ASTÉRISQUE 251

NOMBRE ET RÉPARTITION DE POINTS DE HAUTEUR BORNÉE

édité par

Emmanuel Peyre

Société Mathématique de France 1998

Publié avec le concours du Centre National de la Recherche Scientifique

E. Peyre
Institut de Recherche Mathématique Avancée, Université Louis Pasteur et C.N.R.S., 7 rue René-Descartes, 67084 Strasbourg, France.
E-mail : peyre@irma.u-strasbg.fr

***Classification mathématique par sujets* (2000). —** Primaire 11G35 ; secondaires 14G05, 11E76, 14M25, 14G10.

***Mots clefs*. —** Point rationnel, hauteur, surface cubique, variété torique, mesure de Tamagawa.

NOMBRE ET RÉPARTITION DE POINTS DE HAUTEUR BORNÉE

édité par Emmanuel Peyre

***Résumé.* —** Si les points rationnels d'une variété définie sur un corps de nombres sont denses pour la topologie de Zariski, il est naturel de munir cette variété de hauteurs qui, du point de vue de la géométrie d'Arakelov, s'interprètent comme degrés d'intersection avec des fibrés en droites munis de métriques. L'objectif est alors d'étudier de manière asymptotique l'ensemble des points dont la hauteur est inférieure à un nombre réel donné, et cela en des termes aussi géométriques que possible.

Ce volume est issu de deux séminaires qui ont eu lieu en avril et en mai 1996. Il contient des articles de Slater et Swinnerton-Dyer, de Heath-Brown, de Fouvry et de de la Bretèche centrés sur le cas des surfaces cubiques, un texte de Billard sur les modèles minimaux des surfaces rationnelles, ainsi que des contributions de Salberger, de Peyre et de Batyrev et Tschinkel dont le principal objet est l'interprétation du terme dominant dans l'étude asymptotique du nombre de points de hauteur bornée.

***Abstract* (Number and distribution of points of bounded height)**

If the rational points of a variety over a number field are Zariski dense, then it is natural to equip this variety with heights, which one can interpret as intersection degrees relative to metrized line bundles. The aim is to study the set of points of bounded height asymptotically and to relate the results to the geometry of the variety.

This volume is the outcome of two seminars held in Paris in April and May 1996. It contains articles by Slater and Swinnerton-Dyer, Heath-Brown, Fouvry, and de la Bretèche on cubic surfaces, a text by Billard on minimal models of rational surfaces and contributions of Batyrev and Tschinkel, Salberger, and Peyre on the conjectural interpretation of the dominant term in the asymptotic behaviour of the number of points with bounded height.

PRÉFACE

Si les points rationnels d'une variété algébrique définie sur un corps de nombres sont denses pour la topologie de Zariski, il est naturel de munir cette variété de hauteurs qui, du point de vue de la géométrie d'Arakelov, s'interprètent comme degré d'intersection avec un fibré en droites muni de métriques. Il s'agit donc de fonctions définies sur l'ensemble des points rationnels à valeurs réelles strictement positives. L'objectif est alors d'étudier de manière asymptotique l'ensemble des points dont la hauteur est inférieure à un nombre réel donné, et cela en des termes aussi géométriques que possible. Cette étude a connu ces dernières années un regain important dont Manin a été un des principaux instigateurs. Ce renouveau a en particulier porté sur une meilleure compréhension du terme dominant dans le nombre de points de hauteur bornée. Ceci passe par une prise en compte des problèmes de répartition tels que l'existence de fermés accumulateurs susceptibles d'occulter le reste de la variété dans l'étude asymptotique.

Ce volume est issu de deux séminaires qui ont eu lieu en avril et en mai 1996 et où furent présentés divers développements récents de ce domaine.

Une des richesses de cette théorie est la variété des points de vue et des méthodes mises en œuvre. Nous avons tenté d'en donner une palette aussi large que possible. La cas des surfaces cubiques est particulièrement révélateur à cet égard ; en effet il apparaît dans la plupart des textes présentés ici, mais avec un regard à chaque fois différent.

Les premiers articles leur sont entièrement consacrés : Slater et Swinnerton-Dyer donnent une minoration du nombre des points de hauteur bornée sur le complémentaire des 27 droites pour une vaste classe de surfaces cubiques, tandis que Heath-Brown en présente une majoration. Les deux textes suivants étudient de manière fine le cas particulier de la surface cubique singulière d'équation

$$X_1X_2X_3 = T^3,$$

mais avec des procédés de théorie analytique des nombres « élémentaire » dans celui de Fouvry et d'analyse complexe dans celui de de la Bretèche. Cet exemple qui a fait l'objet de plusieurs discussions informelles lors des rencontres de 1996 apparaît également dans les textes de Salberger et de Batyrev et Tschinkel. La contribution de Billard porte sur la conjecture de Batyrev et Manin pour les modèles minimaux des surfaces rationnelles. Les trois derniers articles mettent plutôt l'accent sur l'interprétation conjecturale du terme dominant dans l'estimation asymptotique. Salberger exprime la constante apparaissant dans ce terme à l'aide des torseurs universels. Cette approche lui permet en outre de redémontrer un résultat de Batyrev et Tschinkel sur les variétés toriques projectives, lisses et déployées. Ce relèvement aux torseurs universels est également l'objet du texte qui le suit. Batyrev et Tschinkel quant à eux considèrent ces interprétations conjecturales à la lumière de travaux récents de Fujita concernant le programme des modèles minimaux pour les variétés algébriques polarisées. Ils développent également une généralisation de la notion de nombre de Tamagawa associé à une hauteur.

Nous tenons à remercier les organisateurs du séminaire sur les variétés rationnelles, le responsable des lundis arithmétiques ainsi que le réseau européen sur les formes automorphes pour les séminaires à l'origine de ce livre.

Emmanuel Peyre

TABLE DES MATIÈRES

RÉSUMÉS DES ARTICLES

Soient V une surface cubique non singulière définie sur $\mathbb{Q}$ et U le complémentaire des 27 droites de V. On note $N(U,H)$ le nombre de points rationnels de U de hauteur plus petite que H. Manin a conjecturé que, si $V(\mathbb{Q})$ n'est pas vide, alors

$$N(U,H) = C_1 H(\log H)^{r-1}(1+o(1)) \tag{1}$$

où C_1 est une constante strictement positive et r le rang du groupe de Néron-Severi $NS(V/\mathbb{Q})$ de V sur $\mathbb{Q}$. Dans cet article nous considérons le cas particulier où V contient deux droites rationnelles disjointes ; et nous prouvons qu'il existe une constante $C_2 > 0$ telle que pour tout réel H assez grand on ait

$$N(U,H) > C_2 H(\log H)^{r-1}.$$

Ceci donne la minoration dans l'estimation asymptotique (1). Il est probable que les arguments de ce texte puissent être modifiés pour montrer le résultat correspondant quand V contient deux droites disjointes conjuguées sur $\mathbb{Q}$ de sorte que chacune soit définie sur une extension quadratique de $\mathbb{Q}$. Mais nous n'avons pas entrepris de rédiger les détails.

Soient $F[W,X,Y,Z]$ une forme cubique rationnelle et $N^{(0)}(R)$ le nombre de zéros rationnels de F de hauteur inférieure ou égale à R, qui ne sont sur aucune des droites rationnelles de la surface $F=0$. Nous montrons que

$$N^{(0)}(R) \ll_{\varepsilon,F} R^{4/3+\varepsilon}$$

pour tout réel fixé $\epsilon > 0$, sous une hypothèse sur la taille du rang des courbes elliptiques. Pour la démonstration on compte les points sur les courbes cubiques obtenues comme sections hyperplanes de la surface $F = 0$.

Par une paramétrisation des solutions de l'équation

$$x_1x_2x_3 = t^3, \qquad (x_1, x_2, x_3, t) = 1$$

et par des méthodes élémentaires de théorie analytique des nombres, on montre que le nombre de points de coordonnées non nulles, de hauteur canonique inférieure à X, de la surface cubique projective de $\mathbb{P}_3(\mathbb{Q})$ d'équation

$$X_1X_2X_3 = T^3,$$

est, pour $X \longrightarrow \infty$, équivalent à $c_0 X \log^6 X$, où c_0 est une certaine constante > 0.

En développant des méthodes d'intégration complexe, nous établissons une formule asymptotique très précise sur le nombre de points de hauteur inférieure à X d'une certaine variété torique. Nous estimons le cardinal

$$V(X) := \mathrm{card}\{(x, y, z, t) \in (\mathbb{N} \cap [1, X])^4 \ : \ (x, y, z, t) = 1, \ xyz = t^3\}.$$

Soit $N(X) := (\log X)^{3/5}(\log_2 X)^{-1/5}$. Alors il existe un polynôme Q de $\mathbb{R}[X]$ de degré 6 et une constante $c > 0$ tels que l'on ait, pour X tendant vers l'infini, l'estimation

$$V(X) = XQ(\log X) + O\big(X^{1-1/8} \exp\{-cN(X)\}\big)$$

De plus, le coefficient dominant de Q est égal à

$$\frac{1}{4 \times 6!} \prod_p \Big\{\Big(1 - \frac{1}{p}\Big)^7 \Big(1 + \frac{7}{p} + \frac{1}{p^2}\Big)\Big\}.$$

Nous étudions la répartition des points rationnels des modèles minimaux des surfaces rationnelles et vérifions que ces surfaces satisfont les conjectures de Batyrev-Manin sur le corps des rationnels. Pour ce faire, nous rappelons d'abord quelques propriétés et descriptions géométriques de telles surfaces. Ensuite, pour chaque plongement considéré, une hauteur naturelle apparaissant, nous établissons directement le comportement asymptotique des points rationnels de hauteur bornée. Finalement, nous regardons les sous-variétés accumulatrices.

Soit X une variété de Fano sur un corps de nombres. Un système d'Arakelov de métriques v-adiques sur le faisceau anticanonique de X définit une hauteur et une nouvelle sorte de mesures sur les torseurs universels sur X.

Le but de cet article est d'établir un lien entre la croissance asymptotique du nombre de points rationels de hauteur bornée sur X et certains volumes adéliques des torseurs universels sur X.

Soient V une variété de Fano et $\mathbf{H}$ un système de hauteurs d'Arakelov définissant un accouplement entre le groupe de Picard $\operatorname{Pic} V$ et les points rationnels de V à valeur dans $\mathbf{R}$. Soit $\zeta_{\mathbf{H}}$ la fonction zêta associée sur $\operatorname{Pic} V \otimes \mathbf{C}$. Batyrev, Manin et Tschinkel ont conjecturé que cette fonction est holomorphe sur un cône de sommet le faisceau anticanonique ω_V^{-1}. Il est en outre possible de donner une expression conjecturale du terme principal de cette fonction $\zeta_{\mathbf{H}}$ au voisinage de ce sommet. Le but de ce texte est de montrer comment cette expression conjecturale peut s'écrire naturellement en passant aux torseurs universels au-dessus de V.

Soit $\mathcal{L} = (L, \|\cdot\|_v)$ un faisceau inversible ample muni de métriques au-dessus d'une variété algébrique lisse quasi-projective V sur un corps de nombres F. Notons $N(V, \mathcal{L}, B)$ le nombre de points rationnels de V ayant une $\mathcal{L}$-hauteur $\leqslant B$. Dans ce texte nous considérons le problème de l'interprétation géométrique et arithmétique du comportement asymptotique de $N(V, \mathcal{L}, B)$

lorsque $B \to \infty$ à la lumière de conjectures récentes de Fujita concernant le programme des modèles minimaux pour les variétés algébriques polarisées.

Nous introduisons également des notions de *variétés $\mathcal{L}$-primitives* et de *fibrations $\mathcal{L}$-primitives*. Pour toute variété $\mathcal{L}$-primitive V sur F nous proposons une méthode pour définir un nombre de Tamagawa adélique $\tau_{\mathcal{L}}(V)$ qui est une généralisation du nombre de Tamagawa $\tau(V)$ défini par Peyre pour des variétés de Fano lisses. Notre méthode nous permet de construire des nombres de Tamagawa pour des $\mathbf{Q}$-variétés de Fano ayant au plus des singularités canoniques.

Dans une série d'exemples de variétés polarisées lisses et de variétés de Fano singulières nous montrons que nos nombres de Tamagawa rendent compte de la dépendance du comportement asymptotique de $N(V, \mathcal{L}, B)$ en fonction du choix des métriques v-adiques sur $\mathcal{L}$.

ABSTRACTS

Counting points on cubic surfaces, I

Let V be a nonsingular cubic surface defined over $\mathbb{Q}$, let U be the open subset of V obtained by deleting the 27 lines, and denote by $N(U,H)$ the number of rational points in U of height less than H. Manin has conjectured that if $V(\mathbb{Q})$ is not empty then

$$N(U,H) = C_1 H(\log H)^{r-1}(1+o(1)) \tag{1}$$

for some $C_1 > 0$, where r is the rank of $NS(V/\mathbb{Q})$, the Néron-Severi group of V over $\mathbb{Q}$. In this note we consider the special case when V contains two rational skew lines ; and we prove that for some $C_2 > 0$ and all large enough H,

$$N(U,H) > C_2 H(\log H)^{r-1}.$$

This is the one-sided estimate corresponding to (1). It seems probable that the arguments in this paper could be modified to prove the corresponding result when V contains two skew lines conjugate over $\mathbb{Q}$ and each defined over a quadratic extension of $\mathbb{Q}$; but we have not attempted to write out the details.

Counting Rational Points on Cubic Surfaces

Let $F[W,X,Y,Z]$ be a rational cubic form, and let $N^{(0)}(R)$ be the number of rational zeros of F of height at most R, which do not lie on any rational line in the surface $F = 0$. We show that

$$N^{(0)}(R) \ll_{\varepsilon,F} R^{4/3+\varepsilon}$$

for any fixed $\varepsilon > 0$, subject to a suitable hypothesis on the size of the rank of elliptic curves. For the proof one counts points on the cubic curves obtained from hyperplane sections of the surface $F = 0$.

By a parametrisation of the solutions of the equation

$$x_1 x_2 x_3 = t^3, \qquad (x_1, x_2, x_3, t) = 1$$

and by elementary methods of analytic number theory, we show that the number of points, with non–zero coordinates, with canonical height less than X, of the cubic projective surface of $\mathbb{P}_3(\mathbb{Q})$ defined by the equation

$$X_1 X_2 X_3 = T^3,$$

is, for $X \longrightarrow \infty$, asymptotically equivalent to $c_0 X \log^6 X$,where c_0 is some positive constant.

By complex integration methods, we prove a very precise asymptotic formula about the number of rational points of height at most X on a certain toric variety. We estimate the cardinality

$$V(X) := \mathrm{card}\{(x, y, z, t) \in (\mathbb{N} \cap [1, X])^4 \ : \ (x, y, z, t) = 1 : xyz = t^3\}.$$

Let $N(X) := (\log X)^{3/5}(\log_2 X)^{-1/5}$. Then there exists a polynomial $Q \in \mathbb{R}[X]$ of degree 6 and a constant $c > 0$ such that, for $X \to +\infty$, we have

$$V(X) = XQ(\log X) + O\big(X^{1-1/8} \exp\{-cN(X)\}\big).$$

Futhermore, the leading coefficient of Q is

$$\frac{1}{4 \times 6!} \prod_p \Big\{ \Big(1 - \frac{1}{p}\Big)^7 \Big(1 + \frac{7}{p} + \frac{1}{p^2}\Big) \Big\}.$$

We verify that the minimal models of rational surfaces satisfy the Batyrev-Manin conjecture for a family of ample divisors on the field of rationals.

Let X be a Fano variety over a number field. An Arakelov system of v-adic metrics on the anticanonical line bundle on X gives rise to a height function on the set of rational points and to a new kind of adelic measures on the universal torsors over X.

The aim of the paper is to relate the asymptotic growth of the number of rational points of bounded height on X to volumes of adelic spaces corresponding to the universal torsors over X.

Let V be a Fano variety and $\mathbf{H}$ a system of Arakelov's heights which defines a pairing between the Picard group Pic V and the set of rational points of V with values in $\mathbf{R}$. Let $\zeta_{\mathbf{H}}$ be the corresponding zeta function on Pic $V \otimes \mathbf{C}$. Batyrev, Manin and Tschinkel conjectured that this function is holomorphic on a cone with appex at the anticanonical sheaf ω_V^{-1}. Moreover it is possible to give a conjectural formula for the principal term of this function in a neighbourhood of this point. The aim of this paper is to give new evidence for this formula by lifting it to the universal torsors over V.

Let $\mathcal{L} = (L, \|\cdot\|_v)$ be an ample metrized invertible sheaf on a smooth quasi-projective algebraic variety V over a number field F. Denote by $N(V, \mathcal{L}, B)$ the number of rational points in V having $\mathcal{L}$-height $\leq B$. In this paper we consider the problem of a geometric and arithmetic interpretation of the asymptotic for $N(V, \mathcal{L}, B)$ as $B \to \infty$ in connection with recent conjectures of Fujita concerning the Minimal Model Program for polarized algebraic varieties.

We introduce the notions of *$\mathcal{L}$-primitive varieties* and *$\mathcal{L}$-primitive fibrations*. For $\mathcal{L}$-primitive varieties V over F we propose a method to define an adelic Tamagawa number $\tau_{\mathcal{L}}(V)$ which is a generalization of the Tamagawa number $\tau(V)$ introduced by Peyre for smooth Fano varieties. Our method allows us to construct Tamagawa numbers for $\mathbf{Q}$-Fano varieties with at worst canonical singularities.

In a series of examples of smooth polarized varieties and singular Fano varieties we show that our Tamagawa numbers express the dependence of the asymptotic of $N(V, \mathcal{L}, B)$ on the choice of v-adic metrics on $\mathcal{L}$.

Astérisque
251, 1998, p. 1–12

COUNTING POINTS ON CUBIC SURFACES, I

by

John B. Slater & Sir Peter Swinnerton-Dyer

Abstract. — Let V be a nonsingular cubic surface defined over $\mathbb{Q}$, let U be the open subset of V obtained by deleting the 27 lines, and denote by $N(U,H)$ the number of rational points in U of height less than H. Manin has conjectured that if $V(\mathbb{Q})$ is not empty then

$$N(U,H) = C_1 H (\log H)^{r-1}(1+o(1)) \tag{1}$$

for some $C_1 > 0$, where r is the rank of $NS(V/\mathbb{Q})$, the Néron-Severi group of V over $\mathbb{Q}$. In this note we consider the special case when V contains two rational skew lines; and we prove that for some $C_2 > 0$ and all large enough H,

$$N(U,H) > C_2 H (\log H)^{r-1}.$$

This is the one-sided estimate corresponding to (1). It seems probable that the arguments in this paper could be modified to prove the corresponding result when V contains two skew lines conjugate over $\mathbb{Q}$ and each defined over a quadratic extension of $\mathbb{Q}$; but we have not attempted to write out the details.

Let V be a nonsingular cubic surface defined over $\mathbb{Q}$, let U be the open subset of V obtained by deleting the 27 lines, and denote by $N(U,H)$ the number of rational points in U of height less than H, and by k the least field of definition of the 27 lines. Once we have chosen our coordinate system, we shall define the *bad primes* for V as those p for which V has bad reduction at p or which ramify in any of the fields k_i defined below. In what follows we shall use $A_1, A_2, \ldots$ and $C_1, C_2, \ldots$ to denote positive constants depending only on V; the distinction between the A_j and the C_j is that the A_j will be rational and will be determined by divisibility considerations. Similarly $B_1, B_2, \ldots$ will each belong to a finite set of elements of k^* and $\mathfrak{b}_1, \mathfrak{b}_2, \ldots$ will belong to a finite set of non-zero fractional ideals of k, in each case depending only on V. The

2000 ***Mathematics Subject Classification.*** — Primary 11G25; Secondary 14G25.
Key words and phrases. — Cubic surfaces, Manin conjecture.

A_j, B_j and $\mathfrak{b}_j$ will always be units outside the bad primes, though this is not important. Letters A, B, C without subscripts will have the same properties, but will not necessarily have the same values from one occurrence to the next.

Manin has conjectured that if $V(\mathbb{Q})$ is not empty then

$$N(U, H) = C_1 H (\log H)^{r-1}(1 + o(1)) \tag{1}$$

for some $C_1 > 0$, where r is the rank of $NS(V/\mathbb{Q})$, the Néron-Severi group of V over $\mathbb{Q}$. In this note, which is the first of a sequence of papers concerned with various aspects of this conjecture, we consider the special case when V contains two rational skew lines; and we prove

Theorem 1. — *Suppose that V contains two rational skew lines. Then for some $C_2 > 0$ and all large enough H,*

$$N(U, H) > C_2 H (\log H)^{r-1}.$$

This is the one-sided estimate corresponding to (1). It seems probable that the arguments in this paper could be modified to prove the corresponding result when V contains two skew lines conjugate over $\mathbb{Q}$ and each defined over a quadratic extension of $\mathbb{Q}$; but we have not attempted to write out the details.

The truth or falsehood of Theorem 1 is not affected by a linear transformation of variables, though the value of C_2 may be; so without loss of generality we can assume that the two given skew lines on V have the form

$$L' : \ X_0 = X_1 = 0 \qquad \text{and} \qquad L'' : \ X_2 = X_3 = 0,$$

and that their five transversals on V have the form

$$L_i : \ X_0 = \alpha_i X_1, \ X_2 = \beta_i X_3 \quad \text{for} \quad 1 \leqslant i \leqslant 5$$

where the α_i, β_i are integers in k. We shall denote by k_i the least field of definition of L_i, so that k is the compositum of the k_i; since the α_i are all distinct, as are the β_i, we have $k_i = \mathbb{Q}(\alpha_i) = \mathbb{Q}(\beta_i)$. Since L', L'' and the L_i are a base for $NS(V/\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$, their traces are a base for $NS(V/\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$; and it follows at once that the L_i form $r - 2$ complete sets of conjugates over $\mathbb{Q}$. Because V contains L' and L'', its equation can be written in the form

$$f_1(X_0, X_1, X_2, X_3) = f_2(X_0, X_1, X_2, X_3) \tag{2}$$

where f_1 is homogeneous quadratic in X_0, X_1 and homogeneous linear in X_2, X_3 and the opposite is true for f_2. We can assume that the coefficients of f_1 and f_2 are rational integers and that we cannot take out an integer factor from (2). With these conditions, the bad primes for V include those which divide $\prod_{i<j}(\alpha_i - \alpha_j)^2$ or $\prod_{i<j}(\beta_i - \beta_j)^2$ and those which divide one side or other

of (2). The resultant of f_1 and f_2, considered as homogeneous polynomials in X_0 and X_1, has degree 5 in X_2 and X_3; so it has the form

$$A_1 \prod (X_2 - \beta_i X_3).$$

Similarly the resultant of f_1 and f_2, considered as homogeneous polynomials in X_2 and X_3, has the form

$$A_2 \prod (X_0 - \alpha_i X_1). \tag{3}$$

Moreover $f_1(\alpha_i, 1, X_2, X_3)$ is the product of $(X_2 - \beta_i X_3)$ and a non-zero integer in k_i; and $f_2(\alpha_i, 1, X_2, X_3)$ is divisible by $(X_2 - \beta_i X_3)$. In particular, for each i both f_1 and f_2 are in the ideal in $\mathfrak{o}_i[X_0, \ldots, X_3]$ generated by $X_0 - \alpha_i X_1$ and $X_2 - \beta_i X_3$, where $\mathfrak{o}_i$ is the ring of integers of k_i.

Our argument depends on the following recipe for generating the rational points on V, subject to certain anomalies. Let

$$P' = (\xi_0, \xi_1, 0, 0) \quad \text{and} \quad P'' = (0, 0, \xi_2, \xi_3)$$

be any rational points on L' and L'' respectively, expressed in lowest terms; thus ξ_0, ξ_1 are coprime integers, as are ξ_2, ξ_3. Note that each point P' corresponds to two pairs ξ_0, ξ_1 and similarly for P''. The third intersection of $P'P''$ with V is

$$P = (\xi_0 f_2(\xi), \xi_1 f_2(\xi), \xi_2 f_1(\xi), \xi_3 f_1(\xi)). \tag{4}$$

The expression (4) is not necessarily in lowest terms; indeed the highest factor which we can take out is precisely $(f_1(\xi), f_2(\xi))$, where the bracket denotes the highest common factor. The point P is geometrically well-defined unless $\xi_0 - \alpha_i \xi_1 = \xi_2 - \beta_i \xi_3 = 0$ for some i; and every rational point of V can be uniquely obtained in this way except for those which lie on some L_i. More generally, if we drop the condition that P' and P'' are rational we can in this way generate every point on V except those that lie on some L_i.

Since we wish to exclude from our count the rational points on the 27 lines, it will be important to know how they are generated under this recipe. We have already dealt with the L_i. If P is to be on L' for example, we must choose P' to be P and P'' to be the unique point satisfying $f_1(P', P'') = 0$ in an obvious notation; since

$$\begin{aligned} f_1(X_0, X_1, X_2, X_3) = {} & (X_0 - \alpha_i X_1) g_i(X_0, X_1, X_2, X_3) \\ & + (X_2 - \beta_i X_3) h_i(X_0, X_1) \end{aligned}$$

and the highest common factor of $(\xi_0 - \alpha_i \xi_1)$ and $h_i(\xi_0, \xi_1)$ in k_i divides the resultant of $(X_0 - \alpha_i X_1)$ and $h_i(X_0, X_1)$ and is therefore bounded, there is a rational integer A_3 such that $A_3(\xi_2 - \beta_i \xi_3)$ is divisible by $(\xi_0 - \alpha_i \xi_1)$ for each i. Next, let L_i' be the third intersection of V with the plane containing L' and L_i; since the one point of L'' in this plane is $(0,0,\beta_i,1)$, the points of L_i' are generated precisely when this point is taken to be P''. A similar argument holds for the line L_i'' which is the third intersection of V with the plane containing L'' and L_i. The remaining ten lines are the ones other than L' and L'' which meet three of the L_i. To fix ideas, consider the line L_{123} which meets L_1, L_2 and L_3. The condition that P is on L_{123} induces a one-one correspondence between P' and P'' in which three of the pairs are given by

$$P' = (\alpha_i, 1, 0, 0),\ \ P'' = (0, 0, \beta_i, 1) \text{ for } i = 1, 2, 3;$$

so this correspondence has the form

$$\frac{(\alpha_3-\alpha_2)(\xi_0-\alpha_1\xi_1)}{(\alpha_3-\alpha_1)(\xi_0-\alpha_2\xi_1)} = \frac{(\beta_3-\beta_2)(\xi_2-\beta_1\xi_3)}{(\beta_3-\beta_1)(\xi_2-\beta_2\xi_3)}.$$

For any pair P', P'' each fraction is in lowest terms, up to a factor belonging to a finite set of ideals depending only on V; so for $i = 1$

$$(\xi_0 - \alpha_i\xi_1) = \mathfrak{b}_1(\xi_2 - \beta_i\xi_3)$$

as ideals, where $\mathfrak{b}_1$ belongs to a finite set of principal ideals of k_i depending only on V. A similar result holds for $i = 2, 3$.

Assume that $\xi_0 - \alpha_i\xi_1$ and $\xi_2 - \beta_i\xi_3$ do not both vanish for any i, and denote by $\mathfrak{a}_i$ the ideal

$$\mathfrak{a}_i = (\xi_0 - \alpha_i\xi_1, \xi_2 - \beta_i\xi_3).$$

Conversely, suppose that the $\mathfrak{a}_i$ are integral ideals, each $\mathfrak{a}_i$ lying in k_i and two $\mathfrak{a}_i$ being conjugate over $\mathbb{Q}$ if the corresponding L_i are. In what follows, sets $\mathfrak{a}_i$ will always be assumed to have these properties. We shall say that the $\mathfrak{a}_i$ are *allowable* for V if there exist coprime pairs ξ_0, ξ_1 and ξ_2, ξ_3 which give rise to this set of $\mathfrak{a}_i$. If the $\mathfrak{a}_i$ are allowable then their product is an ideal in $\mathbb{Z}$, and we can therefore define a positive integer Λ such that $(\Lambda) = \prod \mathfrak{a}_i$.

Lemma 1. — (i) *Suppose that the $\mathfrak{a}_i$ are allowable for V. Then the only primes in k which can divide more than one of the $\mathfrak{a}_i$ are those which lie above bad primes in $\mathbb{Q}$, and the highest common factor of any two of the $\mathfrak{a}_i$ in k belongs to a finite set depending only on V. If p is a good prime, then for given i there is at most one prime $\mathfrak{p}$ in k_i which divides both p and $\mathfrak{a}_i$, and it is a first degree prime in k_i. Moreover $(f_1(\xi), f_2(\xi)) = B_1\Lambda$ where B_1 belongs to a finite set of rationals depending only on V.*
(ii) *Conversely, a sufficient condition for the $\mathfrak{a}_i$ to be allowable is that they are coprime in k and that none of them is divisible by any prime in k above a bad prime.*

Proof. — Since $(\mathfrak{a}_i, \mathfrak{a}_j)$ divides $(\xi_0 - \alpha_i\xi_1, \xi_0 - \alpha_j\xi_1)$ and therefore also $(\alpha_i - \alpha_j)$, the first assertion in (i) is trivial. Since $k_i = \mathbb{Q}(\alpha_i)$, a prime $\mathfrak{p}$ in k_i which is not first degree can only divide $\xi_0 - \alpha_i\xi_1$ if either the prime p below it in $\mathbb{Q}$ divides both ξ_0 and ξ_1 or $\alpha_i \equiv c \bmod \mathfrak{p}$ for some c in $\mathbb{Z}$. In the latter case there is an automorphism σ of k not fixing k_i elementwise and such that $\sigma\mathfrak{p}$ is not prime to $\mathfrak{p}$; and $(\sigma\mathfrak{p}, \mathfrak{p})$ divides $(\alpha_i - \sigma\alpha_i)$, whence p is a bad prime. If there were two primes $\mathfrak{p}'$ and $\mathfrak{p}''$ above p in k_i both of which divide $\mathfrak{a}_i$, then there would similarly be a σ such that $\sigma\mathfrak{p}''$ was not prime to $\mathfrak{p}'$ in k, and $\mathfrak{a}_i$ and $\sigma\mathfrak{a}_i$ would both be divisible by $(\mathfrak{p}', \sigma\mathfrak{p}'')$; hence again p would be a bad prime. This proves the second assertion in (i). As for the third, we know that the resultant of f_1 and f_2, considered as functions of X_2 and X_3, is (3); so $(f_1(\xi), f_2(\xi))$ divides $A_2 \prod(\xi_0 - \alpha_i\xi_1)$. It is therefore enough to prove that

$$(f_1(\xi), f_2(\xi), \xi_0 - \alpha_i\xi_1) = \mathfrak{b}_2(\xi_0 - \alpha_i\xi_1, \xi_2 - \beta_i\xi_3)$$

where $\mathfrak{b}_2$ belongs to a finite set of ideals in k_i depending only on V. But using the remarks after (3) we have

$$f_1(\xi) \equiv \xi_1^2 f_1(\alpha_i, 1, \xi_2, \xi_3) = A\xi_1^2(\xi_2 - \beta_i\xi_3) \mod (\xi_0 - \alpha_i\xi_1)$$

and

$$f_2(\xi) \equiv \xi_1 f_2(\alpha_i, 1, \xi_2, \xi_3) \mod (\xi_0 - \alpha_i\xi_1),$$

and $f_2(\alpha_i, 1, \xi_2, \xi_3)$ is divisible by $(\xi_2 - \beta_i\xi_3)$.

Now suppose that the $\mathfrak{a}_i$ satisfy (ii). Since the $\mathfrak{a}_i$ form complete sets of conjugates over $\mathbb{Q}$ and are coprime, the argument in the first half of the proof shows that each prime factor of $\mathfrak{a}_i$ in k_i is a first degree prime. Moreover, if $\mathfrak{p}_i | \mathfrak{a}_i$ and $\mathfrak{p}_j | \mathfrak{a}_j$ with $\mathfrak{p}_i$ and $\mathfrak{p}_j$ lying above the same good prime p, then we could find σ as above such that $\sigma\mathfrak{p}_i$ was not prime to $\mathfrak{p}_j$; this would imply that $(\sigma\mathfrak{p}_i, \mathfrak{p}_j)$ divides $(\sigma\alpha_i - \alpha_j)$, whence $\sigma\alpha_i = \alpha_j$ and therefore $L_j = \sigma L_i$ and $\mathfrak{p}_j = \sigma\mathfrak{p}_i$. Now choose ξ_1 divisible by every bad prime and by no good prime, and choose ξ_0 not divisible by any bad prime and such that for each prime $\mathfrak{p}_i$ dividing $\mathfrak{a}_i$ with $\mathfrak{p}_i^n \parallel \mathfrak{a}_i$ we have $\mathfrak{p}_i^n \parallel (\xi_0 - \alpha_i\xi_1)$. All this is possible by the Chinese Remainder Theorem, for every prime dividing $\mathfrak{a}_i$ is first degree in k_i and we are imposing on ξ_0 one p-adic condition for each p dividing Λ. Moreover a prime $\mathfrak{q}_i$ which divides $\xi_0 - \alpha_i\xi_1$ but not $\mathfrak{a}_i$ is good, and hence is prime to every other $\mathfrak{a}_j$ and to Λ. We choose ξ_2, ξ_3 by a similar recipe, but with the additional condition that no $\mathfrak{q}_i$ divides $\xi_2 - \beta_i\xi_3$. $\square$

Let h denote the height function; then in the notation above we have

$$h(P') = \max(|\xi_0|, |\xi_1|), \; h(P'') = \max(|\xi_2|, |\xi_3|).$$

It follows from (4) and Lemma 1 that

$$h(P) \leqslant C_3(h(P')h(P''))^2/\Lambda. \tag{5}$$

In order to prove Theorem 1, we shall obtain a lower bound for the number of rational points P in U with preassigned allowable $\mathfrak{a}_i$ such that $h(P) < H$. For simplicity, we confine ourselves to the case when none of the $\mathfrak{a}_i$ is divisible by any bad prime; hence in particular they are coprime. The natural way to proceed is to perform the count on $V \setminus \cup L_i$ and then allow for those rational points which lie on one of the lines other than the L_i. The latter step presents no difficulties because almost all the rational points on these lines come either from very special sets of ξ_i or from values of Λ much larger in terms of N than those which we shall be considering.

Lemma 2. — (i) *If P is a rational point on L'_i or L''_i then we have $\xi_2 - \beta_i\xi_3 = 0$ or $\xi_0 - \alpha_i\xi_1 = 0$ respectively.*
(ii) *The number of rational points on L', L'' or some L_{ijk} with preassigned $\mathfrak{a}_1, \dots, \mathfrak{a}_5$ and with $h(P')h(P'') < N$ is bounded by $C(\log N)^4$.*

Proof. — We have already proved (i). Suppose for example that P is on L'; then the ideals $(\xi_0 - \alpha_i\xi_1)$ and $\mathfrak{a}_i$ differ by a factor drawn from a finite set depending only on V. This means that $\xi_0 - \alpha_i\xi_1 = \epsilon\gamma_i$ where γ_i is an integer in k_i^* drawn from a finite set depending for preassigned $\mathfrak{a}_i$ only on V and ϵ is a unit in k_i. By multiplying γ_i by a suitable unit in k_i we can ensure that $|\sigma\gamma_i| > C$ for every embedding $\sigma : k_i \to \mathbb{C}$. This and $h(P') < N$ imply that $|\sigma\epsilon| < CN$ for every σ; since $[k_i : \mathbb{Q}] \leqslant 5$, there are at most $C(\log N)^4$ units ϵ with this property. If α_i is not in $\mathbb{Z}$ a knowledge of ϵ and γ_i uniquely determines ξ_0 and ξ_1. If however each α_i is in $\mathbb{Z}$ then there are only two possible values of ϵ for each γ_i, and a knowledge of more than one of the $\xi_0 - \alpha_i\xi_1$ determines ξ_0 and ξ_1. In either case $P = P'$, so a knowledge of P' determines P.

Finally suppose for example that P is on L_{123}. For $i = 1, 2, 3$ we know that $(\xi_0 - \alpha_i\xi_1)$, $(\xi_2 - \beta_i\xi_3)$ and $\mathfrak{a}_i$ differ by factors drawn from a finite set of ideals depending only on V. Since if L_{123} is rational the set $\{L_1, L_2, L_3\}$ is a union of complete sets of conjugates over $\mathbb{Q}$, an argument like that of the previous paragraph shows that for any preassigned $\mathfrak{a}_i$ there are at most $C(\log N)^2$ possible P' with $h(P') < N$, and a similar result holds for P''. $\square$

The next step will be to estimate, under suitable conditions, the number of rational points in U which satisfy

$$h(P') \leqslant N', \quad \tfrac{1}{2}N'' \leqslant h(P'') < N'' \tag{6}$$

for given $\mathfrak{a}_i$ and given N', N''. The pairs P', P'' which satisfy

$$\xi_0 - \alpha_i\xi_1 \equiv \xi_2 - \beta_i\xi_3 \equiv 0 \mod \mathfrak{a}_i \tag{7}$$

are precisely those given by

$$\xi_0 = \Lambda\eta_0 + a\eta_1,\ \xi_1 = \eta_1,\ \xi_2 = \Lambda\eta_2 + b\eta_3,\ \xi_3 = \eta_3 \tag{8}$$

for some integers η_i, where a, b have been chosen so that

$$a - \alpha_i \equiv b - \beta_i \equiv 0 \mod \mathfrak{a}_i$$

for each i. (That we can find a, b with these properties depends on the facts that the $\mathfrak{a}_i$ are coprime and divisible only by first degree primes.) We can clearly assume that both a and b are absolutely bounded by Λ. The condition that ξ_0, ξ_1 are coprime is equivalent to η_1 being prime to both Λ and η_0; and similarly for ξ_2, ξ_3. The first condition (6) is equivalent to

$$|\eta_1| \leqslant N', \quad |\eta_0 + a\Lambda^{-1}\eta_1| \leqslant \Lambda^{-1}N' \tag{9}$$

and the second condition (6) is equivalent to either

$$\tfrac{1}{2}N'' \leqslant |\eta_3| < N'', \quad |\eta_2 + b\Lambda^{-1}\eta_3| < \Lambda^{-1}N'' \tag{10}$$

or

$$|\eta_3| < \tfrac{1}{2}N'', \quad \tfrac{1}{2}\Lambda^{-1}N'' \leqslant |\eta_2 + b\Lambda^{-1}\eta_3| < \Lambda^{-1}N''. \tag{11}$$

To obtain all the rational points on V which satisfy (6) and have the given values of the $\mathfrak{a}_i$, we let the η_i run through all sets satisfying (9) and either (10) or (11), and reject all those for which ξ_0, ξ_1 are not coprime, or ξ_2, ξ_3 are not coprime, or $(\xi_0 - \alpha_i\xi_1, \xi_2 - \beta_i\xi_3)$ is a strict multiple of $\mathfrak{a}_i$ for some i. We shall say that a set of η_i *fails* at a prime p if p divides both ξ_0 and ξ_1, or both ξ_2 and ξ_3, or if there is a prime factor $\mathfrak{p}$ of p in k_i such that $\mathfrak{p}\mathfrak{a}_i$ divides both $\xi_0 - \alpha_i\xi_1$ and $\xi_2 - \beta_i\xi_3$ for some i. We shall call a set of η_i *allowable* if it does not fail for any p; note that such a set may still give rise to a point on one of the 27 lines. Whether the η_i fail at p only depends on the values of the $\eta_i \bmod p$. For any prime p, let n_p be the number of sets $\eta_0, \ldots, \eta_3 \bmod p$ which fail at p. Here n_p depends on the $\mathfrak{a}_i$ as well as on V and p. It is not hard to give an explicit formula for n_p, but all we shall need are the obvious estimates, for p a good prime, that $n_p = 2p^3 + O(p^2)$ if $p|\Lambda$ and $n_p = O(p^2)$ otherwise. Because the $\mathfrak{a}_i$ are allowable, $n_p < p^4$ for all p.

Lemma 3. — *Assume that the $\mathfrak{a}_i$ are coprime and that none of them is divisible by any prime in k lying above a bad prime. Suppose that*

$$\Lambda < n^{1/2} \quad \textit{and} \quad \log\max(N', N'') < n,$$

where $n = \min(N', N'')$. *Then, provided that* N' *and* N'' *are large enough, the number of rational points in* U *which satisfy* (6) *and give rise to these* $\mathfrak{a}_i$ *differs from*

$$3(N'N''/\Lambda)^2 \prod_{p<T} (1 - p^{-4} n_p)$$

by at most $C(N'N''/\Lambda)^2 (\log\log n)^{-1/2}$, *for any* T *satisfying*

$$3 \log n / \log\log\log n > T > \log n / \log\log\log n.$$

Proof. — It is well known that there is an absolute constant C_4 such that

$$\sum_{p<x} \log p < C_4 x \quad \text{provided} \quad x > 2; \tag{12}$$

thus the condition on T ensures that $T \log T$ is large compared to $\log n$ and that $Q < n^{C/\log\log\log n}$ where Q denotes the product of all the primes less than T. We assume n so large that all the bad primes are less than T, and to fix ideas we assume that $n = N'$. In what follows we shall use square brackets to denote integral parts. The set of pairs η_0, η_1 satisfying (9) contains

$$[2N'/Q][2\Lambda^{-1} N'/Q]$$

disjoint parallelograms each of which consists of a complete set of pairs incongruent mod$Q$; and the pairs left over are contained in a further

$$[2N'/Q] + [2\Lambda^{-1}N'/Q] + 1 \leqslant CN'/Q$$

such parallelograms. A similar argument holds for the pairs η_2, η_3 which satisfy either (10) or (11); this time the number of filled parallelograms differs from

$$3N''^2/\Lambda Q^2$$

by at most CN''/Q, and the pairs left over are contained in a further CN''/Q such parallelograms. Each product of two filled parallelograms contains $\prod_{p<T}(p^4 - n_p)$ sets which do not fail at any prime less than T. Hence the number of sets of η_i which satisfy (9) and either (10) or (11) and which do not fail at any prime less than T differs from

$$12(N'N''/\Lambda)^2 \prod_{p<T} (1 - p^{-4} n_p) \tag{13}$$

by at most $CN'N''^2 Q \Lambda^{-1}$. In going from sets η_i to pairs P', P'' we lose a factor 4. The sets with $\eta_0 = \eta_1 = 0$ or $\eta_2 = \eta_3 = 0$ have already been rejected because they fail at every p. The number of other sets η_i which satisfy $\xi_0 - \alpha_i \xi_1 = 0$ or $\xi_2 - \beta_i \xi_3 = 0$ for some i with α_i, β_i in $\mathbb{Z}$ is bounded by $CN'N''^2\Lambda^{-1}$, which can be absorbed into the previous error term; since those of them which are allowable give rise to a point on some L_i' or L_i'', we can rule them out now.

Next let p be a prime satisfying $T \leqslant p \leqslant N'$. By an argument similar to that in the previous paragraph, the number of sets η_i which satisfy (9) and either (10) or (11) but which fail at p is at most

$$C(p^{-2}\Lambda^{-1}N'^2 + p^{-1}N')(p^{-2}\Lambda^{-1}N''^2 + p^{-1}N'')n_p. \tag{14}$$

There are at most $C\log\Lambda/\log T$ primes not less than T which divide Λ, and for each of them $p \leqslant \Lambda$ and $n_p < Cp^3$. Thus the sum of the expressions (14) taken over all such primes is bounded by

$$CN'^2N''^2\Lambda^{-2}(\log\Lambda/\log T)T^{-1}.$$

If p does not divide Λ then $n_p < Cp^2$. The sum of the expressions (14) over primes p not dividing Λ and such that $T \leqslant p \leqslant N'$ is bounded by

$$CN'^2N''^2\Lambda^{-2}T^{-1} + CN'N''^2\Lambda^{-1}\log\log N' + CN'^2N''/\log N',$$

because $\sum p^{-1}$ taken over these primes is bounded by $C\log\log N'$.

Now suppose that $p > N'$. If p divides ξ_0 and ξ_1 we must have $\xi_0 = \xi_1 = 0$, and we have already ruled out all η_i with this property. If there is a prime $\mathfrak{p}$ above p such that $\mathfrak{p}\mathfrak{a}_i$ divides $\xi_0 - \alpha_i\xi_1$ and $\xi_2 - \beta_i\xi_3$ for some i, then p must divide $\prod(\xi_0 - \alpha_i\xi_1)$; we have already ruled out all η_i for which this vanishes and it is absolutely bounded by CN'^5, so for preassigned η_0, η_1 at most five primes $p > N'$ come into consideration. For each of these p there are at most

$$CN''(p^{-1}\Lambda^{-1}N'' + 1)$$

pairs η_2, η_3 such that $\mathfrak{p}\mathfrak{a}_i$ divides $\xi_2 - \beta_i\xi_3$. Hence the number of sets η_i which fail in this way for some $p > N'$ but which have not been previously rejected is bounded by

$$CN'^2\Lambda^{-1}N''(N'^{-1}\Lambda^{-1}N'' + 1) \;=\; CN'N''\Lambda^{-2}(N'' + \Lambda N').$$

Again, p can only divide ξ_2 and ξ_3 without them both vanishing if $p \leqslant N''$; and in this case there are at most

$$C(p^{-2}\Lambda^{-1}N''^2 + p^{-1}N'')$$

pairs η_2, η_3 with this property. Multiplying by $CN'^2\Lambda^{-1}$ to allow for the choice of η_0, η_1 and summing over all p with $N' < p \leqslant N''$, we find that the number of sets η_i which fail in this way for some $p > N'$ but which have not been previously rejected is at most

$$CN'N''\Lambda^{-2}(N'' + N'\Lambda\log\log N'').$$

Gathering these estimates together, we find that the number of sets η_i which do not fail at any prime differs from (13) by a sum of terms each of which is dominated by

$$C\left(\frac{N'N''}{\Lambda}\right)^2 \frac{\log T + \log\Lambda}{T\log T}. \tag{15}$$

We must also rule out the sets η_i which do not fail at any p but which give rise to a point on one of the 27 lines. But we have already ruled out the η_i which satisfy the condition in Lemma 2(i), and those which satisfy the condition in Lemma 2(ii) can be absorbed into the error term (15). □

Corollary 1. — *Assume that the $\mathfrak{a}_i$ are coprime, that none of them is divisible by any bad prime, that H is large enough and that $\Lambda < H^{1/11}$. Write*

$$T = \log H/3\log\log\log H.$$

Then the number of rational points in U of height at most H which give rise to these $\mathfrak{a}_i$ is at least

$$\{C\prod_{p<T}(1-p^{-4}n_p) - C(\log\log H)^{-1/2}\}H\Lambda^{-1}\log H. \tag{16}$$

Proof. — For each r with $0 \leqslant r \leqslant \log H/6\log 2$ apply Lemma 3 with $N' = 2^r\Lambda^{1/4}H^{1/6}$ and $N'' = 2^{-r}C_3^{-1/2}\Lambda^{1/4}H^{1/3}$, where C_3 is as in (5). It follows from (5) that every point with these $\mathfrak{a}_i$ satisfying (6) has $h(P) < H$; and we have

$$C\Lambda^{1/4}H^{1/4} \geqslant r = \min(N', N'') \geqslant C\Lambda^{1/4}H^{1/6}$$

and $\max(N', N'') \leqslant C\Lambda^{1/4}H^{1/3}$, so that all the conditions of Lemma 3 are satisfied. Moreover the sets defined by (6) as r varies are disjoint. Hence the number of rational points in U of height at most H and with these $\mathfrak{a}_i$ is at least

$$\sum_r H\Lambda^{-1}\{C\prod_{p<T}(1-p^{-4}n_p) - C(\log\log H)^{-1/2})\},$$

which is the result claimed □

Remark 1. — *The most we can say about $\prod_{p<T}(1-p^{-4}n_p)$ in general is that it is at least $(\log T)^{-C}$; so the error term in Lemma 3 is not always smaller than the leading term, and (16) is not always positive. But this blemish is coped with by the averaging process employed in the proof of Theorem 1.*

To complete the proof of Theorem 1, we have to sum the expression (16) over a sufficiently large collection of allowable sets $\mathfrak{a}_i$. Recall that the L_i form $r-2$ complete sets of conjugates over $\mathbb{Q}$. After renumbering, we can assume that no two of $L_1, \ldots, L_{r-2}$ are conjugate over $\mathbb{Q}$. For each i let $\lambda_i = \operatorname{Norm} \mathfrak{a}_i$; thus a good prime p divides at most one of $\lambda_1, \ldots, \lambda_{r-2}$ and such a p can divide λ_i only if it has a first degree factor in k_i. Moreover $\Lambda = \lambda_1 \ldots \lambda_{r-2}$. We take T as in the Corollary to Lemma 3, and assume H so large that all the bad primes are less than T. Now write $M = H^{1/11(r-2)}$; bearing in mind Lemma 1(ii), we shall allow the $\mathfrak{a}_i$ to run through all sets such that each pair $\mathfrak{a}_i$, $\mathfrak{a}_j$ are coprime in k, no $\mathfrak{a}_i$ is divisible by any prime in k above a bad prime, and each $\lambda_i \leqslant M$. In view of the Corollary to Lemma 3, in order to prove Theorem 1 it is only necessary to prove the following result:

Lemma 4. — *We have*

$$\sum \Lambda^{-1} < C(\log H)^{r-2}, \quad \sum \Lambda^{-1} \prod_{p<T} (1 - p^{-4} n_p) > C(\log H)^{r-2}, \tag{17}$$

where the sum is taken over the sets $\mathfrak{a}_i$ specified above.

Proof. — It is well known that for any algebraic number field K and any $X > 1$ there is a constant $C(K)$ such that

$$\sum (\operatorname{Norm} \mathfrak{a})^{-1} = (C(K) + o(1)) \log X, \tag{18}$$

where the sum is taken over all integer ideals $\mathfrak{a}$ with $\operatorname{Norm} \mathfrak{a} < X$. Now

$$\sum \Lambda^{-1} = \sum (\lambda_1 \ldots \lambda_{r-2})^{-1} \leqslant \prod_{i=1}^{r-2} (\sum \lambda_i^{-1}) \tag{19}$$

where the sums on the left and in the middle are taken over the same collection as in (17) and the sum on the right is taken over all integer ideals in k_i of norm at most M. By (18), the expression on the right of (19) is bounded by $C(\log H)^{r-2}$.

The proof of the second inequality (17) is similar but more complicated. We can multiply each term in the sum on the left by an expression of the form $\prod(1 + c_p)$, where the c_p vary from one term to another but are uniformly $O(p^{-2})$; for in doing so we multiply the terms by uniformly bounded factors. Bearing in mind the information about the n_p which appears just before the statement of Lemma 3, it is therefore enough to prove that

$$\sum \Lambda^{-1} \prod (1 - p^{-1})^2 > C(\log H)^{r-2}, \tag{20}$$

where the product is over those $p < T$ which divide Λ and where the sum is over the same collection as in (17). Recall that $\sum_1^{r-2} [k_i : \mathbb{Q}] = 5$. If we multiply the left hand side of (20) by $(\sum n^{-2})^{10}$, which does not affect the truth or falsehood of the inequality provided we make a compensating adjustment of the value of C, we obtain an expression which exceeds $\sum \Lambda^{-1} \prod (1-p^{-1})^2$ where now the sum is taken over all sets of $\mathfrak{a}_i$ with each $\lambda_i \leqslant M$ and the product is taken over all first degree primes $\mathfrak{p}_i$ in some k_i which divide $\mathfrak{a}_i$ and have $p = \operatorname{Norm} \mathfrak{p}_i < T$. This expression is equal to

$$\prod_{i=1}^{r-2} \left(\sum \lambda_i^{-1} \prod (1-p^{-1})^2\right);$$

so to prove the second inequality (17) it is enough to prove

$$S = \sum \lambda_i^{-1} \prod (1-p^{-1})^2 > C \log H.$$

But if we choose a first degree prime $\mathfrak{p}_i$ in k_i with $p = \operatorname{Norm} \mathfrak{p}_i < T$, the left hand side can be written as

$$(1 + p^{-1}(1-p^{-1})^2) S_1 + S_2 \tag{21}$$

plus some terms arising from $\mathfrak{p}_i^2 | \mathfrak{a}_i$ which we ignore; here S_1 consists of those terms for which $\mathfrak{p}_i$ does not divide $\mathfrak{a}_i$ and $\operatorname{Norm} \mathfrak{a}_i \leqslant Mp^{-1}$, and S_2 consists of those terms for which $\mathfrak{p}_i$ does not divide $\mathfrak{a}_i$ and $\operatorname{Norm} \mathfrak{a}_i > Mp^{-1}$. If we multiply the first term in (21) by

$$(1 + p^{-1}(1-p^{-1})^2)^{-1}(1-p^{-1}) = 1 + O(p^{-2})$$

and delete certain terms, the effect is precisely to delete the factors $(1-p^{-1})^2$ in S corresponding to $\mathfrak{p}_i$. We can do this for each $\mathfrak{p}_i$ in turn, and this reduces the inequality which we are seeking to prove to (18). □

It may appear that by a slight refinement of the argument we could obtain a two-sided estimate for $N(U, H)$ under the hypotheses of the Theorem. But this seems not to be so. The problem is that we need estimates like those of Lemma 3 in three essentially different kinds of case: when $N'' \geqslant N' \geqslant \Lambda^{1/2}$, when $N'' \geqslant \Lambda^{1/2} \geqslant N'$ and when $\Lambda^{1/2} \geqslant N'' \geqslant N'$. The first can be dealt with by the methods of Lemma 3, and we can also deal with the second though the arguments are more complicated. But we do not at present see how to deal with the third case.

Received 23 September 1996

JOHN B. SLATER, Computing Laboratory, University of Kent, Canterbury CT2 7NF, United Kingdom • *E-mail :* `J.B.Slater@ukc.ac.uk`

SIR PETER SWINNERTON-DYER, Isaac Newton Institute, 20 Clarkson Rd, Cambridge CB3 0EH, United Kingdom • *E-mail :* `hpfs100@newton.cam.ac.uk`

Astérisque
251, 1998, p. 13–30

COUNTING RATIONAL POINTS ON CUBIC SURFACES

by

Roger Heath-Brown

Abstract. — Let $F[W, X, Y, Z]$ be a rational cubic form, and let $N^{(0)}(R)$ be the number of rational zeros of F of height at most R, which do not lie on any rational line in the surface $F = 0$. We show that

$$N^{(0)}(R) \ll_{\varepsilon,F} R^{4/3+\varepsilon}$$

for any fixed $\varepsilon > 0$, subject to a suitable hypothesis on the size of the rank of elliptic curves. For the proof one counts points on the cubic curves obtained from hyperplane sections of the surface $F = 0$.

1. Introduction

For any cubic form $F(W, X, Y, Z) \in \mathbb{Z}[W, X, Y, Z]$ let

$$N_F(R) = N(R) = \#\{\mathbf{x} \in \mathbb{Z}^4 : F(\mathbf{x}) = 0, |\mathbf{x}| \leqslant R, \mathbf{x} \text{ primitive}\},$$

where $|\mathbf{x}|$ is the Euclidean length of $\mathbf{x}$, and an integer vector $\mathbf{x} = (x_1, \ldots, x_n)$ is defined to be primitive if $\mathbf{x} \neq \mathbf{0}$ and $x_1, \ldots, x_n$ have no common factor. We are concerned here with the size of $N(R)$ as R tends to infinity. If the surface $F = 0$ contains a rational line, then there will be $cR^2 + O_\varepsilon(R^{1+\varepsilon})$ primitive points on that line, counted by $N(R)$, for any positive ε and an appropriate constant $c > 0$. One would expect that such 'trivial' points greatly outnumber the remaining 'non-trivial' points and we therefore define $N^{(0)}(R)$ to be the number of points $\mathbf{x}$ counted by $N(R)$, such that $\mathbf{x}$ does not lie on any rational line in the surface $F = 0$. There are some very precise conjectures about the size of $N^{(0)}(R)$, (see Franke, Manin, and

2000 ***Mathematics Subject Classification.*** — Primary 11G35; Secondary 11D41, 11E76, 11G05.
Key words and phrases. — Cubic Surface, Rational Points, Height, Upper Bound, Elliptic Curve, Rank.

Tschinkel [2], for example). However very little has been proved in general. Indeed, as far as the author is aware the following question is still open.

Question 1. — *Is it true that* $N_F(R) \ll_{\varepsilon,F} R^{2+\varepsilon}$, *for any irreducible* F *and any* $\varepsilon > 0$ *?*

The notation $\ll_{\varepsilon,F}$ means that the implied constant may depend on both ε and F. A very general result has been given by Pila [8] which shows in particular that

$$N_F(R) \ll_{\varepsilon} R^{7/3+\varepsilon},$$

for any $\varepsilon > 0$, uniformly for all absolutely irreducible cubic forms F.

One might indeed be more ambitious and ask for a positive answer to the following.

Question 2. — *Is there a constant* $\theta < 2$ *such that*

$$N^{(0)}(R) \ll_F R^{\theta},$$

for every F*?*

This would demonstrate that points on rational lines really do dominate the rate of growth of $N_F(R)$. Progress has been made in certain special cases, and the author has recently shown [3] that

$$N^{(0)}(R) \ll_{\varepsilon,F} R^{4/3+\varepsilon} \tag{1}$$

for any $\varepsilon > 0$, and any non-singular F such that the surface $F = 0$ contains three coplanar rational lines. In particular (1) holds for

$$F(W,X,Y,Z) = W^3 + X^3 + Y^3 + Z^3.$$

Ideally however one would hope for an affirmative answer to the following question.

Question 3. — *Is it true that* $N^{(0)}(R) \ll_{\varepsilon,F} R^{1+\varepsilon}$, *for any* F *and any* $\varepsilon > 0$ *?*

This is only known to be true in rather trivial cases, such as those in which $N^{(0)}(R)$ is equal to 0, or forms of the shape $W^3 - XYZ$, for example.

We shall be concerned with Question 2, and it is our goal to describe an approach which yields a satisfactory answer, subject to the following natural hypothesis about elliptic curves.

Rank Hypothesis. — *For any rational elliptic curve* E *let* C_E *denote the conductor and let* r_E *denote the rank. Then we have*

$$r_E = o(\log C_E) \quad \textit{as} \quad C_E \to \infty.$$

To put this into context, we observe that

$$r_E = O(\log C_E) \tag{2}$$

for all rational elliptic curves, and

$$r_E \ll \frac{\log C_E}{\log\log C_E} = o(\log C_E) \tag{3}$$

for any rational elliptic curve with at least one rational point of order 2. These assertions ought to be well-known. However we include proofs in §6, for the sake of completeness. We remark that Mestre [7] has shown, subject to the conjecture of Birch and Swinnerton-Dyer, that (3) holds for every modular rational elliptic curve E, providing the associated L-function satisfies the Riemann Hypothesis.

We can now state our principal result.

Theorem 1. — *If the Rank Hypothesis holds then*

$$N^{(0)}(R) \ll_{\varepsilon,F} R^{4/3+\varepsilon}$$

for any non-singular form F and any $\varepsilon > 0$.

The proof of Theorem 1 involves taking hyperplane sections through the surface $F = 0$, to produce a number of cubic curves, most of which will be non-singular. We then estimate the number of points on each of these curves. We remark that this line of attack can be considerably generalized. Thus one can take a completely arbitrary variety, and attempt to count how many points may lie on each of its plane sections. However we shall not explore this possibility here.

To count points on our cubic curves we introduce the following definitions. Let $G(X, Y, Z) \in \mathbb{Z}[X, Y, Z]$ be a cubic form, and let $||G||$ denote the maximum modulus of the coefficients of G. Now define $N(G, R)$ to be the number of primitive points $\mathbf{x} \in \mathbb{Z}^3$ in the sphere $|\mathbf{x}| \leqslant R$ for which $G(\mathbf{x}) = 0$, with the proviso that if $L(\mathbf{x})$ is a rational linear factor of G, then any points on $L(\mathbf{x}) = 0$ are to be excluded. Of course this latter case can only arise when G is singular.

Our principal results on $N(G, R)$ are then the following.

Theorem 2. — *Let $G(X, Y, Z) \in \mathbb{Z}[X, Y, Z]$ be an absolutely irreducible singular cubic form. Then*

$$N(G, R) \ll_\varepsilon R^{2/3+\varepsilon}||G||^\varepsilon, \tag{4}$$

for any $\varepsilon > 0$.

Theorem 3. — *Let $G(X, Y, Z) \in \mathbb{Z}[X, Y, Z]$ be a non-singular cubic form. Then if the Rank Hypothesis holds we will have*

$$N(G, R) \ll_\varepsilon R^\varepsilon ||G||^\varepsilon,$$

for any $\varepsilon > 0$.

In Theorem 2 the exponent $2/3$ is best possible, as the example $G(X,Y,Z) = XY^2 - Z^3$ shows. This vanishes at (a^3, b^3, ab^2), which takes at least $\gg R^{2/3}$ primitive values in the sphere of radius R.

For Theorem 3, it is easy to show that

$$N(G,R) \ll_{\varepsilon,G} R^{\varepsilon},$$

and the real difficulty lies in establishing a good dependence on $||G||$. We shall also have to handle the case in which G is reducible, but the estimates we shall obtain (in §5) depend on the coefficients of G in a more delicate manner than in the cases above. It would be interesting to obtain unconditional bounds of the type given by Theorem 3, with R^{ε} replaced by a term with a larger exponent. The best such result currently available seems to be the estimate

$$N(G,R) \ll_{\varepsilon} R^{4/3+\varepsilon},$$

due to Pila [8]. This holds uniformly in G, for any fixed $\varepsilon > 0$.

2. Proof of Theorem 2

We begin our treatment of Theorem 2 by observing that G has exactly one singular point, which is therefore rational. We take the singular point to be the primitive integer vector $\mathbf{x}_0$. Elimination theory shows that $|\mathbf{x}_0| \ll ||G||^A$ for a suitable absolute constant A.

At this point it is convenient to introduce a convention concerning the large number of absolute constants which will occur in what follows. All such constants will be denoted by the same letter A, which therefore has a potentially different meaning at each occurence. This notation allows us to write $|\mathbf{x}_0|^A \ll ||G||^A$, for example, given that $|\mathbf{x}_0| \ll ||G||^A$. Such a convention needs to be treated with caution, but it avoids the notational complications of introducing A, A', A''' etc.

Proceeding with our argument, there will be an invertible integer change of variables, T say, which sends $\mathbf{x}_0$ to the point $(1,0,0)$ and with coefficients which are $O(||G||^A)$. This takes G to a form G' with $||G'|| \ll ||G||^A$. The form $G'(X,Y,Z)$ may now be written as $XQ(Y,Z) + C(Y,Z)$ where Q and C are quadratic and cubic forms respectively, and $||Q||, ||C|| \ll ||G||^A$. If $G(\mathbf{x}) = 0$ for some primitive vector $\mathbf{x}$, the corresponding triple $T\mathbf{x} = (X,Y,Z)$ will also be primitive. Thus if $Y = Z = 0$, then $X = \pm 1$. Otherwise we may set $Y = ry$, $Z = rz$ with y, z coprime, so that

$$XQ(y,z) + rC(y,z) = 0. \tag{5}$$

The polynomials Q and C are coprime, since the original cubic G is supposed to be absolutely irreducible. It follows from an application of the Euclidean algorithm that there are a quadratic form Q', a linear form L', and a non-zero constant K_1, all of which are integral, which satisfy

$$Q(Y,Z)Q'(Y,Z)+C(Y,Z)L'(Y,Z)=K_1Y^4$$

identically. Here K_1 may depend on the original form G, and on the linear transformation T. It is clear, when one examines the algorithm, that the constant K_1 will satisfy $K_1 \ll \max\{||Q||\,,\,||C||\}^A$. Since $Q(y,z)|rC(y,z)$ we conclude that $Q(y,z)|rK_1y^4$, with $K_1 \ll ||G||^A$.

In a precisely analogous way one finds that $Q(y,z)|rK_2z^4$, for some non-zero integer constant $K_2 \ll ||G||^A$. Thus $Q(y,z)$ divides both $rK_1K_2y^4$ and $rK_1K_2z^4$, and since y and z are coprime it follows that $Q(y,z)$ must be a factor of rK_1K_2. On the other hand r and X must be coprime, since our original vector $\mathbf{x}$ was assumed to be primitive, and hence (5) shows that $r|Q(y,z)$. We therefore conclude that $Q(y,z)=rK$ for some divisor K of K_1K_2. The number of factors of K_1K_2 is $O_\varepsilon(||G||^\varepsilon)$, by virtue of the well known estimate for the divisor function and our bounds for K_1 and K_2. Since K_1K_2 is independent of our original vector $\mathbf{x}$, we conclude that K only takes $O_\varepsilon(||G||^\varepsilon)$ values. Equation (5) now produces $KX = -C(y,z)$, $KY = yQ(y,z)$, and $KZ = zQ(y,z)$. We can therefore reverse the linear transformation T to obtain

$$K\mathbf{x}=(C_1(y,z),C_2(y,z),C_3(y,z)),$$

where the cubic forms C_i are obtained from $-C(y,z)$, $yQ(y,z)$ and $zQ(y,z)$ by T^{-1}.

It remains to estimate, for each of $O_\varepsilon(||G||^\varepsilon)$ values of K, how many primitive vectors $\mathbf{x}$, formed as above, lie in the sphere $|\mathbf{x}| \leqslant R$. Our principal tool in doing this is the following result.

Lemma 1. — *Let $f(y,z)\in\mathbb{Z}[y,z]$ be an irreducible cubic form. Then for any $\varepsilon>0$ we have*

$$\#\{\mathbf{x}\in\mathbb{Z}^2 : 0<|f(\mathbf{x})|\leqslant\kappa\} \ll_\varepsilon \kappa^{2/3+\varepsilon}$$

uniformly in f, for $\kappa \geqslant 0$.

This follows from Theorem 1C of Schmidt [9; Chapter III].

We will require a corresponding result when f is reducible, but does not have a repeated factor.

Lemma 2. — *Let $f(y,z) \in \mathbb{Z}[y,z]$ be a reducible cubic form, with no repeated factor. Then for any $\varepsilon > 0$ we have*

$$\#\{\mathbf{x} \in \mathbb{Z}^2 : 0 < |f(\mathbf{x})| \leqslant \kappa\} \ll_\varepsilon \kappa^{2/3+\varepsilon}$$

uniformly in f, for $\kappa \geqslant 0$.

This will be proved in the next section.

We may now apply Lemmas 1 and 2 to our situation. Suppose that C' and C'' are rational cubic forms, neither of which has 3 distinct linear factors. Thus, in particular, neither C' nor C'' can vanish. Then either they both contain the same repeated factor, or they may be written in the shape $C'(x,y) = x^2(ax+by)$ and $C''(x,y) = y^2(cx+dy)$ after a suitable change of variable. In the latter case the condition for the form $\lambda C' + \mu C''$ to have a repeated factor is that the discriminant

$$-27(\lambda a)^2(\mu d)^2 + 18(\lambda a)(\lambda b)(\mu c)(\mu d) \\ + (\lambda b)^2(\mu c)^2 - 4(\lambda a)(\mu c)^3 - 4(\lambda b)^3(\mu d)$$

is non-zero. Thus we get a form with no repeated factor from at least one of

$$(\lambda, \mu) = (1,1),\ (1,-1),\ (1,2)$$

unless C or C'' vanishes, a case already excluded.

Since the forms $C(y,z)$ and $Q(y,z)$ must be coprime, there can be no repeated factor common to $C(y,z), yQ(y,z), zQ(y,z)$. The above argument then shows that there is some combination $\lambda C_i + \mu C_j$, with $\lambda, \mu \ll 1$, which has no repeated factor. We shall choose such a combination and denote it $C_0(y,z)$. We therefore need to know how many primitive pairs (y,z) have $K|C_0(y,z)$ and $|K^{-1}C_0(y,z)| \ll R$. In order to remove the factor K from the form C_0 we may follow the procedure used by Schmidt [9; Chapter III, §6]. (It should be noted that Schmidt's presentation contains the misprint '$p|k$' both in the statement of Propsition 6B, and in equation (6.3). In each case this should read $p \nmid k$.) Let K' be a positive integer, all of whose prime factors divide K, and write

$$\mathcal{N}(C_0, m) = \#\{(y,z) \in \mathbb{Z}^2 : C_0(y,z) = m,\ \text{h.c.f.}(y,z) = 1\}.$$

Schmidt's argument shows that there are at most $3^{\omega(K)}$ integral forms $C^{(i)}(y,z)$ say, each of which is equivalent over the rationals to a rational multiple of C_0, such that

$$\mathcal{N}(C_0, KK'm) \leqslant \sum_i \mathcal{N}(C^{(i)}, m)$$

for any m coprime to K.

If $C_0(y,z) = Kn$ with $0 < |n| \ll R$, we may write n as $K'm$ with m coprime to K and satisfying $0 < |m| \ll R/K'$. Now, since none of the forms $C^{(i)}$ above has a repeated factor, Lemmas 1 and 2 yield

$$\#\{(y,z) \in \mathbb{Z}^2 : C_0(y,z) = Kn,\ \text{h.c.f.}(y,z) = 1,\ 0 < |n| \ll R\}$$
$$\ll_\varepsilon 3^{\omega(K)} \sum_{K'} (\frac{R}{K'})^{2/3+\varepsilon}.$$

However, since K' runs over those integers composed solely of primes which divide K, we have

$$\sum_{K'} (\frac{R}{K'})^{\sigma} = R^\sigma \prod_{p|K} (1 - p^{-\sigma})^{-1} \ll R^\sigma 2^{\omega(K)}$$

for any fixed $\sigma > 0$.

Since $C_0(y,z) = 0$ has $O(1)$ solutions with (y,z) primitive, these considerations therefore produce $O_\varepsilon(K^\varepsilon R^{2/3+\varepsilon})$ pairs (y,z), for each of $O_\varepsilon(||G||^\varepsilon)$ values of $K \ll ||G||^A$. Theorem 2 now follows on redefining ε.

3. The proof of Lemma 2

The argument we shall use to establish Lemma 2 is related to, but simpler than, that given by Schmidt [9; Chapter III, §§2& 5]. We observe at the outset that it suffices to establish the result for primitive points $\mathbf{x}$, since the stated form of the lemma follows trivially from the corresponding version for primitive solutions.

After a suitable change of variable we may write f in the shape $xq(x,y)$, where

$$q(x,y) = Ax^2 + Bxy + Cy^2$$

is a non-singular integral quadratic form, with $C \neq 0$. We shall write $D(\neq 0)$ for the discriminant. Without loss of generality we may assume that x and C are positive. We shall also assume throughout the proof that κ is sufficiently large. This is clearly permissable.

We now set $Q = 8\kappa^{2/3}$ and we write N for the number of primitive solutions of $0 < x|q(x,y)| \leqslant \kappa$. For any prime $p \in (Q, 2Q]$ we write $N_1(p)$ for the number of solutions for which $p|x$, and $N_2(p)$ for the remainder. Then

$$\{\pi(2Q) - \pi(Q)\}N = \sum_{Q<p\leqslant 2Q} N_1(p) + \sum_{Q<p\leqslant 2Q} N_2(p).$$

Since x is clearly at most κ the first sum on the right may be written in the form

$$\sum_{x,y} \#\{p \in (Q, 2Q] :\ p|x\} \leqslant N \log \kappa,$$

the summation on the left being over pairs (x,y) counted by N. It follows that

$$\sum_{Q<p\leqslant 2Q} N_1(p) \leqslant \frac{1}{2}\{\pi(2Q)-\pi(Q)\}N,$$

and hence that

$$\{\pi(2Q)-\pi(Q)\}N \leqslant 2\sum_{Q<p\leqslant 2Q} N_2(p).$$

We may therefore fix a prime $p \in (Q, 2Q]$ such that $N \leqslant 2N_2(p)$. For each pair (x,y) counted by $N_2(p)$ there is an integer a in the range $1 \leqslant a \leqslant p$ such that $y \equiv ax \pmod{p}$. Substituting $y = ax + pz$ we find that $0 < x|q_0(x,z)| \leqslant \kappa$, for an appropriate form $q_0(x,z) = A_0x^2 + B_0xz + C_0z^2$, with discriminant $D_0 = p^2D$ and with $C_0 = p^2C$. It follows that there is some value of a such that

$$N \leqslant 2N_2(p) \leqslant 2pN_0 \leqslant 32\kappa^{2/3}N_0, \tag{6}$$

where N_0 counts primitive solutions of $0 < x|q_0(x,z)| \leqslant \kappa$.

We now observe that for each value of x the range for z is given by

$$|(z+\frac{B_0}{2C_0}x)^2 - \frac{D_0x^2}{4C_0^2}| \leqslant \frac{\kappa}{xC_0}.$$

This inequality specifies at most two possible intervals in which z must lie, each having length $O(\sqrt{\kappa/xC_0})$. Since

$$C_0 = p^2C \geqslant Q^2C = 64\kappa^{4/3}C \geqslant \kappa$$

the intervals have length $O(1)$, and it follows that there are $O(1)$ values of z for each possible x.

We shall factor $q_0(x,z)$ as $C_0(z-\theta x)(z-\phi x)$ where

$$\theta = \frac{-B_0+\sqrt{D_0}}{2C_0}, \qquad \phi = \frac{-B_0-\sqrt{D_0}}{2C_0}.$$

It follows that

$$|(z-\theta x)(z-\phi x)| \leqslant \frac{\kappa}{xC_0} \tag{7}$$

We proceed to count those solutions for which $|z-\theta x| \leqslant |z-\phi x|$, the alternative case being treated in an exactly analogous manner. For such solutions we have

$$|z-\theta x| \leqslant |(z-\theta x)(z-\phi x)|^{1/2} \leqslant \{\frac{\kappa}{xC_0}\}^{1/2}. \tag{8}$$

We claim that one also has

$$|z-\theta x| \leqslant \frac{2\kappa}{x^2\sqrt{|D_0|}}. \tag{9}$$

This follows from (8) unless

$$\frac{2\kappa}{x^2\sqrt{|D_0|}} \leqslant \{\frac{\kappa}{xC_0}\}^{1/2},$$

as we now assume. We then have

$$\frac{1}{2}|\theta-\phi|x = \frac{|D_0|^{1/2}x}{2C_0} \geqslant \{\frac{\kappa}{xC_0}\}^{1/2},$$

whence (8) implies that

$$|z-\phi x| \geqslant |\theta-\phi|x - |z-\theta x| \geqslant |\theta-\phi|x - \{\frac{\kappa}{xC_0}\}^{1/2}$$

$$\geqslant \frac{1}{2}|\theta-\phi|x = \frac{|D_0|^{1/2}x}{2C_0}.$$

We therefore see that (9) is a consequence of (7). This completes the proof of (9).

Now suppose we have two distinct primitive solutions (x_1, z_1) and (x_2, z_2) counted by N_0, both with $|z-\theta x| \leqslant |z-\phi x|$. We assume further that $x_1 \leqslant x_2$. Then $x_1 z_2 \neq x_2 z_1$, whence

$$1 \leqslant |x_1 z_2 - x_2 z_1| = |x_1(z_2-\theta x_2) - x_2(z_1-\theta x_1)|.$$

It follows from (8) that

$$1 \leqslant x_1\{\frac{\kappa}{x_2 C_0}\}^{1/2} + x_2\{\frac{\kappa}{x_1 C_0}\}^{1/2} \leqslant 2x_2\{\frac{\kappa}{x_1 C_0}\}^{1/2}.$$

Thus

$$x_1 \leqslant 4x_2^2\kappa C_0^{-1} \leqslant \frac{1}{2}x_2,$$

providing that

$$x_2 \leqslant \frac{C_0}{8\kappa}. \tag{10}$$

Similarly, using (9) we find that

$$1 \leqslant 2x_2\frac{2\kappa}{x_1^2\sqrt{|D_0|}},$$

whence

$$x_1 \leqslant 2(x_2\kappa)^{1/2}|D_0|^{-1/4} \leqslant \frac{1}{2}x_2,$$

providing that

$$x_2 \geqslant \frac{16\kappa}{\sqrt{|D_0|}}. \tag{11}$$

However

$$\frac{16\kappa}{\sqrt{|D_0|}} \leqslant \frac{C_0}{8\kappa}$$

since our initial choice of p and Q ensures that

$$C_0\sqrt{|D_0|} = p^3C\sqrt{|D|} \geqslant p^3 \geqslant Q^3 = 512\kappa^2.$$

It follows that the ranges (10) and (11) cover all possibilities for x_2, so that $x_1 \leqslant \frac{1}{2}x_2$ for any two solutions of the type under consideration. Now, since we always have $x \leqslant \kappa$, we may divide the available range into $O(\log\kappa)$ subintervals of the form $\kappa_0 < x \leqslant 2\kappa_0$, each of which can contain at most one relevent value for x. In this way we see that there can be at most $O(\log\kappa)$ possible values for x, and, as has previously been observed, there are $O(1)$ corresponding values of z for each of these. Thus $N_0 \ll \log\kappa$, and (6) yields $N \ll \kappa^{2/3}\log\kappa$, which proves the lemma.

4. Proof of Theorem 3

If there is no point $\mathbf{x} \neq \mathbf{0}$ on the curve $G(\mathbf{x}) = 0$ lying in the sphere $|\mathbf{x}| \leqslant R$, then of course $N(G, R) = 0$. Otherwise we can use such a point $\mathbf{x}_0$ as a base point to transform the curve into Weierstrass normal form. Thus there is a birational transformation, θ say, which takes G into

$$E : x_2^2x_3 = 4x_1^3 - g_2x_1x_3^2 - g_3x_3^3.$$

By abuse of notation we shall also write E for the cubic form

$$E(\mathbf{x}) = x_2^2x_3 - (4x_1^3 - g_2x_1x_3^2 - g_3x_3^3).$$

The map θ is given by formulae in which the coefficients are rational functions involving the coefficents of the original form G and the coordinates of $\mathbf{x}_0$. We may clear the denominators in these formulae so that g_2 and g_3 are integers, and so that θ takes integer vectors to integer vectors. It follows that any integer solution $\mathbf{x}$ of $G(\mathbf{x}) = 0$ leads to a rational point $P(\mathbf{x})$ on E. Moreover there will be an absolute constant A for which

$$||P(\mathbf{x})|| \leqslant A||G||^A|\mathbf{x}_0|^A|\mathbf{x}|^A,$$

where $||P|| = |(x_1, x_2, x_3)|$ for $P = (x_1, x_2, x_3)$. (Here we should recall the convention concerning the symbol A, which was introduced in §2.) Since θ is a birational transformation, points $\mathbf{x}$ on $G = 0$ which are distinct up to projective equivalence, map to points $P(\mathbf{x})$ on E, which are also distinct up to projective equivalence. It follows that

$$N(G, R) \ll N(E, A||G||^AR^A).$$

Moreover, since $g_2, g_3 \ll ||G||^A|\mathbf{x}_0|^A$, we have

$$||E|| \ll ||G||^AR^A.$$

These considerations therefore yield the following conclusion.

Lemma 3. — *To establish Theorem 3 it suffices to prove it for the case of curves in Weierstrass normal form.*

We are now in a position to use some of the standard results on elliptic curves. We begin by translating our problem into one concerning the canonical height function $h(P)$. This is related to the 'logarithmic height' via the estimate

$$h(P) = \log ||P|| + O(\log ||E||)$$

for primitive points P. This follows from the work of Zimmer [12] for example. It follows there is an absolute constant A such that $h(P) \leqslant A\log(||E||R)$ for all points P to be counted. If the rank of E is r and $P_1, \ldots, P_r$ are generators for the Mordell-Weil group, our point P may be expressed as

$$P = T + \sum_{i=1}^{r} n_i P_i,$$

where T is a rational torsion point. Then

$$h(P) = h(n_1P_1 + \cdots + n_rP_r) = Q(n_1, \ldots, n_r),$$

say, where Q is a positive definite quadratic form. Since there are at most 16 values for T, by the theorem of Mazur [6], it follows that

$$N(E, R) \ll \#\{(n_1, \ldots, n_r) \in \mathbb{Z}^r : \; Q(n_1, \ldots, n_r) \leqslant A\log(||E||R)\}.$$

We therefore call on the following lemma.

Lemma 4. — *Let $Q(x_1, \ldots, x_r) \in \mathbb{R}[x_1, \ldots, x_r]$ be a positive definite quadratic form, and suppose that $Q(\mathbf{n}) \geqslant B_0$ for every non-zero vector $\mathbf{n} \in \mathbb{Z}^r$. Then*

$$\#\{(n_1, \ldots, n_r) \in \mathbb{Z}^r; \; Q(n_1, \ldots, n_r) \leqslant B\} \leqslant 1 + (9B/B_0)^{r/2}.$$

The proof of this will be given at the end of this section.

At this point we shall require an admissable value for B_0. This is given by the following corollary of a result of Hindry and Silverman [4; Theorem 0.3].

Lemma 5. — *Let E be a rational elliptic curve of conductor C_E and discriminant D_E. Then there is an absolute constant A such that the canonical height on E satisfies*

$$h(P) \geqslant (\log |D_E|) \exp\{-A\frac{\log |D_E|}{\log C_E}\}$$

for every non-torsion rational point P on E.

On comparing our various estimates we may now conclude that

$$N(E,R) \ll 1 + \left\{ \frac{A\log(||E||R)}{\log|D_E|} \exp(A\frac{\log|D_E|}{\log C_E}) \right\}^{r/2}.$$

We now set $\lambda = A(\log|D_E|)/(\log C_E)$, whence our bound becomes

$$N(E,R) \ll 1 + \{\frac{A\log(||E||R)}{\log C_E} e^{\lambda}\lambda^{-1}\}^{r/2}. \tag{12}$$

We clearly have $\log|D_E| \ll \log||E||$, so that

$$\lambda \leqslant A\frac{\log||E||}{\log C_E}.$$

On the other hand every prime factor of C_E occurs in $6D_E$, and the exponent to which a prime may occur in C_E is absolutely bounded. It therefore follows that $\log|D_E| \gg \log C_E$, and hence that $\lambda \gg 1$.

We now observe that the function e^{λ}/λ is increasing with respect to λ, for $\lambda \geqslant 1$. It then follows from our bounds for λ that

$$\frac{e^{\lambda}}{\lambda} \leqslant \exp\{A\frac{\log||E||}{\log C_E}\}$$

whether $\lambda \geqslant 1$ or not. On substituting into (12) we therefore obtain

$$N(E,R) \ll 1 + \{A\mu e^{A\mu}\}^{r/2},$$

where $\mu = (\log(||E||R))/(\log C_E)$. Since $A\mu \leqslant \exp(A\mu)$ this leads to the bound

$$N(E,R) \ll \exp\{Ar\mu\}.$$

We are now ready to prove Theorem 3 for the curve E. Let $\varepsilon > 0$ be given. Then, according to the Rank Hypothesis, there is a constant $C(\varepsilon)$ such that

$$r \leqslant \frac{\varepsilon}{A}(\log C_E)$$

for $C_E \geqslant C(\varepsilon)$. Thus

$$N(E,R) \ll \exp\{\varepsilon\log(||E||R)\} = (||E||R)^{\varepsilon},$$

as required, for $C_E \geqslant C_\varepsilon$. In the remaining case, when $C_E \leqslant C(\varepsilon)$, the height $||E||$ of the curve E, and also the rank r and the discriminant D_E, can all be bounded in terms of ε. This follows from the fact (Shafarevich [10]) that there are only finitely many curves E for each value of C_E. The estimate (12) therefore becomes

$$N(E,R) \ll_\varepsilon (\log R)^{r(\varepsilon)/2} \ll_\varepsilon R^{\varepsilon},$$

where $r(\varepsilon)$ is an upper bound for the ranks of all curves with conductor at most $C(\varepsilon)$. This completes the proof of Theorem 3.

It remains to establish Lemma 4. Since $Q(\mathbf{x})$ is positive definite there is a non-singular $r \times r$ real matrix M such that $Q(\mathbf{x}) = |M\mathbf{x}|^2$. Thus, on taking $C = \sqrt{B/B_0}$, it suffices to show that

$$\#\{\mathbf{n} \in \mathbb{Z}^r;\ |M\mathbf{n}| \leqslant C\} \leqslant 1 + (3C)^r, \tag{13}$$

under the assumption that $|M\mathbf{n}| \geqslant 1$ for every non-zero vector $\mathbf{n} \in \mathbb{Z}^r$. However we may observe that the ellipsoids

$$S(\mathbf{n}) = \{\mathbf{x} \in \mathbb{R}^r :\ |M(\mathbf{x}+\mathbf{n})| < \frac{1}{2}\}$$

are disjoint for distinct $\mathbf{n}$, and lie inside the ellipsoid

$$\{\mathbf{x} \in \mathbb{R}^r :\ |M(\mathbf{x})| \leqslant C + \frac{1}{2}\}$$

for the vectors $\mathbf{n}$ to be counted. Thus if

$$V_r = \mathrm{Meas}\{\mathbf{x} \in \mathbb{R}^r :\ |M(\mathbf{x})| \leqslant 1\}$$

we see that

$$\frac{V_r}{2^r} \cdot \#\{\mathbf{n} \in \mathbb{Z}^r;\ |M\mathbf{n}| \leqslant C\} \leqslant V_r(C + \frac{1}{2})^r,$$

whence

$$\#\{\mathbf{n} \in \mathbb{Z}^r;\ |M\mathbf{n}| \leqslant C\} \leqslant (2C+1)^r.$$

Thus (13) follows if $C \geqslant 1$, since $2C + 1 \leqslant 3C$, while if $C < 1$ then only $\mathbf{n} = \mathbf{0}$ can be counted. This completes the proof of Lemma 4.

5. Deduction of Theorem 1

In order to cover our cubic surface with plane sections we use the following result, whose proof is a trivial application of the pigeonhole principle.

Lemma 6. — *Let $\mathbf{x} \in \mathbb{Z}^n$ lie in the sphere $|\mathbf{x}| \leqslant R$. Then there is a primitive vector $\mathbf{y} \in \mathbb{Z}^n$, for which $\mathbf{x} \cdot \mathbf{y} = 0$, and such that $|\mathbf{y}| \ll_n R^{1/(n-1)}$.*

In the case $n = 4$ each set $\{\mathbf{x} \in \mathbb{Z}^4 :\ \mathbf{x} \cdot \mathbf{y} = 0\}$ is a lattice of rank 3, and we can choose a basis $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ say such that if $\mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3$ then

$$|\lambda_1||\mathbf{x}_1| + |\lambda_2||\mathbf{x}_2| + |\lambda_3||\mathbf{x}_3| \ll |\mathbf{x}|. \tag{14}$$

This follows from Davenport [1; Lemma 5].

We therefore see that the points on $F(\mathbf{x}) = 0$ which also lie in the plane $\mathbf{x} \cdot \mathbf{y} = 0$ are in 1–1 correspondence with points on the curve $G_{\mathbf{y}}(\lambda_1, \lambda_2, \lambda_3) = 0$, where

$$G_{\mathbf{y}}(\lambda_1, \lambda_2, \lambda_3) = F(\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3).$$

Moreover primitive points on $F = 0$ correspond to primitive points on $G_{\mathbf{y}} = 0$, and vice-versa. Since (14) implies that $|\lambda_i| \ll R$ and $|\mathbf{x}_i| \ll R$ in all relevent cases, we deduce that

$$|(\lambda_1, \lambda_2, \lambda_3)| \ll R$$

and that

$$||G_{\mathbf{y}}|| \ll_F R^3. \tag{15}$$

These estimates can be improved somewhat, but they suffice for our purposes. Finally we observe that if $G_{\mathbf{y}}$ factors, with $(\lambda_1, \lambda_2, \lambda_3)$ being a zero of a rational linear factor, then the 'curve' $G_{\mathbf{y}} = 0$ is reducible, and includes a rational line, on which $(\lambda_1, \lambda_2, \lambda_3)$ lies. The corresponding point $\mathbf{x}$ of the surface $F = 0$ therefore lies on a rational line in the surface.

The above remarks show that there is an absolute constant A for which

$$N^{(0)}(R) \leqslant \sum_{\mathbf{y}} N(G_{\mathbf{y}}, AR),$$

where $\mathbf{y}$ runs over primitive integer vectors in the sphere $|\mathbf{y}| \leqslant AR^{1/3}$. Here we recall that points lying on rational lines are to be excluded from $N(G_{\mathbf{y}}, AR)$ when $G_{\mathbf{y}}$ is reducible. Theorems 2 and 3, taken in conjunction with the bound (15) therefore lead to the conclusion that

$$N^{(0)}(R) \ll_{\varepsilon,F} \{R^{4/3} + R^{2/3}N^{(1)}(AR^{1/3})\}R^{4\varepsilon} + \sum_{\mathbf{y}}{}^{*}\, N(G_{\mathbf{y}}, AR). \tag{16}$$

Here $N^{(1)}(P)$ is the number of non-zero integer vectors $\mathbf{y}$ in the sphere $|\mathbf{y}| \leqslant P$, for which $G_{\mathbf{y}}$ is singular, and Σ^* denotes a sum over primitive vectors for which $G_{\mathbf{y}}$ is reducible.

To estimate $N^{(1)}(P)$ we note that $G_{\mathbf{y}}$ is singular if and only if $\hat{F}(\mathbf{y}) = 0$, where $\hat{F} = 0$ is the surface dual to $F = 0$. Moreover, if $\hat{F}(\mathbf{y}) = 0$, then $\mathbf{y} = \nabla F(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{C}^4$ on the surface $F = 0$. This vector $\mathbf{x}$ will, according to Lemma 2 of Hooley [5], be a scalar multiple of $\nabla\hat{F}(\mathbf{y})$, unless $\mathbf{y}$ is a singular point of $\hat{F}$. Since the form $\hat{F}$ may be taken to have integer coefficients we conclude in the former case that $\mathbf{y} = \lambda\nabla F(\mathbf{z})$ for some primitive vector $\mathbf{z} \in \mathbb{Z}^4$ on the surface $F(\mathbf{z}) = 0$. The singular locus of $\hat{F}$ has dimension 1 at most, whence the number of integer vectors $\mathbf{y}$ in the sphere $|\mathbf{y}| \leqslant P$, for which $\nabla\hat{F}(\mathbf{y}) = \mathbf{0}$, can be at worst $O_F(P^2)$.

We proceed to investigate the alternative possibility, that $\mathbf{y} = \lambda\nabla F(\mathbf{z})$ for some primitive vector $\mathbf{z} \in \mathbb{Z}^4$. We begin by noting that $p^e|\nabla F(\mathbf{z})$ implies $p|\mathbf{z}$ if either the prime p or the exponent e is large enough, since F is non-singular. Since $\mathbf{y}$ is integral and primitive, the set of possible values of the scalar λ is finite and is determined by F. In order to bound $|\mathbf{z}|$ we observe that the function $|\nabla F(\mathbf{u})|$ is continuous on the compact set $|\mathbf{u}| = 1$, under the usual topology, and hence attains its lower bound. Since F is non-singular this lower bound must be strictly positive, whence $|\nabla F(\mathbf{u})| \gg_F 1$ for $|\mathbf{u}| = 1$. We therefore conclude in general that $|\nabla F(\mathbf{z})| \gg_F |\mathbf{z}|^2$. In the situation under consideration we have $|\nabla F(\mathbf{z})| \ll P$, whence $|\mathbf{z}| \ll_F P^{1/2}$.

We may now conclude that there are $O_F(P^2)$ possible points $\mathbf{y}$ for the case under consideration, and hence that

$$N^{(1)}(AR^{1/3}) \ll_F R^{2/3}. \tag{17}$$

It remains to consider the contribution from the sum Σ^*. Since the forms $G_{\mathbf{y}}$ are singular in this case, there will be $O_F(R^{2/3})$ possible values of $\mathbf{y}$, by (17). Suppose firstly that $G_{\mathbf{y}}$ has a factor H, say, which is not proportional to a rational form. Then if $H(\mathbf{x}) = 0$ for some $\mathbf{x} \in \mathbb{Z}^3$ we must also have $\tilde{H}(\mathbf{x}) = 0$ for any conjugate $\tilde{H}$ of H. These two equations have $O(1)$ solutions in primitive vectors $\mathbf{x}$, and this produces a total contribution $O_F(R^{2/3})$ to Σ^*.

In the alternative case $G_{\mathbf{y}}$ has a rational linear factor, so that the plane $\mathbf{x}.\mathbf{y} = 0$ must contain a rational line l, say, in the surface $F = 0$. There are at most 27 such lines, so it suffices to restrict attention to values of $\mathbf{y}$ for which the corresponding plane includes a specific line l_0, say. It is convenient to change coordinates so that l_0 is given by $x_1 = x_2 = 0$. Then the form F takes the shape

$$F(\mathbf{x}) = x_1 Q_1(x_1, \ldots, x_4) + x_2 Q_2(x_1, \ldots, x_4),$$

and the vector $\mathbf{y}$ takes the shape $\mathbf{y} = (\alpha, \beta, 0, 0)$, where α and β are coprime integers. Since $|\mathbf{y}| \ll R^{1/3}$ we now have $|\alpha|, |\beta| \ll R^{1/3}$. Moreover primitive vectors $\mathbf{x}$ in the plane $\mathbf{x}.\mathbf{y} = 0$ necessarily have the form $(\beta t_1, -\alpha t_1, t_2, t_3)$ with (t_1, t_2, t_3) being a primitive integer vector. Since $|\mathbf{x}| \ll R$ it follows that

$$|t_1| \ll R/\max(|\alpha|, |\beta|), \quad |t_2|, |t_3| \ll R. \tag{18}$$

Now if $F(\mathbf{x}) = 0$ then either $t_1 = 0$, in which case $\mathbf{x}$ is on the line l_0, or $Q(\mathbf{t}) = 0$, where

$$Q(\mathbf{t}) = Q(\mathbf{t}, \alpha, \beta) = \beta Q_1(\beta t_1, -\alpha t_1, t_2, t_3) - \alpha Q_2(\beta t_1, -\alpha t_1, t_2, t_3).$$

We now call on the following result on solutions of quadratic forms, see Heath-Brown [3; Theorem 3], for example.

Lemma 7. — *Let Q be a non-singular integral ternary quadratic form, and suppose that the binary form $Q(0, x_2, x_3)$ is also non-singular. Then for any integer k the equation $Q(\mathbf{x}) = 0$ has $O_\varepsilon((||Q||R)^\varepsilon)$ primitive integer solutions in the sphere $|\mathbf{x}| \leqslant R$, with $x_1 = k$.*

In our application we have $||Q|| \ll_F R$, so that there are $O_{\varepsilon,F}(R^{2\varepsilon})$ solutions t_2, t_3 for each value of t_1, providing that Q and $Q(0, t_2, t_3)$ are non-singular. If Q were singular, with two quadratic conjugate factors, then $Q(\mathbf{x}) = 0$ has $O(1)$ primitive integer solutions. On the other hand, if Q has a rational linear factor L say, then the points for which $L(\mathbf{t}) = 0$ produce a rational line on the surface $F = 0$, and hence are not counted in $N(G_{\mathbf{y}}, AR)$.

It remains to consider the possibility that

$$Q(0, t_2, t_3) = \beta Q_1(0, 0, t_2, t_3) - \alpha Q_2(0, 0, t_2, t_3)$$

is singular. This can happen for $O(1)$ coprime pairs of integers α, β, except in the case which $Q_1(0, 0, t_2, t_3)$ and $Q_2(0, 0, t_2, t_3)$ are both proportional to the square of the same linear form. In the latter case we find that

$$F(\mathbf{x}) = x_1(x_1L_1(\mathbf{x}) + x_2L_2(\mathbf{x}) + c_1L(x_3, x_4)^2) \\ + x_2(x_1L_3(\mathbf{x}) + x_2L_4(\mathbf{x}) + c_2L(x_3, x_4)^2).$$

This has a singular point when $x_1 = x_2 = L(x_3, x_4) = 0$, contradicting our original assumption.

Summarizing the above considerations, we find that the equation $Q(\mathbf{t}) = 0$ has $O_{\varepsilon,F}(R^{1+2\varepsilon}/\max(|\alpha|, |\beta|))$ primitive solutions in the region given by (18), with the possible exception of $O(1)$ pairs α, β. To cover these exceptional cases we observe that if Q is non-singular, at least one of the binary forms

$$Q(x_1, x_2, 0), \quad Q(x_1, 0, x_3), \quad Q(0, x_2, x_3), \quad Q(x_1, x_2, x_2)$$

must also be non-singular. Thus, for the exceptional pairs α, β, Lemma 7 produces $O_{\varepsilon,F}(R^{1+2\varepsilon})$ primitive solutions in the region given by (18).

We may now estimate $\Sigma^* N(G_{\mathbf{y}}, AR)$ as

$$\ll_{\varepsilon,F} R^{1+2\varepsilon} + \sum_{|\alpha|, |\beta| \ll R^{1/3}} R^{1+2\varepsilon}/\max(|\alpha|, |\beta|) \ll_{\varepsilon,F} R^{4/3+2\varepsilon}.$$

Taken in conjunction with the bounds (16) and (17), this completes the proof of Theorem 1.

6. Bounds for the rank of elliptic curves

In this section we shall sketch the proof of the estimates (2) and (3). We write our curve in the form

$$E : y^2 = x^3 - ax - b,$$

where a, b are integers, and the discriminant $D = 4a^3 - 27b^2$ is non-zero. We may assume, without loss of generality, that the equation is in global minimal form, to the extent that there is no integer $d \geqslant 2$ for which $d^4|a$ and $d^6|b$. We now estimate the rank of E by the familiar 2-descent process. If K is the field generated by the roots of the cubic $X^3 - aX - b$, then

$$r_E \leqslant 2\{\omega_K(|D|) + h_2(K)\} + O(1),$$

where $\omega_K(|D|)$ is the number of distinct prime ideal factors of D, and $h_2(K)$ is the 2-rank of the ideal class group of K.

Since K has degree at most 6, we see that $\omega_K(|D|) \leqslant 6\omega(|D|)$, where $\omega(|D|)$ is the number of distinct rational primes dividing $|D|$. Moreover, we have

$$\omega(|D|) = \omega(C_E) + O(1)$$

since, with the possible exceptions $p = 2$ and $p = 3$, a prime p divides D if and only if it divides C_E. We note also that

$$\omega(C_E) \ll \frac{\log C_E}{\log \log C_E} \ll \log C_E.$$

We must now consider the discriminant D_K of the field K. The curve E will have bad reduction at every prime $p \geqslant 6$ which ramifies in K. Moreover since the degree of K is at most 6, the exponent to which p occurs in D_K is absolutely bounded. It follows from these considerations that

$$\log |D_K| \ll \log C_E.$$

It is easy to bound the class number $h(K)$ of K in terms of D_K, given that the degree is at most 6. For example, an estimate of Weyl [11; page 166] gives $h(K) \ll |D_K|^{7/2}$. Since $2^{h_2(K)} \leqslant h(K)$, we conclude that

$$h_2(K) \ll \log |D_K| \ll \log C_E.$$

Taken in conjunction with our earlier estimates this last bound now shows that

$$r_E \ll \log C_E$$

as claimed in (2).

When the curve has a rational point of order 2, the field K is quadratic, or may even reduce to $\mathbb{Q}$. In this case the theory of genera shows that

$$h_2(K) \ll \omega(|D_K|) \ll \omega(C_E).$$

In this case it therefore follows that

$$r_E \ll \omega(C_E) \ll \frac{\log C_E}{\log \log C_E},$$

as claimed in (3).

7. Acknowledgement

This work was initiated while the author was visiting the Institute for Advanced Study, Princeton. The hospitality of the Institute, and its financial support, are gratefully acknowledged.

References

[1] H. Davenport, Cubic forms in 16 variables, *Proc. Roy. Soc. A,* 272 (1963), 285-303.

[2] J. Franke, Yu.I. Manin, and Yu Tschinkel, Rational points of bounded height on Fano varieties, *Invent. Math.,* 95 (1989) 421-435.

[3] D.R. Heath-Brown, The density of rational points on cubic surfaces, *Acta Arithmetica*, 272 (1997) 17-30.

[4] M. Hindry and J.H. Silverman, The canonical height and integral points on elliptic curves, *Invent. Math.,* 93 (1988), 419-450.

[5] C.Hooley, On nonary cubic forms, *J. reine angew. Math.,* 386 (1988), 32-98.

[6] B. Mazur, Modular curves and the Eisenstein ideal, *Publ. Math. IHES,* 47 (1977).

[7] J.-F. Mestre, Courbes elliptiques et formules explicites, *Séminaire de théorie des nombres, Paris 1981-82,* 179-182 Progress in Math., 38, (Birkhäuser, Boston, Mass., 1983).

[8] J. Pila, The density of integral and rational points on varieties, *Astérisque,* 228 (1995), 183-187.

[9] W.M. Schmidt, *Diophantine Approximations and Diophantine Equations,* (Springer, Berlin, 1991).

[10] I.R. Shafarevich, Algebraic number fields, *Proc. Internat. Cong. Math. Stockholm, 1982,* 163-176.

[11] H. Weyl, *Algebraic theory of numbers,* (Princeton Univ. Press, N.J., 1940).

[12] H.G. Zimmer, On the difference of the Weil height and the Néron height, *Math. Zeit.,* 147 (1976), 35-51.

Received: 3 July 1996

ROGER HEATH-BROWN, Magdalen college, Oxford • *E-mail :* math3@hermine.ox.ac.uk

Astérisque
251, 1998, p. 31–49

SUR LA HAUTEUR DES POINTS D'UNE CERTAINE SURFACE CUBIQUE SINGULIÈRE

par

Etienne Fouvry

Résumé. — Par une paramétrisation des solutions de l'équation

$$x_1x_2x_3 = t^3, \qquad (x_1, x_2, x_3, t) = 1$$

et par des méthodes élémentaires de théorie analytique des nombres, on montre que le nombre de points de coordonnées non nulles, de hauteur canonique inférieure à X, de la surface cubique projective de $\mathbb{P}_3(\mathbb{Q})$ d'équation

$$X_1X_2X_3 = T^3,$$

est, pour $X \longrightarrow \infty$, équivalent à $c_0 X \log^6 X$, où c_0 est une certaine constante > 0.

1. Introduction

Cet article a été motivé par le récent travail de Batyrev et Tschinkel ([B–T1], [B–T2], [B–T3]), concernant le nombre de points de hauteur inférieure à une certaine borne X tendant vers l'infini, sur des variétés toriques projectives lisses sur un corps de nombres quelconque, par rapport à des plongements projectifs arbitraires.

Nous traitons ici, le cas très particulier de la variété $\mathcal{V}$ de $\mathbb{P}^3(\mathbb{Q})$, d'équation

$$X_1X_2X_3 = T^3.$$

Soit $V(X)$ le cardinal de l'ensemble des points de $\mathcal{V}$ de coordonnées non nulles et de hauteur au plus égale à X.

L'étude de $V(X)$ est un cas particulier de ([B–T2] Theorem 1.4) mais aussi de ([B–T1] Corollary 7.4). Toutefois, il faut remarquer que, dans ces deux articles, il est question de variétés toriques lisses. Le traitement de la cubique singulière $\mathcal{V}$, se fait par une résolution de singularités qui fournit ainsi une variété torique lisse. On utilise

***Classification mathématique par sujets* (2000).** — Primaire 11D72; secondaire 14M25.
Mots clefs. — Surface cubique, hauteur.

la fonctorialité des hauteurs pour se ramener au résultat de [B–T2]. Cette délicate réduction est développée, dans un cadre plus général, dans [B–T3].

L'objet de cet article est de montrer que des méthodes classiques de théorie analytique des nombres conduisent elles–aussi à un équivalent asymptotique de $V(X)$ pour $X \longrightarrow \infty$. Nous montrerons le

Théorème 1.1. — *Pour $X \longrightarrow \infty$, on a la relation*

$$V(X) = (c_0 + o(1))X \log^6 X, \tag{1}$$

où c_0 est une constante absolue, strictement positive.

Remarquons, dans un premier temps, que $V(X)$ est aussi le cardinal des quadruplets d'entiers relatifs (x, y, z, t) vérifiant

$$(x, y, z, t) = 1; \qquad 1 \leqslant x,\ |y|,\ |z|,\ |t| \leqslant X \qquad \text{et} \qquad xyz = t^3.$$

En jouant sur les signes, on voit qu'on a l'égalité

$$V(X) = 4V^+(X),$$

avec

$$V^+(X) = \sharp\big\{(x, y, z, t) \ \big| \ 1 \leqslant x,\ y,\ z,\ t \leqslant X,\ (x, y, z, t) = 1,\ xyz = t^3\big\}.$$

Il est intéressant de noter que des altérations, apparemment anodines, de la définition de $V^+(X)$, perturbent radicalement son ordre de grandeur asymptotique : on a d'une part

$$V_1^+(X) := \sharp\left\{(x, y, z, t) \ \middle| \ \begin{cases} (x, y) = (y, z) = (z, x) = 1, \\ 1 \leqslant x,\ y,\ z,\ t \leqslant X \\ xyz = t^3 \end{cases}\right\} \sim c_1 X$$

(avec c_1 constante strictement positive, en effet les restrictions précédentes donnent pour paramétrisations des solutions à $xyz = t^3$, les quadruplets de la forme (u^3, v^3, w^3, uvw) avec $(u, v) = (v, w) = (w, u) = 1$ et $1 \leqslant u,\ v,\ w \leqslant X^{\frac{1}{3}}$) et d'autre part

$$\begin{aligned} V_2^+(X) &:= \sharp\big\{(x, y, z, t) \ \big| \ 1 \leqslant t \leqslant X;\ (x, y, z, t) = 1 \text{ et } xyz = t^3\big\} \\ &\sim c_2 X \log^8 X \end{aligned}$$

(pour ce faire, on passe par la fonction arithmétique α, qui sera définie au paragraphe 2.1, par le Lemme 2.1 et par un modeste théorème taubérien pour écrire

$$V_2^+(X) = \sum_{t \leqslant X} \alpha(t) \sim c_2 X \log^8 X,$$

où c_2 est la valeur en 1 de la série de Dirichlet $(\sum_n \alpha(n) n^{-s})\zeta^{-9}(s)$.)

Ces variations ont le mérite de convaincre de l'importance intrinsèque des conditions de la définition de $V^+(X)$.

Pour revenir à la formule (1), la constante c_0 peut être décrite explicitement, de même que le $o(1)$ peut être précisé, tout au moins en théorie. En effet, les méthodes qui suivent sont absolument classiques, n'apportent rien de vraiment nouveau en théorie analytique des nombres, mais sont extrêmement lourdes par le nombre très élevé de variables de sommation engendrées par les différentes transformations. Ce qui suit peut, en principe, être transposé à l'équation

$$X_1X_2X_3X_4 = T^4,$$

mais la démarche devient totalement asphyxiante par le nombres de variables de sommation qu'il faut gérer.

Cet article a été écrit à la cordiale invite de V. Batyrev, J–L. Colliot–Thélène, E. Peyre et Y. Tschinkel et a bénéficié de conversations avec J. Brüdern et H. Iwaniec. Que ces différentes personnes soient ici remerciées !

2. Transformations de $V^+(X)$.

L'idée de base des fastidieuses transformations qui vont suivre, est de se ramener à une équation de la forme $xyz = t^3$ mais avec les conditions plus fortes de coprimalité $(x, y) = (y, z) = (z, x) = 1$. On a alors une représentation paramétrique des solutions en $(x, y, z, t) = (u^3, v^3, w^3, uvw)$ avec $(u, v) = (v, w) = (w, u) = 1$, ce qui est beaucoup plus facile à compter (voir $V_1^+(X)$ ci–dessus).

Puisque les conditions $1 \leqslant x,\ y,\ z \leqslant X$ entraînent $t \leqslant X$ et que $(x, y, z) = 1$ implique $(x, y, z, t) = 1$, le cardinal étudié s'écrit aussi, après un changement de notations, sous la forme

$$V^+(X) = \sharp \left\{ (x_1, x_2, x_3, t) \;\middle|\; \begin{cases} 1 \leqslant x_1,\ x_2,\ x_3 \leqslant X, \\ (x_1, x_2, x_3) = 1, \\ x_1x_2x_3 = t^3 \end{cases} \right\}.$$

Soit $d_1 = (x_2, x_3)$, alors on a l'égalité

$$V^+(X) = \sharp \left\{ (d_1, x_1, x_2', x_3', t) \;\middle|\; \begin{cases} 1 \leqslant d_1,\ x_1 \leqslant X, \\ 1 \leqslant d_1x_2',\ d_1x_3' \leqslant X, \\ (x_2', x_3') = (d_1, x_1) = 1, \\ d_1^2x_1x_2'x_3' = t^3 \end{cases} \right\},$$

puis posant $d_2 = (x_1, x_3')$, on a

$$V^+(X) = \sharp \left\{ (d_1, d_2, x_1', x_2', x_3'', t) \;\middle|\; \left\{ \begin{array}{l} 1 \leqslant d_1,\, d_2 \leqslant X, \\ 1 \leqslant d_2 x_1',\, d_1 x_2',\, d_1 d_2 x_3'' \leqslant X, \\ (x_1' x_2', x_3'') = 1, \\ (d_1, d_2) = (d_1, x_1') = (d_2, x_2') = 1, \\ d_1^2 d_2^2 x_1' x_2' x_3'' = t^3 \end{array} \right. \right\},$$

enfin, en posant $d_3 = (x_1', x_2')$, on parvient, après un changement évident de notations, à

(2)

$$V^+(X) = \sharp \left\{ (d_1, d_2, d_3, x_1, x_2, x_3, t) \;\middle|\; \left\{ \begin{array}{l} 1 \leqslant d_2 d_3 x_1, d_1 d_3 x_2, d_1 d_2 x_3 \leqslant X, \\ (x_2, x_3)=(x_3, x_1)=(x_1, x_2)=1, \\ (d_1, d_2)=(d_1, d_3)=(d_2, d_3)=1, \\ (d_1, x_1)=(d_2, x_2)=(d_3, x_3)=1, \\ d_1^2 d_2^2 d_3^2 x_1 x_2 x_3 = t^3 \end{array} \right. \right\}.$$

2.1. Hypothèses sur les d_i. — L'objet de ce qui suit est de prouver qu'on peut, pour ainsi dire, se ramener au cas où d_1, d_2 et d_3 sont sans facteur carré.

Notons $\alpha(t)$ le nombre de solutions à l'équation

$$t^3 = xyz, \qquad (x, y, z) = 1;$$

sans aucune contrainte de taille sur x, y et z. Nous aurons besoin du

Lemme 2.1. — *On a les propriétés suivantes :*
- *La fonction α est multiplicative.*
- *Pour tout entier $a \geqslant 1$ et tout p premier, on a $\alpha(p^a) = 9a$.*
- *Pour tout m et n on a l'inégalité $\alpha(mn) \leqslant \alpha(m)\alpha(n)$.*

Démonstration. — Seule la deuxième assertion mérite un peu de soin. On voit que $\alpha(p^a)$ est aussi le nombre de solutions à l'équation $3a = a_1 + a_2 + a_3$ avec les a_i entiers positifs ou nuls, vérifiant aussi $a_1 a_2 a_3 = 0$. Un dénombrement direct donne le résultat. □

Pour $d \geqslant 1$, on dissocie d en $d = d^\diamond d^\dagger$, où $d^\diamond$ est le plus grand entier *squarefull* divisant d (rappelons qu'un entier n est *squarefull*, s'il vérifie l'implication $p|n \Rightarrow p^2|n$). Ainsi $d^\dagger$ est sans facteur carré, premier avec $d^\diamond$ et, par exemple, on a $(600)^\diamond = 200$ et $(600)^\dagger = 3$. On pose

$$\mathcal{L} = \log X, \qquad L = \mathcal{L}^7.$$

Nous allons montrer que la contribution (notée $A(L)$) à (2), des 7–uplets tels que $d_1^\diamond \geqslant L$ est négligeable. En effet, cette contribution vérifie

$$A(L) \leqslant \sum_{\ell \textit{ squarefull } \geqslant L} \sum_{\substack{t \leqslant X \\ \Xi(\ell) | t}} \alpha(t); \tag{3}$$

où Ξ est la fonction multiplicative définie sur les puissances de nombres premiers par $\Xi(p) = p$, $\Xi(p^2) = p^2$, $\Xi(p^3) = p^2$,… et plus généralement $\Xi(p^a) = p^b$ où b est le plus petit entier tel que $3b \geqslant 2a$. En effet, si $p^a | d_1$, l'égalité $d_1^2 d_2^2 d_3^2 x_1 x_2 x_3 = t^3$ entraîne $p^{2a} | t^3$, donc la valuation p–adique de t^3 est supérieure ou égale au plus petit multiple de 3 qui dépasse $2a$.

D'après le Lemme 2.1 et (3), on a l'inégalité

$$A(L) \leqslant \sum_{\ell \textit{ squarefull } \geqslant L} \alpha(\Xi(\ell)) \sum_{n \leqslant X/\Xi(\ell)} \alpha(n),$$

puis, en sommant la fonction multiplicative α, on a

$$A \ll X \mathcal{L}^8 \sum_{\ell \textit{ squarefull } > L} \frac{\alpha(\Xi(\ell))}{\Xi(\ell)}.$$

Par la méthode de Rankin qui, rappelons–le, consiste à majorer la fonction caractéristique de l'intervalle $[L, \infty[$ par $(\frac{\ell}{L})^\varepsilon$, on a, pour tout $0 < \varepsilon < 1/3$, la relation

$$\begin{aligned} A(L) &\ll X \mathcal{L}^8 \sum_{\ell \textit{ squarefull}} \frac{\alpha(\Xi(\ell))}{\Xi(\ell)} \Big(\frac{\ell}{L}\Big)^\varepsilon \\ &\ll X \mathcal{L}^8 L^{-\varepsilon} \prod_p \Big(1 + \frac{18}{p^{2-2\varepsilon}} + \frac{18}{p^{2-3\varepsilon}} + \frac{27}{p^{3-4\varepsilon}} + \cdots\Big) \\ &\ll X \mathcal{L}^8 L^{-\varepsilon}. \end{aligned}$$

On a donc la relation

$$A(L) = o(X \mathcal{L}^6), \tag{4}$$

en prenant ε très proche de $1/3$.

En raisonnant de même sur d_2 et d_3, on peut, avec une erreur négligeable, insérer dans la définition (2) de $V^+(X)$, les conditions supplémentaires

$$d_1^\diamond,\ d_2^\diamond,\ d_3^\diamond < L,$$

ce qui implique que tout diviseur premier d'un d_i, supérieur à $\mathcal{L}^{\frac{7}{2}}$ apparaît avec l'exposant 1.

2.2. Deuxième étape de transformation.— Cette étape peut paraître bien absconse si on n'a pas pris le temps de saisir sa motivation sur un cas particulier : Imaginons que nous soyons intéressés par la contribution dans (2) des cas où

$$d_1 = p^2 (= d_1^\diamond), \qquad d_2 = d_3 = 1.$$

L'équation est donc $x_1.(p^2x_2).(p^2x_3) = t^3$. Puisque le second membre est un cube on a, pour des raisons de coprimalité, l'une ou l'autre des deux éventualités suivantes :
• p^2 divise x_2 (alors $v_p(x_2) \equiv 2$ modulo 3 et $v_p(x_3) = 0$),
• p^2 divise x_3 (alors $v_p(x_3) \equiv 2$ modulo 3 et $v_p(x_2) = 0$).

Par contre si

$$d_1 = p^3 (= d_1^\diamond), \qquad d_2 = d_3 = 1,$$

l'équation est $x_1 \cdot (p^3x_2) \cdot (p^3x_3) = t^3$ qui conduit alors à trois éventualités qui s'excluent mutuellement :
• p ne divise pas x_2x_3 (alors $v_p(x_1) = v_p(x_2) = v_p(x_3) = 0$) ;
• p^3 divise x_2 (alors $v_p(x_1) = v_p(x_3) = 0$, $v_p(x_2) \equiv 0$ modulo 3 et $v_p(x_2p^{-3}) \geqslant 0$) ;
• p^3 divise x_3 (alors $v_p(x_1) = v_p(x_2) = 0$, $v_p(x_3) \equiv 0$ modulo 3 et $v_p(x_3p^{-3}) \geqslant 0$).

Il reste à mettre dans un cadre formel toutes ces éventualités lorsque les $d_i^\diamond$ ont un nombre quelconque de facteurs premiers. Soient ℓ_1, ℓ_2 et ℓ_3 les parties *squarefull* de d_1, d_2 et d_3, les ℓ_i sont donc premiers deux à deux, d'après (2). Nous sommes en présence de l'équation

$$(\ell_2\ell_3x_1).(\ell_1\ell_3x_2).(\ell_1\ell_2x_3) = t^3, \tag{5}$$

avec (x_2, x_3), (x_3, x_1), (x_1, x_2) sans facteur carré,

$$(\ell_1, x_2, x_3) = (x_1, \ell_2, x_3) = (x_1, x_2, \ell_3) = 1$$
$$(\ell_1, x_1) = (\ell_2, x_2) = (\ell_3, x_3) = 1$$
$$(x_1, x_2, x_3) = 1.$$

Dans la formule (5), chaque terme à l'intérieur des parenthèses $(\cdots)$ est inférieur à X.

Soit $\lambda(n)$ le plus petit entier tel que $n\lambda(n)$ soit le cube d'un entier. On décompose $\lambda(\ell_1^2)$ sous la forme

$$\lambda(\ell_1^2) = \mu_1^{(2)}\mu_1^{(3)} \text{ avec } (\mu_1^{(2)}, \mu_1^{(3)}) = 1.$$

On désigne par $\mathfrak{P}_1$ l'ensemble des nombres premiers divisant ℓ_1^2 mais pas $\lambda(\ell_1^2)$. On désigne par $(\mathfrak{P}_1^{(2)}, \mathfrak{P}_1^{(3)})$ un couple quelconque de sous–ensembles de $\mathfrak{P}_1$ dont la réunion est $\mathfrak{P}_1$. On note par $P_1^{(2)}$ le produit des éléments de $\mathfrak{P}_1^{(2)}$ et par $\overline{\mathfrak{P}_1^{(2)}}$ le complémentaire de $\mathfrak{P}_1^{(2)}$ dans $\mathfrak{P}_1$.

Grâce à (4), on voit qu'on a l'égalité

$$V^+(X) = \sum_{\ell_i} \sum_{\mu_i^{(j)}} \sum_{\mathfrak{P}_i^{(j)}} V(\ell_i, \mu_i^{(j)}, \mathfrak{P}_i^{(j)}) + o(X\mathcal{L}^6), \tag{6}$$

où

• $V(\ell_i, \mu_i^{(j)}, \mathfrak{P}_i^{(j)})$ désigne le nombre de (y_1, y_2, y_3, t) vérifiant les équations :

$$\begin{aligned} t^3 = \big(\ell_2\ell_3\mu_2^{(1)}\mu_3^{(1)}\overline{P_2^{(1)}}^3\overline{P_3^{(1)}}^3 y_1\big) \cdot \big(\ell_1\ell_3\mu_1^{(2)}\mu_3^{(2)}\overline{P_1^{(2)}}^3\overline{P_3^{(2)}}^3 y_2\big) \\ \cdot \big(\ell_1\ell_2\mu_1^{(3)}\mu_2^{(3)}\overline{P_1^{(3)}}^3\overline{P_2^{(3)}}^3 y_3\big) \end{aligned} \tag{7}$$

avec $(y_1, y_2, y_3) = 1$, (y_1, y_2), (y_2, y_3), (y_3, y_1) sans facteur carré,

$$(\ell_1, y_2, y_3) = (y_1, \ell_2, y_3) = (y_1, y_2, \ell_3) = 1$$

et

$$\begin{aligned} (y_1, \ell_1\mu_2^{(3)}\mu_3^{(2)}P_2^{(1)}P_3^{(1)}) &= (y_2, \ell_2\mu_1^{(3)}\mu_3^{(1)}P_1^{(2)}P_3^{(2)}) \\ &= (y_3, \ell_3\mu_1^{(2)}\mu_2^{(1)}P_1^{(3)}P_2^{(3)}) = 1, \end{aligned}$$

chaque terme à gauche de (7), situé à l'intérieur d'une parenthèse $(\cdots)$ étant inférieur à X.

• la somme est faite sur les ℓ_1, ℓ_2 et ℓ_3 entiers *squarefull* inférieurs à L, premiers entre eux deux à deux, les $\mu_i^{(j)}$ et $\mathfrak{P}_i^{(j)}$ parcourant toutes les décompositions possibles de $\lambda(\ell_i^2)$ et $\mathfrak{P}_i$, comme ci–dessus.

Puisque le nombre

$$\big(\ell_2\ell_3\mu_2^{(1)}\mu_3^{(1)}\overline{P_2^{(1)}}^3\overline{P_3^{(1)}}^3\big).\big(\ell_1\ell_3\mu_1^{(2)}\mu_3^{(2)}\overline{P_1^{(2)}}^3\overline{P_3^{(2)}}^3\big).\big(\ell_1\ell_2\mu_1^{(3)}\mu_2^{(3)}\overline{P_1^{(3)}}^3\overline{P_2^{(3)}}^3\big)$$

est un cube parfait, on voit que chaque $V(\ell_i, \mu_i^{(j)}, \mathfrak{P}_i^{(j)})$ est en fait de la forme

$$\mathcal{K}\Big(\mathbf{X}; \mathbf{b}; \mathbf{c}\Big) = \mathcal{K}\Big(X_1, X_2, X_3; b_1, b_2, b_3; c_1, c_2, c_3\Big),$$

qui désigne le nombre de solutions à l'équation

$$x_1x_2x_3 = t^3,$$

avec $x_i \leqslant X_i$, $(x_1, x_2, x_3) = (x_1, b_1) = (x_2, b_2) = (x_3, b_3) = 1$ et (x_1, x_2), (x_2, x_3) et (x_1, x_3) sans facteur carré et respectivement premiers avec c_3, c_1 et c_2, les b_i et c_i étant des entiers donnés tels que les c_i soient premiers entre eux deux à deux.

Signalons que dans notre application, les X_i seront égaux à X à une puissance de $\mathcal{L}$ près, puisque chacun des ℓ_i est inférieur à L. Pour des raisons évidentes de sobriété dans les notations, nous nous sommes restreints à l'étude soignée de

$$\mathcal{K}(X) := \mathcal{K}\Big(X, X, X; 1, 1, 1; 1, 1, 1\Big).$$

Au paragraphe 6, sera ébauchée la façon de passer au cas général, pour pouvoir effectuer la sommation (6).

3. Étude de $\mathcal{K}(X)$.

Rappelons que $\mathcal{K}(X)$ est le nombre de solutions à l'équation

$$x_1x_2x_3 = t^3,$$

avec $d_1 = (x_2, x_3)$, $d_2 = (x_1, x_3)$ et $d_3 = (x_1, x_2)$ sans facteur carré, x_1, x_2, x_3 premiers entre eux et chaque x_i inférieur à X. On est ramené à une équation apparaissant déjà dans (2).

Pour parfaire le travail de coprimalité, on extrait de x_1/d_2, le produit $\delta_1^{(2)}$ des facteurs premiers apparaissant aussi dans d_2 et aussi le produit $\delta_1^{(3)}$ des facteurs premiers de x_1/d_3 apparaissant aussi dans d_3. On opère de même pour les autres variables. Finalement, après un nouveau changement de notations, on voit que $\mathcal{K}(X)$ est le cardinal de l'ensemble des 13–uplets

$$(d_1, d_2, d_3, \delta_1^{(2)}, \delta_1^{(3)}, \delta_2^{(1)}, \delta_2^{(3)}, \delta_3^{(1)}, \delta_3^{(2)}, x_1, x_2, x_3, t)$$

d'entiers compris entre 1 et X vérifiant

(8) $\quad (d_1, d_2) = (d_2, d_3) = (d_3, d_1) = 1, \qquad d_1,\ d_2$ et d_3 sans facteur carré ,

(9) $$\delta_2^{(1)}\delta_3^{(1)}|d_1^{\infty}, \qquad \delta_1^{(2)}\delta_3^{(2)}|d_2^{\infty}, \qquad \delta_1^{(3)}\delta_2^{(3)}|d_3^{\infty},$$

(10) $$(\delta_2^{(1)}, \delta_3^{(1)}) = (\delta_1^{(2)}, \delta_3^{(2)}) = (\delta_1^{(3)}, \delta_2^{(3)}) = 1;$$

et une autre liste de conditions où apparaissent x_1, x_2 et x_3 :

$$(x_1x_2x_3, d_1d_2d_3) = (x_1, x_2) = (x_1, x_3) = (x_2, x_3) = 1,$$
$$\delta_1^{(2)}\delta_1^{(3)}d_2d_3x_1,\ d_1\delta_2^{(1)}\delta_2^{(3)}d_3x_2,\ d_1d_2\delta_3^{(1)}\delta_3^{(2)}x_3 \leqslant X;$$
$$d_1^2d_2^2d_3^2\delta_1^{(2)}\delta_1^{(3)}\delta_2^{(1)}\delta_2^{(3)}\delta_3^{(1)}\delta_3^{(2)}x_1x_2x_3 = t^3.$$

Les conditions de coprimalité entraînent que tous les entiers

$$x_1,\ x_2,\ x_3 \text{ et } d_1^2d_2^2d_3^2\delta_1^{(2)}\delta_1^{(3)}\delta_2^{(1)}\delta_2^{(3)}\delta_3^{(1)}\delta_3^{(2)}$$

sont les cubes d'entiers, notés respectivement u_1, u_2, u_3 et w, premiers entre eux deux à deux.

Le problème de l'évaluation de $\mathcal{K}(X)$ est ainsi réduit à celui du nombre des 13–uplets

$$(d_1, d_2, d_3, \delta_1^{(2)}, \delta_1^{(3)}, \delta_2^{(1)}, \delta_2^{(3)}, \delta_3^{(1)}, \delta_3^{(2)}, x_1, x_2, x_3, t),$$

vérifiant les conditions (8), (9), (10) et les autres conditions

$$(u_1u_2u_3, d_1d_2d_3) = (u_1, u_2) = (u_3, u_1) = (u_2, u_3) = 1, \tag{11}$$

$$d_1^2d_2^2d_3^2\delta_1^{(2)}\delta_1^{(3)}\delta_2^{(1)}\delta_2^{(3)}\delta_3^{(1)}\delta_3^{(2)} = w^3, \tag{12}$$

$$\begin{cases} 1 \leqslant u_1 \leqslant \big(X/(\delta_1^{(2)}\delta_1^{(3)}d_2d_3)\big)^{1/3}, \\ 1 \leqslant u_2 \leqslant \big(X/(d_1\delta_2^{(1)}\delta_2^{(3)}d_3)\big)^{1/3}, \\ 1 \leqslant u_3 \leqslant \big(X/(d_1d_2\delta_3^{(1)}\delta_3^{(2)})\big)^{1/3}. \end{cases} \tag{13}$$

Le nombre de u_3 vérifiant les conditions précédentes, lorsque les 12 autres variables sont fixées, est

(14)
$$\sum_{\substack{1\leqslant u_3\leqslant \left(X/(d_1d_2\delta_3^{(1)}\delta_3^{(2)})\right)^{1/3} \\ (u_3,u_1u_2d_1d_2d_3)=1}} 1 = \psi_3(u_1u_2d_1d_2d_3)\Big(\frac{X}{d_1d_2\delta_3^{(1)}\delta_3^{(2)}}\Big)^{\frac{1}{3}}$$
$$+ O\Big(\min\{\tau(u_1u_2d_1d_2d_3), \big(\frac{X}{d_1d_2\delta_3^{(1)}\delta_3^{(2)}}\big)^{\frac{1}{3}}\}\Big),$$

en utilisant le résultat classique :

$$\sum_{\substack{n\leqslant y \\ (n,q)=1}} 1 = \psi_3(q)y + O(\min(\tau(q), y)),$$

avec $\tau(q)$ nombre de diviseurs de l'entier q et $\psi_3(q) = \varphi(q)/q$. Rappelons aussi la majoration

$$\tau(q) = O\big(\exp(\frac{\log q}{\log\log q})\big). \tag{15}$$

4. Minoration de $\mathcal{K}(X)$.

Il nous a semblé plus clair de séparer minoration et majoration de $\mathcal{K}(X)$ pour accéder à l'équivalent asymptotique de cette quantité.

Soit
$$Z = \exp\big(\frac{\log X}{\sqrt{\log\log X}}\big).$$

Pour minorer $\mathcal{K}(X)$ nous ajoutons à la liste de conditions (8), ..., (13) la condition

$$\delta_1^{(2)}\delta_1^{(3)}d_2d_3,\ d_1\delta_2^{(1)}\delta_2^{(3)}d_3,\ d_1d_2\delta_3^{(1)}\delta_3^{(2)} \leqslant Y := XZ^{-1}, \tag{16}$$

qui implique donc, que dans les inégalités (13), chaque variable u_i se déplace dans un intervalle assez long, à savoir de longueur au moins égal à $Z^{\frac{1}{3}}$.

La relation (14) et la remarque (15) conduisent donc à la minoration

$$\sum_{\substack{1\leqslant u_3\leqslant \left(X/(d_1d_2\delta_3^{(1)}\delta_3^{(2)})\right)^{1/3}\\ (u_3,u_1u_2d_1d_2d_3)=1}} 1 \geqslant (1-o(1))\cdot\psi_3(d_1d_2d_3)\cdot\psi_3(u_1)\cdot\psi_3(u_2)\cdot\Big(\frac{X}{d_1d_2\delta_3^{(1)}\delta_3^{(2)}}\Big)^{\frac13}, \tag{17}$$

uniformément sous la condition (16).

Pour sommer sur u_2, nous utilisons la formule

$$\sum_{\substack{n\leqslant y\\ (n,q)=1}} \psi_3(n)=\frac{6}{\pi^2}\,\psi_2(q)y+O_\varepsilon\big(\min(y^{\frac12+\varepsilon}\tau(q),y)\big), \tag{18}$$

où $\psi_2(q)$ est définie par

$$\psi_2(q)=\prod_{p\mid q}\Big(1+\frac1p\Big)^{-1}.$$

La formule précédente s'obtient, par exemple, par la théorie des séries de Dirichlet et un peu d'analyse complexe, qui consiste à intégrer

$$\int_{2-i\infty}^{2+i\infty}\Big(\sum_{(n,q)=1}\psi_3(n)n^{-s}\Big)\frac{y^s}{s}\mathrm{d}s$$

et à déformer le parcours d'intégration pour récupérer le résidu en $s=1$.

