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# Lattices in semi-simple Lie groups, and multipliers of group $C^*$- algebras

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## 1 Introduction, and some history.

Let $G$ be a locally compact group, and $H$ be a closed subgroup. Viewing $L^1(G)$ as a two-sided ideal in the measure algebra $M(G)$, and viewing elements of $L^1(H)$ as measures on $G$ supported inside $H$, we obtain an action of $L^1(H)$ on $L^1(G)$ as double centralizers. It is easy to check (see e.g. proposition 4.1 in [Rie]) that this action extends to an action of the full group $C^*$-algebra $C^*(H)$ as double centralizers on $C^*$(G); this corresponds to a $*$-homomorphism $j_H : C^*(H) \to M(C^*(G))$, where $M(C^*(G))$ denotes the multiplier $C^*$-algebra of $C^*(G)$. We now quote from p. 209 of Rieffel's Advances paper [Rie]:

*It does not seem to be known whether this homomorphism $j_H$ is injective. It will be injective if and only if every unitary representation of $H$ is weakly contained in the restriction to $H$ of some unitary representation of $G$ [Fe2]. J.M.G. Fell has pointed out to us that the example that he gave in which this appeared to fail (p. 445 of [Fe2]) depended on the completeness of the classification of the irreducible representations of $SL_3(\mathbb{C})$ given in [GeN], and there is now some doubt that this classification is complete [Ste].*

Probably this quotation requires some word of explanation. In [Fe2], Fell studies extensions to the topological framework of Frobenius reciprocity for finite groups. Thus he introduces a list of weak Frobenius properties, the last and weakest one being (WF3):

The locally compact group $G$ satisfies property (WF3) if, for any closed subgroup $H$ of $G$, every representation $\sigma$ in the dual $\hat{H}$ is weakly contained in the restriction $\pi|_H$ of some unitary representation $\pi$ of $G$.

Property (WF3) is indeed equivalent to the injectivity of $j_H$ for any closed subgroup $H$; for completeness, we shall give a proof in Proposition 2.1 below. In §6 of [Fe2], Fell wishes to show that even (WF3) may fail, by taking $G = SL_3(\mathbb{C})$ and $H = SL_2(\mathbb{C})$; to this end he appeals to the incomplete description of $\hat{G}$ given in [GeN]; Fell's proof was recently corrected in Remark 1.13(i) of [BLS].

In this paper, we take for $G$ a semi-simple Lie group with finite centre and without compact factor, and as closed subgroup a lattice $\Gamma$. In section 3, we prove:

**THEOREM 1.1** *Let $G$ be a semi-simple Lie group without compact factors, with finite centre and with Kazhdan's property (T). Let $\Gamma$ be an irreducible lattice in $G$, and let $\sigma$ be a non-trivial irreducible unitary representation of $\Gamma$ of finite dimension $n$. Then $\sigma$*

*determines a direct summand of $C^*(\Gamma)$ which is contained in the kernel of $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$; this direct summand is isomorphic to the algebra $M_n(\mathbb{C})$ of n-by-n matrices.*

If $G$ is a non-compact *simple* Lie group with finite centre, then $G$ has property $(T)$ unless $G$ is locally isomorphic either to $SO_o(n,1)$ or $SU(n,1)$ (see [HaV]). For these two families, we prove in section 4:

**THEOREM 1.2** *Let $G$ be locally isomorphic either to $SO_o(n,1)$ or $SU(n,1)$, for some $n \geq 2$. Let $\Gamma$ be a lattice in $G$. Denote by $\hat{\Gamma}_f$ the set of (classes of) irreducible, finite-dimensional unitary representations of $\Gamma$. If the trivial representation $1_\Gamma$ is not isolated in $\hat{\Gamma}_f$ (for the induced Fell-Jacobson topology), then infinitely many elements of $\hat{\Gamma}_f$ are not weakly contained in the restriction to $\Gamma$ of any unitary representation of $G$. In particular $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$ is not injective.*

In view of Theorems 1.1 and 1.2, it seems natural to formulate the following

**Conjecture.** If $\Gamma$ is a lattice in a non-compact semi-simple Lie group $G$, then $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$ is not injective.

This conjecture means that, if $\rho$ is a representation of $G$ which is faithful on $M(C^*(G))$ (e.g. take for $\rho$ either the universal representation of $G$, or the direct sum of all its irreducible representations), then $\rho|_\Gamma$ is never faithful on $C^*(\Gamma)$; this has bearing on a question of de la Harpe in his paper in these Proceedings (see immediately after Problem 13 in [Har]). In §5, we give examples of lattices $\Gamma$ in $SO_o(n,1)$ or $SU(n,1)$ such that $1_\Gamma$ is not isolated in $\hat{\Gamma}_f$; this is the case for any lattice in $SL_2(\mathbb{R})$, any non-uniform lattice in $SL_2(\mathbb{C})$, and any arithmetic lattice in $SO_o(n,1)$ for $n \neq 3$, 7.

