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Ergodic Actions of Compact Matrix Pseudogroups on C*-algebras

Florin P. Boca

Dedicated to Professor Masamichi Takesaki
on the ocasion of his 60th birthday

Let G be a compact group acting on a unital C^*-algebra $\mathcal{M}$. The action is said to be ergodic if the fixed point algebra $\mathcal{M}^G$ reduces to scalars. The first breakthrough in the study of such actions was the finiteness theorem of Høegh-Krohn, Landstad and Størmer [HLS]. They proved that the multiplicity of each $\pi \in \widehat{G}$ in $\mathcal{M}$ is at most $\dim(\pi)$ and the unique G-invariant state on $\mathcal{M}$ is necessarily a trace. When combined with Landstad's result [L] that finite-dimensionality for the spectral subspaces of actions of compact groups implies that the crossed product is a type I C^*-algebra (and in fact, as pointed out in [Wa1] is a direct sum of algebras of compact operators), the finiteness theorem shows that the crossed product of a unital C^*-algebra by an ergodic action of a compact group is necessarily equal to $\oplus_i \mathcal{K}(\mathcal{H}_i)$.

The study of such actions was essentialy pushed forward by Wassermann. He developed an outstanding machinery based on the notion of multiplicity maps, establishing a remarkable connection with the equivariant K-theory. This approach allowed him to prove, among other important things, the strong negative result that $SU(2)$ cannot act ergodically on the hyperfinite II_1 factor [Wa3].

The aim of this note is to study ergodic actions of Woronowicz's compact matrix pseudogroups on unital C^*-algebras, extending some of the previous results. In this insight we prove in §1 the analogue of the finiteness theorem. More precisely, if $G = (A, u)$ is a compact matrix pseudogroup acting ergodically on a unital C^*-algebra by a coaction $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$ such that $\sigma(\mathcal{M})(1_{\mathcal{M}} \otimes A)$ is dense in $\mathcal{M} \otimes A$, then there is a decomposition of the *-algebra of σ-finite elements into isotypic subspaces $\mathcal{M}_0 = \oplus_{\alpha \in \widehat{G}} \mathcal{M}_\alpha$, orthogonal with respect to the scalar product induced on $\mathcal{M}$ by the unique σ-invariant state ω. Moreover, the spectral subspaces $\mathcal{M}_\alpha$ are finite dimensional and $\dim(\mathcal{M}_\alpha) \leq M_\alpha^2$, M_α being the *quantum dimension* of $\alpha \in \widehat{G}$. If the Haar measure is faithful on A, then $\mathcal{M}_0$ is dense in the GNS Hilbert space $\mathcal{H}_\omega$.

Although ω is not in general a trace, we prove the existence of a multiplicative linear map $\Theta : \mathcal{M}_0 \to \mathcal{M}_0$ such that $\omega(xy) = \omega(\Theta(y)x)$ for all $x \in \mathcal{M}, y \in \mathcal{M}_0$ and Θ is a scalar multiple of the modular operator F_α when restricted to each irreducible σ-invariant subspace of the spectral subspace $\mathcal{M}_\alpha$.

The crossed products by such coactions are studied in §2 where we prove, using the Takesaki-Takai type duality theorem of Baaj and Skandalis [BS], that they are isomorphic all the time to a direct sum of C^*-algebras of compact operators. As a corollary, if a compact matrix pseudogroup with underlying nuclear C^*-algebra acts ergodically on a unital C^*-algebra $\mathcal{M}$, then $\mathcal{M}$ follows nuclear.

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§1. The isotypic decomposition and finiteness of multiplicities for ergodic actions

We start with a couple of definitions.

Definition 1 ([BS]) *A coaction of a unital Hopf C^*-algebra (A, Δ_A) on a unital C^*-algebra $\mathcal{M}$ is a unital one-to-one *-homomorphism $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$ (the tensor product will be all the time the minimal C^*-one) that makes the following diagram commutative*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\sigma} & \mathcal{M} \otimes A \\ \sigma \uparrow & & \uparrow \quad id_{\mathcal{M}} \otimes \Delta_A \\ \mathcal{M} \otimes A & \xrightarrow{\sigma \otimes id_A} & \mathcal{M} \otimes A \otimes A \end{array}$$

A C^*-algebra $\mathcal{M}$ with a coaction σ of (A, Δ_A) is called an A-algebra if σ is one-to-one and $\sigma(\mathcal{M})(1_{\mathcal{M}} \otimes A)$ is dense in $\mathcal{M} \otimes A$.

Definition 2 *The fixed points of the coaction $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$ are the elements of $\mathcal{M}^\sigma = \{x \in \mathcal{M} \mid \sigma(x) = x \otimes 1_A\}$. The coaction σ is called ergodic if $\mathcal{M}^\sigma = \mathbf{C}1_{\mathcal{M}}$.*

We denote by $\mathcal{M}^*$ the set of continuous linear functionals on $\mathcal{M}$.

Definition 3 *$\phi \in \mathcal{M}^*$ is called σ-invariant if*
$(\phi \otimes \psi)(\sigma(x)) = (\phi \otimes \psi)(x \otimes 1_A) = \psi(1_A)\phi(x)$ *for all* $\psi \in A^*$.

Let $G = (A, u)$ be a compact matrix pseudogroup group with comultiplication $\Delta_A : A \to A \otimes A$, smooth structure $\mathcal{A}$ and coinverse $\kappa : \mathcal{A} \to \mathcal{A}$ (cf [Wor]). Then A^* is an algebra with respect to the convolution $\phi * \psi = (\phi \otimes \psi)\Delta_A$, $\phi, \psi \in A^*$ and there exists a unique state h on A, called the Haar measure of G, so that $\phi * h = h * \phi = \phi(1_A)h$ for all $\phi \in A^*$. Let $\mathcal{M}$ be a unital C^*-algebra which is an A-algebra via the coaction $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$ and consider $\theta = (id_{\mathcal{M}} \otimes h)\sigma$.

Lemma 4 *i) $\theta(x) \in \mathcal{M}^\sigma$ for all $x \in \mathcal{M}$. Moreover θ is a conditional expectation from $\mathcal{M}$ onto $\mathcal{M}^\sigma$.*

ii) If σ is ergodic, then $\theta(x) = \omega(x)1_{\mathcal{M}}$ and ω is the only σ-invariant state on $\mathcal{M}$.

Proof. i) Note that

$$\begin{aligned}\sigma(\theta(x)) &= \sigma((id_{\mathcal{M}} \otimes h)(\sigma(x)) = (\sigma \otimes h)(\sigma(x))\\ &= (id_{\mathcal{M}} \otimes id_A \otimes h)(\sigma \otimes id_A)(\sigma(x))\\ &= (id_{\mathcal{M}} \otimes id_A \otimes h)(id_{\mathcal{M}} \otimes \Delta_A)(\sigma(x))\\ &= (id_{\mathcal{M}} \otimes (id_A \otimes h)\Delta_A)(\sigma(x)), \qquad x \in \mathcal{M}.\end{aligned}$$

Since $(id_A \otimes h)(\Delta_A(a)) = h * a = h(a)1_A$, $a \in A$ ([Wor, 4.2]), we obtain further :

$$\begin{aligned}&(\psi_1 \otimes \psi_2)((id_{\mathcal{M}} \otimes (id_A \otimes h)\Delta_A)(y \otimes a))\\ &\quad= (\psi_1 \otimes \psi_2)(y \otimes (id_A \otimes h)(\Delta_A(a)))\\ &\quad= (\psi_1 \otimes \psi_2)(y \otimes h(a)1_A) = \psi_1(y)\psi_2(1_A)h(a)\\ &\quad= (\psi_1 \otimes \psi_2)((id_{\mathcal{M}} \otimes h)(y \otimes a) \otimes 1_A),\end{aligned}$$

for all $y \in \mathcal{M}, a \in A, \psi_1 \in \mathcal{M}^*, \psi_2 \in A^*$. Therefore for $x \in \mathcal{M}, \psi_1 \in \mathcal{M}^*, \psi_2 \in A^*$ we have :

$$(\psi_1 \otimes \psi_2)(\sigma(\theta(x)) = (\psi_1 \otimes \psi_2)((id_{\mathcal{M}} \otimes h)\sigma(x) \otimes 1_A) = (\psi_1 \otimes \psi_2)(\theta(x) \otimes 1_A)$$

and consequently $\sigma(\theta(x)) = \theta(x) \otimes 1_A$ for all $x \in \mathcal{M}$. θ is a norm one projection since

$$\theta(x) = (id_{\mathcal{M}} \otimes h)(\sigma(x)) = (id_{\mathcal{M}} \otimes h)(x \otimes 1) = x, \qquad x \in \mathcal{M}^\sigma.$$

ii) Let $\psi \in A^*$. Then we get for all $x \in \mathcal{M}$:

$$\begin{aligned}(\omega \otimes \psi)(\sigma(x)) &= ((id_{\mathcal{M}} \otimes h)\sigma \otimes \psi)(\sigma(x))\\ &= (id_{\mathcal{M}} \otimes \psi)((id_{\mathcal{M}} \otimes h)\sigma \otimes id_A)(\sigma(x))\\ &= (id_{\mathcal{M}} \otimes \psi)(id_{\mathcal{M}} \otimes h \otimes id_A)((\sigma \otimes id_A)\sigma(x))\\ &= (id_{\mathcal{M}} \otimes \psi)(id_{\mathcal{M}} \otimes h \otimes id_A)((id_{\mathcal{M}} \otimes \Delta_A)\sigma(x))\\ &= (id_{\mathcal{M}} \otimes \psi)(id_{\mathcal{M}} \otimes (h \otimes id_A)\Delta_A)(\sigma(x))\\ &= \psi(1_A)(id_{\mathcal{M}} \otimes h)(\sigma(x)) = \psi(1_A)\omega(x),\end{aligned}$$

therefore ω is a σ-invariant state on $\mathcal{M}$. Finally, assume that ϕ is a σ-invariant state on $\mathcal{M}$. Then for all $x \in \mathcal{M}$:

$$\phi(x) = (\phi \otimes h)(\sigma(x)) = \phi((id_{\mathcal{M}} \otimes h)(\sigma(x))) = \phi(\theta(x)) = \phi(\omega(x)1_{\mathcal{M}}) = \omega(x). \qquad \square$$

Remarks. 1). The proof of the previous statement doesn't use the faithfulness of σ but only the equality $(\sigma \otimes id_A)\sigma = (id_{\mathcal{M}} \otimes \Delta_A)\sigma$.