Nous appliquons (18) avec $y=\big(X/(d_1\delta_2^{(1)}\delta_2^{(3)}d_3)\big)^{1/3}$, $q=u_1d_1d_2d_3$. Les restrictions (16) permettent de transformer (18) en une inégalité en introduisant un nouveau facteur $(1-o(1))$. Nous sommes ainsi parvenus à

$$\sum_{\substack{1\leqslant u_2\leqslant \left(X/(d_1\delta_2^{(1)}\delta_2^{(3)}d_3)\right)^{1/3}\\ (u_2,u_1d_1d_2d_3)=1}}\ \sum_{\substack{1\leqslant u_3\leqslant \left(X/(d_1d_2\delta_3^{(1)}\delta_3^{(2)})\right)^{1/3}\\ (u_3,u_1u_2d_1d_2d_3)=1}} 1 \geqslant \frac{6}{\pi^2}(1-o(1)).(\psi_2.\psi_3)(d_1d_2d_3)\ (\psi_2.\psi_3)(u_1)\frac{X^{\frac23}}{(d_1^2d_2d_3\delta_2^{(1)}\delta_2^{(3)}\delta_3^{(1)}\delta_3^{(2)})^{\frac13}}. \tag{19}$$

On somme maintenant sur u_1, en utilisant la formule

$$\sum_{\substack{n\leqslant y\\ (n,q)=1}} (\psi_2.\psi_3)(n)=\alpha_0\psi_1(q)y+O_\varepsilon\big(\min(y^{\frac12+\varepsilon}\tau(q),y)\big), \tag{20}$$

avec α_0 constante absolue définie par

$$\alpha_0=\prod_p\Big(1-\frac{2}{p(p+1)}\Big),$$

et

$$\psi_1(q) = \prod_{p|q}\Big(1 + \frac{1}{p+1}\Big)^{-1}.$$

Ici aussi, on utilise (20) sous forme d'une minoration uniforme, toujours grâce à (16) et la formule (19) fournit la minoration

(21)

$$\begin{aligned}\mathcal{K}(X) &\geqslant (1-o(1))\cdot\frac{6}{\pi^2}\cdot\alpha_0\cdot X \\ &\cdot \sum_{(d_1,d_2,d_3,\delta_1^{(2)},\delta_1^{(3)},\delta_2^{(1)},\delta_2^{(3)},\delta_3^{(1)},\delta_3^{(2)})}\psi(d_1)\psi(d_2)\psi(d_3)\Big(\frac{1}{d_1^2d_2^2d_3^2\delta_1^{(2)}\delta_1^{(3)}\delta_2^{(1)}\delta_2^{(3)}\delta_3^{(1)}\delta_3^{(2)}}\Big)^{\frac{1}{3}} \\ &\geqslant (1-o(1))\frac{6}{\pi^2}\cdot\alpha_0\cdot X\cdot\mathcal{B}(Y);\end{aligned}$$

par définition, la somme étant faite sous les conditions (8), (9), (10), (12) et (16) et ψ est le produit $\psi = \psi_1\cdot\psi_2\cdot\psi_3$. Signalons que, dans $\mathcal{B}(Y)$, les variables apparaissent au dénominateur, ce qui laisse supposer que cette expression a un comportement plus docile.

4.1. Exploitation de la condition (12).**—** Nous utilisons les conditions de coprimalité qui lient les d_i et les $\delta_i^{(j)}$ pour écrire que (12) est équivalente aux trois équations

$$\delta_2^{(1)}\delta_3^{(1)}d_1^2 = w_1^3,\quad \delta_1^{(2)}\delta_3^{(2)}d_2^2 = w_2^3,\quad \delta_1^{(3)}\delta_2^{(3)}d_3^2 = w_3^3. \tag{22}$$

C'est ici que nous utilisons le fait que d_1, d_2 et d_3 sont sans facteur carré (ce qui entraîne, avec les notations du paragraphe 2, que $\lambda(d_i^2) = d_i$). Ecrivons $d_1 = \nu_2^{(1)}\nu_3^{(1)}$, $\nu_2^{(1)}$ et $\nu_3^{(1)}$ sont premiers entre eux, sans facteur carré. L'équation

$$\delta_2^{(1)}\delta_3^{(1)}d_1^2 = w_1^3$$

de (22) admet pour uniques solutions les

$$\delta_2^{(1)} = \nu_2^{(1)}(v_2^{(1)})^3,\qquad \delta_3^{(1)} = \nu_3^{(1)}(v_3^{(1)})^3,$$

où $v_2^{(1)}$ et $v_3^{(1)}$ sont des entiers premiers entre eux, divisant d_1^∞. On opère de même sur les autres variables. La somme $\mathcal{B}(Y)$ est ainsi remaniée en

$$\mathcal{B}(Y) = \sum_{\nu}\frac{\psi(\nu_2^{(1)})\psi(\nu_3^{(1)})\psi(\nu_1^{(2)})\psi(\nu_3^{(2)})\psi(\nu_1^{(3)})\psi(\nu_2^{(3)})}{\nu_2^{(1)}\nu_3^{(1)}\nu_1^{(2)}\nu_3^{(2)}\nu_1^{(3)}\nu_2^{(3)}}\Lambda(\nu,Y), \tag{23}$$

la somme étant faite sur les $\nu = (\nu_i^{(j)})$ $(1\leqslant i\neq j\leqslant 3)$, où les $\nu_i^{(j)}$ sont premiers entre eux deux à deux, sans facteur carré.

Dans (23), la quantité $\Lambda(\nu, Y)$ est définie par

$$\Lambda(\nu, Y) = \sum_{\mathbf{v}} \frac{1}{v_2^{(1)} v_3^{(1)} v_1^{(2)} v_3^{(2)} v_1^{(3)} v_2^{(3)}};$$

où les $\mathbf{v} = (v_i^{(j)})$ ($1 \leqslant i \neq j \leqslant 3$), sont tels que les $v_i^{(j)}$ sont premiers entre eux deux à deux, sont tels que

$$v_2^{(1)} v_3^{(1)} | \big(\nu_2^{(1)} \nu_3^{(1)}\big)^\infty, \quad v_1^{(2)} v_3^{(2)} | \big(\nu_1^{(2)} \nu_3^{(2)}\big)^\infty, \quad v_1^{(3)} v_2^{(3)} | \big(\nu_1^{(3)} \nu_2^{(3)}\big)^\infty; \tag{24}$$

et sont tels que les inégalités suivantes (qui sont des réécritures de (16))

$$(\nu_1^{(2)})^2 (\nu_1^{(3)})^2 \nu_3^{(2)} \nu_2^{(3)} (v_1^{(2)})^3 (v_1^{(3)})^3 \leqslant Y; \tag{25}$$

$$(\nu_2^{(1)})^2 (\nu_2^{(3)})^2 \nu_3^{(1)} \nu_1^{(3)} (v_2^{(1)})^3 (v_2^{(3)})^3 \leqslant Y; \tag{26}$$

$$(\nu_3^{(1)})^2 (\nu_3^{(2)})^2 \nu_1^{(2)} \nu_2^{(1)} (v_3^{(1)})^3 (v_3^{(2)})^3 \leqslant Y \tag{27}$$

soient satisfaites.

4.2. Minoration de $\Lambda(\nu, Y)$.— Maintenant, on pose $Y_1 = YZ^{-1}$ et $Y_2 = YZ^{-2}$ et soient les six autres conditions sur les variables de sommation :

$$(\nu_1^{(2)})^2 (\nu_1^{(3)})^2 \nu_3^{(2)} \nu_2^{(3)} (v_1^{(2)})^3 \leqslant Y_1; \tag{28}$$

$$(\nu_2^{(1)})^2 (\nu_2^{(3)})^2 \nu_3^{(1)} \nu_1^{(3)} (v_2^{(1)})^3 \leqslant Y_1; \tag{29}$$

$$(\nu_3^{(1)})^2 (\nu_3^{(2)})^2 \nu_1^{(2)} \nu_2^{(1)} (v_3^{(1)})^3 \leqslant Y_1; \tag{30}$$

et

$$(\nu_1^{(2)})^2 (\nu_1^{(3)})^2 \nu_3^{(2)} \nu_2^{(3)} \leqslant Y_2; \tag{31}$$

$$(\nu_2^{(1)})^2 (\nu_2^{(3)})^2 \nu_3^{(1)} \nu_1^{(3)} \leqslant Y_2; \tag{32}$$

$$(\nu_3^{(1)})^2 (\nu_3^{(2)})^2 \nu_1^{(2)} \nu_2^{(1)} \leqslant Y_2. \tag{33}$$

Soit $\Lambda(\nu, Y, Y_1, Y_2)$ défini de façon semblable à $\Lambda(\nu, Y)$ mais avec les conditions de sommation (24), ..., (33). Puisque $\Lambda(\nu, Y)$ est à termes positifs, on a l'inégalité

$$\Lambda(\nu, Y) \geqslant \Lambda(\nu, Y, Y_1, Y_2);$$

l'introduction des conditions supplémentaires (28), ..., (33), assure que, si on somme dans l'ordre $v_1^{(3)}$, $v_2^{(3)}$, $v_3^{(2)}$, $v_1^{(2)}$, $v_2^{(1)}$ et $v_3^{(1)}$, chacune des variables $v_i^{(j)}$ a un intervalle de variation au moins égal à $Z^{\frac{1}{3}}$, ce qui permet de faire apparaître une minoration uniforme grâce au

Lemme 4.1. — *Soit a un entier au moins égal à 1, $y \geqslant 1$ et f une fonction fortement multiplicative (c'est–à–dire vérifiant $f(p^k) = f(p)$ pour tout $k \geqslant 1$), vérifiant l'inégalité $0 \leqslant f(p) \leqslant 1$. Uniformément sous les conditions précédentes, on a l'égalité*

$$\sum_{\substack{k|a^\infty\\ k\leqslant y}} \frac{f(k)}{k} = \prod_{p|a}\Big(1+\frac{f(p)}{p-1}\Big) + O\big(y^{-\frac12}\tau(a)\big).$$

Démonstration. — C'est une illustration de la méthode de Rankin ; on a l'égalité

$$\sum_{\substack{k|a^\infty\\ k\leqslant y}} \frac{f(k)}{k} = \prod_{p|a}\Big(1+\frac{f(p)}{p-1}\Big) - \sum_{\substack{k|a^\infty\\ k> y}} \frac{f(k)}{k},$$

la dernière somme est inférieure à

$$\sum_{k|a^\infty} \frac{f(k)}{k}(\frac{k}{y})^{\frac12} = y^{-\frac12}\prod_{p|a}\Big(1+\frac{f(p)}{\sqrt{p}-1}\Big) = O\big(y^{-\frac12}\tau(a)\big). \quad \square$$

Ainsi, pour minorer $\Lambda(\nu, Y, Y_1, Y_2)$, nous commençons par sommer sur $v_1^{(3)}$, on applique le Lemme 4.1 avec un y défini par l'inégalité (25) – mais en tout état de cause, vérifiant $y \geqslant Z^{\frac13}$, d'après (28) – et avec $a = \nu_1^{(3)}\nu_2^{(3)}/(v_2^{(3)}, \nu_1^{(3)}\nu_2^{(3)})$, d'où la minoration uniforme

$$\sum_{v_1^{(3)}} \frac{1}{v_1^{(3)}} \geqslant (1-o(1)) \prod_{p|\nu_1^{(3)}\nu_2^{(3)}} \Big(1+\frac{1}{p-1}\Big) \prod_{p|v_2^{(3)}} \Big(1-\frac{1}{p}\Big),$$

puis en appliquant de nouveau le Lemme 4.1, on a la minoration

$$\sum_{v_2^{(3)}}\sum_{v_1^{(3)}} \frac{1}{v_1^{(3)}v_2^{(3)}} \geqslant (1-o(1)) \prod_{p|\nu_1^{(3)}\nu_2^{(3)}} \Big(1+\frac{2}{p-1}\Big).$$

Par un raisonnement identique sur les quatre autres variables $v_3^{(2)}$, $v_1^{(2)}$, $v_2^{(1)}$ et $v_3^{(1)}$, nous accédons à la minoration

$$\Lambda(\nu, Y, Y_1, Y_2) \geqslant (1-o(1)) \frac{1}{(\psi_2.\psi_3)(\nu_2^{(1)}\nu_3^{(1)}\nu_1^{(2)}\nu_3^{(2)}\nu_1^{(3)}\nu_2^{(3)})},$$

où les ψ_i sont définis aux paragraphes 3 et 4, puis en reportant dans (23) on a

$$\mathcal{B}(Y) \geqslant (1-o(1)) \sum_{\nu} \frac{\psi_1(\nu_2^{(1)})\psi_1(\nu_3^{(1)})\psi_1(\nu_1^{(2)})\psi_1(\nu_3^{(2)})\psi_1(\nu_1^{(3)})\psi_1(\nu_2^{(3)})}{\nu_2^{(1)}\nu_3^{(1)}\nu_1^{(2)}\nu_3^{(2)}\nu_1^{(3)}\nu_2^{(3)}};$$

les $\nu_i^{(j)}$ étant sans facteur carrés, premiers entre eux deux à deux et vérifiant les relations (31), (32) et (33).

4.3. Minoration de $\mathcal{B}(Y)$.— La fin de la minoration de $\mathcal{B}(Y)$ est une application itérée de la formule suivante

Lemme 4.2. — *Soit f une fonction multiplicative vérifiant $f(p) \geqslant 0$, $1 - \frac{A}{p} \leqslant f(p) \leqslant 1$ (pour une certaine constante $A > 0$) et $f(p^k) = 0$ pour $k \geqslant 2$. On a alors, pour $y \to \infty$, l'égalité*

$$\sum_{\substack{n\leqslant y\\(n,a)=1}} \frac{f(n)}{n} = \frac{\varphi(a)}{a}\prod_{p\nmid a}\Big(1+\frac{f(p)-1}{p}-\frac{f(p)}{p^2}\Big)\log y + O_A\Big((\log y)^{\frac{9}{10}}\log\log a + \exp\big((\log a)^{\frac12}-(\log y)^{\frac{9}{10}}\big)\Big).$$

Remarque 4.3. — Les exposants proposés n'ont aucun caractère d'optimalité, mais sont suffisants pour notre application. Les méthodes d'analyse complexe évoquées pour (18) et (20) donnent un développement asymptotique pour

$$\sum_{n\leqslant y,\,(n,a)=1} f(n).$$

Une intégration par parties fournit une formule plus précise pour $\sum \frac{f(n)}{n}$. Les méthodes employées seraient alors un peu plus profondes que celles qui suivent.

Démonstration. — Soit $\tilde f$ la fonction multiplicative définie par $\tilde f(n) = f(n)$ si $(n,a) = 1$ et $\tilde f(n) = 0$ autrement. En utilisant la convolution $*$ de Möbius, la somme étudiée s'écrit

$$\sum_{n\leqslant y}\frac{(\tilde f*\mu*\mathbf{1})(n)}{n} = \sum_{k\leqslant y}\frac{(\tilde f*\mu)(k)}{k}\sum_{m\leqslant y/k}\frac1m = \log y\sum_{k\leqslant y}\frac{(\tilde f*\mu)(k)}{k} + O\Big(\sum_{k\leqslant y}\frac{|(\tilde f*\mu)(k)|}{k}\log k\Big).$$

Le terme principal provient de la série complète

$$\sum_k \frac{(\tilde f*\mu)(k)}{k}.$$

Pour contrôler l'erreur, nous utilisons la majoration

$$|(\tilde f*\mu)(k)| \leqslant \prod_{\substack{p\| k\\ p\nmid a}}\frac{A}{p},$$

qu'il est facile de vérifier. En utilisant la méthode de Rankin et en posant $y_0 = \exp\big(4(\log y)^{\frac{9}{10}}\big)$, on majore chacun des trois termes d'erreur comme suit :

$$\begin{aligned}\log y \sum_{k>y} \frac{|(\tilde{f}*\mu)(k)|}{k} &\leqslant \log y \sum_{k} \frac{|(\tilde{f}*\mu)(k)|}{k}\Big(\frac{k}{y}\Big)^{\frac{1}{3}} \\ &\ll_A \log y\; y^{-\frac{1}{3}} \prod_{p|a}\Big(1+\frac{1}{p^{\frac{2}{3}}}\Big) \\ &\ll_A y^{-\frac{1}{4}} \exp\big((\log a)^{\frac{1}{2}}\big);\end{aligned}$$

puis

$$\begin{aligned}\sum_{k\leqslant y_0} \frac{|(\tilde{f}*\mu)(k)|}{k}\log k &\ll \log y_0 \prod_{p\leqslant y_0}\Big(1+\frac{A}{p^2}\Big)\prod_{p|a}\Big(1+\frac{1}{p}\Big) \\ &\ll_A \log y_0 \log\log a\end{aligned}$$

enfin

$$\sum_{y_0<k<y} \frac{|(\tilde{f}*\mu)(k)|}{k}\log k \leqslant \log y \sum_{k>y_0} \frac{|(\tilde{f}*\mu)(k)|}{k} \ll_A y_0^{-\frac{1}{4}} \exp\big((\log a)^{\frac{1}{2}}\big).$$

□

On applique ce lemme six fois, avec des choix à chaque fois différents pour la fonction f, l'entier a et le nombre y défini par (31), (32) et (33). On opère en outre une sommation par partie pour tenir compte des lentes variations des facteurs $\log$, $\log^2$,... $\log^5$, d'où la minoration

$$\mathcal{B}(Y) \geqslant (1-o(1))\alpha_1 \log^6 Y_2 = (1-o(1))\alpha_1 \log^6 X,$$

avec α_1 constante qu'il est tout à fait possible d'évaluer avec du soin. En reportant dans (21), on obtient finalement

$$\mathcal{K}(X) \geqslant (1-o(1))\frac{6}{\pi^2}\alpha_0\alpha_1 X \log^6 X. \tag{34}$$

5. Majoration de $\mathcal{K}(X)$.

Il faut reprendre les calculs de minoration de $\mathcal{K}(X)$ et évaluer chacune des pertes.

L'introduction des conditions restrictives (16) sur les d_i et les $\delta_i^{(j)}$ entraîne au niveau de $\mathcal{K}(X)$, une erreur $\mathcal{E}(X,Y)$ en

$$\mathcal{E}(X,Y) = O\Big(\sum_{\substack{d_1,d_2,d_3,\\ \delta_1^{(2)},\delta_1^{(3)},\delta_2^{(1)},\delta_2^{(3)},\delta_3^{(1)},\delta_3^{(2)}}} \Big(\frac{X}{d_1^2d_2^2d_3^2\delta_1^{(2)}\delta_1^{(3)}\delta_2^{(1)}\delta_2^{(3)}\delta_3^{(1)}\delta_3^{(2)}}\Big)^{\frac{1}{3}}\Big), \tag{35}$$

la somme étant faite sur les d_i et $\delta_i^{(j)}$ vérifiant

$$Y < \delta_1^{(2)}\delta_1^{(3)}d_2d_3 \leqslant X,\ d_1\delta_2^{(1)}\delta_2^{(3)}d_3 \leqslant X,\ d_1d_2\delta_3^{(1)}\delta_3^{(2)} \leqslant X, \tag{36}$$

et les conditions (8), (9), (10), (12) et chacun des d_i sans facteur carré. Pour établir cette majoration, on a utilisé la majoration triviale contenue dans (14), on a tiré avantage de la symétrie des variables et du fait que chaque fonction ψ_i est comprise entre 0 et 1. Autrement dit, on a la majoration

$$\mathcal{K}(X) \leqslant (1+o(1)) \cdot \frac{6}{\pi^2} \cdot \alpha_0 \cdot X \cdot \mathcal{B}(Y) + O\big(\mathcal{E}(X,Y)\big). \tag{37}$$

À partir de (35), on a facilement la relation

$$\mathcal{E}(X,Y) = O\Big(X\mathcal{F}(X) - X\mathcal{F}(Y)\Big); \tag{38}$$

avec

$$\mathcal{F}(X) = \sum_{(d_1,d_2,d_3,\delta_1^{(2)},\delta_1^{(3)},\delta_2^{(1)},\delta_2^{(3)},\delta_3^{(1)},\delta_3^{(2)})} \Big(\frac{1}{d_1^2d_2^2d_3^2\delta_1^{(2)}\delta_1^{(3)}\delta_2^{(1)}\delta_2^{(3)}\delta_3^{(1)}\delta_3^{(2)}}\Big)^{\frac{1}{3}};$$

les intervalles de sommation (36) étant maintenant remplacés par

$$\delta_1^{(2)}\delta_1^{(3)}d_2d_3 \leqslant X, \qquad d_1\delta_2^{(1)}\delta_2^{(3)}d_3 \leqslant X, \qquad d_1d_2\delta_3^{(1)}\delta_3^{(2)} \leqslant X.$$

La quantité $\mathcal{F}(Y)$ ressemble beaucoup à la quantité $\mathcal{B}(Y)$, introduite en (21), à la différence près que les fonctions multiplicatives ψ_i ont été remplacées par 1. Par une méthode identique, nous pouvons énoncer la minoration

$$\mathcal{F}(Y) \geqslant (1-o(1))\alpha_2 \log^6 Y_2 = (1-o(1))\alpha_2 \log^6 X, \tag{39}$$

où α_2 est une constante absolue positive.

La quantité $\mathcal{F}(X)$ doit être majorée et c'est en quelque sorte plus facile que de la minorer. Nous écrivons, par similitude avec (23), l'égalité

$$\mathcal{F}(X) = \sum_{\nu} \frac{1}{\nu_2^{(1)}\nu_3^{(1)}\nu_1^{(2)}\nu_3^{(2)}\nu_1^{(3)}\nu_2^{(3)}}\Lambda(\nu, X),$$

où $\Lambda(\nu, X)$ est défini au paragraphe 4, mais en remplaçant, naturellement, dans (25), (26) et (27), le paramètre Y par X. La série $\Lambda(\nu, X)$ est à termes positifs, donc on a l'inégalité

$$\Lambda(\nu, X) \leqslant \tilde{\Lambda}(\nu, X),$$

où $\tilde{\Lambda}(\nu, X)$ est défini comme $\Lambda(\nu, X)$ à la différence près que les inégalités (25), (26) et (27) sont remplacées par

$$\begin{aligned}(\nu_1^{(2)})^2(\nu_1^{(3)})^2\nu_3^{(2)}\nu_2^{(3)} &\leqslant X;\\ (\nu_2^{(1)})^2(\nu_2^{(3)})^2\nu_3^{(1)}\nu_1^{(3)} &\leqslant X;\\ (\nu_3^{(1)})^2(\nu_3^{(2)})^2\nu_1^{(2)}\nu_2^{(1)} &\leqslant X.\end{aligned}$$

Il n'y a donc plus aucune contrainte sur la taille des $v_i^{(j)}$. Les sommes sur les $v_i^{(j)}$ sont des séries complètes, le Lemme 4.1 est donc inutile. Par une méthode absolument identique, on parvient à majoration

$$\mathcal{F}(X) \leqslant (1 + o(1))\alpha_2 \log^6 X,$$

qui, reportée dans (38), donne grâce à (39), la relation

$$\mathcal{E}(X, Y) = o(X \log^6 X). \tag{40}$$

Pour terminer la majoration de $\mathcal{K}(X)$, il reste à majorer $\mathcal{B}(Y)$ (voir (37)). Par l'argument précédent on a l'inégalité

$$\Lambda(\nu, Y) \leqslant \tilde{\Lambda}(\nu, Y),$$

la série sur les $v_i^{(j)}$ est complète, on a donc l'égalité

$$\tilde{\Lambda}(\nu, Y) = \frac{1}{(\psi_2.\psi_3)(\nu_2^{(1)}\nu_3^{(1)}\nu_1^{(2)}\nu_3^{(2)}\nu_1^{(3)}\nu_2^{(3)})},$$

d'où

$$\mathcal{B}(Y) \leqslant (1 + o(1))\alpha_1 \log^6 Y \leqslant (1 + o(1))\alpha_1 \log^6 X,$$

et, finalement, grâce à (37) et (40), on a l'inégalité

$$\mathcal{K}(X) \leqslant (1 + o(1))\frac{6}{\pi^2}\alpha_0\alpha_1 X \log^6 X. \tag{41}$$

6. Fin de la démonstration du théorème.

En combinant (34) et (41), nous avons donc démontré

$$\mathcal{K}(X, X, X; 1, 1, 1; 1, 1, 1) = (1 + o(1))\alpha_3 X \log^6 X.$$

Une étude de la démonstration indique que la formule précédente s'étend à

$$\mathcal{K}(\frac{X}{a_1}, \frac{X}{a_2}, \frac{X}{a_3}; 1, 1, 1; 1, 1, 1) = (1 + o(1))\alpha_3 \frac{X}{\sqrt[3]{a_1a_2a_3}}. \log^6 X,$$

uniformément pour $a_i \leqslant \mathcal{L}^A$, avec A constante positive fixée –il suffit de remplacer X dans les parties droites des formules (17), (19) et (21) respectivement par $\dfrac{X}{a_3}$, $\dfrac{X}{\sqrt{a_2 a_3}}$ et $\dfrac{X}{\sqrt[3]{a_1 a_2 a_3}}$.

Plus généralement, on a l'égalité

$$\mathcal{K}:\left(\frac{X}{a_1},\frac{X}{a_2},\frac{X}{a_3};\mathbf{b};\mathbf{c}\right)=(1+o(1))\alpha_3\cdot\Theta(\mathbf{b};\mathbf{c})\frac{X}{\sqrt[3]{a_1a_2a_3}}\cdot\log^6 X,$$

uniformément pour les a_i, les b_i et les c_i inférieurs à $\mathcal{L}^A$. C'est ici qu'apparaît l'obstacle à un calcul explicite et assez aisé de la constante c_0 apparaissant en (1). En effet la constante $\Theta(\mathbf{b};\mathbf{c})$ est produit de facteurs locaux sur les diviseurs premiers des b_i, des c_j et de leurs p.g.c.d. Pour se rendre compte de la difficulté d'écriture– mais non de théorie–il est bon de voir ce que donne le début du calcul de $\Theta(\mathbf{b};\mathbf{c})$, par exemple, lors de la démonstration d'un analogue de (21) pour $\mathcal{K}(\frac{X}{a_1},\frac{X}{a_2},\frac{X}{a_3};\mathbf{b};\mathbf{c})$. La minoration (21) reste valable pour la quantité précédente à condition de

• remplacer X par $\dfrac{X}{\sqrt[3]{a_1a_2a_3}}$ (déjà signalé plus haut) ;

• sommer sur les d_i et les $\delta_i^{(j)}$ non seulement sous les contraintes (8), (9), (10), (12) et (16) mais aussi sous la condition

$$(d_1, c_1b_2b_3) = (d_2, b_1c_2b_3) = (d_3, b_1b_2c_3) = 1;$$

• remplacer le facteur $\psi(d_1d_2d_3)$ par

$$\psi(d_1d_2d_3)\cdot\frac{\psi_1(b_1b_2b_3)\psi_2((b_1,b_2))\psi_2((b_1,b_3))\psi_2((b_2,b_3))\psi_3((b_1,b_2,b_3))}{\psi_1((d_1,b_1))\psi_1((d_2,b_2))\psi_1((d_3,b_3))\psi_2^2((b_1,b_2,b_3))}.$$

Ce calcul et tous ceux qui conduisent à la valeur de $\Theta(\mathbf{b};\mathbf{c})$, utilisent des estimations de sommes de la forme

$$\sum_{\substack{n\leqslant y\\(n,u)=1}} f(vn),$$

(pour u et v entiers et pour f fonction complètement multiplicative telle que $1-\frac{A}{p}\leqslant f(p)\leqslant 1$) et des généralisations de telles sommes.

On retiendra uniquement la propriété que $\Theta(\mathbf{b};\mathbf{c})$ est compris entre 0 et 1 : c'est une conséquence directe de l'inégalité triviale

$$\mathcal{K}(\mathbf{X};\mathbf{b};\mathbf{c})\leqslant\mathcal{K}(\mathbf{X};1,1,1;1,1,1)$$

Par retour à la formule (6), on voit que, lorsque le triplet de nombres *square-full* (ℓ_1,ℓ_2,ℓ_3) premiers entre eux inférieurs à L est fixé, il y a, pour tout $\varepsilon>0$,

$O\big((\ell_1\ell_2\ell_3)^\varepsilon\big)$ 12-uplets de $(\mu_i^{(j)},\mathfrak{P}_i^{(j)})$ associés et qu'on a, avec les notations précédentes $a_1 = \ell_2\ell_3\mu_2^{(1)}\mu_3^{(1)}\overline{P_2^{(1)}}^3\overline{P_3^{(1)}}^3$, a_2 et a_3 définis de façon similaire (voir (7)), l'égalité

$$V^+(X) = \alpha_3.X.\log^6 X\sum_{\ell_i}\sum_{\mu_i^{(j)}}\sum_{\mathfrak{P}_i^{(j)}}\frac{\Gamma(\ell_i,\mu_i^{(j)},\mathfrak{P}_i^{(j)})}{\sqrt[3]{a_1a_2a_3}}(1+o(1))+o(X\mathcal{L}^6),$$

avec $\Gamma(\cdots)$ coefficient vérifiant $0<\Gamma(\cdots)\leqslant 1$ et avec, pour conditions de sommation, celles de (6).

La série sur les ℓ_i est alors convergente puisqu'elle est

$$\ll \sum_{\ell_i}\frac{(\ell_1\ell_2\ell_3)^\varepsilon}{\sqrt[3]{\ell_1^2\ell_2^2\ell_3^2}} \ll \prod_p\Big(1+\frac{p^\varepsilon}{p^{\frac{4}{3}}}+\frac{p^{2\varepsilon}}{p^2}+\cdots\Big)^3 \ll 1,$$

ce qui termine la preuve du Théorème.

Remarque 6.1. — On voit que l'analyse complexe ne fait qu'une timide apparition dans ce travail. Un usage plus important des séries de Dirichlet ne peut qu'améliorer les résultats ci–dessus, tant en précision qu'en généralité. C'est l'objet d'un article, actuellement en préparation, de R. de la Bretèche. Il sera alors intéressant de confronter ses méthodes et ses résultats avec ceux de Batyrev et Tschinkel.

Note ajoutée lors de la correction des épreuves : Quelques mois après l'acceptation de cet article, nous avons reçu un preprint de D.R.Heath–Brown et B.Z.Moroz, dans lequel les auteurs donnent, par une démarche différente mais toujours élémentaire, un équivalent asymptotique de la quantité $V(X)$ étudiée ici. Ils donnent même, explicitement, la valeur de la constante c_0. Le travail de R. de la Bretèche, évoqué ci-dessus, est publié dans le présent ouvrage.

Références

[B–T1] V. V. Batyrev and Y. Tschinkel : Manin's Conjecture for Toric Varieties ; J. of Algebraic Geometry (à paraître).

[B–T2] ______ : Height Zeta functions of Toric Varieties ; Preprint de l'Ecole Normale Supérieure, Mars 1996, (à paraître dans Manin's Festschrift).

[B–T3] ______ : Tamagawa Numbers of Polarized Algebraic Varieties (ce volume).

reçu le 14 octobre 1996

ETIENNE FOUVRY, Mathématique, Bâtiment 425, Université de Paris–Sud, F–91405 ORSAY Cedex, France • *E-mail :* `Etienne.Fouvry@math.u-psud.fr`

Astérisque
251, 1998, p. 51–77

SUR LE NOMBRE DE POINTS DE HAUTEUR BORNÉE D'UNE CERTAINE SURFACE CUBIQUE SINGULIÈRE

par

Régis de la Bretèche

Résumé. — En développant des méthodes d'intégration complexe, nous établissons une formule asymptotique très précise sur le nombre de points de hauteur inférieure à X d'une certaine variété torique. Nous estimons le cardinal

$$V(X) := \mathrm{card}\{(x,y,z,t) \in (\mathbb{N}\cap[1,X])^4 \; : \; (x,y,z,t)=1, \; xyz=t^3\}.$$

Soit $N(X) := (\log X)^{3/5}(\log_2 X)^{-1/5}$. Alors il existe un polynôme Q de $\mathbb{R}[X]$ de degré 6 et une constante $c>0$ tels que l'on ait, pour X tendant vers l'infini, l'estimation

$$V(X) = XQ(\log X) + O\big(X^{1-1/8}\exp\{-cN(X)\}\big)$$

De plus, le coefficient dominant de Q est égal à

$$\frac{1}{4\times 6!}\prod_p \Big\{\Big(1-\frac{1}{p}\Big)^7\Big(1+\frac{7}{p}+\frac{1}{p^2}\Big)\Big\}.$$

1. Introduction

1.1. Énoncé des résultats. — Soit l'ensemble

$$E_\infty := \big\{(x,y,z)\in(\mathbb{N}^*)^3 \; : \; (x,y,z)=1, \; \exists t\in\mathbb{N} \quad xyz=t^3\big\}.$$

L'objet de ce travail est d'évaluer asymptotiquement $V(X)$ le cardinal de l'ensemble des éléments de E_∞ de coordonnées toutes inférieures à X lorsque X tend vers l'infini. Nous avons le résultat suivant

Théorème 1.1. — *Il existe un polynôme Q de $\mathbb{R}[X]$ de degré 6 et une constante $c>0$ tels que l'on ait, pour X tendant vers l'infini, l'estimation*

$$V(X) = XQ(\log X) + O\big(X^{7/8}\exp\{-cN(X)\}\big) \tag{1}$$

où l'on a noté $N(X) := (\log X)^{3/5}(\log_2 X)^{-1/5}$. De plus, le coefficient dominant de Q est égal à

$$C = \frac{1}{4\times 6!}\prod_p \Big\{\Big(1-\frac{1}{p}\Big)^7\Big(1+\frac{7}{p}+\frac{1}{p^2}\Big)\Big\}. \tag{2}$$

***Classification mathématique par sujets* (2000). —** Primaire 14M25; secondaire 11M41.
***Mots clefs.* —** Variété torique, Conjecture de Manin, Fonction zêta des hauteurs, Fonction de compte.

On en déduit un prolongement de la fonction zêta des hauteurs associée. Soit $c(m) := \operatorname{card}\{(x,y,z) \in E_\infty \,:\, \max\{x,y,z\} = m\}$ et

$$Z(s) := \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

Théorème 1.2. — *La fonction* $s \to (s-1)^7 Z(s)$ *admet un prolongement holomorphe au demi plan* $\Re\mathrm{e}\, s > \frac{7}{8}$ *et prend la valeur* $6!C$ *en* 1.

Commençons par exposer les motivations de ce problème. Nous montrerons au paragraphe 1.3. que le cardinal considéré est directement lié au dénombrement des points d'une variété torique non lisse de hauteur inférieure à X.

1.2. Origine du problème en géométrie algébrique. — Les variétés toriques sont un sujet d'étude privilégié en géométrie algébrique qui fournit de nombreux exemples pour tester des théories plus générales et plus abstraites. C'est aussi le cas pour l'évaluation asymptotique du nombre de points de hauteur bornée par X. Cette question de dénombrement, qui est clairement naturelle, permet, comme nous le verrons, de faire apparaître des invariants géométriques très intéressants.

Soit k un corps de nombres et une variété V de Fano sur k à laquelle on associe une fonction $\mathbf{h}$ des hauteurs anticanoniquement. Manin a conjecturé que, pour U un ouvert de V convenablement choisi, on a

$$N_U(X) := \big|\{P \in U \,:\, \mathbf{h}(P) \leqslant X\}\big| \sim CX(\log X)^{r-1} \tag{3}$$

où r est le rang du groupe de Picard de V et C est une constante non nulle. La restriction à un ouvert U permet d'éviter les sous-variétés accumulatrices. Par exemple, dans le cas de variétés cubiques, on exclut les points des droites contenues dans V. Dans [BT96-1], Batyrev et Tschinkel ont montré qu'en fait cette conjecture n'est pas pertinente pour certaines variétés de Fano. Cependant, elle a été vérifiée pour plusieurs classes d'exemples.

Dans [P93], Peyre raffine cette conjecture en donnant une expression de C en fonction de la mesure de Tamagawa. Il vérifie sur certains exemples la valeur de C conjecturée. Dans [BT96-2], [BT98-1] & [BT98-2], Batyrev et Tschinkel établissent cette formule pour toutes les variétés toriques sur un corps quelconque. Ils utilisent une série de Dirichlet associée à ce comptage exprimée dans un langage adélique. Dans le cas de $k = \mathbb{Q}$, Salberger [S98] retrouve ce résultat pour toute variété torique définie par un éventail régulier et complet. De plus, il donne un terme d'erreur. Il utilise une méthode différente de celle de Batyrev et Tschinkel initiée par Schanuel [S79] et Peyre [P93]. Fouvry [F98] obtient aussi (3) dans le même cas particulier que l'auteur. Pour ce faire, par une étude arithmétique fine, il réussit à se ramener au dénombrement des triplets $(x,y,z) \in E_\infty$ mais satisfaisant aux conditions supplémentaires de coprimalité $(x,y) = (x,z) = (y,z) = 1$. Cela lui permet alors d'avoir une représentation paramétrique des solutions en $(x, y, z) = (u^3, v^3, w^3)$ avec $(u, v) = (u, w) = (v, w) = 1$ ce qui facilite grandement le dénombrement.

Nous nous proposons ici d'examiner le cas d'une variété torique particulière et de montrer sur cet exemple que l'on obtient une estimation bien plus précise que tous les résultats connus précédemment.

1.3. Énoncé du problème en terme de variété torique. — Une variété torique peut être construite, par une méthode de type combinatoire, à partir de la donnée d'un réseau N et un éventail Δ.

Définition 1.3. — Un éventail est un ensemble fini de cônes strictement convexes polyhédraux rationnels dans $N_{\mathbb{R}} = N \otimes \mathbb{R}$ satisfaisant à :

(i) Chaque cône de Δ contient l'origine,

(ii) Chaque face d'un cône de Δ est aussi un cône de Δ,

(iii) L'intersection de deux cônes de Δ est une face de chacun des deux cônes.

De plus,

(iv) Δ est dit complet si $N_{\mathbb{R}}$ est la réunion des cônes de Δ,

(v) Δ est dit régulier si chaque cône de Δ est engendré par une partie d'une $\mathbb{Z}$-base de N.

Lorsque Δ n'est pas régulier, il est possible de construire un *raffinement* Δ' de Δ (*i.e.* chaque cône de Δ est une réunion de cônes de Δ') tel que Δ' soit régulier et tel que l'application induite entre Δ' et Δ soit propre et birationnelle (voir le livre de Oda [O88], paragraphe 1.5). L'objectif d'une telle manipulation est de pouvoir se ramener à une variété torique $V_{\Delta'}$ non singulière.

On appelle rayon les cônes de dimension 1 et on note $\Delta(1)$ leur cardinal. Lorsque l'éventail est régulier et complet, cette construction permet aisément de calculer le rang du groupe de Picard $\operatorname{Pic}(V_\Delta)$:

$$\operatorname{rang}(\operatorname{Pic}(V_\Delta)) = \operatorname{card}\Delta(1) - \dim\Delta. \tag{4}$$

L'intérêt des variétés toriques est que leurs propriétés géométriques se traduisent en termes de donnée combinatoire qu'est l'éventail.

Expliquons brièvement la construction d'une variété torique à partir de son éventail Δ. Pour plus de détails, nous renvoyons le lecteur intéressé au livre de Fulton [Fu93]. Soit $\sigma \in \Delta$. Le cône dual détermine un semi-groupe S_σ. Celui-ci est engendré de manière finie, donc on peut lui associer une $\mathbb{Q}$-algèbre de groupe $\mathbb{Q}[S_\sigma]$. À une telle algèbre correspond une variété affine $U_\sigma := \operatorname{Spec}(\mathbb{Q}[S_\sigma])$. Si τ est une face de σ, alors S_σ est contenue dans S_τ et donc $\mathbb{Q}[S_\sigma]$ est une sous-algèbre de $\mathbb{Q}[S_\tau]$ ce qui fournit une application $U_\sigma \mapsto U_\tau$. Avec cette identification, ces variétés affines recollées forment une variété algébrique V_Δ.

Construisons maintenant *notre* variété. Soient le réseau $N = \mathbb{Z}^2$ et trois points de ce réseau $n_1 := (-1, 2)$, $n_2 := (-1, -1)$, $n_3 := (2, -1)$. On définit Δ l'éventail composé de trois rayons τ_i engendrés par les n_i et des trois cônes σ_i de dimension deux engendrés par n_{i+1}, n_{i+2}. Notons tout de suite que Δ n'est pas régulier donc la variété considérée n'est pas lisse.

Soit (e_1, e_2) la base du réseau N et (e_1^*, e_2^*) la base duale dans $\mathrm{Hom}(N, \mathbb{Z})$. Le semi-groupe S_{σ_1} est engendré par $e_1^* + 2e_2^*$, $2e_1^* + e_2^*$, $e_1^* + e_2^*$. On lui associe $\mathbb{Q}[S_{\sigma_1}]$ l'algèbre de groupe $\mathbb{Q}[XY^2, X^2Y, XY]$. On obtient ainsi

$$U_{\sigma_1} = \{(y, z, t) \in \mathbb{Q}^3 \text{ tel que } yz = t^3\}.$$

On fait la même chose pour les autres cônes de Δ. Les trois variétés affines U_{σ_i} obtenues correspondent en fait aux points de V_Δ tels que respectivement $x \neq 0$, $y \neq 0$, $z \neq 0$. Dans la géométrie algébrique moderne, une variété algébrique est vue comme un cas particulier de schéma. Ici on définit la variété V_Δ de $\mathbb{P}^3_{\mathbb{Q}}$ comme l'ensemble des zéros dans $\mathbb{P}_3(\mathbb{Q})$ du polynôme homogène de degré trois $XYZ - T^3$. Le plongement dans $\mathbb{P}_3(\mathbb{Q})$ définit la hauteur $h : h(P) = \max\{|x|, |y|, |z|, |t|\}$ lorsque le point P de $\mathbb{P}_3(\mathbb{Q})$ est représenté par (x, y, z, t) et $\mathrm{pgcd}(x, y, z, t) = 1$.

Soit U le sous-ensemble de V_Δ composé des quadruplets (x, y, z, t) tels que $xyz \neq 0$. On note que U est bien un ouvert pour la topologie de Zariski. De plus, U évite les sous-variétés accumulatrices de V_Δ. On peut donc affirmer la formule

$$V(X) = \tfrac{1}{4} N_U(X). \tag{5}$$

Cette transformation du problème en un dénombrement de points entiers positifs est faite d'une manière générale dans [S98].

Pour utiliser la formule (4), il faut que l'éventail soit régulier. Nous contruisons un raffinement de Δ qui le sera. On pose $n_4 := (1, 0)$ $n_5 := (0, 1)$, $n_6 := (-1, 1)$ $n_7 := (-1, 0)$, $n_8 := (0, -1)$ et $n_9 := (1, -1)$. L'éventail Δ' défini à partir des points n_i avec $i \in \{1, \ldots, 9\}$ est un raffinement de Δ qui est lui régulier et complet. On a alors

$$\mathrm{rang}(\mathrm{Pic}(V_{\Delta'})) = \mathrm{card}\,\Delta'(1) - \dim \Delta' = 9 - 2 = 7.$$

L'application induite de $V_{\Delta'}$ dans V_Δ, qui est un isomorphisme sur l'ouvert U, permet de se ramener à un compte sur une variété lisse. La hauteur donnée par le maximum des valeurs absolues des coordonnées correspond à une hauteur sur $V_{\Delta'}$ associée à un faisceau anticanonique. Le degré 6 du polynôme Q de la formule (1) est donc attendu. On remarque aussi que $4C$ est la constante conjecturée par Peyre [P93], obtenue par Batyrev et Tschinkel [BT98-2] et par Salberger [S98].

1.4. Présentation arithmétique du problème. — Notons Φ la fonction qui a un entier générique n associe $\Phi(n)$ le plus petit entier tel que $n\Phi(n)$ soit un cube. Cette fonction est multiplicative (*i.e.* $\Phi(mn) = \Phi(m)\Phi(n)$ pour tout entier m, n tels que $(m, n) = 1$) et peut être aussi définie par

$$\Phi(p) = p^2, \quad \Phi(p^2) = p, \quad \Phi(p^{\nu+3\mu}) = \Phi(p^\nu)$$

pour tout entier ν, μ et p premier.

Pour (x,y) fixé, il existe un entier z tel que $(x,y,z) \in E_\infty$ si et seulement si $(d, \Phi(xy)) = 1$ où l'on note $d = (x,y)$.

L'assertion $(x,y,z) \in E_\infty$ équivaut à

$$(d, \Phi(xy)) = 1, \quad \Phi(xy) \mid z, \quad \frac{z}{\Phi(xy)} \text{ est un cube}, \quad \Big(d, \frac{z}{\Phi(xy)}\Big) = 1. \tag{6}$$

On en déduit que $(x,y,z) \in E_\infty$ équivaut à

$$\begin{cases} v_p(x) + v_p(y) + v_p(z) \equiv 0 \pmod 3, \\ v_p(x)v_p(y)v_p(z) = 0 \end{cases} \quad \text{pour tout } p \text{ premier.} \tag{7}$$

Grâce à cette remarque, nous allons estimer

$$V(X) = \sum_{\substack{x \leqslant X \\ (x,y,z)\in E_\infty}} \sum_{y \leqslant X} \sum_{z \leqslant X} 1.$$

La démonstration du Théorème 1.1 consiste à appliquer de manière attentive des méthodes d'intégration complexe classiques en théorie des nombres. Cependant, nous cherchons à évaluer une somme triple. Quelques lemmes techniques d'intérêt intrinsèque sont alors nécessaires.

C'est un devoir et un plaisir d'exprimer ma gratitude à E. Fouvry, E. Peyre, P. Salberger, J.-L. Colliot-Thélène pour leurs conseils, leurs suggestions et leurs encouragements. Qu'ils soient ici remerciés !

2. Démonstration des résultats

2.1. Préliminaires. — Pour estimer $V(X)$, nous aurons besoin de la formule de Perron énoncée sous la forme suivante (voir le livre de Tenenbaum [T95], théorème II.2.3) :

***Lemme 2.1* (Une formule de Perron).** — *Soit $\mathcal{A}(s) := \sum_{n=1}^\infty a_n n^{-s}$ la série de Dirichlet associée à la suite $\{a_n\}_{n=1}^\infty$ d'abscisse de convergence σ_c. Sous les conditions $\kappa > \max\{0, \sigma_c\}$ et $X \geqslant 1$, on a*

$$\int_1^X \sum_{n \leqslant x} a_n \,\mathrm{d}x = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \mathcal{A}(s) \frac{X^{s+1}}{s(s+1)} \,\mathrm{d}s. \tag{8}$$

Le passage de l'évaluation de $\int_1^X \sum_{n \leqslant x} a_n \,\mathrm{d}x$ à celle de $\sum_{n \leqslant X} a_n$ est classique (voir le chapitre II.5 sur la méthode de Selberg–Delange du livre de Tenenbaum [T95]). En considérant la moyenne de la somme, on fait apparaître une intégrale absolument convergente dans la formule de Perron.

Posons

$$S(X_1, X_2, X_3) := \sum_{\substack{x \leqslant X_1 \\ (x,y,z)\in E_\infty}} \sum_{y \leqslant X_2} \sum_{z \leqslant X_3} 1$$

de sorte que $V(X) = S(X, X, X)$. Pour évaluer $S(X, X, X)$, nous considérons tout d'abord

$$M = M(X_1, X_2, X_3) := \int_1^{X_1} \int_1^{X_2} \int_1^{X_3} S(U, V, W)\, \mathrm{d}W\, \mathrm{d}V\, \mathrm{d}U \tag{9}$$

lorsque $X_1 \asymp X_2 \asymp X_3$ i. e. lorsqu'il existe des constantes absolues $c_1 > 0$ et $c_2 > 0$ telles que $c_1 X_1 \leqslant X_2, X_3 \leqslant c_2 X_1$.

Pour passer d'une évaluation de M à S, nous introduisons un opérateur $\mathfrak{T}$ défini de l'ensemble des fonctions à trois variables $\mathcal{E}_3$ dans l'ensemble des fonctions à deux variables $\mathcal{E}_2$:

$$(\mathfrak{T}f) : \quad \begin{aligned} (\mathfrak{T}f)(X, Y) \; := \; & f(Y, Y, Y) - f(X, Y, Y) \\ & - f(Y, X, Y) - f(Y, Y, X) \\ & + f(Y, X, X) + f(X, Y, X) \\ & + f(X, X, Y) - f(X, X, X). \end{aligned}$$

L'introduction de cet outil permet de minimiser le terme d'erreur dans S. Nous avons rassemblé dans le lemme suivant les propriétés de $\mathfrak{T}$ dont nous aurons besoin.

Lemme 2.2. — (i) *Soit une fonction $f \in \mathcal{E}_3$ de classe C^3. On a*

$$\begin{aligned} (\mathfrak{T}f)(X, Y) &= \int_X^Y \int_X^Y \int_X^Y \frac{\partial^3 f}{\partial X_1 \partial X_2 \partial X_3}(X_1, X_2, X_3)\, \mathrm{d}X_1\, \mathrm{d}X_2\, \mathrm{d}X_2 \\ &\ll |Y - X|^3 \sup_{\substack{X_1 \in [X,Y] \\ X_2, X_3 \in [X,Y]}} \Big| \frac{\partial^3 f}{\partial X_1 \partial X_2 \partial X_3}(X_1, X_2, X_3) \Big|. \end{aligned}$$

(ii) *Soit une fonction $f \in \mathcal{E}_3$ de classe C^4. On a*

$$\begin{aligned} (\mathfrak{T}f)(X, Y) = (Y - X)^3 & \frac{\partial^3 f}{\partial X_1 \partial X_2 \partial X_3}(X, X, X) \\ & + O\bigg(|Y - X|^4 \sum_{\{i,j,k\} = \{1,2,3\}} \sup_{\substack{X_1 \in [X,Y] \\ X_2, X_3 \in [X,Y]}} \Big| \frac{\partial^4 f}{\partial X_i^2 \partial X_j \partial X_k}(X_1, X_2, X_2) \Big| \bigg). \end{aligned}$$

(iii) *Soient f_1, f_2, f_3 des fonctions d'une seule variable et $f \in \mathcal{E}_3$ définie par $f(X_1, X_2, X_3) = f_1(X_1) f_2(X_2) f_3(X_3)$. On a*

$$(\mathfrak{T}f)(X, Y) = (f_1(Y) - f_1(X))(f_2(Y) - f_2(X))(f_3(Y) - f_3(X)).$$

(iv) *Pour $H \leqslant X$, on a*

$$(\mathfrak{T}M)(X-H,X) \leqslant H^3 S(X,X,X) \leqslant (\mathfrak{T}M)(X,X+H).$$

Démonstration. — Les trois premiers points reposent sur des notions de base de calcul différentiel. Le quatrième se déduit de la croissance de $S(X_1,X_2,X_3)$ par rapport à chaque variable et de la formule

$$(\mathfrak{T}M)(X,X+H) = \int_X^{X+H}\int_X^{X+H}\int_X^{X+H} S(U,V,W)\,\mathrm{d}W\,\mathrm{d}V\,\mathrm{d}U.$$

□

Proposition 2.3. — *Pour $i=1,2,3$, soient $X_i \in [1,\infty[$. Soit*

$$H := \max_i\{|X_i - X|\}.$$

Il existe un polynôme P de $\mathbb{R}[X_1,X_2,X_3]$ de degré total égal à 6, une constante $c>0$ et une fonction $R \in \mathcal{E}_3$ tels que l'on ait l'estimation

$$\begin{aligned} M(X_1,X_2,X_3) = {} & X_1^{4/3}X_2^{4/3}X_3^{4/3}P(\log X_1,\log X_2,\log X_3) \\ & + R(X_1,X_2,X_3) \end{aligned} \tag{10}$$

avec pour $\varepsilon>0$ suffisamment petit et $1 \leqslant H \leqslant \frac{1}{2}X$

$$\left.\begin{matrix}(\mathfrak{T}R)(X,X+H)\\ (\mathfrak{T}R)(X-H,X)\end{matrix}\right\} \ll X^{4-1/2-\varepsilon} + H^4(\log X)^7 + H^3X^{7/8}\exp\{-cN(X)\}. \tag{11}$$

De plus, la somme des coefficients des monômes de P de degré total égale à 6 est égale à $\left(\frac{3}{4}\right)^3 C$ où C est la constante définie dans l'énoncé du Théorème 1.1.

Remarque 2.4. — On peut montrer, sans avoir recours à l'opérateur $\mathfrak{T}$, que

$$R(X_1,X_2,X_3) \ll X^{7/8}\exp\{-cN(X)\}$$

lorsque $X_1 \asymp X_2 \asymp X_3 \asymp X$. Mais cela ne donnerait qu'une forme affaiblie du Théorème 1 avec un exposant $\frac{31}{32}$ au lieu de $\frac{7}{8}$. Cela justifie l'utilisation de l'opérateur $\mathfrak{T}$.

En utilisant le Lemme 2.2 nous montrons qu'il suffit d'établir la Proposition 2.3 pour démontrer le Théorème 1.1.

Lemme 2.5. — *La Proposition 2.3 implique le Théorème 1.1 avec Q défini par la relation*

$$XQ(\log X) = \frac{\partial^3}{\partial X_1\partial X_2\partial X_3}\big(X_1^{4/3}X_2^{4/3}X_3^{4/3}P(\log X_1,\log X_2,\log X_3)\big)_{X_\ell=X}. \tag{12}$$

De plus, on vérifie que le coefficient dominant de Q est égal à C où C est la constante définie dans l'énoncé du Théorème.

Démonstration du Lemme 2.5. — En utilisant le Lemme 2.2 et la Proposition 2.3, on obtient

$$\begin{aligned}(\mathfrak{T}M)(X,X+H)&\\ &= H^3\frac{\partial^3}{\partial X_1\partial X_2\partial X_3}\big(X_1^{4/3}X_2^{4/3}X_3^{4/3}P(\log X_1,\log X_2,\log X_3)\big)_{X_\ell=X}\\ &\quad+(\mathfrak{T}R)(X,X+H)+O(H^4(\log X)^6).\end{aligned}$$

Au vu de (12) et (11), on a pour $\varepsilon>0$ suffisamment petit

$$\begin{aligned}\frac{(\mathfrak{T}M)(X,X+H)}{H^3}&-XQ(\log X)\\ &=O\Big(H(\log X)^7+(X/H)^3X^{1-1/2-\varepsilon}+X^{7/8}\exp\{-cN(X)\}\Big).\end{aligned}$$

On a la même estimation pour $(\mathfrak{T}M)(X-H,X)/H^3$. Le point (iv) du Lemme 2.2 fournit alors

$$\begin{aligned}S(X,X,X)-XQ(\log X)\ll H(\log X)^7&+(X/H)^3X^{1-1/2-\varepsilon}\\ &+X^{7/8}\exp\{-cN(X)\}.\end{aligned}$$

On conclut la démonstration en prenant

$$H=\frac{X^{7/8}}{(\log X)^7}\exp\{-cN(X)\}.\quad\square$$

2.2. Application de la formule de Perron. — Nous commençons l'évaluation de M en appliquant successivement trois fois la formule de Perron.

Lemme 2.6. — *Lorsque* $\frac12X\leqslant X_i\leqslant\frac32X$, $T_i\geqslant1$ *et* $\kappa_i-\frac13\geqslant1/\log X$, *on a*

$$\begin{aligned}M=&\frac{1}{(2\pi i)^3}\int_{\kappa_1-iT_1}^{\kappa_1+iT_1}\int_{\kappa_2-iT_2}^{\kappa_2+iT_2}\int_{\kappa_3-iT_3}^{\kappa_3+iT_3}F(s_1,s_2,s_3)\frac{X_1^{s_1+1}X_2^{s_2+1}X_3^{s_3+1}\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1}{s_1(s_1+1)s_2(s_2+1)s_3(s_3+1)}\\ &+O\Big(X^4\frac{(\log X)^9}{\min_i T_i}\Big)\end{aligned}\tag{13}$$

avec

$$\begin{aligned}F(s_1,s_2,s_3)=&\,\zeta(3s_1)\zeta(3s_2)\zeta(3s_3)\\ &\times\prod_p\Big(1+\sum_{\{i,j,k\}=\{1,2,3\}}(1-p^{-3s_k})p^{-2s_i-s_j}-p^{-3s_1-3s_2-3s_3}\Big).\end{aligned}\tag{14}$$

Remarque 2.7. — Le terme d'erreur de (13) sera englobé dans celui de la Proposition 2.3 dès que $T\geqslant X^{1/2+\varepsilon}$.

La fonction F est étudiée à la sous-section 3.2.

Démonstration du Lemme 2.6. — Nous calculons tout d'abord

$$F(s_1, s_2, s_3) = \sum_{(x,y,z)\in E_\infty} \frac{1}{x^{s_1} y^{s_2} z^{s_3}}$$

pour $\Re e\, s_i > 1/3$ où $i = 1, 2, 3$. On remarque que cette somme est absolument convergente lorsque $\Re e\, s_i \geqslant \kappa > 1/3$. En effet, on a

$$\sum_{(x,y,z)\in E_\infty} \Big| \frac{1}{x^{s_1} y^{s_2} z^{s_3}} \Big| \leqslant \sum_{t=1}^{\infty} \frac{d_3(t^3)}{t^{3\kappa}} \ll_\kappa 1$$

où l'on a utilisé la notation classique $d_3(t) := \mathrm{card}\{(t_1, t_2, t_3) \in \mathbb{N}^3 \,:\, t_1 t_2 t_3 = t\}$.

L'équivalence (7) permet d'écrire F sous la forme d'un produit eulérien. Notant

$$J := \big\{(n_1, n_2, n_3) \in \mathbb{N}^3 \,:\, n_1 + n_2 + n_3 \equiv 0 \,(\mathrm{mod}\, 3), \qquad n_1 n_2 n_3 = 0\big\},$$

on a

$$F(s_1, s_2, s_3) = \prod_p \Big(\sum_{(n_1,n_2,n_3)\in J} p^{-n_1 s_1 - n_2 s_2 - n_3 s_3} \Big).$$

Pour calculer la somme sur J, on scinde cet ensemble en une partition de dix ensembles. Nous posons

$$\begin{aligned}
J_{i,j,k} &:= \big\{(n_1, n_2, n_3) \in \mathbb{N}^3 \,:\, n_i \equiv 2 \,(\mathrm{mod}\, 3),\ n_j \equiv 1 \,(\mathrm{mod}\, 3),\ n_k = 0\big\},\\
J'_0 &:= \big\{(n_1, n_2, n_3) \in \mathbb{N}^3 \,:\, n_3 \in 3\mathbb{N},\ n_1 = 0,\ n_2 = 0\big\},\\
J'_1 &:= \big\{(n_1, n_2, n_3) \in \mathbb{N}^3 \,:\, n_1 \in 3\mathbb{N}^*,\ n_3 \in 3\mathbb{N},\ n_2 = 0\big\},\\
J'_2 &:= \big\{(n_1, n_2, n_3) \in \mathbb{N}^3 \,:\, n_2 \in 3\mathbb{N}^*,\ n_3 \in 3\mathbb{N},\ n_1 = 0\big\},\\
J'_3 &:= \big\{(n_1, n_2, n_3) \in \mathbb{N}^3 \,:\, n_1 \in 3\mathbb{N}^*,\ n_2 \in 3\mathbb{N}^*,\ n_3 = 0\big\}.
\end{aligned}$$

Un simple calcul fournit

$$\begin{aligned}
\sum_{(n_1,n_2,n_3)\in J_{i,j,k}} p^{-n_1 s_1 - n_2 s_2 - n_3 s_3} &= \frac{p^{-2s_i - s_j}}{(1 - p^{-3s_i})(1 - p^{-3s_j})},\\
\sum_{(n_1,n_2,n_3)\in J'_0} p^{-n_1 s_1 - n_2 s_2 - n_3 s_3} &= \frac{1}{(1 - p^{-3s_3})},\\
\sum_{(n_1,n_2,n_3)\in J'_1} p^{-n_1 s_1 - n_2 s_2 - n_3 s_3} &= \frac{p^{-3s_1}}{(1 - p^{-3s_1})(1 - p^{-3s_3})},\\
\sum_{(n_1,n_2,n_3)\in J'_2} p^{-n_1 s_1 - n_2 s_2 - n_3 s_3} &= \frac{p^{-3s_2}}{(1 - p^{-3s_2})(1 - p^{-3s_3})},\\
\sum_{(n_1,n_2,n_3)\in J'_3} p^{-n_1 s_1 - n_2 s_2 - n_3 s_3} &= \frac{p^{-3s_1 - 3s_2}}{(1 - p^{-3s_1})(1 - p^{-3s_2})}.
\end{aligned}$$

En faisant la somme de ces quantités, nous obtenons

$$\sum_{(n_1,n_2,n_3)\in J} p^{-n_1 s_1 - n_2 s_2 - n_3 s_3}$$
$$= \frac{1 + \sum_{\{i,j,k\}=\{1,2,3\}} (1-p^{-3s_k})p^{-2s_i - s_j} - p^{-3s_1-3s_2-3s_3}}{(1-p^{-3s_1})(1-p^{-3s_2})(1-p^{-3s_3})},$$

ce qui fournit l'expression voulue de $F(s_1, s_2, s_3)$.

Pour $\Re e\, s_i = \kappa_i$, $(i \in \{1,2,3\})$ vérifiant les hypothèses du lemme, on a la majoration

$$|F(s_1, s_2, s_3)| \ll (\log X)^9. \tag{15}$$

En appliquant trois fois la formule de Perron, nous obtenons

$$M = \frac{1}{(2\pi i)^3} \int_{\kappa_1 - i\infty}^{\kappa_1 + i\infty} \int_{\kappa_2 - i\infty}^{\kappa_2 + i\infty} \int_{\kappa_3 - i\infty}^{\kappa_3 + i\infty} F(s_1, s_2, s_3) \frac{X_1^{s_1+1} X_2^{s_2+1} X_3^{s_3+1}\, \mathrm{d}s_3\, \mathrm{d}s_2\, \mathrm{d}s_1}{s_1(s_1+1)s_2(s_2+1)s_3(s_3+1)}.$$

On déduit de (15) l'estimation du Lemme 2.6. □

Nous appliquons le Lemme 2.6 en prenant

$$\begin{gathered} \kappa_1 = \tfrac{1}{3} + 25/\log X, \quad \kappa_2 = \tfrac{1}{3} + 5/\log X, \quad \kappa_3 = \tfrac{1}{3} + 1/\log X, \\ T_1 = T, \quad T_2 = 3T, \quad T_3 = 9T, \quad \tfrac{1}{2}X \leq X_i \leqslant \tfrac{3}{2}X. \end{gathered} \tag{16}$$

Dans toute la suite de ce travail, nous utilisons un paramètre $\varepsilon > 0$ aussi petit qu'il est nécessaire. On note $\delta := \frac{1}{4} - \varepsilon \in]0, \frac{1}{4}[$. Nous utiliserons aussi la notation classique

$$\sigma_i = \Re e\, s_i, \quad \tau_i = \Im m\, s_i.$$

Pour estimer l'intégrale triple du Lemme 2.6, nous décalons successivement vers la gauche les droites d'intégration en appliquant le théorème des résidus. Nous avons à faire face à deux sortes de problèmes. L'un est de calculer les résidus des pôles rencontrés ; l'autre est de montrer que la contribution de certains contours peut être englobée par le terme d'erreur (11).

3. Application du théorème des résidus

3.1. Préliminaires. — Nous aurons besoin d'une majoration de la fonction ζ de Riemann dans la bande $\Re e\, s \in [0, 1]$.

Lemme 3.1. — (i) *Soit $\varepsilon > 0$. Pour $\sigma \in [\frac{1}{2}, +\infty[$ et $|\sigma + i\tau - 1| \geqslant 1/\log X$, on a*

$$|\zeta(\sigma + i\tau)| \ll_\varepsilon (\log X)(1 + |\tau|)^{\max\{(1-\sigma)/3+\varepsilon, 0\}}. \tag{17}$$

Pour $\sigma \in [0, \frac{1}{2}]$, on a

$$|\zeta(\sigma + i\tau)| \ll_\varepsilon (1 + |\tau|)^{\frac{(3-4\sigma)}{6}+\varepsilon}. \tag{18}$$

(ii) *Il existe une constante* $c_0 > 0$ *telle que la fonction* ζ *n'admette pas de zéro dans la région définie par*

$$\sigma \geqslant 1 - c_0(\log(|\tau| + 3))^{-2/3}(\log_2(|\tau| + 3))^{-1/3}.$$

Dans cette région, on a

$$1/\zeta(s) \ll (\log(|\tau| + 3))^{2/3}(\log_2(|\tau| + 3))^{1/3}.$$

Pour une démonstration du Lemme 3.1(i), nous renvoyons le lecteur au livre de Tenenbaum [T95], théorème II.3.6. On peut trouver une démonstration détaillée de la majoration de Korobov et Vinogradov énoncée au point (ii) dans le chapitre 6 du livre d'Ivić [I85].

Nous énonçons, sous la forme d'un lemme, un résultat de nature purement technique afin de pouvoir s'y référer ultérieurement

Lemme 3.2. — (i) *Soit* $\beta \in [0, 1]$ *et* $\varepsilon > 0$. *On a pour* $1 \leqslant H \leqslant X$ *et* $\sigma \in [0, 2]$ *la majoration*

$$\big|(X + H)^{\sigma+i\tau+1} - X^{\sigma+i\tau+1}\big| \ll X^{\sigma+1}\Big((|\tau| + 1)\frac{H}{X}\Big)^{\beta}. \tag{19}$$

(ii) *Soient* $a \geqslant 0$, $b \geqslant 0$ *tels que* $\frac{1}{4}a + 2b > 1$. *Il existe* $\varepsilon_0 > 0$ *tel que pour chaque* $\varepsilon \in]0, \varepsilon_0]$ *on ait uniformément pour* $1 \leqslant H \leqslant X$ *la majoration*

$$\Big(\frac{H}{X}\Big)^{a} X^{-b} \ll \Big(\frac{H}{X}\Big)^{4} + X^{-1/2-\varepsilon}. \tag{20}$$

Démonstration. — La partie à majorer est inférieure à la fois à $X^{\sigma+1}$ et à

$$X^{\sigma+1}(|\tau| + 1)(H/X).$$

En prenant le produit du premier majorant pris à la puissance $1 - \beta$ et du second pris à la puissance β, on obtient le majorant annoncé.

La majoration (20) peut se démontrer en distinguant le cas $X^{-1/2} \leqslant (H/X)^4$ et le cas $(H/X)^4 \leqslant X^{-1/2}$. □

3.2. Étude de $F(s_1, s_2, s_3)$**.** — Soit G la fonction définie implicitement par

$$\begin{aligned} F(s_1, s_2, s_3) = {} & \zeta(3s_3)\zeta(s_3 + 2s_2)\zeta(2s_3 + s_2)\zeta(s_3 + 2s_1)\zeta(2s_3 + s_1) \\ & \times \zeta(3s_2)\zeta(3s_1)\zeta(s_2 + 2s_1)\zeta(2s_2 + s_1)G(s_1, s_2, s_3). \end{aligned} \tag{21}$$

On peut écrire

$$G(s_1, s_2, s_3) = \prod_p \big(1 + \Delta(p^{-s_1}, p^{-s_2}, p^{-s_3})\big) \tag{22}$$

où Δ est un polynôme en trois variables qui est une somme de monômes qui sont chacun de degré total supérieur à 6. À partir d'une expression de Δ, on affirme que la fonction G est une fonction holomorphe en s_3 pour $\Re e\, s_3 > \frac{1}{12}$ pour s_1 et s_2 fixés tels que $\Re e\, s_1 = \kappa_1$, $\Re e\, s_2 = \kappa_2$. La fonction $s_3 \mapsto F(s_1, s_2, s_3)$ est donc méromorphe sur le demi-plan $\Re e\, s_3 > \frac{1}{12}$ et admet cinq pôles :

$$\begin{aligned} u_1(s_1,s_2) := \tfrac{1}{3},\, u_2(s_1,s_2) := \tfrac{1}{2} - \tfrac{1}{2}s_2,\ u_3(s_1,s_2) := \tfrac{1}{2} - \tfrac{1}{2}s_1, \\ u_4(s_1,s_2) := 1 - 2s_2,\ u_5(s_1,s_2) := 1 - 2s_1, \end{aligned} \tag{23}$$

d'ordre de multiplicité égal à 1. Nos hypothèses sur κ_2 et κ_1 permettent d'affirmer que tous les pôles u_i sont distincts. Il est aussi intéressant de noter que

$$F(s_1, s_2, s_3) = F(s_i, s_j, s_k)$$

lorsque $\{i, j, k\} = \{1, 2, 3\}$. La constante

$$G(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}) = \prod_p \Big\{ \Big(1 - \frac{1}{p}\Big)^7 \Big(1 + \frac{7}{p} + \frac{1}{p^2}\Big) \Big\} \tag{24}$$

est le produit eulérien apparaissant dans la définition de la constante C du Théorème 1.1.

On aura également besoin d'une bonne majoration de $F(s_1, s_2, s_3)$ sur les lignes verticales. Le Lemme 3.1 fournit lorsque

$$\sigma_1 = \kappa_1, \quad \sigma_2 = \kappa_2, \quad \sigma_3 \in [\tfrac{1}{3} - \delta, \kappa_3], \quad |s_3 - u_i(s_1, s_2)| \geqslant 1/\log X,$$

la majoration

$$|F(s_1, s_2, s_3)| \ll (\log X)^9 (1 + |\tau_3|)^{\frac{1}{3}} \big(1 + |\tau_3| + |\tau_1|\big)^{\frac{1}{4}} (1 + |\tau_3| + |\tau_2|)^{\frac{1}{4}}. \tag{25}$$

3.3. Première application du théorème des résidus. — Nous appliquons une première fois le théorème des résidus pour estimer la quantité $M(X_1, X_2, X_3)$.

Lemme 3.3. — *Nous nous plaçons sous les hypothèses* (16). *Il existe une fonction* $R_0 \in \mathcal{E}_3$ *vérifiant*

$$\left.\begin{aligned} (\mathfrak{T}R_0)(X, X+H) \\ (\mathfrak{T}R_0)(X-H, X) \end{aligned}\right\} \ll X^{4-1/2-\varepsilon} + H^4. \tag{26}$$

telle que

$$\begin{aligned} \frac{1}{(2\pi i)^3} \int_{\kappa_1 - iT}^{\kappa_1 + iT} \int_{\kappa_2 - 3iT}^{\kappa_2 + 3iT} \int_{\kappa_3 - 9iT}^{\kappa_3 + 9iT} F(s_1, s_2, s_3) \frac{X_1^{s_1+1} X_2^{s_2+1} X_3^{s_3+1}\, \mathrm{d}s_3\, \mathrm{d}s_2\, \mathrm{d}s_1}{s_1(s_1+1)s_2(s_2+1)s_3(s_3+1)} \\ = \sum_{i=1}^{5} M_i(X_1, X_2, X_3) + R_0(X_1, X_2, X_3) + O\Big(X^4 \frac{(\log X)^9}{T^{1/6}}\Big), \end{aligned} \tag{27}$$

avec

$$\begin{cases} F_1(s_1,s_2) & := \frac{1}{3}\zeta(\frac{1}{3}+2s_2)\zeta(\frac{2}{3}+s_2)\zeta(3s_2)\zeta(\frac{1}{3}+2s_1)\zeta(\frac{2}{3}+s_1)\zeta(3s_1) \\ & \quad \times \zeta(s_2+2s_1)\zeta(2s_2+s_1)G(s_1,s_2,\frac{1}{3}), \\ F_2(s_1,s_2) & := \frac{1}{2}\zeta(-\frac{3}{2}s_2+\frac{3}{2})\zeta(s_1-s_2+1)\zeta(\frac{3}{2}s_2+\frac{1}{2})\zeta(2s_1-\frac{1}{2}s_2+\frac{1}{2}) \\ & \quad \times \zeta(3s_2)\zeta(3s_1)\zeta(s_2+2s_1)\zeta(2s_2+s_1)G(s_1,s_2,\frac{1}{2}-\frac{1}{2}s_2), \\ F_3(s_1,s_2) & := F_2(s_2,s_1), \\ F_4(s_1,s_2) & := 2F_2(s_1,1-2s_2), \\ F_5(s_1,s_2) & := F_4(s_2,s_1) = 2F_2(s_2,1-2s_1), \end{cases}$$

et

$$\begin{aligned} & M_i(X_1,X_2,X_3) \\ & \quad := \frac{1}{(2\pi i)^2}\int_{\kappa_1-iT}^{\kappa_1+iT}\int_{\kappa_2-3iT}^{\kappa_2+3iT} \frac{F_i(s_1,s_2)X_3^{u_i(s_1,s_2)+1}X_2^{s_2+1}X_1^{s_1+1}\,\mathrm{d}s_2\,\mathrm{d}s_1}{u_i(s_1,s_2)(u_i(s_1,s_2)+1)s_2(s_2+1)s_1(s_1+1)}. \end{aligned}$$

Remarque 3.4. — Par commodité, nous ne préciserons pas dans la démonstration la dépendance en X_1, X_2, X_3 des $M_i(X_1, X_2, X_3)$.

Démonstration. — Nous décalons la droite d'intégration de l'abscisse κ_3 jusqu'à l'abscisse $\frac{1}{3}-\delta$. Nous appliquons le théorème des résidus au contour $C_0(\kappa_3)$ joignant les points d'affixe

$$\kappa_3 - 9iT,\ \kappa_3 + 9iT,\ \tfrac{1}{3} - \delta + 9iT,\ \tfrac{1}{3} - \delta - 9iT,\ \kappa_3 - 9iT.$$

Pour majorer la contribution sur les droites horizontales, on utilise la formule issue de (25)

$$|F(s_1,s_2,s_3)| \ll (\log X)^9(1+T)^{\frac{5}{6}}$$

valable lorsque

$$\sigma_1 = \kappa_1,\ \sigma_2 = \kappa_2,\ \sigma_3 \in [\tfrac{1}{3}-\delta,\kappa_3],\ |\tau_3| \leqslant 9T,\ |\tau_2| \leqslant 3T,\ |\tau_1| \leqslant T.$$

Il vient

$$\begin{aligned} \frac{1}{(2\pi i)^3}\int_{\kappa_1-iT}^{\kappa_1+iT}\int_{\kappa_2-3iT}^{\kappa_2+3iT}\int_{\kappa_3-9iT}^{\kappa_3+9iT} F(s_1,s_2,s_3)\frac{X_1^{s_1+1}X_2^{s_2+1}X_3^{s_3+1}\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1}{s_1(s_1+1)s_2(s_2+1)s_3(s_3+1)} \\ = \sum_{i=1}^{5} M_i(X_1,X_2,X_3) + R_0(X_1,X_2,X_3) + O\Big(X^4\frac{(\log X)^9}{T^{1/6}}\Big), \end{aligned}$$

où l'on a posé

$$R_0(X_1, X_2, X_3)$$

$$= \frac{1}{(2\pi i)^3} \int_{\kappa_1 - iT}^{\kappa_1 + iT} \int_{\kappa_2 - 3iT}^{\kappa_2 + 3iT} \int_{\frac{1}{3} - \delta - 9iT}^{\frac{1}{3} - \delta + 9iT} F(s_1, s_2, s_3) \frac{X_1^{s_1+1} X_2^{s_2+1} X_3^{s_3+1} \, \mathrm{d}s_3 \, \mathrm{d}s_2 \, \mathrm{d}s_1}{s_1(s_1+1)s_2(s_2+1)s_3(s_3+1)}.$$

Il reste à établir (26). Le Lemme 2.2(ii) implique

$$(\mathfrak{T} R_0)(X, X+H)$$

$$= \frac{1}{(2\pi i)^3} \int_{\kappa_1 - iT}^{\kappa_1 + iT} \int_{\kappa_2 - 3iT}^{\kappa_2 + 3iT} \int_{\frac{1}{3} - \delta - 9iT}^{\frac{1}{3} - \delta + 9iT} F(s_1, s_2, s_3) \frac{r_{X,X+H}(s_1, s_2, s_3) \, \mathrm{d}s_3 \, \mathrm{d}s_2 \, \mathrm{d}s_1}{s_1(s_1+1)s_2(s_2+1)s_3(s_3+1)}$$

avec

$$r_{X,X+H}(s_1, s_2, s_3) := \prod_{i \in \{1,2,3\}} \left((X+H)^{s_i+1} - X^{s_i+1}\right).$$

La majoration (25) de F peut s'écrire sous la forme

$$F(s_1, s_2, s_3) \ll (\log X)^9 \sum_j \prod_{i=1,2,3} (|\tau_i| + 1)^{\alpha_{i,j}}$$

où les coefficients $\alpha_{i,j}$ vérifient

$$\alpha_{i,j} \in [0, 1[, \qquad \max_j \Big\{ \sum_i \alpha_{i,j} \Big\} = \tfrac{5}{6}.$$

On en déduit

$$(\mathfrak{T} R_0)(X, X+H) \ll (\log X)^9$$

$$\times \sum_j \int_{-T}^{T} \int_{-3T}^{3T} \int_{-9T}^{9T} \frac{\left| r_{X,X+H}(\kappa_1 + i\tau_1, \kappa_2 + i\tau_2, \kappa_3 + i\tau_3) \right|}{\prod_{i=1,2,3} \big(|\tau_i| + 1 \big)^{2-\alpha_{i,j}}} \, \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 \, \mathrm{d}\tau_3.$$

La majoration

$$\left| r_{X,X+H}(\kappa_1 + i\tau_1, \kappa_2 + i\tau_2, \kappa_3 + i\tau_3) \right|$$

$$\ll X^{4-\delta} \prod_{i=1,2,3} \big((|\tau_i| + 1)(H/X) \big)^{1-\alpha_{i,j}}$$

issue du Lemme 3.2(i) implique alors

$$\begin{aligned} (\mathfrak{T} R_0)(X, X+H) &\ll X^{4-\delta} (\log X)^9 (\log T)^3 (H/X)^{3 - \max_j \{\sum_i \alpha_{i,j}\}} \\ &\ll X^{4-1/4+2\varepsilon} (H/X)^{\frac{13}{6}}. \end{aligned}$$

Grâce au Lemme 3.2(ii), il vient

$$(\mathfrak{T}R_0)(X, X+H) \ll X^{4-\frac{1}{2}-\varepsilon} + H^4$$

puisque pour ε suffisamment petit $\frac{1}{4}(\frac{13}{6}) + 2(\frac{1}{4} - 2\varepsilon) > 1$. Ceci achève la démonstration du Lemme 3.3. $\square$

Il reste à évaluer les quantités $M_i = M_i(X, X_2, X_3)$.

3.4. Estimation de $M_1(X_1, X_2, X_3)$. — Notons

$$G_1(s_1, s_2) := G(s_1, s_2, \tfrac{1}{3}) \quad \text{et} \quad H_1(s_1) := \zeta(\tfrac{1}{3} + 2s_1)\zeta(\tfrac{2}{3} + s_1)\zeta(3s_1)$$

de sorte que

$$F_1(s_1, s_2) = \frac{1}{3} H_1(s_1) H_1(s_2) \zeta(s_2 + 2s_1)\zeta(2s_2 + s_1) G_1(s_1, s_2).$$

La fonction $s_2 \to G_1(s_1, s_2)$ est une fonction holomorphe dans le demi plan $\Re e\, s_2 > \frac{1}{12}$ lorsque $\Re e\, s_1 \geqslant \frac{1}{3}$ et uniformément bornée lorsque $\Re e\, s_2 \geqslant \frac{1}{3} - \delta$. La fonction $s_2 \mapsto F_1(s_1, s_2)$ est donc méromorphe sur le demi-plan $\Re e\, s_2 > \frac{1}{12}$ lorsque $\Re e\, s_1 > \frac{1}{3}$ et admet trois pôles : $1 - 2s_1$, $\frac{1}{2} - \frac{1}{2}s_1$ d'ordre de multiplicité égal à 1, et $\frac{1}{3}$ d'ordre de multiplicité égal à 3. Il est aussi intéressant de noter que $F_1(s_1, s_2) = F_1(s_2, s_1)$.

Des deux premières majorations du Lemme 3.1, on déduit la majoration

$$|F_1(s_1, s_2)| \ll (\log X)^8 (1 + |\tau_2|)^{\frac{7}{12}} (1 + |\tau_2| + |\tau_1|)^{\frac{1}{4}} \tag{28}$$

valable lorsque

$$\begin{aligned} &\Re e\, s_1 = \kappa_1,\ \Re e\, s_2 \in [\tfrac{1}{3} - \delta, \kappa_2], \\ &\min\{|s_2 - (1 - 2s_1)|, |s_2 - (\tfrac{1}{2} - \tfrac{1}{2}s_1)|, |s_2 - \tfrac{1}{3}|\} \geqslant 1/\log X. \end{aligned} \tag{29}$$

Décalons la droite d'intégration d'abscisse κ_2 jusqu'à l'abscisse $\frac{1}{3} - \delta$. Le théorème des résidus et la majoration (28) fournissent

$$M_1 = \tfrac{9}{4} \sum_{i=1}^{3} M_{1i}(X_1, X_2, X_3) + \tfrac{9}{4} R_{10}(X_1, X_2, X_3) + O\Big(\frac{X^4 (\log X)^9}{T^{1/6}}\Big) \tag{30}$$

avec

$$R_{10}(X_1, X_2, X_3) := \frac{X_3^{4/3}}{(2\pi i)^2} \int_{\kappa_1 - iT}^{\kappa_1 + iT} \int_{\frac{1}{3} - \delta - i3T}^{\frac{1}{3} - \delta + i3T} F_1(s_1, s_2) \frac{X_1^{s_1+1} X_2^{s_2+1}\, \mathrm{d}s_2\, \mathrm{d}s_1}{s_2(s_2+1)s_1(s_1+1)}$$

et

$$(31)\quad \begin{cases} M_{11}(X_1,X_2,X_3), \\ \qquad = \dfrac{X_3^{4/3}}{2\pi i}\displaystyle\int_{\kappa_1-iT}^{\kappa_1+iT} \operatorname*{Res}_{s_2=1-2s_1}\Big(\frac{F_1(s_1,s_2)X_2^{s_2+1}}{s_2(s_2+1)}\Big)\frac{X_1^{s_1+1}\,\mathrm{d}s_1}{s_1(s_1+1)}, \\ M_{12}(X_1,X_2,X_3), \\ \qquad = \dfrac{X_3^{4/3}}{2\pi i}\displaystyle\int_{\kappa_1-iT}^{\kappa_1+iT} \operatorname*{R}es_{s_2=1/2-s_1/2}\Big(\frac{F_1(s_1,s_2)X_2^{s_2+1}}{s_2(s_2+1)}\Big)\frac{X_1^{s_1+1}\,\mathrm{d}s_1}{s_1(s_1+1)}, \\ M_{13}(X_1,X_2,X_3) \\ \qquad = \dfrac{X_3^{4/3}}{2\pi i}\displaystyle\int_{\kappa_1-iT}^{\kappa_1+iT} \operatorname*{Res}_{s_2=1/3}\Big(\frac{F_1(s_1,s_2)X_2^{s_2+1}}{s_2(s_2+1)}\Big)\frac{X_1^{s_1+1}\,\mathrm{d}s_1}{s_1(s_1+1)}. \end{cases}$$

Nous majorons $\mathfrak{T}R_{10}(X,X+H)$ grâce à la majoration (28). D'après le Lemme 2.2 et le Lemme 3.2, on a

$$\begin{aligned}(\mathfrak{T}R_{10})(X,X+H) &\ll X^{4-\delta}(\log X)^9(\log T)^2(H/X)^{13/6}\\ &\ll X^4X^{-1/4+2\varepsilon}(H/X)^{13/6}.\end{aligned}$$

La deuxième partie du Lemme 3.2 permet d'affirmer que

$$(\mathfrak{T}R_{10})(X,X+H) \ll X^4\Big(X^{-1/4-\varepsilon}+(H/X)^4\Big)$$

puisque, pour ε suffisamment petit, on a l'inégalité $\frac14\big(\frac{13}{6}\big)+2\big(\frac14-2\varepsilon\big)>1$. Donc $(\mathfrak{T}R_{10})(X,X+H)$ (ainsi que $(\mathfrak{T}R_{10})(X-H,X)$) peut être englobé dans le terme d'erreur (11) de la Proposition 2.3.

Calculons maintenant les résidus apparaissant dans (31).

Lemme 3.5. — *Notant*

$$F_{12}(s_1) := \frac13 H_1(s_1)H_1(1-2s_1)\zeta(2-3s_1)G_1(s_1,1-2s_1)$$

on a

$$\begin{aligned}\operatorname*{Res}_{s_2=1-2s_1}\Big(\frac{F_1(s_1,s_2)X_2^{s_2+1}}{s_2(s_2+1)}\Big) &= \frac{F_{12}(s_1)}{(2-2s_1)(1-2s_1)}X_2^{2-2s_1},\\ \operatorname*{Res}_{s_2=1/2-s_1/2}\Big(\frac{F_1(s_1,s_2)X_2^{s_2+1}}{s_2(s_2+1)}\Big) &= \frac12\,\frac{F_{12}(\frac12-\frac12 s_1)}{(\frac12-\frac12 s_1)(\frac32-\frac12 s_1)}X_2^{3/2-s_1/2}.\end{aligned}$$

Il existe trois fonctions g_1, g_2, g_3 holomorphes dans le demi-plan $\Re e\, s_1 > \frac{1}{12}$ telles que

$$\operatorname*{Res}_{s_2=1/3}\Big(\frac{F_1(s_1,s_2)X_2^{s_2+1}}{s_2(s_2+1)}\Big) = X_2^{4/3}\Big(\frac{g_1(s_1)(\log X_2)^2}{(s_1-\frac13)^5}+\frac{g_2(s_1)\log X_2}{(s_1-\frac13)^6}+\frac{g_3(s_1)}{(s_1-\frac13)^7}\Big), \tag{32}$$

avec

$$\begin{cases} g_1(\frac13) = \dfrac{G_1(\frac13,\frac13)}{48}\times\dfrac{1}{4}, \\ g_2(\frac13) = \dfrac{G_1(\frac13,\frac13)}{48}\times\dfrac{-5}{4}, \\ g_3(\frac13) = \dfrac{G_1(\frac13,\frac13)}{48}\times\dfrac{21}{8}. \end{cases}$$

De plus, lorsque $\Re e\, s_1 \in [\frac13-\delta,\kappa_1]$, $|s_1-\frac13| \geqslant 1/\log X$, *on a*

$$\begin{aligned} |g_1(s_1)| &\ll (1+|\tau_1|)^{5+5/6}(\log X)^6, \\ |g_2(s_1)| &\ll (1+|\tau_1|)^{6+5/6}(\log X)^7, \\ |g_3(s_1)| &\ll (1+|\tau_1|)^{7+5/6}(\log X)^8. \end{aligned} \tag{33}$$

Démonstration du Lemme 3.5. — Le calcul des résidus en $1-2s_1$ et $\frac12-\frac12 s_1$ est direct. Nous détaillons maintenant le calcul du résidu en $\frac13$ qui n'est pas immédiat.

On a

$$F_1(s_1,s_2) = \tfrac13 H_1(s_1)H_1(s_2)\frac{\Theta(s_2+2s_1)\Theta(2s_2+s_1)}{(s_2+2s_1-1)(2s_2+s_1-1)}G_1(s_1,s_2) \tag{34}$$

où la fonction Θ, définie par $\Theta(z)=\zeta(z)(z-1)$, est holomorphe sur $\mathbb{C}$. Utilisant la notation $v=(s_2-1/3)/(s_1-1/3)$, nous obtenons la formule exacte

$$\frac{1}{(s_2+2s_1-1)(2s_2+s_1-1)} = \frac{1}{2(s_1-\frac13)^2}\Big\{1-\tfrac52 v+\tfrac{21}{4}v^2+v^3\Big(\frac{19-30v-12v^2}{2(2+v)(1+2v)}\Big)\Big\}. \tag{35}$$

Pour alléger l'écriture des calculs, nous utilisons dans la suite la notation

$$\mathrm{Res}_{1/3} = \operatorname*{Res}_{s_2=1/3}\Big(\frac{F_1(s_1,s_2)X_2^{s_2+1}}{s_2(s_2+1)}\Big).$$

La conjonction de (34) et (35) fournit

$$\operatorname{Res}_{1/3} = \frac{X_2 H_1(s_1)}{6(s_1-\frac13)^2} \operatorname*{Res}_{s_2=1/3} \Big(\frac{H_1(s_2)X_2^{s_2}}{s_2(s_2+1)} \Theta(s_2+2s_1) \\ \times \Theta(2s_2+s_1) G_1(s_1,s_2) \Big\{ 1 - \tfrac52 v + \tfrac{21}{4} v^2 \Big\} \Big).$$

Un développement de Taylor fournit l'existence des fonctions h_1, h_2, h_3 et h_4 telles que

$$\frac{(s_2-1/3)^3 H_1(s_2)}{s_2(s_2+1)} \Theta(s_2+2s_1)\Theta(2s_2+s_1) G_1(s_1,s_2) \\ = h_1(s_1) + h_2(s_1)(s_2-\tfrac13) + h_3(s_1)(s_2-\tfrac13)^2 + (s_2-\tfrac13)^3 h_4(s_1,s_2).$$

Pour $i = 1,\ 2,\ 3$, la fonction h_i est holomorphe dans le demi-plan $\Re e\, s_1 > \frac{1}{12}$. Pour $\Re e\, s_1 = \kappa_1$, la fonction $s_2 \to h_4(s_1,s_2)$ est holomorphe au voisinage de $s_2 = \frac13$. De plus, $h_1(\frac13) = \frac38 G_1\big(\frac13,\frac13\big)$.

Ces notations étant introduites, nous obtenons

(36)

$$\operatorname*{Res}_{s_2=1/3} \Big(X_2^{s_2-1/3} \frac{H_1(s_2)}{s_2(s_2+1)} \Theta(s_2+2s_1)\Theta(2s_2+s_1)G_1(s_1,s_2) \Big) \\ = \tfrac12 h_1(s_1)(\log X_2)^2 + h_2(s_1)\log X_2 + h_3(s_1),$$

$$-\frac52 \operatorname*{Res}_{s_2=1/3} \Big(\frac{s_2-\frac13}{s_1-\frac13} X_2^{s_2-1/3} \frac{H_1(s_2)}{s_2(s_2+1)} \Theta(s_2+2s_1)\Theta(2s_2+s_1)G_1(s_1,s_2) \Big) \\ = -\frac{5}{2(s_1-\frac13)} \big(h_1(s_1)\log X_2 + h_2(s_1) \big),$$

$$\frac{21}{4} \operatorname*{Res}_{s_2=1/3} \Big(\frac{(s_2-\frac13)^2}{(s_1-\frac13)^2} X_2^{s_2-1/3} \frac{H_1(s_2)}{s_2(s_2+1)} \Theta(s_2+2s_1)\Theta(2s_2+s_1)G_1(s_1,s_2) \Big) \\ = \frac{21}{4(s_1-\frac13)^2} h_1(s_1).$$

On obtient alors la relation (32) en posant

$$\begin{cases} g_1(s_1) = \frac{1}{12}(s_1-\frac13)^3 H_1(s_1) h_1(s_1), \\ g_2(s_1) = (s_1-\frac13)^3 H_1(s_1) \big(-\frac{5}{12} h_1(s_1) + \frac16 (s_1-\frac13) h_2(s_1) \big), \\ g_3(s_1) = (s_1-\frac13)^3 H_1(s_1) \\ \qquad \times \big(\frac{21}{24} h_1(s_1) - \frac{5}{12}(s_1-\frac13) h_2(s_1) + \frac16 (s_1-\frac13)^2 h_3(s_1) \big). \end{cases}$$

La valeur de $g_i(\frac{1}{3})$ est obtenue à partir de celle de $h_1(\frac{1}{3}) = \frac{3}{8}G_1(\frac{1}{3}, \frac{1}{3})$ et de l'équivalent

$$(s_1 - \tfrac{1}{3})^3 H_1(s_1) \underset{(s_1 \sim 1/3)}{\sim} \tfrac{1}{6}.$$

Pour terminer la démonstration du Lemme 3.5, nous vérifions les majorations annoncées de g_i qui découlent de la majoration de h_i suivante

$$\begin{aligned}|h_i(s_1)| &\ll \sum_{0 \leqslant j+k \leqslant i} |\Theta^{(j)}(s_1 + \tfrac{2}{3})| \cdot |\Theta^{(k)}(2s_1 + \tfrac{1}{3})| \\ &\ll (\log X)^6 (\log(2 + |\tau_1|))^i (1 + |\tau_1|)^{2 - \Re e\, s_1}\end{aligned}$$

pour $\Re e\, s_1 \in]\frac{1}{3} - \delta, \frac{1}{3} + 25/\log X]$, $|s_1 - \frac{1}{3}| \geqslant 1/\log X$, $|\tau_1| \leqslant T$. □

Pour achever l'estimation de M_1, nous évaluons maintenant successivement les quantités M_{1i} grâce au Lemme 3.5.

Commençons par étudier M_{11} et M_{12} les quantités faisant intervenir la fonction F_{12}. Pour ce faire, nous étudions plus précisément la fonction G_2 définie par

$$G_2(s_1) := G_1(s_1, 1 - 2s_1) = G(s_1, 1 - 2s_1, \tfrac{1}{3}).$$

La définition (22) de Δ permet d'affirmer que G_2 s'écrit sous la forme d'un produit eulérien

$$G_2(s_1) = \prod_p \big(1 + \Delta(p^{-s_1}, p^{-1+2s_1}, p^{-1/3})\big).$$

Cherchons un développement de $\Delta(Z_1, Z_2, Z_3)$ uniforme dans le domaine

$$\mathcal{D} = \{(Z_1, Z_2, Z_3) \in \mathbb{C}^3 \,:\, |Z_1| \leqslant 1,\, |Z_2| \leqslant 1,\, |Z_3| \leqslant 1\}$$

en négligeant les termes majorés par

$$\begin{aligned}&\Delta_1(Z_1, Z_2, Z_3) \\ &\quad := \sum_{\{i,j,k\} = \{1,2,3\}} \big(|Z_i|^3 |Z_j|^2 |Z_k| + |Z_i|^4 |Z_j||Z_k|\big) + |Z_1|^2 |Z_2|^2 |Z_3|^2.\end{aligned}$$

La formule (14) implique

$$\begin{aligned}&1 + \Delta(Z_1, Z_2, Z_3) \\ &\quad = \prod_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} (1 - Z_i^2 Z_j) \\ &\qquad\quad \times \Bigg(1 + \sum_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} Z_i^2 Z_j - \sum_{\{i,j,k\} = \{1,2,3\}} Z_i^3 Z_j^2 Z_k - Z_1^3 Z_3^3 Z_3^3\Bigg) \\ &\quad = \prod_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} (1 - Z_i^2 Z_j) \Big(1 + \sum_{\substack{\ell,k \in \{1,2,3\} \\ \ell \neq k}} Z_\ell^2 Z_k\Big) + O(\Delta_1(Z_1, Z_2, Z_3)).\end{aligned}$$

La majoration $|Z_i^2 Z_j Z_\ell^2 Z_k| \leqslant \Delta_1(Z_1, Z_2, Z_3)$ est vérifiée dans $\mathcal{D}$ sauf pour $(i,j) = (\ell, k)$. On en déduit que

$$\Delta(Z_1, Z_2, Z_3) = -\sum_{\substack{\ell,k\in\{1,2,3\}\\ \ell\neq k}} Z_\ell^4 Z_k^2 + O(\Delta_1(Z_1, Z_2, Z_3)).$$

En contrôlant les termes apparaissant dans l'écriture de $\Delta_1(p^{-s_1}, p^{-1+2s_1}, p^{-1/3})$, on obtient l'existence d'une fonction G_3 holomorphe dans une bande $\Re e\, s \in]\frac{1}{3}, \frac{1}{3} + \frac{15}{112}[$ telle que

$$G_2(s_1) = G_1(s_1, 1-2s_1) = \zeta^{-1}\big(2 - 8(s_1 - \tfrac{1}{3})\big) G_3(s_1)$$

(On note que $\frac{15}{112} = \frac{1}{2}(\frac{1}{7} + \frac{1}{8}) \in]\frac{1}{8}, \frac{1}{7}[$) et $|G_3(s_1)| \ll 1$ dans ce domaine.

Notant $\sigma(\tau) := c_0(\log(|\tau| + 3))^{-2/3}(\log_2(|\tau| + 3))^{-1/3}$, on déduit du lemme 3.1 la majoration

$$\begin{aligned} F_{12}(s_1) &\ll (\log X)^7 (1 + |\tau_1|)^{5(\sigma_1 - 1/3)+\varepsilon} \\ &\ll (\log X)^7 (1 + |\tau_1|)^{3/4} \qquad (\varepsilon = 1/32) \end{aligned}$$

valable lorsque

$$\sigma_1 \in [\kappa_1, \tfrac{1}{3} + \tfrac{1}{8} + \sigma(\tau_1)], \qquad |s_1 - \tfrac{1}{3}| \geqslant 1/\log X.$$

Le lemme 3.5 et la définition permettent d'écrire

$$M_{11} := \frac{(X_1 X_2 X_3)^{4/3}}{2\pi i} \int_{\kappa_1 - iT}^{\kappa_1 + iT} \frac{F_{12}(s_1)(X_1/X_2^2)^{s_1 - 1/3}}{s_1(s_1+1)(1-2s_1)(2-2s_1)}\, \mathrm{d}s_1.$$

On applique le théorème des résidus pour majorer la quantité $(\mathfrak{T}M_{11})(X, X+H)$. On choisit le contour $C_1(\kappa_1)$ joignant les points d'affixe

$$\begin{gathered} \kappa_1 - iT,\ \kappa_1 + iT,\ \tfrac{1}{3} + \tfrac{1}{8} + iT,\ \tfrac{1}{3} + \tfrac{1}{8} + iT',\ \tfrac{1}{3} + \tfrac{1}{8} + iT' - \sigma(T'), \\ \tfrac{1}{3} + \tfrac{1}{8} - iT' - \sigma(T'),\ \tfrac{1}{3} + \tfrac{1}{8} - iT',\ \tfrac{1}{3} + \tfrac{1}{8} - iT,\ \kappa_1 - iT, \end{gathered}$$

où T' est un paramètre. En utilisant la méthode utilisée pour majorer $\mathfrak{T}R_0$, il vient

$$(\mathfrak{T}M_{11})(X, X+H) \ll X^{7/8} H^3 \Big(\exp\Big\{ -\frac{\log X}{\sigma(T')} \Big\} + T'^{-1/4} \Big) + X^4 T^{-1/4}$$

On montre que ce terme est englobé par le membre de droite de (11) en choisissant $\log T' = (\log X)^{3/5} (\log_2 X)^{1/5}$ et en prenant $T \geqslant X^9$.

La contribution du résidu en $s_2 = \frac{1}{2} - \frac{1}{2}s_1$ est estimée à l'aide de la même méthode. Le lemme 3.5 et (31), puis un changement de variable, fournissent

$$\begin{aligned} M_{12} &= \frac{(X_1 X_2 X_3)^{4/3}}{2\pi i} \int_{\kappa_1 - iT}^{\kappa_1 + iT} \frac{1}{2} \frac{F_{12}(\frac{1}{2} - \frac{1}{2}s_1)\,(X_1/\sqrt{X_2})^{s_1 - 1/3}}{(\frac{1}{2} - \frac{1}{2}s_1)(\frac{3}{2} - \frac{1}{2}s_1) s_1(s_1+1)}\, \mathrm{d}s_1 \\ &= \frac{(X_1 X_2 X_3)^{4/3}}{2\pi i} \int_{\frac{1}{2} - \frac{1}{2}\kappa_1 - iT/2}^{\frac{1}{2} - \frac{1}{2}\kappa_1 + iT/2} \frac{F_{12}(s_1)\,(X_2/X_1^2)^{s_1 - 1/3}}{s_1(s_1+1)(1-2s_1)(2-2s_1)}\, \mathrm{d}s_1. \end{aligned}$$

La seule différence avec le calcul précédent est que le pôle $s_1 = \frac{1}{3}$ est à l'intérieur du contour $C(\frac{1}{2} - \frac{1}{2}\kappa_1)$. Il existe donc un polynôme P_{11} de degré égal à 6 et d'une fonction $R_{12} \in \mathfrak{E}_3$ vérifiant

$$\left.\begin{aligned}&(\mathfrak{T}R_{12})(X, X+H)\\&(\mathfrak{T}R_{12})(X-H, X)\end{aligned}\right\} \ll X^{7/8}H^3\exp\{-cN(X)\} + X^{4-1/2-\varepsilon}$$

tels que

$$M_{12} = (X_1X_2X_3)^{4/3}P_{11}(\log(X_1^2/X_2)) + R_{12}(X_1, X_2, X_3), \tag{37}$$

De plus, le polynôme P_{11} est de degré 6 et de coefficient dominant

$$\frac{9}{48}\frac{G_1(\frac{1}{3}, \frac{1}{3})}{6!\,4} \times \Big(-\frac{1}{24}\Big). \tag{38}$$

Enfin pour le résidu en $s_2 = \frac{1}{3}$, nous décalons la droite d'intégration de l'abscisse κ_1 jusqu'à $\frac{1}{3} - \delta$. Le théorème des résidus implique l'existence de trois polynômes P_{12}, P_{13}, P_{14} tels que M_{13} soit égal à

$$\begin{aligned}\frac{(X_2X_3)^{4/3}}{2\pi i}\int_{\kappa_1-iT}^{\kappa_1+iT}\Big(\frac{g_1(s_1)(\log X_2)^2}{(s_1-\frac{1}{3})^5} + \frac{g_2(s_1)\log X_2}{(s_1-\frac{1}{3})^6} + \frac{g_3(s_1)}{(s_1-\frac{1}{3})^7}\Big)\frac{X_1^{s_1+1}\,\mathrm{d}s_1}{s_1(s_1+1)}\\ = (X_1X_2X_3)^{4/3}\big\{P_{12}(\log X_1)(\log X_2)^2 + P_{13}(\log X_1)\log X_2\\ + P_{14}(\log X_1)\big\} + R_{13}(X_1, X_2, X_3) + O\Big(X^4\frac{(\log X)^8}{T^{1/4}}\Big),\end{aligned}$$

avec

$$\begin{aligned}R_{13}(X_1, X_2, X_3) := \frac{(X_2X_3)^{4/3}}{2\pi i}\\ \times\int_{1/3-\delta-iT}^{1/3-\delta+iT}\Big(\frac{g_1(s_1)(\log X_2)^2}{(s_1-\frac{1}{3})^5} + \frac{g_2(s_1)\log X_2}{(s_1-\frac{1}{3})^6} + \frac{g_3(s_1)}{(s_1-\frac{1}{3})^7}\Big)\frac{X_1^{s_1+1}\,\mathrm{d}s_1}{s_1(s_1+1)}.\end{aligned}$$

De plus, P_{12} est de degré 4, P_{13} de degré 5, P_{14} de degré 6, et

$$\left\{\begin{aligned}&\text{le coefficient dominant de } P_{12} \text{ est } \frac{9}{4!\,4}g_1(\tfrac{1}{3}) = \frac{9}{48}\frac{G_1(\frac{1}{3}, \frac{1}{3})}{6!\,4} \times \frac{15}{2},\\&\text{le coefficient dominant de } P_{13} \text{ est } \frac{9}{5!\,4}g_2(\tfrac{1}{3}) = \frac{9}{48}\frac{G_1(\frac{1}{3}, \frac{1}{3})}{6!\,4} \times \frac{-15}{2},\\&\text{le coefficient dominant de } P_{14} \text{ est } \frac{9}{6!\,4}g_3(\tfrac{1}{3}) = \frac{9}{48}\frac{G_1(\frac{1}{3}, \frac{1}{3})}{6!\,4} \times \frac{21}{8}.\end{aligned}\right. \tag{39}$$

On majore $\mathfrak{T}R_{13}$ par la méthode utilisée pour $\mathfrak{T}R_0$. On déduit des majorations (33) que

$$\begin{aligned}(\mathfrak{T}R_{13})(X, X+H) &\\ &\ll H^2 X^{2/3}(\log X)^8 \int_{-T}^{T} \frac{\left|(X+H)^{4/3-\delta+i\tau_1} - X^{4/3-\delta+i\tau_1}\right|}{(\tau_1+1)^{2-5/6}} \,\mathrm{d}\tau_1 \\ &\ll X^{4-\delta}(H/X)^{13/6}(\log X)^8(\log T).\end{aligned}$$

Le Lemme 3.2(ii) permet alors d'affirmer que

$$(\mathfrak{T}R_{13})(X, X+H) \ll H^4 + X^{4-1/2-\varepsilon}. \tag{40}$$

En sommant les M_{1i}, on en déduit que

$$\begin{aligned}M_1 = (X_1X_2X_3)^{4/3}P_1(\log X_1, \log X_2)& \\ &+ R_1(X_1, X_2, X_3) + O\Big(\frac{X^4(\log X)^9}{T^{1/6}}\Big)\end{aligned} \tag{41}$$

avec

$$P_1(X_1, X_2) := \tfrac{9}{4}\big(P_{11}(X_1 - \tfrac{1}{2}X_2) + P_{12}(X_1)X_2^2 + P_{13}(X_1)X_2 + P_{14}(X_1)\big)$$

et

$$R_1 := \tfrac{9}{4}R_{10} + \tfrac{9}{4}M_{11} + \tfrac{9}{4}R_{12} + \tfrac{9}{4}R_{13}.$$

La fonction R_1 satisfait bien à (11). La somme des coefficients des monômes apparaissant dans l'écriture de P_1 de degré total égal à 6 se déduit alors facilement de (38), (39) et (24). Elle est égale à

$$\frac{31}{12}\Big(\frac{3}{4}\Big)^3 C.$$

3.5. Estimation de $M_2(X_1, X_2, X_3)$. — Notons

$$H_2(s_2) := \zeta(3s_2)\zeta(-\tfrac{3}{2}s_2 + \tfrac{3}{2})\zeta(\tfrac{3}{2}s_2 + \tfrac{1}{2})$$

de sorte que

$$\begin{aligned}F_2(s_1, s_2) &:= \tfrac{1}{2}H_2(s_2)\zeta(s_1 - s_2 + 1)\zeta(2s_1 - \tfrac{1}{2}s_2 + \tfrac{1}{2}) \\ &\quad\times \zeta(3s_1)\zeta(s_2 + 2s_1)\zeta(2s_2 + s_1)G(s_1, s_2, \tfrac{1}{2} - \tfrac{1}{2}s_2).\end{aligned}$$

Nous intégrons d'abord par rapport à la variable s_1 ce qui fournit des résidus uniquement d'ordre de multiplicité égal à 1. La fonction $s_1 \to G(s_1, s_2, \frac{1}{2} - \frac{1}{2}s_2)$ est une fonction holomorphe dans le demi-plan $\Re e\, s_1 > \frac{1}{12}$ pour s_2 fixé tel que $\Re e\, s_2 = \kappa_2$. La fonction $s_1 \mapsto F_2(s_1, s_2)$ est donc méromorphe sur le demi-plan $\Re e\, s_1 > \frac{1}{12}$ et admet cinq pôles :

$$\begin{aligned}v_1(s_2) &:= \tfrac{1}{3}, & v_2(s_2) &:= \tfrac{1}{2} - \tfrac{1}{2}s_2, & v_3(s_2) &:= 1 - 2s_2, \\ v_4(s_2) &:= s_2, & v_5(s_2) &:= \tfrac{1}{4}s_2 + \tfrac{1}{4}\end{aligned}$$

d'ordre de multiplicité égal à 1. Les pôles sont tous distincts d'après la taille relative de κ_1 et κ_2.

On déduit de (17) une majoration de $F_2(s_1,s_2)$. Lorsque

$$\Re e\, s_2=\kappa_2,\ \Re e\, s_1\in[\tfrac13-\delta,\kappa_1],\ |\tau_1|\leqslant T,\ \min_i\{|s_1-v_i(s_2)|\}\geqslant 1/\log X,$$

on a

$$|F_2(s_1,s_2)|\ll(\log X)^9(1+|\tau_1|+|\tau_2|)^{1/2}(1+|\tau_1|)^{1/3}. \tag{42}$$

On applique le théorème des résidus au contour $C_2(\kappa_1)$ joignant les points d'affixe

$$\kappa_1+iT,\ \kappa_1+9iT,\ \tfrac13-\delta+9iT,\ \tfrac13-\delta-9iT,\ \kappa_1-9iT,\ \kappa_1-iT,\ \kappa_1+iT.$$

Grâce à (42), on obtient

$$\begin{aligned}M_2&=\frac{1}{(2\pi i)^2}\int_{\kappa_2-3iT}^{\kappa_2+3iT}\int_{\kappa_1-iT}^{\kappa_1+iT}\frac{F_2(s_1,s_2)\,X_1^{s_1+1}X_2^{s_2+1}X_3^{3/2-s_2/2}}{s_1(s_1+1)s_2(s_2+1)(\frac12-\frac12s_2)(\frac32-\frac12s_2)}\,\mathrm{d}s_2\,\mathrm{d}s_1\\&=\sum_{i=1}^{5}M_{2i}(X_1,X_2,X_3)+R_{20}(X_1,X_2,X_3)+O\Big((\log X)^9\frac{X^4}{T^{1/6}}\Big),\end{aligned}$$

avec

$$\begin{cases}F_{21}(s_2)\ :=\tfrac32F_{12}(\tfrac12-\tfrac12s_1),\\ F_{22}(s_2)\ :=\tfrac12\zeta(3s_2)\zeta(-\tfrac32s_2+\tfrac32)^4\zeta(\tfrac32s_2+\tfrac12)^2G(\tfrac12-\tfrac12s_2,s_2,\tfrac12-\tfrac12s_2),\\ F_{23}(s_2)\ :=\tfrac12H_2(s_2)\zeta(3-6s_2)\zeta(-3s_2+2)^2\\ \qquad\qquad\times\zeta(-\tfrac92s_2+\tfrac52)G(1-2s_2,s_2,\tfrac12-\tfrac12s_2),\\ F_{24}(s_2)\ :=\tfrac12\zeta(3s_2)^4\zeta(\tfrac32s_2+\tfrac12)^2\zeta(-\tfrac32s_2+\tfrac32)G(s_2,s_2,\tfrac12-\tfrac12s_2),\\ F_{25}(s_2)\ :=F_{23}(\tfrac12-\tfrac12s_2),\end{cases}$$

$$\begin{aligned}&M_{2i}(X_1,X_2,X_3)\\&\qquad:=\frac{1}{2\pi i}\int_{\kappa_2-3iT}^{\kappa_2+3iT}\frac{F_{2i}(s_2)\,X_1^{v_i(s_2)+1}X_2^{s_2+1}X_3^{3/2-s_2/2}\,\mathrm{d}s_2}{v_i(s_2)(v_i(s_2)+1)s_2(s_2+1)(\frac12-\frac12s_2)(\frac32-\frac12s_2)}\end{aligned}$$

et

$$\begin{aligned}&R_{20}(X_1,X_2,X_3)\\&\quad:=\frac{1}{(2\pi i)^2}\int_{\kappa_2-3iT}^{\kappa_2+3iT}\int_{1/3-\delta-iT}^{1/3-\delta+iT}\frac{F_2(s_1,s_2)\,X_1^{s_1+1}X_2^{s_2+1}X_3^{3/2-s_2/2}\,\mathrm{d}s_2\,\mathrm{d}s_1}{s_1(s_1+1)s_2(s_2+1)(\frac12-\frac12s_2)(\frac32-\frac12s_2)}.\end{aligned}$$

On majore $\mathfrak{T}R_{20}$ par la méthode utilisée pour $\mathfrak{T}R_0$. On déduit des majorations (42) et du lemme 3.2(ii) que

$$\begin{aligned}(43)\qquad &(\mathfrak{T}R_{20})(X,X+H)\\ &\quad\ll H^2X^{2/3}(\log X)^9\int_{-T}^{T}\frac{\big|(X+H)^{4/3-\delta+i\tau_1}-X^{4/3-\delta+i\tau_1}\big|}{(\tau_1+1)^{2-5/6}}\,\mathrm{d}\tau_1\\ &\quad\ll X^{4-\delta}(H/X)^{13/6}(\log X)^9(\log T)\\ &\quad\ll H^4+X^{4-1/2-\varepsilon}.\end{aligned}$$

Pour terminer le calcul lié à $u_2(s_1,s_2)$, il reste à estimer les $M_{2i}(X_1,X_2,X_3)$. Comme pour les étapes précédentes, cela se fait en appliquant le théorème des résidus grâce aux majorations connues sur les valeurs de la fonction ζ. Pour estimer M_{21}, on peut se reporter à l'estimation de F_{12}.

Pour M_{23}, on décale la droite d'intégration à droite jusquà $\frac{1}{3}+\frac{1}{2}\delta$ et on utilise la majoration

$$F_{23}(s_2)\ll(\log X)^7(1+|\tau_2|)^{1/2}$$

valable lorsque $\sigma_2\in[\kappa_2,\frac{1}{3}+\frac{1}{2}\delta]$.

Pour M_{24}, on décale la droite d'intégration à droite jusquà $\frac{1}{3}-\frac{2}{3}\delta$ et on utilise la majoration

$$F_{23}(s_2)\ll(\log X)^7(1+|\tau_2|)^{5/6}$$

valable lorsque $\sigma_2\in[\frac{1}{3}-\frac{2}{3}\delta,\kappa_2]$.

L'estimation de M_{25} est traitée comme celle de M_{23}.

Nous ne détaillons que l'évaluation de M_{22}. Posant $\xi:=\log X_2-\frac{1}{2}\log X_1-\frac{1}{2}\log X_3$, on peut écrire

$$M_{22}=\frac{(X_1X_2X_3)^{4/3}}{2\pi i}\int_{\kappa_2-3iT}^{\kappa_2+3iT}\frac{F_{22}(s_2)\,\mathrm{e}^{\xi(s_2-1/3)}}{(\frac{1}{2}-\frac{1}{2}s_2)^2(\frac{3}{2}-\frac{1}{2}s_2)^2s_2(s_2+1)}\,\mathrm{d}s_2.$$

L'estimation

$$|F_{22}(s_2)|\ll(\log X)^7,\qquad(\Re e\,s_2=\kappa_2),$$

et le développement

$$\mathrm{e}^{\xi(s_2-1/3)}=1+\xi(s_2-\tfrac{1}{3})+\tfrac{1}{2}\xi^2(s_2-\tfrac{1}{3})^2+\tfrac{1}{6}\xi^3(s_2-\tfrac{1}{3})^3+O(|\xi|^4|s_2-\tfrac{1}{3}|^4)$$

(uniforme puisque $\mathrm{e}^{\xi(\Re e\,s_2-1/3)}\ll 1$), fournissent

$$M_{22}=(X_1X_2X_3)^{4/3}\sum_{j=0}^{3}C_j\xi^j+R_{22}(X_1,X_2,X_3)+O\Big(\frac{X^4(\log X)^7}{T^5}\Big),$$

avec

$$C_j:=\frac{1}{j!}\frac{1}{2\pi i}\int_{\kappa_2-i\infty}^{\kappa_2+i\infty}\frac{F_{22}(s_2)\,(s_2-1/3)^j}{(\frac{1}{2}-\frac{1}{2}s_2)^2(\frac{3}{2}-\frac{1}{2}s_2)^2s_2(s_2+1)}\,\mathrm{d}s_2,$$

et $R_{22}(X_1, X_2, X_3) \ll X^4(\log X)^7|\xi|^4$. Les quantités C_j sont des constantes ne dépendant pas de κ_2. Nous remarquons que $\mathfrak{T}R_{22}(X, X+H)$ est une combinaison linéaire finie de valeurs de $R_{22}(X, X_2, X_3)$ pour lesquelles $\xi \ll H/X$.

Il existe donc un polynôme P_2 de $\mathbb{R}[X_1, X_2, X_3]$ de degré total 6 et une fonction $R_2 \in \mathcal{E}_3$ satisfaisant à la majoration (11) tels que l'on ait l'estimation

$$M_2 = (X_1X_2X_3)^{4/3}P_2(\log X_1, \log X_2, \log X_3) \\ + R_2(X_1, X_2, X_3) + O\Big(\frac{X^4(\log X)^{10}}{T^{1/6}}\Big).$$

De plus, la somme des coefficients des monômes de degré 6 de P_2 est

$$-\frac{23}{12}\Big(\frac{3}{4}\Big)^3 C.$$

3.6. Estimation de $M_3(X_1, X_2, X_3)$. — La contribution M_3 s'étudie de la même manière que M_2 puisque $F_3(s_1, s_2) = F_2(s_2, s_1)$. Cependant, du fait de la taille relative de κ_1 et κ_2, le terme principal dans le résultat final est différent (*i.e.* il ne suffit de changer X_1 en X_2). Seuls trois pôles de $s_2 \to F_3(s_1, s_2)$ sont contenus dans la bande $\Re e\, s_2 \in [\frac{1}{3} - \delta, \kappa_2]$. Il existe donc un polynôme P_3 de $\mathbb{R}[X]$ de degré 6 et une fonction $R_3 \in \mathcal{E}_3$ satisfaisant à la majoration (11) tels que l'on ait l'estimation

$$M_3 = (X_1X_2X_3)^{4/3}P_3(\log X_1, \log X_2, \log X_3) \\ + R_3(X_1, X_2, X_3) + O\Big(\frac{X^4(\log X)^{10}}{T^{1/6}}\Big).$$

De plus, la somme des coefficients des monômes de degré 6 de P_3 est

$$-\frac{1}{24}\Big(\frac{3}{4}\Big)^3 C.$$

3.7. Estimation de $M_4(X_1, X_2, X_3)$. — La formule

$$F_4(s_1, s_2) = 2F_1(s_1, 1-2s_2)$$

et un changement de variable en s_2 impliquent

$$\begin{aligned} M_4 &= \frac{1}{(2\pi i)^2}\int_{\kappa_1-iT}^{\kappa_1+iT}\int_{\kappa_2-3iT}^{\kappa_2+3iT}\frac{F_4(s_1,s_2)X_1^{s_1+1}X_2^{s_2+1}X_3^{2-2s_2}\,\mathrm{d}s_2\,\mathrm{d}s_1}{s_2(s_2+1)(1-2s_2)(2-2s_2)s_1(s_1+1)} \\ &= \frac{1}{(2\pi i)^2}\int_{\kappa_1-iT}^{\kappa_1+iT}\int_{1-2\kappa_2-i6T}^{1-2\kappa_2+6iT}\frac{F_2(s_1,s_2)X_1^{s_1+1}X_2^{3/2-s_2/2}X_3^{s_2+1}\,\mathrm{d}s_2\,\mathrm{d}s_1}{(\frac{1}{2}-\frac{1}{2}s_2)(\frac{3}{2}-\frac{1}{2}s_2)s_2(s_2+1)s_1(s_1+1)} \end{aligned}$$

La contribution de F_4 s'estime donc de la même manière que celle de F_2. Cependant, le terme principal diffère puisque l'abscisse $1-2\kappa_2$ est à inférieure à $\frac{1}{3}$. Il existe donc un polynôme P_4 de $\mathbb{R}[X]$ de degré 6 et une fonction $R_4 \in \mathcal{E}_3$ satisfaisant à la majoration (11) tels que l'on ait l'estimation

$$M_4 = (X_1X_2X_3)^{4/3}P_4(\log X_1, \log X_2, \log X_3) \\ + R_4(X_1,X_2,X_3) + O\Big(\frac{X^4(\log X)^{10}}{T^{1/6}}\Big).$$

De plus, la somme des coefficients des monômes de degré 6 de P_4 est

$$\frac{3}{8}\Big(\frac{3}{4}\Big)^3 C.$$

3.8. Estimation de $M_5(X_1,X_2,X_3)$**. —** La relation $F_5(s_1,s_2) = F_4(s_2,s_1)$ permet de nous reporter à l'estimation du paragraphe précédent. Ici, étant donné les tailles relatives de κ_1 et κ_2, aucun des pôles de la fonction $s_1 \to F_5(s_1,s_2)$ est dans la bande comprise entre les deux droites d'intégration. Sous la condition $T^{1/6} \geqslant X^{1/2+\varepsilon}$, on a donc l'estimation

$$\left.\begin{matrix}(\mathfrak{T}M_5)(X,X+H)\\ (\mathfrak{T}M_5)(X-H,X)\end{matrix}\right\} \ll X^{4-1/2-\varepsilon} + H^4(\log X)^7 + H^3X^{7/8}\exp\{-cN(X)\}.$$

En sommant les M_i et en prenant $T = X^9$, nous achevons la démonstration de la Proposition 2.3.

Références

[BT96-1] V.V. Batyrev & Y. Tschinkel, « Rational points on some Fano cubic variety », *C. R. Acad. Sci.*, Paris, **323**, ser. I (1996), 41-46.

[BT96-2] ———, « Height Zeta functions of Toric Varieties », *Journal Math. Sciences*, **82** n°1 (1996), 3220-3229.

[BT98-1] ———, « Manin's Conjecture for Torics Varieties », *J. of Algebraic Geometry* à paraître.

[BT98-2] ———, « Saturated Varieties », ce volume.

[F98] E. Fouvry, « Sur la hauteur des points d'une certaine surface cubique singulière », ce volume.

[Fu93] W. Fulton, *Introduction to toric varieties*, Annal. of Math. Studies 131, Priceton University Press 1993.

[I85] A. Ivić, *The Riemann zeta-function,* John Wiley, New-York, Chichester, Brisbane, Toronto, Singapour, (1985).

[O88] T. Oda, *Convex Bodies and Algebraic Geometry*, Springer (1988).

[P95] E. Peyre, « Hauteurs et nombres de Tamagawa sur les variétés de Fano », *Duke Math. J.* **79** (1995), 101-217.

[S79] S.H. Schanuel, « Heights in number field », Bull. Soc. Math. France **107** (1979), 433-449.

[S98] P. Salberger, « Tamagawa measures on universal torsors and points of bounded height on Fano varieties », ce volume.

[T95] G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres,* 2ème édition, Cours Spécialisés, no. 1, Société Mathématique de France (1995).

Reçu le 2 avril 1997

RÉGIS DE LA BRETÈCHE, Département de Mathématiques, Bâtiment 425, Université de Paris XI-Orsay, 91405 Orsay cedex, France • *E-mail :* `breteche@geo.math.u-psud.fr`

Astérisque
251, 1998, p. 79–90

RÉPARTITION DES POINTS RATIONNELS DES SURFACES GÉOMÉTRIQUEMENT RÉGLÉES RATIONNELLES

par

Hervé Billard

Résumé. — Nous étudions la répartition des points rationnels des modèles minimaux des surfaces rationnelles et vérifions que ces surfaces satisfont les conjectures de Batyrev-Manin sur le corps des rationnels. Pour ce faire, nous rappelons d'abord quelques propriétés et descriptions géométriques de telles surfaces. Ensuite, pour chaque plongement considéré, une hauteur naturelle apparaissant, nous établissons directement le comportement asymptotique des points rationnels de hauteur bornée. Finalement, nous regardons les sous-variétés accumulatrices.

1. Introduction

Lorsque V est une variété algébrique définie sur un corps de nombres k, il est naturel de s'intéresser à la répartition des points k-rationnels de V. Pour étudier $V(k)$, on définit d'abord une hauteur H qui permet de « mesurer la taille » d'un point k-rationnel de V (cf. [La 83] chap. 3 et 4 ou chap. 6 de [Co-Si 86]) ; ensuite on s'intéresse au comportement asymptotique de $\operatorname{card}\{P \in U(k) \mid H(P) \leqslant B\}$ pour tout ouvert Zariski dense U de V. Ce comportement asymptotique illustre parfaitement la répartition des points k-rationnels de V (voir [Ba-Ma 90]). Pour essayer de le déterminer, il existe actuellement, à notre connaissance, quatre méthodes. La première est la méthode du cercle pour certaines intersections complètes (voir par exemple [Bir 62]) ; la deuxième consiste à définir « correctement » la hauteur H et de considérer la fonction zêta associée, $\sum_{P\in U(k)} H(P)^{-s}$ (cf. [Fr-Ma-Ts 89], [Ba-Ma 90], [Pe 95], [Ba-Ts 95], entre autres) ; la troisième est de construire des hauteurs canoniques à la Néron-Tate (cf. [Ne 65], [Si 91], [Bi 97]) ; la quatrième consiste « à expliciter » un plongement de la variété V, à considérer une hauteur associée et à déterminer « directement » le comportement asymptotique

***Classification mathématique par sujets* (2000).** — 11G35, 14G05, 14G25, 14J26.
Mots clefs. — Points rationnels, hauteur, surface de Hirzebruch.

de card$\{P \in U(k) \mid H(P) \leqslant B\}$ (cf. [Ba-Ma 90], [Th 93], [Pe 95], par exemple). C'est cette dernière méthode que nous emploierons par la suite.

D'autre part, lorsqu'on s'intéresse à la répartition des points rationnels d'une classe de variétés, on est tenté de regarder cette répartition sur les modèles minimaux. Nous proposons dans ce travail d'étudier cette question sur les modèles minimaux des surfaces rationnelles, et donc sur les surfaces géométriquement réglées F_m (voir §2), qui sont également connues comme les surfaces de Hirzebruch. Signalons que les surfaces F_m sont des variétés toriques. Les résultats que nous présentons affinent, par une tout autre méthode, un théorème plus général sur les variétés toriques dû à Batyrev et Tschinkel [Ba-Ts 96] restreint à notre cas.

Dans un premier temps, nous rappellerons la définition de telles surfaces, ainsi que quelques-unes de leurs propriétés géométriques. Au deuxième paragraphe, nous déterminons le comportement asymptotique de

$$\operatorname{card}\{P \in U(\mathbb{Q}) \mid H_D(P) \leqslant B\}$$

pour une famille de diviseurs amples, et verrons, que s'ils sont uniformément répartis pour les plongements considérés, ils ne le sont pas pour d'autres.

Nous tenons à remercier Marc Hindry avec qui nous avons eu de nombreuses et fructueuses discussions lors de l'élaboration de ce travail, ainsi qu'Emmanuel Peyre, Philippe Satgé et le rapporteur de ce travail.

2. Géométrie des surfaces de Hirzebruch

Dans ce paragraphe, nous travaillons sur $\overline{\mathbb{Q}}$.

Rappelons tout d'abord la définition d'une surface réglée et d'une surface géométri quement réglée.

Définition 2.1

(a) Une surface S est réglée si elle est birationnellement isomorphe à $C \times \mathbb{P}^1$, où C est une courbe lisse. Si C est isomorphe à $\mathbb{P}^1$, S est dite rationnelle.

(b) Une surface est géométriquement réglée de base C, où C est une courbe lisse, s'il existe un morphisme lisse $p : S \to C$ dont les fibres sont isomorphes à $\mathbb{P}^1$.

Un Théorème de Noether-Enriques nous assure qu'une surface géométriquement réglée est réglée, ce qui n'est pas évident *a priori* ([Be 78], chap. 3). D'autre part les modèles minimaux de $C \times \mathbb{P}^1$, où C est une courbe lisse non rationnelle, sont les surfaces géométriquement réglées de base C. Intéressons-nous maintenant aux surfaces géométriquement réglées rationnelles (voir [Be 78] chap. 3 et 4, ou [Ha 77] chap. V, §2).

Proposition 2.2. — *Les seules surfaces géométriquement réglées rationnelles sur $\overline{\mathbb{Q}}$ sont les surfaces F_m définies par :*

$$F_m = \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(m)) \qquad (m \geqslant 0).$$

Les surfaces F_m sont minimales sauf pour $m = 1$ et F_m n'est pas isomorphe à F_ℓ si $m \neq \ell$.

Rappelons que F_0 est isomorphe à $\mathbb{P}^1 \times \mathbb{P}^1$, F_1 est isomorphe à $\mathbb{P}^2$ éclaté en un point et les surfaces rationnelles minimales sont les surfaces F_m, sauf pour $m = 1$, et $\mathbb{P}^2$.

Notons h (respectivement f) la classe dans $\mathrm{Pic}(F_m)$ du fibré $O_{F_m}(1)$ (respectivement d'une fibre).

Proposition 2.3. — *Soit une surface F_m.*

a) *On a* $\mathrm{Pic}(F_m) = \mathbb{Z}h + \mathbb{Z}f$ *avec :*

$$f^2 = 0, \qquad h^2 = m, \qquad f \cdot h = 1.$$

b) *Si $m \geqslant 1$ il existe une unique courbe irréductible C_m sur F_m de carré négatif. La courbe C_m est rationnelle et si l'on note c sa classe dans* $\mathrm{Pic}(F_m)$, *on a :*

$$c = h - mf, \qquad c^2 = -m.$$

c) *Notons ω_m la classe du diviseur canonique dans* $\mathrm{Pic}(F_m)$, *alors :*

$$\omega_m = -2h + (m-2)f.$$

d) *Soit $D = ah + bf$. Alors :*

$$D \text{ est très ample} \iff D \text{ est ample} \iff a > 0, \quad b > 0.$$

e) *Notons $\overline{NE}(F_m)$ le cône fermé dans* $\mathrm{Pic}(F_m) \otimes \mathbb{R}$ *engendré par les classes des courbes irréductibles de F_m, alors :*

$$\overline{NE}(F_m) = \{D \in \mathrm{Pic}(F_m) \otimes \mathbb{R} \mid D = ah + bf \text{ avec } a \geqslant 0 \text{ et } b \geqslant -ma\}.$$

Soient $m \geqslant 1$, C une courbe irréductible sur F_m et $\alpha h + \beta f$ sa classe dans $\mathrm{Pic}(F_m)$. *Alors :*

$$C \neq C_m \Rightarrow \alpha \geqslant 0 \text{ et } \beta \geqslant 0,$$

en particulier $\alpha m + \beta \geqslant 1$.

f) *Soient $D = ah + bf$ une classe de diviseurs amples et*

$$\alpha(D) = \min\{t \in \mathbb{R} \mid tD + \omega_m \in \overline{NE}(F_m)\}.$$

Alors, pour $m \geqslant 2$:

$$\alpha(D) = \frac{2}{a}.$$

La démonstration des assertions a), b), c), d), et e) peut être consultée dans [Ha 77] chap. V, §2, ou [Be 78] chap. 3 et 4. L'assertion f), est une conséquence immédiate de e). Notons que $-\omega_m$, pour $m \geqslant 2$, appartient à l'intérieur de $\overline{NE}(F_m)$, mais qu'il n'est pas ample.

Géométriquement, les surfaces F_m peuvent être interprétées comme des transformations élémentaires de $\mathbb{P}^2$ (c'est-à-dire comme une succession d'éclatements et de contractions, [Be 78] chap. 3 exo 1, ou [Ha 69]), ou encore comme une réunion de courbes rationnelles. Notons que c'est surtout du point de vue de la répartition asymptotique que F_m doit être vue comme la réunion de courbes rationnelles, d'un point de vue géométrique le fait que F_m soit réglée est plus intéressant et donné par hypothèse.

Proposition 2.4. — *Soient $m \geqslant 0$ et $b \geqslant 1$ deux entiers, et posons $d = m + 2b$. Soient R_b et R_{d-b} les deux courbes rationnelles de $\mathbb{P}^{d+1}$ définies par :*

$$R_b = \{(U_1^b, U_1^{b-1}V_1, \ldots, U_1V_1^{b-1}, V_1^b, 0, \ldots, 0) \mid (U_1, V_1) \in \mathbb{P}^1\}$$

$$R_{d-b} = \{(0, \ldots, 0, U_2^{d-b}, U_2^{d-b-1}V_2, \ldots, U_2V_2^{d-b-1}, V_2^{d-b}) \mid (U_2, V_2) \in \mathbb{P}^1\},$$

et ψ un isomorphisme de R_b sur R_{d-b}. Soit alors $F_{m,b}$ définie par :

$$F_{m,b} = \left\{ (SU_1^b, SU_1^{b-1}V_1, \ldots, SV_1^b, TU_2^{d-b}, TU_2^{d-b-1}V_2, \ldots, TV_2^{d-b}) \middle| \begin{array}{l} (S,T) \in \mathbb{P}^1,\ (U_1, V_1) \in \mathbb{P}^1,\ (U_2, V_2) \in \mathbb{P}^1 \\ \psi(U_1^b, ..., V_1^b, 0, ..., 0) = (0, ..., 0, U_2^{d-b}, ..., V_2^{d-b}) \end{array} \right\}.$$

Alors la surface $F_{m,b}$ est isomorphe à la surface F_m plongée par le système $|h + bf|$ dans $\mathbb{P}^{d+1}$; notons φ_b le plongement associé tel que $\varphi_b(F_m) = F_{m,b}$.

