ANNE BAUVAL

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Introduction

Woronowicz ([W1], [W2]) defined a family (in the set-theoretical sense) of quantum groups $(SU_\mu(2))_{\mu\in\mathbf{R}^*}$. (For $\mu = 1$, the C^*-algebra A_μ underlying $SU_\mu(2)$ is merely the algebra $C(SU(2))$ of continuous functions from the classical group $SU(2)$ into $\mathbf{C}$).

"Forgetting" the group structure of $SU(2)$, Sheu ([S1]) used the Weyl calculus to construct a continuous deformation of the Poisson structure of $C^\infty(SU(2))$, where the fibres are precisely the C^*-algebras A_μ.

Unifying these two points of view, we shall do the following :

(§1) put Woronowicz's $SU_\mu(2)$'s together into a continuous field of quantum groups,

(§2) construct a deformation of Poisson-$SU(2)$ in the underlying continuous field of C^*-algebras A_μ,

(§3) prove that such a deformation is unique among deformations fulfilling suitable requirements,

(§4) prove that Sheu's deformation fulfills these requirements, and compare it in detail with our deformation.

Paragraphs 2, 3 and 4 will be achieved by working, as Sheu did, at the more elementary level of Poisson-deformations of the disc, which is a "slice" of $SU(2)$.

1 Continuous structure on the family of quantum groups $SU_\mu(2)$

Definition 1.1 *([W1], [W2]) For any $\mu \in \mathbf{R}$, A_μ is the enveloping C^*-algebra of the involutive $\mathbf{C}$-algebra $\mathcal{A}_\mu$ defined by two generators α_μ, γ_μ and relations :*

$$\begin{aligned} \alpha_\mu^*\alpha_\mu + \gamma_\mu^*\gamma_\mu &= 1 \quad (1_\mu) \\ \alpha_\mu\alpha_\mu^* + \mu^2\gamma_\mu^*\gamma_\mu &= 1 \quad (2_\mu) \end{aligned} \qquad \begin{aligned} \gamma_\mu^*\gamma_\mu &= \gamma_\mu\gamma_\mu^* \quad (3_\mu) \\ \alpha_\mu\gamma_\mu &= \mu\gamma_\mu\alpha_\mu \quad (4_\mu) \\ \alpha_\mu\gamma_\mu^* &= \mu\gamma_\mu^*\alpha_\mu \quad (5_\mu) \end{aligned}$$

and if $\mu \neq 0$, the quantum group $SU_\mu(2)$ is defined by the unitary matrix

$$\begin{pmatrix} \alpha_\mu & -\mu\gamma_\mu^* \\ \gamma_\mu & \alpha_\mu^* \end{pmatrix}$$

In order to endow the family of these $SU_\mu(2)$'s with a structure of continuous field of quantum groups ([B1], [B2]), we just have to endow the family $(A_\mu)_{\mu\in\mathbf{R}^*}$ with a structure of continuous field of C^*-algebras in such a way that the sections $\mu \mapsto \alpha_\mu$, $\mu \mapsto \gamma_\mu$ are continuous. (We shall even do a little more : this field will also be defined at $\mu = 0$).

Definition 1.2 *A is the universal C^*-algebra defined by three generators α, γ, μ and relations :*

$$\begin{array}{rcll} \alpha^*\alpha + \gamma^*\gamma &=& 1 & (1) \\ \alpha\alpha^* + \mu^2\gamma^*\gamma &=& 1 & (2) \end{array} \qquad \begin{array}{rcll} \gamma^*\gamma &=& \gamma\gamma^* & (3) \\ \alpha\gamma &=& \mu\gamma\alpha & (4) \\ \alpha\gamma^* &=& \mu\gamma^*\alpha & (5) \end{array}$$

$$\begin{array}{cccl} \mu & \textit{commutes with} & \alpha, \gamma & (6) \\ -1 & \leq \mu \leq & 1 & (7). \end{array}$$

$\mathcal{A}$ is the *-subalgebra generated by α, γ, μ.

The restriction of the parameter μ to $[-1;1]$ is harmless since for $\mu \neq 0$, there is an isomorphism of *quantum groups* (not only of C^*-algebras) between $SU_\mu(2)$ and $SU_{1/\mu}(2)$ (sending α_μ to $\alpha_{1/\mu}{}^*$ and γ_μ to $-\frac{1}{\mu}\gamma_{1/\mu}{}^*$) : using this isomorphism it is then easy to extend to $\mathbf{R}$ the field on $[-1;1]$ which we are going to construct.

Moreover, such a restriction of the parameter is necessary, otherwise the generator μ would not be bounded, hence the involutive algebra defined by these generators and relations would not have a C^*-envelope.

We shall construct a field of C^*-algebras over $[-1;1]$, using the natural morphism from $C([-1;1])$ into the center of A. (This morphism is given by relations (6) and (7)). By a slight generalization of the Dauns-Hofmann theorem, proved by Dupré and Gilette ([DG], proposition 1.3 and corollary 2.2) and quoted in [Ri], there is a unique upper semi-continuous field related to the $C([-1;1])$-C^*- algebra A in the following way.

Definition 1.3 *ξ is the upper semi-continuous field of C^*-algebras on $[-1;1]$ such that :*

- *the fiber of ξ at x is A/xA (x denotes here both a point in $[-1;1]$ and the ideal of functions in $C([-1;1])$ vanishing at this point)*
- *the total space $\sqcup_{x\in[-1;1]}A/xA$ of ξ is endowed with a topology such that the continuous sections of ξ are the sections of the form $x \mapsto a \bmod xA$, for any $a \in A$.*

Using the universal properties of A and of the A_μ's, one easily proves the following relationship between our field ξ and Woronowicz's family $(A_\mu)_{\mu\in[-1;1]}$.

Proposition 1.4 *For any $\mu \in [-1;1]$, the fiber at μ of the field ξ is naturally isomorphic to the C^*-algebra A_μ. This family of isomorphisms identifies the two continuous sections of the field ξ associated to $\alpha, \gamma \in A$ with the two sections $\mu \mapsto \alpha_\mu, \mu \mapsto \gamma_\mu$ of the family $(A_\mu)_{\mu\in[-1;1]}$.*

Before introducing another field ζ with more elementary fibers, and proving the (lower) continuity of both fields ξ and ζ, let us first get rid of the case $\mu \leq 0$: we shall prove that the study of $\xi_{|[-1,0]}$ may be reduced to the study of $\xi_{|[0;1]}$ (and conversely), by a property with "fractal" flavour.

Proposition 1.5 *Let $\varepsilon_{i,j}$ $(1 \le i,j \le 2)$ be the canonical generators of $M_2(\mathbf{C})$ and $i_\mu : A_{-\mu} \to M_2(\mathbf{C}) \otimes A_\mu$ the morphism defined by :*

$$i_\mu(\alpha_{-\mu}) = (\varepsilon_{1,2} + \varepsilon_{2,1}) \otimes \alpha_\mu, \qquad i_\mu(\gamma_{-\mu}) = (\varepsilon_{1,1} - \varepsilon_{2,2}) \otimes \gamma_\mu.$$

For any $\mu \in \mathbf{R}$, i_μ is an embedding.

Proof. Let D be the subalgebra of $M_2(\mathbf{C}) \otimes M_2(\mathbf{C})$ generated by the two elements $P = \varepsilon_{1,1} \otimes \varepsilon_{1,1} + \varepsilon_{2,2} \otimes \varepsilon_{2,2}$ and $Q = \varepsilon_{1,2} \otimes \varepsilon_{1,2} + \varepsilon_{2,1} \otimes \varepsilon_{2,1}$ and similarly, D' the subalgebra generated by $P' = \varepsilon_{1,1} \otimes \varepsilon_{2,2} + \varepsilon_{2,2} \otimes \varepsilon_{1,1}$ and $Q' = \varepsilon_{1,2} \otimes \varepsilon_{2,1} + \varepsilon_{2,1} \otimes \varepsilon_{1,2}$. Let $\varphi : D \to \mathbf{C}$ be the morphism such that $\varphi(P) = \varphi(Q) = 1$. One easily checks that the image of $j = (\mathrm{id}_{M_2(\mathbf{C})} \otimes i_{-\mu}) \circ i_\mu$ is included in $(D \oplus D') \otimes A_{-\mu}$ and that $((\varphi \oplus 0) \otimes \mathrm{id}_{A_{-\mu}}) \circ j = \mathrm{id}_{A_{-\mu}}$.