2). Since the tensor product of two faithful completely positive maps is still faithful [T], it follows that if σ is one-to-one and h is faithful on A, then ω is a faithful state on $\mathcal{M}$.

3). Although one can easily pass from a compact matrix pseudogroup to the reduced one, which has faithful Haar measure, as indicated at page 656 in [Wor], it turns out that the Haar measure is faithful in several important examples (e.g. on commutative CMP, on reduced cocommutative CMP or, cf. [N], on $SU_\mu(N)$).

Denote by $\mathcal{H}_\omega$ the completion of $\mathcal{M}$ with respect to the inner product $\langle x,y\rangle_2 = \omega(y^*x)$ and let $\mathcal{M}$ acting on $\mathcal{H}_\omega$ in the GNS representation. Consider the C^*-Hilbert module $\mathcal{H}_\omega \otimes A$ with the A-valued inner product $\langle x_\omega \otimes a, y_\omega \otimes b\rangle_A = \omega(y^*x)b^*a$, for $x,y \in \mathcal{M}$, $a,b \in A$, which can be viewed as $\mathcal{M}\otimes A$ in the Stinespring representation of the completely positive map $\omega \otimes id_A$. Define also $V : \mathcal{H}_\omega \otimes A \to \mathcal{H}_\omega \otimes A$ by :

$$V(\sum_i (x_i)_\omega \otimes a_i) = \sum_i \sigma(x_i)(1_\omega \otimes a_i), \quad x_i \in \mathcal{M}, a_i \in A.$$

Lemma 5 *V is a unitary in $\mathcal{L}(\mathcal{H}_\omega \otimes A) = M(\mathcal{K}(\mathcal{H}_\omega)\otimes A)$ (the multipliers of the C^*-algebra $\mathcal{K}(\mathcal{H}_\omega)\otimes A$) and $\sigma(x) = V(x\otimes 1_A)V^*$, $x \in \mathcal{M}$.*

Proof. For any $\phi \in A^*, a, b \in A$, denote by $\phi(a\cdot b) \in A^*$ the linear functional $\phi(a\cdot b)(x) = \phi(axb)$, $x \in A$. The σ-invariance of ω yields :

$$\begin{aligned}&\phi((\omega\otimes id_A)((1_\mathcal{M}\otimes b^*)\sigma(x)(1_\mathcal{M}\otimes a)) = (\omega\otimes\phi(b^*\cdot a))(\sigma(x))\\ &\quad = \phi(b^*\cdot a)(1_A)\omega(x) = \phi(b^*a)\omega(x), \quad x\in\mathcal{M}, a,b\in A, \phi\in A^*,\end{aligned}$$

therefore we have for all $a,b \in A, x,y \in \mathcal{M}$:

$$\begin{aligned}&\langle V(x_\omega\otimes a), V(y_\omega\otimes b)\rangle_A = \langle\sigma(x)(1_\omega\otimes a), \sigma(y)(1_\omega\otimes b)\rangle_A\\ &\quad = (\omega\otimes id_A)((1_\mathcal{M}\otimes b^*)\sigma(y^*x)(1_\mathcal{M}\otimes a)) = \omega(y^*x)b^*a = \langle x_\omega\otimes a, y_\omega\otimes b\rangle_A\end{aligned}$$

and V follows isometry on $\mathcal{H}_\omega\otimes A$. Furthermore V is unitary since $\sigma(\mathcal{M})(1_\mathcal{M}\otimes A)$ is dense in $\mathcal{M}\otimes A$ and the relation $V(x\otimes 1_A) = \sigma(x)V$, $x\in\mathcal{M}$, is obvious. □

Definition 6 ([BS]) *A corepresentation of the Hopf C^*-algebra (A,Δ_A) is a unitary $V \in \mathcal{L}(\mathcal{H}_V\otimes A) = M(\mathcal{K}(\mathcal{H}_V)\otimes A)$ such that*

$$V_{12}V_{13} = (id_{\mathcal{L}(\mathcal{H}_V)}\otimes\Delta_A)(V).$$

All the corepresentations throughout this paper will be unitary unless specified otherwise. Note that in the case when $\dim\mathcal{H}_V < \infty$ V is called in [Wor] a (finite dimensional) representation of the quantum matrix pseudogroup $G = (A,u)$. Thus the representations of the quantum matrix pseudogroup $G = (A,u)$ are the corepresentations of (A,Δ_A) and we will call them simply the corepresentations of A.

Lemma 7 *The unitary V from Lemma 5 is a corepresentation of A.*

Proof. Let $T = (id_{\mathcal{L}(\mathcal{H}_\omega)} \otimes \Delta_A)(V) \in \mathcal{L}(\mathcal{H}_\omega \otimes A \otimes A) = M(\mathcal{K}(\mathcal{H}_\omega) \otimes A \otimes A)$. Since $1_\omega \otimes 1_A$ is fixed by V, then $\langle V(1_\omega \otimes 1_A), x_\omega^* \otimes 1_A \rangle_A = (\omega(x\cdot) \otimes id_A)(V) = \omega(x)1_A$ and consequently :

$$\begin{aligned}(\omega(x\cdot) \otimes id_{A\otimes A})(id_{\mathcal{L}(\mathcal{H}_\omega)} \otimes \Delta_A)(V) &= (\omega(x\cdot) \otimes \Delta_A)(V) \\ &= \Delta_A((\omega(x\cdot) \otimes id_A)(V)) = \Delta_A(\omega(x)1_A) = \omega(x)1_{A\otimes A}, \quad x \in \mathcal{M}.\end{aligned}$$

Then, for any $a, a', b, b' \in A$:

$$\begin{aligned}\langle T(1_\omega \otimes a \otimes b), x_\omega \otimes a' \otimes b' \rangle_A &= (\omega \otimes id_{A\otimes A})((x^* \otimes a'^* \otimes b'^*)T(1_M \otimes a \otimes b)) \\ &= (a'^* \otimes b'^*)((\omega(x^*\cdot) \otimes id_{A\otimes A})(T))(a \otimes b) \\ &= \omega(x^*)a'^*a \otimes b'^*b = \langle 1_\omega \otimes a \otimes b, x_\omega \otimes a' \otimes b' \rangle_A,\end{aligned}$$

therefore $T(1_\omega \otimes a \otimes b) = 1_\omega \otimes a \otimes b$. Furthermore, since T is unitary, we also have $T^*(1_\omega \otimes a \otimes b) = 1_\omega \otimes a \otimes b$ and for any $x \in \mathcal{M}, a, b \in A$:

$$\begin{aligned}V_{12}V_{13}(x_\omega \otimes a \otimes b) &= V_{12}(\sigma(x)_{13}(1_\omega \otimes a \otimes b)) = (\sigma \otimes id_A)(\sigma(x))(1_\omega \otimes a \otimes b) \\ &= (id_{\mathcal{M}} \otimes \Delta_A)(\sigma(x))(1_\omega \otimes a \otimes b) \\ &= T(x \otimes 1_A \otimes 1_A)T^*(1_\omega \otimes a \otimes b) = T(x_\omega \otimes a \otimes b).\end{aligned}$$

Remark 8 *Let $W \in \mathcal{L}(\mathcal{H}_W \otimes A)$ be a corepresentation, $\mathcal{H}_V \subset \mathcal{H}_W$ be a closed subspace of $\mathcal{H}_W$ and denote by P the orthogonal projection from $\mathcal{H}_W$ onto $\mathcal{H}_V$. If $(P \otimes id_A)W = W(P \otimes id_A)$, then $V = W(P \otimes id_A) \in \mathcal{L}(\mathcal{H}_V \otimes A)$ is by definition a subcorepresentation of W. Clearly $V^\perp = W(P^\perp \otimes id_A)$, where $P^\perp = I_{\mathcal{L}(\mathcal{H}_W)} - P$, is also a subcorepresentation of W.*

For $V \in M(\mathcal{K}(\mathcal{H}_V) \otimes A)$ and $\rho \in A^*$ define as in [Wor] $V_\rho = (id_{\mathcal{L}(\mathcal{H}_V)} \otimes \rho)V \in \mathcal{L}(\mathcal{H}_V)$. Then Lemma 6 yields for any $\rho, \rho' \in A^*$:

$$\begin{aligned}V_\rho V_{\rho'} &= (id \otimes \rho)(V)(id \otimes \rho')(V) = (id \otimes \rho \otimes \rho')(V_{12}V_{13}) \\ &= (id \otimes \rho \otimes \rho')(id \otimes \Delta_A)(V) = (id \otimes (\rho * \rho'))(V) = V_{\rho*\rho'}.\end{aligned}$$

One checks immediately as in [Wor, 4.3] that $E = V_h$ is the projection of $\mathcal{H}_V$ onto the subspace $\{\xi \in \mathcal{H}_V \mid V_\rho \xi = \rho(1_A)\xi \ , \forall \rho \in A^*\}$ of all V-invariant vectors of $\mathcal{H}_V$.

The tensor product of the corepresentations $V \in \mathcal{L}(\mathcal{H}_V \otimes A)$ and $W \in \mathcal{L}(\mathcal{H}_W \otimes A)$ is defined as in the finite dimensional case by $V \odot W = V_{13}W_{23} \in \mathcal{L}(\mathcal{H}_V \otimes \mathcal{H}_W \otimes A)$ and is still a corepresentation since

$$\begin{aligned}(id \otimes \Delta_A)(V \odot W) &= (id \otimes \Delta_A)(V_{13})(id \otimes \Delta_A)(W_{23}) \\ &= (V_{13}W_{23})_{12}(V_{13}W_{23})_{13} = (V \odot W)_{12}(V \odot W)_{13}.\end{aligned}$$

When $\dim \mathcal{H}_W < \infty$, W^c denotes as in [Wor] its contragradient corepresentation, acting on the conjugate Hilbert space $\mathcal{H}'_W$.