Donner l'image d'une fibre de la surface F_m (resp. de C_m) par φ_b revient à fixer U_1 et V_1 (resp. à fixer $T = 0$) ; notons $F_{(U_1,V_1)}$ son image.

On trouvera une démonstration de ce résultat dans [Gr-Ha 78] p. 523-524, ou [Be 78] chap. 4, exo 2, ou [Ha 77] chap. V, Corollaire 2.19. Terminons par une remarque.

Remarque 2.5. — Toute surface lisse de degré d dans $\mathbb{P}^{d+1}$ qui n'est pas contenue dans un hyperplan est une surface F_m, ou la surface de Veronese dans $\mathbb{P}^5$, ou $\mathbb{P}^2$.

3. Répartition des points rationnels des surfaces de Hirzebruch

Rappelons tout d'abord quelques propriétés vérifiées par la fonction de Möbius μ et l'indicatrice d'Euler φ (voir par exemple [Ap 76] chap. 2) qui nous seront très utiles lors du décompte des points $\mathbb{Q}$-rationnels de hauteur bornée des surfaces F_m. Quelque soit le réel x notons $[x]$ sa partie entière.

Lemme 3.1. — *Soit* $x \geqslant 2$.

a)
$$\sum_{d\leqslant x}\frac{\mu(d)}{d^\alpha}=\frac{1}{\zeta(\alpha)}+O\Big(\frac{1}{x^{\alpha-1}}\Big)\ \textit{pour}\ \alpha>1,$$
$$\sum_{d\leqslant x}\frac{\mid\mu(d)\mid}{d}=O(\operatorname{Log}x),$$
$$\sum_{d\leqslant x}\mu(d)\ \Big[\frac{x}{d}\Big]^2=\frac{x^2}{\zeta(2)}+O(x\operatorname{Log}x).$$

b)
$$\sum_{n\leqslant x}\frac{\varphi(n)}{n}=\frac{x}{\zeta(2)}+O(\operatorname{Log}x),$$
$$\sum_{n\leqslant x}\frac{\varphi(n)}{n^2}=\frac{1}{\zeta(2)}\operatorname{Log}x+O(1),$$
$$\sum_{n\leqslant x}\frac{\varphi(n)}{n^\alpha}=\frac{\zeta(\alpha-1)}{\zeta(\alpha)}+O(x^{2-\alpha})\ \textit{pour}\ \alpha>2.$$

On pourra consulter [Ap 76], chap. 3, pour les démonstrations de ces égalités. Signalons également que les deux égalités
$$\sum_{n\leqslant x}\frac{\mu(n)}{n}=o(1)\qquad\text{et}\qquad\sum_{n\leqslant x}\mu(n)=o(x)$$
sont équivalentes au Théorème des nombres premiers.

Soient F_m une surface $\mathbb{Q}$-rationnelle, φ_b le plongement associé à $h+bf$ tel que $\varphi_b(F_m)$ est égal à $F_{m,b}$ (cf. §2) et un point $P\in F_m(\mathbb{Q})$. Quite à changer l'isomorphisme ψ de R_b sur R_{d-b}, on peut supposer dans la proposition 2.4 que $U_1=U_2$ et $V_1=V_2$; posons donc :
$$\varphi_b(P)=(SU^b,\ldots,SV^b,TU^{d-b},\ldots,TV^{d-b}),$$
avec $(S,T)\in\mathbb{Z}^2\setminus\{0\}$, $(U,V)\in\mathbb{Z}^2\setminus\{0\}$, $\operatorname{pgcd}(S,T)=1$ et $\operatorname{pgcd}(U,V)=1$.

Dans ces conditions, on peut définir la hauteur de P associée à $h+bf$ comme étant :
$$H_{h+bf}(P)=\max\ (|S|\max\ (|U|,|V|)^b,|T|\ \max\ (|U|,|V|)^{d-b}).$$

En effet, si un entier q divise S il ne peut diviser U et V, donc également, TU^{d-b} et TV^{d-b}. La hauteur naturelle sur $\mathbb{P}^{d+1}$ induit donc bien la métrique H_{h+bf}.

Posons aussi $H(U,V) = \max(|U|,|V|)$ et notons :

$$N(X,g,B) = \operatorname{card}\{x \in X \mid g(x) \leqslant B\}.$$

Etudions maintenant la répartition des points $\mathbb{Q}$-rationnels des surfaces F_m pour $m \geqslant 2$. Nous nous restreignons aux cas $m \geqslant 2$ pour deux raisons ; la première est que la répartition des points rationnels de F_0 et F_1 est déjà connue sur tout corps de nombres. La deuxième est par souci de clarté. En effet, l'estimation de $N(F_m(\mathbb{Q}), H_{h+bf}, B)$ admet trois cas particuliers, $b = 1$ et $m = 0$, $b = 2$ et $m = 0$, $b = 1$ et $m = 1$, par la méthode que nous emploierons. Nous laissons en exercice la vérification que la méthode qui suit, appliquée aux surfaces F_0 et F_1 redonne les résultats escomptés.

Théorème 3.2. — *Soient les entiers $b \geqslant 1$, $m \geqslant 2$, F_m une surface $\mathbb{Q}$-rationnelle, $F_{(t)}$ une fibre de F_m qui par le plongement φ_b a pour image $F_{(U,V)}$ et C_m la courbe irréductible de F_m de carré négatif. Alors :*

(**a**) $$N(F_m(\mathbb{Q}),\ H_{h+bf}, B) = \frac{8}{\zeta(2)} \cdot \frac{\zeta(2b+m-1)}{\zeta(2b+m)} B^2 + \begin{cases} O(B \operatorname{Log} B) & \textit{pour } b \geqslant 2 \\ O(B \operatorname{Log}^2 B) & \textit{pour } b = 2 \\ O(B^{1+\frac{1}{m+1}} \operatorname{Log} B) & \textit{pour } b = 1 \end{cases}$$

(**b**) $$N(F_{(t)}(\mathbb{Q}), H_{h+bf}, B) = \frac{2B^2}{\zeta(2)} H(U,V)^{-2b-m} + O(B \operatorname{Log} B)$$

(**c**) $$N(C_m(\mathbb{Q}), H_{h+bf}, B) = \frac{2B^{\frac{2}{b}}}{\zeta(2)} + O(B^{\frac{1}{b}} \operatorname{Log} B).$$

Remarque 3.3. — Ainsi que l'a remarqué le rapporteur de ce travail, suite à la démonstration de ce théorème, on peut interpréter la constante asymptotique pour F_m de la manière suivante. Pour chaque fibre $F_{(U,V)}$ de F_m donnée par deux entiers U et V premiers entre eux et paramétrée par $(S:T)$ décrivant $\mathbb{P}^1_{\mathbb{Q}}$, la restriction de la hauteur est donnée par

$$H_{F_{(U,V)}}((S:T)) = \max(|S| \max(|U|,|V|)^b, |T| \max(|U|,|V|)^{d-b})$$

si $\operatorname{pgcd}(S,T)=1$. La constante correspondante est donnée par

$$\begin{aligned}C_H(F_{(U,V)}) &= \frac{1}{2}\operatorname{Vol}(F_{(U,V)}(\mathbb{R}))\prod_p L_p(1,\operatorname{Pic}\overline{F_{(U,V)}})\operatorname{Vol}(F_{(U,V)}(\mathbb{Q}_p))\\ &= \frac{1}{2}\cdot\frac{4}{H(U,V)^b H(U,V)^{d-b}}\cdot\prod_p(1-\frac{1}{p})(1+\frac{1}{p})\\ &= \frac{2}{\zeta(2)}H(U,V)^{-2b-m}.\end{aligned}$$

La constante pour F_m est alors la somme sur toutes les fibres de ce terme.

Avant de présenter un corollaire, rappelons une notation usitée.

Si g' et g sont deux fonctions réelles nous noterons, $g' \gg\ll g$, s'il existe deux constantes strictement positives c_1 et c_2 vérifiant pour tout réel x :

$$c_1 g'(x) \leqslant g(x) \leqslant c_2 g'(x).$$

Corollaire 3.4. — *Soient a et b deux réels vérifiant $b \geqslant a > 0$, alors :*

$$N(F_m(\mathbb{Q}), H'_{ah+bf}, B) \gg\ll B^{2/a}$$

où H'_{ah+bf} est une hauteur de Weil associée à $ah+bf$. De même, pour tout ouvert Zariski dense U de F_m

$$N(U(\mathbb{Q}), H'_{ah+bf}, B) \gg\ll B^{2/a}.$$

Remarquons que lorsque b/a est entier, on obtient en fait une formule exacte en considérant la hauteur $(H_{h+\frac{b}{a}f})^a$.

Nous discuterons des sous-variétés accumulatrices de F_m après la démonstration de ce théorème et de son corollaire.

Démonstration. — Commençons par démontrer les assertions a) et b) du Théorème 3.2. Elles sont conséquences des deux lemmes suivants :

Lemme 3.5. — *Soient $B \geqslant 2$, $n \geqslant 1$, $\alpha > 0$, $\beta > 0$ avec $\beta - \alpha \geqslant 1$, et posons :*

$$\begin{aligned}G(B,n,d,\alpha,\beta) = \operatorname{card}\{(S,T)\in\mathbb{Z}^2\setminus\{0\} \mid |S| \leqslant |T|n^\alpha,\\ |T|n^\beta \leqslant B \text{ et } \operatorname{pgcd}(|S|,|T|)=d\}\end{aligned}$$

$$\begin{aligned}J(B,\alpha,\beta) = \operatorname{card}\{(S,T),(U,V)\in(\mathbb{Z}^2\setminus\{0\})^2 \mid |S| \leqslant |T|\max(|U|,|V|)^\alpha,\\ |T|\max(|U|,|V|)^\beta \leqslant B, \operatorname{pgcd}(S,T)=1 \text{ et } \operatorname{pgcd}(U,V)=1\}.\end{aligned}$$

Alors :

$$G(B,n,1,\alpha,\beta) = \frac{2B^2}{\zeta(2)}n^{\alpha-2\beta} + O(Bn^{\alpha-\beta}\operatorname{Log} Bn^{-\beta}),$$

et

$$J(B,\alpha,\beta)=\frac{16}{\zeta(2)}\cdot\frac{\zeta(2\beta-\alpha-1)}{\zeta(2\beta-\alpha)}\cdot B^2+\begin{cases}O(B\operatorname{Log}B) & si\ \beta-\alpha>2\\ O(B\operatorname{Log}^2 B) & si\ \beta-\alpha=2\\ O(B^{1+\frac{1}{\beta}}\operatorname{Log}B) & si\ \beta-\alpha=1\end{cases}$$

Lemme 3.6. — *Soient $B\geqslant 2$, $m\geqslant 1$, $\gamma\geqslant 2$, $\delta\geqslant 1$, et posons :*

$$G'(B,n,d,\gamma,\delta)=\operatorname{card}\{(S,T)\in\mathbb{Z}^2\setminus\{0\}\mid |T|n^\gamma<|S|,\\ |S|n^\delta\leqslant B \ et\ \operatorname{pgcd}(|S|,|T|)=d\}$$

$$J'(B,\gamma,\delta)=\operatorname{card}\{(S,T),(U,V)\in(\mathbb{Z}^2\setminus\{0\})^2\mid\ |T|\max(|U|,|V|)^\gamma<|S|,\\ |S|\max(|U|,|V|)^\delta\leqslant B,\operatorname{pgcd}(S,T)=1\ et\ \operatorname{pgcd}(U,V)=1\}.$$

Alors :

$$G'(B,n,1,\gamma,\delta)=\frac{2B^2}{\zeta(2)}n^{-\gamma-2\delta}+O(Bn^{-\gamma-\delta}\operatorname{Log}Bn^{-\delta}),$$

et

$$J'(B,\gamma,\delta)=\frac{16}{\zeta(2)}\cdot\frac{\zeta(2\delta+\gamma-1)}{\zeta(2\delta+\gamma)}B^2+O(B\operatorname{Log}B).$$

Admettons provisoirement ces deux lemmes, et vérifions que les assertions a) et b) du Théorème 3.2 en découlent. En effet on a :

$$N(F_m(\mathbb{Q}),H_{h+bf},B)=\frac{1}{4}(J(B,m,m+b)+J'(B,m,b)),$$

$$N(F_{(t)}(\mathbb{Q}),H_{h+bf},B)=\frac{1}{2}(G(B,H(U,V),1,m,m+b)\\ +G'(B,H(U,V),1,m,b)).$$

Pour démontrer les assertions a) et b), il reste donc à démontrer ces deux lemmes. Commençons par le premier, le Lemme 3.5. Posons

$$F(B,n,\alpha,\beta)=\operatorname{card}\{(S,T)\in\mathbb{Z}^2\setminus\{0\}\mid |S|\leqslant|T|n^\alpha,\ |T|n^\beta\leqslant B\}.$$

Ainsi :

$$F(B,n,\alpha,\beta)=2\sum_{1\leqslant T\leqslant Bn^{-\beta}}(2[Tn^\alpha]+1)=2B^2n^{\alpha-2\beta}+O(Bn^{\alpha-\beta}).$$

Or :

$$F(B,n,\alpha,\beta)=\sum_{1\leqslant d\leqslant Bn^{-\beta}}1\cdot G(B,n,d,\alpha,\beta).$$

Comme 1 est Dirichlet inversible, d'inverse μ, on en déduit :

$$\begin{aligned} G(B,n,1,\alpha,\beta) &= \sum_{1\leqslant d\leqslant Bn^{-\beta}} \mu(d)F\Big(\frac{B}{d},n,\alpha,\beta\Big) \\ &= \sum_{1\leqslant d\leqslant Bn^{-\beta}} \mu(d)\left\{\frac{2B^2}{d^2}n^{\alpha-2\beta}+O\Big(\frac{B}{d}n^{\alpha-\beta}\Big)\right\} \\ &= \frac{2B^2}{\zeta(2)}n^{\alpha-2\beta}+O(Bn^{\alpha-\beta}\operatorname{Log} Bn^{-\beta}) \end{aligned}$$

où la dernière égalité découle du Lemme 3.1.

Déterminons maintenant $J(B,\alpha,\beta)$. On a :

$$\begin{aligned} J(B,\alpha,\beta) &= \sum_{n\leqslant B^{1/\beta}} 8\varphi(n)G(B,n,1,\alpha,\beta) \\ &= \sum_{n\leqslant B^{1/\beta}} 8\varphi(n)\{\frac{2B^2}{\zeta(2)}n^{\alpha-2\beta}+O(Bn^{\alpha-\beta}\log Bn^{-\beta})\} \\ &= \frac{16}{\zeta(2)}\frac{\zeta(2\beta-\alpha-1)}{\zeta(2\beta-\alpha)}B^2+\begin{cases} 0(B\log B) & \text{si } \beta-\alpha>2 \\ 0(B\log^2 B) & \text{si } \beta-\alpha=2 \\ 0(B^{1+\frac{1}{\beta}}\log B) & \text{si } \beta-\alpha=1 \end{cases} \end{aligned}$$

d'après le Lemme 3.1.

La démonstration du Lemme 3.5 étant terminée et celle du Lemme 3.6 étant identique à celle ci-dessus (en faisant attention), les assertions a) et b) sont démontrées. Pour l'assertion c), on a :

$$\begin{aligned} N(C_m(\mathbb{Q}),H_{h+bf},B) &= \frac{1}{2}\operatorname{card}\ \{(S,T),(U,V)\in(\mathbb{Z}^2\setminus\{0\})^2 \mid S=1\ ,\ T=0 \\ &\qquad \text{et } \max\ (|S|H^b(U,V)\ ,\ |T|\ H(U,V)^{b+m})\leqslant B\} \\ &= \operatorname{card}\{(U,V)\in\mathbb{P}^1(\mathbb{Q}) \mid H^b(U,V)\leqslant B\} \end{aligned}$$

L'assertion c) est donc conséquence du Théorème de Schanuel [Sc 79].

Le Théorème 3.2 est donc démontré. Démontrons le Corollaire 3.4. Soient a et b deux réels, b_1 et b_2 deux entiers vérifiant $b\geqslant a>0$ et $1\leqslant b_2\leqslant b\leqslant b_1$. En particulier donc, il existe deux constantes strictement positives c_3 et c_4 tel que

$$c_3H_{h+b_2f}\leqslant H'_{h+bf}\leqslant c_4H_{h+b_1f}.$$

Ainsi :

$$N(F_m(\mathbb{Q}),H_{h+b_1f},B)\ll N(F_m(\mathbb{Q}),H'_{h+bf},B)\ll\ N(F_m(\mathbb{Q}),H_{h+b_2f},B)$$

ou encore, d'après le Théorème 3.2 :

$$N(F_m(\mathbb{Q}), H'_{h+bf}, B) \gg\ll B^2 \qquad (b \text{ réel}, \ b \geqslant 1).$$

D'autre part :

$$\begin{aligned} N(F_m(\mathbb{Q}), H'_{ah+bf}, B) &\gg\ll N\left(F_m(\mathbb{Q}), H'_{h+\frac{b}{a}f}, B^{1/a}\right) \\ &\gg\ll B^{2/a} \end{aligned}$$

L'estimation de $N(U(\mathbb{Q}), H'_{ah+bf}, B)$ est identique, d'où notre corollaire, terminant ainsi notre démonstration. □

Cherchons maintenant les sous-variétés accumulatrices des surfaces F_m pour $m \geqslant 2$. Rappelons qu'une sous-variété W de F_m sera dite accumulatrice, pour $ah + bf$, s'il existe un ouvert Zariski dense U de F_m tel que

$$N(U(\mathbb{Q}), H'_{ah+bf}, B) = o(N(W(\mathbb{Q}), H'_{ah+bf}, B)).$$

D'autre part, lorsqu'il n'existe qu'un nombre fini de sous-variétés accumulatrices, voire aucune, une sous-variété W' sera dite faiblement accumulatrice si pour tout ouvert Zariski dense U' de F_m suffisamment petit (*i.e.* ne contenant pas les sous-variétés accumulatrices)

$$N(U'(\mathbb{Q}), H'_{ah+bf}, B) \gg\ll N(W'(\mathbb{Q}), H'_{ah+bf}, B).$$

D'après le Théorème 3.2 les fibres de F_m sont faiblement accumulatrices pour les classes de diviseurs $h + bf$ $(b \geqslant 1)$ et la courbe C_m l'est pour la classe de diviseur $h + f$. En fait ce sont les seules.

Proposition 3.7. — *Soient* $a > 0$ *et* $b > 0$.

(i) *Si $b > a$, alors F_m n'admet pas de courbes accumulatrices, les fibres de F_m sont faiblement accumulatrices et ce sont les seules pour $ah + bf$.*

(ii) *Si $b = a$, alors F_m n'admet pas de courbes accumulatrices, les fibres de F_m et la courbe C_m sont faiblement accumulatrices et ce sont les seules pour $ah + bf$.*

(iii) *Si $b < a$, la courbe C_m est l'unique courbe irréductible accumulatrice, les fibres F_m sont faiblement accumulatrices et ce sont les seules pour $ah + bf$.*

Démonstration. — Soient $a > 0$, $b > 0$, une courbe irréductible C de F_m et $\alpha h + \beta f$ sa classe dans $\mathrm{Pic}(F_m)$. On a :

$$(ah + bf) \cdot (\alpha h + \beta f) = a(\alpha m + \beta) + \alpha b.$$

Démontrons les assertions (i) et (ii). D'après le Corollaire 3.4 il n'existe pas de courbes accumulatrices. Pour que C soit une courbe faiblement accumulatrice, pour $ah + bf$, il est nécessaire, d'après le Théorème de Schanuel [Sc 79] et le Corollaire 3.4, que

$$a(\alpha m + \beta) + \alpha b \;=\; a, \tag{1}$$

outre le fait d'être rationnelle.

Or $\alpha \geqslant 0$, $\alpha m + \beta \geqslant 0$, $\alpha m + \beta = 0$ si et seulement si $C = C_m$ (donc $\alpha = 1$, $\beta = -m$), et si $C \neq C_m$ alors $\alpha \geqslant 0$ et $\beta \geqslant 0$ (cf. Proposition 2.3). Ainsi les fibres F_m sont faiblement accumulatrices ($\alpha = 0, \beta = 1$), la courbe C_m l'est pour $b = a$ et ne l'est pas si $b > a$. D'autre part, si C n'est pas une fibre et n'est pas C_m, C étant irréductible, nécessairement $\alpha > 0$, et donc (1) n'est pas vérifiée, d'où les assertions (i) et (ii).

Démontrons maintenant l'assertion (iii).

D'après le Théorème de Schanuel [Sc 79], le Théorème de Batyrev-Tschinkel [Ba-Ts 96], pour que C soit accumulatrice, il est nécessaire que

$$a(\alpha m + \beta) + \alpha b \;<\; a,$$

et pour que C soit faiblement accumulatrice l'égalité (1) doit être vérifiée. Ainsi comme pour (i) et (ii), nous en déduisons l'assertion (iii).

Terminons par un début de justificatif du fait que la courbe C_m soit accumulatrice, et pas seulement faiblement accumulatrice, dans le cas (iii) : le système linéaire $|h|$ sur F_m définit un morphisme $F_m \to \mathbb{P}^{m+1}$, qui est un plongement en dehors de C_m et contracte C_m sur un point ([Be 78], chap. 4). □

Les Conjectures de Batyrev et Manin sont donc satisfaites par les surfaces F_m.

Références

[Ap 76] T. Apostol, *Introduction to analytic number theory*, Springer-Verlag (1976).

[Ba–Ma 90] V. Batyrev, Y. Manin, *Sur le nombre des points rationnels de hauteur bornée des variétés algébriques*, Math. Ann. **286** (1990), p. 27–43.

[Ba–Ts 95] V. Batyrev, Y. Tschinkel, *Manin's conjecture for toric varieties*, prépublication IHES/M/95/93.

[Ba–Ts 96] V. Batyrev, Y. Tschinkel, *Height zeta functions of toric varieties*, Algebraic geometry 5 (Manin's Festschrift), Journal Math. Sciences **82** n°1 (1996), p. 3220-3239.

[Be 78] A. Beauville, *Surfaces algébriques complexes*, Astérisque **54** (1978).

[Bi 97] H. Billard, *Propriétés arithmétiques d'une famille de surfaces* $K3$, Compo. Math. **108** (1997), p. 247–275.

[Bir 62] B. Birch, *Forms in many variables*, Proc. Royal. Soc. London **265** (1962), p. 245–263.

[Co–Si 86] G. Cornell, J. Silverman (Eds), *: Arithmetic geometry*, Springer Verlag (1986).

[Fr–Ma–Ts 89] J. Franke, Y. Manin, Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. **95**, (1989) p. 421–435.

[Gr–Ha 78] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Wiley Interscience (1978).

[Ha 69] R. Hartshorne, *Curves with high self-intersection on algebraic surfaces*, Publ. Math. IHES **36** (1969) p. 111–125.

[Ha 77] R. Hartshorne, *Algebraic geometry*, Springer Verlag (1977).

[La 83] S. Lang, *Fundamentals of diophantine geometry*, Springer Verlag (1983).

[Ne 65] A. Néron, *Quasi-fonctions et hauteurs sur les variétés abéliennes*, Ann. of Math. **82**, N°2 (1965) p. 249–331.

[Pe 95] E. Peyre, *Hauteurs et mesures de Tamagawa sur les variétés de Fano*, Duke Math. J. **79**, N°1 (1995) p. 101–218.

[Sc 79] S. Schanuel, *Heights in number fields*, Bull. Soc. Math. France **107** (1979), p. 433–449.

[Si 91] J. Silverman, *Rational points on* $K3$ *surfaces : a new canonical height*, Invent. Math. **105** (1991), p. 347–373.

[Th 93] J. Thunder, *Asymptotic estimates for rational points of bounded height on flag varieties*, Compo. Math. **88** (1993), p. 155–186.

HERVÉ BILLARD, Université de Brest, Département de Mathématiques, 6 av. Le Gorgeu, B.P.809, 29285 Brest Cedex, France • *E-mail :* `Herve.Billard@univ-brest.fr`

Astérisque
251, 1998, p. 91–258

TAMAGAWA MEASURES ON UNIVERSAL TORSORS AND POINTS OF BOUNDED HEIGHT ON FANO VARIETIES

by

Per Salberger

Abstract. — Let X be a Fano variety over a number field. An Arakelov system of v-adic metrics on the anticanonical line bundle on X gives rise to a height function on the set of rational points and to a new kind of adelic measures on the universal torsors over X.

The aim of the paper is to relate the asymptotic growth of the number of rational points of bounded height on X to volumes of adelic spaces corresponding to the universal torsors over X.

Introduction

One important but very difficult problem in diophantine geometry is to count the number $f(B)$ of rational points of height at most B on a projective variety over a number field k and to study the asymptotic growth of the counting function when $B \to \infty$. If $A \subset \mathbb{P}^n$ is an abelian variety, then it was proved by Néron (cf. [**62**]) that $f(B)/(\log B)^{\operatorname{rk} A(k)/2}$ converges to a constant depending on $A \subset \mathbb{P}^n$ and the height function. There is a precise adelic conjecture about the rank of $A(k)$, but this has only been established for classes of elliptic curves E over $\mathbb{Q}$ for which $E(\mathbb{Q})$ is of rank 0 or 1.

For Fano varieties there is a (mostly conjectural) theory of counting functions. This theory was initiated by Manin who made some striking observations (cf. [**23**], [**3**], [**42**], [**43**]) about the counting functions for special classes of Fano varieties X under their anticanonical embeddings. Similar observations for other linear systems were made by Batyrev and Manin [**3**]. The heights involved are the "usual" multiplicative heights of Weil depending on the ground

2000 ***Mathematics Subject Classification.*** — 14G05, 11G35, 28C99.
Key words and phrases. — Fano varieties, counting functions, universal torsors, adelic measures.

field k and the choice of coordinates (see [**37**, p. 50]). There may exist accumulating closed subsets with many rational points (e.g. lines on cubic surfaces). Manin therefore counts the number $f_U(B)$ of rational points of height at most B on sufficiently small Zariski open k-subsets U of X. He notices that for a number of Fano varieties X there are open k-subsets U of X such that

$$(*) \qquad f_U(B) = CB(\log B)^{\operatorname{rk}\operatorname{Pic} X - 1}(1 + o(1))$$

for an anticanonical height function. Very recently Batyrev and Tschinkel [**5**] have found that this cannot be true for all Fano varieties. But there are important classes of Fano varieties (cf. [**23**], [**52**], [**7**], [**4**]) for which asymptotic formulas of the above form have been established, and it is interesting to find common features of these results. The motivation for this paper is to obtain a better understanding of the constant C in these formulas and to develop a framework which might be useful for the study of counting functions of other classes of Fano varieties.

The first systematic attempt to understand the constant C in the asymptotic formulas is due to Peyre [**52**], who introduced several new ideas. He defined Tamagawa numbers for Fano varieties, thereby generalizing the classical Tamagawa numbers studied by Weil [**67**]. To define these, Peyre uses a system of ν-adic "metrics" for the places ν of k on the analytic anticanonical line bundle over $X(k_\nu)$ satisfying an adelic condition. He associates to any such "adelic metric" a height function H on $X(k)$ and a measure on the adelic space $X(A_k)$ and suggests that the constant C in $(*)$ should be equal to the product of the Tamagawa number $\tau(X)$ of $X(A_k)$ and an invariant $\alpha(X)$ depending only on effective cone in $\operatorname{Pic} X$.

Peyre assumes that the Picard group of X_K for an algebraic closure K of k contains a $\mathbb{Z}$-basis which is invariant under the action of the Galois group $\operatorname{Gal}(K/k)$. It is clear from the work of Batyrev and Tschinkel in [**7**] and [**4**] that some restriction of this kind is needed. They prove $(*)$ for toric varieties and obtain the constant

$$(**) \qquad C = \alpha(X)\tau(X)h^1(X)$$

where $h^1(X)$ is the order of $H^1(\operatorname{Gal}(K/k), \operatorname{Pic} X_K)$.

We shall in the paper "explain" the appearance of $h^1(X)$ in $(**)$ by means of a new kind of Tamagawa numbers for universal torsors over X. The main idea is that the constant C is related to the volumes of some adelic spaces defined by the universal torsors $\mathcal{T}$ over X. Universal torsors were introduced by Colliot-Thélène and Sansuc [**15**] as a generalization of the classical descent varieties of elliptic curves studied by Fermat, Mordell and Weil. The main applications of this theory so far have been in the study of the Hasse principle and weak approximation for various classes of rational varieties.

We shall use universal torsors as a natural tool when counting rational points on Fano varieties. One central idea will be to extend the height function on $X(k)$ to suitable subquotients of the adelic spaces $\mathcal{T}(A_k)$ by means of certain adelic splittings associated to the universal torsors $\pi : \mathcal{T} \to X$. For toric varieties this reduces the original counting problem to an adelic lattice point problem. We obtain thereby another proof of the asymptotic formula of Batyrev and Tschinkel [**4**] for toric varieties over $\mathbb{Q}$.

We now give a description of the content of the 11 sections of the paper.

In the first section, we study in detail the analytic manifold structure $X_{\mathrm{an}}(k_\nu)$ on $X(k_\nu)$ and metrics (which we will call *norms* from now on) on analytic line bundles on $X_{\mathrm{an}}(k_\nu)$. For anticanonical line bundles with a norm we recall the measure constructed by Peyre and explain its relation to the classical construction of a measure from a global differential form. For submersions

$$Y_{\mathrm{an}}(k_\nu) \longrightarrow X_{\mathrm{an}}(k_\nu)$$

we give a relative version of Peyre's construction and associate a positive linear map

$$\Lambda : C_c(Y_{\mathrm{an}}(k_\nu)) \longrightarrow C_c(X_{\mathrm{an}}(k_\nu))$$

to norms on relative anticanonical line bundles.

In section 2, we assume that ν is non-archimedean and study norms for the compact open subsets $\Xi_\nu(o_\nu) \subseteq X_{\mathrm{an}}(k_\nu)$ defined by models Ξ_ν of $X \times k_\nu$ over the valuation ring o_ν in k_ν. We relate in (2.14) the volume of $\Xi_\nu(o_\nu)$ with respect to the measure determined by the above norm to the density of the reduction of $\Xi_\nu(o_\nu)$ modulo finite powers of the maximal ideal m_ν in o_ν. As a consequence, we get an explicit formula (cf. (2.14)(b)) for the volume of $\Xi_\nu(o_\nu)$ with respect to any measure defined by a norm on $\Xi_\nu(o_\nu)$. This formula holds also when Ξ_ν has bad reduction and X is non proper. If X is proper, then $\Xi_\nu(o_\nu) = X_{\mathrm{an}}(k_\nu)$ and we get a formula for the volume of $X_{\mathrm{an}}(k_\nu)$. But there are also important applications of this formula to universal torsors and other non-proper varieties.

In section 3, we study invariant norms over local fields on the relative anticanonical line bundles for X-torsors $\pi : \mathcal{T} \to X$ under arbitrary algebraic groups G. We then concentrate on the norms defined by relative differential forms. If G is a torus T, then there is a canonical norm of this kind which we will baptize the *order norm.* Now using this relative norm we obtain for each norm on $X_{\mathrm{an}}(k_\nu)$ an "induced" norm on $\mathcal{T}_{\mathrm{an}}(k_\nu)$ which in its turn defines an "induced" measure on $\mathcal{T}_{\mathrm{an}}(k_\nu)$.

In section 4, we consider varieties X over number fields k and the adelic topological space $X(A_k)$. We have not found any modern rigorous version of Weil's account [**66**] and we therefore explain how to use schemes of finite presentation in EGA to develop the foundations for adelic spaces. We then generalize Peyre's notion of "adelic metric" and his adelic measures in many ways. We introduce e.g. relative adelic norms for smooth morphisms $\pi : Y \to X$ over X and positive linear maps $\Lambda : C_c(Y(A_k)) \to C_c(X(A_k))$. It is thereby necessary to consider convergence factors which vary among the fibres and to consider fibres over A_k not defined over k even if $\pi : Y \to X$ is defined over k.

In section 5, we restrict to torsors $p : \mathcal{T} \to X$ for varieties over number fields and study adelic norms and measures for them. When X is smooth and proper with

$$H^1_{\mathrm{Zar}}(X, \mathcal{O}_X) = H^2_{\mathrm{Zar}}(X, \mathcal{O}_X) = 0$$

and with torsion-free Néron-Severi group, then there is a notion of universal torsors $\pi : \mathcal{T} \to X$. The ν-adic order norms form an adelic norm which in its turn gives rise to a positive linear map

$$\Lambda : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(X(A_k)).$$

This map depends on the choice of convergence factors. But one can choose these to be inverse to the convergence factors for $X(A_k)$ so that the induced measure on $C_c(\mathcal{T}(A_k))$ requires no convergence factors. Using this measure on $\mathcal{T}(A_k)$, we define Tamagawa numbers for universal torsors.

In section 6, we recall the Brauer group and torsor obstructions to weak approximation of Manin, Colliot-Thélène and Sansuc. By using their theory together with results of Ono on the arithmetic of tori, we relate our Tamagawa numbers for universal torsors to the Tamagawa numbers of Peyre. The factor $h^1(X)$ enters naturally.

In section 7, we modify the original growth conjectures of Manin in order to exclude the Fano varieties containing infinitely many weakly accumulating subvarieties. This is not so original and closely related to notions of Manin and Peyre (cf. [**51**]). We also generalize and refine Peyre's Tamagawa conjecture for Fano varieties by means of our Tamagawa numbers for universal torsors.

In section 8, we study the geometry of universal torsors $\pi : \mathcal{T} \to X$ over smooth complete toric varieties which are trivial over the unit element of the k-torus U in X. We identify them with the toric varieties studied by Cox in his article [**16**] and find that they are open subsets of affine spaces.

In section 9, we consider toric varieties over local fields k_ν. We give an explicit description of the norms for universal torsors obtained by inducing the norms of Batyrev-Tschinkel [**7**] and of the corresponding measures (cf. (9.12)). Another central idea is the introduction of a *canonical toric splitting*

$$\check{\psi}_\nu : X(k_\nu) \longrightarrow \mathcal{T}(k_\nu)/T(k_\nu)_{\mathrm{cp}}$$

of the map

$$\check{\pi}_\nu : \mathcal{T}(k_\nu)/T(k_\nu)_{\mathrm{cp}} \longrightarrow X(k_\nu).$$

induced by the principal universal torsor $\pi : \mathcal{T} \to X$. Here $T(k_\nu)_{\mathrm{cp}}$ is the maximal compact subgroup of the analytic group $T(k_\nu)$ defined by the Néron-Severi torus.

In section 10, we consider toric varieties over number fields and the induced adelic norm on the universal torsor obtained from the induced ν-adic norms in section 9. The product map of all $\check{\psi}_\nu$ gives rise to a continuous canonical toric splitting

$$\check{\psi}_A : X(A_k) \longrightarrow \mathcal{T}(A_k)/T(A_k)_{\mathrm{cp}}$$

of the map from $\mathcal{T}(A_k)/T(A_k)_{\mathrm{cp}}$ to $X(A_k)$ induced by π. By means of $\check{\psi}$ we give a new torsor theoretic interpretation of the heights of Batyrev-Tschinkel [**7**] and an interpretation of the constant $C = \alpha(X)\tau(X)h^1(X)$. There are several analogies with Bloch's use of torsors (cf. [**8**], [**47**]) to interpret the Néron-heights and the Birch/Swinnerton-Dyer/Tate conjecture for abelian varieties. The universal torsors over toric varieties play the same role as the biextensions do for abelian varieties. The constant $\alpha(X)\tau(X)h^1(X)$ is interpreted by means of the Laurent expansion of an adelic integral which is a continuous approximation on the universal torsors of the zeta functions for X considered by Manin, Batyrev and Tschinkel. We also indicate how these zeta functions can be studied for toric varieties without making use of group structure of the torus.

In section 11, we give a proof of the conjectures of Manin and Peyre for split toric Fano varieties over $\mathbb{Q}$ (first proved in [**4**]). Our method is essentially an extension of the method developed by Schanuel [**56**] for projective spaces and by Peyre [**52**] for certain special blow-ups $X \to \mathbb{P}^n$ of projective spaces. But we make a more systematical use of the universal torsors than in [**52**] and we use geometric invariants of the fans in the study of the main terms and the error terms. Our asymptotic formulas are slightly more precise than in [**52**], [**7**], [**4**] and of the type

$$CB(\log B)^{r-1} + O(B(\log B)^{r-3/2+\varepsilon}), \quad r = \operatorname{rk}\operatorname{Pic} X.$$

while the formulas in (op. cit.) are of the type $CB(\log B)^{r-1}(1+o(1))$. But the real motivation for giving another proof of the theorem of Batyrev and Tschinkel is that many of the arguments can be used for other classes of Fano varieties like moduli spaces of ordered sets of points on the projective line.

The use of universal torsors in the study of Manin's conjectures is quite natural. We used them to give an upper bound for the counting functions of del Pezzo surfaces of degree 5 in a talk at Bern at the Borel seminars in the summer semester 1993. Some months later the author received a preliminary version of Peyre's important paper [**52**] which has had a strong influence on this paper. There he made implicit use of descent varieties for some classes of toric varieties without emphasizing their toric structure and without making use of the descent theory of Colliot-Thélène and Sansuc. Most of the theory of this paper has been developed in an attempt to use universal torsors to count points on toric varieties and to refine Peyre's Tamagawa number conjecture for general Fano varieties. It is thus not surprising that Peyre himself recently considered universal torsors. We understood from his visit at ETH 1996 that he has also found that the use of universal torsors leads to a term $h^1(X)$ in the conjectured asymptotic formula by means of adelic zeta-functions on the universal torsors. His line of thoughts was somewhat different, however, and did not use the induced measures of this paper. We refer to the paper of Peyre [**51**] in this volume for his vision of the role of universal torsors in the theory of counting functions for Fano varieties. We have only given some brief comments here about his work since we received it when this paper was almost completed. There is some overlapping between [**51**] and sections 5, 6 and 10 of this paper. But there are also many differences. Peyre consider certain equivariant partial compactifications of the universal torsors while we systematically avoid compactifications. He does not consider adelic splittings of the universal torsors, but uses instead the notion of "system of heights" for arbitrary linear systems. He uses complete intersections as a prototype for Fano varieties while we have chosen to dicuss toric varieties in detail.

We have learned much about the arithmetic of toric varieties from the impressive papers of Batyrev and Tschinkel [**7**] and [**4**]. Their approach using adelic harmonic analysis for Manin's zeta functions is elegant and conceptual, but depends strongly on the underlying group action on the toric variety. The advantage with universal torsors is that they exist for all Fano varieties. One can e.g. prove the best upper bound $O(B(\log B)^4)$ for del Pezzo surfaces of degree 5 by means of the descent theoretic approach in this paper.

We have after this paper was completed received the paper [**6**] of Batyrev and Tschinkel. The measure theory of our paper (e.g. (1.22) and (4.25)-(4.28)) has applications to the theory of L-primitive fibrations in [**6**]. One can also avoid the use of compactifications and the reference to Denef's work in (op. cit.) and give more intrinsic constructions of the measures involved by means of the results here.

We would like to thank Batyrev, Manin, Peyre and Tschinkel for discussions on counting problems for Fano varieties.

1. Analytic manifolds over locally compact fields

Throughout this section k denotes a non-discrete locally compact field of characteristic zero. It is well known (cf. [**66**]) that k is either totally disconnected (in which case it is a finite extension of the p-adic number field $\mathbb{Q}_p$) or connected (in which case it is either $\mathbb{R}$ or $\mathbb{C}$). Denote by $|\ |: k \to \mathbb{R}$ the absolute value *normalized* in the following way. If $k = \mathbb{R}$, let $|\alpha| = \max(\alpha, -\alpha)$

and if $k = \mathbb{Q}_p$, let $|up^l| = p^{-l}$ for units u in $\mathbb{Z}_p$. If k is a finite extension of $k_0 = \mathbb{R}$ or $k_0 = \mathbb{Q}_p$, let $| \ | : k \to \mathbb{R}$ be the map obtained by composing the norm $N : k \to k_0$ with $| \ | : k_0 \to \mathbb{R}$. The topology of k is induced by the absolute value $| \ | : k \to \mathbb{R}$ and k is complete with respect to $| \ | : k \to \mathbb{R}$ in all cases. We shall assume that all norms and all seminorms on vector spaces over k are compatible with the normalized absolute values $| \ | : k \to \mathbb{R}$ under scalar multiplication ([**36**, p. 33], [**38**, p. 44]).

If k is a finite extension of $k_0 = \mathbb{Q}_p$, let o be the maximal $\mathbb{Z}_p$-order (cf. [**54**]) in k. Then o is a complete discrete valuation ring. The inverse different o^D is defined by

$$o^D := \{\alpha \in k : \mathrm{Tr}(\alpha\beta) \in \mathbb{Z}_p \text{ for all } \beta \in o\}.$$

Denote by μ the Haar measure on the additive locally compact group k normalized in the following way. If $k = \mathbb{R}$, let μ be the usual Lebesgue measure dx and if $k = \mathbb{C}$, let $\mu = dx$ be the measure $2dudv$ for the real (resp. imaginary) part u and v. If k is a finite extension of $\mathbb{Q}_p$, choose μ to be self-dual as in [**66**, Ch. VII, §2]. This means that $\mu(o)\mu(o^D) = 1$. Hence $\mu(o) = 1$ if and only if k is a unramified over $\mathbb{Q}_p$. The Haar measure is sometimes (cf. e.g. [**67**], [**52**]) normalized such that $\mu(o) = 1$ for all finite extensions of $\mathbb{Q}_p$. The normalization chosen here implies that we get no discriminant factors for volumes of adelic measures. If K is a number field, and A_K the adelic group [**66**] of K, then $\mu(A_K/K) = 1$ for the restricted product measure of the local measures above (cf. [**67**, 2.1.3] for more details). Also, we get no discriminant factor (cf. (5.21), (6.12)) in our definition of Tamagawa numbers. Instead, we get a discriminant factor in the local results in (2.14) and (2.15).

Most of the theory of analytic manifolds over locally compact fields are obvious generalizations of the theory of analytic manifolds over $\mathbb{R}$ and $\mathbb{C}$ in differential and analytic geometry. We first recall some definitions (1.1)-(1.3) from Serre's book [**59**, Part II].

Definition 1.1. — Let $V \subset k^n$ be open and let $f : V \to k$ be a function. Then f is said to be analytic in V if for each point $P = (a_1, a_2, \ldots, a_n) \in V \subset k^n$, we may find an open neighbourhood of P

$$N_{P,r} = \{(x_1, x_2, \ldots, x_n) \in k^n : |x_i - a_i| < r, \quad i = 1, \ldots, n\} \subset V$$

such that f is defined by a convergent power series in $k\,[\![x_1 - a_1, \ldots, x_n - a_n]\!]$ on $N_{P,r}$.

Definition 1.2. — Let T be a topological space.

(a) An *n-chart* (or *n-coordinate map*) on T is a homeomorphism $\phi : U \to V$ between open subsets $U \subset T$ and $V \subset k^n$.

(b) Two n-charts $\phi : U \to V$ and $\phi' : U' \to V'$ are *compatible* if the maps $\phi' \circ \phi^{-1}|\phi(U \cap U')$ and $\phi \circ \phi'^{-1}|\phi'(U \cap U')$ are analytic.

(c) An *n-atlas* A on T is a family of n-charts $\phi_j : U_j \to V_j$, $j \in J$ such that $T = \cup_{j\in J} U_j$ and such that ϕ_{j_1} and ϕ_{j_2} are *compatible* for any $j_1, j_2 \in J$.

(d) Two n-atlases

$$A_i := \{\psi_i : U_i \to V_i,\ i \in I\}, A_j := \{\phi_j : U_j \to V_j,\ j \in J\}$$

are said to be *compatible* if ψ_i and ϕ_j are compatible for any $\psi_i \in A_i$, $\phi_j \in A_j$.

Compatibility of n-atlases is an equivalence relation (cf. LG 3.2 in op. cit.).

Definition 1.3. — Let T be a topological space. An analytic n-manifold structure on T (over k) is an equivalence class of compatible n-atlases on T.

Note that the analytic manifold structures considered in this paper are such that the dimension n is the same at all points of the manifold.

Let M_1 and M_2 be two analytic manifolds (over k) of possibly different dimensions. A continuous map $f : M_1 \to M_2$ is said to be an analytic morphism if it is "locally given by analytic functions" (cf. LG 3.6 in op. cit. for a precise definition).

Definition 1.4. — An *analytic vector bundle of rank r over an n-manifold* M consists of a family $\{E_P\}_{P\in M}$ of r-dimensional vector spaces over k parametrized by M, together with an analytic $(n+r)$-manifold structure on $E = \cup_{P\in M} E_P$ such that:

(i) The projection map $\pi : E \to M$ taking E_P to P is analytic.

(ii) For every $P_0 \in M$, there is an open neighbourhood $U \subset M$ of P_0 and an analytic isomorphism $\phi_U : \pi^{-1}(U) \to U \times k^r$ which restricts to a linear homeomorphism from E_P to $\{P\} \times k^r$ for each $P \in U$.

The tangent bundle $\mathrm{Tan}(M) \to M$ and the cotangent bundle $\mathrm{Cot}(M) \to M$ are analytic bundles of rank n on an analytic n-manifold M (see [**12**], [**38**, Ch. XXII] for the global properties and [**59**, Ch. 3, §8] for the local properties of these bundles).

Let $p : E \to M$ be a vector bundle and let $f : N \to M$ be an analytic morphism. The *fibre product*

$$N \times_M E = \{(Q, s) \in N \times E : f(Q) = p(s)\}$$

is an analytic submanifold of $N \times E$ (see [**59**, LG 3.26]) and the projection map

$$p_N : N \times_M E \longrightarrow N$$

is a vector bundle of the same rank as $p : E \to M$. (The trivializations of $p_N : N \times_M E \to N$ and their transition functions are defined by pulling back the trivializations of the bundle on M and their transition functions, cf. e.g.

[**25**, p. 68] for the case $k = \mathbb{C}$.) We shall in the sequel write f^*E for $N \times_M E$ and call $p_N : N \times_M E \to N$ the *pullback bundle* of $p : E \to M$ under $f : N \to M$.

Operations on vector spaces induce operations on vector bundles (cf. e.g. [**25**, pp. 66-67]). There exists thus for each vector bundle $p : E \to M$ a *dual* vector bundle $p^* : E^* \to M$ and exterior product bundles $\Lambda^s E \to M$. The *cotangent bundle* $\mathrm{Cot}(M) \to M$ is dual to the *tangent bundle* $\mathrm{Tan}(M) \to M$. We shall write $\mathbf{det}\, E$ for the line bundle $\Lambda^r E$ when $r = \mathrm{rk}\, E$ and call $\mathbf{det}\,\mathrm{Cot}(M)$ the *canonical bundle* and $\mathbf{det}\,\mathrm{Tan}(M)$ the *anticanonical bundle* of M.

If $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ are two analytic vector bundles, then there are analytic bundles $E_1 \oplus E_2 \to M$ and $E_1 \otimes E_2 \to M$. The analytic manifold $E_1 \oplus E_2$ is equal to the fibre product $E_1 \times_M E_2$ and hence an analytic submanifold of $E_1 \times E_2$ (see [**59**, LG 3.26]). If $g : E_1 \to E_2$ is an analytic map with $\pi_1 = \pi_2 g$ such that corresponding maps between the fibres $g_P : E_{1,P} \to E_{2,P}$ are linear for each $P \in M$, then we define vector bundles $\ker g \to M$ (resp. $\mathrm{coker}\, g \to M$) when g is surjective (resp. injective).

Definition 1.5. — Let $p : E \to M$ be an (analytic) vector bundle over an analytic manifold M (over k). A *norm* (resp. *seminorm*) on $p : E \to M$ is a continuous map

$$\| \ \| : E \longrightarrow [0, \infty)$$

such that the restriction to E_P is a norm (resp. seminorm) on the vector space E_P (over k) for each $P \in M$. A *normed vector bundle* is a vector bundle with a norm.

Remarks 1.6

(a) Silverman [**63**] and Peyre [**52**] define a metric

$$\| \ \| : E \longrightarrow [0, \infty)$$

on a line bundle $p : E \to M$ to be a collection of vector space norms

$$\| \ \|_x : E_x \longrightarrow [0, \infty), \quad x \in M$$

for which the map

$$P \in U \longrightarrow \|s(P)\| \in [0, \infty)$$

is continuous for any local analytic section $s : U \to p^{-1}(U)$ of $p : E \to M$. It is obvious that a norm on a line bundle $p : E \to M$ is a metric. Conversely, each metric $\| \ \| : E \to [0, \infty)$ is a norm. To see this, one works locally at M and reduces to the case $E = M \times k$ by means of (1.4)(ii). Then use the fact

$$P \in M \longrightarrow s(P) = (P, 1) \in E$$

is an analytic section of $pi : E \to M$ with

$$\|(P, \alpha)\| = |\alpha| \|s(P)\|$$

for all $P \in M$, $\alpha \in k$.

(b) Suppose we are given one norm $\| \ \|_1$ and one seminorm $\| \ \|_2$ on a line bundle $p : E \to M$ over an analytic manifold M. Then there is a unique continuous function $r : M \to [0, \infty)$ such that

$$\|s\|_2 = r(p(s)) \|s\|_1$$

for all $s \in E$. If M is compact and $\| \ \|_2$ is a norm then there are positive constants C and D such that

$$C\|s\|_1 \leq \|s\|_2 \leq D\|s\|_1$$

for all $s \in E$.

Any vector space norm on k^r defines a norm on the trivial vector bundle $M \times k^r$ by means of the projection $M \times k^r \to k^r$. We now give some further examples of norms and seminorms on (analytic) vector bundles.

Examples 1.7

(a) Let $p : E \to M$ be a line bundle and let ω be a continuous global section of the dual line bundle $p^\vee : E^\vee \to M$. Then ω may be regarded as a continuous map $E \to M \times k$ which restricts to the linear map $E_P \to P \times k$ for each $P \in M$. We obtain a seminorm $\| \ \| : E \to [0, \infty)$ by sending $s \in E$ to the absolute value of the image of $\omega(s)$ under the projection $M \times k \to k$. In particular, any continuous differential form ω of maximal degree on M defines a seminorm on the anticanonical line bundle $\mathbf{det}\,\mathrm{Tan}(M)$. This seminorm is a norm if and only if ω vanishes nowhere.

(b) Let $p_1 : E_1 \to M$ and $p_2 : E_2 \to M$ be two line bundles and

$$\| \ \|_1 : E_1 \longrightarrow [0, \infty), \qquad \| \ \|_2 : E_2 \longrightarrow [0, \infty)$$

be seminorms on E_1 resp. E_2. Then there exists a product seminorm

$$\| \ \| : E_1 \otimes E_2 \longrightarrow [0, \infty)$$

such that

$$\|(s_1 \otimes s_2)\| = \|s_1\|_1 \cdot \|s_2\|_2$$

for two sections $s_1 \in E_1$, $s_2 \in E_2$ with $p_1(s_1) = p_2(s_2)$. This seminorm $\| \ \|$ is a norm if and only if $\| \ \|_1$ and $\| \ \|_2$ are norms.

(c) Let $p : E \to M$ be a vector bundle and let $p_N : N \times_M E \to N$ be the pullback bundle of $p : E \to M$ under $f : N \to M$. Then there is a seminorm

$$f^*\| \ \| : N \times_M E \longrightarrow [0, \infty)$$

on the pullback bundle associated to each seminorm $\| \;\|$ on $p: E \to M$. This is obtained by composing the projection map $N \times_M E \to E$ with $\| \;\| : E \to [0, \infty)$. The seminorm $f^*\| \;\|$ is a norm if $\| \;\|$ is a norm. We shall then call $f^*\| \;\|$ the pullback norm of $\| \;\|$.

In the next example we shall need the concept of a lattice in a vector bundle.

Definition 1.8. — Let o be a complete discrete valuation ring as above with quotient field k. Let M be an analytic manifold over k and $p : E \to M$ be an analytic vector bundle of rank r. An *analytic lattice* of $p : E \to M$ is a submanifold L of E such that

(a) $L_P := L \cap E_P$ is an o-lattice in E_P for each $P \in M$.
(b) For every $P_0 \in M$, there is an open neighbourhood $U \subset M$ of P_0 and an analytic isomorphism $\lambda_U : L \cap p^{-1}(U) \to U \times o^r$ which restricts to an o-module isomorphism from L_P to $\{P\} \times o^r$ for each $P \in U$.

Example 1.9. — Let $p : E \to M$, L, o, k be as in (1.8) and let p be a uniformizing parameter of o. If $P \in M$ and $s \in E_P$, let

$$\|s\| = \inf\{|\pi^m| : \pi^m s \in L_P\}.$$

Then,

$$\| \;\| : E \longrightarrow [0, \infty)$$

is a norm. (The verification is local on the base so it suffices (see (1.8)(b) to treat the trivial case where p is the projection from $E = M \times k^r$ to M and $L = M \times o^r$.)

The following result is a reformulation of a result due to Peyre [**52**].

Theorem 1.10. — *Let M be an analytic Hausdorff n-manifold and let $\| \;\|$ be a seminorm on the anticanonical bundle $\mathbf{det}\,\mathrm{Tan}(M)$ of M. Then there is a unique positive linear functional Λ on the vector space $C_c(M)$ of real valued continuous functions with compact support on M such that for any n-chart*

$$\phi : U \longrightarrow V \subset k^n,$$

$$\phi(x) = (\phi_1(x), \dots, \phi_n(x)) = (x_1, \dots, x_n), x \in U \subset M$$

and any $f \in C_c(M)$ with support in U the following equality holds:

$$\Lambda f = \int_V f(\phi^{-1}(x_1, \dots, x_n)) \left\| \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} \right\| dx_1 dx_2 \cdots dx_n.$$

Moreover, if $\| \;\|$ is a norm and $f : M \to \mathbb{R}_{\geq 0}$ is a non-negative function in $C_c(M)$, then $\Lambda f = 0$ if and only if $f = 0$.

Proof. — Let $\phi_\alpha : U_\alpha \to V_\alpha \subset k^n$, $\phi_\beta : U_\beta \to V_\beta \subset k^n$ be two charts and suppose that $\phi \in C_c(M)$ have support in $U_a \cap U_b$. Then Peyre (op. cit.) deduces from the Jacobian formula for change of variables in a multiple integral ([**67**, p. 14]) that the two integrals over V_a and V_b coincide so that Λf is well defined. His argument uses only that $\| \ \|$ is a seminorm although it is formulated for a metric (cf. (1.6)(a)). Now once the compatibility of integrals under transition of charts has been established, one concludes by a standard argument with partitions of unity (see th. 5.1 in [**38**, Ch. IX] and note that no paracompactness assumption is needed).

To prove the last statement, it suffices by partition of unity to consider the case when there is an isomorphism $\phi : M \to V$ onto an open subset V of k^n. But then the assertion follows from the integral formula above, thereby completing the proof. □

Since M is locally compact and Hausdorff there exists according to Riesz representation theorem [**38**, Ch. IX, §2] a unique positive Borel measure m on M such that:

1.11 (i) If W is open, then

$$m(W) = \sup \Lambda f,$$

where $f \in C_c(M)$ runs over all functions with support in W such that $0 \leq f(x) \leq 1$ for all $x \in M$.

1.11 (ii) If B is a Borel set, then $m(B) = \inf m(W)$, W open $\supset M$.

This measure m has the following properties:

1.11 (iii) If K is compact, then $m(K)$ is finite.

1.11 (iv) If B is open or a union of countably many Borel sets of finite measure, then $m(B) = \sup m(K)$, K compact $\subset B$.

1.11 (v) $\Lambda f = \int_M f dm$ for any $f \in C_c(M)$.

Definition 1.12. — We shall call m *the positive Borel measure on M determined by the seminorm* $\| \ \| : \mathbf{det}\,\mathrm{Tan}(M) \to [0, \infty)$ and $\Lambda : C_c(M) \to \mathbb{R}$ the *positive functional determined* by $\| \ \| : \mathbf{det}\,\mathrm{Tan}(M) \to [0, \infty)$.

If $\| \ \|$ is a norm, then it follows from (1.11) and the last assertion in (1.10) that $m(W) > 0$ for all non-empty open subsets W of M.

Example 1.13. — Let $\| \ \| : \mathbf{det}\,\mathrm{Tan}(M) \to [0, \infty)$ be a seminorm defined by a continuous differential form ω on M (see 1.7). Then the measure m is just the "usual" positive Borel measure $|\omega|$ on M determined by ω (see e.g. [**38**, Ch. XXIII, §3]) for the case $k = \mathbb{R}$ and [**12**, §10] for general locally compact fields k). The construction of m in (1.10), (1.11) is thus a generalization of the classical volume form construction.

Definition 1.14. — A positive Borel measure m on a locally compact Hausdorff space M will be called σ-regular if it satisfies the properties (1.11)(ii), (1.11)(iii) and (1.11)(iv) in (1.11). It will be called regular if it satisfies (1.11)(ii), (1.11)(iii) and the following stronger version of (1.11)(iv):

$(*)$ If W is a Borel set, then $m(W) = \sup m(K)$, K compact W.

The space M is called σ-finite with respect to m if M is a union of countably many Borel sets $B \subseteq M$ of finite measure $m(B)$.

Remarks 1.15

(a) It follows from the definitions that (cf. [**38**, p. 257]) a σ-regular positive Borel measure m on M is regular if M is σ-finite with respect to m. The σ-finiteness condition is satisfied (see (1.11)(iii)) if M is σ-compact (i.e. a union of countably many compact subsets). It is sometimes useful for integration with respect to product measures.

(b) The positive Borel measure m determined by a seminorm

$$\| \ \| : \mathbf{det}\,\mathrm{Tan}(M) \to [0, \infty)$$

on an analytic Hausdorff n-manifold M is σ-regular (by 1.11) and regular if M is σ-compact.

Lemma 1.16. — *Let M_1 and M_2 be two locally compact Hausdorff spaces and let m_1 (resp. m_2) be a σ-regular positive Borel measure on M_1 (resp. M_2). Then there exists a unique σ-regular positive Borel measure m on $M := M_1 \times M_2$ such that*

$$\int_M h dm = \int_{M_1} h_1 dm_1 \int_{M_2} h_2 dm_2$$

for any function $h \in C_c(M)$ which is a product

$$h(P_1, P_2) = h_1(P_1) h_2(P_2)$$

of two functions $h_1 \in C_c(M_1), h_2 \in C_c(M_2)$.

Moreover, if $h(P_1, P_2) \in C_c(M)$, then

$$g(P_2) := \int_{M_1} h(P_1, P_2) dm_1 \in C_c(M_2)$$

and

$$\int_M h dm = \int_{M_2} g dm_2 = \int_{M_2} \left(\int_{M_1} h(P_1, P_2) dm_1 \right) dm_2.$$

Proof. — This follows from [**11**, Ch. III, §5] and Riesz representation theorem. $\square$

Let M_1 and M_2 be two analytic manifolds over k. The topological space $M := M_1 \times M_2$ carries a natural analytic manifold structure described in [**59**, p. LG 3.7]. Let $pr_i : M \to M_i$, $i = 1, 2$ be the two projections and let $pr_i^* \operatorname{Tan}(M_i)$, $i = 1, 2$ be the pull back bundles (cf. (1.7(c)) of the tangent bundles on M_1 and M_2. Then there exists a natural identification

$$\operatorname{Tan}(M) = pr_1^* \operatorname{Tan}(M_1) \oplus pr_2^* \operatorname{Tan}(M_2)$$

of vector bundles over M which induces a canonical isomorphism

$$\mathbf{det} \operatorname{Tan}(M) = pr_1^*(\mathbf{det} \operatorname{Tan}(M_1)) \otimes pr_2^*(\mathbf{det} \operatorname{Tan}(M_2)).$$

Suppose that we are given seminorms:

$$\| \ \|_{M_i} : \mathbf{det} \operatorname{Tan}(M_i) \longrightarrow [0, \infty), \quad i = 1, 2.$$

Then there are pullback seminorms $pr_i^* \| \ \|_{M_i}$ on $pr_i^*(\mathbf{det} \operatorname{Tan}(M_i))$, $i = 1, 2$ (cf. (1.7)(c)) which are norms if (and only if) $\| \ \|_{M_i}$ are norms.

Theorem 1.17. — *Let M_1 and M_2 be two Hausdorff analytic manifolds over k and let M be the product manifold $M = M_1 \times M_2$. Let*

$$\| \ \|_{M_i} : \mathbf{det} \operatorname{Tan}(M_i) \longrightarrow [0, \infty), \quad i = 1, 2$$

be two seminorms and let

$$\| \ \|_M : \mathbf{det} \operatorname{Tan}(M) \longrightarrow [0, \infty)$$

be the product seminorm of the two pullback seminorms

$$pr_i^* \| \ \|_{M_i} : pr_i^*(\mathbf{det} \operatorname{Tan}(M_i)) \longrightarrow [0, \infty).$$

Let m_i, $i = 1, 2$ (resp. m) be the σ-regular positive Borel measures determined by $\| \ \|_{M_i}$, $i = 1, 2$ (resp. $\| \ \|_M$). Then the following assertions hold.

(a)
$$\int_M h dm = \int_{M_1} h_1 dm_1 \int_{M_2} h_2 dm_2$$
for any function $h \in C_c(M)$ which is a product $h(P_1, P_2) = h_1(P_1) h_2(P_2)$ of two functions $h_1 \in C_c(M_1)$, $h_2 \in C_c(M_2)$.

(b) *If $h(P_1, P_2) \in C_c(M)$, then*
$$g(P_2) = \int_{M_1} h(P_1, P_2) dm_1 \in C_c(M_2)$$
and
$$\int_M h dm = \int_{M_2} g dm_2 = \int_{M_2} \left(\int_{M_1} h(P_1, P_2) dm_1 \right) dm_2.$$

Proof

(a) Using charts and partitions of unity and the definition of the functionals on $C_c(M_1)$, $C_c(M_2)$ and $C_c(M)$ corresponding to m_i, $i = 1, 2$ (see 1.10) it is clear that it suffices to treat the case when $M_1 = k^q$ and $M_2 = k^t$, $q, t \in \mathbb{Z}_{>0}$.

Let us choose coordinates $(x_1, \dots, x_q)$ for M_1 and $(x_{q+1}, \dots, x_{q+t})$ for M_2. Then by definition we have that

$$\int_M hdm =$$
$$\int_{k^{q+t}} h(x_1, \dots, x_{q+t}) \left\| \tfrac{\partial}{\partial x_1} \wedge \dots \wedge \tfrac{\partial}{\partial x_q} \right\| \left\| \tfrac{\partial}{\partial x_{q+1}} \wedge \dots \wedge \tfrac{\partial}{\partial x_{q+t}} \right\| dx_1 dx_2 \cdots dx_{q+t}$$

where the seminorms should be read as $\| \ \|_{M_1}$ resp. $\| \ \|_{M_2}$.

Let

$$f_1(x_1, \dots, x_q) = h_1(x_1, \dots, x_q) \left\| \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_q} \right\|$$

and

$$f_2(x_{q+1}, \dots, x_{q+t}) = h_2(x_{q+1}, \dots, x_{q+t}) \left\| \frac{\partial}{\partial x_{q+1}} \wedge \dots \wedge \frac{\partial}{\partial x_{q+t}} \right\|.$$

Then (cf. (1.6)(a)) $f_1 \in C_c(k^q)$ and $f_2 \in C_c(k^t)$. Therefore, by an elementary version of Fubini's theorem it follows that

$$\int_M hdm$$
$$= \int_{k^q} f_1(x_1, \dots, x_q) dx_1 dx_2 \cdots dx_q \int_{k^{q+t}} f_2(x_{q+1}, \dots, x_{q+t}) dx_{q+1} \cdots dx_{q+t}$$

where the right hand side is equal to $\int_{M_1} h_1 dm_1 \int_{M_2} h_2 dm_2$ by definition.

(b) This follows from (a) and the previous lemma. □

Now let $f : N \to M$ be an analytic morphism from an m-manifold to an n-manifold. Then $f^* \operatorname{Tan}(M) = N \times_M \operatorname{Tan}(M)$ is an analytic vector bundle over N of rank m. It is endowed with a canonical morphism (cf. [**59**, LG 3.12] and [**12**])

$$f' : \operatorname{Tan}(N) \longrightarrow N \times_M \operatorname{Tan}(M)$$

of vector bundles over N. The analytic morphism $N \to M$ is called an *immersion* (resp. a *submersion*) if it "locally looks like" a linear injection (resp. surjection) $k^n \to k^m$ (cf. LG 3.12-14 in (op. cit.) for the precise meaning of this phrase). There is also a proof in (op. cit.) that f is an immersion (resp. a submersion) if and only if f' is injective (resp. surjective).

Definition 1.18

(a) If $f : N \to M$ is an immersion, then the *normal bundle* $\mathrm{Nor}(N/M)$ is defined to be the cokernel of $f' : \mathrm{Tan}(N) \to N \times_M \mathrm{Tan}(M)$.

(b) If $f : N \to M$ is a submersion, then the *relative tangent bundle* $\mathrm{Tan}(N/M)$ is defined to be the kernel of f(see [**12**, 8.1]). The *relative cotangent bundle* $\mathrm{Cot}(N/M)$ is the dual vector bundle of $\mathrm{Tan}(N/M)$.

If $f : N \to M$ is a submersion, then $\mathbf{det}\,\mathrm{Tan}(N/M)$ is an (analytic) line bundle over N, which we shall call the *relative anticanonical line bundle* of $f : N \to M$. The restriction of $\mathbf{det}\,\mathrm{Tan}(N/M)$ to a fibre $N_P, P \in M$ of $f : N \to M$ is equal to the anticanonical line bundle $\mathbf{det}\,\mathrm{Tan}(N_P)$ of N_P. There is also a canonical isomorphism

$$\mathbf{det}\,\mathrm{Tan}(N) = \mathbf{det}(\mathrm{Tan}(N/M)) \otimes f^*(\mathbf{det}\,\mathrm{Tan}(M)) \tag{1.19}$$

of line bundles over N induced by the exact sequence of vector bundles over N

$$0 \longrightarrow \mathrm{Tan}(N/M) \longrightarrow \mathrm{Tan}(N) \xrightarrow{f'} N \times_M \mathrm{Tan}(M) \longrightarrow 0. \tag{1.20}$$

Notation 1.21. — Let $f : N \to M$ be a submersion between two analytic Hausdorff manifolds and let

$$\|\ \|_{N/M} : \mathbf{det}\,\mathrm{Tan}(N/M) \longrightarrow [0, \infty)$$

be a seminorm on $\mathbf{det}\,\mathrm{Tan}(N/M) \to N$.

If $P \in f(N)$, let

$$\Lambda_P : C_c(N_P) \longrightarrow \mathbb{R}$$

be the positive linear functional on the fibre N_P of $f : N \to M$ at P determined by the restriction

$$\|\ \|_P : \mathbf{det}\,\mathrm{Tan}(N_P) \longrightarrow [0, \infty)$$

of $\|\ \|_{N/M}$ to N_P (see (1.10)) and let $h_P \in C_c(N_P)$ be the restriction of $h \in C_c(N)$. Then we denote by

$$\Lambda_{N/M}(h) : M \longrightarrow \mathbb{R}$$

the function with value $\Lambda_P(h_P)$ for $P \in f(N)$ and with value 0 if $P \notin f(N)$.

A linear map

$$\Lambda_{N/M} : C_c(N) \longrightarrow C_c(M)$$

is said to be positive if any non-negative function $h \in C_c(N; \mathbb{R})$ is sent to a non-negative function $g \in C_c(M; \mathbb{R})$.

The following result is stated (without proof) by Serre [**61**, p. 83] in the special case of seminorms $\|\ \|_{N/M}$ and $\|\ \|_M$ defined by global sections of $\mathbf{det}\,\mathrm{Cot}(N/M)$ and $\mathbf{det}\,\mathrm{Cot}(M)$ (cf. (1.7)(a)).

Theorem 1.22. — *Let $f : N \to M$ be a submersion between two analytic Hausdorff manifolds. Let*

$$\| \ \|_{N/M} : \mathbf{det}\,\mathrm{Tan}(N/M) \longrightarrow N$$

$$\| \ \|_{M} : \mathbf{det}\,\mathrm{Tan}(M) \longrightarrow M$$

be seminorms and let

$$\| \ \|_{N} : \mathbf{det}\,\mathrm{Tan}(N/M) \otimes f^{*}\,\mathbf{det}(\mathrm{Tan}(M)) \longrightarrow [0, \infty)$$

be the seminorm on $\mathbf{det}\,\mathrm{Tan}(N) \to N$ *obtained by taking the product of* $\| \ \|_{N/M}$ *and* $f^{*}\| \ \|_{M}$ *(cf. (1.7)(b), (1.7)(c) and (1.19)).*

Then the following holds,

(a) $\Lambda_{N/M}$ *is a positive linear map from* $C_c(N)$ *to* $C_c(M)$.

(b) *Let* $\Lambda_M : C_c(M) \to \mathbb{R}$ *(resp.* $\Lambda_N : C_c(N) \to \mathbb{R}$*) be the positive functionals determined by* $\| \ \|_M$ *(resp.* $\| \ \|_N$*). Then* $\Lambda_N = \Lambda_M \circ \Lambda_{N/M}$.

(c) *Let m resp. n be the positive σ-regular Borel measures on M (resp. N) determined by* $\| \ \|_M$ *(resp.* $\| \ \|_N$. *Let $\theta(P)$, $P \in f(M)$ be the positive σ-regular Borel measure on N_P (cf. (1.11)) corresponding to the positive functional $\Lambda_P : C_c(N_P) \to \mathbb{R}$ determined by the restriction to the fibre N_P of $f : N \to M$ at P of* $\| \ \|_{N/M}$ *to N_P (cf. (1.10), (1.21)). Then,*

$$\int_N h dn = \int_M \Lambda_{N/M}(h) dm = \int_{P \in f(M)} \left(\int_{N_P} h_P d\theta(P) \right) dm$$

for any $h \in C_c(N)$. The integral over N_P is defined to be 0 if N_P is empty.

Proof

(a) It is obvious from the definition in (1.21) that $\Lambda_{N/M}$ is linear and positive. It thus remains to show that $\Lambda_{N/M}(h) \in C_c(M)$ for $h \in C_c(N)$. To show this, we first note that $f(N)$ is an open subset of M (since any submersion is open) and that the support of $\Lambda_{N/M}(h)$ is contained in the compact subset $f(\mathrm{Supp}\, h)$ of $f(N)$. It is therefore sufficient to prove that $\Lambda_{N/M}(h)$ is continuous in an open neighbourhood of $P = f(Q) \in M$ for each $Q \in N$.

Let t resp. s be the dimensions of N resp. M. Then since f is a submersion there exist open analytic neighbourhoods U of Q, V of P and W of 0 in k^{t-s} and an analytic isomorphism $\psi : U \to V \times W$ such that $f(U) = V$ and the following diagram commutes (cf. [**59**, LG 3.16])

(1.23)
$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\psi"'] & V \\ V \times W \arrow[ur, "pr_1"'] & \end{tikzcd}$$

If we cover N with such subsets U and apply the linearity of Λ_P for all $P \in M$, then it is clear from the "usual" argument with partitions of unity [**38**, p. 270] that it suffices to show that $g := \Lambda_{N/M}(h) \in C_c(M)$ for $h \in C_c(N)$ with support in an open subset of U as above. Further, if $h_U \in C_c(U)$ is the restriction of h, then $\Lambda_{N/M}(h) = \Lambda_{U/V}(h_U)$ on V and $\Lambda_{N/M}(h) = 0$ outside V. We may thus assume that $N = U$ and $M = V$.

Since ψ is an isomorphism and (1.23) commutes and the constructions of Λ_P (see (1.10)) are functorial under isomorphisms we may and shall further assume that $U = V \times W$ and $f = pr_1$. Then $N = M \times W$ and $\mathbf{det}\,\mathrm{Tan}(N/M) = pr_2^*(\mathbf{det}\,\mathrm{Tan}(W))$ for the projection $pr_2 : M \times W \to W$.

If $\| \ \|_{N/M} : \mathbf{det}\,\mathrm{Tan}(N/M) \to [0,\infty)$ is the pullback $pr_2^*\| \ \|_W$ of a seminorm $\| \ \|_W$ on $\mathbf{det}\,\mathrm{Tan}(W) \to W$, then it follows from (1.17)(b) that $\Lambda_{N/M}$ is a positive linear map from $C_c(N)$ to $C_c(M)$.

In the general case, choose a norm $\| \ \|_W$ on $\mathbf{det}\,\mathrm{Tan}(W) \to W$, and let $\lambda_{N/M}$ be the linear map $C_c(M) \to C_c(N)$ determined by the norm $pr_2^*\| \ \|_W$ on

$$\mathbf{det}\,\mathrm{Tan}(N/M) \to N.$$

Then there exists a unique continuous function $r : N \to [0,\infty)$ such that $\|s\|_{N/M}$ is equal to $r(Q)pr_2^*\|s\|_W$ for any section $s \in \mathbf{det}\,\mathrm{Tan}(N/M)$ above $Q \in N$ (see (1.6)(b)). Therefore, $\Lambda_{N/M}(h) = \lambda_{N/M}(rh)$ for all $h \in C_c(N)$. Therefore, the general case follows from the already known case where $\| \ \|_{N/M} = pr_2^*\| \ \|_W$.

(b) One reduces immediately to the case $N = M \times W$ and $f = pr_1$ by means of the same arguments as in the proof of (a). If $\| \ \|_{N/M}$ is the pullback $pr_2^*\| \ \|_W$ of a seminorm $\| \ \|_W$ on $\mathbf{det}\,\mathrm{Tan}(W) \to W$, then it follows from (1.17)(b) that $\Lambda_N = \Lambda_M \circ \Lambda_{N/M}$.

In the general case, choose a norm $\| \ \|_W$ on $\mathbf{det}\,\mathrm{Tan}(W) \to W$ and define the linear map $\Lambda_{N/M} : C_c(M) \to C_c(N)$ as above. Let $r : N \to [0,\infty)$ be the function described in the proof of (a) and let $\Lambda_N : C_c(N) \to \mathbb{R}$ be the functional defined by the product seminorm $\| \ \|_{M\times W}$ of $pr_1^*\| \ \|_M$ and $pr_2^*\| \ \|_W$. Then $\|s\|_N = r(Q)\|s\|_{M\times W}$ for any section $s \in \mathbf{det}\,\mathrm{Tan}(N)$ above. Therefore $\Lambda_N(h) = \lambda_N(rh)$ and $\Lambda_{N/M}(h) = \lambda_{N/M}(rh)$ for all $h \in C_c(N)$ (see (a)). This combined with $\lambda_N = \Lambda_M \circ \lambda_{N/M}$ (see (1.17)(b)) implies that $\Lambda_N = \Lambda_M \circ \Lambda_{N/M}$.

(c) This follows from (b) and Riesz representation theorem. □

2. Measures and densities for algebraic varieties over local fields

Let k denote a non-discrete locally compact field of characteristic zero and let X be a smooth k-variety. It is well known that the set of k-points $X(k)$ on X can be given a natural analytic manifold structure $X_{\mathrm{an}}(k)$ and that this construction is functorial. The best reference for the applications here seems to be chapter III of the book [**53**].

We first define the underlying topology of $X(k)$. If X is an affine k-scheme of finite type, then the k-topology on $X(k)$ is defined to be the coarsest topology for which all maps $X(k) \to k$ defined by regular functions on X are continuous. We shall say that a subset of $X(k)$ is k-open (resp. k-closed) if it is open (resp. closed) in the k-topology.

In the following result (2.1) we always refer to the k-topology when regarding the set of k-points $X(k)$ on an affine k-scheme X as a topological space.

Proposition 2.1

(a) *Let $g : X \to Y$ be a closed k-immersion of two affine schemes of finite type over k. Then g induces a closed immersion $X(k) \to Y(k)$ of topological spaces*

(b) *Let $Z = X \times_k Y$ be a product of two affine schemes of finite type over k. Then $Z(k) = X(k) \times Y(k)$ as topological spaces.*

(c) *Let $g : X \to Y$ be an open k-immersion of two affine schemes of finite type over k. Then g induces an open immersion $X(k) \to Y(k)$ of topological spaces.*

(d) *The k-topology on $A^r_k(k)$ is equal to the direct product topology on $k^r = A^r_k(k)$.*

Proof. — (a) and (b) are obvious and the proofs left to the reader.

(c) X may be identified with the closed k-subvariety of $Y \times A^1_k = Y \times \operatorname{Spec} k[T]$ defined by the equation $hT = 1$ for some invertible function h in the coordinate ring of Y. By (a) and (b) one gets that $X(k)$ is the closed subset of the product space $Y(k) \times A^1_k(k)$ defined by $hT = 1$. Now use the fact that the projection of $X(k) \subset Y(k) \times A^1_k(k)$ onto $Y(k)$ is an open immersion of topological spaces.

(d) It suffices by (b) to consider the case $X = A^1_k = \operatorname{Spec} k[T]$. Then any regular function on X is a polynomial in T with coefficients in k. This combined with the fact that k is a topological ring implies that the k-topology on $X(k)$ is the coarsest topology for which $T : X(k) \to k$ is continuous. Hence $T : X(k) \to k$ is a homeomorphism, as was to be proved. □

Proposition and definition 2.2. — *Let X be a separated scheme of finite type over k. Then the family of all k-open subsets of sets of k-points on all Zariski open affine k-subschemes form the base of a topology of $X(k)$. If X is affine this topology is the k-topology above. If X is not affine we call this topology the k-topology of $X(k)$.*

Proof. — The statement follows immediately from (2.1)(c). □

It is obvious from the definition that a k-morphism $g : Y \to X$ between two separated schemes of finite type over k induces a continuous map $Y(k) \to X(k)$ with respect to the k-topologies. It is also clear that the assertions in (2.1)(a)-(c) hold in the non-affine case.

Proposition 2.3

(a) *$X(k)$ is a Hausdorff space with respect to the k-topology.*

(b) *There is a countable base of open subsets with compact support for the k-topology. In particular, $X(k)$ is σ-compact and paracompact with respect to the k-topology.*

(c) *If $X = P_k^n$, then the k-topology of $X(k)$ is the quotient topology arising from the canonical map $k^{n+1}\backslash(0) \to P_k^n$.*

(d) *If X is a closed k-subscheme of P_k^n for some n, then $X(k)$ is compact with respect to the k-topology.*

(e) *If X is a smooth, proper and geometrically connected k-scheme, then $X(k)$ is compact with respect to the k-topology.*

Proof

(a) X is Zariski closed in $X \times_k X$ by assumption. The diagonal of $X(k) \times X(k)$ is therefore closed in the k-topology.

(b) It suffices to prove this for a finite set of open affine subvarieties in a covering of X. To show this, use (2.1) and the fact that the locally compact field k has such a base.

(c) This is checked locally using the standard covering of P_k^n by $n+1$ affine n-spaces.

(d) It suffices by the non-affine version of (2.1)(a) to prove this in the case $X = P_k^n$. Then use the fact that the compact subset of $k^{n+1}\backslash(0)$ consisting of $(n+1)$-tuples $(x_0, x_1, \dots, x_n)$ with $\sup_i |x_i| = 1$ is mapped onto P_k^n under the canonical map from $k^{n+1}\backslash(0)$ to P_k^n.

(e) There exists by Chow's lemma [**32**, p 107] a proper k-morphism $g : Y \to X$ from a projective k-variety Y and a Zariski non-empty open k-subvariety U of X such that g induces an isomorphism of $g^{-1}(U)$ to U. This implies by (a) and(c) that $g(Y(k))$ is a k-closed subset of $X(k)$ containing $U(k)$. But X is smooth and hence $U(k)$ dense in $X(k)$ (cf. [**53**, lemma 3.2]). Therefore $g(Y(k)) = X(k)$, thereby completing the proof. □

The smoothness condition in (2.3)(e) is not necessary, but will be satisfied for all the applications in this paper. One can prove that any proper morphism $g : Y \to X$ between k-varieties gives rise to a proper continuous map $Y(k) \to X(k)$ between topological spaces as remarked by Serre in [**61**, Ch. 2].

Next, let o be a complete discrete valuation ring and $\Phi : \Xi \to \operatorname{Spec} o$ be a separated morphism of finite type. We shall in the sequel write $\Xi(o)$ for the set of o-morphisms $\sigma : \operatorname{Spec} o \to \Xi$ such that $\Phi\sigma$ is the identity morphism on $\operatorname{Spec} o$.

Proposition 2.4. — *Let k be the quotient field of a complete discrete valuation ring o. Let $\iota : \Xi_1 \to \Xi_2$ be a closed immersion of separated o-schemes of finite type over o and let $\Xi_1 \to \Xi_2$ be the closed immersion of their generic fibres. Then the natural maps $\Xi_i(o) \to X_i(k)$, $i = 1, 2$ are injective and $\Xi_1(o) = X_1(k) \cap \Xi_2(o)$ in $X_2(k)$.*

Proof. — The assertion follows from Grothendieck's valuative criterion for separated resp. proper morphisms (cf. e.g. [**32**, Ch. 2 §4]). If $\Xi_1 \to \Xi_2$ is an arbitrary proper o-morphism between separated schemes of finite type over o, then the diagram

$$\begin{array}{ccc} \Xi_1(o) & \longrightarrow & \Xi_2(o) \\ \downarrow & & \downarrow \\ X_1(k) & \longrightarrow & X_2(k) \end{array}$$

is set-theoretically cartesian. □

Corollary 2.5. — *Let k be a finite extension of $\mathbb{Q}_p$, and let o be the maximal $\mathbb{Z}_p$-order in k. Also, let Ξ be a separated scheme of finite type over o. Then the following holds.*

(a) *$\Xi(o)$ is a compact and open subset of $X(k)$ in the k-topology.*
(b) *Let X_{cl} be the scheme theoretical closure of X in Ξ. Then the natural map from $X_{\mathrm{cl}}(o)$ to $\Xi(o)$ is bijective.*

Proof

(a) There is a covering of Ξ by finitely many Zariski open affine o-schemes. We may therefore assume that Ξ is affine over o and choose a closed o-immersion $\iota : \Xi \to A^n_o$. We may then by (2.1) and (2.4) reduce to the obvious case $\Xi = A^n_o$.

(b) X_{cl} is also separated and of finite type over o with X as generic fibre. The assertion therefore follows from (2.4). □

We shall not need (2.5)(b) in the sequel, but it makes it possible to reduce to the case of reduced models over o.

Definition 2.6. — A *k-variety* X is a geometrically connected, separated scheme of finite type over k.

We shall in this paper use the word k-variety in this sense also for other fields than locally compact fields. Thus a smooth k-variety will always be assumed to be geometrically integral in this paper.

The definition of the k-topology works for arbitrary separated schemes of finite type over k. For the definition of an analytic manifold structure on $X(k)$ we assume from now on that X is a *smooth* k-variety.

To define an n-chart ($n = \dim X$) around a k-point X of $X(k)$ choose an affine neighbourhood U equipped with an étale k-morphism $h : U \to A^n_k$. It follows from the inverse function theorem [**59**, part II, Ch. 3, §9] that there is a k-open neighbourhood N_x in $U(k)$ of X such that the restriction of h to N_x defines an homeomorphism between N_x and a k-open neighbourhood of $y = h(x)$ in $k^n = A^n_k(k)$. Moreover, any two such n-charts defined in this way are compatible (see [**53**, p. 111]). Also, any two n-atlases consisting of such n-charts are compatible (in the sense of (1.2)). There is therefore a (canonical) analytic manifold structure on $X(k)$ (see p. 112 in op. cit.). If k is a complete discrete valuation field, then it follows from (2.3)(b) and a result of Serre [**58**] that $X_{\mathrm{an}}(k)$ is a disjoint union of "balls".

Each k-morphism $g : Y \to X$ gives rise to a (unique) analytic morphism $g_{\mathrm{an}} : Y(k) \to X(k)$ between the corresponding analytic manifolds so that one becomes a covariant functor from the category of smooth algebraic k-varieties to the category of analytic manifolds over k (cf. e.g. [**62**] for the case $k = \mathbb{C}$).

The following result is well-known.

Proposition 2.7

(a) *Let $g : Y \to X$ be a smooth k-morphism between smooth k-varieties of constant relative dimension d. Then $g_{\mathrm{an}} : Y(k) \to X(k)$ is a submersion of relative dimension d. In particular, g_{an} is open.*

(b) *Let $g : Y \to X$ be a closed immersion of smooth k-varieties of codimension d. Then $g_{\mathrm{an}} : Y(k) \to X(k)$ is a closed immersion of codimension d.*

Proof

(a) Let $\Omega_{Y/X}$ (resp. $\Omega_{Y/k}$) be the relative cotangent sheaf of Y/X (resp. Y/k). Then the natural sequence

$$0 \longrightarrow g^*\Omega_{X/k} \longrightarrow \Omega_{Y/k} \longrightarrow \Omega_{Y/X} \longrightarrow 0$$

is an exact sequence of locally free coherent sheaves on Y which corresponds to an exact sequence of vector bundles over Y (cf. [**30**, 16.5.12 and 17.2.3])

$$0 \longrightarrow V(\Omega_{Y/X}) \longrightarrow V(\Omega_{Y/k}) \longrightarrow V(\Omega_{X/k}) \times_X Y \longrightarrow 0$$

where $V(\mathcal{E})$ means the spectrum of the quasi-coherent symmetric algebra $S(\mathcal{E})$ on $\mathcal{E}$ as in [**29**, 1.7.8]. Now note that the analytic tangent bundle $\mathrm{Tan}(Y_{\mathrm{an}}(k)) \to Y_{\mathrm{an}}(k)$ is equal to the analytic morphism $V(\Omega_{Y/k})_{\mathrm{an}}(k) \to$

$Y_{\mathrm{an}}(k)$ associated to the algebraic tangent bundle $V(\Omega_{Y/k}) \to Y$ [**53**, p. 113]. Hence the last sequence gives rise to an exact sequence of analytic vector bundles as in (1.20) with $N = Y_{\mathrm{an}}(k)$ and $M = X_{\mathrm{an}}(k)$. This implies that g is a submersion by the criterion mentioned just before (1.18).

(b) The proof is similar, but one uses the exact sequence (cf. [**30**, 17.2.5])

$$0 \longrightarrow V(\Omega_{Y/k}) \longrightarrow V(\Omega_{Y/k}) \times_X Y \longrightarrow V(\mathcal{C}_{Y/X}) \longrightarrow 0$$

where $V(\mathcal{C}_{Y/X})$ is the normal bundle of Y/X corresponding to the algebraic conormal sheaf of $g : Y \to X$. □

The main application of (2.7)(a) will be in connection with the integral formula in (1.22).

For the rest of this section we assume the following:

(A) k is a finite extension of $\mathbb{Q}_p$, o is the maximal $\mathbb{Z}_p$-order in k

(B) $\Xi : X \to \operatorname{Spec} o$ is a separated morphism of finite type (e.g. a proper morphism) such that the generic fibre X is smooth and geometrically connected.

Then, by (2.5)(a) there exists a natural compact analytic manifold structure over k on the k-open subset $\Xi(o)$ of $X_{\mathrm{an}}(k)$. We shall denote this compact analytic manifold by $\Xi_{\mathrm{an}}(o)$.

Next, let $\Omega_{\Xi/o}$ be the relative algebraic cotangent sheaf of $\Phi : \Xi \to \operatorname{Spec} o$. It is a coherent (but not necessarily locally free) sheaf on X and one may form the spectrum $V(\Omega_{\Xi/o})$ of the quasi-coherent symmetric $\mathcal{O}_\Xi$-algebra $S(\Omega_{\Xi/o})$ of $\Omega_{\Xi/o}$ [**29**, 1.7]. Following [**30**, 16.5.12] we will write $T_{\Xi/o} = V(\Omega_{\Xi/o})$. The morphism $T_{\Xi/o} \to \Xi$ is affine, hence separated. The composite map $T_{\Xi/o} \to \operatorname{Spec} o$ is thus also separated and its generic fibre is equal to the algebraic tangent bundle $T_{X/k} = V(\Omega_{X/k})$ of X/k.

The analytic tangent bundle $\operatorname{Tan}(X_{\mathrm{an}}(k)) \to X_{\mathrm{an}}(k)$ is equal to the analytic morphism $T_{X/k}(k) \to X(k)$ associated to the algebraic tangent bundle $T_{X/k} \to X$ (cf. [**53**, p. 113]). Also, the analytic tangent bundle $\operatorname{Tan}(\widetilde{M})$ over $\widetilde{M} = \Xi_{\mathrm{an}}(o)$ is equal to the inverse image of the analytic tangent bundle $\operatorname{Tan}(X_{\mathrm{an}}(k)) \to X_{\mathrm{an}}(k)$ over $\widetilde{M}$. One may therefore regard $T_{\Xi/o}(o)$ as a compact open subset of the analytic manifold $\operatorname{Tan}(\widetilde{M})$.

Proposition 2.8. — *Let Ξ/o be as in (A) and (B) and let $\operatorname{Tan}(\widetilde{M}) \to \widetilde{M}$ denote the analytic tangent bundle of $\widetilde{M} = \Xi_{\mathrm{an}}(o) \subseteq X_{\mathrm{an}}(k)$. Then $L := T_{\Xi/o}(o)$ is an analytic o-lattice of $\operatorname{Tan}(\widetilde{M})$ in the sense of (1.18).*

Proof. — The question is local so we may reduce to the affine case. Choose a closed o-immersion $\iota : \Xi \to A_o^m$ and make use of the natural epimorphism (see

[**32**, II.8])

$$\iota^*\Omega_{A_o^m/o} \longrightarrow \Omega_{\Xi/o}.$$

This defines in its turn (cf. [**30**, 16.5.12]) a closed immersion

$$T_{\Xi/o} \longrightarrow T_{A_o^m/o} \times_{A_o^m} \Xi$$

and hence by (2.5) an equality of subsets of $T_{A_k^m/k}$:

$$T_{\Xi/o}(o) = T_{X/k}(k) \cap T_{A_o^m/o}.$$

But it is noted in [**57**, 3.3] that the right hand side is "un champ localement constant de résaux" for affine k-varieties $X \subseteq A_o^m \times_o k$. This means in our language that it is an analytic o-lattice in $\mathrm{Tan}(\widetilde{M})$. The proof is thus complete. □

By taking exterior products of L one gets an analytic o-lattice $\mathbf{det}\, L$ in $\mathbf{det}\, \mathrm{Tan}(\widetilde{M})$.

Definition 2.9. — We shall call

$$\mathbf{det}\, L \subset \mathbf{det}\, \mathrm{Tan}(\widetilde{M})$$

the analytic o-lattice defined by Ξ/o. The norm

$$\| \ \| : \mathbf{det}\, \mathrm{Tan}(\widetilde{M}) \longrightarrow [0, \infty)$$

determined by $\mathbf{det}\, L \subset \mathbf{det}\, \mathrm{Tan}(\widetilde{M})$ (cf. (1.9)) will be called the *model norm determined by* Ξ/o. The positive Borel measure (see (1.12)) m on $\widetilde{M}$ defined by the model norm

$$\| \ \| : \mathbf{det}\, \mathrm{Tan}(\widetilde{M})[0, \infty)$$

above will be called the *model measure on* $\widetilde{M} = \Xi_{\mathrm{an}}(o)$ *determined by* Ξ/o.

Remark 2.10. — It follows from simple functoriality properties of algebraic cotangent sheaves that the analytic o-lattice in $\mathrm{Tan}(M)$ is uniquely determined by the closure X_{cl} of X in Ξ. This implies that the model norms and the model measures on $\widetilde{M} = \Xi_{\mathrm{an}}(o) = X_{\mathrm{cl,an}}(o)$ determined by Ξ/o and X_{cl}/o coincide (cf. (2.5)(b)).

We now partition $\Xi(o)$ into congruence classes. Let π be a uniformizing parameter of o and let $o_r := o/(\pi^r)$ for each positive integer r. Denote by $\Xi(o_r)$ the set of morphisms $\sigma_r : \mathrm{Spec}(o_r) \longrightarrow \Xi$ such that $\Phi\sigma_r : \mathrm{Spec}(o_r) \to \mathrm{Spec}(o)$ corresponds to the reduction map $o \to o_r$. We shall also write F for o_1 and $\Xi(F)$ for $\Xi(o_1)$.

Lemma 2.11. — *Let $y \in \Xi(F)$. Then there is a natural bijection between sections $\sigma \in \Xi(o)$ which are sent to $y \in \Xi(F)$ under the reduction map $\Xi(o) \to \Xi(F)$ and morphisms $\operatorname{Spec} o \to \operatorname{Spec} \mathcal{O}_{\Xi,y}$ which are sections of the morphism $\operatorname{Spec} \mathcal{O}_{\Xi,y} \to \operatorname{Spec} o$ induced by Φ. This is also true if we replace the stalk $\mathcal{O}_{\Xi,y}$ by the completion along its maximal ideal.*

Proof. — By definition of morphisms of schemes any morphism $\sigma : \operatorname{Spec} o \to \Xi$ which is a lifting of $y \in \Xi(F)$ factorizes over an affine morphism $\operatorname{Spec} o \to \operatorname{Spec} \mathcal{O}_{\Xi,y}$ which is a section of the morphism $\operatorname{Spec} \mathcal{O}_{\Xi,y} \to \operatorname{Spec} o$ induced by Φ. This proves the first statement. The second follows from the first and the fact that o is complete. $\square$

We shall need the following explicit version of the inverse function theorem on power series $h(y_1, \dots, y_m) \in o[\![y_1, \dots, y_m]\!]$. We will write $y = (y_1, \dots, y_m)$, $y^\varepsilon = \prod_{i=1}^m y_i^{\varepsilon_i}$ for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{Z}^{(m)}$ and $\deg(\varepsilon)$ for the degree $\varepsilon_1 + \dots + \varepsilon_m$.

Lemma 2.12. — *Let*

$$h_j(y_1, \dots, y_m) \in o[\![y_1, \dots, y_m]\!], \quad j = 1, \dots, m$$

be formal power series of the form

$$h_j(y) = y_j + \sum \alpha_\varepsilon y^\varepsilon$$

where $\alpha_\varepsilon \in (\pi^{\deg(\varepsilon)-1})$ and $\varepsilon \in \mathbb{Z}^{(m)}$ runs over exponents $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ with $\varepsilon_1, \dots, \varepsilon_m \geq 0$ of degree ≥ 2.

Then there are unique formal power series

$$\gamma_j(y) = y_j + \sum \beta_\varepsilon y^\varepsilon \in o[\![y_1, \dots, y_m]\!]$$

such that:

$$h_j(\gamma_1(y_1, \dots, y_m), \dots, \gamma_m(y_1, \dots, y_m)) = y_j$$

for $j = 1, \dots, m$.

Moreover, $\varepsilon \in \mathbb{Z}^{(m)}$ only runs over exponents $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$; $\varepsilon_1, \dots, \varepsilon_m \geq 0$ of degree $\varepsilon_1 + \dots + \varepsilon_m \geq 2$ and the coefficients $\beta_\varepsilon \in (\pi^{\deg(\varepsilon)-1})$. In particular, all power series converge.

Proof. — This is an immediate consequence of a more general result proved on pp. 11-12 in chapter LG 2 of [**59**]. See in particular the statements 1. and 2. on the top of page LG 2.12. $\square$

Theorem 2.13. — *Let $\Phi : \Xi \to \operatorname{Spec} o$ be a separated morphism of finite type with smooth geometrically connected generic fibre of dimension d and let m be the model measure on $\widetilde{M} = \Xi_{\mathrm{an}}(o)$. Let $\sigma_0 \in \Xi(o)$ and let e be the largest integer such that $H^0(\operatorname{Spec} o, \sigma_0^* \Omega_{\Xi/o})$ contains a torsion subgroup isomorphic* *to* $o/(\pi^e)$. *Finally, let* $D(\sigma_0, \Xi, r) \subset \Xi_{\mathrm{an}}(o)$ *be the* k*-open subset of all* $\sigma \in \Xi(o)$ *with the same reduction as* σ_0 *in* $\Xi_r(o_r)$.

Then,

$$m(D(\sigma_0, \Xi, r)) = (\mu(o)/\operatorname{Card}(o_r))^d$$

for $r > e$.

Proof. — It follows from (2.11) that $D(\sigma_0, \Xi, 1)$ is contained in $\sum(o)$ for each Zariski open neighbourhood $\sum$ of σ_0 in Ξ. It is therefore sufficient to treat the affine case and we shall choose a closed immersion $\iota : \Xi \to A_o^m$ and coordinates $x_1, \ldots, x_m$ such that

$$\sigma_0\iota : \operatorname{Spec} o[x_1, \ldots, x_m]/(x_1, \ldots, x_m) \longrightarrow \operatorname{Spec} o[x_1, \ldots, x_m]$$

is the obvious morphism. Then $F := H^0(\operatorname{Spec} o, (\sigma_0\iota)^*\Omega_{A_o^m/o})$ is a free o-module generated by $dx_1, \ldots, dx_m$.

Let $(f_1, \ldots, f_k) \in o[x_1, \ldots, x_m], k \geq m - d$ generate the ideal defining Ξ. Then $W := H^0(\operatorname{Spec} o, \sigma_0^*\Omega_{\Xi/o})$ is the quotient module F/R of F by the o-submodule R generated by the k elements:

(i) $$df_j = (\partial f_j/\partial x_1)dx_1 + \cdots + (\partial f_j/\partial x_m)dx_m, \quad j = 1, \ldots, k$$

The generic fibre is smooth of dimension d and the o-module R is therefore generated by $m - d$ of these elements, say $df_1, \ldots, df_{m-d}$. By the elementary divisor theorem we may after a linear coordinate change of write:

(ii) $$f_j(x_1, \ldots, x_m) = \pi^{e(j)}x_{d+j} + \sum \xi_\varepsilon x^\varepsilon, \xi_\varepsilon \in o$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{Z}^{(m)}$ runs over exponents $\varepsilon_1, \ldots, \varepsilon_m \geq 0$ of total degree $\varepsilon_1 + \cdots + \varepsilon_m \geq 2$ and where $\varepsilon = \varepsilon(1) \geq \varepsilon(2) \geq \cdots \geq \varepsilon(m - d) \geq 0$.

The polynomials $f_j(x_1, \ldots, x_m), j = 1, \ldots, m - d$ define $\Xi \subset A_o^m$ in an open Zariski neighbourhood of σ_0, which by (2.11) contains $D(\sigma_0, 1, \Xi)$. We may and shall therefore assume that $\Xi \subset A_o^m$ is defined by the polynomials

$$f_j(x_1, \ldots, x_m) = 0, \quad j = 1, \ldots, m - d.$$

We now consider the analytic map

$$p : D(\sigma_0, \Xi, r) \longrightarrow D(0, A_o^d, r)$$

obtained by projecting onto the d first coordinates $x_1, \ldots, x_d$. Let us first show that this map is an analytic isomorphism when $r > e$ by means of an explicit construction of the implicit functions involved.

Consider the morphism $H : A_o^m \to A_o^m$ sending $(y_1, \ldots, y_m)$ to $(h_1, \ldots, h_m)$ where

$$\begin{aligned} h_j(y_1, \ldots, y_m) &= y_j & 1 \leq j \leq d \\ h_j(y_1, \ldots, y_m) &= \pi^{-e(j)-r} f_{j-d}(\pi^r y_1, \ldots, \pi^r y_m) & d + 1 \leq j \leq m \end{aligned}$$

Then each $h_j(y)$, $j = 1, \ldots, m$ is of the form

(iii) $$h_j(y) = y_j + \sum \alpha_\varepsilon y^\varepsilon \in o[y_1, \ldots, y_m]$$

where $\alpha_\varepsilon(\pi^{\deg(\varepsilon)-1})$ and $\varepsilon \in Z^{(m)}$ runs over non-negative m-tuples of degree ≥ 2.

We may thus apply the inverse function theorem described in (2.12) and find unique convergent power series

$$\gamma_j(y_1, \ldots, y_m) \in o\,[\![y_1, \ldots, y_m]\!]\,, \quad j = 1, \ldots, m$$

such that

(iv) $$\gamma_j(y) = y_j + \sum \beta_\varepsilon y^\varepsilon$$

where $\beta_\varepsilon \in (\pi^{\deg(\varepsilon)-1})$ and $\varepsilon \in \mathbb{Z}^{(m)}$ runs over non-negative m-tuples of degree ≥ 2

(v) $$h_j(\gamma_1(y_1, \ldots, y_m), \ldots, \gamma_m(y_1, \ldots, y_m)) = y_j$$

for $j = 1, \ldots, m$.

We now consider $\Gamma : A_o^m \to A_o^m$ sending $(y_1, \ldots, y_m)$ to $(\gamma_1, \ldots, \gamma_m)$. Then (v) says that $H\Gamma$ is the identity map. By applying (2.12) to Γ instead of H one gets an analytic map $\Psi : A_o^m \to A_o^m$ such that $\Gamma\Psi$ is the identity map. But then $H = H\Gamma\Psi = \Psi$ such that:

(vi) $$\gamma_j(h_1(y_1, \ldots, y_m), \ldots, h_m(y_1, \ldots, y_m)) = y_j$$

for $j = 1, \ldots, m$.

From $h_j(y_1, \ldots, y_m) = y_j$ for $j = 1, \ldots, d$ and (vi) one deduces further that:

(vii) $$\gamma_j(y_1, \ldots, y_d, 0, \ldots, 0) = y_j, \quad j = 1, \ldots, m$$

so that $h_j(y_1, \ldots, y_m) = 0$ for $j = d+1, \ldots, m$. Set

$$\phi_j(x_1, \ldots, x_d) = \pi^r \gamma_j(\pi^{-r} x_1, \ldots, \pi^{-r} x_d, 0, \ldots, 0)$$

for $j = 1, \ldots, m$. Then, from (iv) (resp. (v), resp. (viii)) it follows that:

(viii) f_j is analytic on $D(0, A_o^d, r)$ with image in $D(0, A_o^1, r)$.

(ix) $f_j(\phi_1(x_1, \ldots, x_d), \ldots, \phi_m(x_1, \ldots, x_d)) = 0$ for $j = 1, \ldots, m - d$.

(x) $\phi_j(x_1, \ldots, x_d) = x_j$ for $j = 1, \ldots, m$ and $(x_1, \ldots, x_m) \in D(\sigma_0, \Xi, r)$.

Let

$$\Phi(x_1, \ldots, x_d) = (\phi_1(x_1, \ldots, x_d), \ldots, \phi_m(x_1, \ldots, x_d)).$$

Then from (viii), (ix) and (x) it follows that Φ defines a bijective analytic map

$$D(0, A_o^d, r) \longrightarrow D(\sigma_0, \Xi, r)$$

which is inverse to the projection map

$$p : D(\sigma_0, \Xi, r) \longrightarrow D(0, A_o^d, r).$$

We now prove that the analytic isomorphism given by Φ and p between $D(\sigma_0, \Xi, r)$ and $D(0, A_o^d, r)$ is an isometry.

Let $\sigma \in D(\sigma_0, \Xi, r)$. Then it follows from (i) that $H^0(\operatorname{Spec} o, \sigma^*\Omega_{\Xi/o})$ is the o-module generated by $dx_1, \ldots, dx_m$ with relations

$$df_j = (\partial f_j/\partial x_1)dx_1 + \cdots + (\partial f_j/\partial x_m)dx_m, \quad j = 1, \ldots, m-d.$$

Moreover, by (ii) we obtain that

$$\text{(xi)} \qquad df_j = \pi^{e(j)} dx_{d+j} \quad (\text{modulo } \pi^r), \quad j = 1, \ldots, m-d.$$

where by assumption $r > e(j)$ for $j = 1, \ldots, m-d$. This implies that the isomorphism between the analytic tangent bundles of $D(\sigma_0, \Xi, r)$ and $D(0, A_o^d, r)$ induced by p and Φ preserves the o-lattices of these tangent bundles described in (2.8). Hence (see (2.9)) the analytic isomorphism induced by p and Φ respects the model measures on $D(\sigma_0, \Xi, r)$ and $D(0, A_o^d, r)$. It is therefore sufficient to prove the theorem for $\Xi = A_o^d$ and $\sigma_0 = 0$. But then $m(D(0, A_o^d, r)) = (\mu(o)/\operatorname{Card}(o_r))^d$ for $r > 0$. This completes the proof. □

Theorem 2.14. — *Let $\Phi : \Xi \to \operatorname{Spec} o$ be a separated morphism of finite type with smooth geometrically connected generic fibre of dimension d and let m be the model measure on $\widetilde{M} = \Xi_{\mathrm{an}}(o)$. Then there is an integer E such that the torsion part of*

$$H^0(\operatorname{Spec} o, \sigma^*\Omega_{\Xi/o})$$

is annihilated under multiplication by π^E for all $\sigma \in \Xi(o)$. Moreover, if $r > E$, then

(a)
$$m(\Xi_{\mathrm{an}}(o)) = \operatorname{Card}(\operatorname{Im}(\Xi(o) \to X(o_r)))(\mu(o)/\operatorname{Card}(o_r))^d$$

(b)
$$\int_{\Xi(o)} h dm = \sum_{P \in \operatorname{Im}(\Xi(o) \to X(o_r))} h(P)(\mu(o)/\operatorname{Card}(o_r))^d$$

for any $h \in C_c(\Xi_{\mathrm{an}}(o))$ which is constant on the fibres of the reduction map from $\Xi(o)$ to $\Xi(o_r)$.

Proof. — The first statement is an obvious consequence of the assumptions on Φ and the reader may find a proof of a stronger result in [**10**, 3.3.3]. The fact that (a) and (b) hold is a consequence of the previous theorem. □

It follows from (1.6)(b) that any measure on $\widetilde{M} = \Xi_{\mathrm{an}}(o)$ determined by a seminorm on $\Xi_{\mathrm{an}}(o)$ on $\mathbf{det}\operatorname{Tan}(M)$ is of the form hdm for a non-negative continuous function $h \in C_c(X_{\mathrm{an}}(o))$. We may thus use (2.14)(b) to compute the volume of $\Xi_{\mathrm{an}}(o)$ with respect to any such measure.

The theorem (2.14)(a) is due to Serre [**57**, p. 147] (cf. also [**47**]) in case Ξ is affine and $o = \mathbb{Z}_p$. He proves that the equality holds for all sufficiently large r

without giving any explicit bound for those r. The following result is due to Peyre [**52**, 2.2.1] in case Φ is projective and $r = 1$.

Corollary 2.15. — *Let $\Phi : \Xi \to \operatorname{Spec} o$ be a smooth separated morphism of finite type with geometrically connected generic fibre of dimension d. Then,*

$$m(\Xi_{\mathrm{an}}(o)) = \operatorname{Card}(\Xi(o_r))(\mu(o)/\operatorname{Card}(o_r))^d$$

for all $r > 0$.

Proof. — It follows from the smoothness of Φ that $H^0(\operatorname{Spec} o, \sigma^*\Omega_{Xi/o})$ is torsion-free for all $\sigma \in \Xi(o)$ (see [**10**, 3.3.1]). Therefore the equality $(*)$ above holds for all $r > 0$. To complete the proof, note that $\Xi(o) \to \Xi_r(o_r)$ is surjective by Hensel's lemma. □

Our next goal is to study families of measures along the fibres of an analytic morphism $\pi_{\mathrm{an}} : Y_{\mathrm{an}}(k) \to X_{\mathrm{an}}(k)$ associated to a smooth morphism $\pi : Y \to X$ of k-varieties. We shall thereby write $\Omega_{Y/X}$ for the relative cotangent sheaf and $T_{Y/X} = V(\Omega_{Y/X})$ for the relative tangent bundle (cf. [**30**, 16.5.12] and [**29**, 1.7.8] for the affine morphism $V(\mathcal{E}) \to Z$ associated to a coherent sheaf $\mathcal{E}$ on a scheme Z).

Proposition 2.16. — *Let $\widetilde{X}$ (resp. $\widetilde{Y}$) be a smooth separated o-scheme of finite type with geometrically connected fibre X (resp. Y). Let $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ be a smooth o-morphism of constant relative dimension d and let $\pi : Y \to X$ be the corresponding k-morphism between the generic fibres. Then the following holds.*

(a) *The relative cotangent sheaf $\Omega_{\widetilde{Y}/\widetilde{X}}$ is locally free of constant rank d and its restriction to the generic fibre of $\widetilde{Y}/o$ is equal to $\Omega_{Y/X}$.*

(b) *The relative tangent bundles define a commutative diagram with injective vertical maps*

$$\begin{array}{ccc} T_{\widetilde{Y}/\widetilde{X}}(o) & \longrightarrow & \widetilde{Y}(o) \\ \downarrow & & \downarrow \\ T_{Y/X}(k) & \longrightarrow & Y(k) \end{array}$$

(c) *Let $N := Y_{\mathrm{an}}(k)$, $M := X_{\mathrm{an}}(k)$ and $\pi_{\mathrm{an}} : N \to M$ be the analytic map defined by π. Then π_{an} is an analytic submersion of relative dimension d and the relative analytic tangent bundle $\operatorname{Tan}(N/M) \to N$ is equal to the algebraic map $T_{Y/X}(k) \to Y(k)$ in (b).*

(d) *Let $\widetilde{N} = \widetilde{Y}(o)$ and let $\widetilde{M} = \widetilde{X}(o)$. Then $\widetilde{N}$ (resp. $\widetilde{M}$) is a compact open subset of N (resp. M) and π_{an} restricts to an analytic submersion*

$\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ of relative dimension d. Further, $T_{\widetilde{Y}/\widetilde{X}}(o)$ is an analytic o-lattice of the relative analytic tangent bundle $\mathrm{Tan}(\widetilde{N}/\widetilde{M})$.

Proof

(a) This is well known (cf. [**30**, Ch. XVI]).

(b) $V(\Omega_{\widetilde{Y}/\widetilde{X}}) \to \widetilde{Y}$ is separated since it is affine and $\widetilde{Y}$ is separated over o by assumption. Hence $T_{\widetilde{Y}/\widetilde{X}} := V(\Omega_{\widetilde{Y}/\widetilde{X}})$ is also separated over o. Further, $V(\Omega_{Y/X}) = V(\Omega_{\widetilde{Y}/\widetilde{X}}) \times_{\widetilde{Y}} Y$ by (a).

(c) The submersion statement follows from the implicit function theorem. The last assertion follows from a straightforward comparison of the definitions of algebraic and analytic relative tangent bundles.

(d) The first assertion is proved in (2.5)(a) and the submersion statement also follows from (2.5)(a). The o-lattice statement is local and easy to deduce from (a) and (b). It suffices thereby to treat the case where $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ is an affine o-morphism between two affine o-schemes. □

Now set $\widetilde{L} = T_{\widetilde{Y}/\widetilde{X}}(o)$. Then $\mathbf{det}\,\widetilde{L} \to \widetilde{N}$ is an analytic o-lattice of the relative anticanonical line bundle $\mathbf{det}\,\mathrm{Tan}(\widetilde{N}/\widetilde{M}) \to \widetilde{N}$.

Definition 2.17. — Let $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ and $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ be as in (2.16) and let $\widetilde{L} = T_{\widetilde{Y}/\widetilde{X}}(o)$. Then $\mathbf{det}\,\widetilde{L}$ will be called *the analytic o-lattice of* $\mathbf{det}\,\mathrm{Tan}(\widetilde{N}/\widetilde{M}) \to \widetilde{N}$ *defined by* $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$. The norm

$$\|\ \|_{\widetilde{N}/\widetilde{M}} : \mathbf{det}\,\mathrm{Tan}(\widetilde{N}/\widetilde{M}) \longrightarrow [0, \infty)$$

determined by this o-lattice (cf. (1.9)) will be called *the relative model norm determined by* $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$.

Note that the restriction of $\|\ \|_{\widetilde{N}/\widetilde{M}}$ to the tangent bundle of a fibre $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ is equal to the (absolute) model norm for this fibre.

Proposition 2.18. — *Let $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ and $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ be as in (2.16) and let $\|\ \|_{\widetilde{N}/\widetilde{M}}$ be the relative model norm determined by $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$. Further, let*

$$\|\ \|_{\widetilde{N}} : \mathbf{det}\,\mathrm{Tan}(\widetilde{N}) \longrightarrow [0, \infty)$$

$$\|\ \|_{\widetilde{M}} : \mathbf{det}\,\mathrm{Tan}(\widetilde{M}) \longrightarrow [0, \infty)$$

be the model norms determined by $\widetilde{Y}/o$ resp. $\widetilde{X}/o$ and

$$\widetilde{\pi}_{\mathrm{an}}^{*}\|\ \|_{\widetilde{M}} : \widetilde{\pi}_{\mathrm{an}}^{*}(\mathbf{det}\,\mathrm{Tan}(\widetilde{M})) \longrightarrow [0, \infty)$$

be the pullback norm of $\| \ \|_{\widetilde{M}}$. *Then* $\| \ \|_{\widetilde{N}}$ *corresponds to the product norm of* $\| \ \|_{\widetilde{N}/\widetilde{M}}$ *and* $\widetilde{\pi}_{\mathrm{an}}^{*}\| \ \|_{\widetilde{M}}$ *under the canonical isomorphism (cf. (1.19))*

$$\mathbf{det}\,\mathrm{Tan}(\widetilde{N}) = \mathbf{det}\,\mathrm{Tan}(\widetilde{N}/\widetilde{M}) \otimes \widetilde{\pi}_{\mathrm{an}}^{*}(\mathbf{det}\,\mathrm{Tan}(\widetilde{M})).$$

Proof. — The smoothness of $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ and $\widetilde{X}/o$ implies that the canonical sequence of coherent sheaves on:

$$0 \longrightarrow \widetilde{\pi}^{*}\Omega_{\widetilde{X}/o} \longrightarrow \Omega_{\widetilde{Y}/o} \longrightarrow \Omega_{\widetilde{Y}/\widetilde{X}} \longrightarrow 0 \tag{2.19}$$

is exact and that all sheaves involved are locally free.

There is therefore a "dual" exact sequence of (algebraic) vector bundles over

$$0 \longrightarrow T_{\widetilde{Y}/\widetilde{X}} \longrightarrow T_{\widetilde{Y}/o} \longrightarrow T_{\widetilde{X}/o} \times_{\widetilde{X}} \widetilde{Y} \longrightarrow 0 \tag{2.20}$$

and an exact sequence of analytic o-lattices over

$$0 \longrightarrow T_{\widetilde{Y}/\widetilde{X}}(o) \longrightarrow T_{\widetilde{Y}/o}(o) \longrightarrow (T_{\widetilde{X}/o} \times_{\widetilde{X}} \widetilde{Y})(o) \longrightarrow 0 \tag{2.21}$$

inside the canonical exact sequence of analytic vector bundles over $\widetilde{N}$ described in (1.20). The assertion is now an obvious consequence of the compatibility between these two exact sequence. □

Corollary 2.22. — *Let* $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ *and* $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ *be as in (2.16). Let*

$$\Lambda_{\widetilde{M}} : C_c(\widetilde{M}) \longrightarrow \mathbb{R}$$

$$\Lambda_{\widetilde{N}} : C_c(\widetilde{N}) \longrightarrow \mathbb{R}$$

be the positive functional (cf. (1.10)) defined by the model norms ($\| \ \|_{\widetilde{M}}$ *resp.* $\| \ \|_{\widetilde{N}}$) *and let*

$$\Lambda_{\widetilde{N}/\widetilde{M}} : C_c(\widetilde{N}) \longrightarrow C_c(\widetilde{M})$$

be the positive linear map (cf. (1.22)(a)) defined by the relative model norm $\| \ \|_{\widetilde{N}/\widetilde{M}}$. *Then,*

$$\Lambda_{\widetilde{N}} = \Lambda_{\widetilde{M}} \circ \Lambda_{\widetilde{N}/\widetilde{M}}.$$

Proof. — This follows from (2.18) and (1.22)(b). □

The following corollary of (2.15) will be useful in the theory of adelic measures.

Corollary 2.23. — *Let* $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$, $\pi : Y \to X$, $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ *and* $\pi_{\mathrm{an}} : N \to M$ *be as in (2.16) and hence of constant relative dimension* d. *Let* $\Lambda_{\widetilde{N}} : C_c(\widetilde{N}) \to \mathbb{R}$, $\Lambda_{\widetilde{M}} : C_c(\widetilde{M}) \to \mathbb{R}$ *and* $\Lambda_{\widetilde{Y}/\widetilde{X}} : C_c(\widetilde{N}) \to C_c(\widetilde{M})$ *be the positive linear maps defined by the model norms.*

Let F be the (finite) residue field of o and $1_{\widetilde{N}} \in C_c(\widetilde{N})$ resp. $1_{\widetilde{M}} \in C_c(\widetilde{M})$ be the constant functions with value 1. Suppose that all fibres of π are geometrically connected and that all fibres of $\widetilde{\pi}$ over F-points have the same positive number r of F-points. Then,

$$\Lambda_{\widetilde{N}/\widetilde{M}}(1_{\widetilde{N}}) = r(\mu(o)/\operatorname{Card}(F))^d 1_{\widetilde{M}} = (\Lambda_{\widetilde{N}}(1_{\widetilde{N}})/\Lambda_{\widetilde{M}}(1_{\widetilde{M}}))1_{\widetilde{M}}.$$

Proof. — Let $P \in \widetilde{M} = \widetilde{X}(o)$, $\widetilde{N}_P$ be the fibre of $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ over P. Then the restriction of $\mathbf{det}\operatorname{Tan}(\widetilde{N}/\widetilde{M})$ over $\widetilde{N}_P$ is equal to $\mathbf{det}\operatorname{Tan}(\widetilde{N}_P) \to \widetilde{N}_P$ and the restriction of the relative model norm $\| \ \|_{\widetilde{N}/\widetilde{M}}$, to $\mathbf{det}\operatorname{Tan}(\widetilde{N}_P)$ is equal to the model norm defined $\| \ \|_P$ by the smooth o-model $\widetilde{Y}_P$ of Y_P. This implies (cf. (1.21)) that the value of $\Lambda_{\widetilde{N}/\widetilde{M}}(1_{\widetilde{N}})$ at P is equal to $n_P(\widetilde{N}_P)$ for the model measure n_P on $\widetilde{N}_P$ determined by $\widetilde{Y}_P$. But it follows from (2.15) and the assumptions that $n_P(\widetilde{N}_P) = r(\mu(o)/\operatorname{Card}(F))^d$. Hence $\Lambda_{\widetilde{N}/\widetilde{M}}(1_{\widetilde{N}})$ is constant on $\widetilde{M}$ and it is then a formal consequence of (2.22) that it is equal to $(\Lambda_{\widetilde{N}}(1_{\widetilde{N}})/\Lambda_{\widetilde{M}}(1_{\widetilde{M}}))1_{\widetilde{M}}$. This completes the proof. □

3. Invariant norms on torsors over local fields

The purpose of this section is to study norms and measures on torsors which are invariant under the group action. The word K-variety will always mean a *geometrically connected* separated scheme of finite type over a field K. We shall use the following assumptions and notations throughout the section.

3.1 (a) K is an arbitrary field with separable closure $\overline{K}$.
3.1 (b) X is a smooth K-variety with structure morphism $h : X \to \operatorname{Spec} K$.
3.1 (c) G is a smooth K-variety which is an algebraic group over K.

We shall write $e : \operatorname{Spec} K \to G$ for the unit section and

$$c : G \times_K G \longrightarrow G$$

for the morphism defining the group multiplication. We shall for a K-scheme Y write

$$e_Y : Y \longrightarrow G \times_K Y$$

for the morphism obtained from e by base extension.

Definition 3.2

(a) Let $\mathcal{T}$ be a K-scheme. A K-morphism $\sigma : G \times_K \mathcal{T} \to \mathcal{T}$ is a (left) G-action on $\mathcal{T}$ if the following conditions hold.
  (i) The composition $\sigma e_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ is the identity map.

(ii) The following diagram commutes

$$\begin{tikzcd} G \times_K G \times_K \mathcal{T} \arrow[r, "\mathrm{id} \times \sigma"] \arrow[d, "c \times \mathrm{id}"'] & G \times_K \mathcal{T} \arrow[d, "\sigma"] \\ G \times_K \mathcal{T} \arrow[r, "\sigma"] & \mathcal{T} \end{tikzcd}$$

(b) Let $\pi : \mathcal{T} \to X$ be a K-morphism. Then a K-morphism $\sigma : G \times_K \mathcal{T} \to \mathcal{T}$ is a (left) G-action on the fibres of $\pi : \mathcal{T} \to X$ if (i) and (ii) hold and if the following diagram commutes:

(iii)
$$\begin{tikzcd} G \times_K \mathcal{T} \arrow[r, "\sigma"] \arrow[d, "pr_2"'] & \mathcal{T} \arrow[d, "\pi"] \\ \mathcal{T} \arrow[r, "\pi"] & X \end{tikzcd}$$

The map pr_2 is the projection map onto the second factor.

Definition 3.3. — By a (left) X-torsor under G (with respect to the fppf-topology) we shall mean a K-morphism

$$\pi : \mathcal{T} \longrightarrow X$$

endowed with a G-action

$$\sigma : G \times_K \mathcal{T} \longrightarrow \mathcal{T}$$

on the fibres of $\pi : \mathcal{T} \to X$ satisfying the following conditions.

(a) The structural morphism $\pi : \mathcal{T} \to X$ is faithfully flat and locally of finite presentation.

(b) The morphism

$$\rho = (\sigma, pr_2) : G \times_K \mathcal{T} \longrightarrow \mathcal{T} \times_X \mathcal{T}$$

is an isomorphism.

Let $\mathcal{E}$ be a locally free sheaf on $\mathcal{T}$ endowed with an isomorphism $\phi : \sigma^*\mathcal{E} \to pr_2^*\mathcal{E}$ of sheaves over $G \times_K \mathcal{T}$ (cf. (3.2)(iii)). From ϕ we get the following three isomorphisms of sheaves over $G \times_K G \times_K \mathcal{T}$:

$$(c \times \mathrm{id}_{\mathcal{T}})^*\phi : (\mathrm{id}_G \times \sigma)^*(\sigma^*\mathcal{E}) = (c \times \mathrm{id}_{\mathcal{T}})^*(\sigma^*\mathcal{E}) \longrightarrow (c \times \mathrm{id}_{\mathcal{T}})^*(pr_2^*\mathcal{E})$$
$$(\mathrm{id}_G \times \sigma)^*\phi : (\mathrm{id}_G \times \sigma)^*(\sigma^*\mathcal{E}) \longrightarrow (\mathrm{id}_G \times \sigma)^*(pr_2^*\mathcal{E}) = p_{23}^*(\sigma^*\mathcal{E}) \text{ (cf. (3.2)(ii))}$$
$$p_{23}^*\phi : p_{23}^*(\sigma^*\mathcal{E}) \longrightarrow p_{23}^*(pr_2^*\mathcal{E}) = (c \times \mathrm{id}_{\mathcal{T}})^*(pr_2^*\mathcal{E})$$

where p_{23} is the projection onto the last two factors of $G \times_K G \times_K \mathcal{T}$.

ϕ is called an S-linearization of $\mathcal{E}$ (see [**45**, Ch I §3]) if

$$(c \times \mathrm{id}_{\mathcal{T}})^*\phi = (p_{23}^*\phi) \circ (\mathrm{id}_G \times \sigma)^*\phi.$$

The symmetric algebra $S(\mathcal{E})$ on $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_{\mathcal{T}}$-algebra and defines a vector bundle

$$q : V(\mathcal{E}) := \operatorname{Spec} S(\mathcal{E}) \longrightarrow \mathcal{T}$$

(see [**29**, 1.7.8] or [**32**, pp. 128-9]). The isomorphisms

$$\phi : \sigma^*\mathcal{E} \longrightarrow pr_2^*\mathcal{E}$$

of locally free sheaves over $G \times_K \mathcal{T}$ corresponds canonically to vector bundle isomorphisms over $G \times_K \mathcal{T}$

$$\Phi : (G \times_K \mathcal{T}) \times_{\mathcal{T}} V(\mathcal{E}) \longrightarrow (G \times_K \mathcal{T}) \times_{\mathcal{T}} V(\mathcal{E})$$

where the maps from $G \times_K \mathcal{T}$ to $\mathcal{T}$ in the fibre products

$$(G \times_K \mathcal{T}) \times_{\mathcal{T}} V(\mathcal{E})$$

are given by pr_2 resp. σ. Let

$$\Sigma : G \times_K V(\mathcal{E}) \longrightarrow V(\mathcal{E})$$

be the composition of Φ with the projection onto $V(\mathcal{E})$. Then there is a commutative diagram

$$\begin{tikzcd} G \times_K V(\mathcal{E}) \arrow[r, "\Sigma"] \arrow[d, "\mathrm{id} \times \pi"'] & V(\mathcal{E}) \arrow[d, "q"] \\ G \times_K \mathcal{T} \arrow[r, "\sigma"] & \mathcal{T} \end{tikzcd} \tag{3.4}$$

The cocycle condition on ϕ is equivalent to the condition that Σ is a G-action on $V(\mathcal{E})$. This gives (see [**45**, p. 32]) a bijection between G-linearizations of $\mathcal{E}$ and liftings of the G-action on $\mathcal{T}$ to G-actions on $V(\mathcal{E})$.

The isomorphism

$$\rho = (\sigma, pr_2) : G \times_K \mathcal{T} \longrightarrow \mathcal{T} \times_X \mathcal{T}$$

defines a bijection between covering data η of $\mathcal{E}$ (cf. [**10**, p. 133]) and isomorphisms

$$\phi = \rho^*\eta : \sigma^*\mathcal{E} \longrightarrow pr_2^*\mathcal{E}$$

such that η is a descent datum (cf. [**10**, 6.1]) if and only if ϕ is an G linearization of $\mathcal{E}$. There exists therefore by Grothendieck's theory of faithfully flat descent [**28**, exp. VIII] an equivalence between the categories of G-linearized locally free sheaves $(\mathcal{E}, \phi)$ on $\mathcal{T}$ and locally free sheaves ϕ on X (this is indicated at the end on p. 32 in [**45**]). Alternatively, one may use descent for vector bundles [**45**, I. 2.23] and establish the following result.

Lemma 3.5. — *There is an equivalence of the category of vector bundles*

$$q : V(\mathcal{E}) \longrightarrow \mathcal{T}$$

endowed with a G-action

$$\Sigma : G \times_K V(\mathcal{E}) \longrightarrow V(\mathcal{E})$$

lifting

$$\sigma : G \times_K \mathcal{T} \longrightarrow \mathcal{T}(cf.\ (3.4))$$

and the category of vector bundles

$$p : V(\mathcal{F}) \longrightarrow X.$$

The pair (q, Σ) is obtained from (p, σ) by means of base extension under $\pi : \mathcal{T} \to X$. Conversely, $V(\mathcal{F})$ is the geometric quotient $V(\mathcal{E})/G$ (cf. [**45**, p. 4]*) and p is the map*

$$V(\mathcal{E})/G \longrightarrow \mathcal{T}/G$$

induced by q.

We shall in the sequel by a G-vector bundle over $\mathcal{T}$ mean a vector bundle $q : V(\mathcal{E}) \to \mathcal{T}$ endowed with a G-action Σ as in (3.5).

Example 3.6. — Let $\mathcal{J}_X$ (resp. $\mathcal{J}_\mathcal{T}$) be the ideal sheaves of the closed immersions

$$e_X : X \longrightarrow G \times_K X$$

$$e_\mathcal{T} : \mathcal{T} \longrightarrow G \times_K \mathcal{T}$$

and let

$$\pi_G : G \times_K X \longrightarrow G \times_K X$$

be the morphism induced by π under the base extension from K to G. Then

$$e_X \pi = \pi_G e_\mathcal{T}$$

and

$$\mathcal{J}_\mathcal{T}/\mathcal{J}_\mathcal{T}^2 = \pi_G^*(\mathcal{J}_X/\mathcal{J}_X^2).$$

The conormal sheaf

$$\mathcal{E} = e_\mathcal{T}^*(\mathcal{J}_\mathcal{T}/\mathcal{J}_\mathcal{T}^2)$$

of $e_\mathcal{T}$ is therefore equal to the inverse image $\mathcal{E} = \pi^*\mathcal{F}$ of the conormal sheaf

$$\mathcal{F} = e_X^*(\mathcal{J}_X/\mathcal{J}_X^2)$$

of e_X. The corresponding G-linearization

$$\phi : \sigma^*\mathcal{E} \longrightarrow pr_2^*\mathcal{E}$$

of sheaves over $G \times_K \mathcal{T}$ is the identity map beetween the inverse images of $\mathcal{J}_X/\mathcal{J}_X^2$ along $\pi_G e_\mathcal{T} \sigma = \pi_G e_\mathcal{T} pr_2$.

Since $\pi : \mathcal{T} \to X$ is locally of finite presentation it is also separated. Write $\mathcal{J}$ for the ideal sheaf on $\mathcal{T} \times_X \mathcal{T}$ of the closed diagonal immersion

$$\delta : \mathcal{T} \longrightarrow \mathcal{T} \times_X \mathcal{T}$$

and $\Omega_{\mathcal{T}/X}$ for the relative cotangent sheaf $\delta^*(\mathcal{J}/\mathcal{J}^2)$. Then the following identities hold.

3.7 (a) $\delta = \rho e_{\mathcal{T}}$
3.7 (b) $\rho^*(\mathcal{J}/\mathcal{J}^2) = \mathcal{J}_{\mathcal{T}}/\mathcal{J}_{\mathcal{T}}^2 = \pi_G^*(\mathcal{J}_X/\mathcal{J}_X^2)$
3.7 (c) $\Omega_{\mathcal{T}/X} = e_{\mathcal{T}}^*(\mathcal{J}_{\mathcal{T}}/\mathcal{J}_{\mathcal{T}}^2) = \pi^*(e_X^*(\mathcal{J}_X/\mathcal{J}_X^2))$

We now consider the tangent bundle (cf. [**30**, 16.5.12]) $V(\Omega_{\mathcal{T}/X})$ of $\pi : \mathcal{T} \to X$.

Proposition 3.8

(a) *There exists a canonical isomorphism*

$$V(\Omega_{\mathcal{T}/X}) = V(e_{\mathcal{T}}^*(\mathcal{J}_{\mathcal{T}}/\mathcal{J}_{\mathcal{T}}^2))$$

between the tangent bundle of $\pi : \mathcal{T} \to X$ and the normal bundle of $e_{\mathcal{T}} : \mathcal{T} \to G \times_K \mathcal{T}$.

(b) *There exists a canonical $\mathcal{T}$-isomorphism*

$$V(e_X^*(\mathcal{J}_X/\mathcal{J}_X^2)) = \mathcal{T} \times_X V(e_X^*(\mathcal{J}_X/\mathcal{J}_X^2))$$

for the normal bundle $V(e_X^*(\mathcal{J}_X/\mathcal{J}_X^2))$ of $e_X : X \to G \times_K X$.

Proof. — This follows from (3.7)(c) and the contravariant equivalence between locally free sheaves and "geometric" vector bundles described in [**32**, p. 129]. □

It follows from (3.8) that there is a natural G-action on the tangent bundle

$$V(\Omega_{\mathcal{T}/X}) \to \mathcal{T}$$

given by the G-linearization of $e_{\mathcal{T}}^*(\mathcal{J}_{\mathcal{T}}/\mathcal{J}_{\mathcal{T}}^2) = \pi^*(e_X^*(\mathcal{J}_X/\mathcal{J}_X^2))$ in (3.6).