In the final §6, we come back to property (WF3) and show that it always fails for almost connected, non-amenable groups:

**THEOREM 1.3** *Let $G$ be an almost connected, locally compact group. The following properties are equivalent:*

*(i) $G$ has Fell's property (WF3);*
*(ii) $G$ is amenable.*

Observe that Theorem 1.3 cannot hold for any locally compact group. Indeed, any discrete group $G$ satisfies property (WF3) since, given a subgroup $H$ of $G$, one checks easily that $C^*(H)$ is a $C^*$-subalgebra of $C^*(G) = M(C^*(G))$.

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A word about terminology: as usual, semi-simple Lie groups are assumed to be connected and non-trivial; group representations are assumed to be unitary, strongly continuous, and on non-zero Hilbert spaces.

## 2 On multipliers of $C^*$-algebras.

For a $C^*$-algebra $B$, we denote by $M(B)$ its multiplier algebra.

**PROPOSITION 2.1** *Let $A$, $B$ be $C^*$-algebras, and let $j : A \to M(B)$ be a $*$-homomorphism. The following properties are equivalent:*

*(i) $j$ is one-to-one;*
*(ii) for any $\sigma \in \hat{A}$, there exists a non-degenerate $*$-representation $\pi$ of $B$ such that $\sigma$ is weakly contained in $\tilde{\pi} \circ j$, where $\tilde{\pi}$ denotes the extension of $\pi$ to $M(B)$;*
*(iii) any $\sigma \in \hat{A}$ is weakly contained in $\{\tilde{\pi} \circ j \mid \pi \in \hat{B}\}$.*

Fell's property (WF3), mentioned in §1, is deduced from property (ii) above by taking $B = C^*(G)$ and $A = C^*(H)$, for any closed subgroup $H$ of the locally compact group $G$.

**Proof of Proposition 2.1.** (i) $\Rightarrow$ (ii) Let us assume that $j$ is injective, so that we may identify $A$ with a $C^*$-subalgebra of $M(B)$. Let $\pi$ be a faithful representation of $B$. It is known that the extension $\tilde{\pi}$ of $\pi$ to $M(B)$ is also faithful ([Ped], 3.12.5). Thus any representation of $A$ is weakly contained in the restriction of $\tilde{\pi}$ to $A$.

(ii) $\Rightarrow$ (iii) This follows from decomposition theory.

(iii) $\Rightarrow$ (i) Assume that (iii) holds. Fix a non-zero element $x$ of $A$; choose $\sigma \in \hat{A}$ such that $\sigma(x) \neq 0$. Our assumption says that $Ker\,\sigma$ contains $\bigcap_{\pi\in\hat{B}} Ker\tilde{\pi} \circ j = Ker(\bigoplus_{\pi\in\hat{B}} \tilde{\pi} \circ j)$; in particular $x \notin Ker(\bigoplus_{\pi\in\hat{B}} \tilde{\pi} \circ j)$. It follows that $j(x) \neq 0$, i.e. that $j$ is one-to-one.

## 3 Proof of theorem 1.1

We slightly generalize Theorem 1.1 in the following form:

**THEOREM 3.1** *Let $G$ be a non-compact semi-simple Lie group with finite centre and with Kazhdan's property (T). Let $\Gamma$ be an irreducible lattice in $G$, and let $\sigma$ be an irreducible representation of $\Gamma$ of finite dimension $n$, which is not contained in the restriction to $\Gamma$ of a unitary, finite-dimensional representation of $G$. Then $\sigma$ determines a direct summand of $C^*(\Gamma)$ isomorphic to the algebra $M_n(\mathbb{C})$ of $n$-by-$n$ matrices, which moreover is contained in the kernel of $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$.*

Observe that Theorem 1.1 is an immediate consequence of Theorem 3.1: indeed, if $G$ has no compact factor, then any unitary, finite-dimensional representation of $G$ is trivial.

**Proof of Theorem 3.1.** Since $G$ has property ($T$), so has $\Gamma$ (see [HaV], Théorème 4 in Chapter 3). Let $\sigma$ be an irreducible representation of $\Gamma$, of finite dimension $n$. By Theorem 2.1 in [Wan], $\sigma$ is isolated in the dual $\hat{\Gamma}$, hence determines a direct sum decomposition of $C^*(\Gamma)$:

$$C^*(\Gamma) = J \oplus M_n(\mathbb{C})$$

where $J$ is the $C^*$-kernel of $\sigma$.