Lemma 9 *If the Haar measure is faithful on A and $V \in M(\mathcal{K}(\mathcal{H}_V) \otimes A)$ is a corepresentation of A such that $(V \odot \alpha^c)_h = 0$ for all $\alpha \in \widehat{G}$, then $V = 0$.*

Proof. Denote d_α =dim(α). Then $\alpha = \sum_{i,j=1}^{d_\alpha} e_{ij} \otimes u_{ij}^\alpha$, $u_{ij} \in \mathcal{A}$, where $\{e_{ij}\}_{1\leq i,j\leq d_\alpha}$ is the matrix unit of $\mathcal{L}(\mathcal{H}_\alpha)$ and $\alpha^c = \sum_{i,j=1}^{d_\alpha} e_{ij}^T \otimes u_{ji}^{\alpha *} \in \mathcal{L}(\mathcal{H}'_\alpha) \otimes \mathcal{A}$. If ϕ_{ij} denotes the linear functional on $\mathcal{L}(\mathcal{H}'_\alpha)$ defined by $\phi_{ij}(e_{kl}^T) = \delta_{ik}\delta_{jl}$ we obtain for all $\phi \in \mathcal{L}(\mathcal{H}_V)_*$:

$$(\phi \otimes \phi_{ij} \otimes id_A)(id_{\mathcal{L}(\mathcal{H}_V \otimes \mathcal{H}'_\alpha)} \otimes h)(V \odot \alpha^c) = h((\phi \otimes id_A)(V) u_{ji}^{\alpha *}).$$

Since $\{u_{ij}^\alpha\}_{1\leq i,j\leq d_\alpha, \alpha \in \widehat{G}}$ is a linear basis in $\mathcal{A}$, then $h((\phi \otimes id_A)(V)a) = 0$ for all $a \in \mathcal{A}$ and therefore for all $a \in A$. But h is faithful on A hence $(\phi \otimes id_A)(V) = 0$ for all $\phi \in \mathcal{L}(\mathcal{H}_V)_*$ and therefore $V = 0$. □

Corollary 10 *i) If the Haar measure is faithful on A, then there are no irreducible infinite dimensional corepresentations of A.*

ii) All the finite dimensional corepresentations (not necessarily unitary) of A are smooth (compare with a related question at page 636 in [Wor]).

Proof. i) Let V be an infinite dimensional corepresentation of A on $\mathcal{H}_V$. By the previous Lemma, there exists $\alpha \in \widehat{G}$ such that $(V \odot \alpha^c)_h \neq 0$. Pick a nonzero element $S \in \text{Ran}(V \otimes \alpha^c)_h$. Since α is finite dimensional, $S \in \text{Mor}(\alpha, V)$. But Ker$S$ is α-invariant and α irreducible, thus S is one-to-one and we get the V-invariant finite-dimensional subspace Ran$S \subset \mathcal{H}_V$, contradicting the irreducibility of V.

ii) Consider now a finite dimensional corepresentation V of A, not necessarily unitary. The statement in the previous Lemma holds true, thus either V is completely degenerate or there exists $\alpha \in \widehat{G}$ such that $(V \odot \alpha^c)_h \neq 0$ and $0 \neq S \in \text{Mor}(\alpha, V)$. It follows that $\mathcal{H}_V$ contains a V-invariant subspace $\mathcal{K} = \text{Ran}S$ and $V\,|_{\mathcal{K}}$ is equivalent to α, therefore it is smooth, irreducible and nondegenerate. By [Wor, Prop.4.6] $\mathcal{K}$ has a V-invariant complement and the process continues until we write $\mathcal{H}_V$ as a direct sum of linear subspaces $\mathcal{H}_V = \oplus_i \mathcal{H}_i \oplus \mathcal{H}_0$, each subspace being V-invariant, $V\,|_{\mathcal{H}_i}$ being equivalent to a corepresentation from $\widehat{G}$ and $V\,|_{\mathcal{H}_0}$ completely nondegenerate. □

The statements in the previous Corollary are implicit in S. L. Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups, *Invent. Math. 93(1988), 35-76*, as the referee informed as.

For any $\alpha \in \widehat{G}$ consider as in [Wor, §5] $\rho_\alpha \in A^*$, $\rho_\alpha(a) = M_\alpha h((f_1 * \chi_\alpha)^* a)$. Then $\rho_\alpha * \rho_\beta = \delta_{\alpha\beta}\rho_\alpha$ for all $\alpha, \beta \in \widehat{G}$. Therefore, if $U \in M(\mathcal{K}(\mathcal{H}_U) \otimes A)$ is a representation of A, $P_\alpha = U_{\rho_\alpha} = (id_{\mathcal{L}(\mathcal{H}_U)} \otimes \rho_\alpha)(U)$ are mutually orthogonal projections, but they are not self-adjoint in general. Consider also the bounded linear maps $P_\alpha : \mathcal{M} \to \mathcal{M}$, $P_\alpha(x) = (id_{\mathcal{M}} \otimes \rho_\alpha)(\sigma(x))$. Then we have for all $x \in \mathcal{M}$:

$$\begin{aligned}
P_\alpha P_\beta(x) &= (id_{\mathcal{M}} \otimes \rho_\alpha)(\sigma((id_{\mathcal{M}} \otimes \rho_\beta)(\sigma(x)))\\
&= (id_{\mathcal{M}} \otimes \rho_\alpha)((id_{\mathcal{M}\otimes\mathcal{A}} \otimes \rho_\beta)(\sigma \otimes id_A)(\sigma(x)))\\
&= (id_{\mathcal{M}} \otimes \rho_\alpha)((id_{\mathcal{M}\otimes\mathcal{A}} \otimes \rho_\beta)(id_{\mathcal{M}} \otimes \Delta_A)(\sigma(x)))\\
&= (id_{\mathcal{M}} \otimes \rho_\alpha(id_A \otimes \rho_\beta)\Delta_A)(\sigma(x))\\
&= (id_{\mathcal{M}} \otimes (\rho_\alpha * \rho_\beta))(\sigma(x)) = \delta_{\alpha\beta} P_\alpha(x).
\end{aligned}$$

We denote by $\mathcal{M}_\alpha$ the closed subspace $P_\alpha(\mathcal{M})$ of $\mathcal{M}$, $\alpha \in \widehat{G}$ and call it the *spectral subspace* associated with α.

The following lemma contains a couple of properties of the functionals $\rho_\alpha \in A^*$, $\alpha \in \widehat{G}$.

Lemma 11 *i)* $\mathcal{M}_{\alpha^c} = \mathcal{M}_\alpha^*$ *for all* $\alpha \in \widehat{G}$.
ii) $(h \otimes \rho_{\beta^c} \otimes \rho_\alpha)(\Delta_A(a)_{12}\Delta_A(b)_{13}) = 0$ *for all* $a, b \in A, \alpha \neq \beta$ *in* $\widehat{G}$.

Proof. i) Let $x \in \mathcal{M}_\alpha$. Then $x = (id_\mathcal{M} \otimes \rho_\alpha)(\sigma(x))$ and we have to check that $x^* = (id_\mathcal{M} \otimes \rho_{\alpha^c})(\sigma(x^*))$. Obvious computations which are implicit in [Wor, 5.6] show that $(f_1 * \chi_{\alpha^c})^* = \chi_\alpha * f_{-1}$ and $M_\alpha = M_{\alpha^c}$, where $M_\alpha = f_{-1}(\chi_\alpha) = f_1(\chi_\alpha)$, therefore we have for any $a \in A$:

$$\begin{aligned}\overline{\rho_\alpha(a)} &= M_\alpha \overline{h((f_1 * \chi_\alpha)^* a)} = M_\alpha h(a^*(f_1 * \chi_\alpha)) \\ &= M_\alpha h(a^*(f_1 * \chi_\alpha * f_{-1} * f_1)) = M_\alpha h((\chi_\alpha * f_{-1})a^*) \\ &= M_\alpha h((f_1 * \chi_{\alpha^c})^* a^*) = M_{\alpha^c} h((f_1 * \chi_{\alpha^c})^* a^*) = \rho_{\alpha^c}(a^*).\end{aligned}$$

This shows that $x^* = (id_\mathcal{M} \otimes \rho_\alpha)(\sigma(x))^* = (id_\mathcal{M} \otimes \rho_{\alpha^c})(\sigma(x)^*) = (id_\mathcal{M} \otimes \rho_{\alpha^c})(\sigma(x^*))$.

ii) Since $span\{u_{ij}^\gamma\}_{1 \le i,j \le d_\gamma, \gamma \in \widehat{G}}$ is norm dense in A it is enough to take $a = u_{ij}^\gamma, b = u_{kl}^\theta$ for some $\gamma, \theta \in \widehat{G}$ and to remark that :

$$\begin{aligned}&(h \otimes \rho_{\beta^c} \otimes \rho_\alpha)(\Delta_A(u_{ij}^\gamma)_{12}\Delta_A(u_{kl}^\theta)_{13}) \\ &\quad = (h \otimes \rho_{\beta^c} \otimes \rho_\alpha)(\sum_{r=1}^{d_\gamma}\sum_{s=1}^{d_\theta} u_{ir}^\gamma u_{ks}^\theta \otimes u_{rj}^\gamma \otimes u_{sl}^\theta) \\ &\quad = \sum_{r=1}^{d_\gamma}\sum_{s=1}^{d_\theta} h(u_{ir}^\gamma u_{ks}^\theta)\delta_{\beta^c\gamma}\delta_{rj}\delta_{\alpha\theta}\delta_{sl} \\ &\quad = h(u_{ij}^{\beta^c} u_{kl}^\alpha) = h(u_{ij}^{\beta *} u_{kl}^\alpha) = 0. \qquad \square\end{aligned}$$

Corollary 12 $\omega(P_\beta(y)^* P_\alpha(x)) = \omega(P_\alpha(x)P_\beta(y)^*) = 0$ *for all* $x, y \in \mathcal{M}, \alpha \neq \beta$ *in* $\widehat{G}$.