Proposition 3.9. — *Let $\mathcal{J}$ be the ideal sheaf of the closed immersion e : $\operatorname{Spec} K \to G$ and let*

$$V(e^*(\mathcal{J}/\mathcal{J}^2))$$

be the tangent space of G at e regarded as a K-variety. Then there is a canonical isomorphism

$$V(e_X^*(\mathcal{J}_X/\mathcal{J}_X^2)) = X \times_K V(e^*(\mathcal{J}/\mathcal{J}^2))$$

from the normal bundle of $e_X : X \to G \times_K X$.

Proof. — Let $h_G : G \times_K X \to G$ be the first projection map. Then $\mathcal{J}_X/\mathcal{J}_X^2 = h_G^*(\mathcal{J}/\mathcal{J}^2)$ and $h_G e_X = eh$. Hence $e_X^*(\mathcal{J}_X/\mathcal{J}_X^2) = h^*(e^*(\mathcal{J}/\mathcal{J}^2))$ as was to be proved. □

Now let k denote a non-discrete locally compact field of characteristic zero and let $\pi : \mathcal{T} \to X$ be a torsor over a smooth k-variety X under an algebraic k-group G as in (3.1).

The algebraic morphism π gives rise to an analytic morphism $\pi_{\mathrm{an}} : \mathcal{T}_{\mathrm{an}}(k) \to X_{\mathrm{an}}(k)$. Set

$$N = \mathcal{T}_{\mathrm{an}}(k),$$
$$M = \pi_{\mathrm{an}}(\mathcal{T}_{\mathrm{an}}(k)).$$

Then M is a k-open subset of $X_{\mathrm{an}}(k)$ by the implicit function theorem. It is thus endowed with a natural analytic manifold structure. We shall from now on change notation and write

$$\pi_{\mathrm{an}} : N \longrightarrow M$$

for the *surjective* analytic morphism sending $Q \in N$ to $\pi_{\mathrm{an}}(Q)$. This map is a submersion in the sense of [**59**, LG 3.16].

The fibre product

$$N \times_M N := \{(Q_1, Q_2) \in N \times N : \pi_{\mathrm{an}}(Q_1) = \pi_{\mathrm{an}}(Q_2)\}$$

is an analytic submanifold of $N \times N$ since π_{an} is a submersion (cf. [**59**, LG3.26]). It is the analytic immersion defined by the algebraic immersion $\mathcal{T} \times_X \mathcal{T} \to \mathcal{T} \times_k \mathcal{T}$.

Let $\Gamma = G_{\mathrm{an}}(k)$. The group law $c : G \times_k G \to G$ gives rise to an analytic morphism

$$c_{\mathrm{an}} : \Gamma \times \Gamma \longrightarrow \Gamma$$

which makes Γ to an analytic group in the sense of [**59**, Chap. IV] with $e \in G(k)$ as neutral element.

The G-action $\sigma : G \times_k \mathcal{T} \to \mathcal{T}$ (cf. (3.4)) induces an analytic (left) G-action

$$\sigma_{\mathrm{an}} : \Gamma \times N \longrightarrow N$$

as in [**59**, LG 4.11]. This Γ-action satisfies $\pi_{\mathrm{an}}(\sigma_{\mathrm{an}}(g, n)) = \pi_{\mathrm{an}}(n)$ and we shall therefore say that σ_{an} is a Γ-action on $\pi_{\mathrm{an}} : N \to M$ (cf. (3.2)(iii)).

Definition 3.10. — A submersion

$$\alpha : N \longrightarrow M$$

between analytic manifolds with an analytic G-action

$$\beta : \Gamma \times N \longrightarrow N$$

on α is called an analytic M-torsor under Γ if the map

$$\gamma : \Gamma \times N \longrightarrow N \times_M N$$

sending (g, n) to $(\beta(g, n), n)$ is an isomorphism.

The torsor isomorphism $\rho : G \times_k \mathcal{T} \to \mathcal{T} \times_X \mathcal{T}$ induces an analytic isomorphism

$$\rho_{\mathrm{an}} : \Gamma \times N \longrightarrow N \times_M N$$

such that the pair $(\pi_{\mathrm{an}} : N \to M,\ \sigma_{\mathrm{an}} : \Gamma \times N \to N)$ obtained from the pair $(\pi : \mathcal{T} \to X,\ \sigma : G \times_k \mathcal{T} \to \mathcal{T})$ is an analytic M-torsor under Γ.

Proposition 3.11. — *Let $\alpha : N \to M$ be an analytic M-torsor under the Γ-action $\beta : \Gamma \times N \to N$.*

(a) *There exists for each $P \in M$ an analytic section $s : U \to N$ of α for some open neighbourhood $U \subset M$ of P.*

(b) *Let $V = \alpha^{-1}(U)$ for the open subset $U \subset M$ in (a). Then the restriction of β to $\Gamma \times s(U)$ defines an analytic isomorphism between $\Gamma \times s(U)$ and V.*

Proof

(a) This follows from the implicit function theorem (cf. [**59**, LG 3.1.6]).

(b) ρ_{an} restricts to an analytic isomorphism between $\Gamma \times s(U)$ and $V \times_U s(U)$ which preserves the second coordinate. The projection map $V \times_U s(U) \to V$ is also an isomorphism. This completes the proof. □

By (3.11) any analytic M-torsor under Γ is a locally trivial analytic principal G-bundle over M as defined in [**59**, Ch. IV]. The converse is also true. Note that (3.11) implies that M as a topological space is homeomorphic to the quotient space N/Γ.

Let $q : V(\mathcal{E}) \to \mathcal{T}$ be an (algebraic) vector bundle as in (3.5) equipped with a G-action $\Sigma : G \times_k V(\mathcal{E}) \to V(\mathcal{E})$ lifting $\sigma : G \times k\mathcal{T} \to \mathcal{T}$. Let $E = V(\mathcal{E})_{\mathrm{an}}(k)$ and let $q_{\mathrm{an}} : E \to N$ be the analytic vector bundle defined by q. Then Σ defines an analytic G-action

$$\Sigma_{\mathrm{an}} : \Gamma \times E \longrightarrow E$$

which lifts the Γ-action on N. We shall in the sequel write gs instead of $\Sigma_{\mathrm{an}}(g, s)$ for each pair $(g, s) \in \Gamma \times E$ whenever it is clear what the Γ-action is. A seminorm

$$\| \ \| : E \longrightarrow [0, \infty)$$

on $q_{\mathrm{an}} : E \to N$ (cf. (1.5)) will be called G-invariant if $\|gs\| = \|s\|$ for all $(g, s) \in \Gamma \times E$.

Let $p : V(\mathcal{F}) \to X$ be a vector bundle corresponding to the G-vector bundle $q : V(\mathcal{E}) \to \mathcal{T}$ under the equivalence of categories in (3.5) and let us identify $V(\mathcal{E})$ and $\mathcal{T} \times_X V(\mathcal{F})$. To p one may associate an analytic vector bundle

$$p_{\mathrm{an}} : V(\mathcal{F})_{\mathrm{an}}(k) \longrightarrow X_{\mathrm{an}}(k).$$

Let $F \subseteq V(\mathcal{F})_{\text{an}}(k)$ be the inverse image of $M := \pi_{\text{an}}(\mathcal{T}_{\text{an}}(k))$ under p_{an}. Then p_{an} restricts to an analytic vector bundle $p_{\text{an},M} : F \to M$. The following result is trivial.

Proposition 3.12

(a) *The analytic bundle* $q_{\text{an}} : E \to N$ *is the pullback bundle of* $p_{\text{an},M} : F \to M$ *under* $\pi_{\text{an}} : N \to M$.

(b) *The pullback seminorm on* $q_{\text{an}} : E \to N$ *(cf. (1.7)(c)) of a seminorm on* $p_{\text{an},M} : F \to M$ *is* G*-invariant.*

(c) *Any* G*-invariant seminorm (resp. norm) on* $q_{\text{an}} : E \to N$ *is the pullback of a unique seminorm (resp. norm) on* $p_{\text{an},M} : F \to M$.

Now recall that $V(\Omega_{\mathcal{T}/X}) = \mathcal{T} \times_k V(e^*(\mathcal{J}/\mathcal{J}^2))$ is endowed with a G-action (cf. (3.5), (3.8)). As a corollary of the corresponding algebraic results result we obtain:

Corollary 3.13. — *The analytic tangent bundle*

$$\operatorname{Tan}(N/M) \longrightarrow N$$

is canonically isomorphic to the pullback of the analytic normal bundle

$$\operatorname{Nor}(M/\Gamma \times M) \longrightarrow M$$

along $\pi_{\text{an}} : N \to M$.

There is a natural analytic G*-action on* $\operatorname{Tan}(N/M) \to N$. *The normal bundle*

$$\operatorname{Nor}(M/\Gamma \times M) \longrightarrow M$$

is canonically isomorphic to the constant bundle

$$M \times V \longrightarrow M$$

for the analytic tangent space V *of* G *at* e.

By considering the corresponding determinant bundles one deduces immediately the following result.

Main proposition 3.14. — *Let* $\pi_{\text{an}} : N \to M$ *be an analytic torsor under the analytic group* G *as above and let* V *be the analytic tangent space* V *of* Γ *at* e. *Then the following holds:*

(a) *The analytic anticanonical bundle*

$$\mathbf{det}\operatorname{Tan}(N/M) \longrightarrow N$$

is canonically isomorphic to the constant bundle

$$N \times \mathbf{det}\, V \longrightarrow N.$$

(b) *Let*

$$\| \ \|_M : M \times \mathbf{det}\, V \longrightarrow [0, \infty)$$

be a norm on $M \times \mathbf{det}\, V \to M$. *Then the pullback norm*

$$\pi_{\mathrm{an}}^* \| \ \|_M : N \times \mathbf{det}\, V \longrightarrow [0, \infty)$$

is a Γ*-invariant norm on* $\mathbf{det}\, \mathrm{Tan}(N/M) \to N$.

(c) *Any* Γ*-invariant norm on* $\mathbf{det}\, \mathrm{Tan}(N/M) \to N$ *is the pullback norm of a unique norm on* $M \times \mathbf{det}\, V \to M$.

Definition 3.15. — A constant norm on $\mathbf{det}\, \mathrm{Tan}(N/M) \to N$ is a pullback norm of a vector space norm on $\mathbf{det}\, V$ under the projection

$$\mathbf{det}\, \mathrm{Tan}(N/M) = N \times \mathbf{det}\, V \to \mathbf{det}\, V.$$

The constant norms on $\mathbf{det}\, \mathrm{Tan}(N/M) \to N$ are clearly Γ-invariant and unique up to multiplication by a positive real number.

Remark 3.16. — Let ω_0 be a non-zero section of $\mathbf{det}\, V^\vee$ for the cotangent space $V^\vee$ of Γ at e. Then ω_0 defines a vector space norm $\| \ \| : \mathbf{det}\, V \to [0, \infty)$ (cf. (1.7)) and any norm on $\mathbf{det}\, V$ is obtained in this way. Further, the bundle

$$\mathbf{det}\, \mathrm{Cot}(N/M) \to N$$

is canonically isomorphic to the constant bundle $N \times \mathbf{det}\, V^\vee \to N$ by (3.14)(a). There is thus a global Γ-invariant section ω of $\mathbf{det}\, \mathrm{Cot}(N/M) \to N$ corresponding to the pullback of ω_0 along the projection $N \times \mathbf{det}\, V^\vee \to N$. The constant norm on $\mathbf{det}\, \mathrm{Tan}(N/M) \to N$ corresponding to $\| \ \| : \mathbf{det}\, V \to [0, \infty)$ is the norm associated to ω in (1.7).

We now formulate an analog of (3.7) for schemes satisfying the following hypothesis.

3.17 (a) o is a henselian discrete valuation ring, o^{sh} is a strict henselization of o.

3.17 (b) $\widetilde{h} : \widetilde{X} \to \mathrm{Spec}\, o$ is a smooth separated morphism of finite type with geometrically connected fibre.

3.17 (c) $\widetilde{G}$ is a smooth separated group scheme of finite type over o with geometrically connected fibres.

There is thus a unit section $\widetilde{e} : \mathrm{Spec}\, o \to \widetilde{G}$ and a morphism

$$\widetilde{c} : \widetilde{G} \times_o \widetilde{G} \longrightarrow \widetilde{G}$$

defining the group multiplication. We denote by $\widetilde{\mathcal{J}}$ the ideal sheaf on $\widetilde{G}$ of the closed immersion $\widetilde{e}$ and by $\widetilde{\mathcal{J}}_{\widetilde{X}}$ the ideal sheaf on $\widetilde{G}_{\widetilde{X}}$ of the closed immersion

$$\widetilde{e}_{\widetilde{X}} : \widetilde{X} \longrightarrow \widetilde{G} \times_o \widetilde{X}.$$

Let $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ be an o-morphism between o-schemes. A $\widetilde{G}$-action

$$\widetilde{\sigma} : \widetilde{G} \times_o \widetilde{\mathcal{T}} \longrightarrow \widetilde{\mathcal{T}}$$

on the fibres of $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ is defined just as in (3.2)(b). A (left) $\widetilde{X}$-torsor under $\widetilde{G}$ (with respect to the fppf-topology) is an o-morphism

$$\widetilde{\pi} : \widetilde{\mathcal{T}} \longrightarrow \widetilde{\mathcal{X}}$$

with an action

$$\widetilde{\sigma} : \widetilde{G} \times_o \widetilde{\mathcal{T}} \longrightarrow \widetilde{\mathcal{T}}$$

on the fibres of $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ satisfying the following conditions.

3.18 (a) The structural morphism $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ is faithfully flat and locally of finite presentation.

3.18 (b) The morphism $\widetilde{\rho} = (\widetilde{\sigma}, pr_2) : \widetilde{G} \times_o \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}} \times_{\widetilde{X}} \widetilde{\mathcal{T}}$ is an isomorphism.

Proposition 3.19. — *Assume (3.17) and let $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ be an $\widetilde{X}$-torsor under $\widetilde{G}$. Then there is a canonical $\mathcal{T}$-isomorphism*

$$V(\Omega_{\widetilde{\mathcal{T}}/\widetilde{X}}) = \widetilde{\mathcal{T}} \times_{\widetilde{X}} V(\widetilde{e}^*_{\widetilde{X}}(\widetilde{\mathcal{J}}_{\widetilde{X}}/\widetilde{\mathcal{J}}_{\widetilde{X}^2}))$$

and a canonical $\widetilde{X}$-isomorphism

$$V(\widetilde{e}^*_{\widetilde{X}}(\widetilde{\mathcal{J}}_{\widetilde{X}}/\widetilde{\mathcal{J}}_{\widetilde{X}^2})) = \widetilde{X} \times_o V(\widetilde{e}^*(\widetilde{\mathcal{J}}/\widetilde{\mathcal{J}}^2)).$$

Proof. — The proofs are almost identical to those of (3.8), (3.9). □

We now add the following hypothesis to (3.17).

3.20 (a) k is a finite extension of $\mathbb{Q}_p$, o is the maximal $\mathbb{Z}_p$-order in k.

3.20 (b) $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ is an $\widetilde{X}$-torsor under with action $\widetilde{\sigma} : \widetilde{G} \times_o \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$.

3.20 (c) The restrictions to the generic fibres of $\widetilde{h}$, $\widetilde{\pi}$ and $\widetilde{\sigma}$ are equal to the k-morphisms $h : X \to \operatorname{Spec} k$, $\pi : \mathcal{T} \to X$ and $\sigma : G \times_K \mathcal{T} \to \mathcal{T}$ in (3.1) and (3.3).

Then $\widetilde{N} = \widetilde{\mathcal{T}}(o)$ (resp. $\widetilde{\Gamma} = \widetilde{G}(o)$) is a compact open subset of $N = \mathcal{T}_{\mathrm{an}}(k)$ (resp. $\Gamma = G_{\mathrm{an}}(k)$) in the k-topology (cf. (2.5)(a)). Also, $\widetilde{M} = \widetilde{X}(o)$ is a compact open subset of $X_{\mathrm{an}}(k)$. This provides $\widetilde{M}$ and $\widetilde{N}$ with analytic manifold structures and $\widetilde{\Gamma}$ with an analytic group structure. The algebraic morphisms $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ and $\widetilde{\sigma} : \widetilde{G} \times_o \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$ induce analytic morphisms

$$\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \longrightarrow \widetilde{M},$$

$$\widetilde{\sigma}_{\mathrm{an}} : \widetilde{\Gamma} \times \widetilde{N} \longrightarrow \widetilde{N}.$$

Lemma 3.21

(a) $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ *is a surjective submersion.*

(b) $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ *is an analytic $\widetilde{M}$-torsor under the $\widetilde{\Gamma}$-action given by $\widetilde{\sigma}$.*

(c) *There is a canonical isomorphism of analytic vector bundles over $\widetilde{N}$:*

$$\operatorname{Tan}(\widetilde{N}/\widetilde{M}) = \widetilde{N} \times V$$

for the analytic tangent space of Γ *at* $e \in \Gamma(k)$. *This isomorphism is compatible with the isomorphism*

$$\operatorname{Tan}(N/M) = N \times V$$

in (3.14).

Proof

(a) The pullback of $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ with respect to a section $\operatorname{Spec} o \to \widetilde{X}$ is a torsor over $\operatorname{Spec} o$ under $\widetilde{G}$. The closed fibre of such a torsor is a torsor under the *connected* algebraic group $\widetilde{G} \times_o o/m$ and therefore trivial by a well-known result of Lang (cf. e.g. [**64**]). It follows from Hensel's lemma that $\widetilde{\pi}_{\mathrm{an}} : \widetilde{\mathcal{T}}(o) \to \widetilde{X}(o)$ is surjective. It is thus a submersion since $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ is smooth.

(b) The torsor isomorphism $\widetilde{\rho} = (\widetilde{\sigma}, pr_2) : \widetilde{G} \times_o \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}} \times_X \widetilde{\mathcal{T}}$ (cf. (3.18)(b)) induces an analytic isomorphism $\widetilde{\rho}_{\mathrm{an}} : \widetilde{\Gamma} \times \widetilde{N} \to \widetilde{N} \times_{\widetilde{M}} \widetilde{N}$.

(c) This follows from the fact that the relative analytic tangent bundle $\operatorname{Tan}(\widetilde{N}/\widetilde{M})$ is equal to the inverse image over $\widetilde{N}$ of $\operatorname{Tan}(N/M) \to N$. □

Definition 3.22. — The tangent lattice of $\widetilde{G}/o$ at $\widetilde{e} \in \widetilde{G}(o)$ is the o-module

$$L_1 := \mathbf{V}(\widetilde{e}^*(\widetilde{\mathcal{J}}/\widetilde{\mathcal{J}}^2))(o)$$

of sections $\operatorname{Spec} o \to \mathbf{V}(\widetilde{e}^*(\widetilde{\mathcal{J}}/\widetilde{\mathcal{J}}^2))$ of $\mathbf{V}(\widetilde{e}^*(\widetilde{\mathcal{J}}/\widetilde{\mathcal{J}}^2)) \to \operatorname{Spec} o$.

Note that L_1 is an o-lattice of the k-vector space $V = \mathbf{V}(\widetilde{e}^*(\mathcal{J}/\mathcal{J}^2))$.

Now let $T_{\widetilde{\mathcal{T}}/\widetilde{X}} = \mathbf{V}(\Omega_{\widetilde{\mathcal{T}}/\widetilde{X}})$ and recall that $T_{\widetilde{\mathcal{T}}/\widetilde{X}}(o)$ is an analytic o-lattice of the vector bundle $\operatorname{Tan}(\widetilde{N}/\widetilde{M})$ (see (2.16)(d)).

Proposition 3.23. — *Assume (3.17) and (3.20) and let* $\widetilde{N}$, $\widetilde{M}$ *be as above. Let* L_1 *be the tangent lattice of* $\widetilde{G}/o$ *at* $\widetilde{e}$ *in the tangent space* V *of* G/k *at* $\widetilde{e}$. *Then* $T_{\widetilde{\mathcal{T}}/\widetilde{X}}(o)$ *is mapped isomorphically onto* $\widetilde{N} \times L_1$ *under the canonical isomorphism*

$$\operatorname{Tan}(\widetilde{N}/\widetilde{M}) = \widetilde{N} \times V.$$

of analytic vector bundles over in (3.21)(c).

Proof. — This follows from (3.19). □

Corollary 3.24. — *Assume (3.17) and (3.20) and let $\widetilde{N}$, $\widetilde{M}$ be as above. Then the relative model norm*

$$\| \ \|_{\widetilde{N}/\widetilde{M}} : \mathbf{det}\,\mathrm{Tan}(\widetilde{N}/\widetilde{M}) \to [0,\infty)$$

determined by $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ (cf. (2.17)) is equal to the constant norm

$$\widetilde{N} \times \mathbf{det}\,V \longrightarrow [0,\infty)$$

obtained by pulling back the vector space norm on $\mathbf{det}\,V$ given by $\mathbf{det}\,L_1$ for the tangent lattice L_1 of $\widetilde{G}/o$ at $\widetilde{e}$.

Proposition 3.25. — *Assume (3.17) and (3.20). Let μ be the model measure on $\Delta_{\mathrm{an}}(o)$ determined by Δ/o for any of the three smooth o-schemes $\Delta = \widetilde{X}$, $\widetilde{G}$ or $\widetilde{\mathcal{T}}$. Then*

$$\mu(\widetilde{\mathcal{T}}(o)) = \mu(\widetilde{X}(o))\mu(\widetilde{G}(o)).$$

Proof. — Let $\widetilde{N} = \widetilde{\mathcal{T}}(o)$, $\widetilde{M} = \widetilde{X}(o)$, $\widetilde{\Gamma} = \widetilde{G}(o)$ and $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ be as above. Then it follows from the theorem of Lang already used in (3.21) that each fibre of $\widetilde{\pi}$ over a F-point contains the same number of F-points as $\widetilde{G}$. This implies by (2.23) that $\mu(\widetilde{N})/\mu(\widetilde{M})$ is equal to $\mathrm{Card}\,\widetilde{G}(F)(\mu(o)/\mathrm{Card}(F))^{\dim G}$ which in its turn is equal to $\mu(\widetilde{G}(o))$ by (2.15). This finishes the proof. □

We now restrict to the case when G is a K-torus. This means that G is a commutative algebraic group which after a finite separable base field extension is isomorphic to the product of a finite number of $\mathbb{G}_m$. The K-torus is said to be split if there is such an isomorphism defined over K.

Let $\hat{G}$ be the group of characters defined over K. Then there is a homomorphism:

$$d\log : \hat{G} \longrightarrow H^0(G, \Omega^1_{G/K}) \tag{3.26}$$

which sends a character $f : G \to \mathbb{G}_m$ to the G-invariant differential form df/f.

If G is a product of r copies of $\mathbb{G}_m$, then $\hat{G}$ is a free $\mathbb{Z}$-module of rank r and one obtains by taking the r-th exterior products a homomorphism

$$\bigwedge^r d\log : \bigwedge^r \hat{G} \longrightarrow H^0(G, \Omega^1_{G/K}). \tag{3.27}$$

Definition 3.28. — Let G be a split K-torus of dimension r. Then the image of any generator of $\bigwedge^r \hat{G}$ under $\bigwedge^r d\log$ is called an algebraic differential r-form of minimal $d\log$-type. We shall also say that the corresponding analytic differential r-form is of minimal $d\log$-type.

Proposition and definition 3.29. — *Let k be a non-discrete locally compact field and let G be an r-dimensional k-torus. Then there exists a unique vector space* *norm* $\| \ \|$ *on* $\bigwedge^r(T_{G,e}(k))$ *for the tangent space* $T_{G,e}(k)$ *of* G *at* e *defined as follows.*

Let E *be a finite field extension of* k *such that the absolute Galois group of* E *acts trivially on* $\bigwedge^r(\hat{T})$ *and let* ω_1 *be an differential* r*-form of minimal* $d\log$*-type on* G_E*. Then if* $s \in \bigwedge^r(T_{G,e}(k))$*, put*

$$\|s\| := \sqrt[d]{|\omega_1(s)|_E}$$

where $d := \dim_k E$ *and* $| \ |_E$ *is the normalized absolute value of* E*.*

This definition does not depend on the choice of E *and* ω_1*. The number* $\|s\|$ *will be called the order of* s *and*

$$\| \ \| : \bigwedge^r(T_{G,e}(k)) \longrightarrow \mathbb{R}$$

the order norm.

Proof. — The proof of the independence statement is obvious and left to the reader. $\square$

Definition 3.30. — Let k be a non-discrete locally compact field and let $\pi : \mathcal{T} \to X$ be a torsor under a k-torus G. Then the constant norm (cf. (3.15))

$$\| \ \|_{\mathcal{T}/X} : \mathbf{det}\,\mathrm{Tan}(\mathcal{T}(k)/X(k)) \longrightarrow [0, \infty)$$

obtained by pulling back the order norm on $\bigwedge^r(T_{G,e}(k))$ is called the order norm on $\mathbf{det}\,\mathrm{Tan}(\mathcal{T}(k)/X(k))$. If

$$\| \ \|_X : \mathbf{det}\,\mathrm{Tan}(X(k)) \longrightarrow [0, \infty)$$

is a norm for X, then the product norm on $\mathbf{det}\,\mathrm{Tan}(\mathcal{T}(k))$ of the order norm $\| \ \|_{\mathcal{T}/X}$ and the pullback norm $\pi^*\| \ \|_X$ (cf. (1.19)) is called the norm on $\mathbf{det}\,\mathrm{Tan}(\mathcal{T}(k))$ induced by $\| \ \|_X$.

We shall in the sequel write $\| \ \|_{X \cdot \to \mathcal{T}}$ for the norm induced by $\| \ \|_X$.

Remark 3.31. — Let k be the quotient of a complete discrete valuation ring o. Suppose that G extends to a commutative group scheme $\widetilde{G}$ and that π extends to a torsor $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ under $\widetilde{G}$ as in (3.20). Then the order norm

$$\| \ \|_{\mathcal{T}/X} : \mathbf{det}\,\mathrm{Tan}(\mathcal{T}(k)/X(k)) \to [0, \infty)$$

restricts to the model norm on $\mathbf{det}\,\mathrm{Tan}(\widetilde{\mathcal{T}}(o)/\widetilde{X}(o))$. (To see this, use (3.24).) Hence, if a norm $\| \ \|_X$ restricts to the model norm on $\mathbf{det}\,\mathrm{Tan}(\widetilde{X}(o))$, then the induced norm $\| \ \|_{X \cdot \to \mathcal{T}}$ restricts to the model norm on $\mathbf{det}\,\mathrm{Tan}(\widetilde{\mathcal{T}}(o))$ by (2.22).

4. Adelic norms and measures

The aim of the section is to define norms and measures on the adele spaces of varieties over number fields. This section is inspired by [**52**] but our adelic notions are more general than in that paper.

We shall in this section use the following notations.

Notations 4.1

(a) k denotes a number field and o denotes the maximal $\mathbb{Z}$-order in k.

(b) If ν is a non-archimedean place of k, then o_ν denotes the complete discrete valuation ring corresponding to ν and F_ν the residue field of o_ν.

(c) If Σ is a finite closed subset of $\operatorname{Spec} o$, then $o_{(\Sigma)}$ denotes the Dedekind domain of elements in k which are integral with respect to non-archimedean places outside Σ.

(d) W denotes the set of all places ν of k and W_∞ the subset of all archimedean places of k.

(e) W_{fin} denotes the set of all non-archimedean places of k, which we will identify with the set of closed points of $\operatorname{Spec} o$.

(f) If Σ be a finite subset of W_{fin}, then

$$A(\Sigma) = A_k(\Sigma) = \prod_{\nu \in T} k_\nu \times \prod_{\nu \in W - T} o_\nu, \qquad T := \Sigma \cup W_\infty.$$

(g) $A = A_k$ is the adele ring

$$A = \varinjlim A(S), \quad S \subset W_{\mathrm{fin}}$$

where S is finite.

(h) Let $\Sigma \subseteq S$ be finite subsets of W_{fin}, $\widetilde{o} = o_{(\Sigma)}$, $\widetilde{A} = A_k(\Sigma)$ and $\widetilde{X}$ (resp. Ξ) be a scheme over $\widetilde{o}$ (resp. $\widetilde{A}$). Then

$$\widetilde{X}_{(S)} := \widetilde{X} \times_{\widetilde{o}} o(S),$$

$$\Xi_{A(S)} := \Xi \times_{\widetilde{A}} A(S),$$

and

$$\Xi_A := \Xi \times_{\widetilde{A}} A.$$

(i) Let $\Sigma \subseteq S$ be finite subsets of W_{fin} and $\widetilde{P}$, $\widetilde{X}$ (resp. Π, Ξ) be schemes over $\widetilde{o} = o_{(\Sigma)}$ (resp. $\widetilde{A} = A_k(S)$). Then,

$$\operatorname{Hom}_{(S)}(\widetilde{P}_{(S)}, \widetilde{X}_{(S)})$$

is the set of all $o_{(S)}$-morphisms from $\widetilde{P}_{(S)}$ to $\widetilde{X}_{(S)}$ and

$$\operatorname{Hom}_{A(S)}(\Pi_{A(S)}, \Xi_{A(S)})$$

the set of all $A(S)$-morphisms from $\Pi_{A(S)}$ to $\Xi_{A(S)}$.

We shall in the sequel also consider the embedding $k \subset A_k$ obtained from the inductive limit of the embeddings $o_{(S)} \subset A_{(S)}$ for all finite subsets S of W_{fin}.

Proposition 4.2

(a) *Let X be a scheme of finite type over k and let X_A be a scheme of finite presentation over A_k. Then there exists a scheme $\widetilde{X}$ (resp. Ξ) of finite presentation over $\widetilde{o} = o_{(\Sigma)}$ (resp. $\widetilde{A} = A_k(\Sigma)$) for some finite closed subset Σ of* $\operatorname{Spec} o$, *such that*

$$X = \widetilde{X} \times_{\widetilde{o}} k$$

and

$$X_A = \Xi \times_{\widetilde{A}} A.$$

(b) *Let Σ be a finite closed subset of* $\operatorname{Spec} o$. *Let $\widetilde{P}$, $\widetilde{X}$ be two schemes of finite type over $\widetilde{o} = o_{(\Sigma)}$ with generic k-fibres P and X and let Π, Ξ be two schemes of finite presentation over $\widetilde{A} = A_k(\Sigma)$ with*

$$P_A = \Pi \times_{\widetilde{A}} A,$$
$$X_A = \Xi \times_{\widetilde{A}} A.$$

Then the obvious maps

$$e : \varinjlim \operatorname{Hom}_{(S)}(\widetilde{P}_{(S)}, \widetilde{X}_{(S)}) \longrightarrow \operatorname{Hom}_k(P, X), \quad \Sigma \subseteq S \subset W_{\mathrm{fin}},$$

$$e_A : \varinjlim \operatorname{Hom}_{A(S)}(\Pi_{A(S)}, \Xi_{A(S)}) \longrightarrow \operatorname{Hom}_A(P_A, X_A), \quad \Sigma \subseteq S \subset W_{\mathrm{fin}},$$

over finite subsets S are bijective.

(c) *Let*

$$(p_{(S)}) \in \varinjlim \operatorname{Hom}_{(S)}(\widetilde{P}_{(S)}, \widetilde{X}_{(S)})$$

resp.

$$(\pi_{A(S)}) \in \varinjlim \operatorname{Hom}_{A(S)}(\Pi_{A(S)}, \Xi_{A(S)})$$

be the element corresponding to

$$p \in \operatorname{Hom}_k(P, X) \quad \textit{resp.} \quad \pi_A \in \operatorname{Hom}_A(P_A, X_A)$$

under the bijection e (resp. e_A) in (b). Let $\mathcal{P}$ be any of the following properties.

(i) *p resp. π_A is an isomorphism,*
(ii) *p resp. π_A is an open immersion,*
(iii) *p resp. π_A is a closed immersion,*
(iv) *p resp. π_A is separated,*
(v) *p resp. π_A is surjective,*
(vi) *p resp. π_A is affine,*
(vii) *p resp. π_A is proper,*
(viii) *p resp. π_A is projective,*

(ix) *p resp. π_A is quasi-projective,*

(x) *p resp. π_A is smooth.*

Then there exists a closed subset $T \supseteq \Sigma$ of $\operatorname{Spec} o$ *such that $\mathcal{P}$ holds for $p_{(S)}$ resp. $\pi_{A(S)}$ for all finite closed subsets S of W_{fin} containing T.*

Proof. — Let R be a Noetherian ring. Then a scheme of finite type over R is also of finite presentation over R (cf. [**31**, 6.3.7]). Therefore, (a) resp. (b) is a special case of part (ii) resp. (i) of Théorème 8.8.2 in [**30**] and part (i)-(ix) of (c) is a special case of Théorème 8.10.5 in (op. cit.) and (x) is a special case of Proposition 17.7.8 in (op. cit.). □

Lemma 4.3. — *Let Σ be a finite closed subset of* $\operatorname{Spec} o$ *and let $f : \widetilde{X} \to$* $\operatorname{Spec} o_{(\Sigma)}$ *be a morphism of finite type. Then there exists a finite closed subset $T \supseteq \Sigma$ of* $\operatorname{Spec} o$ *such that the morphism*

$$f_{(S)} : \widetilde{X}_{(S)} \longrightarrow \operatorname{Spec} o_{(S)}$$

is flat for all finite subsets $S \subset W_{\text{fin}}$ containing T.

Proof. — See Théorème 6.9.1 in [**30**]. □

Definition 4.4

(a) Let X be a separated scheme of finite type over k and let Σ be a finite closed subset of $\operatorname{Spec} o$. An *$o_{(\Sigma)}$-model of* X is an $o_{(\Sigma)}$-scheme $\widetilde{X}$ which is separated and of finite type over $o_{(\Sigma)}$ and for which the generic fibre of $\widetilde{X}/o_{(\Sigma)}$ is equal to X. A *model of* X is an $o_{(\Sigma)}$-model of X for some finite subset $\Sigma \subset W_{\text{fin}}$.

(b) Let X_A be an A_k-scheme which is separated and of finite presentation over A_k. Let Σ be a finite closed subset of $\operatorname{Spec} o$. An *$A_k(\Sigma)$-model of* X_A/A_k is an $A_k(\Sigma)$-scheme $\widetilde{X}$ which is separated and of finite presentation over $A_k(\Sigma)$. A *model of* X_A/A_k is an $A_k(\Sigma)$-model of X_A for some finite subset $\Sigma \subset W_{\text{fin}}$.

Remarks 4.5

(a) It follows from (4.2)(a) that any k-scheme X (resp. any A_k-scheme X) in (4.4) has a model. Also, by (4.3) there exists a model $\widetilde{X}$ of X which is flat over its base ring. If X/k (resp. X_A/A_k) is smooth, then there exists a model which is smooth over the base ring (cf. (4.2)(c)(x)). The models are not unique, but any two such models become isomorphic after a base extension to $o_{(S)}$ resp. $A_k(S)$ for a sufficiently large finite subset

$$S \subset W_{\text{fin}}$$

(see (b) and (c)(i) in (4.2)).

(b) Let X be a separated scheme of finite type over k and let $\widetilde{X}$ be a model over $\widetilde{o} = o_{(\Sigma)}$ of X. Further, let

$$\widetilde{A} = A_k(\Sigma),$$
$$\Xi = \widetilde{X} \times_{\widetilde{o}} \widetilde{A}$$

for the diagonal embeddings $\widetilde{o} \subset \widetilde{A}$, $k \subset A$ and

$$X_A = X \times k_A.$$

Then Ξ is a model of X_A/A_k.

The adele ring A_k is a locally compact topological ring. The underlying topological space is the restricted direct product

$$A_k = \prod\nolimits'_{\nu \in W} k_\nu$$

of the locally compact fields k_ν with respect to the compact integer rings

$$o_\nu \subset k_\nu, \quad \nu \in W_{\text{fin}}.$$

The subspace topology of $A_k(\Sigma)$ for finite subsets $\Sigma \subset W_{\text{fin}}$ is the product topology of the ν-adic topologies. The topology of A_k is therefore the inductive limit topology of the product topologies of $A_k(\Sigma)$ for finite subsets $\Sigma \subset W_{\text{fin}}$.

Let X_A be an A_k-scheme such that X_A/A_k is separated and of finite presentation. Write $X_A(k_\nu)$ (resp. $X_A(A_k)$) for the set of all A_k-morphisms

$$\operatorname{Spec} k_\nu \longrightarrow X_A,$$
$$\operatorname{Spec} A_k \longrightarrow X_A.$$

The embedding

$$A_k \subset \prod_{\nu \in W} k_\nu$$

induces an injection

$$X_A(A_k) \subset \prod_{\nu \in W} X_A(k_\nu).$$

If $\Xi/\widetilde{A}$, $\widetilde{A} = A_k(\Sigma)$ is a model of X_A, then Ξ is the direct sum (cf. [**31**, 3.1]) of the o_ν-schemes

$$\Xi_\nu := \Xi \times_{\widetilde{A}} o_\nu, \quad \nu \in W_{\text{fin}} \backslash \Sigma$$

and of the k_ν-schemes

$$X_\nu := X_A \times_A k_\nu, \quad \nu \in \Sigma \cup W_\infty.$$

The set $\Xi(o_\nu)$ of all $\widetilde{A}$-morphisms

$$\operatorname{Spec} o_\nu \longrightarrow \widetilde{X}$$

form a compact open subset of $X_A(k_\nu)$ for $\nu \in W_{\text{fin}} \backslash \Sigma$ (cf. (2.6)).

Lemma 4.6. — *Let X_A be an A_k-scheme such that X_A/A_k is separated and of finite presentation and let $\Xi/\widetilde{A}$, $\widetilde{A} = A_k(\Sigma)$ be a model of X_A. Further, let*

$$\prod\nolimits'_{\nu\in W} X_A(k_\nu)$$

be the set of elements

$$\{P_\nu\} \in \prod_{\nu\in W} X_A(k_\nu)$$

such that $P_\nu \in \Xi(o_\nu)$ for all but finitely many places. Then $X_A(A_k)$ is mapped onto $\prod'_{\nu\in W} X_A(k_\nu)$ under the embedding

$$X_A(A_k) \subset \prod_{\nu\in W} X_A(k_\nu)$$

If $\Xi'/o_{(\Sigma')}$ is another model of X_A, then

$$\Xi(o_\nu) = \Xi'(o_\nu)$$

in $X(k_\nu)$ for all but finitely many places.

Proof. — The first statement is a consequence of the isomorphisms e_A (see (4.2)(b)) between $\varinjlim \mathrm{Hom}_{A(S)}(\mathrm{Spec}\, A(S), \Xi_{A(S)})$ and $\mathrm{Hom}_A(\mathrm{Spec}\, A, X_A)$. To prove the second assertion we may replace Σ and Σ' by $\Sigma \cup \Sigma'$ and reduce to the case $\Sigma = \Sigma'$. Then the assertion follows from (4.2)(c)(i) for $\Pi = \Xi'$. □

Definition and proposition 4.7. — *Let X_A be an A_k-scheme such that X_A/A_k is separated and of finite presentation and let $\Xi/A_k(\Sigma)$ be a model of X. The adelic topology on $X_A(A_k)$ is the restricted product topology on*

$$X_A(A_k) = \prod\nolimits'_{\nu\in W} X_A(k_\nu)$$

with respect to the compact open subsets

$$\Xi(o_\nu) \subset X(k_\nu), \quad \nu \in W_{\mathrm{fin}}\backslash\Sigma.$$

This topology is independent of the choice of model.

If X is a separated scheme of finite type over k and $X(A_k)$ is the set of all k-morphisms

$$\mathrm{Spec}\, A_k \longrightarrow X,$$

then we define the adelic topology on $X(A_k)$ to be the topology induced by the adelic topology on $X_A(A_k)$ for

$$X_A = X \times_k A_k$$

under the obvious bijection between $X(A_k)$ and $X_A(A_k)$. The adelic space of X is the topological space $X(A_k)$ endowed with the adelic topology.

Proof. — The restricted topological product

$$\prod\nolimits'_{\nu\in W} X(k_\nu)$$

does not change if we change or omit $\widetilde{X}(o_\nu)$ (resp. $\Xi(o_\nu)$) at finitely many places. It therefore follows from (4.6) that the adelic topology is independent of the choice of model. □

Examples 4.8

(a) Let X be a separated scheme of finite type over k and let $\widetilde{X}/o_{(\Sigma)}$ be a model of X. If $\nu \in W$ (resp. $\nu \in W_{\mathrm{fin}}\backslash\Sigma$), let $X_A(k_\nu)$ (resp. $\widetilde{X}(o_\nu)$) denote the set of all k-morphisms

$$\operatorname{Spec} k_\nu \longrightarrow X$$

resp. $o_{(\Sigma)}$-morphisms

$$\operatorname{Spec} o_\nu \longrightarrow \widetilde{X}.$$

Let

$$X(A_k) \subset \prod_{\nu\in W} X(k_\nu)$$

be the injection induced by the embedding

$$A_k \subset \prod_{\nu\in W} k_\nu.$$

Then, the adelic space of X is equal to the restricted topological product of all $X(k_\nu)$, $\nu \in W$ with respect to the compact open subsets $\widetilde{X}(o_\nu) \subseteq X(k_\nu)$ (cf. (2.5)) for all $\nu \in W_{\mathrm{fin}}\backslash\Sigma$. In particular, $X(A_k)$ is locally compact.

(b) Let $X = \mathbb{A}_k^r$ and choose $\Xi = \mathbb{A}_o^r$ as o-model of X. The adelic space $X(A_k)$ may be identified with the topological product $A_k^{(r)}$ of r copies of A_k. If X is an affine k-variety, then the adelic topology of $X(A_k)$ is the coarsest topology such that all the maps $X(A_k) \to A_k$ determined by regular k-functions on X are continuous.

(c) Let X_A be a smooth proper A-scheme. There exists (cf. (4.2)) a proper model $\Xi/A_k(\Sigma)$ of X over $o_{(\Sigma)}$ for some finite subset Σ of W_{fin}. Moreover,

$$\Xi(o_\nu) = X(k_\nu)$$

for all $\nu \in W_{\mathrm{fin}}\backslash\Sigma$ by Grothendieck's valuative criterion for properness so that

$$X_A(A_k) = \prod_{\nu\in W} X(k_\nu)$$

as a topological space. By Tychonoff's theorem we conclude from (2.3)(b) that

$$X_A(A_k) = \prod_{\nu \in W} X(k_\nu)$$

is compact if X is smooth and proper.

Remarks 4.9

(a) Let X_A be an A_k-scheme such that X_A/A_k is separated and of finite presentation and let $\Xi/A_k(\Sigma)$ be a model of X. Then it follows from the definition of the adelic topology that the sets of the form:

$$\prod_{\nu \in S} U_\nu \times \prod_{\nu \notin S} \Xi(o_\nu), \quad \Sigma \cup W_\infty \subseteq S \subseteq W$$

for finite S and open subsets

$$U_\nu \subset X(k_\nu), \quad \nu \in S$$

form an open base for the topology on $X_A(A_k)$. Also, there exists for each place $\nu \in W$ a countable base $\mathcal{F}_\nu$ of open subsets with compact support for the k_ν-topology on $X(k_\nu)$ (cf. (2.3)). Let $\mathcal{F}$ be the family of open subsets of $X(A_k)$ as above with the additional condition that $U_\nu \in \mathcal{F}$ for all $\nu \in S$. Then $\mathcal{F}$ is a countable base of open subsets with compact support for the adelic topology on $X_A(A_k)$. It follows that the adelic space $X_A(A_k)$ is a locally compact, σ-compact, paracompact Hausdorff space.

(b) Let $\pi_A : P_A \to X_A$ be an A-morphism between two A_k-schemes as in (a). Then π_A determines a map

$$\pi_A(A_k) : P(A_k) \longrightarrow X(A_k)$$

between the adelic spaces. It is an easy consequence of (4.2) and the continuity of the maps

$$P(k_\nu) \longrightarrow X(k_\nu)$$

that this map is continuous.

We now define adelic norms for A-schemes X_A, thereby generalizing Peyre's notion of adelic metrics [Pe1]. If $X_A = X \times_k A_k$ for a k-variety X, then

$$X_A \times_A k_\nu = X \times_k k_\nu$$

and $X_A(k_\nu)$ corresponds bijectively to the set $X(k_\nu)$ of all k-morphisms

$$\operatorname{Spec} k_\nu \longrightarrow X.$$

We shall by *abuse of notation* write

$$X_\nu = X_A \times_A k_\nu,$$

$$X(k_\nu) = X_A(k_\nu)$$

also in the case where X_A is *not* a fibre product $X \times_k A_k$ for a k-variety X.

The scheme X_A will from now on be smooth, separated and of finite type over A. This implies that X_A is of finite presentation over A since any smooth morphism is locally of finite presentation by definition. Hence there exist smooth models of X_A by (4.2).

The set $X(k_\nu) := X_A(k_\nu)$ is canonically isomorphic to $X_\nu(k_\nu)$ and will always be endowed with a manifold structure. We shall write $X_{\mathrm{an}}(k_\nu)$ for this analytic manifold over k_ν. If X is an $A_k(\Sigma)$-model of X and $\nu \in W_{\mathrm{fin}} \backslash \Sigma$, then the compact open subset

$$\Xi(o_\nu) \subseteq X(k_\nu)$$

inherits an analytic manifold structure from $X(k_\nu)$, which we denote by $\Xi_{\mathrm{an}}(o_\nu)$.

Definition 4.10

(a) A smooth A_k-variety is an A_k-scheme which is smooth, separated and of finite type over $A = A_k$ such that X_ν is geometrically connected for all $\nu \in W$.

(b) Let X_A be a smooth A-variety. An *adelic norm for* X_A is a family of norms

$$\| \ \| = \{\| \ \|_\nu : \mathbf{det}\, \mathrm{Tan}\, X_{\mathrm{an}}(k_\nu) \longrightarrow [0, \infty), \quad \nu \in W\}$$

on the analytic tangent bundles of $X_{\mathrm{an}}(k_\nu)$, $\nu \in W$ with the following property. There exists a finite subset Σ of W_{fin} and an $A_k(\Sigma)$-model Ξ of X such that the restriction of $\| \ \|_\nu$ to the analytic anticanonical line bundle on $\Xi(o_\nu)$ is equal to the model norm determined by X_ν/o_ν for all $\nu \in W_{\mathrm{fin}} \backslash \Sigma$.

If $X_A = X \times_k A_k$ for a smooth k-variety X, then $\| \ \|$ is said to be *an adelic norm for* X.

It is clear from (4.2) that the last condition in (4.10) is true for any model of X as soon as it is true for one model of X. We may therefore assume that Ξ is a smooth model.

Let M be a topological space. We shall in the sequel write $C(M)_{>0}$ for the vector space of all continuous functions

$$f : M \longrightarrow (0, \infty).$$

Definition 4.11. — Let X_A be smooth A-variety and let

$$\gamma = \{\gamma_\nu \in C(X(k_\nu))_{>0},\ \nu \in W\}$$

be a set of positive continuous functions. Then γ is called a set of convergence factors for X_A if the following condition holds.

There exists a finite subset Σ of W_{fin} and an $A_k(\Sigma)$-model Ξ of X such that the product

$$\prod_{W_{\mathrm{fin}}\backslash\Sigma}\left(\int_{\Xi(o_\nu)}\gamma_\nu d\mu_\nu\right)$$

converges absolutely to a positive real number for the model measures μ_ν on $\Xi(o_\nu)$ determined by Ξ_ν/o_ν (see (2.9)).

If all functions γ_ν are constant, then $\gamma = \{\gamma_\nu,\ \nu \in W\}$ is called a *constant set of convergence factors for* $X_A(A_k)$ if

$$\prod_{W_{\mathrm{fin}}\backslash\Sigma}\gamma_\nu\mu_\nu(\Xi(o_\nu))$$

converges absolutely to a positive real number.

If $X_A = X \times_k A_k$ for a smooth k-variety X, then

$$\gamma = \{\gamma_\nu \in C(X(k_\nu))_{>0},\ \nu \in W\}$$

is said to be a *set of convergence factors for* X.

Remarks 4.12

(a) The absolute convergence of the product is not affected by a change of finitely many γ_ν. We shall therefore use the term "set of convergence factors" also in cases where γ_ν is not defined for a finite set S of places of k. One can then put $\gamma_\nu = 1$ for $\nu \in S$.

(b) It is clear from (4.2)(c)(i) that the absolute convergence of the products above only depends on γ and not on the choice of model.

(c) If X is a smooth k variety and $\widetilde{X}$ is a model over $\widetilde{o} = o_{(\Sigma)}$, then $\widetilde{X}(o_\nu)$ is non-empty for almost all $\nu \in W_{\mathrm{fin}}\backslash\Sigma$ (cf. (4.19)). One may thus find a set of constant convergence factors for X with

$$\gamma_\nu = \mu_\nu(\widetilde{X}(o_\nu))^{-1}$$

for almost all ν. This is not true for arbitrary smooth A_k-varieties since there are smooth A_k-varieties such that $X_A(k_\nu) = \varnothing$ for infinitely many places.

Definition 4.13

Let X_A be an A_k-variety as in (4.10).

(a) Let S be a finite set of places of k. Then X_S is the topological space

$$X_S = \prod_{\nu \in S} X(k_\nu)$$

endowed with the product topology of the analytic k_ν-topologies, $\nu \in S$. A subset B_S of X_S is said to be decomposable if it is of the form

$$B_S := \prod_{\nu \in S} B_\nu$$

for some Borel subsets

$$B_\nu \subset X_{\mathrm{an}}(k_\nu), \quad \nu \in S.$$

(b) A decomposable subset of $X_A(A_k)$ is a subset B of the form

$$B := \prod_{\nu \in W} B_\nu$$

for Borel subsets

$$B_\nu \subset X_{\mathrm{an}}(k_\nu)$$

such that

$$B_\nu = \Xi(o_\nu)$$

for all $\nu \in W_{\mathrm{fin}} \backslash \Sigma$ for some model $\Xi / o_{(\Sigma)}$ of X.

(c) Let $B = \prod_{\nu \in W} B_\nu$ be a compact decomposable subset of $X(A_k)$. For each $\nu \in W$, let

$$f_\nu : X(k_\nu) \longrightarrow \mathbb{R}$$

be a continuous function with support in B_ν. Suppose that there exists a finite subset $T \subset W$ such that f_ν is (strictly) positive on B_ν for all $\nu \in W \backslash T$ and such that

$$\prod_{W \backslash T} f_\nu$$

converges absolutely to a strictly positive function on $\prod_{W \backslash T} B_\nu$. Let $f \in C_c(X_A(A_k))$ be the function with support in B, such that

$$f = \prod_{\nu \in W} f_\nu$$

on B. Then f is called *the restricted product of* $\{f_\nu,\ \nu \in W\}$ and any such function $f \in C_c(X_A(A_k))$ is said to be *decomposable*. If f_ν can be chosen to be the characteristic function of B_ν for almost all $\nu \in W$, then f is said to be *finitely decomposable*.

We now define adelic measures following Weil [**67**] (cf. also [**61**] and [**52**]).

Theorem 4.14. — *Let X_A be a smooth A-variety as in (4.10) and let*

$$\gamma = \{\gamma_\nu \in C(X(k_\nu))_{>0},\ \nu \in W\}$$

be a set of convergence factors for $X(A_k)$. Let

$$\|\ \| = \{\|\ \|_\nu : \mathbf{det}\, \mathrm{Tan}\, X(k_\nu) \longrightarrow [0, \infty), \quad \nu \in W\}$$

be an adelic norm for X_A *and let*

$$\{m_\nu,\ \nu \in W\}$$

be the regular positive Borel measures on $X(k_\nu)$ *determined by these norms (see (1.12), (1.14)). Then the following holds.*

(a) *Let* T *be a finite subset of* W. *Then there exists a unique* σ*-regular positive Borel measure* $m_{T,\gamma}$ *on* $X(k_\nu)$ *such that*

$$m_{T,\gamma}(B) = \prod_{\nu \in T} \left(\int_{B_\nu} \gamma_\nu dm_\nu \right)$$

for any decomposable subset

$$B = \prod_{\nu \in T} B_\nu$$

of $X(k_\nu)$.

(b) *There exists a unique* σ*-regular positive Borel measure* $m_{A,\gamma}$ *on* $X_A(A_k)$ *such that*

$$(*) \qquad m_{A,\gamma}(B) = \prod_{\nu \in W} \left(\int_{B_\nu} \gamma_\nu dm_\nu \right)$$

for any decomposable subset

$$B = \prod_{\nu \in W} B_\nu$$

of $X_A(A_k)$ *for which all* $m_\nu(B_\nu)$ *are finite.*

(c) *Let* $\{f_\nu \in C_c(X(k_\nu)),\ \nu \in W\}$, $T \subset W$, $f \in C_c(X(A_k))$ *be as in (4.13)(c). Then,*

$$\prod_{W \setminus T} \left(\int_{X(k_\nu)} f_\nu \gamma_\nu dm_\nu \right)$$

converges absolutely to a positive real number and

$$\int_{X(A_k)} f dm_{A,\gamma} = \prod_{\nu \in W} \left(\int_{X(k_\nu)} f_\nu \gamma_\nu dm_\nu \right).$$

Proof

(a) Let $\| \ \|_\nu : \mathbf{det}\, \mathrm{Tan}\, X(k_\nu) \to [0,\infty)$, $\nu \in W$, be the new norms obtained by multiplying $\| \ \|_\nu$ with γ_ν and let m_ν be the Borel measure on $X(k_\nu)$ determined by $\| \ \|_\nu$. Then,

$$\int_{B_\nu} \gamma_\nu dm_\nu = m_\nu(B_\nu).$$

The assertion is therefore equivalent to the existence and uniqueness of the product measure of m_ν over all $\nu \in T$. This follows from repeated use of Theorem 8.2 in [**38**, Ch. VI] (the reader can also consult Chap. III, §5, No 4 in [**11**]).

(b) Let Ξ be an $A_k(\Sigma)$-model of X_A for some finite subset S of W_{fin} such that $\| \|_\nu$ restricts to the model norm on $\mathbf{det}\,\mathrm{Tan}\,\Xi(o_\nu)$ for all $\nu \in W_{\text{fin}} \backslash \Sigma$. We shall call a set of functions

$$\{g_\nu \in C_c(X(k_\nu),\ \nu \in W\}$$

adelic if g_ν is the characteristic function of

$$\Xi(o_\nu) \subseteq X(k_\nu)$$

for all but finitely many $\nu \in W_{\text{fin}} \backslash \Sigma$. It is then by Riesz' representation theorem (cf. (1.11) and the references there) sufficient to show that there exists a unique positive functional Λ on $X_A(A_k)$ such that:

$$\Lambda(g) = \prod_{\nu \in W} \left(\int_{B_\nu} g_\nu \gamma_\nu dm_\nu \right) \tag{”}$$

for the restricted product g of any adelic set $\{g_\nu \in C_c(X(k_\nu)),\ \nu \in W\}$.

The product of two finitely decomposable function is again finitely decomposable. This implies that finite sums of decomposable functions form a subalgebra $\mathcal{A}$ of $C_c(X(A_k))$. Also, if $g_1, \dots, g_n$ are finitely decomposable functions on $X_A(A_k)$ with non-negative sum, then it follows from (a) that

$$\Lambda(g_1) + \cdots + \Lambda(g_n) \geq 0.$$

There is thus a unique well-defined positive functional Λ on $\mathcal{A}$ satisfying (”).

Next, let $f \in C_c(X_A(A_k))$. There is then an open decomposable neighbourhood

$$U = \prod_{\nu \in W} U_\nu$$

of the support of f with compact closure

$$J = \prod_{\nu \in W} J_\nu$$

in $\prod_{\nu \in W} X(k_\nu)$.

The restrictions to J of functions in $C_c(X(A_k))$ form a subalgebra

$$\mathcal{A}(J) \subseteq C_c(J)$$

which separates points and contains the constant functions. There exists therefore by the Stone-Weierstrass theorem [**38**, p. 52] a sequence $(g_i)_{i=1}^{\infty}$ of functions in $\mathcal{A}(J)$ which converges uniformly to f on J. There exists further by Urysohn's lemma [**38**, IX. 2.1] a finitely decomposable function $H \in C_c(X(A_k))$ such that

$$\begin{aligned} 0 \le H \le 1 &\quad \text{on } X_A(A_k) \\ H = 0 &\quad \text{outside } U \\ H = 1 &\quad \text{on } \operatorname{Supp} f. \end{aligned}$$

Hence

$$(f_i)_{i=1}^{\infty} := (Hg_i)_{i=1}^{\infty}$$

is a sequence of functions in $\mathcal{A}$ with support in J which converges uniformly to f on $X(A_k)$. The same is true for $(H^2g_i^2)_{i=1}^{\infty}$ with respect to f^2. We may thus for non-negative f assume that all functions in the sequence $(f_i)_{i=1}^{\infty}$ are non-negative.

Now let $(f_i)_{i=1}^{\infty}$ be an arbitrary sequence of functions in $\mathcal{A}$ which converges uniformly to f on $X(A_k)$ and such that there exists a compact subset K of $X(A_k)$ containing the supports of all f_i. There exists then by Urysohn's lemma a finitely decomposable non-negative function G such that $G = 1$ on K. Then,

$$-G \sup |f_i - f_j| \le (f_i - f_j) \le G \sup |f_i - f_j|$$

on $X_A(A_k)$ so that:

$$|\Lambda(f_i) - \Lambda(f_j)| = |\Lambda(f_i - f_j)| \le \Lambda(G) \sup |f_i - f_j|$$

by the linearity and positivity of Λ for functions in $\mathcal{A}$. Also,

$$|\Lambda(f_i)| \le \Lambda(G) \sup |f_i|$$

by the same argument.

This implies that $(\Lambda f_i)_{i=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ for any sequence $(f_i)_{i=1}^{\infty}$ as above and that $(\Lambda f_i)_{i=1}^{\infty}$ converges to zero if $(f_i)_{i=1}^{\infty}$ converges to zero. There exists therefore a unique extension of the positive functional Λ on $\mathcal{A}$ to a positive functional Λ on $C_c(X(A_k))$. This completes the proof of (b).

(c) Choose a bijective function $\mathbb{Z}_{>0} \to W \backslash T$ and write W_i for the image of $\{1, \ldots, i\}$. Let $g_i \in C_c(X(A_k))$, $i \in \mathbb{Z}_{>0}$ be the restricted product of f_ν for $\nu \in T \cup W_i$ and of the characteristic function of B_ν for $\nu \in W \backslash (T \cup W_i)$. Then $(g_i)_{i=1}^{\infty}$ is a sequence of functions of finitely decomposable functions with supports in B such that $(g_i)_{i=1}^{\infty}$ converges uniformly to f on $X(A_k)$. Therefore, $(\Lambda g_i)_{i=1}^{\infty}$ converges to Λf so that the result follows from the validity of the formula for finitely decomposable functions established in (b). This completes the proof. □

Remarks 4.15

(a) All the Borel measures in (4.14) are regular since $X_A(A_k)$ is σ-compact (see (1.15) and (4.9)).

(b) Suppose that $X_A := X \times_k A_k$. We may then regard $m_{A,\gamma}$ as a measure on the adelic space $X(A_k)$ for X (cf. (4.7) and (4.8)(a)).

The following example is due Peyre [**52**, 2.2] except for the fact that (2.15) makes it possible to consider non-projective varieties as well.

Example 4.16. — Let X be a smooth proper k-variety. Then there exists a smooth proper model $\widetilde{X}/o_{(\Sigma)}$ of X (cf. (4.2), (4.4)) such that all reductions

$$Y_\nu := \Xi \times F_\nu, \quad \nu \in W_{\text{fin}} \backslash \Sigma$$

are geometrically integral (cf. [**30**, 12.2.1] or (4.18) below).

Let $\overline{F}_\nu$ be an algebraic closure of F_ν, and let

$$\overline{Y}_\nu = \Xi \times \overline{F}_\nu, \quad \nu \in W_{\text{fin}} \backslash \Sigma.$$

Then the geometric Frobenius Fr_ν acts on $\operatorname{Pic} \overline{Y} \otimes \mathbb{Q}$ and one defines the local L-function for $\nu \in W_{\text{fin}} \backslash \Sigma$ by:

$$L_\nu(s, \operatorname{Pic} \overline{Y}_\nu) = 1/\det(1 - q_\nu^{-s} \operatorname{Fr}_\nu \mid \operatorname{Pic} \overline{Y}_\nu \otimes \mathbb{Q}).$$

Now let $H^1_{\text{Zar}}(X, \mathcal{O}_X) = H^2_{\text{Zar}}(X, \mathcal{O}_X) = 0$. Then

$$H^1_{\text{Zar}}(Y_\nu, \mathcal{O}_{Y_\nu}) = H^2_{\text{Zar}}(Y_\nu, \mathcal{O}_{Y_\nu}) = 0$$

for all but finitely many $\nu \in W_{\text{fin}} \backslash \Sigma$. Hence Deligne's theorem [**19**, 3.3.9] implies that (cf. [**52**, p. 117])

$$\operatorname{Card}(Y_\nu(F_\nu))/q_\nu^{\dim X} = 1 + \operatorname{Tr}(\operatorname{Fr}_\nu \mid \operatorname{Pic} \overline{Y}_\nu \otimes \mathbb{Q})/q_\nu + O(1/q_\nu^{3/2})$$

for $\nu \in W_{\text{fin}} \backslash \Sigma$.

Moreover, by (2.15) one has

$$\mu_\nu(\Xi(o_\nu)) = \operatorname{Card}(Y_\nu(F_\nu))(\mu_\nu(o_\nu)/q_\nu)^{\dim X}$$

with $\mu_\nu(o_\nu) = 1$ for all but finitely many $\nu \in W_{\text{fin}}$.

Set

$$\gamma_\nu := \begin{cases} 1/L_\nu(1, \operatorname{Pic} \overline{Y}_\nu) & \text{for } \nu \in W_{\text{fin}} \backslash \Sigma \\ 1 & \text{for } \nu \in W_\infty \cup \Sigma. \end{cases}$$

Then it follows from the equalities above that (γ_ν) is a set of convergence factors for X.

Example 4.17. — Serre [**61**, p. 655] applies Deligne's theorem [**19**, 3.3.4] to smooth (not necessarily proper) varieties X for which the first two Betti numbers

$$B_i := \dim_{\mathbb{Q}} H^i_{\text{Sing}}(X_{\text{an}}(\mathbb{C}), \mathbb{Q}), \quad i = 1, 2$$

vanish (for some embedding $k \subset \mathbb{C}$). He concludes that any model $\Xi/o_{(\Sigma)}$ of X satisfies

$$\operatorname{Card}(\Xi(F_\nu))/q_\nu^{\dim X} = 1 + O(1/q_\nu^{3/2}), \quad \nu \in W_{\text{fin}} \backslash \Sigma.$$

One may thus choose $\lambda_\nu := 1, \quad \nu \in W$ as a set of convergence factors for such X.

It is possible to formulate a result for smooth (not necessarily proper) varieties with $B_1 = 0$ which contains the results of Peyre and Serre as special cases. The convergence factors are given by $1/L_\nu(1)$ for local L-functions $L_\nu(s)$ defined by the means of the second ℓ-adic cohomology groups for a fixed ℓ. One expects that the local L-functions $L_\nu(s)$ are independent of the choice of ℓ, but this is known only in the proper case (cf. [**34**, pp. 27-28]).

Our next goal is to describe a relative version of theorem (4.14) for families of smooth varieties. We first need some results on models.

Lemma 4.18. — *Let Σ be a finite subset of W_{fin} and let $\widetilde{P}$, $\widetilde{X}$ be two schemes of finite type over $o_{(\Sigma)}$ with generic fibres P resp. X. Let $p : P \to X$ be a morphism such that all geometric fibres of p are irreducible (resp. connected, reduced or integral).*

Then there exists a finite subset $S \subset W_{\text{fin}}$ containing Σ and an extension of $p : P \to X$ to a morphism.

$$p_{(S)} : \widetilde{P}_{(S)} \longrightarrow \widetilde{X}_{(S)}$$

for which all geometric fibres are irreducible (resp. connected, reduced or integral).

Proof. — Let

$$p_{(S)} : \widetilde{P}_{(S)} \longrightarrow \widetilde{X}_{(S)}, \quad \Sigma \subseteq S \subset W_{\text{fin}}$$

be a morphism in the inductive limit (cf. (4.2))

$$(p_{(S)}) \in \varinjlim \operatorname{Hom}_{(S)}(\widetilde{P}_{(S)}, \widetilde{X}_{(S)}), \quad \Sigma \subseteq S \subset W_{\text{fin}}$$

corresponding to p and let E be the set of closed points Q of $\widetilde{X}_{(S)}$ such that the geometric fibre of $p_{(S)}$ over Q is irreducible (resp. connected, reduced or integral).

Then $p_{(S)}$ is of finite presentation since: $\widetilde{P}_{(S)}$ and $\widetilde{X}_{(S)}$ are of finite presentation over $o_{(S)}$ [**31**, 6.3.7-8]. Hence by [**30**, Th. 9.7.7], E must be a locally constructible subset of $\widetilde{X}_{(S)}$. Then it follows from a theorem of Chevalley [**31**, 7.1.4] that $\widetilde{X}_{(S)} \backslash E$ is mapped onto a locally constructible subset of $\operatorname{Spec} o_{(S)}$ under the structure morphism of $\widetilde{X}_{(S)}/o_{(S)}$. But any locally constructible subset of an affine Dedekind scheme is either an open or closed subset. Thus since $\widetilde{X}_{(S)} \backslash E$ and the generic fibre of $\widetilde{X}_{(S)}/o_{(S)}$ are disjoint by assumption we conclude that $\widetilde{X}_{(S)} \backslash E$ is contained in finitely many closed fibres. This completes the proof. □

Recall that a k-variety X in this paper means a *geometrically connected* separated scheme of finite type over k. Hence if X is smooth, then it is geometrically integral.

Lemma 4.19. — *Let X be a smooth k-variety and let $\pi : Y \to X$ be a smooth k-morphism of constant relative dimension on X with geometrically connected fibres. Then there exists two smooth models $\widetilde{Y}$, $\widetilde{X}$ over some integer ring $\widetilde{o} = o_{(\Sigma)}$ and an extension of π to a smooth $\widetilde{o}$-morphism $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ with the following properties:*

(i) all fibres of $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ are smooth and geometrically integral,
(ii) $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ is of constant relative dimension on $\widetilde{Y}$,
(iii) the map from $\widetilde{Y}(o_\nu)$ to $\widetilde{X}(o_\nu)$ defined by composition with $\widetilde{\pi}$ is surjective for all places.

Proof. — It follows from (4.2) that π has an extension to a smooth $\widetilde{o}$-morphism $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ between smooth models over some integer ring $\widetilde{o} = o_{(\Sigma)}$. We now prove (i)-(iii)

(i) It follows from (4.18) that the fibres of $\widetilde{\pi}$ are geometrically integral after an enlargement of Σ since the fibres of $\pi : Y \to X$ are geometrically integral

(ii) It follows from the irreducibility of $\widetilde{Y}$ and the fact that $\widetilde{\pi}$ is flat and of locally finite presentation that $\widetilde{\pi}$ is of constant relative dimension on $\widetilde{Y}$ (cf. [**30**, 14.2.2 and 2.4.6]).

(iii) We may (and shall) assume that (i) holds and that there exists a prime number ℓ which is invertible in all residue fields F_ν of $\widetilde{o}$. Then by the Lefschetz formula of Grothendieck and Deligne (cf. [**34**]) one has for each variety Z over F_ν, $\nu \in W_{\text{fin}} \backslash \Sigma$ an equality

$$\operatorname{Card} Z(F_\nu) = \sum_i (-1)^i \operatorname{Tr}(F^n, H_c^i(Z \times \overline{F}_\nu, \mathbb{Q}_l)) = \sum_{i,j} (-1)^i \alpha_{i,j}^n$$

where $H_c^i(Z, \mathbb{Q}_l)$ are the ℓ-adic cohomology groups $0 \leq i \leq 2 \dim Z$ with compact support and $\alpha_{i,j}$ are the eigenvalues of the Frobenius endomorphsim on $H_c^i(Z, \mathbb{Q}_l)$. Moreover, by the fundamental result of Deligne [**19**, 3.3.4] one has that $|\alpha_{i,j}| \leq q_\nu^{i/2}$ for $q_\nu = \operatorname{Card} F_\nu$ with a unique eigenvalue $\alpha_i = \pm q^{\dim Z}$ when $i = 2 \dim Z$ (use Poincaré duality and the integrality of $Z \times \overline{F}_\nu$). Finally, since (cf. [**18**, 6.2]) $R^i \widetilde{\pi}_! \mathcal{F}$ is constructible for any constructible sheaf $\mathcal{F}$ on $\widetilde{Y}$ it follows that the dimensions of the ℓ-adic cohomology groups

$$H_c^i(Z \times \overline{F}_\nu, \mathbb{Q}_l)$$

are uniformly bounded for the closed fibres Z of $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$. Therefore,

$$\operatorname{Card} Z(F_\nu) > 0$$

for all F_ν-fibres Z of $\widetilde{\pi}$ if q_ν is sufficiently large (cf. [**62**, p. 184] for the case $X = k$). To complete the proof, use Hensel's lemma. $\square$

It is worth noting that it follows from (4.19) and (2.7)(a) that the map

$$\pi_A : Y(A_k) \longrightarrow X(A_k)$$

between adelic spaces is *open* for $\pi : Y \to X$ as in (4.19).

Definition 4.20. — Let X be a smooth k-variety and let $\pi : Y \to X$ be a smooth k-morphism with geometrically connected fibres. An *adelic norm for* $\pi : Y \to X$ is a family of norms

$$\| \ \| = \{\| \ \|_\nu : \mathbf{det}\,\mathrm{Tan}(Y_{\mathrm{an}}(k_\nu)/X_{\mathrm{an}}(k_\nu)) \longrightarrow [0,\infty), \quad \nu \in W\}$$

on the relative analytic anticanonical bundles of $\pi_{\nu,\mathrm{an}} : Y_{\mathrm{an}}(k_\nu) \to X_{\mathrm{an}}(k_\nu)$, $\nu \in W$ with the following property.

There exists a finite subset Σ of W_{fin} and an extension of π to a smooth $o_{(\Sigma)}$-morphism $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ between two smooth models $\widetilde{Y}$ and $\widetilde{X}$ over $\widetilde{o} = o_{(\Sigma)}$ such that the restriction of $\| \ \|_\nu$ to the relative analytic anticanonical line bundle of $\widetilde{Y}(o_\nu) \to \widetilde{X}(o_\nu)$ is equal to the model norm determined by $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ (cf. (2.17)) for all $\nu \in W_{\mathrm{fin}} \backslash \Sigma$.

The notion of adelic norms is similar to notions in Arakelov theory and it is possible to use (4.2) to define adelic norms for arbitrary vector bundles. Here we introduce adelic norms only in order to define adelic measures.

Definition 4.21. — Let X be a smooth k-variety and $\pi : Y \to X$ be a smooth k-morphism with geometrically connected fibres. Then a set

$$\beta = \{\beta_\nu \in C(Y(k_\nu))_{>0},\ \nu \in W\}$$

of positive continuous functions is called a *set of convergence factors for* $\pi : Y \to X$ if the following holds.

There exists an extension of $\pi : Y \to X$ to a smooth $o_{(\Sigma)}$-morphism

$$\widetilde{\pi} : \widetilde{Y} \longrightarrow \widetilde{X}$$

as in (4.19) satisfying the following condition:

($*$) The product

$$\prod_{W_{\mathrm{fin}} \backslash \Sigma} \left(\int_{Z_\nu(o_\nu)} \beta_\nu d\mu_\nu \right)$$

converges absolutely to a positive real number for the model measure μ_ν on the o_ν-schemes

$$Z_\nu, \quad \nu \in W_{\mathrm{fin}} \backslash \Sigma$$

obtained by base extension of $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ along any

$$\{P_\nu\} \in \prod_{W_{\mathrm{fin}} \setminus \Sigma} \widetilde{X}(o_\nu).$$

Remarks 4.22

(a) The condition $(*)$ is clearly valid for *all* smooth morphisms $\widetilde{\pi}$ satisfying the conditions in (4.19) as soon as it is valid for one such morphism.

(b) Let $\nu \in W_{\mathrm{fin}} \setminus \Sigma$. Then the map

$$P_\nu \longrightarrow \int_{Z_\nu(o_\nu)} \beta_\nu d\mu_\nu$$

is a continuous real-valued function on the compact space $\widetilde{X}(o_\nu)$ (cf. (1.22)(a) and (2.5)). The product of integrals in $(*)$ defines therefore a continuous function on

$$\prod_{W_{\mathrm{fin}} \setminus \Sigma} \widetilde{X}(o_\nu).$$

(c) One may construct constant sets of convergence factors for $\pi : Y \to X$ when the first two Betti numbers B_1 and B_2 vanish for all geometric fibres of $\pi : Y \to X$. This follows from the arguments in the proofs of (4.17) and (4.19)(iii).

Example 4.23. — Let $\pi : Y \to X$ be as in (4.21) and suppose in addition that X is proper. There is then an extension of $\pi : Y \to X$ to a smooth $o_{(\Sigma)}$-morphism $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ as in (4.19) such that $\widetilde{X}$ is proper over $o_{(\Sigma)}$. For $\nu \in W_{\mathrm{fin}} \setminus \Sigma$, let

$$\widetilde{\Lambda}_\nu : C_c(\widetilde{Y}(o_\nu)) \longrightarrow C_c(\widetilde{X}(o_\nu))$$

be the positive linear map of $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ (cf. (1.22)(a)) defined by the relative model norm (2.17) and let

$$\varepsilon_\nu \in C_c(\widetilde{X}(o_\nu)) = C_c(X(k_\nu))$$

be the image of the constant map on $\widetilde{Y}(o_\nu)$ with value 1.

Then,

$$\varepsilon_\nu \in C(X(k_\nu))_{>0}$$

since

$$\widetilde{\pi}_\nu : \widetilde{Y}(o_\nu) \longrightarrow \widetilde{X}(o_\nu)$$

is surjective (cf. (4.19) and the last part of (1.10)). Hence any set

$$\beta = \{\beta_\nu \in C(Y(k_\nu))_{>0},\ \nu \in W\}$$

with

$$\beta_\nu = 1/(\varepsilon_\nu \circ \pi_\nu)$$

for all $\nu \in W_{\mathrm{fin}} \setminus \Sigma$ form a set of convergence factors for $\pi : Y \to X$.

Note also that

$$\beta_\nu = 1/(\varepsilon_\nu \circ \pi_\nu), \quad \nu \in W_{\mathrm{fin}} \backslash \Sigma$$

is constant on $Y(k_\nu)$ if the number of F_ν-points in the F_ν-fibres of $\widetilde{\pi}$ is a constant function on $\widetilde{X}(F_\nu)$.

Notation 4.24. — Let X be a smooth k-variety and let $\pi : Y \to X$ be a smooth k-morphism with geometrically connected fibres. Let

$$\| \ \| = \{\| \ \|_\nu : \mathbf{det}\,\mathrm{Tan}(Y_{\mathrm{an}}(k_\nu)/X_{\mathrm{an}}(k_\nu)) \longrightarrow [0, \infty),\ \nu \in W\}$$

be an adelic norm and let

$$\beta = \{\beta_\nu \in C(Y(k_\nu))_{>0},\ \nu \in W\}$$

be a set of convergence factors for $\pi : Y \to X$. If $P \in \pi(Y(A_k)) \subseteq X(A_k)$, let Z_P be the smooth A_k-variety which is the fibre product of $\pi : Y \to X$ and $P : \mathrm{Spec}\, A_k \to X(A_k)$. Then $\theta(P)$ is the positive σ-regular Borel measure on $Z_P(A_k)$ determined by the restrictions of $\| \ \|$ and β to Z_P (cf. (4.14)(b)).

Theorem 4.25. — *Let X, π, $\| \ \|$, β, $\theta(P)$, Z_P, $P \in \pi(Y(A_k))$ be as in (4.24). Let f_P be the restriction of $f \in C_c(Y(A_k))$ to $Z_A(A_k)$ and let $\Lambda_\beta(f) : X(A_k) \to \mathbb{R}$ be the function such that*

$$\Lambda_\beta(f)(P) = \int_{Z_P(A)} f_P d\theta(P)$$

for $P \in \pi(Y(A_k))$ and such that $\Lambda_\beta(f)(P) = 0$ for $P \notin \pi(Y(A_k))$.

Then Λ_β is a positive linear map from $C_c(Y(A_k))$ to $C_c(X(A_k))$.

Proof. — The proof is very similar to the proof of (4.14)(b) and we shall use the notions introduced there without further comments. In particular, $\mathcal{A}$ will denote the subalgebra of $C_c(Y(A_k))$ of finite sums of finitely decomposable functions. It is obvious from the definition that Λ_β is positive and linear. It is thus sufficient to prove that $\Lambda_\beta(f) \in C_c(X(A_k))$ for all $f \in C_c(Y(A_k))$. Moreover, since $\pi_A : Y(A_k) \to X(A_k)$ is continuous it suffices to show that $\Lambda_\beta(f)$ is continuous for all $f \in C_c(Y(A_k))$.

We first treat the case where $f \in C_c(Y(A_k))$ is the restricted product of an adelic set

$$\{f_\nu \in C_c(Y(k_\nu)),\ \nu \in W\}.$$

Choose a finite subset $\Sigma \subset W_{\mathrm{fin}}$ and two smooth models $\widetilde{Y}$, $\widetilde{X}$ over $\widetilde{o} = o_{(\Sigma)}$ and an extension of π to a smooth $\widetilde{o}$-morphism

$$\widetilde{\pi} : \widetilde{Y} \longrightarrow \widetilde{X}$$

satisfying all the conditions of (4.19) and such that f_ν is the characteristic function of

$$\widetilde{Y}(o_\nu) \subseteq Y(k_\nu)$$

for all $\nu \in W_{\mathrm{fin}} \backslash \Sigma$.

Let

$$\Lambda_\nu : C_c(Y(k_\nu)) \longrightarrow C_c(X(k_\nu))$$

be the positive linear map corresponding to $\| \ \|_\nu$ and let

$$P = \{P_\nu\}_{\nu \in W} \in \widetilde{X}(A_k(\Sigma)) = \prod_{W_{\mathrm{fin}} \backslash \Sigma} \widetilde{X}(o_\nu) \times \prod_{W_\infty \cup \Sigma} X(k_\nu).$$

Finally, let Z_ν, $\nu \in W_{\mathrm{fin}} \backslash \Sigma$ be the o_ν-scheme obtained from $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ after base extension along P_ν and let μ_ν be the model measure on $Z_\nu(o_\nu)$. Then,

$$\Lambda_\beta(f)(P) = \prod_{W_{\mathrm{fin}} \backslash \sigma} \left(\int_{Z_\nu(o_\nu)} \beta_\nu d\mu_\nu \right) \times \prod_{W_\infty \cup \Sigma} \Lambda_\nu(\beta_\nu f_\nu).$$

Hence by (4.22)(b) and (1.22)(a) we get that the restriction of $\Lambda_\beta(f)$ to $\widetilde{X}(A_k(\Sigma))$ is continuous. Moreover, since $\Lambda_\beta(f) = 0$ outside the compact open subset

$$\widetilde{X}(A_k(\Sigma)) \subseteq X(A_k)$$

we obtain that $\Lambda_\beta(f) \in C_c(X(A_k))$ for all decomposable functions $f \in C_c(Y(A_k))$. Moreover, by the additivity of Λ_β, one has the same result for any function f in $\mathcal{A}$.

For arbitrary $f \in C_c(Y(A_k))$, there exists by the proof of (4.14)(b) a compact subset K of $Y(A_k)$ and a sequence $(f_i)_{i=1}^\infty$ of functions in $\mathcal{A}$ with support in K which converges uniformly to f. There exists also a continuous decomposable non-negative function G in $C_c(Y(A_k))$ such that $G = 1$ on K. From the linearity and positivity of Λ_β one gets (cf. the proof of op. cit.) an inequality:

$$\sup |\lambda_\beta(f) - \Lambda_\beta(f_i)| = \sup |\lambda_\beta(f - f_i)| \leq \sup \lambda(G) \sup |(f - f_i)|$$

for all positive integers i. Therefore $(\Lambda_\beta(f_i))_{i=1}^\infty$ is a sequence in $C_c(X(A_k))$ which converges uniformly to $\Lambda_\beta(f)$. Hence $\Lambda_\beta(f) \in C_c(X(A_k))$, thereby completing the proof. $\square$

Now recall the canonical isomorphism described in (1.22):

(4.26)

$$\mathbf{det}\,\mathrm{Tan}(Y_{\mathrm{an}}(k_\nu)) = \mathbf{det}\,\mathrm{Tan}(Y_{\mathrm{an}}(k_\nu)/X_{\mathrm{an}}(k_\nu)) \otimes \pi^*_{\nu,\mathrm{an}}(\mathbf{det}\,\mathrm{Tan}(X_{\mathrm{an}}(k_\nu)))$$

Lemma 4.27. — *Let X be a smooth k-variety and let $\pi : Y \to X$ be a smooth k-morphism with geometrically connected fibres. Let*

$$\| \ \|_X = \{\| \ \|_{X,\nu}, \ \nu \in W\}$$

and

$$\| \ \|_{Y/X} = \{\| \ \|_{Y/X,\nu}, \ \nu \in W\}$$

be adelic norms for X resp. $\pi : Y \to X$.

Finally, for $\nu \in W$, *let*

$$\| \ \|_{Y,\nu} = \| \ \|_{Y/X,\nu} \cdot \pi^*_{\nu,\mathrm{an}} \| \ \|_{X,\nu}$$

be the product norm of $\| \ \|_{Y/X,\nu}$ *and the pullback norm* $\pi^*_{\nu,\mathrm{an}}\| \ \|_{X,\nu}$ *(cf. (1.7)(a), (b) and (4.26)). Then*

$$\| \ \|_Y = \{\| \ \|_{Y,\nu}, \ \nu \in W\}$$

is an adelic norm for Y.

Proof. — It is clear from (1.7) and (4.26) that $\| \ \|_{Y,\nu}$ is a norm on $\mathbf{det}\,\mathrm{Tan}(Y_{\mathrm{an}}(k_\nu))$ for each ν. Moreover, if $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ is an extension of π to a smooth $o_{(\Sigma)}$-morphism $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ between two smooth models $\widetilde{Y}$, $\widetilde{X}$ over $\widetilde{o} = o_{(\Sigma)}$, then

$$\| \ \|_{Y,\nu} : \mathbf{det}\,\mathrm{Tan}(Y_{\mathrm{an}}(k_\nu)) \longrightarrow [0,\infty)$$

restricts to the model norm on $\mathbf{det}\,\mathrm{Tan}(\widetilde{Y}_{\mathrm{an}}(o_\nu))$ for each $\nu \in W_{\mathrm{fin}} \backslash \Sigma$ by (2.18). This completes the proof. □

Theorem 4.28. — *Let* X *be a smooth* k*-variety and let* $\pi : Y \to X$ *be a smooth* k*-morphism with geometrically connected fibres. Let* $\| \ \|_X$ *(resp.* $\| \ \|_{Y/X}$*) be an adelic norm for* X *(resp. an adelic norm for* $\pi : Y \to X$*) and let*

$$\gamma = \{\gamma_\nu \in C(X(k_\nu))_{>0}, \ \nu \in W\}$$

resp.

$$\beta = \{\beta_\nu \in C(Y(k_\nu))_{>0}, \ \nu \in W\}$$

be a set of convergence factors for X *(resp.* $\pi : Y \to X$*).*

Let $m_{A,\gamma}$ *be the positive* σ*-regular Borel measure on* $X(A_k)$ *determined by* $\| \ \|_X$ *and* γ *and let*

$$\Lambda_\beta : C_c(Y(A_k)) \longrightarrow C_c(X(A_k))$$

be the positive linear map determined by $\| \ \|_{Y/X}$ *and* β *(cf. (4.26)).*

Finally, let

$$\| \ \|_Y = \{\| \ \|_{Y/X,\nu}\pi^*_{\nu,\mathrm{an}}\| \ \|_{X,\nu}, \ \nu \in W\}$$

be the adelic product norm for Y *constructed in (4.27).*

Then the following holds.

(a) *The products*

$$\alpha_\nu = \beta_\nu(\gamma_\nu \circ \pi_\nu), \quad \nu \in W$$

form a set α *of convergence factors for* Y.

(b) *Let $n_{A,\alpha}$ be the positive σ-regular Borel measure on $Y(A_k)$ determined by $\| \ \|_Y$ and α. Then,*

$$\int_{Y(A_k)} f dn_{A,\alpha} = \int_{X(A_k)} \Lambda_\beta(f) dm_{A,\gamma}$$

for any $f \in C_c(Y(A_k))$.

Proof

(a) Let $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ be a smooth $o_{(\Sigma)}$-morphism between smooth $o_{(\Sigma)}$-schemes which extends $\pi : Y \to X$ and satisfies all the conditions in (4.19) and such that the restriction of $\| \ \|_\nu$ to $\mathbf{det}\,\mathrm{Tan}(\widetilde{Y}_{\mathrm{an}}(o_\nu)/\widetilde{X}_{\mathrm{an}}(o_\nu))$ is the model norm for each $\nu \in W_{\mathrm{fin}} \backslash \Sigma$.

Let $\nu \in W_{\mathrm{fin}} \backslash \Sigma$, and write μ_ν for all model measures. Let

$$\widetilde{\Lambda}_\nu : C_c(\widetilde{Y}(o_\nu)) \longrightarrow C_c(\widetilde{X}(o_\nu))$$

be the positive linear map defined by the relative model norm. Then, by (2.22),

$$\int_{\widetilde{Y}(o_\nu)} \alpha_\nu d\mu_\nu = \int_{\widetilde{X}(o_\nu)} \widetilde{\Lambda}(\alpha_\nu) d\mu_\nu = \int_{\widetilde{X}(o_\nu)} \gamma_\nu \widetilde{\Lambda}_\nu(\beta_\nu) d\mu_\nu.$$

Choose $P_\nu \in \widetilde{X}(o_\nu)$ for each $\nu \in W_{\mathrm{fin}} \backslash \Sigma$ such that $|\log(\Lambda_\nu \beta_\nu(P_\nu))|$ is maximal. Then,

$$\left| \log \int_{\widetilde{Y}(o_\nu)} \alpha_\nu d\mu_\nu \right| \leq \left| \log \int_{\widetilde{X}(o_\nu)} \gamma_\nu d\mu_\nu \right| + |\log(\Lambda_\nu \beta_\nu(P_\nu))|$$

by the positivity of the model measure on $\widetilde{X}(o_\nu)$.

Now sum over all $\nu \in W_{\mathrm{fin}} \backslash \Sigma$ and use the assumptions on $\{\gamma_\nu\}$ and $\{\beta_\nu\}$.

(b) First, let f be finitely decomposable. One may adapt the choice of $\widetilde{Y}$, $\widetilde{X}$ and $\widetilde{\pi}$ in (a) to f such that in addition f is the restricted product of an adelic set

$$\{f_\nu \in C_c(Y(k_\nu)),\ \nu \in W\},$$

where f_ν is the characteristic function of

$$\widetilde{Y}(o_\nu) \subseteq Y(k_\nu)$$

for all $\nu \in W_{\mathrm{fin}} \backslash \Sigma$. Now combine the calculation of $\Lambda_\beta(f)$ in the proof of (4.25) with the formula in (4.14)(c) for the integral over $X(A_k)$ for a decomposable function. Then the formula for finitely decomposable functions follows from the corresponding assertion over local fields established in (1.22)(c).

Let $\mathcal{A}$ denote the subalgebra of $C_c(Y(A_k))$ of finite sums of finitely decomposable functions. By additivity of the integrals and Λ_β, we get the formula also for functions f in $\mathcal{A}$.

Finally, let f be an arbitrary function in $C_c(Y(A_k))$. Then, by the proof of (4.25), there exists a sequence $(f_i)_{i=1}^{\infty}$ of functions in $\mathcal{A}$ which converges uniformly to f on $X(A_k)$ and such that the supports of all f_i are contained in one common compact subset K of $X(A_k)$. Also, by (op. cit.), $(\Lambda_\beta(f_i))_{i=1}^{\infty}$ is a sequence in $C_c(X(A_k))$ with supports in $f(K)$ converging uniformly to $\Lambda_\beta(f)$. The uniform convergence and the compactness statement now implies that

$$\int_{Y(A_K)} f dn_{A,\alpha} = \lim_{i\to\infty} \int_{Y(A_K)} f_i dn_{A,\alpha}$$

$$\int_{X(A_K)} \Lambda_\beta(f) dm_{A,\gamma} = \lim_{i\to\infty} \int_{X(A_K)} \Lambda_\beta(f_i) dm_{A,\gamma}$$

so that the formula also holds for arbitrary functions $f \in C_c(Y(A_k))$. This completes the proof. □

5. Torsors over global fields and Tamagawa measures

The purpose of this section is to define Tamagawa measures on universal torsors. It is thereby important to generalize some of the constructions of Colliot-Thélène and Sansuc in [**15**] and to define universal torsors over schemes. We shall keep the notations in part 4 and use the word k-variety for a geometrically connected, separated scheme of finite type over k. Also, k will denote a number field and K a general field.

Hypothesis 5.1

(a) k is a number field
(b) X is a smooth k-variety with structure morphism $h : X \to \operatorname{Spec} k$
(c) G is a smooth k-variety which is an algebraic group over k
(d) $\pi : \mathcal{T} \to X$ is a (left) X-torsor under G (with respect to the fppf-topology) with G-action $\sigma : G \times_k \mathcal{T} \to \mathcal{T}$ (cf. (3.3))

Lemma 5.2. — *Assume (5.1). Then there exists schemes $\widetilde{X}$, $\widetilde{G}$, $\widetilde{\mathcal{T}}$ of finite presentation over $\widetilde{o} := o_{(\Sigma)}$ for some finite set Σ of closed points of* $\operatorname{Spec} o$ *with the following properties.*

(a) *$\widetilde{X}$ is a smooth $\widetilde{o}$-model of X (see (4.4)(a)).*
(b) *$\widetilde{h} : \widetilde{X} \to \operatorname{Spec} \widetilde{o}$ is a smooth separated morphism with geometrically connected fibres.*
(c) *$\widetilde{G}$ is a smooth separated group scheme over $\widetilde{o}$ with geometrically connected fibres.*
(d) *$\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ is a (left) $\widetilde{X}$-torsor under $\widetilde{G}$ with $\widetilde{G}$-action $\widetilde{\sigma} : \widetilde{G} \times_o \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$.*

(e) *The generic fibres of $\widetilde{X}$, $\widetilde{G}$, $\widetilde{\mathcal{T}}$ are equal to X, G and $\mathcal{T}$ and the restrictions to the generic fibres of $\widetilde{h}$, $\widetilde{\pi}$ and $\widetilde{\sigma}$ are equal to the k-morphisms h, π, σ in (5.1).*

Any other set of such $\widetilde{o}$-schemes and $\widetilde{o}$-morphisms are related to the given ones by means of a canonical isomorphism after base extension to $o_{(S)}$ for some finite set $S \supseteq \Sigma$ of places of k.

Proof. — This result is a formal consequence of (4.2). The reader may also consult [**45**, III.4.3] and [**1**, VII.5]. □

We denote the unit sections of G/k and $\widetilde{G}/\widetilde{o}$ by e resp. $\widetilde{e}$. There is thus a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} \widetilde{o} & \xrightarrow{\widetilde{e}} & \widetilde{G} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{e} & G \end{array}$$

We shall in the sequel consider o_ν-schemes and o_ν-morphisms obtained by base extension from the $\widetilde{o}$-schemes and $\widetilde{o}$-morphisms in (5.2). We will add a lower index ν for places defined by prime ideals of $\widetilde{o}$ to indicate that we have made a base extension to the ν-adic completion o_ν of o and $\widetilde{o}$. We denote by F_ν the residue field of o_ν for $\nu \in W_{\text{fin}}$.

Let $\pi : \mathcal{T} \to X$ be an X-torsor under G as in (5.1) with G-action $\sigma : G \times_k \mathcal{T} \to \mathcal{T}$. Then σ determines a (continuous) left $G(A_k)$-action

$$\sigma_A : G(A_k) \times \mathcal{T}(A_k) \longrightarrow \mathcal{T}(A_k)$$

on the fibres of

$$\pi_A : \mathcal{T}(A_k) \longrightarrow X(A_k)$$

(cf. (3.2)(iii)).

Let

$$\mathcal{T}(A_k) \times_{X(A)} \mathcal{T}(A_k) \subset \mathcal{T}(A_k) \times \mathcal{T}(A_k)$$

be the inverse image of the diagonal under

$$(\pi_A, \pi_A) : \mathcal{T}(A_k) \times \mathcal{T}(A_k) \to X(A_k) \times X(A_k).$$

It is a closed subspace of $\mathcal{T}(A_k) \times \mathcal{T}(A_k)$ since $X(A_k)$ is a Hausdorff space. The condition (3.3)(b) for torsors implies that the map

$$(5.3) \qquad \rho_A := (\sigma_A, pr_2) : G(A_k) \times \mathcal{T}(A_k) \longrightarrow \mathcal{T}(A_k) \times_{X(A)} \mathcal{T}(A_k)$$

is a homeomorphism of adelic spaces.

Proposition 5.4. — *Let $\pi : \mathcal{T} \to X$ be a (left) X-torsor under G as in (5.1) and let $\pi_A : \mathcal{T}(A_k) \to X(A_k)$ be the continuous map induced by $\pi : \mathcal{T} \to X$. Then*

(a) π_A *is open.*
(b) $\pi_A(\mathcal{T}(A_k))$ *is homeomorphic to the homogeneous space*

$$\mathcal{T}(A_k)/G(A_k)$$

of $\mathcal{T}(A_k)$ *with respect to* σ_A.

Proof

(a) Let $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ be a torsor of $o_{(\Sigma)}$-schemes as in (5.2). Then the open decomposable sets of the form

$$B := \prod_{\nu \in W} B_\nu = \left(\prod_{\nu \notin S} \mathcal{T}(o_\nu) \right) \times \left(\prod_{\nu \in S} B_\nu \right), \quad S \supseteq \Sigma \cup W_\infty$$

for finite subsets S of W form a base for the adelic topology. Also, by Lang's theorem (cf. (3.21)(a)) one has

$$\pi_\nu(B_\nu) = \widetilde{X}(o_\nu)$$

for all places outside S. This combined with the openness of π_ν for $\nu \in S$ (cf. (3.11)(b)) implies that $\pi_A(B)$ is open in $X(A_k)$.

(b) π_A is open and continuous. A subset U of $\pi_A(\mathcal{T}(A_k))$ is therefore open in $X(A_k)$ if and only if $\pi_A^{-1}(U)$ is open in $X(A_k)$ (cf.[**38**, p. 311]). □

We shall now consider positive Borel measures on $\mathcal{T}(A_k)$ which are invariant under the left action of $G(A_k)$.

Recall (cf. (3.8), (3.9)) that the relative algebraic tangent bundle $T_{\mathcal{T}/X} \to \mathcal{T}$ is equal to the fibre product

$$\mathcal{T} \times_k T_{G,e} \longrightarrow \mathcal{T}$$

for the fibre $T_{G,e}$ at $e \in G(k)$ of the algebraic tangent bundle of G. There is also a canonical isomorphism where $T_{\widetilde{G},\widetilde{e}}$ is the fibre at $\widetilde{e} \in \widetilde{G}(\widetilde{o})$ of the algebraic tangent bundle of $\widetilde{G}/\widetilde{o} = o_{(\Sigma)}$. The $\widetilde{o}$-points of $T_{\widetilde{G},\widetilde{e}}$ form an $\widetilde{o}$-lattice $T_{\widetilde{G},\widetilde{e}}(\widetilde{o})$ in the tangent space $T_{G,e}(k)$ of G at e.

Definition 5.5. — A constant adelic norm for $\pi : \mathcal{T} \to X$ is a set of constant norms (cf. (3.15)

$$\| \ \| = \{\| \ \|_\nu : \mathbf{det}(\mathrm{Tan}(\mathcal{T}_{\mathrm{an}}(k_\nu)/X_{\mathrm{an}}(k_\nu))) \longrightarrow [0, \infty), \nu \in W\}$$

such that $\| \ \|_\nu$ is the pullback of the ν-adic norm on $\mathbf{det}(T_{G,e}(k_\nu))$ determined by the o_ν-lattice $\mathbf{det}(T_{\widetilde{G},\widetilde{e}}(o_\nu))$ for all but finitely many places in $W_{\mathrm{fin}} \backslash \Sigma$.

Example 5.6. — Let ω_0 be a local non-zero section at e of the canonical line bundle of G. Then ω_0 defines ν-adic vector space norms on $\mathbf{det}(T_{G,e}(k_\nu))$ for all places ν which are equal to the lattice norms given by $\mathbf{det}(T_{\widetilde{G},\widetilde{e}}(o_\nu))$ for almost all ν. The constant norms

$$\| \ \|_\nu : \mathbf{det}(\mathrm{Tan}(\mathcal{T}_{\mathrm{an}}(k_\nu)/X_{\mathrm{an}}(k_\nu))) \longrightarrow [0,\infty), \quad \nu \in W$$

therefore form an adelic norm for $\pi : \mathcal{T} \to X$.

To define adelic measures, we shall need sets of convergence factors.

Lemma 5.7. — *Let $\pi : \mathcal{T} \to X$ be an X-torsor under G as in (5.1) and let*

$$\beta : W \longrightarrow \mathbb{R}_{>0}$$

be a set of constant convergence factors for G. Then β is also a set of convergence factors for $\pi : \mathcal{T} \to X$.

Proof. — Choose $\widetilde{o} = o_{(\Sigma)}$, $\widetilde{G}/\widetilde{o}$, $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ as in (5.2) and let $\nu \in W_{\mathrm{fin}}\backslash\Sigma$. Then $\widetilde{\pi}(\widetilde{\mathcal{T}}(o_\nu)) = \widetilde{X}(o_\nu)$ (cf. (3.21)). Moreover, if $P_\nu \in \widetilde{X}(o_\nu)$, then

$$\widetilde{\mathcal{T}} \times_{\widetilde{X}} P_\nu \approx \widetilde{G} \times_{\widetilde{o}} o_\nu$$

as o_ν-schemes by the torsor isomorphism (3.18)(b). This completes the proof. □

The following result is fundamental for the applications of torsors to problems concerning counting functions of rational points.

Theorem 5.8. — *Let $\pi : \mathcal{T} \to X$ be an X-torsor under G with left action $\sigma : G \times \mathcal{T} \to \mathcal{T}$ as in (5.1). Also, let*

$$\{\| \ \|_\nu, \ \nu \in W\}$$

be a constant adelic norm for $\pi : \mathcal{T} \to X$ defined by a local differential form $\omega_0 \neq 0$ of G at e and let

$$\{\Lambda_\nu : C_c(\mathcal{T}(k_\nu)) \longrightarrow C_c(X(k_\nu)), \quad \nu \in W\}$$

be the positive linear maps defined by these norms. Finally, let

$$\beta : W \to \mathbb{R}_{>0}$$

be a set of constant convergence factors for G. Then there exists a unique positive linear map

$$\Lambda_\beta : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(X(A_k))$$

which is independent of the choice of ω_0 such that

$$(**) \qquad \Lambda_\beta(f_A) = \prod_{\nu \in W} \beta_\nu \Lambda_\nu(f_\nu)$$

for the restricted product

$$f_A = \prod\nolimits'_{\nu \in W} f_\nu$$

of any adelic set of functions

$$\{f_\nu \in C_c(Y(k_\nu)),\ \nu \in W\}.$$

Moreover, if $\sigma(g) : \mathcal{T}(A_k) \to \mathcal{T}(A_k)$ is the left translation defined by an element $g \in G(A_k)$, then

$$\Lambda_\beta(f) = \Lambda_\beta(f \circ \sigma(g))$$

for any $f \in C_c(\mathcal{T}(A_k))$.

Proof. — The existence and the uniqueness of Λ_β follows from (4.25) and (5.7). To prove that Λ_β is independent of the choice of ω_0, let $\alpha \neq 0$ be an element of k. Then the corresponding adelic norms is given by $\{|\alpha|_\nu \| \ \|_\nu,\ \nu \in W\}$ and the corresponding ν-adic linear maps (cf. (1.21)) by $|\alpha|_\nu \Lambda_\nu$. The independence therefore follows from the formula

$$\prod_{\nu \in W} |\alpha|_\nu = 1.$$

To prove the last assertion use $(**)$ to reduce to the statement

$$\Lambda_\nu(f_\nu) = \Lambda_\nu(f_\nu \circ \sigma(g_\nu))$$

for the left translations

$$\sigma(g_\nu) : \mathcal{T}(k_\nu) \longrightarrow \mathcal{T}(k_\nu)$$

defined by $g_\nu \in G(k_\nu)$. The equality follows from the fact that $\| \ \|_\nu$ is invariant under the left action of $g_\nu \in G(k_\nu)$ (cf. (3.15)), thereby completing the proof. □

Remarks 5.9

(a) It is immediate from the definitions of Λ_ν that $\Lambda_\beta(f)$ has support in the open subset $\pi_A(\mathcal{T}(A_k))$ of $X(A_k)$. Therefore (cf. (5.4)) Λ_β may also be seen as a positive linear map

$$\Lambda_\beta : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(\mathcal{T}(A_k)/G(A_k)).$$

This can be used to give another construction of Λ_β starting with a Haar measure on $G(A_k)$ (cf. [**11**, Ch. VII, §2, n°2]), thereby avoiding the general result in (4.25).

But we shall in other papers give applications of (4.25), which cannot be deduced from the theory of homogenous spaces in (op. cit). The results (4.25)-(4.28) can e.g. be used to answer a question in [**6**] about L-primitive fibrations. Also, even for torsors it is useful to consider measures which are not invariant under the group action and to define measures on (partial) compactifications of torsors (see [**51**]).

(b) Let G be an r-dimensional k-torus and let $\pi : \mathcal{T} \to X$ be an X-torsor under G. Let

$$\| \ \|_{\mathcal{T}/X,\nu} : \mathbf{det}(\mathrm{Tan}(\mathcal{T}(k_\nu)/X(k_\nu)) \longrightarrow [0,\infty), \quad \nu \in W$$

be the order norms (see (3.30)). It follows from (3.31) that these ν-adic norms form an adelic norm. There is, therefore, for each set $\{\beta_\nu,\ \nu \in W\}$ of constant convergence factors for G a unique positive linear map

$$\Lambda_\beta : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(X(A_k))$$

satisfying the same property $(**)$ as in (5.8). This map coincides with the positive linear map in (5.8) for split k-tori G (since the order norms are defined by means of a differential form of minimal $d\log$-type). The two maps also coincide for non-split k-tori G since $\prod_{\nu \in W} \|s\|_\nu = 1$ for the order norms $\| \ \|_\nu : \bigwedge^r(T_{G,e}(k_\nu)) \to \mathbb{R}$ of a section $s \neq 0$ of $\bigwedge^r(T_{G,e}(k_\nu))$.

Definition 5.10

(a) A *torus T over a scheme B* is a commutative group scheme of finite type over B which locally in the fpqc-topology (cf. [**21**, exp. VIII]) is isomorphic to the group scheme

$$G_{m,B} \times_B \cdots \times_B G_{m,B}$$

over B for a finite number of copies $G_{m,B}$. The torus is said to be split over B if T is isomorphic to

$$G_{m,B} \times_B \cdots \times_B G_{m,B}.$$

(b) A *finitely generated torsion free twisted constant group scheme* Π *over* B is a commutative group scheme over B which locally in the fpqc-topology (cf. [**21**, exp. VIII]) is isomorphic to the constant group scheme

$$Z_B \times_B \cdots \times_B Z_B$$

over B for a finite number of copies Z_B. The f.g. torsion free twisted constant group scheme Π over B is said to be *split over* B if T is isomorphic to

$$Z_B \times_B \cdots \times_B Z_B.$$

It is known by a theorem of Grothendieck (cf. [**21**, exp. X]) that any torus and any twisted constant torsion free group scheme split after some surjective étale base extension. One can therefore replace the fpqc-topology by the *fppf-topology* or by the *étale topology* in the definition above. In particular, if $B = \mathrm{Spec}\, K$ for a field K, then all tori and all f.g. torsion free twisted constant group schemes split over a separable closure $\overline{K}$ of K.

Notations 5.11. — If T is a B-torus, then

$$\hat{T} := \mathrm{Hom}_{Gr}(T, G_{m,B})$$

If Π is a f.g. torsion free twisted constant B-group scheme, then

$$\mathbf{D}(\Pi) := \mathrm{Hom}_{Gr}(\Pi, G_{m,B}).$$

Here $\mathrm{Hom}_{Gr}(\cdot, G_{m,B})$ means the group scheme representing the Hom-functor in the category of group schemes. It is easily seen that $\hat{T}$ is a f.g. torsion free twisted constant B-group scheme and that $\mathbf{D}(M)$ is a B-torus in (5.11).

These two constructions are inverse to each other in the sense that they define a contravariant equivalence of categories between tori T over B and f.g. twisted constant torsion free group schemes Π over B (cf. [**21**, X.5]).

If the base scheme B is a field K, and $\overline{K}$ is a separable closure over K, then the functor sending Π to the $\mathrm{Gal}(\overline{K}/K)$-module $P = \Pi(\overline{K})$ defines an equivalence between the category of f.g. torsion free twisted constant group schemes over K and the category of f.g. torsion free continuous discrete $\mathrm{Gal}(\overline{K}/K)$-modules (see [**45**, p. 52]). There is therefore in this case a contravariant equivalence between the category of K-tori and the category of f.g. torsion free continuous discrete $\mathrm{Gal}(\overline{K}/K)$-modules for which a K-torus is sent to its character $\mathrm{Gal}(\overline{K}/K)$-module $\hat{T} := \mathrm{Hom}(\overline{T}, G_{m,\overline{X}})$. Conversely, if M is a f.g. torsion free continuous discrete $\mathrm{Gal}(\overline{K}/K)$-module, form the K-Hopf algebra $\overline{K}[M]^G$ of $\mathrm{Gal}(\overline{K}/K)$-invariant elements in $\overline{K}[M]$ and let $D(P) = \mathrm{Spec}\,\overline{K}[M]^G$. Then $D(P)$ is an K-torus (cf. Ch. III, §8 in Borel's book [**9**]).

Now assume the following:

5.12 (a) B is a Noetherian scheme.

5.12 (b) $f : X \to B$ is a smooth proper surjective morphism of constant relative dimension with geometrically connected fibres.

5.12 (c) $R^2 f_* \mathcal{O}_X = 0$.

5.12 (d) The relative Picard functor (cf. [**10**, Ch. 8]) of f is representable by a f.g. torsion free twisted constant B-group scheme $\mathbf{Pic}_{X/B}$.

It follows from more general results of Grothendieck and Murre (cf. [**10**, pp. 210-211]) that $\mathbf{Pic}_{X/B}$ is representable when f is projective or B is the spectrum of a field. It is easy to show that the assumptions in (5.12) also hold for smooth proper toric schemes over B.

Let $g : T \to B$ be a B-torus. Let $\mathcal{F}_T$ (resp. $\mathcal{G}_T$) be the fppf-sheaves of abelian groups on $(\mathbf{Sch}/B)^{\mathrm{opp}}$ associated to the functor:

$$Y \longrightarrow H^1_{\mathrm{fppf}}(X \times_B Y, T_Y)$$

resp.

$$Y \longrightarrow \mathrm{Hom}_Z(\mathrm{Hom}_Y(T_Y, G_{m,Y}), H^1_{\mathrm{fppf}}(X \times_B Y, G_{m,Y}))$$

where Hom_Y means the abelian group of Y-homomorphisms in the category of group schemes over Y. The covariant functoriality of $H^1_{\mathrm{fppf}}(X \times_B Y, \cdot)$ under homomorphisms of commutative group schemes over $X \times_B Y$ gives rise to a morphism of functors:

$$\mathcal{F} \longrightarrow \mathcal{G}.$$

It is easy to verify that this is an isomorphism of functors since T is locally isomorphic to $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$ in the fppf-topology. One can also according to a theorem of Grothendieck [**27**, pp. 171-183] define $\mathcal{F}_T$ (resp. $\mathcal{G}_T$) by means of the étale topology instead of the fppf-topology.

If $T = \mathbb{G}_m$, then $\mathcal{F}_T$ coincides with the relative Picard functor. We may thus for an arbitrary torus $g : T \to B$ regard $\mathcal{G}_T$ as the fppf-sheaves of abelian groups on $(\mathbf{Sch}\,/B)^{\mathrm{opp}}$, where

$$Y \longrightarrow \mathrm{Hom}_Y(\hat{T}_Y, (\mathbf{Pic}_{X/B})_Y).$$

There is also an interpretation

$$\mathcal{F}_T(Y) = H^0(Y, R^1 g_*(T))$$

where the right hand side can be read both with respect to the fppf-topology and the étale topology (cf. [**10**, p. 202-203]). The isomorphism $\mathcal{F}_T(B) = \mathcal{G}_T(B)$ may thus be interpreted as an isomorphism:

$$H^0(B, R^1 g_*(T)) = \mathrm{Hom}_B(\hat{T}, \mathbf{Pic}_{X/B}).$$

There is further by Leray's spectral sequence an exact sequence

$$0 \to H^1(B,T) \to H^1(X,T) \to H^0(B, R^1 g_*(T)) \to H^2(B,T) \to H^2(X,T)$$

of cohomology groups in the fppf-topology or the étale topology, which reduces to a short exact sequence

$$0 \longrightarrow H^1(B,T) \longrightarrow H^1(X,T) \longrightarrow H^0(B, R^1 g_*(T)) \longrightarrow 0$$

if there is a section $s : B \to X$ to $f : X \to B$.

There are thus isomorphic exact sequences generalizing the sequence of Colliot-Thélène and Sansuc (cf. [**15**, 2.0.2])

5.13 (a)

$$1 \to H^1_{\text{ét}}(B,T) \to H^1_{\text{ét}}(X,T) \xrightarrow{t} \mathrm{Hom}_B(\hat{T}, \mathbf{Pic}_{X/B}) \to H^2_{\text{ét}}(B,T) \to H^2_{\text{ét}}(X,T),$$

5.13 (b)

$$1 \longrightarrow H^1_{\mathrm{fppf}}(B,T) \longrightarrow H^1_{\mathrm{fppf}}(X,T) \xrightarrow{t} \mathrm{Hom}_B(\hat{T}, \mathbf{Pic}_{X/B}) \longrightarrow H^2_{\mathrm{fppf}}(B,T) \longrightarrow H^2_{\mathrm{fppf}}(X,T).$$

The following definitions are equivalent to those of Colliot-Thélène and Sansuc [**15**] when B is the spectrum of a field. We shall write $[\mathcal{T}]$ for the class in $H^1_{\text{ét}}(X,T) = H^1_{\mathrm{fppf}}(X,T)$ of an X-torsor $\pi : \mathcal{T} \to X$ under T.

Definition 5.14. — Assume (5.12).

(a) Let $\pi : \mathcal{T} \to X$ be a torsor under T. Then the image

$$t([\mathcal{T}]) \in \mathrm{Hom}_B(\hat{T}, \mathbf{Pic}_{X/B})$$

is called *the type of* $\pi : \mathcal{T} \to X$.

(b) The B-torus

$$g : \mathbf{D}(\mathbf{Pic}_{X/B}) \longrightarrow B$$

is called *the Néron-Severi torus of* $f : X \to B$.

(c) Let $T = \mathbf{D}(\mathbf{Pic}_{X/B})$ be the Néron-Severi torus of $f : X \to B$. A *universal torsor over* X is a torsor

$$\pi : \mathcal{T} \longrightarrow X$$

under T of identity type $t([\mathcal{T}]) : \hat{T} \to \mathbf{Pic}_{X/B}$.

(We use here the reflexive property of the group scheme $\mathbf{Pic}_{X/B}$, cf. [**21**, Exp. VIII]).

The following result in (op. cit.) is an easy consequence of (5.13)(a)

Proposition 5.15. — *Suppose that there exists a section $s : B \to X$ to $f : X \to B$. Then there are universal torsors over X. The isomorphism classes of the universal torsors over X are parametrized by the $H^1_{\text{ét}}(B,T)$-orbit in $H^1_{\text{ét}}(X,T)$ defined by* $t^{-1}(\mathrm{id})$.

Proof. — The map $H^2_{\text{ét}}(B,T) \to H^2_{\text{ét}}(X,T)$ in (5.13)(a) is the contravariant functorial map. There is also a contravariant map $H^2_{\text{ét}}(X,T) \to H^2_{\text{ét}}(B,T)$ induced by s. There are thus classes in $H^1_{\text{ét}}(X,T) = H^1_{\text{fppf}}(X,T)$ of identity type. It is well-known that these classes are represented by torsors (cf. [**45**, p. 121]). This completes the proof. □

For the rest of this section let k be a number field and let X be a smooth proper k-variety as in (5.13). Denote by T the Néron-Severi k-torus of X. The following result shows a remarkable property of universal torsors.

Lemma 5.16. — *Let k be a number field and let X be a smooth proper k-variety satisfying $H^1_{\text{Zar}}(X,\mathcal{O}_X) = H^2_{\text{Zar}}(X,\mathcal{O}_X) = 0$ and such that the Néron-Severi group of $\overline{X} := \overline{k} \times X$ is torsion-free. Let T be the Néron-Severi k-torus of X and let $\pi : \mathcal{T} \to X$ be a universal torsor or an arbitrary X-torsor under T. Then*
$\{\alpha_\nu = 1,\ \nu \in W\}$ form a set of convergence factors for $\mathcal{T}$.

Proof. — Choose $\widetilde{o} = o_{(\Sigma)}$, $\widetilde{T}/\widetilde{o}$, $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ as in (5.2). The image H of the representation

$$\rho : \mathrm{Gal}(\overline{k}/k) \longrightarrow Gl(\hat{T})$$

is finite and the subfield $K \subset \overline{k}$ of H-invariant elements is unramified over k at all places $\nu \in W_{\text{fin}} \backslash \Sigma$. For these places, define the local L-function by

$$L_\nu(s,T) = 1/\det(1 - q_\nu^{-s}\rho(\mathrm{Fr}_\nu) \mid \hat{T}).$$

The reduction $\widetilde{T} \times F_\nu$ is an F_ν-torus for each $\nu \in W_{\text{fin}} \backslash \Sigma$ and $\widetilde{T}(F_\nu)$ of cardinality $(\operatorname{Card} F_\nu)^{\dim T}/L_\nu(1,T)$ by a result of Ono [**49**, 3.3]. Further, by (2.15) one has

$$\mu_\nu(\widetilde{T}(o_\nu)) = \operatorname{Card}(\widetilde{T}(F_\nu)) \left(\mu_\nu(\widetilde{T}(o_\nu))/(\operatorname{Card} F_\nu)\right)^{\dim T}$$

for $\nu \in W_{\text{fin}} \backslash \Sigma$. Therefore,

$$\beta_\nu = \begin{cases} L_\nu(1,T) = 1/\mu_\nu(\widetilde{T}(o_\nu)), & \nu \in W_{\text{fin}} \backslash \Sigma \\ 1, & \nu \in W_\infty \cup \Sigma \end{cases}$$

form a set of convergence factors for T (this is due to Weil [**67**]).

Hence by (5.7) one gets that $\beta : W \to \mathbb{R}_{>0}$ is a set of convergence factors for $\pi : \mathcal{T} \to X$. But $\widehat{T} = \operatorname{Pic}(\overline{X})$ by assumption. Therefore,

$$\gamma_\nu = \begin{cases} 1/L_\nu(1,T), & \nu \in W_{\text{fin}} \backslash \Sigma \\ 1, & \nu \in W_\infty \cup \Sigma \end{cases}$$

form a set of convergence factors for X by (4.16). Hence $\{\beta_\nu\gamma_\nu,\ \nu \in W\}$ form a set of convergence factors for $\mathcal{T}$, as was to be proved. □

Theorem 5.17. — *Let k be a number field and let X be a smooth proper k-variety as in (5.16). Let T be the Néron-Severi k-torus of X and let $\pi : \mathcal{T} \to X$ be a universal torsor or an arbitrary torsor under T. Let*

$$\|\ \|_X = \{\|\ \|_{X,\nu} : \mathbf{det}\operatorname{Tan} X(k_\nu) \longrightarrow [0,\infty),\ \nu \in W\}$$

be an adelic norm for X and let

$$\|\ \|_{X\to\mathcal{T}} = \{\|\ \|_{\mathcal{T},\nu} : \mathbf{det}\operatorname{Tan}(\mathcal{T}(k_\nu)) \longrightarrow [0,\infty),\ \nu \in W\}$$

be the induced adelic norm for (cf. (3.30), (3.31)). Let

$$\Lambda_\nu : C_c(\mathcal{T}(k_\nu)) \longrightarrow \mathbb{R}, \quad \nu \in W$$

be the positive functionals determined by the norms $\|\ \|_{\mathcal{T},\nu}$ (cf. (1.10)). Then there exists a unique positive linear map

$$\Lambda : C_c(\mathcal{T}(A_k)) \longrightarrow \mathbb{R}$$

such that

$$\Lambda(f_A) = \prod_{\nu\in W} \Lambda_\nu(f_\nu)$$

for the restricted product

$$f_A = {\prod}'_{\nu\in W} f_\nu$$

of any adelic set of functions

$$\{f_\nu \in C_c(Y(k_\nu)),\ \nu \in W\}.$$

Proof. — It follows from (3.31) that the order norms $\{\| \ \|_{\mathcal{T}/X,\nu} : \nu \in W\}$ form an adelic norm for $\pi : \mathcal{T} \to X$. The assertion therefore follows from (5.8) and (5.16). □

The group of characters of the Néron-Severi torus T of X defined over k is canonically isomorphic to $\operatorname{Pic} X$. Therefore, the map

$$\{\alpha_\nu\}_{\nu\in W} \in G_{m,k}(A_k) \longrightarrow \log(\prod_{\nu\in W} |\alpha|_\nu) \in \mathbb{R}$$

induces a pairing $T(A_k)\times \operatorname{Pic} X \to \mathbb{R}$ which we may reinterpret as a continuous epimorphism

$$T(A_k) \longrightarrow \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R}).$$

Let $T^1(A_k)$ be the kernel of this map. Then there is an exact sequence:

$$1 \longrightarrow T^1(A_k) \longrightarrow T(A_k) \longrightarrow \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R}) \longrightarrow 0 \tag{5.18}$$

where $T(k) \subseteq T^1(A_k)$ by the Artin-Whaples product formula for number fields.

We now define a Haar measure $\overline{\Theta}^1_\Sigma$ on $T^1(A_k)/T(k)$. To do this, we recall the following result.

Lemma 5.19. — *Let G be a locally compact group and let N be a closed normal subgroup of G. Let Θ_N be a Haar measure on N and let $f \in C_c(G)$. Then there exists a unique function $f^N \in C_c(G/N)$ such that*

$$f^N(gN) := \int_N f(gn)d\Theta_N$$

for all $g \in G$.

Moreover, if Θ_H is a Haar measure on $H := G/N$ and Θ_G is a Haar measure on G, then there exists a unique positive real number $c > 0$ such that

$$\int_G f(g)d\Theta_G = c\int_H f^N(h)d\Theta_H \tag{$*$}$$

for all $f \in C_c(G)$.

Proof. — See [**38**, Ch. XII]. □

We shall say that the three Haar measures Θ_G, Θ_N, Θ_H are compatible if $(*)$ holds with $c = 1$.

Now endow $T(k)$ with the counting measure and

$$V := T(A_k)/T^1(A_k) = \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})$$

with the unique Haar measure such that $\operatorname{Vol}(V/L) = 1$ for the $\mathbb{Z}$-lattice

$$L := \operatorname{Hom}(\operatorname{Pic} X, \mathbb{Z})$$

in V. Then apply (5.19) twice to the chain of normal closed subgroups

$$T(k) \subseteq T^1(A_k) \subseteq T(A_k).$$

This defines a bijection between Haar measures on $T^1(A_k)/T(k)$ and Haar measures on $T(A_k)$.

Proposition 5.20. — *Let X be as in (5.1) and let B_A be a compact open subset of $X(A_k)$. Let T be the Néron-Severi torus and let $\pi : \mathcal{T} \to X$ be a universal torsor under T. Let $\| \ \|_X$ be an adelic norm for X and let m_ν, $\nu \in W$ (resp. n_ν, $\nu \in W$) be the Borel measure on $X(k_\nu)$ determined by by $\| \ \|_{X(k)}$ (resp. $\| \ \|_{X(k)\to\mathcal{T}(k)}$).*

Choose Σ, $\widetilde{o}$, $\widetilde{X}$, $\widetilde{T}$, $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ as in (5.2) and such that there exists a compact open subset

$$B_S \subseteq \prod_{\nu \in S} X(k_\nu), \quad S = W_\infty \cup \Sigma$$

for which

$$(\%) \qquad B_A = B_S \times \prod_{\nu \in W_{\mathrm{fin}} - \Sigma} \widetilde{X}(o_\nu).$$

Let Θ_S be the Haar measure on $T(A_k)$ given by the adelic order norm (cf. (5.9)(b) and the convergence factors

$$\begin{aligned} \beta_\nu &= 1/\mu_\nu(\widetilde{T}(o_\nu)), \quad && \nu \in W_{\mathrm{fin}} \backslash \Sigma \\ \beta_\nu &= 1, && \nu \in W_\infty \cup \Sigma \end{aligned}$$

and let $\overline{\Theta}^1_\Sigma$ be the corresponding Haar measure on $T^1(A_k)/T(k)$ under the bijection above. Also, let $m_{A,\Sigma}$ be the Borel measures on $X(A_k)$ determined by $\| \ \|_X$ and $\gamma = \{1/\beta_\nu\}_{\nu \in W}$. Then the following holds.

(a) *$\pi_A(\mathcal{T}(A_k))$ is a compact open subset of $X(A_k)$*

(b) *The product*

$$\tau_\varepsilon(X, B_A, \| \ \|_X) := \overline{\Theta}^1_\Sigma(T^1(A_k)/T(k)) \cdot m_{A,\Sigma}(B_A \cap \pi_A(\mathcal{T}(A_k)))$$

is independent of the choices of Σ and $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$.

(c) *Suppose that Σ, $\widetilde{o}$, $\widetilde{X}$ satisfy the additional condition that $\| \ \|_{X(k_\nu)}$ is the model norm defined by $\widetilde{X}_\nu/o_\nu$ for $\nu \in W_{\mathrm{fin}} \backslash \Sigma$. Then,*

$$\tau_\varepsilon(X, B_A, \| \ \|_X) = \overline{\Theta}^1_\Sigma(T^1_A/T(k)) m_S(B_S \cup \pi((\mathcal{T}_S))) \prod_{\nu \in W_{\mathrm{fin}} - \Sigma} n_\nu(\widetilde{\mathcal{T}}(o_\nu)).$$

Proof

(a) It follows from the implicit function theorem that $\pi(\mathcal{T}(k_\nu))$ is an open subset of $X(k_\nu)$. By twisting the torsor with elements in $H^1_{\text{ét}}(k_\nu, T_\nu)$ one concludes (cf. [**15**]) that the functorial map $X(k_\nu) \to H^1_{\text{ét}}(k_\nu, T_\nu)$ defined by

$\mathcal{T}_\nu$ is locally constant. Therefore, since $H^1_{\text{ét}}(k_\nu, T_\nu)$ is finite, it follows that $\pi(\mathcal{T}(k_\nu))$ is a closed subset of $X(k_\nu)$.

Also, $X(k_\nu)$ is compact since X is proper (see (2.3)) so that $\pi(\mathcal{T}(k_\nu))$ is compact for all places ν of k. Hence $\pi_A(\mathcal{T}(A_k))$ is a compact open subset of $X(A_k)$.

(b) It suffices by the last assertion in (5.2) to show that the product does not change if we replace Σ by $\Phi = \Sigma \cup \{w\}$ and $\widetilde{\mathcal{T}}$ by $\widetilde{\mathcal{T}} \times o_{(\Phi)}$ for $w \in W_{\text{fin}} \backslash \Sigma$. It follows from the definitions of Θ_Σ and $\overline{\Theta}^1_\Sigma$ that

$$\overline{\Theta}^1_\Phi(T^1(A_k)/T(k)) = \overline{\Theta}^1_\Sigma(T^1(A_k)/T(k))\mu_w(\widetilde{T}(o_w)).$$

It is therefore sufficient to show that:

$$m_{A,\Sigma}(B_A \cap \pi_A(\mathcal{T}(A_k))) = m_{A,\Phi}(B_A \cap \pi_A(\mathcal{T}(A_k)))\mu_w(\widetilde{T}(o_w)).$$

To show this, we first note that

$$\widetilde{\pi}(\widetilde{\mathcal{T}}(o_\nu)) = \widetilde{X}(o_\nu), \qquad \widetilde{X}(o_\nu) = X(k_\nu)$$

for $\nu \in W_{\text{fin}} \backslash \Sigma$ by (3.21) resp. the properness of $\widetilde{X}_w/o_w$. Hence

$$\pi(\mathcal{T}(k_\nu)) = \widetilde{X}(o_\nu)$$

for $\nu \in W_{\text{fin}} \backslash \Sigma$ and

$$B_A \cap \pi_A(\mathcal{T}(A_k)) = B_S \cap \pi(\mathcal{T}_S) \times \prod_{\nu \in W_{\text{fin}} - \Sigma} \widetilde{X}(o_\nu)$$

for

$$\pi(\mathcal{T}_S) := \prod_{\nu \in S} \pi(\mathcal{T}(k_\nu)).$$

Now from (%) and the definition of $m_{A,\Sigma}$, we get

(5.21)
$$m_{A,\Sigma}(B_A \cap \pi_A(\mathcal{T}(A_k))) = m_S(B_S \cap \pi(\mathcal{T}_S)) \prod_{\nu \in W_{\text{fin}} - \Sigma} \mu_\nu(\widetilde{T}(o_\nu)) m_\nu(\widetilde{X}(o_\nu))$$

where m_S is the product measure of m_ν for $\nu \in S$.

Let $F = S \cup \{w\} = \Phi \cup W_\infty$ and let m_F be the product measure of m_ν for $\nu \in F$. Moreover, let

$$\pi(\mathcal{T}_F) = \prod_{\nu \in F} \pi(\mathcal{T}(k_\nu)), \qquad B_F = \widetilde{X}(o_w) \times B_S.$$

Then, by the definition of $m_{A,\Phi}$, we get

$$m_{A,\Phi}(B_A \cap \pi_A(\mathcal{T}(A_k))) = m_F(B_F \cap \pi(\mathcal{T}_F)) \prod_{\nu \in W_{\text{fin}} - \Phi} \mu_\nu(\widetilde{T}(o_\nu)) \cdot m_\nu(\widetilde{X}(o_\nu)).$$

Now note that

$$B_F \cap \pi(\mathcal{T}_F) = \widetilde{X}(o_w) \times (B_S \cap \pi(\mathcal{T}_S))$$

$$m_F(B_F \cap \pi(\mathcal{T}_F)) = m_w(\widetilde{X}(o_w))m_S(B_S \cap \pi(\mathcal{T}_F)).$$

Hence,

$$m_{A,\Sigma}(B_A \cap \pi_A(\mathcal{T}(A_k))) = m_{A,\Phi}(B_A \cap \pi_A(\mathcal{T}(A_k)))\mu_w(\widetilde{\mathcal{T}}(o_w))$$

thereby completing the proof of (b).

(c) Let $\nu \in W_{\text{fin}} \backslash \Sigma$. Then $\| \ \|_{X(k_\nu) \to \mathcal{T}(k_\nu)}$ restricts to the model norm on $\mathbf{det}\,\mathrm{Tan}(\widetilde{\mathcal{T}}(o_\nu))$ by (3.21). Hence m_ν, n_ν are equal to the model measures on $\widetilde{X}(o_\nu)$ (resp. $\widetilde{\mathcal{T}}(o_\nu)$) and

$$\mu_\nu(\widetilde{\mathcal{T}}(o_\nu))m_\nu(\widetilde{X}(o_\nu)) = n_\nu(\widetilde{\mathcal{T}}(o_\nu))$$

by (3.25). Now apply (5.21) and the equality above. This finishes the proof. □

Definition 5.22. — Let X be as in (5.1) and let B_A be a compact and open subset of $X(A_k)$. Let $\varepsilon \in H^1_{\text{ét}}(X, T)$ be the class of a universal torsor $\pi : \mathcal{T} \to X$ and let $\| \ \|_X$ be an adelic norm for X. Then the *Tamagawa number* $\tau_\varepsilon(X, B_A, \| \ \|_X)$ is the product

$$\tau_\varepsilon(X, B_A, \| \ \|_X) = \overline{\Theta}^1_\Sigma(T^1(A_k)/T(k))m_{A,\Sigma}(B_A \cap \pi_A(\mathcal{T}(A_k)))$$

in (5.20).

In order to understand this definition we shall need the following corollary of (4.28).

Corollary 5.23. — *Let $\pi : \mathcal{T} \to X$, $\| \ \|_X$, $\| \ \|_{X \to \mathcal{T}}$, $\beta = \{\beta_\nu\}_{\nu \in W}$ be as in (5.20). Let*

$$\Lambda_\beta : C_c(\mathcal{T}(A_k)) \to C_c(X(A_k))$$

be the positive linear map described in (5.8) and let m_A resp. n_A be the Borel measures on $X(A_k)$ resp. $\mathcal{T}(A_k)$ determined by $\| \ \|_X$ and $\gamma = \{1/\beta_\nu\}_{\nu \in W}$ (cf. (4.14), (4.16)) resp. $\| \ \|_{X \to \mathcal{T}}$ and the convergence factors 1 *(cf. (5.17)).*

Let

$$\mathrm{Tr} : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(\mathcal{T}(A_k)/T(k))$$

be the trace map obtained by summing over all $T(k)$-translates of a function in $C_c(\mathcal{T}(A_k))$. Then the following holds.

(a) *There exists a unique positive σ-regular Borel measure $\overline{n}_A$ on $\mathcal{T}(A_k)/T(k)$ with the following property:*

If M is a Borel subset of $\mathcal{T}(A_k)$ with $tM \cap M = \varnothing$ for all $t \in T(k)\backslash\{1\}$ and $\overline{M}$ is the image of M in $\mathcal{T}(A_k)/T(k)$, then

$$\overline{n}_A(\overline{M}) = n_A(M).$$

(b) *There exists a unique positive linear map*

$$\overline{\Lambda}_\beta : C_c(\mathcal{T}(A_k)/T(k)) \longrightarrow C_c(X(A_k))$$

with $\Lambda_\beta = \overline{\Lambda}_\beta \circ \mathrm{Tr}$.

(c)

$$\int_{\mathcal{T}(A_k)/T(k)} f d\overline{n}_A = \int_{X(A_k)} \overline{\Lambda}_\beta(f) dm_{A,\gamma}$$

for each $f \in C_c(\mathcal{T}(A_k)/T(k))$.

Proof

(a) This follows from the $T(k)$-invariance of n_A (cf. e.g [**11**, Ch. VII, §2 prop. 4 b]).