We shall now reduce the study of the A_μ's (quantum $SU(2)$) to the study of more elementary C^*-algebras B_μ (quantum disc). Let us recall the two results which naturally led us to this reduction.

Theorem 1.6 *([W2] appendix 2) A_μ is isomorphic to A_0, for any $\mu \in]-1,1[$.*

The isomorphism $T_\mu : A_\mu \xrightarrow{\sim} A_0$ was defined by Woronowicz as follows :

$$\begin{aligned} T_\mu(\alpha_\mu) &= \textstyle\sum_{n=0}^{\infty} \frac{(1-\mu^2)\mu^{2n}}{\sqrt{1-\mu^{2n+2}}+\sqrt{1-\mu^{2n}}} \alpha_0^{*n} \alpha_0^{n+1} \\ \text{and} \quad T_\mu(\gamma_\mu) &= \textstyle\sum_{n=0}^{\infty} \mu^n \alpha_0^{*n} \gamma_0 \alpha_0^n \end{aligned}$$

Theorem 1.7 *([S1] proposition 1.1) Let*

$$0 \longrightarrow C(D) \longrightarrow C(\overline{D}) \xrightarrow{\sigma_1} C(\mathbf{T}) \longrightarrow 0$$

be the exact sequence of the unit disc and

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(\mathcal{S}) \xrightarrow{\sigma_0} C(\mathbf{T}) \longrightarrow 0$$

be the Tœplitz exact sequence. Set $B_1 = C(\overline{D})$ and $B_0 = C^*(\mathcal{S})$. For $\mu = 1$ or 0, A_μ is isomorphic to the algebra of continuous functions $f : \mathbf{T} \to B_\mu$ such that $\sigma_\mu(f(u))$ does not depend on $u \in \mathbf{T}$.

For $\mu = 1$, the isomorphism consists in identifying $SU(2)$ with a family of discs $(D_u)_{u \in \mathbf{T}}$, glued together along their boundary circle :

$$(u, Z) \in \mathbf{T} \times \overline{D} \text{ is identified to } \begin{pmatrix} Z & -\overline{u}c \\ uc & \overline{Z} \end{pmatrix}, \text{ with } c = \sqrt{1 - |Z|^2}.$$

(This "slicing" of $SU(2)$ is compatible with the Poisson structure, cf §3 and 4).

For $\mu = 0$, let us recall the Tœplitz exact sequence. $C^*(\mathcal{S})$ is the C^*-algebra generated by the unilateral shift operator $\mathcal{S}$. $\mathcal{S}\mathcal{S}^*$ is equal to $1 - p$, p being a rank one projection. The closed ideal of $C^*(\mathcal{S})$ generated by p is the algebra $\mathcal{K}$ of compact operators, and $C^*(\mathcal{S})/\mathcal{K}$ is isomorphic to $C(\mathbf{T})$, the isomorphism sending the unitary generator $(\mathcal{S} \bmod \mathcal{K}) \in C^*(\mathcal{S})/\mathcal{K}$ to $\mathrm{id}_\mathbf{T}$.

In both cases $\mu = 1$ or 0, the embedding $A_\mu \to C(\mathbf{T}, B_\mu)$ sends

α_μ to $(u \mapsto \overline{\alpha}_\mu)$ and γ_μ to $(u \mapsto u\overline{\gamma}_\mu)$, with

$$\begin{aligned} \overline{\alpha}_1 &= (Z \mapsto Z), & \overline{\gamma}_1 &= (Z \mapsto \sqrt{1 - |Z|^2}) \\ \overline{\alpha}_0 &= \mathcal{S}^*, & \overline{\gamma}_0 &= p. \end{aligned}$$

Using the proof of theorem 1.6, one gets the following "generalization" of theorem 1.7 for free (we pass from the case $\mu = 0$ to the "more general" case $|\mu| < 1$ by a rather silly renaming ; the only nontrivial assertion of the following corollary is the first one, which justifies this renaming).

Corollary 1.8 *For $|\mu| < 1$, let us denote by*

- $\overline{\alpha}_\mu, \overline{\gamma}_\mu$ *the elements of $C^*(\mathcal{S})$ defined as series in $\overline{\alpha}_0, \overline{\gamma}_0$ by the same formulas as in theorem 1.6, where $T_\mu(\alpha_\mu), T_\mu(\gamma_\mu)$ were defined as series in α_0, γ_0*
- $\mathcal{B}_\mu$ *the involutive subalgebra generated by $\overline{\alpha}_\mu, \overline{\gamma}_\mu$ and*
- B_μ *its closure in $C^*(\mathcal{S})$.*

For $|\mu| < 1$, B_μ is equal to $C^*(\mathcal{S})$ and the closed ideal of B_μ generated by $\overline{\gamma}_\mu$ is $\mathcal{K}$. Moreover, for $-1 < \mu \leq 1$, there is a morphism $\sigma_\mu : B_\mu \to C(\mathbf{T})$ such that $\sigma_\mu(\overline{\alpha}_\mu^*) = \mathrm{id}_{\mathbf{T}}$ and such that the sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow B_\mu \xrightarrow{\sigma_\mu} C(\mathbf{T}) \longrightarrow 0$$

is exact, and A_μ is isomorphic to the algebra of continuous functions $f : \mathbf{T} \to B_\mu$ such that $\sigma_\mu(f(u))$ does not depend on $u \in \mathbf{T}$.

Remark. For $\mu = 1$, the morphism $\sigma_1 : C(\overline{D}) \to C(\mathbf{T})$ which we are choosing is *not the mere restriction* but (in order to make the notations fit together) $\sigma_1(f)(v) = f(\overline{v})$. This remark will be important in lemma 4.5.

Since $\overline{\gamma}_0 = p \geq 0$, the definition of $\overline{\gamma}_\mu$ as a series makes it self adjoint. If $\mu \geq 0$ we even get : $\overline{\gamma}_\mu \geq 0$, hence (under the identification of A_μ given above) $|\gamma_\mu| = \left|(u \mapsto u\overline{\gamma}_\mu)\right| = (u \mapsto \overline{\gamma}_\mu)$. Using this fact and the universal property of A_μ, one easily proves the following proposition.

Proposition 1.9 *For $0 \leq \mu \leq 1$, B_μ is isomorphic to the universal C^*-algebra defined by generators $\overline{\alpha}_\mu, \overline{\gamma}_\mu$ and relations :*

the relations (1_μ)–(5_μ) (cf definition 1.1)

the additional relation : $\overline{\gamma}_\mu \geq 0$.

Remark. Instead of adding a relation and looking at B_μ as a quotient of A_μ, one may also prove (but this will not be used) that B_μ is isomorphic to the C^*-subalgebra of A_μ generated by α_μ, and characterize B_μ as the universal C^*-algebra defined by one generator α_μ and one relation (deduced from relations (1_μ) and (2_μ) by eliminating γ_μ) ([NN]).

Paraphrasing definitions 1.2 and 1.3 and proposition 1.4, we can now define the field of B_μ's.

Definition 1.10 • *B is the C^*-algebra defined by generators $\overline{\alpha}, \overline{\gamma}$ and $\overline{\mu}$ and relations :*

$$\begin{array}{rclcrcll} \overline{\alpha}^*\overline{\alpha} + \overline{\gamma}^2 & = & 1 & (\overline{1}) & \overline{\gamma} & \geq & 0 & (\overline{3}) \\ \overline{\alpha}\,\overline{\alpha}^* + \overline{\mu}^2\overline{\gamma}^2 & = & 1 & (\overline{2}) & \overline{\alpha}\overline{\gamma} & = & \overline{\mu}\overline{\gamma}\overline{\alpha} & (\overline{4}) \end{array}$$

$$\begin{array}{cccc} \overline{\mu} & \textit{commutes with} & \overline{\alpha}, \overline{\gamma} & (\overline{6}) \\ 0 & \leq \overline{\mu} \leq & 1 & (\overline{7}) \end{array}$$

- *$\mathcal{B}$ is the involutive subalgebra of B generated by $\overline{\alpha}, \overline{\gamma}, \overline{\mu}$ and*

- *ζ is the (upper semi-continuous) field on* $[0;1]$ *associated to B, with fibers B_μ.*

Proposition 1.11 *ζ is trivial on* $[0;1)$ *and ξ is trivial on* $(-1;1)$.