Proof. Denote $x_0 = P_\alpha(x), y_0 = P_\beta(y)$. By the previous lemma $x_0^* \in \mathrm{Ran}P_{\alpha^c}$ and $y_0^* \in \mathrm{Ran}P_{\beta^c}$. Then we obtain :

$$\begin{aligned}\sigma(x_0) &= \sigma((id_\mathcal{M} \otimes \rho_\alpha)(\sigma(x))) = (id_{\mathcal{M}\otimes A} \otimes \rho_\alpha)(id_\mathcal{M} \otimes \Delta_A)(\sigma(x)) \\ &= (id_\mathcal{M} \otimes (id_A \otimes \rho_\alpha)\Delta_A)(\sigma(x))\end{aligned}$$

and the similar relations for the pairs $(x_0^*, \alpha^c), (y_0, \beta), (y_0^*, \beta^c)$. The σ-invariance of ω and the previous relations yield :

$$\begin{aligned}\omega(y_0^* x_0) &= (\omega \otimes h)(\sigma(y_0^*)\sigma(x_0)) \\ &= (\omega \otimes h)((id_\mathcal{M} \otimes (id_A \otimes \rho_{\beta^c})\Delta_A)(\sigma(y^*))(id_\mathcal{M} \otimes (id_A \otimes \rho_\alpha)\Delta_A)(\sigma(x))),\end{aligned}$$

which is equal to 0 by Lemma 11 ii) since we have for any $a, b \in A$:

$$h((id_A \otimes \rho_{\beta^c})(\Delta_A(a))(id_A \otimes \rho_\alpha)(\Delta_A(b)) = (h \otimes \rho_{\beta^c} \otimes \rho_\alpha)(\Delta_A(a)_{12}\Delta_A(b)_{13}) = 0.$$

The equality $\omega(x_0 y_0^*) = 0$ follows similarly. $\square$

The previous corollary actually shows that the spectral subspaces $\mathcal{M}_\alpha$, $\alpha \in \widehat{G}$ are mutually orthogonal with respect to both scalar products $\langle x, y\rangle_{2,\omega} = \omega(y^* x)$ and $\langle x, y\rangle_{1,\omega} = \omega(x y^*)$ on $\mathcal{M}$.

The next statement is the analogue of the decomposition of a representation of a compact Lie group into isotypic subrepresentations.

Proposition 13 *i) The spectral subspaces $\mathcal{M}_\alpha$, $\alpha \in \widehat{G}$ are σ-invariant. Moreover*

$$\sigma(\mathcal{M}_\alpha) \subset \mathcal{M}_\alpha \otimes \mathcal{A}_\alpha = span\{x_{ij} \otimes u_{ij}^\alpha \mid x_{ij} \in \mathcal{M}_\alpha, \quad 1 \le i, j \le d_\alpha\}$$

and if V is a finite dimensional σ-invariant subspace of $\mathcal{M}_\alpha$, then α is the only irreducible subcorepresentation of $\sigma|_V$.

ii) Any finite dimensional subspace of $\mathcal{M}_\alpha$ is contained in a finite dimensional σ-invariant subspace of $\mathcal{M}_\alpha$.

iii) If the Haar measure is faithful on A, then $\mathcal{M}_0 = span\{\mathcal{M}_\alpha \mid \alpha \in \widehat{G}\}$ is dense in the Hilbert space $\mathcal{H}_\omega$.

Proof. i) Remark first that

$$\begin{aligned} \sigma(P_\alpha(x)) &= \sigma((id_{\mathcal{M}} \otimes \rho_\alpha)(\sigma(x))) = (id_{\mathcal{M}\otimes\mathcal{A}} \otimes \rho_\alpha)(id_{\mathcal{M}} \otimes \Delta_A)(\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes (id_A \otimes \rho_\alpha)\Delta_A)(\sigma(x)), \qquad x \in \mathcal{M}. \end{aligned}$$

Since $(id_A \otimes \rho_\alpha)(\Delta_A(A)) \subset \mathcal{A}_\alpha$ it follows that $\sigma(\mathcal{M}_\alpha) \subset \mathcal{M} \otimes \mathcal{A}_\alpha$. Using again the density of $span\{\mathcal{A}_\beta \mid \beta \in \widehat{G}\}$ in A and the equality

$$(\rho_\alpha \otimes id_A)(\Delta_A(u_{ij}^\beta)) = \delta_{\alpha\beta} u_{ij}^\beta = (id_A \otimes \rho_\alpha)(\Delta_A(u_{ij}^\beta)),$$

we obtain $(\rho_\alpha \otimes id_A)\Delta_A = (id_A \otimes \rho_\alpha)\Delta_A$ and furthermore

$$\begin{aligned} (P_\alpha \otimes id_A)(\sigma(x)) &= ((id_{\mathcal{M}} \otimes \rho_\alpha)\sigma \otimes id_A)(\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes \rho_\alpha \otimes id_A)(\sigma \otimes id_A)(\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes \rho_\alpha \otimes id_A)(id_{\mathcal{M}} \otimes \Delta_A)(\sigma(x)) \\ &= (id_{\mathcal{M}} \otimes (\rho_\alpha \otimes id_A)\Delta_A)(\sigma(x)) = \sigma(P_\alpha(x)), \qquad x \in \mathcal{M}. \end{aligned}$$

This shows in particular that $\sigma(\mathcal{M}_\alpha) \subset \mathcal{M}_\alpha \otimes \mathcal{A}_\alpha$. If $V \subset \mathcal{M}_\alpha$ is finite dimensional and σ-invariant, then $\sigma|_V$ contains only copies of α since $(id_{\mathcal{M}} \otimes \rho_\alpha)(\sigma(x)) = x$, $x \in V$.

ii) It is enough to prove that for any $x \in \mathcal{M}_\alpha$, there exists $V_x \subset \mathcal{M}_\alpha$ finite dimensional σ-invariant space that contains x. Set $V_x = \{(id_{\mathcal{M}} \otimes \rho)(\sigma(x)) \mid \rho \in A^*\}$, subspace of $\mathcal{M}_\alpha$ which contains x since $x = (id_{\mathcal{M}} \otimes \rho_\alpha)(\sigma(x))$ and is finite dimensional since

$\sigma(x) = \sum_{i,j=1}^{d_\alpha} x_{ij} \otimes u^\alpha_{ij}$ for some $x_{ij} \in \mathcal{M}_\alpha$. Therefore $V_x \subset span\{x_{ij} \mid 1 \le i,j \le d_\alpha\}$ and finally V_x is σ-invariant since for any $\rho, \rho' \in A^*$, $x \in \mathcal{M}$ we have :

$$\begin{aligned}(id_{\mathcal{M}} \otimes \rho')(\sigma((id_{\mathcal{M}} \otimes \rho)(\sigma(x))) &= (id_{\mathcal{M}} \otimes \rho')((id_{\mathcal{M}\otimes\mathcal{A}} \otimes \rho)(id_{\mathcal{M}} \otimes \Delta_A)(\sigma(x))) \\ &= (id_{\mathcal{M}} \otimes \rho'((id_A \otimes \rho) \circ \Delta_A))(\sigma(x)) = (id_{\mathcal{M}} \otimes (\rho' * \rho))(\sigma(x)).\end{aligned}$$

iii) One can easily check as in the case when both corepresentations are finite dimensional that for any $\alpha \in \widehat{G}$ and any corepresentation $V \in \mathcal{L}(\mathcal{H}_V \otimes A)$ of A we still have:

$$\begin{aligned}Mor(\alpha, V) &= \{\xi \in \mathcal{H}_V \otimes \mathcal{H}'_\alpha = \mathcal{L}(\mathcal{H}_\alpha, \mathcal{H}_V) \mid (V \odot \alpha^c)_\rho \xi = \rho(1_A)\xi, \quad \forall \rho \in A^*\} \\ &= (V \odot \alpha^c)_h(\mathcal{H}_V \odot \mathcal{H}'_\alpha).\end{aligned}$$

Assume now that $\overline{\mathcal{M}_0} \neq \mathcal{H}_\omega$. Then, since $U \mid_{\overline{\mathcal{M}_0}}$ is a subcorepresentation of U it follows that there exists $V \in \mathcal{L}(\mathcal{H}_V \otimes A)$ corepresentation of A with $0 \neq \mathcal{H}_V \subset \mathcal{H}_\omega \ominus \overline{\mathcal{M}_0}$. By Lemma 9 there exists $\beta \in \widehat{G}$ such that $(V \odot \beta^c)_h \neq 0$ and we find a nonzero $S \in \mathcal{L}(\mathcal{H}_\beta, \mathcal{H}_V)$ with $(S \otimes id_A)\beta = V(S \otimes id_A)$. Since $KerS$ is β-invariant and β irreducible, S follows one-to-one. But $RanS$ is a finite dimensional V-invariant subspace of $\mathcal{H}_V$, thus $\beta \preceq V$. Since $\mathcal{H}_V$ is orthogonal to $\mathcal{M}_0$ we have $(id_{\mathcal{L}(\mathcal{H}_\beta)} \otimes \rho_\alpha)(\beta) = 0$ for all $\alpha \in \widehat{G}$, therefore $\beta = 0$ by [Wor, 5.8]. But σ is one-to-one, thus V is nondegenerate and we get a contradiction. Consequently $\overline{\mathcal{M}_0} = \mathcal{H}_\omega$. □

Proposition 14 *$\mathcal{M}_0$ is a *-algebra and $\mathcal{M}_\alpha \mathcal{M}_\beta \subset span\{\mathcal{M}_\gamma \mid \gamma \preceq \alpha \odot \beta\}$ for all $\alpha, \beta \in \widehat{G}$.*