(b) It suffices by the usual argument with partitions of unity [**38**, p. 270] to prove the existence and uniqueness of such a map $C_c(\overline{M}) \to C_c(X(A_k))$ for each open subset $\overline{M} \subset \mathcal{T}(A_k)/T(k)$ of the type described in (a).

(c) One reduces again to the case $f \in C_c(\overline{M})$ and applies (4.28)(b). □

Remark 5.24. — The motivation for the definition (5.22) is the following. The continuous map

$$\pi_A : \mathcal{T}(A_k) \longrightarrow X(A_k)$$

and the continuous $T(A_k)$-action

$$\sigma_A : T(A_k) \times \mathcal{T}(A_k) \longrightarrow \mathcal{T}(A_k)$$

determine (cf. (5.4)) a continuous map

$$\mathcal{T}(A_k)/T^1(A_k) \longrightarrow \pi_A(\mathcal{T}(A_k))$$

and a continuous $T(A_k)/T^1(A_k)$-action

$$T(A_k)/T^1(A_k) \times \mathcal{T}(A_k)/T^1(A_k) \longrightarrow \mathcal{T}(A_k)/T^1(A_k)$$

such that $\mathcal{T}(A_k)/T^1(A_k)$ becomes a topological $\pi_A(\mathcal{T}(A_k))$-torsor under

$$T(A_k)/T^1(A_k) = \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R}).$$

Now assume that there exists a continuous section

$$\psi : \pi_A(\mathcal{T}(A_k)) \longrightarrow \mathcal{T}(A_k)/T^1(A_k)$$

of this map. This gives rise to an isomorphism:

$$I : \pi_A(\mathcal{T}(A_k)) \times \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R}) \longrightarrow \mathcal{T}(A_k)/T^1(A_k)$$

of topological $\pi_A(\mathcal{T}(A_k))$-torsors under $T(A_k)/T^1(A_k) = \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$.

Let

$$\mathcal{F}_0 \subset \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$$

be a fundamental domain (cf. e.g. [**53**, p. 163]) with respect to

$$\mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{Z}) \subset \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$$

and let $\mathcal{U}$ be the inverse image of

$$I((B_A \cap \pi_A(\mathcal{T}(A_k))) \times \mathcal{F}_0)$$

under the rest class map

$$\mathcal{T}(A_k)/T(k) \longrightarrow \mathcal{T}(A_k)/T^1(A_k).$$

Then $\mathcal{U}$ is a Borel subset of $\mathcal{T}(A_k)/T(k)$ which is invariant under $T^1(A_k)/T(k)$ for the action

$$\overline{\sigma}_A : T(A_k)/T(k) \times \mathcal{T}(A_k)/T(k) \longrightarrow \mathcal{T}(A_k)/T(k)$$

induced by σ_A. Moreover,

Assertion 5.25. — *We have*

$$\tau_\varepsilon(X, B_A, \| \ \|_X) = \overline{n}_A(\mathcal{U}).$$

To see this, one can assume that $\mathcal{F}_0$ has compact closure in $\mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$. Then $\mathcal{U}$ has compact closure $\mathcal{U}_c$ in $\mathcal{T}(A_k)/T(k)$ with

$$\overline{n}_A(\mathcal{U}) = \overline{n}_A(\mathcal{U}_c).$$

The characteristic function of

$$\mathcal{U}_c \subset \mathcal{T}(A_k)/T(k)$$

is the infimum of the set J of all $T^1(A_k)/T(k)$-invariant non-negative functions in $C_c(\mathcal{T}(A_k)/T(k))$ which are equal to 1 on $\mathcal{U}_c$ (cf. [**38**, p. 256]). Now regard these functions as functions on $\mathcal{T}(A_k)/T^1(A_k)$ and make use of (5.23)(c). Then,

$$\overline{n}_A(\mathcal{U}_c) = \inf_{h \in J} \int_{\mathcal{T}(A_k)/T(k)} h d\overline{n}_A = \int_{X(A_k)} (\inf_{h \in J} \overline{\Lambda}_\beta(h)) dm_{A,\gamma}.$$

Moreover, from the isomorphism I above, it follows that

$$\inf_{h \in J} \overline{\Lambda}_\beta(h) = \begin{cases} \overline{\Theta}^1_\Sigma(T^1(A_k)/T(k)) & \text{on } B_A \cap \pi_A(\mathcal{T}(A_k)) \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\overline{n}_A(\mathcal{U}) = \overline{\Theta}^1_\Sigma(T^1(A_k)/T(k)) \cdot m_{A,\Sigma}(B_A \cap \pi_A(\mathcal{T}(A_k)))$$

as asserted.

We shall in section 10 construct a canonical "toric" section ψ as above for universal torsors over toric varieties. One can more generally construct a continuous section ψ as above for varieties with a "system of heights" in the sense of Peyre [**51**].

6. Reciprocity conditions and Tamagawa numbers

The aim of this section is to relate the Tamagawa numbers for the universal torsors defined in the previous section to the Tamagawa numbers defined by Peyre [**52**]. To do this, we shall need Manin's reciprocity condition for adelic points on X which is defined by means of étale cohomology.

If R is commutative ring and F is an abelian sheaf on the étale site of $\operatorname{Spec} R$ (cf. [**45**]), we shall write $H^i_{\text{ét}}(R, F)$ instead of $H^i_{\text{ét}}(\operatorname{Spec} R, F)$. If G_Y is a commutative group scheme over a scheme Y we let G_Y denote also the abelian sheaf F on the étale site $Y_{\text{ét}}$ associated to this group scheme (cf. [**45**, p. 52]). In particular, we shall write $H^1_{\text{ét}}(Y, G_Y) = H^1_{\text{ét}}(Y, F)$. If $G_Y = G_{m,Y}$ we omit the index Y.

There is a canonical isomorphsims of schemes:

$$\operatorname{Spec} A_k = \varprojlim \operatorname{Spec} A_k(\Sigma)$$

where Σ runs over all finite subsets of W_{fin}. Since "étale cohomology commutes with inverse limits of schemes" (cf. [**1**, VII.5.9]) we get:

$$H^2_{\text{ét}}(A_k, \mathbb{G}_m) = \varprojlim H^2_{\text{ét}}(A_k(\Sigma), \mathbb{G}_m).$$

Next, note that $\operatorname{Spec} A_k(\Sigma)$ is a direct sum (cf. [**31**, 3.1]) of schemes

$$\operatorname{Spec} A_k(\Sigma) = (\oplus_{\nu \in T} \operatorname{Spec} k_\nu) \oplus \big(\oplus_{\nu \in W \setminus T} \operatorname{Spec} o_\nu\big)$$

where $T := \Sigma \cup W_\infty$. Hence:

$$H^2_{\text{ét}}(A_k(\Sigma), \mathbb{G}_m) = \big(\oplus_{\nu \in T} H^2_{\text{ét}}(k_\nu, \mathbb{G}_m)\big) \oplus \big(\oplus_{\nu \in W \setminus T} H^2_{\text{ét}}(o_\nu, \mathbb{G}_m)\big)\,.$$

Moreover, $H^2_{\text{ét}}(\operatorname{Spec} o_\nu, \mathbb{G}_m) = 0$ for $\nu \in W_{\text{fin}}$ (see [**45**, p. 108]). Hence,

$$H^2_{\text{ét}}(A_k, \mathbb{G}_m) = \oplus_{\nu \in W} H^2_{\text{ét}}(k_\nu, \mathbb{G}_m).$$

The Hasse invariant

$$i_\nu : H^2_{\text{ét}}(k_\nu, \mathbb{G}_m) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is defined as follows. If k_ν is the quotient field of a complete discrete valuation ring o_ν with residue field F_ν, then the Hasse-Witt residue map (cf. [**27**, III.2]) gives a canonical isomorphism

$$H^2_{\text{ét}}(k_\nu, \mathbb{G}_m) = H^1_{\text{ét}}(F_\nu, \mathbb{Q}/\mathbb{Z}).$$

The map i_ν is obtained by evaluating

$$H^1_{\text{ét}}(F_\nu, \mathbb{Q}/\mathbb{Z}) = \operatorname{Hom}(G_\nu, \mathbb{Q}/\mathbb{Z})$$

at the Frobenius element of the absolute Galois group G_ν of k_ν.

If $k_\nu = \mathbb{R}$, then i_ν is the group monomorphism

$$H^2_{\text{ét}}(k_\nu, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

If $k_\nu = \mathbb{C}$, then $H^2_{\text{ét}}(k_\nu, \mathbb{G}_m) = 0$.

There is a fundamental exact sequence of Albert-Brauer-Hasse-Noether (cf. e.g. [**65**, p, 196])

$$0 \longrightarrow H^2_{\text{ét}}(k, \mathbb{G}_m) \longrightarrow H^2_{\text{ét}}(A_k, \mathbb{G}_m) \xrightarrow{i} \mathbb{Q}/\mathbb{Z} \longrightarrow 0. \tag{6.1}$$

The map from $H^2_{\text{ét}}(k, \mathbb{G}_m)$ to $H^2_{\text{ét}}(A_k, \mathbb{G}_m)$ is the functorial map and the reciprocity map

$$i : \oplus_{\nu \in W} H^2_{\text{ét}}(k_\nu, \mathbb{G}_m) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is the sum of the Hasse invariants. The fact that the sum of the Hasse invariants is zero for elements in $H^2_{\text{ét}}(k, \mathbb{G}_m)$ is called the reciprocity law.

The other ingredient in Manin's reciprocity condition is the (cohomological) Brauer group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ of X. The contravariant functoriality of $H^2_{\text{ét}}(\cdot, \mathbb{G}_m)$ yields pairings:

6.2 (a) $X(k_\nu) \times H^2_{\text{ét}}(X, \mathbb{G}_m) \longrightarrow H^2_{\text{ét}}(k_\nu, \mathbb{G}_m)$
6.2 (b) $X(A_k) \times H^2_{\text{ét}}(X, \mathbb{G}_m) \longrightarrow H^2_{\text{ét}}(A_k, \mathbb{G}_m)$.

Notation 6.3. — Let $\mathcal{A} \in H^2_{\text{ét}}(X, \mathbb{G}_m)$ be an element of the Brauer group of X.

(a) If $Q_\nu \in X(k_\nu)$, then $\mathcal{A}(Q_\nu) \in H^2_{\text{ét}}(k_\nu, \mathbb{G}_m)$ is the element defined by the pairing (6.2)(a).
(b) If $Q_A \in X(A_k)$, then $\mathcal{A}(Q_A) \in H^2_{\text{ét}}(A_k, \mathbb{G}_m)$ is the element defined by the pairing (6.2)(b).

Proposition 6.4. — *Let X be a smooth variety over a number field k and let $\mathcal{A} \in H^2_{\text{ét}}(X, \mathbb{G}_m)$. Then the following holds.*

(a) *The map which sends $Q_\nu \in X(k_\nu)$ to $\mathcal{A}(Q_\nu) \in H^2_{\text{ét}}(k_\nu, \mathbb{G}_m)$ is locally constant for all places $\nu \in W$ of k. It has finite image if X is proper.*
(b) *The map which sends $Q_A \in X(A_k)$ to $\mathcal{A}(Q_A) \in H^2_{\text{ét}}(A_k, \mathbb{G}_m)$ is locally constant. It has finite image if X is proper.*

Proof
(a) We change notation and write $k = k_\nu$, $X = X_\nu$, $\mathcal{A} = \mathcal{A}_\nu$. Also, let $\delta \in H^2_{\text{ét}}(k, \mathbb{G}_m)$ and choose $\mathcal{B} \in H^2_{\text{ét}}(X, \mathbb{G}_m)$ such that $\mathcal{B} - \mathcal{A}$ is equal to the image δ under the contravariant map

$$H^2_{\text{ét}}(k, \mathbb{G}_m) \longrightarrow H^2_{\text{ét}}(X, \mathbb{G}_m)$$

corresponding to the structure morphism from X to $\operatorname{Spec} k$.

Then the subset of $Q \in X(k)$ with $\mathcal{A}(Q) = \delta$ is equal to the kernel of the specialization map

$$X(k) \longrightarrow H^2_{\text{ét}}(k, \mathbb{G}_m)$$

defined by $\mathcal{B}$. To study this kernel, represent $\mathcal{B}$ locally (in the Zariski topology) by Azumaya algebras (cf. [**45**, IV.2.16]) and apply the implicit function theorem to the associated Severi-Brauer schemes. It then follows that the kernel of the specialization map defined by $\mathcal{B}$ is open in the k-topology (cf. sec. 2), thereby proving the first assertion.

The second assertion follows from the first and the fact that $X(k)$ is compact (see (2.3)(d)).

(b) Each element $\mathcal{A} \in H^2_{\text{ét}}(X, \mathbb{G}_m)$ is the restriction of an element $\widetilde{\mathcal{A}} \in H^2_{\text{ét}}(\widetilde{X}, \mathbb{G}_m)$ for some smooth $o_{(\Sigma)}$-model $\widetilde{X}$ of X (cf. [**45**, III.1.16]). This combined with the vanishing of $H^2_{\text{ét}}(o_\nu, \mathbb{G}_m)$ implies that

$$\mathcal{A}(Q_\nu) = 0$$

for

$$Q_\nu \in \operatorname{Im}(\widetilde{X}(o_\nu) \longrightarrow X(k_\nu)), \quad \nu \in W_{\text{fin}} \backslash \Sigma.$$

Hence by (a) the map from $X(A_k)$ to $H^2_{\text{ét}}(A_k, \mathbb{G}_m)$ is locally constant. The finiteness of the image follows from the fact (see (4.9)(c)) that $X(A_k)$ is compact. $\square$

The pairing in (6.2)(b) is part of a commutative diagram:

(6.5)
$$\begin{array}{ccc} X(A_k) \times H^2_{\text{ét}}(k, \mathbb{G}_m) & \longrightarrow & H^2_{\text{ét}}(k, \mathbb{G}_m) \\ \downarrow & & \downarrow \\ X(A_k) \times H^2_{\text{ét}}(X, \mathbb{G}_m) & \longrightarrow & H^2_{\text{ét}}(A_k, \mathbb{G}_m) \end{array}$$

where the top map is the projection onto the second factor.

By the reciprocity law (6.1), one deduces from (6.5) a pairing

(6.6)
$$X(A_k) \times H^2_{\text{ét}}(X, \mathbb{G}_m)/\operatorname{Im} H^2_{\text{ét}}(k, \mathbb{G}_m) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

which is a homomorphism with respect to the second factor.

Corollary 6.7. — *Suppose that $H^2_{\text{ét}}(X, \mathbb{G}_m)/\operatorname{Im} H^2_{\text{ét}}(k, \mathbb{G}_m)$ is a finite group. Then the map*

$$X(A_k) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H^2_{\text{ét}}(X, \mathbb{G}_m)/\operatorname{Im} H^2_{\text{ét}}(k, \mathbb{G}_m) \longrightarrow \mathbb{Q}/\mathbb{Z})$$

induced by (6.6) is locally constant. In particular, if X is proper, then the inverse image of any element of group on the right hand side is a compact open subset of $X(A_k)$.

Proof. — This is an immediate consequence of (6.4)(b). $\square$

Notation 6.8. — $X(A_k)^0 \subseteq X(A_k)$ is the inverse image of 0 under the map in (6.7).

We now examine the group $H^2_{\text{ét}}(X, \mathbb{G}_m)/\operatorname{Im} H^2_{\text{ét}}(k, \mathbb{G}_m)$.

Notation 6.9

(a) $H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{alg}} := \ker(H^2_{\text{ét}}(X, \mathbb{G}_m) \to H^2_{\text{ét}}(\overline{X}, \mathbb{G}_m))$
(b) $H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{trans}} := H^2_{\text{ét}}(X, \mathbb{G}_m)/H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{alg}}$
(c) $H^3_{\text{sing}}(X(\mathbb{C})_{\text{an}}, \mathbb{Z})_{\text{tors}}$

is the torsion subgroup of $H^3_{\text{sing}}(X(\mathbb{C})_{\text{an}}, \mathbb{Z})$ where $X(\mathbb{C})$ depends on the choice of an embedding $k \subset \mathbb{C}$.

It follows from $H^2_{\text{ét}}(\overline{k}, \mathbb{G}_m) = 0$, that the image of $H^2_{\text{ét}}(k, \mathbb{G}_m)$ in $H^2_{\text{ét}}(X, \mathbb{G}_m)$ is a subgroup of $H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{alg}}$. The following result is well-known (cf. [**40**], [**41**] for (a))

Lemma 6.10. — *Let k be a number field and let X be a smooth (geometrically connected) proper k-variety satisfying $H^1_{\text{Zar}}(X, \mathcal{O}_X) = H^2_{\text{Zar}}(X, \mathcal{O}_X) = 0$ and for which the Néron-Severi group of $\overline{X} := \overline{k} \times X$ is torsion-free. Then*

(a) $H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{alg}}/\operatorname{Im} H^2_{\text{ét}}(k, \mathbb{G}_m) \cong H^1(\text{Gal}(\overline{k}/k), \text{NS}(\overline{X}))$
(b) $H^2_{\text{ét}}(\overline{X}, \mathbb{G}_m) \cong H^3_{\text{sing}}(X(\mathbb{C})_{\text{an}}, \mathbb{Z})_{\text{tors}}$ *for any embedding* $k \subset \mathbb{C}$.
(c) $H^2_{\text{ét}}(X, \mathbb{G}_m)/\operatorname{Im} H^2_{\text{ét}}(k, \mathbb{G}_m)$ *is a finite group.*

Proof

(a) The spectral sequence (cf. [**27**, II.2.4])

$$H^p(\text{Gal}(\overline{k}/k), H^q_{\text{ét}}(\overline{X}, \mathbb{G}_m)) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathbb{G}_m)$$

of Hochschild-Serre gives rise to an exact sequence

$$H^2_{\text{ét}}(k, \mathbb{G}_m) \to H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{alg}} \to H^1(\text{Gal}(\overline{k}/k), H^1_{\text{ét}}(\overline{X}, \mathbb{G}_m)) \to H^3_{\text{ét}}(k, \mathbb{G}_m).$$

For global fields k it is known from class field theory that $H^3_{\text{ét}}(k, \mathbb{G}_m) = 0$. Hence it only remains to note that $\text{Pic}(\overline{X}) = \text{NS}(\overline{X})$ by the assumption on X.

(b) This follows from results of Grothendieck (cf. [**45**, VI.4.3], [**27**, I.3.1] and [**26**, 1.35]). It is also known from the work of Artin-Mumford [**2**].

(c) This follows from (a) and (b). □

Peyre makes use of the following lemma (cf. [**52**, 2.1.1]) in his definition of Tamagawa numbers.

Lemma 6.11. — *Let k be a number field and let X be a k-variety as in (6.10). Then there is an $o_{(\Sigma)}$-model Ξ of X for some finite set $\Sigma \subset \operatorname{Spec} o$ such that Ξ is smooth and proper over $\operatorname{Spec} o_{(\Sigma)}$. Let $\overline{Y}_\nu := \Xi \times \overline{F}_\nu$ and let (cf. (4.16))*

$$L_\nu(s, \operatorname{Pic} \overline{Y}_\nu) = 1/\det(1 - q_\nu^{-s} \operatorname{Fr}_\nu \mid \operatorname{Pic} \overline{Y}_\nu \otimes \mathbb{Q}), \nu \in W_{\text{fin}} \backslash \Sigma.$$

Then,

$$L_\Sigma(s, \operatorname{Pic} \overline{X}) := \prod_{\nu \in W_{\text{fin}} \backslash \Sigma} L_\nu(s, \operatorname{Pic} \overline{Y}_\nu)$$

converges absolutely and uniformly on compact subsets of $\Re s > 1$ *and defines a holomorphic function for* $\Re s > 1$.

The function $L_\Sigma(s, \operatorname{Pic} \overline{X})$ *has a meromorphic continuation to* $\mathbb{C}$ *with a pole of order* $r := \operatorname{rk} \operatorname{Pic} X$ *at* $s = 1$.

Definition and proposition 6.12. — *Let* k *be a number field and let* X *be a* k-*variety as in (6.10). Let*

$$\| \ \|_X = \{\| \ \|_{X(k_\nu)} : \mathbf{det}(\operatorname{Tan} X(k_\nu)) \longrightarrow [0, \infty), \ \nu \in W\}$$

be an adelic norm for X. *Then there is an* $o_{(\Sigma)}$-*model* Ξ *of* X *for some finite set* $\Sigma \subset W_{\text{fin}}$ *such that* Ξ *is smooth and proper over* $\operatorname{Spec} o_{(\Sigma)}$ *and such that* $\| \ \|_{X(k_\nu)}$ *is the model norm determined by* Ξ_ν / o_ν *for* $\nu \in W_{\text{fin}} \backslash \Sigma$.

Let m_ν, $\nu \in W$, *be the positive Borel measure on* $X_\nu(k_\nu)$ *defined by the* ν-*adic norm* $\| \ \|_{X(k)}$ *and let*

$$\gamma_\nu = \begin{cases} 1/L_\nu(1, \operatorname{Pic} \overline{Y}_\nu) & \text{for } \nu \in W_{\text{fin}} \backslash \Sigma, \\ 1 & \text{for } \nu \in W_\infty \cup \Sigma. \end{cases}$$

If $X(A_k) \neq \varnothing$, *then* $\tau(X, \| \ \|)$ *is defined to be the product:*

$$\lim_{s \to 1} (s-1)^{\operatorname{rk} \operatorname{Pic} X} L_\Sigma(s, \operatorname{Pic} \overline{X})(m_{A,\Sigma}(X(A_k)^0))$$

where $m_{A,\Sigma} := m_{A,\gamma}$ *is the regular positive Borel measure described in (4.14)(b).*

This number is positive and independent of the choices of Σ *and* Ξ.

Proof. — The proof of the statement is clear from the definitions of $L_\Sigma(s, \operatorname{Pic} \overline{X})$ and $m_{A,\gamma}$ (compare [**52**, 2.2.4], but note that there is an additional discriminant factor there due to differences in the normalization of the Haar measures). □

Remark 6.13. — Peyre [**52**] uses the topological closure $\overline{X(k)}$ of $X(k)$ instead of $X(A_k)^0$ in his definition of Tamagawa numbers. It follows from the reciprocity law that $X(k)$ is mapped into $X(A_k)^0$ under the functorial embedding $X(k) \subset X(A_k)$ (cf. [**40**], [**41**]).

$X(A_k)^0$ is a closed subset of $X(A_k)$ (see (6.7) and (6.10)(c)). Therefore, $\overline{X(k)} \subseteq X(A_k)^0$. It has been conjectured in [**15**] that $\overline{X(k)} = X(A_k)^0$ when X is rational. We shall in (7.8) explain why we find it more natural to consider $m_A(X(A_k)^0)$ than $m_A(\overline{X(k)})$.

The following result is due to Colliot-Thélène and Sansuc [**15**, §3] in the case of rational varieties.

Lemma 6.14. — *Let* k *be a number field and let* X *be a* k-*variety as in (6.10). Let* $(Q_\nu)_{\nu \in W} \in X(A_k)$ *be an adelic point on* X. *Then the following assertions are equivalent*

(i) $\sum_{\nu \in W} i_\nu(\mathcal{A}(Q_\nu)) = 0$ *for all* $\mathcal{A} \in H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{alg}}$.

(ii) *There exists a universal torsor* $\pi : \mathcal{T} \to X$ *such that*

$$(P_\nu)_{\nu \in W} \in \pi_A(\mathcal{T}(A_k)).$$

Proof. — This is proved in [**15**, §3] for rational varieties by means of an explicit computation of cocycles. An examination of the proof in (op. cit.) reveals that it works also under the weaker hypothesis in (6.10). For a conceptual proof, which does not use brutal force, see [**55**]. □

Lemma 6.15. — *Let k be a number field and let X be a k-variety as in (6.10). Let $\pi : \mathcal{T} \to X$ be a universal torsor. Then the restriction to*

$$\pi(\mathcal{T}(A_k)) \times H^2_{\text{ét}}(X, \mathbb{G}_m)$$

of Manin's pairing (6.3) factorizes to give a pairing

$$\pi(\mathcal{T}(A_k)) \times H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{trans}} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

Proof. — This is a corollary of the previous lemma. □

Notation 6.16

$$\text{Ш}^1(k,T) := \ker(H^1_{\text{ét}}(k,T) \longrightarrow \prod_{\nu \in W} H^1_{\text{ét}}(k_\nu, T)).$$

It is known from class field theory that this (Shafarevich) group is finite.

Lemma 6.17. — *Let k be a number field and let X be a k-variety as in (6.10) with $X(A_k) \neq \varnothing$. Let*

$$\| \ \|_X = \{\| \ \|_{X(k_\nu)} : \mathbf{det}\,\mathrm{Tan}\, X(k) \longrightarrow [0,\infty),\ \nu \in W\}$$

be an adelic norm for X and let $m_{A,\gamma}$ be the Borel measure on $X(A_k)$ (cf. (4.14)(b)) determined by $\| \ \|_X$ and the convergence factors γ in (6.12). Let $I \subset H^1_{\text{ét}}(X,T)$ be the subset parametrizing isomorphism classes of universal torsors $\pi : \mathcal{T} \to X$ such that $\pi(\mathcal{T}(A_k)) \cap X(A_k)^0 \neq \varnothing$. Then the following holds.

(a) *I is finite*

(b) *$\pi(\mathcal{T}(A_k)) \cap X(A_k)^0$ is a compact open subset of $X(A_k)$.*

(c)

$$\mathrm{Card}(\text{Ш}^1(k,T)) m_{A,\gamma}(X(A_k)^0)) = \sum_{\varepsilon \in I} m_{A,\gamma}(\pi_\varepsilon(\mathcal{T}_\varepsilon(A_k) \cap X(A_k)^0)).$$

Proof

(a) Let $J \subset H^1_{\text{ét}}(X,T)$ be the subset parametrizing isomorphism classes of universal torsors $\pi : \mathcal{T}(A_k) \to X$ such that $\mathcal{T}(A_k) \neq \varnothing$. It then follows by a weak Mordell-Weil argument (cf. [**15**, Th. 2.7.3]) that J is finite. Hence $I \subseteq J$ is also finite.

(b) It was shown in the proof of (5.20) that $\pi(\mathcal{T}(k_\nu))$ is a compact open subset of $X(k_\nu)$ for all places ν of k and that $\pi(\mathcal{T}(k_\nu)) = X(k_\nu)$ for all but finitely many places. Hence $\pi(\mathcal{T}(A_k))$ is a compact open subset of $X(A_k)$. By (6.7) $X(A_k)^0$ is also a compact open subset of $X(A_k)$. Therefore $\pi(\mathcal{T}(A_k)) \cap X(A_k)^0$ is a compact open subset of $X(A_k)$.

(c) Let $\pi_\alpha : \mathcal{T}_\alpha \to X$, $\pi_\beta : \mathcal{T}_\beta \to X$ be two universal torsors and α, β their classes in $H^1_{\text{ét}}(X,T)$. Then from (5.13), (5.15) it follows that:

$$\begin{aligned}\pi_\alpha(\mathcal{T}_\alpha(A_k)) = \pi_\beta(\mathcal{T}_\beta(A_k)) &\iff \alpha - \beta \in \operatorname{Im}(\text{Ш}^1(k,T) \to H^1_{\text{ét}}(X,T)),\\ \pi_\alpha(\mathcal{T}_\alpha(A_k)) \cap \pi_\beta(\mathcal{T}_\beta(A_k)) = \varnothing &\iff \alpha - \beta \notin \operatorname{Im}(\text{Ш}^1(k,T)).\end{aligned}$$

This combined with $\pi(\mathcal{T}(A_k)^0) = \pi(\mathcal{T}(A_k)) \cap X(A_k)^0$ implies that

$$\operatorname{Card}(\text{Ш}^1(k,T)) m_{A,\gamma}(\bigcup_{\varepsilon\in I} \pi_{\varepsilon,A}(\mathcal{T}_\varepsilon(A_k)^0)) = \sum_{\varepsilon\in I} m_{A,\gamma}(\pi_{\varepsilon,A}(\mathcal{T}_\varepsilon(A_k)^0)).$$

But

$$X(A_k)^0 = \bigcup_{\varepsilon\in I} \pi_{\varepsilon,A}(\mathcal{T}_\varepsilon(A_k))^0$$

by (6.14). This completes the proof. □

Definition 6.18. — Let X be as in (6.10) and let $\varepsilon \in H^1_{\text{ét}}(X,T)$ be the class of a universal torsor $\pi : \mathcal{T} \to X$. Let $\| \,\|_X$ be an adelic norm for X. Then the Tamagawa number $\tau_\varepsilon(X, \| \,\|_X)$ is the number $\tau_\varepsilon(X, X(A_k)^0, \| \,\|_X)$ defined in (5.22).

The following theorem is one of the main results of this paper.

Theorem 6.19. — *Let k be a number field and let X be a k-variety as in (6.10). Let*

$$\| \,\|_X = \{\| \,\|_{X(k_\nu)} : \mathbf{det}(\operatorname{Tan} X(k_\nu)) \longrightarrow [0,\infty),\ \nu \in W\}$$

be an adelic norm for X. Then $\mathcal{T}(A_k) = \varnothing$ and $\tau_\varepsilon(X, \| \,\|_X) = 0$ for all but finitely many isomorphism classes $\varepsilon \in H^1_{\text{ét}}(X,T)$ of universal torsors $\pi_\varepsilon : \mathcal{T}_\varepsilon \to X$. Further,

$$(*) \qquad \sum_\varepsilon \tau_\varepsilon(X, \| \,\|_X) = \operatorname{Card} H^1(\operatorname{Gal}(\overline{k}/k), \operatorname{Pic} \overline{X}) \tau(X, \| \,\|_X)$$

where $\varepsilon \in H^1_{\text{ét}}(X,T)$ runs over all elements of identity type $\chi(\varepsilon)$ in

$$\operatorname{Hom}_G(\hat{T}, H^1_{\text{ét}}(\overline{X}, \mathbb{G}_m)).$$

In particular, if $\mathrm{Pic}(\overline{X})$ *is a direct summand of a permutation* $\mathrm{Gal}(\overline{k}/k)$*-module, then*

$$\tau(X, \|\ \|) = \tau_\varepsilon(X, \|\ \|)$$

for the unique element $\varepsilon \in H^1_{\text{ét}}(X, T)$ *of identity type.*

Proof. — Let $I \subset H^1_{\text{ét}}(X, T)$ be the subset parametrizing isomorphism classes of universal torsors $\pi : \mathcal{T} \to X$ such that $\pi(\mathcal{T}(A_k))^0 \neq \varnothing$. Then I is finite by (6.17)(a). To prove $(*)$, choose Σ, $\widetilde{o} := o_{(\Sigma)}$, Ξ, $\widetilde{T}$, $\widetilde{\pi}_\varepsilon : \widetilde{\mathcal{T}}_\varepsilon \to \Xi$ as in (5.2) for each representative $\pi_\varepsilon : \mathcal{T}_\varepsilon \to X$ of an isomorphism class $\varepsilon \in I$ of universal torsors with $\mathcal{T}(A_k) \neq \varnothing$.

We may clearly choose the same finite set Σ and $\widetilde{o} = o_{(\Sigma)}$ for all $\varepsilon \in I$ since I is finite. Also, by the definition of adelic norms, we may assume that $\|\ \|_{X(k_\nu)}$ is the model norm determined by Ξ_ν/o_ν for $\nu W_{\text{fin}} \backslash \Sigma$ and by (6.7) we may chose the finite set $\Sigma \subset W_{\text{fin}}$ such that

$$X(A_k)^0 = X_S(A_k)^0 \times \prod_{\nu \notin S} X(k_\nu)$$

for some compact open subset

$$X_S(A_k)^0 \subseteq \prod_{\nu \in S} X(k_\nu), \quad S := \Sigma \cup W_\infty.$$

By definition of $\tau_\varepsilon(X, \|\ \|_X)$ we have (cf. (5.20), (6.18)):

$$\tau_\varepsilon(X, \|\ \|_X) = \Theta^1_\Sigma(T^1(A_k)/T(k)) \cdot m_{A,\Sigma}(\pi_\varepsilon(\mathcal{T}_\varepsilon(A_k)^0)).$$

Hence by (6.17)(c) we get:

$$\sum_{\varepsilon \in I} \tau_\varepsilon(X, \|\ \|_X) = \Theta^1_\Sigma(T^1(A_k)/T(k)) \cdot m_{A,\Sigma}(X(A_k)^0) \cdot \mathrm{Card}(\text{Ш}^1(k, T)).$$

By the main theorem of Ono [**50**, §5] on Tamagawa numbers for tori we have firther

$$\Theta^1_\Sigma(T^1(A_k)/T(k)) = (h(T)/\,\mathrm{Card}\,\text{Ш}^1(k,T)) \lim_{s \to 1} (s-1)^{\mathrm{rk}\,\mathrm{Pic}\,X} L_\Sigma(s, \mathrm{Pic}\,\overline{X})$$

where

$$h(T) := \mathrm{Card}\, H^1(\mathrm{Gal}(\overline{k}/k), \hat{T}) = \mathrm{Card}\, H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Pic}\,\overline{X}).$$

Hence,

$$\sum_{\varepsilon \in I} \tau_\varepsilon(X, \|\ \|_X) = h(T)(\lim_{s \to 1} (s-1)^{\mathrm{rk}\,\mathrm{Pic}\,X} L_\Sigma(s, \mathrm{Pic}\,\overline{X}) m_{A,\Sigma}(X(A_k)^0)$$

as was to be proved. □

7. Counting functions of Fano varieties

We shall in this section study the asymptotic growth of the number of rational points of bounded anticanonical height on Fano varieties X over number fields. This theory was initiated by Manin. He suggested (cf. [**23**], [**3**], [**42**], [**43**]) that the asymptotic growth of the counting function should be of the form

$$CB(\log B)^{\operatorname{rk}\operatorname{Pic}X-1}(1+o(1))$$

on sufficiently small Zariski open subsets of X.

Peyre [**52**] gave a conjectural interpretation of the constant C by means of his Tamagawa number $\tau(X,\|\ \|)$ and a geometrical invariant $\alpha(X)$ depending only on the effective cone in $\operatorname{Pic}X$. His interpretation of the constant C concerns the case where $\operatorname{Pic}\overline{X}$ is a permutation module. It does not cover the case of general toric varieties which have recently been studied by Batyrev and Tschinkel ([**7**], [**4**]). The aim of this section is to use the Tamagawa numbers of universal torsors over X to reinterpret and to refine the conjecture of Peyre on the value of C.

Definition 7.1. — Let k be a field and let X be a smooth proper geometrically connected scheme over k. Then X is said to be a Fano variety if the anticanonical sheaf ω_X^{-1} of X is very ample. A Fano variety of dimension 2 is called a del Pezzo surface.

It is usually only required that ω_X^{-1} is ample in the definition of Fano varieties and del Pezzo surfaces. In particular, "our" del Pezzo surfaces are always of degree ≥ 3 and we do not consider surfaces of degree 1 or 2 (cf. [**41**], [**44**]). It is known (cf. e.g. [**52**, 1.2.1]) that Fano varieties satisfy all the conditions in (6.10) and (5.12). We can therefore apply all the constructions and results made in the previous two sections to them.

We now turn to the arithmetic of Fano varieties and assume for the rest of this section that k is a number field. Let

$$\|\ \| = \{\|\ \|_\nu : \mathbf{det}\operatorname{Tan}X(k_\nu) \longrightarrow [0,\infty),\ \nu\in W\}$$

be an adelic norm for X. It is well-known (cf. e.g. [**63**]) that $\|\ \|$ defines a height function

$$H : X(k) \longrightarrow [0,\infty)$$

although the "metrics" $\|\ \|_\nu$ usually only occur explicitly for the archimedean places.

Definition 7.2. — The *height function* $H : X(k) \to (0,\infty)$ *defined by* $\|\ \|$ is given by

$$H(P) = \prod_{\nu\in W} \|s(P)\|_\nu^{-1}$$

where s is a local section of ω_X^{-1} at P with $s(P) \neq 0$.

This function is well defined since $H(P)$ does not depend on the choice of s by the product formula for number fields.

Remarks 7.3

(a) It follows from the compactness of all $X(k_\nu)$ that $\log H_1 - \log H_2$ is bounded on $X(k)$ for the height functions H_1 and H_2 defined by two adelic norms $\| \, \|_1$, $\| \, \|_2$ for X (cf. (1.6)(b), (2.3)(e) and (4.10)).

(b) Let X be a Fano variety. Then

$$\mathcal{C}(B) := \{P \in X(k) : H(P) \leq B\}$$

is *finite* for any height H defined by an adelic norm. It suffices by (a), to prove this for *one* adelic norm on X and it is natural to consider a norm defined by means of a finite set of global sections generating ω_X^{-1} (cf. [**52**, pp. 107-8]). One can then apply the classical arguments for heights defined by projective coordinates (cf. e.g. [**37**, Ch. 3, §1]).

Notation 7.4. — Let X be a Fano variety and let H be a height defined by an adelic norm $\| \, \|$. Let U be a constructible subset defined over k. Then,

(a)
$$\mathcal{C}_U(B, \| \, \|) = \operatorname{Card}\{P \in U(k) : H(P) \leq B\}.$$

(b)
$$\beta_U := \limsup_{B \to \infty} \log \mathcal{C}_U(B, \| \, \|) / \log B.$$

(c)
$$\gamma_U(\| \, \|) := \limsup_{B \to \infty} \mathcal{C}_U(B) / B(\log B)^{r-1} > 0$$

for $r = \operatorname{rk} \operatorname{Pic} X$.

It is clear from (7.3)(a) that β_U is independent of the choice of adelic norm $\| \, \|$. It also follows from (7.3)(a) that the condition $\gamma_U(\| \, \|) > 0$ is independent of the choice of adelic norm and we shall therefore write $\gamma_U > 0$ for this condition. The function sending B to $\mathcal{C}_U(B, \| \, \|)$ will be called the counting function of U with respect to $\| \, \|$ and β_U the growth order of U.

The following definitions are inspired by notions of Manin (cf. [**42**]) and [**51**]). But the reader should observe that our definitions are not identical with the definitions in (op. cit.) and that we only consider Fano varieties.

Definition 7.5. — Let X be a Fano variety.

(a) A closed proper subset $F \subset X$ of X is said to be accumulating if for each non-empty subset V of F one has $\beta_V > 1$.

(b) A closed proper subset $F \subset X$ of X is said to be weakly accumulating if $\gamma_V > 0$ for each non-empty open subset V of F.

We are now in a position to formulate a version of Manin's conjecture on $\mathcal{C}_U(H)$ (cf. [**23**], [**3**], [**42**], [**43**], [**52**], [**51**]).

Conjecture 7.6. — *Let X be a rational Fano variety over a number field k for which $X(A_k)^0$ is non-empty and let $\| \ \|$ be an adelic norm for X. Suppose that the complement U in X of the union of all weakly accumulating (proper) subsets is a Zariski open non-empty set defined over k. Then there is a positive constant $C = C(\| \ \|) > 0$ such that*

$$\lim_{B\to\infty} \mathcal{C}_U(B, \| \ \|)/B(\log B)^{r-1} = C \quad \textit{for } r = \operatorname{rk} \operatorname{Pic} X.$$

Manin assumes that $X(k)$ is Zariski dense instead of just assuming that $X(A_k)^0$ is non-empty. To compensate this we have added the condition that X is rational. This excludes cubic three-folds which are unirational but not rational. A more optimistic version of (7.6) would be to assume that X is unirational. This is still a restriction since some Fano varieties are not even uniruled [**35**, V.5].

We have in our definition of U used weakly accumulating (proper) subsets instead of accumulating (proper) subsets. It has recently been shown (cf. [**5**]) by Batyrev and Tschinkel that (7.6) is false when U is defined as the complement of the union of all accumulating subsets. The counterexamples are given by smooth hypersurfaces in $\mathbb{P}^3_k \times \mathbb{P}^n_k$ of bidegree $(3,1)$.

To formulate a conjecture for the constant $C(\| \ \|) > 0$, Peyre [**52**, p. 120] defines for Fano varieties an invariant $\alpha(X)$ which we will often denote by $\alpha_{\mathrm{Peyre}}(X)$. It depends only on the effective cone in $\operatorname{Pic} X$. To describe this invariant, let

$$L = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{Z})$$
$$V = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{R})$$

and let $d\nu$ be the unique Haar measure on V such that volume $\mathrm{Vol}(V/L) = 1$.

Further, let $\sigma_{\mathrm{eff}}(X) \subset V$ be the cone of all homomorphisms

$$\varphi : \operatorname{Pic} X \longrightarrow \mathbb{R}$$

such that

$$\varphi([D]) \geq 0$$

for the class $[D] \in \operatorname{Pic} X$ of each effective divisor on X. Let $\lambda : V \to \mathbb{R}$ be the linear form obtained by evaluating at the anticanonical class and let $V_x = \lambda^{-1}(x)$ for $x \in \mathbb{R}$ and note that $\lambda : V \to \mathbb{R}$ is a trivial analytic torsor under V_0.

There is then a unique positive linear map $\Lambda : C_c(V) \to C_c(\mathbb{R})$ such that:

$$\int_V g d\nu = \int_{\mathbb{R}} \Lambda(g) dx$$

for any function $g \in C_c(V)$. Λ restricts to a positive functional Λ_x on $V_x = \lambda^{-1}(x)$ for each $x \in \mathbb{R}$. Let $d\nu_x$ be the corresponding positive Borel measure on V_x. Then,

$$\alpha_{\text{Peyre}}(X) := \int_B d\nu_1 \quad \text{for } B := V_1 \cap \sigma_{\text{eff}}(X).$$

Batyrev and Tschinkel [**7**] define a similar invariant which they also call $\alpha(X)$, but which we will denote by $\alpha_{BT}(X)$. It follows from one of the lemmas in [**7**] that their invariant

$$\alpha_{BT}(X) = (r-1)! \cdot \alpha_{\text{Peyre}}(X).$$

The α_{BT}-invariant is multiplicative (see [Pe1, lemme 4.2])

$$\alpha_{BT}(X \times Y) = \alpha_{BT}(X)\alpha_{BT}(Y)$$

for varieties as in (6.10). It is more natural when one considers Manin s zeta-functions (see [**7**] and [**4**]).

The following conjecture is due to Peyre [**52**, 2.3.1].

Conjecture 7.7. — *Suppose that the assumptions in (7.6) are satisfied and suppose that* $\operatorname{Pic} \overline{X}$ *has a* $\mathbb{Z}$*-basis which is invariant under the action of the absolute Galois group* G_k *of* k*. Then the constant* $C(\| \ \|)$ *in Manin's conjecture is of the form:*

$$C(\| \ \|) = \alpha_{\text{Peyre}}(X)\tau(X, \| \ \|).$$

Remark 7.8. — Peyre defines (cf. (6.13)) "his" $\tau(X, \| \ \|)$ by means of $m_A(\overline{X(k)})$ instead of $m_A(X(A_k)^0)$ as we have done. The two definitions are expected (cf. (6.13)) to be identical when $\overline{X}$ is rational and perhaps also when $\overline{X}$ is unirational. We find it more natural to give an adelic interpretation of C (with an extra twist given by reciprocity pairing of Manin).

It is not difficult to compute $m_A(X(A_k)^0)$. The group

$$H^2_{\text{ét}}(X, \mathbb{G}_m)/\operatorname{Im} H^2_{\text{ét}}(k, \mathbb{G}_m)$$

is finite (cf. (6.10)) and it is clear from the proof of (6.4)(b) that it can be decided if an adelic point belongs to $X(A_k)^0$ by looking at finitely many places $S \subset W$. Also, if $\| \ \|_\nu$ is the model norm of a smooth model $\widetilde{X}/o_\nu$, then

$$m_\nu(X_\nu) = \operatorname{Card}(\widetilde{X}(F_\nu))(m_\nu(o_\nu)/\operatorname{Card}(F_\nu))^{\dim X}$$

by (2.15). It is thus possible, in principle, to compute $m_A(X(A_k)^0)$ in finitely many steps although the answer is given as an infinite product.

The only reasonable way to compute $m_A(\overline{X(k)})$, however, is to prove that $\overline{X(k)} = X^0_A$ and then use the computation for $m_A(X(A_k)^0)$. This is a much more difficult problem than the computation of $m_A(X(A_k)^0)$.

Fraenke, Manin and Tschinkel [**23**] proved (7.6) for generalized flag spaces (using Langlands' Eisenstein series). They also noticed that one can deduce (7.6) for smooth complete intersections whenever asymptotic results on the affine cone are available by the Hardy-Littlewood circle method. In these cases the asymptotic formulas hold for X itself. Peyre (op. cit) completed their results and proved his conjecture (7.7) for these varieties. He also proved (7.6) and (7.7) some toric surfaces over $\mathbb{Q}$. The open subset U is then the open subset defined by the underlying torus. These results are proved for counting functions defined by one special adelic norm.

Peyre formulates his conjecture (7.7) in the case where $\operatorname{Pic}\overline{X}$ is permutation G_k-module. It is natural to weaken this condition and also allow varieties for which $\operatorname{Pic}\overline{X}$ is a G_k-direct summand of a permutation G_k-module.

One cannot expect to omit such a condition completely. Batyrev and Tschinkel study in two recent papers [**7**], [**4**] $\mathcal{C}_U(H)$ for smooth projective U-equivariant compactifications X of tori U. They choose a natural adelic norm (cf. (9.2) of this paper) and make strong use of the group action. Using the abstract Poisson formula, they conclude that (7.6) holds with the constant

$$C(\| \;\|) = \alpha_{\mathrm{Peyre}}(X)\tau(X, \| \;\|)h^1(\operatorname{Pic}\overline{X}) \tag{7.9}$$

where $h^1(\operatorname{Pic}\overline{X}) = \operatorname{Card} H^1(G_k, \operatorname{Pic}\overline{X})$.

This does not contradict (7.7) since $H^1(G_k, \operatorname{Pic}\overline{X}) = 0$ when $\operatorname{Pic}\overline{X}$ is a G_k-direct summand of a permutation G_k-module. It is tempting to reformulate (7.7) and conjecture that (7.9) holds for general Fano varieties satisfying (7.6). Numerical work of Heath-Brown [**33**] on the two diagonal cubic surfaces defined by $x^3 + y^3 + z^3 + 2w^3$ and $x^3 + y^3 + z^3 + 3w^3$ seems to indicate that such a generalization of (7.9) is true. Note that

$$H^1(G_k, \operatorname{Pic}\overline{X}) \neq 0$$

for both these surfaces.

Definition 7.10. — Let G be a profinite group. Let M be a finitely generated torsion-free G-module and let $N = \operatorname{Hom}(M, \mathbb{Z})$. Then M is said to be flasque [**14**] if

$$H^1(H, N) = 0$$

for each closed subgroup H of G.

It is known from work of Colliot-Thélène /Sansuc [**14**] and Voskresenskii that $\operatorname{Pic}\overline{X}$ is a flasque G_k-module for toric varieties. It is obvious that $\operatorname{Pic}\overline{X}$ is flasque if it is a G_k-direct summand of a permutation G_k-module. The conjecture (7.6) has sofar only been verified for classes of k-varieties X for which $\operatorname{Pic}\overline{X}$ is a flasque G_k-module. This is not surprising since there are only finitely many isomorphism classes of universal torsors for such varieties.

The asymptotic formulas obtained by Batyrev, Fraenke, Manin, Peyre and Tschinkel all satisfy the following conjecture:

Conjecture 7.11. — *Suppose that the assumptions in (7.6) are satisfied and suppose that* $\operatorname{Pic}\overline{X}$ *is a flasque* G_k*-module for the absolute Galois group* G_k *of* k. *Then the constant* $C(\|\ \|)$ *in Manin's conjecture is of the form:*

$$C(\|\ \|) = \alpha_{\mathrm{Peyre}}(X)\tau(X, \|\ \|)h^1(\operatorname{Pic}\overline{X}).$$

This conjecture is compatible with products. To see this, one uses the arguments in [**23**] and [**52**] to prove compatibility of (7.6) and (7.7) under products. The only new ingredient needed is the equality

$$h^1(\operatorname{Pic}\overline{X \times Y}) = h^1(\operatorname{Pic}\overline{X})h^1(\operatorname{Pic}\overline{Y})$$

which follows from the canonical isomorphism

$$\operatorname{Pic}\overline{X \times Y} = (\operatorname{Pic}\overline{X}) \times (\operatorname{Pic}\overline{Y})$$

already used in [**52**, 4.1].

The following refinement of (7.6) and (7.11) is natural even if there is not much evidence for it.

Conjecture 7.12. — *Let* X *be a rational Fano variety over a number field* k *and suppose that* $\operatorname{Pic}\overline{X}$ *is a flasque* G_k*-module. Let* $\|\ \|$ *be an adelic norm for* X *and let* H *be the height function on* $X(k)$ *defined by* $\|\ \|$. *Let* $\varepsilon \in H^1_{\text{ét}}(X, T)$ *be the class of a universal torsor* $\pi : \mathcal{T} \to X$ *such that* $\mathcal{T}(A_k) \neq \varnothing$. *Suppose that the complement* U *in* X *of the union of all weakly accumulating subsets is Zariski open and non-empty and defined over* k *and let*

$$\mathcal{C}_{U,\varepsilon}(B) = \operatorname{Card}\{P \in \pi(\mathcal{T}(k)) : H(P) \leq B\}.$$

Then

$$\lim_{B\to\infty} \mathcal{C}_{U,\varepsilon}(B)/B(\log B)^{r-1} = \alpha_{\mathrm{Peyre}}(X)\tau_\varepsilon(X, \|\ \|)$$

where $r = \operatorname{rk}\operatorname{Pic} X$.

It follows immediately from theorem (6.19) that (7.12) implies (7.11). If $\operatorname{Pic}\overline{X}$ is a direct summand of a permutation G_k-module, then there is only one isomorphism class of universal torsors since $H^1_{\text{ét}}(k, T) = 0$ (see (5.4)). Thus in this case (7.11) and (7.12) are equivalent by (6.19). If one examines the proof of conjecture (7.11) for toric varieties in [**7**] and [**4**] then it is likely that (7.12) will follow if the Poisson formula is applied to smaller discrete subgroups than in (op. cit.).

All numerical work on Manin's conjecture on (7.6) so far concerns surfaces. For surfaces the intersection pairing

$$\operatorname{Pic}\overline{X} \times \operatorname{Pic}\overline{X} \longrightarrow \mathbb{Z}$$

is perfect and induces a canonical isomorphism

$$\mathrm{Hom}(\mathrm{Pic}\,\overline{X}, \mathbb{Z}) = \mathrm{Pic}\,\overline{X}$$

of G_k-modules. In particular,

$$H^1(G_k, \mathrm{Pic}\,\overline{X}) = 0$$

if X is a surface and $\mathrm{Pic}\,\overline{X}$ is flasque.

The numerical work to date is insufficient to make any conjectures when $\mathrm{Pic}\,\overline{X}$ is not flasque The only work known to the author concerns the two diagonal cubic surfaces of Heath-Brown [**33**] described above.

8. Torsors over toric varieties

We shall in this section study universal torsors $\pi : \mathcal{T} \to X$ over toric varieties X. The aim is to prove that the universal torsors are the toric morphisms described by Cox in [**16**]. We shall in this section use the word complete instead of proper for toric varieties corresponding to complete fans.

Let M be a free finitely generated abelian group of rank $d \geq 1$ and let $N := \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice, with dual pairing denoted by $\langle\ ,\ \rangle$. Let us recall some basic facts on toric varieties and refer to [**24**] and the references there for more background.

Definition 8.1. — A finite set Δ consisting of convex rational polyhedral cones in $N_R = N \otimes \mathbb{R}$ is called a fan if the following conditions are satisfied

(i) Each cone in Δ contains $0 \in N_{\mathbb{R}}$;
(ii) Each face of a cone in Δ is also a cone in Δ;
(iii) The intersection of two cones in Δ is a face of both cones

A fan Δ in N is called complete (resp. regular) if

(a) $N_{\mathbb{R}}$ is the union of cones in Δ resp.
(b) Each cone in Δ is generated by a part of a $\mathbb{Z}$-basis of N.

Each cone σ in Δ determines a finitely generated commutative semigroup:

$$S_\sigma = \check{\sigma} \cap M = \{m \in M : \langle m, n\rangle \geq 0 \text{ for all } n \in \sigma\}.$$

The group ring $\mathbb{Q}[S_\sigma]$ is a finitely generated commutative $\mathbb{Q}$-algebra corresponding to an affine $\mathbb{Q}$-variety $U_\sigma = \mathrm{Spec}\,\mathbb{Q}[S_\sigma]$. If $\sigma = \{0\}$, then $S_\sigma = M$ is an abelian group and $\mathbb{Q}[S_\sigma]$ a Hopf algebra. This provides U_σ with a natural structure of algebraic group when $\sigma = \{0\}$ (cf. [**9**]) and we shall denote this $\mathbb{Q}$-torus by U. It is by definition the $\mathbb{Q}$-torus $\mathbf{D}(M)$ mentioned in section 5 and there is a canonical isomorphism between M and the group $\hat{U}$ of characters of U. We shall in the sequel write $\chi^m : U \to \mathbb{G}_m$ for the character corresponding to $m \in M$.

There is a natural U-action on U_σ for each cone σ in Δ and any inclusion $\rho \subset \sigma$ of cones corresponds to a U-equivariant open $\mathbb{Q}$-immersion $U_\rho \subset U_\sigma$. By gluing these one obtains for each complete regular fan Δ in $N := \mathrm{Hom}(M, \mathbb{Z})$ a smooth complete $\mathbb{Q}$-variety X_Δ containing $U = \mathrm{Spec}\,\mathbb{Q}[M]$ as an open Zariski-dense subvariety. This $\mathbb{Q}$-variety X_Δ is equipped with a $\mathbb{Q}$-morphism $U \times X_\Delta \to X_\Delta$ extending the group multiplication on U.

It is possible to do the whole construction over $\mathbb{Z}$ (cf. [**20**]) and start with the affine schemes $\widetilde{U}_\sigma = \mathrm{Spec}\,\mathbb{Z}[S_\sigma]$. If $\sigma = \{0\}$, then $\sigma = \mathrm{Spec}\,\mathbb{Z}[M]$ and any inclusion $\rho \subset \sigma$ of cones gives rise to an open embedding $\widetilde{U}_\rho \subset \widetilde{U}_\sigma$. We may therefore glue $\widetilde{U}_\sigma$ for all cones $\sigma \in \Delta$ and obtain a scheme $\widetilde{X}_\Delta$ which is smooth (resp. proper) over $\mathbb{Z}$ if Δ is regular (resp. complete). The open affine scheme $\widetilde{U} = \mathrm{Spec}\,\mathbb{Z}[M]$ of $\widetilde{X}_\Delta$ has a canonical structure of a $\mathbb{Z}$-torus which comes from the canonical isomorphism (cf. [**21**, exp. I, 4.4]) between $\mathrm{Spec}\,\mathbb{Z}[M]$ and $\mathbf{D}(M_\mathbb{Z}) = \mathrm{Hom}_{Gr}(M_\mathbb{Z}, \mathbb{G}_m)$ for the constant group scheme (cf. [**45**, p. 52])

$$M_\mathbb{Z} = \coprod_{m \in M} \mathrm{Spec}\,\mathbb{Z}.$$

The group scheme morphism $\widetilde{U} \times_\mathbb{Z} \widetilde{U} \to \widetilde{U}$ extends to a (left) $\widetilde{U}$- action

$$\widetilde{U} \times_\mathbb{Z} \widetilde{X}_\Delta \longrightarrow \widetilde{X}_\Delta.$$

We shall call a 1-dimensional cone a ray. If Δ is complete, then the set $\Delta(1)$ of rays of Δ spans $N_\mathbb{R}$. We shall for a given $\rho \in \Delta(1)$, let n_ρ denote the unique generator of $\rho \cap N$. We shall write $\sigma(1)$ for the set of one-dimensional faces of σ for any cone $\sigma \in \Delta$.

The affine toric variety U_ρ defined by a ray ρ of Δ has two U-orbits. Let D_ρ denote the Zariski closure of the orbit given by the complement of U in U_ρ. This defines a bijection between rays $\rho \in \Delta(1)$ and irreducible U-invariant Weil divisors D_ρ in X_Δ. The free abelian group of U-invariant Weil divisors will be denoted by $\mathbb{Z}^{\Delta(1)}$.

Any U-invariant Cartier divisor on the affine toric variety U_σ, $\sigma \in \Delta$ is represented by a character $\chi^{m(\sigma)} : U \to \mathbb{G}_m$, $m(\sigma) \in M$ which is unique up to an element in $M(\sigma) = \sigma^\perp \cap M$. This defines a canonical isomorphism between the group $\mathrm{Div}_U(X)$ of U-invariant Cartier divisor and $\lim_{\leftarrow} M/M(\sigma)$. If we order the maximal cones σ_i, $1 \leq i \leq s$ then the latter group is equal to (cf. [**24**, 3.3])

$$\ker(\oplus_i M/M(\sigma_i) \longrightarrow \oplus_{i<j} M/M(\sigma_i \cap \sigma_j)).$$

If Δ is complete, then all maximal cones of Δ are of the same dimension as $N_\mathbb{R}$. Hence $m(\sigma)$ is unique for each maximal cone σ of Δ in this case.

There is a commutative diagram with exact sequences [**24**, 3.4]

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Q}[U]^*/\mathbb{Q}^* \arrow[r] \arrow[d, "\wr"] & \mathrm{Div}_U(X) \arrow[r] \arrow[d, "\wr"] & \mathrm{Pic}\, X \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathbb{Z}^{\Delta(1)} \arrow[r] & \mathrm{CH}^1(X) \arrow[r] & 0
\end{tikzcd}
\tag{8.2}
$$

The maps are defined as follows. The map from $\mathbb{Q}[U]^*/\mathbb{Q}^*$ to $\mathrm{Div}_U(X)$ is defined by representing a class in $\mathbb{Q}[U]^*/\mathbb{Q}^*$ by the unique character χ^m belonging to this class.

The map from $\mathrm{Div}_U(X) \to \mathrm{Pic}\, X$ is the standard map (cf. [**32**, 2.6]) sending a U-invariant Cartier divisor represented by $\chi^{m(\sigma)} : U \to \mathbb{G}_m$ on U_σ, $\sigma \in \Delta$ to the subsheaf of the constant sheaf $k(X)^*$ on X generated by $\chi^{m(\sigma)}$. We shall denote this invertible sheaf by $\mathcal{O}(D)$ where D is the associated Weil divisor of the second vertical map. If D is effective, then $\mathcal{O}(D)$ is the inverse of the ideal sheaf of D.

The map from M to $\mathbb{Z}^{\Delta(1)}$ is defined by sending $m \in M$ to $\sum_{\rho\in\Delta(1)} \langle m, n_\rho\rangle D_\rho$ and the map from $\mathbb{Z}^{\Delta(1)}$ to $\mathrm{CH}^1(X)$ sends a divisor to its linear equivalence class.

The first vertical map is defined by sending $\chi^m \in \mathbb{Q}[U]^*/\mathbb{Q}^*$ to $m \in M$.

The second vertical map sends a U-invariant Cartier divisor defined by characters $\chi^{m(\sigma)} : U \to \mathbb{G}_m$ on U_σ, $\sigma \in \Delta$ to the Weil divisor $\Delta = \sum_{\rho\in\Delta(1)} a_\rho D_\rho$ where $a_\rho = \langle m(\sigma), n_\rho\rangle$ for any cone $\sigma \in \Delta$ for which $\rho \in \sigma(1)$. (This map is well-defined since $\langle m(\sigma), n_\rho\rangle$ is independent of the cone σ for which $\rho \in \sigma(1)$.) The map is the same as the "usual" map from Weil divisors to Cartier divisors [**32**, 2.6] since the order of vanishing of the Cartier divisor along D_ρ is equal to $\langle m(\sigma), n_\rho\rangle$ by [**24**, 3.3]. Any toric variety X_Δ is normal, so the map from $\mathrm{Div}_U(X)$ to $\mathbb{Z}^{\Delta(1)}$ is injective. It is also surjective when Δ is regular.

The third vertical map is defined by the exactness of the rows and the commutativity of the first square.

Definition 8.3

(a) Let $\phi : N' \to N$ be a homomorphism of lattices and Δ be a fan in N, Δ' be a fan in N' satisfying the condition:
For each cone σ' in Δ', there is some cone σ in Δ such that $\phi(\sigma') \subseteq \sigma$.
Then ϕ is called a *morphism of fans*.

(b) Let $\pi : X' \to X$ be a morphism of toric varieties and let $U' \to U$ be the corresponding homomorphism of tori obtained by restricting π. Then π is said to be a *toric morphism* if π is equivariant with respect to the toric actions of U' and U. This means that the following diagram commutes

$$\begin{array}{ccc} U' \times X' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ U \times X & \longrightarrow & X \end{array}$$

It is easy to see that any homomorphism $\phi : N' \to N$ of lattices with $\phi(\sigma') \subseteq \sigma$ for two rational strongly convex polyhedral cones $\sigma' \subset N'_{\mathbb{R}}$ and $\sigma \subset N_{\mathbb{R}}$ determines a toric morphism between affine toric varieties $U_{\sigma'} \to U_{\sigma}$. It follows by gluing (cf. [**24**, p. 23]) that any morphism $(N', \Delta') \to (N, \Delta)$ of fans gives rise to a toric morphism from $X' = X_{\Delta'}$ to $X = X_{\Delta}$.

There is also a notion of toric morphism for toric schemes defined just as in (8.3)(b) and it is clear that the proof in (op. cit.) implies that any morphism of fans defines not only a toric morphism between toric varieties but also a toric morphism between toric schemes over $\mathbb{Z}$.

We now consider torsors over $X = X_{\Delta}$ for a regular complete fan Δ in $N := \mathrm{Hom}(M, \mathbb{Z})$. We shall by $D_{\rho}^{\vee}$, $\rho \in \Delta(1)$ denote the basis of $\mathrm{Hom}(\mathbb{Z}^{\Delta(1)}, \mathbb{Z})$ which is dual to the base D_{ρ}, $\rho \in \Delta(1)$ of irreducible Weil divisors in $\mathbb{Z}^{\Delta(1)}$. Note that $D_{\rho}^{\vee}$ is sent to n_{ρ} under the map from $\mathrm{Hom}(\mathbb{Z}^{\Delta(1)}, \mathbb{Z})$ to $N = \mathrm{Hom}(M, \mathbb{Z})$ induced by the inclusion $M \subseteq \mathbb{Z}^{\Delta(1)}$ in (8.2).

Proposition 8.4. — *Let $M_0 \subseteq \mathbb{Z}^{\Delta(1)}$ be a sublattice containing M and let $N_0 = \mathrm{Hom}(M_0, \mathbb{Z})$. Let $n_{0,\rho} \in N_0$, $\rho \in \Delta(1)$ be the image of $D_{\rho}^{\vee}$ under the restriction map $\mathrm{Hom}(\mathbb{Z}^{\Delta(1)}, \mathbb{Z}) \to N_0$. Moreover, if $\sigma \in \Delta$, let σ_0 be the cone in $N_{0,\mathbb{R}}$ generated by $n_{0,\rho}$ for all one-dimensional faces ρ of σ. Then the following holds.*

(a) *The set of all these cones $\sigma_0 \subseteq N_{0,\mathbb{R}}$ form a regular fan Δ_0 of $N_{0,\mathbb{R}}$ and any $\sigma_0 \in \Delta_0$ is sent isomorphically onto the cone $\sigma \in \Delta$ defining it under the restriction map from $\mathrm{Hom}(M_0, \mathbb{R})$ to $N = \mathrm{Hom}(M, \mathbb{R})$.*

(b) *Let $U_{0,\sigma} := \mathrm{Spec}[S_{\sigma_0}]$, $S_{\sigma_0} = \check{\sigma_0} \cap M_0$ be the affine toric variety defined by $\sigma_0 \in \Delta_0$ and let $\pi_{\sigma} : U_{0,\sigma} \to U_{\sigma}$ be the toric morphism defined by the map from σ_0 to σ. Then these toric morphisms glue to a toric morphism $\pi : X_0 \to X$ from the toric variety X_0 defined by (Δ_0, N_0) to the toric variety X defined by (Δ, N). This morphism is a torsor under the the torus $\mathbf{D}(M/M_0)$.*

Proof

(a) This is a consequence of the fact that each cone $\sigma \in \Delta$ is generated by its one-dimensional faces.

(b) The last assertion is essentially a special case of the exercise on p. 41 in [**24**]. See also [**46**, §1.5]. □

Proposition 8.5. — *Let Δ be a complete regular fan in $N := \mathrm{Hom}(M, \mathbb{Z})$ and let (Δ_0, N_0) be the fan defined in (8.4) by a sublattice $M_0 \subseteq \mathbb{Z}^{\Delta(1)}$. Let P_0 be the image of M_0 in $\mathrm{Pic}\, X$ and let $S_0 = \mathbf{D}(P_0)$ be the dual $\mathbb{Q}$-torus. Then the corresponding morphism $X_0 \to X$ of toric varieties is an torsor under S_0 of type $P_0 \subseteq \mathrm{Pic}\, X$. In particular, if $M_0 = \mathbb{Z}^{\Delta(1)}$, then $X_0 \to X$ is a universal torsor.*

Proof. — The type of an S_0-torsor over X is uniquely determined by the types of the $\mathbb{G}_m$-torsors induced by the characters $S_0 \to \mathbb{G}_m$ of S_0. Using this and the functoriality of the construction of X_0 under restriction of the lattice M_0 to smaller lattices containing M, one reduces to the case $S_0 = \mathbb{G}_m$. Now note that by the completeness of Δ one may extend any group embedding $n : \mathbb{G}_m \to U$ to an equivariant closed embedding $\mathbb{P}^1 \to X$ of toric varieties. Since the one-parameter subgroups corresponds bijectively to elements of $N := \mathrm{Hom}(M, \mathbb{Z})$ it follows that a class in $\mathrm{Pic}\, X$ is determined by its restriction to these closed subschemes.

It may happen that P_0 is sent to zero under some of these restrictions, in which case the pullback of the $\mathbb{G}_m$-torsor is trivial for trivial reasons. It this is not the case, then we reduce to the case when M is of rank 1 and $X = \mathbb{P}^1_{\mathbb{Q}}$ and $P_0 = \mathrm{Pic}\, X$. Then (Δ_0, N_0) simply defines the affine cone of $\mathbb{P}^1_{\mathbb{Q}}$ and it is known and easy to prove that the affine cone of $\mathbb{P}^1_{\mathbb{Q}}$ is a universal torsor. This completes the proof. □

Remarks 8.6

(a) One can also give a proof of (8.5) based on purity and look at the restriction of the torsor over a toric variety defined by the cones in Δ of dimension ≤ 1.

(b) There is a version of (8.5) for smooth proper toric $\mathbb{Z}$-schemes $\widetilde{X}_\Delta$ obtained by gluing the affine schemes $\widetilde{U}_\sigma = \mathrm{Spec}\, \mathbb{Z}[S_\sigma]$ for the complete regular fan Δ in N. In particular, one can construct a universal torsor $\widetilde{X}_0 \to \widetilde{X}$ (cf. (5.14)) which extends the homomorphism of $\mathbb{Z}$-tori $\widetilde{U}_0 = \mathbf{D}(M_{0,\mathbb{Z}}) \to \widetilde{U} = \mathbf{D}(M_{\mathbb{Z}})$ associated to the monomorphism $M \subseteq M_0 = \mathbb{Z}^{\Delta(1)}$. We leave the details to the reader since the statement and the proof is almost identical to (8.5). One can now make base extensions and obtain versions of (8.5) for toric schemes over arbitrary base schemes B. One can e.g. choose $B = \mathrm{Spec}\, \mathbb{Z}/p\mathbb{Z}$ and consider toric varieties over $\mathbb{Z}/p\mathbb{Z}$.

We shall in the sequel fix an arbitrary field k and let $X = X_\Delta$ be the k-variety determined by the complete regular fan (N, Δ).

Let $M_0 = \mathbb{Z}^{\Delta(1)}$ in (8.4). The fan Δ_0 and the morphism $(N_0, \Delta_0) \to (N, \Delta)$ of fans described there defines a toric variety X_0 and an equivariant morphism

$\pi : X_0 \to X$ which according to (8.5) is a universal torsor. We shall call this universal torsor $\pi : X_0 \to X$ the *principal* universal X-torsor.

Let $N_1 = N_0$ and let Δ_1 be the fan in N_1 consisting of all cones generated by $n_{0,\rho} = D_\rho^\vee$, for ρ belonging to *any* subset of $\Delta(1)$. Then the toric variety X_1 determined by (N_1, Δ_1) is the affine n-space $\mathbb{A}^n$, $n = \operatorname{Card}\Delta(1)$ and the morphism of fans $(N_0, \Delta_0) \to (N_1, \Delta_1)$ defines an open equivariant embedding $X_0 \subseteq X_1$.

We now give a more concrete description of this embedding following Cox [**16**]. We introduce one variable x_ρ for each $\rho \in \Delta(1)$ and extend this to a bijection between monomials

$$x^D = \prod_{\rho\in\Delta(1)} x_\rho^{a_\rho}, \quad \rho \in \Delta(1)$$

and effective Weil divisors

$$D = \sum_{\rho\in\Delta(1)} a_\rho D_\rho \in \mathbb{Z}^{\Delta(1)}$$

with support outside U. For a cone $\sigma \in \Delta$, let $\underline{\sigma}$ be the divisor

$$\underline{\sigma} = \sum_{\rho\notin\sigma(1)} D_\rho.$$

Then $U_{0,\sigma}$ (cf. (8.4)(b)) is the open subvariety of $X_1 = \operatorname{Spec} k[x_\rho]$, $\rho \in \Delta(1)$ for which $x^{\underline{\sigma}} \neq 0$.

The open affine toric subvarieties $U_{0,\sigma}$, $\sigma \in \Delta_{\max}$ form a covering of X_0. Hence X_0 is the open subvariety in $X_1 = \operatorname{Spec} k[x_\rho]$, $\rho \in \Delta(1)$ for which not all the monomials $x^{\underline{\sigma}}$, $\sigma \in \Delta_{\max}$ vanish.

Proposition 8.7. — *Let Δ be a complete regular fan and let $\Delta_0 = \sum_{\rho\in\Delta(1)} D_\rho$. Let σ be a maximal cone of Δ and let $\chi^{m(\sigma)}$, $m(\sigma) \in M$ be the unique character of U such that $\chi^{-m(\sigma)}$ generates $\mathcal{O}(D_0)$ on U_σ. Let*

$$D(\sigma) = D_0 + \sum_{\rho\in\Delta(1)} \langle -m(\sigma), n_\rho\rangle D_\rho.$$

Then the following holds for any maximal cone σ of Δ.

(a) *If $\mathcal{O}(D_0)$ is generated by its global sections, then $\chi^{-m(\sigma)}$ is a global section of $\mathcal{O}(D_0)$ and $D(\sigma)$ is an effective divisor with support contained in $\cup_{\rho\notin\sigma(1)} D_\rho$.*

(b) *If $\mathcal{O}(D_0)$ is ample, then $\mathcal{O}(D_0)$ is very ample and $D(\sigma)$ an effective divisor with support $\cup_{\rho\notin\sigma(1)} D_\rho$.*

Proof

(a) If $\mathcal{O}(D_0)$ is generated by its global sections, then there exists for each $\sigma \in \Delta$ a global section of $\mathcal{O}(D_0)$ which generates $\mathcal{O}(D_0)$ on U_σ. But U_σ is an affine space for maximal cones σ in a complete fan Δ. There is thus up to multiplication with an element in k^* only one local section which generates $\mathcal{O}(D_0)$ on U_σ for $\sigma \in \Delta_{\max}$. This implies that for maximal cones σ, $\chi^{-m(\sigma)}$ is not only a local section on U_σ but a global section of $\mathcal{O}(D_0)$.

The Weil divisor of the rational function $\chi^{-m(\sigma)}$ is $\sum_{\rho\in\Delta(1)}\langle -m(\sigma), n_\rho\rangle D_\rho$ (cf. [**24**, p. 61]). Therefore, since $\chi^{-m(\sigma)}$ is a global section of $\mathcal{O}(D_0)$ we must have that $\langle -m(\sigma), n_\rho\rangle \geq -1$ for all rays ρ of Δ [**24**, p. 68]. Also, $\langle -m(\sigma), n_\rho\rangle = -1$ for $\rho \in \sigma(1)$, since $\chi^{-m(\sigma)}$ generates the fractional ideal $\mathcal{O}(D_0)$ on U_ρ so that $\langle -m(\sigma), n_\rho\rangle = -1$. Hence $\Delta(\sigma)$ is effective with support contained in $\cup_{\rho\notin\sigma(1)} D_\rho$.

(b) The very ampleness of $\mathcal{O}(D_0)$ is part of a more general result of Demazure [**24**, p. 71]. Also, if D_0 is ample, then the function $\psi : N_\mathbb{R} \to \mathbb{R}$ defined by $\psi(n) = \langle -m(\sigma), n\rangle$ is strictly convex [**24**, p. 70] and $\langle -m(\sigma), n_\rho\rangle > -1$ for all $\rho \notin \sigma(1)$. This completes the proof. $\square$

Corollary 8.8. — *Let Δ be a complete regular fan such that $D_0 = \sum_{\rho\in\Delta(1)} D_\rho$ is ample. Let $D(\sigma)$, $\sigma \in \Delta_{\max}$ be the U-invariant Weil divisors on X described in (8.7).*

(a) *If $\mathcal{O}(D_0)$ is generated by its global sections and σ a maximal cone of Δ, then $x^{D(\sigma)} \neq 0$ on $U_{0,\sigma} \subseteq X_0$ (cf. (8.4)).*
(b) *If $\mathcal{O}(D_0)$ is ample and σ a maximal cone of Δ, then $U_{0,\sigma}$ is the open subset of $X_1 = \operatorname{Spec} k[x_\rho]$, $\rho \in \Delta(1)$ defined by $x^{D(\sigma)} \neq 0$. Hence X_0 is the open subvariety in $X_1 = \operatorname{Spec} k[x_\rho]$, $\rho \in \Delta(1)$ for which not all $x^{D(\sigma)}$, $\sigma \in \Delta_{\max}$ vanish.*

Proof. — This follows from (8.7) and the description of $U_{0,\sigma} \subset X_1$ by means of $x^{\underline{\sigma}}$. $\square$

The following lemma will be used to prove (9.10) and in the proof of the asymptotic formulas in section 11. Recall that a facet τ of a cone σ is a face of codimension one (cf. [**24**, Ch. 1]).

Lemma 8.9. — *Let Δ be a complete regular d-dimensional fan and let $\sigma^{(0)}$ be a maximal (and hence d-dimensional) cone of Δ with facets $\tau^{(1)}, \ldots, \tau^{(d)}$. Then there exist unique d-dimensional cones $\sigma^{(1)}, \ldots, \sigma^{(d)}$ such that $\tau^{(i)} = \sigma^{(0)} \cap \sigma^{(i)}$ for $i = 1, \ldots, d$.*

Moreover, for any such set of cones the following holds:

(i) There exists exactly one 1-dimensional face $\rho^{(i)}$ of $\sigma^{(0)}$ such that $\rho^{(i)} \cap \sigma^{(i)} = \{0\}$ for each $i = 1, \ldots, d$. Moreover, any one-dimensional face of $\sigma^{(0)}$ is equal to $\rho^{(i)}$ for exactly one integer $i = 1, \ldots, d$.

(ii) *Let $\{n^{(i)} : 1 \leq i \leq d\}$ be the $\mathbb{Z}$-basis of N defined by the generators of the rays $\rho^{(1)}, \ldots, \rho^{(d)} \in \sigma(1)$ and let $\{m^{(i)} : 1 \leq i \leq d\}$ be the dual $\mathbb{Z}$-basis of M. Let b_i be the multiplicity of D_i in $D(\sigma^{(i)})$, $1 \leq i \leq d$. Then,*

$$m(\sigma^{(i)}) - m(\sigma^{(0)}) = b_i m^{(i)}, \quad i \in \{1, \ldots, d\}.$$

(iii) *Let $m = m^{(1)} + \cdots + m^{(d)}$ and let $D_0 = \sum_{\rho \in \Delta(1)} D_\rho$. Then,*

$$D_0 - D(\sigma^{(0)}) = \sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle D_\rho.$$

(iv) *Suppose that $D_0 = \sum_{\rho \in \Delta(1)} D_\rho$ is ample. Then $b_i > 0$ for $i \in \{1, \ldots, d\}$.*

(v) *Suppose that there is only one 1-dimensional cone $n^{(0)} \in \Delta$ not contained in $\sigma(0)$. Then $n^{(0)} + n^{(1)} + \cdots + n^{(d)} = 0$.*

Proof. — Let $\tau \in \Delta$ be a cone of dimension $d-1$ and let H_τ be the hyperplane generated by τ and $-\tau$. Then τ is part of the boundary of any d-dimensional cone $\sigma \in \Delta$ containing τ [**24**, p. 10]. Hence any such cone $\sigma \in \Delta$ must lie in one of the closed half spaces of $N_{\mathbb{R}}$ defined by H_τ. Let R be one of these closed half spaces and let $\Omega \subset R$ be the open subset of interior points of R which do not lie on any cone of dimension $< d$. Then Ω is contained in the union $\sigma_1 \cup \cdots \cup \sigma_s$ of the maximal cones $\sigma_1, \ldots, \sigma_s$ with a point in Ω by completeness of Δ. Hence the closure R of Ω is also contained in the closed subset $\sigma_1 \cup \cdots \cup \sigma_s$ so that $\tau = \tau \cap (\sigma_1 \cup \cdots \cup \sigma_s)$ is a finite union of cones $(\tau \cap \sigma_1) \cup \cdots \cup (\tau \cap \sigma_s)$. Therefore, $\tau = \tau \cap \sigma$ for some d-dimensional cone $\sigma \in \Delta$ with $\sigma \subset R$. This cone is clearly unique since any such cone σ must contain $O \cap R$ for any sufficiently small neighbourhood O around an interior point of τ. There are thus exactly two d-dimensional cones $\sigma \in \Delta$ containing τ.

The statement (i) is a trivial consequence of the regularity assumption on Δ. To prove (ii), let $m(\sigma) \in M$ be the unique element such that the character $\chi^{-m(\sigma)}$ of U generates $\mathcal{O}(D_0)$ on U_σ. Then (cf. (8.7)):

$$D(\sigma) := \sum_{\rho \in \Delta(1)} (1 + \langle -m(\sigma), n_\rho \rangle) D_\rho$$

where $\langle -m(\sigma), n_\rho \rangle = -1$ for any maximal cone $\sigma \in \Delta$ and any ray ρ of σ.

Hence

$$\langle m(\sigma^{(0)}) - m(\sigma^{(i)}), n^{(i)} \rangle = b_i$$

and

$$\langle m(\sigma^{(0)}) - m(\sigma^{(i)}), n^{(j)} \rangle = 0$$

for the generators $n^{(j)}$, $j \in \{1, \ldots, d\}, j \neq i$ of the rays of $\mathcal{T}^{(i)}$. This proves (ii).

To show (iii), note that the multiplicity of the D_i in

$$D_0 - D(\sigma^{(0)}) = \sum_{\rho \in \Delta(1)} \langle m(\sigma^{(0)}), n_\rho \rangle D_\rho$$

and

$$\sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle D_\rho$$

is equal to 1 for $1 \leq i \leq d$. But then $\langle m(\sigma^{(0)}), n_\rho \rangle = \langle m, n_\rho \rangle$ for all rays ρ of Δ, since $\{n^{(i)} : 1 \leq i \leq d\}$ is a $\mathbb{Z}$-basis of N.

To show (iv), use the fact that $1 + \langle -m(\sigma), n_\rho \rangle > 0$ for any maximal cone $\sigma \in \Delta$ and any ray ρ of σ if D_0 is ample.

To show (v), note that any subset of d elements in $\{n^{(0)}, n^{(1)}, \ldots, n^{(d)}\}$ form a $\mathbb{Z}$-basis of N by the regularity of the maximal cones $\{\sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(d)}\}$. Hence we have $\langle m^{(i)}, n^{(0)} \rangle = \pm 1$ for $1 \leq i \leq d$. Also, since $\sigma^{(i)} \neq \sigma^{(0)}$, we must have $\langle m^{(i)}, n^{(0)} \rangle \neq 1$ for $1 \leq i \leq d$. This completes the proof. □

Proposition 8.10. — *Let $D_0 = \sum_{\rho \in \Delta(1)} D_\rho$ and let $\omega_{X/k}$ be the canonical sheaf. Let du be a non-vanishing U-invariant section of $\omega_{X/k}$ on U. Then there exists a unique extension of du to a global section of $\omega_{X/k} \otimes \mathcal{O}(D_0)$. This global section generates $\omega_{X/k} \otimes \mathcal{O}(D_0)$.*

Proof. — This is proved in [**24**, 4.3] when du is of minimal $d\log$-type (cf. (3.28)). If $d\nu$ is another U-invariant section, then $d\nu/du$ is a U-invariant regular function on U and hence $d\nu = \alpha du$ for some $\alpha \in k$. This proves the assertion. □

We end this section with some comments on twisted toric varieties.

Suppose that $M_0 = \mathbb{Z}^{\Delta(1)}$ and let G be a finite group of automorphisms of (Δ, N). Then G acts also on the fan (Δ_0, N_0) so that we get a morphism of G-fans from (Δ_0, N_0) to (D, N). This implies that the corresponding morphism $\pi : X_0 \to X$ is a G-equivariant morphism between toric varieties with G-actions.

An important case is when G is the Galois group of a finite Galois extension $k \subset K$ of fields. If we regard $X = X_\Delta$ as a variety over the base field K, then the G-actions on (Δ, N) and K define a G-action on X which is compatible with the G-action on K. This may be interpreted as a descent datum (see [**10**, pp. 139-141]). The descent datum is effective if and only if each Galois orbit is contained in a quasi-affine variety (e.g. if X is projective). It then follows from results of Weil and Grothendieck (cf. op. cit) that there is a k-variety X^G with a G-equivariant K-isomorphism λ between $X^G \times_k K$ and X. The G-action on $X^G \times_k K$ is induced by the Galois action on K. The pair (X^G, λ) is unique up to unique isomorphism.

There is also a G-action of X_0 which defines a k-variety X_0^G and an isomorphism λ_0 between $X_0^G \times_k K$ and X_0. The corresponding descent datum is always effective since X_0 is quasi-affine. By Galois descent there exists further a unique k-morphism $\pi^G : X_0^G \to X^G$ such that the induced K-morphism $\pi^G \times K : X_0^G \times_k K \to X^G \times_k K$ belongs to the following commutative diagram of G-morphisms

$$
\begin{tikzcd}
X_0^G \times_k K \arrow[r, "\pi^G \times K"] \arrow[d, "\lambda_0"', "\wr"] & X^G \times_k K \arrow[d, "\lambda", "\wr"'] \\
X_0 \arrow[r, "\pi"] & X
\end{tikzcd}
\tag{8.11}
$$

Let $\pi_U : U_0 \to U$ be the restriction of π corresponding to the morphism of fans from $(\{0\}, N_0)$ to $(\{0\}, N)$. Then π^G restricts to a homomorphism $(\pi_U)^G : U_0^G \to U^G$ of k-tori which may also be obtained directly from the G-morphism of fans from $(\{0\}, N_0)$ to $(\{0\}, N)$. This morphism is nothing but the map from $\mathbf{D}(M_0) = U_0^G$ to $\mathbf{D}(M) = U^G$ dual to the G-monomorphism from $M = \ker(\mathbb{Z}^{\Delta(1)} \to \mathrm{CH}^1(X))$ to $M_0 = \mathbb{Z}^{\Delta(1)}$. The kernel of $(\pi_U)^G$ is thus the Néron-Severi k-torus T^G dual to the G-module $\mathrm{Pic}\, X = \mathrm{CH}^1(X)$.

The k-torus U_0^G (resp. U^G) acts on X_0^G (resp. X^G) so that X_0^G and X^G become twisted toric varieties. The k-morphism $\pi^G : X_0^G \to X^G$ is equivariant under these torus actions. It is clear from the G-equivariant commutative diagram (8.11) that $\pi^G : X_0^G \to X^G$ is a torsor under T^G. The type of this torsor is uniquely determined by the type of the torsor after a base extension. It must therefore be a universal torsor since $\pi^G : X_0^G \to X^G$ is a universal torsor. The fibre of this torsor at the neutral element e of $U^G \subset X^G$ is trivial since it contains the neutral element e_0 of U_0^G. We have thus (cf. (5.13)) determined the isomorphism class of the universal torsor $\pi^G : X_0^G \longrightarrow X^G$. We shall call this universal torsor *the principal* universal torsor.

9. Norms on toric varieties over local fields

Let $X = X_\Delta$ be a smooth complete toric variety over a locally compact field. There is a natural norm $\| \ \|_D$ for each U-invariant Weil divisor D on X described in [**7**]. We shall in this section give a new interpretation of these norms by means of a "canonical toric splitting" for the principal universal torsor $X_0 \to X$. This interpretation may be seen as an analog of Bloch's (cf. [**8**], [**48**]) approach to local Néron heights for abelian varieties. We shall also show that the induced norm on X_0 of $\| \ \|_D$ for the anticanonical system is very natural and simpler than $\| \ \|_D$.

We shall keep all the notations in section 8. We shall thus by X denote the smooth complete k-variety defined by a regular complete fan Δ in $N =$

$\mathrm{Hom}(M, \mathbb{Z})$. The only difference is that k will denote a non-discrete *locally compact field* of characteristic 0 throughout the section.

Let $|\ | : k^* \to \mathbb{R}_{>0}$ be the normalized absolute value defined in section 1. The additive valuation $\log |\ | : k^* \to \mathbb{R}$ induces a homomorphism

$$L : U(k) = \mathrm{Hom}(M, k^*) \longrightarrow N_{\mathbb{R}} = \mathrm{Hom}(M, \mathbb{R})$$

from the multiplicative group $U(k)$ to the additive group $N_{\mathbb{R}}$. If $\sigma \in \Delta$ is a cone, then $L^{-1}(-\sigma)$ is a closed subset of $U(k)$ in the k-topology.

Notation 9.1. — Let $\sigma \in \Delta$. Then $C_\sigma(k)$ is the closure of $L^{-1}(-\sigma)$ in $X(k)$.

Batyrev and Tschinkel [**7**] use the compact subsets $C_\sigma(k)$, $\sigma \in \Delta$ to define a norm $\|\ \|_D$ on the line bundle $\mathcal{O}(D)$ for any U-invariant Weil divisor D. We now give a slightly different, but equivalent, definition of their norm.

Proposition and definition 9.2. — *Let D be a U-invariant Weil divisor on X and let s be a local analytic section of $\mathcal{O}(D)$ defined at $P \in X(k)$. Then any $P \in X(k)$ belongs to $C_\sigma(k)$ for some cone $\sigma \in \Delta$. Let $\chi^{m(\sigma)}$ be a character which on U_σ represents the Cartier divisor with Weil divisor D. Then the expression*

$$\|s(P)\|_D := |s(P)\chi^{m(\sigma)}(P)|$$

is independent of the choice of cone σ with $P \in C_\sigma(k)$. It defines a norm in the sense of (1.5) on the analytic line bundle $V(\mathcal{O}(-D))_{\mathrm{an}}(k) \to X_{\mathrm{an}}(k)$ of sections of $\mathcal{O}(D)$. If D is effective, then

$$|\chi^{m(\sigma)}(P)| \leq 1$$

and

$$\|s(P)\|_D \leq |s(P)|.$$

Proof. — It suffices to prove the first two statements for the dense subset $U(k)$. To show the first, use the completeness of Δ. For the second, note that

$$|\chi^{m(\sigma)}(P)| = \exp\langle m(\sigma), L(P)\rangle$$

for $P \in U(k)$. If $P \in C_\sigma(k) \cap C_\tau(k)$ for two cones σ and τ, then there are unique non-negative real numbers λ_ρ, $\rho \in (\sigma \cap \tau)(1)$ such that

$$-L(P) = \sum_{\rho \in (\sigma\cap\tau)(1)} \lambda_\rho n_\rho.$$

Therefore $|\chi^{m(\sigma)}(P)|$ and $|\chi^{m(\tau)}(P)|$ are equal since $\langle m(\sigma), n_\rho\rangle = \langle m(\tau), n_\rho\rangle$ is the multiplicity of D along D_ρ for each ray (cf. (8.2)). The third statement is obvious since $|s(P)\chi^{m(\sigma)}(P)|$ is a norm for the line bundle over $C_\sigma(k)$ for each cone σ. To prove the inequality, note that $\langle m(\sigma), L(P)\rangle \leq 0$ since $\langle m(\sigma), n_\rho\rangle \geq 0$ for each $\rho \in \sigma(1)$. This completes the proof. $\square$

Let T be the Néron-Severi torus of X and let $\pi : \mathcal{T} \to X$, $\mathcal{T} = X_0$ be the principal universal X-torsor constructed from the fan (N_0, Δ_0) (see (8.5)). There is an analytic torsor

$$\pi_{\text{an}} : \mathcal{T}_{\text{an}}(k) \longrightarrow X_{\text{an}}(k)$$

under $T_{\text{an}}(k)$ associated to π. In particular, we may identify the topological space $X(k)$ with the quotient space $X_0(k)/T(k)$ (cf. (3.11)).

Notation 9.3

(a) $T(k)_{\text{cp}}$ (resp. $U_0(k)_{\text{cp}}$) is the maximal compact subgroup of the locally compact group $T(k)$ (resp. $U_0(k)$).
(b) $\breve{\pi} : X_0(k)/T(k)_{\text{cp}} \to X(k)$ is the unique continuous map sending the $T(k)_{\text{cp}}$-orbit of a point $Q \in \mathcal{T}(k)$ to $\pi(Q)$.

The existence of the norms in (9.2) is related to the existence of a canonical splitting of $\breve{\pi} : X_0(k)/T(k)_{\text{cp}} \to X(k)$. To construct such a splitting, we first define local splittings of π_{an}.

Lemma 9.4. — *Let σ be a maximal cone of Δ and let $X_1 = \operatorname{Spec} k[x_\rho]$, $\rho \in \Delta(1)$. If D_ρ is a U-invariant Weil divisor on X corresponding to a ray $\rho \in \Delta(1)$, let $\chi_\rho^{m(\sigma)}$, $\sigma \in \Delta_{\max}$ be the unique character on U which generates $\mathcal{O}(-D_\rho)$ on U_σ. Let $\psi_\sigma : U_\sigma \to X_1$ be the affine k-morphism such that $\psi_\sigma^* x_\rho = \chi_\rho^{m(\sigma)}$, $\rho \in \Delta(1)$ and let $\pi_\sigma : U_{0,\sigma} \to U_\sigma$ be the open affine toric morphism described in (8.4)(b).*

Then, $\psi_\sigma(U_\sigma) \subseteq U_{0,\sigma}$ and $\pi_\sigma \circ \psi_\sigma : U_\sigma \to U_\sigma$ is the identity morphism.

Proof. — $\chi_\rho^{m(\sigma)}$ generates $\mathcal{O}(-D_\rho)$ on U_σ. Therefore, $x_\rho \neq 0$ on U_σ for all $\rho \notin \sigma(1)$. This proves the first statement since $U_{0,\sigma}$ is the open subset of X_1 such that $x_\rho \neq 0$ for all rays $\rho \in \Delta(1)$ not in σ.

The ring $k[U_\sigma]$ of regular functions is generated by characters $\chi^m \in \hat{U}$ for $m \in M$ in the dual cone $\sigma^\vee$ of σ. Let

$$D = \sum_{\rho \in \Delta(1)} a_\rho D_\rho \in \mathbb{Z}^{\Delta(1)}, \quad a_\rho = \langle m, n_\rho \rangle$$

be the corresponding Weil divisor (cf. (8.2)) and let

$$\begin{aligned} \pi_\sigma^{\#} : k[U_\sigma] &\longrightarrow k[U_{0,\sigma}], \\ \psi_\sigma^{\#} : k[U_{0,\sigma}] &\longrightarrow k[U_\sigma] \end{aligned}$$

be the homomorphisms defined by π_σ and ψ_σ. Then,

$$\pi_\sigma^{\#}(\chi^m) = x^D := \prod_{\rho \in \Delta(1)} x_\rho^{a_\rho}$$

and
$$\psi_\sigma^\#(x^D) = \prod_{\rho\in\Delta(1)} \chi_\rho^{m(\sigma)a_\rho} = \chi^m.$$
This completes the proof. □

Proposition 9.5

(a) *There exists a unique continuous map $\check{\psi} : X(k) \to X_0(k)/T(k)_{\mathrm{cp}}$ such that $\check{\psi}(P)$ is the class of $\psi_\sigma(P) \in X_0(k)$ in $X_0(k)/T(k)_{\mathrm{cp}}$ for $P \in C_\sigma(k)$, $\sigma \in \Delta_{\max}$.*

(b) *The map $\check{\psi}$ is a section of the continuous map $\breve{\pi} : X_0(k)/T(k)_{\mathrm{cp}} \to X(k)$.*

Proof

(a) Suppose that $P \in C_\sigma(k) \cap C_\tau(k)$ for two maximal cones σ, τ of Δ. It then follows from the proof of (9.2) that $|\chi_\rho^{m(\sigma)}(P)| = |\chi_\rho^{m(\tau)}(P)|$ for each ray ρ of Δ. This is equivalent to the existence of an element $u \in U_0(k)_{\mathrm{cp}}$ such that $\psi_\sigma(P) = u\psi_\tau(P)$ under the toric action of $U_0(k)$ on $X_0(k)$. But $\pi(\psi_\sigma(P)) = \pi(\psi_\tau(P)) = P$ by the previous lemma. Therefore, since π is a toric morphism, $P = \pi(u)P$ for the toric action of $U(k)$ on $X(k)$. Hence $\pi(u) = 1$ in the group $U(k)$ and u an element of $T(k) = \ker(U_0(k) \to U(k))$. But then $u \in T(k) \cap U_0(k)_{\mathrm{cp}} = T(k)_{\mathrm{cp}}$ so that $\psi_\sigma(P) = \psi_\tau(P)$ in $\mathcal{T}(k)/T(k)_{\mathrm{cp}}$. This proves that $\check{\psi}$ is well defined.

The continuity of $\check{\psi}$ is immediate from the continuity of $\psi_{\sigma,\mathrm{an}}$ for each maximal cone σ of Δ.

(b) This follows from the corresponding property of ψ_σ (cf. (9.4)). □

We shall in the following sections call $\check{\psi} : X(k) \to X_0(k)/T(k)_{\mathrm{cp}}$ *the canonical toric section* (or *splitting*) of $\breve{\pi} : X_0(k)/T(k)_{\mathrm{cp}} \to X(k)$.

Notation and remark 9.6

(a) Let $D = \sum_{\rho\in\Delta(1)} a_\rho D_\rho \in \mathbb{Z}^{\Delta(1)}$ be a U-invariant Weil divisor on X. Then $|\ |_D : X_1(k) \to \mathbb{R}$ is the map which sends the n-tuple (β_ρ), $\rho \in \Delta(1)$ in $X_1(k)$ to $|\beta^D|$ where $\beta^D = \prod_{\rho\in\Delta(1)} \beta_\rho^{a_\rho}$. We shall also write $|\ |_D$ for the restrictions of this map to $\mathcal{T}(k) = X_0(k)$ and $U_0(k)$.

(b) $|\ |_D : X_1(k) \to \mathbb{R}$ is constant on the $U_0(k)_{\mathrm{cp}}$-orbits under the toric action of $U_0(k)$ on $X_1(k)$. Therefore $|\ |_D : \mathcal{T}(k) \to \mathbb{R}$ factorizes to give a map $\mathcal{T}(k)/T(k)_{\mathrm{cp}} \to \mathbb{R}$ which we also denote by $|\ |_D$ by abuse of notation.

Proposition 9.7. — *Let D_1, $D_2 \in \mathbb{Z}^{\Delta(1)}$ be two effective U-invariant Weil divisors on X and let $D = D_1 - D_2$. Then 1 is a local section of $\mathcal{O}(D)$ at each k-point $P \in X(k)$ outside the support of D_2. Moreover,*
$$\|1(P)\|_D = |\widetilde{\psi}(P)|_D$$
for each k-point outside the support of D_2.

Proof. — Let $\sigma \in \Delta$ be a maximal cone such that $P \in C_\sigma(k)$ and let $D = \sum_{\rho\in\Delta(1)} a_\rho D_\rho$. Further, let $\chi^{m(\sigma)}$ resp. $\chi_\rho^{m(\sigma)}$, $\rho \in \Delta(1)$ be the unique character on U which generates $\mathcal{O}(-D)$ resp. $\mathcal{O}(-D_\rho)$ on U_σ. Then,

$$\|1(P)\|_D = |\chi^{m(\sigma)}(P)| = |\prod_{\rho\in\Delta(1)} \chi_\rho^{m(\sigma)a_\rho}| = |\widetilde{\psi}(P)|_D$$

as was to be proved. □

Proposition 9.8. — *Let D be a U-invariant Weil divisor on X such that $\mathcal{O}(D)$ is generated by its global sections. Let $\chi^{-m(\sigma)}$, $\sigma \in \Delta_{\max}$ be the unique character on U which generates $\mathcal{O}(D)$ on U_σ. Then $\chi^{-m(\sigma)}$ is a global section of $\mathcal{O}(D)$ and $\chi^{-m(\sigma)}(P) \neq 0$ for points $P \in U_\sigma(k)$. If σ is a local section of $\mathcal{O}(D)$ defined at $P \in X(k)$, then*

$$\|s(P)\|_D = \inf_\sigma |s(P)\chi^{m(\sigma)}(P)|$$

where σ runs over all maximal cones in Δ.

Moreover, if D is ample and σ is a maximal cone in Δ, then $C_\sigma(k)$ is equal to the subset of $P \in X(k)$ such that

$$|\chi^{m(\sigma)-m(\tau)}(P)| \leq 1$$

for all maximal cones $\tau \in \Delta$.

Proof. — It follows from the proof of (8.7)(a), that $\chi^{-m(\sigma)}$ is global section of $\mathcal{O}(D)$.

Next, let $D = \sum_{\rho\in\Delta(1)} a_\rho D_\rho$. Then, by [**24**, p. 68], one has

(i) $$\langle m(\sigma), n_\rho\rangle \leq a_\rho \text{ for all } \sigma \in \Delta_{\max},\ \rho \in \Delta(1),$$

(ii) $$\langle m(\sigma), n_\rho\rangle = a_\rho \text{ for all } \sigma \in \Delta_{\max},\ \rho \in \sigma(1).$$

Here (i) follows from the fact that $\chi^{-m(\sigma)}$ is a global section of $\mathcal{O}(D)$ while (ii) follows from the assumption that $\chi^{-m(\sigma)}$ generates $\mathcal{O}(D)$ on U_σ.

To prove the formula for $\|s(P)\|_D$ it suffices by continuity to treat the case $P \in U(k)$. Let $\sigma \in \Delta_{\max}$ and suppose that $P \in C_\sigma(k) \cap U(k)$. Then $-L(P)$ is a linear combination $\sum_{\rho\in\sigma(1)} \lambda_\rho n_\rho$ with non-negative real coefficients λ_ρ (cf. (9.1)). By (i), (ii) we get that

$$\langle m(\sigma), -L(P)\rangle = \sum_{\rho\in\sigma(1)} \lambda_\rho \langle m(\sigma), n_\rho\rangle = \sum_{\rho\in\sigma(1)} \lambda_\rho a_\rho, \quad \sigma \in \Delta_{\max},$$

$$\langle m(\tau), -L(P)\rangle = \sum_{\rho\in\sigma(1)} \lambda_\rho \langle m(\tau), n_\rho\rangle \leq \sum_{\rho\in\sigma(1)} \lambda_\rho a_\rho, \quad \sigma, \tau \in \Delta_{\max}.$$

From this we conclude that if $P \in C_\sigma(k) \cap U(k)$, then

$$|\chi^{m(\sigma)}(P)| = \exp\langle m(\sigma), L(P)\rangle \leq \exp\langle m(\tau), L(P)\rangle = |\chi^{m(\tau)}(P)|$$

for any other maximal cone τ of Δ. This proves the first statement. Further, if $P \in C_\sigma(k)$, it follows by continuity that

$$|\chi^{m(\sigma)-m(\tau)}(P)| \leq 1$$

for all maximal cones $\tau \in \Delta$.

Now assume that D is ample and that

$$|\chi^{m(\sigma)-m(\tau)}(P)| \leq 1$$

for all maximal cones $\tau \in \Delta_{\max}$ (cf. (8.9)). Then (cf. [**24**, p. 70]),

(iii) $$\langle m(\tau), n_\rho \rangle < a_\rho \text{ for all } \tau \in \Delta_{\max},\ \rho \notin \tau(1).$$

Therefore, if $P \in C_\sigma(k) \cap U(k)$ and $\tau \in \Delta_{\max}$, then

$$|\chi^{m(\sigma)}(P)| = |\chi^{m(\tau)}(P)| \Leftrightarrow \lambda_\rho = 0 \text{ for all } \rho \in \sigma(1)\backslash\tau(1) \Leftrightarrow P \in C_\tau(k).$$

Hence by continuity if $P \in C_\sigma(k)$ then $|\chi^{m(\sigma)-m(\tau)}(P)| = 1$ if and only if $P \in C_\tau(k)$. This completes the proof of the last assertion. □

We now concentrate on the anticanonical system.

Definition 9.9. — Let Δ be a complete regular fan and σ be a maximal cone. Then another maximal cone τ of Δ is said to be *adjacent* to σ if $\sigma \cap \tau$ is a facet of σ.

Lemma 9.10. — *Let Δ be a complete regular fan such that $D_0 = \sum_{\rho \in \Delta(1)} D_\rho$ is ample. Let σ be a maximal cone in Δ and let P be a k-point on $X_\Delta(k)$. Then $P \in C_\sigma(k)$ if and only if*

$$|\chi^{m(\sigma)-m(\tau)}(P)| \leq 1$$

for all maximal adjacent cones $\tau \in \Delta$.

Proof. — $-L(P)$ is a linear combination $\sum_{\rho \in \sigma(1)} \lambda_\rho n_\rho$ with real coefficients λ_ρ and $P \in C_\sigma(k)$ if and only if $\lambda_\rho \geq 0$ for all rays ρ of σ. Let $\{m_\rho,\ \rho \in \sigma(1)\}$ be the $\mathbb{Z}$-basis of M dual to the $\mathbb{Z}$-basis $\{n_\rho,\ \rho \in \sigma(1)\}$ of N. Let σ_ρ be the maximal adjacent cone corresponding to $\rho \in \sigma(1)$ under the bijection in (8.9). Then (cf. op. cit.) there exists a positive integer b_ρ, such that

$$b_\rho m_\rho = m(\sigma_\rho) - m(\sigma).$$

Hence,

$$\log |\chi^{m(\sigma)-m(\sigma_\rho)}(P)| = \langle m(\sigma) - m(\sigma_\rho), L(P) \rangle = -b_\rho \lambda_\rho$$

so that $\lambda_\rho \geq 0$ if and only if $|\chi^{m(\sigma)-m(\tau)}(P)| \leq 1$ for $\tau = \sigma_\rho$. This completes the proof. □

Proposition and definition 9.11

(a) *Let $X = X_\Delta$ be a smooth k-variety defined by a regular fan Δ and let $D = \sum_{\rho\in\Delta(1)} D_\rho$. Then there exists a unique extension of the order norm (3.30) on* $\mathbf{det}\,\mathrm{Tan}(U_{\mathrm{an}}(k)) \to U_{\mathrm{an}}(k)$ *to a norm*

$$\| \ \|^\# : \mathbf{det}(\mathrm{Tan}\, X_{\mathrm{an}}(k)) \otimes V(\mathcal{O}(D))_{\mathrm{an}}(k) \longrightarrow \mathbb{R}$$

which we shall call the order norm on

$$\mathbf{det}(\mathrm{Tan}\, X_{\mathrm{an}}(k)) \otimes V(\mathcal{O}(D))_{\mathrm{an}}(k) \longrightarrow X_{\mathrm{an}}(k)$$

and denote by $\| \ \|^\#$.

(b) *Let $X = X_\Delta$ be a smooth complete k-variety defined by a regular complete fan Δ and let $D = \sum_{\rho\in\Delta(1)} D_\rho$. Then we define the toric norm on*

$$\mathbf{det}(\mathrm{Tan}\, X_{\mathrm{an}}(k)) \longrightarrow X_{\mathrm{an}}(k)$$

to be the product norm of the order norm in (a) and the norm $\| \ \|_D$ on

$$V(\mathcal{O}(-D))_{\mathrm{an}}(k) \longrightarrow X_{\mathrm{an}}(k)$$

described in (9.2).

Proof. — The uniqueness follows from the density of $U(k)$ in $X_{\mathrm{an}}(k)$ (cf. (1.6)(b)). To show the existence, let du be an analytic differential form of minimal $d\,$log-type on $U_{\mathrm{an}}(k)$ (3.28). Then du extends to a global nowhere vanishing section on $\mathbf{det}(\mathrm{Cot} X_{\mathrm{an}}(k)) \otimes V(\mathcal{O}(-D))_{\mathrm{an}}(k)$ by (8.10). We may thus define $\|s\|^\#$ to be the absolute value of $du(s) \in k$ (cf. (1.6)(a)). This completes the proof. □

The toric norm depends on the toric structure of $X = X_\Delta$, but is otherwise a "canonical" norm.

Choose an ordering $\rho_1, \ldots, \rho_n$ of the one-dimensional cones in Δ and let $x_1, \ldots, x_n$ be the corresponding variables. Recall that (8.7) the principal universal X-torsor X_0 is the open subset of $X_1 = \mathrm{Spec}\, k[x_1, \ldots, x_n]$ for which not all the monomials

$$x^{\underline{\sigma}} := \prod_{\rho\notin\sigma(1)} x_\rho, \quad \sigma \in \Delta_{\max}$$

vanish.

Theorem 9.12. — *Let k be a non-discrete locally compact field of characteristic zero and let $X = X_\Delta$ be a toric variety over k obtained from a regular complete fan (N, Δ) such that the anticanonical sheaf is generated by its global sections. Let $D = \sum_{\rho\in\Delta(1)} D_\rho$ and let*

$$D(\sigma) := D + \sum_{\rho\in\Delta(1)} \langle -m(\sigma), n_\rho\rangle D_\rho, \qquad \sigma \in \Delta_{\max}$$

be the effective anticanonical U-invariant Weil divisors described in (8.7).

Let T be the Néron-Severi torus of X and let $\pi : \mathcal{T} \to X$, $\mathcal{T} = X_0$ be the universal torsor constructed from the fan (N_0, Δ_0) (see (8.4)).

Let $\| \ \| : \mathbf{det}\,\mathrm{Tan}(X_{\mathrm{an}}(k)) \to [0,\infty)$ be the toric norm on the analytic anticanonical line bundle and let $\| \ \|_0 : \mathbf{det}\,\mathrm{Tan}(\mathcal{T}_{\mathrm{an}}(k)) \to [0,\infty)$ be the induced norm on $\mathcal{T}_{\mathrm{an}}(k)$. Then,

$$\|s(P)\|_0 := |f(P)|/(\sup_\sigma |x^{D(\sigma)}(P)|), \quad \sigma \in \Delta_{\max}$$

for any local analytic section $s = f(x_1,\ldots,x_n)\dfrac{\partial}{\partial x_1} \wedge \cdots \wedge \dfrac{\partial}{\partial x_n}$ defined at $P \in \mathcal{T}_{\mathrm{an}}(k)$.

Proof. — Let du be a U-invariant global section of $\mathbf{det}\,\Omega^1_{U/k}$ of minimal $d\log$-type (see (3.28)) and ω be a T-invariant global section of $\mathbf{det}\,\Omega^1_{U/k}$ of minimal $d\log$-type (cf. (3.16)). Then $du_0 := \omega \otimes \pi^* du$ is a U_0-invariant global section of $\mathbf{det}\,\Omega^1_{U_0/k}$ of minimal $d\log$-type. The order norm of $\mathbf{det}\,\mathrm{Tan}(U_{0,\mathrm{an}}(k))$ is therefore (cf. (3.30)) the product norm of the order norm on

$$\mathbf{det}\,\mathrm{Tan}(U_{0,\mathrm{an}}(k)/U_{\mathrm{an}}(k))$$

and the pullback norm of the order norm on $\mathbf{det}\,\mathrm{Tan}(U_{\mathrm{an}}(k))$. Also, by definition, $\| \ \|_0$ is the product norm on $\mathbf{det}\,\mathrm{Tan}(\mathcal{T}_{\mathrm{an}}(k)$ of the order norm $\| \ \|_{\mathcal{T}/X}$ on $\mathbf{det}\,\mathrm{Tan}(\mathcal{T}_{\mathrm{an}}(k)/X_{\mathrm{an}}(k))$ and the pullback norm $\pi^*\| \ \|$ on $\pi^*_{\mathrm{an}}(\mathbf{det}\,\mathrm{Tan}(X_{\mathrm{an}}(k)))$. Hence by the definition of $\| \ \|$ (cf. (9.11)) one concludes that the restriction of $\| \ \|_0$ to $\mathbf{det}\,\mathrm{Tan}(U_{0,\mathrm{an}}(k)) \subset \mathbf{det}\,\mathrm{Tan}(X_{0,\mathrm{an}}(k))$ is the product norm of the order norm $\| \ \|_1$ on $\mathbf{det}\,\mathrm{Tan}(U_{0,\mathrm{an}}(k))$ and the restriction of $\pi^*_{\mathrm{an}}\| \ \|_D$ to the trivial line bundle over $U_{0,\mathrm{an}}(k)$.

Therefore, since $dx_1 \wedge \cdots \wedge dx_n/x^D$ is of minimal $d\log$-type it follows that:

$$\begin{aligned}\left\|\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} x^D(P)\right\|_0 &= \left\|\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} x^D(P)\right\|_1 \pi^*_{\mathrm{an}}\|1(P)\|_D \\ &= \pi^*_{\mathrm{an}}\|1(P)\|_D.\end{aligned}$$

Hence, if $s = f(x_1,\ldots,x_n)\dfrac{\partial}{\partial x_1} \wedge \cdots \wedge \dfrac{\partial}{\partial x_n}$ is defined at $P \in U_0(k)$, then

$$\begin{aligned}\|s(P)\|_0 &= |f(P)| \left\|\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} x^D(P)\right\|_0 /|x^D(P)| \\ &= |f(P)| \|1(\pi_{\mathrm{an}}(P))\|_D/|x^D(P)|\end{aligned}$$

where $\|1(\pi(P))\|_D$ means the norm of 1 regarded as a section of the analytic line bundle $V(\mathcal{O}(-D))_{\mathrm{an}}(k) \to X_{\mathrm{an}}(k)$ at the point $\pi(P) \in U(k)$. But it follows from (9.8) that

$$\|1(\pi(P))\|_D = \inf_\sigma |\chi^{m(\sigma)}(\pi(P))| = \inf_\sigma |x^D(P)/x^{D(\sigma)}(P)|$$

where σ runs over all maximal cones in Δ.

Therefore, if $s = f(x_1, \dots, x_n)\dfrac{\partial}{\partial x_1} \wedge \cdots \wedge \dfrac{\partial}{\partial x_n}$ is defined at $P \in U_0(k)$, then

$$\|s(P)\|_0 = |f(P)| \inf_{\sigma} |1/x^{D(\sigma)}(P)|, \quad \sigma \in \Delta_{\max}.$$

The same statement for $P \in X_0(k)$ follows by continuity since $U_0(k)$ is dense in $X_0(k)$ and both sides define a norm on the anticanonical line bundle on $\mathcal{T}_{\text{an}}(k)$. This completes the proof. $\square$

The preceding proposition is related to the following description of the canonical splitting $\check{\psi} : X(k) \to X_0(k)/T(k)_{\text{cp}}$ of $\check{\pi} : X_0(k)/T(k)_{\text{cp}} \to X(k)$ (cf. (9.5)) for non-archimedean fields k.

Proposition 9.13. — *Let k be a finite extension of $\mathbb{Q}_p$, let o be the maximal $\mathbb{Z}_p$-order in k and let (Δ, N) be a complete regular fan. Let $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ be the toric o-morphism of toric o-schemes defined by the obvious morphism $(\Delta_0, N_0) \to (\Delta, N)$ of fans (cf. (8.6)). Then the following holds.*

(a) *$T(k)_{\text{cp}}$ is the image of $\widetilde{T}(o)$ under the natural embedding of $\widetilde{T}(o)$ in $T(k)$.*

(b) *The obvious maps $\widetilde{X}_0(o)/\widetilde{T}(o) \to \widetilde{X}(o)$ and $\widetilde{X}(o) \to X(k)$ are isomorphsisms. $\check{\psi} : X(k) \to X_0(k)/T(k)_{\text{cp}}$ is the unique map such that the following diagram commutes*

$$\begin{array}{ccc}
\widetilde{X}(o) & \xleftarrow{\ \sim\ } & \widetilde{X}_0(o)/\widetilde{T}(o) \\
\wr\downarrow & & \downarrow \\
\check{X}(k) & \xrightarrow{\ \check{\psi}\ } & X_0(k)/T(k)_{\text{cp}}
\end{array}$$

Proof

(a) This is true for any split o-torus $\widetilde{T} = \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ with generic fibre T.

(b) It follows from the assumptions on Δ that $\widetilde{X}$ is a smooth proper scheme over o and hence that $\widetilde{X}(o) = X(k)$. The morphism $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ is a (universal) torsor under $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$. Thus, by Grothendieck's version of Hilbert 90, it follows that $\widetilde{\pi}(\widetilde{X}_0(o)) = \widetilde{X}(o)$ and $\widetilde{X}_0(o)/\widetilde{T}(o) = \widetilde{X}(o)$. The map $\check{\psi}$ is defined (cf. (9.5)) by gluing the restrictions of the maps

$$\psi_{\sigma,\text{an}} : U_\sigma(k) \longrightarrow U_{0,\sigma}(k)$$

to $C_\sigma(k)$ modulo elements in $\widetilde{T}(o)$. But the algebraic maps $\psi_\sigma : U_\sigma \to U_{0,\sigma}$ and $\psi_\sigma : U_{0,\sigma} \to U_\sigma$ are actually defined over o (i.e. they extend to toric morphisms between affine toric schemes over o). Therefore, ψ_σ sends a point in $U_\sigma(k)$ to a lifting to $\widetilde{X}(o) \subset X_0(k)$ of the corresponding point in $\widetilde{X}(o) = X(k)$. But this defines its class in $X_0(k)/T(k)_{\text{cp}}$ uniquely, thereby proving the assertion. $\square$

Proposition 9.14. — *Let k be a finite extension of $\mathbb{Q}_p$ and let X, o, $\widetilde{X}$ be as above. Then the following holds.*

(a) *The toric norm (cf. (9.11)) of the analytical anticanonical line bundle on $\widetilde{X}(o) = X(k)$ is equal to the model norm determined by $\widetilde{X}/o$.*

(b) *The restriction to $\widetilde{X}_0(o)$ of the induced norm $\| \ \|_0$ for $X_0(k)$ of the toric norm $\| \ \|$ coincides with the model norm for $\widetilde{X}_0(o)$.*

Proof

(a) The quotients of two norms on the analytical anticanonical line bundle on $X(k)$ define a continuous function $f : X(k) \to (0, \infty)$ (cf. (1.6)(b)). Hence it suffices to prove that the two norms coincide for the local section 1 of $\mathcal{O}(D)$ on the dense set $U(k)$. Then,

$$\|1(P)\|_D = |\widetilde{\psi}(P)|_D$$

by (9.7) so that the statement is an immediate consequence of (9.13)(b).

(b) This follows from (a) and (3.31). □

Remarks 9.15

(a) It follows from (9.14)(a) that the toric ν-adic norms on a smooth complete (split) toric variety over a number field form an adelic norm which we shall call the *toric adelic norm.* Also, by (4.7) or (9.14)(b) we have that the induced ν-adic norms on the universal torsor form an adelic norm on the principal universal torsor which we shall call the *induced toric adelic norm.*

(b) One can define toric norms on *twisted* toric varieties (cf. sec. 8) over locally compact fields. Batyrev and Tschinkel [**7**] define their norms $\| \ \|_D$ for arbitrary (twisted) toric varieties by means of the corresponding norms for the (split) toric varieties obtained after a base extension. It is also clear that one can define order norms (cf. (9.11)(a)) on twisted toric varieties by means of $d\log$-forms of minimal type over a splitting field (cf. (3.29), (3.30) for the case of non-split tori). We may therefore, just as in (9.11)(b), define toric norms on twisted toric varieties over locally compact fields as a product of the Batyrev-Tschinkel norm $\| \ \|_D$ and the order norm. One can deduce from (9.14) that the toric norms form an adelic norm also for twisted toric varieties.

(c) One can construct a canonical toric splitting of the map $\breve{\pi}$ (cf. (9.3)) for principal universal torsors over twisted toric varieties (cf. the end of section 8).

Proposition 9.16. — *Let $X = X_\Delta$ be a toric variety over k defined by a regular complete d-dimensional fan (N, Δ) and let m be the Borel measure on $X(k)$ determined by the toric norm. Then,*

(a) $m(X(k)) = 2^{\dim \Delta} \operatorname{Card}(\Delta_{\max})$ *for* $k = \mathbb{R}$,
(b) $m(X(k)) = (2\pi)^{\dim \Delta} \operatorname{Card}(\Delta_{\max})$ *for* $k = \mathbb{C}$.

Proof. — Let $\sigma \in \Delta$ be a maximal cone and let $\{m^{(j)},\ 1 \leq j \leq d\}$ be the $\mathbb{Z}$-basis of M which is dual to the $\mathbb{Z}$-basis $\{n^{(j)},\ 1 \leq j \leq d\}$ of N, consisting of generators of the rays $\rho_1, \dots, \rho_d$ of σ. Then the d characters $\chi^{(j)} \in k[U]$, $1 \leq j \leq d$ corresponding to $\{m^{(j)},\ 1 \leq j \leq d\}$ form a set of coordinates $(z_1, \dots, z_d)$ for the affine toric variety $U_\sigma = \operatorname{Spec} k[z_1, \dots, z_d]$ with $U = \operatorname{Spec} k[z_1, z_1^{-1}, \dots, z_d, z_d^{-1}]$. Moreover, by definition (cf. the proof of (9.10))

$$C_\sigma(k) = \{(z_1, \dots, z_d) \in k^{(d)} : |z_j| \leq 1 \text{ for } j = 1, \dots, d\}.$$

Finally, note that $\prod_{j=1}^d z_i = \chi^{-m(\sigma)}$ by (8.9)(iv).

The norm $\|\ \|_D$ for $D = D_1 + \cdots + D_n$ is defined by (cf. (9.2))

$$\|s(P)\|_D = \left|s(P)\chi^{m(\sigma)}(P)\right| = \left|\left(s \prod_{j=1}^d z_i\right)(P)\right|$$

for a local section σ of $\mathcal{O}(D)$, $D = D_1 + \cdots + D_n$ defined at $P \in C_\sigma(k)$. This implies that the toric norm (cf. (9.11)) of the section $\dfrac{\partial}{\partial z_1} \wedge \cdots \wedge \dfrac{\partial}{\partial z_d}$ of $\mathbf{det}\operatorname{Tan} X_{\mathrm{an}}(k)$ is equal to 1 for all $P \in C_\sigma(k)$. Hence the Borel measure m determined by the toric norm (cf. (1.12)) is given by $dz_1 \cdots dz_d$, so that

$$m(C_\sigma(k)) = \prod_{j=1}^d \int_{|z_j| \leq 1} dz_j = (\int_{|z| \leq 1} dz)^d = 2^d (\textit{resp. } (2\pi)^d) \tag{9.17}$$

for $k = \mathbb{R}$ (resp. $k = \mathbb{C}$).

We now show that

$$m(C_\sigma(k) \cap C_\tau(k)) = 0 \tag{9.18}$$

for any pair of (different) maximal cones σ, τ of Δ. This implies that

$$m(X(k)) = \sum_{\sigma \in \Delta_{\max}} m(C_\sigma(k)) \tag{9.19}$$

since $\bigcup_{\sigma \in \Delta_{\max}} C_\sigma(k) = X(k)$.

Let $\chi^{-m(\sigma)}$ (resp. $\chi^{-m(\tau)}$) σ, $\tau \in \Delta_{\max}$ be the unique character on U which generates $\mathcal{O}(D)$ on U_σ (resp. U_τ). Then $\chi^{m(\sigma)-m(\tau)}$ is a Laurent monomial in $(z_1, z_1^{-1}, \dots, z_d, z_d^{-1})$ with $|\chi^{m(\sigma)-m(\tau)}| = 1$ on $C_\sigma(k) \cap C_\tau(k)$ (cf. (9.2)).

Now suppose that $\sigma \neq \tau$. Then $\chi^{m(\sigma)-m(\tau)} \neq 1$ so that one of the variables, say z_1 occurs in the Laurent monomial $\chi^{m(\sigma)-m(\tau)}$. Let $r_j = |z_j|$ for $j =$

$1, \ldots, d$. Then there are rational numbers α_j for $j = 2, \ldots, d$ such that

$$r_1 = \prod_{j=2}^{d} r_j^{\alpha_j}$$

for each point on $C_\sigma(k) \cap C_\tau(k)$ with $\prod_{j=1}^{d} z_j \neq 0$. This implies that in case $k = \mathbb{R}$ there exists finitely many bounded subsets A_i of $\mathbb{R}^{d-1}$ and finitely many C^1-maps $f_i : A_i \to C_\sigma(\mathbb{R}) \cap C_\tau(\mathbb{R})$ satisfying a Lipschitz condition (cf. [**39**, Chap. XI, §1]) such that

$$C_\sigma(k) \cap C_\tau(k) = \bigcup_i f_i(A_i).$$

Therefore, $m(C_\sigma(k) \cap C_\tau(k)) = 0$ by lemma 1.3 in (op. cit).

For $k = \mathbb{C}$, we regard $U_\sigma(k) = \mathbb{C}^d$ as a real analytic manifold $\mathbb{R}^{2d}$ with coordinates $x_j = \Re z_j$, $y_j = \Im z_j$ and introduce polar coordinates $x_j = r_j \cos\theta_j$, $y_j = r_j \sin\theta_j$.

There is then a parametrization of the real analytic subset $C_\sigma(k) \cap C_\tau(k) \subset \mathbb{R}^{2d}$ by finitely many Lipschitz C^1-maps $f_i : A_i \to C_\sigma(k) \cap C_\tau(k)$ from bounded subsets A_i of $\mathbb{R}^{2d-1}$ so that $m(C_\sigma(k) \cap C_\tau(k)) = 0$ by lemma 1.3 in [**39**, Chap. XI]. Hence (9.19) holds also for $k = \mathbb{C}$. To complete the proof of the proposition, combine (9.17) and (9.19). □

Note that $m(C_\sigma(k) \cap C_\tau(k)) \neq 0$ in the non-archimedean case. The volume $m(X(k))$ can then computed by means of (2.15) and (9.14)(a). We shall in (11.50) consider these volumes in a concrete case.

10. Toric height functions and Tamagawa volumes of universal torsors

Let $X = X_\Delta$ be a smooth complete toric variety over a number field k defined by a complete regular fan. We shall in this section use the canonical toric splittings of universal torsors over such varieties to define heights and to interpret the main term $\alpha(X)\tau_\varepsilon(X, \| \, \|)$ in the conjectured asymptotic formula (7.12) as an adelic volume using the induced measures on universal torsors. We shall not need to assume that $X = X_\Delta$ is a Fano variety. In (10.14) we assume that the anticanonical sheaf of $X = X_\Delta$ is generated by its global sections. This is used to give a very concrete description of the anticanonical height function and of the adelic volume corresponding to the main term $\alpha(X)\tau_\epsilon(X, \| \, \|)$. But the remaining arguments of the section go through without this assumption if one works with the original definition (10.4) of the anticanonical height function.

We shall in this section let k denote a number field and o denote the maximal $\mathbb{Z}$-order in k. Otherwise we will keep the notations in section 8. We shall thus by Δ denote a complete regular fan in $N = \mathrm{Hom}(M, \mathbb{Z})$ and put $M_0 = \mathbb{Z}^{\Delta(1)}$, $N_0 = \mathrm{Hom}(M_0, \mathbb{Z})$. We denote by Δ_0 the "pullback" fan in N_0 described in (8.4) and by Δ_1 the fan in $N_1 = N_0$ described after (8.5). There is a morphism of fans from (Δ_0, N_0) to (Δ, N) which defines a toric k-morphism $\pi : X_0 \to X$ of split toric k-varieties. This morphism is defined over $\mathbb{Q}$, but we shall in this section let $X = X_\Delta$ denote the corresponding (geometrically integral) k-variety obtained by base extension. The morphism π is by (8.5) a universal torsor over X and we shall also write $\mathcal{T}$ instead of X_0.

The morphism of fans from (Δ_0, N_0) to (Δ, N) defines also a toric morphism $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ of toric schemes over $\mathbb{Z}$. But we shall in this section let $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ be the toric o-morphism obtained by base extension (cf. (8.6)(b)). The restriction of $\widetilde{\pi}$ to the generic fibres over k is thus the morphism $\pi : X_0 \to X$ of toric varieties over k. The morphism $\widetilde{\pi}$ is by (8.5) a universal torsor over $\widetilde{X}$ under the Néron-Severi o-torus $\widetilde{T} = \mathbf{D}(\mathbf{Pic}_{\widetilde{X}/o})$ as defined in (5.14). We shall call $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ the *principal* universal torsor.

Recall (cf. (8.2)) that there is an exact sequence of finitely generated free $\mathbb{Z}$-modules

$$1 \longrightarrow M \longrightarrow \mathbb{Z}^{\Delta(1)} \longrightarrow \mathrm{Pic}\, X \longrightarrow 1. \tag{10.1}$$

We may endow $\mathrm{Hom}(L, \mathbb{Z})$ with the trivial fan consisting of the cone $\{0\}$ for each of the three lattices $L = M$, $\mathbb{Z}^{\Delta(1)}$, $\mathrm{Pic}\, X$. The morphism between these fans defines toric k-morphisms between toric varieties over k as well as toric o-morphisms between toric o-schemes. We obtain thereby an exact sequence of split k-tori:

$$1 \longrightarrow T \longrightarrow U_0 \longrightarrow U \longrightarrow 1 \tag{10.2}$$

as well as an exact sequence of split o-tori $G_{m,o} \times \cdots \times G_{m,o}$:

$$1 \longrightarrow \widetilde{T} \longrightarrow \widetilde{U}_0 \longrightarrow \widetilde{U} \longrightarrow 1. \tag{10.3}$$

The exact sequence of f.g. torsion-free abelian groups in (10.1) gives rise to an exact sequence of constant group schemes over k (resp. o). The sequences in (10.2) (resp (10.3)) are the "dual" exact sequences of tori over k (resp. o) under the contravariant equivalence between torsion-free constant group schemes and tori described in section 5.

The assumption that Δ is complete and regular implies that $\widetilde{X}/o$ is proper and smooth. There is a natural $\widetilde{U}_0$-equivariant open toric immersion of toric schemes $\widetilde{X}_0 \subset \widetilde{X}_1$ corresponding to the inclusion of fans $\Delta_0 \subset \Delta_1$. It was shown in section 8 that the generic fibre X_1 of $\widetilde{X}_1$ is canonically isomorphic to an affine space $\mathbb{A}^n_k = \mathrm{Spec}\, k[x_\rho]$ where n is the cardinality of $\Delta(1)$ and the variables are indexed by the rays ρ of Δ. Moreover, $X_0 \subset X_1 = \mathbb{A}_k^n$ is the open complement of the closed subscheme defined by the monomials

$$x^{\underline{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho, \quad \sigma \in \Delta_{\max}.$$

There is a similar description of $\widetilde{X}_0 \subset \widetilde{X}_1$. The latter scheme is equal to $\mathbb{A}_o^n = \operatorname{Spec} o[x_\rho]$ where ρ runs over $\Delta(1)$ and $\widetilde{X}_0$ is the open complement in $\widetilde{X}_1$ of the closed subscheme defined by the monomials $x^{\underline{\sigma}}$, $\sigma \in \Delta_{\max}$.

Let W be the set of places of k and denote by $\| \ \|_\nu$, $\nu \in W$ the toric ν-adic norm (see (9.11)) of the analytic anticanonical line bundle of $X_{\mathrm{an}}(k_\nu)$. It follows from (9.14) that the toric norms $\| \ \|_\nu$, $\nu \in W$ form an adelic norm $\| \ \|$ for X, which we shall call the *adelic toric norm* for $X = X_\Delta$.

The adelic toric norm $\| \ \|$ gives rise to a height function $H : X(k) \to (0, \infty)$ with

$$H(P) = \prod_{\nu \in W} \|s(P)\|_\nu^{-1} \tag{10.4}$$

for a local section s of ω_X^{-1} at $P \in X(k)$ with $s(P) \neq 0$ (cf. (7.2)). We shall call this function the *toric anticanonical height function* or simply the *toric height function* when it is clear that we consider the anticanonical linear system.

We now give alternative descriptions of the toric height function. We first represent ω_X as the tensor product of the invertible sheaves

$$\omega_X = \mathcal{O}(-D_0) \otimes (\omega_X \otimes \mathcal{O}(D_0)), D_0 = \sum_{\rho \in \Delta(1)} D_\rho.$$

Let $\| \ \|_{D,\nu}$, $\nu \in W$ be the ν-adic norm on

$$V(\mathcal{O}(-D_0))_{\mathrm{an}}(k_\nu) \longrightarrow X_{\mathrm{an}}(k_\nu)$$

described in (9.2). The toric norm $\| \ \|_\nu$ on $\mathbf{det} \operatorname{Tan} X_{\mathrm{an}}(k_\nu) \to X_{\mathrm{an}}(k_\nu)$ is by definition the product of this norm and the order norm (cf. (9.11)) $\| \ \|_\nu^{\#}$ on

$$\mathbf{det}(\operatorname{Tan} X_{\mathrm{an}}(k_\nu)) \otimes V(\mathcal{O}(D_0))_{\mathrm{an}}(k_\nu) \longrightarrow X_{\mathrm{an}}(k_\nu).$$

Proposition 10.5. — *The toric height function $H : X(k) \to (0, \infty)$ is equal to the product*

$$\prod_{\nu \in W} \|s_1(P)\|_{D,\nu}^{-1}$$

where s_1 is a local section of $\mathcal{O}(D)$ for $D = D_0$ at $P \in X(k)$ with $s_1(P) \neq 0$.

Proof. — Choose local sections s_1 resp. s_2 at the k-point P (defined over k) of $\mathcal{O}(D)$ resp. $\omega_X^{-1} \otimes \mathcal{O}(-D)$ and put

$$H_1(P) = \prod_{\nu \in W} \|s_1(P)\|_{D,\nu}^{-1},$$

$$H_2(P) = \prod_{\nu \in W} \|s_2(P)\|_\nu^{\sharp-1}.$$

Then $H_1(P)$ and $H_2(P)$ are independent of the choice of s_1 and s_2 and give well defined height functions $H_1 : X(k) \to (0, \infty)$ and $H_2 : X(k) \to (0, \infty)$ such that $H(P) = H_1(P)H_2(P)$.