We assume from now on that $\sigma$ is not contained in the restriction to $\Gamma$ of a unitary, finite-dimensional representation of $G$, and wish to prove that the direct summand $M_n(\mathbb{C})$ lies in the kernel of $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$. Suppose by contradiction that $j_\Gamma$ is non-zero on $M_n(\mathbb{C})$. Choose $\pi \in \hat{G}$ such that $\tilde{\pi} \circ j_\Gamma$ is non-zero, hence faithful on $M_n(\mathbb{C})$ (here $\tilde{\pi}$ denotes the extension of $\pi$ to $M(C^*(G))$, as in Proposition 2.1). Then the $C^*$-kernel of $\tilde{\pi} \circ j_\Gamma$ is contained in $J$, which means that $\sigma$ is weakly contained in the restriction $\pi|_\Gamma$ . As $\sigma$ is isolated in $\hat{\Gamma}$, this implies that $\sigma$ is actually a subrepresentation of $\pi|_\Gamma$ (see Corollary 1.9 in [Wan]). Our assumption shows that $\pi$ is infinite-dimensional. Two cases may occur:

(a) $\pi$ is a discrete series representation of $G$ (if any); this would imply that $\sigma$ is an irreducible subrepresentation of the left regular representation of $\Gamma$, which in turn implies that $\Gamma$ is finite - and this is absurd.

(b) $\pi$ is not in the discrete series of $G$; then, by a result of Cowling and Steger (Proposition 2.4 in [CoS]), the restriction $\pi|_\Gamma$ is irreducible, which contradicts the fact that $\sigma$ is a finite-dimensional subrepresentation.

With a contradiction reached in both cases, the proof of Theorem 3.1 is complete. We thank G. Skandalis for a helpful conversation that led to a more explicit version of Theorem 3.1.

**Remark.** Let us show that there are countably many finite-dimensional elements $\sigma \in \hat{\Gamma}$ satisfying the assumptions of Theorem 3.1.

Thus, let $G/Z(G)$ be the adjoint group of $G$; this is a linear group. Denote by $\Gamma_1$ the image of $\Gamma$ in $G/Z(G)$; as a finitely generated linear group, $\Gamma_1$ is residually finite (see [Mal]); a non-trivial irreducible representation $\sigma$ of $\Gamma$ that factors through a finite quotient of $\Gamma_1$ cannot be contained in the restriction to $\Gamma$ of a finite-dimensional unitary representation of $G$.

This argument shows that $Ker[j_\Gamma : C^*(\Gamma) \to M(C^*(G))]$ contains the $C^*$-direct sum of countably many matrix algebras.

# 4 The cases $SO_o(n,1)$ and $SU(n,1)$.

We begin with the following result, which is certainly known to many experts (see [Moo], Proposition 3.6; compare also with [Mar], Chap. III, (1.12), Remark 1).

**PROPOSITION 4.1** *Let G be a simple Lie group with finite centre, and let $\Gamma$ be a lattice in G. Denote by $\gamma$ the quasi-regular representation of G on $L^2(G/\Gamma)$, and by $\gamma_0$ the restriction of $\gamma$ to $L^2_o(G/\Gamma) = \{f \in L^2(G/\Gamma) | < f|1 >= 0\}$.*

*(a) There exists $N \in \mathbb{N}$ such that the N-fold tensor product $\gamma_0^{\otimes N}$ is weakly contained in the left regular representation $\lambda_G$ of G.*

*(b) The trivial representation $1_G$ is not weakly contained in $\gamma_0$.*

**Proof.** (a) Suppose first that G has Kazhdan's property $(T)$. Then, by Theorems 2.4.2 and 2.5.3 in [Cow], there exists $N \in \mathbb{N}$ such that $\pi^{\otimes N}$ is weakly contained in

$\lambda_G$ for any unitary representation $\pi$ of $G$ which does not contain $1_G$. This implies the result.

Suppose now that $G$ is locally isomorphic either to $SO_o(n,1)$ or to $SU(n,1)$. Let $K$ be a maximal compact subgroup of $G$. Let $\hat{G}_1 = \{\pi \in \hat{G} \mid \pi|_K \text{ contains } 1_K\}$ be the set of all spherical representations of $G$. Observe that $\hat{G}_1$ is open in $\hat{G}$ (because $\pi \in \hat{G}_1$ if and only if there exists $\xi \in H_\pi$ such that $\int_K < \pi(k)\xi|\xi > dk \neq 0$). For a unitary representation $\sigma$ of $G$, set $Supp\,\sigma = \{\pi \in \hat{G} \mid \pi \text{ is weakly contained in } \sigma\}$. By Proposition 3.6 in [Moo], the existence of $N \in \mathbb{N}$ such that $\sigma^{\otimes N}$ is weakly contained in $\lambda_G$ is equivalent to $1_G \notin Supp\,\sigma \cap \hat{G}_1$ (the proof of this uses the explicit description of the unitary duals of $SO_o(n,1)$ and $SU(n,1)$). So we must prove that $1_G$ is not in $Supp\,\gamma_0 \cap \hat{G}_1$ or, equivalently, that $1_G$ is isolated in $Supp\gamma \cap \hat{G}_1$.

