Astérisque

F. COMETS

A spherical bound for the Sherrington-Kirkpatrick model

Astérisque, tome 236 (1996), p. 103-108

<http://www.numdam.org/item?id=AST_1996__236__103_0>

A Spherical Bound for the Sherrington-Kirkpatrick Model

F. Comets

Abstract. — We prove existence of a phase transition for the Sherrington-Kirkpatrick model at $\beta = 1$: making use of the domination by the spherical model, we derive a bound for the pressure as well as for the ground state energy.

1. Introduction. The Sherrington-Kirkpatrick (SK) model consists of a set of N binary spins $\sigma_i \in \{-1, +1\}$ with Hamiltonian

$$H_N(\sigma) = -\frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} J_{ij}\sigma_i\sigma_j - \frac{1}{\sqrt{2N}} \sum_{1 \le i \le N} J_{ii}\sigma_i^2 \tag{1.1}$$

where the couplings J_{ij} are independent Gaussian random variables with mean zero and unit variance. Compared with the original definition [SK] we have added the second summand in (1.1) which does not depend on σ in the binary case and does not change the results below. The partition function at inverse temperature β and the pressure of the model are the random variables

$$Z_N^{\rm SK} = {\rm E}_\sigma \exp\{-\beta H_N(\sigma)\}\ , \qquad p_N^{\rm SK} = \frac{1}{N} \log Z_N^{\rm SK} \tag{1.2}$$

depending on the J_{ij}'s; in (1.2) we have used the notation ${\rm E}_\sigma = 2^{-N} \sum_{\sigma \in \{-1,+1\}^N}$ to denote the average over the spin configurations, and we keep the symbol $\mathbb{E}$ for expectations over the couplings J_{ij}.

Among the few mathematical results for this model it is known that, at high temperature, the "quenched" and "annealed" behaviors coincide:

$$p_N^{\rm SK} \underset{N\to\infty}{\longrightarrow} \frac{\beta^2}{4} = \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\, Z_N^{\rm SK}\ . \tag{1.3}$$

Convergence in probability and in L_p, $1 \le p < \infty$, follows from the fluctuations results of [ALR] or [CN], and convergence with probability one follows using in addition the concentration property proved in [T] from general considerations, or in [BGP] more directly. On the other hand, the convergence (1.3) cannot hold for large β since the entropy per spin is at most $\log 2$ for binary spins.

In this note we show that the convergence in (1.3) only happens when $\beta \leq 1$ and hence phase transition takes place at $\beta_c = 1$. For this we prove below a (weak) bound for the limit points of p_N^{SK}, which is a self-averaging quantity [PS]. This bound implies another one for the ground state energy.

These bounds reflect the domination of the SK model by the spherical model, where the uniform probability measure E_σ on the N-dimensional hypercube $\{-1,+1\}^N$ is replaced with the uniform probability measure E_η on the sphere

$$\{\eta \in \mathbb{R}^N \ ; \quad |\eta|^2 := \sum_{i=1}^N \eta_i^2 = N\} \ .$$

The partition function and pressure of this second model

$$Z_N^{\text{S}} = \mathrm{E}_\eta \exp\{-\beta H_N(\eta)\} \ , \qquad p_N^{\text{S}} = \frac{1}{N} \log Z_N^{\text{S}} \tag{1.4}$$

depend only on the eigenvalues of the quadratic form H_N. Spherical models, introduced by Berlin and Kac [BK], are completely solvable in many instances. They are commonly analysed via the method of steepest descent, or more simply via the mean-spherical approximation (i.e., as Gaussian models where the spherical constraint is satisfied in the mean). The asymptotics of (1.4) are studied in [KTJ] and in [Th], but for completeness of this note, we give below a quick derivation of the bound.

Acknowledgements: I thank Olivier Catoni and Leonid Pastur for stimulating conversations and useful remarks about these bounds.