Proof. By uniqueness of the field defined by a total family of sections ([DG], proposition 1.3), in order to prove the first assertion it suffices to show that for any $b \in \mathcal{B}$ (viewed as a continuous section b of ζ) the restriction of b to $[0;1)$ is a continuous section of the trivial field $C^*(\mathcal{S}) \times [0;1)$ i.e. b is a continuous map from $[0;1)$ to $C^*(\mathcal{S})$. It is sufficient to prove this for $b = \overline{\alpha}$ or $b = \overline{\gamma}$. This is then a consequence of the definition of $\overline{\alpha}_\mu, \overline{\gamma}_\mu$ in corollary 1.8. This proves the first assertion. The trivialness of ξ on $[0;1)$ is then a consequence of corollary 1.8, and its trivialness on $(-1;0]$ may be deduced from this and from proposition 1.5. (There is also a direct proof of the second assertion, analogous to that of the first one).

Before studying the behaviour of these two fields at the extreme points, let us make explicit the representations of the B_μ's and A_μ's deduced from the canonical representation of $C^*(\mathcal{S})$ by corollary 1.8. $\mathcal{S}$ acts on the space $\mathcal{H}$ with Hilbert basis $(e_n)_{n\in\mathbf{N}}$ by $\mathcal{S}(e_n) = e_{n+1}$, $\overline{\alpha}_0 = \mathcal{S}^*$ and $\overline{\gamma}_0 = p$ hence by definition of $\overline{\alpha}_\mu$, $\overline{\gamma}_\mu$ (corollary 1.8) we get, for $|\mu| < 1$:

$$\overline{\alpha}_\mu(e_n) = \sqrt{1-\mu^{2n}}e_{n-1} \qquad \overline{\gamma}_\mu(e_n) = \mu^n e_n.$$

¿From this faithful representation of B_μ on $\mathcal{H}$ we deduce a faithful representation π_μ of $C(\mathbf{T}, B_\mu)$ (hence of A_μ) on $L^2(\mathbf{T}, \mathcal{H})$: since $\alpha_\mu(u) = \overline{\alpha}_\mu$ and $\gamma_\mu(u) = u\overline{\gamma}_\mu$ ($\forall u \in \mathbf{T}$), we get, on the basis $(\psi_{n,k})_{k\in\mathbf{Z},n\in\mathbf{N}}$ of $L^2(\mathbf{T},\mathcal{H})$ defined by $\psi_{n,k}(u) = u^k e_n$:

$$\alpha_\mu(\psi_{n,k}) = \sqrt{1-\mu^{2n}}\psi_{n-1,k} \qquad \gamma_\mu(\psi_{n,k}) = \mu^n\psi_{n,k+1}.$$

($\pi_{-\mu}$ and π_μ are related by the morphism i_μ defined in proposition 1.5 : $\pi_\mu \circ i_\mu$ is equivalent to $\pi_{-\mu} \oplus \pi_{-\mu}$). Let us treat similarly the case $\mu = \pm 1$. (π_{-1} and π_1 will be related by the same formula hence it is sufficient to check that π_1 is faithful, which will be obvious by construction). The representations of B_μ and A_μ for $\mu = \pm 1$ will be constructed as direct sums : $\bigoplus\limits_{t\in[0;1]} \sigma_{\mu,t}$ for B_μ and $\pi_\mu = \bigoplus\limits_{t\in[0;1]} \sigma'_{\mu,t}$ for A_μ. Let us first define $\sigma_{1,t}$ and $\sigma'_{1,t}$. For any $t \in [0;1]$, let $\sigma_{1,t}$ be the representation of $B_1 = C(\overline{D})$ on $\mathcal{H}' = L^2(\mathbf{T})$ given by restricting an element of $C(\overline{D})$ to the circle of radius $\sqrt{1-t^2}$ and making this element of $C(\mathbf{T})$ act on $\mathcal{H}'$ by multiplication. Since $\overline{\alpha}_1(Z) = Z$ and $\overline{\gamma}_1(Z) = \sqrt{1-|Z|^2}$, we get, on the basis $(e'_n)_{n\in\mathbf{Z}}$ of $\mathcal{H}'$ defined by $e'_n(v) = v^{-n}$:

$$\sigma_{1,t}(\overline{\alpha}_1)(e'_n) = \sqrt{1-t^2}e'_{n-1} \qquad \sigma_{1,t}(\overline{\gamma}_1)(e'_n) = te'_n.$$

¿From this representation $\sigma_{1,t}$ of B_1 on $\mathcal{H}'$ we deduce a representation $\sigma'_{1,t}$ of $C(\mathbf{T}, B_1)$ (hence of A_1) on $L^2(\mathbf{T}, \mathcal{H}')$. On the basis $(\psi'_{n,k})_{k,n\in\mathbf{Z}}$ of $L^2(\mathbf{T}, \mathcal{H}')$ defined by $\psi'_{n,k}(u) = u^k e'_n$,

$$\sigma'_{1,t}(\alpha_1)(\psi'_{n,k}) = \sqrt{1-t^2}\psi'_{n-1,k} \qquad \sigma'_{1,t}(\gamma_1)(\psi'_{n,k}) = t\psi'_{n,k+1}.$$

The definitions of $\sigma_{-1,t}$ and $\sigma'_{-1,t}$ are analogous, but with t multiplied by $(-1)^n$.

These representations of the A_μ's (for $|\mu| \leq 1$) are the ones used in [W2] (theorem 1.2) to show that $\{\alpha_\mu^k \gamma_\mu^m \gamma_\mu^{*n} | k, m, n \in \mathbf{N}\} \cup \{\alpha_\mu^{*k} \gamma_\mu^m \gamma_\mu^{*n} | k, m, n \in \mathbf{N}, k \neq 0\}$ is a linear basis of $\mathcal{A}_\mu$, for $0 < |\mu| \leq 1$. This theorem is deducible from the following lemma, which may be proved by the same arguments, using the representations of the B_μ's which we just presented.

Lemma 1.12 *Let* $\beta_\mu^+ = \{\overline{\alpha_\mu^p \gamma_\mu^q} | p, q \in \mathbf{N}\}$ *and* $\beta_\mu^- = \{\overline{\alpha_\mu^{*p} \gamma_\mu^q} | p, q \in \mathbf{N}, p \neq 0\}$. *For* $0 < |\mu| \leq 1$, $\beta_\mu := \beta_\mu^+ \cup \beta_\mu^-$ *is a linear basis of* $\mathcal{B}_\mu$.

There remains to prove the lower semi-continuity of ζ and ξ at the extreme points. It is not really necessary to prove it here, since it will be a corollary of our theorem 4.1. But a direct proof, applying the ideas of [Ri] to the continuous field of Haar measures, may be found in the two preprints [S2] and [NN], which I did not yet know at Algop in July. I thank E.Blanchard for explaining to me [B1] and [B2], and A.Sheu, G.Skandalis and G.Nagy for sending me [S2], [NN], and [N1], [N2], and I include the sketch of this proof, for the reader's convenience.

Proposition 1.13 *([S2], [NN]) The field* ζ *is lower semi-continuous at* 1 *and the field* ξ *is lower semi-continuous at* ± 1.

(Sketch of) proof. As in the proof of proposition 1.11, the properties of ξ may be either deduced from those of ζ or proved directly by the same method. So we just have to prove the lower semi-continuity of ζ at 1. For $|\mu| < 1$, let $\overline{h_\mu}$ be the state on B_μ defined by the same formulas as the Haar measure h_μ on A_μ ([W1], appendix 1) :

$$\overline{h_\mu}(b) = (1-\mu^2) \sum_{n=0}^{\infty} \mu^{2n} \langle e_n | b(e_n) \rangle .$$

Let $\overline{h_1}$ be the normalised Lebesgue measure on the disc. (The state h_μ on $A_\mu \subset C(\mathbf{T}, B_\mu)$ is obtained from the state $\overline{h_\mu}$ on B_μ by integrating along the circle $\mathbf{T}$. Conversely, viewing B_μ as a subalgebra of A_μ as in the remark after proposition 1.9, $\overline{h_\mu}$ is obtained from h_μ by restriction). One proves that $\overline{h_\mu}$ is a continuous field of states by checking continuity on polynomials $b \in \mathcal{B}$. By faithfulness of the associated G.N.S. representations, this yields lower semi-continuity.