Recall the standard parametrization of $\hat{G}_1$. Let $\rho$ be half the sum of the positive roots associated with a maximal split torus of $G$. Then $\hat{G}_1$ identifies (topologically) with $i\mathbb{R}^+ \cup [0,\rho]$, the representations $\pi_s$ with $s \in i\mathbb{R}^+$ being the spherical principal series representations, those $\pi_s$ with $s \in ]0,\rho[$ being the spherical complementary series representations, and $\pi_\rho$ being the trivial representation $1_G$.

Let $X$ be the Riemannian symmetric space associated with $G$. The Laplace-Beltrami operator $\Delta$ on $X$ is invariant for the left action of $G$, so it descends to a positive, unbounded operator on $L^2(\Gamma\backslash X)$. It is well-known that $\pi_s$ is weakly contained in $\gamma$ if and only if $\rho^2 - s^2$ belongs to the spectrum of $\Delta$ on $L^2(\Gamma\backslash X)$ (see §4 of Chap. I in [GGP] for $G = SL_2(\mathbb{R})$ and $\Gamma$ uniform, or Theorem 1.7.10 in [GaV] for the general case; note that this Theorem is stated there for the quasi-regular representation of $G$ on $L^2(X)$, but the proof extends word for word to our representation $\gamma$).

Denoting by $\lambda_1(\Gamma\backslash X)$ the bottom of the spectrum of the restriction of $\Delta$ to the orthogonal of constants in $L^2(\Gamma\backslash X)$, we see that our result follows from $\lambda_1(\Gamma\backslash X) > 0$. In turn, this is a consequence of the facts that the continuous spectrum of $\Delta$ on $L^2(\Gamma\backslash X)$ is the half-line $[\rho^2, \infty[$ (see [OsW]), and that its discrete spectrum is a sequence increasing to $\infty$ (see Theorem 3 in [BoG]). In our case, $\lambda_1(G\backslash X) > 0$ can also be deduced from the fact that $\lambda_1(M) > 0$ for any complete Riemannian manifold $M$ with finite volume and pinched negative sectional curvature (see [Dod]).

(b) This follows from (a) and non-amenability of $G$.

**Proof of Theorem 1.2**

We shall use several times Fell's inner hull-kernel topology, which is defined on sets of unitary (not necessarily irreducible) representations of a locally compact group (cf. [Fel], section 2): a net $(\pi_i)_{i\in I}$ of representations converges to a representation $\pi$ if and only if $\pi$ is weakly contained in $\{\pi_j \mid j \in J\}$ for each subnet $(\pi_j)_{j\in J}$ of $(\pi_i)_{i\in I}$.

Assume that $G$ and $\Gamma$ satisfy the assumptions of Theorem 1.2. We are going to show that Fell's property (WF3) fails for the pair $(G,\Gamma)$; i.e., we shall produce some $\sigma \in \hat{\Gamma}$ such that $\sigma$ is not weakly contained in the set $\{\pi|_\Gamma \mid \pi \in \hat{G}\}$.

Since $1_\Gamma$ is not isolated in $\hat{\Gamma}_f$, there exists a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $\hat{\Gamma}_f - \{1_\Gamma\}$ that converges to $1_\Gamma$ .

**1st step:** There exists a sequence of integers $n_1 < n_2 < \ldots$, and spherical complementary series representations $\pi_{n_k}$ of $G$ such that $\pi_{n_k}$ is weakly contained in $Ind_\Gamma^G \sigma_{n_k}$ for any $k$, and $\lim_{k\to\infty} \pi_{n_k} = 1_G$ .

Indeed, by continuity of induction ([Fel], Theorem 4.1),

$$\lim_{n\to\infty} Ind_\Gamma^G \sigma_n = Ind_\Gamma^G 1_\Gamma = \gamma.$$

Since $1_G$ is a subrepresentation of $\gamma$, we also have

$$\lim_{n\to\infty} Ind_\Gamma^G \sigma_n = 1_G.$$

This implies that there exists integers $n_1 < n_2 < ...$, and irreducible representations $\pi_{n_k}$ of $G$ such that $\pi_{n_k}$ is weakly contained in $Ind_\Gamma^G \sigma_{n_k}$ for any $k$, and such that $\lim_{k\to\infty} \pi_{n_k} = 1_G$ (cf. proof of Lemme 2, §1, in [Bur]). Since the spherical dual $\hat{G}_1$ is open in $\hat{G}$, and since $G$ is not amenable, we can clearly assume that $\pi_{n_k}$ is either $1_G$, or a spherical complementary series representation. To exclude the case $\pi_{n_k} = 1_G$, we are going to show that $1_G$ is not weakly contained in $Ind_\Gamma^G \sigma_{n_k}$; this can be viewed as a form of Frobenius reciprocity.

Indeed, since $\sigma_{n_k}$ is finite-dimensional, $\sigma_{n_k}$ does not contain $1_G$ weakly. Moreover, we know by Proposition 4.1(b) that $1_G$ is isolated in $Supp\,\gamma$. Hence, by a result of Margulis ([Mar], Chap. III, (1.11)(b)), $1_G$ is not weakly contained in $Ind_\Gamma^G \sigma_{n_k}$. This proves the 1st step.

