathbb{R}|S}$ and check easily that

$$\mathcal{M}_{\mathbb{R}|S} = \mathcal{A}^{0,n}_{Z\times S|S} \otimes_{\mathcal{O}_{Z\times S}} \mathcal{M} = \delta^{-1}(\mathcal{M} \boxtimes \Omega_{\overline{Z}}).$$

Parametric realification is a powerful tool to simplify problems dealing with $\mathcal{D}_{Z\times S|S}$-modules thanks to the following result.

Proposition 3.1 *Let K be a compact subset of Z and let Δ be a closed polydisc of S. Assume $\mathcal{M}$ is a good $\mathcal{D}_{Z\times S|S}$-module. Then, in a neighborhood of $K \times \Delta$, $\mathcal{M}_{\mathbb{R}|S}$ has a left resolution by finite free $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-modules .*

Proof: The assumption insures that $\mathcal{M} \boxtimes \mathcal{O}_{\overline{Z}}$ is a good $\mathcal{D}_{Z\times\overline{Z}\times S|S}$-module. Hence it is generated by a coherent $\mathcal{O}_{Z\times\overline{Z}\times S}$-module in a neighborhood of the Stein compact subset $\delta(K \times \Delta)$ of the complex analytic manifold $Z \times \overline{Z} \times S$. By Cartan's Theorem A, it is thus finitely generated in a neighborhood of $\delta(K \times \Delta)$. The conclusion follows easily. □

In order to be able to use effectively the preceding proposition in the sequel, we need to understand the links between parametric realification and the finiteness and duality results. These links are made explicit in the following five lemmas. Since the proofs are just easy computational verifications we leave them to the reader. Recall that $\mathcal{H}om^{\cdot}$ denotes as usual the internal Hom functor of the category of complexes of sheaves.

Lemma 3.2 *a) The sheaf $\mathcal{A}^{0,p}_{Z\times S|S}(\mathcal{D}_{Z^{\mathbb{R}}\times S|S})$ is naturally endowed with a structure of left $\mathcal{D}_{Z\times S|S}$-module and a structure of right $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-module and the differential $\overline{\partial}$ is compatible with these two structures.*

b) As a complex of $(\mathcal{D}_{Z\times S|S}, \mathcal{D}^{\mathrm{op}}_{Z^{\mathbb{R}}\times S|S})$-bimodules $\mathcal{A}^{0,\cdot}_{Z\times S|S}(\mathcal{D}_{Z^{\mathbb{R}}\times S|S})$ is quasi-isomorphic to $(\mathcal{D}_{Z\times S|S})_{\mathbb{R}|S}[-n]$ (where the realification uses the right module structure of $\mathcal{D}_{Z\times S|S}$)

Lemma 3.3 *The map*

$$\begin{array}{rcl} \mathcal{A}^{0,\cdot}_{Z\times S|S}(\mathcal{D}_{Z^{\mathbb{R}}\times S|S}) \otimes_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S|S} & \longrightarrow & \mathcal{F}^{0,\cdot}_{Z\times S|S} \\ (a^{0,p} \otimes Q) \otimes u & \mapsto & a^{0,p} \wedge Qu \end{array}$$

is an isomorphism of complexes of left $\mathcal{D}_{Z\times S|S}$-modules. Combined with the Dolbeault quasi-isomorphism

$$\mathcal{O}_{Z\times S} \longrightarrow \mathcal{F}^{0,\cdot}_{Z\times S|S},$$

it induces in the derived category the isomorphism

$$\mathcal{M} \otimes^{L}_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S} \xrightarrow{\sim} \mathcal{M}_{\mathbb{R}|S}[-n] \otimes^{L}_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S|S}$$

for any complex of right $\mathcal{D}_{Z\times S|S}$-modules $\mathcal{M}$. We have also similar results with $\mathcal{F}$ replaced by $\mathcal{D}b$ or $\mathcal{A}$.

Lemma 3.4 *The map*

$$\begin{array}{rcl} \mathcal{D}b^{n,\cdot}_{Z\times S|S}[-n] & \longrightarrow & \mathcal{H}om^{\cdot}_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}(\mathcal{A}^{0,\cdot}_{Z\times S|S}(\mathcal{D}_{Z^{\mathbb{R}}\times S|S}), \mathcal{D}b^{n,n}_{Z\times S|S}) \\ \varphi^{n,r+n} & \mapsto & [\omega^{0,-r}\otimes Q \mapsto (\varphi^{n,r+n}\wedge\omega^{0,-r})\cdot Q] \end{array}$$

is an isomorphism of complexes of right $\mathcal{D}_{Z\times S|S}$-modules. Combined with the Dolbeault quasi-isomorphism

$$\Omega^n_{Z\times S|S} \longrightarrow \mathcal{D}b^{n,\cdot}_{Z\times S|S},$$

it induces in the derived category the isomorphism

$$R\mathcal{H}om_{\mathcal{D}_{Z\times S|S}}(\mathcal{M}, \Omega^n_{Z\times S|S}[n]) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}(\mathcal{M}_{\mathbb{R}|S}[-n], \mathcal{D}b^{n,n}_{Z\times S|S})$$

for any complex of right $\mathcal{D}_{Z\times S|S}$-modules $\mathcal{M}$. We have also similar results with $\mathcal{D}b$ replaced by $\mathcal{A}$ or $\mathcal{F}$.

Lemma 3.5 *The natural arrow*

$$\begin{array}{rcl} \Omega^{\cdot}_{Z\times S|S}(\mathcal{D}_{Z\times S|S}) & \longrightarrow & \Omega^n_{Z\times S|S}[-n] \\ (\text{resp. } \mathcal{D}b^{\cdot}_{Z\times S|S}(\mathcal{D}_{Z^{\mathbb{R}}\times S|S}) & \longrightarrow & \mathcal{D}b^{2n}_{Z\times S|S}[-2n] \quad) \end{array}$$

of complexes of right $\mathcal{D}_{Y\times S|S}$ (resp. $\mathcal{D}_{Y^{\mathbb{R}}\times S|S}$) modules is a quasi-isomorphism. Together with the relative de Rham quasi-isomorphism

$$\begin{array}{rcl} \epsilon^{-1}\mathcal{O}_S & \xrightarrow{\sim} & \Omega^{\cdot}_{Z\times S|S} \\ (\text{resp. } \epsilon^{-1}\mathcal{O}_S & \xrightarrow{\sim} & \mathcal{D}b^{\cdot}_{Z\times S|S} \quad) \end{array}$$

it induces the $\epsilon^{-1}\mathcal{O}_S$ linear pairing

$$\begin{array}{rcl} \Omega^n_{Z\times S|S}[n]\otimes^L_{\mathcal{D}_{Z\times S|S}}\mathcal{O}_{Z\times S} & \longrightarrow & \epsilon^{-1}\mathcal{O}_S[2n] \\ (\text{resp. } \mathcal{D}b^{2n}_{Z\times S|S}\otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}\mathcal{F}_{Z\times S|S} & \longrightarrow & \epsilon^{-1}\mathcal{O}_S[2n] \quad). \end{array}$$

Lemma 3.6 *Assume $\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{Z\times S|S})$. We have the commutative diagram*

$$\begin{array}{ccc} (\mathcal{M}\otimes^L_{\mathcal{D}_{Z\times S|S}}\mathcal{O}_{Z\times S})\otimes^L_{\epsilon^{-1}\mathcal{O}_S} R\mathcal{H}om_{\mathcal{D}_{Z\times S|S}}(\mathcal{M},\Omega^n_{Z\times S}[n]) & \longrightarrow & \epsilon^{-1}\mathcal{O}_S[2n] \\ \downarrow & & \downarrow \\ (\mathcal{M}_{\mathbb{R}|S}\otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}\mathcal{F}_{Z\times S|S})\otimes^L_{\epsilon^{-1}\mathcal{O}_S} R\mathcal{H}om_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}(\mathcal{M}_{\mathbb{R}|S},\mathcal{D}b^{2n}_{Z\times S|S}) & \longrightarrow & \epsilon^{-1}\mathcal{O}_S[2n] \end{array}$$

where the horizontal arrows are constructed by contraction followed by the pairings of the preceding lemma, the first vertical arrow being the tensor product of the isomorphisms of Lemma 3.3 and 3.4 while the second vertical arrow is the identity.

3.3 Trivialization of $\mathbb{R}$-constructible sheaves

In this section, we follow the notations of [12, Ch. VIII]. As in the classical theory of simplicial complexes, the sets $U(x)$ of [loc. cit.] are called open stars. Let us first point out some basic facts about the topology of polyhedra.

Lemma 3.7 *Let (S,Σ) be a simplicial set and let $x \in |S|$. Then*

(a) $y \in U(x) \quad \Rightarrow \quad [x,y] \subset U(x)$

(b) $y \in \partial U(x) \quad \Rightarrow \quad [x,y[\subset U(x)$

where $U(x)$ denotes the open star of x in $|S|$.

Proof: (a) One knows that

$$U(x) = \bigcup_{\sigma \supset \sigma(x)} |\sigma|.$$

Thus, if $y \in U(x)$, there is a $\sigma \supset \sigma(x)$ such that $y \in |\sigma|$. Since it is clear that $]x,y] \subset |\sigma|$ and that $x \in U(x)$, one gets that $[x,y] \subset U(x)$.

(b) The set $\{\sigma \in \Sigma : \sigma \supset \sigma(x)\}$ being finite, one has the equality

$$\overline{U(x)} = \bigcup_{\sigma \supset \sigma(x)} \overline{|\sigma|}.$$

Hence, since $y \in \partial U(x)$, there is a simplex $\sigma \supset \sigma(x)$ such that $y \in \overline{|\sigma|} \setminus |\sigma|$. Let σ' be a simplex included in σ such that $y \in |\sigma'|$. If $\sigma' \supset \sigma(x)$ then $]x,y[\subset |\sigma'| \subset U(x)$ as requested. If $\sigma' \not\supset \sigma(x)$ then $\sigma'' = \sigma' \cup \sigma(x)$ is a simplex of Σ included in σ and $]x,y[\subset |\sigma''| \subset U(x)$ and the conclusion follows. □

Lemma 3.8 *If (S,Σ) is a simplicial set and if $x \in |S|$ then one has the following commutative diagram*

$$\begin{array}{ccc} \partial U(x) \times]0,1] / \partial U(x) \times \{1\} & \xrightarrow{\sim} & U(x) \\ \downarrow & & \downarrow \\ \partial U(x) \times [0,1] / \partial U(x) \times \{1\} & \xrightarrow{\sim} & \overline{U(x)} \end{array}$$

where the horizontal arrows are homeomorphisms, the vertical arrows being the natural inclusions.

Proof: Let us define the continuous application

$$f : \partial U(x) \times [0,1] \longrightarrow \overline{U(x)}$$

by setting $f(u,t) = (1-t)u + tx$. The preceding lemma shows that $f(u,t) = f(u',t')$ if either $t = t' = 1$ or $(u,t) = (u',t')$. Moreover, it is clear that for every $u \in \overline{U(x)}$ there is $v \in \partial U(x)$ such that $u \in [x,v]$. From these facts, one deduces that the continuous map

$$g : \partial U(x) \times [0,1] / \partial U(x) \times \{1\} \longrightarrow \overline{U(x)}$$

associated to f is bijective. Since $\partial U(x) \times [0,1]$ is a compact space, g is an homeomorphism. To conclude, it remains to note that $f^{-1}(U(x)) = \partial U(x) \times]0,1]$. □

Proposition 3.9 *If (S, Σ) is a simplicial set, then for every open star $U(x)$ of $x \in |S|$ one has*

$$D'_{|S|}(\mathbb{C}_{U(x)}) = \mathbb{C}_{\overline{U(x)}}$$

Proof: It is clear that

$$\begin{aligned} D'_{|S|}(\mathbb{C}_{U(x)}) &= R\mathcal{H}om(\mathbb{C}_{U(x)}, \mathbb{C}_{|S|}) \\ &= Rj_{U(x)_*}(\mathbb{C}_{U(x)}). \end{aligned}$$

It remains to prove that the canonical arrow

$$\mathbb{C}_{\overline{U(x)}} \longrightarrow Rj_{U(x)_*}(\mathbb{C}_{U(x)})$$

is a quasi-isomorphism on $\overline{U(x)}$. Thanks to the preceding lemma, there is a neighborhood ω of $\partial U(x)$ in $\overline{U(x)}$ and an homeomorphism

$$\phi : \omega \longrightarrow \partial U(x) \times [0, \epsilon[$$

such that $\phi(\omega \cap U(x)) = \partial U(x) \times]0, \epsilon[$. We are thus reduced to show that the canonical arrows

$$\begin{aligned} \mathbb{C} &\longrightarrow \varinjlim_{V \in \mathcal{V}, \eta > 0} H^0(V \times]0, \eta[; \mathbb{C}) \\ 0 &\longrightarrow \varinjlim_{V \in \mathcal{V}, \eta > 0} H^k(V \times]0, \eta[; \mathbb{C}) \quad (k \geq 1) \end{aligned}$$

are isomorphisms when $\mathcal{V}$ is a fundamental system of neighborhoods of $y \in \partial U(x)$. But, using homotopy, it is clear that

$$H^k(V \times]0, \eta[; \mathbb{C}) \xrightarrow{\sim} H^k(V; \mathbb{C})$$

and the proof is complete. □

The following proposition is the main result of this section and will be used as a basic tool in the sequel.

Proposition 3.10 *An $\mathbb{R}$-constructible sheaf F on a real analytic manifold M is quasi-isomorphic to a bounded complex $T^{\cdot}$ of the form*

$$\cdots 0 \longrightarrow \cdots \bigoplus_{i_a \in I_a} \mathbb{C}_{W_{a,i_a}} \longrightarrow \cdots \bigoplus_{i_k \in I_k} \mathbb{C}_{W_{k,i_k}} \longrightarrow \cdots \bigoplus_{i_b \in I_b} \mathbb{C}_{W_{b,i_b}} \longrightarrow 0 \cdots$$

where each family $(W_{k,i_k})_{i_k \in I_k}$ is locally finite, the open subsets W_{k,i_k} being subanalytic, relatively compact, connected and such that

$$D'_M(\mathbb{C}_{W_{k,i_k}}) \simeq \mathbb{C}_{\overline{W}_{k,i_k}},$$

the differential $d^k_{T^\cdot}$ being such that the induced map

$$(d^k_{T^\cdot})_{ji} : \mathbb{C}_{W_{k,i}} \longrightarrow \mathbb{C}_{W_{k+1,j}}$$

is either 0 *if* $W_{k,i} \not\subset W_{k+1,j}$ *or a complex multiple* $c^k_{ji} I_{W_{k+1,j}W_{k,i}}$ *of the canonical inclusion map if* $W_{k,i} \subset W_{k+1,j}$.

Moreover, if F *has compact support, we may assume that the set* I_k *is finite for every* $k \in \mathbb{Z}$.

Proof: From the theory of $\mathbb{R}$-constructible sheaves, one knows that there is a simplicial set (S, Σ) and an homeomorphism $i : |S| \longrightarrow M$ such that $i^{-1}F$ is a simplicialy constructible sheaf. From a construction due to M. Kashiwara [11] one knows that such a sheaf is quasi-isomorphic to a bounded complex $T^\cdot$ such that each T^k is a locally finite direct sum of the sheaves $\mathbb{C}_{U(\sigma)}$ associated to the open stars of the simplexes of Σ where F is non zero. Since we have just proven in the preceding lemma that for such a sheaf one has

$$D'(\mathbb{C}_{U(\sigma)}) \simeq \mathbb{C}_{\overline{U(\sigma)}}$$

the first part of the proposition is clear.

Concerning the differential of the complex, we note that if σ, σ' are two simplexes of Σ then

$$\mathrm{Hom}(\mathbb{C}_{U(\sigma)}, \mathbb{C}_{U(\sigma')}) \simeq \Gamma(U(\sigma); \mathbb{C}_{U(\sigma)\cap U(\sigma')})$$

hence the conclusion since $U(\sigma)$ is a connected open set.

In case F has compact support K, the open stars $U(\sigma)$ meeting K are in finite number and since only these stars appear in the components of $T^\cdot$, the sets I_k are finite. □

3.4 Topological $\mathcal{O}_S$-modules

Let S be a complex analytic manifold. Recall that the sheaf $\mathcal{O}_S$ of holomorphic functions on S is a multiplicatively convex sheaf of Fréchet algebras over S (see [7, 25]). Also recall that if V is a relatively compact open subset of a Stein open subset U of X then the restriction map

$$\Gamma(U; \mathcal{O}_S) \longrightarrow \Gamma(V; \mathcal{O}_S)$$

is $\mathbb{C}$-nuclear. From this it follows easily that $\Gamma(U; \mathcal{O}_U)$ is a Fréchet nuclear (FN) space and that $\Gamma(\overline{V}, \mathcal{O}_S)$ is a dual Fréchet nuclear (DFN) space.

As in [7], we will consider $\mathcal{O}_S$ as a sheaf of complete bornological algebras and deal with the category $\mathrm{Born}(\mathcal{O}_S)$ of complete bornological modules over $\mathcal{O}_S$. Recall that Houzel has shown that $\mathrm{Born}(\mathcal{O}_S)$ has a natural internal hom functor denoted by $\mathcal{L}_{\mathcal{O}_S}(\cdot, \cdot)$ and an associated tensor product functor denoted by $\cdot \mathbin{\hat{\otimes}_{\mathcal{O}_S}} \cdot$. They are linked by the adjunction formula

$$\mathrm{Hom}_{\,\mathrm{Born}(\mathcal{O}_S)}(\mathcal{M} \mathbin{\hat{\otimes}_{\mathcal{O}_S}} \mathcal{N}, \mathcal{P}) = \mathrm{Hom}_{\,\mathrm{Born}(\mathcal{O}_S)}(\mathcal{M}, \mathcal{L}_{\mathcal{O}_S}(\mathcal{N}, \mathcal{P})).$$

We denote by $\mathrm{L}_{\mathcal{O}_S}(\cdot,\cdot)$ the global sections of $\mathcal{L}_{\mathcal{O}_S}(\cdot,\cdot)$ considered as a bornological vector space. For any $\mathcal{M}$ in $\mathrm{Born}(\mathcal{O}_S)$, the functor $\mathrm{L}_{\mathcal{O}_S}(\mathcal{M},\cdot)$ has a left adjoint. We denote it by $\cdot\,\hat{\otimes}\,\mathcal{M}$.

Following [15], an FN-free (resp. a DFN-free) $\mathcal{O}_S$-module is a module isomorphic to $E\hat{\otimes}\mathcal{O}_S$ for some Fréchet nuclear (resp. dual Fréchet nuclear) space E. It is easily shown that the $\mathcal{O}_S$ topological dual $\mathcal{L}_{\mathcal{O}_S}(\mathcal{M},\mathcal{O}_S)$ of an FN-free (resp. a DFN-free) $\mathcal{O}_S$-module $\mathcal{M}$ is DFN-free (resp. FN-free). Moreover both FN-free and DFN-free $\mathcal{O}_S$-modules are $\mathcal{O}_S$ reflexive.

The results needed for the proof of the finiteness, duality and base change theorems for relative elliptic pairs are summarized in the three following propositions. The first one is Corollary 5.1 of [25] and the next two ones are easily deduced from the results in §1–2 of [15] (see also [16]).

Proposition 3.11 *Let $\mathcal{M}^{\cdot}$ (resp. $\mathcal{N}^{\cdot}$) be a complex of DFN-free (resp. FN-free) $\mathcal{O}_S$-modules. Assume $\mathcal{M}^{\cdot}$ and $\mathcal{N}^{\cdot}$ are bounded from above and*

$$u^{\cdot}:\mathcal{M}^{\cdot}\longrightarrow\mathcal{N}^{\cdot}$$

is a continuous $\mathcal{O}_S$-linear morphism. Assume moreover that $u^{\cdot}$ is a quasi-isomorphism forgetting the topology. Then $\mathcal{M}^{\cdot}$ and $\mathcal{N}^{\cdot}$ have $\mathcal{O}_S$-coherent cohomology.

Proposition 3.12 *Let $\mathcal{M}^{\cdot}$ be a complex of FN-free $\mathcal{O}_S$-modules and let $\mathcal{N}$ be a DFN $\mathcal{O}_S$-module. Assume $\mathcal{M}^{\cdot}$ is bounded from above and has $\mathcal{O}_S$-coherent cohomology. Then the natural morphism of $\mathbf{D}^+(\mathcal{O}_S)$*

$$\mathcal{L}_{\mathcal{O}_S}(\mathcal{M}^{\cdot},\mathcal{N})\longrightarrow R\mathcal{H}om_{\mathcal{O}_S}(\mathcal{M}^{\cdot},\mathcal{N})$$

is an isomorphism.

Proposition 3.13 *Let $\mathcal{M}^{\cdot}$ be a complex of FN-free (resp. DFN-free) $\mathcal{O}_S$-modules and let $\mathcal{N}$ be an FN (resp. DFN) $\mathcal{O}_S$-module. Assume $\mathcal{M}^{\cdot}$ is bounded from above and has $\mathcal{O}_S$-coherent cohomology. Then the natural morphism of $\mathbf{D}^-(\mathcal{O}_S)$*

$$\mathcal{M}^{\cdot}\otimes^L_{\mathcal{O}_S}\mathcal{N}\longrightarrow\mathcal{M}^{\cdot}\hat{\otimes}_{\mathcal{O}_S}\mathcal{N}$$

is an isomorphism.

In the sequel, when applying the preceding propositions, we will use the following well-known result.

Proposition 3.14 *Assume Z, S are complex manifolds. Denote by $\epsilon:Z\times S\longrightarrow S$ the second projection. Then, we have the following isomorphisms:*

$$\begin{aligned}\epsilon_*\mathcal{F}_{Z\times S|S} &\simeq \Gamma(Z;\mathcal{F}_Z)\,\hat{\otimes}\,\mathcal{O}_S\\ \epsilon_!\mathcal{D}b^{dz,dz}_{Z\times S|S} &\simeq \Gamma_c(Z;\mathcal{D}b^{dz,dz})\,\hat{\otimes}\,\mathcal{O}_S=\mathcal{L}_{\mathcal{O}_S}(\epsilon_*\mathcal{F}_{Z\times S|S},\mathcal{O}_S).\end{aligned}$$

Hence, $\epsilon_! \mathcal{D}b^{dz,dz}_{Z\times S|S}$ is a DFN-free $\mathcal{O}_S$-module which is the topological dual over $\mathcal{O}_S$ of the FN-free $\mathcal{O}_S$-module $\epsilon_* \mathcal{F}_{Z\times S|S}$. Moreover,

$$\epsilon_* \mathcal{O}_{Z\times S} = \Gamma(Z; \mathcal{O}_Z) \hat{\otimes} \mathcal{O}_S.$$

Hence, if K is a compact subset of Z, we have

$$\epsilon_*[(\mathcal{A}_{Z\times S|S})_{K\times S}] \simeq \Gamma(K; \mathcal{A}_Z) \hat{\otimes} \mathcal{O}_S$$

and $\epsilon_*[(\mathcal{A}_{Z\times S|S})_{K\times S}]$ is a DFN-free topological $\mathcal{O}_S$-module. Finally, if T is another complex manifold and $p : Z \times T \times S \longrightarrow S$ and $q : T \times S \longrightarrow S$ denotes the canonical projections, we have

$$p_* \mathcal{F}_{Z\times T\times S|T\times S} \simeq \epsilon_* \mathcal{F}_{Z\times S|S} \hat{\otimes}_{\mathcal{O}_S} q_* \mathcal{O}_{T\times S}.$$

4 Finiteness

4.1 The case of a projection

Proposition 4.1 *Let Z, S be complex analytic manifolds. Consider $Z\times S$ as a relative analytic manifold over S through the second projection ϵ. Let G be an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(Z)$ and set $F = G \boxtimes \mathbb{C}_S$. Assume that $(\mathcal{M}, F)$ is a good relative elliptic pair with ϵ-proper support on $Z \times S|S$. Then*

$$R\epsilon_!(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S})$$

is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S)$.

Proof: By "dévissage", it is obviously sufficient to prove the result when $\mathcal{M}$ is a $\mathcal{D}_{Z\times S|S}$-module which admits a good filtration on a neighborhood of any compact subset of $Z \times S$.

It follows from the relative regularity theorem (Theorem 2.15) that the canonical map

$$F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S} \longrightarrow R\mathcal{H}om(D'F, \mathcal{M} \otimes^L_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S})$$

is an isomorphism. Using Lemma 3.3, we get the isomorphism

$$\begin{aligned} &R\epsilon_*(F \otimes \mathcal{M}_{\mathbb{R}|S} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{A}_{Z\times S|S}) \\ &\quad\longrightarrow\ R\epsilon_* R\mathcal{H}om(D'F, \mathcal{M}_{\mathbb{R}|S} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S|S}). \end{aligned} \tag{4.1}$$

Let V be the interior of a closed polydisc Δ of S. Since $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F) \cap \epsilon^{-1}(\Delta)$ is compact, we can find a compact subset K in Z such that $K \times V$ is a neighborhood of $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F) \cap \epsilon^{-1}(V)$. Replacing S by V and G by G_K shows that we may assume from the beginning that G has compact support. Moreover, by Proposition 3.1 we may also assume that $\mathcal{M}_{\mathbb{R}|S}$ is quasi-isomorphic to complex $\mathcal{L}^\cdot$ whose components are free $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-modules of finite rank. Since $\mathcal{M}_{\mathbb{R}|S}$ has bounded amplitude, we may even assume $\mathcal{L}^k = 0$ for $k \gg 0$.

We know by Proposition 3.10 that $D'G$ is isomorphic to a bounded complex $T^{\cdot}$ of the form

$$\cdots \longrightarrow \underset{i\in I_k}{\oplus} \mathbb{C}_{W_{k,i}} \longrightarrow \cdots$$

where $W_{k,i}$ is relatively compact subanalytic subset of Z such that $D'(\mathbb{C}_{W_{k,i}}) = \mathbb{C}_{\overline{W}_{k,i}}$. Thus $G \simeq D'D'G$ is quasi-isomorphic to a complex $C^{\cdot}$ of the form

$$\cdots \longrightarrow \underset{i\in I_k}{\oplus} \mathbb{C}_{\overline{W}_{k,i}} \longrightarrow \cdots$$

It is clear that the sheaf $(\mathcal{A}_{Z\times S})_{|K\times S}$ (resp. $(\mathcal{F}_{Z\times S})_{|U\times S}$) is acyclic for the the functor $\epsilon_{|K\times S_*}$ (resp. $\epsilon_{|U\times S_*}$) for any compact subset K (resp. any open subset U) of Z. Hence we may view isomorphism (4.1) more explicitly as the morphism

$$\begin{aligned} &\epsilon_*((C^{\cdot} \boxtimes \mathbb{C}_S) \otimes \mathcal{L}^{\cdot} \otimes_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{A}_{Z\times S|S}) \\ &\qquad \longrightarrow \epsilon_*\mathcal{H}om(T^{\cdot} \boxtimes \mathbb{C}_S, \mathcal{L}^{\cdot} \otimes_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S|S}) \end{aligned} \tag{4.2}$$

in the category of complexes of $\mathcal{O}_S$-modules (not the derived category). Let us denote by $\mathcal{R}_1^{\cdot}$ (resp. $\mathcal{R}_2^{\cdot}$) the source (resp. target) of the preceding arrow.

The components of $\mathcal{R}_1^{\cdot}$ (resp. $\mathcal{R}_2^{\cdot}$) are easily seen to be finite sums of the sheaves

$$\epsilon_{|\overline{W}_{k,i}\times S_*}(\mathcal{A}_{Z\times S|S|\overline{W}_{k,i}\times S}) \qquad (\text{resp. } \epsilon_{|W_{k,i}\times S_*}(\mathcal{F}_{Z\times S|S|W_{k,i}\times S})\ ).$$

Hence, $\mathcal{R}_1^{\cdot}$ (resp. $\mathcal{R}_2^{\cdot}$) is naturally a complex of DFN-free (resp. FN-free) topological $\mathcal{O}_S$-modules. For these natural topologies, the regularity quasi-isomorphism is clearly continuous. Applying Proposition 3.11 we conclude that $\mathcal{R}_2^{\cdot}$ has $\mathcal{O}_S$-coherent cohomology and the proof is complete. □

4.2 The general case

Theorem 4.2 *Let $f : X|S \longrightarrow Y|S$ be a morphism of relative analytic manifolds over S. Assume ($\mathcal{M}$, F) is a good relative f-elliptic pair with f-proper support; i.e.*

- *$\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{X|S})$,*
- *F is an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$,*
- *$\phi^{-1}\mathrm{char}_{f|S}(\mathcal{M}) \cap SS(F) \subset T^*_X X$,*
- *$\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ is f-proper.*

Then $\underline{f}_{|S!}(\mathcal{M} \otimes F)$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{Y|S})$.

Proof: By "dévissage", it is obviously sufficient to prove the result when $\mathcal{M}$ is a $\mathcal{D}_{X|S}$-module which admits a good filtration on a neighborhood of any compact subset of X.

Decomposing f through its graph embedding

$$i : X \longrightarrow X \times Y$$

shows that it is sufficient to prove the finiteness theorem for the second projection

$$p_2 : X \times Y|S \longrightarrow Y|S$$

and the pair $\underline{i}_{|S!}(\mathcal{M}) \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{X\times Y|S}))$, $F \boxtimes \mathbb{C}_Y \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X \times Y))$ since by the projection formula we have

$$\underline{i}_{|S!}(\mathcal{M}) \otimes (F \boxtimes \mathbb{C}_Y) \simeq \underline{i}_{|S!}(\mathcal{M} \otimes F).$$

From the definition of $\mathrm{char}_{f|S}(\mathcal{M})$, it is clear that

$$\mathrm{char}_{p_2|S}(\underline{i}_{|S!}(\mathcal{M})) \cap SS(F \boxtimes \mathbb{C}_Y)$$

is contained in the zero section of $T^*(X \times Y|S)$. So if $Y = S$, the theorem is a consequence of the results obtained in the case of a projection.

To conclude, we will show that if f is a relative submersion and the theorem is true for $f : X|Y \longrightarrow Y|Y$ then it is also true for $f : X|S \longrightarrow Y|S$.

We will use a device introduced in [8] and extended in Lemma 2.10.

Let Δ be a polydisc in Y and denote by K the compact subset of X defined by

$$K = \mathrm{supp}(\mathcal{M}) \cap SS(F) \cap f^{-1}(\overline{\Delta}).$$

Using Lemma 2.10, it is easy to see that, in a neighborhood of K, $\mathcal{M}$ is isomorphic to a complex of right $\mathcal{D}_{X|S}$-modules of the form $\mathcal{R} \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}$ where $\mathcal{R}$ is a coherent right $\mathcal{D}_{X|Y}$-submodule which admits a good filtration and is such that

$$\phi^{-1}\mathrm{char}_{X|Y}(\mathcal{R}) \subset \mathrm{char}_{f|S}(\mathcal{M}).$$

Moreover, this complex may be assumed to be bounded from above.

Since the functor $\underline{f}_{|S!}$ has finite cohomological dimension, it is thus sufficient to prove the coherence on Δ of the cohomology of $\underline{f}_{|S!}(F \otimes \mathcal{M})$ when $\mathcal{M}$ has the special form $\mathcal{M} = \mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}$ where $\mathcal{M}_0$ is a coherent $\mathcal{D}_{X|Y}$-module which admits a good filtration.

In this case, one knows that the complex $\underline{f}_{|Y!}(F \otimes \mathcal{M}_0)$ has $\mathcal{O}_Y$-coherent cohomology, and the chain of isomorphisms

$$\begin{aligned}
&Rf_!(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \\
&\qquad = \quad Rf_!(F \otimes (\mathcal{M}_0 \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}) \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \\
&\qquad \xrightarrow{\sim} \quad Rf_!(F \otimes \mathcal{M}_0 \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S\to Y|S}) \\
&\qquad \xrightarrow{\sim} \quad Rf_!(F \otimes \mathcal{M}_0 \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{O}_X) \otimes^L_{\mathcal{O}_Y} \mathcal{D}_{Y|S}
\end{aligned}$$

shows that $\underline{f}_{|S!}(F \otimes \mathcal{M})$ belongs to $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{Y|S})$. □

Corollary 4.3 *In the situation of the preceding theorem, the well known formula:*

$$\underline{f}_{|S!}(F \otimes \mathcal{M}) \otimes^L_{\mathcal{D}_{Y|S}} \mathcal{O}_Y \xrightarrow{\sim} Rf_!(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X)$$

gives the inclusion:

$$\mathrm{char}_{Y|S}(\underline{f}_{|S!}(F \otimes \mathcal{M})) \subset f_\pi {}^t f'^{-1} \mathrm{char}_{X|S}(\mathcal{M})$$

Proof: By Theorem 2.13 and Proposition 5.4.4 and 5.4.14 of [12], we know that

$$\begin{aligned} p_{Y|S}^{-1}\mathrm{char}_{Y|S}(\underline{f}_{|S!}(F \otimes \mathcal{M})) &= SS(\underline{f}_{|S!}(F \otimes \mathcal{M}) \otimes^L_{\mathcal{D}_{Y|S}} \mathcal{O}_Y) \\ &= SS(Rf_!(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X)) \\ &\subset f_\pi {}^t f'^{-1} SS(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X) \\ &\subset f_\pi {}^t f'^{-1} (SS(F) + p_{X|S}^{-1}\mathrm{char}_{X|S}(\mathcal{M})). \end{aligned}$$

Note that by hypothesis:

$$p_{X|S}^{-1}\mathrm{char}_{f|S}(\mathcal{M}) \cap SS(F) \subset T^*_X X.$$

Moreover, one has:

$$p_{X|S}^{-1}\mathrm{char}_{X|S}(\mathcal{M}) + {}^t f'(X \times_Y T^*Y) \subset p_{X|S}^{-1}\mathrm{char}_{f|S}(\mathcal{M}).$$

Hence,

$$\left[p_{X|S}^{-1}\mathrm{char}_{X|S}(\mathcal{M}) + SS(F)\right] \cap {}^t f'(X \times_Y T^*Y) \subset p_{X|S}^{-1}\mathrm{char}_{X|S}(\mathcal{M}) \cap {}^t f'(X \times_Y T^*Y)$$

so

$${}^t f'^{-1}(SS(F) + p_{X|S}^{-1}\mathrm{char}_{X|S}(\mathcal{M})) \subset {}^t f'^{-1}(p_{X|S}^{-1}\mathrm{char}_{X|S}(\mathcal{M}))$$

and the proof is complete. □

5 Duality

Let $f : X|S \longrightarrow Y|S$ be a morphism of relative complex manifolds. Our aim in this section is to prove that, under suitable hypotheses, duality commutes with direct images (see Theorem 5.15 for a precise statement). The proof will use the graph decomposition of f and various "dévissages". Hence, it is necessary to construct first the natural transformation:

$$\underline{f}_{|S!} \circ \underline{D}_{X|S} \longrightarrow \underline{D}_{Y|S} \circ \underline{f}_{|S!}$$

and to check its compatibility with respect to composition in f. This will be a consequence of the explicit construction of the trace morphism for $\mathcal{D}$-modules given in the next section. We follow the lines of [24] (see also [11, 17]).

5.1 The relative duality morphism

Recall that if $f : X \longrightarrow Y$ is a holomorphic map then it induces integration maps

$$\int_*^{p,q} : f_! \mathcal{D}b_X^{p+d_X,q+d_X} \longrightarrow \mathcal{D}b_Y^{p+d_Y,q+d_Y}$$

commuting with the Dolbeault operators. We will use this fact and the machinery of distributional Dolbeault complexes of §3.1 to construct canonically the duality map for right $\mathcal{D}$-modules.

Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. To simplify notations, we will set

$$\mathcal{D}b_X^{\cdot,\cdot}(\mathcal{M}) = \mathcal{D}b_{X\times\{\mathrm{pt}\}|\{\mathrm{pt}\}}^{\cdot,\cdot}(\mathcal{M}_{\mathbb{R}|\{\mathrm{pt}\}}).$$

Hence, the components are

$$\mathcal{D}b_X^{p,q}(\mathcal{M}) = \mathcal{D}b_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M}$$

and the differentials are given in a local coordinate system $z : U \longrightarrow \mathbb{C}^{d_Z}$ by

$$\begin{array}{rcl} \partial^{p,q} : \mathcal{D}b_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M} & \longrightarrow & \mathcal{D}b_X^{p+1,q} \otimes_{\mathcal{O}_X} \mathcal{M} \\ u \otimes P & \mapsto & \partial^{p,q} u \otimes P + \sum_{i=1}^{d_X} dz_i \wedge u \otimes D_{z_i} P \end{array}$$

and

$$\begin{array}{rcl} \overline{\partial}^{p,q} : \mathcal{D}b_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M} & \longrightarrow & \mathcal{D}b_X^{p,q+1} \otimes_{\mathcal{O}_X} \mathcal{M} \\ u \otimes P & \longrightarrow & \overline{\partial}^{p,q} u \otimes P \end{array}$$

respectively. Also recall that we denote by $\mathcal{D}b_X^{\cdot}(\mathcal{M})$ the simple complex associated with $\mathcal{D}b_X^{\cdot,\cdot}(\mathcal{M})$.