2. Spherical bound and consequences.

The $N\times N$ symmetric matrix M, with coefficients $M_{ij} = \frac{1}{2\sqrt{N}} J_{ij}$ if $i < j$ and $M_{ii} = \frac{1}{\sqrt{2N}} J_{ii}$, has a.s. simple eignevalues $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ with normalized eigenvectors $\phi_1 = (\phi_{1,j})_{j\leq N}, \ldots, \phi_N$. Since the distribution of M is invariant under orthogonal transformations, the diagonal matrix Λ of the eigenvalues $\lambda_1, \ldots, \lambda_N$ is independent of the orthogonal matrix $\phi = (\phi_1, \ldots, \phi_N)$, and we may clearly choose the frame ϕ such that it is uniformly distributed on the set $O(N)$ of orthogonal matrices; in particular one has for any positive measurable function F

$$\mathbb{E}\{F(M)|\Lambda\} = \mathbb{E}_\phi \ F(\phi\Lambda\phi^t) \ , \qquad \mathbb{P}\text{-a.s.} \tag{2.1}$$

with $\mathbb{E}_\phi$ the expectation in ϕ uniformly distributed on $O(N)$. This implies that the spherical model dominates the SK model: more precisely, for any fixed binary spin σ, the distribution of the scalar products $(\sigma.\phi_i)_{i\leq N}$ under $\mathbb{E}_\phi$ is the uniform measure E_η on the sphere, and then

$$\mathbb{E}\{Z_N^{\text{SK}}|\Lambda\} = Z_N^{\text{S}} \ . \qquad \mathbb{P}\text{-a.s.} \tag{2.2}$$

On the other hand, by Wigner's semicircle law, the empirical measure of eigenvalues converges weakly [G]

$$\frac{1}{N}\sum_{k=1}^{N}\delta_{\lambda_k} \underset{N\to\infty}{\longrightarrow} w(\lambda)\,d\lambda := \frac{2}{\pi}(1-\lambda^2)^{\frac{1}{2}}\,\mathbb{1}_{]-1,1[}(\lambda)\,d\lambda \qquad \mathbb{P}\text{-a.s.} \qquad (2.3.a)$$

and moreover the maximal and minimal eigenvalues do not deviate [BY]

$$\lim_{N\to\infty}\lambda_N = -\lim_{N\to\infty}\lambda_1 = 1\,. \qquad \mathbb{P}\text{-a.s.} \qquad (2.3.b)$$

Let us now state our main result which implies non-analyticity of the pressure at $\beta = 1$.

Proposition: *a) For all $\beta \geq 1$,*

$$\limsup_{N\to\infty} p_N^{\mathrm{SK}} \leq \beta - \tfrac{1}{2}\log\beta - \tfrac{3}{4}\,. \qquad \mathbb{P}\text{-}a.s.$$

b) In particular, $\limsup_{N\to\infty} p_N^{\mathrm{SK}} < \beta^2/4$ *$\mathbb{P}$-a.s. when $\beta > 1$, though the limit* (1.3) *holds when* $0 \leq \beta \leq 1$.

In fact the bound in a) is equal to the limit of p_N^{S} [Th]. For $\beta > 1$ but close to 1, the bound is larger than the Sherrington-Kirkpatrick solution for the pressure of the SK model, and then it does not prove absence of self-averaging of the Edwards-Anderson order parameter [PS] for these values of β.

Proof. We will prove that

$$\limsup_{N\to\infty} p_N^{\mathrm{SK}} \leq \inf_{s>\beta}\left\{s - \tfrac{1}{2} - \tfrac{1}{2}\int \log 2(s-\beta\lambda)\,w(\lambda)\,d\lambda\right\} \qquad \mathbb{P}\text{-a.s.} \qquad (2.4)$$

Then, noticing that the function between braces in (2.4) is convex in s, and using the identities

$$\begin{cases}\int (s-\beta\lambda)^{-1}\,w(\lambda)\,d\lambda = 2\beta^{-2}\left[s-(s^2-\beta^2)^{\frac{1}{2}}\right] & (2.5.a)\\ \int \log(s-\beta\lambda)\,w(\lambda)\,d\lambda = \beta^{-2}s\left[s-(s^2-\beta^2)^{\frac{1}{2}}\right] + \log\left(\left[s+(s^2-\beta^2)^{\frac{1}{2}}\right]/2\right) - 1/2 & (2.5.b)\end{cases}$$

valid for $s > \beta$, it can be checked that the bound in (2.4) is achieved for $\beta < 1$ at the saddle-point $s = (\beta^2+1)/2$ and is equal to $\beta^2/4$, but corresponds to $s = \beta$ when $\beta \geq 1$ and is equal to $\beta - \frac{1}{2}\log\beta - \frac{3}{4}$.