Let $\pi_{n_k} \in \hat{G}$ be a sequence as above. By Proposition 4.1(a), there exists $N \in \mathbb{N}$ such that $\gamma_0^{\otimes N}$ is weakly contained in $\lambda_G$. Since $\lim_{k\to\infty} \pi_{n_k} = 1_G$ , we see that $\pi_{n_l}^{\otimes N}$ is not weakly contained in $\lambda_G$ for $l \in \mathbb{N}$ big enough. Fix such an $l$, and set $\sigma = \sigma_{n_l}$ and $\pi = \pi_{n_l}$ .

**2nd step:** $\sigma$ is not weakly contained in $\{\rho|_\Gamma : \rho \in \hat{G}\}$. Indeed, assume by contradiction that there exists a sequence $\rho_n \in \hat{G}$ with $\lim_{n\to\infty} \rho_n|_\Gamma = \sigma$.

Then $\lim_{n\to\infty} Ind_\Gamma^G \rho_n|_\Gamma = Ind_\Gamma^G \sigma$. Hence, since $\pi$ is weakly contained in $Ind_\Gamma^G \sigma$ :

$$\lim_{n\to\infty} Ind_\Gamma^G (\rho_n|_\Gamma) = \pi\,.$$

But

$$Ind_\Gamma^G (\rho_n|_\Gamma) = \rho_n \otimes Ind_\Gamma^G 1_\Gamma = \rho_n \oplus (\rho_n \otimes \gamma_0).$$

Since $\pi$ is irreducible, this implies (upon passing to a subsequence) that either $\lim_{n\to\infty} \rho_n \otimes \gamma_0 = \pi$ or $\lim_{n\to\infty} \rho_n = \pi$.

We first exclude the case $\lim_{n\to\infty} \rho_n \otimes \gamma_0 = \pi$. Indeed, $(\rho_n \otimes \gamma_0)^{\otimes N} = \rho_n^{\otimes N} \otimes \gamma_0^{\otimes N}$ is weakly contained in $\lambda_G$. Hence, $\lim_{n\to\infty} \rho_n \otimes \gamma_0 = \pi$ would imply that $\pi^{\otimes N} = \lim_{n\to\infty} (\rho_n \otimes \gamma_0)^{\otimes N}$ is weakly contained in $\lambda_G$; this would contradict our choice of $\pi$.

It remains to exclude the case $\lim_{n\to\infty} \rho_n = \pi$. Since the set $\hat{G_1}^c = \{\pi_s \mid s \in ]0, r[\}$ of all spherical complementary series representations is open in $\hat{G}$ and since $\pi \in \hat{G_1}^c$, we can clearly assume that $\rho_n \in \hat{G_1}^c$ for all $n$. Then, there exists $s_o \in ]0, r[$ such that, for all $n$:

$$\rho_n \in \{\pi_s : 0 < s < s_o\}.$$

Therefore, there exists $M \in \mathbb{N}$ such that $\rho_n^{\otimes M}$ is weakly contained in $\lambda_G$, for all $n \in \mathbb{N}$. Hence

$$\sigma^{\otimes M} = \lim_{n\to\infty} (\rho_n^{\otimes M})|_\Gamma$$

is weakly contained in $\lambda_\Gamma$. Since $\sigma^{\otimes M}$ is finite-dimensional, this contradicts non-amenability of $G$. This concludes the proof of Theorem 1.2.

**Remark:** In our previous paper [BeV], Theorem 1.2 was already proved for $G = PSL_2(\mathbb{R})$ and $\Gamma$ the fundamental group of a closed Riemann surface of genus 2.

# 5 Some examples of lattices in $SO_o(n,1)$ and $SU(n,1)$.

Let $\Gamma$ be a lattice in a simple Lie group locally isomorphic either to $SO_o(n,1)$ or $SU(n,1)$. Let us denote by $\hat{\Gamma}_{fq}$ the set of elements of $\hat{\Gamma}$ that factor through some finite quotient of $\Gamma$.

**DEFINITION 5.1** *We say that $\Gamma$ satisfies property (*) if the trivial representation $1_\Gamma$ is not isolated in $\hat{\Gamma}_{fq}$, for the induced Fell-Jacobson topology.*

Our property (*) is precisely the negation of property $(T;R(\Im))$ in the notation of Lubotzky-Zimmer [LuZ], where a lucid discussion of this property appears on pp. 291-292. Since $\hat{\Gamma}_{fq}$ is a subset of $\hat{\Gamma}_f$, it is clear that, if $\Gamma$ satisfies (*), then $1_\Gamma$ is not isolated in $\hat{\Gamma}_f$; note the question at the bottom of p. 291 of [LuZ] whether or not the converse implication holds.

The purpose of this section is to give examples of lattices with property (*), i.e. for which Theorem 1.2 is true. We begin with a sufficient condition for property (*).