2 Poisson-deformation of $SU(2)$

Let us reformulate the two definitions of a "strict deformation" and of an "operator deformation" ([S1] p. 223). The first one corresponds to Rieffel's definition, the second one is a more flexible version, allowing to consider Sheu's construction as a deformation.

Definition 2.1 *Let $\mathcal{C}$ be an involutive Poisson* $\mathbf{C}$*-algebra, endowed with a C^*-norm. A Poisson-deformation of $\mathcal{C}$ is :*

(a) *a continuous field of C^*-algebras $(C_h)_{0\leq h<\varepsilon}$, such that C_0 is the completion of $\mathcal{C}$,*

(b) *for any $f \in \mathcal{C}$, a section $(h \mapsto \rho_h(f))$ such that $\rho_0(f) = f$*

satisfying :

(c) *the $\rho(f)$'s form a total family of continuous sections of this field*

(d) $\lim\limits_{h\to 0} \frac{[\rho_h(f),\rho_h(g)]}{ih} = \{f,g\} \qquad (\forall f,g \in \mathcal{C})$

(e) *the map ρ is linear*

(f) *for any h, the map ρ_h is injective*

and satisfying moreover one of the two conditions :

(SD) *for any h, $\rho_h(\mathcal{C})$ is closed under multiplication ("strict deformation")*

(OD) *ρ preserves the involution ("operator deformation").*

Remarks.

- $\mathcal{C}$ is then necessarily commutative
- In Rieffel's original definition ([S1]), one starts from a Poisson manifold M and chooses to deform some Poisson subalgebra $\mathcal{C}$ of $C_b^\infty(M)$ (smooth bounded functions), such that $\mathcal{C}$ contains the subalgebra $C_c^\infty(M)$ (smooth functions with compact support). In our case $\mathcal{C}$ will only be dense in $C(M)$ (and M will be compact), but $\mathcal{C}$ will not contain $C^\infty(M)$.
- In this reformulation, the data (a) and (b) are redundant : the topology of the field is fully determined by the total family of continuous sections.
- The limit in condition (d) has a meaning in the total space of the field.
- Condition (a) is much stronger than Rieffel's original definition ([S1]). One part of this strength is irrelevant : we could have required the field to be continuous only as a field of Banach spaces outside $h = 0$ (this would have caused no change in the rest of this paper). But the other part is crucial : at $h = 0$ we really want the multiplication and the involution to be continuous.
- This way of reformulating the definition forced us to make condition (f) explicit, whereas the injectivity of the ρ_h's was originally implicit. This condition does not seem relevant (the rest of the paper is true if we drop it), but we shall keep it since it will be fulfilled in our case. (This property is sometimes technically useful, to check condition (SD) : see [S1], [N1]).

We shall construct such a deformation for $M = SU(2)$ and $\mathcal{C} = \mathcal{A}_1$. It will be both a strict- and operator- deformation, in contrast with Sheu's one which was only an operator deformation. But Sheu deformed the whole algebra $C^\infty(SU(2))$, whereas our $\mathcal{A}_1$ is only dense in $C(SU(2))$.

"The" Poisson structure on $SU(2)$ will always be the one chosen by Sheu [S1] and described by Lu and Weinstein [LW]. In order to deform it, we first deform the unit disc D, which is a "slice" of it (theorem 1.7). The restriction to $\mathcal{A}_1$ of the Poisson bracket on $C^\infty(SU(2))$ is completely determined by :

$$\begin{array}{rclrcl} \{\alpha_1,\alpha_1^*\} & = & -\mathrm{i}\gamma_1\gamma_1^*, & \{\alpha_1,\gamma_1\} & = & \frac{\mathrm{i}}{2}\gamma_1\alpha_1, \\ \{\alpha_1,\gamma_1^*\} & = & \frac{\mathrm{i}}{2}\gamma_1^*\alpha_1, & \{\gamma_1,\gamma_1^*\} & = & 0. \end{array}$$

Hence it is induced by the Poisson bracket on $\mathcal{B}_1$ defined by :

$$\{\overline{\alpha}_1,\overline{\alpha}_1^*\} \;=\; -\mathrm{i}\overline{\gamma}_1^2, \quad \{\overline{\alpha}_1,\overline{\gamma}_1\} \;=\; \tfrac{\mathrm{i}}{2}\overline{\gamma}_1\,\overline{\alpha}_1.$$

(These two brackets are related by : $\{f,g\}(u) = \{f(u),g(u)\} \quad \forall f,g \in \mathcal{A}_1, \forall u \in \mathbf{T}$.)

The main trick to deform these brackets is to notice that in the field ζ, if we choose a good change of parameter like $h(\mu) = 1-\mu^2$, condition (d) is fulfilled for $(f,g)=(\overline{\alpha}_1,\overline{\alpha}_1^*)$ or $(\overline{\alpha}_1,\overline{\gamma}_1)$ as soon as $\rho_\mu(\overline{\alpha}_1) = \overline{\alpha}_\mu$, $\rho_\mu(\overline{\alpha}_1^*) = \overline{\alpha}_\mu^*$ and $\rho_\mu(\overline{\gamma}_1) = \overline{\gamma}_\mu$. (This comes from the relations between $\overline{\alpha}_\mu$ and $\overline{\gamma}_\mu$).

There remains to choose $\rho(f)$ for any polynomial f in the (commuting) variables $\overline{\alpha}_1,\overline{\alpha}_1^*,\overline{\gamma}_1$. By lemma 1.12, it suffices to define $\rho(f)$ for any element f of the linear basis β_1. $\rho(f)$ may be chosen as a polynomial in the (non commuting) variables $\overline{\alpha},\overline{\alpha}^*,\overline{\gamma}$, and must be such that $\rho(f)_1 = f$. We shall present the simplest example of such a ρ, and prove that it yields a deformation. (The proof will use a combinatoric property of this choice, but many other choices satisfy this property).

Proposition 2.2 *Let $\rho : \mathcal{B}_1 \to \mathcal{B}$ be the linear involutive map such that*

$$\rho(\overline{\alpha}_1^p\overline{\gamma}_1^q) = \overline{\alpha}^p\overline{\gamma}^q \qquad (\forall p,q \in \mathbf{N}).$$

(ζ,ρ) is a strict- and operator- deformation of $\mathcal{B}_1$, for $\mu \in (0;1]$ and for any change of parameter $h(\mu)$ equivalent to $1-\mu^2$ when $\mu \to 1^-$.

Proof. All conditions except (d) are fulfilled by construction. ((c) and (f) are true by lemma 1.12). Let us prove (d). It suffices to prove it for $f,g \in \beta_1$, and we shall do this by induction on the length $\ell(fg) = \ell(f)+\ell(g)$ of the "word" fg on the "alphabet" $\{\overline{\alpha}_1,\overline{\alpha}_1^*,\overline{\gamma}_1\}$. If $\ell(fg) \le 2$, (d) holds by construction. If $\ell(fg) > 2$ we have $\ell(f) \ge 2$ or $\ell(g) \ge 2$, let us say for instance $\ell(f) \ge 2$. Take $x,y \in \beta_1$ of lengths $< \ell(f)$ such that $f = xy$ and $\rho(f) = \rho(x)\rho(y)$. From these two equalities we deduce :

$$\{f,g\} = x\{y,g\} + \{x,g\}y,$$

$$\text{and} \qquad [\rho(f),\rho(g)] = \rho(x)[\rho(y),\rho(g)] + [\rho(x),\rho(g)]\rho(y).$$

Since $\ell(yg) < \ell(fg)$, we may assume (by induction) that $\lim_{h\to 0} \frac{[\rho_h(y),\rho_h(g)]}{\mathrm{i}h} = \{y,g\}$. Since $\rho(x)$ is a continuous section of ζ such that $\rho_0(x) = x$, the upper semi-continuity of ζ at $h = 0$ entails :

$$\lim_{h\to 0} \frac{\rho_h(x)[\rho_h(y),\rho_h(g)]}{\mathrm{i}h} = x\{y,g\}.$$

Similarly (using $\ell(xg) < \ell(fg)$), we get :

$$\lim_{h\to 0} \frac{[\rho_h(x), \rho_h(g)]\rho_h(y)}{ih} = \{x, g\}y,$$

and add up these two equalities.