We now prove (2.4). Let $s > s_0 > \beta$; and assume that $\beta\lambda_N \leq s_0$. Then,

$$a_N := \int_{\mathbb{R}^N} \exp\left\{\beta\zeta^t\Lambda\zeta - s|\zeta|^2\right\}d\zeta = \exp\left\{\tfrac{N}{2}\log 2\pi - \tfrac{1}{2}\log\det 2(s\mathrm{I}-\beta\Lambda)\right\}. \qquad (2.6)$$

For $\epsilon \in\,]0,1[$, let $v_{N,\epsilon}$ be the Euclidean volume of $\left\{\zeta \in \mathbb{R}^N, |\zeta|^2/N \in [1-\epsilon, 1]\right\}$. The uniform probability measure on this domain makes the vectors $\eta = \sqrt{N}(\zeta_i/|\zeta|)_{i\leq N}$

and $|\zeta|^2/N$ independent, the first one with distribution $\mathbb{E}_\eta$ and the second one with mean $m_{N,\epsilon} \in [1-\epsilon, 1]$. Therefore, we have

$$\begin{aligned} a_N &\geq \int \exp \frac{|\zeta|^2}{N} \{\beta\eta^t\Lambda\eta - sN\} \, 1\!\!1_{N^{-1}|\zeta|^2 \in [1-\epsilon,1]} \, d\zeta \, v_{N,\epsilon}^{-1} . v_{N,\epsilon} \\ &\geq v_{N,\epsilon} . \mathrm{E}_\eta \exp m_{N,\epsilon} \{\beta\eta^t\Lambda\eta - sN\} \end{aligned}$$

from Jensen inequality for E_η. Since $s > s_0 \geq \beta\lambda_N$, the last term in braces is negative, the function $m \mapsto \mathrm{E}_\eta \exp m\{\beta\eta^t\Lambda\eta - sN\}$ is decreasing and

$$a_N \geq v_{N,\epsilon} \, \mathrm{E}_\eta \exp \{\beta\eta^t\Lambda\eta - sN\} = V_{N,\epsilon} \exp\{-Ns\} \, Z_N^{\mathrm{S}} \, .$$

Combining (2.6) and the estimate

$$\begin{aligned} v_{N,\epsilon} &= \pi^{N/2}\Gamma(1+N/2)^{-1} N^{N/2} \left[1 - \exp \frac{N}{2} \log(1-\epsilon)\right] \\ &= \exp \frac{N}{2} \{\log 2\pi + 1 + \mathcal{O}_\epsilon(1)\} \end{aligned}$$

with some (deterministic) sequence $\mathcal{O}_\epsilon(1)$ going to zero, we obtain finally for $\beta\lambda_N \leq s_0$

$$p_N^{\mathrm{S}} \leq s - \tfrac{1}{2} - \frac{1}{2N} \sum_{i=1}^{N} \log 2(s - \beta\lambda_k) + \mathcal{O}_\epsilon(1) \, . \tag{2.7}$$

Letting $b = s - \frac{1}{2} - \frac{1}{2} \int \log 2(s-\beta\lambda) \, w(\lambda) \, d\lambda$ we define the following set A_N of couplings $A_N = \left\{\lambda_N \leq s_0/\beta, \lambda_1 > -2, s - \frac{1}{2} - \frac{1}{2N} \sum_{i=1}^N \log 2(s-\beta\lambda_k) < b+\epsilon\right\}$. We have

$$\begin{aligned} \mathbb{E}\, Z_N^{\mathrm{SK}} \, 1\!\!1_{A_N} &= \mathbb{E}\, 1\!\!1_{A_N} \mathrm{E}(Z_N^{\mathrm{SK}} | \Lambda) \\ &= \mathbb{E}\, 1\!\!1_{A_N} \, Z_N^{\mathrm{S}} && \text{(from (2.2))} \\ &\leq \exp N(b+2\epsilon) && \text{for large } N \quad \text{(from (2.7))}, \end{aligned}$$

$$\mathbb{P}\big(\{p_N^{\mathrm{SK}} \geq b+3\epsilon\} \cap A_N\big) \leq \exp -N\epsilon$$

and the Borel-Cantelli lemma implies that $\mathbb{P}(\limsup_{N\to\infty} \{p_N^{\mathrm{SK}} \geq b+3\epsilon\} \cap A_N) = 0$. On the other hand we know from (2.3.a and b) that $\mathbb{P}(\liminf_{N\to\infty} A_N) = 1$, therefore $\mathbb{P}(\limsup_{N\to\infty} \{p_N^{\mathrm{SK}} \geq b+3\epsilon\}) = 0$ for all $\epsilon > 0$. We have finally $\limsup_{N\to\infty} p_N^{\mathrm{SK}} \leq b$ $\mathbb{P}$-a.s., which shows (2.4).