**PROPOSITION 5.2** *If $\Gamma$ has a finite index subgroup $\Gamma_o$ that maps homomorphically onto $\mathbb{Z}$, then $\Gamma$ has property (*).*

**Proof.** Let $(\chi_m)_{m\in\mathbb{N}}$ be a sequence of non-trivial characters of finite order of $\mathbb{Z}$, viewed as characters of $\Gamma_o$, that converges to the trivial character. Set:

$$\pi_m = Ind_{\Gamma_o}^{\Gamma}\chi_m \, .$$

**Claim:** $\pi_m$ factors through some finite quotient of $\Gamma$. Indeed, since $\chi_m$ has finite order, the subgroup $Ker\,\chi_m$ of $\Gamma_o$ has finite index in $\Gamma$, so there exists a normal subgroup $N_m$ of $\Gamma$, of finite index and contained in $Ker\,\chi_m$. Then $\pi_m$ factors through the finite group $\Gamma/N_m$ , which establishes the claim.

The rest of the proof is similar in spirit to the first step of the proof of Theorem 1.2, but considerably easier: by continuity of induction, the sequence $(\pi_m)_{m\in\mathbb{N}}$ converges to the quasi-regular representation $\lambda_o$ of $\Gamma$ on $l^2(\Gamma/\Gamma_o)$. Since $\lambda_o$ contains the trivial representation $1_\Gamma$, we may select for any $m \in \mathbb{N}$ an irreducible component $\sigma_m$ of $\pi_m$ in such a way that the sequence $(\sigma_m)_{m\in\mathbb{N}}$ converges to $1_\Gamma$ in $\hat{\Gamma}$. By the claim, each $\sigma_m$ lies in $\hat{\Gamma}_{fq}$; finally, no $\sigma_m$ may be trivial, by Frobenius reciprocity. This shows that $1_\Gamma$ is not isolated in $\hat{\Gamma}_{fq}$.

Because $\Gamma$ is finitely generated, the condition that $\Gamma_o$ maps homomorphically onto $\mathbb{Z}$ is equivalent to the non-vanishing of the first cohomology $H^1(\Gamma_o, \mathbb{C})$. This is known to have deep representation-theoretic consequences, as it gives information on the decomposition of $L^2(G/\Gamma_o)$ into irreducibles (see the whole of Chapter VII in [BoW], and especially Propositions 4.9 and 4.11). There is a conjecture, sometimes attributed to Thurston (see e.g. [Bor], 2.8), according to which any uniform lattice $\Gamma$ in $SO_o(n,1)$ ($n \geq 2$) admits a finite index subgroup $\Gamma_o$ such that $H^1(\Gamma_o, \mathbb{C}) \neq 0$. Next proposition summarizes what we know about this problem, both in the uniform and non-uniform cases.

**PROPOSITION 5.3** *The following lattices $\Gamma$ in $SO_o(n,1)$ admit a finite index subgroup $\Gamma_o$ such that $H^1(\Gamma_o, \mathbb{C}) \neq 0$, and hence satisfy property (*):*

*(i) any lattice in $PSL_2(\mathbb{R}) \simeq SO_o(2,1)$;*

*(ii) any non-uniform lattice in $PSL_2(\mathbb{C}) \simeq SO_o(3,1)$;*

*(iii) any uniform lattice $\Gamma$ in $PSL_2(\mathbb{C})$ such that, for some $x$ in the 3-dimensional real hyperbolic space $H_3(\mathbb{R})$, the orbit $\Gamma.x$ is invariant under some orientation-reversing involutive isometry of $H_3(\mathbb{R})$;*

*(iv) any arithmetic lattice, provided $n \neq 3$, 7; any non-uniform arithmetic lattice, without restriction on $n$.*

**Proof.** The proof is compilation; however, it makes constant use of Selberg's lemma asserting that any lattice has a torsion-free subgroup of finite index.

(i) A torsion-free lattice in $PSL_2(\mathbb{R})$ is either a surface group (in the uniform case) or a non-abelian free group (in the non-uniform case); in any case, it surjects onto $\mathbb{Z}$.

(ii) Any torsion-free non-uniform lattice in $PSL_2(\mathbb{C})$ surjects onto $\mathbb{Z}$, by Propositions 5.1 and 3.1 in [Lub].

(iii) See Theorem 3.3 and Corollary 3.4 in [Hem]. Explicit examples of such lattices are given in §4 of [Hem].

(iv) The first statement is the main result of [LiM]. For the second one, combine the main result in [Mil] with the remarks on p. 365 of [LiM].