The invertible sheaf $\omega_X^{-1} \otimes \mathcal{O}(-D)$ is trivial and we may choose s_2 to be the dual of an algebraic differential form du of minimal $d\,$log-type on $U_{\mathrm{an}}(k)$ (cf. (3.28)) for all points $P \in X(k)$. Then $\|s_2(P)\|_\nu^\sharp = |du(s_2(P))|_\nu = 1$ for all $\nu \in W$ (cf. the proof of (9.11)(a)) and $H_2(P) = 1$ for all $P \in X(k)$. This completes the proof. $\square$

Definition 10.6

(a) Let $D \in \mathbb{Z}^{\Delta(1)}$ be a U-invariant Weil divisor on X and $P \in X(k)$. Then,

$$H_D(P) := \prod_{\nu \in W} \|s(P)\|_\nu^{-1},$$

$$h_D(P) := \log H_D(P) = -\sum_{\nu \in W} \log \|s(P)\|_\nu$$

where s is a local section of $\mathcal{O}(D)$ at $P \in X(k)$ with $s(P) \neq 0$.

(b)

$$H_\Delta : X(k) \times \mathbb{Z}^{\Delta(1)} \longrightarrow (0, \infty)$$

resp.

$$h_\Delta : X(k) \times \mathbb{Z}^{\Delta(1)} \longrightarrow \mathbb{R}$$

is the pairing

$$H_\Delta(P, D) := H_D(P)$$

resp.

$$h_\Delta(P, D) := h_D(P).$$

Note that h_Δ is additive on the right hand side.

One can extend the restriction of H_D to $U(k)$ to a height function

$$F_{D,A} : U(A_k) \longrightarrow (0, \infty)$$

by choosing $s = 1$ as local section. To see this, we make use of the local ν-adic sections

$$\check{\psi}_\nu : X(k_\nu) \longrightarrow X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}}$$

of

$$\check{\pi}_\nu : X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \longrightarrow X(k_\nu)$$

in (9.5) and the map

$$|\ |_{D,\nu} : X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \longrightarrow \mathbb{R}$$

described in (9.6). Then (cf. (9.7)):

$$\|1(P_\nu)\|_{D,\nu}^{-1} = |\check{\psi}_\nu(P_\nu)|_{D,\nu}^{-1} \tag{10.7}$$

for all $P_\nu \in U(k_\nu)$.

Let $\widetilde{X}$ be the smooth proper toric o-model above and let $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ be the principal universal torsor. This model can be used to define the splitting $\check{\psi}_\nu$ at non-archimedean places ν (see (9.13)). It is the unique map for which the following diagram commutes:

$$\begin{tikzcd} \widetilde{X}(o_\nu) \arrow[d, "\wr"'] & \widetilde{X}_0(o_\nu)/\widetilde{T}(o_\nu) \arrow[l, "\sim"'] \arrow[d] \\ \check{X}(k_\nu) \arrow[r, "\check{\psi}_\nu"] & X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \end{tikzcd} \tag{10.8}$$

There is further an obvious commutative diagram:

$$\begin{tikzcd} \widetilde{U}(o_\nu) \arrow[d] & \widetilde{U}_0(o_\nu)/\widetilde{T}(o_\nu) \arrow[l, "\sim"'] \arrow[d] \\ \widetilde{X}(o_\nu) & \widetilde{X}_0(k_\nu)/\widetilde{T}(o_\nu) \arrow[l, "\sim"'] \end{tikzcd} \tag{10.9}$$

Hence, if $P_\nu \in \mathrm{Im}(\widetilde{U}(o_\nu) \to U(k_\nu))$, then

$$\check{\psi}(P_\nu) \in \mathrm{Im}(\widetilde{U}(o_\nu) \to \mathcal{T}(k_\nu)/T(k_\nu)_{\mathrm{cp}})$$

so that $\|1(P_\nu)\|_{D,\nu}^{-1} = |\check{\psi}_\nu(P_\nu)|_{D,\nu}^{-1}$ by the remark in (9.6)(b). This implies the following result:

Proposition 10.10. — *Let $P_A \in U(A_k)$ be an adelic point and let*

$$\{P_\nu\}_{\nu\in W} \in \prod_{\nu\in W} U(k_\nu)$$

be the corresponding set of k_ν-points. Then the products

$$F_{D,A}(P_A) := \prod_{\nu\in W} \|1(P_\nu)\|_{D,\nu}^{-1} = \prod_{\nu\in W} |\check{\psi}_\nu(P_\nu)|_{D,\nu}^{-1}$$

are absolute convergent with at most finitely many factors different from 1.

Let

$$f_{D,A}(P_A) = \log F_{D,A}(P_A) = -\sum_{\nu\in W} \log \|1(P_\nu)\|_{D,\nu}.$$

It is an immediate consequence of the definition that the pairing

$$f_{\Delta,A} : U(A_k) \times \mathbb{Z}^{\Delta(1)} \to (0,\infty) \tag{10.11}$$

with $f_{\Delta,A}(P_A, D) := f_{D,A}(P_A)$ is additive on the right hand side. One has also the following result:

Proposition 10.12. — *Let $P \in U(k)$. Then $h_D(P)$ and $H_D(P)$, $D \in \mathbb{Z}^{\Delta(1)}$ depend only on the linear equivalence class of D in* $\operatorname{Pic} X$.

Proof. — h_Δ is additive with respect to $\mathbb{Z}^{\Delta(1)}$. It therefore suffices to prove that $H_D(P) = 1$ for principal divisors

$$D = \sum_{\rho \in \Delta(1)} a_\rho D_\rho \in \mathbb{Z}^{\Delta(1)}, \quad a_\rho = \langle m, n_\rho \rangle,\ m \in M.$$

But $x^D(\check{\psi}_\nu(P)) = \chi^m(P)$ for all ν since $\check{\psi}_\nu$ is a section to $\check{\pi}_\nu$ (cf. the proof of (9.4)).

This implies in its turn that $|\check{\psi}_\nu(P_\nu)|_{D,\nu} = |\chi^m(P)|_\nu$ and hence that

$$H_D(P) := \prod_{\nu \in W} \|1(P)\|_{D,\nu}^{-1} = \prod_{\nu \in W} |\check{\psi}_\nu(P)|_{D,\nu}^{-1} = \prod_{\nu \in W} |\chi^m(P)|_\nu^{-1} = 1$$

by the product formula in algebraic number theory. This finishes the proof. □

Remarks 10.13

(a) The properties (10.11) and (10.12) of the height functions are essentially due to Batyrev and Tschinkel [**7**] (cf. also [**4**]) although they state them in a somewhat different language. They make essential use of these properties in their study of Manin's zeta functions for counting functions on $U(k)$ (cf. the comments after (10.28)). We shall not use the pairing in (10.11) further. Instead, we will extend the restriction of h_Δ to

$$U(k) \times \mathbb{Z}^{\Delta(1)} = U_0(k)/T(k) \times \mathbb{Z}^{\Delta(1)}$$

to a pairing

$$h_{\Delta,A} : X_0(A_k)/T(k) \times \mathbb{Z}^{\Delta(1)} \longrightarrow \mathbb{R}$$

which factorizes over $X_0(A_k)/T(k) \times \operatorname{Pic} X$.

(b) The treatment of Tamagawa volumes and local heights by means of universal torsors and the canonical toric splitting of

$$\check{\psi}_\nu : X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \to X(k_\nu)$$

has some similarities with Bloch's torsor theoretic approach to the Birch-Swinnerton-Dyer conjecture and to the local Néron symbols (cf. [**8**], [**48**]) for abelian varieties. The analogy is not perfect since the pairing $H_{\Delta,A}$: $U(A_k) \times \mathbb{Z}^{\Delta(1)} \to (0, \infty)$ is not multiplicative on the left hand side. But it is still suggestive to think of universal torsors for toric varieties as the analog of biextensions for abelian varieties.

Proposition 10.14. — *Let $X = X_\Delta$ be a smooth complete k-variety defined by a regular complete fan Δ. Let $D = \sum_{\rho \in \Delta(1)} D_\rho$ and suppose that $\mathcal{O}(D)$ is generated by its global sections. Let $\chi^{m(\sigma)}(P)$, $\sigma \in \Delta_{\max}$ be the unique character on U such that $\chi^{-m(\sigma)}$ generates $\mathcal{O}(D)$ on U_σ and let*

$$D(\sigma) = D + \sum_{\rho \in \Delta(1)} \langle -m(\sigma), n_\rho \rangle D_\rho$$

be the effective Weil divisors in (8.7). Let

$$H_0 : X_0(k) \longrightarrow (0, \infty)$$

be the composition of $\pi : X_0(k) \to X(k)$ and the toric anticanonical height function $H : X(k) \to (0, \infty)$. Then,

$$H_0(P_0) = \prod_{\nu \in W} \sup_{\sigma \in \Delta_{\max}} |x^{D(\sigma)}(P_0)|_\nu.$$

Proof. — Let $P \in X(k)$ be the image of $P_0 \in X_0(k)$ and let $\tau \in \Delta_{\max}$ be chosen such that $P \in U_\tau$. Further, let du be a U-invariant global section of $\omega_{U/k}$ of minimal $d\log$-type regarded as a global section of $\omega_{X/k} \otimes \mathcal{O}(D)$ (cf. (8.8)). Then $\chi^{m(\tau)} du$ is a local section of $\omega_{X/k}$ which does not vanish at P. Let s be local section of ω_X^{-1} at P which is the inverse of $\chi^{m(\tau)} du$. Then, by definition of the toric ν-adic norm $\| \ \|$ (cf. (9.11)) and (9.8)), we have:

$$\|s(P)\|_\nu = \|\chi^{-m(\tau)}(P)\|_{D,\nu} = \inf_{\sigma \in \Delta_{\max}} |\chi^{m(\sigma)-m(\tau)}(P)|_\nu$$

for any $P \in X(k)$.

From $P \in U_\tau$ it follows that $x^{D(\tau)}(P_0) \neq 0$ (cf. (8.8)(a)). Hence

$$x^{D(\sigma)}(P_0)/x^{D(\tau)}(P_0)$$

is defined and equal to $\chi^{m(\tau)-m(\sigma)}(P)$. We may thus rewrite the preceding formula in the following form:

$$\|s(P)\|_\nu^{-1} = \sup_{\nu \in \Delta_{\max}} |x^{D(\sigma)}(P_0)|_\nu / |x^{D(\tau)}(P_0)|_\nu.$$

By the Artin-Whaples product formula for number fields we have:

$$\prod_{\nu \in W} |x^{D(\tau)}(P_0)|_\nu = 1.$$

The assertion follows from the last two equalities. □

Notation 10.15

(a) Let B be a positive integer. Then

$$c(B) := \mathrm{Card}\{P \in U(k) : H(P) = B\}.$$

(b) Let $B \geq 1$ be a real number. Then

$$\mathcal{C}(B) := \operatorname{Card}\{P \in U(k) : H(P) \leq B\}.$$

(c) Let $B \geq 1$ be a real number. Then

$$\mathcal{D}(B) := \sum_{m=1}^{[B]} c(m)/m.$$

These numbers are finite (cf. the proof of (10.28) below).

Since X is a *split* toric variety, it follows that the Néron-Severi torus T is a split k-torus $T = \mathbb{G}_m^{(r)}$ and that $H^1_{\text{ét}}(k, T) = 0$. The principal universal torsor $\pi : X_0 \to X$ is thus the only universal torsor up to isomorphism. In particular, $\pi(X_0(k)) = X(k)$.

The conjecture (7.12) is therefore equivalent to the conjecture:

$$\mathcal{C}(B) = \alpha(X)\tau_\varepsilon(X, \| \,\|)B(\log B)^{r-1}(1 + o(1)) \tag{10.16}$$

where $r = \operatorname{rk} \operatorname{Pic} X$, $\alpha(X) = \alpha_{\text{Peyre}}(X)$ is the constant of Peyre (cf. the lines before (7.7)) and $\tau_\varepsilon(X, \| \,\|)$ is the Tamagawa number defined in (6.18) for the trivial (and only) class ε in $H^1_{\text{ét}}(k, T)$.

Moreover, $\tau_\varepsilon(X, \| \,\|) = \tau(X, \| \,\|)$, since $\operatorname{Pic}(\overline{X})$ is a Galois permutation module (cf. (6.19)). Therefore (cf. (5.21))

$$\tau_\varepsilon(X, \| \,\|) = \overline{\Theta}^1_\Sigma(T(A_k)/T(k)) m_S(\pi(\mathcal{T}_S)) \prod_{\nu \in W_{\text{fin}} - \Sigma} n_\nu(\widetilde{\mathcal{T}}(o_\nu)). \tag{10.17}$$

Peyre [**52**] verified that (7.12) for holds for some classes of toric varieties over $\mathbb{Q}$ and Batyrev and Tschinkel [**4**] establish a version of (7.12) for arbitrary toric varieties. Their proof depends on the claim that any effective divisor on X is linearly equivalent to an effective divisor with support outside U. This fact is well-known (cf. e.g. the lemma on p. 66 in [**24**]). We may therefore use the following equivalent definition of $\alpha(X)$ for smooth complete toric varieties.

Definition 10.18

Let $V = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{R})$, let L be the $\mathbb{Z}$-lattice $\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{Z})$ in V and let $d\nu$ be the unique Haar measure on V such that $\operatorname{Vol}(V/L) = 1$.

Let $\lambda : V \to \mathbb{R}$ be the linear form obtained by evaluating at the anticanonical class and let $\Lambda : C_c(V) \to C_c(\mathbb{R})$ be the unique positive linear map such that:

$$\int_V g\, d\nu = \int_{\mathbb{R}} \Lambda(g)\, dx$$

for each function $g \in C_c(V)$.

Let $V_x = \lambda^{-1}(x)$ for $x \in \mathbb{R}$ and let $d\nu_x$ be the positive Borel measure on V_x corresponding to the restriction $\Lambda_x : C_c(V_x) \to \mathbb{R}$ of Λ to V_x. Finally, let $\sigma_{\text{eff}}(X, U) \subset V$ be the cone of all homomorphisms $\varphi : \operatorname{Pic} X \to \mathbb{R}$ such that $\varphi([D]) \geq 0$ for the class $[D] \in \operatorname{Pic} X$ of each effective divisor on X with support outside U. Then,

$$\alpha(X) = \int_{D_1} g d\nu_1$$

for $D_1 = \sigma_{\text{eff}}(X, U) \cap V_1$.

Remark 10.19. — Let $x \geq 0$ and let $D_x = \sigma_{\text{eff}}(X, U) \cap V_x$. Then an easy substitution of variables implies that:

$$\int_{D_x} d\nu_x = x^{r-1}\alpha(X)$$

for $r = \operatorname{rk} \operatorname{Pic} X$.

Therefore, if E_b is the set of all $\varphi \in \sigma_{\text{eff}}(X, U)$ with $0 \leq \lambda(\varphi) \leq b$, and g is the characteristic function of E_b, then

$$\operatorname{Vol}(E_b) = \int_V g d\nu = \int_0^b \Lambda(g) dx = \alpha(X) \int_0^b x^{r-1} dx = \alpha(X) b^r / r.$$

In particular, $\alpha(X) = \operatorname{Vol}(E_1)/r$. This completes the remark.

Now suppose that (10.16) holds. Then, by partial summation we get that:

$$\begin{aligned} \mathcal{D}(B) &= \mathcal{C}(B)/(B+1) + \sum_{m=1}^{B} \mathcal{C}(m)/m(m+1) \\ &= \sum_{m=1}^{B} \mathcal{C}(m)/m^2 + (\log B)^{r-1} o(1). \end{aligned}$$

Also, when (10.16) can be proved one usually has enough information to deduce that

$$\sum_{m=1}^{B} \mathcal{C}(m)/m^2 = \alpha(X)\tau_\varepsilon(X, \| \ \|) \sum_{m=1}^{B} (\log m)^{r-1}/m + (\log B)^r o(1).$$

We may further replace

$$\sum_{m=1}^{B} (\log m)^{r-1}/m \quad \text{by} \quad \int_1^B (\log x)^{r-1} \frac{dx}{x}$$

and still have an error term of order $(\log B)^r o(1)$. This suggests the following conjecture, which in a sense is a weak version of (10.16):

$$\mathcal{D}(B) = \operatorname{Vol}(E_1)\tau_\varepsilon(X, \| \ \|)(\log B)^{\operatorname{rk} \operatorname{Pic} X}(1 + o(1)). \tag{10.20}$$

One advantage with this alternative conjecture is that it is easy to give an interpretation of

$$\mathrm{Vol}(E_1)\tau_\varepsilon(X, \|\ \|)(\log B)^r$$

as an adelic volume (cf. (10.22)) of a compact subset of $X_0(A_k)/X_0(k)$. To see this, we need some further notations.

Notations 10.21

(a)

$$T(A_k)_{\mathrm{cp}} = \prod_{\nu \in W} T(k_\nu)_{\mathrm{cp}}$$

is the maximal compact subgroup of $T(A_k)$.

(b)

$$\check{\psi}_A : X(A_k) \longrightarrow X_0(A_k)/T(A_k)_{\mathrm{cp}}$$

is the continuous product map of all $\check{\psi}_\nu : X(k_\nu) \to X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}}$ (cf. (10.8)).

(c)

$$\check{\sigma}_\nu : T(k_\nu)/T(k_\nu)_{\mathrm{cp}} \times X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \longrightarrow X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}}$$

is the continuous map induced be the analytic action

$$\sigma_\nu : T(k_\nu) \times X_0(k_\nu) \longrightarrow X_0(k_\nu)$$

(cf. the lines before (3.10)).

(d)

$$\begin{aligned}\check{\rho}_\nu = (\check{\sigma}_\nu, pr_2) : T(k_\nu)/T(k_\nu)_{\mathrm{cp}} &\times X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \\ &\longrightarrow X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \times_{X(k_\nu)} X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}}\end{aligned}$$

is the homeomorphism induced by the analytic isomorphism

$$\rho_\nu = (\sigma_\nu, pr_2) : T(k_\nu) \times X_0(k_\nu) \longrightarrow X_0(k_\nu) \times_{X(k_\nu)} X_0(k_\nu)$$

(cf. (3.3)(b) and the lines before (3.11)).

(e)

$$\check{\xi}_\nu : X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \longrightarrow T(k_\nu)/T(k_\nu)_{\mathrm{cp}}$$

is the continuous map composed of (cf. (9.5))

$$((\check{\psi}_\nu \check{\pi}_\nu), \mathrm{id}) : X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \longrightarrow X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \times_{X(k_\nu)} X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}}$$

$\check{\rho}_\nu^{-1}$ and $pr_1 : T(k_\nu)/T(k_\nu)_{\mathrm{cp}} \times X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \longrightarrow T(k_\nu)/T(k_\nu)_{\mathrm{cp}}$.

(f)

$$\check{\sigma}_A : T(A_k)/T(A_k)_{\mathrm{cp}} \times X_0(A_k)/T(A_k)_{\mathrm{cp}} \longrightarrow X_0(A_k)/T(A_k)_{\mathrm{cp}}$$

is the continuous map induced by (cf. (5.3))

$$\sigma_A : T(A_k) \times X_0(A_k) \longrightarrow X_0(A_k).$$

(g)

$$\check{\rho}_A = (\check{\sigma}_A, pr_2) : T(A_k)/T(A_k)_{\mathrm{cp}} \times X_0(A_k)/T(A_k)_{\mathrm{cp}} \longrightarrow X_0(A_k)/T(A_k)_{\mathrm{cp}} \times_{X(A_k)} X_0(A_k)/T(A_k)_{\mathrm{cp}}$$

is the homeomorphism induced by the homeomorphism

$$\rho_A = (\sigma_A, pr_2) : T(A_k) \times X_0(A_k) \longrightarrow X_0(A_k) \times_{X(A_k)} X_0(A_k)$$

in (5.3).

(h)

$$\check{\xi}_A : X_0(A_k)/T(A_k)_{\mathrm{cp}} \longrightarrow T(A_k)/T(A_k)_{\mathrm{cp}}$$

is the continuous composite map obtained by replacing all the maps in (e) by their adelic counterparts.

(i)

$$\xi_A^1 : X_0(A_k)/T^1(A_k) \longrightarrow T(A_k)/T^1(A_k)$$

is the unique continuous map such that

$$\begin{array}{ccc} \check{\xi}_A : X_0(A_k)/T(A_k)_{\mathrm{cp}} & \longrightarrow & T(A_k)/T(A_k)_{\mathrm{cp}} \\ \downarrow & & \downarrow \\ \xi_A^1 : X_0(A_k)/T^1(A_k) & \longrightarrow & T(A_k)/T^1(A_k) \end{array}$$

commutes.

(j)

$$\overline{\xi}_A : X_0(A_k)/T(k) \longrightarrow \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$$

is the map obtained by composing the obvious map

$$X_0(A_k)/T(k) \longrightarrow X_0(A_k)/T^1(A_k)$$

with

$$\xi_A^1 : X_0(A_k)/T^1(A_k) \longrightarrow T(A_k)/T^1(A_k)$$

and (cf. (5.18))

$$T(A_k)/T^1(A_k) = \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R}).$$

Proposition 10.22. — *Let B be a positive real number, $b = \log B$ and let $\mathbf{K}_B$ be the inverse image of $E_b \subset \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$ under $\overline{\xi}_A : X_0(A_k)/T(k) \to \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$. Let n_ν, $\nu \in W$ be the Borel measure on $X_0(k_\nu)$ determined by the induced norms of the toric norms on $X(k_\nu)$ (cf. (9.12)) and let n_A be the (restricted) product measure on $X_0(A_k)$ (cf. (5.16)) of these measures. Finally, let $\overline{n}_A$ be the quotient measure of n_A on $X_0(A_k)/T(k)$ (cf. (5.23)). Then,*

$$\overline{n}_A(\mathbf{K}_B) = \mathrm{Vol}(E_1)\tau_\varepsilon(X, \|\ \|)(\log B)^r.$$

Proof. — The translation invariance of n_A under $\sigma_A : T(A_k) \times X_0(A_k) \to X_0(A_k)$ implies that $\overline{n}_A$ is translation invariant under the action of $T(A_k)/T(k)$. Let $\mathcal{F} \subset \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})$ be a fundamental domain with respect to

$$\operatorname{Hom}(\operatorname{Pic} X, \mathbb{Z}) \subset \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R}).$$

Then,

$$\overline{n}_A(\mathbf{K}_B)/\overline{n}_A(\overline{\xi}_A^{-1}(\mathcal{F})) = \operatorname{Vol}(E_{\log B})/\operatorname{Vol}(\mathcal{F}) = (\log B)^r \operatorname{Vol}(E_1).$$

Also, $\tau_\varepsilon(X, \| \ \|) = \overline{n}_A(\overline{\xi}_A^{-1}(\mathcal{F}))$ by (5.24). This completes the proof. □

Thus, by (10.22), (10.20) is equivalent to the conjecture:

$$\mathcal{D}(B) = \overline{n}_A(\mathbf{K}_B)(1 + o(1)). \tag{10.23}$$

To understand the relation between the adelic volume $\overline{n}_A(\mathbf{K}_B)$ and the counting function

$$\mathcal{D}(B) := \sum_{\{P \in U(k) : H(P) \leq B\}} 1/H(P)$$

we extend the toric height function

$$H : U(k) = U_0(k)/T(k) \longrightarrow \mathbb{R}$$

to a function on $X_0(A_k)/T(k)$ by means of the following result.

Proposition 10.24. — *Let $P \in U(k) = U_0(k)/T(k) \subset X_0(A_k)/T(k)$ and let $D \in \mathbb{Z}^{\Delta(1)}$ be a U-invariant Weil divisor. Let $V = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{R})$ and let $\lambda_D : V \to \mathbb{R}$ be the linear form obtained by evaluating at the class of D in* $\operatorname{Pic} X$. *Then,*

$$h_D(P) = \lambda_D \overline{\xi}_A(P).$$

In particular, $\overline{\xi}_A(P) \in \sigma_{\text{eff}}(X, U)$.

Proof

$$H_D(P) = \prod_{\nu \in W} |\check{\psi}_\nu(P)|_{D,\nu}^{-1} = \prod_{\nu \in W} \|1(P)\|_{D,\nu}^{-1}$$

by (10.7) and (10.10). Let $P_0 \in U_0(k)$ be a lifting of P. Then $P_0 = \check{\xi}_\nu(P_0)\check{\psi}_\nu(P)$ in $U_0(k_\nu)/T(k_\nu)_{\text{cp}}$ for each place ν of k by the definition of $\check{\xi}_\nu$. Therefore, since

$$\prod_{\nu \in W} |P_0|_{D,\nu}^{-1} = 1$$

(cf. (9.6)), we conclude that

$$H_D(P) = \prod_{\nu \in W} |\check{\xi}_\nu(P)|_{D,\nu}.$$

Hence,

$$h_D(P) = \log H_D(P) = \lambda_D(\overline{\xi}_A(P))$$

by the definitions of $|\ |_{D,\nu}$ (cf. (9.6)) λ_D and ξ_A^1. To prove the last statement, it suffices to show that $\log H_D(P) \geq 0$ for all effective divisors. But this follows from the first formula for $H_D(P)$ since $\|1(P)\|_D \leq 1$ by (9.2). This completes the proof. $\square$

To understand the conjecture $\lim_{B\to\infty} \mathcal{D}(B)/\overline{n}_A(\mathbf{K}_B) = 1$ in (10.23), recall that the measure $\overline{n}_A$ on $X_0(A_k)/T(k)$ is the quotient measure of the measure n_A on $X_0(A_k)$ which is the restricted product of the measures

$$dx_1 dx_2 \cdots dx_n/(\sup_\sigma |x^{D(\sigma)}(P_0)|_\nu), \sigma \in \Delta_{\max}, P_0 \in X_0(k_\nu)$$

with notations as in (9.12). Here we make use of the toric open embedding

$$X_0 \subset X_1 = \operatorname{Spec} k[x_\rho] = \mathbb{A}_k^n$$

and an ordering of the rays $\sigma \in \Delta(1)$.

Recall that

$$\mathcal{D}(B) := \sum 1/H_0(P_0)$$

where P_0 runs over all classes in $U_0(k)/T(k)$ such that (cf. (10.14)):

$$(') \qquad x_1 x_2 \cdots x_n \neq 0,$$

$$('') \qquad H_0(P_0) = \prod_{\nu\in W} \sup_{\sigma\in\Delta_{\max}} |x^{D(\sigma)}(P_0)|_\nu \leq B.$$

These two conditions can be reformulated by means of (10.24). Since $\overline{\xi}_A(P_0) \in \sigma_{\text{eff}}(X, U)$ for all $P_0 \in U_0(k)/T(k)$ one gets the following equivalent conditions:

10.25 (i) $x_1 x_2 \cdots x_n \neq 0$,
10.25 (ii) $P_0 \in \mathbf{K}_B$.

One can therefore regard $\overline{n}_A(\mathbf{K}_B)$ as an adelic integral approximating the sum $\mathcal{D}(B)$. To understand the link recall that the local measures in this paper are normalized (cf. sec.1) such that $\mu(A_k/k) = 1$ for the adelic product measure on A_k. This implies that the volume of the additive quotient group $X_1(A_k)/X_1(k)$ with respect to $dx_1 dx_2 \cdots dx_n$ is equal to 1.

One can now modify this discussion to give a similar interpretation of $\mathcal{C}(B)$ by means of (10.19)-(10.20). But it is better to introduce generating Dirichlet series following Manin, Batyrev and Tschinkel [**3**], [**7**], [**4**].

Notation 10.26

(a) Let $\{1,\ldots,n\} \to \Delta(1)$ be a bijection (i.e. an ordering of the rays of Δ) sending i to ρ_i. Then we shall write

$$H_i : X(k) \longrightarrow (0,\infty)$$

for the height function defined by the irreducible U-invariant Weil divisor D_i (cf. (10.6)). If $D \in \mathbb{C}^{\Delta(1)}$ is a formal sum $s_1D_1 + \cdots + s_nD_n$, $s_1,\ldots,s_n \in \mathbb{C}$ and $P \in U(k)$, put

$$h_D(P) = \sum_{i=1}^{n} s_i h_i(P),$$

$$H_D(P) = \prod_{i=1}^{n} H_i(P)^{s_i} = \exp h_D(P).$$

(b)

$$\begin{array}{rcl} h_\Delta : X(k) \times \mathbb{C}^{\Delta(1)} & \longrightarrow & \mathbb{C}, \\ H_\Delta : X(k) \times \mathbb{C}^{\Delta(1)} & \longrightarrow & \mathbb{C} \end{array}$$

are the pairings for which

$$h_\Delta(D,P) = h_D(P),$$

$$H_\Delta(D,P) = H_D(P).$$

Remarks 10.27

(a) The pairings in (10.26)(b) extend the pairings

$$\begin{array}{rcl} h_\Delta : X(k) \times \mathbb{Z}^{\Delta(1)} & \longrightarrow & \mathbb{R}, \\ H_\Delta : X(k) \times \mathbb{Z}^{\Delta(1)} & \longrightarrow & (0,\infty) \end{array}$$

defined earlier.

(b) Suppose that $D = s_1D_1 + \cdots + s_nD_n$ comes from $M_\mathbb{C} = M \otimes_\mathbb{Z} \mathbb{C}$ under the map induced by (8.2). Then $H_\Delta(D,P) = 0$ by the proof of (10.12). Therefore, the pairings in (10.26)(b) factorize over

$$U(k) \times (\operatorname{Pic} X)_\mathbb{C} \longrightarrow \mathbb{C}$$

for $(\operatorname{Pic} X)_\mathbb{C} := \operatorname{Pic} X \otimes_\mathbb{Z} \mathbb{C}$. We shall by abuse of notation write

$$h_\Delta : U(k) \times (\operatorname{Pic} X)_\mathbb{C} \longrightarrow \mathbb{C}$$

and

$$H_\Delta : U(k) \times (\operatorname{Pic} X)_\mathbb{C} \longrightarrow \mathbb{C}$$

also for these pairings.

Proposition 10.28. — *$Z_\Delta(s) := \sum_{P\in U(k)} H_\Delta(P, -s_1D_1 - \cdots - s_nD_n)$ converges absolutely and uniformly in the domain $\Re s = (\Re s_1, \ldots, \Re s_n) \in \mathbb{R}^n_{>1+\delta}$ for any $\delta > 0$.*

Proof. — This is due to Batyrev and Tschinkel [**4**, Ch. 4]. First note that

$$\begin{aligned} |H_\Delta(P, -s_1D_1 - \cdots - s_nD_n)| &= \prod_{i=1}^n |H_i(P)|^{-\Re s_i} \\ &\leq |\prod_{i=1}^n H_i(P)|^{-(1+\delta)} \\ &= H(P)^{-(1+\delta)} \end{aligned}$$

where $H : X(k) \to (0,\infty)$ is the toric height function. Then choose a finite closed subset S of $\operatorname{Spec} o$, such that $o_{(S)}$ is a principal ideal domain and consider the toric $o_{(S)}$-morphism $\widetilde{\pi}_{(S)} : \widetilde{X}_{0(S)} \to \widetilde{X}_{(S)}$ obtained by base extension from the toric o-morphism $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$. The natural map from $\widetilde{X}_0(o_{(S)})$ to $X(k)$ is surjective and

$$H_0(P_0) = \prod_{\nu\in W} \sup_{\sigma\in\Delta_{\max}} |x^{D(\sigma)}(P_0)|_\nu = \prod_{S\cup W_\infty} \sup_{\sigma\in\Delta_{\max}} |x^{D(\sigma)}(P_0)|_\nu \geq \prod_{S\cup W_\infty} |x^D|_\nu$$

for $D = D_1 + \cdots + D_n$ (cf. (11.20) for the inequality). Also, $x_1x_2\cdots x_n \neq 0$ for any lifting to $\widetilde{X}_0(o_{(S)})$ of a k-point in $U(k)$. Therefore,

$$|Z_\Delta(s)| \leq (\sum_{x\in o_{(S)}\setminus\{0\}} (\prod_{S\cup W_\infty} |x|_\nu^{-1-\delta}))^n$$

so that the desired assertion follows from standard results on S-integers of bounded height (cf. e.g. [**37**, th. 5.2(i)] for the case $S = \varnothing$). □

The deepest contribution of the authors of [**7**] and [**4**] is the construction of a meromorphic continuation of $Z_\Delta(s)$ to a region with $\Re s \in \mathbb{R}^n_{>1-\delta}$ for some $\delta > 0$. They also prove that the function $\zeta_\Delta(s) := Z_\Delta(s, \ldots, s)$ has no other poles in the region $\Re s \in \mathbb{R}_{>1-\delta}$ than at $s = 1$ where it has a pole of order $r = \operatorname{rk}\operatorname{Pic} X$. Now once this is known it follows from the Tauberian theorem of Delange [**17**] that

(10.29) $$\mathcal{C}(B) = cB(\log B)^{r-1}(1 + o(1))$$

where $c = a_{-r}/(r-1)!$ for the leading coefficient in the Laurent expansion

$$\zeta_\Delta(s) = \sum_{k=-r}^{\infty} a_k(s-1)^k.$$

Batyrev and Tschinkel prove that $c = a_{-r}/(r-1)!$ is equal to the constant $\alpha(X)\tau_\varepsilon(X, \|\ \|)$ in (10.17). We shall here give an interpretation of the leading constant as an adelic density and describe an alternative approach to derive the meromorphic continuation of $Z_\Delta(s)$ which apart from the use of multidimensional zeta functions is closer to the method of Schanuel [**56**] and Peyre [**52**].

To describe the differences between the two methods, recall that there are (at least) two methods to find a meromorphic continuation of Riemann's zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

to a region $\Re s > 1 - \delta$. The most elegant and group theoretic approach is to use the Poisson formula and integral transforms. This method was formulated in an adelic language by Tate and Iwasawa and this is the route followed by Batyrev and Tschinkel. But the use of harmonic analysis on $U(A_k)$ depends strongly on the group structure of $U(A_k)$ and it is hard to see how this "idelic" method can be applied for varieties without a transitive group action.

An alternative method to prove that $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ has a meromorphic continuation is by means of the integral representation

$$\sum_{n=1}^{\infty} n^{-s} = \int_1^{\infty} [x]^{-s} dx = \int_1^{\infty} x^{-s} dx + \sum_{n=1}^{\infty} (\int_n^{n+1} ([x]^{-s} - x^{-s}) dx) \tag{10.30}$$

where the last sum converges absolutely and uniformly on compact subsets with $\Re s > 0$. It defines therefore a holomorphic function in the right half plane. This should be seen as the "error term" while the main term $\int_1^{\infty} x^{-s} dx = 1/(s-1)$.

The idea is now to apply a multidimensional version of the last argument and approximate sums over lattices by integrals. This suffices over $\mathbb{Q}$ and the reader will find some of the techniques to bound the error terms in the next section although we shall not consider zeta functions there. Over arbitrary number fields, it is more systematical to use an adelic approach and approximate sums over the discrete set $U(k) = U_0(k)/T(k)$ by integrals over $X_0(A_k)/T(k)$. To find the adelic integral approximating $Z_\Delta(s)$ we extend the restrictions to $U(k) \times (\operatorname{Pic} X)_\mathbb{C}$ of the pairings

$$\begin{aligned} h_\Delta : X(k) \times (\operatorname{Pic} X)_\mathbb{C} &\longrightarrow \mathbb{C}, \\ H_\Delta = \exp h_\Delta : X(k) \times (\operatorname{Pic} X)_\mathbb{C} &\longrightarrow \mathbb{C} \end{aligned}$$

in (10.27)(b) to pairings:

$$\begin{aligned} h_{\Delta,A} : X_0(A_k)/T(k) \times (\operatorname{Pic} X)_\mathbb{C} &\longrightarrow \mathbb{C}, \\ H_{\Delta,A} = \exp h_{\Delta,A} : X_0(A_k)/T(k) \times (\operatorname{Pic} X)_\mathbb{C} &\longrightarrow \mathbb{C}. \end{aligned}$$

To define these, we use the canonical toric splitting. Let

$$\overline{\xi}_A : X_0(A_k)/T(k) \longrightarrow \operatorname{Hom}_\mathbb{Z}(\operatorname{Pic} X, \mathbb{R})$$

be the continuous map described in (10.21)(i) and let $(\operatorname{Pic} X)_{\mathbb{R}} = \operatorname{Pic} X \otimes \mathbb{R}$. The obvious $\mathbb{R}$-linear pairing $\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{R}) \times (\operatorname{Pic} X)_{\mathbb{R}} \to \mathbb{R}$ of dual real vector spaces extends uniquely to a pairing:

$$\langle\ ,\ \rangle : \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{R}) \times (\operatorname{Pic} X)_{\mathbb{C}} \longrightarrow \mathbb{C} \tag{10.31}$$

which is $\mathbb{C}$-linear on the right hand side.

Notations 10.32. — Let $(\overline{P}_0, [D]) \in X_0(A_k)/T(k) \times (\operatorname{Pic} X)_{\mathbb{C}}$. Then,

$$h_{\Delta,A}(\overline{P}_0, [D]) = \langle \overline{\xi}_A(\overline{P}_0), [D]\rangle,$$
$$H_{\Delta,A}(\overline{P}_0, [D]) = \exp\langle \overline{\xi}_A(\overline{P}_0), [D]\rangle.$$

Proposition 10.33. — *The pairings*

$$\begin{array}{rcl} h_{\Delta,A} : X_0(A_k)/T(k) \times (\operatorname{Pic} X)_{\mathbb{C}} & \longrightarrow & \mathbb{C}, \\ H_{\Delta,A} : X_0(A_k)/T(k) \times (\operatorname{Pic} X)_{\mathbb{C}} & \longrightarrow & \mathbb{C} \end{array}$$

coincide on $U_0(k)/T(k) = U(k)$ with the pairings

$$\begin{array}{rcl} h_{\Delta} : X(k) \times (\operatorname{Pic} X)_{\mathbb{C}} & \longrightarrow & \mathbb{C}, \\ H_{\Delta} : X(k) \times (\operatorname{Pic} X)_{\mathbb{C}} & \longrightarrow & \mathbb{C} \end{array}$$

in (10.27)(b).

Proof. — The pairings $h_{\Delta,A}$ and h_{Δ} are $\mathbb{C}$-linear on the right hand side. It is therefore sufficient to prove that $h_{\Delta,A}(P, [D]) = h_{\Delta}(P, [D])$, $P \in U_0(k)/T(k) = U(k)$ for $D \in \mathbb{Z}^{\Delta(1)}$. But this has already been established in (10.24). □

Remark 10.34. — The restriction of $h_{\Delta,A}$ to

$$X_0(k)/T(k) \times (\operatorname{Pic} X)_{\mathbb{C}} = X(k) \times (\operatorname{Pic} X)_{\mathbb{C}}$$

does not coincide with h_{Δ} outside $U(k) \times (\operatorname{Pic} X)_{\mathbb{C}}$. This is not surprising since the conjectures of Manin and Peyre according to our reinterpretation will give a link between the counting functions on $U(k)$ and adelic integrals on $X(A_k)/T(k)$ whereas on $X(k)\backslash U(k)$ there are accumulating subsets (cf. (7.5)).

To study the adelic integral which approximates $Z_{\Delta}(s)$, we shall need a positive linear map

$$\Lambda_{\Delta} : C_c(X_0(A_k)/T(k)) \longrightarrow C_c(\operatorname{Hom}(\operatorname{Pic} X, \mathbb{R}))$$

obtained by integrating along the fibres of

$$\overline{\xi}_A : X_0(A_k)/T(k) \longrightarrow \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R}).$$

To define this map we choose (cf. the proof of (5.16)) two sets

$$\{\beta_\nu,\ \nu \in W\},$$
$$\{\gamma_\nu,\ \nu \in W\}$$

of convergence factors for T resp. X such that $\beta_\nu\gamma_\nu = 1$ for all $\nu \in W$. Let Θ_Σ be the Haar measure on $T(A_k)$ given by the adelic order norm (cf. (5.9)(b)) and the convergence factors $\{\beta_\nu,\ \nu \in W\}$ and let $\overline{\Theta}{}^1_\Sigma$ be the corresponding Haar measure on $T^1(A_k)/T(k)$ under the bijection between Haar measures on $T(A_k)$ and $T^1(A_k)/T(k)$ described after (5.19). Finally, let m be the measure on $X(A_k)$ given by the toric adelic norm and the convergence factors $\{\gamma_\nu,\ \nu \in W\}$.

Then, there are two positive linear maps

$$\begin{aligned}\overline{\Lambda}_\beta : C_c(X_0(A_k)/T(k)) &\longrightarrow C_c(X_0(A_k)/T^1(A_k)),\\ \Lambda_\gamma : C_c(X_0(A_k)/T^1(A_k)) &\longrightarrow C_c(\operatorname{Hom}(\operatorname{Pic} X, \mathbb{R}))\end{aligned}$$

defined as follows. Let $f \in C_c(X_0(A_k)/T(k))$ and $x \in X_0(A_k)/T(k)$ be a lifting of $x^1 \in X_0(A_k)/T^1(A_k)$. Then

$$\overline{\Lambda}_\beta(f)(x^1) := \int_{T^1(A_k)/T(k)} f(tx)d\overline{\Theta}{}^1_\Sigma. \tag{10.35}$$

Here we integrate with respect to $t \in T^1(A_k)/T(k)$. The notation $tx \in X_0(A_k)/T(k)$ refers to the the translation

$$T^1(A_k)/T(k) \times X_0(A_k)/T(k) \longrightarrow X_0(A_k)/T(k)$$

induced by

$$\sigma_A : T(A_k) \times X_0(A_k) \longrightarrow X_0(A_k)$$

(cf. (5.3)). This map makes

$$X_0(A_k)/T(k) \longrightarrow X_0(A_k)/T^1(A_k)$$

into a topological torsor under the compact group $T^1(A_k)/T(k)$. We may therefore apply the results in [**11**, Ch. VII, §2 n°2] and conclude that $\overline{\Lambda}_\beta(f)(x^1)$ is a $T^1(A_k)/T(k)$-invariant function on $X_0(A_k)/T(k)$ with compact support. It is clear that $\overline{\Lambda}_\beta$ is a positive linear map.

To define Λ_γ, we make use of the canonical splitting

$$\xi^1_A : X_0(A_k)/T^1(A_k) \longrightarrow \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})$$

in (10.21)(i) (cf. also (5.24)) and the corresponding canonical isomorphism

$$X_0(A_k)/T^1(A_k) = X(A_k) \times \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R}). \tag{10.36}$$

Then each $g \in C_c(X_0(A_k)/T^1(A_k))$ can be regarded as a function on $X(A_k) \times \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})$. Further, by (1.16) and [**11**, Ch. III, §5], the integral

$$\int_{P\in X(A_k)} g(P,\lambda)dm, \quad \lambda \in \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})$$

defines a continuous function on $\operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})$. There is, therefore, a positive linear map

$$\Lambda_\gamma : C_c(X_0(A_k)/T^1(A_k)) \longrightarrow C_c(\operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})) \tag{10.37}$$

with

$$\Lambda_\gamma(g)(\lambda) = \int_{P\in X(A_k)} g(P,\lambda)dm$$

for $\lambda \in \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$.

Definition 10.38. — The *toric positive linear map*

$$\Lambda_\Delta : C_c(X_0(A_k)/T(k)) \longrightarrow C_c(\mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R}))$$

is the composition of $\overline{\Lambda}_\beta$ and Λ_γ.

The map Λ_Δ does not depend on the choice of the convergence factors as long as $\beta_\nu\gamma_\nu = 1$ for all $\nu \in W$. It depends only on the fan Δ.

Lemma 10.39. — *Let n_A be the measure on $\mathcal{T}(A_k) = X_0(A_k)$ determined by the induced adelic norm (cf. (5.16)) of the toric adelic norm on $X(A_k)$ and let $\overline{n}_A$ be the quotient measure of n_A on $X_0(A_k)/T(k)$ (cf. (5.23)). Let*

$$V = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}\, X, \mathbb{R}), \qquad L = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}\, X, \mathbb{Z}) \subset V$$

and $d\nu$ be the unique Haar measure on V such that the volume $\mathrm{Vol}(V/L) = 1$. *Then, the following holds.*

(a)

$$\int_{\mathcal{T}(A_k)/T(k)} g d\overline{n}_A = \int_{\mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})} \Lambda_\Delta(g) d\nu$$

for any $g \in C_c(\mathcal{T}(A_k)/T(k))$.

(b) *Let $h \in C_c(V)$. Then,*

$$h \circ \overline{\xi}_A \in C_c(\mathcal{T}(A_k)/T(k)),$$

$$\Lambda_\Delta(h \circ \overline{\xi}_A) = \tau_\varepsilon(X, \|\ \|)h.$$

Proof

(a) Let

$$\mathrm{Tr} : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(\mathcal{T}(A_k)/T(k))$$

be the trace map obtained by summing over all $T(k)$-translates of a function in $C_c(\mathcal{T}(A_k))$. It suffices to prove the equality in the case $f = \mathrm{Tr}(g)$ for each $g \in C_c(\mathcal{T}(A_k))$. Let

$$\beta = \{\beta_\nu,\ \nu \in W\}, \qquad \gamma = \{\gamma_\nu,\ \nu \in W\}$$

be two sets of convergence factors for T resp. X such that $\beta_\nu\gamma_\nu = 1$ for all $\nu \in W$ and let

$$\Lambda_\beta : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(X(A_k))$$

be the positive linear map defined by the adelic order norm on T and β (cf. (5.9)(b)). Then, by (5.23) and (4.28) there are equalities:

$$\int_{\mathcal{T}(A_k)/T(k)} g d\overline{n}_A = \int_{\mathcal{T}(A_k)} f dn_A = \int_{X(A_k)} \Lambda_\beta(f) dm_A \tag{10.40}$$

for the measure m_A on $X(A_k)$ defined by the toric norm and γ.

Now make use of the canonical isomorphism

$$\mathcal{T}(A_k)/T^1(A_k) = X(A_k) \times \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})$$

in (10.36). Then by (1.16) we obtain a positive linear map

$$\Lambda^1 : C_c(\mathcal{T}(A_k)/T^1(A_k)) \longrightarrow C_c(X(A_k))$$

by integrating over the fibres of the projection $\mathcal{T}(A_k)/T^1(A_k) \to X(A_k)$. We may also change the order of integration (cf. op. cit.) such that

$$\int_{X(A_k)} \Lambda^1(h) dm_A = \int_{\operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})} \Lambda_\gamma(h) d\nu \tag{10.41}$$

for any $h \in C_c(\mathcal{T}(A_k)/T^1(A_k))$. Finally, the following positive linear maps are equal:

$$\Lambda_\beta = \Lambda^1 \circ \overline{\Lambda}_\beta \circ \operatorname{Tr}. \tag{10.42}$$

This is a consequence of the compatibility of the Haar measures of

$$T(A_k), \quad T(A_k)/T^1(A_k), \quad T^1(A_k)/T(k) \quad \text{and} \quad T(k)$$

(cf. the discussion after (5.19)) and the alternative description of Λ_β. Now let $h = \overline{\Lambda}_\beta(g)$. Then, by (10.40) and (10.42) it follows that:

$$\int_{\mathcal{T}(A_k)/T(k)} g d\overline{n}_A = \int_{X(A_k)} \Lambda_\beta(f) dm_A = \int_{X(A_k)} \Lambda^1(h) dm_A. \tag{10.43}$$

Therefore, (a) follows from (10.43) and (10.41) and the identity $\Lambda_\gamma(h) = \Lambda_\Delta(g)$.

(b) It is clear that $h \circ \overline{\xi}_A \in C_c(\mathcal{T}(A_k)/T(k))$ since $\overline{\xi}_A : X_0(A_k)/T(k) \to V$ is continuous and proper (in the topological sense). The equality is a direct consequence of the definitions of Λ_Δ and $\tau_\varepsilon(X, \| \|)$ (see (5.24) for more details). This completes the proof. □

Notation 10.44. — Let $\sigma_{\text{eff}}(X, U) \subset V = \operatorname{Hom}(\operatorname{Pic} X \to \mathbb{R})$ be the (dual) effective cone described in (10.18) and let $\overline{\xi}_A : X_0(A_k)/T(k) \to \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})$ be the map described in (10.21)(j). Then,

$$(X_0(A_k)/T(k))_{\text{eff}} := \overline{\xi}_A^{-1}(\sigma_{\text{eff}}(X, U)).$$

Definition and proposition 10.45. — *Let n_A be the (restricted) product measure on $X_0(A_k)$ of the induced adelic norm (cf. (5.16)) for X_0 of the toric adelic norm for X and let $\overline{n}_A$ be the quotient measure on $X_0(A_k)/T(k)$ (cf. (5.23)). Then,*

$$I_\Delta(s) := \int_{\overline{P}_0 \in (X_0(A_k)/T(k))_{\mathrm{eff}}} H_\Delta(\overline{P}_0, -s_1D_1 - \cdots - s_nD_n) d\overline{n}_A$$

converges absolutely in the domain $\Re s = (\Re s_1, \ldots, \Re s_n) \in \mathbb{R}^n_{>1}$. Moreover, if $d\nu$ is the normalized Haar measure of $V = \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$ in (10.18), and

$$\langle\ ,\ \rangle : V \times (\mathrm{Pic}\, X)_{\mathbb{C}} \longrightarrow \mathbb{C}$$

is the obvious pairing (cf. (10.31)), then

$$I_\Delta(s) = \tau_\varepsilon(X, \|\ \|) \int_{\lambda \in \sigma_{\mathrm{eff}}(X,U)} \exp\langle \lambda, [-s_1D_1 - \cdots - s_nD_n]\rangle d\nu \tag{10.46}$$

in the domain $\Re s = (\Re s_1, \ldots, \Re s_n) \in \mathbb{R}^n_{>1}$.

Proof. — Fix $s \in \mathbb{R}^n_{>1}$ and let $f_s : V \to \mathbb{R}$ be the function with support in $\sigma_{\mathrm{eff}}(X, U)$ such that $f_s(\lambda) = \exp\langle \lambda, [-s_1D_1 - \cdots - s_nD_n]\rangle$ for $\lambda \in \sigma_{\mathrm{eff}}(X, U)$. Then,

$$f_s(\overline{\xi}_A(\overline{P}_0)) = \begin{cases} H_\Delta(\overline{P}_0, -s_1D_1 - \cdots - s_nD_n) & \text{if } \overline{P}_0 \in (X_0(A_k)/T(k))_{\mathrm{eff}}, \\ 0 & \text{if } \overline{P}_0 \notin (X_0(A_k)/T(k))_{\mathrm{eff}}. \end{cases}$$

so that

$$I_\Delta(s) := \int_{X_0(A_k)/T(k)} (f_s \circ \overline{\xi}_A) d\overline{n}_A.$$

Now recall that $C_c(V)$ is dense in $L^1(d\nu)$ (see [**38**, Ch. IX, §3]). There exists thus an L^1-Cauchy sequence $(h_i)_{i=1}^\infty$ in $C_c(V)$ converging to h. Hence, by (10.39), $(h_i \circ \overline{\xi}_A)_{i=1}^\infty$ is a L^1-Cauchy sequence in $C_c(X_0(A_k)/T(k))$ converging to $f_s \circ \overline{\xi}_A$ with

$$\lim_{i\to\infty} \int_{X_0(A_k)/T(k)} (h_i \circ \overline{\xi}_A) = \tau_\varepsilon(X, \|\ \|) \lim_{i\to\infty} \int_V h_i d\nu = \tau_\varepsilon(X, \|\ \|) \int_V f_s d\nu.$$

Hence $I_\Delta(s)$ converges absolutely and takes the value:

$$\begin{aligned} I_\Delta(s) &= \tau_\varepsilon(X, \|\ \|) \int_V f_s d\nu \\ &= \tau_\varepsilon(X, \|\ \|) \int_{\lambda \in \sigma_{\mathrm{eff}}(X,U)} \exp\langle \lambda, [-s_1D_1 - \cdots - s_nD_n]\rangle d\nu \end{aligned}$$

for $s \in \mathbb{R}^n_{>1}$. But then $I_\Delta(s)$ converges absolutely for $s \in \mathbb{C}^n$ with

$$(\Re s_1, \ldots, \Re s_n) \in \mathbb{R}^n_{>1}$$

since

$$|H_\Delta(\overline{P}_0, -s_1D_1 - \cdots - s_nD_n)| = H_\Delta(\overline{P}_0, -(\Re s_1)D_1 - \cdots - (\Re s_n)D_n).$$

It is also clear how to extend the proof to complex $\mathbf{s}$ by means of complex valued L^1-Cauchy sequences $(h_i)_{i=1}^{\infty}$ in $C_c(V, \mathbb{C})$. This completes the proof. □

Corollary 10.47. — *Let $I_\Delta(\mathbf{s})$ be as above and let $Q_\Delta(\mathbf{s})$ be a product of linear forms defining the codimension one faces of the cone in* $\operatorname{Pic} X$*, generated by the classes $[D] \in \operatorname{Pic} X$ of effective U-invariant Weil divisors $D \in \mathbb{Z}^{\Delta(1)}$. Then $I_\Delta(\mathbf{s})Q_\Delta(\mathbf{s})$ is a polynomial in $s_1, \ldots, s_n$.*

Proof. — This follows from a result of Batyrev and Tschinkel [**4**, prop. 5.4] applied to the characteristic transform:

$$\int_{\lambda \in \sigma_{\mathrm{eff}}(X,U)} \exp\langle \lambda, [-s_1D_1 - \cdots - s_nD_n]\rangle d\nu$$

of the effective cone in $\operatorname{Pic} X$. □

Now recall that $U(k) = U_0(k)/T(k)$ is a discrete subset of $(X_0(A_k)/T(k))_{\mathrm{eff}}$ (cf. (10.24)) and that the volume of the additive quotient group $X_1(A_k)/X_1(k)$ with respect to $dx_1dx_2\cdots dx_n$ is equal to 1, because of the normalizations of the Haar measures in section 1. It is therefore reasonable to regard

$$I_\Delta(s) := \int_{\overline{P}_0 \in (X_0(A_k)/T(k))_{\mathrm{eff}}} H_\Delta(\overline{P}_0, -s_1D_1 - \cdots - s_nD_n) d\overline{n}_A$$

as a continuous approximation of

$$Z_\Delta(s) := \sum_{P \in U(k)} H_\Delta(P, -s_1D_1 - \cdots - s_nD_n)$$

in the domain $\Re s = (\Re s_1, \ldots, \Re s_n) \in \mathbb{R}^n_{>1}$.

To make this precise, one has to choose a fundamental domain in $X_0(A_k)$ under the action of $T(k)$. Suppose for simplicity that the o is a principal domain. Then the natural map from $T(k)$ to the direct sum $\oplus T(k_\nu)/T(k_\nu)_{\mathrm{cp}}$ over all non-archimedean places ν of k is surjective. This together with the canonical isomorphisms in (10.8) implies that there is a canonical isomorphism:

$$\left(\prod_{W_{\mathrm{fin}}} \widetilde{X}_0(o_\nu) \times \prod_{W_\infty} X_0(k_\nu)\right) \Big/ T(o) = X_0(A_k)/T(k)$$

so that we are reduced to a choice of a fundamental domain modulo $T(o)$. We refer to the papers of Schanuel [**56**] and Peyre [**52**], [**51**] for more details about the choice of fundamental domains. The zeta function $Z_\Delta(s)$ may be reinterpreted as a sum over a subset of $U_0(k)$ lying in the fundamental domain

$\mathcal{F}$ and the goal is to approximate this by the corresponding integral over the subset of $\mathcal{F}$ defined by the inverse image of $(X_0(A_k)/T(k))_{\text{eff}}$.

If we restrict to the diagonal $s_1 = \cdots = s_n$ and put $\zeta_\Delta(s) := Z_\Delta(s, \ldots, s)$ for $\Re s > 1$, this leads to proving that $\zeta_\Delta(s) - I_\Delta(s, \ldots, s)$ is small compared to $I_\Delta(s, \ldots, s)$.

Batyrev and Tschinkel prove in [**7**] that the characteristic transform in (10.46) has a pole of order r and that the leading coefficient in the Laurent expansion is equal to $(r-1)! \cdot \alpha_{\text{Peyre}}(X)$. If one applies the Tauberian theorem in $[D]$ to $\zeta_\Delta(s)$, one thus concludes that if

$$(s-1)^{r-1}(\zeta_\Delta(s) - I_\Delta(s, \ldots, s))$$

has a holomorphic continuation to a right half plane $\Re s > 1 - \delta$, $\delta > 0$, then (7.6) and (7.7) holds for the counting function defined by the toric height. (This is how Batyrev and Tschinkel proceeds in [**7**] and [**4**] although there is no volume-theoretic interpretation of I_Δ there.)

We shall in the next section prove (7.6) and (7.7) for toric varieties over $\mathbb{Q}$ without making use of Manin's zeta function $\zeta_\Delta(s)$. But the bounds of the error terms in section 11 can be reinterpreted as bounds for $\zeta_\Delta(s) - I_\Delta(s, \ldots, s)$.

11. Asymptotic formulas for counting functions on toric $\mathbb{Q}$-varieties

We shall in this section study the asymptotic growth of counting functions for (split) toric varieties X_Δ over $\mathbb{Q}$ defined by complete regular fans Δ. We shall assume that the base field $k = \mathbb{Q}$, but otherwise keep the notations in section 10. We will thus write H for the toric (anticanonical) height function and $\mathcal{C}(B)$ for the number of rational points of toric height at most B on the open $\mathbb{Q}$-torus U in $X = X_\Delta$. Also, r will denote the rank of $\operatorname{Pic} X$ as in the previous sections. Hence $r = \operatorname{Card} \Delta(1) - \dim \Delta$ by (8.2). We assume throughout this section that ω_X^{-1} is generated by its global sections.

The aim of this section is to prove the asymptotic formula

$$\mathcal{C}(B) = CB(\log B)^{r-1} + O(B(1 + \log B)^{r-3/2+\varepsilon}) \tag{11.1}$$

where

$$C = \alpha(X)\tau(X, \|\ \|) > 0$$

is the constant of Peyre (cf. (7.7)) and $\varepsilon > 0$ is a positive number which may be arbitrarily small. This improves somewhat upon the earlier results of Peyre [**52**] for special classes of toric $\mathbb{Q}$-varieties and the results of Batyrev and Tschinkel [**7**], [**4**] for arbitrary toric varieties. They prove asymptotic formulas of the type

$$\mathcal{C}(B) = \alpha(X)\tau(X, \|\ \|)B(\log B)^{r-1}(1 + o(1))$$

Batyrev and Tschinkel deduce this result from the meromorphic continuation of the Manin zeta function $\zeta_\Delta(s)$ (cf. the end of sec. 10). By using their estimates for the growth of $\zeta_\Delta(s)$ along suitable vertical strips with $\Re s < 1$ and a method of Landau (cf. [**60**]), one can probably deduce the more precise result

$$\mathcal{C}(B) = BP(\log B) + O(B^{1-\delta})$$

for some real polynomial P and some positive number δ.

Our proof of (11.1) has its origin in the papers of Schanuel [**56**] and Peyre [**52**] on projective spaces resp. certain special blow-ups of projective spaces. The main idea is to count integer points on models of finite type over $\mathbb{Z}$ of universal torsors over X instead of rational points on X. If $r = 1$, then $X_\Delta = \mathbb{P}^n$ (cf. (8.9)(v) and [**24**, p. 22]) and the toric height is equal to the standard height. There is, then, only one (isomorphism class of) universal torsors given by the affine cone $\mathbb{A}^{n+1}\backslash(0, \dots, 0)$ of $\mathbb{P}^n$. Hence our method reduces to Schanuel's method when $r = 1$. We shall therefore omit this trivial case in some lemmas.

Let W be the set of places of $\mathbb{Q}$ and let

$$\| \ \| = \{\| \ \|_\nu \ , \ \nu \in W\}$$

be the toric adelic norm for X. This adelic norm gives rise to the toric height function H on $X(\mathbb{Q})$ defined by

$$H(P) = \prod_{\nu \in W} \|s(P)\|_\nu^{-1} \tag{11.2}$$

where s is a local section of the anticanonical sheaf ω_X^{-1} at P with $s(P) \neq 0$ (cf. (7.2)).

Let $\pi : X_0 \to X$ (resp. $\pi : \widetilde{X}_0 \to \widetilde{X}$) be the principal universal torsor over $\mathbb{Q}$ (resp. $\mathbb{Z}$) described in the beginning of section 10. Let

$$D_0 = \sum_{\rho \in \Delta(1)} D_\rho.$$

If σ is a maximal cone of Δ, let $\chi^{m(\sigma)}$, $m(\sigma) \in M$ be the unique character of U such that $\chi^{-m(\sigma)}$ generates $\mathcal{O}(D_0)$ on U_σ.

Let

$$D(\sigma) = D_0 + \sum_{\rho \in \Delta(1)} \langle -m(\sigma), n_\rho \rangle, \quad \sigma \in \Delta_{\max}$$

be the Weil divisors described in (8.7) and recall that these are effective divisors when ω_X^{-1} is generated by its global sections. Let

$$x^D = \prod_{\rho \in \Delta(1)} x_\rho^{a_p}$$

be the monomial in the indeterminates x_ρ, $\rho \in \Delta(1)$ corresponding to the effective Weil divisor

$$D = \sum_{\rho\in\Delta(1)} a_\rho D_\rho \in \mathbb{Z}^{\Delta(1)}.$$

We now give a concrete description of the composite map $H_0 = H \circ \pi$ from $X_0(\mathbb{Q})$ to $\mathbb{R}_{>0}$ in the case where $\mathcal{O}(D_0)$ is generated by its global sections.

Proposition 11.3. — *Let P_0 be a $\mathbb{Q}$-point on X_0 which extends to a $\mathbb{Z}$-point on $\widetilde{X}_0$. Then*

$$H_0(P_0) = \sup_{\sigma\in\Delta_{\max}} |x^{D(\sigma)}(P_0)|$$

for the usual archimedean absolute value $|\ |$ of $\mathbb{R}$.

Proof. — Let $\widetilde{P}_0 \in \widetilde{X}_0(\mathbb{Z})$ be the extension of P_0 and let $\widetilde{X}_0 \subset \widetilde{X}_1 = \operatorname{Spec}\mathbb{Z}[x_\rho]$, $\rho \in \Delta(1)$ be the toric open immersion described after 10.3. Let p be a prime and let $Y_0 \subset \operatorname{Spec}\mathbb{Z}/p\mathbb{Z}[x_\rho]$, $\rho \in \Delta(1)$ be the reduction modulo p of $\widetilde{X}_0$. Then, by applying (8.8)(a) to Y_0, we conclude that $x^{D(\sigma)}(\widetilde{P}_0) \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$ for some maximal cone σ of Δ. Hence the product formula for $H_0(P_0)$ in (10.14) reduces to its archimedian factor. This completes the proof. $\square$

Lemma 11.4

(a) *Let $c(m) = \operatorname{Card}\{P \in U(\mathbb{Q}) : H(P) = m\}$ and*

$$c_0(m) = \operatorname{Card}\{P_0 \in \widetilde{X}_0(\mathbb{Z}) \cap U_0(\mathbb{Q}) : H_0(P_0) = m\}.$$

Then, $c(m) = c_0(m)/\operatorname{Card}\widetilde{T}(\mathbb{Z})$.

(b) *Let $U_0(\mathbb{R})^+$ be the real connected component of $U_0(\mathbb{R})$ containing 1 and let*

$$U_0(\mathbb{Q})^+ = U_0(\mathbb{Q}) \cap U_0(\mathbb{R})^+,$$

$$c_0(m)^+ = \operatorname{Card}\{P_0 \in \widetilde{X}_0(\mathbb{Z}) \cap U_0(\mathbb{Q})^+ : H_0(P) = m\}.$$

Then,

$$c_0(m) = \operatorname{Card}(U_0(\mathbb{R})/U_0(\mathbb{R})^+)c_0(m)^+.$$

Proof

(a) It follows from Grothendieck's version of Hilbert 90 (cf. [**45**, p. 124]) and the fact that $\mathbb{Z}$ is a principal ideal domain that $H^1_{\text{ét}}(\operatorname{Spec}\mathbb{Z}, \widetilde{T}) = 0$. The fibres of $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ at $\mathbb{Z}$-points on $\widetilde{X}$ are therefore trivial $\operatorname{Spec}\mathbb{Z}$-torsors under $\widetilde{T}$. Hence $\widetilde{\pi}(\widetilde{X}_0(\mathbb{Z})) = \widetilde{X}(\mathbb{Z})$ and there are exactly $\operatorname{Card}\widetilde{T}(\mathbb{Z})$ points in $\widetilde{X}_0(\mathbb{Z})$ with a given image in $\widetilde{X}(\mathbb{Z})$.

(b) We regard $\widetilde{U}_0(\mathbb{Z})$ as a subgroup of $U_0(\mathbb{Q})$. Then, since the group action $U_0 \times U_0 \to U_0$ extends to a toric action $\widetilde{U}_0 \times \widetilde{X}_0 \to \widetilde{X}_0$ we conclude that $\widetilde{X}_0(\mathbb{Z}) \cap U_0(\mathbb{Q})$ is a union of $\widetilde{U}_0(\mathbb{Z})$-cosets in $U_0(\mathbb{Q})$. Also, each $\widetilde{U}_0(\mathbb{Z})$-coset in $U_0(\mathbb{Q})$ contains exactly one element in $U_0(\mathbb{Q})^+$ and the height function

H_0 takes the same value for each $\mathbb{Q}$-point in a $\widetilde{U}_0(\mathbb{Z})$-orbit of $X_0(\mathbb{Q})$. Hence $c_0(m) = c_0(m)^+ \operatorname{Card} \widetilde{U}_0(\mathbb{Z})$. To complete the proof, note that there is a canonical group isomorphism between $U_0(\mathbb{R})$ and $\widetilde{U}_0(\mathbb{Z}) \times U_0(\mathbb{R})^+$. $\square$

We may thus instead of counting $\mathbb{Q}$-points on U count $\mathbb{Z}$-points on the universal torsor $\widetilde{X}_0$ such that the corresponding $\mathbb{Q}$-points lie on $\pi^{-1}(U)$. The number of such $\mathbb{Z}$-points on the universal torsor $\widetilde{X}_0$ of a given height is the same in each real connected component of U_0.

To study $c_0(m)$, let $\rho_1, \dots, \rho_n$ be an ordering of the rays in Δ. Let $D_1, \dots, D_n$ be the corresponding U-invariant irreducible Weil divisors and let $x_1, \dots, x_n$ be the variables corresponding to these prime divisors. Each $x^{D(\sigma)}$, $\sigma \in \Delta_{\max}$ is a monomial in $x_1, \dots, x_n$ with exponents $\varepsilon_1, \dots, \varepsilon_n$ corresponding to the multiplicities of $D(\sigma)$ along $D_1, \dots, D_n$. If $r = (r_1, \dots, r_n)$ is an n-tuple with components in an arbitrary commutative ring R and $\sigma \in \Delta_{\max}$, let $r^{D(\sigma)}$ be the element in R obtained by evaluating $x^{D(\sigma)}$ at $x = r$.

Now recall (cf. the comments after (8.6) and (10.3)) that the open toric embedding

$$\widetilde{X}_0 \subset \widetilde{X}_1 = \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$$

is the complement in $\widetilde{X}_1$ of the closed subscheme defined by the monomials $x^{\underline{\sigma}}$, $\sigma \in \Delta_{\max}$. This implies that $\widetilde{X}_0(\mathbb{Z}) \subset \widetilde{X}_1(\mathbb{Z}) = \mathbb{Z}^{(n)}$ is the subset of n-tuples $\mathbf{q}$ of integers for which the greatest common divisor

$$\gcd_{\sigma \in \Delta_{\max}} (\mathbf{q}^{\underline{\sigma}}) = 1.$$

Hence, by (11.3) it follows that $c_0(m)$ is the number of n-tuples $\mathbf{q}$ of non-zero integers satisfying the following two conditions

11.5 (i) $$\sup_{\sigma \in \Delta_{\max}} |\mathbf{q}^{D(\sigma)}| = m,$$

11.5 (ii) $$\gcd_{\sigma \in \Delta_{\max}} (\mathbf{q}^{\underline{\sigma}}) = 1.$$

Moreover, $c_0(m)^+$ is the number of n-tuples $\mathbf{q}$ of *positive* integers satisfying (11.5).

If $\mathcal{O}(D_0)$ is generated by its global sections, then we have seen in the proof of (11.2) that (11.5)(ii) implies that

11.5 (ii)' $$\sup_{\sigma \in \Delta_{\max}} |\mathbf{q}^{D(\sigma)}| = 1,$$

If $\mathcal{O}(D_0)$ is ample, then these two conditions are equivalent (cf. (8.8)(b)).

It is difficult to make use of (11.5)(11.5 (ii)) directly. We shall therefore first study a simpler counting function $A_{\mathbf{d}}(H)$ and then make use of Möbius inversion. This idea is central in the papers of Schanuel [**56**] and Peyre [**52**].

Notation 11.6

(a) $a(m)$ is the number of n-tuples $\mathbf{q}$ of positive integers such that

$$\sup_{\sigma \in \Delta_{\max}} |\mathbf{q}^{D(\sigma)}| = m$$

$A(B)$ is the number of n-tuples $\mathbf{q}$ of positive integers such that

$$\sup_{\sigma \in \Delta_{\max}} |\mathbf{q}^{D(\sigma)}| \leq B.$$

(b) Let $\mathbf{d}$ be an n-tuple of positive integers. Then, $a_{\mathbf{d}}(m)$ is the number of n-tuples $\mathbf{q}$ of positive integers such that

$$\sup_{\sigma \in \Delta_{\max}} |\mathbf{q}^{D(\sigma)}| = m, \qquad \mathbf{d} \mid \mathbf{q}$$

(i.e. d_i divides q_i for $i = 1, \ldots, n$) whereas $A_{\mathbf{d}}(B)$ is the number of n-tuples $\mathbf{q}$ of positive integers such that

$$\sup_{\sigma \in \Delta_{\max}} |\mathbf{q}^{D(\sigma)}| \leq B, \qquad \mathbf{d} \mid \mathbf{q}.$$

(c) $\mathcal{C}_0(B)$ (resp. $\mathcal{C}_0(B)^+$) is the set of all n-tuples $\mathbf{q}$ of non-zero integers (resp. the set of all n-tuples $\mathbf{q}$ of positive integers) such that
   (i) $\sup_{\sigma \in \Delta_{\max}} |\mathbf{q}^{D(\sigma)}| \leq B$,
   (ii) $\gcd_{\sigma \in \Delta_{\max}} (\mathbf{q}^{\underline{\sigma}}) = 1$.

(d)

$$\Pi(\mathbf{d}) := \prod_{i=1}^{n} d_i$$

for any n-tuple of integers $\mathbf{d} = (d_1, \ldots, d_n)$.

All the cardinalities above are finite since (cf. (11.20))

$$\sup_{\sigma \in \Delta_{\max}} |\mathbf{q}^{D(\sigma)}| \geq |\Pi(\mathbf{q})|$$

Lemma 11.7. — *Let* $\mathbf{d}$ *be an* n*-tuple of positive integers. Then,*

$$0 \leq \Pi(\mathbf{d}) A_{\mathbf{d}}(B) \leq A(B).$$

Proof. — We introduce an equivalence relation $\sim$ on the set of all positive n-tuples $\mathbf{q} = (q_1, \ldots, q_n)$ such that

$$\sup_{\sigma \in \Delta_{\max}} |\mathbf{q}^{D(\sigma)}| \leq B.$$

Let

$$\mathbf{q} = (q_1, \ldots, q_n) \sim \mathbf{r} = (r_1, \ldots, r_n)$$

if and only if

$$([(q_1-1)/d_1],\dots,[(q_n-1)/d_n]) = ([(r_1-1)/d_1],\dots,[(r_n-1)/d_n]).$$

An equivalence class consists of $\Pi(\mathbf{d})$ elements if and only if it contains a positive n-tuple $\mathbf{q}\in\mathbb{Z}^n_{>0}$ such that $\mathbf{d}\mid\mathbf{q}$. Also, no equivalence class contains more than one such n-tuple. Therefore, $A_{\mathbf{d}}(B)$ is the number of equivalence classes with $\Pi(\mathbf{d})$ elements. This implies the assertion. □

Notations 11.8

(a)

$$\chi : \mathbb{Z}^n_{>0} \longrightarrow \{0,1\}$$

is the function such that $\chi(\mathbf{e}) = 1$ if and only if the greatest common divisor of all $\mathbf{e}^{\underline{\sigma}}$, $\sigma\in\Delta_{\max}$ is 1.

(b)

$$\chi^{(p)} : \mathbb{Z}^n_{>0} \longrightarrow \{0,1\}$$

is the function such that $\chi^{(p)}(\mathbf{e}) = 1$ if and only if the greatest common divisor of all $\mathbf{e}^{\underline{\sigma}}$, $\sigma\in\Delta_{\max}$ is not divisible by p.

(c) Let $\mathbf{d}\in\mathbb{Z}^n_{>0}$. Then,

$$\chi_{\mathbf{d}} : \mathbb{Z}^n_{>0} \longrightarrow \{0,1\}$$

is the function such that $\chi_d(e) = 1$ if and only if $\mathbf{d}\mid\mathbf{e}$.

We have also denoted characters of tori by χ. But no confusion should occur since the indices are not the same.

We now define a Möbius function

$$\mu : \mathbb{Z}^n_{>0} \longrightarrow \mathbb{Z}$$

recursively with respect to the relation $|$. This Möbius function was defined by Peyre [**52**, 7.1.7] for special classes of toric varieties.

Definition and proposition 11.9. — *There exists a unique function*

$$\mu : \mathbb{Z}^n_{>0} \longrightarrow \mathbb{Z}$$

such that

$$\chi(\mathbf{e}) = \sum_{\mathbf{d}\mid\mathbf{e}} \mu(\mathbf{d})$$

for $\mathbf{d}$ *running over all n-tuples of positive integers dividing* $\mathbf{e}\in\mathbb{Z}^n_{>0}$.

Proof. — Use the finiteness of the set $\{\mathbf{d}\in\mathbb{Z}^n_{>0} : \mathbf{d}\mid\mathbf{e}\}$ for all $\mathbf{e}\in\mathbb{Z}^n_{>0}$. □

It is unfortunate that we also use the letter μ for measures, but it will be clear from the context if μ is a measure or a Möbius function.

From the definition of μ, it follows that:

$$\chi = \sum_{\mathbf{d}} \mu(\mathbf{d})\chi_{\mathbf{d}} \tag{11.10}$$

where $\mathbf{d}$ runs over all n-tuples of positive integers. Now consider the sum of the values of all $\mathbf{q} \in (\mathbb{Z}_{\neq 0})^n$ such that $H_0(\mathbf{q}) = m$. Then,

$$c_0(m)^+ = \sum_{\mathbf{d}} \mu(\mathbf{d})a_{\mathbf{d}}(m), \mathbf{d} \in \mathbb{Z}^n_{>0}. \tag{11.11}$$

The sum is finite since $a_{\mathbf{d}}(m) = 0$ if $\sup(d_1, \ldots, d_n) > m$.

Notation 11.12. — Let p be a prime number. By

$$\sum_{\mathbf{d}}^{(p)}$$

we shall mean a sum over all n-tuples $\mathbf{d}$ of positive integers of the form

$$\mathbf{d} = (p^{e_1}, \ldots, p^{e_n})$$

for non-negative integers $e_1, \ldots, e_n$.

There is a local version of (11.10)

$$\chi^{(p)} = \sum_{\mathbf{d}}^{(p)} \mu(\mathbf{d})\chi_{\mathbf{d}} \tag{11.13}$$

where $\mu(\mathbf{d}) = 0$ if $p^2 \mid d_i$ for some component of $\mathbf{d} = (d_1, \ldots, d_n)$.

The values of the functions $\chi^{(p)}(\mathbf{q})$ and $\chi_{\mathbf{d}}(\mathbf{q})$ for n-tuples $\mathbf{d}$ as in (11.12), depend only on the residue class of $\mathbf{q} = (q_1, \ldots, q_n)$ in $(\mathbb{Z}/p\mathbb{Z})^n$. We can therefore compute the cardinality of

$$\widetilde{X}_0(\mathbb{Z}/p\mathbb{Z}) = \{\mathbf{r} = (r_1, \ldots, r_n) \in (\mathbb{Z}/p\mathbb{Z})^n : \exists \sigma \in \Delta_{\max} \text{ with } \mathbf{r}^{\underline{\sigma}} \neq 0\}$$

by means of (11.13). If we consider the sum of the values of all $\mathbf{r} \in (\mathbb{Z}/p\mathbb{Z})^n$, then we obtain

$$\operatorname{Card} \widetilde{X}_0(\mathbb{Z}/p\mathbb{Z}) = \sum_{\mathbf{d}}^{(p)} \mu(\mathbf{d})(p^n/\Pi(\mathbf{d})). \tag{11.14}$$

Now write

$$\mathbf{de} := (d_1e_1, \ldots, d_ne_n)$$

for n-tuples $\mathbf{d} = (d_1, \ldots, d_n)$, $\mathbf{e} = (e_1, \ldots, e_n)$ of integers.

The following result is due to Peyre for special classes of toric varieties (cf. [**52**, Ch. 7]).

Lemma 11.15

(a) *Let* $\mathbf{d} = (d_1, \ldots, d_n)$*,* $\mathbf{e} = (e_1, \ldots, e_n)$ *be* n*-tuples of positive integers. Let*

$$\delta = \gcd(\mathbf{d}^{\underline{\sigma}}), \quad \sigma \in \Delta_{\max},$$
$$\varepsilon = \gcd(\mathbf{e}^{\underline{\sigma}}), \quad \sigma \in \Delta_{\max}$$

and suppose that δ *and* ε *are relatively prime. Then*

$$\mu(\mathbf{de}) = \mu(\mathbf{d})\mu(\mathbf{e}).$$

(b) *Let* $(e_1, \ldots, e_n)$ *be an* n*-tuple of non-negative integers. Then* $\mu(p^{e_1}, \ldots, p^{e_n})$ *is independent of the prime number* p*.*

(c) *Let* $(e_1, \ldots, e_n)$ *be an* n*-tuple of non-negative integer, not all* 0 *and suppose that there exists a cone* $\sigma \in \Delta$ *such that* $e_i = 0$ *for all rays* $\rho_i \in \Delta$ *outside* σ*. Then*

$$\mu(p^{e_1}, \ldots, p^{e_n}) = 0.$$

(d) *Let* f *be the smallest integer such that there exist* f *rays of* Δ *not contained in a cone of* Δ*. Then the product*

$$\prod_p \left(\sum_{\mathbf{d}}^{(p)} |\mu(\mathbf{d})|/\Pi(\mathbf{d})^s \right)$$

over all prime numbers p *is absolute convergent for* $s > 1/f$*.*

(e) *The sum*

$$\sum_{\mathbf{d}} |\mu(\mathbf{d})|/\Pi(\mathbf{d})^s, \quad \mathbf{d} \in \mathbb{Z}^n_{>0}$$

is convergent for $s > 1/f$ *and equal to*

$$\prod_p \left(\sum_{\mathbf{d}}^{(p)} (|\mu(\mathbf{d})|/\Pi(\mathbf{d})^s \right)$$

for $s > 1/f$*.*

(f) *The product* $\prod_p \left(\sum_{\mathbf{d}}^{(p)} \mu(\mathbf{d}) / \prod(\mathbf{d}) \right)$ *over all prime numbers* p *is absolute convergent and equal to* $\sum_{\mathbf{d}} \mu(\mathbf{d})/\Pi(\mathbf{d})$*.*

Proof. — (a) and (b) are easy consequences of the definition of μ.

(c) Let $\mathbf{q} = (p^{e_1}, \ldots, p^{e_n})$. Then the greatest common divisor of all integers $\mathbf{q}^{\underline{\sigma}}$, $\sigma \in \Delta_{\max}$ is 1 if and only if there exists a cone $\sigma \in \Delta$ with $e_i = 0$ for all rays $\rho_i \in \Delta$ outside σ (cf. the discussion before (11.5)). The desired statement now follows from the recursive definition of μ.

(d) It follows from (b) and (c) that there is a polynomial $Q(T)$ with non-negative integer coefficients such that $\sum_{\mathbf{d}}^{(p)} (|\mu(\mathbf{d})|/\Pi(\mathbf{d})^s) = 1 + p^{-fs} Q(1/p^s)$

for all prime numbers p. Now use the fact that the sum of all $p^{-f}Q(1/p) \leq p^{-2}Q(1)$ is absolute convergent.

(e) This follows from (d) and the multiplicativity of $(|\mu(\mathbf{d})|/\Pi(\mathbf{d})^s)$ (cf. (a))

(f) This follows from (e) and the multiplicativity of $\mu(\mathbf{d})/\Pi(\mathbf{d})$. □

Lemma 11.16. — *Let $f : \mathbb{Z} > 0 \to \mathbb{R}$ be a function such that $f(B) > 0$ for all sufficiently large integers $B > 0$ and such that*

$$\Pi(\mathbf{d})A_{\mathbf{d}}(B) = f(B)(1 + o(1))$$

for all $\mathbf{d} \in \mathbb{Z}^n_{>0}$.

Then there exists a positive constant

$$\kappa = \prod_p \left(\sum_{\mathbf{d}}^{(p)} (\mu(\mathbf{d})/\Pi(\mathbf{d})) \right)$$

such that

$$\mathcal{C}_0(B)^+ = \kappa f(B)(1 + o(1)).$$

Proof. — Choose $k > 0$ such that $f(B)$ is positive for $B > k$ and let

$$g(\mathbf{d}, B) = |\mu(\mathbf{d})A_{\mathbf{d}}(B)/f(B) - \mu(\mathbf{d})/\Pi(\mathbf{d})|$$

for all $B > k$ and all $\mathbf{d} \in \mathbb{Z}^n_{>0}$. Then

$$|\mathcal{C}_0(B)^+/f(B) - \kappa| = \left| \sum_{\mathbf{d}} A_{\mathbf{d}}(B)/f(B) - \kappa \right| \leq \sum_{\mathbf{d}} g(\mathbf{d}, B)$$

for $B > k$. It is therefore sufficient to prove that

$$\lim_{B\to\infty} \sum_{\mathbf{d}} g(\mathbf{d}, B) = 0.$$

Also, since $0 \leq \Pi(\mathbf{d})A_{\mathbf{d}}(B) \leq A(B)$ there exists a constant $E > 0$ such that

$$|\Pi(\mathbf{d})A_{\mathbf{d}}(B)/f(B) - 1| \leq |A(B)/f(B)| + |A(B)/f(B) - 1| \leq E$$

for all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ and all $B > k$.

Hence by (11.15)(f), it follows that $\sum_{\mathbf{d}} g(\mathbf{d}, B)$ converges uniformly with respect to $B > k$ (this means that it for each $\varepsilon > 0$ exists a finite subset S of $\mathbb{Z}^n_{>0}$ such that $\sum_{\mathbf{d}\notin S} g(\mathbf{d}, B) < \varepsilon$ for all $B > k$).

Therefore,

$$\lim_{B\to\infty} \sum_{\mathbf{d}} g(\mathbf{d}, B) = \sum_{\mathbf{d}} (\lim_{B\to\infty} g(\mathbf{d}, B)) = 0$$

as was to be proved. □

We can now extend the method of Peyre [**52**] for certain blow-ups of projective spaces to arbitrary toric varieties. It can be described as follows. One first proves asymptotic formulas of the type

$$\Pi(\mathbf{d})A_{\mathbf{d}}(B) = \varpi B(\log B)^{r-1}(1+O(1/\log B)) \tag{11.17}$$

with the same positive constant ϖ for all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ and compares ϖ with a volume. Then from (11.16) one gets the asymptotic formula

$$\mathcal{C}_0(B)^+ = \kappa\varpi B(\log B)^{r-1}(1+o(1)) \tag{11.18}$$

with κ as in (op. cit.).

Peyre computes the main term $\kappa f(B)$ by Möbius inversion as in (11.11)-(11.15). He bounds the error term by means of an argument with uniform convergence as in (11.16) although he never mentions the role of uniform convergence. His proof of (11.17) is less direct than here (cf. his comment at the end of p. 171 in op. cit.). It also makes use of special properties of the class of blow-ups of projective spaces he considers.

We shall in this paper apart from the generalization to arbitrary toric varieties improve the treatment of error terms under Möbius inversion in [**52**]. One of the tools to do this will be the following result about the Möbius function $\mu : \mathbb{Z}^n_{>0} \to Z$ introduced in (11.9).

Lemma 11.19. — *Let f be the smallest integer such that there exist f rays of Δ not contained in a cone of Δ. Let ε be any posititve real number. Then the following holds.*

(a) $\sum_{\Pi(\mathbf{d})\leq b} |\mu(\mathbf{d})| = O(b^{1/f+\varepsilon})$

(b) $\sum_{\Pi(\mathbf{d})\geq b}(|\mu(\mathbf{d})|/\Pi(\mathbf{d})) = O(b^{1/f-1+\varepsilon})$

Proof

(a) This follows from the fact (cf. (11.15)(e)) that the sum $\sum_{\mathbf{d}} |\mu(\mathbf{d})|/\Pi(\mathbf{d})^s$ over all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ converges for $s > 1/f$ (cf. e.g. [**68**, p. 5]).

(b) Let $y(m) = \sum_{\Pi(\mathbf{d})=m} |\mu(\mathbf{d})|$ where $\mathbf{d} \in \mathbb{Z}^n_{>0}$. Then we have to prove that

$$\sum_{y\geq b} y(m)/m = O(b^{1/f-1+\varepsilon})$$

for any $\varepsilon > 0$.

To show this, put $y(0) = 0$ and $Y(m) = y(0) + \cdots + y(m)$ for $m \geq 0$. Then, by partial summation we obtain:

$$\sum_{m\geq b} y(m)/m = -Y(b-1)/b + \sum_{m\geq b} Y(m)/m(m+1)$$

for any positive integer b.

But $Y(m) = O(m^{1/f+\varepsilon})$ by (a). Hence

$$Y(b-1)/b = O(b^{1/f-1+\varepsilon})$$

and

$$\sum_{m\geq b} Y(m)/m(m+1) \leq \sum_{m\geq b} Y(m)/m^2 = O(\sum_{m\geq b} m^{1/f-2+\varepsilon}) = O(b^{1/f-1+\varepsilon}).$$

This completes the proof. □

Proposition 11.20. — *Let $D_0 = \sum_{\rho\in\Delta(1)} D_\rho$ and suppose that $\mathcal{O}(D_0)$ is generated by its global sections. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in U_0(\mathbb{R})$ be an n-tuple of real numbers different from* 0. *Then,*

$$|\Pi(\alpha)| \leq \sup_{\sigma\in\Delta_{\max}} |\alpha^{D(\sigma)}|.$$

Proof. — Let $P \in U_0(\mathbb{R})$ be the image of $\alpha \in U_0(\mathbb{R})$ and let σ be a maximal cone such that $P \in C_\sigma(\mathbb{R})$ and let $\chi^{m(\sigma)}$ be the character which on U_σ represents the Cartier divisor with Weil divisor D_0. Then $|\chi^{m(\sigma)}(P)| \leq 1$ (cf. (9.2)). Now note that $\chi^{m(\sigma)}(P) = \alpha^{D_0}/\alpha^{D(\sigma)}$ by the definition of $D(\sigma)$. This completes the proof. □

Note that the argument is valid for arbitrary locally compact local fields.

Notation 11.21

(a) Let $\pi_{\mathbb{R},\mathrm{an}} : X_0(\mathbb{R}) \to X(\mathbb{R})$ be the analytic map defined by $\pi : X_0 \to X$ and let $C_\sigma(\mathbb{R}) \subset X(\mathbb{R})$ be the compact subset defined in (9.1). Then, $C_{0,\sigma}(\mathbb{R})$ is the inverse image of $C_\sigma(\mathbb{R})$ under $\pi_{\mathbb{R},\mathrm{an}}$.

(b) Let ρ_i, $1 \leq i \leq n$ be the rays of Δ, let D_i, $1 \leq i \leq n$ be U-invariant irreducible Weil divisors corresponding to these rays and let x_i, $1 \leq i \leq n$ be indeterminates indexed by the rays. Let

$$D = l_1 D_1 + \cdots + l_n D_n, \quad l_1, \dots, l_n \in \mathbb{Z}$$

be a U-invariant Weil divisor on X_Δ. Then $\mathbf{x}^D$ denotes the Laurent monomial $\prod_{i=1}^n x_i^{l_i}$.

Proposition 11.22. — *Let $\sigma \in \Delta$ be a maximal cone of the complete regular fan Δ and let*

$$(\rho_1, \dots, \rho_{r+d})$$

$d = \dim\Delta$, $r + d = \mathrm{Card}\,\Delta(1)$ *be an ordering of the rays of Δ such that*

$$(\rho_{r+1}, \dots, \rho_{r+d})$$

are the rays of σ. Let

$$\{n^{(j)},\ 1 \leq j \leq d\}$$

be the $\mathbb{Z}$-basis of N, consisting of the generators of the rays ρ_{r+j} of σ and let

$$\{m^{(j)},\ 1 \le j \le d\}$$

be the dual $\mathbb{Z}$-basis of M. Finally, let

$$D(j) = \sum_{\rho \in \Delta(1)} \langle m^{(j)}, n_\rho \rangle D_\rho$$

for $1 \le j \le d$.

Then $C_{0,\sigma}(\mathbb{R})$ is the subset of $X_0(\mathbb{R})$ of real n-tuples $\mathbf{x} = (x_1, \dots, x_{r+d})$ for which

$$(*) \qquad |x^{D(j)}| \le 1, \quad j \in \{1, \dots, d\}.$$

Proof. — It suffices by continuity to show that an $\mathbb{R}$-point P_0 on U_0 satisfies

$$|\mathbf{x}^{D(j)}(P_0)| \le 1 \quad \text{for} \quad 1 \le j \le d$$

if and only if $P = \pi(P_0) \in C_\sigma(\mathbb{R})$. To see this, let

$$-L(P) = \sum_{j=1}^{d} \lambda_j n^{(j)}, \quad \lambda_j \in \mathbb{R}.$$

Then $P \in C_\sigma(\mathbb{R})$ if and only if

$$\langle m^{(j)}, L(P) \rangle \ge 0$$

for $1 \le j \le d$ (cf. (9.1)). Now note that

$$|\mathbf{x}^{D(j)}(P_0)| = \exp\langle m^{(j)}, L(P) \rangle$$

by the definitions of $m^{(j)}$, $D(j)$ and L. This completes the proof. $\square$

Remark 11.23. — Let $\tau = \sigma^{(j)}$ be the (unique) adjacent maximal cone with

$$\tau \cap \rho_{r+j} = \{0\},$$

and let b_j be the multiplicity of D_{r+j} in $D(\tau)$. Then,

$$D(\tau) - D(\sigma) = b_j D(j)$$

by (8.9) so that

$$|\mathbf{x}^{D(\tau)-D(\sigma)}| = |\mathbf{x}^{D(j)}|^{b_j}.$$

Also, if $\mathbf{x} \in C_{0,\sigma}(\mathbb{R})$, then $|\mathbf{x}^{D(\tau)-D(\sigma)}| \le 1$ for all maximal cones $\tau \in \Delta$ by (9.8).

If $\sum_{\rho \in \Delta(1)} D_\rho$ is ample, then $b_j > 0$ for $1 \le j \le d$ (cf. (8.9)). We may therefore in this case (cf. also (9.10)) replace $(*)$ by the condition:

$$(*') \qquad |\mathbf{x}^{D(\tau)-D(\sigma)}| \le 1, \quad \tau \in \Delta_{\max}.$$

Sublemma 11.24. — *Let* $(e_1, \dots, e_r)$ *be non-negative integers and let* e *be a positive integer. Let*

$$S(B, e; e_1, \dots, e_r) = \sum (\prod_{i=1}^{r} g_i^{e_i/e-1})$$

be the sum over all r*-tuples* $(g_1, \dots, g_r)$ *of positive integers satisfying*

$$\prod_{i=1}^{r} g_i^{e_i} \leq B$$

and

$$\max(g_1, \dots, g_r) \leq B.$$

Then,

$$S(B, e; e_1, \dots, e_r) = O(B^{1/e}(1 + \log B)^{r-1}).$$

Proof. — Suppose that $e_r = 0$. Then,

$$S(B, e; e_1, \dots, e_r) \leq \begin{cases} S(B, e; e_1, \dots, e_{r-1}) \left(\sum_{i=1}^{[B]} 1/i\right) & \text{if } r > 1 \\ \left(\sum_{i=1}^{[B]} 1/i\right) & \text{if } r = 1. \end{cases}$$

We may and shall therefore assume that $(e_1, \dots, e_r)$ is a positive r-tuple. The statement is easy to verify for $r = 1$ by means of a comparison with an integral. Suppose $r > 1$ and put $m = g_r$, $a = e_r$. Then

$$S(B, e; e_1, \dots, a) = \sum_{m=1}^{[\sqrt[a]{B}]} m^{a/e-1} S([B/m^a], e; e_1, \dots, e_{r-1}).$$

The induction assumption for $S([B/m^a], e; e_1, \dots, e_{r-1})$, $1 \leq m \leq [\sqrt[a]{B}]$ gives

$$S(B, e; e_1, \dots, a) = O\left(\sum_{m=1}^{[\sqrt[a]{B}]} m^{a/e-1} (B/m^a)^{1/e} (1 + \log(B/m^a)^{r-2})\right),$$

$$S(B, e; e_1, \dots, a) = O\left(B^{1/e} \sum_{m=1}^{[\sqrt[a]{B}]} m^{-1} (1 + \log(B/m^a)^{r-2})\right)$$

and the desired result follows. □

Lemma 11.25

(a) $A(B) = O(B(1 + \log B)^{r-1})$.

(b) *Let*

$$u_k = (0, \dots, 0, 1, 0, \dots, 0), \quad 1 \leq k \leq n$$

be the k*-th unit vector in the standard basis for* $\mathbb{R}^n$ *and let* $\delta_k(B)$ *be the set of* n*-tuples* $\mathbf{g}$ *of positive integers such that*

(i) $\sup_{\sigma\in\Delta_{\max}} |\mathbf{g}^{D(\sigma)}| \leq B$,
(ii) $\sup_{\sigma\in\Delta_{\max}} |(\mathbf{g}+u_k)^{D(\sigma)}| > B$.
Suppose that $\operatorname{Card}\Delta(1) - \dim\Delta > 1$. *Then,*

$$\operatorname{Card}\delta(B) = O(B(1+\log B)^{r-2}).$$

Proof

(a) Let $\sigma \in \Delta$ be a maximal cone and let $A_\sigma(B)$ be the number of positive n-tuples $\mathbf{g}$ of integers such that

$$\mathbf{g} = (g_1, \ldots, g_n) \in C_{0,\sigma}(\mathbb{R})$$

and

$$\sup_{\sigma\in\Delta_{\max}} |\mathbf{g}^{D(\sigma)}| \leq B.$$

It suffices to prove that

$$A_\sigma(B) = O(B(1+\log B)^{r-1})$$

for each maximal cone σ of Δ.

Fix $\sigma \in \Delta_{\max}$ and let $(\rho_1, \ldots, \rho_n)$ be an ordering of the rays of Δ such that the last $d = n - r$ rays are the one-dimensional faces of σ. Then

$$D(\sigma) = \sum_{i=1}^{r} e_i D_i$$

for some non-negative integers $e_1, \ldots, e_r$. (We use here the assumption that $\mathcal{O}(D_0)$ is generated by its global sections.)

Let $\{m^{(j)} \in M,\ 1 \leq j \leq d\}$ be the $\mathbb{Z}$-basis of M in (11.22) and let

$$E(j) = D_{r+j} - \sum_{\rho\in\Delta(1)} \langle m^{(j)}, n_\rho\rangle D_\rho, \quad j \in \{1, \ldots, d\}.$$

Then each $\mathbf{x}^{E(j)}$ is a Laurent monomial in $(x_1, \ldots, x_r)$.

Moreover, by (11.22)-(11.23) it follows that $A_\sigma(B)$ is the cardinality of the set of $r+d$-tuples $\mathbf{g} = (g_1, \ldots, g_{r+d})$ of positive integers such that

$$(\#) \qquad \mathbf{g}^{D(\sigma)} = \prod_{i=1}^{r} g_i^{e_i} \leq B$$

$$(\#\#) \qquad g_{r+j} \leq \mathbf{g}^{E(j)}, \quad j \in \{1, \ldots, d\}$$

Now note that by (8.9)(iii):

$$D(\sigma) - (D_1 + \cdots + D_r) = \sum_{j=1}^{d} E(j)$$

and hence that

$$\prod_{j=1}^{d} \mathbf{g}^{E(j)} = \prod_{i=1}^{r} g_i^{e_i-1}.$$

There is, therefore, for each r-tuple $(g_1, \ldots, g_r) \in \mathbb{Z}_{>0}^r$ at most $\prod_{i=1}^r g_i^{e_i-1}$ d-tuples satisfying (##). Hence,

$$A_\sigma(B) \leq \sum (\prod_{i=1}^{r} g_i^{e_i-1})$$

where in the sum $(g_1, \ldots, g_r)$ runs over all r-tuples of positive integers satisfying $\prod_{i=1}^r g_i^{e_i} \leq B$. Finally, note that

$$\max(g_1, \ldots, g_r) \leq |\Pi(\mathbf{g})| \leq \sup_{\sigma \in \Delta_{\max}} |\mathbf{g}^{D(\sigma)}| \leq B$$

by (11.20). Hence $A_\sigma(B) = O(B(1 + \log B)^{r-2})$ by the sublemma above.

(b) It suffices to count the set $\delta_\sigma(B)$ of n-tuples $\mathbf{g} = (g_1, \ldots, g_n)$ satisfying (i) and (ii) and the additional condition

(iii) $$\mathbf{g} + u_k \in C_{0,\sigma}(\mathbb{R})$$

for some maximal cone $\sigma \in \Delta$.

Now order the rays as in (a) with

$$\mathbf{g}^{D(\sigma)} = \prod_{i=1}^{r} g_i^{e_i}$$

and recall that

$$\mathbf{g}^{D(\sigma)} < (\mathbf{g} + u_k)^{D(\sigma)}$$

by (i)-(iii). Then $k \in \{1, \ldots, r\}$ and $e_k > 0$. After a permutation of the first r rays we may thus assume that $k = 1$. Moreover, from the arguments in (a), it follows that $\delta_\sigma(B)$ is the number of n-tuples $\mathbf{g}$ of positive integers satisfying:

(I) $\sup_{\sigma \in \Delta_{\max}} |\mathbf{g}^{D(\sigma)}| \leq B$
(II) $g_{r+j} \leq (\mathbf{g} + u_1)^{E(j)},\ j \in \{1, \ldots, d\}$
(III) $(g_1 + 1)^{e_1} \prod_{i=2}^r g_i^{e_i} > B$

Also, by (8.9)(iii) we have just as in (a) that

$$\prod_{j=1}^{d} (\mathbf{g} + u_1)^{E(j)} = (g_1 + 1)^{e_1 - 1} \prod_{i=2}^{r} g_i^{e_i-1} \leq 2^{e_1-1} \prod_{i=1}^{r} g_i^{e_i-1}.$$

There are therefore for each r-tuple $(g_1, \dots, g_r) \in \mathbb{Z}^r_{>0}$ at most $2^{e_1-1} \prod_{i=1}^r g_i^{e_i-1}$ d-tuples satisfying (II). Moreover, from (I) and (III) we conclude that

$$g_1 = [(B/\prod_{j=2}^r g_i^{e_i})^{1/e_1}] \le (B/\prod_{i=2}^r g_i^{e_i})^{1/e_1}.$$

Hence,

$$\begin{aligned} \operatorname{Card} \delta_\sigma(B) &\le 2^{e_1-1} \sum \left((B/\prod_{i=2}^r g_i^{e_i})^{1-1/e_1} \prod_{i=2}^r g_i^{e_i-1} \right) \\ &= 2^{e_1-1} B^{1-1/e_1} \sum \left(\prod_{i=2}^r g_i^{e_i/e_1-1} \right) \end{aligned}$$

where in the sums $(g_2, \dots, g_r)$ runs over all r-tuples of positive integers satisfying

$$\prod_{i=1}^r g_i^{e_i} \le B.$$

We therefore obtain from the sublemma above that

$$\operatorname{Card} \delta_\sigma(B) = O(B(1+\log B)^{r-2})$$

as was to be proved. □

Lemma 11.26. — *Suppose that $r > 1$. Then there exists for each $\varepsilon > 0$ a positive constant $C = C(\varepsilon) > 0$ depending only on ε such that*

$$0 \le A(B) - A_{\mathbf{d}}(B)\Pi(\mathbf{d}) \le C\Pi(\mathbf{d})B(1+\log B)^{r-2}$$

for all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ and all $B \ge 1$.

Proof. — Fix $\mathbf{d} \in \mathbb{Z}^n_{>0}$ and $B \ge 1$. Then (cf. the proof of (11.7)) $A(B) - A_{\mathbf{d}}(B)\Pi(\mathbf{d})$ is equal to the cardinality of the set $\Omega(B)$ of $\mathbf{q} \in \mathbb{Z}^n_{>0}$ such that

(i) $\sup_{\sigma\in\Delta_{\max}} |(\mathbf{q} + \mathbf{d} - (1, \dots, 1))^{D(\sigma)}| > B$

(ii) $\sup_{\sigma\in\Delta_{\max}} |\mathbf{q}^{D(\sigma)}| \le B$

But there exists for each $\mathbf{q} \in \Omega(B)$ an n-tuple $\mathbf{e}$ of positive integers with

$$e_i \le d_i, \quad i \in \{1, \dots, n\}$$

and an integer $k \in \{1, \dots, n\}$ such that $\mathbf{q} + \mathbf{e}$ belongs to the set $\delta_k(B)$ in (11.25). Hence

$$\operatorname{Card} \Omega(B) \le \Pi(\mathbf{d}) \sum_{k=1}^n \operatorname{Card} \delta_k(B)$$

so that the desired assertion follows from the previous lemma. □

Main lemma 11.27. — *Let Δ be a complete regular d-dimensional fan with n rays such that $r := n - d > 1$ and let f be the smallest integer such that there exist f rays of Δ not contained in a cone of Δ. Then the Euler product*

$$\prod_p \kappa_p = \prod_p \operatorname{Card} \widetilde{X}_0(\mathbb{Z}/p\mathbb{Z})/p^n$$

converges absolutely to a positive constant κ and

$$\mathcal{C}_0(B)^+ - \kappa A(B) = O(B(1+\log B)^{r-2+1/f+\varepsilon})$$

for each $\varepsilon > 0$.

Proof. — The first assertion about κ is just a restatement of (11.15) since

$$\kappa_p = \sum_d^{(p)} (\mu(\mathbf{d})/\Pi(\mathbf{d}))$$

by (11.14). To prove the last assertion, we fix $\varepsilon > 0$ and use the identity (cf. (11.11))

$$\mathcal{C}_0^+(B) = \sum_{\mathbf{d}} \mu(\mathbf{d}) A_{\mathbf{d}}(B), \quad \mathbf{d} \in \mathbb{Z}^n_{>0}.$$

Then,

$$\mathcal{C}_0^+(B) - \kappa A(B) = \sum_{\mathbf{d}} \mu(\mathbf{d}) A_{\mathbf{d}}(B) - \sum_{\mathbf{d}} \mu(\mathbf{d}) A(B)/\Pi(\mathbf{d}), \quad \mathbf{d} \in \mathbb{Z}^n_{>0}.$$

where the last sum is absolute convergent by (11.15).

It is therefore sufficient to prove that

$$\sum_{\mathbf{d}} |\mu(\mathbf{d})| (A(B)/\Pi(\mathbf{d}) - A_{\mathbf{d}}(B)) = O(B(1+\log B)^{r-2+1/f+\varepsilon}).$$

But there exists by (11.26) a constant C such that

$$0 \leq A(B)/\Pi(\mathbf{d}) - A_{\mathbf{d}}(B) \leq CB(1+\log B)^{r-2}$$

for all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ and all $B \in \mathbb{Z}_{>0}$.

Hence by (11.19)(a) we get that

$$\begin{aligned} \sum_{\mathbf{d}} |\mu(\mathbf{d})| (A(B)/\Pi(\mathbf{d}) - A_{\mathbf{d}}(B)) &\leq CB(1+\log B)^{r-2}) \sum_{\mathbf{d}} |\mu(\mathbf{d})| \\ &= O(B(1+\log B)^{r-2+1/f+\varepsilon}) \end{aligned}$$

for the partial sum of all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ such that $\Pi(\mathbf{d}) \leq (1+\log B)$.

For $\mathbf{d} \in \mathbb{Z}^n_{>0}$ with $\Pi(\mathbf{d}) > (1+\log B)$, we use the fact that (cf. (11.25))

$$0 \leq A(B) - A_{\mathbf{d}}(B)\Pi(\mathbf{d}) \leq A(B) = O(B(1+\log B)^{r-1}).$$

Hence, by (11.19)(b) we get that

$$\begin{aligned}\sum_{\mathbf{d}} |\mu(\mathbf{d})|(A(B)/\Pi(\mathbf{d}) - A_{\mathbf{d}}(B)) &= O(B(1+\log B)^{r-1}) \sum_{\mathbf{d}} |\mu(\mathbf{d})|/\Pi(\mathbf{d}) \\ &= O(B(1+\log B)^{r-2+1/f+\varepsilon})\end{aligned}$$

for the partial sum of all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ such that $\Pi(\mathbf{d}) > (1+\log B)$. We therefore obtain the desired upper bound for the sum of all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ by combing the two partial sums. This completes the proof. □

Let m_∞ be the Borel measure associated to the toric norm on $X(\mathbb{R})$ (cf. (9.11)(b)) and let n_∞ be the Borel measure on $X_0(\mathbb{R})$ determined by the induced norm of the toric norm (cf. (3.30) and (9.12)).

Notation 11.28. — Let $B \geq 1$ be a real number. Then,

(a) $D(B) \subset U_0(\mathbb{R})$ is the set of n-tuples $\mathbf{x} = (x_1, \dots, x_n)$ such that
  (i) $\inf(x_1, \dots, x_n) \geq 1$,
  (ii) $\sup_{\sigma \in \Delta_{\max}} |\mathbf{x}^{D(\sigma)}| \leq B$.

(b) $I(B) = \int_{D(B)} \sup_{\sigma \in \Delta_{\max}} |\mathbf{x}^{D(\sigma)}| dn_\infty$.

Lemma 11.29. — *We have*

$$A(B) = I(B) + O(B(1+\log B)^{r-2}).$$

Proof. — Let $\widetilde{X}_1 = \operatorname{Spec} \mathbb{Z}[x_\rho]$, $\rho \in \Delta(1)$. Then $\widetilde{X}_1(\mathbb{Z})$ is a $\mathbb{Z}$-lattice in $\widetilde{X}_1(\mathbb{R}) = X_1(\mathbb{R}) = \mathbb{R}^n$. Let

$$d\mathbf{x} = dx_1 \cdots dx_n$$

be the Haar measure on $X_1(\mathbb{R})$ such that the volume of a fundamental domain of $\widetilde{X}_1(\mathbb{Z}) \subset X_1(\mathbb{R})$ is equal to 1. Then (cf. (9.12)) the following equality holds on $X_0(\mathbb{R}) \subset X_1(\mathbb{R})$

$$\int_{X_0(\mathbb{R})} f dn_\infty = \int_{X_0(\mathbb{R})} \left(f(\mathbf{x}) / \sup_{\sigma \in \Delta_{\max}} |\mathbf{x}^{D(\sigma)}| \right) d\mathbf{x}$$

for any continuous function f on $X_0(\mathbb{R})$ with compact support. It is therefore also true for the characteristic function g of $D(B) \subset X_0(\mathbb{R})$ since there exists an L^1-Cauchy sequence $(f_i)_{i=1}^\infty$ in $C_c(X_1(\mathbb{R}))$ converging to g (cf. [**38**, Ch IX, §3]). Hence,

$$I(B) = \int_{D(B)} d\mathbf{x}.$$

while $A(B)$ is the number of lattice points in $D(B)$. Let $\gamma(B)$ be the set of n-tuples $\mathbf{g} = (g_1, \dots, g_n)$ of positive integers such that

(i) $\sup_{\sigma \in \Delta_{\max}} |\mathbf{g}^{D(\sigma)}| \leq B$,
(ii) $\sup_{\sigma \in \Delta_{\max}} |(\mathbf{g} + \mathbf{1})^{D(\sigma)}| > B$ for $\mathbf{1} = (1, \dots, 1)$.

Then, $D(B)$ contains a union of of $A(B) - \operatorname{Card}\gamma(B)$ disjoint n-dimensional unit cubes and is contained in a union of $A(B)$ n-dimensional unit cubes. Hence, by an argument due to Archimedes one has

$$0 \leq A(B) - I(B) \leq \operatorname{Card}\gamma(B).$$

There is for each $\mathbf{g} \in \gamma(B)$ a binary n-tuple $\mathbf{e}$ such that $\mathbf{g} + \mathbf{e}$ belongs to one of the sets $\delta_k(B)$ described in (11.25). Therefore,

$$\operatorname{Card}\gamma(B) \leq 2^n \sum_{k=1}^{n} \operatorname{Card}\delta_k(B)$$

so that the desired assertion follows from (op. cit.). □

Let n_p be the measure on $X_0(\mathbb{Q}_p)$ determined by the induced norm of the toric norm on $X(\mathbb{Q}_p)$. Then,

$$\operatorname{Card}\widetilde{X}_0(\mathbb{Z}/p\mathbb{Z})/p^n = n_p(\widetilde{X}_0(\mathbb{Z}_p))$$

by (9.14) and (2.15). We may thus summarize the results obtained so far as follows.

***Theorem 11.30* (preliminary form).** — *Let Δ be a complete regular d-dimensional fan with n rays such that $r := n - d > 1$ and let $X = X_\Delta$ be the toric $\mathbb{Q}$-variety defined by Δ. Suppose that the anticanonical sheaf of X is generated by its global sections. Let H be the height function on $X(\mathbb{Q})$ defined by the toric adelic norm $\| \ \|$ for X and let $\mathcal{C}(B)$ be the number of $\mathbb{Q}$-points on U of toric height at most B.*

Let $\pi : X_0 \to X$ be the principal universal torsor of X and let n_p and n_∞ be the Borel measures on $X_0(\mathbb{Q}_p)$ resp. $X_0(\mathbb{R})$ determined by the induced norm of the toric norms. Finally, let $D(B) \subset U_0(\mathbb{R})$ be as in (11.28) and let f be the smallest integer such that there exist f rays of Δ not contained in a cone of Δ. Then,

$$\prod_p n_p(\widetilde{X}_0(\mathbb{Z}_p))$$

converges absolutely to a positive constant κ and

$$\mathcal{C}(B) - \kappa \frac{\operatorname{Card}(U_0(\mathbb{R})/U_0(\mathbb{R}^+)}{\operatorname{Card}\widetilde{T}(\mathbb{Z})} \int_{D(B)} \sup_{\sigma \in \Delta_{\max}} |\mathbf{x}^{D(\sigma)}| dn_\infty$$
$$= O(B(1 + \log B)^{r-2+1/f+\varepsilon})$$

for any $\varepsilon > 0$.

Proof. — We have

$$\begin{aligned} c(m) &= c_0(m)/\operatorname{Card}(\widetilde{T}(\mathbb{Z})) \\ &= c_0(m)^+ \operatorname{Card}(U_0(\mathbb{R})/U_0(\mathbb{R})^+)/\operatorname{Card}(\widetilde{T}(\mathbb{Z})) \end{aligned}$$

(cf. (11.4)) and hence

$$\mathcal{C}(B) = \mathrm{Card}(U_0(\mathbb{R})/U_0(\mathbb{R})^+)\,\mathrm{Card}(\widetilde{T}(\mathbb{Z}))^{-1}\mathcal{C}_0(B)^+.$$

The theorem is therefore a consequence of (11.27) and (11.29). □

It remains to study the integral:

$$\int_{D(B)} \sup_{\sigma\in\Delta_{\max}} |\mathbf{x}^{D(\sigma)}| dn_\infty = \int_{D(B)} d\mathbf{x}.$$

We shall do this by means of the toric canonical splitting over $\mathbb{R}$.

Notation 11.31. — Let $B \geq 1$ be a real number, $\sigma \in \Delta$ be a cone and $D(B)$ be as in (11.28). Then,

$$D(B,\sigma) = D(B) \cap C_\sigma(\mathbb{R}).$$

It follows immediately from the definition of $C_\sigma(\mathbb{R})$ that

$$D(B,\sigma) \cap D(B,\tau) = D(B,\sigma\cap\tau)$$

for any two cones $\sigma, \tau \in \Delta$.

Proposition 11.32. — *We have*

$$\int_{D(B)} \mathbf{x} = \sum_{\sigma\in\Delta_{\max}} \left(\int_{D(B,\sigma} \mathbf{dx}\right)$$

where $\mathbf{dx}$ *is the Lebesgue measure on* $\widetilde{X}_1(\mathbb{R}) = \mathbb{R}^{r+d}$ *normalized by* $\widetilde{X}_1(\mathbb{Z}) = \mathbb{Z}^{r+d}$.

Proof. — It follows from the definition that the sets $D(B,\sigma)$, $\sigma \in \Delta_{\max}$ form a covering of $D(B)$. It is therefore sufficient to prove that

$$\int_{D(B,\sigma\cap\tau)} \mathbf{dx} = 0$$

for each pair (σ,τ) of maximal cones in Δ. But this is done exactly as in the proof of $m(C_\sigma(\mathbb{R}) \cap C_\tau(\mathbb{R})) = 0$ in (9.16). □

Now fix a maximal d-dimensional cone $\sigma \in \Delta$ and choose an ordering $(\rho_1,\ldots,\rho_n)$ of the rays in Δ such that the last d rays are the one-dimensional faces of σ and such that (cf. (8.7))

$$D(\sigma) = \sum_{i=1}^r e_i D_i$$

for some non-negative integers $e_1,\ldots,e_r$, $r = n-d$.

Let $\{m^{(j)},\ 1 \leq j \leq d\}$ be the $\mathbb{Z}$-basis of M which is dual to the $\mathbb{Z}$-basis $\{n^{(j)},\ 1 \leq j \leq d\}$ of N, consisting of generators of the rays ρ_{r+j}, $1 \leq j \leq d$ of σ. Also, let

$$E(j) = D_{r+j} - \sum_{\rho\in\Delta(1)} \langle m^{(j)}, n_\rho\rangle D_\rho, \quad j \in \{1,\ldots,d\}$$

as in the proof of (11.25). Recall that $\mathbf{x}^{E(j)}$, $1 \leq i \leq d$ is a Laurent monomial in $(x_1,\ldots,x_r)$ and that (cf. (8.9)(iii))

$$\prod_{j=1}^{d} \mathbf{x}^{E(j)} = \mathbf{x}^{D(\sigma)} / \prod_{i=1}^{r} x_i \tag{11.33}$$

$D(B,\sigma)$ is then by (11.22)-(11.23) the set of $\mathbf{x} = (x_1,\ldots,x_{r+d})$ such that

(i) $\min(x_1,\ldots,x_{r+d}) \geq 1$

(ii) $\prod_{j=1}^{r} x_j^{e_j} \leq B$

(iii) $x_{r+j} \leq \mathbf{x}^{E(j)}$, $j \in \{1,\ldots,d\}$

Notations 11.34. — $\Omega(B,\sigma)$ is the set of all real r-tuples $(x_1,\ldots,x_r)$ such that

(i) $\min(x_1,\ldots,x_r) \geq 1$,

(ii) $\mathbf{x}^{D(\sigma)} \leq B$,

(iii) $\mathbf{x}^{E(j)} \geq 1$, $j \in \{1,\ldots,d\}$.

We now first integrate with respect to $(x_{r+1},\ldots,x_{r+d})$ and then with respect to $(x_1,\ldots,x_r)$. By Fubini's theorem it follows that:

$$\int_{D(B,\sigma)} \mathbf{dx} = \int_{\Omega(B,\sigma)} \left(\prod_{j=1}^{d} (\mathbf{x}^{E(j)} - 1\right) dx_1 \cdots dx_r \tag{11.35}$$

and by (11.33) that

$$\int_{D(B,\sigma)} \mathbf{dx} = \int_{\Omega(B,\sigma)} \mathbf{x}^{D(\sigma)} \left(\prod_{j=1}^{d} (1 - \mathbf{x}^{-E(j)}\right) \frac{dx_1}{x_1} \cdots \frac{dx_r}{x_r}. \tag{11.36}$$

We may regard $\omega = \dfrac{dx_1}{x_1} \cdots \dfrac{dx_r}{x_r}$ as a global T-invariant differential form on $T(\mathbb{R}) \subset U_0(\mathbb{R})$. Moreover, since $D(\sigma) - D_0$ and $D_{r+j} - E(j)$ are principal divisors one has

$$\mathbf{x}^{D(\sigma)} = \mathbf{x}^{D_0}$$

and

$$\mathbf{x}^{E_j} = x_{r+j}, \quad j \in \{1,\ldots,d\}$$

on $T(\mathbb{R})$.

Let $F(B) \subset T(\mathbb{R})$ be the subset of $T(\mathbb{R}) \subset U_0(\mathbb{R})$ satisfying

$$(*) \qquad \min(x_1, \dots, x_{r+d}) \geq 1,$$

$$(**) \qquad \mathbf{x}^{D_0} \leq B.$$

Then the projection onto the r first coordinates defines an analytic isomorphism between $\Omega(B,\sigma)$ and $F(B,\sigma)$. Hence from (11.36) we deduce that

$$\int_{D(B,\sigma)} \mathbf{dx} = \int_{F(B,\sigma)} \mathbf{x}^{D_0} \left(\prod_{j=1}^{d} (1 - 1/x_{r+j}) \right) \frac{dx_1}{x_1} \cdots \frac{dx_r}{x_r} \tag{11.37}$$

Lemma 11.38. — *We have*

$$\int_{D(B,\sigma)} \mathbf{dx} = \int_{F(B)} \mathbf{x}^{D_0} \frac{dx_1}{x_1} \cdots \frac{dx_r}{x_r} + O(B(1 + \log B)^{r-2})$$

for $B \geq 1$.

Proof. — It suffices by (11.37) to show that

$$\int_{F(B)} (\mathbf{x}^{D_0}/x_{r+j}) \frac{dx_1}{x_1} \cdots \frac{dx_r}{x_r} = O(B(1 + \log B)^{r-2})$$

for $j = 1, \dots, d$. But the measure $|\omega|$ on $T(\mathbb{R})$ defined by $\omega = \dfrac{dx_1}{x_1} \cdots \dfrac{dx_r}{x_r}$ (cf. (1.13)) does not depend on the maximal cone σ used to define $\{x_1, \dots, x_r\}$. We may thus in the description of $|\omega|$ replace the coordinates $(x_1, \dots, x_r)$ by any set of r coordinates corresponding to the rays outside a given maximal cone τ of Δ. We shall choose τ such that $-n_{r+j} \in \tau$. Then D_{r+j} has multiplicity ≥ 2 in $D(\tau)$ (cf. (8.7)). After a permutation of the rays we may change the index of ρ_{r+j} to 1 and assume that $(\rho_1, \dots, \rho_r)$ are the rays not in τ. Then it suffices to show that

$$\int_{F(B)} (\mathbf{x}^{D(\tau)}/x_1) \frac{dx_1}{x_1} \cdots \frac{dx_r}{x_r} = O(B(1 + \log B)^{r-2}).$$

We shall in fact consider the integral over the larger subset $G(B)$ defined by

$$\begin{array}{rcl} x_1 & \geq & 0 \\ x_i & \geq & 1, \quad i \in \{2, \dots, r\} \\ \mathbf{x}^{D(\tau)} & \leq & B. \end{array}$$

We shall also assume that $r \geq 2$ and leave the trivial case $r = 1$ to the reader.

Now note that

$$D(\tau) = f_1 D_1 + \cdots + f_r D_r$$

for some non-negative integers $e = f_1 \geq 2,\ f_2, \ldots, f_r$ (cf. (8.7)). Let $E(\tau) = D(\tau) - f_1 D_1$ and let $H(B)$ be the subset of $\mathbb{R}^{r-1}$ defined by

$$\mathbf{x}^{E(\tau)} \leq B \quad \text{and} \quad \min(x_2, \ldots, x_r) \geq 1.$$

Then by integrating with respect to $x_1 \in [0, (B/\mathbf{x}^{E(t)})^{1/e}]$ we obtain

$$\begin{aligned}\int_{G(B)} (\mathbf{x}^{D(\tau)}/x_1) \frac{dx_1}{x_1} \cdots \frac{dx_r}{x_r} &= \frac{B^{1-1/e}}{e-1} \int_{H(B)} \mathbf{x}^{E(\tau)/e} \frac{dx_2}{x_2} \cdots \frac{dx_r}{x_r} \\ &= e^{r-1} \frac{B^{1-1/e}}{e-1} \int_{H(B^{1/e})} \mathbf{x}^{E(\tau)} \frac{dx_2}{x_2} \cdots \frac{dx_r}{x_r}.\end{aligned}$$

The latter integral can be calculated by means of the variable substitution $y_i = f_i \log x_i$, $2 \leq i \leq n$. This gives (cf. (11.39)-(11.40) below):

$$\int_{H(B^{1/e})} \mathbf{x}^{E(\tau)} \frac{dx_2}{x_2} \cdots \frac{dx_r}{x_r} = O(B^{1/e}(1 + \log \frac{B}{e})^{r-2})$$

thereby completing the proof. □

Now let $b = \log B \geq 0$ and $y_i = \log x_i$ for $i = 1, \ldots, n$. This gives an analytic isomorphism between $T(\mathbb{R})^+$ and $V = \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})$ which sends the subset $T(\mathbb{R})_{\geq 1} \subset T(\mathbb{R})^+$ with all coordinates $x_1, \ldots, x_{r+d} \geq 1$ onto the cone $\sigma_{\text{eff}}(X, U) \subset V$ described in (10.18).

Let E_b be the set of all $\varphi \in \sigma_{\text{eff}}(X, U)$ with $0 \leq \varphi(D) \leq b$ and let $d\nu$ be the unique Haar measure on V such that $\operatorname{Vol}(V/L) = 1$ for the $\mathbb{Z}$-lattice $L = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{Z})$ in V. Then, by the variable substitution $y_i = \log x_i$, we get

$$\int_{D(B,\sigma)} \mathbf{dx} = \int_{E_b} \exp(y_1 + \cdots + y_n) \prod_{j=1}^{d} (1 - \exp(-y_{r+j})) d\nu$$

$$\int_{F(B)} \mathbf{x}^D \frac{dx_1}{x_1} \cdots \frac{dx_r}{x_r} = \int_{E_b} \exp(y_1 + \cdots + y_n) d\nu. \tag{11.39}$$

Let $\lambda : V \to \mathbb{R}$ be the linear form $\lambda(y_1, \ldots, y_n) = y_1 + \cdots + y_n$ obtained by evaluating at the anticanonical class. By integrating along the fibres of λ (cf. (10.18)-(10.19)), we obtain

$$\int_{E(b)} \exp(y_1 + \cdots + y_n) d\nu = \alpha(X) \int_0^b \exp(y) y^{r-1} dy \tag{11.40}$$

where $\alpha(X)$ is Peyre's constant. The integral on the right hand side is equal to

$$B \sum_{k=0}^{r-1} (-1)^k \frac{(r-1)!}{(r-1-k)!} (\log B)^{r-1-k}.$$

We therefore conclude from (11.38)-(11.40) that

$$\int_{D(B,\sigma)} \mathbf{dx} = \alpha(X)B(\log B)^{r-1} + O(B(1+\log B)^{r-2}) \tag{11.41}$$

for any maximal cone σ of Δ and hence by (11.32) that

$$\int_{D(B)} \mathbf{dx} = \alpha(X)\operatorname{Card}(\Delta_{\max})B(\log B)^{r-1} + O(B(1+\log B)^{r-2}). \tag{11.42}$$

If we combine this with (11.30) then we obtain the asymptotic formula

$$\mathcal{C}(B) = CB(\log B)^{r-1} + O(B(1+\log B)^{r-2+1/f+\varepsilon}) \tag{11.43}$$

for any $\varepsilon > 0$ with the constant

$$C = (\operatorname{Card}\widetilde{T}(\mathbb{Z}))^{-1}\operatorname{Card}(U_0(\mathbb{R})/U_0(\mathbb{R})^+)\alpha(X)\operatorname{Card}(\Delta_{\max})\Pi_p n_p(\widetilde{X}_0(\mathbb{Z}_p)).$$

We now verify that this result is compatible with Peyre's conjecture (7.7)

$$\mathcal{C}(B) = \alpha_{\text{Peyre}}(X)\tau(X, \|\ \|)B(\log B)^{r-1}(1+o(1)) \tag{11.44}$$

This is by theorem (6.19) (cf. also (5.21)-(5.22)) true if the following identity holds:

$$\begin{aligned} \overline{\Theta}^1_{\{\infty\}}(T^1(A_{\mathbb{Q}})/T(\mathbb{Q}))m_\infty(X(\mathbb{R})) \\ = (\operatorname{Card}\widetilde{T}(\mathbb{Z}))^{-1}\operatorname{Card}(U_0(\mathbb{R})/U_0(\mathbb{R})^+)\operatorname{Card}(\Delta_{\max}). \end{aligned} \tag{11.45}$$

Here $\overline{\Theta}^1_{\{\infty\}}$ is the Haar measure on $T^1(A_{\mathbb{Q}})/T(\mathbb{Q})$ corresponding to the Haar measure $\Theta_{\{\infty\}}$ on $T(A_{\mathbb{Q}})$ given by the adelic order norm (5.9)(b) and the convergence factors $\beta_p = \mu_p(\widetilde{T}(\mathbb{Z}_p))^{-1}$ and $\beta_\infty = 1$ under the bijection described after (5.19). Hence if $\mathbf{K} \subset T(\mathbb{R})$ is a Borel set, and $\overline{\mathbf{K}}$ the image of $\prod_p \widetilde{T}(\mathbb{Z}_p) \times \mathbf{K}$ in $T(A_{\mathbb{Q}})/T^1(A_{\mathbb{Q}}) = \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})$, then

$$\Theta_{\{\infty\}}\left(\prod_p \widetilde{T}(\mathbb{Z}_p) \times \mathbf{K}\right) = \int_{\overline{K}} d\nu$$

for the Haar measure $d\nu$ on $T(A_{\mathbb{Q}})/T^1(A_{\mathbb{Q}}) = \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})$ used in (11.38)-(11.40).

Now recall that the bijection betwen measures on $T^1(A_{\mathbb{Q}})/T(\mathbb{Q})$ and $T(A_{\mathbb{Q}})$ is established by applying (5.19) twice to the chain of normal closed subgroups

$$T(\mathbb{Q}) \subseteq T^1(A_{\mathbb{Q}}) \subseteq T(A_{\mathbb{Q}})$$

with the counting measure on $T(\mathbb{Q})$ and the measure $d\nu$ on $T(A_{\mathbb{Q}})/T^1(A_{\mathbb{Q}})$. Note also that $T(\mathbb{R})^+ \times \prod_p \widetilde{T}(\mathbb{Z}_p)$ is a fundamental domain for the $T(\mathbb{Q})$-action on $T(A_{\mathbb{Q}})$. It is straightforward from this description that:

$$\overline{\Theta}^1_{\{\infty\}}(T^1(A_{\mathbb{Q}})/T(\mathbb{Q})) = \operatorname{Card}(T(\mathbb{R})/T(\mathbb{R})^+)(\operatorname{Card}\widetilde{T}(\mathbb{Z}))^{-1} \tag{11.46}$$

which is equal to 1. It is easy to verify that there is a commutative diagram with exact sequences

$$
\begin{tikzcd}
1 \arrow[r] & T(\mathbb{R})^+ \arrow[r] \arrow[d] & U_0(\mathbb{R})^+ \arrow[r] \arrow[d] & U(\mathbb{R})^+ \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & T(\mathbb{R}) \arrow[r] & U_0(\mathbb{R}) \arrow[r] & U(\mathbb{R}) \arrow[r] & 1
\end{tikzcd}
\tag{11.47}
$$

where the upper +-index means the subgroup given by the real connected component containing 1. One can e.g. make use of the fact that the subgroups in the first row are generated by squares of elements in the groups below.)

Finally, by (9.16) we have:

$$
m_\infty(X(\mathbb{R})) = \mathrm{Card}(U(\mathbb{R})/U(\mathbb{R})^+)\,\mathrm{Card}(\Delta_{\max})
\tag{11.48}
$$

Therefore, (11.45) follows from (11.46)-(11.48). We have thus given a new very different proof of the following result of Batyrev-Tschinkel [**4**].

Theorem 11.49. — *Let Δ be a complete regular d-dimensional fan with n rays and let $X = X_\Delta$ be the toric $\mathbb{Q}$-variety defined by Δ. Suppose that the anticanonical sheaf of X is generated by its global sections. Let H be the toric anticanonical height function on $X(\mathbb{Q})$ defined by the toric adelic norm $\| \ \|$ for X (cf. (11.2)) and let $\mathcal{C}(B)$ be the number of $\mathbb{Q}$-points on U of toric anticanonical height at most B. Let $\alpha_{\mathrm{Peyre}}(X)$ and $\tau(X, \| \ \|)$ be the numbers in (7.7) and let f be the smallest integer such that there exist f rays of Δ not contained in a cone of Δ. Then,*

$$
\mathcal{C}(B) = \alpha_{\mathrm{Peyre}}(X)\tau(X, \| \ \|)B(\log B)^{r-1} + O(B(1+\log B)^{r-2+1/f+\varepsilon})
$$

for any $\varepsilon > 0$.

The following toric surface has been discussed in talks by Batyrev and Tschinkel:

Example 11.50. — Let $N = \mathbb{Z}^2$, $n_1 = (1,0)$, $n_2 = (0,1)$, $n_3 = (-1,2)$, $n_4 = (-1,1)$, $n_5 = (-1,0)$, $n_6 = (-1,-1)$, $n_7 = (0,-1)$, $n_8 = (1,-1)$, $n_9 = (2,-1)$. Then there exists a unique complete regular fan (N, Δ) with 9 two-dimensional cones σ_i, $1 \le i \le 9$ and 9 one-dimensional cones ρ_i, $1 \le i \le 9$ where σ_i, $i \in \mathbb{Z}/9\mathbb{Z}$ is generated by n_{i+4} and n_{i+5} and ρ_i is generated by n_i.

Let

$$
D_0 = \sum_{\rho \in \Delta(1)} D_\rho
$$

and let U_i, $1 \le i \le 9$ be the affine toric plane defined by σ_i. Let χ^{m_i}, $m_i = m(\sigma_i) \in M$ be the unique character of $U \subset X_\Delta$ which on U_i represents the Cartier divisor with Weil divisor D_0 and let

$$D(\sigma_i) := D_0 + \sum_{\rho\in\Delta(1)} \langle -m(\sigma_i), n_\rho\rangle D_\rho.$$

Finally, let D_i be the prime divisor which corresponds to ρ_i. Then,

$$D(\sigma_{i-1}) = D(\sigma_i) = D(\sigma_{i+1}) = D_{i-2} + 2D_{i-1} + 3D_i + 2D_{i+1} + D_{i+2}$$

for all $i \in \mathbb{Z}/9\mathbb{Z}$ with $i \equiv 0 \bmod 3$.