Lemma 5.1 *The differential of $\mathcal{D}b_X^{\cdot}(\mathcal{D}_X)$ is compatible with the right $\mathcal{D}_X$-module structure of its components and, in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$, one has a canonical isomorphism:*

$$\mathcal{D}b_X^{\cdot}(\mathcal{D}_X)[d_X] \xrightarrow{\sim} \Omega_X.$$

Proof: The compatibility of the differential of $\mathcal{D}b_X^{\cdot}(\mathcal{D}_X)$ with the right $\mathcal{D}_X$-module structure of its components is a direct consequence of the local forms of ∂ and $\overline{\partial}$ recalled above.

Using the fact that $\mathcal{D}_X$ is flat over $\mathcal{O}_X$ and the Dolbeault resolution of Ω_X^p, we get the quasi-isomorphisms

$$\Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{\sim} \mathcal{D}b_X^{p,\cdot} \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{\sim} \mathcal{D}b_X^{p,\cdot}(\mathcal{D}_X).$$

Hence, Weil's lemma shows that the natural morphism

$$DR_X^{\cdot}(\mathcal{D}_X) \longrightarrow \mathcal{D}b_X^{\cdot}(\mathcal{D}_X)$$

from the holomorphic to the distributional de Rham complex of $\mathcal{D}_X$ is a quasi-isomorphism of complexes of right $\mathcal{D}_X$-modules and the conclusion follows from the Spencer quasi-isomorphism

$$DR^{\cdot}_X(\mathcal{D}_X) \simeq \Omega_X[-d_X].$$

□

Lemma 5.2 *To any morphism $f : X \longrightarrow Y$ of analytic manifolds one can associate a canonical integration morphism*

$$f_* : f_! \mathcal{D}b^{\cdot}_X(\mathcal{D}_{X\to Y})[2d_X] \longrightarrow \mathcal{D}b^{\cdot}_Y(\mathcal{D}_Y)[2d_Y].$$

in the category of bounded complexes of right $\mathcal{D}_Y$-modules.

Proof: At the level of components the integration morphism is obtained as the following chain of morphisms:

$$\begin{aligned} f_!\mathcal{D}b^{p+d_X,q+d_X}(\mathcal{D}_{X\to Y}) &\xrightarrow{\sim} f_!(\mathcal{D}b_X^{p+d_X,q+d_X} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \\ &\xrightarrow{\sim} f_!\mathcal{D}b_X^{p+d_X,q+d_X} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \\ &\longrightarrow \mathcal{D}b_Y^{p+d_Y,q+d_Y} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \\ &\xrightarrow{\sim} \mathcal{D}b_Y^{p+d_Y,q+d_Y}(\mathcal{D}_Y). \end{aligned}$$

To get the second morphism one has used the projection formula, the fact that $\mathcal{D}b_X$ is a soft sheaf and the fact that $\mathcal{D}_Y$ is locally free over $\mathcal{O}_Y$. The third arrow is deduced from the integration of distributions along the fibers of f.

To conclude, we need to show that the integration morphism is compatible with the differentials of the complexes involved. Thanks to the local forms of the differentials, this is an easy computational verification and we leave it to the reader. □

Lemma 5.3 *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are morphisms of complex analytic manifolds, one has the following commutative diagram:*

$$\begin{array}{ccc} g_!(f_!\mathcal{D}b^{\cdot}_X(\mathcal{D}_{X\to Y})[2d_X] \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z}) & \xrightarrow{1} & g_!(\mathcal{D}b^{\cdot}_Y(\mathcal{D}_Y)[2d_Y] \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z}) \\ \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 3} \\ g_!f_!\mathcal{D}b^{\cdot}_X(\mathcal{D}_{X\to Z})[2d_X] & \xrightarrow{4} & \mathcal{D}b^{\cdot}_Z(\mathcal{D}_Z)[2d_Z]. \end{array}$$

In this diagram, arrow (1) is deduced by tensor product and proper direct image from f_*, arrow (2) is an isomorphism deduced from the projection formula, arrow (3) is g_* and arrow (4) is equal to $(g \circ f)_*$.

Proof: Going back to the definition of the various morphisms, one sees easily that the commutativity of the preceding diagram is a consequence of the Fubini theorem for distributions, that is, the formula

$$(g \circ f)_*(u) = g_*(f_*(u))$$

where g_* and f_* denotes the push-forward of distributions along g and f respectively, u being a distribution with $g \circ f$ proper support. □

Proposition 5.4 *If $f : X \longrightarrow Y$ is a morphism of complex analytic manifolds then there is a canonical integration arrow*

$$\int_f : \underline{f}_!(\Omega_X[d_X]) \longrightarrow \Omega_Y[d_Y]$$

in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y^{\mathrm{op}})$. Moreover, if $g : Y \longrightarrow Z$ is a second morphism of complex analytic manifolds then

$$\int_{g \circ f} = \int_g \circ \underline{g}_!(\int_f)$$

Proof: One gets the arrow $\int_f$ by composing the morphisms:

$$\begin{aligned} \underline{f}_!(\Omega_X[d_X]) &\xrightarrow{\sim} Rf_!(\Omega_X[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \\ &\xrightarrow{\sim} Rf_!(\mathcal{D}b^{\cdot}_X(\mathcal{D}_{X\to Y})[2d_X]) \\ &\xrightarrow{\sim} f_!(\mathcal{D}b^{\cdot}_X(\mathcal{D}_{X\to Y})[2d_X]) \\ &\longrightarrow \mathcal{D}b^{\cdot}_Y(\mathcal{D}_Y)[2d_Y] \\ &\xrightarrow{\sim} \Omega_Y[d_Y] \end{aligned}$$

Let us point out that the second and last isomorphisms come from Lemma 5.1, that the third one is deduced from the fact that $\mathcal{D}b^{p,q}_X(\mathcal{D}_{X\to Y})$ is c-soft and that the fourth arrow is given by Lemma 5.2.

The compatibility of integration with composition is then a direct consequence of Lemma 5.3. □

Corollary 5.5 *If $f : X|S \longrightarrow Y|S$ is a morphism of relative analytic manifolds over S then there is a canonical arrow*

$$\int_{f|S} : \underline{f}_{|S!}(\Omega_{X|S}[d_{X|S}]) \longrightarrow \Omega_{Y|S}[d_{Y|S}].$$

Moreover, if $g : Y|S \longrightarrow Z|S$ is another morphism of relative analytic manifolds over S then

$$\int_{g\circ f|S} = \int_{g|S} \circ \underline{g}_{|S!}(\int_{f|S})$$

Proof: Using the canonical morphism

$$\Omega_X[d_X] \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X\to Y} \longrightarrow \Omega_X[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$$

and the integration morphism

$$\int_f : Rf_!(\Omega_X[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \longrightarrow \Omega_Y[d_Y]$$

we get the morphism

$$Rf_!(\Omega_X[d_X] \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X\to Y}) \longrightarrow \Omega_Y[d_Y].$$

Since

$$\mathcal{D}_{X\to Y} \simeq \mathcal{D}_{X|S\to Y|S} \otimes_{f^{-1}\mathcal{D}_{Y|S}} f^{-1}\mathcal{D}_Y$$

as a $(\mathcal{D}_{X|S},f^{-1}\mathcal{D}_Y^{\mathrm{op}})$-bimodule and Ω_Y is a right $\mathcal{D}_Y$-module, we get a $\mathcal{D}_{Y|S}$-linear morphism:

$$Rf_!(\Omega_X[d_X]\otimes^L_{\mathcal{D}_{X|S}}\mathcal{D}_{X|S\to Y|S})\longrightarrow\Omega_Y[d_Y].$$

Tensoring on both sides by $\epsilon_Y^{-1}\Omega_S^{\otimes -1}[-d_S]$ and using the projection formula, we get the requested relative integration map:

$$Rf_!(\Omega_{X|S}[d_{X|S}]\otimes^L_{\mathcal{D}_{X|S}}\mathcal{D}_{X|S\to Y|S})\longrightarrow\Omega_{Y|S}[d_{Y|S}].$$

The last part of the corollary is then an easy consequence of the similar result for $S=\{\mathrm{pt}\}$. □

Definition 5.6 One defines the direct image of a right $(\mathcal{D}_{X|S},\mathcal{D}_{X|S})$-bimodule $\mathcal{M}$ by setting:

$$\underline{\underline{f}}_{|S!}(\mathcal{M}) \;=\; Rf_!((\mathcal{M}\otimes^L_{\mathcal{D}_{X|S}}\mathcal{D}_{X|S\to Y|S})\otimes^L_{\mathcal{D}_{X|S}}\mathcal{D}_{X|S\to Y|S})$$

Lemma 5.7 *There is a canonical isomorphism*

$$\begin{aligned}&[(\mathcal{D}_{X|S}\otimes_{\mathcal{O}_X}\mathcal{D}_{X|S})\otimes^L_{\mathcal{D}_{X|S}}\mathcal{D}_{X|S\to Y|S}]\otimes^L_{\mathcal{D}_{X|S}}\mathcal{D}_{X|S\to Y|S}\\ &\qquad\xrightarrow{\sim}\mathcal{D}_{X|S\to Y|S}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_{Y|S}\end{aligned}$$

compatible both with the structure of left $\mathcal{D}_{X|S}$-module and the structure of right $(f^{-1}\mathcal{D}_{Y|S},f^{-1}\mathcal{D}_{Y|S})$-bimodule.

Proof: One has the chain of isomorphisms

$$\begin{aligned}&[(\mathcal{D}_{X|S}\otimes_{\mathcal{O}_X}\mathcal{D}_{X|S})\otimes^L_{\mathcal{D}_{X|S}}\mathcal{D}_{X|S\to Y|S}]\otimes^L_{\mathcal{D}_{X|S}}\mathcal{D}_{X|S\to Y|S}\\ &\xrightarrow{\sim}[\mathcal{D}_{X|S}\otimes_{\mathcal{O}_X}\mathcal{D}_{X|S\to Y|S}]\otimes^L_{\mathcal{D}_{X|S}}\mathcal{D}_{X|S\to Y|S}\\ &\xrightarrow{\sim}\mathcal{D}_{X|S\to Y|S}\otimes^L_{\mathcal{O}_X}\mathcal{D}_{X|S\to Y|S}\\ &\xrightarrow{\sim}\mathcal{D}_{X|S\to Y|S}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_{Y|S}\\ &\xrightarrow{\sim}\mathcal{D}_{X|S\to Y|S}\otimes_{f^{-1}\mathcal{D}_{Y|S}}(f^{-1}\mathcal{D}_{Y|S}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_{Y|S})\\ &\xrightarrow{\sim}\mathcal{D}_{X|S\to Y|S}\otimes_{f^{-1}\mathcal{D}_{Y|S}}(f^{-1}\mathcal{D}_{Y|S}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_{Y|S})\end{aligned}$$

In the second isomorphism we have used the exchange lemma. In the fourth line, the last tensor product uses the structure of left $f^{-1}\mathcal{O}_Y$-module of $f^{-1}\mathcal{D}_{Y|S}$. In the fifth isomorphism the last tensor product uses the structure of right $f^{-1}\mathcal{O}_Y$-module of the first factor and the structure of left $f^{-1}\mathcal{O}_Y$-module of the second one. Finally, in the last line, we have used the exchange lemma again. □

Proposition 5.8 *Let $\mathcal{M}$ be a right $\mathcal{D}_{X|S}$-module and let $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_{X|S}$ be the associated right $\mathcal{D}_{X|S}$-bimodule. If $f:X\longrightarrow Y$ is a morphism of relative analytic manifolds over S then one has the following canonical isomorphism*

$$\underline{\underline{f}}_{|S!}(\mathcal{M}\otimes^L_{\mathcal{O}_X}\mathcal{D}_{X|S})\xrightarrow{\sim}\underline{f}_{|S!}(\mathcal{M})\otimes^L_{\mathcal{O}_Y}\mathcal{D}_{Y|S}$$

in the derived category $\mathbf{D}(\mathcal{D}^{\mathrm{op}}_{Y|S}\otimes\mathcal{D}^{\mathrm{op}}_{Y|S})$.

Proof: This is a direct consequence of the preceding lemma. □

Definition 5.9 The differential trace map associated to a morphism

$$f : X|S \longrightarrow Y|S$$

of relative analytic manifolds over S is defined to be the arrow

$$\mathrm{tr}_f : \underline{\underline{f}}_{|S!}\mathcal{K}_{X|S} \longrightarrow \mathcal{K}_{Y|S}$$

in the derived category $\mathbf{D}(\mathcal{D}^{\mathrm{op}}_{Y|S} \otimes \mathcal{D}^{\mathrm{op}}_{Y|S})$ obtained by composing the following arrows:

$$\begin{aligned} \underline{\underline{f}}_{|S!}\mathcal{K}_{X|S} &= \underline{\underline{f}}_{|S!}(\Omega_{X|S}[d_{X|S}] \otimes^{L}_{\mathcal{O}_X} \mathcal{D}_{X|S}) \\ &\xrightarrow{\sim} (\underline{f}_{|S!}\Omega_{X|S}[d_{X|S}]) \otimes^{L}_{\mathcal{O}_Y} \mathcal{D}_{Y|S} \\ &\xrightarrow{\sim} \Omega_{Y|S}[d_{Y|S}] \otimes^{L}_{\mathcal{O}_Y} \mathcal{D}_{Y|S} \end{aligned}$$

where the first arrow comes from the definition of $\mathcal{K}_{X|S}$ (see p. 11), the second one being a consequence of the preceding proposition and the third one being constructed by tensor product with the integration arrow of Corollary 5.5. By construction, tr_f is compatible with the composition of maps.

Proposition 5.10 *Assume $f : X|S \longrightarrow Y|S$ is a morphism of relative analytic manifolds over S. Then the differential trace map*

$$\mathrm{tr}_f : \underline{\underline{f}}_{|S!}K^{\cdot}_{X|S} \longrightarrow K^{\cdot}_{Y|S}$$

induces a morphism

$$\mathrm{du}_f : \underline{f}_{|S!}\underline{D}_{X|S}(\mathcal{M}) \longrightarrow \underline{D}_{Y|S}(\underline{f}_{|S!}\mathcal{M})$$

for any object $\mathcal{M}$ of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}^{\mathrm{op}}_{X|S})$. Moreover, this morphism is functorial in $\mathcal{M}$ and compatible with composition in f.

Proof: Since, by definition,

$$\underline{D}_{X|S}(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_{X|S}}(\mathcal{M}, \mathcal{K}_{X|S})$$

we have a canonical morphism

$$\underline{f}_{|S!}\underline{D}_{X|S}(\mathcal{M}) \longrightarrow R\mathcal{H}om_{\mathcal{D}_{Y|S}}(\underline{f}_{|S!}\mathcal{M}, \underline{\underline{f}}_{|S!}\mathcal{K}_{X|S})$$

in $\mathbf{D}(\mathcal{D}^{\mathrm{op}}_{Y|S})$. Composing this morphism with the morphism

$$R\mathcal{H}om_{\mathcal{D}_{Y|S}}(\underline{f}_{|S!}\mathcal{M}, \underline{\underline{f}}_{|S!}\mathcal{K}_{X|S}) \longrightarrow R\mathcal{H}om_{\mathcal{D}_{Y|S}}(\underline{f}_{|S!}\mathcal{M}, \mathcal{K}_{Y|S})$$

associated to tr_f gives the requested duality morphism. The construction shows that it is natural in $\mathcal{M}$. The compatibility with composition in f comes from the corresponding property of tr_f. □

To conclude this section, we will show that the differential duality morphism is compatible with the duality morphism of complex analytic geometry.

Recall that since $\mathcal{D}_{X|S}$ is an $\mathcal{O}_X$-module, we have a well defined scalar extension functor

$$\begin{array}{rcl} E_{X|S} : \mathrm{Mod}(\mathcal{O}_X) & \longrightarrow & \mathrm{Mod}(\mathcal{D}^{\mathrm{op}}_{X|S}) \\ \mathcal{F} & \mapsto & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}. \end{array}$$

The image by this functor of an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent as a right $\mathcal{D}_{X|S}$-module if and only if $\mathcal{F}$ is coherent as an $\mathcal{O}_X$-module. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$, one sets

$$D_{X|S}(\mathcal{F}) = R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \Omega_{X|S}[d_{X|S}])$$

hence, we have the canonical isomorphism:

$$D_{X|S}(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S} \xrightarrow{\sim} \underline{D}_{X|S}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}).$$

Moreover, if $f : X|S \longrightarrow Y|S$ is a morphism of relative analytic manifolds and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module we have the canonical isomorphism:

$$(Rf_!\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y|S} \xrightarrow{\sim} \underline{f}_{|S!}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}).$$

With these facts in mind, we can now state:

Proposition 5.11 *Let $f : X|S \longrightarrow Y|S$ be a morphism of relative analytic manifolds. Assume $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. Then we have the commutative diagram:*

$$\begin{array}{ccc} Rf_!D_{X|S}(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y|S} & \longrightarrow & D_{Y|S}(Rf_!\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y|S} \\ \downarrow & & \downarrow \\ \underline{f}_{|S!}\underline{D}_{X|S}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}) & \longrightarrow & \underline{D}_{Y|S}(\underline{f}_{|S!}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S})) \end{array}$$

where the first and second horizontal arrows come respectively from the geometric and differential duality morphisms, and the vertical arrows isomorphisms are deduced from the compatibility with direct image and duality of the scalar extensions functors $E_{X|S}$ and $E_{Y|S}$.

Proof: Consider the morphism

$$\begin{array}{rcl} \Omega_{X|S}[d_{X|S}] & \longrightarrow & \mathcal{K}_{X|S} \\ \omega & \mapsto & \omega \otimes 1_{X|S}. \end{array} \tag{5.1}$$

It follows easily from the definition of the differential integration map that we have the following commutative diagram:

$$\begin{array}{ccc} Rf_!\Omega_{X|S}[d_{X|S}] & \longrightarrow & \Omega_{Y|S}[d_{Y|S}] \\ \downarrow & & \downarrow \\ \underline{f}_{|S!}\mathcal{K}_{X|S} & \longrightarrow & \mathcal{K}_{Y|S}. \end{array}$$

In the preceding diagram the horizontal arrows are the geometric and differential trace maps, the first vertical arrow is deduced from (5.1) by using the canonical section $1_{X|S \to Y|S}$ of $\mathcal{D}_{X|S \to Y|S}$ and the second vertical arrow is (5.1) with X replaced by Y.

Since the geometric and differential duality morphisms are directly constructed from the corresponding trace maps the result is easily reduced to the commutativity of the preceding diagram. □

5.2 The case of a closed embedding

Proposition 5.12 *Let $i : X|S \longrightarrow Y|S$ be a closed relative embedding. Then, for every coherent right $\mathcal{D}_{X|S}$-module $\mathcal{M}$, the canonical morphism*

$$\underline{i}_{|S!}\underline{D}_{X|S}(\mathcal{M}) \longrightarrow \underline{D}_{Y|S}(\underline{i}_{|S*}\mathcal{M})$$

is an isomorphism.

Proof: Since the problem is local on X, we may assume $\mathcal{M}$ has a bounded resolution by finite free right $\mathcal{D}_{X|S}$-modules. Thus it is sufficient to prove the result for $\mathcal{M} = \mathcal{D}_{X|S}$. Since we have

$$\mathcal{D}_{X|S} = \mathcal{O}_X \otimes_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S}$$

it follows from Proposition 5.11 that the result is a direct consequence of the corresponding result for $\mathcal{O}$-modules. Since we do not have a precise direct reference for this well known result we recall it in the following lemma. □

Lemma 5.13 *If $i : Z \longrightarrow X$ is a closed embedding of analytic manifolds, then for any object $\mathcal{F}$ of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Z)$ the complex $Ri_!\mathcal{F}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ and the geometric duality morphism*

$$Ri_! R\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{F}, \Omega_Z[d_Z]) \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(Ri_!\mathcal{F}, \Omega_X[d_X])$$

is an isomorphism.

Proof: Since the result is of local nature and the duality morphism is compatible with composition, it sufficient to consider the case when $\mathcal{F} = \mathcal{O}_Z$ and

$$\begin{array}{rcl} i : U' & \longrightarrow & U' \times U'' \\ z' & \mapsto & (z', 0) \end{array}$$

where U' (resp. U'') is an open neighborhood of 0 in $\mathbb{C}^{d_Z}$ (resp. $\mathbb{C}$).

In this case, the arrow

$$i_!\Omega_Z[d_Z] \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(i_!\mathcal{O}_Z, \Omega_X[d_X])$$

corresponds up to shift to the arrow

$$i_!\Omega_Z \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(i_!\mathcal{O}_Z, \Omega_X[1])$$

deduced from the arrow

$$\begin{array}{rcl} i_!\Omega_Z & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(i_!\mathcal{O}_Z, \mathcal{D}b_X^{d_X,\cdot}[1]) \\ i_!\omega & \mapsto & (i_!h \mapsto i_*(h\omega)) \end{array} \tag{5.2}$$

by using the natural map from $\mathcal{H}om_{\mathcal{O}_X}$ to $R\mathcal{H}om_{\mathcal{O}_X}$ and the Dolbeault resolution $\Omega_X \xrightarrow{\sim} \mathcal{D}b_X^{d_X,\cdot}$. Using the negative Koszul complex $K.(z'', \mathcal{O}_X)$ as a free resolution of $i_!\mathcal{O}_Z$, the map (5.2) corresponds to the morphism of complexes

$$i_!\Omega_Z \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(K.(z'', \mathcal{O}_X), \mathcal{D}b_X^{d_X,\cdot}) \tag{5.3}$$

which associates to $i_!\omega$ the morphism of complexes which sends h to $i_*(h_{|Z}\omega)$ in degree zero and is zero in other degrees.

The target of the preceding arrow is the simple complex associated to the double complex $K^{\cdot,\cdot}$ below

$$\begin{array}{ccccccc} \mathcal{D}b_X^{d_X,0} & \longrightarrow & \mathcal{D}b_X^{d_X,1} & \cdots & \longrightarrow & \mathcal{D}b_X^{d_X,d_X} \\ \uparrow{\scriptstyle z''} & & \uparrow{\scriptstyle z''} & \cdots & & \uparrow{\scriptstyle z''} \\ \mathcal{D}b_X^{d_X,0} & \longrightarrow & \mathcal{D}b_X^{d_X,1} & \cdots & \longrightarrow & \mathcal{D}b_X^{d_X,d_X} \end{array}$$

where the horizontal maps are the $\overline{\partial}$ Dolbeault operators, the vertical ones being multiplication by z''. In the preceding diagram the term of bidegree $(0,0)$ is in the upper left corner and the image of $i_!\omega$ by the arrow (5.3) corresponds to the section $i_*\omega$ of $\mathcal{D}b_X^{d_X,1}$ in bidegree $(-1,1)$.

Since the canonical inclusion of $K.(z'', \Omega_X)$ in the simple complex $sK^{\cdot,\cdot}$ is a quasi-isomorphism, it follows that the cohomology of $sK^{\cdot,\cdot}$ is concentrated in degree zero and that $H^0(sK^{\cdot,\cdot})$ is isomorphic to $\Omega_X/z''\Omega_X$.

Now we have successively

$$\begin{array}{rcl} i_*\omega & = & \omega(z') \wedge \delta(z'')dx'' \wedge dy'' \\ & = & \omega(z') \wedge \overline{\partial}\left(\dfrac{dz''}{2i\pi z''}\right) \\ & = & \overline{\partial}\left(\omega(z') \wedge \dfrac{dz''}{2i\pi z''}\right) \end{array}$$

and this shows that $i_*\omega$ has the same cohomology class in $H^0(sK^{\cdot,\cdot})$ as the section $\omega(z') \wedge (dz''/2i\pi)$ of $K^{0,0}$. Hence the arrow (**) corresponds at the level of H^0 to the isomorphism

$$\begin{array}{rcl} i_!\Omega_Z & \longrightarrow & \Omega_X/z''\Omega_X \\ i_!\omega & \mapsto & \left[\omega(z') \wedge \dfrac{dz''}{2i\pi}\right]_{z''\Omega_X} \end{array}$$

and the conclusion follows. □

5.3 The case of a projection

As for the finiteness theorem our starting point will be the case of a projection.

Proposition 5.14 *Let Z, S be complex analytic manifolds and denote by n the complex dimension of Z. Consider $Z \times S$ as a relative analytic manifold over S through the second projection ϵ. Let G be an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(Z)$ and set $F = G \boxtimes \mathbb{C}_S$. Assume that $(\mathcal{M}, F)$ is a good relative elliptic pair with ϵ-proper support on $Z \times S|S$. Then the natural pairing*

$$R\epsilon_*(\mathcal{M} \otimes F \otimes^L_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S}) \otimes^L_{\mathcal{O}_S} R\epsilon_!(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{Z\times S|S}}(\mathcal{M}, \Omega^n_{Z\times S|S}[n])) \longrightarrow \mathcal{O}_S$$

identifies each complex with the $\mathcal{O}_S$ dual of the other.

Proof: Since the dual of a relative elliptic pair is a relative elliptic pair, we need only to show that the map

$$\begin{aligned} & R\epsilon_!(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{Z\times S|S}}(\mathcal{M}, \Omega^n_{Z\times S|S}[n]) \\ & \quad \longrightarrow \; R\mathcal{H}om_{\mathcal{O}_S}(R\epsilon_*(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S}), \mathcal{O}_S) \end{aligned}$$

deduced from the duality pairing is an isomorphism in the derived category.

Using Lemmas 3.3, 3.4 and 3.6 and the regularity quasi-isomorphism, it is equivalent to prove that the canonical map

$$\begin{aligned} & R\epsilon_!(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}(\mathcal{M}_{\mathbb{R}|S}, \mathcal{D}b^{n,n}_{Z\times S|S})) \\ & \quad \longrightarrow \; R\mathcal{H}om_{\mathcal{O}_S}(R\epsilon_* R\mathcal{H}om(D'F, \mathcal{M}_{\mathbb{R}|S} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S}), \mathcal{O}_S)) \end{aligned} \tag{5.4}$$

is an isomorphism in the derived category.

We will work as in the proof of Proposition 4.1 and use the notations introduced there. Using the resolution $\mathcal{L}^\cdot$ of $\mathcal{M}_{\mathbb{R}|S}$ and the resolution $T^\cdot$ of $D'G$, we will compute explicitly the preceding morphism. We already know that

$$\begin{aligned} & R\epsilon_* R\mathcal{H}om(D'F, \mathcal{M}_{\mathbb{R}|S} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S}) \\ & \quad \xrightarrow{\sim} \; \epsilon_* \mathcal{H}om(T^\cdot \boxtimes \mathbb{C}_S, \mathcal{L}^\cdot \otimes_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S}) = \mathcal{R}_2^\cdot. \end{aligned}$$

Since $\mathcal{D}b_{W_{k,i}\times S}$ is acyclic for the functor $\epsilon_{|W_{k,i}\times S_!}$, we get

$$\begin{aligned} & R\epsilon_!(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}(\mathcal{M}_{\mathbb{R}|S}, \mathcal{D}b^{n,n}_{Z\times S|S})) \\ & \quad \xrightarrow{\sim} \; \epsilon_!((T^\cdot \boxtimes \mathbb{C}_S) \otimes \mathcal{H}om_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}(\mathcal{L}^\cdot, \mathcal{D}b^{n,n}_{Z\times S|S})). \end{aligned}$$

We denote by $\mathcal{R}_3^\cdot$ this last complex.

The components of $\mathcal{R}_2^\cdot$ (resp. $\mathcal{R}_3^\cdot$) are finite sums of the sheaves

$$\epsilon_{|W_{k,i}\times S_*}(\mathcal{F}_{W_{k,i}\times S}) \qquad (\text{resp. } \epsilon_{|W_{k,i}\times S_!}(\mathcal{D}b^{n,n}_{W_{k,i}\times S}) \;)$$

which are FN-free (resp. DFN-free) $\mathcal{O}_S$-modules. Hence, $\mathcal{R}_2^{\cdot}$ and $\mathcal{R}_3^{\cdot}$ are naturally complexes of topological $\mathcal{O}_S$-modules.

For any open subset U of Z, we know that

$$\epsilon_{U\times S|S_!}\mathcal{D}b^{n,n}_{U\times S|S} = \mathcal{L}_{\mathcal{O}_S}(\epsilon_{|U\times S|S_*}\mathcal{F}_{U\times S|S}, \mathcal{O}_S)$$

where the second member of the preceding equality is the sheaf of continuous $\mathcal{O}_S$-linear homomorphisms between the FN-free $\mathcal{O}_S$-module $\epsilon_{|U\times S|S_*}\mathcal{F}_{U\times S|S}$ and $\mathcal{O}_S$. Hence we get the canonical isomorphism

$$\mathcal{R}_3^k \xrightarrow{\sim} \mathcal{L}_{\mathcal{O}_S}(\mathcal{R}_2^{-k}, \mathcal{O}_S)$$

for any integer k. One checks easily that these maps define an isomorphism of complexes

$$\mathcal{R}_3^{\cdot} \xrightarrow{\sim} \mathcal{L}_{\mathcal{O}_S}(\mathcal{R}_2^{\cdot}, \mathcal{O}_S).$$

Moreover, the composition of this morphism with the natural morphism

$$\mathcal{L}_{\mathcal{O}_S}(\mathcal{R}_2^{\cdot}, \mathcal{O}_S) \longrightarrow R\mathcal{H}om_{\mathcal{O}_S}(\mathcal{R}_2^{\cdot}, \mathcal{O}_S) \tag{5.5}$$

gives the map (5.4).

Since $\mathcal{R}_2^{\cdot}$ has $\mathcal{O}_S$-coherent cohomology, Proposition 3.12 shows that (5.5) is a quasi-isomorphism and the proof is complete. □

5.4 The general case

Theorem 5.15 *Let $f : X|S \longrightarrow Y|S$ be a morphism of relative analytic manifolds over S. Assume $(\mathcal{M}, F)$ is a good relative f-elliptic pair with f-proper support; i.e.*

- *$\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{X|S})$,*
- *F is an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$,*
- *$\phi^{-1}\mathrm{char}_{f|S}(\mathcal{M}) \cap SS(F) \subset T^*_X X$,*
- *$\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ is f-proper.*

Then the duality morphism

$$\underline{f}_{|S!}(D'F \otimes \underline{D}_{X|S}(\mathcal{M})) \longrightarrow \underline{D}_{Y|S}(\underline{f}_{|S!}(F \otimes \mathcal{M}))$$

is an isomorphism.

Proof: Using the factorization of f through its graph embedding

$$i : X \longrightarrow X \times Y$$

we deduce from the results obtained for closed embeddings that the theorem will be true if it is true for the second projection

$$q : X \times Y|S \longrightarrow Y|S$$

and the pair

$$(\underline{i}_{|S!}(\mathcal{M}), F \boxtimes \mathbb{C}_Y) \in Ob(\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{X\times Y|S}) \times \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X \times Y)).$$

From the definition of $\mathrm{char}_{f|S}(\mathcal{M})$, it is clear that

$$\mathrm{char}_{q|S}(\underline{i}_{|S!}(\mathcal{M})) \cap SS(F \boxtimes \mathbb{C}_Y)$$

is in the zero section of $T^*(X \times Y|S)$. So if $Y = S$, the theorem is a consequence of the results obtained in the product case.

To conclude, we will show that if f is a relative submersion and the theorem is true for $f : X|Y \longrightarrow Y|Y$ then it is also true for $f : X|S \longrightarrow Y|S$.

Let us assume first that there is a coherent right $\mathcal{D}_{X|Y}$-module $\mathcal{M}_0$ such that

$$\mathcal{M} = \mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}.$$

One get successively:

$$\begin{aligned}
&\underline{D}_{Y|S}(\underline{f}_!(F \otimes \mathcal{M}))\\
&\quad\xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_{Y|S}}([Rf_!(F \otimes \mathcal{M}_0 \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{O}_X)] \otimes^L_{\mathcal{O}_Y} \mathcal{D}_{Y|S}, \mathcal{K}_{Y|S})\\
&\quad\xrightarrow{\sim} R\mathcal{H}om_{\mathcal{O}_Y}(\underline{f}_{|Y!}(F \otimes \mathcal{M}_0), \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{K}_{Y|S}\\
&\quad\xrightarrow{\sim} \underline{f}_{|Y!}(D'F \otimes \underline{D}_{X|Y}(\mathcal{M}_0)) \otimes_{\mathcal{O}_Y} \mathcal{K}_{Y|S}\\
&\quad\xrightarrow{\sim} Rf_!(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{X|Y}}(\mathcal{M}_0, \mathcal{K}_{X|Y}) \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{K}_{Y|S})\\
&\quad\xrightarrow{\sim} Rf_!(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{X|Y}}(\mathcal{M}_0, \mathcal{K}_{X|S}) \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S})\\
&\quad\xrightarrow{\sim} \underline{f}_{|S!}(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{X|S}}(\mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}, \mathcal{K}_{X|S}))\\
&\quad\xrightarrow{\sim} \underline{f}_{|S!}(D'F \otimes \underline{D}_{X|S}(\mathcal{M}))
\end{aligned}$$

and the theorem is proved.

The general case is reduced to the preceding case by using Lemma 2.10 as in the proof of Theorem 4.2.

□

6 Base change and Künneth formula

6.1 Base change

Recall that to any morphism $b : S_b \longrightarrow S$ of complex manifolds is associated a base change functor

$$\begin{aligned}
(\cdot)_b : \mathrm{Man}(S) &\longrightarrow \mathrm{Man}(S_b)\\
X|S &\longrightarrow X \times_S S_b|S_b
\end{aligned}$$

which transforms relative manifolds over S into relative manifolds over S_b. The aim of this section is to study the behavior of relative elliptic pairs under this functor. The main result is Theorem 6.5.

Let us fix a base change map $b : S_b \longrightarrow S$. For any relative manifold $X|S$, we denote by $X_b|S_b$ its image by the base change functor $(\cdot)_b$ and by b_X the projection from X_b to X. By construction, we have the cartesian square:

$$\begin{array}{ccc} X & \xrightarrow{\epsilon_X} & S \\ \uparrow{\scriptstyle b_X} & \square & \uparrow{\scriptstyle b} \\ X_b & \xrightarrow[\epsilon_{X_b}]{} & S_b. \end{array}$$

Hence, there is a canonical ring morphism

$$b_X^{-1}\mathcal{D}_{X|S} \longrightarrow \mathcal{D}_{X_b|S_b}$$

and we may introduce the following definition.

Definition 6.1 The *base change functor for relative right $\mathcal{D}$-modules* is the functor

$$\begin{array}{ccc} \mathbf{D}(\mathcal{D}_{X|S}^{\mathrm{op}}) & \longrightarrow & \mathbf{D}(\mathcal{D}_{X_b|S_b}^{\mathrm{op}}) \\ \mathcal{M} & \mapsto & b_X^{-1}\mathcal{M} \otimes^L_{b_X^{-1}\mathcal{D}_{X|S}} \mathcal{D}_{X_b|S_b}. \end{array}$$

This functor clearly induces a functor from $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{X|S}^{\mathrm{op}})$ to $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{X_b|S_b}^{\mathrm{op}})$.

The *base change functor for sheaves of $\mathbb{C}$-vector spaces* is the functor

$$\begin{array}{ccc} \mathbf{D}(X) & \longrightarrow & \mathbf{D}(X_b) \\ F & \mapsto & b_X^{-1}F. \end{array}$$

This functor clearly induces a functor from $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$ to $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X_b)$. Since the context will avoid any possible confusion, we denote all these functors by $(\cdot)_b$.

Let us consider now a morphism $f : X|S \longrightarrow Y|S$ of relative manifolds. We denote by $f_b : X_b|S_b \longrightarrow Y_b|S_b$ the image of f by the base change associated with b. One checks easily that the square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle b_X} & & \uparrow{\scriptstyle b_Y} \\ X_b & \xrightarrow[f_b]{} & Y_b \end{array} \tag{6.1}$$

is cartesian.

Proposition 6.2 *Using the notations introduced above:*

a) There is a canonical morphism

$$b_X^{-1}\mathcal{D}_{X|S\to Y|S} \otimes^L_{f_b^{-1}b_Y^{-1}\mathcal{D}_{Y|S}} f_b^{-1}\mathcal{D}_{Y_b|S_b} \longrightarrow \mathcal{D}_{X_b|S_b\to Y_b|S_b}$$

in $\mathbf{D}^{\mathrm{b}}(b_X^{-1}\mathcal{D}_{X|S} \otimes f_b^{-1}\mathcal{D}_{Y_b|S_b}^{\mathrm{op}})$.

b) The preceding morphism induces a natural morphism

$$(\underline{f}_{|S!}(\mathcal{M}))_b \longrightarrow \underline{f_b}_{|S_b!}(\mathcal{M}_b)$$

for $\mathcal{M}$ in $\mathbf{D}(\mathcal{D}^{\mathrm{op}}_{X|S})$.

c) The morphisms in (a) and (b) are isomorphisms if either f or b is a closed embedding.