Corollary 2.3 *Let $\rho' : \mathcal{A}_1 \to \mathcal{A}$ be the linear involutive map such that*

$$\rho'(\alpha_1^p \gamma_1^q \gamma_1^{*r}) = \alpha^p \gamma^q \gamma^{*r} \qquad (\forall p, q, r \in \mathbf{N}).$$

(ξ, ρ') is a strict- and operator- deformation of $\mathcal{A}_1$, for $\mu \in (0; 1]$ and for any change of parameter $h(\mu)$ equivalent to $1 - \mu^2$ when $\mu \to 1^-$.

Proof. By proposition 2.2, $\forall f, g \in \mathcal{A}_1$, $\lim_{h\to 0} \frac{[\rho_\mu(f(u)), \rho_\mu(g(u))]}{ih} = \{f(u), g(u)\}$, hence $\lim_{h\to 0} \frac{[\rho'_\mu(f), \rho'_\mu(g)]}{ih}(u) = \{f, g\}(u)$. A careful inspection of the proof of proposition 2.2 reveals that this holds *uniformly* in $u \in \mathbf{T}$.

Remark. Extending the field as explained just after definition 1.2 and letting μ tend to 1 from above, one gets a deformation of the opposite Poisson structure of $SU(2)$.

3 Relative uniqueness of the Poisson-deformation of R^2

We shall prove that the Poisson-deformation of $\mathcal{B}_1$ we have just constructed is, in some sense, "the only one". (We referred to $\mathbf{R}^2$ in the title because, as will be explained in the next paragraph, $\mathcal{B}_1$ is isomorphic to a Poisson-subalgebra of $C^\infty(\mathbf{R}^2)$).

Theorem 3.1 *Let (Θ, τ) be an operator deformation of $\mathcal{B}_1$ such that :*

- *Θ is trivial on $\mathbf{R}^*$, with fiber $C^*(\mathcal{S})$*
- *$\sigma_0(\tau_h(\overline{\alpha}_1)) = \sigma_1(\overline{\alpha}_1)$ for any $h \neq 0$*
- *$\tau_h(\overline{\alpha}_1) = R_h \mathcal{S}^*$, for any $h \neq 0$, with R_h selfadjoint, diagonal in the basis $(e_n)_{n\in\mathbf{N}}$ for which $\mathcal{S}(e_n) = e_{n+1}$.*

Then Θ is isomorphic to ζ.

Remark. In fact we shall use the continuity of Θ (in lemma 3.2) but only the upper semi-continuity of ζ, hence we do not need proposition 1.13.

Proof. Let us set $h = 1 - \mu^2$ and abbreviate $\lim_{h\to 0} x_h - y_h = 0$ by $x_h \asymp y_h$. We must prove that $\forall b \in \mathcal{B}_1, \tau_h(b) \asymp \rho_\mu(b)$. Let $w_h = \tau_h(\overline{\alpha}_1)$, and $v_h = \tau_h(\overline{\gamma}_1)$. It suffices to prove that (1) $w_h \asymp \overline{\alpha}_\mu$ and (2) $v_h \asymp \overline{\gamma}_\mu$. Let us first assume (1) and deduce (2) from it. *Using only the upper semi-continuity of ζ and Θ*, since $\overline{\gamma}_1 = \sqrt{1 - \overline{\alpha}_1^* \overline{\alpha}_1}$ we get : $\overline{\gamma}_\mu^2 \asymp 1 - \overline{\alpha}_\mu^* \overline{\alpha}_\mu \asymp 1 - w_h^* w_h \asymp v_h^2$ hence $\overline{\gamma}_\mu \asymp |v_h|$, and since $v_0 = \overline{\gamma}_1 \geq 0$ we get : $v_h \asymp |v_h|$. This yields (2). Let us now prove (1). Let $y_n(h)$ be $\sqrt{1 - \mu^{2(n+1)}}$ and Q_μ be the operator such that $Q_\mu(e_n) = y_n(h)e_n$. Since τ_h is a Poisson-deformation, we have : $\frac{[w_h, w_h^*]}{h} \asymp 1 - w_h^* w_h$, hence $(1 - w_h w_h^*) - \mu^2(1 - w_h^* w_h) = o(h)$, which (by elementary calculus) leads to : $R_h^2 \asymp Q_\mu^2$. From this, using the hypothesis that R_h is equal to 1 modulo $\mathcal{K}$, we deduce $R_h \asymp Q_\mu$ by the following elementary lemma. Since $\overline{\alpha}_\mu = Q_\mu \mathcal{S}^*$ by §1 (before lemma 1.12), this yields (1).

Lemma 3.2 *Let $x_n(h), y_n(h) \in \mathbf{R}$, for $n \in \mathbf{N}$ and $h \in [0, \varepsilon]$, be such that :*

- *$x_n^2(h) - y_n^2(h) \to 0$ when $h \to 0$, uniformly in n,*
- *for any fixed n, $y_n(h)$ is an increasing positive function of h, with limit 0 at $h = 0$, and $x_n(h)$ is a continuous function of h,*
- *for any fixed h, the sequence $x_n(h)$ is eventually positive.*

Then $x_n(h) - y_n(h) \to 0$ when $h \to 0$, uniformly in n.

4 Comparision with Sheu's deformation

Let us recall the deformation defined in [S1]. Let the unit disc D be equipped with the Poisson structure deduced from the usual one on $\mathbf{R}^2$ by the change of variables :

$$(z \in \mathbf{R}^2) \mapsto (Z = \frac{z}{|z|}\sqrt{1 - \mathrm{e}^{-|z|^2/2}} \in D).$$

The Poisson structure on $SU(2)$ is related to this Poisson-D by the "slicing" described after theorem 1.7, and Sheu's deformation of Poisson-$SU(2)$ is naturally deduced from the following deformation of Poisson-$\mathbf{R}^2$.

Let S^0 be the algebra of Weyl symbols of order 0 on $\mathbf{R}^2$. For any $a \in S^0$ and any $h > 0$, let $W_h(a)$ be the operator on $L^2(\mathbf{R})$ defined by

$$W_h(a)u(y) = \frac{1}{2\pi h}\int \mathcal{A}(x, (y + y')/2)\, \exp(\frac{\mathrm{i}x(y - y')}{h})\; u(y')\; dx\; dy',$$

$$\text{with} \quad \mathcal{A}(x, y) = a(y, x).$$

(In [S1], $\mathcal{A} = a$. We rectify the formula according to [Vo], in order to really get, as claimed by Sheu, a deformation of the Poisson-bracket on $\mathbf{R}^2$ defined by $\{f, g\} = \partial_1 f\ \partial_2 g - \partial_2 f\ \partial_1 g$ and not of the opposite one. The only point of [S1] spoiled by this mistake is proposition 2.1, which we shall put right in our lemma 4.5).

$W_h(a)$ is unitarily equivalent to $W_h'(a) := W(a_h)$, with $W = W_1$ and $a_h(z) = a(\sqrt{h}z)$ ([S1] p.224), and Sheu proved that W_h is a deformation of S^0. Hence W_h' is also a deformation of S^0. We call ζ' the continuous field associated to W_h' and ξ' the continuous field associated to the induced deformation of $C^\infty(SU(2))$.

Theorem 4.1 *The fields ζ and ζ' are isomorphic ; so are the fields ξ and ξ'.*

Proof. The second point is a consequence of the first one, which comes from theorem 3.1 and from lemma 4.2, 4.5 and 4.6 below.

Lemma 4.2 *ζ' is trivial on $\mathbf{R}^{+*}$, with fibers $C^*(\mathcal{S})$.*

Proof. The fibers are $C^*(\mathcal{S})$ by [S1] (proposition 2.1). By Howe's proof of theorem 3.1.3 in [Ho], there exists a constant C such that for any $a \in S^0$, if we define $\|a\|_{sup4} = \max_{|\gamma|\leq 4} \|\partial^\gamma a\|_\infty$, we get : $\|W(a)\| \leq C\|a\|_{sup4}$, hence for any $h > 0$, $\|W'_h(a) - W'_t(a)\| = \|W(a_h - a_t)\| \leq C\|a_h - a_t\|_{sup4} \xrightarrow[t\to h]{} 0$, hence the map $(t \mapsto W'_t(a))$ is continuous at h.