Proof. Fix $\alpha, \beta \in \widehat{G}, u^\alpha \in \alpha, u^\beta \in \beta$ and $\mathcal{H}_\alpha \subset \mathcal{M}_\alpha$, $\mathcal{H}_\beta \subset \mathcal{M}_\beta$ irreducible finite dimensional σ-invariant subspaces. Let $\{e_i\}_{1\le i\le d_\alpha}$ and $\{f_r\}_{1\le r\le d_\beta}$ be orthonormal basis in $\mathcal{H}_\alpha$ and respectively $\mathcal{H}_\beta$ such that $\sigma(e_i) = \sum_{j=1}^{d_\alpha} e_j \otimes u^\alpha_{ji}$ and $\sigma(f_r) = \sum_{s=1}^{d_\beta} f_s \otimes u^\beta_{sr}$. The restriction of σ to $\mathcal{H}_\alpha$ (respectively to $\mathcal{H}_\beta$), $\widehat{V}_\alpha : \mathcal{H}_\alpha \to \mathcal{H}_\alpha \otimes A$ (respectively $\widehat{V}_\beta : \mathcal{H}_\beta \to \mathcal{H}_\beta \otimes A$) implements u^α (respectively u^β). Then $\widehat{V} : \mathcal{H}_\alpha \otimes \mathcal{H}_\beta \to \mathcal{H}_\alpha \otimes \mathcal{H}_\beta \otimes A$, $\widehat{V}(x \otimes y) = \widehat{V}_\alpha(x)_{13}\widehat{V}_\beta(y)_{23} = \sigma(x)_{13}\sigma(y)_{23}$ implements $u^\alpha \odot u^\beta$. Consider the onto linear operator $S : \mathcal{H}_\alpha \otimes \mathcal{H}_\beta \to span\mathcal{H}_\alpha\mathcal{H}_\beta$, $S(x \otimes y) = xy$. Since :

$$\begin{aligned}\widehat{V}(e_i \otimes f_r) &= \sum_{j=1}^{d_\alpha}\sum_{s=1}^{d_\beta} e_j f_s \otimes u^\alpha_{ji} u^\beta_{sr} = (\sum_{j=1}^{d_\alpha} e_j \otimes u^\alpha_{ji})(\sum_{s=1}^{d_\beta} e_s \otimes u^\beta_{sr}) \\ &= \sigma(e_i)\sigma(f_r) = \sigma(e_i f_r) = \sigma S(e_i \otimes f_r),\end{aligned}$$

the following diagram is commutative

$$\begin{array}{ccc} \mathcal{H}_\alpha \otimes \mathcal{H}_\beta & \xrightarrow{\widehat{V}} & \mathcal{H}_\alpha \otimes \mathcal{H}_\beta \otimes \mathcal{A} \\ S \uparrow & & \uparrow S \otimes id_A \\ span\mathcal{H}_\alpha\mathcal{H}_\beta & \xrightarrow{\sigma} & span\mathcal{H}_\alpha\mathcal{H}_\beta \otimes A. \end{array}$$

Therefore $S \in Mor(u^\alpha \odot u^\beta, \sigma |_{span\mathcal{H}_\alpha\mathcal{H}_\beta})$ and any irreducible subcorepresentation of $\sigma |_{span\mathcal{H}_\alpha\mathcal{H}_\beta}$ should appear in $\alpha \odot \beta$, hence decomposing the last corepresentation into irreducible components we get $\mathcal{M}_\alpha\mathcal{M}_\beta \subset span\{\mathcal{M}_\gamma \mid \gamma \preceq \alpha \odot \beta\}$. $\mathcal{M}_0$ is *-closed since $\mathcal{M}_{\alpha^c} = \mathcal{M}_\alpha^*$ by Lemma 11. □

Denote by $\mathcal{H}_h$ the completion of A with respect to the scalar product $\langle a, b\rangle_{2,h} = h(b^*a)$, $a, b \in A$ and consider $W(x_\omega \otimes a_h) = (\sigma(x)(1_\mathcal{M} \otimes a))(1_\omega \otimes 1_h)$, $x \in \mathcal{M}, a \in A$. Clearly $W \in \mathcal{L}(\mathcal{H}_\omega \otimes \mathcal{H}_h)$ is a unitary. Since $\rho_\alpha = M_\alpha h((f_1 * \chi_\alpha)^* \cdot)$ and $f_1 * \chi_\alpha \in A$, it follows that $\rho_\alpha \in \mathcal{L}(\mathcal{H}_h)_*$ and ρ_α coincides on $\mathcal{L}(\mathcal{H}_h)$ with the vector form $M_\alpha\langle \quad \cdot 1_h, (f_1 * \chi_\alpha)_h\rangle_{2,h}$. Therefore it makes sense to consider $p(\alpha) = (id_{\mathcal{L}(\mathcal{H}_\omega)} \otimes \rho_\alpha)(W) \in \mathcal{L}(\mathcal{H}_\omega)$. A straightforward computation yields for any $x, y \in \mathcal{M}$:

$$\begin{aligned}
&\langle P_\alpha(x)_\omega, y_\omega\rangle_2 = \omega(y^* P_\alpha(x)) = \omega(y^*(id_\mathcal{M} \otimes \rho_\alpha)(\sigma(x))) \\
&= \omega((id_\mathcal{M} \otimes \rho_\alpha)((y^* \otimes 1_A)\sigma(x))) \\
&= (\omega \otimes \rho_\alpha)((y^* \otimes 1_A)\sigma(x)) = (\omega(y^* \cdot) \otimes h(M_\alpha(f_1 * \chi_\alpha)^* \cdot))\sigma(x) \\
&= \langle \sigma(x)(1_\omega \otimes 1_h), y_\omega \otimes M_\alpha((f_1 * \chi_\alpha)^*)_h\rangle_{2,\omega\otimes h} \\
&= \langle W(x_\omega \otimes 1_h), y_\omega \otimes M_\alpha((f_1 * \chi_\alpha)^*)_h\rangle_{2,\omega\otimes h} \\
&= \langle (id_{\mathcal{L}(\mathcal{H}_\omega)} \otimes \rho_\alpha)(W)x_\omega, y_\omega\rangle_2 = \langle p(\alpha)(x_\omega), y_\omega\rangle_2.
\end{aligned}$$

We obtain :

Remark 15 *P_α extends to the bounded operator $p(\alpha) \in \mathcal{L}(\mathcal{H}_\omega)$ and $p(\alpha)$ is a projection from $\mathcal{H}_\omega$ onto $\mathcal{H}_\alpha = \{x_\omega \mid x \in \mathcal{M}_\alpha\}$. Moreover, it is easy to see that $p(\alpha)$ is self-adjoint.*

Lemma 16 $\sum\limits_{i=1}^{d_\alpha} u_{pi}^{\alpha^*} \kappa^2(u_{qi}^\alpha) = \delta_{pq} 1_A = \sum\limits_{i=1}^{d_\alpha} \kappa^2(u_{ip}^\alpha) u_{iq}^{\alpha^*}$ *for all $\alpha \in \widehat{G}, 1 \le p, q \le d_\alpha$.*

Proof. Using the antimultiplicativity of κ and $\kappa(u_{ij}^\alpha) = u_{ji}^{\alpha^*}$ we obtain :

$$\sum_{i=1}^{d_\alpha} u_{pi}^{\alpha^*} \kappa^2(u_{qi}^\alpha) = \sum_{i=1}^{d_\alpha} \kappa(u_{ip}^\alpha)\kappa^2(u_{qi}^\alpha) = \sum_{i=1}^{d_\alpha} \kappa(\kappa(u_{qi}^\alpha)u_{ip}^\alpha) = \kappa(\sum_{i=1}^{d_\alpha} \kappa(u_{qi}^\alpha)u_{ip}^\alpha) = \delta_{pq}1_A$$

$$\sum_{i=1}^{d_\alpha} \kappa^2(u_{ip}^\alpha)u_{iq}^{\alpha^*} = \sum_{i=1}^{d_\alpha} \kappa(u_{qi}^\alpha \kappa(u_{ip}^\alpha)) = \delta_{pq}1_A. \qquad \square$$

Theorem 17 *If $G = (A, u)$ is a compact matrix pseudogroup and $\mathcal{M}$ is a unital C^*-algebra with an A-algebra structure given by the ergodic coaction $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$, then the spectral subspaces $\mathcal{M}_\alpha$ are finite dimensional and $dim\mathcal{M}_\alpha \le M_\alpha^2$.*

Proof. Let $\alpha \in \widehat{G}$ and fix $u^\alpha \in \alpha$ acting on $\mathcal{H}_\alpha$ and an orthonormal basis $\xi_1, ..., \xi_{d_\alpha}$ in $\mathcal{H}_\alpha$ such that $u^\alpha(\xi_i) = \sum_{r=1}^{d_\alpha} \xi_r \otimes u^\alpha_{ri}$. Let $V_1, ..., V_N \subset \mathcal{M}_\alpha$ be mutually $\langle\ ,\ \rangle_{2,\omega}$ orthogonal irreducible σ-invariant subspaces and let $U_k \in \mathcal{L}(\mathcal{H}_\alpha, V_k)$, $1 \le k \le N$ be unitaries such that $\sigma U_k = (U_k \otimes id_A)\alpha$. Denoting $e_i^{(k)} = U_k\xi_i$ it follows that $e_1^{(k)}, ..., e_{d_\alpha}^{(k)}$ is an orthonormal basis in V_k and

$$\sigma(e_i^{(k)}) = \sum_{r=1}^{d_\alpha} e_r^{(k)} \otimes u^\alpha_{ri}, \tag{1}$$

thus

$$\sigma\Big(\sum_{i=1}^{d_\alpha} e_i^{(k)} e_i^{(l)*}\Big) = \sum_{i,r,s=1}^{d_\alpha} e_r^{(k)} e_s^{(l)*} \otimes u^\alpha_{ri} u^{\alpha*}_{si} = \sum_{r=1}^{d_\alpha} e_r^{(k)} e_r^{(l)*} \otimes 1_A$$

and $\sum_{i=1}^{d_\alpha} e_i^{(k)} e_i^{(l)*} \in \mathbf{C}1_{\mathcal{M}}$.