Proof: Since

$$b_X^{-1}\mathcal{D}_{X|S\to Y|S} \otimes^L_{f_b^{-1}b_Y^{-1}\mathcal{D}_{Y|S}} f_b^{-1}\mathcal{D}_{Y_b|S_b} \xrightarrow{\sim} b_X^{-1}\mathcal{O}_X \otimes^L_{b_X^{-1}f^{-1}\mathcal{O}_Y} f_b^{-1}\mathcal{D}_{Y_b|S_b}$$

and

$$\mathcal{D}_{X_b|S_b\to Y_b|S_b} \xrightarrow{\sim} \mathcal{O}_{X_b} \otimes_{f_b^{-1}\mathcal{O}_{Y_b}} f_b^{-1}\mathcal{D}_{Y_b|S_b}$$

the canonical morphism $b_X^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_{X_b}$ induces the morphism in (a).

For any $\mathcal{M}$ in $\mathbf{D}(\mathcal{D}^{\mathrm{op}}_{X|S})$, we construct the morphism in (b) as the chain of morphisms:

$$\begin{aligned}
(\underline{f}_{|S!}\mathcal{M})_b &= b_Y^{-1}Rf_!(\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \otimes^L_{b_Y^{-1}\mathcal{D}_{Y|S}} \mathcal{D}_{Y_b|S_b}\\
&\xrightarrow{\sim} Rf_{b!}b_X^{-1}(\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \otimes^L_{b_Y^{-1}\mathcal{D}_{Y|S}} \mathcal{D}_{Y_b|S_b}\\
&\xrightarrow{\sim} Rf_{b!}(b_X^{-1}\mathcal{M} \otimes^L_{b_X^{-1}\mathcal{D}_{X|S}} b_X^{-1}\mathcal{D}_{X|S\to Y|S} \otimes^L_{f_b^{-1}b_Y^{-1}\mathcal{D}_{Y|S}} f_b^{-1}\mathcal{D}_{Y_b|S_b})\\
&\longrightarrow Rf_{b!}(b_X^{-1}\mathcal{M} \otimes^L_{b_X^{-1}\mathcal{D}_{X|S}} \mathcal{D}_{X_b|S_b\to Y_b|S_b})\\
&\xrightarrow{\sim} \underline{f_b}_{|S_b!}(\mathcal{M}_b).
\end{aligned}$$

This chain of morphisms is obtained using the definition of the base change functor, the fact that the square (6.1) is cartesian, the projection formula, the morphism constructed in (a) and again the definition of the base change functor.

To conclude the proof, it is sufficient to show that if either f or b is a closed embedding then the morphism constructed in (a) is an isomorphism.

Assume f is a closed embedding. The problem being local, we may assume there are open neighborhoods U and V of zero in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively with $X = U \times S$, $Y = U \times V \times S$ and

$$\begin{aligned}
f : U \times S &\longrightarrow U \times V \times S\\
(u, s) &\longrightarrow (u, 0, s).
\end{aligned}$$

Then, we get $X_b = U \times S_b$, $Y_b = U \times V \times S_b$ and

$$\begin{aligned}
f_b : U \times S_b &\longrightarrow U \times V \times S_b\\
(u, s_b) &\longrightarrow (u, 0, s_b).
\end{aligned}$$

Moreover, $b_X(u, s_b) = (u, b(s_b))$ and $b_Y(u, v, s_b) = (u, v, b(s_b))$. In this simple geometric situation, we have the Koszul quasi-isomorphisms:

$$\begin{aligned}
K.(\mathcal{O}_Y; v_1, \dots, v_n) &\xrightarrow{\sim} f_*\mathcal{O}_X\\
K.(\mathcal{O}_{Y_b}; v_1 \circ b_Y, \dots, v_n \circ b_Y) &\xrightarrow{\sim} f_{b*}\mathcal{O}_{X_b}
\end{aligned}$$

where $v_1, \ldots, v_n$ denotes the functions on Y induced by the standard coordinates on V. Hence, we get the isomorphisms:

$$\begin{aligned} b_X^{-1}\mathcal{O}_X \otimes^L_{b_X^{-1}f^{-1}\mathcal{O}_Y} f_b^{-1}\mathcal{D}_{Y_b|S_b} &\simeq f_b^{-1}K.(\mathcal{D}_{Y_b|S_b}; v_1 \circ b_Y, \ldots, v_n \circ b_Y) \\ \mathcal{O}_{X_b} \otimes^L_{f_b^{-1}\mathcal{O}_{Y_b}} f_b^{-1}\mathcal{D}_{Y_b|S_b} &\simeq f_b^{-1}K.(\mathcal{D}_{Y_b|S_b}; v_1 \circ b_Y, \ldots, v_n \circ b_Y) \end{aligned}$$

and the conclusion follows.

The case where b is a closed embedding is treated in a similar way. □

The following easy lemma will be useful in the sequel. We leave its proof to the reader.

Lemma 6.3 *Let $f : X|S \longrightarrow Y|S$ be a relative submersion and let $b : S_b \longrightarrow S$ be a base map. Consider X as a relative manifold over Y through the map f and assume $\mathcal{N}$ is an object of $\mathbf{D}(\mathcal{D}^{\mathrm{op}}_{X|Y})$. Then*

$$f_b : X_b|Y_b \longrightarrow Y_b|Y_b$$

is the image of $f : X|Y \longrightarrow Y|Y$ by the base change associated with b_Y and

$$(\mathcal{N} \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S})_b \simeq \mathcal{N}_{b_Y} \otimes_{\mathcal{D}_{X_b|Y_b}} \mathcal{D}_{X_b|S_b}.$$

The behavior of the characteristic variety under base change is given by the following result.

Proposition 6.4 *Let $f : X|S \longrightarrow Y|S$ be a morphism of relative manifolds. In the diagram*

$$T^*X_b|S_b \xleftarrow[{}^t(b_X)']{} X_b \times_X T^*X|S \xrightarrow[(b_X)_\pi]{} T^*X|S$$

the first arrow is an isomorphism and for any object $\mathcal{M}$ of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{X|S})$ we have

$$\mathrm{char}_{f_b|S_b}(\mathcal{M}_b) \subset {}^t(b_X)'(b_X)_\pi^{-1}\mathrm{char}_{f|S}(\mathcal{M}).$$

Proof: Using the graph factorization of f and part (c) of Proposition 6.2 we are reduced to the case where f is a relative submersion. In this case, assume $\mathcal{M}$ is generated as a right $\mathcal{D}_{X|S}$-module by a coherent right $\mathcal{D}_{X|Y}$-module $\mathcal{M}_0$. Thanks to Lemma 6.3 the epimorphism

$$\mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S} \longrightarrow \mathcal{M} \longrightarrow 0$$

induces the epimorphism

$$(\mathcal{M}_0)_{b_Y} \otimes_{\mathcal{D}_{X_b|Y_b}} \mathcal{D}_{X_b|S_b} \longrightarrow \mathcal{M}_b \longrightarrow 0.$$

Hence,

$$\mathrm{char}_{f_b|S_b}(\mathcal{M}_b) \subset \phi_f^{-1}\mathrm{char}_{X_b|Y_b}((\mathcal{M}_0)_{b_Y})$$

and the result will be true for $f : X|S \longrightarrow Y|S$ and the base change by b if it is true for $f : X|Y \longrightarrow Y|Y$ and the base change by b_Y. In other words, we are reduced to the obvious case where $Y = S$. □

Theorem 6.5 *Let $f : X|S \longrightarrow Y|S$ be a morphism of relative manifolds and let $b : S_b \longrightarrow S$ be a base map. Denote by $f_b : X_b|S_b \longrightarrow Y_b|S_b$ the image of f by the base change associated to b. Assume $(\mathcal{M},F)$ is a good relative f-elliptic pair. Then*

a) $(\mathcal{M}_b,F_b)$ is an f_b-elliptic pair in some neighborhood of $\operatorname{supp}\mathcal{M}_b$,

b) the canonical morphism

$$[\underline{f}_{|S!}(F \otimes \mathcal{M})]_b \longrightarrow \underline{f_b}_{|S_b!}(F_b \otimes \mathcal{M}_b)$$

is an isomorphism.

Proof: Since $(\mathcal{M},F)$ is a relative elliptic pair, F is non characteristic for b_X in a neighborhood of $\operatorname{supp}\mathcal{M}$ and [12, Proposition 5.4.13] gives us an estimate of the micro-support of F_b which together with the preceding proposition gives us (a).

To prove part (b), we will use the graph factorization of f and part (c) of Proposition 6.2 to reduce the problem to the case where $f : X|S \longrightarrow Y|S$ is a relative submersion.

As in the preceding proposition, it is sufficient to treat the case $Y = S$. Assume $\mathcal{M}_0$ is a right $\mathcal{D}_{X|Y}$-module and set $\mathcal{M} = \mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}$. We have successively:

$$\begin{aligned}[\underline{f}_{|S!}(\mathcal{M})]_b &\simeq [\underline{f}_{|Y!}(\mathcal{M}_0) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y|S}]_b \\ &\simeq [\underline{f}_{|Y!}(\mathcal{M}_0)]_{b_Y} \otimes_{\mathcal{O}_{Y_b}} \mathcal{D}_{Y_b|S_b}\end{aligned}$$

and

$$\begin{aligned}\underline{f_b}_{|S_b!}(\mathcal{M}_b) &\simeq \underline{f_b}_{|S_b!}[(\mathcal{M}_0)_{b_Y} \otimes_{\mathcal{D}_{X_b|Y_b}} \mathcal{D}_{X_b|S_b}] \\ &\simeq \underline{f_b}_{|Y_b!}[(\mathcal{M}_0)_{b_Y}] \otimes_{\mathcal{O}_{Y_b}} \mathcal{D}_{Y_b|S_b}.\end{aligned}$$

Hence, using Lemma 2.10, we see that the theorem will be true for $f : X|S \longrightarrow Y|S$ and the base change by b if it is true for $f : X|Y \longrightarrow Y|Y$ and the base change by b_Y.

Finally, factorizing f and b through their graphs and using once more part (c) of Proposition 6.2, we see that it is sufficient to treat the case where $f : Z \times S|S \longrightarrow S|S$ is the second projection. We may also assume that the corresponding f-elliptic pair is of the form $(\mathcal{M}, G \boxtimes \mathbb{C}_S)$ where G and $\mathcal{M}$ are objects of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(Z)$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{Z\times S|S})$ respectively, and that $b : T \times S \longrightarrow S$ is the first projection. This product case is treated in Proposition 6.6 below. □

Proposition 6.6 *Let Z, S, T be complex analytic manifolds. Consider $Z \times S$ as a relative analytic manifold over S through the second projection ϵ. Let G be an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(Z)$ and set $F = G \boxtimes \mathbb{C}_S$. Consider the cartesian square*

$$\begin{array}{ccc} Z \times S & \xrightarrow{\epsilon} & S \\ \uparrow{\scriptstyle p} & \square & \uparrow{\scriptstyle b} \\ Z \times T \times S & \xrightarrow{\eta} & T \times S \end{array}$$

where the maps are the canonical projections. Assume that $(\mathcal{M},F)$ is a relative elliptic pair with ϵ-proper support on $Z \times S|S$. Then the canonical map

$$\begin{aligned} b^{-1}R\epsilon_*(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S}) \otimes_{b^{-1}\mathcal{O}_S} \mathcal{O}_{T\times S} \\ \longrightarrow\ R\eta_*(p^{-1}F \otimes p^{-1}\mathcal{M} \otimes^L_{p^{-1}\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times T\times S}) \end{aligned}$$

is an isomorphism.

Proof: Thanks to the regularity theorem 2.15 and Lemma 3.3, it is equivalent to prove that the canonical morphism:

$$\begin{aligned} b^{-1}R\epsilon_*R\mathcal{H}om(D'F, \mathcal{M}_{\mathbb{R}} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S}) \otimes_{b^{-1}\mathcal{O}_S} \mathcal{O}_{T\times S} \\ \longrightarrow\ R\eta_*R\mathcal{H}om(p^{-1}D'F, p^{-1}\mathcal{M}_{\mathbb{R}} \otimes^L_{p^{-1}\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times T\times S|T\times S}) \end{aligned}$$

is an isomorphism. For short, let us denote by $\mathcal{S}_1^{\cdot}$ (resp. $\mathcal{S}_2^{\cdot}$) the source (resp. target) of the preceding arrow. Clearly, it is sufficient to prove that for any open polydisc Δ of T, the induced morphism

$$Rb_*(\mathcal{S}^{\cdot}_{1|\Delta\times S}) \longrightarrow Rb_*(\mathcal{S}^{\cdot}_{2|\Delta\times S}) \tag{6.2}$$

is an isomorphism. We will compute this morphism explicitly as in the proof of Proposition 4.1. Using the notations introduced there, we already know that

$$\begin{aligned} R\epsilon_*R\mathcal{H}om(D'F, \mathcal{M}_{\mathbb{R}|S} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S}) \\ \xrightarrow{\sim}\ \epsilon_*\mathcal{H}om(T^{\cdot} \boxtimes \mathbb{C}_S, \mathcal{L}^{\cdot} \otimes_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S}) = \mathcal{R}_2^{\cdot}. \end{aligned}$$

Since $\mathcal{R}_2^{\cdot}$ has $\mathcal{O}_S$-coherent cohomology,

$$\begin{aligned} Rb_*(\mathcal{S}^{\cdot}_{1|\Delta\times S}) &\simeq Rb_*(b^{-1}\mathcal{R}_2^{\cdot} \otimes^L_{b^{-1}\mathcal{O}_S} \mathcal{O}_{\Delta\times S}) \\ &\simeq \mathcal{R}_2^{\cdot} \otimes^L_{\mathcal{O}_S} b_*\mathcal{O}_{\Delta\times S}. \end{aligned}$$

Moreover, we have:

$$\begin{aligned} Rb_*(\mathcal{S}^{\cdot}_{2|\Delta\times S}) &\simeq R\epsilon_*R\mathcal{H}om(D'F, \mathcal{M}_{\mathbb{R}} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} p_*\mathcal{F}_{Z\times\Delta\times S|\Delta\times S}) \\ &\simeq \epsilon_*\mathcal{H}om(T^{\cdot} \boxtimes \mathbb{C}_S, \mathcal{L}^{\cdot} \otimes_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} p_*\mathcal{F}_{Z\times\Delta\times S|\Delta\times S}). \end{aligned}$$

Let us denote $\mathcal{R}_4^{\cdot}$ this last complex. Since we have the isomorphism

$$\epsilon_*p_*i_{W\times\Delta\times S*}i^{-1}_{W\times\Delta\times S}\mathcal{F}_{Z\times\Delta\times S|\Delta\times S} \simeq \Gamma(W;\mathcal{F}_W) \hat{\otimes} \Gamma(\Delta;\mathcal{O}_\Delta) \hat{\otimes} \mathcal{O}_S$$

for any open subset W of Z, a direct computation shows that

$$\mathcal{R}_4^{\cdot} \xrightarrow{\sim} \mathcal{R}_2^{\cdot} \hat{\otimes}_{\mathcal{O}_S} b_*\mathcal{O}_{\Delta\times S}.$$

Clearly, the morphism (6.2) corresponds to the canonical morphism

$$\mathcal{R}_2^{\cdot} \otimes^L_{\mathcal{O}_S} b_*\mathcal{O}_{\Delta\times S} \longrightarrow \mathcal{R}_2^{\cdot} \hat{\otimes}_{\mathcal{O}_S} b_*\mathcal{O}_{\Delta\times S}.$$

Since $\mathcal{R}_2^{\cdot}$ has $\mathcal{O}_S$-coherent cohomology, Proposition 3.13 shows that it is an isomorphism and the conclusion follows. □

6.2 Künneth formula

Theorem 6.7 *Let $f_1 : X_1|S \longrightarrow Y_1|S$ and $f_2 : X_2|S \longrightarrow Y_2|S$ be two morphisms of relative manifolds. Assume*

i) $(\mathcal{M}_1,F_1)$ is a good relative f_1-elliptic pair with f_1-proper support,

ii) $(\mathcal{M}_2,F_2)$ is a good relative f_2-elliptic pair with f_2-proper support.

Then:

a) $(\mathcal{M}_1 \underline{\boxtimes}_S \mathcal{M}_2, F_1 \boxtimes_S F_2)$ is a good relative $f_1 \times_S f_2$-elliptic pair with $f_1 \times_S f_2$-proper support,

b) the natural morphism

$$\underline{f_1}_{|S!}(F_1 \otimes \mathcal{M}_1) \underline{\boxtimes}_S \underline{f_2}_{|S!}(F_2 \otimes \mathcal{M}_2) \longrightarrow \underline{f_1 \times_S f_2}_{|S!}[(F_1 \boxtimes_S F_2) \otimes (\mathcal{M}_1 \underline{\boxtimes}_S \mathcal{M}_2)]$$

is an isomorphism.

Proof: Part (a) being obvious, we skip directly to part (b). Since

$$\underline{f_1}_{|S!}(F_1 \otimes \mathcal{M}_1)$$

has $\mathcal{D}_{Y_1|S}$-coherent cohomology, the formula

$$f_1 \times_S f_2 = (\mathrm{id}_{X_1} \times_S f_2) \circ (f_1 \times_S \mathrm{id}_{X_2})$$

allows us to restrict to the case $f_2 = \mathrm{id}_{X_2}$. So, we need only to prove that the canonical map

$$\underline{f_1}_{|S!}(F_1 \otimes \mathcal{M}_1) \underline{\boxtimes}_S (F_2 \otimes \mathcal{M}_2) \longrightarrow \underline{f_1 \times_S \mathrm{id}_{X_2}}_{|S!}(F_1 \boxtimes_S F_2) \otimes (\mathcal{M}_1 \underline{\boxtimes}_S \mathcal{M}_2)$$

is an isomorphism. Using the projection formula, we may get rid of F_2. So we assume $F_2 = \mathbb{C}_{X_2}$. The problem being local on $Y_1 \times_S X_2$, we may further assume that $\mathcal{M}_2$ is equal to $\mathcal{D}_{X_2|S}$.

The image of $f_1 : X_1|S \longrightarrow Y_1|S$ under the base change associated with $\epsilon_2 : X_2 \longrightarrow S$ is

$$f_1 \times_S \mathrm{id}_{X_2} : X_1 \times_S X_2|X_2 \longrightarrow Y_1 \times_S X_2|X_2.$$

Hence, by Theorem 6.5, we have the isomorphism:

$$[\underline{f_1}_{|S!}(F_1 \otimes \mathcal{M}_1)]_{\epsilon_2} \xrightarrow{\sim} \underline{(f_1 \times_S \mathrm{id}_{X_2})}_{|X_2!}[(F_1 \otimes \mathcal{M}_1)_{\epsilon_2}].$$

By scalar extension, we get the isomorphism:

$$\underline{f_1}_{|S!}(F_1 \otimes \mathcal{M}_1) \underline{\boxtimes}_S \mathcal{D}_{X_2|S} \xrightarrow{\sim} \underline{f_1 \times_S \mathrm{id}_{X_2}}_{|S!}[(F_1 \boxtimes_S \mathbb{C}_{X_2}) \otimes (\mathcal{M}_1 \underline{\boxtimes}_S \mathcal{D}_{X_2|S})]$$

and the conclusion follows. □

7 Microlocalization

Here, we shall prove that direct image commutes with microlocalization. More precisely, denote by $\mathcal{E}_X$ the sheaf of (finite order) microdifferential operators on T^*X (see [18] or [19] for a detailed exposition).

Consider a morphism $f : X \longrightarrow Y$ of complex analytic manifolds and the associated diagram:

$$T^*X \xleftarrow[{}^t f']{} X \times_Y T^*Y \xrightarrow[f_\pi]{} T^*Y$$

and recall that the microlocal proper direct image of a right $\mathcal{E}_X$-module $\mathcal{M}$ is defined through the formula

$$\underline{f}_!(\mathcal{M}) = Rf_{\pi!}({}^t f'^{-1}\mathcal{M} \otimes^L_{{}^t f'^{-1}\mathcal{E}_X} \mathcal{E}_{X\to Y}),$$

where $\mathcal{E}_{X\to Y}$ denotes the micro-differential transfer module associated to f.

Also recall that the microlocalization of a right $\mathcal{D}_X$-module $\mathcal{M}$ is the right $\mathcal{E}_X$-module $\mathcal{M}\mathcal{E}$ defined on T^*X by setting

$$\mathcal{M}\mathcal{E} = \pi_X^{-1}\mathcal{M} \otimes_{\pi_X^{-1}\mathcal{D}_X} \mathcal{E}_X.$$

In this section, we prove that, under the hypothesis of the finiteness theorem, we have

$$[\underline{f}_!(\mathcal{M} \otimes F)]\mathcal{E} \simeq \underline{f}_![(\mathcal{M} \otimes F)\mathcal{E}].$$

This result was established by Kashiwara [9] when $F = \mathbb{C}_X$ and f is projective. It was also announced in a non proper case in [8].

7.1 The topology of the sheaf $\mathcal{C}_{Y|X}(0)$

Let us show that the sheaf $\mathcal{C}_{Y|X}(0)$ of [18] is naturally a sheaf of topological vector spaces and that its sections on a compact subset of T^*_YX form a DFN space.

Proposition 7.1 *Let X be a complex analytic manifold. Assume Y is a complex submanifold of X and denote by $\mathcal{C}_{Y|X}(0)$ the sheaf of holomorphic microfunctions of order 0 on T^*_YX. Then, for any compact subset $K \subset T^*_YX$, the space*

$$\Gamma(K; \mathcal{C}_{Y|X}(0))$$

has a canonical DFN topology.

Proof: Locally, we may use a coordinate system $(x_1, \ldots, x_d, y_1, \ldots, y_{n-d})$ where Y is defined by the equations

$$x_1 = 0, \cdots, x_d = 0.$$

Denote by $(y_1, \cdots, y_{n-d}, \xi_1, \ldots, \xi_d)$ the corresponding coordinates on T^*_YX. It follows from [18, Theorem 1.4.5] that, for any open subset U of T^*_YX, the formula

$$\int \delta(p - \langle x, \xi\rangle)u(x, y)dx = \sum_{j=-\infty}^{0} a_j(y, \xi)\delta^{(j)}(p) \tag{7.1}$$

establishes a one to one correspondence between holomorphic microfunctions

$$u(x,y) \in \Gamma(U; \mathcal{C}_{Y|X}(0))$$

and sequences of homogeneous holomorphic functions

$$a_j(x,\xi) \in \Gamma(U; \mathcal{O}_{T_Y^*X}(j)) \qquad (j \leq 0)$$

such that for any compact subset $K \subset U$

$$\sum_{j=-\infty}^{0} |a_j(x,\xi)|_K \frac{\epsilon^{-j}}{(-j)!} < +\infty$$

for some $\epsilon > 0$.

Let us first construct the requested DFN topology in two special cases.

Case a. Assume K is a convex compact subset of T_Y^*X on which $\xi_k \neq 0$. Denote by $p : \dot{T}_Y^*X \longrightarrow P_Y^*X$ the canonical projection. The preceding discussion shows that the map

$$\begin{array}{rcl} \Gamma(K; \mathcal{C}_{Y|X}(0)) & \longrightarrow & \Gamma(p(K) \times \{0\}; \mathcal{O}_{P_Y^*X \times \mathbb{C}}) \\ u(x,y) & \mapsto & f_k(y,\xi,\tau) = \displaystyle\sum_{j=0}^{+\infty} a_{-j}(y, \xi/\xi_k) \frac{\tau^j}{j!} \end{array}$$

is an isomorphism. Using this isomorphism, we endow $\Gamma(K; \mathcal{C}_{Y|X}(0))$ with the usual DFN topology of $\Gamma(p(K) \times \{0\}; \mathcal{O}_{P_Y^*X \times \mathbb{C}})$. If, moreover, $\xi_\ell \neq 0$ on K, one has

$$f_k(y,\xi,\tau) = f_\ell(y,\xi,\tau\xi_k/\xi_\ell).$$

Hence, the DFN topology of $\Gamma(K; \mathcal{C}_{Y|X}(0))$ does not depend on k.

Case b. Let π denote the canonical projection of the bundle T_Y^*X on its base Y identified to the zero section. Assume K is a convex compact subset of T_Y^*X such that $\pi(K) \subset K$. It follows from (7.1) that

$$\begin{array}{rcl} \Gamma(K; \mathcal{C}_{Y|X}(0)) & \longrightarrow & \Gamma(\pi(K); \mathcal{O}_Y) \\ u(x,y) & \mapsto & a_0(y,0) \end{array}$$

is an isomorphism. We use this isomorphism to transport on $\Gamma(K; \mathcal{C}_{Y|X}(0))$ the usual DFN topology of $\Gamma(\pi(K); \mathcal{O}_Y)$.

One checks easily that, if $K_1 \subset K_2$ are two compact subsets of T_Y^*X of the kind treated in case (a) or (b) above, then the restriction map

$$\Gamma(K_2; \mathcal{C}_{Y|X}(0)) \longrightarrow \Gamma(K_1; \mathcal{C}_{Y|X}(0))$$

is continuous.

Let K be an arbitrary compact subset of $T^*_Y X$. The preceding discussion shows that we can find a finite covering $(K_i)_{i\in I}$ of K by compact subsets such that $\Gamma(K_i;\mathcal{C}_{Y|X}(0))$ and $\Gamma(K_i\cap K_j;\mathcal{C}_{Y|X}(0))$ are DFN spaces. Thanks to the exact sequence

$$0 \longrightarrow \Gamma(K;\mathcal{C}_{Y|X}(0)) \xrightarrow{\alpha} \prod_{i\in I}\Gamma(K_i;\mathcal{C}_{Y|X}(0)) \xrightarrow{\beta} \prod_{i,j\in I}\Gamma(K_i\cap K_j;\mathcal{C}_{Y|X}(0)),$$

we may use α to transport on $\Gamma(K;\mathcal{C}_{Y|X}(0))$ the DFN topology of $\ker\beta$. To show that this topology is independent of the chosen covering, it is sufficient to show that it is equivalent to the topology induced by a finer covering. Since such a topology is obviously weaker, the conclusion follows from the closed graph theorem.

Since a direct computation shows that the above defined topology is independent of the chosen coordinate systems, the conclusion follows easily. □

Corollary 7.2 *Let X be a complex analytic manifold. Assume K is a compact subset of T^*X. Then*

$$\Gamma(K;\mathcal{E}_X(0))$$

has a canonical DFN topology.

Proof: Apply the preceding proposition to $\mathcal{C}_{\Delta_X|X\times X}(0)$. □

Proposition 7.3 *Let X, Z be complex analytic manifolds and let Y be a complex submanifold of X. We identify $T^*_{(Z\times Y)}(Z\times X)$ and $Z\times T^*_Y X$. We denote by q : $Z\times T^*_Y X \longrightarrow T^*_Y X$ the second projection. Then, for any Stein compact subset $K\subset Z$, one has*

$$Rq_![(\mathcal{C}_{Z\times Y|Z\times X}(0))_{K\times T^*_Y X}] \simeq \Gamma(K;\mathcal{O}_Z)\mathbin{\hat{\otimes}}\mathcal{C}_{Y|X}.$$

Proof: Let S be a complex manifold. Denote by $p_S : Z\times S \longrightarrow S$ the second projection. By classical results of analytic geometry, we know that

$$Rp_{S!}[(\mathcal{O}_{Z\times S})_{K\times S}] \simeq \Gamma(K;\mathcal{O}_Z)\mathbin{\hat{\otimes}}\mathcal{O}_S.$$

Using the explicit isomorphisms constructed in the proof of the preceding proposition, the conclusion follows easily. □

Corollary 7.4 *Let Z, Y be complex analytic manifolds and denote by*

$$f : Z\times Y \longrightarrow Y$$

the second projection. Assume K is a Stein compact subset of Z. Then

$$Rf_{\pi!}[(\mathcal{E}_{Z\times Y\to Y}(0))_{K\times T^*Y}] \simeq \Gamma(K;\mathcal{O}_Z)\mathbin{\hat{\otimes}}\mathcal{E}_Y(0).$$

Proof: Apply the preceding proposition to $\mathcal{C}_{Z\times\Delta_Y|Z\times(Y\times Y)}(0)$. □

7.2 Direct image and microlocalization

Theorem 7.5 *Assume $f : X \longrightarrow Y$ is a morphism of complex analytic manifolds and $(\mathcal{M}, F)$ is an f-elliptic pair on X with f-proper support. Then the canonical map*

$$[\underline{f}_!(\mathcal{M} \otimes F)]\mathcal{E} \longrightarrow \underline{f}_!([\mathcal{M} \otimes F]\mathcal{E})$$

is an isomorphism in $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_Y)$.

Proof: Recall that we have the commutative diagram

$$\begin{tikzcd}
T^*X \arrow[d, "\pi_X"'] & X \times_Y T^*Y \arrow[l, "{}^t f'"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"] & T^*Y \arrow[d, "\pi_Y"] \\
X & X \arrow[l, "\sim"'] \arrow[r, "f"'] & Y
\end{tikzcd}$$

Hence, we have successively

$$\begin{aligned}
&\pi_Y^{-1}[\underline{f}_!(\mathcal{M} \otimes F)] \otimes_{\pi_Y^{-1}\mathcal{D}_Y} \mathcal{E}_Y \\
&\quad = \; Rf_{\pi !}[\pi^{-1}(\mathcal{M} \otimes F \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y})] \otimes_{\pi_Y^{-1}\mathcal{D}_Y} \mathcal{E}_Y \\
&\quad = \; Rf_{\pi !}[\pi^{-1}(\mathcal{M} \otimes F \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{D}_Y} f_\pi^{-1}\mathcal{E}_Y] \\
&\quad = \; Rf_{\pi !}[\pi^{-1}(\mathcal{M} \otimes F) \otimes^L_{\pi^{-1}\mathcal{D}_X} (\pi^{-1}\mathcal{D}_{X\to Y} \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{D}_Y} f_\pi^{-1}\mathcal{E}_Y)].
\end{aligned}$$

Note that there is a canonical map

$$\pi^{-1}\mathcal{D}_{X\to Y} \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{D}_Y} f_\pi^{-1}\mathcal{E}_Y \longrightarrow \mathcal{E}_{X\to Y}. \tag{7.2}$$

Hence, we get a canonical morphism

$$\pi_Y^{-1}[\underline{f}_!(\mathcal{M} \otimes F)] \otimes_{\pi_Y^{-1}\mathcal{D}_Y} \mathcal{E}_Y \longrightarrow \underline{f}_![\pi_X^{-1}(\mathcal{M} \otimes F) \otimes_{\pi_X^{-1}\mathcal{D}_X} \mathcal{E}_X]. \tag{7.3}$$

When f is a closed embedding, (7.2) is an isomorphism. Hence (7.3) is an isomorphism for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ and any $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$.

In the general case, consider the graph embedding

$$i : X \longrightarrow X \times Y$$

and the projection

$$p : X \times Y \longrightarrow Y.$$

Since $(\mathcal{M}, F)$ is an f-elliptic pair, the pair $(\underline{i}_!\mathcal{M}, F \boxtimes \mathbb{C}_Y)$ is p-elliptic. Since our result holds for closed embeddings and

$$\underline{i}_!\mathcal{M} \otimes (F \boxtimes \mathbb{C}_Y) \simeq \underline{i}_!(\mathcal{M} \otimes F),$$

we are reduced to prove the theorem for the pair $(\underline{i}_!\mathcal{M}, F \boxtimes \mathbb{C}_Y)$ and the map p.

We may thus assume that f is the second projection from $X = Z \times Y$ to Y and that $F = G \boxtimes \mathbb{C}_Y$ where G is an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(Z)$. Moreover, working as in §4, we may also assume that $\mathcal{M} = \mathcal{N} \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_X$ where $\mathcal{N}$ is a coherent $\mathcal{D}_{X|Y}$-module. In this case,

$$
\begin{aligned}
&\pi_Y^{-1}[\underline{f}_!(\mathcal{M} \otimes F)] \otimes_{\pi_Y^{-1}\mathcal{D}_Y} \mathcal{E}_Y \\
&\qquad = \; Rf_{\pi !}[\pi^{-1}(\mathcal{M} \otimes F) \otimes^L_{\pi^{-1}\mathcal{D}_X} (\pi^{-1}\mathcal{D}_{X\to Y} \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{D}_Y} f_\pi^{-1}\mathcal{E}_Y)] \\
&\qquad = \; Rf_{\pi !}[\pi^{-1}(\mathcal{N} \otimes (G \boxtimes \mathbb{C}_Y)) \otimes^L_{\pi^{-1}\mathcal{D}_{X|Y}} (\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y)]
\end{aligned}
$$

and

$$
\underline{f}_![\pi_X^{-1}(\mathcal{M} \otimes F) \otimes_{\pi_X^{-1}\mathcal{D}_X} \mathcal{E}_X] \;=\; Rf_{\pi !}[\pi^{-1}(\mathcal{N} \otimes (G \boxtimes \mathbb{C}_Y)) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} \mathcal{E}_{X\to Y}].
$$

Hence, we are reduced to show that the canonical arrow

$$
\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0) \longrightarrow \mathcal{E}_{X\to Y}(0)
$$

induces an isomorphism

$$
\begin{aligned}
&Rf_{\pi !}[\pi^{-1}(\mathcal{N} \otimes (G \boxtimes \mathbb{C}_Y)) \otimes^L_{\pi^{-1}\mathcal{D}_{X|Y}} (\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0))] \qquad (7.4)\\
&\qquad \xrightarrow{\sim} \; Rf_{\pi !}[\pi^{-1}(\mathcal{N} \otimes (G \boxtimes \mathbb{C}_Y)) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} \mathcal{E}_{X\to Y}(0)]
\end{aligned}
$$

As a matter of fact, $\mathcal{E}_{X\to Y} \simeq \mathcal{E}_{X\to Y}(0) \otimes_{f_\pi^{-1}\mathcal{E}_Y(0)} f_\pi^{-1}\mathcal{E}_Y$ as a $(\mathcal{D}_{X|Y},\mathcal{E}_Y)$-bimodule and a scalar extension of (7.4) gives the theorem.

Using the realification process as in §4, we may assume from the beginning that Z is a complexification of a real analytic manifold M and that G is supported by M.

Since the result is local on T^*Y (hence on Y), we may assume also that $\mathcal{N}$ has a projective resolution $\mathcal{L}^\cdot$ by finite free $\mathcal{D}_{X|Y}$-modules (see Proposition 3.1).

As for G, we may assume it is isomorphic to a bounded complex $T^\cdot$ of the type

$$
0 \longrightarrow \cdots \bigoplus_{i_a \in I_a} \mathbb{C}_{K_{a,i_a}} \longrightarrow \cdots \bigoplus_{i_k \in I_k} \mathbb{C}_{K_{k,i_k}} \longrightarrow \cdots \bigoplus_{i_b \in I_b} \mathbb{C}_{K_{b,i_b}} \longrightarrow 0
$$

where the sets I_k are finite and K_{k,i_k} is a subanalytic compact subset of M (see Proposition 3.10).

Hence,

$$
\mathcal{N} \otimes (F \boxtimes \mathbb{C}_Y) \simeq \mathcal{L}^\cdot \otimes (T^\cdot \boxtimes \mathbb{C}_Y)
$$

and the components of this last complex are finite direct sums of sheaves of the type

$$
\mathcal{D}_{X|Y} \otimes \mathbb{C}_{K\times Y}
$$

where K is a subanalytic compact subset of M.