Remark. The above proof is inspired by [S1], page 225, first paragraph, which contains a hint of proof for the fact that the map $(t \mapsto W_t(a))$ is also continuous (Sheu uses a weaker seminorm on S^0 but this does not matter). So the field associated to W_h is, like ζ', trivial on $\mathbf{R}^{+*}$. But these two fields on $\mathbf{R}$ are different (see remark 2 after lemma 4.5).

Lemma 4.3 *ζ' is upper semi-continuous at* 0.

Proof. (This fact was already proved in [S1] p.225, hence this lemma is not necessary, but it is worth noticing that the inequality we just mentionned yields a much simpler proof). Call $\sharp_h$ the product on S_0 such that $W_h(f)W_h(g) = W_h(f\sharp_h g)$. Let $a \in S^0$ and $\varepsilon > \|a\|_\infty$. Choose M such that $\|a\|_\infty < M < \varepsilon$ and let $c = \sqrt{M^2 - |a|^2}$ and $r(h) = c^*\sharp_h c + a^*\sharp_h a - M^2$. When $h \to 0$, $\frac{r(h)}{h} = \frac{r(h)-r(0)}{h}$ is bounded in S^{-2} ([Vo] p. 124), hence $r(h) \to 0$ for the semi-norms of S^{-2} and a fortiori for the semi-norm $\| \ \|_{sup4}$ defined in the proof of the previous lemma. Therefore, we also have $r(h)_h \to 0$ for this semi-norm. This yields :

$$\begin{aligned}
\|W_h(a)\|^2 - M^2 &\leq \|W_h(a)^*W_h(a)\| - \|M^2 - W_h(c)^*W_h(c)\| \\
&\leq \|W_h(a)^*W_h(a) + W_h(c)^*W_h(c) - M^2\| \\
&= \|W_h(r(h))\| = \|W'_h(r(h))\| = \|W(r(h)_h)\| \\
&\leq C\|r(h)_h\|_{sup4} \xrightarrow[h\to 0]{} 0,
\end{aligned}$$

so for h small enough, $\|W'_h(a)\| = \|W_h(a)\| < \varepsilon$.

Let us recall ([Gu], §7 and 8) some facts and notations needed for the next three lemmas. Let m be the gaussian measure on $\mathbf{C}$, defined by :

$$\mathrm{d}m(x,y) = \frac{1}{\pi}\mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\ \mathrm{d}y.$$

(By the change of variable described at the beginning of this paragraph, m corresponds to the normalized Lebesgue measure on D). The Fock space $\mathcal{F}$ is the Hilbert subspace of $L^2(\mathbf{C}, m)$ of holomorphic functions. For any $\lambda \in \mathbf{C}$, let $\varepsilon_\lambda(z) = \mathrm{e}^{\overline{\lambda}z}$: this defines $\varepsilon_\lambda \in \mathcal{F}$. The Bargmann transform is an isomorphism between $\mathcal{F}$ and $L^2(\mathbf{R})$, which sends the "Berezin basis" $(\varepsilon_\lambda)_{\lambda\in\mathbf{C}}$ of $\mathcal{F}$ to the "Berezin basis" $(c_\lambda f_\lambda)_{\lambda\in\mathbf{C}}$ of $L^2(\mathbf{R})$, with $f_\lambda(x) = \exp(\sqrt{2}\ \overline{\lambda}x - \frac{x^2}{2})$ and c_λ is a constant (such that the Bargmann transform is unitary). The Bargmann transform also sends the Hilbert basis $(e'_n)_{n\in\mathbf{N}}$ of $\mathcal{F}$ defined by $e'_n(z) = z^n/\sqrt{n!}$ to the Hilbert basis $(e_n)_{n\in\mathbf{N}}$ of Hermite functions in $L^2(\mathbf{R})$.

Any bounded operator $T^\sharp$ on $\mathcal{F}$ is completely determined by its "Berezin symbol" $\sigma_{T^\sharp}$ defined by :

$$\sigma_{T^\sharp}(\lambda,\mu) \quad = \quad \frac{\langle T^\sharp \varepsilon_\lambda, \varepsilon_\mu\rangle}{\langle \varepsilon_\lambda, \varepsilon_\mu\rangle} \quad = \quad \mathrm{e}^{-\overline{\lambda}\mu}\ \langle T^\sharp \varepsilon_\lambda, \varepsilon_\mu\rangle.$$

(We take Guillemin's notations, where $\langle f, g\rangle$ is linear in f and antilinear in g). (Since the map $\sigma_{T^\sharp}(\lambda,\mu)$ is holomorphic in μ and antiholomorphic in λ, it is determined by its restriction to the diagonal, denoted by $\sigma_{T^\sharp}$ again).

If T is the operator on $L^2(\mathbf{R})$ related to $T^\sharp$ by the Bargmann transform, the Berezin symbol of $T^\sharp$ may be calculated by :

$$\sigma_{T^\sharp}(\lambda,\mu) \quad = \quad \frac{\langle T(f_\lambda), f_\mu\rangle}{\langle f_\lambda, f_\mu\rangle} \quad = \quad \frac{\mathrm{e}^{-(\overline{\lambda}+\mu)^2/2}}{\sqrt{\pi}}\ \langle T(f_\lambda), f_\mu\rangle.$$

This holds in particular when $T = \mathcal{A}_\tau$ is the operator associated to a Weyl symbol $\mathcal{A} \in S^0$ (i.e. $\mathcal{A}_\tau = W(a)$, with $\mathcal{A}(x,y) = a(y,x)$).

Lemma 4.4 *With the preceeding notations,*

i)

$$\sigma_{W(a)^\sharp}(\lambda,\mu) = \mathrm{e}^{-2\overline{\lambda}\mu}\int a(\overline{z})\mathrm{e}^{\sqrt{2}(\overline{\lambda}z+\mu\overline{z})}\ \mathrm{d}m(z)$$

ii)

$$\langle W(a)(e_m), e_n\rangle = \sqrt{m!n!}\sum_{k=0}^{\inf(m,n)} \frac{(-1)^k I_{m-k,n-k}}{k!(m-k)!(n-k)!},$$

with

$$I_{r,s} = \int a(\overline{z})(\sqrt{2}z)^r(\sqrt{2}\overline{z})^s\ \mathrm{d}m(z).$$

Proof. i) Let us generalize and rectify Guillemin's calculus ([Gu] pp. 186–187). By definition of $\mathcal{A}_\tau = W(a)$ and of f_λ,

$$\langle \mathcal{A}_\tau(f_\lambda), f_\mu\rangle = \frac{1}{2\pi}\int \mathcal{A}(p,\frac{x+y}{2})\exp(\sqrt{2}(\overline{\lambda}y+\mu x) - \frac{x^2+y^2}{2} + \mathrm{i}p(x-y))\ \mathrm{d}x\mathrm{d}y\mathrm{d}p.$$

If we make the change of variable $q = \frac{x+y}{2}$ with x and p fixed, the integral above becomes

$$\frac{1}{\pi}\int \mathcal{A}(p,q)[\ldots]\exp(2\sqrt{2}\ \overline{\lambda}q - 2q^2 - 2\mathrm{i}pq)\ \mathrm{d}q\ \mathrm{d}p,$$

$$\text{with} \qquad [\ldots] = \int \exp((\sqrt{2}(\mu-\overline{\lambda}) + 2(q+\mathrm{i}p))x - x^2)\ \mathrm{d}x =$$

$$\sqrt{\pi}\ \exp((\frac{\mu-\overline{\lambda}}{\sqrt{2}} + q + \mathrm{i}p)^2),$$

hence (after some simplifications) $\langle \mathcal{A}_\tau(f_\lambda), f_\mu\rangle =$

$$\frac{\exp((\overline{\lambda}+\mu)^2/2)}{\sqrt{\pi}}\int \mathcal{A}(p,q)\exp(-\overline{(\sqrt{2}\lambda - (q-\mathrm{i}p))}(\sqrt{2}\mu - (q-\mathrm{i}p))\ \mathrm{d}q\ \mathrm{d}p,$$

and the result follows.
ii) From i) we deduce :

$$\langle \mathcal{A}^\sharp_\tau \varepsilon_\lambda, \varepsilon_\mu \rangle = \mathrm{e}^{-\overline{\lambda}\mu} \int a(\overline{z}) \sum_{r,s\geq 0} \frac{(\sqrt{2}\,\overline{\lambda}z)^r}{r!} \frac{(\sqrt{2}\mu\overline{z})^s}{s!} \,\mathrm{d}m(z).$$