Moreover, the equality $(F_\alpha \otimes id_A)\alpha = \alpha^{cc} F_\alpha$ and the previous Lemma yield :

$$\begin{aligned}
\sigma\Big(\sum_{i=1}^{d_\alpha} (e_i^{(k)*} U_l F_\alpha^{-1} U_l^*(e_i^{(l)})\Big) &= \sigma\Big(\sum_{i=1}^{d_\alpha} e_i^{(k)*} U_l F_\alpha^{-1}(\xi_i)\Big) \\
&= \sum_{i=1}^{d_\alpha} \sigma(e_i^{(k)*})(U_l F_\alpha^{-1} U_l^* \otimes id_A)\alpha^{cc}(\xi_i) \\
&= \sum_{i=1}^{d_\alpha} \sigma(e_i^{(k)*}) \sum_{s=1}^{d_\alpha} U_l F_\alpha^{-1} U_l^*(e_s^{(l)}) \otimes \kappa^2(u^\alpha_{si}) \\
&= \sum_{r,s=1}^{d_\alpha} e_r^{(k)*} U_l F_\alpha^{-1} U_l^*(e_s^{(l)}) \otimes \sum_{i=1}^{d_\alpha} u^{\alpha*}_{ri} \kappa^2(u^\alpha_{si}) \\
&= \sum_{r=1}^{d_\alpha} e_r^{(k)*} U_l F_\alpha^{-1} U_l^*(e_r^{(l)}) \otimes 1_A,
\end{aligned}$$

thus

$$\begin{aligned}
\sum_{i=1}^{d_\alpha} e_i^{(k)*} U_l F_\alpha^{-1} U_l^*(e_i^{(l)}) &= \omega\Big(\sum_{i=1}^{d_\alpha} e_i^{(k)*} U_l F_\alpha^{-1} U_l^*(e_i^{(l)})\Big)1_{\mathcal{M}} \\
&= \sum_{i=1}^{d_\alpha} \langle U_l F_\alpha^{-1} U_l^*(e_i^{(l)}), e_i^{(k)}\rangle_{2,\omega} 1_{\mathcal{M}} = \delta_{kl} Tr(F_\alpha^{-1}) 1_{\mathcal{M}} = \delta_{kl} M_\alpha 1_{\mathcal{M}}.
\end{aligned} \tag{2}$$

Since the modular operator F_α associated with u^α is positive ([Wor, 5.4]), the matrix $C = (\langle F_\alpha^{-1}\xi_i, \xi_j\rangle)_{ij} \in Mat_{d_\alpha}(\mathbf{C})$ is positive definite.

Denote $C^{1/2} = (\lambda_{ij})_{ij}$, $x_{rl} = \sum_{j=1}^{d_\alpha} \lambda_{rj} e_j^{(l)}$ and $X = (x_{rl})_{rl} \in Mat_{d_\alpha, N}(\mathcal{M})$. Then

$$\sum_{i=1}^{d_\alpha} e_i^{(k)*} U_l F_\alpha^{-1} \xi_i = \sum_{i,j=1}^{d_\alpha} \langle F_\alpha^{-1}\xi_i, \xi_j\rangle e_i^{(k)*} e_j^{(l)} = \sum_{r,i,j=1}^{d_\alpha} \overline{\lambda_{ri}} \lambda_{rj} e_i^{(k)*} e_j^{(l)} = \sum_{r=1}^{d_\alpha} x^*_{rk} x_{rl},$$

or in other words $X^*X = M_\alpha I$ in $\mathcal{M} \otimes Mat_N(\mathbf{C})$. This yields $XX^* \le M_\alpha I$ in $\mathcal{M} \otimes Mat_{d_\alpha}(\mathbf{C})$.

Consider the positive linear functional $\phi \in Mat_{d_\alpha}(\mathbf{C})_*$, $\phi(Y) = Tr(C^{-1}Y)$, Tr being the normalized trace on $Mat_{d_\alpha}(\mathbf{C})$. Then

$$\|\phi\| = \phi(I) = Tr(C^{-1}) = \sum_{i=1}^{d_\alpha}(C^{-1})_{ii} = \sum_{i=1}^{d_\alpha}\langle F_\alpha \xi_i, \xi_i\rangle = Tr(F_\alpha) = M_\alpha$$

and the last inequality yields :

$$\begin{aligned} M_\alpha^2 &\geq \|\omega \otimes \phi\| \cdot \|XX^*\| \geq (\omega \otimes \phi)(XX^*) = \sum_{i,j=1}^{d_\alpha}\sum_{k=1}^{N} \phi(e_{ij})\omega(x_{ik}x_{jk}^*) \\ &= \sum_{k=1}^{N}\sum_{i,j,r,s=1}^{d_\alpha} \lambda_{sj}(C^{-1})_{ji}\lambda_{ir}\omega(e_r^{(k)}e_s^{(k)*}) = \sum_{k=1}^{N}\sum_{r=1}^{d_\alpha}\omega(e_r^{(k)}e_r^{(k)*}) \\ &= \sum_{k=1}^{N}\sum_{r=1}^{d_\alpha}\|e_r^{(k)}\|_{1,\omega}^2. \end{aligned}$$

This inequality plays a crucial role since it shows that if $W_\alpha \subset \mathcal{M}_\alpha$ is a σ-invariant subspace and A_{W_α} the positive invertible operator on W_α (with respect to $\langle\ ,\ \rangle_{2,\omega}$) such that $\langle A_{W_\alpha}(x), y\rangle_{2,\omega} = \langle x, y\rangle_{1,\omega}$, $x, y \in W_\alpha$, then

$$(3) \qquad Tr(A_{W_\alpha}) \leq M_\alpha^2.$$

Consider now a finite-dimensional σ-invariant subspace $W \subset \mathcal{M}_0$ and let A_W be the unique positive invertible operator on W with $\langle A_W(x), y\rangle_{2,\omega} = \langle x, y\rangle_{1,\omega}$, $x, y \in W$. Since $\langle \mathcal{M}_\alpha, \mathcal{M}_\beta\rangle_{1,\omega} = 0$ for $\alpha \neq \beta$, it follows that A_W invariates each spectral subspace W_α of W and

$$(4) \qquad Tr(A_W) = \sum_{\alpha \prec W} Tr(A_{W_\alpha}) \leq \sum_{\alpha \prec W} M_\alpha^2.$$

If in addition A_W is *-invariant and $Jx = x^*$, $x \in \mathcal{M}$, then

$$\langle JA_W^{-1}Jx, y\rangle_{2,\omega} = \langle y^*, A_W^{-1}(x^*)\rangle_{1,\omega} = \langle y^*, x^*\rangle_{2,\omega} = \langle x, y\rangle_{1,\omega} = \langle A_W(x), y\rangle_{2,\omega}, \quad x, y \in W.$$

Therefore $JA_W^{-1}J = A_W$ and in particular if λ is an eigenvalue of A_W with multiplicity m_λ, so is λ^{-1} with the same multiplicity. It follows that

$$(5) \qquad Tr(A_W) = \sum_\lambda m_\lambda \lambda = m_1 + \sum_{\lambda>1} m_\lambda(\lambda + \lambda^{-1}) \geq m_1 + \sum_{\lambda>1} 2m_\lambda = dim(W).$$

Finally let $V \subset \mathcal{M}_\alpha$ be a finite-dimensional σ-invariant subspace. Since $V^* \subset \mathcal{M}_{\alpha^c}$ it follows that $W = V + V^*$ is σ-invariant. If α is self-conjugate, then $W \subset \mathcal{M}_\alpha$, $W = W^*$ and therefore $dim(V) \leq dim(W) \leq M_\alpha^2$. In the case when $\alpha \neq \alpha^c$ in $\widehat{G}$ the sum $V + V^*$ is orthogonal and by (4) and (5) we get $2dim(V) = dim(W) \leq M_\alpha^2 + M_{\alpha^c}^2 = 2M_\alpha^2$. □

The next statement describes the modularity of ω.

Proposition 18 *There exists $\Theta : \mathcal{M}_0 \to \mathcal{M}_0$ linear multiplicative map such that $\omega(x\Theta(y)) = \omega(yx)$, $x \in \mathcal{M}, y \in \mathcal{M}_0$. Moreover Θ invariates all the irreducible σ-invariant subspaces of $\mathcal{M}_0$ and is a scalar multiple of the modular operator F_α on such a subspace of $\mathcal{M}_\alpha$ for all $\alpha \in \widehat{G}$.*

Proof. Let $V_1, ..., V_N$ be the mutually orthogonal irreducible σ-invariant subspaces of $\mathcal{M}_\alpha$ and let $e_1^{(k)}, ..., e_d^{(k)}$ be an orthonormal basis in V_k ($d = dim(\alpha)$) such that $\sigma(e_i^{(k)}) = \sum_{r=1}^{d} e_r^{(k)} \otimes u_{ri}^\alpha$, $1 \le i \le d, 1 \le k \le N$. Then $\sum_{i=1}^{d} e_i^{(k)} e_i^{(l)*} \in \mathcal{M}^\sigma$, therefore there exist $\lambda_{kl} \in \mathbf{C}$, $1 \le k, l \le N$ such that $\sum_{i=1}^{d} e_i^{(k)} e_i^{(l)*} = \lambda_{kl} 1_{\mathcal{M}}$. The matrix $\Lambda = (\lambda_{kl})_{1 \le k, l \le 1, N}$ is positively definite, hence there exists a unitary $S = (s_{kl})_{1 \le k, l \le N} \in Mat_N(\mathbf{C})$ and $\lambda_1, ..., \lambda_N > 0$ such that $S \Lambda S^* = diag(\lambda_1, ..., \lambda_N)$. Replacing $e_i^{(k)}$ by $\sum_{p=1}^{N} s_{kp} e_i^{(p)}$ we get :

$$(6) \qquad \sum_{i=1}^{d} e_i^{(k)} e_i^{(l)*} = \delta_{kl} \lambda_k 1_{\mathcal{M}}, \qquad 1 \le k, l \le N.$$