Concerning uniform lattices in $PSL_2(\mathbb{C})$, it seems appropriate to mention here the connection with a somewhat (in)famous question which is for sure due to Thurston (question 18 in [Thu]): does any complete, finite-volume, hyperbolic 3-manifold have a finite-sheeted cover that fibers over the circle $S^1$? An affirmative answer would imply that any lattice $\Gamma$ in $PSL_2(\mathbb{C})$ satisfies property (*) (indeed, let $\Gamma_1$ be a torsion-free subgroup of finite index in $\Gamma$; then $\Gamma_1$ is the fundamental group, $\pi_1(M)$, of a complete finite-volume hyperbolic 3-manifold $M$; if $N$ is a finite-sheeted cover of $M$ which fibers over $S^1$, then $\Gamma_o = \pi_1(N)$ is a finite-index subgroup of $\Gamma_1$ that maps onto $\pi_1(S^1) = \mathbb{Z}$). For an example of a compact hyperbolic 3-manifold that does not fiber over $S^1$ but with a finite-sheeted cover that does, see example 2.1 in [Gab][1].

[1]Clearly, for a 3-dimensional closed hyperbolic manifold $M$, fibering over $S^1$ is a much stronger condition than having non-zero first Betti number. Algebraically, this can be seen by Stallings'fibration theorem [Sta]: if $N$ is a normal subgroup of $\pi_1(M)$ such that $\pi_1(M)/N = \mathbb{Z}$, then $N$ comes from a fibration of $M$ over $S^1$ if and only if $N$ is a finitely generated subgroup. Also, surface groups in $\pi_1(M)$ that come from some finite-sheeted cover of $M$ fibering over $S^1$ (so-called virtual fibre groups) have been characterized algebraically in Corollary 1 of [Som].

In contrast with Proposition 5.3, we are not aware of any "large" class of lattices in $SU(n,1)$ that satisfies property (*). Essentially the only result we know is that, for any $n \geq 2$, there exists a uniform arithmetic lattice $\Gamma$ in $SU(n,1)$ such that $H^1(\Gamma, \mathbb{C}) \neq 0$ (see Theorem 1 in [Kaz], or Theorem 1.4(b) in [Li]).

To conclude, let us indicate why, for a given lattice $\Gamma$ in $SO_o(n,1)$, it is usually difficult to check that $\Gamma$ satisfies property (*). Assume that $\Gamma$ is arithmetic. It is then easy to construct elements of $\hat{\Gamma}_{fq}$ : take a congruence subgroup $\Gamma(\wp)$ and consider irreducible representations of $\Gamma$ that factor through the finite group $\Gamma/\Gamma(\wp)$ (for $\Gamma = SL_2(\mathbb{Z})$, these are representations that factor through some $SL_2(\mathbb{Z}/n\mathbb{Z})$). Denote by $\hat{\Gamma}_{arith}$ the subset of elements in $\hat{\Gamma}_{fq}$ that factor through some $\Gamma/\Gamma(\wp)$; then it follows from Selberg's inequality (see [Sel] for $n = 2$, and corollary 1.3 in [BuS] for $n > 2$) that the trivial representation $1_\Gamma$ is isolated in $\hat{\Gamma}_{arith}$. Thus, if $\Gamma$ verifies property (*), any non-stationary net in $\hat{\Gamma}_{fq}$ that converges to $1_\Gamma$ will have to leave $\hat{\Gamma}_{arith}$ eventually.

We have been informed by M. Burger that, in unpublished work with P. Sarnak, similar phenomena have been obtained for a large class of arithmetic lattices in $SU(n,1)$.

# 6 Proof of Theorem 1.3

We begin with hereditary properties of the class of groups satisfying Fell's property (WF3).

**LEMMA 6.1** *Let $G$ be a locally compact group with property (WF3).*

*(a) Any closed subgroup of $G$ has property (WF3).*

*(b) Let $K$ be a compact normal subgroup of $G$; then $G/K$ has property (WF3).*

**Proof.** (a) is obvious. To see (b), denote by $p : G \to G/K$ the quotient map. Let $L$ be a closed subgroup of $G/K$; fix $\tau \in \hat{L}$. Set $H = p^{-1}(L)$ and $\sigma = \tau \circ (p|_H)$. Let $\pi$ be a representation of $G$ on a Hilbert space $\mathcal{H}$ such that $\pi|_H$ weakly contains $\sigma$. Let $\mathcal{H}^K$ be the space of $K$-fixed vectors in $\mathcal{H}$. Since $K$ is a normal subgroup, $\mathcal{H}^K$ is an invariant subspace of $\pi$, and we denote by $\pi_o$ the restriction of $\pi$ to $\mathcal{H}^K$. Since $K$ is compact and $\sigma$ is irreducible, it is easy to see that $\sigma$ is weakly contained in $\pi_o|_H$. But $\pi_o|_H$ can be viewed as a representation of $L = H/K$, that weakly contains $\tau$.

Next lemma is probably well-known.