Since a is bounded and $\int \sum_{r,s\geq 0} \frac{|\sqrt{2}\,\overline{\lambda}z|^r}{r!} \frac{|\sqrt{2}\mu\overline{z}|^s}{s!} \,\mathrm{d}m(z) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{(|\lambda|+|\mu|)^2/2} < \infty$, this expression becomes :

$$\mathrm{e}^{-\overline{\lambda}\mu} \sum_{r,s\geq 0} \frac{\overline{\lambda}^r \mu^s I_{r,s}}{r!s!} = \sum_{r,s,k\geq 0} \frac{(-1)^k \overline{\lambda}^{r+k} \mu^{s+k} I_{r,s}}{r!s!k!},$$

hence $\langle \mathcal{A}^\sharp_\tau \varepsilon_\lambda, \varepsilon_\mu \rangle$ is equal to the series (with infinite bi-radius of convergence)

$$\sum_{m,n\geq 0} \overline{\lambda}^m \mu^n \sum_{k=0}^{\inf(m,n)} \frac{(-1)^k I_{m-k,n-k}}{k!(m-k)!(n-k)!}.$$

But it is also equal to

$$\sum_{m,n\geq 0} \langle \varepsilon_\lambda, e'_m \rangle \langle \mathcal{A}^\sharp_\tau (e'_m), e'_n \rangle \langle e'_n, \varepsilon_\mu \rangle = \sum_{m,n\geq 0} \frac{\overline{\lambda}^m}{\sqrt{m!}} \langle \mathcal{A}^\sharp_\tau (e'_m), e'_n \rangle \frac{\mu^n}{\sqrt{n!}},$$

with bi-radius at least $(1,1)$ (since $|\langle \mathcal{A}^\sharp_\tau (e'_m), e'_n \rangle| \leq \|\mathcal{A}^\sharp_\tau\|$). The result follows by identification.

Lemma 4.5 *For any $a \in S^0$ and $h > 0$, $\sigma_0(W'_h(a)) = \sigma_1(a)$.*

Proof. This lemma rectifies the proposition 2.1 of [S1], which states an analogous formula but with σ instead of σ_1 : $\sigma(a)(v) = \lim_{r\to\infty} a(rv)$, $\sigma_1(a)(v) = \sigma(a)(\overline{v})$ (see our remark after corollary 1.8). The mistake in [S1] came from two reasons : the wrong orientation (which we compensated by replacing a by $\mathcal{A}$ in the definition of W), and the use of an erroneous formula in [Gu], which we just rectified in lemma 4.4.i : in [Gu] p. 187, the result must be replaced by

$$\sigma_{\mathcal{A}^\sharp_\tau}\left(\frac{a+ib}{\sqrt{2}}\right) = \int \mathcal{A}(p,q) \mathrm{e}^{-(q-a)^2-(p+b)^2} \,\mathrm{d}m(q-\mathrm{i}p) = (\mathrm{e}^{-\Delta}\mathcal{A})(-b,a),$$

$$\text{hence} \qquad \sigma(\mathcal{A}^\sharp_\tau) = \mathrm{e}^{-\Delta/2}\mathcal{B}, \qquad \text{with} \qquad \mathcal{B}(x,y) = \mathcal{A}(-\sqrt{2}y, \sqrt{2}x).$$

In formula 8.20 p.187 of [Gu] and in the subsequent pages $\mathcal{A}$ must be replaced by $\mathcal{B}$. Apart from this, the proof is the same as in [S1] so we do not repeat it.

Remarks.

1. If we do not replace a by $\mathcal{A}$, i.e. if we deform the opposite Poisson bracket, (as Sheu did) we find $\sigma_0(W'_h(a))(z) = \sigma(a)(\mathrm{i}\,z)$, not $\sigma(a)(z)$.

2. From this lemma we deduce : $\sigma_0(W_h(a)) = \sigma_1(a(.,h.))$. This shows that one cannot hope $\lim_{h\to 0} W_h(a) - W'_h(a) = 0$ to hold in general (i.e. for all $a \in S^0$),

because this would imply $\lim_{h\to 0} b(\frac{\cos\theta + \mathrm{i}\ h\sin\theta}{\sqrt{\cos^2\theta + h^2\sin^2\theta}}) = b(\cos\theta + \mathrm{i}\ \sin\theta)$ (uniformly in θ) for all $b = \sigma_1(a) \in C(\mathbf{T})$. This may be roughly expressed by saying that the two (isomorphic) fields on $\mathbf{R}^+$ associated to W_h and W'_h, which both consist of the same trivial field on $\mathbf{R}^{+*}$, "glued" to the same C^*-algebra at $h = 0$, are "not glued in the same way".

3. The isomorphism between $\overline{W'_h(S^0)}$ and $C^*(\mathcal{S})$, proved in [S1] proposition 2.1 and used in ourlemma 4.2, can now be made explicit in its correct version : it is simply an equality, when $C^*(\mathcal{S})$ is realized as a concrete algebra of operators on $L^2(\mathbf{R})$, letting $\mathcal{S}$ be the unilateral shift operator on the Hilbert basis of Hermite functions e_n. Moreover, $\sigma_0(W'_h(\overline{\alpha}_1)) = \sigma_1(\overline{\alpha}_1) = (z \mapsto \overline{z}) = \sigma_0(\mathcal{S}^*)$, thus modulo compact operators, $W'_h(\overline{\alpha}_1)$ is congruent to $\mathcal{S}^* = \overline{\alpha}_0$. By definition of B_μ (corollary 1.8), $\overline{W'_h(S^0)}$ is also equal to B_μ and $W'_h(\overline{\alpha}_1)$ is congruent to $\overline{\alpha}_\mu$ (modulo compact operators), for any values $h > 0$ and $\mu \in (-1;1)$.

Lemma 4.6 *Let $(e_n)_{n\in\mathbf{N}}$ be the Hilbert basis of $L^2(\mathbf{R})$ of Hermite functions and $\mathcal{S}$ be the unilateral shift on this basis. $W'_h(\overline{\alpha}_1)$ is of the form $R_h\mathcal{S}^*$ with R_h selfadjoint and diagonal in this basis.*

Proof. $\overline{\alpha}_1(z) = \frac{z}{|z|}\sqrt{1 - \mathrm{e}^{-|z|^2/2}}$, hence (taking the polar decomposition $z = \rho u$) $(\overline{\alpha}_1)_h(\rho u) = u\ b(h\rho^2)$ with $b(t) = \sqrt{1 - \mathrm{e}^{-t/2}}$. More generally, let us apply the result of lemma 4.4.ii to a symbol a_h such that $a_h(\rho u) = u^d b(h\rho^2)$ for some integer d and some function b. Under this hypothesis, $I_{r,s}$ will be equal to

$$\int u^{r-s-d} b(h\rho^2)(\sqrt{2}\ \rho)^{r+s}\ \mathrm{d}m(\rho u),$$

hence it will be 0 if $r \neq s + d$, and

$$I_{s+d,d} = \int b(ht)(2t)^{s+\frac{d}{2}}\mathrm{e}^{-t}\ \mathrm{d}t.$$

Hence $\langle W(a_h)(e_m), e_n\rangle = 0$ if $m \neq n + d$ and

$$\langle W(a_h)(e_{n+d}), e_n\rangle = \sqrt{(n+d)!n!}\sum_{k=0}^{\inf(n+d,n)} \frac{(-1)^k I_{n+d-k,n-k}}{k!(n+d-k)!(n-k)!}$$

will be real if b takes real values.