There exists thus for each closed point P on X_Δ a divisor $D(\sigma)$, $\sigma \in \Delta_{\max}$ with $P \notin \operatorname{Supp} D(\sigma)$ so that $\mathcal{O}(D_0)$ is generated by its global sections. Also, it is easy to see that $m = 0$, m_3, m_6, m_9 are the only elements $m \in M$ such that

$$D_0 + \sum_{\rho\in\Delta(1)} \langle -m, n_\rho\rangle D_\rho$$

is effective. Therefore, by [**24**, p. 66] it follows that:

$$H^0(X_\Delta, \mathcal{O}(D_0)) = \mathbb{Q} \oplus \mathbb{Q}\chi^{-m_3} \oplus \mathbb{Q}\chi^{-m_6} \oplus \mathbb{Q}\chi^{-m_9}.$$

This means that the morphism $g : X_\Delta \to \mathbb{P}^3_{\mathbb{Q}}$ defined by the linear sytem $|D_0|$ restricts to the open immersion restriction

$$g_U : U \to \mathbb{P}^3_{\mathbb{Q}} \quad \text{defined by} \quad (1, \chi^{-m_3}, \chi^{-m_6}, \chi^{-m_9}).$$

If we introduce projective coordinates (z_0, z_3, z_6, z_9) for $\mathbb{P}^3_{\mathbb{Q}}$, then the scheme-theoretic image X' of g is given by the closed subscheme with equation $z_0^3 = z_3z_6z_9$. The morphism $g : X \to X'$ is a toric morphism corresponding to a morphism of fans $(N, \Delta) \to (N, \Delta')$. The fan Δ' is the complete non-regular fan with three two-dimensional cones τ_{3i}, $i = 1, 2, 3$ generated by n_{3i+3} and n_{3i+6}.

Therefore, the number $\mathcal{C}(B)$ of $\mathbb{Q}$-points on $U \subset X_\Delta$ of toric height at most B is equal to the number of integral points $(z_0, z_3, z_6, z_9) \in \mathbb{P}^3(\mathbb{Z})$ with $z_0^3 = z_3z_6z_9 \neq 0$ of height:

$$\max(|z_0|, |z_3|, |z_6|, |z_9|) \leq B.$$

The Picard group of X_Δ is of rank $r = \operatorname{Card}\Delta(1) - \dim\Delta = 9 - 2 = 7$ (cf. (8.2)). Further, if p is a prime number, then we can determine the cardinality of $\widetilde{X}_\Delta(\mathbb{Z}/p\mathbb{Z})$ by means of the bijection between $\widetilde{U}(\mathbb{Z}/p\mathbb{Z})$-orbits under the action of $\widetilde{U}(\mathbb{Z}/p\mathbb{Z})$ on $\widetilde{X}_\Delta(\mathbb{Z}/p\mathbb{Z})$ and the cones in Δ (see [**24**, p. 94]). This gives

$$\operatorname{Card}\widetilde{X}_\Delta(\mathbb{Z}/p\mathbb{Z}) = 1(p-1)^2 + 9(p-1) + 9 = p^2 + 7p + 1.$$

Therefore, the principal universal torsor $\widetilde{X}_0$ over $\widetilde{X} = \widetilde{X}_\Delta$ satisfies

$$\operatorname{Card}\widetilde{X}_0(\mathbb{Z}/p\mathbb{Z}) = \operatorname{Card}\widetilde{X}(\mathbb{Z}/p\mathbb{Z})\operatorname{Card}\widetilde{T}(\mathbb{Z}/p\mathbb{Z}) = (p^2 + 7p + 1)(p-1)^7,$$

and (cf. (2.15))

$$n_p(\widetilde{X}_0(\mathbb{Z}_p)) = (1 + 7/p + 1/p^2)(1 - 1/p)^7.$$

Also, Peyre's constant $\alpha(X)$ is easily seen to be equal to the $\mathrm{Vol}(P)/9$, where P is the polytope in $\mathbb{R}^6_{\geq 0}$ with coordinates $(y_1, y_2, y_4, y_5, y_7, y_8)$ defined by

$$\max(y_1 + 2y_2 + 2y_4 + y_5, y_4 + 2y_5 + 2y_7 + y_8, y_7 + 2y_8 + 2y_1 + y_2) \leq 1$$

and a further calculation gives $\alpha(X) = 1/25920$.

Hence by (11.43) we get that

$$\mathcal{C}(B) = CB(\log B)^6 + O(B(1 + \log B)^{6-1/2+\varepsilon}) \tag{11.51}$$

for any $\varepsilon > 0$ with

$$C = \frac{1}{720} \prod_p (1 + 7/p + 1/p^2)(1 - 1/p)^7.$$

This asymptotic formula was first established by Batyrev and Tschinkel as a corollary of [**4**] (cf. [**6**] which we received after completing this paper). Another treatment of this asymptotic formula, but without a discussion of the value for C, can be found in the paper of Fouvry [**22**]. We have also after the completion of this paper received the paper [**13**] of de la Bretèche on the counting function for $z_0^3 = z_3 z_6 z_9 \neq 0$. He obtains a more precise asymptotic formula than in (11.51). He has also just before this book goes to press improved the estimates of the error terms for other toric varieties.

References

[1] M. Artin, A. Grothendieck & J.-L. Verdier – *Théorie des topos et cohomologie étale des schémas*, Lecture notes in Math., vol. 269, 270, 305, Springer-Verlag, 1972-1973.

[2] M. Artin & D. Mumford – Some elementary examples of unirational varieties which are not rational, *Proc. London Math. Soc.* **25** (1972), p. 75–95.

[3] V. Batyrev & Y. I. Manin – Sur les nombres des points rationnels de hauteur bornée des variétés algébriques, *Math. Ann.* **286** (1980), p. 27–43.

[4] V. Batyrev & Y. Tschinkel – Manin's conjecture for toric varieties, to appear in J. Alg. Geometry.

[5] ______, Rational points on some fano cubic bundles, Preprint.

[6] ______, Saturated Varieties, this volume.

[7] ______, Rational points of bounded height on compactifications of anisotropic tori, *Int. Math. Research Notes* **12** (1995), p. 591–635.

[8] S. Bloch – A note on height pairings, Tamagawa numbers and the Birch-Swinnerton-Dyer conjecture, *Invent. Math.* **58** (1980), p. 65–76.

[9] A. Borel – *Linear algebraic groups*, 2nd ed., Graduate texts in Math., vol. 126, Springer-Verlag, Berlin, 1991.

[10] S. Bosch, W. Lüthkebohmert & M. Raynaud – *Néron models*, Springer-Verlag, 1990.

[11] N. Bourbaki – *Intégration*, Hermann, Paris, Chap I-VII.

[12] ———, *Variétés differentielles et analytiques*, Paris, 1971, Fascicule de resultats, paragraphes 8 à 15.

[13] R. de la Bretèche – Sur le nombre de points de hauteur bornée d'une certaine surface cubique singulière, this volume.

[14] J.-L. Colliot-Thélène & J.-J. Sansuc – La R-équivalence sur les tores, *Ann. Sci. École Norm. Sup. (4)* **10** (1977), p. 175–229.

[15] ———, La descente sur les variétés rationnelles II, *Duke Math. J.* **54** (1987), p. 375–492.

[16] D. A. Cox – The homogenous coordinate ring of a toric variety, *J. Alg. Geom.* **4** (1995), p. 17–50.

[17] H. Delange – Généralisation du théorème de Ikehara, *Ann. Sci. École Norm. Sup. (4)* **3** (1971), p. 213–242.

[18] P. Deligne – *Cohomologie étale*, Springer-Verlag, Berlin, 1977, Séminaire de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier, Lecture Notes in Mathematics, Vol. 569.

[19] P. Deligne – La conjecture de Weil II, *Inst. Hautes Études Sci. Publ. Math.* **52** (1981), p. 313–428.

[20] M. Demazure – Sous-groupes algébriques de rang maximum du groupe de Cremona, *Ann. Sci. École Norm. Sup. (4)* (1970).

[21] M. Demazure & A. Grothendieck – *Schémas en groupes*, Lecture notes in Math., vol. 151, 152, 153, Springer-Verlag, 1970.

[22] E. Fouvry – Sur la hauteur des points d'une certaine surface cubique singulière, this volume.

[23] J. Fraenke, Y. I. Manin & Y. Tschinkel – Rational points of bounded height of Fano varieties, *Invent. Math.* **95** (1989), p. 425–35.

[24] W. Fulton – *Introduction to toric varieties*, Ann. of Math. Studies, vol. 131, Princeton University Press, 1993.

[25] P. Griffiths & J. Harris – *Principles of algebraic geometry*, Wiley Interscience, 1978.

[26] A. Grothendieck – Classes de Chern et représentations linéaires des groupes discrets, Adv. Stud. Pure Math., vol. 3, Adv. Stud. Pure Math., North-Holland, Amsterdam, 1968.

[27] ———, Le groupe de Brauer I-III, in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968.

[28] ———, *Séminaire de Géométrie Algébrique: Revêtements étales et groupe fondamental*, Lecture notes in Math., vol. 224, Springer-Verlag, Heidelberg, 1971.

[29] A. Grothendieck & J. Dieudonné – Éléments de Géométrie Algébrique, *Inst. Hautes Études Sci. Publ. Math.* **8** (1961).

[30] ———, Éléments de Géométrie Algébrique, *Inst. Hautes Études Sci. Publ. Math.* **20, 24, 28, 32** (1964, 1966).

[31] ———, *Éléments de Géométrie Algébrique I*, Springer, 1971.

[32] R. Hartshorne – *Algebraic geometry*, Graduate texts in Math., vol. 52, Springer-Verlag, Berlin, 1991.

[33] D. R. Heath-Brown – The density of zeros of forms for which weak approximation fails, *Math. Comp.* **59** (1992), p. 613–623.

[34] N. Katz – Review of ℓ-adic cohomology, in *Motives*, vol. Volume 55, Symp. in pure Mathematics, no. 1, American Mathematical Society.

[35] J. Kollar – *Rational curves on algebraic varieties*, Ergebnisse der Mathematik 3 Folge Band 32, Springer-Verlag, 1996.

[36] S. Lang – *Algebraic number theory*, Addison-Wesley, 1970.

[37] ———, *Fundamentals of Diophantine Geometry*, Springer-Verlag, 1983.

[38] ———, *Real and Functional Analysis*, 3 ed., Springer-Verlag, 1993.

[39] ———, *Differential and Riemannian Manifolds*, Springer-Verlag, 1995.

[40] Y. I. Manin – Le groupe de Brauer-Grothendieck en géométrie diophantienne, in *Actes du congrès intern. Math. Nice 1 (1970)*, Gauthiers-Villars, Paris, 1971, p. 401–411.

[41] ———, *Cubic forms : Algebra, geometry and arithmetic*, 2nd ed., North-Holland, Amsterdam, 1974 and 1986.

[42] ———, Notes on arithmetic on fano three-folds, *Compositio Math.* **85** (1993), p. 37–55.

[43] ———, *Problems on rational points and rational curves on algebraic varieties*, Surveys in Differential Geometry, vol. II, International Press, 1995.

[44] Y. I. Manin & M. A. Tsfasman – Rational varieties : Algebra, geometry and arithmetic, *Russian mathematical surveys* **41** (1986), no. 2.

[45] J. S. Milne – *Etale Cohomology*, Princeton University Press, Princeton, 1980.

[46] T. Oda – *Convex Bodies and Algebraic Geometry*, Springer-Verlag, Berlin, 1988.

[47] J. Oesterlé – Construction de hauteurs archimédienne et p-adiques suivant la méthode de Bloch, in *Séminaire de théorie des nombres 1980-81*, Birkhäuser, Boston, 1980.

[48] ———, Réduction modulo p^n des sous-ensembles analytiques fermés de $\mathbb{Z}_p^N$, *Invent. Math.* **66** (1982).

[49] T. Ono – Arithmetic of algebraic tori, *Ann. of Math.* **74** (1961), p. 101–139.

[50] ———, On the Tamagawa number of algebraic tori, *Ann. of Math.* **78** (1963), p. 47–73.

[51] E. Peyre – Terme principal de la fonction zêta des hauteurs et torseurs universels, these proceedings.

[52] ———, Hauteurs et mesures de Tamagawa sur les variétés de Fano, *Duke Math. J.* **79** (1995), p. 101–218.

[53] V. Platonov & A. Rapinchuk – *Algebraic Groups and Number Theory*, Academic Press, 1994.

[54] I. Reiner – *Maximal Orders*, Academic Press, 1975.

[55] P. Salberger – Manuscript on universal torsors, 1993.

[56] S. Schanuel – Heights in number fields, *Bull. Soc. Math. France* **107** (1979), p. 433–449.

[57] J.-P. Serre – Quelques applications du théorème du densité de Chebotarev, *Inst. Hautes Études Sci. Publ. Math.* **54**, p. 123–201.

[58] ______, *Classification des variétés analytiques p-adiques compactes*, vol. 3, 1965.

[59] ______, *Lie algebras and Lie groups*, Benjamin, New York, 1965.

[60] ______, Divisibilité de certaines fonctions arithmétiques, *l'Ens. Math.* **22** (1976), p. 227–260.

[61] ______, *Résumé de Cours au Collège de France*, 1980.

[62] ______, *Lectures on the Mordell-Weil theorem*, Viehweg-Verlag, 1989.

[63] J. H. Silverman – The Theory of Height Functions, in *Arithmetic Geometry* (G. Cornell & J. Silverman, eds.), Springer-Verlag, Berlin, 1986, p. 151–166.

[64] T. A. Springer – Galois cohomology of linear groups, in *Algebraic Groups and Discontinuous Subgroups* (A. Borel & G. Mostow, eds.), Proc. of Symposia in Pure Math., vol. 9, Amer. Math. Society, 1966, p. 149–158.

[65] J. Tate – Global class field theory, in *Algebraic number theory* (Cassels & Fröhlich, eds.), Academic Press, 1967.

[66] A. Weil – *Basic number theory*, Springer-Verlag, Berlin, 1967.

[67] ______, *Adeles and algebraic groups*, Birkhäuser, Boston, 1982.

[68] D. B. Zagier – *Zeta Funktionen und quadratische Körper*, Springer-Verlag, Berlin, 1991.

Per Salberger, Chalmers University of Technology

Astérisque
251, 1998, p. 259–297

TERME PRINCIPAL DE LA FONCTION ZÊTA DES HAUTEURS ET TORSEURS UNIVERSELS

par

Emmanuel Peyre

Résumé. — Soient V une variété de Fano et $\mathbf{H}$ un système de hauteurs d'Arakelov définissant un accouplement entre le groupe de Picard $\operatorname{Pic} V$ et les points rationnels de V à valeur dans $\mathbf{R}$. Soit $\zeta_{\mathbf{H}}$ la fonction zêta associée sur $\operatorname{Pic} V \otimes \mathbf{C}$. Batyrev, Manin et Tschinkel ont conjecturé que cette fonction est holomorphe sur un cône de sommet le faisceau anticanonique ω_V^{-1}. Il est en outre possible de donner une expression conjecturale du terme principal de cette fonction $\zeta_{\mathbf{H}}$ au voisinage de ce sommet. Le but de ce texte est de montrer comment cette expression conjecturale peut s'écrire naturellement en passant aux torseurs universels au-dessus de V.

1. Introduction

Soit V une variété projective, lisse et géométriquement intègre sur un corps de nombres k telle que $H^i(V, \mathcal{O}_V)$ soit nul pour $i = 1$ ou 2 et telle que la classe du faisceau anticanonique ω_V^{-1} soit à l'intérieur du cône effectif. On suppose en outre que les points rationnels de V sont Zariski denses. Soit $\mathbf{h}$ une hauteur sur V correspondant à la donnée d'une paire $(L, (\|.\|_v)_{v \in M_k})$ où L est un fibré en droites dont la classe est dans le cône effectif et $(\|.\|_v)_{v \in M_k}$ un système adélique de métriques sur L. On s'intéresse alors au comportement asymptotique de

$$n_{U,\mathbf{h}}(H) = \#\{x \in U(k) \mid \mathbf{h}(x) \leqslant H\}$$

où U est un ouvert dense de V.

À la connaissance de l'auteur, dans les exemples considérés à ce jour ce comportement est de la forme

$$n_{U,\mathbf{h}}(H) \sim C H^a (\log H)^{b-1}$$

où C est une constante réelle strictement positive, $a \geqslant 0$ et $b \geqslant 1$. Diverses conjectures à des degrés de précision divers ont été faites sur a et b par Batyrev et Manin

***Classification mathématique par sujets* (2000).** — Primaire 14G05; secondaires 14L30, 11D72.
Mots clefs. — Torseurs universels, mesure de Tamagawa.

(cf. [**FMT**], [**BM**] et [**Man**]) puis, lorsque L est le faisceau anticanonique, sur C (cf. [**Pe**]). Les principales familles de variétés pour lesquelles des résultats ont été effectivement démontrés peuvent être regroupées en deux groupes : d'une part les intersections complètes lisses de grande dimension sur $\mathbf{Q}$, pour lesquelles on dispose de la méthode du cercle (cf. [**Bi**], [**FMT**]) et d'autre part les variétés sur lesquelles agissent des groupes algébriques avec une orbite ouverte pour lesquelles on utilise des techniques d'analyse harmonique fine (cf. [**FMT**], [**BT1**] et [**BT2**]). Peu de cas ont été traités en dehors de ces deux grands groupes, faute d'avoir une autre méthode générale. Toutefois Salberger a montré une majoration de la forme souhaitée pour la surface de Del Pezzo obtenue en éclatant quatre points rationnels en position générale sur $\mathbf{P}^2_{\mathbf{Q}}$, surface qui ne rentre dans aucun des deux groupes indiqués. La méthode qu'il utilise souligne le rôle du torseur universel pour attaquer le problème en général.

Ces torseurs universels, introduits par Colliot-Thélène et Sansuc en liaison avec les problèmes du principe de Hasse et de l'approximation faible (cf. [**CTS1**] et [**CTS2**]) apparaissent implicitement dans le cas des intersections complètes lisses en tant que cône au-dessus de la variété ainsi que dans des démonstrations directes de la formule asymptotique dans quelques cas particuliers de variétés toriques (cf. [**Pe**] et [**Ro**]). Tout récemment, Salberger a démontré dans [**Sa**] comment la constante

$$\#H^1(k, \operatorname{Pic} \overline{V}) C_{\mathbf{h}}(V)$$

où $C_{\mathbf{h}}(V)$ a été défini dans [**Pe**], constante qui apparaît dans la plupart des cas considérés à ce jour, pouvait s'interpréter comme somme de nombres de Tamagawa associés aux torseurs universels ayant un point rationnel.

Le but de ce texte est similaire : il s'agit de montrer comment la conjecture actuellement la plus précise sur le terme principal de la fonction zêta des hauteurs

$$\zeta_{\mathbf{H}} : \operatorname{Pic} V \otimes \mathbf{C} \to \mathbf{C}$$

peut s'exprimer de manière naturelle comme passage d'une sommation sur les points rationnels à une intégrale sur un domaine de type adélique muni d'une mesure canonique.

Les outils développés permettent également d'interpréter la conjecture de Manin raffinée sur $n_{U,\mathbf{h}}(H)$ pour une hauteur $\mathbf{h}$ associée a ω_V^{-1} en termes des torseurs universels.

Dans la partie 2, nous introduisons les notations qui nous sont nécessaires puis rappelons la formule empirique obtenue pour le terme principal de la fonction zêta des hauteurs. La partie 3 est consacrée à des rappels sur les torseurs universels. La partie 4 décrit le relèvement de divers objets à ces torseurs. Enfin la partie 5 contient les deux résultats principaux et leurs démonstrations.

2. Le terme principal de la fonction zêta des hauteurs

2.1. Notations. — Nous allons tout d'abord fixer un certain nombre de notations qui seront utilisées dans l'ensemble du texte.

Notations 2.1.1. — Pour tout corps E, on note $\overline{E}$ une clôture algébrique de E et E^s la clôture séparable de E dans $\overline{E}$. Pour tout $\mathrm{Gal}(E^s/E)$-module discret M, on désigne par $H^i(E, M)$ le groupe de cohomologie $H^i(\mathrm{Gal}(E^s/E), M)$.

Dans la suite k désigne un corps de nombres, $\mathcal{O}_k$ son anneau des entiers et d son discriminant. L'ensemble des places de k est noté M_k, l'ensemble des places finies M_f et l'ensemble des places archimédiennes M_∞. Pour tout $v \in M_k$, on note k_v le complété de k en v, $|.|_v$ la norme associée normalisée par

$$\forall v|p, \quad \forall x \in k_v, \quad |x|_v = \left|N_{k_v/\mathbf{Q}_p}(x)\right|_p.$$

Si v appartient à M_f, $\mathcal{O}_v$ désigne l'anneau des entiers de k_v et $\mathbf{F}_v$ le corps résiduel. Pour tout v de M_∞ tel que $[k_v : \mathbf{R}] = 2$, on fixe un isomorphisme de k_v sur $\mathbf{C}$. Pour toute place v, la mesure de Haar dx_v sur k_v est normalisée comme dans [**We**, § 2.1.1] ; autrement dit,

- si $v \in M_f$, alors $\int_{\mathcal{O}_v} dx_v = 1$,
- si $k_v = \mathbf{R}$, alors dx_v est la mesure de Lebesgue usuelle,
- si $k_v = \mathbf{C}$, alors $dx_v = i\, dz\, d\bar{z} = 2\, dx\, dy$.

Soit $\mathscr{V}$ un schéma sur un anneau A. Alors pour toute A-algèbre B, $\mathscr{V}(B)$ désigne l'ensemble $\mathrm{Hom}_{\mathrm{Spec}\, A}(\mathrm{Spec}\, B, \mathscr{V})$ et $\mathscr{V}_B$ le produit $\mathscr{V} \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, B$. Si V est défini sur k et $v \in M_k$, on note V_v le schéma V_{k_v}.

Si V est une variété lisse sur un corps E, son groupe de Picard est noté $\mathrm{Pic}\, V$, son groupe de Neron-Severi $\mathrm{NS}(V)$ et son faisceau canonique ω_V. On note également $C_{\mathrm{eff}}(V)$ le cône dans $\mathrm{NS}(V) \otimes_{\mathbf{Z}} \mathbf{R}$ engendré par les classes de diviseurs effectifs. Si V est une variété sur k, $V(\boldsymbol{A}_k)$ désigne l'espace adélique associé (cf. [**We**, § 1]).

Si A est une partie de B, on note $\mathbf{1}_A$ sa fonction indicatrice.

Hypothèses 1. — Dans la suite V désigne une variété projective, lisse et géométriquement intègre sur k qui satisfait les conditions suivantes :

(i) $H^i(V, \mathcal{O}_V) = 0$ pour $i = 1$ ou 2,
(ii) $\mathrm{Pic}\, \overline{V} = \mathrm{NS}\, \overline{V}$ est sans torsion,
(iii) ω_V^{-1} appartient à l'intérieur du cône $C_{\mathrm{eff}}(V)$ et
(iv) $V(k)$ est Zariski dense dans V.

Ces hypothèses suffisent pour définir les objets décrits dans [**Pe**]. Toutefois nous serons amenés par la suite à supposer que la variété vérifie d'autres conditions plus techniques.

2.2. Hauteurs. — Dans la suite nous utiliserons les hauteurs d'Arakelov qui sont définies de la manière suivante :

Définitions 2.2.1. — Soit X une variété projective, lisse et géométriquement intègre sur k, L un faisceau inversible sur X. Pour toute place v de k, une *métrique v-adique* sur L est une application qui à tout point x de $X(k_v)$ associe une norme $\|.\|_v$ sur $L(x) = L_x \otimes_{\mathcal{O}_{X,x}} k_v$ de sorte que pour tout ouvert W de X et toute section s de L sur W l'application
$$x \mapsto \|\mathrm{s}(x)\|_v$$
soit continue.

Une *métrique adélique* sur L est une famille de métriques $(\|.\|_v)_{v \in M_k}$ de sorte qu'il existe un ensemble fini $S \subset M_f$, un modèle projectif et lisse $\mathscr{X}$ de X sur $\mathcal{O}_S$ et un modèle $\mathscr{L}$ de L sur $\mathscr{X}$ de sorte que pour tout $v \in M_f - S$, pour tout $x \in X(k_v)$, correspondant à $\tilde{x} \in \mathscr{X}(\mathcal{O}_v)$, la norme $\|.\|_v$ sur $L(x)$ soit définie à l'aide d'un générateur y_0 du $\mathcal{O}_v$-module de rang un $\tilde{x}^*(\mathscr{L})$ par la formule
$$\forall y \in L(x), \quad \|y\|_v = \left|\frac{y}{y_0}\right|_v.$$

Une *hauteur d'Arakelov* $\mathbf{h}$ sur X est la donnée d'une paire $(L, (\|.\|_v)_{v \in M_k})$ où L est un fibré en droites et $(\|.\|_v)_{v \in M_k}$ une métrique adélique sur le fibré en droites L. Si $\mathbf{h}$ est une hauteur sur X et si x est un point rationnel de X, alors la *hauteur de x relativement à* $\mathbf{h}$ est définie par
$$\mathbf{h}(x) = \prod_{v \in M_k} \|\mathrm{s}(x)\|_v^{-1}$$
où s est une section de L sur un voisinage de x avec $\mathrm{s}(x) \neq 0$. Par la formule du produit cela est indépendant du choix de la section.

Pour tout sous-espace constructible W de V et tout $H > 0$ on pose
$$n_{W,\mathbf{h}}(H) = \#\{x \in W(k) \mid \mathbf{h}(x) \leqslant H\} \leqslant +\infty.$$

Remarques 2.2.1. — (i) Si $[L]$ appartient à l'intérieur du cône effectif, alors il existe un ouvert non vide U de X tel que $n_{U,\mathbf{h}}(H)$ soit fini pour tout H.

(ii) On a une notion évidente de produit tensoriel de hauteurs et
$$\forall x \in X(k), \quad \mathbf{h}_1 \otimes \mathbf{h}_2(x) = \mathbf{h}_1(x)\mathbf{h}_2(x).$$

2.3. Mesures de Tamagawa. — Nous allons maintenant rappeler comment toute métrique adélique sur ω_V^{-1} définit une mesure sur $V(\boldsymbol{A}_k)$.

Notations 2.3.1. — Soit $\mathbf{h} = (\omega_V^{-1}, (\|.\|_v)_{v \in M_k})$ une hauteur sur V. Pour toute place v de k, on lui associe la mesure $\boldsymbol{\omega}_{\mathbf{h},v}$ sur $V(k_v)$ définie par la relation (cf. [**We**], [**Pe**, § 2.2.1])

$$\boldsymbol{\omega}_{\mathbf{h},v} = \left\|\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}\right\|_v dx_{1,v} \dots dx_{n,v} \tag{2.3.1}$$

où $x_1, \dots, x_n$ sont des coordonnées locales analytiques au voisinage d'un point x de $V(k_v)$ et $\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$ est vu comme section locale de ω_V^{-1}.

Comme dans [**Pe**, Lemme 2.1.1], il existe un ensemble fini de places $S_0 \subset M_f$ et un modèle projectif et lisse $\mathscr{V}$ de V sur $\mathcal{O}_{S_0}$ dont les fibres sont géométriquement intègres et tel que, pour tout $\mathfrak{p} \in M_f - S_0$, il existe un isomorphisme de $\operatorname{Pic}(\overline{V})$ sur $\operatorname{Pic}(\mathscr{V}_{\overline{\mathbf{F}}_\mathfrak{p}})$ compatible aux actions des groupes de Galois et tel que pour tout nombre premier l non divisible par $\mathfrak{p}$ la partie l-primaire du groupe de Brauer $\operatorname{Br}(\mathscr{V}_{\overline{\mathbf{F}}_\mathfrak{p}})$ soit finie.

On considère alors les termes locaux de la fonction L associée à $\operatorname{Pic}\overline{V}$, définis pour toute place $\mathfrak{p} \in M_f - S_0$ par

$$L_\mathfrak{p}(s, \operatorname{Pic}\overline{V}) = \frac{1}{\operatorname{Det}(1 - (\#\mathbf{F}_\mathfrak{p})^{-s}\operatorname{Fr}_\mathfrak{p} \mid \operatorname{Pic}\mathscr{V}_{\overline{\mathbf{F}}_\mathfrak{p}} \otimes \mathbf{Q})}$$

où $\operatorname{Fr}_\mathfrak{p}$ désigne le Frobenius géométrique en $\mathfrak{p}$. La fonction L globale est alors définie par le produit eulérien

$$L_{S_0}(s, \operatorname{Pic}\overline{V}) = \prod_{\mathfrak{p} \in M_f - S_0} L_\mathfrak{p}(s, \operatorname{Pic}\overline{V}).$$

Comme dans [**Pe**, lemme 2.2.5], on montre que ce produit converge absolument pour $\operatorname{Re} s > 1$ et s'étend en une fonction méromorphe sur $\mathbf{C}$ avec un pôle d'ordre $t = \operatorname{rg}\operatorname{Pic} V$ en 1.

On utilise les facteurs de convergences $(\lambda_v)_{v \in M_k}$ définis par

$$\lambda_v = \begin{cases} L_v(1, \operatorname{Pic}\overline{V}) & \text{si } v \in M_f - S_0, \\ 1 & \text{sinon.} \end{cases}$$

Il résulte alors des conjectures de Weil démontrées par Deligne que la mesure adélique $\prod_{v \in M_k} \lambda_v^{-1} \boldsymbol{\omega}_{\mathbf{h},v}$ converge (cf. [**Pe**, proposition 2.2.2]).

Remarque 2.3.1. — Il est également possible d'utiliser la série L associée au groupe de cohomologie $H^2_{\text{ét}}(\overline{V}, \mathbf{Q}_l(1))$ comme le propose Swinnerton-Dyer dans [**S-D**].

Définition 2.3.2. — Avec les notations qui précèdent, la *mesure de Tamagawa* associée à $\mathbf{h}$ est donnée par

$$\boldsymbol{\omega}_\mathbf{h} = \Big(\lim_{s \to 1} (s-1)^t L_{S_0}(s, \operatorname{Pic}\overline{V})\Big) \frac{1}{\sqrt{d}^{\dim V}} \prod_{v \in M_k} \lambda_v^{-1} \boldsymbol{\omega}_{\mathbf{h},v},$$

où t est le rang de $\operatorname{Pic} V$. Elle est indépendante du choix de S_0.

On pose également

$$\tau_\mathbf{h}(V) = \boldsymbol{\omega}_\mathbf{h}\big(\overline{V(k)}\big).$$

2.4. Notions de répartition. — Une des premières questions qu'on est amené à se poser lorsqu'on étudie le comportement asymptotique du nombre de points de hauteur bornée sur une variété est la question de la répartition asymptotique des points rationnels. Ce phénomène peut être considéré à différents niveaux de précision.

La notion la plus stricte, introduite par Manin, est celle de sous-variété accumulatrice.

Définition 2.4.1. — Soit F un sous-espace constructible de V. Alors pour toute hauteur $\mathbf{h} = (L, (\|.\|_v)_{v \in M_k})$ avec $[L]$ appartenant à l'intérieur du cône $C_{\text{eff}}(V)$, on note

$$a_F(L) = \overline{\lim_{H \to +\infty}} \log(n_{F,\mathbf{h}}(H))/\log H \leqslant +\infty$$

qui donne la puissance de H apparaissant dans le comportement asymptotique de $n_{F,\mathbf{h}}(H)$.

Un fermé irréductible F de V est dit *L-accumulateur* (ou simplement *accumulateur* quand $L = \omega_V^{-1}$) si et seulement si, pour tout ouvert non vide W de F, il existe un ouvert non vide U de V avec $a_W(L) > a_U(L)$.

Remarques 2.4.1. — (i) Par [**BM**, proposition 1.4.a], le fait d'être L-accumulateur ne dépend pas du choix de la métrique adélique sur L.

(ii) À la connaissance de l'auteur, dans les exemples étudiés à ce jour, si $[\omega_V^{-1}]$ appartient à l'intérieur de $C_{\text{eff}}(V)$, le complémentaire des sous-variétés $[L]$-accumulatrices dans V est un ouvert de Zariski non vide de V.

(iii) Batyrev et Manin ont défini dans [**BM**, §2.5] une notion de sous-variété géométriquement accumulatrice de la manière suivante : on considère pour toute sous-variété irréductible F de V, une normalisation $\overline{F}$ de F d'ouvert lisse $\overline{F}_0$; si $\phi : \overline{F}_0 \to V$ est l'application induite, on pose

$$a_F^{\text{géo}}(L) = \inf\left\{\frac{p}{q}, p \in \mathbf{Z}, q \in \mathbf{Z}_{>0} \,\middle|\, \Gamma(\overline{F}_0, \phi^*(L)^{\otimes p} \otimes \omega_{\overline{F}_0}^{\otimes q}) \neq \{0\}\right\}$$

et la sous-variété est dite géométriquement accumulatrice si $a_F^{\text{géo}}(L) > a_V^{\text{géo}}(L)$. Il est envisageable que les deux notions coïncident lorsque le corps est suffisamment gros (*i.e.* contient un sous-corps fixé).

En ce qui concerne les sous-variétés accumulatrices, le faisceau anticanonique ne joue pas un rôle vraiment particulier. Il n'en est plus de même lorsqu'on considère la notion suivante :

Définition 2.4.2. — Soit $\mathbf{h}$ une hauteur dont le fibré est à l'intérieur de $C_{\text{eff}}(V)$. Un fermé irréductible strict F de V est dit *modérément accumulateur* pour $\mathbf{h}$ si et seulement si pour tout ouvert non vide W de F, il existe un ouvert non vide U de V tel que

$$\overline{\lim_{H \to +\infty}} \frac{n_{W,\mathbf{h}}(H)}{n_{U,\mathbf{h}}(H)} > 0.$$

Remarques 2.4.2. — (i) Si pour une hauteur $\mathbf{h} = (L, (\|.\|_v)_{v\in M_k})$, pour tout ouvert U de V et tout nombre reel $B > 0$

$$\varliminf_{H\to+\infty} n_{U,\mathbf{h}}(BH)/n_{U,\mathbf{h}}(H) > 0,$$

ce qui est vérifié si $n_{U,\mathbf{h}}(H)$ est de la forme $CH^a(\log H)^{b-1}$ avec $a > 0$ alors un fermé est modérément accumulateur pour $\mathbf{h}$ si et seulement s'il l'est pour toute hauteur $\mathbf{h}'$ correspondant à une métrique sur L.

(ii) Il est aisé d'obtenir des variétés V telles que si $[L] \notin \mathbf{R}_{>0}[\omega_V^{-1}]$, alors les sous-variétés modérément accumulatrices pour L sont Zariski denses. Ainsi pour $V = \mathbf{P}^1_k \times \mathbf{P}^1_k$ et $L = \mathcal{O}(m, m')$ avec $m > m' > 0$, toutes les fibres de la projection sur le premier facteur sont modérément accumulatrices pour L.

(iii) Lorsque $L = \omega_V^{-1}$, on ne connaît pas d'exemple où l'on ait montré que les sous-variétés modérément accumulatrices sont Zariski denses. Toutefois, dans le contre-exemple à la conjecture de Batyrev et Manin sur la puissance du logarithme, construit par Batyrev et Tschinkel dans [**BT3**] et qui est une variété fibrée en surfaces cubiques, il est vraisemblable que toutes les fibres dont le groupe de Picard est de rang 7 sur k sont faiblement accumulatrices. Or ces fibres sont Zariski denses.

(iv) Si F est une sous-variété lisse et si $0 < a_F^{\text{géo}}(L) < +\infty$, on peut également définir $b_F^{\text{géo}}(L)$ comme la codimension de la face du cône $[\omega_F^{-1}] + C_{\text{eff}}(F)$ contenant $a_F^{\text{géo}}(L)\left[L_{|F}\right]$ dans $\mathrm{NS}(F)$. On peut alors se poser la question de savoir si les sous-variétés modérément accumulatrices minimales sont données par les conditions :

$$a_F^{\text{géo}}(L) > a_V^{\text{géo}}(L),$$

auquel cas la sous-variété est géométriquement accumulatrice, ou

$$a_F^{\text{géo}}(L) = a_V^{\text{géo}}(L) \quad \text{et} \quad b_F^{\text{géo}}(L) \geqslant b_V^{\text{géo}}(L).$$

Il faut noter que le contre-exemple de Batyrev et Tschinkel est basé sur cette idée géométrique.

Si le complémentaire des sous-variétés modérément accumulatrices est un ouvert de Zariski U non vide de V, ce qui semble être assez général pour $L = \omega_V^{-1}$, lorsque V vérifie les hypothèses 1, on peut se demander comment sont répartis les points rationnels d'un point de vue adélique. Une réponse semble être donnée à cette question par la notion d'équidistribution, définie dans [**Pe**, § 3] de la manière suivante :

Définition 2.4.3. — On dit qu'un ouvert W de $V(\boldsymbol{A}_k)$ est *bon* si et seulement s'il existe une hauteur $\mathbf{h}$ correspondant à une métrique sur ω_V^{-1} telle que $\boldsymbol{\omega}_{\mathbf{h}}(\partial W) = 0$ où ∂W désigne $\overline{W} - W$. Cela est alors vrai pour toute métrique.

On dit alors que les points de V sont *équidistribués* sur U ouvert de Zariski de V si et seulement s'il existe une métrique adélique sur ω_V^{-1} telle que pour tout bon ouvert

W de $V(\boldsymbol{A}_k)$

$$\frac{\#\{x \in U(k) \cap W \mid \mathbf{h}(x) \leqslant H\}}{n_{U,\mathbf{h}}(H)} \underset{H \to +\infty}{\longrightarrow} \frac{\boldsymbol{\omega}_{\mathbf{h}}(W \cap \overline{V(k)})}{\boldsymbol{\omega}_{\mathbf{h}}(\overline{V(k)})}$$

Remarques 2.4.3. — (i) Dans [**Pe**], on montre que les points de V sont équidistribués sur V si c'est une variété de drapeau généralisée, c'est-à-dire de la forme $P\backslash G$ où G est un groupe algébrique linéaire semi-simple et P un k-sous-groupe parabolique de G, ou si V est une intersection complète lisse définie par m polynômes homogènes de degré d en N variables avec

$$N > 2^{d-1}m(m+1)(d-1).$$

(ii) Comme dans [**Pe**, remarque 3.1], on montre que si les points sont équidistribués dans U alors cet ouvert est inclus dans le complémentaire des sous-variétés modérément accumulatrices pour la hauteur correspondante.

On peut alors poser la question suivante :

Question 2.4.4. — *Fixons une hauteur* $\mathbf{h}$ *définie par une métrique sur le faisceau* ω_V^{-1}. *Si* U*, le complémentaire des sous-variétés modérément accumulatrices de* V *est un ouvert de Zariski non vide, les points rationnels de* V *sont-ils équidistribués dans* U *?*

Hypothèse 2. — Dans la suite on suppose en outre que le complémentaire U des sous-variétés modérément accumulatrices pour toute hauteur relative à ω_V^{-1} est un ouvert de Zariski non vide de V.

2.5. Fonction zêta des hauteurs

Définition 2.5.1. — On note $\mathscr{H}(V)$ l'ensemble des classes d'isomorphismes de hauteurs d'Arakelov modulo la relation d'équivalence $\sim$ engendrée par

$$\forall (\lambda_v)_{v \in M_k} \in \bigoplus_{v \in M_k} \mathbf{R}_{>0},$$
$$\prod_{v \in M_k} \lambda_v = 1 \Rightarrow (L, (\|.\|_v)_{v \in M_k}) \sim (L, (\lambda_v \|.\|_v)_{v \in M_k}).$$

L'ensemble $\mathscr{H}(V)$ muni du produit tensoriel est un groupe commutatif et on a un morphisme canonique $\boldsymbol{o} : \mathscr{H}(V) \to \operatorname{Pic}(V)$.

On appelle *système de hauteurs* un morphisme $\mathbf{H} : \operatorname{Pic} V \to \mathscr{H}(V)$ tel que $\boldsymbol{o} \circ \mathbf{H} = \operatorname{Id}_{\operatorname{Pic} V}$.

Remarques 2.5.1. — (i) Si on se donne une base $([L_i])_{1\leqslant i\leqslant t}$ de $\operatorname{Pic} V \otimes \mathbf{R}$ où pour tout i, L_i est un faisceau très ample sur V et si on fixe des métriques m_i sur L_i données par des bases de $\Gamma(V, L_i)$ (cf. [**Pe**, page 107]), alors l'application

$$\begin{array}{rcl} \{[L_i], 1 \leqslant i \leqslant t\} & \to & \mathscr{H}(V) \\ [L_i] & \mapsto & [(L_i, m_i)] \end{array}$$

s'étend de manière unique en un système de hauteurs sur V.

(ii) Si $\mathbf{h} = (\omega_V^{-1}, (\|.\|_v)_{v\in M_k})$ est un représentant de $\mathbf{H}([\omega_V^{-1}])$, on notera, par abus de langage, $n_{U,\mathbf{H}}$ pour $n_{U,\mathbf{h}}$, $\boldsymbol{\omega}_{\mathbf{H}}$ pour $\boldsymbol{\omega}_{\mathbf{h}}$, etc. Il faut noter que, par la formule du produit, cela est indépendant du choix du représentant.

Définition 2.5.2 **(**[**Ar**, page 408]**,** [**FMT**, §2.1]**).** — Soit $\mathbf{H}$ un système de hauteurs sur V, il induit un accouplement $\mathbf{H} : \operatorname{Pic} V \otimes \mathbf{C} \times V(k) \to \mathbf{C}$ qui est l'exponentielle d'une fonction linéaire en la première variable et tel que

$$\forall L \in \operatorname{Pic} V, \quad \forall x \in V(k), \quad \mathbf{H}(L, x) = \mathbf{H}(L)(x).$$

La fonction zêta des hauteurs associée au système $\mathbf{H}$ est définie par

$$\forall s \in \operatorname{Pic} V \otimes \mathbf{C}, \quad \zeta_{\mathbf{H}}(s) = \sum_{x\in U(k)} \mathbf{H}(s, x)^{-1}$$

qui converge absolument dans un cône ouvert de $\operatorname{Pic} V \otimes \mathbf{C}$.

2.6. Une question optimiste

Notations 2.6.1. — Comme dans [**BT1**, §2.4], on utilise la *fonction caractéristique* du cône effectif, qui remonte à [**Kö**], définie par

$$\forall s \in \operatorname{Pic} V \otimes \mathbf{C}, \quad \chi_{C_{\text{eff}}(V)}(s) = \int_{C_{\text{eff}}(V)^\vee} e^{-\langle s,y\rangle} dy$$

où $C_{\text{eff}}(V)^\vee$ est le cône dual de $C_{\text{eff}}(V)$, c'est-à-dire

$$C_{\text{eff}}(V)^\vee = \{x \in (\operatorname{Pic} V \otimes \mathbf{R})^\vee \mid \forall y \in C_{\text{eff}}(V), \langle x, y\rangle \geqslant 0\}$$

et dy est la mesure de Haar sur $\operatorname{Pic} V \otimes \mathbf{R}$ normalisée par le dual du réseau naturel de $\operatorname{Pic} V \otimes \mathbf{R}$.

On pose également $\alpha(V) = \chi_{C_{\text{eff}}(V)}(\omega_V^{-1})/(t-1)!$ où $t = \operatorname{rg} \operatorname{Pic} V$ qui coïncide avec la constante définie dans [**Pe**, définition 2.4] et, comme dans [**BT1**, corollary 1.4.16], $\beta(V) = \#H^1(k, \operatorname{Pic} \overline{V})$.

Les divers exemples connus amènent à se poser la question suivante qui est une version raffinée d'une conjecture de Batyrev et Manin [**BM**, conjecture B′] :

Question 2.6.1. — *Pour quelles variétés V vérifiant les hypothèses 1 et 2, existe-t-il un système de hauteurs $\mathbf{H}$ tel que la fonction zêta $\zeta_{\mathbf{H}}$ converge sur*

$$\mathscr{C} = \omega_V^{-1} + \overbrace{C_{\text{eff}}(V)}^{\circ} + i \operatorname{Pic} V \otimes \mathbf{R}$$

et tel que la fonction $s \mapsto \zeta_{\mathbf{H}}(s)/\chi_{C_{\mathrm{eff}}(V)}(s-\omega_V^{-1})$ *s'étende en une fonction holomorphe sur un ouvert contenant* $\overline{\mathscr{C}}$ *et prenne la valeur* $\beta(V)\tau_{\mathbf{H}}(V)$ *en* ω_V^{-1} *?*

Remarques 2.6.2. — (i) Par un théorème taubérien standard (cf. [**BT1**, theorem 3.3.2]), une réponse positive à cette question implique que

$$n_{U,\mathbf{H}}(H) \sim \alpha(V)\beta(V)\tau_{\mathbf{H}}(V)H(\log H)^{t-1} \text{ quand } H \to +\infty,$$

où $t = \operatorname{rg} \operatorname{Pic} V$.

(ii) Par [**FMT**, §2] et [**Pe**, théorème 6.2.2], la réponse est positive si V est une variété de drapeaux généralisée grâce aux travaux de Langlands sur les séries d'Eisenstein. Par [**BT1**] et [**BT2**] elle est également positive pour les variétés toriques lisses.

(iii) Dans [**BT1**], Batyrev et Tschinkel donnent un exemple où la fonction $\zeta_{\mathbf{H}}$ ne peut pas avoir d'extension méromorphe à $\operatorname{Pic} V \otimes \mathbf{C}$.

3. Rappels sur les torseurs universels

3.1. Les tores. — Le but de ce paragraphe est de rappeler quelques notions sur les tores algébriques ainsi que le résultat d'Ono sur leur nombre de Tamagawa.

Définition 3.1.1. — Soit E un corps. Un *tore* sur E est un groupe algébrique commutatif T tel que

$$\overline{T} \xrightarrow{\sim} \mathbf{G}_{m,\overline{E}}^{\dim T}.$$

On dit qu'une extension F de E *déploie* T si et seulement s'il existe un isomorphisme $T_F \xrightarrow{\sim} (\mathbf{G}_{m,F})^{\dim T}$. Par [**Ono2**, proposition 1.2.1], il existe une telle extension de E qui est séparable et finie.

***Théorème 3.1.1* (cf. [Ono2**, remark 1.2.1]). — *Il existe une équivalence de catégories entre les tores algébriques sur un corps* E *et les* $\mathrm{Gal}(E^s/E)$*-réseaux, c'est-à-dire les* $\mathrm{Gal}(E^s/E)$*-modules qui sont libres et de rang fini en tant que* $\mathbf{Z}$*-modules. Cette équivalence de catégories est donnée par le foncteur contravariant qui à* T *associe le groupe des caractères de* T

$$X^*(T) = \operatorname{Hom}_{E^s}(T_{E^s}, \mathbf{G}_m).$$

Définition 3.1.2. — Soit $(\xi_i)_{1 \leqslant i \leqslant \dim T}$ une base de $X^*(T)$ alors la forme différentielle

$$\omega_T = \bigwedge_{i=1}^{\dim T} \xi_i^{-1} d\xi_i$$

est, au signe près, indépendante du choix de la base. Elle est invariante sous l'action de T et définie sur E. On l'appelle la *forme canonique* de T.

Passons maintenant au cas des tores sur le corps de nombres k.

Notations 3.1.3. — Soit T un tore sur k et K une extension séparable finie qui déploie T. Soit $\mathscr{G} = \mathrm{Gal}(\overline{k}/k)$, $X^*(T)_k = X^*(T)^{\mathscr{G}}$, et $X_*(T)$ le $\mathscr{G}$-réseau dual $\mathrm{Hom}(X^*(T), \mathbf{Z})$.

Pour toute place v de k, soit $\mathscr{G}_v \subset \mathscr{G}$ le groupe de décomposition de v, $X^*(T)_v$ le groupe $X^*(T)^{\mathscr{G}_v}$ et $X_*(T)_v = X_*(T)^{\mathscr{G}_v}$. Par abus de notations, le sous-groupe compact maximal de $T(k_v)$ est noté $T(\mathcal{O}_v)$. On a alors une injection canonique (cf. [**Ono2**, §2.1]).

$$\log_v : T(k_v)/T(\mathcal{O}_v) \hookrightarrow X_*(T)_v \otimes \mathbf{R}$$

définie par $\forall r \in T(k_v)$, $\forall \xi \in X^*(T)_v$, $|\xi(r)|_v = q_v^{\log_v([r])(\xi)}$, où $q_v = \#\mathbf{F}_v$ si $v \in M_f$, $q_v = e$ si $k_v = \mathbf{R}$ et $q_v = e^2$ sinon. Par [**Ono2**, §2.1], $\log_v$ est un isomorphisme si v est archimédienne, d'après [**Dr**, page 449] l'image de $\log_v$ est $X_*(T)_v$ si v est finie et non ramifiée dans K/k et c'est un sous-module d'indice fini de $X_*(T)_v$ dans le cas restant.

L'espace adélique associé, $T(\boldsymbol{A}_k)$ est, par définition, le produit restreint des $T(k_v)$ relativement aux $T(\mathcal{O}_v)$.

On note $K_T = \prod_{v \in M_k} T(\mathcal{O}_v)$, $W(T) = K_T \cap T(k)$ et

$$T^1(\boldsymbol{A}_k) = \{\, (r_v)_{v \in M_k} \in T(\boldsymbol{A}_k) \mid \forall \xi \in X^*(T)_k,\ \prod_{v \in M_k} |\xi(r_v)| = 1 \,\}.$$

Si S est une partie finie de M_k, on note

$$T_S(\boldsymbol{A}_k) = \prod_{v \in S} T(k_v) \times \prod_{v \in M_k - S} T(\mathcal{O}_v) \subset T(\boldsymbol{A}_k).$$

Proposition 3.1.2 **(Ono, [Ono1**, theorem 4] **et [Ono2**, pages 120–122]**)**

Avec les notations qui précèdent, on a alors :

(i) $T(\boldsymbol{A}_k)/T^1(\boldsymbol{A}_k) \xrightarrow{\sim} (X^*(T)_k)^\vee \otimes \mathbf{R}$,

(ii) $T^1(\boldsymbol{A}_k)/T(k)$ *est compact,*

(iii) $W(T)$ *est le groupe fini des éléments de torsion dans* $T(k)$,

(iv) Il existe un ensemble fini de places S_1 *contenant* M_∞, *tel que pour tout* S *contenant* S_1

$$T_S(\boldsymbol{A}_k).T(k) = T(\boldsymbol{A}_k).$$

Autrement dit, il existe S_1 *tel que l'application naturelle*

$$T(k) \to \bigoplus_{v \in M_k - S_1} X_*(T)_v$$

soit surjective,

(v) Soit S *comme dans* (iv) *et* $\mathcal{O}_S$ *l'anneau des* S*-entiers. On a une suite exacte*

$$0 \to W(T) \to T(\mathcal{O}_S) \xrightarrow{\log_S} \prod_{v \in S} X_*(T)_v \otimes \mathbf{R}$$

et l'image de $\log_S$ *est un réseau du noyau du morphisme naturel*

$$\begin{array}{rcl} \prod_{v\in S} X^*(T)_v^\vee \otimes \mathbf{R} & \to & X^*(T)_k^\vee \otimes \mathbf{R} \\ (\lambda_v)_{v\in S} & \mapsto & \sum_{v\in S} (\log q_v)\lambda_v. \end{array}$$

Nous passons maintenant à la définition du nombre de Tamagawa de T.

Notations 3.1.4. — Pour toute place v de k, soit $\boldsymbol{\omega}_{T,v}$ la mesure de Haar sur $T(k_v)$ définie par la forme différentielle ω_T (cf. [**We**]).

Soit S un ensemble fini de places contenant les places archimédiennes, les places ramifiées dans K/k et vérifiant l'assertion (iv) de la proposition. Pour tout $v \in M_f - S$, le terme local de la fonction L associé à $X^*(T)$ est défini par

$$L_v(s, X^*(T)) = \frac{1}{\det(1 - (\#\mathbf{F}_v)^{-s}\operatorname{Fr}_v \mid X^*(T))}$$

où Fr_v est un morphisme de Frobenius en v. La fonction L globale associée est donnée par

$$L_S(s, X^*(T)) = \prod_{v\in M_f - S} L_v(s, X^*(T)).$$

Comme dans [**Ono2**, §3.3], on utilise les facteurs de convergence

$$\lambda_v = \begin{cases} L_v(1, X^*(T))^{-1} & \text{si } v \in M_f - S \\ 1 & \text{sinon} \end{cases}$$

et on considère la mesure adélique

$$\boldsymbol{\omega}_T = \left(\lim_{s\to 1}(s-1)^{\operatorname{rg} X^*(T)_k} L_S(s, X^*(T))\right)^{-1} \frac{1}{\sqrt{d}^{\dim T}} \prod_{v\in M_f} \lambda_v^{-1}\boldsymbol{\omega}_{T,v}$$

qui est indépendante du choix de S. En outre le réseau $X^*(T)_k^\vee$ fournit une mesure canonique $\boldsymbol{\omega}_{T/T^1}$ sur $X^*(T)_k^\vee \otimes \mathbf{R}$. Il existe alors une unique mesure $\boldsymbol{\omega}_{T^1}$ sur $T^1(\boldsymbol{A}_k)$ telle qu'on ait la formule

$$\int_{T(\boldsymbol{A}_k)/T^1(\boldsymbol{A}_k)} \boldsymbol{\omega}_{T/T^1}(x) \int_{xT^1(\boldsymbol{A}_k)} f(y)\boldsymbol{\omega}_{T^1}(y) = \int_{T(\boldsymbol{A}_k)} f(y)\boldsymbol{\omega}_T(y).$$

Définition 3.1.5. — Le *nombre de Tamagawa* de T est alors défini par

$$\tau(T) = \boldsymbol{\omega}_{T^1}(T^1(\boldsymbol{A}_k)/T(k))$$

où on note également $\boldsymbol{\omega}_{T^1}$ la mesure induite sur $T^1(\boldsymbol{A}_k)/T(k)$.

Notations 3.1.6. — On définit

$$Ш^1(k, T) = \operatorname{Ker}(H^1(k, T) \to \prod_{v\in M_k} H^1(k_v, T))$$

et on pose $h(T) = \#H^1(k, X^*(T))$ et $i(T) = \#Ш^1(k, T)$.

***Théorème 3.1.3* (Ono, [Ono3, §5]).** — *Avec les notations ci-dessus, on a*

$$\tau(T) = h(T)/i(T).$$

3.2. Cônes et structures associées. — Dans ce paragraphe, on se donne en outre un cône dans $X^*(T)$, ce qui correspond à une variété torique affine pour T. Dans cette situation diverses notions qui nous seront utiles ont été introduites par Draxl dans [**Dr**].

Notations 3.2.1. — Soit T un tore sur le corps de nombres k déployé par une extension finie K et soit C un cône rationnel polyédrique strictement convexe de l'espace vectoriel $X^*(T)\otimes\mathbf{R}$; c'est-à-dire qu'il existe une famille finie $(\xi_i)_{1\leqslant i\leqslant m}$ de $X^*(T)$ telle que $C = \sum_{i=1}^m \mathbf{R}_{\geqslant 0}\xi_i$ et $C\cap -C = \{0\}$. On suppose en outre que C est invariant sous l'action du groupe de Galois $\mathscr{G} = \mathrm{Gal}(\overline{k}/k)$ et que C est d'intérieur non vide.

Par la théorie des variétés toriques (cf. [**Da**, chapter 1]) cela correspond à la variété torique affine $\mathbf{A}_{C,k}$ sur k définie par

$$\mathbf{A}_{C,k} = \mathrm{Spec}(\overline{k}[C\cap X^*(T)])^{\mathscr{G}}$$

où $\overline{k}[C\cap X^*(T)]$ est l'algèbre associée au monoïde $C\cap X^*(T)$. Cette variété a un modèle naturel, à savoir le schéma affine

$$\mathbf{A}_{C,\mathcal{O}_k} = \mathrm{Spec}(\mathcal{O}_{\overline{k}}[C\cap X^*(T)])^{\mathscr{G}}$$

où $\mathcal{O}_{\overline{k}}$ désigne l'anneau des entiers algébriques.

D'autre part pour toute place $\mathfrak{p}$ de k, Draxl définit [**Dr**, page 447]

$$T(C,\mathcal{O}_\mathfrak{p}) = T(C,\mathcal{O}_\mathfrak{P})\cap T(k_\mathfrak{p})$$

où $\mathfrak{P}$ désigne une place de K au-dessus de $\mathfrak{p}$ et $T(C,\mathcal{O}_\mathfrak{P})$ est défini par

$$T(C,\mathcal{O}_\mathfrak{P}) = \{\, r\in T(K_\mathfrak{P}) \mid \forall\xi\in C\cap X^*(T),\ \xi(r)\in\mathcal{O}_\mathfrak{P}\}.$$

Lemme 3.2.1. — *Il existe un entier N tel que si A est une $\mathcal{O}_k$-algèbre dans laquelle N est inversible, alors les A-points de $\mathbf{A}_{C,\mathcal{O}_k}$ sont donnés par*

$$\mathbf{A}_{C,\mathcal{O}_k}(A) = \mathrm{Hom}_{\mathrm{Gal}(K/k)}(C\cap X^*(T), (A\otimes\mathcal{O}_K, .))$$

où $\mathrm{Hom}_{\mathrm{Gal}(K/k)}$ *désigne l'ensemble des homomorphismes de monoïdes invariants sous le groupe* $\mathrm{Gal}(K/k)$. *En particulier, pour toute place v de k ne divisant pas N,*

$$T(C,\mathcal{O}_v) = \mathbf{A}_{C,\mathcal{O}_k}(\mathcal{O}_v)\cap T(k_v).$$

Démonstration. — Le schéma $\mathbf{A}_{C,\mathcal{O}_k}$ peut aussi se décrire comme

$$\mathbf{A}_{C,\mathcal{O}_k} = \mathrm{Spec}(\mathcal{O}_K[C\cap X^*(T)])^{\mathrm{Gal}(K/k)}.$$

Comme il existe une base de K sur k invariante sous l'action de Galois, il existe un entier N_0 tel que $\mathcal{O}_K[1/N_0]$ ait une base invariante sur $\mathcal{O}_k[1/N_0]$. D'autre part il résulte du lemme sans nom qu'il existe un isomorphisme naturel

$$K[C \cap X^*(T)]^{\mathrm{Gal}(K/k)} \otimes_k K \xrightarrow{\sim} K[C \cap X^*(T)]$$

il existe donc un entier N_1 tel que

$$\mathcal{O}_K[C \cap X^*(T)]^{\mathrm{Gal}(K/k)} \otimes_{\mathcal{O}_k} \mathcal{O}_K[1/N_1] \xrightarrow{\sim} \mathcal{O}_K[1/N_1][C \cap X^*(T)].$$

Si N est un multiple de N_0 et N_1, et si N est inversible dans A, alors on a une suite d'isomorphismes naturels

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Gal}(K/k)}(C \cap X^*(T), (A \otimes \mathcal{O}_K, .))\\
&\quad \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[1/N][C \cap X^*(T)], A \otimes \mathcal{O}_K)^{\mathrm{Gal}(K/k)}\\
&\quad \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_K}((\mathcal{O}_K[C \cap X^*(T)])^{\mathrm{Gal}(K/k)} \otimes \mathcal{O}_K[1/N], A \otimes \mathcal{O}_K)^{\mathrm{Gal}(K/k)}\\
&\quad \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_k}((\mathcal{O}_K[C \cap X^*(T)])^{\mathrm{Gal}(K/k)}, A)\\
&\quad = \mathbf{A}_{C,\mathcal{O}_k}(A),
\end{aligned}$$

ce qui montre la première assertion. La seconde s'en déduit à partir des définitions. □

Définition 3.2.2. — Soient T et C comme ci-dessus. Pour toute place finie v de k, on définit un accouplement

$$\langle \cdot, \cdot \rangle_v : X^*(T)_v \otimes \mathbf{C} \times T(k_v) \to \mathbf{C}^\times$$

qui est l'exponentielle d'une fonction linéaire en la première variable et tel que pour tout ξ de $X^*(T)_v$ et tout r de $T(k_v)$ on ait

$$\langle \xi \otimes 1, r \rangle_v = |\xi(r)|_v.$$

La *fonction L associée à C* est alors définie pour tout s de $(\mathring{C} + iX^*(T) \otimes \mathbf{R})^{\mathscr{G}_v}$ par la formule

$$L_v(s, T, C) = \frac{1}{\boldsymbol{\omega}_{T,v}(T(\mathcal{O}_v))} \int_{T(C,\mathcal{O}_v)} \langle s, r \rangle_v \, \boldsymbol{\omega}_{T,v}(r).$$

Il résulte de [**Dr**, proposition 2] que cette intégrale converge bien sur le domaine indiqué.

Cette fonction L peut également s'exprimer de la manière suivante :

Proposition 3.2.2 **(Draxl,** [**Dr**, proposition 4]). — *Si v est une place finie non ramifiée dans l'extension K/k alors*

$$L_v(s, T, C) = \sum_{y \in C^\vee \cap X_*(T)_v} (\#\mathbf{F}_v)^{-\langle y, s \rangle}.$$

Définition 3.2.3. — La fonction L globale associée est alors définie par le produit eulérien
$$L_S(s,T,C) = \prod_{v\in M_f - S} L_v(s,T,C)$$
pour tout ensemble fini de places S.

***Théorème 3.2.3* (Draxl, [Dr, proposition 5]).** — *On considère le domaine $\mathcal{D}(C)$ de l'espace vectoriel $X^*(T)_k \otimes \mathbf{R}$ défini comme l'ensemble des y tels que*
$$\forall x \in C^\vee - \{0\} \cap X_*(T), \quad \langle x,y\rangle > 1$$
où $C^\vee$ est le cône dual de C dans $X_*(T)\otimes\mathbf{R}$. Alors le produit eulérien définissant $L_S(s,T,C)$ converge sur $\mathcal{D}(C) + iX^*(T)_k \otimes \mathbf{R}$.

3.3. Torseurs universels. — La notion de torseur universel a été introduite par Colliot-Thélène et Sansuc dans [**CTS1**] et [**CTS2**] en liaison avec le principe de Hasse et l'approximation faible. Nous allons en rappeler la définition et en donner quelques propriétés. Nous nous restreindrons ici au cas où le groupe de Picard géométrique est sans torsion.

Définition 3.3.1. — Si G est un groupe algébrique linéaire sur un corps E, X et Y deux variétés sur E, $\pi : X \to Y$ un morphisme fidèlement plat et $\mu : G\times X \to X$ une action de G sur X, alors X est appelé un *G-torseur* ou *espace principal homogène sous G* au-dessus de Y si et seulement si l'application $(g,x)\mapsto(gx,x)$ définit un isomorphisme de variétés de $G\times_E X$ sur $X\times_Y X$.

Si $\eta : G \to H$ est un morphisme de groupes algébriques et X un G-torseur au-dessus de Y, on peut considérer le produit contracté $X\times^G H$ qui est un H-torseur au-dessus de Y et qu'on notera $\eta_*(X)$.

Par [**Mi**, III, 3.9], si G est lisse et abélien, les G-torseurs au-dessus de Y sont classifiés à isomorphisme près par le groupe de cohomologie $H^1_{\text{ét}}(Y,G)$.

***Proposition 3.3.1* (Colliot-Thélène, Sansuc [CTS2, (2.0.2), §2.2])**

Si E est un corps , X une E-variété algébrique propre, lisse et géométriquement intègre et T un tore sur E, alors on a une suite exacte naturelle
$$0\to H^1(E,T)\to H^1_{\text{ét}}(X,T)\xrightarrow{\rho}\operatorname{Hom}_{\operatorname{Gal}(E^s/E)}(X^*(T),\operatorname{Pic}X_{E^s})\xrightarrow{\delta}H^2(k,T)$$
où l'application ρ est définie de la manière suivante : pour tout torseur $\mathcal{T}$ et tout caractère ξ de T, $\rho(\mathcal{T})(\xi)$ est la classe du $\mathbf{G}_m$-torseur $\xi_*(\mathcal{T})$ dans $\operatorname{Pic}X_{E^s}$. En outre δ est nulle si X a un point rationnel.

***Définition 3.3.2* (Colliot-Thélène, Sansuc [CTS2, (2.0.4)])**

Un *torseur universel* pour une variété X propre, lisse et géométriquement intègre sur un corps E de groupe de Picard de type fini et sans torsion sur E^s est un torseur $\mathcal{T}$ au-dessus de X sous le tore T_{NS} associé au $\mathscr{G}$-module $\operatorname{Pic}X_{E^s}$ et tel que $\rho(\mathcal{T}) = \operatorname{Id}_{\operatorname{Pic}X_{E^s}}$.

Par la proposition de tels torseurs existent si la variété X a en outre un point rationnel. De plus, par [**CTS1**, proposition 2], si E est un corps de nombres les classes d'isomorphismes de torseurs universels ayant un point rationnel sont en nombre fini.

Nous allons maintenant en donner quelques exemples.

Exemple 3.3.1. — Soit X une intersection complète lisse de dimension $n \geqslant 3$ dans $\mathbf{P}^N_E$. Par [**SGA2**, exposé XII, corollaire 3.7], le groupe de Picard de X_{E^s} est de rang 1 engendré par $\mathcal{O}_X(1)$. Soit $\mathcal{T}$ le cône épointé de $\mathbf{A}^{N+1}_E$ au-dessus de X muni de l'action naturelle de $\mathbf{G}_m$. Le cône $\mathcal{T}$ est un $\mathbf{G}_m$-torseur au-dessus de X.

On identifie $X^*(\mathbf{G}_m)$ avec $\operatorname{Pic} X$ en envoyant $\mathrm{Id}_{\mathbf{G}_m}$ sur $\mathcal{O}_X(-1)$. Le morphisme $\rho(\mathcal{T})$ envoie alors $\mathcal{O}_X(-1)$ sur la classe de $\mathcal{T}$ donc $\rho(\mathcal{T}) = \mathrm{Id}$ et $\mathcal{T}$ représente l'unique classe d'isomorphisme de torseurs au-dessus de X.

Exemple 3.3.2. — De même si $X \subset \prod_{i=1}^m \mathbf{P}^{N_i}_E$ est intersection normale d'hypersurfaces définies par des équations multihomogènes en les variables

$$((X_{i,j})_{1\leqslant i\leqslant m, 0\leqslant j\leqslant N_i})$$

de sorte qu'on ait un isomorphisme

$$\operatorname{Pic} \prod_{i=1}^m \mathbf{P}^{N_i}_E \xrightarrow{\sim} \operatorname{Pic} X$$

alors l'unique torseur universel au-dessus de X est obtenu comme l'image inverse de X dans $\prod_{i=1}^m (\mathbf{A}^{N_i+1}_E - \{0\})$, l'action de $(\mathbf{G}_m)^m$ se faisant composante par composante.

Exemple 3.3.3. — Si X est une variété torique propre et lisse pour un tore T dont l'orbite ouverte a un point rationnel, autrement dit une compactification équivariante lisse du tore T, alors il résulte de [**Sa**, §8] qu'un torseur universel au-dessus de X est donné par un ouvert d'un espace affine dont la construction remonte à Delzant [**De**]. Plus précisément, soit $D(X)$ le $\mathbf{Z}$-module libre ayant pour base les T_{E^s}-orbites dans X_{E^s} de codimension 1, ou encore, ce qui revient au même, les cônes de dimension 1 dans l'éventail de $X_*(T)$ correspondant à X (cf. [**Da**, §6.5] ou [**Oda1**]). On a alors une suite exacte

$$0 \to X^*(T) \to D(X) \to \operatorname{Pic} X_{E^s} \to 0.$$

En passant aux tores correspondants on a une injection $T_{\mathrm{NS}} \hookrightarrow T(D(X))$. Soit C le cône de $D(X) \otimes \mathbf{R}$ engendré par la base ci-dessus. Alors le tore T_{NS} agit sur l'espace affine $\mathbf{A}_{C,k} = \operatorname{Spec}(E^s(C \cap D(X))^{\mathrm{Gal}(E^s/E)})$. Soit $\mathcal{T}$ l'ouvert de $\mathbf{A}_{C,k}$ correspondant aux points dont l'orbite est de dimension maximale. Alors, par [**De**] ou [**Oda2**], le quotient de $\mathcal{T}$ par T_{NS} est isomorphe à X, cet isomorphisme dépendant du choix d'un point rationnel dans l'orbite ouverte de X. Les différents torseurs universels au-dessus de X sont obtenus en faisant varier ce point rationnel dans l'orbite ouverte.

***Exemple 3.3.4* (Skorobogatov, [Sk1] et [Sk2]).** — On considère maintenant la surface X obtenue en éclatant les quatre points $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, $P_3 = (0 : 0 : 1)$ et $P_4 = (1 : 1 : 1)$ dans $\mathbf{P}^2_E$. Alors $\operatorname{Pic} X$ est isomorphe à $\mathbf{Z}^5$. On considère alors $\mathrm{Gr}(2,5)$, la Grassmanienne des sous-espaces de dimension 2 dans E^5. On a un plongement de $\mathrm{Gr}(2,5)$ dans $\mathbf{P}(\Lambda^2(E^5))$ par les coordonnées de Plücker. Soit $(e_i)_{1\leqslant i\leqslant 5}$ la base canonique de E^5 et pour tout $J \subset \{1,\ldots,5\}$, soit M_J le sous espace $\bigoplus_{j\in J} Ee_j$. Soit $W(2,5)$ l'ouvert de $\mathrm{Gr}(2,5)$ correspondant aux sous-espaces M tels que

$$\dim(M \cap M_J) \leqslant (2/5)\#J$$

et soit $\mathcal{T}$ le cône de $\Lambda^2(E^5)$ au-dessus de $W(2,5)$. Or les diviseurs exceptionnels de X sont les éclatés des quatres points P_i que l'on note $F_{i,5}$ et les relevés stricts des droites (P_iP_j) notés $F_{l,m}$ si $\{i,j,l,m\} = \{1,2,3,4\}$. On identifie les $F_{i,j}$ pour $i < j$ avec la base naturelle de $\Lambda^2(E^5)$. L'épimorphisme du $\mathbf{Z}$ module libre de base $F_{i,j}$ sur $\operatorname{Pic} X$ induit un morphisme $T_{\mathrm{NS}} \to \mathbf{G}_m^{10}$ et une action de T_{NS} sur $\mathcal{T}$. Le quotient $\mathcal{T}/T_{\mathrm{NS}}$ est naturellement isomorphe à X donnant ici l'unique torseur universel au-dessus de X à isomorphisme près.

Du point de vue arithmétique un des intérêts des torseurs universels provient de l'annulation des obstructions de Brauer-Manin au principe de Hasse et à l'approximation faible pour ces variétés, lorsque X est une variété algébrique propre, lisse et géométriquement intègre et rationnelle (cf. [**CTS2**, théorème 2.1.2]). Cela amène à envisager les hypothèses arithmétiques suivantes sur la variété V :

***Hypothèses 3*.** — Désormais, on suppose en outre que

(i) les torseurs universels au-dessus de V vérifient le principe de Hasse, c'est-à-dire pour tout torseur universel $\mathcal{T}$ on a l'implication

$$(\forall v \in M_k,\, \mathcal{T}(k_v) \neq \varnothing) \Rightarrow \mathcal{T}(k) \neq \varnothing,$$

(ii) les torseurs universels au-dessus de V vérifient l'approximation faible, c'est-à-dire, pour tout ensemble fini S de places de k, $\mathcal{T}(k)$ est dense dans le produit $\prod_{v\in S} \mathcal{T}(k_v)$.

4. Montée aux torseurs universels

4.1. Les hauteurs

***Notations 4.1.1*.** — Dans ce paragraphe, on fixe un torseur universel $\mathcal{T}$ au-dessus d'une variété vérifiant les hypothèses 1 à 3. On suppose en outre que $\mathcal{T}$ a un point rationnel y_0.

Si K/k est une extension finie et $\mathbf{H}_K$ un système de hauteurs pour V_K, alors pour tout fibré en droites L sur V, on a une métrique adélique définie par

$$\forall \mathfrak{p} \in M_k,\ \forall x \in V(k_\mathfrak{p}),\ \forall y \in L(x),\ \|y\|_\mathfrak{p} = \left(\prod_{\mathfrak{P}|\mathfrak{p}} \|y\|_\mathfrak{P}\right)^{\frac{1}{[K:k]}} \tag{4.1.1}$$

si $[(L, (\|.\|_\mathfrak{P})_{\mathfrak{P}\in M_K})] = \mathbf{H}_K([L])$ et on obtient ainsi un système de hauteurs $\mathbf{H}$ sur V appelé *système induit*.

Dans la suite, on note K une extension galoisienne finie de k de sorte que $\operatorname{Pic} V_K$ soit isomorphe à $\operatorname{Pic} \overline{V}$ et on choisit un système de hauteurs $\mathbf{H}_K$ sur V_K. On désignera par $\mathbf{H}$ le système induit sur V. Le système $\mathbf{H}_K$ permet également d'obtenir des métriques $\|.\|_\mathfrak{p}$ pour les fibrés en droites sur $V_\mathfrak{p}$ bien définies à une constante multiplicative près.

Soit $\mathfrak{P}$ une place de K et L un fibré en droites sur $V_{K_\mathfrak{P}}$. Le morphisme $\mathbf{Z} \to \operatorname{Pic} \overline{V}$ envoyant 1 sur $[L]$ induit un morphisme $\phi_L : T_{\mathrm{NS}} \to \mathbf{G}_m$; le $\mathbf{G}_m$-torseur $\phi_{L*}(\mathcal{T})$ est isomorphe au $\mathbf{G}_m$-torseur $L^\times$ obtenu en ôtant à L la section nulle. On obtient ainsi un morphisme $\psi_L : \mathcal{T} \to L^\times$ compatible avec ϕ_L. On fixe une place $\mathfrak{p}_0$ de k. La fonction $\|.\|_\mathfrak{P}^L$ qui envoie un élément y de $\mathcal{T}(K_\mathfrak{P})$ sur $\|\psi_L(y)\|_\mathfrak{P} / \|\psi_L(y_0)\|_\mathfrak{P}$ si $\mathfrak{P}$ ne divise pas $\mathfrak{p}_0$ et sur la même expression multipliée par $\mathbf{H}_K(L, \pi(y_0))^{-[K_\mathfrak{P}:k_{\mathfrak{p}_0}]/[K:k]}$ sinon est alors indépendante du choix de l'isomorphisme de $\phi_*(\mathcal{T})$ sur $L^\times$ et du choix du représentant $(L, (\|.\|_\mathfrak{P})_{\mathfrak{P}\in M_K})$ de $\mathbf{H}_K([L])$, mais dépend du choix de y_0 et de $\mathfrak{p}_0$.

On note $\tilde{\mathbf{H}}^K_{\mathcal{T},\mathfrak{P}}$ l'application $(\operatorname{Pic} V_K \otimes \mathbf{C}) \times \mathcal{T}(K_\mathfrak{P}) \to \mathbf{C}^\times$ qui est l'exponentielle d'une fonction linéaire en la première variable et telle que

$$\tilde{\mathbf{H}}^K_{\mathcal{T},\mathfrak{P}}(L, y) = (\|y\|_\mathfrak{P}^L)^{-1}.$$

on définit de manière similaire $\tilde{\mathbf{H}}_{\mathcal{T},\mathfrak{p}} : \operatorname{Pic} V_\mathfrak{p} \otimes \mathbf{C} \times \mathcal{T}(k_\mathfrak{p}) \to \mathbf{C}^\times$.

Remarque 4.1.1. — Soit $\pi : \mathcal{T} \to V$ l'application quotient. Pour tout point y de $\mathcal{T}(k)$ et tout L de $\operatorname{Pic} V \otimes \mathbf{C}$, on a la relation

$$\mathbf{H}(L, \pi(y)) = \prod_{v \in M_k} \tilde{\mathbf{H}}_{\mathcal{T},v}(L, y). \tag{4.1.2}$$

4.2. Un espace de type adélique. — Le but de ce paragraphe est d'utiliser le cône des diviseurs effectifs dans le groupe $\operatorname{Pic} V$ pour construire sur le torseur universel une structure adélique qui, fibre à fibre, est donnée par la construction de Draxl (cf §3.2). Nous en donnerons tout d'abord une construction pratique pour la descente avant d'en donner une interprétation plus intrinsèque. Pour cela, on fera l'hypothèse suivante :

Hypothèse 4. — Le cône $C_{\mathrm{eff}}(V)$ est un cône polyédrique rationnel de l'espace vectoriel $\operatorname{Pic} V \otimes \mathbf{R}$.

Remarque 4.2.1. — Par la théorie de Mori, cela est vérifié pour les surfaces de Del Pezzo. En outre, aucun contre-exemple ne semble être connu pour une variété vérifiant les hypothèses 1.

Notations 4.2.1. — Pour tout torseur universel $\mathcal{T}$ au-dessus de V ayant un point rationnel y_0 et pour tout système de hauteurs $\mathbf{H}_K$ sur V_K, on note pour toute place $\mathfrak{P}$ de K

$$\mathcal{T}_{\mathbf{H}_K}(\mathcal{O}_{K_\mathfrak{P}}) = \{\, y \in \mathcal{T}(K_\mathfrak{P}) \mid \forall L \in C_{\text{eff}}(V_{K_\mathfrak{P}}),\ \tilde{\mathbf{H}}^K_{\mathcal{T},\mathfrak{P}}(L, y) \leqslant 1 \,\}$$

et pour toute place $\mathfrak{p}$ de k,

$$\mathcal{T}_{\mathbf{H}}(\mathcal{O}_\mathfrak{p}) = \mathcal{T}(k_\mathfrak{p}) \cap \bigcap_{\mathfrak{P}|\mathfrak{p}} \mathcal{T}_{\mathbf{H}_K}(\mathcal{O}_{K_\mathfrak{P}}).$$

Définition 4.2.2. — *L'espace adélique associé à $\mathcal{T}$ et $C_{\text{eff}}(\overline{V})$* est alors défini comme le produit restreint des $\mathcal{T}(k_\mathfrak{p})$ relativement aux $\mathcal{T}_{\mathbf{H}}(\mathcal{O}_\mathfrak{p})$. On le notera temporairement $\mathcal{T}_{\mathbf{H}}(\boldsymbol{A}_k)$.

Nous allons maintenant justifier cette terminologie en montrant qu'il s'agit de l'intersection d'un espace adélique avec $\prod_{v \in M_k} \mathcal{T}(k_v)$. L'idée de la construction est donnée par le paragraphe 3.2.

Proposition 4.2.2. — *Soit $\widehat{\mathcal{T}}$ le produit contracté $\mathcal{T} \times^{T_{\text{NS}}} \mathbf{A}_{-C_{\text{eff}}(\overline{V}),k}$ et $\widetilde{\mathcal{T}}$ un $\mathcal{O}_k$-modèle de $\widehat{\mathcal{T}}$. Alors il existe un ensemble fini de places S_2 contenant les places archimédiennes tel que pour tout $\mathfrak{p}$ de $M_k - S_2$, $\mathcal{T}_{\mathbf{H}}(\mathcal{O}_\mathfrak{p})$ coïncide avec $\mathcal{T}(k_\mathfrak{p}) \cap \widetilde{\mathcal{T}}(\mathcal{O}_\mathfrak{p})$.*

Démonstration. — Par définition de $\mathcal{T}_{\mathbf{H}}(\mathcal{O}_\mathfrak{p})$, il suffit de démontrer le résultat sur K. On peut donc supposer que $\operatorname{Pic} V = \operatorname{Pic} \overline{V}$.

Soit L un fibré en droites sur V tel que $\Gamma(V, L) \neq \{0\}$. L'application de $\mathbf{Z}$ dans $\operatorname{Pic} V$ qui envoie 1 sur $-[L]$ induit un morphisme du monoïde $\mathbf{N}$ dans le monoïde $-C_{\text{eff}}(\overline{V}) \cap \operatorname{Pic} \overline{V}$ et donc un morphisme d'algèbres

$$\overline{k}[\mathbf{N}] \to \overline{k}[-C_{\text{eff}}(\overline{V}) \cap \operatorname{Pic} \overline{V}].$$

D'où un morphisme $\hat{\phi}_L : \mathbf{A}_{-C_{\text{eff}}(\overline{V}),k} \to \mathbf{A}^1_k$ compatible avec le morphisme de tores $T_{\text{NS}} \to \mathbf{G}_m$. En prenant le produit contracté par $\mathcal{T}$ relativement à T_{NS}, on obtient un morphisme $\hat{\psi}_{L^\vee} : \widehat{\mathcal{T}} \to L^\vee$ qui étend le morphisme $\psi_{L^\vee}$ défini au paragraphe précédent.

Soient $L_1, \ldots, L_m$ des fibrés en droites tels que les classes d'isomorphismes $[L_i]$ forment un système de générateurs du monoïde $C_{\text{eff}}(V) \cap \operatorname{Pic} V$. Alors on a

$$\mathcal{T}_{\mathbf{H}}(\mathcal{O}_\mathfrak{p}) = \{\, y \in \mathcal{T}(k_\mathfrak{p}) \mid \forall i,\ 1 \leqslant i \leqslant m \Rightarrow \tilde{\mathbf{H}}_{\mathcal{T},\mathfrak{p}}(L_i, y) \leqslant 1 \,\}.$$

Soit S_2 un ensemble fini de places tel que pour toute place $\mathfrak{p}$ de $M_k - S_2$, les métriques sur les fibrés $L_i^\vee$ soient définies par un modèle $\mathscr{L}_i^\vee$ défini sur $\mathcal{O}_{S_2}$ et tel que pour tout i le morphisme $\hat{\psi}_{L_i^\vee}$ s'étende en un morphisme

$$\tilde{\psi}_{L_i^\vee} : \widetilde{\mathcal{T}} \times \operatorname{Spec} \mathcal{O}_{S_2} \to \mathscr{L}_i^\vee.$$

Si $\mathfrak{p} \in M_k - S_2$ et $y \in \mathcal{T}(k_\mathfrak{p}) \cap \widetilde{\mathcal{T}}(\mathcal{O}_\mathfrak{p})$, alors pour tout i compris entre 1 et m, $\tilde{\psi}_{L_i^\vee}(y)$ appartient à $\mathscr{L}_i^\vee(\mathcal{O}_\mathfrak{p})$ et par conséquent $\|\psi_{L_i^\vee}(y)\|_\mathfrak{p} \leqslant 1$. Quitte à augmenter S_2, on peut également supposer que $\|\psi_{L_i^\vee}(y_0)\|_\mathfrak{p} = 1$ et que $\mathfrak{p}_0$ appartient à S_2. Il résulte alors des définitions que $x \in \mathcal{T}_\mathbf{H}(\mathcal{O}_\mathfrak{p})$. On a donc montré l'inclusion

$$\mathcal{T}(k_\mathfrak{p}) \cap \widetilde{\mathcal{T}}(\mathcal{O}_\mathfrak{p}) \subset \mathcal{T}_\mathbf{H}(\mathcal{O}_\mathfrak{p}).$$

D'autre part le morphisme $\bigoplus_{i=1}^m \hat{\psi}_{L_i^\vee} : \widehat{\mathcal{T}} \to \bigoplus_{i=1}^m L_i^\vee$ est une immersion fermée. En effet il suffit de le démontrer pour l'application

$$\bigoplus_{i=1}^m \hat{\phi}_{L_i^\vee} : \mathbf{A}_{-C_{\text{eff}}(\overline{V}),k} \to \bigoplus_{i=1}^m \mathbf{A}_k^1.$$

Mais ceci résulte du fait que le choix des L_i donne une surjection

$$\mathop{\times}_{i=1}^m \mathbf{N} \to -C_{\text{eff}}(V) \cap \operatorname{Pic} V.$$

Quitte à augmenter S_2, on peut supposer que $\widetilde{\mathcal{T}} \times \operatorname{Spec} \mathcal{O}_{S_2}$ est l'adhérence de $\widehat{\mathcal{T}}$ dans $(\bigoplus_{i=1}^m \mathscr{L}_i^\vee) \times \operatorname{Spec} \mathcal{O}_{S_2}$. Soit $\mathfrak{p} \in M_k - S_2$ et $y \in \mathcal{T}_\mathbf{H}(\mathcal{O}_\mathfrak{p})$. Alors pour tout i entre 1 et m, $\tilde{\mathbf{H}}_{\mathcal{T},\mathfrak{p}}(L_i, y) \leqslant 1$ entraîne que $\|\psi_{L_i^\vee}(y)\|_v \leqslant 1$ et donc $\hat{\psi}_{L_i^\vee}(y) \in \mathscr{L}_i^\vee(\mathcal{O}_\mathfrak{p})$. Donc

$$\bigoplus_{i=1}^m \hat{\psi}_{L_i^\vee}(y) \in (\bigoplus_{i=1}^m \mathscr{L}_i^\vee)(\mathcal{O}_\mathfrak{p}).$$

Par conséquent $y \in \widetilde{\mathcal{T}}(\mathcal{O}_\mathfrak{p})$. □

Corollaire 4.2.3. — *L'espace adélique $\mathcal{T}_{\mathbf{H}_K}(\boldsymbol{A}_k)$ est indépendant du choix du système de hauteurs et du choix de y_0.*

Notation 4.2.3. — Dans la suite, on notera $\mathcal{T}_{C_{\text{eff}}(\overline{V})}(\boldsymbol{A}_k)$ cet espace adélique.

Remarque 4.2.4. — Il faut noter qu'en général $C_{\text{eff}}(\overline{V})$ ne définit pas un éventail régulier. Le nombre de raies extrêmales peut, par exemple, être supérieur au rang de $\operatorname{Pic} \overline{V}$. La variété $\mathbf{A}_{-C_{\text{eff}}(\overline{V}),k}$ est alors singulière et il en est a fortiori de même de la variété $\widehat{\mathcal{T}}$.

Exemple 4.2.1. — Soit X une intersection complète lisse de dimension $n \geqslant 3$ dans $\mathbf{P}_k^N$ et $\mathcal{T}$ le cône épointé de $\mathbf{A}_k^{N+1}$ au-dessus de X. Par l'exemple 3.3.1, $\mathcal{T}$ est, à isomorphisme près, l'unique torseur universel au-dessus de X. On peut en outre choisir comme système de hauteurs celui induit par le système générateur $(X_i)_{0 \leqslant i \leqslant N}$

de $\Gamma(X, \mathcal{O}_X(1))$. Autrement dit, pour toute place v de k, tout x de $X(k_v)$ et toute section s de $\mathcal{O}_X(1)$ correspondant à une forme linéaire l sur k^{N+1},

$$\|\mathrm{s}(x)\|_v = \inf_{X_i(x)\neq 0} \left|\frac{l}{X_i}(x)\right|_v .$$

On suppose qu'il existe $\boldsymbol{y}_0 \in \mathcal{T}(k)$. Le morphisme $\psi_{\mathcal{O}(1)}$ envoie le point $\boldsymbol{y}$ de coordonnées $(y_0, \dots, y_N)$ de $\mathcal{T}(k)$ sur $1/y_i X_i(y_0 : \dots : y_N)$ pour un i tel que $y_i \neq 0$. On a donc

$$\left\|\psi_{\mathcal{O}(1)}(\boldsymbol{y})\right\|_v = \inf_{y_i\neq 0} \left|\frac{1}{y_i}\right|_v = \frac{1}{\sup_{0\leqslant i\leqslant N} |y_i|_v}.$$

Donc $\tilde{\mathbf{H}}_{\mathcal{T},v}(\mathcal{O}(1), \boldsymbol{y}) = \sup_{0\leqslant i\leqslant N} |y_i|_v / \sup_{0\leqslant i\leqslant N} |y_{0,i}|_v$ si $v \neq \mathfrak{p}_0$. Par conséquent pour presque toute place v de k

$$\mathcal{T}_{\mathbf{H}}(\mathcal{O}_v) = \mathbf{A}^{N+1}_{\mathcal{O}_k}(\mathcal{O}_v) \cap \mathcal{T}(k_v)$$

et l'espace adélique considéré est égal à $\mathbf{A}_k^{N+1}(\boldsymbol{A}_k) \cap \prod_{v\in M_k} \mathcal{T}(k_v)$. C'est donc celui utilisé pour la méthode du cercle.

Exemple 4.2.2. — Plus généralement, si $X \subset \prod_{i=1}^m \mathbf{P}_k^{N_i+1}$ est donnée comme intersection normale d'hypersurfaces définies par des équations multihomogènes et si $\mathbf{Z}^m = \mathrm{Pic}(\prod_{i=1}^m \mathbf{P}_k^{N_i+1})$ est isomorphe à $\mathrm{Pic}\,\overline{X}$, l'unique torseur universel $\mathcal{T}$ au-dessus de X est l'image inverse de X dans $\prod_{i=1}^m (\mathbf{A}_k^{N_i+1} - \{0\})$. Le cône effectif est donné par $\mathbf{R}^m_{\geqslant 0} \subset \mathbf{R}^m$ et comme dans le cas précédent, on vérifie que

$$\mathcal{T}_{C_{\mathrm{eff}}(\overline{V})}(\boldsymbol{A}_k) = \mathbf{A}^{\Sigma_{i=1}^m (N_i+1)}(\boldsymbol{A}_k) \cap \prod_{v\in M_k} \mathcal{T}(k_v).$$

Remarque 4.2.5. — Une des raisons de l'apparition du cône $-C_{\mathrm{eff}}$ plutôt que C_{eff} dans les construction précédentes est donnée par les exemples 4.2.1 et 4.2.2 : dans ces cas les structures adéliques usuelles sur les cônes sont induites par celle de $\mathbf{A}^{N+1}$ vu au-dessus de $\mathbf{P}^N$ et correspondent donc à $\mathcal{O}(-1)$ et non à $\mathcal{O}(1)$.

Exemple 4.2.3. — Considérons le cas où V est une compactification lisse d'un tore T sur k. Soit $D(V)$ le $\mathbf{Z}$-module libre de base les $\overline{T}$-orbites $(T_j)_{j\in J}$ de codimension 1 dans $\overline{V}$, soit C le cône naturel dans ce module et $\mathbf{A}_{C,k}$ l'espace affine correspondant. La suite exacte

$$0 \to X^*(T) \xrightarrow{j} D(V) \to \mathrm{Pic}\,\overline{V} \to 0$$

induit une action de T_{NS} sur $\mathbf{A}_{C,k}$. Dans ce cas, par l'exemple 3.3.3, les torseurs universels au-dessus de V sont isomorphes en tant que variété à un ouvert $\mathcal{T}$ de $\mathbf{A}_{C,k}$ et les morphismes $\pi : \mathcal{T} \to V$ sont paramétrés par le choix d'un point x_0 de l'orbite ouverte de V et donnés comme les uniques extensions du morphisme équivariant de $T(D(V))$ dans V envoyant l'élément neutre sur x_0.

On prend pour y_0 l'image de l'élément neutre de $T(D(V))$ dans $\mathcal{T}$. En outre, on choisit l'extension K/k de sorte que K trivialise le $\mathrm{Gal}(\overline{k}/k)$-ensemble J. Soit F_j

l'adhérence de T_j pour $j \in J$. On se donne un ensemble fini $S \subset M_f$ et un modèle $\mathscr{V}$ de V sur $\mathcal{O}_S$. On peut supposer que S contient les places ramifiées dans l'extension K/k et que, S_K désignant les places de K au-dessus de celles de S, il existe des modèles $\mathscr{L}_j$ des fibrés associés à F_j pour j décrivant J, de sorte que les métriques sur ces fibrés soient définies en dehors de S_K par les $\mathscr{L}_j$. Enfin on suppose que l'on a $\|\psi_{[F_j]}(y_0)\|_v = 1$ en dehors de S et que l'adhérence $\mathscr{F}_j$ de F_j dans $\mathscr{V}_{\mathcal{O}_{S_K}}$ est définie comme le lieu des zéros d'une section s_j de $\mathscr{L}_j$.

La description de l'espace $\mathcal{T}_{\mathbf{H}}(\mathcal{O}_\mathfrak{p})$ nous amène à considérer des multiplicités d'intersection. Soit $\mathfrak{P}$ une place finie de K en dehors de S_K et $i \in J$. Pour tout $x \in V(K_\mathfrak{P})$ n'appartenant pas à F_i, on note $m_i(x)$ la multiplicité d'intersection de l'adhérence $\tilde{x}$ de x dans $\mathscr{V}_{\mathcal{O}_\mathfrak{P}}$ avec $(\mathscr{F}_i)_{\mathcal{O}_\mathfrak{P}}$. Cette intersection étant de dimension 0 par hypothèse, on peut utiliser la formule de Serre (cf. [**Se**, page V-32])

$$m_i(x) = \sum_{y \in (\mathscr{V}_{\mathcal{O}_\mathfrak{P}})_{(0)}} \sum_{j \geqslant 0} (-1)^j l \operatorname{Tor}_j^{\mathcal{O}_{\mathscr{V},y}}(\mathcal{O}_{\mathscr{V},y}/\mathcal{I}_{y,\tilde{x}}, \mathcal{O}_{\mathscr{V},y}/\mathcal{I}_{y,\mathscr{F}})$$

où $\mathcal{O}_{\mathscr{V},y}$ désigne $\mathcal{O}_{\mathscr{V}_{\mathcal{O}_\mathfrak{P}},y}$ et pour tout sous-schéma fermé Y de $\mathscr{V}$, $\mathcal{I}_{y,Y}$ désigne l'idéal associé à Y dans l'anneau local $\mathcal{O}_{\mathscr{V},y}$ et l la longueur d'un $\mathcal{O}_\mathfrak{P}$-module. Dans ce cas particulier, la formule se réduit à

$$m_i(x) = l_{\mathbf{F}_\mathfrak{P}}(\mathscr{L}_i(\tilde{x})/(\mathrm{s}_i(x))).$$

De manière plus intuitive, on peut également décrire $m_i(x)$ comme le plus grand entier j tel que, modulo $\mathfrak{P}^j$, $\tilde{x}$ appartienne à $\mathscr{F}_i$.

Soit $\mathfrak{p} \in M_f - S$. On a $k_\mathfrak{p} \otimes K = \prod_{\mathfrak{P}|\mathfrak{p}} K_\mathfrak{P}$. Comme $\mathfrak{p}$ n'est pas ramifiée dans K/k, on peut choisir des uniformisantes $\pi_\mathfrak{P}$ de $\mathcal{O}_\mathfrak{P}$ pour $\mathfrak{P}|\mathfrak{p}$ de sorte que $(\pi_\mathfrak{P})_{\mathfrak{P}|\mathfrak{p}}$ soit invariant sous l'action du groupe de Galois. On considère alors pour tout $\mathfrak{P}|\mathfrak{p}$ l'application

$$\eta_{K_\mathfrak{P}} : T(D(V))(K_\mathfrak{P}) \to T(D(V))(K_\mathfrak{P})$$

qui à $\boldsymbol{y} = (y_j)_{j\in J}$ associe $(y_j/\pi_\mathfrak{P}^{v_\mathfrak{P}(y_j) - m_j(\pi(\boldsymbol{y}))})_{j\in J}$. On obtient alors que pour tout j de J, l'entier $v_\mathfrak{P}(\eta_{K_\mathfrak{P}}(\boldsymbol{y})_j)$ vaut $m_j(\pi(\boldsymbol{y}))$. Le produit $\prod_{\mathfrak{P}|\mathfrak{p}} \eta_{K_\mathfrak{P}}$ définit par passage aux invariants une application

$$\eta_\mathfrak{p} : T(D(V))(k_\mathfrak{p}) \to T(D(V))(k_\mathfrak{p}).$$

Proposition 4.2.6. — *Avec les notations qui précèdent, pour tout élément $\boldsymbol{y}$ de l'ensemble $T(D(V))(k_\mathfrak{p})$, l'intersection de $\mathcal{T}_{\mathbf{H}}(\mathcal{O}_\mathfrak{p})$ avec la fibre de π en $\pi(\boldsymbol{y})$ est donnée par*

$$T(C_{\mathrm{eff}}(\overline{V}), \mathcal{O}_\mathfrak{p}).\eta_\mathfrak{p}(\boldsymbol{y}).$$

Démonstration. — Toutes les définitions étant obtenues par descente galoisienne à partir de K, il suffit de montrer le résultat dans le cas où le $\mathrm{Gal}(\overline{k}/k)$-ensemble J est trivial.

Notons tout d'abord que la formule $m_j(x) = l_{\mathbf{F}_\mathfrak{p}}(\mathscr{L}_j(\tilde{x})/(\mathrm{s}_j(x)))$ se réécrit également

$$\|\mathrm{s}_j(x)\|_\mathfrak{p} = (\#\mathbf{F}_\mathfrak{p})^{-m_j(x)}. \tag{4.2.1}$$

Vérifions que, pour tout $\boldsymbol{y}$ de $T(D(V))(k_\mathfrak{p})$, on a $\pi(\eta_\mathfrak{p}(\boldsymbol{y})) = \pi(\boldsymbol{y})$. Pour cela, on étend l'application $F_j \mapsto y_j$ en un morphisme $u_{\boldsymbol{y}} : D(V) \to k_\mathfrak{p}{}^\times$. Alors $\pi(\boldsymbol{y})$ est caractérisé par les relations $\xi(\pi(\boldsymbol{y})) = u_{\boldsymbol{y}}(\xi)$ où ξ décrit $X^*(T)$. Il nous faut donc calculer $u_{\eta_\mathfrak{p}(\boldsymbol{y})}(\xi)$. Mais la fonction $F_j \mapsto \|\mathrm{s}_j(\pi(\boldsymbol{y}))\|_\mathfrak{p}$ s'étend également en un morphisme $w_{\boldsymbol{y}} : D(V) \to \mathbf{R}_{>0}$ qui vérifie

$$\forall \xi \in X^*(T), \quad w_{\boldsymbol{y}}(\xi) = |\xi(\pi(\boldsymbol{y}))|_\mathfrak{p}$$

et, par conséquent, les morphismes $D(V) \to \mathbf{Z}$ qui étendent les applications $F_j \mapsto m_j(\pi(\boldsymbol{y}))$ et $F_j \mapsto v_\mathfrak{p}(y_j)$ coïncident sur $X^*(T)$ et, par définition de $\eta_\mathfrak{p}$, on obtient que $u_{\eta_\mathfrak{p}(\boldsymbol{y})}(\xi) = u_{\boldsymbol{y}}(\xi)$ pour tout caractère ξ de T.

Pour montrer la proposition, il reste à montrer que pour tout j, on a

$$\left\|\psi_{[F_j]}(\eta_\mathfrak{p}(\boldsymbol{y}))\right\|_\mathfrak{p} = 1.$$

Mais il découle des définitions que l'on peut choisir $\psi_{[F_j]}$ de sorte que $\boldsymbol{y}$ soit envoyé sur $y_j^{-1}\mathrm{s}_j(\pi(y))$ et il résulte de (4.2.1) que

$$\left\|\psi_{[F_j]}(\eta_\mathfrak{p}(\boldsymbol{y}))\right\|_\mathfrak{p} = \left\|y_j^{-1}\pi_\mathfrak{p}^{v_\mathfrak{p}(y_j)-m_j(\pi(\boldsymbol{y}))}\mathrm{s}_j(\pi(\boldsymbol{y}))\right\|_\mathfrak{p} = 1. \quad \square$$

Cas particulier. — Si V est la variété torique obtenue en éclatant 3 points rationnels P_1, P_2 et P_3 en position générale sur $\mathbf{P}^2_\mathbf{Q}$, alors les diviseurs F_j sont les images inverses E_1, E_2 et E_3 des points ainsi que les relevés stricts $E_{1,2}$, $E_{2,3}$ et $E_{3,1}$ des droites (P_1P_2), (P_2P_3) et (P_3P_1). On note $(y_1, y_2, y_3, y_{1,2}, y_{2,3}, y_{3,1})$ les coordonnées naturelles sur $D(V)^\vee \otimes k$ et $(x_1 : x_2 : x_3)$ des coordonnées sur $\mathbf{P}^2_\mathbf{Q}$ de sorte que $P_1 = (1:0:0)$, $P_2 = (0:1:0)$ et $P_3 = (0:0:1)$. Soit U le complémentaire des diviseurs F_j que l'on peut identifier avec son image dans $\mathbf{P}^2_\mathbf{Q}$. Au-dessus de U le morphisme $\pi : \mathcal{T} \to V$ est donné par

$$x_1 = y_{2,3}y_2y_3, \quad x_2 = y_{1,3}y_1y_3 \quad \text{et} \quad x_3 = y_{1,2}y_1y_2.$$

Considérons la section ensembliste $s : U(\mathbf{Q}) \to \pi^{-1}(U)(\mathbf{Q})$ qui apparaît notamment dans [**BM**, §5] et [**Pe**, §7.1.2] et définie par, si $(x_1, x_2, x_3) = 1$,

$$s(x_1 : x_2 : x_3) = (y_1, y_2, y_3, y_{1,2}, y_{2,3}, y_{3,1}),$$

où $y_i = (x_j, x_l)$ et $y_{i,j} = x_l y_i^{-1} y_j^{-1}$ pour $\{i, j, l\} = \{1, 2, 3\}$. Il est aisé de vérifier que pour tout nombre premier p, et pour tout $\boldsymbol{x} = (x_1 : x_2 : x_3)$, les valuations en p des coordonnées de son image par s coïncident avec les multiplicités d'intersection de $\boldsymbol{x}$ avec les diviseurs correspondants dans V_p. On en déduit comme dans le cas général que l'intersection de $\mathcal{T}_\mathbf{H}(\mathbf{Z}_p)$ avec la fibre en $\boldsymbol{x}$ est le produit

$$T_{\mathrm{NS}}(C_{\mathrm{eff}}(\overline{V}), \mathbf{Z}_p).s(\boldsymbol{x}).$$

Mais $\operatorname{Pic} V \xrightarrow{\sim} \mathbf{Z}[\Lambda] \oplus \mathbf{Z}[E_1] \oplus \mathbf{Z}[E_3]$ où Λ est le relevé d'une droite ne contenant aucun des points éclatés et pour la décomposition correspondante

$$T_{\mathrm{NS}}(C_{\mathrm{eff}}(V), \mathbf{Z}_p) = \left\{ (r, r_1, r_2, r_3) \in \mathbf{G}_m^4 \;\middle|\; \begin{cases} r_i \in \mathbf{Z}_p & \text{pour } 1 \leqslant i \leqslant 3 \\ r r_i^{-1} r_j^{-1} \in \mathbf{Z}_p & \text{pour } 1 \leqslant i < j \leqslant 3 \end{cases} \right\}.$$

Or, pour tout nombre premier p,

$$s(\pi(p, p, 1, 1, 1, 1)) = s(p : p : p^2) = s(1 : 1 : p) = (1, 1, 1, p, 1, 1)$$

et donc $(p, p, 1, 1, 1, 1) \notin \mathcal{T}_{\mathbf{H}}(\mathbf{Z}_p)$. L'espace adélique $\mathcal{T}_{\mathbf{H}}(\boldsymbol{A}_{\mathbf{Q}})$ est donc différent de l'intersection de $\prod_v \mathcal{T}(\mathbf{Q}_v)$ avec l'espace adélique associé à l'espace affine.

Exemple 4.2.4. — On reprend les notations de l'exemple 3.3.4. On considère donc la surface de Del Pezzo V obtenue en éclatant les quatre points rationnels de coordonnées $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$ et $(1 : 1 : 1)$ dans $\mathbf{P}^2_{\mathbf{Q}}$ et l'unique torseur universel au-dessus de V est décrit comme un ouvert du cône de $\Lambda^2 k^5$ au-dessus de l'image de la grassmannienne $\operatorname{Gr}(2, 5)$ (cf. [**Sk1**], [**Sk2**] et l'exemple 3.3.4). Ce cône est donné par les équation de Plücker :

$$\begin{cases} y_{1,2}y_{3,4} - y_{1,3}y_{2,4} + y_{1,4}y_{2,3} &= 0, \\ y_{1,2}y_{3,5} - y_{1,3}y_{2,5} + y_{1,5}y_{2,3} &= 0, \\ y_{1,2}y_{4,5} - y_{1,4}y_{2,5} + y_{1,5}y_{2,4} &= 0, \\ y_{1,3}y_{4,5} - y_{1,4}y_{3,5} + y_{1,5}y_{3,4} &= 0, \\ y_{2,3}y_{4,5} - y_{2,4}y_{3,5} + y_{2,5}y_{3,4} &= 0. \end{cases}$$

Soit W l'image inverse dans $\mathcal{T}$ du complémentaire des diviseurs exceptionnels. Alors, comme dans le cas des variétés toriques, on peut définir des applications $\eta_{\mathfrak{p}}$: $W(k_{\mathfrak{p}}) \to W(k_{\mathfrak{p}})$ avec $\pi \circ \eta_p = \pi$ et telles que, si $1 \leqslant i < j \leqslant 5$, la composante en i, j de $\eta_p(y)$ ait pour valuation la multiplicité d'intersection de $\pi(y)$ avec le diviseur exceptionnel $F_{i,j}$. On montre alors aisément que l'intersection de la fibre en $\pi(y)$ avec $\mathcal{T}_{\mathbf{H}}(\mathbf{Z}_p)$ est donnée par le produit

$$T_{\mathrm{NS}}(C_{\mathrm{eff}}(\overline{V}), \mathbf{Z}_p).\eta_p(y).$$

Remarque 4.2.7. — Pour presque toute place $\mathfrak{P}$ de K, on peut également décrire l'ensemble $\mathcal{T}_{\mathbf{H}_K}(\mathcal{O}_{K_{\mathfrak{P}}})$ comme le produit

$$T_{\mathrm{NS}}(C_{\mathrm{eff}}(\overline{V}), \mathcal{O}_{\mathfrak{P}}).\mathscr{T}(\mathcal{O}_{\mathfrak{P}})$$

où $\mathscr{T}$ est un modèle de $\mathcal{T}$. Cette description est sous-jacente dans les exemples 4.2.3 et 4.2.4, puisque les applications $\eta_{\mathfrak{p}}$ ont en fait leur image dans $\mathscr{T}(\mathcal{O}_{\mathfrak{p}})$ pour des modèles bien choisis de $\mathcal{T}$.

4.3. Un domaine fondamental sous $\mathrm{NS}(\mathcal{O}_S)$**.** — Comme dans le cas de l'espace projectif étudié par Schanuel dans [**Sc**], ou plus généralement celui des intersections complètes lisses (cf. [**Pe**, §5.1]), le passage aux torseurs universels nécessite la construction d'un domaine fondamental sous l'action des unités. L'objet de ce paragraphe est de définir ce domaine à l'aide des fonctions $\tilde{\mathbf{H}}_{\mathcal{T},v}$ introduites dans le paragraphe 4.1.

Notations 4.3.1. — Dans la suite S désigne un ensemble fini de places contenant l'ensemble S_0 défini au paragraphe 2.3 ainsi que l'ensemble S_2 défini dans la démonstration de la proposition 4.2.2, les places archimédiennes, les places ramifiées dans l'extension K/k, et tel que les conditions (iv) et (v) de la proposition 3.1.2 soient vérifiées pour le tore T_{NS} associé au groupe de Picard.

On a donc, d'une part, une surjection

$$\log'_S : T_{\mathrm{NS}}(k) \to \bigoplus_{v \notin S} X_*(T_{\mathrm{NS}})_v$$

et, d'autre part, une suite exacte

$$0 \to W(T_{\mathrm{NS}}) \to T_{\mathrm{NS}}(\mathcal{O}_S) \xrightarrow{\log_S} \prod_{v \in S} (X_*(T_{\mathrm{NS}})_v) \otimes \mathbf{R}.$$

Soit M l'image de $\log_S$. Le $\mathbf{Z}$-module M est un réseau dans le noyau de l'application canonique

$$\prod_{v \in S} X^*(T_{\mathrm{NS}})_v^\vee \otimes \mathbf{R} \to X^*(T_{\mathrm{NS}})_k^\vee \otimes \mathbf{R}.$$

Soit Δ un domaine fondamental pour M dans ce noyau, donné par une base de M, et pr une projection de $\prod_{v \in S} X^*(T_{\mathrm{NS}})_v^\vee \otimes \mathbf{R}$ sur ce noyau.

Dans la section 4.1, on a construit des fonctions

$$\tilde{\mathbf{H}}_{\mathcal{T},v} : \operatorname{Pic} V_v \otimes \mathbf{C} \times \mathcal{T}(k_v) \to \mathbf{C}^\times$$

pour tout torseur $\mathcal{T}$ ayant un point rationnel. Elles induisent des fonctions

$$\tilde{\mathbf{H}}^{\log}_{\mathcal{T},v} : \mathcal{T}(k_v) \to (\operatorname{Pic} V_v)^\vee \otimes \mathbf{R}$$

données par

$$\forall y \in \mathcal{T}(k_v),\ \forall L \in \operatorname{Pic} V_v,\ \tilde{\mathbf{H}}_{\mathcal{T},v}(y, L) = q_v^{\tilde{\mathbf{H}}^{\log}_{\mathcal{T},v}(y)(L)}$$

où $q_v = \#\mathbf{F}_v$ si $v \in M_f$, $q_v = e$ si $k_v = \mathbf{R}$ et $q_v = e^2$ sinon. On définit alors un domaine $\Delta_{\mathbf{H}_K}(\mathcal{T})$ par

$$\Delta_{\mathbf{H}_K}(\mathcal{T}) = \left\{ \boldsymbol{y} \in \prod_{v \in S} \mathcal{T}(k_v) \,\middle|\, \mathrm{pr}((\tilde{\mathbf{H}}^{\log}_{\mathcal{T},v}(y_v))_{v \in S}) \in \Delta \right\}.$$

Proposition 4.3.1. — *L'ensemble $\Delta_{\mathbf{H}_K}(\mathcal{T})$ est un domaine fondamental de l'espace $\prod_{v\in S}\mathcal{T}(k_v)$ sous l'action de $T_{\mathrm{NS}}(\mathcal{O}_S)/W(T_{\mathrm{NS}})$. Autrement dit, il vérifie les conditions suivantes :*

(i) $\Delta_{\mathbf{H}_K}(\mathcal{T})$ est stable sous l'action de $W(T_{\mathrm{NS}})$,
(ii) $\forall r\in T_{\mathrm{NS}}(\mathcal{O}_S)-W(T_{\mathrm{NS}})$, $r.\Delta_{\mathbf{H}_K}(\mathcal{T})\cap\Delta_{\mathbf{H}_K}(\mathcal{T})=\varnothing$,
(iii) $\bigcup_{r\in T_{\mathrm{NS}}(\mathcal{O}_S)} r.\Delta_{\mathbf{H}_K}(\mathcal{T})=\prod_{v\in S}\mathcal{T}(k_v)$.

Démonstration. — Notons tout d'abord que pour toute place v de k on a

$$\begin{aligned}\forall L\in\operatorname{Pic}V_v,\ \forall y\in\mathcal{T}(k_v),\ \forall r\in T_{\mathrm{NS}}(k_v),\ \tilde{\mathbf{H}}_{\mathcal{T},v}(L,ry)&=(\|ry\|_v^L)^{-1}\\&=|L(r)|_v^{-1}\tilde{\mathbf{H}}_{\mathcal{T},v}(L,y).\end{aligned}$$

Par conséquent la fonction

$$(y_v)_{v\in S}\mapsto(\tilde{\mathbf{H}}^{\log}_{\mathcal{T},v}(y_v))_{v\in S}$$

est compatible avec les actions de $T(\mathcal{O}_S)$, l'action de ce groupe sur l'espace vectoriel

$$\prod_{v\in S}(\operatorname{Pic}V_v)^\vee\otimes\mathbf{R}$$

étant donnée par $-\log_S$.

L'assertion (i) résulte alors du fait que $W(T_{\mathrm{NS}})$ est contenu dans le noyau de $\log_S$ et les assertions (ii) et (iii) du fait que Δ est un domaine fondamental pour l'image de $\log_S$. $\square$

Exemple 4.3.1. — Si V est l'espace projectif $\mathbf{P}^n_k$, alors on peut prendre pour S l'ensemble des places archimédiennes et pour métriques celles associées à la base usuelle de l'espace vectoriel $\Gamma(\mathbf{P}^n_k,\mathcal{O}(1))$. Le domaine fondamental coïncide alors avec celui construit par Schanuel [**Sc**].

Plus généralement, si V est une intersection complète lisse de dimension supérieure à 3 et si on prend $S=M_\infty$, alors, en utilisant les métriques obtenues par restriction à partir de $\mathbf{P}^n_k$, le domaine fondamental est celui décrit dans [**Pe**, §5.1].

En particulier si $k=\mathbf{Q}$ on a simplement dans ce cas

$$\Delta_{\mathbf{H}_K}(\mathcal{T})=\prod_{v\in S}\mathcal{T}(\mathbf{Q}_v).$$

4.4. Mesures sur les torseurs universels. — Notre but dans cette section est de construire une mesure adélique canonique sur les torseurs universels. Là encore, notre fil conducteur sera le cas des intersections complètes lisses pour lesquelles on dispose de la mesure induite par la forme différentielle de Leray (cf. [**Lac**, page 148], [**Pe**, page 130]). Comme pour la mesure introduite par Salberger [**Sa**], il ne sera pas nécessaire d'introduire des facteurs de convergence, ce qui est un des avantages du passage aux torseurs universels. Toutefois l'espace adélique considéré par Salberger diffère de celui utilisé ici. En outre la mesure utilisée ici n'est pas invariante sous l'action du tore T_{NS}.

Notations 4.4.1. — Soit $\mathcal{T}$ un torseur universel au-dessus de V ayant un point rationnel sur k. Pour toute place v de k et tout point x de $V(k_v)$ au-dessus duquel le torseur $\mathcal{T}$ a un point rationnel sur k_v, on considère la mesure $\boldsymbol{\omega}_{\mathcal{T}_x,v}$ définie par la relation :

$$\int_{\mathcal{T}_x(k_v)} f(r)\boldsymbol{\omega}_{\mathcal{T}_x,v}(r) = \int_{T_{\mathrm{NS}}(k_v)} f(r.y)\|r.y\|_v^{\omega_V}\boldsymbol{\omega}_{T_{\mathrm{NS}},v}(r) \tag{4.4.1}$$

où f est une fonction sur $\mathcal{T}_x(k_v)$ et y un point fixé de $\mathcal{T}_x(k_v)$, où $\|.\|_v^{\omega_V}$ a été défini au paragraphe 4.1 et où $\boldsymbol{\omega}_{T_{\mathrm{NS}},v}$ est la mesure canonique sur $T_{\mathrm{NS}}(k_v)$, dont la définition a été rappelée dans le paragraphe 3.1.

On considère alors la mesure $\boldsymbol{\omega}_{\mathcal{T},v}$ définie sur $\mathcal{T}(k_v)$ par la relation :

$$\int_{\mathcal{T}(k_v)} f(y)\boldsymbol{\omega}_{\mathcal{T},v}(y) = \int_{V(k_v)} \boldsymbol{\omega}_{\mathbf{H},v}(x)\int_{\mathcal{T}_x(k_v)} f(y)\boldsymbol{\omega}_{\mathcal{T}_x,v}(y).$$

Remarque 4.4.1. — (i) Pour ces notations on fixe $(\omega_V, (\|.\|_v)_{v\in M_k})$, un représentant de $\mathbf{H}([\omega_V])$. Ce choix n'intervient que dans les termes locaux de la mesure.

(ii) En termes de celle de Salberger, cette mesure peut s'exprimer de la manière suivante :

$$\boldsymbol{\omega}_{\mathcal{T},v}(r) = \|r\|_v^{\omega_V}\boldsymbol{\omega}_{\mathrm{Salberger},v}(r).$$

Hypothèse 5. — Désormais on suppose en outre que le faisceau ω_V^{-1} appartient à l'ouvert $\mathcal{D}(C_{\mathrm{eff}}(V))$ défini dans le théorème 3.2.3.

Proposition 4.4.2. — *Avec les notations ci-dessus, le produit des mesures $\boldsymbol{\omega}_{\mathcal{T},v}$ définit une mesure adélique sur l'espace $\mathcal{T}_{C_{\mathrm{eff}}(\overline{V})}(\boldsymbol{A}_k)$.*

Démonstration. — Il nous faut montrer que, quitte à élargir S, le produit

$$\prod_{v\notin S}\boldsymbol{\omega}_{\mathcal{T},v}(\mathcal{T}_{\mathbf{H}}(\mathcal{O}_v))$$

converge. Mais on a les relations

$$\begin{aligned}\forall v\in M_f,\ \boldsymbol{\omega}_{\mathcal{T},v}(\mathcal{T}_{\mathbf{H}}(\mathcal{O}_v)) &= \int_{\mathcal{T}(k_v)} \mathbf{1}_{\mathcal{T}_{\mathbf{H}}(\mathcal{O}_v)}\boldsymbol{\omega}_{\mathcal{T},v}\\ &= \int_{V(k_v)}\boldsymbol{\omega}_{\mathbf{H},v}(x)\int_{\mathcal{T}_x(k_v)}\mathbf{1}_{\mathcal{T}_{\mathbf{H}}(\mathcal{O}_v)}(y)\boldsymbol{\omega}_{\mathcal{T}_x,v}(y).\end{aligned}$$

Par [**CTS2**, lemme 3.2.3], quitte à augmenter S, pour tout v de $M_f - S$, l'application $\mathcal{T}(k_v)\to V(k_v)$ est surjective.

En outre, quitte à augmenter S, on peut supposer qu'il existe un modèle $\mathscr{T}$ de $\mathcal{T}$ sur $\mathcal{O}_S$ de sorte que pour toute place $\mathfrak{P}$ ne dominant pas un élément de S, tout y de $\mathscr{T}(\mathcal{O}_{\mathfrak{P}})$ et tout $L\in\operatorname{Pic}V_K$, on ait

$$\tilde{\mathbf{H}}^K_{\mathcal{T},\mathfrak{P}}(L,y) = 1.$$

Pour un tel S, si v appartient à $M_f - S$ et x à $V(k_v)$, il existe un élément y de $\mathcal{T}_x(k_v)$ provenant de $\mathscr{T}(\mathcal{O}_v)$. Mais alors

$$\int_{\mathcal{T}_x(k_v)} \mathbf{1}_{\mathcal{T}_{\mathbf{H}}(\mathcal{O}_v)}(y)\boldsymbol{\omega}_{\mathcal{T}_x,v}(y) = \int_{T_{\mathrm{NS}}(k_v)} \mathbf{1}_{T_{\mathrm{NS}}(-C_{\mathrm{eff}}(\overline{V}),\mathcal{O}_v)}|\omega_V(r)|_v \boldsymbol{\omega}_{T_{\mathrm{NS}},v}(r).$$

On obtient donc les égalités

$$\begin{aligned}
&\prod_{v\in M_k - S} \boldsymbol{\omega}_{\mathcal{T},v}(\mathcal{T}_{\mathbf{H}}(\mathcal{O}_v)) \\
&\qquad = \prod_{v\in M_k-S}\left[\int_{T_{\mathrm{NS}}(-C_{\mathrm{eff}}(\overline{V}),\mathcal{O}_v)} |\omega_V(r)|_v \boldsymbol{\omega}_{T_{\mathrm{NS}},v}(r)\boldsymbol{\omega}_{\mathbf{H},v}(V(k_v))\right] \\
&\qquad = \left[\prod_{v\in M_k-S} L_v(T_{\mathrm{NS}}, C_{\mathrm{eff}}(\overline{V}), \omega_V^{-1})\right] \\
&\qquad\qquad \times \left[\prod_{v\in M_k-S} \boldsymbol{\omega}_{T_{\mathrm{NS}},v}(T_{\mathrm{NS}}(\mathcal{O}_v)) L_v(1, \operatorname{Pic}\overline{V})\right] \\
&\qquad\qquad \times \left[\prod_{v\in M_k-S} L_v(1,\operatorname{Pic}\overline{V})^{-1}\boldsymbol{\omega}_{\mathbf{H},v}(V(k_v))\right].
\end{aligned}$$

La convergence du premier terme résulte des travaux de Draxl, (cf. théorème 3.2.3) et de l'hypothèse 5, la convergence du second des travaux de Weil et Ono et celle du troisième se démontre comme dans [**Pe**, proposition 2.2.2]. □

Il résulte de la preuve qu'il est possible de prendre la convention suivante :

Convention 4.4.2. — Dans la suite, on augmente S de façon à ce que pour tout $\mathfrak{p}$ de $M_k - S$ et tout x de $V(k_{\mathfrak{p}})$, il existe $y \in \mathcal{T}_x(k_{\mathfrak{p}})$ tel que

$$\forall \mathfrak{P}|\mathfrak{p},\ \forall L \in \operatorname{Pic} V_K,\ \tilde{\mathbf{H}}^K_{\mathcal{T},\mathfrak{P}}(L, y) = 1.$$

Lemme 4.4.3. — *Le produit des mesures $\boldsymbol{\omega}_{\mathcal{T},v}$ est indépendant du choix du système de hauteurs.*

Démonstration. — Soient $\mathbf{H}^1$ et $\mathbf{H}^2$ deux systèmes de métriques correspondant à deux métriques adéliques $(\|.\|_v^1)_{v\in M_k}$ et $(\|.\|_v^2)_{v\in M_k}$ sur ω_V^{-1}. La seconde peut alors s'écrire comme $(f_v\|.\|_v^1)_{v\in M_k}$ où les fonctions f_v sont continues, partout non nulles et presque toutes constantes et égale à 1 sur $V(k_v)$. Par la formule (2.3.1), la mesure $\boldsymbol{\omega}_{\mathbf{H},v}$ est multipliée par f_v. Mais par (4.4.1) et la définition de $\|.\|_v^{\omega_V}$, les mesures $\boldsymbol{\omega}_{\mathcal{T}_x,v}$ sur $\mathcal{T}_x(k_v)$ sont divisées par $f_v\lambda_v$ où $(\lambda_v)_{v\in M_k}$ est une famille de réels presque tous égaux à 1 et telle que $\prod_{v\in M_k}\lambda_v = 1$. La mesure $\prod_{v\in M_k}\boldsymbol{\omega}_{\mathcal{T},v}$ est donc inchangée. □

Définition 4.4.3. — On définit la mesure adélique canonique $\boldsymbol{\omega}_{\mathcal{T}}$ sur l'espace adélique $\mathcal{T}_{C_{\mathrm{eff}}(\overline{V})}(\boldsymbol{A}_k)$ par la formule

$$\boldsymbol{\omega}_{\mathcal{T}} = \frac{1}{\sqrt{d}^{\dim \mathcal{T}}} \prod_{v \in M_k} \boldsymbol{\omega}_{\mathcal{T},v}.$$

Remarque 4.4.4. — Il est en fait également possible de définir cette mesure en termes d'une section partout non nulle de $\omega_{\mathcal{T}}$, mais la description donnée ici est plus pratique du point de vue de la descente.

Exemple 4.4.1. — Si V est une intersection complète lisse de dimension $n \geqslant 3$ dans $\mathbf{P}_k^N$ définie par l'annulation de m polynômes homogènes f_i de degré d_i et si $\mathcal{T}$ est le cône épointé au-dessus de V dans $\mathbf{A}_k^{N+1}$, alors la forme différentielle de Leray ω_{L} sur $\mathcal{T}$ est définie par la relation suivante : si y est un point rationnel de $\mathcal{T}$ et si $\left(\frac{\partial f_i}{\partial X_{l_j}}(y)\right)_{1\leqslant i,j\leqslant m}$ est inversible pour $l_1 < \cdots < l_m$, alors

$$\begin{aligned}\omega_{\mathrm{L}}(y) =& (-1)^{Nm-\Sigma_{j=1}^m l_j} \det\left(\frac{\partial f_i}{\partial X_{l_j}}(y)\right)^{-1}_{1\leqslant i,j\leqslant m} \\ & dX_0 \wedge \cdots \wedge \widehat{dX_{l_1}} \wedge \cdots \wedge \widehat{dX_{l_m}} \wedge \cdots \wedge dX_N.\end{aligned}$$

Cette forme définit des mesures $\boldsymbol{\omega}_{\mathrm{L},v}$ sur $\mathcal{T}(k_v)$ pour toute place v de k. D'autre part le choix des équations $(f_i)_{1\leqslant i\leqslant m}$ définit un isomorphisme de ω_V^{-1} sur $\mathcal{O}_V(\delta)$ où $\delta = N+1-\sum_{i=1}^m d_i$. De manière plus explicite un polynôme homogène g de degré δ correspond à la forme $\omega(g)$ définie en $\boldsymbol{x} = (1 : x_1 : \ldots : x_N)$ par la formule (cf. [**Pe**, page 134])

$$\omega(g)(\boldsymbol{x}) = g(\boldsymbol{x}) \det\left(\frac{\partial f_j}{\partial X_i}(\boldsymbol{x})\right)_{\substack{N-m+1\leqslant i\leqslant N\\ 1\leqslant j\leqslant m}} \frac{\partial}{\partial X_1} \wedge \cdots \wedge \frac{\partial}{\partial X_{N-m}}.$$

Par ailleurs, le système de hauteurs est défini par le choix d'une métrique adélique sur $\mathcal{O}_V(1)$.

Proposition 4.4.5. — *Pour tout choix de la métrique sur $\mathcal{O}_V(1)$ et toute place v de k la mesure $\boldsymbol{\omega}_{\mathcal{T},v}$ coïncide avec la mesure $\boldsymbol{\omega}_{\mathrm{L},v}$.*

Démonstration. — Il suffit de montrer le résultat au-dessus d'un ouvert W de $V(k_v)$ pour la topologie v-adique sur lequel X_0 et $\det\left(\frac{\partial f_j}{\partial X_i}(x)\right)_{\substack{N-m+1\leqslant i\leqslant N\\ 1\leqslant j\leqslant m}}$ ne s'annulent pas. Soit $\rho : W \to \mathcal{T}$ l'application qui envoie $\boldsymbol{x}$ de coordonnées $(1 : x_1 : \ldots : x_N)$ sur $(1, x_1, \ldots, x_{N-m})$. Alors par [**Pe**, Lemme 5.4.4], l'image de la mesure $\boldsymbol{\omega}_{\mathbf{H},v}$ sur

$\rho(W)$ est donnée par

$$\rho_* \boldsymbol{\omega}_{\mathbf{H},v} = \frac{1}{\|y\|_v^{\omega_V} \left| \det\left(\frac{\partial f_j}{\partial X_i}(y)\right)_{\substack{N-m+1 \leqslant i \leqslant N \\ 1 \leqslant j \leqslant m}} \right|_v} dX_{1,v} \dots dX_{N-m,v}.$$

Par définition, la mesure $\boldsymbol{\omega}_{\mathcal{T},v}$ s'exprime par la formule suivante

$$\boldsymbol{\omega}_{\mathcal{T},v} = \frac{1}{\left| \det\left(\frac{\partial f_j}{\partial X_i}(y)\right)_{\substack{N-m+1 \leqslant i \leqslant N \\ 1 \leqslant j \leqslant m}} \right|_v} dX_{1,v} \dots dX_{N-m,v}$$

qui coïncide avec $\boldsymbol{\omega}_{\mathrm{L},v}$. □

5. Deux résultats de descente

5.1. Une fonction de comptage. — Comme pour la méthode du cercle, il semble raisonnable de chercher à comparer dans certains cas des sommations du type

$$\sum_{y \in \mathcal{T}(k)} \Phi(y)$$

avec des intégrales de la forme

$$\int_{\mathcal{T}_{C_{\mathrm{eff}}(\overline{V})}(\boldsymbol{A}_k)} \Phi \boldsymbol{\omega}_{\mathcal{T}}$$

où Φ est une fonction sur l'espace $\mathcal{T}_{C_{\mathrm{eff}}(\overline{V})}(\boldsymbol{A}_k)$. Or les différentes conjectures sur le nombre de points de hauteur bornée sur la variété V peuvent s'interpréter à l'aide de comparaisons de ce type.

À titre d'exemple, nous allons décrire comment la question 2.6.1 peut se relever. Il nous faut pour cela décrire les fonctions pour lesquelles devraient être effectuées les comparaisons.

Notations 5.1.1. — Si $\mathcal{T}$ est un torseur universel au-dessus de V avec $\mathcal{T}(k) \neq \varnothing$ et si s appartient à $\operatorname{Pic} V \otimes \mathbf{C}$, on définit $\Phi_{\mathcal{T}}^{\mathbf{H}}$ par le produit

$$\Phi_{\mathcal{T}}^{\mathbf{H}}(s, y) = \frac{1}{\# W(T_{\mathrm{NS}})} \Phi_{\mathcal{T},S}^{\mathbf{H}}(s, y) \cdot \prod_{v \in M_k - S} \Phi_{\mathcal{T},v}^{\mathbf{H}}(s, y_v)$$

où pour tout v en dehors de S, $\Phi_{\mathcal{T},v}^{\mathbf{H}}(s, .)$ est la fonction indicatrice de $\mathcal{T}_{\mathbf{H}}(\mathcal{O}_v)$ et $\Phi_{\mathcal{T},S}^{\mathbf{H}}(s, \cdot)$ est la fonction indicatrice de l'ensemble donné par

$$\left\{ (y_v)_{v \in S} \in \Delta_{\mathbf{H}_K}(\mathcal{T}) \,\middle|\, \left\{ \begin{array}{l} \forall v \in S, \pi(y_v) \in U(k_v) \\ \forall L \in C_{\mathrm{eff}}(V), \prod_{v \in S} \tilde{\mathbf{H}}_{\mathcal{T},v}(L, y_v) \geqslant 1 \end{array} \right. \right\}$$

multipliée par la fonction $\prod_{v\in S}\tilde{\mathbf{H}}_{\mathcal{T},v}(s,y_v)^{-1}$ où U désigne le complémentaire dans V des sous-variétés modérément accumulatrices.

5.2. La descente pour la fonction zêta des hauteurs. — Nous nous placerons sous les hypothèses suivantes, qui contiennent toutes les hypothèses déjà effectuées.

Hypothèses. — La lettre V désigne une variété projective, lisse et géométriquement intègre sur k qui satisfait les conditions suivantes :

(1) $H^i(V,\mathcal{O}_V)=0$ pour $i=1$ ou 2,
(2) $\operatorname{Pic}\overline{V}=\operatorname{NS}\overline{V}$ est sans torsion,
(3) ω_V^{-1} appartient à l'ouvert $\mathcal{D}(C_{\text{eff}}(\overline{V}))\subset C_{\text{eff}}(V)$,
(4) $C_{\text{eff}}(\overline{V})$ est un cône polyédrique rationnel,
(5) $V(k)$ est Zariski dense dans V,
(6) le complémentaire dans V des sous-variétés modérément accumulatrices pour toute hauteur de fibré ω_V^{-1} est un ouvert de Zariski non vide de V,
(7) les torseurs universels au-dessus de V vérifient le principe de Hasse et l'approximation faible.

Remarque 5.2.1. — Les hypothèses 1 à 4 sont de nature géométrique alors que les conditions 5 à 7 sont de nature arithmétique. Comme indiqué précédemment, l'auteur ne connaît pas de variété vérifiant les conditions 1 à 3 sans vérifier la condition 4 ; la condition 7 a été étudiée par Colliot-Thélène et Sansuc [**CTS2**, §3.8] et là encore aucun contre-exemple ne semble être connu lorsque V vérifie les conditions 1 à 3. Il n'en est peut-être pas de même de la condition 6 qui pourrait être fausse pour certaines variétés de Fano (cf. l'exemple de Batyrev et Tschinkel [**BT3**]).

Pour simplifier l'énoncé des résultats, nous supposerons que le système de hauteurs $\mathbf{H}_K$ vérifie la condition suivante :

$$\forall x\in V(K),\quad \forall L\in C_{\text{eff}}(\overline{V})\cap\operatorname{Pic}\overline{V},\quad \mathbf{H}_K(L,x)\geqslant 1$$

situation à laquelle on peut toujours se ramener.