Fix now $1 \le k, l \le N$ and consider $T_{kl} : V_k \to V_l$, $T_{kl}(x) = \sum_{j=1}^{d} \omega(x e_j^{(l)*}) e_j^{(l)}$, $x \in V_k$. Combining the σ-invariance of ω, the orthogonality relations (5.25) in [Wor] and (6) we get for $1 \le i \le d$:

$$\begin{aligned} T_{kl} e_i^{(k)} &= \sum_{j=1}^{d} \omega(e_i^{(k)} e_j^{(l)*}) e_j^{(l)} = \sum_{j,p,q=1}^{d} (\omega \otimes h)(e_p^{(k)} e_q^{(l)*} \otimes u_{pi}^\alpha u_{qj}^{\alpha *}) e_j^{(l)} \\ &= \sum_{j,p,q=1}^{d} \omega(e_p^{(k)} e_q^{(l)*}) \tfrac{\delta_{pq}}{M_\alpha} f_1(u_{ji}^\alpha) e_j^{(l)} = \sum_{j,p=1}^{d} \omega(e_p^{(k)} e_p^{(l)*}) \tfrac{f_1(u_{ji}^\alpha)}{M_\alpha} e_j^{(l)} \\ &= \tfrac{\delta_{kl}\lambda_k}{M_\alpha} \sum_{j=1}^{d} f_1(u_{ji}^\alpha) e_j^{(k)} = \tfrac{\delta_{kl}\lambda_k}{M_\alpha} (id_{\mathcal{M}} \otimes f_1)(\sigma(e_i^{(k)})), \end{aligned}$$

thus $T_{kl} = \frac{\delta_{kl}\lambda_k}{M_\alpha} (id_{\mathcal{M}} \otimes f_1) \circ \sigma$. We check now that $T_k = T_{kk} \in Mor(\sigma \mid_{V_k}, \sigma^{cc} \mid_{V_k})$. The previous formula for T_k yields :

$$\sigma T_k = \frac{\lambda_k}{M_\alpha} \sigma((id_{\mathcal{M}} \otimes f_1)\sigma) = \frac{\lambda_k}{M_\alpha} (id_{\mathcal{M}} \otimes ((f_1 \otimes id_A)\Delta_A))\sigma$$

and

$$(T_k \otimes id_A)\sigma = \frac{\lambda_k}{M_\alpha} ((id_{\mathcal{M}} \otimes f_1)\sigma \otimes id_A)\sigma = \frac{\lambda_k}{M_\alpha} (id_{\mathcal{M}} \otimes ((f_1 \otimes id_A)\Delta_A))\sigma.$$

Since $\kappa^2(a) = f_{-1} * a * f_1$, $a \in \mathcal{A}$ we also have :

$$\begin{aligned} (id_{\mathcal{M}} \otimes \kappa^2)(id_{\mathcal{M}} \otimes (id_A \otimes f_1) \circ \Delta_A)(y \otimes a) &= y \otimes \kappa^2(f_1 * a) = y \otimes (a * f_1) \\ = (id_{\mathcal{M}} \otimes (f_1 \otimes id_A) \circ \Delta_A)(y \otimes a), &\qquad y \in \mathcal{M}, a \in \mathcal{A}, \end{aligned}$$

hence $(T_k \otimes id_A)\sigma = (id_{\mathcal{M}} \otimes \kappa^2)\sigma T_k = \sigma^{cc} T_k$ on V_k.

But $Mor(\alpha, \alpha^{cc}) = \{\lambda F_\alpha \mid \lambda \in \mathbf{C}\}$, therefore $\omega(e_i^{(k)} e_j^{(l)*}) = \delta_{kl} \lambda'_k (F_\alpha)_{ji}$. Comparing the previous relation with (7) we get :

$$\langle e_i^{(k)}, e_j^{(l)} \rangle_{1,\omega} = \omega(e_i^{(k)} e_j^{(l)*}) = \frac{\delta_{kl}\lambda_k}{M_\alpha} (F_\alpha)_{ji} = \frac{\delta_{kl}\lambda_k}{M_\alpha} \langle F_\alpha(e_i^{(k)}), e_j^{(l)} \rangle_{2,\omega},$$

thus $\omega(xy^*) = \delta_{kl}\omega(y^*\Theta(x)) = \omega(y^*\Theta(x))$ for all $x \in V_k, y \in V_l$, where we let $\Theta \mid_{V_k} = \frac{\lambda_k}{M_\alpha} F_\alpha \in \mathcal{L}(V_k)$. The orthogonality of the spectral subspaces $\mathcal{M}_\alpha$ implies now that $\omega(xy) = \omega(y\Theta(x))$ for all $x \in \mathcal{M}_0, y \in \mathcal{M}$. Clearly Θ is linear by definition and follows multiplicative by the previous equality. $\square$

§2. The structure of the crossed product

Using the finite dimensionality of the spectral subspaces $\mathcal{M}_\alpha$ we prove that the reduced C^*-crossed product $\mathcal{N} = \mathcal{M} \times_\sigma \widehat{A}$ (as defined in [BS]) of a unital C^*-algebra $\mathcal{M}$ by an ergodic coaction of a compact matrix pseudogroup $G = (A, u)$ with faithful Haar measure h on A, turning $\mathcal{M}$ into an A-algebra, is isomorphic to a direct sum of algebras of compact operators, generalizing Proposition 2 in [L] and Corollary 2 in [Wal, §1.4]. The main ingredient is the Takesaki-Takai duality type theorem of Baaj and Skandalis and the line of the proof follows Wassermann's one for the case when $A = C(G)$, the commutative C^*-algebra of continuous functions on a compact group G and $\mathcal{N} = \mathcal{M} \times_\sigma G$.

Remark first that A is an A-algebra via the comultiplication $\Delta_A : A \to A \otimes A$, that Δ_A is ergodic and that h is the Δ_A-invariant state on A. Denote $\mathcal{H} = \mathcal{H}_h$ and consider as in Remark 15 the operator $V \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$, $V(a_h \otimes b_h) = (\Delta_A(a)(1_A \otimes b))(1_h \otimes 1_h)$, $a, b \in A$. Then V is a biregular irreducible multiplicative unitary. Moreover, the partial isometry U (no relationship with the unitary associated to the coaction σ in §1) in the polar decomposition $T = U \mid T \mid$ of the closure of the preclosed operator T_0 defined by $T_0(a_h) = \kappa(a)_h$, $a \in \mathcal{A}$ is a unitary on $\mathcal{H}$ with $U^2 = I_{\mathcal{H}}$. In fact $Ux_h = (f_1 * \kappa(x))_h$, $x \in \mathcal{A}$ and $(\mathcal{H}, V, U)$ is a Kac system (see [BS, §6]). Consider the reduced C^*-crossed product $\mathcal{N} = \mathcal{M} \times_\sigma \widehat{A}$, defined in [BS, §7] as the C^*-algebra generated by products of type $\sigma_L(x)(1_A \otimes \rho(\omega))$, $x \in \mathcal{M}, \omega \in \mathcal{L}(\mathcal{H})_*$. Then, there exists a dual coaction $\widehat{\sigma} = \sigma_{\mathcal{M} \times_\sigma \widehat{A}}$ on $\mathcal{N}$, which transforms this way into an $\widehat{A}$-algebra. Let A acting in the GNS representation of h on $\mathcal{H}$ and denote $\sigma_R(x) = (id_{\mathcal{M}} \otimes AdU)(\sigma(x))$, $x \in \mathcal{M}$, $\lambda(\omega) = (id_{\mathcal{L}(\mathcal{H})} \otimes \omega)(V)$, $\omega \in \mathcal{L}(\mathcal{H})_*$. Then, by Theorem 7.5 in [BS], $\mathcal{M} \times_\sigma \widehat{A}$ identifies with the norm closure of $span\{\sigma_R(x)(1_{\mathcal{M}} \otimes \lambda(\omega)) \mid x \in \mathcal{M}, \omega \in \mathcal{L}(\mathcal{H})_*\}$, $\mathcal{M} \otimes \mathcal{K}(\mathcal{H})$ with the closure of $span\{\sigma_R(x)(1_{\mathcal{M}} \otimes \lambda(\omega)a) \mid x \in \mathcal{M}, \omega \in \mathcal{L}(\mathcal{H})_*, a \in A\}$ and denoting $\sigma'(x \otimes k) = V_{23}\sigma(x)_{13}(1_{\mathcal{M}} \otimes k \otimes 1_A)V_{23}^*$, $x \in \mathcal{M}$, $k \in \mathcal{K}(\mathcal{H})$, the double crossed-product $((\mathcal{M} \times_\sigma \widehat{A}) \times_{\widehat{\sigma}} A, \widehat{\sigma})$ is isomorphic to $(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}), \sigma')$ as A-algebras. Moreover, the last two equalities in the proof of that theorem show that via the previous identification $\mathcal{M} \times_\sigma \widehat{A} \subset (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'}$.

The proof of the structure of the crossed-product will make use of the next Lemma:

Lemma ([Wal, §1.4]). *Let A be a (not necessarily unital) C^*-algebra with an approximate identity $p_1 \leq p_2 \leq \ldots$ consisting of finite rank projections (i.e. $dim(p_iAp_i) < \infty$). Then A is isomorphic to a direct sum of algebras of compact operators.*

Theorem 19 *Let $G = (A, u)$ be a compact matrix pseudogroup and let $\mathcal{M}$ be a unital C^*-algebra which is an A-algebra via the ergodic coaction $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$. Then $\mathcal{M} \times_\sigma \widehat{A} \simeq \oplus_i \mathcal{K}(\mathcal{H}_i)$.*

Proof. Replacing A by its reduced C^*-algebra, the crossed product $\mathcal{M} \times_\sigma \widehat{A}$ is still unchanged (we owe this remark to the referee), thus one may assume that that the Haar measure is faithful on A. For each $\alpha \in \widehat{G}$ consider the projections $p(\alpha)$ from $\mathcal{H}$ onto $\mathcal{H}_\alpha$ defined in Remark 15. Since $p(\alpha) = (id_{\mathcal{L}(\mathcal{H})} \otimes \rho_\alpha)(V)$ and $\rho_\alpha \in \mathcal{L}(\mathcal{H})_*$ it follows that $p(\alpha) \in \widehat{A}$. But $\mathcal{M}$ is unital, therefore $1_\mathcal{M} \otimes p(\alpha) \in \mathcal{N} = \mathcal{M} \times_\sigma \widehat{A}$. Since $\sum_{\alpha\in\widehat{G}} p(\alpha) = 1_{\widehat{A}}$ in the strict topology, it follows that writing $\widehat{G} = \cup_n F_n$ with $card(F_n) < \infty$ we get an approximate unit $p_n = p(F_n) = \sum_{\alpha \in F_n} p(\alpha)$, $n \geq 1$ of $\mathcal{N}$. Finally we check that each $p(F_n)$ has finite rank, or equivalently that $dim(p(\alpha)\mathcal{N}p(\pi)) < \infty$ for all $\alpha, \pi \in \widehat{G}$. Since $1_\mathcal{M} \otimes p(\alpha) \in \mathcal{N}$ and $\mathcal{N} \subset (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'}$, we obtain :

$$p(\alpha)\mathcal{N}p(\pi) \subset p(\alpha)(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'}p(\pi) \subset (\mathcal{M} \otimes p(\alpha)\mathcal{K}(\mathcal{H})p(\pi))^{\sigma'} = (\mathcal{M} \otimes \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\alpha))^{\sigma'}.$$