Note that

$$
\begin{aligned}
&\pi^{-1}(\mathcal{D}_{X|Y} \otimes \mathbb{C}_{K\times Y}) \otimes^L_{\pi^{-1}\mathcal{D}_{X|Y}} (\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0)) \qquad (7.5)\\
&\qquad \xrightarrow{\sim} \; \pi^{-1}(\mathcal{O}_X)_{K\times Y} \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0)\\
&\pi^{-1}(\mathcal{D}_{X|Y} \otimes \mathbb{C}_{K\times Y}) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} \mathcal{E}_{X\to Y}(0) \qquad (7.6)\\
&\qquad \xrightarrow{\sim} \; (\mathcal{E}_{X\to Y}(0))_{K\times T^*Y}
\end{aligned}
$$

The right hand side of (7.5) is acyclic for $f_{\pi!}$ thanks to usual properties of Stein compact subsets. Moreover, Corollary 7.4 shows that the right hand side of (7.6) is also acyclic for $f_{\pi!}$. Hence, the morphism (7.4) of $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_Y(0))$ is represented in $C^b(\mathcal{E}_Y(0))$ by the morphism

$$\begin{aligned} & f_{\pi!}[\pi^{-1}(\mathcal{L}^{\cdot} \otimes (T^{\cdot} \boxtimes \mathbb{C}_Y)) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} (\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0))] \\ \longrightarrow\ & f_{\pi!}[\pi^{-1}(\mathcal{L}^{\cdot} \otimes (T^{\cdot} \boxtimes \mathbb{C}_Y)) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} \mathcal{E}_{X\to Y}(0)] \end{aligned} \tag{7.7}$$

Let us denote by $R^{\cdot}$ the complex

$$f_![\mathcal{L}^{\cdot} \otimes (T^{\cdot} \otimes \mathbb{C}_Y) \otimes_{\mathcal{D}_{X|Y}} \mathcal{O}_X].$$

Its components are direct sums of sheaves of the type

$$f_![(\mathcal{O}_X)_{K\times Y}] \simeq \Gamma(K;\mathcal{O}_Z)\,\hat{\otimes}\,\mathcal{O}_Y$$

which are DFN-free $\mathcal{O}_Y$-modules. It is easy to check that the $\mathcal{O}_Y$-linear differential of $R^{\cdot}$ is continuous with respect to the these natural topologies. Hence, we may consider $R^{\cdot}$ as a topological complex of DFN-free $\mathcal{O}_Y$-modules. Using Corollary 7.4, we have successively

$$\begin{aligned} Rf_{\pi!}[(\mathcal{E}_{X\to Y}(0))_{K\times T^*Y}] &\simeq \Gamma(K;\mathcal{O}_Z)\,\hat{\otimes}\,\mathcal{E}_Y(0) \\ &\simeq [\Gamma(K;\mathcal{O}_Z)\,\hat{\otimes}\,\pi_Y^{-1}\mathcal{O}_Y]\,\hat{\otimes}_{\pi_Y^{-1}\mathcal{O}_Y}\mathcal{E}_Y(0) \\ &\simeq \pi_Y^{-1}f_![(\mathcal{O}_X)_{K\times Y}]\,\hat{\otimes}_{\pi_Y^{-1}\mathcal{O}_Y}\mathcal{E}_Y(0) \end{aligned}$$

and (7.7) is represented as the canonical morphism

$$\pi^{-1}R^{\cdot} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{E}_Y(0) \longrightarrow \pi^{-1}R^{\cdot}\,\hat{\otimes}_{\pi^{-1}\mathcal{O}_Y}\mathcal{E}_Y(0).$$

Since $R^{\cdot}$ has $\mathcal{O}_Y$-coherent cohomology, Proposition 3.13 allows us to conclude the proof. □

Corollary 7.6 *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module endowed with a good filtration. Assume:*

(i) f is proper on $\operatorname{supp}\mathcal{M}$,

(ii) f_π is finite on ${}^tf'^{-1}(\mathrm{char}\mathcal{M}) \cap (X \times_Y \dot{T}^*Y)$, where $\dot{T}^*Y = T^*Y \setminus T^*_YY$.

Then, for $j \neq 0$, $H^j(\underline{f}_!\mathcal{M})$ is a flat connection (i.e. its characteristic variety is contained in the zero section).

Proof: The second hypothesis implies that $\underline{f}_!(\mathcal{M}\mathcal{E})$ is concentrated in degree zero on $\dot{T}^*Y$. The first hypothesis and Theorem 7.5 imply that

$$(\underline{f}_!\mathcal{M})\mathcal{E} \simeq \underline{f}_!(\mathcal{M}\mathcal{E}).$$

Hence, for $j \neq 0$, $\operatorname{supp} H^j[(\underline{f}_!\mathcal{M})\mathcal{E}]$ is contained in the zero section. Since $\mathcal{E}$ is flat over $\pi^{-1}\mathcal{D}$, the conclusion follows easily. □

This Corollary has important applications when studying correspondences of $\mathcal{D}$-modules, such as, for example, the Penrose correspondence. We refer the interested reader to [5] for more details.

8 Main corollaries

8.1 Extension to the non proper case

In this subsection, we shall generalize Theorems 4.2 and 5.15 to a non proper situation, using the techniques of [8, 12].

Let $f : X|S \longrightarrow Y|S$ be a morphism of complex manifolds over S and let $\varphi : X \longrightarrow \mathbb{R}$ be a real analytic function. Set

$$\Lambda_\varphi = \{(x, d\varphi(x)) : x \in X\}.$$

This is a Lagrangian submanifold of T^*X (which is not conic for a non locally constant φ). We also associate to φ the following subsets of X:

$$\begin{aligned} Z_t &= \{x \in X : \varphi(x) \leq t\}, \\ U_t &= \{x \in X : \varphi(x) < t\}, \end{aligned}$$

and denote by $j_t : U_t \longrightarrow X$ the open embedding. Recall finally that the image of a subset S of T^*X by the antipodal map is denoted by S^a.

Corollary 8.1 *Let $\mathcal{M}$ and F be objects of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{X|S})$ and $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$ respectively and assume:*

i) for each $t \in \mathbb{R}$, f is proper on $\operatorname{supp}\mathcal{M} \cap \operatorname{supp} F \cap Z_t$,

ii) $p^{-1}\mathrm{char}_{f|S}(\mathcal{M}) \cap SS(F) \subset T^*_X X$,

iii) there is $t_0 \in \mathbb{R}$ such that

$$\Lambda_\varphi \cap (p^{-1}\mathrm{char}_{f|S}(\mathcal{M}) + SS(F)^a) \subset \pi^{-1}(Z_{t_0}).$$

Then:

a) setting

$$F_t = j_{t!}j_t^{-1}F \simeq F_{U_t},$$

the canonical morphisms:

$$\begin{aligned} \underline{f}_{|S!}(F_t \otimes \mathcal{M}) &\longrightarrow \underline{f}_{|S!}(F \otimes \mathcal{M}) \\ \underline{f}_{|S*}(D'F_t \otimes \underline{D}_{X|S}(\mathcal{M})) &\longleftarrow \underline{f}_{|S*}(D'F \otimes \underline{D}_{X|S}\mathcal{M}) \end{aligned}$$

are isomorphisms for $t > t_0$,

b) both

$$\underline{f}_{|S!}(F \otimes \mathcal{M}) \qquad \text{and} \qquad \underline{f}_{|S*}(D'F \otimes \underline{D}_{X|S}(\mathcal{M}))$$

are objects of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{Y|S})$, *c) the natural duality morphism:*

$$\underline{f}_{|S*}(D'F \otimes \underline{D}_{X|S}(\mathcal{M})) \longrightarrow \underline{D}_{Y|S}\underline{f}_{|S!}(F \otimes \mathcal{M})$$

is an isomorphism.

Note that replacing $SS(F)$ by $SS(F)^a$ in hypothesis (iii), we get a similar conclusion after interchanging $\underline{f}_{|S!}$ and $\underline{f}_{|S*}$. Also note that it would be possible to generalize to a non proper situation the results of §6 but for the sake of brevity, we leave it to the reader.

Proof: Let $x \in X$. If $x \in \operatorname{supp}\mathcal{M} \cap \operatorname{supp} F$ and $x \notin Z_{t_0}$, then $d\varphi(x) \notin \operatorname{char}(\mathcal{M}) + SS(F)^a$ by hypothesis (iii) and in particular $d\varphi(x) \neq 0$. Applying Proposition 5.4.8 of [12], we find for $t > t_0$:

$$SS(F_t) \subset SS(F) + \mathbb{R}^+\Lambda_\varphi$$

where $\mathbb{R}^+\Lambda_\varphi = \{(x; \lambda d\varphi(x)) : x \in X, \lambda > 0\}$. Since:

$$p^{-1}\operatorname{char}_{f|S}(\mathcal{M}) \cap (SS(F) + \mathbb{R}^+\Lambda_\varphi) \subset T^*_X X \cup \pi^{-1}(Z_{t_0}),$$

again by hypothesis (iii), we obtain that $(\mathcal{M}, F_t)$ satisfies the hypothesis of Theorems 4.2 and 5.15 for $t > t_0$. Hence, the conclusions of these theorems apply to the pair $(\mathcal{M}, F_t)$ and part (b) and (c) are consequences of part (a) which we shall now prove.

First, we consider the morphism

$$\underline{f}_{|S!}(F_t \otimes \mathcal{M}) \longrightarrow \underline{f}_{|S!}(F \otimes \mathcal{M}). \tag{8.1}$$

Set $G = F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}$. By Theorem 2.15, hypothesis (ii) and Proposition 5.4.14 of [12] we have:

$$SS(G) \subset SS(F) + p^{-1}\operatorname{char}_{f|S}(\mathcal{M}).$$

Since

$$p^{-1}\operatorname{char}_{f|S}(\mathcal{M}) = p^{-1}\operatorname{char}_{f|S}(\mathcal{M}) + {}^t f'(X \times_Y T^*Y),$$

the above morphism (8.1) is an isomorphism by Proposition 5.4.17 of [12].

To prove the second isomorphism in (a), consider the chain of isomorphisms which follows from the regularity theorem applied first to F_t, then to F:

$$\begin{aligned}
&Rf_*(D'F_t \otimes \underline{D}_{X|S}\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \\
&\qquad \simeq Rf_* R\mathcal{H}om(F_t, \underline{D}_{X|S}\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \\
&\qquad \simeq Rf_* Rj_{t*}j_t^{-1} R\mathcal{H}om(F, \underline{D}_{X|S}\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \\
&\qquad \simeq Rf_* Rj_{t*}j_t^{-1}(D'F \otimes \underline{D}_{X|S}\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}).
\end{aligned}$$

Set

$$G = D'F \otimes \underline{D}_{X|S}\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}.$$

Isomorphism (8.1) applied to $(D'F_t,\ \underline{D}_{X|S}\mathcal{M})$ tells us in particular that the projective system

$$Rf_*(D'F_t \otimes \underline{D}_{X|S}\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S})$$

is essentially constant for $t > t_0$. Hence, the projective system $Rf_*Rj_{t*}j_t^{-1}G$ is also essentially constant for $t > t_0$ and using the Mittag-Leffler theorem we get the isomorphism

$$Rf_*G \xrightarrow{\sim} Rf_*Rj_{t*}j_t^{-1}G$$

which completes the proof. □

8.2 Special cases and examples

In this subsection, we will consider various special situations and give the corresponding form of Theorem 4.2 and 5.15 leaving the reader do the same thing for Theorem 6.7.

First, let us specialize our results to the non relative case taking $S = \{\mathrm{pt}\}$.

Corollary 8.2 *Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds. Assume $(\mathcal{M},F)$ is a good f-elliptic pair with f-proper support i.e.:*

- *$\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X^{\mathrm{op}})$,*
- *F is an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$,*
- *$\mathrm{char}_f(\mathcal{M}) \cap SS(F) \subset T^*_XX$,*
- *$\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ is f-proper.*

Then

- *$\underline{f}_!(\mathcal{M} \otimes F)$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Y^{\mathrm{op}})$,*
- *$\underline{f}_![\underline{D}_X(\mathcal{M}) \otimes D'F] \xrightarrow{\sim} \underline{D}_Y[\underline{f}_!(\mathcal{M} \otimes F)]$.*

When we take $F = \mathbb{C}_X$ in the preceding corollary we recover the coherence theorem for $\mathcal{D}$-modules of Kashiwara [9] (who treated only projective morphisms). Moreover, using Corollary 8.1, we also recover the finiteness theorem for non proper morphisms of [8] and the corresponding duality result of [24].

It is well known that an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent if and only if the induced $\mathcal{D}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is itself coherent. Moreover, this scalar extension process is compatible with direct images and duality (see Proposition 5.11). Applying the preceding corollary to the pair $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathbb{C}_X)$ we recover Grauert's coherence theorem [6] and Ramis-Ruget-Verdier's relative duality theorem [15, 16] in the important special case of analytic manifolds.

Taking $Y = \{\mathrm{pt}\}$ in the preceding corollary, we get the following absolute result:

Corollary 8.3 *Let X be a complex analytic manifold. Assume $(\mathcal{M},F)$ is a good elliptic pair with compact support i.e.:*

- $\mathcal{M}$ *is an object of* $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X^{\mathrm{op}})$,
- F *is an object of* $\mathbf{D}^{\mathrm{b}}_{\mathrm{IR-c}}(X)$,
- $\mathrm{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X$,
- $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ *is compact.*

Then the complexes

$$\mathrm{R}\Gamma(X; \mathcal{M} \otimes F \otimes^L_{\mathcal{D}_X} \mathcal{O}_X) \qquad \text{and} \qquad \mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes F, \Omega_X[d_X]))$$

have finite dimensional cohomology and are dual one to each other.

In the special case where $F = \mathbb{C}_X$, we get an absolute finiteness and duality result for good $\mathcal{D}_X$-modules which was considered by Mebkhout in [14]. For coherent analytic sheaves, the preceding corollary corresponds to the very classical Cartan-Serre [4] and Serre [26]'s theorems.

In the case $Y = S$, Theorem 4.2 and 5.15 give information on analytic families of absolute elliptic pairs.

Corollary 8.4 *Let $X|S$ be a relative analytic manifold and let $(\mathcal{M}, F)$ be a relative elliptic pair on $X|S$ i.e.:*

- $\mathcal{M}$ *is an object of* $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{X|S}^{\mathrm{op}})$,
- F *is an object of* $\mathbf{D}^{\mathrm{b}}_{\mathrm{IR-c}}(X)$,
- $p^{-1}\mathrm{char}_{X|S}(\mathcal{M}) \cap SS(F) \subset T^*_X X$,
- $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ *is ϵ_X-proper,*

where $p : T^*X \longrightarrow T^*X|S$ is the canonical projection. Then

$$Rf_!(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X) \quad \text{and} \quad Rf_! R\mathcal{H}om_{\mathcal{D}_{X|S}}(F \otimes \mathcal{M}, \Omega_{X|S}[d_{X|S}])$$

are objects of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S)$, dual one to each other, i.e. the canonical morphism:

$$Rf_! R\mathcal{H}om_{\mathcal{D}_{X|S}}(F \otimes \mathcal{M}, \Omega_{X|S}[d_{X|S}]) \longrightarrow R\mathcal{H}om_{\mathcal{O}_S}(Rf_!(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X), \mathcal{O}_S)$$

is an isomorphism.

Combining the preceding corollary with the base change formula, we get:

Corollary 8.5 *Let $X|S$ be a relative analytic manifold. For any $s \in S$, denote by b_s the canonical inclusion of $\{s\}$ in S. Assume $(\mathcal{M},F)$ is a good relative elliptic pair with ϵ-proper support on $X|S$. Then for any $s \in S$, $(\mathcal{M}_{b_s},F_{b_s})$ is a good elliptic pair with compact support (on a neighborhood of $\mathrm{supp}\,\mathcal{M}_{b_s}$ in X_{b_s}, the fiber of X over s) and the Euler-Poincaré index:*

$$\chi(\mathrm{R}\Gamma(X_{b_s}; \mathcal{M}_{b_s} \otimes F_{b_s} \otimes^L_{\mathcal{D}_{X_{b_s}}} \mathcal{O}_{X_{b_s}}))$$

is a locally constant function on S.

Proof: Let us denote by $\mathcal{I}$ the ideal of holomorphic functions vanishing at s. The base change Theorem 6.5 tells us that:

$$\mathrm{R}\Gamma(X_{b_s};\mathcal{M}_{b_s}\otimes F_{b_s}\otimes^L_{\mathcal{D}_{X_{b_s}}}\mathcal{O}_{X_{b_s}})=[\mathcal{O}_S/\mathcal{I}\otimes^L_{\mathcal{O}_S}R\epsilon_*(\mathcal{M}\otimes F\otimes^L_{\mathcal{D}_{X|S}}\mathcal{O}_X)]_s.$$

We know by the finiteness theorem that

$$R\epsilon_*(\mathcal{M}\otimes F\otimes^L_{\mathcal{D}_{X|S}}\mathcal{O}_X)$$

has $\mathcal{O}_S$ coherent cohomology. Hence, it is locally quasi-isomorphic to a bounded complex of finite free $\mathcal{O}_S$-modules. The conclusion follows easily since $[\mathcal{O}_S/\mathcal{I}]_s=\mathbb{C}$. □

Remark 8.6 Let $P_0:E\longrightarrow F$, $P_1:E\longrightarrow F$ be two complex analytic linear differential operators between holomorphic vector bundles on X. Assume that their principal symbols induce the same morphism of fiber bundles

$$\sigma:\pi^{-1}E\longrightarrow\pi^{-1}F.$$

Then, $P_\lambda=(1-\lambda)P_0+\lambda P_1$ is a one parameter analytic family of operators with principal symbols equal to σ. Combining this remark with the preceding corollary, we recover, for example, the fact that the index of an elliptic operator on a compact real analytic manifold depends only on its principal symbol.

Let us now consider a few explicit examples. For the sake of brevity, we only consider non-relative situations.

Example 8.7 Let M be a real analytic manifold with X as a complexification and $\mathcal{M}$ a good $\mathcal{D}_X$-module. Then, as we have already noticed in the introduction, $\mathcal{M}$ is elliptic on M in the classical sense if and only if $(\mathcal{M},\mathbb{C}_M)$ is elliptic. In fact, $SS(\mathbb{C}_M)=T^*_MX$. Since $\mathbb{C}_M\otimes\mathcal{O}_X=\mathcal{A}_M$, the sheaf of real analytic functions on M and

$$R\mathcal{H}om(D'\mathbb{C}_M,\mathcal{O}_X)=\mathcal{B}_M$$

the sheaf of Sato's hyperfunctions, the regularity theorem 2.15 entails the isomorphism:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{A}_M)\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M).\tag{8.2}$$

This is the Petrowski theorem for $\mathcal{D}$-modules which is often proved using micro-differential equations as in [18]. Moreover, if M is compact and $\mathcal{M}$ is good, Corollary 8.3 asserts that the spaces

$$H^j(\mathrm{R}\Gamma(M;R\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M},\mathcal{B}_M)))=\mathrm{Ext}^j_{\mathcal{D}_M}(M;\mathcal{M},\mathcal{B}_M)$$

and

$$H^{n-j}(\mathrm{R}\Gamma(M;\Omega_M\otimes^L_{\mathcal{D}_M}\mathcal{M}))=\mathrm{Tor}^{\mathcal{D}_M}_{j-n}(M;\Omega_M,\mathcal{M}),$$

are finite dimensional and dual to each other. Note that for solutions of elliptic operators the duality and finiteness theorems are well-known results.

Example 8.8 Let X be a complex manifold, U an open subset with real analytic boundary. Then $(\mathcal{M}, \mathbb{C}_U)$ is an elliptic pair if and only if the boundary ∂U is non characteristic for $\mathcal{M}$, that is, $\mathrm{char}(\mathcal{M}) \cap T^*_{\partial U}X \subset T^*_X X$. The regularity theorem yields the isomorphism:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, (\mathcal{O}_X)_{\overline{U}}) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathrm{R}\Gamma_U(\mathcal{O}_X)). \tag{8.3}$$

In other words, the holomorphic solutions on U of the system $\mathcal{M}$ extend holomorphically through the boundary. If U is relatively compact, and $\mathcal{M}$ is good, we get that the spaces $\mathrm{Ext}^j_{\mathcal{D}_X}(U, \mathcal{M}, \mathcal{O}_X)$ and $\mathrm{Tor}^{\mathcal{D}_X}_{j-n}(\overline{U}; \Omega_X, \mathcal{M})$ are finite dimensional and dual to each other.

Note that the regularity theorem is due to Zerner [27] (for the 0-th cohomology) and [2], both in case of one equation with one unknown, then to Kashiwara [10] for systems. The finiteness theorem is due to [2], this last result being extended in various directions by Kawai [13].

Example 8.9 One can generalize both preceding examples as follows. Let M be a real analytic manifold, X being a complexification of M and let U be an open subset of M with real analytic boundary. Then $(\mathcal{M}, \mathbb{C}_U)$ is an elliptic pair if and only if $\mathcal{M}$ is elliptic on M on a neighborhood of $\overline{U}$ and moreover the conormal vectors to ∂U in M are hyperbolic with respect to $\mathcal{M}$. Then we get the isomorphism:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, (\mathcal{A}_M)_{\overline{U}}) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_U(\mathcal{B}_M)).$$

(i.e.: the hyperfunction solutions of $\mathcal{M}$ on U are real analytic and extend analytically through the boundary), and we also get finiteness and duality results that we do not develop here.

Example 8.10 A general situation including the preceding examples is the following.

Let $X = \bigsqcup_\alpha X_\alpha$ be a subanalytic μ-stratification (cf. [12, Chap. VIII]) and assume:

$$\begin{cases} SS(F) & \subset \bigsqcup_\alpha T^*_{X_\alpha}X, \\ \mathrm{char}(\mathcal{M}) \cap T^*_{X_\alpha}X & \subset T^*_X X \quad \forall \alpha. \end{cases} \tag{8.4}$$

(In other words, F is locally constant on the strata X_α and these strata are non characteristic for $\mathcal{M}$.)

Then of course, the pair $(\mathcal{M}, F)$ is elliptic. If, moreover, $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ is compact we may apply Theorem 4.2 and Theorem 5.15 and we obtain new finiteness and duality results.

Example 8.11 For any $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X))$, the pair $(\mathcal{O}_X, F)$ is elliptic. Since $F \simeq \Omega_X \otimes^L_{\mathcal{D}_X} \mathcal{O}_X \otimes F[-n]$ and $D'F \simeq R\mathcal{H}om_{\mathcal{D}_X}(F \otimes \mathcal{O}_X, \mathcal{O}_X)$, one recovers the classical finiteness and duality theorem on constructible sheaves. In fact if M is a real analytic manifold and $i : M \hookrightarrow X$ denote a complexification of M, to $G \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(M))$ one associates the elliptic pair $(\mathcal{O}_X, i_*G)$.

Example 8.12 Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and let $x_0 \in X$. Let $B(x_0, \varepsilon)$ denote the open ball with center x_0 and radius $\varepsilon > 0$ in some local chart at x_0. By a result of Kashiwara [10], the pair $(\mathcal{M}, \mathbb{C}_{B(x_0,\varepsilon)})$ is elliptic for $0 < \varepsilon \ll 1$. If X is open in $\mathbb{C}^n$ and $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X))$ has compact support, one proves similarly that $(\mathcal{M}, F * \mathbb{C}_{B(0,\varepsilon)})$ is elliptic for $0 < \varepsilon \ll 1$. (Here "$*$" denotes the convolution of sheaves; cf. [12] Exercise 2.20.)

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Elliptic Pairs II. Euler Class and Relative Index Theorem

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Contents

1 Introduction

In [14], we introduced the notion of an elliptic pair $(\mathcal{M},F)$ on a complex manifold X. Recall that this is the data of a (let us say, right) coherent $\mathcal{D}_X$-module $\mathcal{M}$ and an $\mathbb{R}$-constructible sheaf F (more precisely, objects of derived categories), these data satisfying:

$$\operatorname{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X, \tag{1.1}$$

where $\operatorname{char}(\mathcal{M})$ is the characteristic variety of $\mathcal{M}$, $SS(F)$ is the micro-support of F, (defined in [7]), and $T^*_X X$ is the zero-section of the cotangent bundle to X. More generally, if $f : X \longrightarrow Y$ is a morphism of complex manifolds, we defined the notion of an f-elliptic pair, replacing in (1.1) $\operatorname{char}(\mathcal{M})$ by $\operatorname{char}_f(\mathcal{M})$, the relative characteristic variety.

In [14], we give four basic results on elliptic pairs: we prove a finiteness theorem (coherence of the direct images of $F \otimes \mathcal{M}$, assuming $(\mathcal{M}, F)$ is an f-elliptic pair with proper support), a duality theorem (in the above situation, duality commutes with direct images), a Künneth formula and we prove that microlocalization commutes with direct images.

In this second paper on elliptic pairs, expanding results announced in [12, 13], we will attach a cohomology class to $(\mathcal{M}, F)$ and prove an index formula. More precisely, let $\Lambda_0 = \mathrm{char}(\mathcal{M})$, $\Lambda_1 = SS(F)$, let $d_X = \dim_{\mathbb{C}} X$ and denote by ω_X the dualizing complex on X (hence $\omega_X \simeq \mathbb{C}_X[2d_X]$, since X is oriented). Assuming $(\mathcal{M}, F)$ is elliptic, we construct a cohomology class:

$$\mu\mathrm{eu}(\mathcal{M}, F) \in H^{2d_X}_{\Lambda_0+\Lambda_1}(T^*X; \mathbb{C}_{T^*X}) \qquad (\simeq H^0_{\Lambda_0+\Lambda_1}(T^*X; \pi^{-1}\omega_X))$$

that we call the "microlocal Euler class" of $(\mathcal{M}, F)$. This class is constructed using a diagonal procedure, like in the proof of the Lefschetz formula for constructible sheaves by Kashiwara [6] (see also [7, Chapter IX]), but working here in the framework of $\mathcal{D}$-modules. Set for short:

$$\begin{aligned} \mu\mathrm{eu}(\mathcal{M}) &= \mu\mathrm{eu}(\mathcal{M}, \mathbb{C}_X), \\ \mu\mathrm{eu}(F) &= \mu\mathrm{eu}(\Omega_X, F). \end{aligned}$$

Then the two main results of this paper may be stated as follows.

1) One has the formula:

$$\mu\mathrm{eu}(\mathcal{M}, F) = \mu\mathrm{eu}(\mathcal{M}) *_\mu \mu\mathrm{eu}(F), \tag{1.2}$$

where the operation $*_\mu$:

$$H^0_{\Lambda_0}(T^*X; \pi^{-1}\omega_X) \times H^0_{\Lambda_1}(T^*X; \pi^{-1}\omega_X) \longrightarrow H^0_{\Lambda_0+\Lambda_1}(T^*X; \pi^{-1}\omega_X)$$

is defined by integration along the fibers of the map:

$$s : T^*X \times_X T^*X \longrightarrow T^*X, \quad s(x; \xi_1, \xi_2) = (x; \xi_1 + \xi_2)$$

(this map is proper, thanks to the ellipticity hypothesis).

2) Assume $(\mathcal{M}, F)$ is f-elliptic with proper support. One knows by [14] that $\underline{f}_!(F \otimes \mathcal{M})$ is $\mathcal{D}_Y$-coherent, and we prove the formula:

$$\mu\mathrm{eu}(\underline{f}_!(F \otimes \mathcal{M})) = f_\mu\, \mu\mathrm{eu}(\mathcal{M}, F), \tag{1.3}$$

where f_μ is the morphism:

$$H^0_{\Lambda_0+\Lambda_1}(T^*X; \pi^{-1}\omega_X) \longrightarrow H^0_{f_\pi {}^t f'^{-1}(\Lambda_0+\Lambda_1)}(T^*Y; \pi^{-1}\omega_Y)$$

deduced from the integration morphism $Rf_!\omega_X \longrightarrow \omega_Y$, (see [7, Chapter IX, §3]).

These two theorems will be proved along the same lines as the corresponding results for constructible sheaves (see [7]). We will use various commutative diagrams in derived categories to express the compatibility of the functors involved, and as usual in these matters we do not distinguish between commutative and anti-commutative diagrams. Hence the results should be understood up to sign.

Using these two formulas, we find in particular that if $(\mathcal{M}, F)$ is an elliptic pair with compact support, then:

$$\chi(R\Gamma(X; F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)) = \int_{T^*X} \mu\mathrm{eu}(\mathcal{M}) \cup \mu\mathrm{eu}(F) \tag{1.4}$$

where $\chi(\cdot)$ denotes the Euler-Poincaré index and $\cup$ the cup product.

If M is a real analytic compact manifold and X is a complexification of M, then $(\mathcal{M}, \mathbb{C}_M)$ is an elliptic pair if and only if $\mathcal{M}$ is elliptic on M in the usual sense. Hence formula (1.4) is similar to the Atiyah-Singer formula [1].

By formula (1.2), we see that to compute $\mu\mathrm{eu}(\mathcal{M}, F)$, it is enough to compute separately $\mu\mathrm{eu}(\mathcal{M})$ and $\mu\mathrm{eu}(F)$. It is easily shown that $\mu\mathrm{eu}(F)$ is nothing but the "characteristic cycle" of F constructed by Kashiwara (loc. cit.). This is a Lagrangian cycle whose calculation is made at generic points and thus offers no difficulties (see [7, Chapter IX, §3]). Hence the remaining problem is to understand $\mu\mathrm{eu}(\mathcal{M})$. At this step our results are essentially conjectural. Assume $\mathcal{M}$ is endowed with a good filtration and denote by $\sigma_\Lambda(\mathcal{M})$ the image of $gr(\mathcal{M})$ in the Grothendieck group of coherent $\mathcal{O}_{T^*X}$-modules supported by Λ, the characteristic variety of $\mathcal{M}$. In the last section we make the two following conjectures (1.5) and (1.6) below:

$$[ch_\Lambda(\sigma_\Lambda(\mathcal{M})) \cup \pi^* td_X(TX)]^j = 0 \quad \text{for} \quad j > 2d_X \tag{1.5}$$

where $ch_\Lambda(\cdot)$ and $td_X(TX)$ denote as usual the local Chern character with support in Λ and the Todd class of X, respectively, and $[\cdot]^j$ is the homogeneous part of degree j in $\oplus_k H^k_\Lambda(T^*X; \mathbb{C}_{T^*X})$,

$$\mu\mathrm{eu}(\mathcal{M}) = [ch_\Lambda(\sigma_\Lambda(\mathcal{M})) \cup \pi^* td_X(TX)]^{2d_X}. \tag{1.6}$$

As an evidence for these conjectures, we prove that both sides of (1.6) are compatible to proper direct images, external products and non-characteristic inverse images, and moreover they coincide in the two extreme cases where $\mathcal{M}$ is holonomic or is induced by a coherent $\mathcal{O}_X$-module.

The Atiyah-Singer theorem, in its K-theoretical version, has recently been generalized to the relative case by Boutet de Monvel and Malgrange [3]. Our results provide a relative index formula in the cohomological setting, and the proof of the above conjectures would give a precise link with the Atiyah-Singer theorem. We hope to come back to these conjectures in a next future.

2 Review on sheaves

In this section, we fix some notations and recall a few results of [7].

Let X be a real analytic manifold. One denotes by $\tau : TX \longrightarrow X$ and $\pi : T^*X \longrightarrow X$ the tangent and cotangent bundles to X, respectively. If Y is a submanifold of X, one denotes by T_YX and T^*_YX the normal and conormal bundles to Y in X, respectively. In particular, T^*_XXdenotes the zero-section of T^*X, that ones identifies to X. If Λ is a subset of T^*X, one denotes by Λ^a its image by the antipodal map.

One denotes by $\delta : X \hookrightarrow X \times X$ the diagonal embedding, and we identify X to its image Δ and T^*X to $T^*_\Delta(X \times X)$ by the first projection defined on $X \times X$ and $T^*(X \times X) \simeq T^*X \times T^*X$, respectively.

If X and Y are two manifolds, one denotes by q_1 and q_2 the first and second projection defined on $X \times Y$.

One denotes by $\mathbf{D}(X)$ the derived category of the category of sheaves of $\mathbb{C}$-vector spaces, and by $\mathbf{D}^{\mathrm{b}}(X)$ the full triangulated subcategory consisting of objects with bounded cohomology. If Z is a subset of X, one denotes by $\mathbb{C}_Z$ the sheaf on X which is constant with stalk $\mathbb{C}$ on Z and zero on $X \setminus Z$.

One denotes by or_X the orientation sheaf on X and by ω_X the dualizing complex on X. Hence:

$$\omega_X \simeq \mathrm{or}_X[\dim X]$$

where $\dim X$ is the real dimension of X. More generally, if f is a morphism from X to Y, one denotes by $\omega_{X/Y}$ the relative dualizing complex. Hence:

$$\omega_{X/Y} \simeq \omega_X \otimes f^{-1}\omega_Y^{\otimes -1}.$$

One denotes by $f^{-1}, Rf_*, Rf_!, f^!, \otimes, R\mathcal{H}om$ the usual classical operations on sheaves and we denote by $\boxtimes$ the external product. We shall use the two duality functors:

$$D'_XF = R\mathcal{H}om(F, \mathbb{C}_X), \tag{2.1}$$

$$D_XF = R\mathcal{H}om(F, \omega_X). \tag{2.2}$$

If there is no risk of confusion, we write D' or D instead of D'_X or D_X.

If F is an object of $\mathbf{D}^{\mathrm{b}}(X)$, one denotes by $SS(F)$ its micro-support, defined in [7], a closed conic involutive subset of T^*X. Moreover, we shall use the functor μ_M of Sato's microlocalization along M. Recall that for F in $\mathbf{D}^{\mathrm{b}}(X)$

$$\operatorname{supp} \mu_M(F) \subset T^*_MX \cap SS(F).$$

Now, recall that an object F of $\mathbf{D}^{\mathrm{b}}(X)$ is called weakly $\mathbb{R}$-constructible (w-$\mathbb{R}$-constructible, for short) if there is a subanalytic stratification $X = \bigsqcup_\alpha X_\alpha$ such that for all α, all j, the sheaves $H^j(F)_{|X_\alpha}$ are locally constant. If moreover, for each $x \in X$, each $j \in \mathbb{Z}$, the stalk $H^j(F)_x$ is finite dimensional, one says that F is $\mathbb{R}$-constructible. One denotes by $\mathbf{D}^{\mathrm{b}}_{w-\mathbb{R}-c}(X)$ (resp. $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$) the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(X)$ consisting of w-$\mathbb{R}$-constructible (resp. $\mathbb{R}$-constructible) objects. It follows from the involutivity of the micro-support that F is w-$\mathbb{R}$-constructible if and only if $SS(F)$ is a closed conic subanalytic Lagrangian subset of T^*X.

Let $f : X \longrightarrow Y$ be a morphism of real analytic manifolds. To f one associates the maps:

$$TX \underset{f'}{\longrightarrow} X \times_Y TY \underset{f_\tau}{\longrightarrow} TY, \tag{2.3}$$
$$T^*X \underset{{}^tf'}{\longleftarrow} X \times_Y T^*Y \underset{f_\pi}{\longrightarrow} T^*Y. \tag{2.4}$$

One says that f is non-characteristic with respect to a closed conic subset Λ of T^*Y if:

$$f_\pi^{-1}(\Lambda) \cap {}^tf'^{-1}(T^*_XX) \subset X \times_Y T^*_YY. \tag{2.5}$$

Let $F \in \mathbf{D}^{\mathrm{b}}(X)$, $G \in \mathbf{D}^{\mathrm{b}}(Y)$. Recall that:

(i) if f is non-characteristic with respect to $SS(G)$, then:

$$SS(f^{-1}G) \subset {}^tf' f_\pi^{-1} SS(G), \tag{2.6}$$

(ii)) if f is proper on $\mathrm{supp}(F)$, then:

$$SS(Rf_*F) \subset f_\pi {}^tf'^{-1}(SS(F)), \tag{2.7}$$

(iii) one has:

$$SS(F \boxtimes G) \subset SS(F) \times SS(G). \tag{2.8}$$

Finally, let us recall some microlocal constructions of [7, Chapter IX] that we shall use.