The statement b) is an obvious consequence of a), except for $\lim_N p_N^{\mathrm{SK}} = \beta^2/4$ if $\beta = 1$; but this follows from convexity of p_N^{SK} in β, from (1.3) and a) for $\beta = 1$. See also [Gu] for the case $\beta = 1$. ∎

Simple entropy considerations yield an improvement of the proposition, which in turn reflects as a bound on the ground state energy (per site)

$$\mathcal{E}_N = \max_{\sigma} \{ \frac{-1}{N} H_N(\sigma) \} .$$

Following [ALR] note that, since the entropy rate does not exceed $\log 2$, the function $(p_N^{\rm SK}(\beta) + \log 2)/\beta$ is non-increasing. Hence if some asymptotic upper bound $\bar{p}(\beta)$ for the pressure $p_N^{\rm SK}(\beta)$ has its tangent line at some $\beta_0 > 0$ which intersect the vertical axis at height $-\log 2$, then this tangent is itself an asymptotic upper bound for $\beta \geq \beta_0$, and its slope is an asymptotic upper bound for $\mathcal{E}_N$. Taking here $\bar{p}(\beta) = \beta^2/4$ for $\beta \leq 1$ and $\bar{p}(\beta) = \beta - \frac{1}{2}\log\beta - \frac{3}{4}$ for $\beta > 1$ we obtain the following.

Corollary: $\mathbb{P}$*-a.s.,* $\limsup_{N\to\infty} \mathcal{E}_N \leq 1 - e^{\frac{1}{2}}/8 = 0.79390\ldots$
and

$$\limsup_{N\to\infty} p_N^{\rm SK} \leq (1 - e^{\frac{1}{2}}/8) - \log 2 , \quad \beta \geq 4e^{-\frac{1}{2}} .$$

This bound on $\mathcal{E}_N$ numerically improves the bound 0.83 in [ALR], and is slightly better than the bound $(2/\pi)^{1/2} = 0.798\ldots$ given in [S] with the use of Slepian's lemma. We recall that the asymptotic value for the ground state energy predicted from numerical studies is 0.76 (see [MPV], p. 2).

References

[ALR] M. AIZENMAN, J.L. LEIBOWITZ, D. RUELLE (1987): Some rigorous results on the Sherrington-Kirkpatrick spin-glass model. *Comm. Math. Phys. 112, 3–20.*

[BGP] A. BOVIER, N. GAYRARD, P. PICCO: Gibbs states of the Hopfield model with extensively many patterns. *Preprint, Marseille 1994.*

[BK] T.H. BERLIN, M. KAC (1952): The spherical model of a ferromagnet. *Phys. Rev. 86, 821–825.*

[BY] Z.D. BAI, Y.Q. YIN (1988): Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a Wigner matrix. *Ann. Prob. 16, 1729–1741.*

[CN] F. COMETS, J. NEVEU (1995): The Sherrington-Kirkpatrick model of spin glasses and stochastic calculus : the high temperature case. *Comm. Math. Phys. 166, 549–564.*

[G] V.L. GIRKO (1989) : Asymptotics of the distribution of the spectrum of random matrices. *Russian Math. Surv. 44, 3–36.*

[Gu] F. GUERRA : Fluctuations and thermodynamic variables in mean field spin glass models. Preprint, Rome 1992.

[KTJ] J. KOSTERLITZ, D. THOULESS, R. JONES (1976) : Spherical models of spin-glass. *Phys. Rev. Lett. 36, 1217–1220.*

[MPV] M. MEZARD, G. PARISI, M.A. VIRASORO (1987) : Spin glass theory and beyond. Lect. Notes in Physics 9, World Scientific, Singapore.

[PS] L. PASTUR, M.V. SHCHERBINA (1991) : Absence of self-averaging of the order parameter in the Sherrington-Kirkpatrick model. *J. Stat. Phys. 62, 1–19.*

[S] D. SZASZ : Private communication on a bound of G. HALASZ, Budapest (1994).

[SK] D. SHERRINGTON, S. KIRKPATRICK (1975) : Solvable model of a spin-glass, *Phys. Rev. Lett. 35, 1792–1796.*

[T] M. TALAGRAND : Concentration of measure and isoperimetric inequalities in product spaces. Preprint, 1994.

[Th] C.J. THOMPSON (1985) : Random spin systems. In : Critical phenomena, 1983 Brasov school (ed. Jaffe, Parisi, Ruelle) 136–186. Progress in Physics II, Birkhäuser, Boston.

Francis Comets
Université Denis Diderot
UFR de Mathématiques
Case 7012
2 place Jussieu
75 251 PARIS Cedex 05