If $\alpha_0 \in \widehat{G}$ is fixed, then $\sigma' |_{\mathcal{M}^c_{\alpha_0} \otimes \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\alpha)}$ is a corepresentation of A that coincides with the tensor product corepresentation $\sigma |_{\mathcal{M}^c_0} \odot \sigma_0 |_{\mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\alpha)}$ and therefore

$$\begin{aligned}(id_{\mathcal{M}\otimes\mathcal{L}(\mathcal{H})} \otimes h)(\sigma'(\mathcal{M}^c_0 \otimes \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\alpha))) &= Ran(\sigma |_{\mathcal{M}^c_0} \odot \sigma_0 |_{\mathcal{L}(\mathcal{H}_\pi,\mathcal{H}_\alpha)})_h \\ &= Mor(\sigma |_{\mathcal{M}^c_0}, \sigma_0 |_{\mathcal{L}(\mathcal{H}_\pi,\mathcal{H}_\alpha)})^T.\end{aligned}$$

Since there are only finitely many $\alpha_0 \in \widehat{G}$ that appear in $\sigma_0 |_{\mathcal{L}(\mathcal{H}_\pi,\mathcal{H}_\alpha)}$, the space $(\mathcal{M} \otimes \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\alpha))^{\sigma'} = (id_\mathcal{M} \otimes h)(\sigma'(\mathcal{M} \otimes \mathcal{L}(\mathcal{H}_\pi, \mathcal{H}_\alpha)))$ follows finite dimensional. □

Remark 20 *Although we didn't use it in proving the last theorem, it is easy to check that in fact* $\mathcal{M} \times_\sigma \widehat{A} = (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'}$ *via the realization of the crossed-product in* $\mathcal{M} \otimes \mathcal{K}(\mathcal{H})$.

This is the case since the canonical conditional expectation $E = (id_{\mathcal{M}\otimes\mathcal{K}(\mathcal{H})} \otimes h)\sigma'$ carries total sets in $\mathcal{M} \otimes \mathcal{K}(\mathcal{H})$ onto total sets in $(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'}$. In particular the set

$$\mathcal{S} = \{E(\sigma_R(x)(1_\mathcal{M} \otimes \lambda(\omega)a)) \mid x \in \mathcal{M}, \omega \in \mathcal{L}(\mathcal{H})_*, a \in A\}$$

is total in $(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'}$. But the formula for σ' and $(id_A \otimes h)\Delta_A(a) = h(a)1_A$, $a \in A$, yield for any $x \in \mathcal{M}, \omega \in \mathcal{L}(\mathcal{H})_*, a \in A$:

$$\begin{aligned}&E(\sigma_R(x)(1_\mathcal{M} \otimes \lambda(\omega)a)) \\ &\quad= (id_{\mathcal{M}\otimes\mathcal{K}(\mathcal{H})} \otimes h)((\sigma_R(x)(1_\mathcal{M} \otimes \lambda(\omega)) \otimes 1_A)(1_\mathcal{M} \otimes \Delta_A(a)) \\ &\quad= \sigma_R(x)(1_\mathcal{M} \otimes \lambda(\omega))(1_\mathcal{M} \otimes (id_{\mathcal{L}(\mathcal{H})} \otimes h)\Delta_A(a)) \\ &\quad= h(a)\sigma_R(x)(1_\mathcal{M} \otimes \lambda(\omega)),\end{aligned}$$

therefore $\mathcal{S} \subset \mathcal{M} \times_\sigma \widehat{A}$. □

Let B and C be unital C^*-algebras, $G = (A, u)$ be a compact matrix pseudogroup with A nuclear and $\sigma : C \to C \otimes A$ be a unital *-morphism. By the associativity of the maximal tensor product and the nuclearity of A, the diagram :

$$\begin{tikzcd}
B \otimes_{\max} C \arrow[r, "\tilde{\sigma} = id_B \otimes_{\max} \sigma"] & B \otimes_{\max} (C \otimes A) = (B \otimes_{\max} C) \otimes A \\
B \otimes_{\max} (C \otimes A) = (B \otimes_{\max} C) \otimes A \arrow[u, "\tilde{\sigma}"] \arrow[r, "id_B \otimes_{\max} (\sigma \otimes id_A)"] & (B \otimes_{\max} C) \otimes A \otimes A = B \otimes_{\max} (C \otimes A \otimes A) \arrow[u, "id_{B\otimes_{\max}C} \otimes \Delta_A = id_B \otimes_{\max} (id_C \otimes \Delta_A)"']
\end{tikzcd}$$

is commutative although $\tilde{\sigma}$ may not be one-to-one in general (note that $\tilde{\sigma} \otimes_{\max} id_A = id_B \otimes_{\max} (\sigma \otimes id_A)$). But Lemma 4.1 i) holds true in such a case, thus $\tilde{E} = (id_{B\otimes_{\max}C} \otimes h)\tilde{\sigma}$ is a conditional expectation from $B \otimes_{\max} C$ onto $(B \otimes_{\max} C)^{\tilde{\sigma}}$ and it turns out that $\tilde{E} = id_B \otimes_{\max} E$, where $E = (id_C \otimes h)\sigma$ is conditional expectation from C onto C^{σ}. In particular, this shows

Remark 21 $(B \otimes_{\max} C)^{\tilde{\sigma}} = B \otimes_{\max} C^{\sigma}$.

The next statement, whose proof follows the line of [Wa2, Lemma22], shows that if A is nuclear, then $\mathcal{M}$ itself follows nuclear.

Proposition 22 *Let $(\mathcal{M}, \sigma)$ be an A-algebra such that $\mathcal{M} \times_{\sigma} \widehat{A}$ and A are nuclear C^*-algebras. Then $\mathcal{M}$ is nuclear.*

Proof. We prove that given any unital C^*-algebra B, the natural *-morphism θ that maps $B \otimes_{\max} \mathcal{M}$ onto $B \otimes \mathcal{M}$ is one-to-one.

Consider the coaction $\sigma' : \mathcal{M} \otimes \mathcal{K}(\mathcal{H}) \to \mathcal{M} \otimes \mathcal{K}(\mathcal{H}) \otimes A$ and define as in the previous remark $\widetilde{\sigma'} : B \otimes_{\max} (\mathcal{M} \otimes \mathcal{K}(\mathcal{H})) \to B \otimes_{\max} (\mathcal{M} \otimes \mathcal{K}(\mathcal{H})) \otimes A$.

The map $\Theta = \theta \otimes id_{\mathcal{K}(\mathcal{H})} : B \otimes_{\max} (\mathcal{M} \otimes \mathcal{K}(\mathcal{H})) \to B \otimes \mathcal{M} \otimes \mathcal{K}(\mathcal{H})$ is A-equivariant, i.e. $(id_B \otimes \sigma')\Theta = \Theta\widetilde{\sigma'}$ and using the canonical conditional expectations onto fixed point algebras

$$E^{\widetilde{\sigma'}} : B \otimes_{\max} (\mathcal{M} \otimes \mathcal{K}(\mathcal{H})) \to (B \otimes_{\max} (\mathcal{M} \otimes \mathcal{K}(\mathcal{H})))^{\widetilde{\sigma'}} = B \otimes_{\max} (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\widetilde{\sigma'}}$$

and

$$E^{id_B \otimes \sigma'} : B \otimes \mathcal{M} \otimes \mathcal{K}(\mathcal{H}) \to (B \otimes \mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{id_B \otimes \sigma'} = B \otimes (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'},$$

we get $\Theta E^{\widetilde{\sigma'}} = E^{id_B \otimes \sigma'}\Theta$. Consequently

$$(Ker\theta \otimes \mathcal{K}(\mathcal{H}))^{\widetilde{\sigma'}} = Ker(B \otimes_{\max} (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'} \xrightarrow{\Theta} B \otimes (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'}).$$

Since $(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}))^{\sigma'} \simeq \mathcal{M} \times_{\sigma} \widehat{A}$ is nuclear, it follows that $(Ker\theta \otimes \mathcal{K}(\mathcal{H}))^{\widehat{\sigma'}} = 0$ and therefore $Ker\theta = 0$. □

Corollary 23 *If $G = (A, u)$ is a compact matrix pseudogroup, the C^*-algebra A is nuclear and $\mathcal{M}$ is a unital C^*-algebra which is an A-algebra via an ergodic coaction, then $\mathcal{M}$ is nuclear.*

References

[BS] S. Baaj, G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres, *Ann. Sci. Ec. Norm. Sup. 26(1993), 425-488.*

[HLS] R. Høegh-Krohn, M. Landstad, E. Størmer, Compact ergodic groups of automorphisms, *Ann. Math. 114(1981), 75-86.*

[L] M. Landstad, Algebras of spherical functions associated with covariant systems over a compact group, *Math.Scandinavica 47(1980), 137-149.*

[N] G. Nagy, On the Haar measure of the twisted SU(N) group, *Comm. Math. Phys. 153(1993), 217-228.*

[T] M. Takesaki, Theory of Operator Algebras I, *Springer-Verlag, New York, 1979.*

[Wa1] A. Wassermann, Ergodic actions of compact groups on operator algebras: I. General theory, *Ann. Math. 130(1989), 273-319.*

[Wa2] A. Wassermann, Classification of full multiplicity actions, *Can. J. Math. 40(1988), 1482-1527.*

[Wa3] A. Wassermann, Classification for $SU(2)$, *Invent. Math. 93(1988), 309-355.*

[Wa4] A. Wassermann, Coactions and Yang-Baxter equations for ergodic actions and subfactors, *L.M.S. Lecture Notes 136(1988), 202-236.*

[Wor] S. L. Woronowicz, Compact matrix pseudogroups, *Comm. Math. Phys. 111(1987), 613-665.*

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