Let Λ_X and Λ_Y be two closed conic subsets of T^*X and T^*Y, respectively, and consider the diagram:

$$\begin{tikzcd} T^*X \arrow[d, "\pi_X"'] & X \times_Y T^*Y \arrow[l, "{}^tf'"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"'] & T^*Y \arrow[d, "\pi_Y"'] \\ X \arrow[r, equal] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Set for short:

$$\begin{aligned} f_\mu(\Lambda_X) &= f_\pi {}^tf'^{-1}(\Lambda_X), && (2.9)\\ f^\mu(\Lambda_Y) &= {}^tf' f_\pi^{-1}(\Lambda_Y). && (2.10) \end{aligned}$$

a) Assume f is proper on $T^*_XX \cap \Lambda_X$, (or equivalently, f_π is proper on Λ_X). Using the morphism:

$$Rf_{\pi!}\pi^{-1}\omega_X \longrightarrow \pi_Y^{-1}Rf_!\omega_X \longrightarrow \pi_Y^{-1}\omega_Y, \tag{2.11}$$

we get the morphisms, for all $j \in \mathbb{Z}$:

$$\begin{aligned} f_\mu : H^j_{\Lambda_X}(T^*X; \pi^{-1}\omega_X) &\longrightarrow H^j_{{}^tf'^{-1}(\Lambda_X)}(X \times_Y T^*Y; \pi^{-1}\omega_X) \\ &\longrightarrow H^j_{f_\mu(\Lambda_X)}(T^*Y; \pi^{-1}\omega_Y). && (2.12) \end{aligned}$$

b) Assume f is non-characteristic for Λ_Y (i.e., ${}^tf'$ is proper on $f_\pi^{-1}(\Lambda_Y)$). Using the natural morphism (see [7]):

$$R^tf'_!\pi^{-1}f^{-1}\omega_Y \longrightarrow \pi_X^{-1}\omega_X, \tag{2.13}$$

we get for all $j \in \mathbb{Z}$, the morphisms:

$$\begin{aligned} f^\mu : H^j_{\Lambda_Y}(T^*Y;\pi^{-1}\omega_Y) &\longrightarrow H^j_{f_\pi^{-1}(\Lambda_Y)}(X\times_Y T^*Y;\pi^{-1}f^{-1}\omega_Y) \\ &\longrightarrow H^j_{f^\mu(\Lambda_Y)}(T^*X;\pi^{-1}\omega_X). \end{aligned} \tag{2.14}$$

Note that the morphism (2.13) may also be obtained as follows. On a manifold Z, there is a natural isomorphism: $\pi_Z^{-1}\omega_Z \simeq \omega_{T^*Z/Z}$. Hence we have the chain of morphisms:

$$\begin{aligned} R^tf'_!\pi^{-1}f^{-1}\omega_Y &\simeq R^tf'_!f_\pi^{-1}\omega_{T^*Y/Y} \\ &\simeq R^tf'_!\omega_{X\times_Y T^*Y/X} \\ &\longrightarrow \omega_{T^*X/X} \\ &\simeq \pi_X^{-1}\omega_X. \end{aligned}$$

c) Using the natural isomorphism:

$$\omega_X \boxtimes \omega_Y \simeq \omega_{X\times Y},$$

we get the morphism:

$$\begin{aligned} \boxtimes : H^j_{\Lambda_X}(T^*X;\pi^{-1}\omega_X) \times H^k_{\Lambda_Y}(T^*Y;\pi^{-1}\omega_Y) \\ \longrightarrow H^{j+k}_{\Lambda_X\times\Lambda_Y}(T^*X\times Y;\pi^{-1}\omega_{X\times Y}). \end{aligned} \tag{2.15}$$

d) Let Λ_0 and Λ_1 be two closed conic subsets of T^*X satisfying:

$$\Lambda_0^a \cap \Lambda_1 \subset T^*_XX. \tag{2.16}$$

Setting:

$$*_\mu = \delta^\mu \circ \boxtimes$$

we get a morphism:

$$*_\mu : H^j_{\Lambda_0}(T^*X;\pi^{-1}\omega_X) \times H^k_{\Lambda_1}(T^*X;\pi^{-1}\omega_X) \longrightarrow H^{j+k}_{\Lambda_0+\Lambda_1}(T^*X;\pi^{-1}\omega_X). \tag{2.17}$$

Note that the morphism $*_\mu$ (which is not the cup-product) may also be defined as the composite of:

$$\begin{aligned} &H^j_{\Lambda_0}(T^*X;\pi^{-1}\omega_X) \times H^k_{\Lambda_1}(T^*X;\pi^{-1}\omega_X) \\ &\underset{\delta_\pi^*}{\longrightarrow} H^{j+k}_{\Lambda_0\times_X\Lambda_1}(T^*X\times_X T^*X;\pi^{-1}\omega_X\otimes\omega_X) \\ &\underset{{}^t\delta'_*}{\longrightarrow} H^{j+k}_{\Lambda_0+\Lambda_1}(T^*X;\pi^{-1}\omega_X) \end{aligned} \tag{2.18}$$

where δ_π^* is associated to the embedding $T^*X\times_X T^*X \underset{\delta_\pi}{\longrightarrow} T^*X\times T^*X$ and ${}^t\delta'_*$ to the map

$$T^*X\times_X T^*X \underset{{}^t\delta'}{\longrightarrow} T^*X, \quad (x;\xi_1,\xi_2)\mapsto(x;\xi_1+\xi_2).$$

3 Euler class of elliptic pairs

From now on, all manifolds and morphisms of manifolds are complex analytic. If X is a complex manifold, we shall often identify X and $X^{\mathbb{R}}$, the real analytic underlying manifold. We shall also identify $(T^*X)^{\mathbb{R}}$ with $T^*X^{\mathbb{R}}$, as in [7]. We denote by d_X the complex dimension of X. Hence,

$$\dim X^{\mathbb{R}} = 2d_X.$$

Since X is oriented, we identify the orientation sheaf or_X with the constant sheaf $\mathbb{C}_X$, and the dualizing complex ω_X with $\mathbb{C}_X[2d_X]$.

We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X, by Ω_X the sheaf of holomorphic d_X-forms and by $\mathcal{D}_X$ the sheaf of rings of (finite order) holomorphic differential operators on X. If Y is another complex manifold and if $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X\times Y}$-modules, one sets:

$$\mathcal{F}^{(0,d_Y)} = \mathcal{F} \otimes_{q_2^{-1}\mathcal{O}_Y} q_2^{-1}\Omega_Y,$$

and one defines similarly $\mathcal{F}^{(d_X,0)}$ or $\mathcal{F}^{(d_X,d_Y)}$.

We shall follow the notations of [7] for $\mathcal{D}$-modules. In particular, $\mathrm{Mod}(\mathcal{D}_X)$ denotes the category of left $\mathcal{D}_X$-modules, $\mathbf{D}(\mathcal{D}_X)$ its derived category, and $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ the full triangulated subcategory of $\mathbf{D}(\mathcal{D}_X)$ consisting of complexes with bounded and coherent cohomology. Replacing $\mathcal{D}_X$ by $\mathcal{D}_X^{op}$, we have similar notations for right $\mathcal{D}_X$-modules. In fact, if there is no risk of confusion, we shall often make no differences between right and left $\mathcal{D}$-modules and write $\mathcal{D}_X$ instead of $\mathcal{D}_X^{op}$.

In the sequel, we will often need to work with bimodule structures. Let k be a field. Recall that if A and B are k-algebras, giving a left (A,B)-bimodule structure on an abelian group M is equivalent to give M a structure of a left $A \otimes_k B$-module. Using this point of view it is easy to extend to bimodules the notions and notations defined usually for modules. For example, we will denote by $\mathrm{Mod}(\mathcal{D}_{X|S} \otimes \mathcal{D}_{X|S})$ the category of left $\mathcal{D}_{X|S}$-bimodules and by $\mathbf{D}(\mathcal{D}_{X|S} \otimes \mathcal{D}_{X|S})$ the corresponding derived category.

The characteristic variety of an object $\mathcal{M}$ of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ is denoted by $\mathrm{char}(\mathcal{M})$. This is a closed conic involutive analytic subset of T^*X [10], and we have the formula [7, Theorem 11.3.3]:

$$\mathrm{char}(\mathcal{M}) = SS(\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X). \tag{3.1}$$

As usual, one denotes by $\mathcal{B}_{Z|X}$ the simple holonomic left $\mathcal{D}_X$-module associated to a closed complex submanifold Z of X. We denote by $\underline{f}^{-1}$, $\underline{f}_!$, $\boxtimes$ the operations of inverse image, proper direct image, and external product for $\mathcal{D}$-modules, and we denote by $\underline{D}_X$ the dualizing functor. Recall that if $\mathcal{M}$ is a right $\mathcal{D}_X$-module, then

$$\underline{D}_X(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{K}_X)$$

where

$$\mathcal{K}_X = \Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

as a right $\mathcal{D}_X \otimes \mathcal{D}_X$-module. Notice the isomorphism of $\mathcal{D}_X \otimes \mathcal{D}_X^{\mathrm{op}}$-modules:

$$\delta_! \mathcal{D}_X \simeq \mathcal{B}^{(0,d_X)}_{\Delta|X\times X}, \tag{3.2}$$

which induces the isomorphism of $\mathcal{D}_X^{\mathrm{op}} \otimes \mathcal{D}_X^{\mathrm{op}}$-modules:

$$\delta_! \mathcal{K}_X \simeq \mathcal{B}^{(d_X,d_X)}_{\Delta|X\times X}.$$

By this isomorphism, $\mathcal{K}_X$ is naturally endowed with a structure of a right $\delta^{-1}\mathcal{D}_{X\times X}$-module and

$$\delta_! \mathcal{K}_X = \underline{\delta_!} \Omega_X[d_X].$$

Let us recall the notion of an elliptic pair introduced in [14].

Definition 3.1 An elliptic pair $(\mathcal{M}, F)$ on X is the data of $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X^{op})$ and $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$ satisfying:

$$\mathrm{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X.$$

The same definition holds for left $\mathcal{D}_X$-modules.

Proposition 3.2 *Let $(\mathcal{M}, F)$ be an elliptic pair on X. Then there are canonical morphisms:*

(i) $\delta_! R\mathcal{H}om_{\mathcal{D}_X}(F \otimes \mathcal{M}, F \otimes \mathcal{M}) \longrightarrow (F \otimes \mathcal{M}) \underline{\boxtimes} (D'F \otimes \underline{D}\mathcal{M}) \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}$,

(ii) $F \otimes \mathcal{M} \underline{\boxtimes} D'F \otimes \underline{D}\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X} \longrightarrow \delta_! \omega_X$.

Proof: (i) Let $\mathcal{D}_X^\infty$ denote the ring of infinite order holomorphic differential operators. Sato's isomorphism:

$$\mathcal{D}_X^\infty \simeq \delta^! \mathcal{O}^{(0,d_X)}_{X\times X}[d_X]$$

entails the morphism:

$$\delta_! \mathcal{D}_X \longrightarrow \mathcal{O}^{(0,d_X)}_{X\times X}[d_X]. \tag{3.3}$$

Set for short:

$$\mathcal{P} = F \otimes \mathcal{M}.$$

Applying the functor $q_1^{-1}\mathcal{P} \otimes^L_{q_1^{-1}\mathcal{D}_X} \cdot$ to (3.3), then the functor $R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, \cdot)$, and using the isomorphism:

$$\delta_! R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}, \mathcal{P}) \simeq R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, \delta_! \mathcal{P}),$$

we get the morphism:

$$\delta_! R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}, \mathcal{P}) \longrightarrow R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, q_1^{-1}\mathcal{P} \otimes^L_{q_1^{-1}\mathcal{D}_X} \mathcal{O}^{(0,d_X)}_{X\times X}[d_X]).$$

Then:

$$\begin{aligned}
&R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, q_1^{-1}\mathcal{P} \otimes^L_{q_1^{-1}\mathcal{D}_X} \mathcal{O}^{(0,d_X)}_{X\times X}[d_X]) \\
&\quad \simeq \; R\mathcal{H}om(q_2^{-1}F, R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{M}, q_1^{-1}\mathcal{P} \otimes^L_{q_1^{-1}\mathcal{D}_X} \mathcal{O}^{(0,d_X)}_{X\times X}[d_X])) \\
&\quad \simeq \; R\mathcal{H}om(q_2^{-1}F, \mathcal{P} \boxtimes \underline{D}_X\mathcal{M} \otimes^L_{\mathcal{D}_X\boxtimes\mathcal{D}_X} \mathcal{O}_{X\times X}) \\
&\quad \simeq \; R\mathcal{H}om(q_2^{-1}F, \mathcal{P} \boxtimes \underline{D}_X\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}).
\end{aligned}$$

The micro-support of $\mathcal{P} \boxtimes \underline{D}_X\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}$ is contained in $T^*X \times \mathrm{char}(\mathcal{M})$, hence it intersects $SS(q_2^{-1}F)$ inside the zero-section of $T^*(X \times X)$. Using [7, Prop. 5.4.14], we get the isomorphisms:

$$\begin{aligned}
&R\mathcal{H}om(q_2^{-1}F, \mathcal{P} \boxtimes \underline{D}_X\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}) \\
&\quad \xleftarrow{\sim} \; q_2^{-1}D'F \otimes \left[\mathcal{P} \boxtimes \underline{D}_X\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}\right] \\
&\quad \xleftarrow{\sim} \; (F \otimes \mathcal{M}) \boxtimes (D'F \otimes \underline{D}\mathcal{M}) \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}.
\end{aligned}$$

(ii) Set for short:

$$L_X = (F \otimes \mathcal{M}) \boxtimes (D'F \otimes \underline{D}\mathcal{M}) \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X} \tag{3.4}$$

Using the $\mathcal{D}_{X\times X}$-linear morphism:

$$F \otimes \mathcal{M} \boxtimes D'F \otimes \underline{D}\mathcal{M} \longrightarrow \delta_!\mathcal{K}_X,$$

we get the sequence of morphisms:

$$\begin{aligned}
L_X \;&\longrightarrow\; \delta_!\mathcal{K}_X \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X} \\
&\simeq\; \underline{\delta_!}\Omega_X[d_X] \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X} \\
&\simeq\; \delta_! \left[\Omega_X[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to X\times X} \otimes^L_{\delta^{-1}\mathcal{D}_{X\times X}} \delta^{-1}\mathcal{O}_{X\times X}\right] \\
&\simeq\; \delta_!\Omega_X[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{O}_X \\
&\simeq\; \delta_!\omega_X.
\end{aligned}$$

□

Using the morphisms defined in the preceding proposition, we can now construct the microlocal Euler class of the elliptic pair $(\mathcal{M}, F)$. Set:

$$\Lambda = \mathrm{char}(\mathcal{M}) + SS(F)$$

Then $SS(L_X) \subset \Lambda \times \Lambda^a$ where L_X is defined in (3.4), and

$$\mathrm{supp}(\mu_\Delta L_X) \subset \Lambda.$$

By paraphrasing Kashiwara's construction of the characteristic cycle of $\mathbb{R}$-constructible sheaves, [6], we obtain the sequence of morphisms:

$$\begin{array}{rcl} R\mathcal{H}om_{\mathcal{D}_X}(F\otimes\mathcal{M},F\otimes\mathcal{M}) & \longrightarrow & \delta^! L_X \\ & \simeq & R\pi_*\mu_\Delta L_X \\ & \simeq & R\pi_* R\Gamma_\Lambda \mu_\Delta L_X \\ & \longrightarrow & R\pi_* R\Gamma_\Lambda \mu_\Delta \delta_! \omega_X \\ & \simeq & R\pi_* R\Gamma_\Lambda \pi^{-1}\omega_X. \end{array}$$

Applying $H^0R\Gamma(X;\cdot)$, we find the morphism:

$$\mathrm{Hom}_{\mathcal{D}_X}(F\otimes\mathcal{M},F\otimes\mathcal{M}) \longrightarrow H^0_\Lambda(T^*X;\pi^{-1}\omega_X). \tag{3.5}$$

(Recall that

$$H^0(T^*X;\pi^{-1}\omega_X)\simeq H^{2d_X}_\Lambda(T^*X;\mathbb{C}_{T^*X}).)$$

Definition 3.3 Let $(\mathcal{M},F)$ be an elliptic pair. The image of $\mathrm{id}_{F\otimes\mathcal{M}}$ by the morphism (3.5) is the *microlocal Euler class of* $(\mathcal{M},F)$

$$\mu\mathrm{eu}(\mathcal{M},F)\in H^0_{\mathrm{char}(\mathcal{M})+SS(F)}(T^*X;\pi^{-1}\omega_X)$$

Its restriction to the zero-section of T^*X is the *Euler class of* $(\mathcal{M},F)$

$$\mathrm{eu}(\mathcal{M},F)\in H^0_{\mathrm{supp}(\mathcal{M})\cap\mathrm{supp}(F)}(X;\omega_X)$$

If $\mathcal{M}$ is a left $\mathcal{D}_X$-module, we define the microlocal Euler class of $(\mathcal{M},F)$ as being that of $(\Omega_X\otimes_{\mathcal{O}_X}\mathcal{M},F)$. We also introduce the following notations. For $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ and $F\in\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$, we set:

$$\begin{array}{rcl} \mu\mathrm{eu}(\mathcal{M}) & = & \mu\mathrm{eu}(\mathcal{M},\mathbb{C}_X), \\ \mu\mathrm{eu}(F) & = & \mu\mathrm{eu}(\Omega_X,F). \end{array}$$

4 The product formula

Let $(\mathcal{M},F)$ be an elliptic pair on the complex manifold X. Set:

$$\Lambda_0=\mathrm{char}(\mathcal{M}),\quad \Lambda_1=SS(F).$$

Then:

$$\begin{array}{rcl} \mu\mathrm{eu}(\mathcal{M}) & \in & H^0_{\Lambda_0}(T^*X;\pi^{-1}\omega_X), \\ \mu\mathrm{eu}(F) & \in & H^0_{\Lambda_1}(T^*X;\pi^{-1}\omega_X), \\ \mu\mathrm{eu}(\mathcal{M},F) & \in & H^0_{\Lambda_0+\Lambda_1}(T^*X;\pi^{-1}\omega_X). \end{array}$$

The operation $*_\mu$ being that defined in §2, the aim of this section is to prove:

Theorem 4.1 *Let $(\mathcal{M}, F)$ be an elliptic pair. Then:*

$$\mu\mathrm{eu}(\mathcal{M}, F) = \mu\mathrm{eu}(\mathcal{M}) *_\mu \mu\mathrm{eu}(F)$$

The proof decomposes into several steps. In Proposition 4.2 below, and its proof, X will denote a real analytic manifold. In this statement and its proof as well as in the proof of Theorem 5.1, we shall not write the symbol "R" of derived functors, for short; e.g. $\mathcal{H}om(\cdot,\cdot)$ means $R\mathcal{H}om(\cdot,\cdot)$, π_* means $R\pi_*$, etc.

We denote by X_i $(i = 1, 2, 3, 4)$ a copy of X and we write

$$(X \times X) \times (X \times X) = X_1 \times X_2 \times X_3 \times X_4.$$

For $J \subset \{1,2,3,4\}$ and any set Z, we introduce the notation

$$\delta_{ij} : \prod_{\ell \in J \setminus \{j\}} X_\ell \times Z \longrightarrow \prod_{\ell \in J} X_\ell \times Z$$

for the diagonal embedding sending $(x_\ell)_{\ell \in J\setminus\{j\}}$ to $(x_\ell)_{\ell \in J}$ with $x_j = x_i$. Similarly, we introduce the notation

$$\delta_{ijk} : \prod_{\ell \in J \setminus \{j,k\}} X_\ell \times Z \longrightarrow \prod_{\ell \in J} X_\ell \times Z$$

for the diagonal embedding sending $(x_\ell)_{\ell \in J\setminus\{j,k\}}$ to $(x_\ell)_{\ell \in J}$ with $x_j = x_k = x_i$. If there is no risk of confusion, we simply write δ for any of these morphisms.

On a product, we denote by q_i the projection to X_i.

We shall make a frequent use of the morphism of functors

$$\delta_{ij}^{-1} \longrightarrow \delta_{ij}^! \otimes \omega_X. \tag{4.1}$$

Now, we assume to be given:

$$F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X), \quad G \in \mathbf{D}^{\mathrm{b}}(X), \quad H \in \mathbf{D}^{\mathrm{b}}(X \times X).$$

We set:

$$K = G \boxtimes DF$$

$$\tilde{\Lambda}_0 = SS(K), \quad \tilde{\Lambda}_1 = SS(H)$$

$$\Lambda_i = T^*_\Delta(X \times X) \cap \tilde{\Lambda}_i, \quad i = 0, 1$$

We identify $T^*_\Delta(X \times X)$ to T^*X by the first projection. We shall assume:

$$\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^a \subset T^*_{X\times X}(X \times X). \tag{4.2}$$

Proposition 4.2 *The diagrams below commute.*

$$\begin{tikzcd}
\mathcal{H}om(F,G)\otimes\delta^!H \arrow[rr] & & \mathcal{H}om(F,G\otimes\delta^!H) \arrow[d] \\
\delta^!K\otimes\delta^!H \arrow[u, "\sim"] \arrow[rr] \arrow[urr, phantom, "(1)"] & & \mathcal{H}om(F,\delta^!(q_1^{-1}G\otimes H)) \\
 & (2) & \delta^!(K\otimes H\otimes\omega_X^{\otimes-1}) \arrow[u, "\sim"] \\
\pi_*\Gamma_{\Lambda_0}\mu_\Delta K\otimes\pi_*\Gamma_{\Lambda_1}\mu_\Delta H \arrow[uu, "\sim"] \arrow[rr] & & \pi_*\Gamma_{\Lambda_0+\Lambda_1}\mu_\Delta(K\otimes H\otimes\omega_X^{\otimes-1}) \arrow[u, "\sim"]
\end{tikzcd}$$

First, we state three lemmas whose proofs are easy verifications left to the reader.

Lemma 4.3 *The diagram:*

$$\begin{tikzcd}
\mathcal{H}om(F,G)\otimes\delta^!H \arrow[r] \arrow[dr] & \mathcal{H}om(F,G\otimes\delta^!H) \arrow[d] \\
 & \mathcal{H}om(F,\delta^!(q_1^{-1}G\otimes H))
\end{tikzcd}$$

is isomorphic to:

$$\begin{tikzcd}
\delta_{13}^{-1}\delta_{12}^!\delta_{34}^!(K\boxtimes H) \arrow[r] \arrow[dr] & \delta_{12}^!\delta_{13}^{-1}\delta_{34}^!(K\boxtimes H) \arrow[d] \\
 & \delta_{12}^!\delta_{14}^!\delta_{13}^{-1}(K\boxtimes H)
\end{tikzcd}$$

Note that the morphisms

$$\delta_{13}^{-1}\delta_{12}^! \longrightarrow \delta_{12}^!\delta_{13}^{-1}$$

or

$$\delta_{13}^{-1}\delta_{34}^! \longrightarrow \delta_{14}^!\delta_{13}^{-1}$$

are defined as follows. Consider a cartesian square:

$$\begin{tikzcd}
Z_1 \arrow[r, "\lambda_1"'] & Z \\
Z_{12} \arrow[u, "\mu_2"] \arrow[r, "\mu_1"'] \arrow[ur, phantom, "\square"] & Z_2 \arrow[u, "\lambda_2"']
\end{tikzcd}$$

Then we have the natural morphism:

$$\mu_2^{-1}\circ\lambda_1^! \longrightarrow \mu_1^!\circ\lambda_2^{-1}$$

defined by:

$$\mu_{1!}\mu_2^{-1}\lambda_1^! \simeq \lambda_2^{-1}\lambda_{1!}\lambda_1^! \longrightarrow \lambda_2^{-1}.$$

Lemma 4.4 *The diagram below commutes:*

$$\begin{tikzcd}
\delta^!_{12}\delta^{-1}_{13}\delta^{-1}_{24}(K\boxtimes H)\otimes\omega_X^{\otimes -1} \arrow[r, "\alpha"] \arrow[dr, "\gamma"'] & \delta^!_{12}\delta^!_{14}\delta^{-1}_{13}(K\boxtimes H) \arrow[d, "\beta"] \\
& \delta^!_{1234}(K\boxtimes H)\otimes\omega_X
\end{tikzcd}$$

Moreover, assuming (4.2), α, β and γ are isomorphisms.

Note that α is defined through:

$$\delta^{-1}_{24}\otimes\omega_X^{\otimes -1}\longrightarrow\delta^!_{24}$$

and

$$\delta^!_{12}\delta^!_{24}\simeq\delta^!_{12}\delta^!_{14},$$

and γ is defined through:

$$\delta^{-1}_{13}\delta^{-1}_{24}\otimes\omega_X^{\otimes -1}\longrightarrow\delta^!_{13}\delta^!_{24}\otimes\omega_X.$$

Lemma 4.5 *The diagram below commutes:*

$$\begin{tikzcd}
\delta^{-1}_{13}\delta^!_{12}\delta^!_{34}(K\boxtimes H) \arrow[r] \arrow[dr] & \delta^!_{12}\delta^!_{14}\delta^{-1}_{13}(K\boxtimes H) \arrow[d] \\
& \delta^!_{1234}(K\boxtimes H)\otimes\omega_X
\end{tikzcd}$$

Proof of Proposition 4.2: Diagram (1) obviously commutes. To prove that (2) commutes, we decompose it in the diagram below, after applying Lemma 4.3:

$$\begin{tikzcd}
\delta^!K\otimes\delta^!H \arrow[rrrr] \arrow[ddrr] & & (6) & & \delta^!_{12}\delta^!_{14}\delta^{-1}_{13}(K\boxtimes H) \arrow[ddll] \\
& & & & \\
& (3) & \delta^!_{1234}(K\boxtimes H)\otimes\omega_X & (5) & \delta^!_{12}(K\otimes H\otimes\omega_X^{\otimes -1}) \arrow[uu, "\sim"'] \\
& & (4) & & \\
\pi_*\Gamma_{\Lambda_0}\mu_\Delta K\otimes\pi_*\Gamma_{\Lambda_1}\mu_\Delta H \arrow[rrrr] \arrow[uuuu, "\sim"] \arrow[uurr] & & & & \pi_*\Gamma_{\Lambda_0+\Lambda_1}\mu_\Delta(K\otimes H\otimes\omega_X^{\otimes -1}) \arrow[uu, "\sim"'] \arrow[uull]
\end{tikzcd}$$

In this diagram, the sub-diagram (6) commutes by Lemma 4.5, the sub-diagram (5) commutes by Lemma 4.4, the sub-diagram (4) commutes by [7, Prop. 4.3.5] and the sub-diagram (3) obviously commutes. Hence the full diagram commutes. □

Proof of Theorem 4.1: We shall apply Proposition 4.2 with $G = F$, $H = \mathcal{M} \boxtimes \underline{D}\mathcal{M} \otimes^{L}_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}$ (hence $K = F \boxtimes DF$). Note that we have trace morphisms:

$$\begin{aligned} K &\longrightarrow \delta_!\omega_X, \\ H &\longrightarrow \delta_!\omega_X. \end{aligned}$$

Consider the diagrams:

$$\begin{tikzcd}
\mathcal{H}om(F,F) \otimes \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{M}) \arrow[r] \arrow[d] \arrow[dr, phantom, "(7)"] & \mathcal{H}om_{\mathcal{D}_X}(F\otimes\mathcal{M}, F\otimes\mathcal{M}) \arrow[d] \\
\mathcal{H}om(F,F) \otimes \delta^! H \arrow[r] & \mathcal{H}om(F, F\otimes\delta^! H) \\
\delta^! K \otimes \delta^! H \arrow[u, "\sim"] \arrow[r, phantom, "(8)"] & \delta^!(K\otimes H\otimes\omega_X^{\otimes -1}) \arrow[u, "\sim"] \\
\pi_*\Gamma_{\Lambda_0}\mu_\Delta K \otimes \pi_*\Gamma_{\Lambda_1}\mu_\Delta H \arrow[u, "\sim"] \arrow[r] \arrow[d] \arrow[dr, phantom, "(9)"] & \pi_*\Gamma_{\Lambda_0+\Lambda_1}\mu_\Delta(K\otimes H\otimes\omega_X^{\otimes -1}) \arrow[u, "\sim"] \arrow[d] \\
\pi_*\Gamma_{\Lambda_0}\mu_\Delta\delta_!\omega_X \otimes \pi_*\Gamma_{\Lambda_1}\mu_\Delta\delta_!\omega_X \arrow[r] \arrow[d, "\sim"] \arrow[dr, phantom, "(10)"] & \pi_*\Gamma_{\Lambda_0+\Lambda_1}\mu_\Delta(\delta_!\omega_X\otimes\delta_!\omega_X\otimes\omega_X^{\otimes -1}) \arrow[d, "\sim"] \\
\pi_*\Gamma_{\Lambda_0}\pi^{-1}\omega_X \otimes \pi_*\Gamma_{\Lambda_1}\pi^{-1}\omega_X \arrow[r] & \pi_*\Gamma_{\Lambda_0+\Lambda_1}\pi^{-1}\omega_X
\end{tikzcd}$$

Diagrams (7) and (10) obviously commute, diagram (8) commutes by Proposition 4.2 and diagram (9) commutes since it is obtained by applying the morphism of functors:

$$\pi_*\Gamma_{\Lambda_0}\mu_\Delta(\cdot)\otimes\pi_*\Gamma_{\Lambda_1}\mu_\Delta(\cdot) \longrightarrow \pi_*\Gamma_{\Lambda_0+\Lambda_1}\mu_\Delta(\cdot\otimes\cdot\otimes\omega_X^{\otimes -1})$$

obtained from [7, Prop. 4.3.5] to $K \longrightarrow \delta_!\omega_X$ and $H \longrightarrow \delta_!\omega_X$. To conclude the proof, it remains to notice that the sequence of morphisms in the second column of the preceding diagrams (7) and (8) is the same as the morphism

$$\delta_!\mathcal{H}om(F\otimes\mathcal{M}, F\otimes\mathcal{M}) \longrightarrow K\otimes H\otimes\omega_X^{\otimes -1} = L_X$$

obtained in 3.2. Then, applying $H^0\mathrm{R}\Gamma(X;\cdot)$ to the preceding diagram, we find the commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(F,F)\otimes\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}_X}(F\otimes\mathcal{M}, F\otimes\mathcal{M}) \arrow[d] \\
H^0_{\Lambda_0}(T^*X;\pi^{-1}\omega_X)\otimes H^0_{\Lambda_1}(T^*X;\pi^{-1}\omega_X) \arrow[r] & H^0_{\Lambda_0+\Lambda_1}(T^*X;\pi^{-1}\omega_X)
\end{tikzcd}$$

$\square$

5 The direct image formula

Let $f : X \longrightarrow Y$ be a morphism of complex manifolds, and let $(\mathcal{M}, F)$ be an elliptic pair on X. Under suitable conditions that we shall recall now, it is proved in [14] that the direct image $\underline{f}_!(F \otimes \mathcal{M})$ belongs to $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y)$. The aim of this section is to prove that in this situation, the microlocal Euler class of this image is the image by the morphism (2.12) of that of $(\mathcal{M}, F)$.

Let us first recall the definition of $\mathrm{char}_f(\mathcal{M})$, the relative characteristic variety of $\mathcal{M}$, (see [11, 14]). If f is smooth, one denotes by $\mathcal{D}_{X|Y}$ the sub-ring of $\mathcal{D}_X$ generated by the vertical vector fields, one locally chooses $\mathcal{M}_0$, a coherent $\mathcal{D}_{X|Y}$-submodule of $\mathcal{M}$ which generates it, and one sets:

$$\mathrm{char}_f(\mathcal{M}) = \mathrm{char}(\mathcal{D}_X \otimes_{\mathcal{D}_{X|Y}} \mathcal{M}_0).$$

One checks easily that this does not depend on the choice of $\mathcal{M}_0$. In the general case (f not necessarily smooth), one decomposes f by its graph as:

$$f : X \underset{i}{\longrightarrow} X \times Y \underset{q}{\longrightarrow} Y$$

and one sets:

$$\mathrm{char}_f(\mathcal{M}) = {}^t i' i_\pi^{-1} \mathrm{char}_q(\underline{i}_! \mathcal{M}).$$

Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ and let $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$. One says that $(\mathcal{M}, F)$ is f-elliptic if

$$\mathrm{char}_f(\mathcal{M}) \cap SS(F) \subset T^*_X X.$$

Since $\mathrm{char}_f(\mathcal{M})$ contains $\mathrm{char}(\mathcal{M})$, an f-elliptic pair is elliptic. Let $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ denote the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ generated by the objects $\mathcal{M}$ such that for all $j \in \mathbb{Z}$ and all compact subset K of X, $H^j(\mathcal{M})$ may be endowed with a good filtration in a neighborhood of K. If $(\mathcal{M}, F)$ is f-elliptic and moreover $\mathcal{M}$ belongs to $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, one says that $(\mathcal{M}, F)$ is a good f-elliptic pair. If moreover f is proper on $\mathrm{supp}\,\mathcal{M} \cap \mathrm{supp}\,F$, one says that $(\mathcal{M}, F)$ has f-proper support. It is proved in [14] that if $(\mathcal{M}, F)$ is a good f-elliptic pair with f-proper support, then $\underline{f}_!(F \otimes \mathcal{M})$ belongs to $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Y)$. Let $\Lambda_0 = \mathrm{char}(\mathcal{M})$, $\Lambda_1 = SS(F)$. We have the canonical morphism:

$$f_\mu : H^0_{\Lambda_0 + \Lambda_1}(T^*X; \pi^{-1}\omega_X) \longrightarrow H^0_{f_\mu(\Lambda_0+\Lambda_1)}(T^*Y; \pi^{-1}\omega_Y).$$

Theorem 5.1 *Assume $(\mathcal{M}, F)$ is an f-elliptic pair with f-proper support. Then:*

$$\mu\mathrm{eu}(\underline{f}_!(F \otimes \mathcal{M})) = f_\mu\, \mu\mathrm{eu}(\mathcal{M}, F) = f_\mu(\mu\mathrm{eu}(\mathcal{M}) *_\mu \mu\mathrm{eu}(F)).$$

Proof: The proof will decompose into several steps. For short, during this proof, we will not write the symbol "R" or "L" of right or left derived functors. We introduce the notations:

$$\tilde{X} = X \times X, \quad \tilde{f} = f \times f : X \times X \longrightarrow Y \times Y.$$

We denote by δ_X the diagonal embedding $X \hookrightarrow X \times X$, and if there is no risk of confusion, we write δ instead of δ_X. We also set for short:

$$\begin{aligned}
L_X &= (F \otimes \mathcal{M}) \boxtimes (D'F \otimes \underline{D}\mathcal{M}) \otimes_{\mathcal{D}_{\tilde{X}}} \mathcal{O}_{\tilde{X}} \\
L_Y &= \underline{f}_!(F \otimes \mathcal{M}) \boxtimes \underline{D}(\underline{f}_!(F \otimes \mathcal{M})) \otimes_{\mathcal{D}_{\tilde{Y}}} \mathcal{O}_{\tilde{Y}}
\end{aligned}$$

By the results of [14], we have the isomorphisms:

$$\begin{aligned}
\tilde{f}_! L_X &\simeq \tilde{f}_!(F \otimes \mathcal{M} \boxtimes D'F \otimes \underline{D}\mathcal{M} \otimes_{\mathcal{D}_{\tilde{X}}} \mathcal{D}_{\tilde{X}\to\tilde{Y}}) \otimes_{\mathcal{D}_{\tilde{Y}}} \mathcal{O}_{\tilde{Y}} \\
&\xleftarrow{\sim} \underline{f}_!(F \otimes \mathcal{M}) \boxtimes \underline{f}_!(D'F \otimes \underline{D}\mathcal{M}) \otimes_{\mathcal{D}_{\tilde{Y}}} \mathcal{O}_{\tilde{Y}} \\
&\simeq L_Y.
\end{aligned}$$

Consider the diagram:

$$\begin{tikzcd}
f_!\mathcal{H}om_{\mathcal{D}_X}(F \otimes \mathcal{M}, F \otimes \mathcal{M}) \arrow[rr] \arrow[d] & {} \arrow[d, phantom, "(1)"] & \mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!(F \otimes \mathcal{M}), \underline{f}_!(F \otimes \mathcal{M})) \arrow[d] \\
f_!\delta^! L_X \arrow[r] & \delta^! \tilde{f}_! L_X \arrow[r] & \delta^! L_Y \\
\pi_* f_{\pi!}{}^t f'^{-1} \Gamma_{\Lambda_X} \mu_{\Delta_X} L_X \arrow[u, "\sim"] \arrow[r] \arrow[d] \arrow[ur, phantom, "(2)"] \arrow[dr, phantom, "(4)"] & \pi_* \Gamma_{\Lambda_Y} \mu_{\Delta_Y} \tilde{f}_! L_X \arrow[u, "\sim"] \arrow[r] \arrow[d] \arrow[ur, phantom, "(3)"] & \pi_* \Gamma_{\Lambda_Y} \mu_{\Delta_Y} L_Y \arrow[u, "\sim"] \arrow[dd] \\
\pi_* f_{\pi!}{}^t f'^{-1} \Gamma_{\Lambda_X} \mu_{\Delta_X} \delta_! \omega_X \arrow[r] \arrow[dd, "\sim"] \arrow[dr, phantom, "(6)"] & \pi_* \Gamma_{\Lambda_Y} \mu_{\Delta_Y} \tilde{f}_! \delta_! \omega_X \arrow[d, "\sim"] \arrow[r, phantom, "(5)"] & {} \\
{} & \pi_* \Gamma_{\Lambda_Y} \mu_{\Delta_Y} \delta_! f_! \omega_X \arrow[r] \arrow[d, "\sim"] \arrow[dr, phantom, "(7)"] & \pi_* \Gamma_{\Lambda_Y} \mu_{\Delta_Y} \delta_! \omega_Y \arrow[d, "\sim"] \\
\pi_* f_{\pi!}{}^t f'^{-1} \Gamma_{\Lambda_X} \pi^{-1} \omega_X \arrow[r] & \pi_* \Gamma_{\Lambda_Y} \pi^{-1} f_! \omega_X \arrow[r] & \pi_* \Gamma_{\Lambda_Y} \pi^{-1} \omega_Y
\end{tikzcd}$$

It is enough to prove it is commutative. In fact, applying $H^0\mathrm{R}\Gamma(Y;\cdot)$ to it we get the commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}_X}(F \otimes \mathcal{M}, F \otimes \mathcal{M}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}_Y}(\underline{f}_!(F \otimes \mathcal{M}), \underline{f}_!(F \otimes \mathcal{M})) \arrow[d] \\
H^0_{\Lambda_X}(T^*X; \pi^{-1}\omega_X) \arrow[r] & H^0_{\Lambda_Y}(T^*Y; \pi^{-1}\omega_Y).
\end{tikzcd}$$

Diagram (2) commutes since

$$f_!\delta^! \longrightarrow \delta^! \tilde{f}_!$$

is the restriction to the zero section of:

$$f_{\pi!}{}^t f'^{-1} \mu_{\Delta_X} \longrightarrow \mu_{\Delta_Y} \tilde{f}_!$$

(see [7, Prop. 4.3.4]).