Now we may consider that ρ and W' are two deformations of Poisson-$\mathcal{B}_1$ within the same field ζ', and that for any $a \in \mathcal{B}_1$, $\rho_h(a) - W'_h(a) \to 0$ when $h \to 0^+$ (using the identification of ζ and ζ' given by theorem 4.1 and the identification of the Poisson-disk and Poisson-$\mathbf{R}^2$). Moreover, the product induced by W'_h (on $\mathcal{B}_1$, and even on S^0) admits an asymptotic expansion. The following last proposition gives an analogous asymptotic expansion for the product induced by ρ_h, and a nice description of $\rho_h(a) - W'_h(a)$ for "good choices" of $\mu \mapsto h(\mu)$ (among functions equivalent to $1 - \mu^2$ when $\mu \to 1^-$, cf proposition 2.2)

Proposition 4.7 *Let $\varphi : \mathcal{B}_1 \to \mathcal{B}_1$ be the linear map such that*

$$\begin{array}{rrcll} & \varphi(\overline{\alpha}_1^p\overline{\gamma}_1^q) & = & \frac{\mathrm{i}}{2}\,(p-1)\,q\,\overline{\alpha}_1^p\overline{\gamma}_1^q & \forall p,q\in\mathbf{N} \\ \text{and} & \varphi(f^*) & = & -\varphi(f)^* & \forall f\in\mathcal{B}_1, \end{array}$$

and $C = \{\ ,\ \} + \partial\varphi$, with $\partial\varphi(f,g) = f\varphi(g) + \varphi(f)g - \varphi(fg)$.

i) $\rho_h(f)\rho_h(g) = \rho_h(fg + \frac{\mathrm{i}\,h}{2}C(f,g)) + \mathrm{o}(h)$

ii) *If $h = 2\frac{1-\mu^2}{1+\mu^2} + \mathrm{o}(h^2)$ (or $\mu = \frac{1-h/4}{1+h/4} + \mathrm{o}(h^2)$, which is equivalent), then*

$$\rho_h(f) = W_h'(f + \frac{\mathrm{i}\,h}{2}\varphi(f)) + \mathrm{o}(h) \qquad \forall f\in\mathcal{B}_1.$$

Proof. For $f,g \in \beta_1$ (cf lemma 1.12), one easily finds $C(f,g)$ such that (i) holds, using the relations between $\overline{\alpha}_\mu$ and $\overline{\gamma}_\mu$ (definition 1.10). (I do not reproduce this calculus here because it is not nice-looking, except that $C(g^*,f^*) = -C(f,g)^*$ since ρ is $*$-preserving). Let $D = C - \{\ ,\ \}$ (hence $D(g^*,f^*) = -D(f,g)^*$). One may check by induction on $p,q,p',q' \in \mathbf{N}$ that

$$\begin{array}{rcll} \{\overline{\alpha}_1^p\overline{\gamma}_1^q, \overline{\alpha}_1^{p'}\overline{\gamma}_1^{q'}\} & = & \frac{\mathrm{i}}{2}(pq'-qp')\overline{\alpha}_1^{p+p'}\overline{\gamma}_1^{q+q'} & \text{and} \\ \{\overline{\alpha}_1^p\overline{\gamma}_1^q, \overline{\alpha}_1^{*p'}\overline{\gamma}_1^{q'}\} & = & \frac{\mathrm{i}}{2}\overline{\gamma}_1^{q+q'}[(pq'+qp'+2pp')\overline{\alpha}_1^p\overline{\alpha}_1^{*p'} - 2pp'\overline{\alpha}_1^{p-1}\overline{\alpha}_1^{*p'-1}]. & \end{array}$$

Then one may calulate $D(f,g)$ for $f \in \beta_1^+$ and $g \in \beta_1$, and check that it is equal to $\partial\varphi(f,g)$. This proves (i). We shall prove (ii) in three steps : prove that it holds for $f = \overline{\gamma}_1$ (step 1) and for $f = \overline{\alpha}_1$ or $\overline{\alpha}_1^*$ (step 2), and then use (i) to extend (ii) to any $f \in \mathcal{B}_1$ (step 3).
Stepl. Applying the formulas of the proof of lemma 4.6 to $a(z) = \overline{\gamma}_1(z) = \mathrm{e}^{-\frac{1}{4}|z|^2}$ we get :

$$W_h'(\overline{\gamma}_1)(e_n) = \frac{(1-\frac{h}{4})^n}{(1+\frac{h}{4})^{n+1}}\,e_n.$$

Comparing this with $\overline{\gamma}_\mu(e_n) = \mu^n e_n$, one easily gets that if $\mu = \frac{1-h/4}{1+h/4} + \mathrm{o}(h^2)$ then

$$\overline{\gamma}_\mu = (1+h/4)W_h'(\overline{\gamma}_1) + \mathrm{o}(h) = W_h'(\overline{\gamma}_1 + \frac{\mathrm{i}\,h}{2}\varphi(\overline{\gamma}_1)) + \mathrm{o}(h).$$

Step 2. Let $w_h = W_h'(\overline{\alpha}_1)$ and let $x_n(h)$ be the real numbers such that $w_h(e_{n+1}) = x_n(h)e_n$. The application of the formulas of the proof of lemma 4.6 leads to rather complicated results for $a = \overline{\alpha}_1$ (in order to get the exact values of $x_n(h)$), but the results are much simpler for $a = \overline{\alpha}_1^2$:

$$W_h'(\overline{\alpha}_1^2)(e_{n+2}) = \frac{2n+3}{2\sqrt{(n+1)(n+2)}}[1 - \frac{(1-h/2)^{n+1}}{(1+h/2)^{n+2}}(1+\frac{h}{2(2n+3)})]e_n.$$

Using the fact that $w_h^2 = W_h'(\overline{\alpha}_1^2) + \mathrm{o}(h)$, the above formula proves that for h sufficiently small, $\forall n \in \mathbf{N}, x_n(h)x_{n+1}(h) > 0$ hence (since $\lim\limits_{n\to\infty} x_n(h) = 1$) the numbers $x_n(h)$ are all positive. This simplifies the proof of theorem 3.1 in the particular case $\tau = W'$ (making lemma 3.2 superfluous). Moreover, this allows a refinement of this proof : if $(1 - w_h w_h^*) - \mu^2(1 - w_h^* w_h) = o(h^2)$ (instead of only $o(h)$) then (by the same calculus) $w_h - \overline{\alpha}_\mu = o(h)$ (instead of only $\varepsilon(h)$). Using asymptotic expansions [Vo], an elementary calculus shows that this condition $(1 - w_h w_h^*) - \mu^2(1 - w_h^* w_h) = o(h^2)$ is obtained when $1 - \mu^2 - \frac{h}{2}(1 + \mu^2) = o(h^2)$. Thus we get

$$\overline{\alpha}_\mu = W'_h(\overline{\alpha}_1) + o(h) = W'_h(\overline{\alpha}_1 + \frac{\mathrm{i}\,h}{2}\varphi(\overline{\alpha}_1)) + o(h).$$

Since W' and ρ are $*$-preserving and $\varphi(\overline{\alpha}_1^*) = -\varphi(\overline{\alpha}_1) = 0$, we deduce the same property for $\overline{\alpha}_1^*$.

Step 3. More generally, let ρ and W' be two deformations, in the same field, of some algebra $\mathcal{B}_1$ generated by some subset X, such that $\forall f, g \in \mathcal{B}_1$,

$$\begin{aligned} \rho_h(f) &= W_h(f) + \varepsilon(h) \\ \rho_h(f)\rho_h(g) &= \rho_h(fg + \tfrac{\mathrm{i}\,h}{2}C(f,g)) + o(h), \\ W'_h(f)W'_h(g) &= W'_h(fg + \tfrac{\mathrm{i}\,h}{2}C'(f,g)) + o(h), \end{aligned}$$

$C = C' + \partial\varphi$ for some linear map φ, and (ii) holds for any generator $f \in X$. In order to extend (ii) to any $f \in \mathcal{B}_1$, it suffices to show that if (ii) holds for f and g then it holds for fg. Let us do it :

$$\begin{aligned} \rho_h(fg) &= \rho_h(f)\rho_h(g) - \rho_h(\tfrac{\mathrm{i}\,h}{2}C(f,g))) + o(h) \\ &= W'_h(f + \tfrac{\mathrm{i}\,h}{2}\varphi(f))W'_h(g + \tfrac{\mathrm{i}\,h}{2}\varphi(g)) - W'_h(\tfrac{\mathrm{i}\,h}{2}C(f,g))) + o(h) \\ &= W'_h(fg + \tfrac{\mathrm{i}\,h}{2}[f\varphi(g) + \varphi(f)g + C'(f,g) - C(f,g)]) + o(h) \\ &= W'_h(fg + \tfrac{\mathrm{i}\,h}{2}\varphi(fg)) + o(h). \end{aligned}$$

Remark. The first assertion (i) in this proposition gives another way to prove proposition 2.2.

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Anne BAUVAL
Laboratoire Topologie et Géométrie,
U.R.A. C.N.R.S. 1408
Université Paul Sabatier, U.F.R. M.I.G.
118, route de Narbonne
31062 Toulouse-Cedex, France