**LEMMA 6.2** *Let $G$ be a Lie group, and let $S$ be a semisimple analytic subgroup. The closure $\bar{S}$ of $S$ is reductive.*

**Proof.** We begin with a

Claim: Let $h$ be a finite-dimensional Lie algebra, and let $s$ be a semisimple ideal; then there exists an ideal $j$ of $h$ such that $h = s \oplus j$. Indeed, let $Der(s)$ be the Lie algebra of derivations of $s$. Since $s$ is an ideal in $h$, we have a Lie algebra homomorphism:

$$\alpha : h \to Der(s) \; : \; X \to ad(X)_{|s}$$

the kernel of which is precisely the centralizer of $s$ in $h$; set $j = Ker\,\alpha$. Since $s$ is semisimple, $Der(s)$ is canonically isomorphic to $s$, so that $\alpha$ is onto and $h = s \oplus j$; this establishes the claim.

To prove Lemma 6.2, denote by $s$ and $h$ the Lie algebras of $S$ and $\bar{S}$ respectively. Clearly $Ad(x)(s) = s$ for any $x$ in $S$, so by density the same is true for any $x$ in $\bar{S}$. This shows that $s$ is an ideal in $h$. By the claim, there exists an ideal $j$ of $h$ such that $h = s \oplus j$. To see that $h$ is reductive, it is enough to prove that $j$ is central in $h$. But, for $X \in j$, we have $Ad(x)(X) = X$ for any $x$ in $S$; again by density, this remains true for any $x$ in $\bar{S}$; so $X$ is central in $h$.

**Proof of Theorem 1.3** It is easy to see that any amenable group $G$ satisfies property (WF3); indeed, for a closed subgroup $H$ of $G$, by amenability of $H$ any representation of $H$ is weakly contained in the left regular representation of $H$, which is itself contained in the restriction to $H$ of the left regular representation of $G$; see also Corollary 1.5 in [BLS] for another proof.

Let us now prove the converse, namely that any almost connected locally compact group $G$ with property (WF3) is amenable. In this proof, the stability of amenability under short exact sequences will be used constantly.

**1st step: reduction to the connected case.** Let $G_0$ be the connected component of the identity of $G$. By Lemma 6.1(a), $G_0$ has property (WF3). If $G_0$ is amenable, then so is $G$, since $G/G_0$ is compact.

**2nd step: reduction to the Lie group case.** Let $G$ be a connected group with property (WF3). By the structure theory for connected groups, $G$ admits a compact normal subgroup $K$ such that $G/K$ is a Lie group. By Lemma 6.1(b), $G/K$ has property (WF3). If $G/K$ is amenable, then so is $G$, since $K$ is compact.

**3rd step: reduction to the reductive case.** Let $G$ be a connected Lie group with property (WF3). Let $G = RS$ be a Levi decomposition, with $R$ the solvable radical and $S$ a semisimple analytic subgroup. Then the closure $\bar{S}$ is reductive with property (WF3), by Lemmas 6.1(a) and 6.2. If $\bar{S}$ is amenable, then so is $\bar{S}/(\bar{S} \cap R) = G/R$, hence so is $G$.

**Coda.** Let $G$ be a connected, reductive Lie group with property (WF3). The adjoint group $G/Z(G)$ is a semisimple Lie group without centre, so it decomposes as a direct product

$$G/Z(G) = G_1 \times \cdots \times G_n$$

of simple Lie groups without centre. To prove that $G$ is amenable, we have to show that $G_j$ is compact for $j = 1, \cdots, n$. So suppose by contradiction that some $G_j$, say $G_1$, is not compact. By root theory, $G_1$ then contains a 3-dimensional analytic subgroup $L$ which is locally isomorphic to $SL_2(\mathbb{R})$. Because $G_1$ is centreless, hence linear, $L$ is closed in $G_1$ (any semisimple analytic subgroup in a linear group is closed, see Theorem 2 in [Got]). By the proof of Theorem 3 in [BeV], there exists a lattice $\Gamma$ in $L$ and a representation $\tau \in \hat{\Gamma}_f$ such that $\tau \otimes \bar{\tau}$ is not weakly contained in the restriction to $\Gamma$ of any unitary representation of $L$. Denote by $p : G \to G_1$ the homomorphism obtained by composing the quotient map $G \to G/Z(G)$ with the projection of $G/Z(G)$ onto $G_1$. Set $H = p^{-1}(\Gamma)$ and $\sigma = \tau \circ (p|_H)$. Because $H$ is closed in $G$, we find by property

(WF3) a net $(\rho_i)_{i\in I}$ in $\hat{G}$ such that

$$lim_i\, \rho_i|_H = \sigma\,.$$

Then:

$$lim_i(\rho_i \otimes \bar{\rho}_i)|_H = \sigma \otimes \bar{\sigma}\,.$$

But, since $\rho_i$ is irreducible, the representation $\rho_i \otimes \bar{\rho}_i$ of $G$ is trivial on $Z(G)$, so it factors through a representation $\pi_i$ of $G/Z(G)$. Last formula then reads:

$$lim_i\, \pi_i|_\Gamma = \tau \otimes \bar{\tau}$$

and this contradicts our choice of $G$ and $\tau$.

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