Diagram (3) commutes since it is obtained by applying the natural transformation $\pi_*\Gamma_{\Lambda_Y}\mu_{\Delta_Y} \longrightarrow \delta^!$ to the morphism $\tilde{f}_! L_X \longrightarrow L_Y$.

Diagram (4) commutes. In fact, it is obtained by applying the morphism of functors:

$$\pi_* f_{\pi !}{}^t f'^{-1}\Gamma_{\Lambda_X}\mu_{\Delta_X} \longrightarrow \pi_*\Gamma_{\Lambda_Y}\mu_{\Delta_Y}\tilde{f}_!$$

to $L_X \longrightarrow \delta_! \omega_X$. Diagrams (6) and (7) obviously commute.

Using the base change formula for elliptic pairs of [14] we see that for $\mathcal{P} = \mathcal{Q} = F \otimes \mathcal{M}$ the map

$$\begin{aligned}
&\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, q_1^{-1}\underline{f}_!\mathcal{Q} \otimes_{q_1^{-1}\mathcal{D}_Y} \mathcal{O}^{(0,d_Y)}_{Y\times Y}[d_Y]) \\
&\quad\longrightarrow \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, f_{2!}(q_1^{-1}\mathcal{Q} \otimes_{q_1^{-1}\mathcal{D}_X} \mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y]))
\end{aligned}$$

appearing in Lemma 5.2 and Lemma 5.3 below is an isomorphism. Moreover, the Künneth formula for elliptic pairs [loc. cit.] shows that the canonical map

$$\underline{f}_!(F\otimes\mathcal{M}) \underline{\boxtimes} \underline{f}_!(D'F\otimes \underline{D}\mathcal{M}) \otimes_{\mathcal{D}_{Y\times Y}} \mathcal{O}_{Y\times Y} \longrightarrow \tilde{f}_!(F\otimes\mathcal{M} \boxtimes D'F \otimes \underline{D}\mathcal{M} \otimes_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X})$$

is also an isomorphism. Hence, the conjunction of Lemma 5.2 and Lemma 5.3 below, gives the commutativity of diagram (1). In the same way, Lemma 5.4 below shows that diagram (5) commutes. □

Let $f : X \longrightarrow Y$ be a morphism of complex manifolds. We will decompose $\tilde{f} = f \times f$ as

$$\begin{tikzcd}
X\times X \arrow[r, "f_1"] & X\times Y \arrow[r, "f_2"] & Y\times Y \\
X \arrow[u, "\delta_X"] \arrow[r, equal] & X \arrow[u, "\delta"] \arrow[r, "f"] & Y \arrow[u, "\delta_Y"']
\end{tikzcd}$$

Lemma 5.2 *Let $\mathcal{P}$ and $\mathcal{Q}$ belong to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$. Then we have the canonical commutative diagram:*

$$\begin{tikzcd}
\tilde{f}_!\delta_{X!}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P},\mathcal{Q}) \arrow[r] \arrow[dd] & \tilde{f}_!\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, q_1^{-1}\mathcal{Q} \otimes_{q_1^{-1}\mathcal{D}_X} \mathcal{O}^{(0,d_X)}_{X\times X}[d_X]) \arrow[d] \\
& \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, f_{2!}(q_1^{-1}\mathcal{Q} \otimes_{q_1^{-1}\mathcal{D}_X} \mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y])) \\
\delta_{Y!}\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{P}, \underline{f}_!\mathcal{Q}) \arrow[r] & \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, q_1^{-1}\underline{f}_!\mathcal{Q} \otimes_{q_1^{-1}\mathcal{D}_Y} \mathcal{O}^{(0,d_Y)}_{Y\times Y}[d_Y]) \arrow[u]
\end{tikzcd}$$

Proof: The kernel representation of differential operators induces the morphism of bimodules:

$$\delta_{X!}\mathcal{D}_X \longrightarrow \mathcal{O}^{(0,d_X)}_{X\times X}[d_X].$$

From the relative integration map

$$\underline{f_1}_{|X!}\Omega_{X\times X|X}[d_X] \longrightarrow \Omega_{X\times Y|X}[d_Y],$$

and the Poincaré-Verdier adjunction formula, we deduce the bimodule morphism:

$$\mathcal{O}^{(0,d_X)}_{X\times X}[d_X]\otimes_{q_2^{-1}\mathcal{D}_X} q_2^{-1}\mathcal{D}_{X\to Y} \longrightarrow f_1^!\mathcal{O}^{(0,d_Y)}_{X\times Y}.$$

Hence, we get the chain of bimodule morphisms:

$$\begin{aligned}
\delta_{X!}\mathcal{D}_{X\to Y} &\longrightarrow \delta_{X!}\mathcal{D}_X\otimes_{q_2^{-1}\mathcal{D}_X} q_2^{-1}\mathcal{D}_{X\to Y}\\
&\longrightarrow \mathcal{O}^{(0,d_X)}_{X\times X}[d_X]\otimes_{q_2^{-1}\mathcal{D}_X} q_2^{-1}\mathcal{D}_{X\to Y}\\
&\longrightarrow f_1^!\mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y].
\end{aligned}$$

This chain of morphisms gives rise to the commutative diagram:

$$\begin{tikzcd}
\delta_{X!}\mathcal{Q}\otimes_{q_2^{-1}\mathcal{D}_X} q_2^{-1}\mathcal{D}_{X\to Y} \arrow[r] \arrow[d] & \delta_{X!}(\mathcal{Q}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}) \arrow[d]\\
q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}^{(0,d_X)}_{X\times X}[d_X]\otimes_{q_2^{-1}\mathcal{D}_X} q_2^{-1}\mathcal{D}_{X\to Y} \arrow[r] & q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X} f_1^!\mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y].
\end{tikzcd}$$

By adjunction of the tensor product, this gives us the commutative diagram:

$$\begin{tikzcd}
\delta_{X!}\mathcal{Q} \arrow[r] \arrow[d] & \delta_{X!}\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(\mathcal{D}_{X\to Y},\mathcal{Q}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}) \arrow[d]\\
q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}^{(0,d_X)}_{X\times X}[d_X] \arrow[r] & \mathcal{H}om_{q_2^{-1}f^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{D}_{X\to Y}, q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X} f_1^!\mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y]).
\end{tikzcd}$$

Applying the functor $\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P},\cdot)$ to this diagram, we get the commutative diagram:

$$\begin{tikzcd}
\delta_{X!}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P},\mathcal{Q}) \arrow[r] \arrow[d] & \delta_{X!}\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(\mathcal{P}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y},\mathcal{Q}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}) \arrow[d]\\
M_X \arrow[r] & \mathcal{H}om_{q_2^{-1}f^{-1}\mathcal{D}_Y}(q_2^{-1}(\mathcal{P}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}), q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X} f_1^!\mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y])
\end{tikzcd}$$

where we have set for short:

$$M_X = \mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}^{(0,d_X)}_{X\times X}[d_X]).$$

Applying $f_{1!}$ and using the Poincaré-Verdier adjunction formula, we get the commutative diagram:

$$\begin{tikzcd}
f_{1!}\delta_{X!}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P},\mathcal{Q}) \arrow[r] \arrow[d] & \delta_!\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(\mathcal{P}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y},\mathcal{Q}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}) \arrow[d]\\
f_{1!}M_X \arrow[r] & M_{XY}
\end{tikzcd}$$

where we have set for short:

$$M_{XY} = \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}^{(0,d_Y)}_{X\times Y}).$$

Finally, apply $f_{2!}$ and note that the diagram below is commutative:

$$\begin{tikzcd}
f_{2!}\delta_!\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(\mathcal{P}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}, \mathcal{Q}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}) \arrow[r] \arrow[dd] & \delta_{Y!}\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{P}, \underline{f}_!\mathcal{Q}) \arrow[d] \\
 & M_Y \arrow[d] \\
f_{2!}M_{XY} \arrow[r] & \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, f_{2!}(q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y])).
\end{tikzcd}$$

Where we have set for short:

$$M_Y = \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, q_1^{-1}\underline{f}_!\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}^{(0,d_Y)}_{Y\times Y}[d_Y]).$$

□

Lemma 5.3 *Let $\mathcal{P}$ and $\mathcal{Q}$ belong to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$. Then, we have the canonical commutative diagram:*

$$\begin{tikzcd}
\tilde{f}_!\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}^{(0,d_X)}_{X\times X}[d_X]) \arrow[d] & \tilde{f}_!(\mathcal{Q}\boxtimes\underline{D}\mathcal{P}\otimes_{\mathcal{D}_{X\times X}}\mathcal{O}_{X\times X}) \arrow[l] \\
\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, f_{2!}(q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y])) & \underline{f}_!\mathcal{Q}\boxtimes\underline{f}_!\underline{D}\mathcal{P}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y} \arrow[u] \arrow[d] \\
\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, q_1^{-1}\underline{f}_!\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}^{(0,d_Y)}_{Y\times Y}[d_Y]) \arrow[u] & \underline{f}_!\mathcal{Q}\boxtimes\underline{D}\underline{f}_!\mathcal{P}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y} \arrow[l]
\end{tikzcd}$$

Proof: Notice that the diagram below is commutative:

$$\begin{tikzcd}
\tilde{f}_![(\mathcal{Q}\boxtimes\underline{D}\mathcal{P})\otimes_{\mathcal{D}_X^{\boxtimes 2}}\mathcal{D}^{\boxtimes 2}_{X\to Y}] \arrow[r] & \tilde{f}_![\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, \mathcal{Q}\boxtimes\mathcal{K}_X)\otimes_{\mathcal{D}_X^{\boxtimes 2}}\mathcal{D}^{\boxtimes 2}_{X\to Y}] \arrow[d] \\
\underline{f}_!\mathcal{Q}\boxtimes\underline{f}_!\underline{D}\mathcal{P} \arrow[u] \arrow[d] & \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, \underline{\tilde{f}}_![(\mathcal{Q}\boxtimes\mathcal{K}_X)\otimes_{\mathcal{D}_X^{\boxtimes 2}}\mathcal{D}^{\boxtimes 2}_{X\to Y}\otimes_{q_2^{-1}\mathcal{D}_X}q_2^{-1}\mathcal{D}_{X\to Y}]) \\
\underline{f}_!\mathcal{Q}\boxtimes\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{P}, \underline{\underline{f}}_!\mathcal{K}_X) \arrow[r] \arrow[d] & \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, \underline{f}_!\mathcal{Q}\boxtimes\underline{\underline{f}}_!\mathcal{K}_X) \arrow[u] \arrow[d] \\
\underline{f}_!\mathcal{Q}\boxtimes\underline{D}\underline{f}_!\mathcal{P} \arrow[r] & \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, \underline{f}_!\mathcal{Q}\boxtimes\mathcal{K}_Y)
\end{tikzcd}$$

Thanks to the isomorphism

$$q_2^{-1}\mathcal{K}_X \otimes_{q_2^{-1}\mathcal{D}_X} \mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_{X\times X}^{(0,d_X)}[d_X],$$

and the canonical morphism

$$\mathcal{D}_{X\to Y}^{\boxtimes 2} \otimes_{\mathcal{D}_Y^{\boxtimes 2}} \mathcal{O}_{Y\times Y} \longrightarrow \mathcal{O}_{X\times X},$$

an application of the functor $\cdot \otimes_{\mathcal{D}_Y^{\boxtimes 2}} \mathcal{O}_{Y\times Y}$ to the preceding diagram allows us to conclude. □

Lemma 5.4 *Let $\mathcal{P}$ belong to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$. Then, we have the canonical commutative diagram:*

$$\begin{tikzcd}
\tilde{f}_!((\mathcal{P} \underline{\boxtimes} \underline{D}\mathcal{P}) \otimes_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}) \arrow[r] & \tilde{f}_! \delta_{X!}\omega_X \arrow[d] \\
(\underline{f}_!\mathcal{P} \underline{\boxtimes} \underline{f}_!\underline{D}\mathcal{P}) \otimes_{\mathcal{D}_{Y\times Y}} \mathcal{O}_{Y\times Y} \arrow[u] \arrow[d] & \delta_{Y!} f_! \omega_X \arrow[d] \\
(\underline{f}_!\mathcal{P} \underline{\boxtimes} \underline{D}\underline{f}_!\mathcal{P}) \otimes_{\mathcal{D}_{Y\times Y}} \mathcal{O}_{Y\times Y} \arrow[r] & \delta_{Y!}\omega_Y
\end{tikzcd}$$

Proof: Recall that the dualizing complex for $\mathcal{D}$-modules

$$\mathcal{K}_X = \Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

has a canonical structure of right $\mathcal{D}_X^{\otimes 2}$-module and that

$$\delta_! \mathcal{K}_X \simeq \underline{\delta}_! \Omega[d_X]$$

as $\mathcal{D}_X^{\boxtimes 2}$-modules. Also recall that:

$$\underline{\underline{f}}_! \mathcal{K}_X = f_!(\mathcal{K}_X \otimes_{\mathcal{D}_X^{\otimes 2}} \mathcal{D}_{X\to Y}^{\otimes 2})$$

and that the trace of the duality morphism associated to f is given by the $\mathcal{D}_Y^{\otimes 2}$-linear integration morphism

$$\underline{\underline{f}}_! \mathcal{K}_X \longrightarrow \mathcal{K}_Y.$$

From the construction of this morphism (see [14]) it is clear that we have the canonical commutative $\mathcal{D}_Y^{\boxtimes 2}$-linear diagram:

$$\begin{tikzcd}
\delta_! \underline{\underline{f}}_! \mathcal{K}_X \arrow[r] \arrow[d] & \tilde{\underline{f}}_! \delta_! \mathcal{K}_X \arrow[d] \\
\delta_! \mathcal{K}_Y \arrow[r, equal] & \delta_! \mathcal{K}_Y.
\end{tikzcd}$$

In this diagram, the first horizontal arrow is deduced from the morphism

$$\mathcal{D}^{\boxtimes 2}_{X\to Y} \longrightarrow D_{X^2\to Y^2},$$

the first vertical arrow is deduced from the duality trace map and the second vertical arrow is deduced from the isomorphism

$$\tilde{\underline{f}}_!\underline{\delta}_!\Omega_X[d_X] \xrightarrow{\sim} \underline{\delta}_!\underline{f}_!\Omega_X[d_X]$$

and the integration map

$$\underline{f}_!\Omega_X[d_X] \longrightarrow \Omega_Y[d_Y].$$

Let us consider the commutative diagram:

$$\begin{tikzcd}
\tilde{f}_!(\mathcal{P}\boxtimes\underline{D}\mathcal{P}\otimes_{\mathcal{D}_X^{\boxtimes 2}}\mathcal{D}^{\boxtimes 2}_{X\to Y}) \arrow[r] & \delta_!\underline{\underline{f}}_!\mathcal{K}_X \arrow[r] \arrow[dd, equal] & \tilde{\underline{f}}_!\delta_!\mathcal{K}_X \arrow[ddd] \\
\underline{f}_!\mathcal{P}\boxtimes\underline{f}_!\underline{D}\mathcal{P} \arrow[u] \arrow[d] & & \\
\underline{f}_!\mathcal{P}\boxtimes\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{P},\underline{\underline{f}}_!\mathcal{K}_X) \arrow[r] \arrow[d] & \delta_!\underline{\underline{f}}_!\mathcal{K}_X \arrow[d] & \\
\underline{f}_!\mathcal{P}\boxtimes\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{P},\mathcal{K}_Y) \arrow[r] & \delta_!\mathcal{K}_Y \arrow[r, equal] & \delta_!\mathcal{K}_Y
\end{tikzcd}$$

By scalar extension, it gives rise to the commutative diagram:

$$\begin{tikzcd}
\tilde{\underline{f}}_!(\mathcal{P}\boxtimes\underline{D}\mathcal{P}) \arrow[r] & \tilde{\underline{f}}_!\delta_!\mathcal{K}_X \arrow[dd] \\
\underline{f}_!\mathcal{P}\boxtimes\underline{f}_!\underline{D}\mathcal{P} \arrow[u, "\alpha"] \arrow[d] & \\
\underline{f}_!\mathcal{P}\boxtimes\underline{D}\underline{f}_!\mathcal{P} \arrow[r] & \delta_!\mathcal{K}_Y
\end{tikzcd} \tag{5.1}$$

Note that α is an isomorphism by the Künneth formula (see [14]).

Recall that for any holomorphic map $f : X \longrightarrow Y$ and any right $\mathcal{D}_X$-module $\mathcal{M}$, we have

$$\underline{f}_!\mathcal{M}\otimes_{\mathcal{D}_Y}\mathcal{O}_Y \xrightarrow{\sim} f_!(\mathcal{M}\otimes_{\mathcal{D}_X}\mathcal{O}_X).$$

Also recall that the compatibility between the duality morphism for $\mathcal{D}$-modules and the Poincaré-Verdier duality morphism may be expressed by the commutative diagram

$$\begin{tikzcd}
(\underline{f}_!\Omega_X[d_X])\otimes_{\mathcal{D}_Y}\mathcal{O}_Y \arrow[r] \arrow[d, "\sim"'] & \Omega_Y[d_Y]\otimes_{\mathcal{D}_Y}\mathcal{O}_Y \arrow[dd, "\sim"'] \\
f_!(\Omega_X[d_X]\otimes_{\mathcal{D}_X}\mathcal{O}_X) \arrow[d, "\sim"'] & \\
f_!\omega_X \arrow[r] & \omega_Y.
\end{tikzcd}$$

With these facts in mind, the conclusion follows easily by applying the functor $\cdot \otimes_{\mathcal{D}_{\tilde{Y}}} \mathcal{O}_{\tilde{Y}}$ to the diagram (5.1). □

As a particular case of Theorem 5.1, we get:

Corollary 5.5 *Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ and assume f is proper on* $\operatorname{supp} \mathcal{M}$. *Then:*

$$\mu\mathrm{eu}(\underline{f}_!\mathcal{M}) = f_\mu(\mu\mathrm{eu}(\mathcal{M})).$$

Now apply these results to the map $f : X \longrightarrow \{\mathrm{pt}\}$. We get:

Corollary 5.6 *Let $(\mathcal{M}, F)$ be a good elliptic pair with compact support, i.e.:*

(i) $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ *and* $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$,

(ii) $\mathrm{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X$,

(iii) $\operatorname{supp} \mathcal{M} \cap \operatorname{supp} F$ *is compact.*

Then the complex $R\Gamma(X; F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)$ has finite dimensional cohomology and its Euler-Poincaré index is given by the formulas:

$$\begin{aligned} \chi(R\Gamma(X; F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)) &= \int_X \mathrm{eu}(\mathcal{M}, F) \\ &= \int_X (\mu\mathrm{eu}(\mathcal{M}) *_\mu \mu\mathrm{eu}(F))|_X \\ &= \int_{T^*X} \mu\mathrm{eu}(\mathcal{M}) \cup \mu\mathrm{eu}(F). \end{aligned}$$

Proof: The first formula follows from Theorem 5.1, the second one follows from Theorem 4.1, and the last one from the equality:

$$\int_{T^*X/X} (\lambda_0^a \cup \lambda_1) = (\lambda_0 *_\mu \lambda_1)|_X,$$

which holds for any $\lambda_j \in H^0_{\Lambda_j}(T^*X; \pi^{-1}\omega_X), j = 0, 1$ and whose proof is left to the reader. □

6 Inverse image and external product formulas

Let $f : X \longrightarrow Y$ be a morphism of complex manifolds and let $(\mathcal{N}, G)$ be an elliptic pair on Y. We shall first study its inverse image by f.

Definition 6.1 We shall say that f is non-characteristic for the elliptic pair $(\mathcal{N}, G)$ if f is non characteristic with respect to the set $\mathrm{char}(\mathcal{N}) + SS(G)$ (see (2.5)).

Proposition 6.2 *Assume f is non-characteristic for the elliptic pair $(\mathcal{N}, G)$. Then $(\underline{f}^{-1}\mathcal{N}, f^{-1}G)$ is an elliptic pair in a neighborhood of $f^{-1}(\operatorname{supp} \mathcal{N} \cap \operatorname{supp} G)$.*

Proof: The hypothesis implies that f is non-characteristic with respect to $\mathcal{N}$ and with respect to G in a neighborhood of $\operatorname{supp}\mathcal{N} \cap \operatorname{supp} G$. In particular, $\underline{f}^{-1}\mathcal{N}$ will be $\mathcal{D}_X$-coherent on a neighborhood of $f^{-1}(\operatorname{supp}\mathcal{N} \cap \operatorname{supp} G)$.

Let $(x;\xi) \in \operatorname{char}(\underline{f}^{-1}\mathcal{N}) \cap SS(f^{-1}G)$, and let $y = f(x)$. Since f is non-characteristic for $\mathcal{N}$ and for G, one knows [5, 7] that there exist $(y;\eta_0) \in \operatorname{char}(\mathcal{N})$ and $(y;\eta_1) \in SS(G)$ such that ${}^t f'(x).\eta_j = \xi$ for $j = 0, 1$. Hence ${}^t f'(x).(\eta_0 - \eta_1) = 0$ which implies by the hypothesis that $\eta_0 - \eta_1 = 0$, hence $\eta_0 = \eta_1 = 0$ since $(\mathcal{N}, G)$ is elliptic. □

In view of the above proposition and Theorem 4.1, in order to calculate the microlocal Euler class of $(\underline{f}^{-1}\mathcal{N}, f^{-1}G)$, it is enough to calculate separately $\mu\mathrm{eu}(\underline{f}^{-1}\mathcal{N})$ and $\mu\mathrm{eu}(f^{-1}G)$. As we shall see below, the microlocal Euler class of an $\mathbb{R}$-constructible sheaf is nothing but its characteristic cycle, and the functorial properties of this cycle have been studied in [7], where it is proved in particular that it commutes to inverse image (and external product). Hence it is enough to calculate the microlocal Euler class of the inverse image (and external product) of coherent $\mathcal{D}$-modules. Notice that such a situation did not appear when studying direct image, where the result obtained when treating simultaneously both $\mathcal{M}$ and F was much stronger than if we would have assumed f proper on $\operatorname{supp}\mathcal{M}$ and on $\operatorname{supp} F$.

Let $f : X \longrightarrow Y$ be a morphism of complex manifolds. We shall use the notations (2.12), (2.13), (2.15) of §2.

Theorem 6.3 *Let $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y^{\mathrm{op}})$ and assume f is non-characteristic with respect to $\mathcal{N}$. Then:*

$$\mu\mathrm{eu}(\underline{f}^{-1}\mathcal{N}) = f^{\mu}(\mu\mathrm{eu}(\mathcal{N})).$$

Proof: The proof is similar to that of Theorem 5.1, and we shall not give here all details.

Set $\tilde{f} = (f, f) : X \times X \longrightarrow Y \times Y$, and decompose $\tilde{f}$ as:

$$\begin{tikzcd}
X \times X \arrow[r, "f_1"] & Y \times X \arrow[r, "f_2"] & Y \times Y \\
\Delta_X \arrow[u, "\delta_X"] \arrow[r, "\sim"] & \Delta \arrow[u, "\delta"] \arrow[r, "f"] & \Delta_Y \arrow[u, "\delta_Y"]
\end{tikzcd}$$

Set:

$$\begin{aligned}
L_Y &= \mathcal{N} \boxtimes \underline{D}\mathcal{N} \otimes^L_{\mathcal{D}_{Y\times Y}} \mathcal{O}_{Y\times Y}, \\
L_X &= \underline{f}^{-1}\mathcal{N} \boxtimes \underline{D}\underline{f}^{-1}\mathcal{N} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}, \\
\Lambda_Y &= \operatorname{char}\mathcal{N}, \Lambda = f_\pi^{-1}\Lambda_Y, \Lambda_X = f^{\mu}\Lambda_Y.
\end{aligned}$$

Since f is non-characteristic for $\mathcal{N}$, the natural morphism:

$$f_2^{-1} R\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N}, \mathcal{O}^{(0,d_Y)}_{Y\times Y}) \longrightarrow R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\underline{f}^{-1}\mathcal{N}, \mathcal{O}^{(0,d_X)}_{Y\times X})$$

is an isomorphism.

On the other-hand, the natural $f_1^{-1}\mathcal{D}_Y$-linear morphism:

$$Rf_{1!}(q_1^{-1}\mathcal{D}_{Y\leftarrow X} \otimes^L_{q_1^{-1}\mathcal{D}_X} \mathcal{O}_{X\times X}[d_X]) \longrightarrow \mathcal{O}_{Y\times X}[d_Y]$$

defines the morphism:

$$Rf_{1!}(f_1^{-1}q_1^{-1}\mathcal{N} \otimes^L_{f_1^{-1}q_1^{-1}\mathcal{D}_Y} q_1^{-1}\mathcal{D}_{Y\leftarrow X} \otimes^L_{q_1^{-1}\mathcal{D}_X} \mathcal{O}_{X\times X}[d_X]) \longrightarrow q_1^{-1}\mathcal{N} \otimes^L_{q_1^{-1}\mathcal{D}_Y} \mathcal{O}_{Y\times X}[d_Y],$$

hence the morphism:

$$q_1^{-1}\underline{f}^{-1}\mathcal{N} \otimes^L_{q_1^{-1}\mathcal{D}_X} \mathcal{O}_{X\times X}[d_X] \longrightarrow f_1^!(q_1^{-1}\mathcal{N} \otimes^L_{q_1^{-1}\mathcal{D}_Y} \mathcal{O}_{Y\times X})[d_Y]$$

and this morphism is an isomorphism when f is non-characteristic for $\mathcal{N}$. Combining these two isomorphisms, we get the isomorphism:

$$f_1^! f_2^{-1} L_Y \xrightarrow{\sim} L_X$$

Then, as for Theorem 5.1, the proof is decomposed by proving the commutativity of the diagrams below. Until the end of the proof, we shall not write the symbols "R" or "L" of derived functors, for short.

$$\begin{tikzcd}
f^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{N}) \arrow[rrr] \arrow[d] & & & \mathcal{H}om_{\mathcal{D}_X}(\underline{f}^{-1}\mathcal{N},\underline{f}^{-1}\mathcal{N}) \arrow[d] \\
f^{-1}\delta_Y^! L_Y \arrow[r] & \delta^! f_2^{-1} L_Y \arrow[r, "\sim"] & \delta_X^! f_1^! f_2^{-1} L_Y \arrow[r] & \delta_X^! L_X \\
\pi_* {}^t f'_! f_\pi^{-1} \Gamma_{\Lambda_Y} \mu_{\Delta_Y} L_Y \arrow[r] \arrow[u, "\sim"] \arrow[d] & \pi_* {}^t f'_! \Gamma_\Lambda \mu_\Delta f_2^{-1} L_Y \arrow[r, "\sim"] \arrow[u, "\sim"] \arrow[d, "\sim"] & \pi_* \Gamma_{\Lambda_X} \mu_{\Delta_X} f_1^! f_2^{-1} L_Y \arrow[r] \arrow[u, "\sim"] \arrow[d] & \pi_* \Gamma_{\Lambda_X} \mu_{\Delta_X} L_X \arrow[u, "\sim"] \arrow[d] \\
\pi_* {}^t f'_! f_\pi^{-1} \Gamma_{\Lambda_Y} \mu_{\Delta_Y} \delta_{Y!} \omega_Y \arrow[r] & \pi_* \Gamma_\Lambda \mu_\Delta \delta_! f^{-1} \omega_Y \arrow[r] & \pi_* \Gamma_{\Lambda_X} \mu_{\Delta_X} \delta_{X!} f^{-1} \omega_Y \otimes \omega_{X/Y} \arrow[r] & \pi_* \Gamma_{\Lambda_X} \mu_{\Delta_X} \delta_{X!} \omega_X
\end{tikzcd}$$

□

The commutativity of the first diagram will follows from Lemma 6.4 and 6.5 below, and that of the last one from Lemma 6.6 below. Since their proofs follow the same lines as for the direct image, we shall omit them. The other diagrams obviously commute.

Note that in lemmas 6.4, 6.5, 6.6 below, the reversed arrows will become isomorphisms when assuming that $\mathcal{M}$ and $\mathcal{N}$ belong to $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y^{op})$ and f is non-characteristic.

Lemma 6.4 *Let $\mathcal{M}$ and $\mathcal{N}$ belong to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y^{op})$. Then the diagram below commutes.*

$$
\begin{tikzcd}
f^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{M}) \arrow[r] \arrow[dddd] & f^{-1}\delta_Y^!\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \arrow[d] \\
 & \delta^! f_2^{-1}\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \arrow[d, "\sim"'] \\
 & \delta_X^! f_1^! f_2^{-1}\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \arrow[d] \\
 & \delta_X^! f_1^!\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\underline{f}^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times X}^{(0,d_X)}[d_Y]) \\
\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathcal{N},\underline{f}^{-1}\mathcal{M}) \arrow[r] & \delta_X^!\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\underline{f}^{-1}\mathcal{N}, q_1^{-1}\underline{f}^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}_{X\times X}^{(0,d_X)}[d_X]) \arrow[u]
\end{tikzcd}
$$

Lemma 6.5 *Let $\mathcal{M}$ and $\mathcal{N}$ belong to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y^{op})$. Then the diagram below commutes.*

$$
\begin{tikzcd}
f^{-1}\delta_Y^!\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \arrow[d] & f^{-1}\delta_Y^!\mathcal{M}\boxtimes\underline{D}\mathcal{N}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y} \arrow[l] \arrow[d] \\
\delta^! f_2^{-1}\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \arrow[d, "\sim"'] & \delta^! f_2^{-1}\mathcal{M}\boxtimes\underline{D}\mathcal{N}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y} \arrow[l] \arrow[d, "\sim"'] \\
\delta_X^! f_1^! f_2^{-1}\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \arrow[d] & \delta_X^! f_1^! f_2^{-1}\mathcal{M}\boxtimes\underline{D}\mathcal{N}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y} \arrow[l] \arrow[d] \\
\delta_X^! f_1^!\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\underline{f}^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times X}^{(0,d_X)}[d_Y]) & \delta_X^! f_1^!\mathcal{M}\boxtimes\underline{D}\underline{f}^{-1}\mathcal{N}\otimes_{\mathcal{D}_{Y\times X}}\mathcal{O}_{Y\times X}[d_{Y/X}] \arrow[l] \\
\delta_X^!\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\underline{f}^{-1}\mathcal{N}, q_1^{-1}\underline{f}^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}_{X\times X}^{(0,d_X)}[d_X]) \arrow[u] & \underline{f}^{-1}\mathcal{M}\boxtimes\underline{D}\underline{f}^{-1}\mathcal{N}\otimes_{\mathcal{D}_{X\times X}}\mathcal{O}_{X\times X} \arrow[l] \arrow[u]
\end{tikzcd}
$$

Lemma 6.6 *Let $\mathcal{N}$ belong to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y^{op})$. Then we have the commutative diagram:*

$$
\begin{tikzcd}
f_1^! f_2^{-1}\mathcal{N}\boxtimes\underline{D}\mathcal{N}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y} \arrow[r] & \underline{f}^{-1}\mathcal{N}\boxtimes\underline{D}\underline{f}^{-1}\mathcal{N}\otimes_{\mathcal{D}_{X\times X}}\mathcal{O}_{X\times X} \arrow[dd] \\
f_1^{-1} f_2^{-1}\mathcal{N}\boxtimes\underline{D}\mathcal{N}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y}\otimes\omega_{X/Y} \arrow[u] \arrow[d] & \\
\delta_{X!}f^{-1}\omega_Y\otimes\omega_{X/Y} \arrow[r] & \delta_{X!}\omega_X
\end{tikzcd}
$$

Now let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ and $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y)$.

Theorem 6.7 *One has:*

$$\mu\mathrm{eu}(\mathcal{M} \underline{\boxtimes} \mathcal{N}) = \mu\mathrm{eu}(\mathcal{M}) \boxtimes \mu\mathrm{eu}(\mathcal{N}).$$

Proof: We set:

$$\begin{aligned} L_X &= \mathcal{M} \underline{\boxtimes} \underline{D}\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}, \\ L_Y &= \mathcal{N} \underline{\boxtimes} \underline{D}\mathcal{N} \otimes^L_{\mathcal{D}_{Y\times Y}} \mathcal{O}_{Y\times Y}, \\ L_{X\times Y} &= (\mathcal{M} \underline{\boxtimes} \mathcal{N}) \underline{\boxtimes} (\underline{D}\mathcal{M} \underline{\boxtimes} \underline{D}\mathcal{N}) \otimes^L_{\mathcal{D}_{X\times Y\times X\times Y}} \mathcal{O}_{X\times Y\times X\times Y}, \end{aligned}$$

$$\Lambda_X = \mathrm{char}(\mathcal{M}), \Lambda_Y = \mathrm{char}(\mathcal{N}), \Lambda = \Lambda_X \times \Lambda_Y.$$

Then the diagram below obviously commutes, which completes the proof.

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{M}) \boxtimes \mathcal{H}om_{\mathcal{D}_X}(\mathcal{N},\mathcal{N}) \arrow[rr] \arrow[d] & & \mathcal{H}om_{\mathcal{D}_{X\times Y}}(\mathcal{M} \underline{\boxtimes} \mathcal{N}, \mathcal{M} \underline{\boxtimes} \mathcal{N}) \arrow[d] \\
\delta^!_X L_X \boxtimes \delta^!_Y L_Y \arrow[r] & \delta^!(L_X \boxtimes L_Y) \arrow[r] & \delta^! L_{X\times Y} \\
\pi_*\Gamma_{\Lambda_X}\mu_{\Delta_X} L_X \boxtimes \pi_*\Gamma_{\Lambda_Y}\mu_{\Delta_Y} L_Y \arrow[r] \arrow[u, "\sim"] \arrow[d] & \pi_*\Gamma_\Lambda\mu_\Delta(L_X \boxtimes L_Y) \arrow[r] \arrow[u, "\sim"] \arrow[d] & \pi_*\Gamma_\Lambda\mu_\Delta L_{X\times Y} \arrow[u, "\sim"] \arrow[d] \\
\pi_*\Gamma_{\Lambda_X}\mu_{\Delta_X}\delta_!\omega_X \boxtimes \pi_*\Gamma_{\Lambda_Y}\mu_{\Delta_Y}\delta_!\omega_Y \arrow[r] \arrow[d, "\sim"] & \pi_*\Gamma_\Lambda\mu_\Delta\delta_!(\omega_X \boxtimes \omega_Y) \arrow[r] \arrow[d, "\sim"] & \pi_*\Gamma_\Lambda\mu_\Delta\delta_!\omega_{X\times Y} \arrow[d, "\sim"] \\
\pi_*\Gamma_{\Lambda_X}\pi^{-1}\omega_X \boxtimes \pi_*\Gamma_{\Lambda_Y}\pi^{-1}\omega_Y \arrow[r] & \pi_*\Gamma_\Lambda\pi^{-1}(\omega_X \boxtimes \omega_Y) \arrow[r, "\sim"] & \pi_*\Gamma_\Lambda\pi^{-1}\omega_{X\times Y}
\end{tikzcd}$$

□

7 Examples

7.1 Euler class of $\mathbb{R}$-constructible sheaves

Let F be an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$, X being still a complex manifold. We shall prove that $\mu\mathrm{eu}(F)$ is nothing but $CC(F)$, the characteristic cycle of F constructed by Kashiwara in [6], (see also [7, Chapter IX]). Recall that $CC(F)$ is obtained as the image of $id_F \in \mathrm{Hom}(F,F)$ in $H^0_\Lambda(T^*X;\pi^{-1}\omega_X)$, (where $\Lambda = SS(F)$), by the sequence of morphisms:

$$\begin{aligned} R\mathcal{H}om(F,F) &\xleftarrow{\sim} \delta^!(F \boxtimes DF) \\ &\xleftarrow{\sim} R\pi_*R\Gamma_\Lambda\mu_\Delta(F \boxtimes DF) \\ &\longrightarrow R\pi_*R\Gamma_\Lambda\mu_\Delta\delta_!\omega_X \\ &\simeq R\pi_*R\Gamma_\Lambda\pi^{-1}\omega_X. \end{aligned}$$

Proposition 7.1 *Let $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$. Then:*

$$\mu\mathrm{eu}(F) = CC(F)$$

Proof: We start with the commutative diagram:

$$\begin{tikzcd}
\delta_!\mathbb{C}_X \arrow[r, "\sim"] \arrow[d] & \delta_! R\mathcal{H}om_{\mathcal{D}_X}(\Omega_X, \Omega_X) \arrow[d] \\
\mathbb{C}_X \boxtimes \omega_X \arrow[r, "\sim"] & \Omega_X \underline{\boxtimes} \underline{D}\Omega_X \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}
\end{tikzcd}$$

Tensoring by $q_1^{-1}F$, then applying $R\mathcal{H}om(q_2^{-1}F, \cdot)$, we get the commutative diagram:

$$\begin{tikzcd}
\delta_! R\mathcal{H}om(F,F) \arrow[r, "\sim"] \arrow[d] & \delta_! R\mathcal{H}om_{\mathcal{D}_X}(F\otimes\Omega_X, F\otimes\Omega_X) \arrow[d] \\
F \boxtimes DF \arrow[r, "\sim"] & F\otimes\Omega_X \underline{\boxtimes} D'F \otimes \underline{D}\Omega_X \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}
\end{tikzcd}$$

Set

$$H = F\otimes\Omega_X \underline{\boxtimes} D'F \otimes \underline{D}\Omega_X \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}$$

We have a commutative diagram:

$$\begin{tikzcd}
F \boxtimes DF \arrow[r] \arrow[d] & H \arrow[d] \\
\delta_!\omega_X \arrow[r, phantom, "="] & \delta_!\omega_X.
\end{tikzcd}$$

Hence we have a commutative diagram, in which $\Lambda = SS(F)$:

$$\begin{tikzcd}
R\mathcal{H}om(F,F) \arrow[r, "\sim"] \arrow[d] & R\mathcal{H}om_{\mathcal{D}_X}(F\otimes\Omega_X, F\otimes\Omega_X) \arrow[d] \\
\delta^! F \boxtimes DF \arrow[r, "\sim"] & \delta^! H \\
R\pi_* R\Gamma_\Lambda \mu_\Delta (F\boxtimes DF) \arrow[r, "\sim"] \arrow[u, "\sim"'] \arrow[d] & R\pi_* R\Gamma_\Lambda \mu_\Delta H \arrow[u, "\sim"'] \arrow[d] \\
R\pi_* R\Gamma_\Lambda \mu_\Delta \delta_!\omega_X \arrow[r, phantom, "="] \arrow[d] & R\pi_* R\Gamma_\Lambda \delta_!\omega_X \arrow[d] \\
R\pi_* R\Gamma_\Lambda \pi^{-1}\omega_X \arrow[r, phantom, "="] & R\pi_* R\Gamma_\Lambda \pi^{-1}\omega_X.
\end{tikzcd}$$

The result follows by applying the functor $H^0 R\Gamma(X; \cdot)$. □

7.2 Euler class of $\mathcal{D}$-modules and $\mathcal{E}$-modules

Let us first recall the construction of the microlocal Euler class of a coherent $\mathcal{D}_X$-module, which of course, is a little easier than that of an elliptic pair.

Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{coh}(\mathcal{D}_X)$, and let $\Lambda = \mathrm{char}(\mathcal{M})$. The isomorphism of $(\mathcal{D}_X, \mathcal{D}_X)$-bimodules

$$\mathcal{D}_X \simeq \mathcal{B}^{(0,d_X)}_{\Delta|X\times X}$$

gives rise to the chain of morphisms:

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{M}) &\simeq R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{M}, q_1^{-1}\mathcal{M} \otimes_{q_1^{-1}\mathcal{D}_X} \mathcal{B}^{(0,d_X)}_{\Delta|X\times X}) \\ &\simeq (\mathcal{M} \boxtimes \underline{D}\mathcal{M}) \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{B}_{\Delta|X\times X}[-d_X] \\ &\longrightarrow R\pi_* R\Gamma_\Lambda \mu_\Delta(\mathcal{M} \boxtimes \underline{D}\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}) \\ &\longrightarrow R\pi_* R\Gamma_\Lambda \mu_\Delta \delta_! \omega_X \\ &\simeq R\pi_* R\Gamma_\Lambda \pi^{-1}\omega_X. \end{aligned}$$

This defines the morphism:

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{M}) \longrightarrow H^0_\Lambda(T^*X; \pi^{-1}\omega_X)$$

and since this morphism is obviously the same as that constructed for elliptic pairs in §3, $\mu\mathrm{eu}(\mathcal{M})$ is the image of $\mathrm{id}_\mathcal{M}$.

Let $\mathcal{E}_X$ denote the sheaf on T^*X of finite order microdifferential operators of [10] (see also [11] for a detailed exposition). We shall adapt our construction of the microlocal Euler class to the case of coherent $\mathcal{E}_X$-modules.

One denotes by $\mathcal{C}_{\Delta|X\times X}$ the simple holonomic $\mathcal{E}_{X\times X}$-module associated to the diagonal embedding $\Delta \hookrightarrow X \times X$, the "microlocalization" of the $\mathcal{D}_X$-module $\mathcal{B}_{\Delta|X\times X}$ encountered above. Isomorphism (3.2) entails the isomorphism of $(\mathcal{E}_X, \mathcal{E}_X)$-bimodules:

$$\mathcal{E}_X \simeq \mathcal{C}^{(0,d_X)}_{\Delta|X\times X}. \tag{7.1}$$

Consider a coherent right $\mathcal{E}_X$-module $\mathcal{N}$ defined on an open subset U of T^*X (or more generally an object of the derived category $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_X^{\mathrm{op}}|_U)$). One can adapt to this situation the construction of the microlocal Euler class of elliptic pairs. Set

$$\underline{D}\mathcal{N} = R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{N}, \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\Omega_X[d_X])$$

and let $\Lambda = \mathrm{supp}\,\mathcal{N}$. Morphism (7.1) gives rise to the chain of morphisms:

$$\begin{aligned} R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{N},\mathcal{N}) &\simeq R\mathcal{H}om_{q_2^{-1}\mathcal{E}_X}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{N} \otimes_{q_1^{-1}\mathcal{E}_X} \mathcal{C}^{(0,d_X)}_{\Delta|X\times X}) \\ &\simeq \mathcal{N} \boxtimes \underline{D}\mathcal{N} \otimes^L_{\mathcal{E}_{X\times X}} \mathcal{C}_{\Delta|X\times X}[-d_X] \\ &\xleftarrow{\sim} \mathrm{R}\Gamma_\Lambda(\mathcal{N} \boxtimes \underline{D}\mathcal{N} \otimes^L_{\mathcal{E}_{X\times X}} \mathcal{C}_{\Delta|X\times X}[-d_X]) \\ &\longrightarrow \mathrm{R}\Gamma_\Lambda(\mathcal{C}^{(2d_X)}_{\Delta|X\times X} \otimes^L_{\mathcal{E}_{X\times X}} \mathcal{C}_{\Delta|X\times X}) \\ &\simeq \mathrm{R}\Gamma_\Lambda \pi^{-1}\omega_X. \end{aligned}$$

Applying the functor $H^0R\Gamma(U;\cdot)$, we obtain the morphism:

$$\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{N},\mathcal{N}) \longrightarrow H^0_\Lambda(U; \pi^{-1}\omega_X). \tag{7.2}$$

Definition 7.2 Let $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_X^{\mathrm{op}}|_U)$. The image of $\mathrm{id}_\mathcal{N}$ by the morphism (7.2) is called the microlocal Euler class of $\mathcal{N}$ and is denoted $\mu\mathrm{eu}(\mathcal{N})$.

This definition is clearly compatible to that we have made for $\mathcal{D}$-modules, which implies that if U is open in T^*X and if $\mathcal{M}$ belongs to $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, then:

$$\mu\mathrm{eu}(\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}|_U) = \mu\mathrm{eu}(\mathcal{M})|_U. \tag{7.3}$$

Remark 7.3 Let $\mathcal{E}_X^{\mathbb{R}}$ denote the sheaf of microlocal operators constructed in [10]. Recall that it is defined as:

$$\mathcal{E}_X^{\mathbb{R}} = \mu_\Delta \mathcal{O}_{X\times X}^{(0,d_X)}[d_X]. \tag{7.4}$$

If $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_X^{\mathrm{op}}|_U)$, we set

$$\mathcal{N}^{\mathbb{R}} = \mathcal{N} \otimes_{\mathcal{E}_X} \mathcal{E}_X^{\mathbb{R}}.$$

Then replacing $\mathcal{C}_{\Delta|X\times X}$ by $\mathcal{C}^{\mathbb{R}}_{\Delta|X\times X}$ in the above construction, one sees it would be possible to define directly the microlocal Euler class of $\mathcal{N}^{\mathbb{R}}$ for $\mathcal{N}^{\mathbb{R}}$ perfect. Using the isomorphism

$$\mathcal{C}^{(2d_X)}_{\Delta|X\times X} \otimes^L_{\mathcal{E}_{X\times X}} \mathcal{C}_{\Delta|X\times X} \simeq \mathcal{C}^{(2d_X)}_{\Delta|X\times X} \otimes^L_{\mathcal{E}_{X\times X}} \mathcal{C}^{\mathbb{R}}_{\Delta|X\times X},$$

one gets that $\mu\mathrm{eu}(\mathcal{N}) = \mu\mathrm{eu}(\mathcal{N}^{\mathbb{R}})$. In particular, $\mu\mathrm{eu}(\mathcal{N})$ depends only on $\mathcal{N}^{\mathbb{R}}$.

7.3 Euler class of holonomic modules

Let $\mathcal{N}$ be a holonomic $\mathcal{E}_X$-module defined on an open subset U of T^*X, and let Λ denotes its support (i.e., its characteristic variety). Then Λ is a closed complex analytic Lagrangian subset of U, conic for the action of $\mathbb{C}^\times$ on T^*X and there is a complex conic smooth submanifold $\Lambda_0 \subset \Lambda$ which is open and dense in Λ. Let $\Lambda_0 = \bigsqcup_\alpha \Lambda_\alpha$, the Λ_α's being locally closed smooth and connected.

On each Λ_α, the $\mathcal{E}_X$-module $\mathcal{N}$ has a well-defined multiplicity m_α, defined by Kashiwara in [5]. Moreover, each Λ_α is closed in

$$U' := U \setminus (\Lambda \setminus \bigsqcup_\alpha \Lambda_\alpha)$$

and defines a Lagrangian cycle $[\Lambda_\alpha]$ in U'. Since $U \setminus U'$ has real codimension at least two in U, the sum $\sum_\alpha m_\alpha[\Lambda_\alpha]$ defines a Lagrangian cycle on U supported by Λ. Let us denote it by $CC(\mathcal{N})$. Then:

$$CC(\mathcal{N}) \in H^0_\Lambda(U; \pi^{-1}\omega_X).$$

Proposition 7.4 *Let $\mathcal{N}$ be a holonomic $\mathcal{E}_X$-module. Then:*

$$\mu\mathrm{eu}(\mathcal{N}) = CC(\mathcal{N}).$$

Proof: Since both terms of the formula are Lagrangian cycles, it is enough to prove the result at generic points of Λ. Hence we may assume $\Lambda = T^*_Z X \cap U$, where Z is a closed complex submanifold of X. Since $\mu\mathrm{eu}(\mathcal{N})$ depends only on $\mathcal{N}^{\mathbb{R}}$ (see Remark 7.3), we may assume that $\mathcal{N}$ is a finite direct sum of sheaves $\mathcal{C}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X$ on U. Hence, it remains to prove the formula:

$$\mu\mathrm{eu}(\mathcal{B}_{Z|X}) = [T^*_Z X] \tag{7.5}$$

This equality is a corollary of our preceding results. In fact, consider the embedding $i : Z \hookrightarrow X$ and the projection $a : Z \longrightarrow \{\mathrm{pt}\}$. Then $\mathcal{B}_{Z|X} = \underline{i}_*(\mathcal{O}_Z)$, and $\mathcal{O}_Z = \underline{a}^{-1}(\mathbb{C}_{\{\mathrm{pt}\}})$. Since the Lagrangian cycle $[T^*_Z X]$ is the direct image by the map i of the inverse image by a of the Lagrangian cycle $\mathbb{C}$ on the manifold $\{\mathrm{pt}\}$, the result follows from Theorems 5.1 and 6.3. □

Corollary 7.5 *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then*

$$\mu\mathrm{eu}(\mathcal{M}) = \mu\mathrm{eu}(\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X).$$

In other words, the microlocal Euler class of a holonomic $\mathcal{D}_X$-module is the same as that of the complex of its holomorphic solutions. (Recall that this last complex is constructible by [4].)

Proof: The result follows from Proposition 3.2 and the equality

$$CC(\mathcal{M}) = CC(\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)$$

proved in [5], but it can also be obtained directly, by considering the commutative diagram below.

$$\begin{tikzcd}
R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{M}) \arrow[r] \arrow[d] & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X, \mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X) \arrow[d] \\
\delta^!(\mathcal{M} \boxtimes \underline{D}\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}) \arrow[d] & \delta^!((\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X) \boxtimes (\underline{D}\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)) \arrow[l, "\sim"'] \arrow[d] \\
\Omega_X[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{O}_X \arrow[r, "\sim"] & \omega_X
\end{tikzcd}$$

□

7.4 Euler class of $\mathcal{O}$-modules

Consider a coherent $\mathcal{O}_X$-module $\mathcal{F}$. To it, one can associate the right coherent $\mathcal{D}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X$. We shall show that the Euler class of this $\mathcal{D}_X$-module is the natural image of a cohomology class which belongs to $H^{d_X}_{\mathrm{supp}\,\mathcal{F}}(X; \Omega_X)$. For that purpose, let us introduce the following notations.

Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules. We set:

$$\begin{aligned}
D_{\mathcal{O}}\mathcal{F} &= R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \Omega_X[d_X]), \\
\mathcal{F} \boxtimes_{\mathcal{O}} \mathcal{G} &= \mathcal{O}_{X\times X} \otimes_{\mathcal{O}_X \boxtimes \mathcal{O}_X} (\mathcal{F} \boxtimes \mathcal{G}).
\end{aligned}$$

Now assume $\mathcal{F}$ is $\mathcal{O}_X$-coherent and consider the chain of morphisms:

$$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \longrightarrow \delta^!(\mathcal{F}\boxtimes_{\mathcal{O}} D_{\mathcal{O}}\mathcal{F}) \longrightarrow \mathcal{F}\otimes^L_{\mathcal{O}} D_{\mathcal{O}}\mathcal{F} \longrightarrow \Omega_X[d_X].$$

This defines:

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \longrightarrow H^{d_X}_{\mathrm{supp}\,\mathcal{F}}(X;\Omega_X). \tag{7.6}$$

Definition 7.6 Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and let S denote its support. The image of $id_{\mathcal{F}}$ in $H^{d_X}_S(X;\Omega_X)$ by the morphism (7.6) is called the *holomorphic Euler class* of $\mathcal{F}$ and denoted by $\mathrm{eu}_{\mathcal{O}}(\mathcal{F})$.

The natural morphism $\Omega_X[d_X] \longrightarrow \omega_X$ defines the morphism:

$$\alpha : H^{d_X}_S(X;\Omega_X) \longrightarrow H^0_S(X;\omega_X). \tag{7.7}$$

Proposition 7.7 *Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\mathrm{eu}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X)$ is the image of $\mathrm{eu}_{\mathcal{O}}(\mathcal{F})$ by the morphism (7.7).*

Proof: We start with the commutative diagram:

$$\begin{array}{ccc}
\delta_!\mathcal{O}_X & \longrightarrow & \delta_!\mathcal{D}_X \\
\downarrow & & \downarrow \\
\mathcal{O}_X\boxtimes_{\mathcal{O}} D_{\mathcal{O}}\mathcal{O}_X & \longrightarrow & (\mathcal{D}_X\boxtimes \underline{D}\mathcal{D}_X)\otimes^L_{\mathcal{D}_X\boxtimes\mathcal{D}_X}\mathcal{O}_{X\times X}.
\end{array}$$

Applying the functor $q_1^{-1}\mathcal{F}\otimes_{q_1^{-1}\mathcal{O}_X}\cdot$, then the functor $R\mathcal{H}om_{q_2^{-1}\mathcal{O}_X}(q_2^{-1}\mathcal{F},\cdot)$, we get the commutative diagram:

$$\begin{array}{ccc}
\delta_!R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) & \longrightarrow & \delta_!R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X,\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X) \\
\downarrow & & \downarrow \\
\mathcal{F}\boxtimes_{\mathcal{O}} D_{\mathcal{O}}\mathcal{F} & \longrightarrow & (\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X)\boxtimes\underline{D}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X)\otimes^L_{\mathcal{D}_X\boxtimes\mathcal{D}_X}\mathcal{O}_{X\times X}.
\end{array}$$

Set

$$H = (\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X)\,\underline{\boxtimes}\,\underline{D}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X)\otimes^L_{\mathcal{D}_{X\times X}}\mathcal{O}_{X\times X}.$$

Then

$$H \simeq (\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X)\boxtimes\underline{D}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X)\otimes^L_{\mathcal{D}_X\boxtimes\mathcal{D}_X}\mathcal{O}_{X\times X}.$$

On the other hand, we have the commutative diagram:

$$\begin{array}{ccc}
\delta^{-1}\mathcal{F}\boxtimes_{\mathcal{O}} D_{\mathcal{O}}\mathcal{F} & \longrightarrow & \delta^{-1}H \\
\downarrow & & \downarrow \\
\Omega_X[d_X] & \longrightarrow & \omega_X.
\end{array}$$

Hence, we get the commutative diagram:

$$\begin{tikzcd} R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \arrow[r] \arrow[d] & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X,\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X) \arrow[d] \\ \delta^!(\mathcal{F}\boxtimes_{\mathcal{O}} D_{\mathcal{O}_X}\mathcal{F}) \arrow[r] \arrow[d] & \delta^! H \arrow[d] \\ \Omega_X[d_X] \arrow[r] & \omega_X \end{tikzcd}$$

which completes the proof. □

Remark 7.8 The holomorphic Euler class of a coherent $\mathcal{O}_X$-module is known for long, and O'Brian, Toledo and Tong [9] have proved that this class can be obtained as the term of degree d_X of the product of the Chern character of $\mathcal{F}$ by the Todd class of X. See §7 below for further comments on this point.

Remark 7.9 One should not confuse the holomorphic Euler class $\mathrm{eu}_{\mathcal{O}_X}(\cdot)$ and the Euler class $\mathrm{eu}(\cdot)$. For example, $\alpha(\mathrm{eu}_{\mathcal{O}_X}(\mathcal{O}_X)) = \mathrm{eu}(\mathcal{D}_X)$ and $\mathrm{eu}(\mathcal{O}_X) = \mathrm{eu}(\mathbb{C}_X)$. If one chooses $X = P^1(\mathbb{C})$, it follows from Theorem 5.1 that:

$$\int_X \mathrm{eu}(\mathcal{O}_X) = 2, \tag{7.8}$$

$$\int_X \mathrm{eu}_{\mathcal{O}_X}(\mathcal{O}_X) = 1. \tag{7.9}$$

This example also shows that the diagram below is not commutative.

$$\begin{tikzcd} \mathbb{C}_X \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[d] \\ \omega_X & \Omega_X[d_X] \arrow[l] \end{tikzcd}$$

Here, the first and second vertical arrows are defined by

$$\delta_!\mathbb{C}_X \longrightarrow \mathbb{C}_X \boxtimes \omega_X,$$

and

$$\delta_!\mathcal{O}_X \longrightarrow \mathcal{O}_X \boxtimes_{\mathcal{O}} D_{\mathcal{O}}\mathcal{O}_X,$$

respectively, as in the proofs of Propositions 3.2 and 3.8.

Remark 7.10 Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and denote by S its support. Then $\mathrm{char}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X) = \pi^{-1}S$, hence:

$$\mu\mathrm{eu}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X) = \pi^*\,\mathrm{eu}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X),$$

where π^* is the isomorphism:

$$H^0_S(X;\omega_X) \xrightarrow{\sim} H^0_{\pi^{-1}S}(T^*X;\pi^{-1}\omega_X).$$

8 A conjectural link with Chern classes

Let X be a complex manifold, Z a complex analytic subset and denote by $K_Z^{an}(X)$ the Grothendieck group of the full subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ consisting of objects supported by Z. In this section we shall assume to be constructed a local Chern character:

$$ch_Z : K_Z^{an}(X) \longrightarrow \bigoplus_j H_Z^{2j}(X; \mathbb{C}_X)$$

such that if we define the local Euler character by the formula:

$$\mathrm{eu}_Z(\mathcal{F}) = ch_Z(\mathcal{F}) \cup td_X(TX)$$

(where $\cup$ is the cup product and $td_X(\cdot)$ is the Todd class), then the local Chern character is compatible to external product and inverse image and the local Euler character is compatible to external product and proper direct image, this last point being what we shall refer as the Grothendieck-Riemann-Roch theorem. Such a construction does exists in the algebraic case (see [2]). In the analytic case, one can construct $ch_Z(\cdot)$ after shrinking X (see [3]). More precisely, let X' be an open relatively compact subset of X. Then one defines the natural morphism:

$$\rho : K_Z^{an}(X) \longrightarrow K_Z^{top}(X')$$

by realification. If $\mathcal{F}$ is a bounded complex of coherent $\mathcal{O}_X$-modules, we associate to it the complex $\mathcal{F}^{\mathbb{R}} := \mathcal{A}_{X^{\mathbb{R}}} \otimes_{\mathcal{O}_X} \mathcal{F}$, where $\mathcal{A}_{X^{\mathbb{R}}}$ denotes the sheaf of real analytic functions on the real analytic manifold $X^{\mathbb{R}}$ underlying X. Applying Cartan's theorem "A" (on the closure of X') we see that $\mathcal{F}^{\mathbb{R}}$ defines an element of $K_Z^{top}(X')$. Unfortunately, the Grothendieck-Riemann-Roch theorem (with supports) has, to our knowledge, never been written in this case. Hence the results of this section should be considered as conjectural, or should be stated with suitable modifications (e.g. assuming we work in the algebraic category).

Now consider a left coherent $\mathcal{D}_X$-module $\mathcal{M}$ endowed with a good filtration and whose characteristic variety is contained in a closed conic analytic subset Λ of T^*X. Let $gr(\mathcal{M})$ denote the associated graded module and set:

$$\tilde{gr}(\mathcal{M}) = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}gr(\mathcal{D}_X)} \pi^{-1} gr(\mathcal{M}).$$

Note that the element $\sigma_\Lambda(\mathcal{M})$ of $K_\Lambda^{an}(T^*X)$ defined by $\tilde{gr}(\mathcal{M})$ locally depends only on $\mathcal{M}$, not on the choice of the good filtration [5].

Let $f : X \longrightarrow Y$ be a morphism of complex manifolds. We shall use the notations introduced in §2, in particular in (2.9), (2.12), (2.13) and (2.15).

First consider a closed conic subset Λ_Y of T^*Y, and assume f is non-characteristic with respect to Λ_Y (i.e. ${}^t f'$ is proper on $f_\pi^{-1}(\Lambda_Y)$). Then the morphisms:

$$f_\pi^* : K_{\Lambda_Y}^{an}(T^*Y) \longrightarrow K_{f_\pi^{-1}\Lambda_X}^{an}(X \times_Y T^*Y),$$

and

$$^t f'_* : K^{an}_{f_\pi^{-1}\Lambda_Y}(X \times_Y T^*Y) \longrightarrow K^{an}_{f^\mu(\Lambda_Y)}(T^*X)$$

are well-defined and if $\mathcal{N}$ is a left coherent $\mathcal{D}_Y$-module whose characteristic variety is contained in Λ_Y, it follows from Kashiwara [5] that:

$$^t f'_* f_\pi^* \sigma_{\Lambda_Y}(\mathcal{N}) = \sigma_{f^\mu(\Lambda_Y)}(\underline{f}^{-1}\mathcal{N}). \tag{8.1}$$

Similarly if f is proper on $\Lambda_X \cap T^*_X X$ (i.e. f_π is proper on ${}^t f'^{-1}(\Lambda_Y)$), then:

$$^t f'^* : K^{an}_{\Lambda_X}(T^*X) \longrightarrow K^{an}_{{}^t f'^{-1}(\Lambda_X)}(X \times_Y T^*Y)$$

and

$$f_{\pi *} : K^{an}_{{}^t f'^{-1}(\Lambda_Y)}(X \times_Y T^*X) \longrightarrow K^{an}_{f_\mu(\Lambda_X)}(T^*Y)$$

are well-defined, and it is shown in Laumon [8] that if $\mathcal{M}$ is a right good $\mathcal{D}_X$-module whose characteristic variety is contained in Λ_X, then:

$$f_{\pi *}{}^t f'^* \sigma_{\Lambda_X}(\mathcal{M}) = \sigma_{f_\mu(\Lambda_X)}(\underline{f}_!\mathcal{M}). \tag{8.2}$$

Finally one shows easily that:

$$\sigma_{\Lambda_X}(\mathcal{M}) \boxtimes \sigma_{\Lambda_Y}(\mathcal{N}) = \sigma_{\Lambda_X \times \Lambda_Y}(\mathcal{M} \underline{\boxtimes} \mathcal{N}). \tag{8.3}$$

Using the Riemann-Roch-Grothendieck Theorem at the level of cotangent bundles, Laumon (loc.cit.) has deduced from (8.2) a formula which computes the Chern characters of $\sigma_{f_\mu(\Lambda_X)}(\underline{f}_!\mathcal{M})$ from that of $\sigma_{\Lambda_X}(\mathcal{M})$. In order to get a class which behaves well both under direct and inverse images, we introduce the following:

Definition 8.1 Let $\mathcal{M}$ (resp. $\mathcal{N}$) be a right (resp. left) coherent $\mathcal{D}_X$-module endowed with a good filtration and whose characteristic variety is contained in a closed conic analytic subset Λ of T^*X. We define the microlocal Chern character of $\mathcal{M}$ and $\mathcal{N}$ along Λ as:

$$\begin{aligned} \mu ch_\Lambda(\mathcal{M}) &= ch_\Lambda(\sigma_\Lambda(\mathcal{M})) \cup \pi^* td_X(TX), \\ \mu ch_\Lambda(\mathcal{N}) &= ch_\Lambda(\sigma_\Lambda(\mathcal{N})) \cup \pi^* td_X(T^*X). \end{aligned}$$

We denote by $\mu ch^j_\Lambda(\mathcal{M})$ the component of $\mu ch_\Lambda(\mathcal{M})$ in $H^j_\Lambda(T^*X; \mathbb{C}_{T^*X})$, and similarly for $\mathcal{N}$.

This definition is motivated by the two following statements.

Proposition 8.2 *The microlocal Chern character of the right $\mathcal{D}_X$-module $\mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X$ is the microlocal Chern character of the left $\mathcal{D}_X$-module $\mathcal{M}$. In other words, if $\mathcal{M}$ is a left $\mathcal{D}_X$-module:*

$$\mu ch_\Lambda(\mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X) = \mu ch_\Lambda(\mathcal{M}).$$

Proof: Recall that if E is a complex vector bundle of rank d, then denoting by $c_1(\cdot)$ the first Chern class:

$$\begin{aligned} td_X(E^*) &= e^{c_1(E^*)} \cup td_X(E), \\ ch(\Lambda^d E) &= e^{c_1(E)}. \end{aligned}$$

Choosing $E = TX$, we get:

$$td_X(T^*X) = ch(\Omega_X) \cup td_X(TX),$$

hence:

$$\begin{aligned} ch_\Lambda(\sigma_\Lambda(\mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X)) \cup \pi^* td_X(TX) &= ch_\Lambda(\sigma_\Lambda(\mathcal{M})) \cup \pi^* ch(\Omega_X) \cup \pi^* td_X(TX) \\ &= ch_\Lambda(\sigma_\Lambda(\mathcal{M})) \cup \pi^* td_X(T^*X). \end{aligned}$$

□

Theorem 8.3 *Let $\mathcal{M}$ (resp. $\mathcal{N}$) be a coherent $\mathcal{D}_X$-module (resp. $\mathcal{D}_Y$-module) endowed with a good filtration, and let Λ_X (resp. Λ_Y) denote its characteristic variety.*

(i) Assume f in non-characteristic for $\mathcal{N}$. Then:

$$f^\mu(\mu ch_{\Lambda_Y}(\mathcal{N})) = \mu ch_{f^\mu(\Lambda_Y)}(\underline{f}^{-1}\mathcal{N}).$$

(ii) Assume f is proper on supp $\mathcal{M}$*. Then:*

$$f_\mu(\mu ch_{\Lambda_X}(\mathcal{M})) = \mu ch_{f_\mu(\Lambda_X)}(\underline{f}_*\mathcal{M}).$$

(iii) One has:

$$\mu ch_{\Lambda_X}(\mathcal{M}) \boxtimes \mu ch_{\Lambda_Y}(\mathcal{N}) = \mu ch_{\Lambda_X \times \Lambda_Y}(\mathcal{M} \boxtimes \mathcal{N}).$$

Notice that in the above statements (i) and (ii), $\mathcal{M}$ or $\mathcal{N}$ can either be a right or a left $\mathcal{D}$-module (of course, in (iii) they need to be of the same type). This follows from Proposition 8.2 since

$$\underline{f}_!(\mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X) = (\underline{f}_!\mathcal{M}) \otimes_{\mathcal{O}_Y} \Omega_Y,$$

and similarly for inverse images.

Proof: In the course of the proof we shall sometimes use the following notations: if W is a manifold, we set for short

$$td(W) = td_W(TW).$$

Then recall that if $p : E \longrightarrow W$ is a complex vector bundle on W, one has:

$$td(E) = td_E(TE) = p^* td_W(E) \cup p^* td(W),$$

which follows from the exact sequence of vector bundles on W:

$$0 \longrightarrow p^{-1}E \longrightarrow TE \longrightarrow p^{-1}TW \longrightarrow 0.$$

Also recall the diagram associated to f:

$$\begin{tikzcd} T^*X \arrow[d, "\pi_X"'] & X \times_Y T^*Y \arrow[l, "{}^tf'"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"] & T^*Y \arrow[d, "\pi_Y"] \\ X \arrow[r, equal] & X \arrow[r, "f"] & Y \end{tikzcd}$$

(i) We may assume $\mathcal{N}$ is a left $\mathcal{D}_Y$-module. Using (8.1) and the Riemann-Roch-Grothendieck theorem applied to the map ${}^tf'$, we obtain:

$$\begin{aligned} ch_{f^\mu(\Lambda_Y)}\sigma_{f^\mu(\Lambda_Y)}(\underline{f}^{-1}\mathcal{N}) \cup td(T^*X) &= ch_{f^\mu(\Lambda_Y)}[{}^tf'_!f_\pi^*\sigma_{\Lambda_Y}(\mathcal{N})] \cup td(T^*X) \\ &= {}^tf'_![f_\pi^*ch_{\Lambda_Y}\sigma_{\Lambda_Y}(\mathcal{N}) \cup td(X \times_Y T^*Y)]. \end{aligned}$$

Hence:

$$\begin{aligned} &ch_{f^\mu(\Lambda_Y)}\sigma_{f^\mu(\Lambda_Y)}(\underline{f}^{-1}\mathcal{N}) \cup \pi_X^*td_X(T^*X) \cup \pi_X^*td_X(TX) \\ &\quad= {}^tf'_![f_\pi^*ch_{\Lambda_Y}\sigma_{\Lambda_Y}(\mathcal{N}) \cup \pi^*td_X(TX) \cup \pi^*f^*td_Y(T^*Y)] \\ &\quad= {}^tf'_!f_\pi^*[ch_{\Lambda_Y}\sigma_{\Lambda_Y}(\mathcal{N}) \cup \pi_Y^*td_Y(T^*Y)] \cup \pi_X^*td_X(TX) \end{aligned}$$

and the result follows since $td_X(TX)$ has an inverse.
(ii) We may assume $\mathcal{M}$ is a right $\mathcal{D}_X$-module. Using (8.2) and the Riemann-Roch-Grothendieck theorem, we get:

$$\begin{aligned} ch_{f_\mu(\Lambda_X)}\sigma_{f_\mu}(\underline{f}_!(\mathcal{M}) \cup td(T^*Y)) &= ch_{f_\mu(\Lambda_X)}[f_{\pi!}{}^tf'^*\sigma_{\Lambda_X}(\mathcal{M})] \cup td(T^*Y) \\ &= f_{\pi!}[{}^tf'^*ch_{\Lambda_X}\sigma_{\Lambda_X}(\mathcal{M}) \cup td(X \times_Y T^*Y)]. \end{aligned}$$

Hence:

$$\begin{aligned} &ch_{f_\mu(\Lambda_X)}\sigma_{f_\mu(\Lambda_X)}(\underline{f}_!\mathcal{M}) \cup \pi_Y^*td_Y(T^*Y) \cup \pi_Y^*td_Y(TY) \\ &\quad= f_{\pi!}[{}^tf'^*ch_{\Lambda_X}\sigma_{\Lambda_X}(\mathcal{M}) \cup \pi^*td_X(TX) \cup \pi^*f^*td_Y(T^*Y)] \\ &\quad= f_{\pi!}{}^tf'^*[ch_{\Lambda_X}\sigma_{\Lambda_X}(\mathcal{M}) \cup \pi_X^*td_X(TX)] \cup \pi_Y^*td_Y(T^*Y) \end{aligned}$$

and the result follows since $td_Y(T^*Y)$ is invertible.
(iii) follows from (8.3) and the fact that $ch(\cdot)$ commutes to external product. □

As a corollary we get that if $\mathcal{M}$ and $\mathcal{N}$ are two $\mathcal{D}_X$-modules with characteristic variety contained in Λ_0 and Λ_1 respectively, and if $\Lambda_0 \cap \Lambda_1 \subset T_X^*X$, then:

$$\mu ch_{(\Lambda_0+\Lambda_1)}(\mathcal{M} \otimes_{\mathcal{O}_X}^L \mathcal{N}) = \mu ch_{\Lambda_0}(\mathcal{M}) *_\mu \mu ch_{\Lambda_1}(\mathcal{N}).$$

In view of Theorem 8.3, the microlocal Chern character has the same functorial properties as the microlocal Euler class.

Let us come back to the situation of Definition 8.1 and set:

$$c_\Lambda = \operatorname{codim}_{\mathbb{C}} \Lambda.$$

It is clear that:

$$\mu ch^j_\Lambda(\mathcal{M}) = 0 \quad \text{for} \quad j < 2c_\Lambda.$$

Proposition 8.4 *The component of degree $2c_\Lambda$ of the microlocal Chern character of $\mathcal{M}$, that is, $\mu ch^{2c_\Lambda}_\Lambda(\mathcal{M})$, is the characteristic cycle of $\mathcal{M}$ along Λ. In particular if $\mathcal{M}$ is holonomic:*

$$\mu ch^{2d_X}_\Lambda(\mathcal{M}) = \mu\mathrm{eu}(\mathcal{M}).$$

Proof: Since $\mu ch^j_\Lambda(\mathcal{M})$ is zero for $j < 2c_\Lambda$,we have

$$\mu ch^{2c_\Lambda}_\Lambda(\mathcal{M}) = [ch_\Lambda(\sigma_\Lambda(\tilde{gr}(\mathcal{M})))]^{2c_\Lambda},$$

and it is well-known that the term on the right-hand side is the analytic cycle on Λ of the $\mathcal{O}_{T^*X}$-coherent module $\tilde{gr}(\mathcal{M})$, that is, the characteristic cycle of $\mathcal{M}$. □

Now we make the following conjectures:

Conjecture 8.5 *(i) $\mu ch^j_\Lambda(\mathcal{M}) = 0$ for $j \notin [2c_\Lambda, 2d_X]$,*

(ii) $\mu ch^{2d_X}_\Lambda(\mathcal{M}) = \mu\mathrm{eu}(\mathcal{M})$.

By Proposition 8.4, Conjecture 8.5 (ii) is true for holonomic $\mathcal{D}_X$-modules. Moreover it follows from Remark 7.10 and the work of O'Brian-Toledo-Tong [9] that Conjecture 8.5 is true for induced $\mathcal{D}_X$-modules, i.e., for modules of the type $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X$, $\mathcal{F}$ being $\mathcal{O}_X$-coherent.

Example 8.6 Let M be a compact n-dimensional real analytic manifold, X a complexification of M, $\mathcal{M}$ a right coherent $\mathcal{D}_X$-module, elliptic on M. By Corollary 5.6, we have:

$$\chi(R\Gamma(M; \mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)) = \int_{T^*X} \mu\mathrm{eu}(\mathcal{M}) \cup \mu\mathrm{eu}(\mathbb{C}_M).$$

Denote by σ_M the zero-section embedding $M \hookrightarrow T^*_M X$ and by j the embedding $T^*_M X \hookrightarrow T^*X$. Since $T^*_M X \cap \operatorname{char}(\mathcal{M})$ is contained in M, we get:

$$\int_{T^*X} \mu\mathrm{eu}(\mathcal{M}) \cup \mu\mathrm{eu}(\mathbb{C}_M) = \int_{T^*_M X} j^* \mu\mathrm{eu}(\mathcal{M}) = \int_M \sigma^*_M j^* \mu\mathrm{eu}(\mathcal{M}).$$

Now assume Conjecture 8.5 (ii) is true. We get, with $\Lambda = \operatorname{char}(\mathcal{M})$:

$$\begin{aligned}\chi(R\Gamma(M; \mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)) &= \int_M \sigma^*_M j^*[ch(\sigma_\Lambda(\mathcal{M})) \cup \pi^* td_X(TX)] \\ &= \int_M \sigma^*_M [j^* ch(\sigma_\Lambda(\mathcal{M}))] \cup td_M(TM^{\mathbb{C}}).\end{aligned}$$

This is the classical Atiyah-Singer index formula.

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A coherence criterion for Fréchet modules

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A Coherence Criterion for Fréchet Modules

JEAN-PIERRE SCHNEIDERS

1 Introduction

In the literature, one finds essentially two general criteria to get the finiteness of the cohomology groups of complexes of locally convex topological vector spaces. They are

(a) If $u^{\cdot} : G^{\cdot} \longrightarrow F^{\cdot}$ is a compact morphism of complexes of Fréchet spaces then $\dim H^k(F^{\cdot}) < +\infty$ for any $k \in \mathbb{Z}$ such that $H^k(u^{\cdot})$ is surjective.

(b) If $u^{\cdot} : G^{\cdot} \longrightarrow F^{\cdot}$ is a continuous morphism between a complex of (DFS) spaces and a complex of Fréchet spaces then $\dim H^k(F^{\cdot}) < +\infty$ for any $k \in \mathbb{Z}$ such that $H^k(u^{\cdot})$ is surjective.

The first of these criterion was used by Cartan-Serre [3] in order to get the finiteness of the cohomology groups of a compact complex analytic manifold with values in a coherent analytic sheaf. It was also used by Kashiwara [5] to get the constructibility of the solution complex associated to a holonomic differential module. The second criterion was proved by Bony-Schapira in [2]. Both are extensions of the Schwartz compact perturbation theorem.

In 1973, Houzel [4] has extended criterion (a) to complexes of modules over some sheaf of bornological algebras $\mathcal{A}$; the finiteness of the cohomology groups being replaced by the pseudo-coherence of the complex in the sense of SGA6 [1] (this is the only cohomological notion of finiteness which works well for a non necessarily coherent base algebra).

More precisely, Houzel assumes that $\mathcal{A}$ is a sheaf of bornological algebras which is complete and multiplicatively convex. The fibers of $\mathcal{A}$ are also assumed to be separated and to possess the homomorphism property. Then, working with complexes null in degree $\geq b$, he shows that in order to get the a-pseudocoherence of a complex $\mathcal{M}^{\cdot}$ of complete bornological $\mathcal{A}$-modules it is sufficient to find a sequence of $r \geq b - a + 1$ complexes of complete bornological $\mathcal{A}$-modules and bounded a-quasi-isomorphisms

$$\mathcal{M}_r \underset{u_r}{\longrightarrow} \mathcal{M}_{r-1} \underset{u_{r-1}}{\longrightarrow} \cdots \underset{u_1}{\longrightarrow} \mathcal{M}$$

such that u_i is $\mathcal{A}$-nuclear in degree $\geq a$ and that the fibers of $\mathcal{M}_i^k$ are separated and possess the homomorphism property.

Using this theorem, Houzel shows that it is possible to give a simple proof of Grauert's coherence theorem. This theorem has also been used by Houzel and Schapira to give a criterion for the coherence of direct images of a coherent $\mathcal{D}$-module.

In [7], we found a situation where Houzel's result was not sufficient to solve the problem. What was needed was a generalization of the criterion (b) above. It is that extension which is the subject of this paper. It is contained in the following theorem.

Theorem 1.1 *Let X be a topological space with countable basis endowed with a multiplicatively convex sheaf of Fréchet algebras $\mathcal{A}$. Let $(\mathcal{M}_i)_{i\in\mathbb{N}}$, $\mathcal{M}$ be complexes of Fréchet $\mathcal{A}$-modules null in degree $\geq b$, let $u_{i+1,i} : \mathcal{M}_i \longrightarrow \mathcal{M}_{i+1}$ be $\mathcal{A}$-nuclear morphisms and let*

$$u : \varinjlim_{i\in\mathbb{N}} \mathcal{M}_i \longrightarrow \mathcal{M}$$

be a continuous morphism of $\mathcal{A}$-modules. Assume that u is an a-quasi-isomorphism and that any $x \in X$ has a fundamental system of open neighborhoods on which $\mathcal{M}$ and each $\mathcal{M}_i$ has enough sections. Then the complex $\mathcal{M}$ is a-pseudo-coherent over $\mathcal{A}$.

We refer the reader to §4 for definitions of the various concepts used in the previous statement. Note that the main difference with Houzel's results is that we only need *one* quasi-isomorphism to get the pseudo-coherence. This is important since, in practice, it is much more difficult to build quasi-isomorphisms than nuclear maps. For example, if X, S are complex analytic manifolds and $U \subset\subset V$ are two Stein open subsets of X then the restriction map

$$\pi_*(\mathcal{O}_{U\times S}) \longrightarrow \pi_*(\mathcal{O}_{V\times S})$$

is $\mathcal{O}_S$-nuclear.

The proof of this theorem will be found in §4. It has essentially the same structure as the proof of Houzel's finiteness theorem [4]. Despite our much weaker hypothesis, a proper use of Baire's theorem allows us to get the pseudo-coherence of the target complex.

2 Nuclear Perturbation Theorem

The basic finiteness tool used in this paper is Houzel's nuclear perturbation theorem. For the reader's convenience, we will give a quick proof of this result for Fréchet modules over a multiplicatively convex Fréchet algebra. We will avoid Houzel's bornological point of view since it is not important for our application (we are in a Fréchet framework). To fix the vocabulary, we first give some definitions.

Definition 2.1 A *Fréchet algebra* is a Fréchet space endowed with a continuous $\mathbb{C}$-bilinear multiplication

$$\cdot : A \times A \longrightarrow A$$

which is associative and admits a unit. We *do not* assume A commutative.

A Fréchet algebra is *multiplicatively convex* if any bounded subset B of A is absorbed by an absolutely convex subset of A stable for the multiplication law.

A (left) *Fréchet module* over a Fréchet algebra A is a Fréchet space M endowed with a structure of (left) A-module in such way that the action

$$\cdot : A \times M \longrightarrow M$$

is continuous.

A *morphism of Fréchet A-modules* is a continuous A-linear map. We denote by $L_A(E, F)$ the set of morphisms of the A-module E to the A-module F. This is naturally a $\mathbb{C}$-vector space. We turn it into a locally convex topological vector space by endowing it with the semi-norms

$$q_B(u) = \sup_{x \in B} q(u(x))$$

associated with the bounded subsets B of E and the semi-norms q of F. With this topology $L_A(E, F)$ is complete.

A morphism $u : E \longrightarrow F$ between two Fréchet A-modules is *A-nuclear* if

$$u(x) = \sum_{m=0}^{+\infty} \lambda_m \langle e^m, x \rangle f_m$$

where

- λ_m is a summable sequence in $\mathbb{C}$,
- e^m is a bounded sequence in $L_A(E, A)$,
- f_m is a bounded sequence in F.

We denote by $N_A(E, F)$ the $\mathbb{C}$-vector space of A-nuclear morphisms from E to F.

A morphism $u : E \longrightarrow F$ between two Fréchet A-modules is *A-finite* if

$$u(x) = \sum_{m=0}^{p} \langle e^m, x \rangle f_m$$

where

$$\begin{array}{rcl} e^0, \ldots, e^p & \in & L_A(E, A) \\ f_0, \ldots, f_p & \in & F \end{array}$$

In the rest of this section, A denotes a Fréchet algebra.

Proposition 2.2 *Let $u : E \longrightarrow F$, $v : F \longrightarrow G$ be two morphisms of Fréchet A-modules. Assume that either u or v is A-nuclear then so is $v \circ u$. Hence $N_A(E, F)$ is naturally a functor in E and F.*

Proof: Obvious. □

Proposition 2.3 [Lifting of nuclear morphisms]
Let $u : E \longrightarrow F$, $v : G \longrightarrow F$ be two morphisms of Fréchet A-modules. Assume u is surjective and v is A-nuclear. Then there is a morphism $w : G \longrightarrow E$ such that $u \circ w = v$.

Proof: Since v is A-nuclear one can find sequences $\lambda_m \in \mathbb{C}$, $e^m \in L_A(G, A)$, $f_m \in F$ such that $\sum_{m=0}^{\infty} |\lambda_m| < +\infty$, f_m being bounded in F and e^m being bounded in $L_A(G, A)$ in such a way that

$$v(x) = \sum_{m=0}^{+\infty} \lambda_m \langle e^m, x\rangle f_m.$$

Lemma 2.5 below shows that one can even assume f_m converges to 0 in F. Since u is surjective it is a strict morphism and one can find by Lemma 2.4 below a sequence $e_m \in E$ converging to 0 and such that $u(e_m) = f_m$. Let us define $w : G \longrightarrow E$ by setting

$$w(x) = \sum_{m=0}^{+\infty} \lambda_m \langle e^m, x\rangle e_m.$$

Obviously w is A-nuclear and $u \circ w = v$ as required. □

Lemma 2.4 *Let $u : E \longrightarrow F$ be a continuous $\mathbb{C}$ linear map between l.c.s. with a countable basis of semi-norms. Then u is a strict epimorphism if and only if for any sequence f_m of F converging to 0 there exist a sequence e_m of E converging to 0 such that $u(e_m) = f_m$.*

Proof: *The condition is necessary.*

Assume f_m converges to 0 in F. Let $(V_m)_{m\in\mathbb{N}}$ be a countable fundamental system of absolutely convex neighborhoods of 0 such that

$$m \in \mathbb{N} \Longrightarrow V_m \supset V_{m+1}$$

Since $u(V_m)$ is a neighborhood of 0 in F one can build a strictly increasing sequence of natural numbers M_k such that

$$m \geq M_k \Longrightarrow f_m \in u(V_k).$$

For $M_k \leq m < M_{k+1}$ let us choose an $e_m \in V_k$ such that $f_m = u(e_m)$. The sequence e_m converges to 0 in E as required.

The condition is sufficient.

Let V be a neighborhood of 0 in E. We need to show that $u(V)$ is a neighborhood of 0 in F. If it is not the case there is a sequence $f_m \in F \backslash u(V)$ which converges to 0 in F. Let e_m be a sequence in E converging to 0 and such that $u(e_m) = f_m$. There is an integer M such that $e_m \in V$ for $m \geq M$. For such an M, $f_M \in u(v)$ and this is impossible. □

Lemma 2.5 *Let λ_m be a sequence of complex numbers such that*

$$\sum_{m=0}^{\infty} |\lambda_m| < +\infty.$$

Then, there is a sequence r_m of positive real numbers converging to 0 such that

$$\sum_{m=0}^{+\infty} \frac{1}{r_m}|\lambda_m| < +\infty.$$

Proof: There is a strictly increasing sequence M_k of integers such that

$$\sum_{m=M_k}^{+\infty} |\lambda_m| < 2^{-2k}.$$

Let us define r_m to be 2^{-k} if $M_k \leq m < M_{k+1}$. We have

$$\sum_{m=M_k}^{M_{k+1}-1} \frac{1}{r_m}|\lambda_m| \leq 2^{-k}.$$

Hence,

$$\sum_{m=M_k}^{+\infty} \frac{1}{r_m}|\lambda_m| \leq 2^{-k} + 2^{-k-1} + \cdots \leq 2^{-k+1},$$

and the conclusion follows. □

Proposition 2.6 *Assume A is multiplicatively convex. Then, every A-nuclear endomorphism $u : E \longrightarrow E$ of the Fréchet A-module E may be written as*

$$u = u' + u''$$

where u' is A-finite and $1 - u''$ is invertible in $L_A(E, E)$.

Proof: Let

$$u(x) = \sum_{m=0}^{+\infty} \lambda_m \langle e^m, x \rangle f_m$$

with λ_m summable, e^m bounded in $L_A(E, A)$, f_m bounded in E. Since e^m is bounded in $L_A(E, A)$ and A is multiplicatively convex, there is a multiplicatively stable absolutely convex subset B_0 of A such that

$$\{\langle e^m, f_n \rangle : m, n \in \mathbb{N}\} \subset \mu B_0$$

where μ is a positive real number. Let us define u_p by setting

$$u_p = \sum_{m=p}^{+\infty} \lambda_m \langle e^m, x \rangle f_m.$$

We have:

$$u_p^k(x) = \sum_{m_k=p}^{+\infty} \cdots \sum_{m_1=p}^{+\infty} \lambda_{m_k} \cdots \lambda_{m_1} \langle e^{m_1}, x \rangle \langle e^{m_2}, f_{m_1} \rangle \cdots \langle e^{m_k}, f_{m_{k-1}} \rangle f_{m_k}.$$

Thus, for any semi-norm q in E and any bounded subset B of E, there is a constant C such that

$$\sup_{x\in B} q(u_p^k(x)) \leq C\Big(\sum_{m=p}^{+\infty} |\lambda_m|\Big)^k \mu^k$$

for any $p, k \in \mathbb{N}$. Choosing p such that $\mu \sum_{m=p}^{+\infty} |\lambda_m| = \varepsilon < 1$, it follows that

$$\sum_{k=0}^{+\infty} u_p^k$$

converges in $L_A(E, E)$ and $(1 - u_p)\sum_{k=0}^{+\infty} u_p^k = 1$. The conclusion follows since

$$u = \sum_{m=0}^{p-1} \lambda_m \langle e^m, x\rangle f_m + u_p.$$

□

Theorem 2.7 [Nuclear perturbation]

Let $u : E \longrightarrow F$, $v : E \longrightarrow F$ be two morphisms of Fréchet A-modules over a multiplicatively convex Fréchet algebra A. Assume u is surjective and v is A-nuclear. Then the A-module coker$(u + v)$ *is finitely generated.*

Proof: Using Proposition 2.3, let us write $-v$ as $u \circ w$ where $w : E \longrightarrow E$ is an A-nuclear morphism. By the preceding proposition,

$$w = w' + w''$$

where w' is A-finite and $1 - w''$ is invertible in $L_A(E, E)$. We have

$$u + v = u - u \circ w = u \circ (1 - w) = u \circ (1 - w'') - u \circ w'.$$

Of course $u' = u \circ (1 - w'')$ is an epimorphism and $v' = u \circ w'$ is A-finite. Since they induce the same morphism from E to coker$(u + v)$, coker$(u + v)$ is finitely generated over A. □

3 Coherence over Fréchet Algebras

We will first consider the case where the base space X is reduced to a point and work with Fréchet modules over a multiplicatively convex Fréchet algebra.

Let us recall the notion of pseudo-coherence introduced in SGA6 [1].

Definition 3.1 Let A be any ring.

A morphism $u^\cdot : E^\cdot \longrightarrow F^\cdot$ of complexes of A-modules is an *a-quasi-isomorphism* if $H^a(u^\cdot)$ is an epimorphism and $H^k(u^\cdot)$ is an isomorphism for $k > a$. Equivalently, we can ask that $H^k(\text{cone}(u)) = 0$ for $k \geq a$.

A complex of A-modules $E^{\cdot}$ is *perfect* if it is quasi-isomorphic to a bounded complex $F^{\cdot}$ such that F^k is a finite free A-module for every $k \in \mathbb{Z}$.

A complex of A-modules is *a-pseudo-coherent* if it is a-quasi-isomorphic to a perfect complex.

One checks that if in a distinguished triangle of complexes of A-modules,

$$E^{\cdot} \longrightarrow F^{\cdot} \longrightarrow G^{\cdot} \underset{+1}{\longrightarrow},$$

$E^{\cdot}$ and $G^{\cdot}$ are a-pseudo-coherent, then so is $F^{\cdot}$.

The global version of our coherence criterion is the following theorem.

Theorem 3.2 *Let A be a multiplicatively convex Fréchet algebra. Assume $(F_i^{\cdot})_{i\in\mathbb{N}}$, $F^{\cdot}$ are complexes of Fréchet A-modules null in degree $\geq b$. Let $u_{i+1,i} : F_i^{\cdot} \longrightarrow F_{i+1}^{\cdot}$ be an A-nuclear morphism and let*

$$u : \varinjlim_{i\in\mathbb{N}} F_i^{\cdot} \longrightarrow F^{\cdot}$$

be a continuous morphism of A-modules. Assume u is an a-quasi-isomorphism. Then $F^{\cdot}$ is a-pseudo-coherent over A.

This result is a consequence of the two following lemmas.

Lemma 3.3 *Under the assumptions and notations of the theorem, let $c \in]a, b]$ and assume $H^k(F^{\cdot}) = 0$ for $k \geq c$. Then*

(i) For $i \gg 0$, there is an A-nuclear homotopy

$$h_i^{\cdot} : F_i^{\cdot} \longrightarrow F^{\cdot}[-1]$$

such that

$$v_i = f_i - d[-1] \circ h_i - h_i[1] \circ d$$

is zero in degrees $\geq c$.

(ii) $H^{c-1}(F^{\cdot})$ is an A-module of finite type.

Proof: (i) We will proceed by decreasing induction on c (the case $c = b$ being clear).

For $i \gg 0$ there is an

$$h_{i+1}^{\cdot} : F_{i+1}^{\cdot} \longrightarrow F^{\cdot}[-1]$$

such that

$$v_{i+1} = f_{i+1} - d[-1] \circ h_{i+1}^{\cdot} - h_{i+1}[1] \circ d$$

is zero in degrees $\geq c + 1$.

We have

$$d^c \circ v_{i+1}^c = v_{i+1}^{c+1} \circ d_{i+1}^c = 0.$$

Thus, we get a morphism
$$v^c_{i+1} : F^c_{i+1} \longrightarrow Z^c.$$
The map
$$v^c_{i+1} \circ f^c_{i+1,i} : F^c_i \longrightarrow Z^c$$
is A-nuclear since $f^c_{i+1,i}$ is A-nuclear.

The differential
$$d^{c-1} : F^{c-1} \longrightarrow Z^c$$
is a continuous epimorphism since $H^c(F^\cdot) = 0$. Since both F^{c-1} and Z^{c-1} are Fréchet A-modules there is an A-nuclear morphism
$$h' : F^c_i \longrightarrow F^{c-1}$$
such that
$$d^{c-1} \circ h' = v^c_{i+1} \circ f^c_{i+1,i}.$$
Let us set
$$h'^k_i = h^k_{i+1} \circ f^k_{i+1,i} \quad \text{if} \quad k \neq c$$
and
$$h'^c_i = h^c_{i+1} \circ f^c_{i+1,i} + h'.$$
By construction the homotopy
$$h'_i : F^\cdot_i \longrightarrow F[-1]$$
is A-nuclear, and
$$v'_i = f_i - d[-1] \circ h'_i - h'_i[1] \circ d$$
is zero in degrees $\geq c$.

(ii) Let us take i great enough so that there is an homotopy
$$h_{i+1} : F^\cdot_{i+1} \longrightarrow F[-1]$$
which is A-nuclear and such that
$$v_{i+1} = f_{i+1} - d[-1] \circ h_{i+1} - h_{i+1}[1] \circ d$$
is zero in degrees $\geq c$. It is clear that v^{c-1}_{i+1} induces a morphism
$$v^{c-1}_{i+1} : F^{c-1}_{i+1} \longrightarrow Z^{c-1}.$$
The arrow
$$v^{c-1}_{i+1} \circ f^{c-1}_{i+1,i} : F^{c-1}_i \longrightarrow Z^{c-1}$$
is A-nuclear since $f^{c-1}_{i+1,i}$ is A-nuclear.

Let us denote by
$$u^i : F^{c-2} \oplus F^{c-1}_i \longrightarrow Z^{c-1}$$

the morphism defined by

$$u^i(a,b) = d^{c-2}(a) + v_{i+1}^{c-1} f_{i+1,i}^{c-1}(b).$$

Since $H^{c-1}(v_{i+1} \circ f_{i+1,i}) = H^{c-1}(f_i)$, the assumptions show that

$$\bigcup_{i\in\mathbb{N}} \operatorname{im} u^i = Z^{c-1}.$$

Since Z^{c-1} and all the $F^{c-2} \oplus F_i^{c-1}$ are Fréchet A-modules, the conjunction of Baire's theorem and Banach's homomorphism theorem shows that

$$\operatorname{im} u^i = Z^{c-1}$$

for some $i \in \mathbb{N}$. The morphism u^i is thus an epimorphism, and, since $u^i(0,b)$ is A-nuclear, the nuclear perturbation theorem shows that $u^i(a,0)$ has a finitely generated cokernel. Hence $H^{c-1}(F^\cdot) = Z^{c-1}/\operatorname{im} d^{c-2}$ is finitely generated over A, and the proof is complete. □

Lemma 3.4 *Under the assumptions and the notations of the theorem, let $c \in [a,b]$ and assume $H^k(F^\cdot) = 0$ for $k \geq c$. Then $F^\cdot$ is a-pseudo-coherent over A.*

Proof: We proceed by increasing induction on c (the case $c = a$ being clear).

We know that $H^{c-1}(F^\cdot)$ is finitely generated over A. Hence, there is an epimorphism

$$u' : A^m \longrightarrow H^{c-1}(F^\cdot).$$

The morphism

$$\varinjlim_{i\in\mathbb{N}} H^{c-1}(F_i^\cdot) \longrightarrow H^{c-1}(F^\cdot)$$

is surjective. Hence, for $i \gg 0$, we can find morphisms

$$v_i : A^m \longrightarrow Z^{c-1}(F_i^\cdot)$$

which are compatible with the $f_{i+1,i}^{c-1}$ (i.e. $f_{i+1,i}^{c-1} \circ v_i = v_{i+1}$) and

$$v : A^m \longrightarrow Z^{c-1}(F^\cdot)$$

such that $f_i^{c-1} \circ v_i = v$. Moreover, we can ask that

$$p_{c-1} \circ v = u'$$

where $p_{c-1} : Z^{c-1}(F^\cdot) \longrightarrow H^{c-1}(F^\cdot)$ is the canonical projection.

This construction gives us an inductive system of morphisms of complexes of Fréchet A-modules

$$w_i : A^m[-(c-1)] \longrightarrow F_i^\cdot$$

and a morphism

$$w^\cdot : A^m[-(c-1)] \longrightarrow F^\cdot$$

such that $w^\cdot = f_i^\cdot \circ w_i^\cdot$ which induce an epimorphism

$$H^{c-1}(w^\cdot) : A^m \longrightarrow H^{c-1}(F^\cdot).$$

It is clear that the mapping cones of w and w_i are complexes of Fréchet A-modules null in degrees $\geq b$, the transition maps

$$\mathrm{cone}^\cdot(w_i) \longrightarrow \mathrm{cone}^\cdot(w_{i+1})$$

induced by $f_{i+1,i}$ being A-nuclear.

We get the following commutative diagram

$$\begin{array}{ccccccc}
A^m[-(c-1)] & \longrightarrow & \varinjlim_{i\in\mathbb{N}} F_i & \longrightarrow & \varinjlim_{i\in\mathbb{N}} \mathrm{cone}^\cdot(w_i) & \underset{+1}{\longrightarrow} \\
\downarrow & & \downarrow & & \downarrow & \\
A^m[-(c-1)] & \longrightarrow & F^\cdot & \longrightarrow & \mathrm{cone}^\cdot(w) & \underset{+1}{\longrightarrow}
\end{array}$$

where the lines are distinguished triangles. Since the first two vertical arrows are a-quasi-isomorphisms, so is the third one. From the second line we get the exact sequence

$$H^{c-1}(A^m[-(c-1)]) \underset{u^\cdot}{\longrightarrow} H^{c-1}(F^\cdot) \longrightarrow H^{c-1}(\mathrm{cone}^\cdot(w)) \longrightarrow 0$$

and since u is surjective, $H^{c-1}(\mathrm{cone}^\cdot(w)) = 0$. Applying the induction hypothesis to $\mathrm{cone}^\cdot(w^\cdot)$, we see that it is an a-pseudo-coherent complex. Since $A^m[-(c-1)]$ is perfect, the conclusion follows easily. □

4 Proof of Theorem 1.1

We will now extend the result established in the preceding section to the case of Fréchet modules over a sheaf of Fréchet algebras. Since the proof follows the same lines as in the absolute case, we shall give, in a few lemmas, the tools needed for the extension, and leave most of the obvious translation process to the reader.

Let X be a topological space with countable basis.

Recall that a *sheaf of Fréchet spaces* $\mathcal{F}$ on X is simply a sheaf with values in the category of Fréchet spaces and continuous linear maps. This means that for any countable covering $\mathcal{U}$ of an open subset U of X, $\Gamma(U;\mathcal{F})$ is the topological kernel of the usual Čech map

$$\prod_{V\in\mathcal{U}} \Gamma(V;\mathcal{F}) \longrightarrow \prod_{V,W\in\mathcal{U}} \Gamma(V\cap W;\mathcal{F}).$$

A *morphism of Fréchet sheaves* on X is a usual morphism of sheaves of $\mathbb{C}$-vector spaces which is continuous on the sections.

A *sheaf of Fréchet algebras* $\mathcal{A}$ on X is a sheaf of Fréchet spaces endowed with a $\mathbb{C}$-bilinear continuous multiplication

$$\cdot : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$$

which is associative and admits a unit $1 \in \Gamma(X; \mathcal{A})$.

A sheaf of Fréchet algebras on X is *multiplicatively convex* if, for any open subset U of X, any bounded subset B of $\Gamma(U; \mathcal{A})$ and any $x \in X$, we can find a neighborhood V of x and a multiplicatively stable absolutely convex bounded subset B' of $\Gamma(V; \mathcal{A})$ absorbing $B_{|V}$.

A *Fréchet module* over a sheaf of Fréchet algebras $\mathcal{A}$ on X is a Fréchet sheaf $\mathcal{F}$ on X endowed with a structure of $\mathcal{A}$-module in such a way that the action map

$$\cdot : \mathcal{A} \times \mathcal{F} \longrightarrow \mathcal{F}$$

is continuous.

Let U be an open subset of X. A Fréchet module $\mathcal{F}$ over a sheaf of Fréchet algebras $\mathcal{A}$ on X *has enough sections on* U if for any $x \in U$ and any $f_x \in \mathcal{F}_x$ we may find a neighborhood V of x in X and

- λ_m : a summable sequence of complex numbers,
- f_m : a bounded sequence in $\Gamma(U; \mathcal{F})$,
- a_m : a bounded sequence in $\Gamma(V; \mathcal{A})$,

such that $f_x = g_x$ where $g \in \Gamma(V; \mathcal{F})$ is defined by

$$g = \sum_{m=0}^{\infty} \lambda_m a_m (f_m)_{|V}.$$

A *morphism of Fréchet modules* over a sheaf of Fréchet algebras $\mathcal{A}$ on X is a morphism of Fréchet sheaves which is $\mathcal{A}$-linear. We denote by $L_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ the set of morphisms from the Fréchet $\mathcal{A}$-module $\mathcal{E}$ to the Fréchet $\mathcal{A}$-module $\mathcal{F}$. It is naturally endowed with a structure of locally convex topological vector space with the semi-norms

$$q_B(u) = \sup_{e \in B} q(u_{|V}(e))$$

where V is an open subset of X, B a bounded subset of $\Gamma(V; \mathcal{E})$ and q a semi-norm of $\Gamma(V; \mathcal{F})$.

A morphism $u : \mathcal{E} \longrightarrow \mathcal{F}$ between two Fréchet $\mathcal{A}$-modules is $\mathcal{A}$*-nuclear* if for any $x \in X$ we can find an open neighborhood V of x and

- λ_m a summable sequence of complex numbers,
- e^m a bounded sequence of $L_{\mathcal{A}_{|V}}(\mathcal{E}_{|V}, \mathcal{A}_{|V})$,
- f_m a bounded sequence of $\Gamma(V; \mathcal{F})$,

such that

$$u(s) = \sum_{m=0}^{+\infty} \lambda_m e^m(s) f_{m|W}$$

for any open subset $W \subset V$ and any $s \in \Gamma(W, \mathcal{E})$. If in the preceding definition we use only finite sequences, we get the notion of $\mathcal{A}$*-finite* morphism.

Lemma 4.1 *Let $u : \mathcal{E} \longrightarrow \mathcal{F}$ be a morphism of Fréchet sheaves on X and let $x \in X$. Assume $u_x : \mathcal{E}_x \longrightarrow \mathcal{F}_x$ is surjective. Then any neighborhood U of x contains a neighborhood V such that*

$$\forall f \in \Gamma(U;\mathcal{F}) \quad \exists e \in \Gamma(V;\mathcal{E}) \qquad f_{|V} = u(e).$$

Proof: Of course, we may assume U is open. Let $(V_m)_{m\in\mathbb{N}}$ denote a countable fundamental system of open neighborhoods of x in U. For any $m \in \mathbb{N}$ consider the cartesian square

$$\begin{tikzcd} \Gamma(V_m;\mathcal{E}) \arrow[r, "u"] & \Gamma(V_m;\mathcal{F}) \\ G_m \arrow[u] \arrow[r, "u_m"'] \arrow[ur, phantom, "\square"] & \Gamma(U;\mathcal{F}). \arrow[u, "|V_m"'] \end{tikzcd}$$

By construction, G_m is a Fréchet space and the hypothesis implies that

$$\bigcup_{m\in\mathbb{N}} \operatorname{im} u_m = \Gamma(U;\mathcal{F}).$$

Hence, there is $m \in \mathbb{N}$ such that $\operatorname{im} u_m = \Gamma(U;\mathcal{F})$. This relation gives us the requested result. □

Using this lemma, the reader will easily check that in the sheaf version of Proposition 2.3, Proposition 2.6 and Theorem 2.7 the conclusions remain true locally.

Lemma 4.2 *Let $\mathcal{A}$ be a sheaf of Fréchet algebras on X, and let*

$$u_i : \mathcal{F}_i \longrightarrow \mathcal{F} \qquad (i \in \mathbb{N})$$

be a family of morphisms of Fréchet $\mathcal{A}$-modules. Assume that

$$\bigcup_{i\in\mathbb{N}} \operatorname{im} u_i = \mathcal{F}.$$

Assume moreover that $x \in X$ has a fundamental system of neighborhoods on which each $\mathcal{F}_i$ has enough sections. Then, for any $x \in X$, there is a neighborhood V of x and an integer $i \in \mathbb{N}$ such that

$$(u_i)_{|V} : (\mathcal{F}_i)_{|V} \longrightarrow \mathcal{F}_{|V}$$

is a sheaf epimorphism.

Proof: Denote by $\mathcal{V}$ a fundamental system of open neighborhoods of x on which each $\mathcal{F}_i$ has enough sections. Notice that $\mathcal{F}$ has also enough sections on any $V \in \mathcal{V}$.

Working as in the preceding lemma, it is easy to show that any $U \in \mathcal{V}$ contains a $V \in \mathcal{V}$ such that the map

$$\Gamma(V,\mathcal{F}_i) \times_{\Gamma(V,\mathcal{F})} \Gamma(U;\mathcal{F}) \longrightarrow \Gamma(U;\mathcal{F})$$

is surjective for some $i \in \mathbb{N}$. Let W be an open subset of V and assume

- λ_m is a summable sequence of complex numbers,
- a_m is a bounded sequence in $\Gamma(W; \mathcal{A})$,
- f_m is a bounded sequence in $\Gamma(U; \mathcal{F})$.

Then, we can find a bounded sequence $g_m \in \Gamma(V, \mathcal{F}_i)$ such that $(f_m)_{|V} = u_i(g_m)$. Hence,

$$\sum_{m=0}^{+\infty} \lambda_m a_m (f_m)_{|W} = u_i \left(\sum_{m=0}^{+\infty} \lambda_m a_m (g_m)_{|W} \right).$$

Combining this fact with the fact that $\mathcal{F}$ has enough sections on U, we see that

$$(u_i)_{|W} : (\mathcal{F}_i)_{|V} \longrightarrow \mathcal{F}_{|V}$$

is a sheaf epimorphism. □

With the preceding lemmas at hand, we can prove Theorem 1.1 by working as in the proof of Theorem 3.2 but in the context of sheaves. For the sake of brevity we leave this straightforward rewriting to the reader.

5 An application to analytic geometry

In this section, we will give an example of application of our finiteness criterion in the case of topological modules over the algebra $\mathcal{O}_S$ of holomorphic functions on a complex manifold S. This corollary is used in [7] to get the relative finiteness theorem for elliptic pairs.

Let S be a complex analytic manifold. Recall that the sheaf $\mathcal{O}_S$ of holomorphic functions on S is a multiplicatively convex sheaf of Fréchet algebras over S (see [4]). Also recall that if V is a relatively compact open subset of a Stein open subset U of X, then the restriction map

$$\Gamma(U; \mathcal{O}_S) \longrightarrow \Gamma(V; \mathcal{O}_S)$$

is $\mathbb{C}$-nuclear. From this it follows easily that $\Gamma(U; \mathcal{O}_U)$ is a Fréchet nuclear (FN) space and that $\Gamma(\overline{V}, \mathcal{O}_S)$ is a dual Fréchet nuclear (DFN) space.

Following [6], an FN-free (resp. a DFN-free) $\mathcal{O}_S$-module is a module isomorphic to $E \hat{\otimes} \mathcal{O}_S$ for some Fréchet nuclear (resp. dual Fréchet nuclear) space E.

Corollary 5.1 *Let $\mathcal{M}^\cdot$ (resp. $\mathcal{N}^\cdot$) be a complex of DFN-free (resp. FN-free) $\mathcal{O}_S$-modules. Assume $\mathcal{M}^\cdot$ and $\mathcal{N}^\cdot$ are bounded from above and*

$$u^\cdot : \mathcal{M}^\cdot \longrightarrow \mathcal{N}^\cdot$$

is a continuous $\mathcal{O}_S$-linear quasi-isomorphism. Then $\mathcal{M}^\cdot$ and $\mathcal{N}^\cdot$ have $\mathcal{O}_S$-coherent cohomology.

Proof: Let E be a (DFN)-space and set $\mathcal{E} = E \hat{\otimes} \mathcal{O}_S$. It is well-known that we can find a countable inductive system $(F_n, f_{mn})_{n\in\mathbb{N}}$ of Fréchet spaces with nuclear transition maps such that

$$\varinjlim_{n\in\mathbb{N}} F_n \xrightarrow{\sim} E.$$

Let us denote by $f_n : F_n \longrightarrow E$ the projection to the limit. Since E is separated, $\ker f_n$ is a Fréchet subspace of F_n, and it follows from the equality

$$\ker f_n = \bigcup_{m\geq n} \ker f_{mn}$$

that $\ker f_n = \ker f_{mn}$ for $m \gg 0$. The sheaf $\mathcal{F}_n = F_n \hat{\otimes} \mathcal{O}_S$ is obviously a sheaf of Fréchet modules over the Fréchet algebra $\mathcal{O}_S$ and each transition map $\phi_{mn} = f_{mn} \hat{\otimes} \mathrm{id}_{\mathcal{O}_S}$ is clearly $\mathcal{O}_S$-nuclear. Using the maps $\phi_n = f_n \hat{\otimes} \mathrm{id}_{\mathcal{O}_S}$, we get the isomorphism

$$\varinjlim_{n\in\mathbb{N}} \mathcal{F}_n \xrightarrow{\sim} \mathcal{E}.$$

Moreover, locally on S,

$$\ker \phi_n = \ker \phi_{mn} \quad \text{for} \quad m \gg 0. \tag{5.1}$$

Now, let E^0, E^1 be two (DFN) spaces and consider a continuous $\mathcal{O}_S$-linear morphism

$$u : \mathcal{E}^0 \longrightarrow \mathcal{E}^1$$

between the associated DFN-free $\mathcal{O}_S$-modules. As above,

$$\mathcal{E}^0 = \varinjlim_{n\in\mathbb{N}} \mathcal{F}_n^0, \quad \mathcal{E}^1 = \varinjlim_{n\in\mathbb{N}} \mathcal{F}_n^1,$$

where $(\mathcal{F}_n^0, \phi_{mn}^0)$, $(\mathcal{F}_n^1, \phi_{mn}^1)$ are inductive systems of FN-free $\mathcal{O}_S$-modules with $\mathcal{O}_S$-nuclear transition maps. Let us fix $n \in \mathbb{N}$. Working as in Lemma 4.1 it is easy to see that for $m \gg 0$, there is a map $u_n : \mathcal{F}_n^0 \longrightarrow \mathcal{F}_m^1$ such that $u \circ \phi_n = \phi_m \circ u_n$. Hence, locally, thanks to (5.1), it is possible to find a strictly increasing sequence $k_n \in \mathbb{N}$ and a morphism

$$(u_n) : (\mathcal{F}_n^0) \longrightarrow (\mathcal{F}_{k_n}^1)$$

of inductive systems such that $\varinjlim\limits_{m\in\mathbb{N}} u_n = u$.

Finally, assume we have a complex $\mathcal{E}^\cdot$ of DFN-free $\mathcal{O}_S$-modules. Using the preceding procedure and (5.1) one sees that it is possible to find an inductive system of complexes of FN-free $\mathcal{O}_S$-modules $(\mathcal{F}_n^\cdot, \phi_{mn}^\cdot)_{n\in\mathbb{N}}$ with $\mathcal{O}_S$-nuclear transition maps such that

$$\varinjlim_{n\in\mathbb{N}} \mathcal{F}_n^\cdot \simeq \mathcal{E}^\cdot.$$

The conclusion then follows easily from Theorem 1.1 by using the well-known fact that FN-free $\mathcal{O}_S$-modules have enough sections on polydiscs. $\square$

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