Théorème 5.2.2. — *Soit V une variété vérifiant les conditions ci-dessus, K une extension galoisienne de k telle que* $\operatorname{Pic}V_K \xrightarrow{\sim} \operatorname{Pic}\overline{V}$, $\mathbf{H}_K$ *un système de hauteurs sur V_K satisfaisant à la condition qui précède, soit $(\mathcal{T}_i)_{i\in I}$ une famille de représentants des classes d'isomorphismes de torseurs universels au-dessus de V ayant un point rationnel sur k. Alors, pour tout s appartenant à un cône ouvert de* $\operatorname{Pic}V\otimes\mathbf{C}$

$$\sum_{i\in I}\sum_{y\in\mathcal{T}_i(k)}\Phi^{\mathbf{H}}_{\mathcal{T}_i}(s,y)=\zeta_{\mathbf{H}}(s)L_S(s,T_{\text{NS}},C_{\text{eff}}(\overline{V})).$$

Démonstration. — On se fixe un élément s de $\operatorname{Pic}V\otimes\mathbf{C}$ pour lequel les suites qui définissent $\zeta_{\mathbf{H}}(s)$ et $L_S(s,T_{\text{NS}},C_{\text{eff}}(\overline{V}))$ convergent. Soit alors x un point rationnel de U et i l'unique élément de I tel que $\mathcal{T}_i$ ait un point rationnel au-dessus de x. Il nous faut donc démontrer la formule suivante :

$$\sum_{y\in\mathcal{T}_{ix}(k)} \Phi^{\mathbf{H}}_{\mathcal{T}_i}(s,y) = \mathbf{H}(s,x)^{-1} L_S(s, T_{\mathrm{NS}}, C_{\mathrm{eff}}(\overline{V})). \tag{5.2.1}$$

Par définition des fonctions $\Phi^{\mathbf{H}}_{\mathcal{T}_i}$ et par la formule (4.1.2), le terme de gauche s'écrit également

$$\frac{1}{\#W(T_{\mathrm{NS}})}\sum_{y\in\mathcal{F}_x}\prod_{v\in S}\tilde{\mathbf{H}}_{\mathcal{T}_i,v}(s,y)^{-1} = \frac{1}{\#W(T_{\mathrm{NS}})}\sum_{y\in\mathcal{F}_x}\frac{\prod_{v\in M_k-S}\tilde{\mathbf{H}}_{\mathcal{T}_i,v}(s,y)}{\mathbf{H}(s,x)}$$

où $\mathcal{F}_x$ est l'ensemble des y de $\mathcal{T}_{ix}(k)$ vérifiant les conditions suivantes

(i) $\forall\mathfrak{P}\in M_K - S_K$, $\forall L\in C_{\mathrm{eff}}(\overline{V})$, $\tilde{\mathbf{H}}^K_{\mathcal{T}_i,\mathfrak{P}}(L,y)\leqslant 1$,

(ii) $\forall L\in C_{\mathrm{eff}}(V)$, $\prod_{\mathfrak{p}\in S}\tilde{\mathbf{H}}_{\mathcal{T}_i,\mathfrak{p}}(L,y)\geqslant 1$,

(iii) $(y_v)_{v\in S}\in\Delta_{\mathbf{H}_K}(\mathcal{T}_i)$.

Mais l'hypothèse faite sur $\mathbf{H}_K$ et la formule (4.1.2) entraînent que la première assertion implique la seconde. Il suffit donc de considérer la première et la dernière.

Il résulte de la convention 4.4.2 que si $v\in M_k - S$, il existe $y_{0,v}\in\mathcal{T}_{ix}(k_v)$ tel que pour tout L de $\operatorname{Pic} V_K$ on ait

$$\tilde{\mathbf{H}}_{\mathcal{T}_i,v}(L, y_{0,v}) = 1. \tag{5.2.2}$$

Mais comme le morphisme

$$T_{\mathrm{NS}}(k)\to\bigoplus_{v\in M_k-S} X_*(T_{\mathrm{NS}})_v$$

est surjectif, il existe en fait $y_0\in\mathcal{T}_{ix}(k)$ tel que, pour tout v de $M_k - S$, l'image de y_0 dans $\mathcal{T}_{ix}(k_v)$ vérifie (5.2.2). Considérons alors la bijection

$$\begin{array}{rcl}\theta : T_{\mathrm{NS}}(k) & \to & \mathcal{T}_{ix}(k)\\ r & \mapsto & r\cdot y_0.\end{array}$$

La somme $\sum_{y\in\mathcal{F}_x}\prod_{v\in M_k-S}\tilde{\mathbf{H}}_{\mathcal{T}_i,v}(s,y)$ s'écrit alors

$$\begin{aligned}&\sum_{\left\{r\in T_{\mathrm{NS}}(k)\,\middle|\,\left\{\begin{smallmatrix}\forall v\in M_k-S,\ r\in T_{\mathrm{NS}}(-C_{\mathrm{eff}}(\overline{V}),\mathcal{O}_v)\\ ry_0\in\Delta_{\mathbf{H}_K}(\mathcal{T})\end{smallmatrix}\right.\right\}}\ \prod_{v\in M_k-S}\langle s,r\rangle_v\\ &\qquad = \#W(T_{\mathrm{NS}})\sum_{\{r\in T_{\mathrm{NS}}(k)\mid\forall v\in M_k-S,\, r\in T_{\mathrm{NS}}(-C_{\mathrm{eff}}(\overline{V}),\mathcal{O}_v)\}/T_{\mathrm{NS}}(\mathcal{O}_S)}\ \prod_{v\in M_k-S}\langle s,r\rangle_v\end{aligned}$$

où l'égalité résulte de la proposition 4.3.1. Mais par la proposition 3.1.2, on a une suite exacte

$$0\to T_{\mathrm{NS}}(\mathcal{O}_S)\to T_{\mathrm{NS}}(k)\to\bigoplus_{v\in M_k-S}X_*(T_{\mathrm{NS}})_v\to 0$$

et donc la somme ci-dessus coïncide avec

$$\prod_{v\in M_k-S}\ \sum_{x\in -C_{\text{eff}}(\overline{V})\cap X_*(T_{\text{NS}})_v} (\#\mathbf{F}_v)^{\langle x,s\rangle}$$

qui par le résultat de Draxl, (cf. proposition 3.2.2) est égale à $L_S(s, T_{\text{NS}}, C_{\text{eff}}(\overline{V}))$. $\square$

5.3. La descente pour l'analogue intégral

Théorème 5.3.1. — *Sous les hypothèses du théorème 5.2.2, pour tout* s *appartenant à un cône ouvert de* $\operatorname{Pic} V \otimes \mathbf{C}$*, on a la formule*

$$\sum_{i\in I}\int_{\mathcal{T}_{i\,C_{\text{eff}}(\overline{V})}(\boldsymbol{A}_k)} \Phi^{\mathbf{H}}_{\mathcal{T}_i}(s,y)\boldsymbol{\omega}_{\mathcal{T}_i}(y)$$
$$= \#H^1(k,\operatorname{Pic}\overline{V}).\tau_{\mathbf{H}}(V).\chi_{C_{\text{eff}}(V)}(s-\omega_V^{-1})L_S(\omega_V^{-1}, T_{\text{NS}}, C_{\text{eff}}(\overline{V})).$$

Démonstration. — On fixe un élément s de $\operatorname{Pic} V \otimes \mathbf{C}$ pour lequel l'expression $\chi_{C_{\text{eff}}(V)}(s-\omega_V^{-1})$ est bien définie. Par définition, le terme de gauche du théorème s'écrit de la manière suivante

$$\sum_{i\in I}\int_{\mathcal{T}_{i\,C_{\text{eff}}(\overline{V})}(\boldsymbol{A}_k)} \frac{1}{\#W(T_{\text{NS}})}\Phi^{\mathbf{H}}_{\mathcal{T}_i,S}(s,y)\prod_{v\in M_k-S}\mathbf{1}_{\mathcal{T}_{i\mathbf{H}}(\mathcal{O}_v)}\frac{1}{\sqrt{d}^{\dim \mathcal{T}_i}}\prod_{v\in M_k}\boldsymbol{\omega}_{\mathcal{T}_i,v}(x)$$

ou encore

$$\frac{1}{\#W(T_{\text{NS}})\sqrt{d}^{\dim \mathcal{T}_i}}\sum_{i\in I}\ \prod_{v\in M_k-S}\boldsymbol{\omega}_{\mathcal{T}_i,v}(\mathcal{T}_{i\mathbf{H}}(\mathcal{O}_v))\int_{\prod_{v\in S}\mathcal{T}_i(k_v)}\Phi^{\mathbf{H}}_{\mathcal{T}_i,S}(s,x)\prod_{v\in S}\boldsymbol{\omega}_{\mathcal{T}_i,v}(x).$$

Soit v une place de k en dehors de S, i un élément de I et x un point de $V(k_v)$. On peut alors choisir dans la fibre un point y_0 tel que

$$\forall L \in \operatorname{Pic}\overline{V},\quad \|y_0\|_v^L = 1.$$

On obtient la formule

$$\int_{\mathcal{T}_{ix}(k_v)\cap\mathcal{T}_{i\mathbf{H}}(\mathcal{O}_v)}\boldsymbol{\omega}_{\mathcal{T}_i,v} = \int_{T_{\text{NS}}(-C_{\text{eff}}(\overline{V}),\mathcal{O}_v)}|\omega_V(r)|_v\boldsymbol{\omega}_{T_{\text{NS}},v}(r)$$
$$= \boldsymbol{\omega}_{T_{\text{NS}},v}(T_{\text{NS}}(\mathcal{O}_v))\cdot L_v(T_{\text{NS}}, C_{\text{eff}}(\overline{V}),\omega_V^{-1})$$

et donc

$$\boldsymbol{\omega}_{\mathcal{T}_i,v}(\mathcal{T}_{i\mathbf{H}}(\mathcal{O}_v)) = \boldsymbol{\omega}_{T_{\text{NS}},v}(T_{\text{NS}}(\mathcal{O}_v))L_v(T_{\text{NS}}, C_{\text{eff}}(\overline{V}),\omega_V^{-1})\boldsymbol{\omega}_{\mathbf{H},v}(V(k_v)).$$

En prenant le produit sur les places v en dehors de S, on obtient

$$\prod_{v\in M_k-S}\boldsymbol{\omega}_{\mathcal{T}_i,v}(\mathcal{T}_{i\mathbf{H}}(\mathcal{O}_v))=\left[\prod_{v\in M_k-S}L_v(T_{\mathrm{NS}},C_{\mathrm{eff}}(\overline{V}),\omega_V^{-1})\right]$$
$$\times\left[\prod_{v\in M_k-S}\boldsymbol{\omega}_{T_{\mathrm{NS}},v}(T_{\mathrm{NS}}(\mathcal{O}_v))L_v(1,\operatorname{Pic}\overline{V})\right]$$
$$\times\left[\prod_{v\in M_k-S}L_v(1,\operatorname{Pic}\overline{V})^{-1}\boldsymbol{\omega}_{\mathbf{H},v}(V(k_v))\right].$$

Soit i un élément de I et $(x_v)_{v\in S}$ un point de l'image de $\prod_{v\in S}\mathcal{T}_i(k_v)$. Comme, par la proposition 3.1.2 (i), le morphisme de $T_{\mathrm{NS}}(\boldsymbol{A}_k)$ vers $(\operatorname{Pic}V)^\vee\otimes\mathbf{R}$ est surjectif, il existe un élément y_0 de $\prod_{v\in S}\mathcal{T}_i(k_v)$ tel que pour tout L de $\operatorname{Pic}V$ on ait $\prod_{v\in S}\|y_0\|_v^L=1$. On a alors la relation

$$\#W(T_{\mathrm{NS}})^{-1}\int_{\prod_{v\in S}\mathcal{T}_{ix_v}(k_v)}\Phi^{\mathbf{H}}_{\mathcal{T}_i,S}(s,y)\left(\prod_{v\in S}\boldsymbol{\omega}_{\mathcal{T}_{ix_v},v}\right)(y)$$
$$=\int_{\prod_{v\in S}T_{\mathrm{NS}}(k_v)/T_{\mathrm{NS}}(\mathcal{O}_S)}\mathbf{1}_{\{(r_v)_{v\in S}|\forall\xi\in C_{\mathrm{eff}}(V),\ \prod_{v\in S}|\xi(r_v)|_v\leqslant 1\}}\prod_{v\in S}\langle s-\omega_V^{-1},r_v\rangle_v\prod_{v\in S}\boldsymbol{\omega}_{T_{\mathrm{NS}},v}(r_v).$$

Notons $T^1_{\mathrm{NS}}(\prod_{v\in S}k_v)$ l'ensemble

$$\{(r_v)_{v\in S}\in\prod_{v\in S}T_{\mathrm{NS}}(k_v)\mid\forall\xi\in\operatorname{Pic}V,\ \prod_{v\in S}|\xi(r_v)|_v=1\,\}.$$

Comme dans le paragraphe 3.1, on définit une mesure $\boldsymbol{\omega}_{T^1_{\mathrm{NS}},S}$ sur ce groupe. Le terme ci-dessus est alors donné par la formule

$$\int_{C_{\mathrm{eff}}(V)^\vee}e^{-\langle s-\omega_V^{-1},y\rangle}dy.\boldsymbol{\omega}_{T^1_{\mathrm{NS}},S}\Big(T^1_{\mathrm{NS}}\Big(\prod_{v\in S}k_v\Big)/T_{\mathrm{NS}}(\mathcal{O}_S)\Big).$$

On obtient donc l'égalité

$$\#W(T_{\mathrm{NS}})^{-1}\int_{\prod_{v\in S}\mathcal{T}_i(k_v)}\Phi^{\mathbf{H}}_{\mathcal{T}_i,S}(s,y)\prod_{v\in S}\boldsymbol{\omega}_{\mathcal{T},v}(y_v)$$
$$=\chi_{C_{\mathrm{eff}}(V)}(s-\omega_V^{-1})\boldsymbol{\omega}_{T^1_{\mathrm{NS}},S}\Big(T^1_{\mathrm{NS}}\Big(\prod_{v\in S}k_v\Big)/T_{\mathrm{NS}}(\mathcal{O}_S)\Big)$$
$$\times\Big(\prod_{v\in S}\boldsymbol{\omega}_{\mathbf{H},v}\Big)\Big(\pi\Big(\prod_{v\in S}\mathcal{T}_i(k_v)\Big)\Big).$$

Or il résulte de la proposition 3.1.2 qu'on a un isomorphisme

$$T^1_{\mathrm{NS}}\Big(\prod_{v\in S} k_v\Big)/T_{\mathrm{NS}}(\mathcal{O}_S) \times \prod_{v\in M_k - S} T_{\mathrm{NS}}(\mathcal{O}_v) \xrightarrow{\sim} T^1_{\mathrm{NS}}(\boldsymbol{A}_k)/T_{\mathrm{NS}}(k).$$

En réunissant le terme obtenu pour les bonnes places à celui obtenu aux mauvaises, on obtient la formule

$$\int_{\mathcal{T}_{i\,C_{\mathrm{eff}}(\overline{V})}(\boldsymbol{A}_k)} \Phi^{\mathbf{H}}_{\mathcal{T}_i}(s,y)\boldsymbol{\omega}_{\mathcal{T}_i}(y)$$
$$= \chi_{C_{\mathrm{eff}}(V)}(s-\omega_V^{-1})\boldsymbol{\omega}_{T^1_{\mathrm{NS}}}(T^1_{\mathrm{NS}}(\boldsymbol{A}_k)/T_{\mathrm{NS}}(k))\boldsymbol{\omega}_{\mathbf{H}}\Big(\pi\Big(\prod_{v\in M_k}\mathcal{T}_i(k_v)\Big)\Big).$$

Par le théorème d'Ono,

$$\boldsymbol{\omega}_{T^1_{\mathrm{NS}}}(T^1_{\mathrm{NS}}(\boldsymbol{A}_k)/T_{\mathrm{NS}}(k)) = \#H^1(k,\operatorname{Pic}\overline{V})/\#\text{Ш}^1(k,T_{\mathrm{NS}}).$$

D'autre part si $x \in V(k)$, alors le nombre de classes d'isomorphisme de torseurs universels au-dessus de V qui ont un point adélique au-dessus de l'image de x dans $V(\boldsymbol{A}_k)$ est précisément $\text{Ш}^1(k,T_{\mathrm{NS}})$. Et il résulte de l'hypothèse (7) du paragraphe précédent que

$$\pi\Big(\prod_{v\in M_k}\mathcal{T}_i(k_v)\Big) \subset \overline{V(k)}.$$

En sommant sur les différents torseurs universels, on obtient la formule recherchée. □

5.4. Conclusion. — En définitive on obtient que, sous les hypothèses faites à la section 5.2, la question optimiste 2.6.1, à savoir la comparaison entre $\zeta_{\mathbf{H}}(s)$ et

$$\chi_{C_{\mathrm{eff}}(V)}(s-\omega_V^{-1})\beta(V)\tau_{\mathbf{H}}(V)$$

au voisinage de ω_V^{-1} se ramène, au niveau des torseurs universels à des majorations de termes de la forme

$$\left|\sum_{y\in\mathcal{T}_i(k)}\Phi(s,y) - \int_{\mathcal{T}_{i\,C_{\mathrm{eff}}(\overline{V})}(\boldsymbol{A}_k)}\Phi(s,y)\boldsymbol{\omega}_{\mathcal{T}_i}(y)\right|$$

où, par le corollaire 4.2.3 et le lemme 4.4.3, $\mathcal{T}_{i\,C_{\mathrm{eff}}(\overline{V})}(\boldsymbol{A}_k)$ et $\boldsymbol{\omega}_{\mathcal{T}_i}$ sont des structures canoniques associées au torseur universel.

De même la conjecture de Manin raffinée (cf. [**Pe**, conjecture 2.3.1] et [**BT1**]) qui relie $n_{U,\mathbf{H}}(H)$ à

$$\alpha(V)\beta(V)\tau_{\mathbf{H}}(V)H(\log H)^{t-1}$$

avec $t = \operatorname{rg}\operatorname{Pic} V$, peut se ramener à des majorations similaires à l'aide des fonctions de comptage

$$\Phi'^{\mathbf{H}}_{\mathcal{T}_i}(H, \mathfrak{b}, y) = \Phi'^{\mathbf{H}}_{\mathcal{T}_i,S}(H, \mathfrak{b}, y) \prod_{v \in M_k - S} \Phi'^{\mathbf{H}}_{\mathcal{T}_i,v}(H, \mathfrak{b}, y)$$

avec $H \in \mathbf{R}_{>0}$ et $\mathfrak{b} \in \bigoplus_{v \in M_k - S} X_*(T_{\mathrm{NS}})_v$ où, pour tout v en dehors de S, $\Phi'^{\mathbf{H}}_{\mathcal{T}_i,v}(H, \mathfrak{b}, y)$ est la fonction indicatrice de $b_v \mathcal{T}_{i\mathbf{H}}(\mathcal{O}_v)$ pour un élément b_v de l'ensemble $T_{\mathrm{NS}}(k_v)$ tel que $\log_v(b_v) = \mathfrak{b}_v$ et où $\Phi'^{\mathbf{H}}_{\mathcal{T}_i,S}(H, \cdot)$ est la fonction indicatrice de l'ensemble défini par

$$\begin{cases} (y_v)_{v \in S} \in \Delta_{\mathbf{H}_K}(\mathcal{T}_i), \\ \forall v \in S,\ \pi(y_v) \in U(k_v), \\ \forall L \in C_{\mathrm{eff}}(V),\ \prod_{v \in S} \tilde{\mathbf{H}}_{\mathcal{T},v}(L, y_v) \geqslant 1, \\ \prod_{v \in S} \tilde{\mathbf{H}}_{\mathcal{T},v}(\omega_V^{-1}, y_v) \leqslant H. \end{cases}$$

Ces réductions sont des généralisations du passage de la variété au cône épointé pour les intersections complètes lisses.

L'étape suivante dans le développement d'une hypothétique machinerie générale serait de passer d'un torseur $\mathcal{T}$ à un espace affine. Pour cela, il nous faut tout d'abord trouver un plongement de $\mathcal{T}$ dans un espace affine qui joue le même rôle que celui considéré dans le cas des variétés multihomogènes (cf. exemple 3.3.2) ou torique (cf. exemple 3.3.3).

Dans ce but, donnons-nous une famille génératrice $([L_j])_{j \in J}$ de $C_{\mathrm{eff}}(\overline{V})$ provenant de $\operatorname{Pic}\overline{V}$ et qui soit invariante sous l'action du groupe de Galois $\operatorname{Gal}(\overline{k}/k)$ et soit $\mathscr{M}(\overline{V})$ le $\mathbf{Z}$-module libre de base J. C'est un module de permutation et on a un épimorphisme naturel

$$p : \mathscr{M}(\overline{V}) \twoheadrightarrow \operatorname{Pic}\overline{V}.$$

Le fibré vectoriel $\overline{E} = \bigoplus_{j \in J} L_j^{\vee}$ provient d'un unique fibré E sur V et l'unique torseur $\mathcal{R}$ sous $T(\mathscr{M}(\overline{V}))$ au-dessus de V dont l'invariant $\rho(\mathcal{R})$ est p est donné comme k-forme de $\times_{V\, j \in J} L_j^{\times}$.

L'application p induit un morphisme de tores

$$j : T_{\mathrm{NS}} \hookrightarrow T(\mathscr{M}(\overline{V}))$$

et on a un morphisme de $\mathcal{T}$ vers $j_*(\mathcal{T})$ qui est isomorphe à $\mathcal{R}$. On obtient ainsi un morphisme

$$\widetilde{j} : \mathcal{T} \to \mathcal{R} \subset E.$$

Comme dans la démonstration de la proposition 4.2.2, on peut montrer que

$$\mathcal{T}_{C_{\mathrm{eff}}(\overline{V})}(\boldsymbol{A}_k) = \prod_{v \in M_k} \mathcal{T}(k_v) \cap E(\boldsymbol{A}_k).$$

Considérons maintenant l'espace vectoriel A obtenu par descente galoisienne à partir de

$$\overline{A} = \bigoplus_{j \in J} \Gamma(\overline{V}, L_j)^\vee.$$

On a un morphisme naturel ev : $E \to A$. On suppose dans la suite que cela définit un plongement de $\mathcal{T}$ dans A. Ce plongement a l'avantage que l'action de T_{NS} sur $\mathcal{T}$ s'étend en une action sur A.

Un premier problème auquel on se heurte est, comme dans le cas des variétés toriques, la différence entre $\mathcal{T}_{C_{\mathrm{eff}}(\overline{V})}(\boldsymbol{A}_k)$ et $\prod_{v \in M_k} \mathcal{T}(k_v) \cap A(\boldsymbol{A}_k)$. Plus précisément, soit v une place finie, $x \in V(k_v)$ au-dessus duquel $\mathcal{T}$ a un point y_0. En dehors de S, on peut choisir y_0 de sorte que

$$\forall L \in C_{\mathrm{eff}}(\overline{V}), \quad \|y_0\|_v^L = 1.$$

et on a un isomorphisme naturel

$$\mathcal{T}_x(k_v)/T_{\mathrm{NS}}(\mathcal{O}_v) \xrightarrow{\sim} X_*(T_{\mathrm{NS}})_v.$$

On se donne des bases $\mathrm{s}_{i,j}$ de $\Gamma(\overline{V}, L_j)$. Quitte à augmenter S, on peut supposer que l'image de $\mathcal{T}_x(K_{\mathfrak{P}}) \cap A(\mathcal{O}_{\mathfrak{P}})$, qui est invariante sous $T_{\mathrm{NS}}(\mathcal{O}_{\mathfrak{P}})$, est donnée par

$$\{\, \xi \in X_*(T_{\mathrm{NS}}) \mid \forall j \in J,\ \langle \xi, L_j \rangle \geqslant \sup\bigl(\log_{q_{\mathfrak{P}}} \|\mathrm{s}_{i,j}(x)\|_{\mathfrak{P}}^j\bigr) \,\}$$

où $(L_j, (\|\cdot\|_{\mathfrak{P}}^j)_{\mathfrak{P} \in M_K})$ représente $\mathbf{H}(L_j)$ pour $j \in J$. Il s'agit d'une réunion finie de translatés de $C_{\mathrm{eff}}(\overline{V})^\vee$ qui n'est autre que l'image de $\mathcal{T}_x(K_{\mathfrak{P}}) \cap \mathcal{T}_{\mathbf{H}_K}(\mathcal{O}_{K_{\mathfrak{P}}})$. En outre le terme

$$\sup(\log_{q_{\mathfrak{P}}} \|\mathrm{s}_{i,j}(x)\|_{\mathfrak{P}}^j)$$

est d'autant plus important que x est proche des points bases des faisceaux L_j, qui, dans plusieurs exemples, coïncident avec les sous-variétés accumulatrices.

Cette difficulté résolue, il faudrait alors passer de $\prod_{v \in M_k} \mathcal{T}(k_v) \cap A(\boldsymbol{A}_k)$ à $A(\boldsymbol{A}_k)$ et de $\mathcal{T}(k)$ à $A(k)$ ce qui devrait être réalisable en s'inspirant des travaux d'Igusa lorsque $\mathcal{T}$ est donné comme intersection complète dans A, puis majorer sur l'analogue des arcs majeurs les différences

$$\sum_{y \in A(k)} \phi(x) - \int_{A(\boldsymbol{A}_k)} \phi$$

pour les fonctions ϕ obtenues. Toutefois il n'est pas clair a l'heure actuelle que l'on puisse obtenir des généralisations du cœur de la méthode du cercle à savoir obtenir des majorations suffisamment fines de sommes d'exponentielles.

Je tiens à remercier Per Salberger et Yuri Tschinkel pour plusieurs dicussions qui ont permis la rédaction de cet article.

Références

[Ar] S. J. Arakelov, *Theory of intersections on the arithmetic surface*, Proceedings of the international congress of mathematicians, Vol. 1 (Vancouver, 1974), Canad. Math. Congress, Montréal, 1975, pp. 405–408.

[BM] V. V. Batyrev et Y. I. Manin, *Sur le nombre des points rationnels de hauteur bornée des variétés algébriques*, Math. Ann. **286** (1990), 27–43.

[BT1] V. V. Batyrev and Y. Tschinkel, *Rational points of bounded height on compactifications of anisotropic tori*, Internat. Math. Res. Notices **12** (1995), 591–635.

[BT2] ______, *Manin's conjecture for toric varieties*, Prépublication M/95/93, IHES, 1995.

[BT3] ______, *Rational points on some Fano cubic bundles*, C. R. Acad. Sci. Paris Sér. I Math. **323** (1996), 41–46.

[Bi] B. J. Birch, *Forms in many variables*, Proc. Roy. Soc. London **265A** (1962), 245–263.

[CTS1] J.-L. Colliot-Thélène et J.-J. Sansuc, *La descente sur les variétés rationnelles*, Journées de géométrie algébrique d'Angers (1979) (A. Beauville, ed.), Sijthoff & Noordhoff, Alphen aan den Rijn, 1980, pp. 223–237.

[CTS2] ______, *La descente sur les variétés rationnelles, II*, Duke Math. J. **54** (1987), n$^{\circ}$ 2, 375–492.

[Da] V. I. Danilov, *The geometry of toric varieties*, Uspekhi. Mat. Nauk **33** (1978), n$^{\circ}$ 2, 85–134 ; English transl. in Russian Math. Surveys **33** (1978), n$^{\circ}$ 2, 97–154.

[De] T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France **116** (1988), 315–339.

[Dr] P. K. J. Draxl, *L-Funktionen algebraischer Tori*, J. Number Theory **3** (1971), 444–467.

[FMT] J. Franke, Y. I. Manin, and Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. **95** (1989), 421–435.

[SGA2] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux.*, Adv. Stud. Pure Math., vol. 2, North Holland, Amsterdam et Masson, Paris, 1968.

[Kö] M. Köcher, *Positivitätsbereiche im* $\mathbf{R}^n$, Amer. J. Math. **79** (1957), 575–596.

[Lac] G. Lachaud, *Une présentation adélique de la série singulière et du problème de Waring*, Enseign. Math. (2) **28** (1982), 139–169.

[Man] Y. I. Manin, *Problems on rational points and rational curves on algebraic varieties*, Surveys in differentiable geometry **2** (1995), 214–245.

[Mi] J. S. Milne, *Étale cohomology*, Princeton Math. Series, vol. 33, Princeton University Press, 1980.

[Oda1] T. Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 15, Springer-Verlag, Berlin, Heidelberg and New York, 1988.

[Oda2] ______, *Recent topics on toric varieties*, Selected papers on number theory and algebraic geometry (K. Nomizu, ed.), Amer. Math. Soc. Trans. (2), vol. 172, AMS, Providence, 1996, pp. 77–91.

[Ono1] T. Ono, *On some arithmetic properties of linear algebraic groups*, Ann. of Math. (2) **70** (1959), n° 2, 266–290.

[Ono2] ———, *Arithmetic of algebraic tori*, Ann. of Math. (2) **74** (1961), n° 1, 101–139.

[Ono3] ———, *On the Tamagawa number of algebraic tori*, Ann. of Math. (2) **78** (1963), n° 1, 47–73.

[Pe] E. Peyre, *Hauteurs et mesures de Tamagawa sur les variétés de Fano*, Duke Math. J. **79** (1995), n° 1, 101–218.

[Ro] M. Robbiani, *Rational points of bounded height on Del Pezzo surfaces of degree six*, Comment. Math. Helv. **70** (1995), 403–422.

[Sa] P. Salberger, *Tamagawa measures on universal torsors and points of bounded height on Fano varieties*, in this volume (1996).

[Sc] S. H. Schanuel, *Heights in number fields*, Bull. Soc. Math. France **107** (1979), 433–449.

[Se] J.-P. Serre, *Algèbre locale, multiplicités (Troisième édition)*, Lecture Notes in Math., vol. 11, Springer-Verlag, Berlin, Heidelberg and New York, 1975.

[Sk1] A. N. Skorobogatov, *On a theorem of Enriques—Swinnerton-Dyer*, Ann. Fac. Sci. Toulouse Math. (6) **2** (1993), n° 3, 429–440.

[Sk2] ———, *Partial exceptional heights and descent on rational surfaces*, Unpublished text (1993).

[S-D] P. Swinnerton-Dyer, *Counting rational points on cubic surfaces*, (L'Aquila, 1992) (C. Ciliberto, E. L. Livorni, and A. J. Sommese, eds.), Contemp. Math., vol. 162, AMS, Providence, 1994, pp. 371–379.

[We] A. Weil, *Adèles and algebraic groups*, Progress in Mathematics, vol. 23, Birkhaüser, Boston, Basel, Stuttgart, 1982.

EMMANUEL PEYRE, Institut de Recherche Mathématique Avancée, Université Louis Pasteur et C.N.R.S., 7 rue René-Descartes, 67084 Strasbourg, France
E-mail : peyre@irma.u-strasbg.fr

Astérisque
251, 1998, p. 299–341

TAMAGAWA NUMBERS OF POLARIZED ALGEBRAIC VARIETIES

by

Victor V. Batyrev & Yuri Tschinkel

Dedicated to Professor Yu. I. Manin on his 60th birthday

Abstract. — Let $\mathcal{L} = (L, \|\cdot\|_v)$ be an ample metrized invertible sheaf on a smooth quasi-projective algebraic variety V over a number field F. Denote by $N(V, \mathcal{L}, B)$ the number of rational points in V having $\mathcal{L}$-height $\leqslant B$. In this paper we consider the problem of a geometric and arithmetic interpretation of the asymptotic for $N(V, \mathcal{L}, B)$ as $B \to \infty$ in connection with recent conjectures of Fujita concerning the Minimal Model Program for polarized algebraic varieties.

We introduce the notions of *$\mathcal{L}$-primitive varieties* and *$\mathcal{L}$-primitive fibrations.* For $\mathcal{L}$-primitive varieties V over F we propose a method to define an adelic Tamagawa number $\tau_{\mathcal{L}}(V)$ which is a generalization of the Tamagawa number $\tau(V)$ introduced by Peyre for smooth Fano varieties. Our method allows us to construct Tamagawa numbers for $\mathbf{Q}$-Fano varieties with at worst canonical singularities.

In a series of examples of smooth polarized varieties and singular Fano varieties we show that our Tamagawa numbers express the dependence of the asymptotic of $N(V, \mathcal{L}, B)$ on the choice of v-adic metrics on $\mathcal{L}$.

1. Introduction

Let F be a number field (a finite extension of $\mathbf{Q}$), $\mathrm{Val}(F)$ the set of all valuations of F, F_v the v-adic completion of F with respect to $v \in \mathrm{Val}(F)$, and $|\cdot|_v : F_v \to \mathbf{R}$ the v-adic norm on F_v normalized by the conditions $|x|_v = |N_{F_v/\mathbf{Q}_p}(x)|_p$ for p-adic valuations $v \in \mathrm{Val}(F)$.

Consider a projective space $\mathbf{P}^m$ with homogeneous coordinates $(z_0, ..., z_m)$ and a locally closed quasi-projective subvariety $V \subset \mathbf{P}^m$ defined over F (we want to stress that V is not assumed to be projective). Let $V(F)$ be the

2000 ***Mathematics Subject Classification.*** — Primary 11G35; Secondary 14G05, 14G10.
Key words and phrases. — Rational points, height zeta functions, Tamagawa numbers.

set of points in V with coordinates in F. A **standard height function** $H : \mathbf{P}^m(F) \to \mathbf{R}_{>0}$ is defined as follows

$$H(x) := \prod_{v \in \mathrm{Val}(F)} \max_{j=0,\ldots,m} \{|z_j(x)|_v\}.$$

A basic fact about the standard height function H claims that the set

$$\{x \in \mathbf{P}^m(F) \,:\, H(x) \leq B\}$$

is finite for any real number B [**29**]. We set

$$N(V,B) = \#\{x \in V(F) \,:\, H(x) \leq B\}.$$

It is an experimental fact that whenever one succeeds in proving an asymptotic formula for the function $N(V,B)$ as $B \to \infty$, one obtains the asymptotic

$$N(V,B) = c(V)B^{a(V)}(\log B)^{b(V)-1}(1+o(1)) \tag{1}$$

with some constants $a(V) \in \mathbf{Q}$, $b(V) \in \frac{1}{2}\mathbf{Z}$, and $c(V) \in \mathbf{R}_{>0}$. We want to use this observation as our starting point. It seems natural to ask the following:

Question A. — *For which quasi-projective subvarieties $V \subset \mathbf{P}^m$ defined over F do there exist constants $a(V) \in \mathbf{Q}$, $b(V) \in \frac{1}{2}\mathbf{Z}$ and $c(V) \in \mathbf{R}_{>0}$ such that the asymptotic formula* (1) *holds?*

Question B. — *Does there exist a quasi-projective variety V over F with an asymptotic which is different from* (1)*?*

In this paper we will be interested not in Questions A and B themselves but in a related to them another natural question:

Question C. — *Assume that V is an irreducible quasi-projective variety over a number field F such that the asymptotic formula* (1) *holds. How to compute the constants $a(V), b(V)$ and $c(V)$ in this formula via some arithmetical properties of V over F and geometrical properties of V over $\mathbf{C}$?*

To simplify our terminology, it will be convenient for us to postulate:

Assumption. — For all quasi-projective V', V with $V' \subset V \subset \mathbf{P}^m$ and with $|V(F)| = \infty$ there exists the limit

$$\lim_{B\to\infty} \frac{N(V',B)}{N(V,B)}.$$

The following definitions have been useful to us:

Definition $\boldsymbol{S_1}$. — A smooth irreducible quasi-projective subvariety $V \subset \mathbf{P}^m$ over a number field F is called **weakly saturated**, if $|V(F)| = \infty$ and if for any locally closed subvariety $W \subset V$ with $\dim W < \dim V$ one has

$$\lim_{B\to\infty} \frac{N(W,B)}{N(V,B)} < 1.$$

It is important to remark that Question C really makes sense *only for weakly saturated* varieties. Indeed, if there were a locally closed subvariety $W \subset V$ with $\dim W < \dim V$ and

$$\lim_{B\to\infty} \frac{N(W,B)}{N(V,B)} = 1,$$

then it would be enough to answer Question C for each irreducible component of W and for all possible intersections of these components (i.e., one could forget about the existence of V and reduce the situation to a lower-dimensional case). In general, it is not easy to decide whether or not a given locally closed subvariety $V \subset \mathbf{P}^m$ is weakly saturated. We expect (and our assumption implies this) that the orbits of connected subgroups $G \subset PGL(m+1)$ are examples of weakly saturated varieties $V \subset \mathbf{P}^m$ (see 3.2.8).

Definition $\boldsymbol{S_2}$. — A smooth irreducible quasi-projective subvariety $V \subset \mathbf{P}^m$ with $|N(V,B)| = \infty$ is called **strongly saturated**, if for all dense Zariski open subsets $U \subset V$, one has

$$\lim_{B\to\infty} \frac{N(U,B)}{N(V,B)} = 1.$$

First of all, if $V \subset \mathbf{P}^m$ is a strongly saturated subvariety, then for any locally closed subvariety $W \subset V$ with $\dim W < \dim V$, one has

$$\lim_{B\to\infty} \frac{N(W,B)}{N(V,B)} = 0,$$

i.e., V is weakly saturated.

On the other hand, if $V \subset \mathbf{P}^m$ is weakly saturated, but not strongly saturated, then there must be an infinite sequence $W_1, W_2, \dots$ of pairwise different locally closed irreducible subvarieties $W_i \subset V$ with $\dim W_i < \dim V$ and $|W_i(F)| = \infty$ such that for an arbitrary positive integer k one has

$$0 < \lim_{B\to\infty} \frac{N(W_1 \cup \cdots \cup W_k, B)}{N(V,B)} < 1.$$

Moreover, in this situation one can always choose the varieties W_i to be strongly saturated (otherwise one could find $W_i' \subset W_i$ with $\dim W_i' < \dim W_i$

with the same properties as W_i etc.). The strong saturatedness of each W_i implies that

$$\lim_{B\to\infty}\frac{N(W_{i_1}\cap\cdots\cap W_{i_l},B)}{N(V,B)}=0$$

for all pairwise different $i_1,\ldots,i_l$ and $l\geqslant 2$. In particular, one has

$$\sum_{i=1}^{k}\lim_{B\to\infty}\frac{N(W_i,B)}{N(V,B)}=\lim_{B\to\infty}\frac{N(W_1\cup\cdots\cup W_k,B)}{N(V,B)}<1\ \ \forall k>0.$$

Definition F. — Let V be a weakly saturated quasi-projective variety in $\mathbf{P}^m$ and $W_1,W_2,\ldots$ an infinite sequence of strongly saturated irreducible subvarieties W_i having the property

$$0<\theta_i:=\lim_{B\to\infty}\frac{N(W_i,B)}{N(V,B)}<1\ \ \forall i>0.$$

We say that the set $\{W_1,W_2,\ldots\}$ forms an **asymptotic arithmetic fibration** on V, if the following equality holds

$$\sum_{i=1}^{\infty}\theta_i=1.$$

The main purpose of this paper is to explain some geometric and arithmetic ideas concerning weakly saturated varieties and their asymptotic arithmetic fibrations by strongly saturated subvarieties. It seems that the cubic bundles considered in [**9**] are examples of such a fibration. We want to remark that most of the above terminology grew out of our attempts to restore a conjectural picture of the interplay between the geometry of algebraic varieties and the arithmetic of the distribution of rational points on them after we have found in [**9**] an example which contradicted general expectations formulated in [**4**].

In section 2 we consider smooth quasi-projective varieties V over $\mathbf{C}$ together with a polarization $\mathcal{L}=(L,\|\cdot\|_h)$ consisting of an ample line bundle L on V equipped with a positive hermitian metric $\|\cdot\|_h$. Our main interest in this section is a discussion of geometric properties of V in connection with the Minimal Model Program [**27**] and its version for polarized algebraic varieties suggested by Fujita [**20, 23, 24**]. We introduce our main geometric invariants $\alpha_{\mathcal{L}}(V)$, $\beta_{\mathcal{L}}(V)$, and $\delta_{\mathcal{L}}(V)$ for an arbitrary $\mathcal{L}$-polarized variety V. It is important to remark that we will be only interested in the case $\alpha_{\mathcal{L}}(V)>0$. The number $\alpha_{\mathcal{L}}(V)$ was first introduced in [**4, 5**], it equals to the opposite of the so called *Kodaira energy* (investigated by Fujita in [**20, 23, 24**]). Our basic geometric notion in the study of $\mathcal{L}$-polarized varieties V with $\alpha_{\mathcal{L}}(V)>0$ is the notion of an *$\mathcal{L}$-primitive* variety. In Fujita's program for polarized varieties with negative Kodaira energy $\mathcal{L}$-primitive varieties play the same role as $\mathbf{Q}$-Fano varieties in Mori's program for algebraic varieties with negative Kodaira dimension. In particular, one expects the existence of so called *$\mathcal{L}$-primitive fibrations*, which are analogous to $\mathbf{Q}$-Fano fibrations in Mori's program. We show that on $\mathcal{L}$-primitive varieties there exists a canonical volume measure. Moreover, this measure allows us to construct a descent of hermitian metrics to the base of $\mathcal{L}$-primitive fibrations. Many geometric ideas of this section are inspired by [**4, 5**].

In section 3 we introduce our main arithmetic notions of *weakly* and *strongly $\mathcal{L}$-saturated* varieties. Our first main diophantine conjecture claims that if an adelic $\mathcal{L}$-polarized quasi-projective algebraic variety V over a number field F is strongly $\mathcal{L}$-saturated, then the corresponding $\mathcal{L}$-polarized complex algebraic variety $V(\mathbf{C})$ is $\mathcal{L}$-primitive. Moreover, we conjecture that if an adelic $\mathcal{L}$-polarized quasi-projective algebraic variety V over a number field F is weakly $\mathcal{L}$-saturated, then the corresponding $\mathcal{L}$-polarized complex algebraic variety $V(\mathbf{C})$ admits an $\mathcal{L}$-primitive fibration having infinitely many fibers W defined over F which form an asymptotic arithmetic fibration. These conjectures allow us to establish a connection between the geometry of $V(\mathbf{C})$ and the arithmetic of V. Following this idea, we explain a construction of an adelic measure on an arbitrary $\mathcal{L}$-primitive variety V with $\alpha_{\mathcal{L}}(V) > 0$ and of the corresponding Tamagawa number $\tau_{\mathcal{L}}(V)$ as a regularized adelic integral of this measure. Our construction generalizes the definition of Tamagawa measures associated with a metrization of the canonical line bundle due to Peyre [**30**]. We expect that for strongly $\mathcal{L}$-saturated varieties V the number $\tau_{\mathcal{L}}(V)$ reflects the dependence of the constant $c(V)$ in the asymptotic formula (1) on the adelic metrization of the ample line bundle L. We discuss the natural question about the behavior of the adelic constant $\tau_{\mathcal{L}}(W)$ for fibers W in $\mathcal{L}$-primitive fibrations on weakly $\mathcal{L}$-saturated varieties.

In section 4 we show that our diophantine conjectures agree with already known examples of asymptotic formulas established for polarized algebraic varieties through the study of analytic properties of height zeta functions.

In Section 5 we illustrate our expectations for the constants $a(V), b(V)$ and $c(V)$ in the asymptotics of $N(V,B)$ on some examples of smooth Zariski dense subsets V in Fano varieties with singularities.

We would like to thank J.-L. Colliot-Thélène for his patience and encouragement. We are very grateful to B. Mazur, Yu. I. Manin, L. Ein and A. Chambert-Loir for their comments and suggestions. We thank the referee for several useful remarks.

2. Geometry of $\mathcal{L}$-polarized varieties

2.1. $\mathcal{L}$-closure. — Let V be a smooth irreducible quasi-projective algebraic variety over $\mathbf{C}$, $V(\mathbf{C})$ the set of closed points of V, L an ample invertible sheaf on V, i.e., $L^{\otimes k} = i^*\mathcal{O}_{\mathbf{P}^m}(1)$ for some $k > 0$ and some embedding $i : V \hookrightarrow \mathbf{P}^m$. Since we don't assume V to be compact, the invertible sheaf L on V itself contains too little information about the embedding $i : V \hookrightarrow \mathbf{P}^m$. For instance, let V be an affine variety of positive dimension. Then the space of global sections of L is infinite dimensional and we don't know anything about the projective closure of V in $\mathbf{P}^m$ even though we know that the invertible sheaf $L^{\otimes k}$ is isomorphic to $i^*\mathcal{O}_{\mathbf{P}^m}(1)$. This situation changes if one considers L together with a positive hermitian metric, i.e., an ample metrized invertible sheaf $\mathcal{L}$ associated with L. Let us choose a positive hermitian metric h on $\mathcal{O}_{\mathbf{P}^m}(1)$ (e.g. Fubini-Study metric) and denote by $\|\cdot\|_h$ the induced metric on $L^{\otimes k}$. Thus we obtain a metric $\|\cdot\|$ on L by putting $\|s(x)\| := \|s^k(x)\|_h^{1/k}$ for any $x \in V(\mathbf{C})$ and any section $s \in \mathrm{H}^0(U, L)$ over an open subset $U \subset V$.

Definition 2.1.1. — We call a pair $\mathcal{L} = (L, \|\cdot\|)$ **an ample metrized invertible sheaf** associated with L. We denote by $\mathcal{L}^{\otimes\nu}$ the pair $(L^{\otimes\nu}, \|\cdot\|^\nu)$.

Our next goal is to show that an ample metrized invertible sheaf contains almost complete information about the projective closure of V in $\mathbf{P}^m$.

Definition 2.1.2. — Let $\mathcal{L} = (L, \|\cdot\|)$ be an ample metrized invertible sheaf on a complex irreducible quasi-projective variety V. We denote by

$$\mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L})$$

the subspace of $\mathrm{H}^0(V, L)$ consisting of those global sections s of L over V such that the corresponding continuous function $x \mapsto \|s(x)\|$ ($x \in V(\mathbf{C})$) is globally bounded on $V(\mathbf{C})$ from above by a positive constant $C(s)$ depending only on s. We call $\mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L})$ the **space of globally bounded sections** of $\mathcal{L}$.

Proposition 2.1.3. — *Let $\overline{V}$ be the normalization of the projective closure of V with respect to the embedding $i : V \hookrightarrow \mathbf{P}^m$ with $L^{\otimes k} = i^*\mathcal{O}_{\mathbf{P}^m}(1)$. Denote by $e : \overline{V} \to \mathbf{P}^m$ the corresponding finite projective morphism. Then one has a natural isomorphism*

$$\mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L}^{\otimes k}) \cong \mathrm{H}^0(\overline{V}, e^*\mathcal{O}_{\mathbf{P}^m}(1)).$$

Proof. — Since $\overline{V}(\mathbf{C})$ is compact, the continuous function $x \mapsto \|s(x)\|$ is globally bounded on $\overline{V}(\mathbf{C})$ for any $s \in \mathrm{H}^0(\overline{V}, e^*\mathcal{O}_{\mathbf{P}^m}(1))$. Therefore, we obtain that $\mathrm{H}^0(\overline{V}, e^*\mathcal{O}_{\mathbf{P}^m}(1))$ is a subspace of $\mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L}^{\otimes k})$.

Now let $f \in \mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L}^{\otimes k})$ be a globally bounded on $V(\mathbf{C})$ section of $i^*\mathcal{O}_{\mathbf{P}^m}(1)$. Since $\mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L}^{\otimes k})$ is a subspace of $\mathrm{H}^0(V, L^{\otimes k})$, the section f uniquely extends to a global meromorphic section $\overline{f} \in \mathrm{H}^0(\overline{V}, e^*\mathcal{O}_{\mathbf{P}^m}(1))$. Since a bounded meromorphic function is holomorphic, $\overline{f}$ is a global regular section of $e^*\mathcal{O}_{\mathbf{P}^m}(1)$ (we apply the theorem of Riemann to some resolution of singularities $\rho : X \to \overline{V}$ and use the fact that $\rho_*\mathcal{O}_X = \mathcal{O}_{\overline{V}}$). Thus we have

$$\mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L}^{\otimes k}) \subset \mathrm{H}^0(\overline{V}, e^*\mathcal{O}(1)).$$

□

Definition 2.1.4. — We define the graded $\mathbf{C}$-algebra

$$\mathrm{A}(V, \mathcal{L}) = \bigoplus_{\nu \geqslant 0} \mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L}^{\otimes \nu}).$$

Using 2.1.3, one immediately obtains:

Corollary 2.1.5. — *The graded algebra* $\mathrm{A}(V, \mathcal{L})$ *is finitely generated.*

Definition 2.1.6. — We call the normal projective variety

$$\overline{V}^{\mathcal{L}} = \operatorname{Proj} \mathrm{A}(V, \mathcal{L}).$$

the $\mathcal{L}$-**closure** of V with respect to an ample metrized invertible sheaf $\mathcal{L}$.

Remark 2.1.7. — By 2.1.3, $\overline{V}^{\mathcal{L}}$ is isomorphic to $\overline{V}$. Therefore, we have obtained a way to define the normalization of the projective closure of V with respect to an $L^{\otimes k}$-embedding via a notion of an ample metrized invertible sheaf $\mathcal{L}$ on V.

2.2. Kodaira energy and $\alpha_{\mathcal{L}}(V)$. — Let X be a normal irreducible algebraic variety of dimension n. We denote by $\mathrm{Div}(X)$ (resp. by $\mathrm{Z}_{n-1}(X)$) the group of Cartier divisors (resp. Weil divisors) on X. An element of $\mathrm{Div}(X) \otimes \mathbf{Q}$ (resp.$\mathrm{Z}_{n-1}(X) \otimes \mathbf{Q}$) is called a $\mathbf{Q}$-Cartier divisor (resp. a $\mathbf{Q}$-divisor). By K_X we denote a divisor of a meromorphic differential n-form on X, where K_X is considered as an element of $\mathrm{Z}_{n-1}(X)$.

Definition 2.2.1. — Let X be a projective variety and L be an invertible sheaf on X. The **Iitaka-dimension** $\kappa(L)$ is defined as

$$\kappa(L) = \begin{cases} -\infty & \text{if } \mathrm{H}^0(X, L^{\otimes \nu}) = 0 \text{ for all } \nu > 0 \\ \operatorname{Max} \dim \phi_{L^{\otimes \nu}}(X) : & \mathrm{H}^0(X, L^{\otimes \nu}) \neq 0 \end{cases}$$

where $\phi_{L^{\otimes \nu}}(X)$ is the closure of the image of X under the rational map

$$\phi_{L^{\otimes \nu}} : X \to \mathbf{P}(\mathrm{H}^0(X, L^{\otimes \nu})).$$

A Cartier divisor L is called **semi-ample** (resp. **effective**), if $L^{\otimes \nu}$ is generated by global sections for some $\nu > 0$ (resp. $\kappa(L) \geqslant 0$).

Remark 2.2.2. — The notions of Iitaka-dimension, ampleness and semi-ampleness obviously extend to **Q**-Cartier divisors. Let L be a Cartier divisor. Then for all $\kappa_1, k_2 \in \mathbf{N}$ we set

$$\kappa(L^{\otimes k_1/k_2}) := \kappa(L),$$

$$L^{\otimes k_1/k_2} \text{ is ample} \Leftrightarrow L \text{ is ample},$$

and

$$L^{\otimes k_1/k_2} \text{ is semi-ample} \Leftrightarrow L \text{ is semi-ample}.$$

Definition 2.2.3. — Let X be a smooth projective variety. We denote by $\mathrm{NS}(X)$ the group of divisors on X modulo numerical equivalence and set

$$\mathrm{NS}(X)_{\mathbf{R}} = \mathrm{NS}(X) \otimes \mathbf{R}.$$

By $[L]$ we denote the class of a divisor L in $\mathrm{NS}(X)$. The **cone of effective divisors** $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{NS}(X)_{\mathbf{R}}$ is defined as the closure of the subset

$$\bigcup_{\kappa(L)\geqslant 0} \mathbf{R}_{\geqslant 0}[L] \subset \mathrm{NS}(X)_{\mathbf{R}}.$$

Definition 2.2.4. — Let V be a smooth quasi-projective algebraic variety with an ample metrized invertible sheaf $\mathcal{L}$, $\overline{V}^{\mathcal{L}}$ the $\mathcal{L}$-closure of V and ρ some resolution of singularities

$$\rho \,:\, X \to \overline{V}^{\mathcal{L}}.$$

We define the number

$$\alpha_{\mathcal{L}}(V) = \inf\{t \in \mathbf{Q} \,:\, t[\rho^* L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)\}.$$

and call it the **$\mathcal{L}$-index** of V.

Remark 2.2.5. — It is easy to see that the $\mathcal{L}$-index does not depend on the choice of the resolution ρ.

Remark 2.2.6. — The $\mathcal{L}$-index $\alpha_{\mathcal{L}}(V)$ for smooth projective varieties V was first introduced in [**4**] and [**5**]. We remark that the opposite number $-\alpha_{\mathcal{L}}(V)$ coincides with the notion of **Kodaira energy** introduced and investigated by Fujita in [**20, 23, 24**]:

$$\kappa\epsilon(V, L) = -\alpha_{\mathcal{L}}(V) = -\inf\{t \in \mathbf{Q} \,:\, \kappa((L)^{\otimes t} \otimes K_V) \geqslant 0\}.$$

From the viewpoint of our diophantine applications it is much more natural to consider $\alpha_{\mathcal{L}}(V)$ instead of its opposite $-\alpha_{\mathcal{L}}(V)$. The only reason that we could see for introducing the number $-\alpha_{\mathcal{L}}(V)$ instead of $\alpha_{\mathcal{L}}(V)$ is some kind of compatibility between the notions of *Kodaira energy* and *Kodaira dimension*, e.g. Kodaira energy must be positive (resp. negative) iff the Kodaira dimension is positive (resp. negative).

The following statement was conjectured in [**4**] (see also [**5, 21, 22**]):

***Conjecture 2.2.7* (Rationality).** — *Assume that $\alpha_{\mathcal{L}}(V) > 0$. Then $\alpha_{\mathcal{L}}(V)$ is rational.*

Remark 2.2.8. — It was shown in [**5**] that this conjecture follows from the Minimal Model Program. In particular, it holds for $\dim V \leqslant 3$. If $\dim V = 1$, then the only possible values of $\alpha_{\mathcal{L}}(V)$ are numbers $2/k$ with ($k \in \mathbf{N}$). If $\dim V = 2$, then $\alpha_{\mathcal{L}}(V) \in \{2/k, 3/l\}$ with ($k, l \in \mathbf{N}$).

Definition 2.2.9. — A normal irreducible algebraic variety W is said to have at worst **canonical** (resp. **terminal**) singularities if K_W is a $\mathbf{Q}$-Cartier divisor and if for some (or every) resolution of singularities

$$\rho \,:\, X \to W$$

one has

$$K_X = \rho^*(K_W) \otimes \mathcal{O}(D)$$

where D is an effective $\mathbf{Q}$-Cartier divisor (resp. the support of the effective divisor D coincides with the exceptional locus of ρ). Irreducible components of the exceptional locus of ρ which are not contained in the support of D are called **crepant divisors** of the resolution ρ.

Definition 2.2.10. — A normal irreducible algebraic variety W is called a **canonical $\mathbf{Q}$-Fano** variety, if W has at worst canonical singularities and K_W^{-1} is an ample $\mathbf{Q}$-Cartier divisor. A maximal positive rational number $r(W)$ such that $K_W^{-1} = L^{\otimes r(W)}$ for some Cartier divisor L is called the **index** of a canonical $\mathbf{Q}$-Fano variety W (obviously, one has $r(W) = \alpha_{\mathcal{L}}(W)$ for some positive metric on L).

The following conjecture is due to Fujita [**20**]:

***Conjecture 2.2.11* (Spectrum Conjecture).** — *Let $S(n)$ be the set all possible values of $\alpha_{\mathcal{L}}(V)$ for smooth quasi-projective algebraic varieties V of dimension $\leqslant n$ with an ample metrized invertible sheaf $\mathcal{L}$. Then for any $\varepsilon > 0$ the set*

$$\{\alpha_{\mathcal{L}}(V) \in S(n) \,:\, \alpha_{\mathcal{L}}(V) > \varepsilon\}$$

is finite.

This conjecture follows from the Minimal Model Program [**27**] and from the following conjecture on the boundedness of index for Fano varieties with canonical singularities:

***Conjecture 2.2.12* (Boundedness of Index).** — *The set of possible values of index $r(W)$ for canonical $\mathbf{Q}$-Fano varieties W of dimension n is finite.*

In particular, both conjectures are true for $\mathbf{Q}$-Fano varieties of dimension $n \leqslant 3$ [**2, 20, 23, 24, 28, 32**].

2.3. $\mathcal{L}$-primitive varieties

Definition 2.3.1. — Let X be a projective algebraic variety. We call an effective $\mathbf{Q}$-divisor D **rigid**, if $\kappa(D) = 0$.

Proposition 2.3.2. — An effective $\mathbf{Q}$-divisor D on X is rigid if and only if there exist finitely many irreducible subvarieties $D_1, \ldots, D_l \subset X$ ($l \geqslant 0$) of codimension 1 such that $D = r_1 D_1 + \cdots + r_l D_l$ with $r_1, \ldots, r_l \in \mathbf{Q}_{>0}$ and

$$\dim \mathrm{H}^0(X, \mathcal{O}(n_1 D_1 + \cdots + n_l D_l)) = 1 \;\; \forall\, (n_1, \ldots, n_l) \in \mathbf{Z}^l_{\geqslant 0}.$$

Proof. — Let D be rigid. Take a positive integer m_0 such that $m_0 D$ is a Cartier divisor and $\dim \mathrm{H}^0(X, \mathcal{O}(m_0 D)) = 1$. Denote by $D_1, \ldots, D_l$ the irreducible components of the divisor (s) of a non-zero section $s \in \mathrm{H}^0(X, \mathcal{O}(mD))$. One has

$$(s) = m_1 D_1 + \cdots + m_l D_l \quad m_1, \ldots, m_l \in \mathbf{N}.$$

Since $\mathcal{O}(D_i)$ admits at least one global non-zero section we obtain that

$$\dim \mathrm{H}^0(X, \mathcal{O}(n_1' D_1 + \cdots + n_l' D_l)) \geqslant \dim \mathrm{H}^0(X, \mathcal{O}(n_1 D_1 + \cdots + n_l D_l)),$$

whenever $n_1 \geqslant n_1', \ldots, n_l \geqslant n_l'$ for $(n_1', \ldots, n_l'),\ (n_1, \ldots, n_l) \in \mathbf{Z}^l_{\geqslant 0}$. This implies that

$$\dim \mathrm{H}^0(X, \mathcal{O}(n_1 D_1 + \cdots + n_l D_l)) \geqslant 1 \;\; \forall\, (n_1, \ldots, n_l) \in \mathbf{Z}^l_{\geqslant 0}.$$

On the other hand, for any $(n_1, \ldots, n_l) \in \mathbf{Z}^l_{\geqslant 0}$ there exists a positive integer n_0 such that $n_0 m_1 \geqslant n_1, \ldots, n_0 m_l \geqslant n_l$. Therefore,

$$\dim \mathrm{H}^0(X, \mathcal{O}(n_1 D_1 + \cdots + n_l D_l)) \leqslant \dim \mathrm{H}^0(X, \mathcal{O}(n_0 m_0 D)) = 1,$$

since $\kappa(n_0 m_0 D) = \kappa(D) = 0$. □

Corollary 2.3.3. — *Let $D_1, \ldots, D_l \subset X$ be all irreducible components of the support of a rigid $\mathbf{Q}$-Cartier divisor D. Then a linear combination*

$$n_1 D_1 + \cdots + n_l D_l, \;\; n_1, \ldots, n_l \in \mathbf{Z}$$

is a principal divisor, iff $n_1 = \cdots = n_l = 0$.

Proof. — Assume that $n_1 D_1 + \cdots + n_l D_l$ is linearly equivalent to 0. Then the effective Cartier divisor $D_0 = \sum_{n_i \geqslant 0} n_i D_i$ is linearly equivalent to the effective Cartier divisor $D_0' = \sum_{n_j < 0} (-n_j) D_j$. Since D_0 and D_0' have different supports we have $\dim \mathrm{H}^0(X, \mathcal{O}(D_0)) \geqslant 2$. Contradiction to 2.3.2. □

Definition 2.3.4. — Let V be a smooth quasi-projective algebraic variety with an ample metrized invertible sheaf $\mathcal{L}$ and $\overline{V}^{\mathcal{L}}$ the projective $\mathcal{L}$-closure of V.

The variety V is called **$\mathcal{L}$-primitive**, if the number $\alpha_{\mathcal{L}}(V)$ is rational and if for some resolution of singularities

$$\rho : X \to \overline{V}^{\mathcal{L}}$$

one has $\rho^*(L)^{\otimes \alpha_{\mathcal{L}}(V)} \otimes K_X = \mathcal{O}(D)$, where D is a rigid effective $\mathbf{Q}$-Cartier divisor on X.

Remark 2.3.5. — It is easy to see that the notion of an $\mathcal{L}$-primitive variety doesn't depend on the choice of a resolution of singularities ρ. Since V is smooth, we can always assume that the natural mapping

$$\rho : \rho^{-1}(V) \to V$$

is an isomorphism.

Example 2.3.6. — Let V_1 and V_2 be two smooth quasi-projective varieties with ample metrized invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ (resp. on V_1 and V_2). Assume that V_1 (resp. V_2) is $\mathcal{L}_1$-primitive (resp. $\mathcal{L}_1$-primitive). Then the product $V = V_1 \times V_2$ is $\mathcal{L}$-primitive, where $\mathcal{L} = \pi_1^*\mathcal{L}_1 \otimes \pi_2^*\mathcal{L}_2$.

Our main list of examples of $\mathcal{L}$-primitive varieties is obtained from canonical $\mathbf{Q}$-Fano varieties:

Example 2.3.7. — Let V be the set of nonsingular points of a canonical $\mathbf{Q}$-Fano variety W with an ample metrized invertible sheaf $\mathcal{L} = (L, \|\cdot\|)$ such that $K_W^{-1} = L^{\otimes r(W)}$ ($W = \overline{V}^{\mathcal{L}}$). Then V is an $\mathcal{L}$-primitive variety with $\mathcal{L}$-index $r(W)$. Indeed, let $\rho : X \to W$ be a resolution of singularities. By 2.2.10, we have

$$\rho^*(L)^{\otimes r(W)} \otimes K_X = \mathcal{O}(D),$$

where D is an effective $\mathbf{Q}$-Cartier divisor. Since the support of D consists of exceptional divisors with respect to ρ, D is rigid (see 2.3.2).

We expect that the above examples cover all $\mathcal{L}$-primitive varieties:

***Conjecture 2.3.8* (Canonical Q-Fano contraction)**. — *Let V be an $\mathcal{L}$-primitive variety with $\alpha_{\mathcal{L}}(V) > 0$. Then there exists a resolution of singularities $\rho : X \to \overline{V}^{\mathcal{L}}$ and a birational projective morphism $\pi : X \to W$ to a canonical $\mathbf{Q}$-Fano variety W such that $\pi^*K_W^{-1} \cong \rho^*(L)^{\alpha_{\mathcal{L}}(V)}$ (i.e., $\alpha_{\mathcal{L}}(V) = r(W)$) and the support of D ($\rho^*(L)^{\otimes r(W)} \otimes K_X = \mathcal{O}(D)$) is contained in the exceptional locus of π.*

The above conjecture is expected to follow from the Minimal Model Program using the existence and termination of flips (in particular, it holds for toric varieties).

The following statement will be important in our construction of Tamagawa numbers for $\mathcal{L}$-primitive varieties defined over a number field:

***Conjecture 2.3.9* (Vanishing)**. — *For V an $\mathcal{L}$-primitive variety such that $\alpha_{\mathcal{L}}(V) > 0$ we have*

$$\mathrm{h}^i(X, \mathcal{O}_X) = 0 \;\; \forall\, i > 0$$

for any resolution of singularities

$$\rho \,:\, X \to \overline{V}^{\mathcal{L}}$$

such that the support of the $\mathbf{Q}$-Cartier divisor $\rho^*(L)^{\otimes \alpha_{\mathcal{L}}(V)} \otimes K_X$ is a $\mathbf{Q}$-Cartier divisor with normal crossings. In particular, $\mathrm{Pic}(X)$ *is a finitely generated abelian group and one has a canonical isomorphism*

$$\mathrm{Pic}(X) \otimes \mathbf{Q} \cong \mathrm{NS}(X) \otimes \mathbf{Q}.$$

Remark 2.3.10. — Theorem 1-2-5 in [**27**] implies the vanishing for the structure sheaf for $\mathbf{Q}$-Fano varieties with canonical singularities (even with log-terminal singularities). All canonical (and log-terminal singularities) are rational and it follows that the higher cohomology of the structure sheaf on any desingularization of a canonical or a log-terminal Fano variety must also vanish (by Leray spectral sequence). Therefore, we would obtain the vanishing 2.3.9 for all $\mathcal{L}$-primitive varieties which are birationally equivalent to a Fano variety with at worst log-terminal singularities. The existence of a canonical $\mathbf{Q}$-contraction 2.3.8 would insure this.

Definition 2.3.11. — Let V be an $\mathcal{L}$-primitive variety with $\alpha_{\mathcal{L}}(V) > 0$, $\rho \,:\, X \to \overline{V}^{\mathcal{L}}$ any resolution of singularities, $D_1, \ldots, D_l$ irreducible components of the support of the rigid effective $\mathbf{Q}$-Cartier divisor D with $\mathcal{O}(D) = \rho^*(L)^{\otimes \alpha_L(V)} \otimes K_X$. We shall call

$$\mathrm{Pic}(V, \mathcal{L}) := \mathrm{Pic}(X \setminus \bigcup_{i=1}^{l} D_i)$$

the $\mathcal{L}$-**Picard group** of V. The number

$$\beta_{\mathcal{L}}(V) := \mathrm{rk}\,\mathrm{Pic}(V, \mathcal{L})$$

will be called the $\mathcal{L}$-**rank** of V. We define the $\mathcal{L}$-**cone of effective divisors**

$$\Lambda_{\mathrm{eff}}(V, \mathcal{L}) \subset \mathrm{Pic}(V, \mathcal{L}) \otimes \mathbf{R}$$

as the image of $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{NS}(X)_{\mathbf{R}} = \mathrm{Pic}(X) \otimes \mathbf{R}$ under the natural surjective $\mathbf{R}$-linear mapping

$$\tilde{\rho} \,:\, \mathrm{Pic}(X) \otimes \mathbf{R} \to \mathrm{Pic}(V, \mathcal{L}) \otimes \mathbf{R}.$$

Remark 2.3.12. — By 2.3.3, one obtains the exact sequence

$$0 \to \mathbf{Z}[D_1] \oplus \cdots \oplus \mathbf{Z}[D_l] \to \mathrm{Pic}(X) \xrightarrow{\tilde{\rho}} \mathrm{Pic}(V, \mathcal{L}) \to 0 \tag{2}$$

and therefore

$$\beta_{\mathcal{L}}(V) = \mathrm{rk}\,\mathrm{Pic}(X) - l.$$

Using these facts, it is easy to show that the group $\mathrm{Pic}(V, \mathcal{L})$ and the cone $\Lambda_{\mathrm{eff}}(V, \mathcal{L})$ do not depend on the choice of a resolution of singularities $\rho : X \to \overline{V}^{\mathcal{L}}$.

Conjecture 2.2.7 holds in dimension $n \leqslant 3$ as a consequence of the Minimal Model Program. More precisely, it is a consequence of Conjecture 2.3.8 and the following weaker statement:

***Conjecture 2.3.13* (Polyhedrality)**. — *Let V be an $\mathcal{L}$-primitive variety with*

$$\alpha_{\mathcal{L}}(V) > 0.$$

Then $\Lambda_{\mathrm{eff}}(V, \mathcal{L})$ is a rational finitely generated polyhedral cone.

Definition 2.3.14. — Let $(A, A_{\mathbf{R}}, \Lambda)$ be a triple consisting of a finitely generated abelian group A of rank k, a k-dimensional real vector space $A_{\mathbf{R}} = A \otimes \mathbf{R}$ and a convex k-dimensional finitely generated polyhedral cone $\Lambda \in A_{\mathbf{R}}$ such that $\Lambda \cap -\Lambda = 0 \in A_{\mathbf{R}}$. For $\mathrm{Re}(\mathbf{s})$ contained in the interior of the cone Λ we define the $\mathcal{X}$-function of Λ by the integral

$$\mathcal{X}_{\Lambda}(\mathbf{s}) := \int_{\Lambda^*} e^{-\langle \mathbf{s}, \mathbf{y} \rangle} \mathbf{dy}$$

where $\Lambda^* \in A^*_{\mathbf{R}}$ is the dual cone to Λ and $\mathbf{dy}$ is the Lebesgue measure on $A^*_{\mathbf{R}}$ normalized by the dual lattice $A^* \subset A^*_{\mathbf{R}}$ where $A^* := \mathrm{Hom}(A, \mathbf{Z})$.

Remark 2.3.15. — If Λ is a finitely generated rational polyhedral cone the function $\mathcal{X}_{\Lambda}(\mathbf{s})$ is a rational function in $\mathbf{s}$. However, the explicit determination of this function might pose serious computational problems.

Definition 2.3.16. — Let V be an $\mathcal{L}$-primitive smooth quasi-projective algebraic variety with a metrized invertible sheaf $\mathcal{L}$ and $\alpha_{\mathcal{L}}(V) > 0$. Let X be any resolution of singularities $\rho : X \to \overline{V}^{\mathcal{L}}$. We consider the triple

$$(\mathrm{Pic}(V, \mathcal{L}), \mathrm{Pic}(V, \mathcal{L})_{\mathbf{R}}, \Lambda_{\mathrm{eff}}(V, \mathcal{L}))$$

and the corresponding $\mathcal{X}$-function. Assuming that $\mathrm{Pic}(V, \mathcal{L})$ is a finitely generated abelian group (cf. 2.3.9) and that $\Lambda_{\mathrm{eff}}(V, \mathcal{L})$ is a polyhedral cone (cf. 2.3.13), we define the constant $\gamma_{\mathcal{L}}(V) \in \mathbf{Q}$ by

$$\gamma_{\mathcal{L}}(V) := \mathcal{X}_{\Lambda_{\mathrm{eff}}(V, \mathcal{L})}(\tilde{\rho}(-[K_X])).$$

2.4. $\mathcal{L}$-primitive fibrations and descent of metrics. — Let V be an $\mathcal{L}$-primitive variety of dimension n. We show that there exists a canonical measure on $V(\mathbf{C})$ which is uniquely defined up to a positive constant.

In order to construct this measure we choose a resolution of singularities $\rho : X \to \overline{V}^{\mathcal{L}}$ and a positive integer k_2 such that k_2D is a Cartier divisor, where $\mathcal{O}(D) \cong (\rho^*L)^{\otimes \alpha_{\mathcal{L}}(V)} \otimes K_X$. Then $k_1 = k_2\alpha_{\mathcal{L}}(V)$ is a positive integer. Let

$$g \in \mathrm{H}^0(X, \mathcal{O}(k_2D))$$

be a non-zero global section (by 2.3.2, it is uniquely defined up to a non-zero constant). We define a measure $\omega_{\mathcal{L}}(g)$ on $X(\mathbf{C})$ as follows. Choose local complex analytic coordinates $z_1, \ldots, z_n$ in some open neighborhood $U_x \subset X(\mathbf{C})$ of a point $x \in V(\mathbf{C})$. We write the restriction of the global section g to U_x as

$$g = s^{k_2\alpha_{\mathcal{L}}(V)}(dz_1 \wedge \cdots \wedge dz_n)^{\otimes k_2} = s^{k_1}(dz_1 \wedge \cdots \wedge dz_n)^{\otimes k_2},$$

where s is a local section of L . Then we set

$$\omega_{\mathcal{L}}(g) := \left(\frac{\sqrt{-1}}{2}\right)^n \|s\|^{\alpha_{\mathcal{L}}(V)}(dz_1 \wedge d\overline{z}_1) \wedge \cdots \wedge (dz_n \wedge d\overline{z}_n).$$

By a standard argument, one obtains that $\omega_{\mathcal{L}}(g)$ doesn't depend on the choice of local coordinates in U_x and that it extends to the whole complex space $X(\mathbf{C})$. It remains to notice that the restriction of the measure $\omega_{\mathcal{L}}(g)$ to $V(\mathbf{C}) \subset X(\mathbf{C})$ does not depend on the choice of ρ. So we obtain a well-defined measure on $V(\mathbf{C})$.

Remark 2.4.1. — We note that the measure $\omega_{\mathcal{L}}(g)$ depends on the choice of $g \in \mathrm{H}^0(X, \mathcal{O}(k_2D))$. More precisely, it multiplies by $|c|^{1/k_2}$ if we multiply g by some non-zero complex number c. Thus we obtain that the mapping

$$\begin{aligned} \|\cdot\|_{\mathcal{L}} \ &: \ \mathrm{H}^0(X, \mathcal{O}(k_2D)) \to \mathbf{R}_{\geqslant 0} \\ g &\mapsto \int_{X(\mathbf{C})} \omega_{\mathcal{L}}(g) = \int_{V(\mathbf{C})} \omega_{\mathcal{L}}(g) \end{aligned}$$

satisfies the property

$$\|cg\|_{\mathcal{L}} = |c|^{1/k_2}\|g\|_{\mathcal{L}} \ \ \forall c \in \mathbf{C}^*.$$

Definition 2.4.2. — Let V be a smooth quasi-projective variety with an ample metrized invertible sheaf $\mathcal{L}$, $\rho : X \to \overline{V}^{\mathcal{L}}$ a resolution of singularities. A regular projective morphism $\pi : X \to Y$ to a projective variety Y ($\dim Y < \dim X$) is called an **$\mathcal{L}$-primitive fibration on** V if there exists a Zariski dense open subset $U \subset Y$ such that the following conditions are satisfied:

(i) for any point $y \in U(\mathbf{C})$ the fiber $V_y = \pi^{-1}(y) \cap V$ is a smooth quasi-projective $\mathcal{L}$-primitive subvariety;

(ii) $\alpha_{\mathcal{L}}(V) = \alpha_{\mathcal{L}}(V_y) > 0$ for all $y \in U(\mathbf{C})$;
(iii) for any $k \in \mathbf{N}$ such that $k\alpha_{\mathcal{L}}(V) \in \mathbf{Z}$,

$$L_k := \mathrm{R}^0\pi_*\left(\rho^*(L)^{\otimes \alpha_L(V)} \otimes K_X\right)^{\otimes k}$$

is an ample invertible sheaf on Y.

We propose the following version of the Fibration Conjecture of Fujita (see [**20**]):

***Conjecture 2.4.3* (Existence of Fibrations)**. — *Let V be an arbitrary smooth quasi-projective variety with an ample metrized invertible sheaf $\mathcal{L}$ and $\alpha_{\mathcal{L}}(V) > 0$. Then there exists a resolution of singularities $\rho : X \to \overline{V}^{\mathcal{L}}$ such that X admits an $\mathcal{L}$-primitive fibration $\pi : X \to Y$ on some dense Zariski open subset $V' \subset V$.*

Remark 2.4.4. — From the viewpoint of the Minimal Model Program, Conjecture 2.4.3 is equivalent to the statement about the conjectured existence of $\mathbf{Q}$-Fano fibrations for algebraic varieties of negative Kodaira-dimension (cf. [**27**]). The existence of an $\mathcal{L}$-primitive fibration is equivalent to the fact that the graded algebra

$$\mathrm{R}(V, \mathcal{L}) = \oplus_{\nu \geqslant 0} \mathrm{H}^0(X, M^{k\nu}), \quad M := \rho^*(L)^{\otimes \alpha_L(V)} \otimes K_X$$

is finitely generated (cf. 2.4 in [**4**]). One can define Y as $\mathrm{Proj}\,\mathrm{R}(V, \mathcal{L})$ and X as a common resolution of singularities of $\overline{V}^{\mathcal{L}}$ and of the indeterminacy locus and generic fiber of the natural rational map $\overline{V}^{\mathcal{L}} \to Y$ (cf. [**26**]).

It is important to observe that a metric $\|\cdot\|$ on L induces natural metrics on all ample invertible sheaves L_k on Y:

Definition 2.4.5. — Let L_k be an ample invertible sheaf on Y as above. We define a metric $\|\cdot\|_{\mathcal{L},k}$ on L_k as follows. Let $y \in Y(\mathbf{C})$ be a closed point, $U \subset Y$ be a Zariski open subset containing y, and $s \in \mathrm{H}^0(U, L_k)$ be a section with $s(y) \neq 0$. Then we set

$$\|s(y)\|_{\mathcal{L},k} := \left(\int_{V_y(\mathbf{C})} \omega_{\mathcal{L}}(\pi^* s)\right)^k,$$

where $V_y(\mathbf{C})$ is the fiber over y of the $\mathcal{L}$-primitive fibration π^*, $\pi^* s$ the π-pullback of s restricted to $\mathcal{L}$-primitive variety $V_y(\mathbf{C})$, and $\omega_{\mathcal{L}}(\pi^* s)$ the corresponding to $\pi^* s$ volume measure on $V_y(\mathbf{C})$. We call $\|\cdot\|_{\mathcal{L},k}$ a k-**adjoint descent** to Y of a metric $\|\cdot\|$ on L.

3. Heights and asymptotic formulas

3.1. Basic terminology and notations. — Let F be a number field, $\mathcal{O}_F \subset F$ the ring of integers in F, $\mathrm{Val}(F)$ the set of all valuations of F, F_v the completion of F with respect to a valuation $v \in \mathrm{Val}(F)$, $\mathrm{Val}(F)_\infty = \{v_1, \ldots, v_r\}$ the set of all archimedean valuations of F. For any algebraic variety X over a field F we denote by $X(F)$ the set of its K-rational points.

Definition 3.1.1. — Let E be a vector space of dimension $m+1$ over F, $\mathcal{O}_E \subset E$ a projective $\mathcal{O}_F$-module of rank $m+1$ and $\|\cdot\|_{v_1}, \ldots, \|\cdot\|_{v_r}$ the set of Banach norms on the real or complex vector spaces $E_{v_i} = E \times_F F_{v_i}$ corresponding to elements of $\mathrm{Val}(F)_\infty = \{v_1, \ldots, v_r\}$. It is well-known that the above data for E define a family $\{\|\cdot\|_v,\ v \in \mathrm{Val}(F)\}$ of v-adic metrics for a standard invertible sheaf $\mathcal{O}(1)$ on $\mathbf{P}(E)$. If $x \in X(F)$ is a point and $s \in \mathrm{H}^0(U, \mathcal{O}(1))$ is a section over an open subset $U \subset \mathbf{P}(E)$ containing x, then we denote by $\|s(x)\|_v$ the corresponding v-adic norm of s at x. We set $\tilde{\mathcal{O}}(1) = (\mathcal{O}(1), \|\cdot\|_v)$ to be the standard invertible sheaf $\mathcal{O}(1)$ on $\mathbf{P}(E)$ together with a family v-adic metrics $\|\cdot\|_v$ defined by the above data and we call $\tilde{\mathcal{O}}(1)$ the **standard ample metrized invertible sheaf** on $\mathbf{P}(E)$.

Definition 3.1.2. — Let X be an algebraic variety over F. For any point $x \in X(F)$ and any regular function $f \in \mathrm{H}^0(U, \mathcal{O}_X)$ on an open subset $U \subset \mathbf{P}(E)$ containing x, we define the v-adic norm $\|f(x)\|_v := |f(x)|_v$. We call this family of v-adic metrics on $\mathcal{O}_X$ the **canonical metrization** of the structure sheaf $\mathcal{O}_X$.

Definition 3.1.3. — Let X be a quasi-projective algebraic variety, L a very ample invertible sheaf on X, $i : X \hookrightarrow \mathbf{P}(E)$ an embedding with $L = i^*\mathcal{O}(1)$. We denote by $\mathcal{L} = (L, \|\cdot\|_v)$ the sheaf L together with v-adic metrics induced from a family of v-adic metrics on the standard ample metrized invertible sheaf $\tilde{\mathcal{O}}(1)$ on $\mathbf{P}(E)$. In this situation we call $\mathcal{L}$ **a very ample metrized sheaf** on X and write $\mathcal{L} = i^*\tilde{\mathcal{O}}(1)$.

Definition 3.1.4. — Let $\mathcal{L} = (L, \|\cdot\|_v)$ be a very ample metrized invertible sheaf on X. Then for any point $x \in X(F)$, the $\mathcal{L}$-**height** of x is defined as

$$H_{\mathcal{L}}(x) = \prod_{v \in \mathrm{Val}(F)} \|s(x)\|_v^{-1},$$

where $s \in \Gamma(U, L)$ is a nonvanishing at x section of L over some open subset $U \subset X$.

Remark 3.1.5. — Using a canonical metrization of the structure sheaf $\mathcal{O}_X$, the linear mapping

$$S^k(\Gamma(U, L)) \to \Gamma(U, L^{\otimes k}) \ \ (k > 0),$$

and the F-bilinear mapping

$$\Gamma(U, L^{\otimes k}) \times \Gamma(U, L^{\otimes -k}) \to \Gamma(U, \mathcal{O}_X),$$

one immediately sees that a family of v-adic metrics on an invertible sheaf L allows to define a family of v-adic metrics on $L^{\otimes k}$ and on any invertible sheaf M such that there exist integers k_1, k_2 ($k_2 \neq 0$) with $L^{\otimes k_1} = M^{\otimes k_2}$. In this situation we write

$$\mathcal{M} = (M, \|\cdot\|_v) := (L^{\otimes k_1/k_2}, \|\cdot\|_v^{k_1/k_2}),$$

or simply $\mathcal{M} = \mathcal{L}^{k_1/k_2}$. Obviously, one obtains

$$H_{\mathcal{M}}(x) = (H_{\mathcal{L}}(x))^{k_1/k_2} \quad \forall\, x \in X(F).$$

Definition 3.1.6. — Let L be an ample invertible sheaf on a quasi-projective variety X and k a positive integer such that $L^{\otimes k}$ is very ample. We define an **ample metrized invertible sheaf** $\mathcal{L} = (L, \|\cdot\|_v)$ on X associated with L by considering $\mathcal{L}^{\otimes k} := (L^{\otimes k}, \|\cdot\|_v^k)$ as a very ample metrized invertible sheaf on X.

3.2. Weakly and strongly $\mathcal{L}$-saturated varieties. — Let V be an arbitrary quasi-projective algebraic variety over F with an ample metrized invertible sheaf $\mathcal{L}$. We always assume that $V(F)$ is infinite and set

$$N(V, \mathcal{L}, B) := \#\{x \in V(F) \ : \ H_{\mathcal{L}}(x) \leq B\}.$$

Here and in 3.4 we will work under the following

Assumption 3.2.1. — *For all quasi-projective V', V and with $V' \subset V$ and $|V(F)| = \infty$ there exists the limit*

$$\lim_{B\to\infty} \frac{N(V', \mathcal{L}, B)}{N(V, \mathcal{L}, B)}.$$

Definition 3.2.2. — We call an irreducible quasi-projective algebraic variety V with an ample metrized invertible sheaf $\mathcal{L}$ **weakly $\mathcal{L}$-saturated** if for any Zariski locally closed subset $W \subset V$ with $\dim W < \dim V$, one has

$$\lim\nolimits_{B\to\infty} \frac{N(W, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} < 1.$$

Definition 3.2.3. — We call an irreducible quasi-projective algebraic variety V with an ample metrized invertible sheaf $\mathcal{L}$ **strongly $\mathcal{L}$-saturated** if for any dense Zariski open subset $U \subset V$, one has

$$\lim\nolimits_{B\to\infty} \frac{N(U, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} = 1.$$

Definition 3.2.4. — Let V be a weakly $\mathcal{L}$-saturated variety, $W \subset V$ a locally closed strongly saturated subvariety of smaller dimension. Then we call W an $\mathcal{L}$-**target** of V, if

$$0 < \lim_{B\to\infty} \frac{N(W,\mathcal{L},B)}{N(V,\mathcal{L},B)} < 1.$$

Theorem 3.2.5. — *Let V be an arbitrary quasi-projective algebraic variety with an ample metrized invertible sheaf $\mathcal{L}$. Assume that $|V(F)| = \infty$ and that 3.2.1 holds. Then we have:*

(i) *if V is strongly $\mathcal{L}$-saturated then V is weakly $\mathcal{L}$-saturated;*

(ii) *V contains finitely many weakly $\mathcal{L}$-saturated subvarieties $W_1, \ldots, W_k$ with*

$$\lim_{B\to\infty} \frac{N(W_1 \cup \cdots \cup W_k,\mathcal{L},B)}{N(V,\mathcal{L},B)} = 1.$$

(iii) *V contains a strongly $\mathcal{L}$-saturated subvariety W having the property*

$$0 < \lim_{B\to\infty} \frac{N(W,\mathcal{L},B)}{N(V,\mathcal{L},B)} < 1.$$

(iv) *if V is weakly saturated and if it doesn't contain a dense Zariski open subset $U \subset V$ which is strongly saturated then V contains infinitely many $\mathcal{L}$-targets.*

Proof. — (i) Let $W \subset V$ be a Zariski closed subset with $\dim W < \dim V$ and $U = V \setminus W$. Then

$$\lim_{B\to\infty} \frac{N(W,\mathcal{L},B)}{N(V,\mathcal{L},B)} + \lim_{B\to\infty} \frac{N(U,\mathcal{L},B)}{N(V,\mathcal{L},B)} = \lim_{B\to\infty} \frac{N(V,\mathcal{L},B)}{N(V,\mathcal{L},B)} = 1.$$

Since V is strongly $\mathcal{L}$-saturated, we have

$$\lim_{B\to\infty} \frac{N(U,\mathcal{L},B)}{N(V,\mathcal{L},B)} = 1$$

and therefore

$$\lim_{B\to\infty} \frac{N(W,\mathcal{L},B)}{N(V,\mathcal{L},B)} = 0 < 1.$$

(ii) Let $W \subset V$ be a minimal Zariski closed subset such that

$$\lim_{B\to\infty} \frac{N(W,\mathcal{L},B)}{N(V,\mathcal{L},B)} = 1$$

and $W_1, \ldots, W_k$ irreducible components of W. It immediately follows from the minimality of W that each W_i is weakly saturated.

(iii) Let $W \subset V$ be an irreducible Zariski closed subset of minimal dimension such that

$$\lim_{B\to\infty} \frac{N(W,\mathcal{L},B)}{N(V,\mathcal{L},B)} = 1.$$

The minimality of W implies that W is weakly saturated.

(iv) By (iii) the set of $\mathcal{L}$-targets is nonempty. Assume that the set of all $\mathcal{L}$-targets is finite: $\{W_1, \dots, W_k\}$. The strong saturatedness of each W_i implies that

$$\lim_{B\to\infty} \frac{N(W_{i_1} \cap \cdots \cap W_{i_l}, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} = 0$$

for all pairwise different $i_1, \dots, i_l \in \{1, \dots, k\}$ and $l \geqslant 2$. In particular, one has

$$\sum_{i=1}^{k} \lim_{B\to\infty} \frac{N(W_i, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} = \lim_{B\to\infty} \frac{N(W_1 \cup \cdots \cup W_k, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} < 1.$$

We set $U := V \setminus (W_1 \cup \cdots \cup W_k)$. Then

$$\lim_{B\to\infty} \frac{N(U, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} > 0.$$

Since U is not strongly saturated, there exists an irreducible Zariski closed subset $W_0 \subset V$ of minimal dimension $< \dim V$ such that

$$\lim_{B\to\infty} \frac{N(W_0 \cap U, \mathcal{L}, B)}{N(U, \mathcal{L}, B)} > 0.$$

It follows from the minimality of W_0 that W_0 is strongly saturated. On the other hand, one has

$$\lim_{B\to\infty} \frac{N(W_0, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} \geqslant \lim_{B\to\infty} \frac{N(W_0 \cap U, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} > 0,$$

i.e., W_0 is an $\mathcal{L}$-target and $W_0 \notin \{W_1, \dots, W_k\}$. Contradiction. □

Definition 3.2.6. — Let V be a weakly $\mathcal{L}$-saturated variety and $W_1, W_2, \dots$ an infinite sequence of strongly saturated irreducible subvarieties W_i having the property

$$0 < \theta_i := \lim_{B\to\infty} \frac{N(W_i, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} < 1 \ \ \forall i > 0.$$

We say that the set $\{W_1, W_2, \dots\}$ forms an **asymptotic arithmetic $\mathcal{L}$-fibration** on V, if the following equality holds

$$\sum_{i=1}^{\infty} \theta_i = 1.$$

We expect that the main source of examples of weakly and strongly saturated varieties should come from the following situation:

Proposition 3.2.7. — *Assume 3.2.1 and let $G \subset PGL(n+1)$ be a connected linear algebraic group acting on $\mathbf{P}^n$ and $V := Gx \subset \mathbf{P}^n$ a G-orbit of a point $x \in \mathbf{P}^n(F)$. Then V is weakly $\tilde{O}(1)$-saturated.*

Proof. — Let $W \subset V$ be an arbitrary locally closed subset with $\dim W < \dim V$, $\overline{W} \subset V$ its Zariski closure in V and $U := V \setminus \overline{W} \subset V$ the corresponding dense Zariski open subset of V. Then V is covered by the open subsets gU, where g runs over all elements in $G(F)$ (this follows from the fact that G is unirational and that $G(F)$ is Zariski dense in G [**10**]). Therefore, the orbit of $x \in V(F)$ under $G(F)$ is Zariski dense in V. Since the Zariski topology is noetherian we can choose a finite subcovering: $V = \bigcup_{i=1}^k g_i U$ (g_i in $G(F)$). Considering $g_i \in G(F)$ as matrices in $PGL(n+1)$ and using standard properties of heights [**29**], one obtains positive constants c_i such that

$$H_{\mathcal{L}}(g_i(x)) \leq c_i H_{\mathcal{L}}(x)$$

for all $x \in \mathbf{P}^n(F)$. It is clear that for $c_0 := \sum_{i=1}^k c_i$ we have

$$N(U, \mathcal{L}, B) \leq N(V, \mathcal{L}, B) \leq c_0 N(U, \mathcal{L}, B).$$

It follows that

$$\frac{1}{c_0} \leqslant \lim_{B\to\infty} \frac{N(U, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} \leqslant 1.$$

Hence

$$\lim_{B\to\infty} \frac{N(W, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} \leqslant \lim_{B\to\infty} \frac{N(\overline{W}, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} \leqslant 1 - \frac{1}{c_0} < 1.$$

□

We can reformulate the statement of 3.2.7 as follows:

Proposition 3.2.8. — *Let G be a connected linear algebraic group, $H \subset G$ a closed subgroup and $V := G/H$. If 3.2.1 holds then V is weakly saturated with respect to any G-equivariant projective embedding of V.*

It is easy to see that $V = G/H$ is not necessarily weakly saturated with respect to projective embeddings which are not G-equivariant:

Example 3.2.9. — Let $S \subset \mathbf{P}^8$ be the anticanonically embedded Del Pezzo surface which is a blow up of a rational point in $\mathbf{P}^2$. Denote by $\mathcal{L}$ the metrized anticanonical sheaf on S. The unique exceptional curve $C \subset S$ is contained in the union of two open subsets $U_0, U_1 \subset S$ where $U_0 \cong U_1 \cong \mathbf{A}^2$. Therefore, S can be considered as a projective compactification of the algebraic group $\mathbf{G}_a^2$ (after an identification of $\mathbf{G}_a^2$ with U_0 or U_1). This compactification is not $\mathbf{G}_a^2$-equivariant. One has

$$a_{\mathcal{L}}(U_0) = a_{\mathcal{L}}(U_1) = a_{\mathcal{L}}(C) = 2,$$

but

$$a_{\mathcal{L}}(U_0 \setminus C) = a_{\mathcal{L}}(U_1 \setminus C) = 1.$$

Hence, U_0 and U_1 are not weakly $\mathcal{L}$-saturated.

It is easy to show that an equivariant compactification of G/H is not necessarily strongly $\mathcal{L}$-saturated:

Example 3.2.10. — Let $V = \mathbf{P}^1 \times \mathbf{P}^1$. Then V is a G-homogeneous variety with $G = GL(2) \times GL(2)$. However, V is not strongly $\mathcal{L}$-saturated for $L := \pi_1^*\mathcal{O}(k_1) \otimes \pi_2^*\mathcal{O}(k_2)$ $(k_1, k_2 \in \mathbf{N})$, if $k_1 \neq k_2$.

3.3. Adelic $\mathcal{L}$-measure and $\tau_{\mathcal{L}}(V)$. — Now we define an adelic measure $\omega_{\mathcal{L}}$ corresponding to an ample metrized invertible sheaf $\mathcal{L}$ on an $\mathcal{L}$-primitive variety V with $\alpha_{\mathcal{L}}(V) > 0$ which satisfies the assumption 2.3.9. This is a generalization of a construction due to Peyre ([**30**]) for V being a smooth projective variety and $\mathcal{L}$ the metrized canonical line bundle, which in its turn is a generalization of the classical construction of Tamagawa measures on the adelic points of algebraic groups.

Let V be an $\mathcal{L}$-primitive variety of dimension n, $\rho : X \to \overline{V}^{\mathcal{L}}$ a resolution of singularities, k_2 a positive integer such that $k_1 = k_2\alpha_{\mathcal{L}}(V) \in \mathbf{Z}$ and

$$(\rho^*(L)^{\alpha_{\mathcal{L}}(V)} \otimes K_X)^{\otimes k_2} \cong \mathcal{O}(D),$$

where D is a rigid effective Cartier divisor on X.

Definition 3.3.1. — Let g be a non-zero element of the 1-dimensional F-vector space $\mathrm{H}^0(X, \mathcal{O}(D))$ (g is defined uniquely up to an element of F^*). Let $v \in \mathrm{Val}(F)$. We define a measure $\omega_{\mathcal{L},v}(g)$ on $V(F_v)$ as follows. Choose local v-analytic coordinates $x_{1,v}, \ldots, x_{n,v}$ in some open neighborhood $U_x \subset X(F_v)$ of a point $x \in X(F_v)$. We write the restriction of the global section g to U_x as

$$g = s^{k_1}(dx_{1,v} \wedge \cdots \wedge dx_{n,v})^{k_2}$$

where s is a local section of L. Define a v-adic measure on U_x as

$$\omega_{\mathcal{L},v}(g) := \|s\|_v^{k_1/k_2} dx_{1,v} \cdots dx_{n,v} = \|s\|_v^{\alpha_{\mathcal{L}}(V)} dx_{1,v} \cdots dx_{n,v},$$

where $dx_{1,v} \cdots dx_{n,v}$ is the usual normalized Haar measure on F_v^n. By a standard argument, one obtains that $\omega_{\mathcal{L},v}(g)$ doesn't depend on the choice of local coordinates in U_x and that it extends to the whole v-adic space $X(F_v)$. The restriction of $\omega_{\mathcal{L},v}(g)$ doesn't depend on the choice of ρ. So we obtain a well-defined v-adic measure on $V(F_v)$.

Remark 3.3.2. — We remark that $\omega_{\mathcal{L},v}(g)$ depends on the choice of a global section $g \in \mathrm{H}^0(X, \mathcal{O}(D))$: if $g' = cg$ ($c \in F^*$) is another global section, then

$$\omega_{\mathcal{L},v}(g') = |c|_v^{1/k}\omega_{\mathcal{L},v}(g).$$

Our next goal is to obtain an explicit formula for the integral

$$d_v(V) := \int_{V(F_v)} \omega_{\mathcal{L},v}(g) = \int_{X(F_v)} \omega_{\mathcal{L},v}(g)$$

for almost all $v \in \mathrm{Val}(F)$. We use the p-adic integral formula of Denef ([**16**] Th. 3.1) in the same way as the p-adic formula of A. Weil ([**35**] Th. 2.2.3) was used by Peyre in [**30**].

Remark 3.3.3. — By [**1, 26**], we can choose ρ in such a way that ρ is defined over F and all irreducible components $D_1, \ldots, D_l$ $(l \geqslant 0)$ of the support of $D = \sum_{i=1}^{l} m_i D_i$ are smooth divisors with normal crossings over the algebraic closure $\overline{F}$.

Definition 3.3.4. — Let G be the image of $\mathrm{Gal}(\overline{F}/F)$ in the symmetric group S_l that acts by permutations on $D_1, \ldots, D_l$. We set $I := \{1, \ldots, l\}$ and denote by I/G the set of all G-orbits in I. For any $J \subset I$ we set

$$D_J := \begin{cases} \bigcap_{j \in J} D_j, & \text{if } J \neq \varnothing \\ X \setminus \bigcup_{j \in I} D_j, & \text{if } J = \varnothing \end{cases},$$

$$D_J^\circ = D_J \setminus \bigcup_{j \notin J} D_j.$$

(D_J is defined over F if and only if J is a union of some of G-orbits in I/G.).

We can extend X to a projective scheme $\mathcal{X}$ of finite type over $\mathcal{O}_F$ and divisors $D_1, \ldots, D_l$ to codimension-1 subschemes $\mathcal{D}_1, \ldots, \mathcal{D}_l$ in $\mathcal{X}$ such that for almost all non-archimedean $v \in \mathrm{Val}(F)$ the reductions of X and $\mathcal{D}_1, \ldots, \mathcal{D}_l$ modulo $\wp_v \subset \mathcal{O}_F$ are smooth projective varieties $\mathcal{X}_v$ and $\mathcal{D}_{v,1}, \ldots, \mathcal{D}_{v,l}$ over the algebraic closure $\overline{k_v}$ of the residue field k_v with $\mathcal{D}_{v,i} \neq \mathcal{D}_{v,j}$ for $i \neq j$. Moreover, we can assume that

$$\mathcal{D}_{v,J} := \begin{cases} \bigcap_{j \in J} \mathcal{D}_{v,j}, & \text{if } J \neq \varnothing \\ X \setminus \bigcup_{j \in I} \mathcal{D}_{v,j}, & \text{if } J = \varnothing \end{cases}$$

are also smooth over $\overline{k_v}$.

Definition 3.3.5. — A non-archimedean valuation $v \in \mathrm{Val}(F)$ which satisfies all the above assumptions will be called a **good valuation** for the pair $(\mathcal{X}, \{\mathcal{D}_i\}_{i \in I})$.

Definition 3.3.6. — Let $G_v \subset G$ be a cyclic subgroup generated by a representative of the Frobenius element in $\mathrm{Gal}(\overline{k_v}/k_v)$. We denote by I/G_v the set of all G_v-orbits in I. If $j \in I/G_v$, then we set b_j to be the length of the corresponding G_v-orbit and put $r_j = m_j/k_2$, where m_j is the multiplicity of irreducible components of D corresponding to the G_v-orbit j.

The following theorem is a slightly generalized version of Th. 3.1 in [**16**]:

Theorem 3.3.7. — *Let $v \in \mathrm{Val}(F)$ be a good non-archimedean valuation for*

$$(\mathcal{X}, \{\mathcal{D}_i\}_{i\in I}).$$

Then

$$d_v(V) = \int_{X(F_v)} \omega_{\mathcal{L},v}(g) = \frac{c_\varnothing}{q_v^n} + \frac{1}{q_v^n} \sum_{\varnothing \neq J \subset I/G_v} c_J \prod_{j\in J} \left(\frac{q_v^{b_j} - 1}{q_v^{b_j(r_j+1)} - 1} \right),$$

where q_v is the cardinality of k_v, and c_J is the cardinality of the set of k_v-rational points in $\mathcal{D}_J^\circ$.

Let us consider an exact sequence of Galois $\mathrm{Gal}(\overline{F}/F)$-modules:

$$0 \to \mathbf{Z}[\mathcal{D}_1] \oplus \cdots \oplus \mathbf{Z}[\mathcal{D}_l] \to \mathrm{Pic}(\mathcal{X}) \xrightarrow{\tilde{\rho}} \mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i) \to 0 \tag{3}$$

Theorem 3.3.8. — *Assume that X has the property* $\mathrm{h}^1(X, \mathcal{O}_X) = 0$. *Then*

$$d_v(V) = 1 + \frac{1}{q_v} \mathrm{Tr}(\Phi_v | \mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i) \otimes \mathbf{Q}_l) + O\left(\frac{1}{q_v^{1+\varepsilon}}\right),$$

where

$$\varepsilon = \min\{1/2, r_1, \ldots, r_l\}$$

and Φ_v is the Frobenius morphism.

Proof. — By conjectures of Weil proved by Deligne [**15**], one has

$$\frac{c_J}{q_v^n} = O\left(\frac{1}{q_v}\right) \quad J \neq \varnothing$$

(since $\dim D_J \leqslant n-1$ for $J \neq \varnothing$) and

$$\frac{c_\varnothing}{q_v^n} = \sum_{k=0}^{2n} (-1)^k \mathrm{Tr}(\Phi_v | \mathrm{H}_c^k(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i, \mathbf{Q}_l)),$$

where $\mathrm{H}_c^k(\cdot, \mathbf{Q}_l)$ denotes the étale cohomology group with compact supports. Using long cohomology sequence of the pair $(\mathcal{X}, \bigcup_i \mathcal{D}_i)$, one obtains isomorphisms

$$\mathrm{H}_c^k(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i, \mathbf{Q}_l) = \mathrm{H}_c^k(\mathcal{X}, \mathbf{Q}_l) \text{ for } k = 2n, 2n-1$$

and the short exact sequence

$$0 \to \mathrm{H}_c^{2n-2}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i, \mathbf{Q}_l) \to \mathrm{H}_c^{2n-2}(\mathcal{X}, \mathbf{Q}_l) \to \bigoplus_{i=1}^{l} \mathrm{H}_c^{2n-2}(\mathcal{D}_i, \mathbf{Q}_l) \to 0.$$

Using isomorphisms

$$\mathrm{H}_c^{2n-2}(\mathcal{D}_i, \mathbf{Q}_l(n-1)) \cong \mathbf{Q}_l, \mathrm{H}_c^{2n}(\mathcal{X}, \mathbf{Q}_l(n)) \cong \mathbf{Q}_l,$$

Poincaré duality

$$\mathrm{H}_c^{2n-2}(\mathcal{X}, \mathbf{Q}_l(n-1)) \times \mathrm{Pic}(\mathcal{X}) \xrightarrow{\sim} \mathbf{Q}_l$$

and the vanishing property $\mathrm{h}^1(X, \mathcal{O}_X) = 0$, we obtain

$$\mathrm{H}_c^{2n-1}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i, \mathbf{Q}_l) = 0$$

and

$$\begin{aligned}\frac{c_\varnothing}{q_v^n} &= 1 + \frac{1}{q_v}\mathrm{Tr}(\Phi_v|\mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i) \otimes \mathbf{Q}_l) \\ &+ \sum_{k=0}^{2n-3} \frac{(-1)^k}{q_v^n}\mathrm{Tr}(\Phi_v|\mathrm{H}_c^k(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i, \mathbf{Q}_l)) \\ &= 1 + \frac{1}{q_v}\mathrm{Tr}(\Phi_v|\mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i) \otimes \mathbf{Q}_l) + O\left(\frac{1}{q_v^{3/2}}\right).\end{aligned}$$

On the other hand, for $J \neq \varnothing$ one has

$$\prod_{j\in J} \left(\frac{q_v^{b_j} - 1}{q_v^{b_j(r_j+1)} - 1} \right) = O\left(\frac{1}{q_v^{1+\varepsilon_0}}\right),$$

where $\varepsilon_0 = \min\{r_1, \ldots, r_l\}$. □

Definition 3.3.9. — We define the convergency factors

$$\lambda_v := \begin{cases} L_v(1, \mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i)), & \text{if } v \text{ is good} \\ 1 & \text{otherwise} \end{cases}$$

where L_v is the local factor of the Artin L-function corresponding to the G-module $\mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i)$ and we set

$$\omega_{\mathcal{L},S} := \sqrt{|\mathrm{disc}(F)|}^{-n} \prod_{v\in\mathrm{Val}(F)} \lambda_v^{-1}\omega_{\mathcal{L},v}(g),$$

where $\mathrm{disc}(F)$ is the absolute discriminant of $\mathcal{O}_F$ and S is the set of bad valuations.

By the product formula, $\omega_{\mathcal{L},S}$ doesn't depend on the choice of g.

Definition 3.3.10. — Denote by $\mathbf{A}_F$ the adele ring of F. Let $\overline{X(F)}$ be the closure of $X(F)$ in $X(\mathbf{A}_F)$ (in direct product topology). Under the vanishing assumption $\mathrm{h}^1(X, \mathcal{O}_X) = 0$, we define the constant

$$\tau_{\mathcal{L}}(V) = \lim_{s\to 1}(s-1)^{\beta_{\mathcal{L}}(V)} L_S(s, \mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i)) \int_{\overline{X(F)}} \omega_{\mathcal{L},S},$$

where $\beta_{\mathcal{L}}(V)$ is the rank of the submodule of $\mathrm{Gal}(\overline{F}/F)$-invariants of the module $\mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i)$.

3.4. Main strategy. — Now we proceed to discuss our main strategy in understanding the asymptotic for the number $N(V, \mathcal{L}, B)$ as $B \to \infty$ for an arbitrary $\mathcal{L}$-polarized quasi-projective variety. Again, we shall make the assumption 3.2.1. Our approach consists in 4 steps including 3 subsequent simplifications of the situation:

Step 1 (reduction to weakly $\mathcal{L}$-saturated varieties): By 3.2.5 (ii), every quasi-projective $\mathcal{L}$-polarized variety V contains a finite number of weakly $\mathcal{L}$-saturated varieties $W_1, \dots W_k$ such that

$$\lim_{B\to\infty} \frac{N(W, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} = 1.$$

Therefore, it would be enough to understand separately the asymptotics of

$$N(W_i, \mathcal{L}, B), i \in \{1, \dots, k\}$$

modulo the asymptotics

$$N(W_{i_1} \cap \dots \cap W_{i_l}, \mathcal{L}, B)$$

for low-dimensional subvarieties $W_{i_1} \cap \dots \cap W_{i_l}$, where $\{i_1, \dots, i_l\} \subset \{1, \dots, k\}$ are subsets of pairwise different elements with $l \geqslant 2$.

For our next reduction step, we need:

Conjecture 3.4.1. — *Let V be a weakly $\mathcal{L}$-saturated variety which doesn't contain an open Zariski dense and strongly $\mathcal{L}$-saturated subset $U \subset V$. Then the set of $\mathcal{L}$-targets of V forms an asymptotic arithmetic $\mathcal{L}$-fibration.*

Step 2 (reduction to strongly $\mathcal{L}$-saturated varieties): Let V be an arbitrary weakly $\mathcal{L}$-saturated variety. Then either V contains a strongly $\mathcal{L}$-saturated Zariski open subset or, according to 3.4.1, we obtain an asymptotic arithmetic $\mathcal{L}$-fibration of V by $\mathcal{L}$-targets. In the first situation, it is enough to understand the asymptotic of $N(U, \mathcal{L}, B)$ for the strongly $\mathcal{L}$-saturated variety U (we note that the complement $V \setminus U$ consists of low-dimensional irreducible components). In the second situation, it is enough to understand the asymptotic of $N(W_i, \mathcal{L}, B)$ for each of the $\mathcal{L}$-targets $W_i \subset V$.

For our next reduction step, we need:

Conjecture 3.4.2. — *Let V be a smooth strongly $\mathcal{L}$-saturated quasi-projective variety. Then the complex analytic variety $V(\mathbf{C})$ is $\mathcal{L}$-primitive.*

Step 3 (reduction to $\mathcal{L}$-primitive varieties): Every quasi-projective algebraic variety V is a disjoint union of finitely many locally closed smooth subvarieties V_i. Therefore, if one knows the asymptotic for each $N(V_i, \mathcal{L}, B)$ then one immediately obtains the asymptotic for $N(V, \mathcal{L}, B)$.

Definition 3.4.3. — Let V be an $\mathcal{L}$-primitive algebraic variety over a number field F, $\rho : X \to \overline{V}^{\mathcal{L}}$ a desingularization over F of the closure of V with the exceptional locus consisting of smooth irreducible divisors $D_1, \ldots, D_l$. We consider $\mathrm{Pic}(X)$ and $\mathrm{Pic}(V, \mathcal{L})$ as $\mathrm{Gal}(\overline{F}/F)$-modules and we denote by $\beta_{\mathcal{L}}(V)$ the rank of $\mathrm{Gal}(\overline{F}/F)$-invariants in $\mathrm{Pic}(V, \mathcal{L})$ and by $\delta_{\mathcal{L}}(V)$ the cardinality of the cohomology group

$$\mathrm{H}^1(\mathrm{Gal}(\overline{F}/F), \mathrm{Pic}(V, \mathcal{L})).$$

Remark 3.4.4. — Using the long exact cohomology sequence associated with (3), one immediately obtains that $\beta_{\mathcal{L}}(V)$ and $\delta_{\mathcal{L}}(V)$ do not depend on the choice of the resolution ρ.

Step 4 (expected asymptotic formula): Let V be a strongly $\mathcal{L}$-saturated (and $\mathcal{L}$-primitive) smooth quasi-projective variety. Assume that $a_{\mathcal{L}}(V) > 0$. Then we expect that the following asymptotic formula holds:

$$N(V, \mathcal{L}, B) = c_{\mathcal{L}}(V) B^{\alpha_{\mathcal{L}}(V)} (\log B)^{\beta_{\mathcal{L}}(V)-1} \left(1 + o(1)\right),$$

where

$$c_{\mathcal{L}}(V) := \frac{\gamma_{\mathcal{L}}(V)}{\alpha_{\mathcal{L}}(V)(\beta_{\mathcal{L}}(V)-1)!} \delta_{\mathcal{L}}(V) \tau_{\mathcal{L}}(V),$$

$\gamma_{\mathcal{L}}(V)$ is an invariant of the triple

$$(\mathrm{Pic}(V, \mathcal{L}), \mathrm{Pic}(V, \mathcal{L})_{\mathbf{R}}, \Lambda_{\mathrm{eff}}(V, \mathcal{L}))$$

(2.3.16), $\delta_{\mathcal{L}}(V)$ is a cohomological invariant of the $\mathrm{Gal}(\overline{F}/F)$-module $\mathrm{Pic}(V, \mathcal{L})$ (3.4.3) and $\tau_{\mathcal{L}}(V)$ is an adelic invariant of a family of v-adic metrics $\{\| \cdot \|_v\}$ on L (3.3.10).

In sections 4 and 5 we discuss some examples which show how the constants

$$\alpha_{\mathcal{L}}(V), \beta_{\mathcal{L}}(V), \delta_{\mathcal{L}}(V), \gamma_{\mathcal{L}}(V), \tau_{\mathcal{L}}(V)$$

appear in asymptotic formulas for the number of rational points of bounded $\mathcal{L}$-height on algebraic varieties. Naturally, we expect that the exhibited behavior is typical. However, we also feel that one should collect more examples which could help to clarify the general situation.

3.5. $\mathcal{L}$-primitive fibrations and $\tau_{\mathcal{L}}(V)$. — We proceed to discuss our observations concerning the arithmetic conjecture 3.4.1 and its relation to the geometric conjecture 2.4.3.

Let V be a weakly $\mathcal{L}$-saturated smooth quasi-projective variety with $a_{\mathcal{L}}(V) > 0$ which is not strongly saturated and which doesn't contain Zariski open dense strongly saturated subvarieties. We distinguish the following two cases:

Case 1. V is not $\mathcal{L}$-primitive. In this case we expect that some Zariski open dense subset $U \subset V$ admits an $\mathcal{L}$-primitive fibration which is defined by a projective regular morphism $\pi : X \to Y$ over F to a low-dimensional normal irreducible projective variety Y satisfying the conditions (i)-(iii) in 2.4.2, for an appropriate smooth projective compactification X of U (see 2.4.3). It seems natural to expect that all fibers satisfy the vanishing assumption 2.3.9. Thus we see that for any $y \in Y(F)$ such that $V_y = \pi^{-1}(y) \cap V$ is $\mathcal{L}$-primitive we can define the adelic number $\tau_{\mathcal{L}}(V_y)$. Furthermore, we expect that every $\mathcal{L}$-target W is contained in an appropriate $\mathcal{L}$-primitive subvariety V_y which is a fiber of the $\mathcal{L}$-primitive fibration $\pi : V \to U$ on V. In particular, Step 4 of our main strategy implies that if every $\mathcal{L}$-target W *coincides* with a suitable $\mathcal{L}$-primitive fiber V_y then one should expect the asymptotic

$$N(V, \mathcal{L}, B) = c_{\mathcal{L}}(V) B^{\alpha_{\mathcal{L}}(V)} (\log B)^{\beta_{\mathcal{L}}(V)-1} (1 + o(1)),$$

where the numbers $\alpha_{\mathcal{L}}(V)$ (resp. $\beta_{\mathcal{L}}(V)$) coincide with the numbers $\alpha_{\mathcal{L}}(V_y)$ (resp. $\beta_{\mathcal{L}}(V_y)$) for the corresponding $\mathcal{L}$-targets V_y and the constant $c_{\mathcal{L}}(V)$ is equal to the sum

$$\sum_y c_{\mathcal{L}}(V_y) = \sum_y \frac{\gamma_{\mathcal{L}}(V_y)}{\alpha_{\mathcal{L}}(V_y)(\beta_{\mathcal{L}}(V_y) - 1)!} \delta_{\mathcal{L}}(V_y) \tau_{\mathcal{L}}(V_y),$$

where y runs over all points in $Y(F)$ such that V_y is an $\mathcal{L}$-target of V. It is natural to try to understand the dependence of $\tau_{\mathcal{L}}(V_y)$ on the choice of a point $y \in Y(F)$. We expect that the number $\tau_{\mathcal{L}}(V_y)$ can be interpreted as a "height" of y. More precisely, the examples we considered suggest the following:

Conjecture 3.5.1. — *There exist a family of v-adic metrics on K_Y and two positive constants $c_2 > c_1 > 0$ such that*

$$c_1 H_{\mathcal{F}}(y) \leqslant \tau_{\mathcal{L}}(V_y) \leqslant c_2 H_{\mathcal{F}}(y) \ \ \forall y \in Y(F) \cap U,$$

where $U \subset Y$ is some dense Zariski open subset and $\mathcal{F}$ is a metrized $\mathbf{Q}$-invertible sheaf associated with the $\mathbf{Q}$-Cartier divisor $L_1^{-1} \otimes K_Y$ (recall that L_1 is the tautological ample $\mathbf{Q}$-Cartier divisor on Y defined by the graded ring $\mathrm{R}(V, \mathcal{L})$ *(see 2.4.4)).*

Case 2. V is $\mathcal{L}$-primitive (but not strongly saturated!). We don't know examples of a precise asymptotic formula in this situation.

Example 3.5.2. — Let V be a Fano diagonal cubic bundle over $\mathbf{P}^3$ with the homogeneous coordinates $(X_0 : X_1 : X_2 : X_3)$ defined as a hypersurface in $\mathbf{P}^3 \times \mathbf{P}^3$ by the equation

$$X_0 Y_0^3 + X_1 Y_1^3 + X_2 Y_2^3 + X_3 Y_3^3 = 0$$

in $\mathbf{P}^3 \times \mathbf{P}^3$ (see [**9**]). We expect that V is not strongly saturated with respect to a metrized anticanonical sheaf $\mathcal{L} := \mathcal{O}(3,1)$ and that the corresponding $\mathcal{L}$-targets are the splitting diagonal cubics in fibers of the natural projection $\pi : V \to \mathbf{P}^3$ (this leads to the failure of the expected asymptotic formula in Step 4 for this example).

The next example was suggested to us by Colliot-Thélène:

Example 3.5.3. — Let V be an analogous diagonal quadric bundle over $\mathbf{P}^3$ defined as a hypersurface in $\mathbf{P}^3 \times \mathbf{P}^3$ by the equation

$$X_0 Y_0^2 + X_1 Y_1^2 + X_2 Y_2^2 + X_3 Y_3^2 = 0.$$

For infinitely many fibers $V_x = \pi_1^{-1}(x)$ $(x \in \mathbf{P}^3(F))$ we have $\operatorname{rk}\operatorname{Pic}(V_x) = 2$. At the same time, we have also $\operatorname{rk}\operatorname{Pic}(V) = 2$. We consider the height function associated to some metrization of the line bundle $\mathcal{L} := \mathcal{O}(3,2)$. On the one hand, we think that the asymptotic on the whole variety is $c(V)B\log B(1 + o(1))$ for $B \to \infty$ with some $c(V) > 0$. On the other hand, if $X_0X_1X_2X_3$ is a square in F we get already about $B\log B$ solutions. Another important observation is the *expected* convergency of the series

$$\sum_{x\in\mathbf{P}^3(F):V_x\cong\mathbf{P}^1\times\mathbf{P}^1} c(V_x).$$

The latter would be a consequence of the following two facts. First, the condition $V_x \cong \mathbf{P}^1\times\mathbf{P}^1$ $(x = (X_0 : X_1 : X_2 : X_3))$ is equivalent to the conditions that V_x contains an F-rational point and that the product $X_0X_1X_2X_3$ is a square in F. The number of F-rational points $x' = (X_0 : X_1 : X_2 : X_3 : Z)$ with $H_{\mathcal{O}(1)}(x') \leqslant B$ lying on the hypersurface with the equation $X_0X_1X_2X_3 = Z^2$ in the weighted projective space $\mathbf{P}^4(1,1,1,1,2)$ can be estimated from above by $B^2(\log B)^3(1 + o(1))$.

Secondly, we expect

$$c(V_x) = H^{-1}_{\mathcal{O}(4)}(x)(1 + o(1)),$$

uniformly over the base $\mathbf{P}^3(F)$. This would imply the claimed convergency.

4. Height zeta-functions

4.1. Tauberian theorem. — One of the main techniques in the proofs of asymptotic formulas for the counting function

$$N(V, \mathcal{L}, B) := \#\{x \in V(F) \ : \ H_{\mathcal{L}}(x) \leq B\}$$

has been the use of *height zeta functions*. Let $\mathcal{L}$ be an ample metrized invertible sheaf on a smooth quasi-projective algebraic variety X. We define the height zeta function by the series

$$Z(X, \mathcal{L}, s) := \sum_{x \in X(F)} H_{\mathcal{L}}(x)^{-s}$$

which converges absolutely for $\mathrm{Re}(s) \gg 0$. After establishing the analytic properties of $Z(X, \mathcal{L}, s)$ one uses the following version of a Tauberian theorem:

Theorem 4.1.1. — *([**14**]) Suppose that there exist an $\varepsilon > 0$ and a real number $\Theta(\mathcal{L}) > 0$ such that*

$$Z(X, \mathcal{L}, s) = \frac{\Theta(\mathcal{L})}{(s-a)^b} + \frac{f(s)}{(s-a)^{b-1}}$$

for some $a > 0$, $b \in \mathbf{N}$ and some function $f(s)$ which is holomorphic for $\mathrm{Re}(s) > a - \varepsilon$. Then we have the following asymptotic formula

$$N(X, \mathcal{L}, B) = \frac{\Theta(\mathcal{L})}{a \cdot (b-1)!} B^a (\log B)^{b-1}(1 + o(1))$$

for $B \to \infty$.

4.2. Products. — Let X_1 and X_2 be two smooth quasi-projective varieties with ample metrized invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ (resp. on X_1 and X_2). Denote by $X = X_1 \times X_2$ the product and by $\mathcal{L}$ the product of $\mathcal{L}_1$ and $\mathcal{L}_2$ (with the obvious metrization). Clearly,

$$Z(X, \mathcal{L}, s) = Z(X_1, \mathcal{L}_1, s) \cdot Z(X_2, \mathcal{L}_2, s).$$

Assume that for $i = 1, 2$ we have

$$Z(X_i, \mathcal{L}_i, s) = \frac{\Theta(\mathcal{L}_i)}{(s - \alpha_{\mathcal{L}_i}(X_i))^{\beta_{\mathcal{L}_i}(X_i)}} + \frac{f_i(s)}{(s - \alpha_{\mathcal{L}_i}(X_i))^{\beta_{\mathcal{L}_i}(X_i)-1}}$$

with some functions $f_i(s)$ which are holomorphic in the domains

$$\mathrm{Re}(s_i) > \alpha_{\mathcal{L}_i}(X_i) - \varepsilon$$

for some $\varepsilon > 0$. There are two possibilities:

Case 1: $\alpha_{\mathcal{L}_1}(X) = \alpha_{\mathcal{L}_2}(X)$. In this situation the constant $\Theta(\mathcal{L})$ at the pole of highest order $\beta_{\mathcal{L}_1}(X_1) + \beta_{\mathcal{L}_2}(X_2)$ is given by $\Theta(\mathcal{L}) = \Theta(\mathcal{L}_1)\Theta(\mathcal{L}_2)$.

Case 2: $\alpha_{\mathcal{L}_1}(X) < \alpha_{\mathcal{L}_2}(X)$. In this situation the constant is a sum

$$\Theta(\mathcal{L}) = \sum_{x \in X_1(F)} H_{\mathcal{L}_1}^{-\alpha_{\mathcal{L}_2}}(x)\Theta(\mathcal{L}_2).$$

Consider the projection $X \to X_1$ and denote by V_x the fiber over $x \in X_1(F)$. We notice

$$\tau_{\mathcal{L}}(V_x) = H_{\mathcal{L}_1}^{-\alpha_{\mathcal{L}_2}}(x)\tau_{\mathcal{L}_2}(X_2).$$

We denote by M the $\mathbf{Q}$-Cartier divisor $\pi_1^*\mathcal{L}_1^{\alpha_{\mathcal{L}_2}} \otimes K_{V_1}$. We obtain that $\pi_1^*\mathcal{L}_1^{\alpha_{\mathcal{L}_2}} = M \otimes K_{V_1}^{-1}$. So we have $\tau_{\mathcal{L}}(V_x) \sim H^{-1}_{M \otimes K_{V_1}^{-1}}$. We observe that Tamagawa numbers of fibers depend on the height of the points on the base.

4.3. Symmetric product of a curve. — Let C be a smooth irreducible curve of genus $g \geq 2$ over F. We denote by $X = C^{(m)}$ the m-th symmetric product of C and by $Y := \mathrm{Jac}(C)$ the Jacobian of C. We fix an $m > 2g - 2$. We have a fibration

$$\pi \,:\, C^{(m)} \to Y,$$

with $\mathbf{P}^{m-g}$ as fibers. We denote by V_y a fiber over $y \in Y(F)$.

Let $\tilde{C} \to \mathrm{Spec}(\mathcal{O}_F)$ be a smooth model of C over the integers and $\mathcal{L}$ an ample hermitian line bundle on $\tilde{C}$. It defines a height function

$$H_{\mathcal{L}} \,:\, C(F) \to \mathbf{R}_{>0}$$

which extends to $X(F)$. Observe that $\alpha_{\mathcal{L}}(X) = (m + 1 - g)/d$, where $d := \deg_{\mathbf{Q}}(\mathcal{L})$. Consider the height zeta function

$$Z(X, \mathcal{L}, s) := \sum_{x \in X(F)} H_{\mathcal{L}}(x)^{-s}.$$

This function was introduced by Arakelov in [**3**].

Theorem 4.3.1 **([17])**. — *Let $\mathcal{L}$ be an ample hermitian line bundle on $\tilde{C}$. There exist an $\varepsilon(\mathcal{L}) > 0$ and a real number $\Theta(\mathcal{L}) \neq 0$ such that the height zeta function has the following representation*

$$Z(X, \mathcal{L}, s) = \frac{\Theta(\mathcal{L})}{(s - \alpha_{\mathcal{L}}(X))} + f(s)$$

with some function $f(s)$ which is holomorphic for $\mathrm{Re}(s) > \alpha_{\mathcal{L}}(X) - \varepsilon(\mathcal{L})$.

This is Theorem 8 in ([**17**], p. 422). Arakelov gives an explicit expression for the constant $\Theta(\mathcal{L})$ ([**3**]). We are very grateful to J.-B. Bost for pointing out to us that Arakelov's formula is not correct and for allowing us to use his notes on the Arakelov zeta function.

Theorem 4.3.2. — *With the notations above we have*

$$\Theta(\mathcal{L}) = \sum_{y \in Y(F)} \tau_{\mathcal{L}}(V_y).$$

Proof. — We outline the proof for $F = \mathbf{Q}$. For $\mathrm{Re}(s) \gg 0$ one can rearrange the order of summation and one obtains

$$Z(X, \mathcal{L}, s) := \sum_{y \in Y(F)} \sum_{x \in V_y(F)} H_{\mathcal{L}}(x)^{-s}.$$

It is proved in ([**17**], p. 420-422) that the sums

$$Z(V_y, \mathcal{L}, s) := \sum_{x \in V_y(F)} H_{\mathcal{L}}(x)^{-s}$$

have simple poles at $s = \alpha_{\mathcal{L}}(X)$ with non-zero residues and that one can "sum" these expressions to obtain a function with a simple pole at $s = \alpha_{\mathcal{L}}(X)$ and meromorphic continuation to $\mathrm{Re}(s) > \alpha_{\mathcal{L}}(X) - \varepsilon(\mathcal{L})$ for some $\varepsilon(\mathcal{L}) > 0$. Moreover, the residue at this pole is obtained as a sum over $y \in Y(F)$ of the residues of $Z(V_y, \mathcal{L}, s)$.

Choosing an element $z(y)$ in the class of $y \in \mathrm{Jac}(C)(F)$, and denoting by

$$E(y) := \Gamma(C, \mathcal{O}(z(y)))$$

one can identify the fiber as $V_y = \mathbf{P}(E(y))$, where $f \in E(y) = \Gamma(C, \mathcal{O}(z(y)))$ is mapped to $z(y) + \mathrm{div}(f)$. The height is given by the formula

$$H_{\mathcal{L}}(z(y) + \mathrm{div}(f)) := H_{\mathcal{L}}(z(y)) \exp\Big(\int_{C(\mathbf{C})} \log |f| c_1(\mathcal{L})\Big).$$

This defines a metrization of the anticanonical line bundle on $V_y = \mathbf{P}(E(y))$. Assuming the Tamagawa number conjecture for $\mathbf{P}^n$ (for suitable metrizations of the line bundle $\mathcal{O}(1)$ on $\mathbf{P}^n$) we obtain

$$\lim_{s \to \alpha_{\mathcal{L}}(X)} (s - \alpha_{\mathcal{L}}(X)) Z(V_y, \mathcal{L}, s) = \tau_{\mathcal{L}}(V_y).$$

□

One can write down an explicit formula for $\tau_{\mathcal{L}}(V_y)$. For $f \in E(y)_{\mathbf{C}} \backslash \{0\}$ we define

$$\Phi(f) := \exp\Big(\frac{1}{d} \int_{C(\mathbf{C})} \log |f| c_1(\mathcal{L})\Big)$$

and we put $\Phi(0) = 0$. It follows that

$$\tau_{\mathcal{L}}(V_y) = \frac{1}{2} \alpha_{\mathcal{L}}(X) H_{\mathcal{L}}(y)^{-\alpha_{\mathcal{L}}(X)} \cdot \frac{\mathrm{vol}(\{f \in E(y)_{\mathbf{R}} \mid \Phi(f) \leq 1\})}{\mathrm{vol}(E(y)_{\mathbf{R}} / E(y))},$$

where the volumes are calculated with respect to some Lebesgue measure on the space $E(z(y))_{\mathbf{R}}$. Arakelov relates this last expression to the Neron-Tate height of $y \in \mathrm{Jac}(C)$. A detailed calculation due to Bost indicates that Arakelov's formula is correct only up-to $O(1)$.

4.4. Homogeneous spaces G/P. — Let G be a split semisimple linear algebraic group defined over a number field F. It contains a Borel subgroup P_0 defined over F and a maximal torus which is split over F. Let P be a standard parabolic. Denote by $Y_P = P\backslash G$ (resp. $X = P_0\backslash G$) the corresponding flag variety.

A choice of a maximal compact subgroup $\mathbf{K}$ such that $G(\mathbf{A}_F) = P_0(\mathbf{A}_F)\mathbf{K}$ defines a metrization on every line bundle L on the flag varieties Y_P ([**19**], p. 426). We will denote by

$$H_{\mathcal{L}} \,:\, P(F)\backslash G(F) \to \mathbf{R}_{>0}$$

the associated height. We consider the height zeta function

$$Z(X, \mathcal{L}, s) := \sum_{x \in X(F)} H_{\mathcal{L}}(x)^{-s}.$$

Theorem 4.4.1. — *Let $\mathcal{L}$ be an ample metrized line bundle on X. There exist an $\varepsilon(\mathcal{L}) > 0$ and a real number $\Theta(\mathcal{L}) \neq 0$ such that the height zeta function has the following representation*

$$Z(X, \mathcal{L}, s) = \frac{\Theta(\mathcal{L})}{(s - \alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)}} + \frac{f(s)}{(s - \alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)-1}}$$

with some function $f(s)$ which is holomorphic for $\mathrm{Re}(s) > \alpha_{\mathcal{L}}(X) - \varepsilon(\mathcal{L})$.

This theorem follows from the identification of the height zeta function with an Eisenstein series and from the work of Langlands. The formula (2.10) in ([**19**], p. 431) provides an expression for $\Theta(\mathcal{L})$ which we will now analyze.

There is a canonical way to identify the faces of the closed cone of effective divisors $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbf{R}}$ with $\Lambda_{\mathrm{eff}}(Y_P)$ as P runs through the set of standard parabolics. A line bundle L such that its class is contained in the interior of the cone $\in \Lambda_{\mathrm{eff}}(X)$ defines a line bundle

$$[L_Y] := \alpha_{\mathcal{L}}(X)[L] + [K_X]$$

which is contained in the interior of the face $\Lambda_{\mathrm{eff}}(Y) \subset \Lambda_{\mathrm{eff}}(X)$ for some $Y = Y_P$. We have a fibration $\pi_{\mathcal{L}} : X \to Y$ with fibers isomorphic to the flag variety $V := P_0\backslash P$. A fiber over $y \in Y(F)$ will be denoted by V_y. Denote by $\mathcal{K}_Y$ the canonical line bundle on Y with the metrization defined above.

Theorem 4.4.2. — *We have*

$$\Theta(\mathcal{L}) = \sum_{y\in Y(F)} \gamma_{\mathcal{L}}(X)\tau_{\mathcal{L}}(V_y).$$

Proof. — In the domains of absolute and uniform convergence we can rearrange the order of summation and we obtain

$$Z(X,\mathcal{L},s) = \sum_{y\in Y(F)} \sum_{x\in P_0(F)\backslash P(F)} H_{\mathcal{L}}(yx)^{-s}.$$

One can check that the sums

$$Z(V_y,\mathcal{L},s) := \sum_{x\in V_y(F)} H_{\mathcal{L}}(x)^{-s} = \sum_{x\in P_0(F)\backslash P(F)} H_{\mathcal{L}}(yx)^{-s}$$

have poles at $s = \alpha_{\mathcal{L}}(X)$ of order $\beta_{\mathcal{L}}(X)$ with non-zero residues, and that they admit meromorphic continuation to $\alpha_{\mathcal{L}}(X) - \varepsilon(\mathcal{L})$ for some $\varepsilon(\mathcal{L}) > 0$. Moreover, the constant $\Theta(\mathcal{L})$ is obtained as a sum over $y \in Y(F)$ of the residues of $Z(V_y,\mathcal{L},s)$. From the Tamagawa number conjecture for $P_0\backslash P$ (with varying metrizations of the anticanonical line bundle) we obtain

$$\lim_{s\to\alpha_{\mathcal{L}}(X)} (s-\alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)} Z(V_y,\mathcal{L},s) = \gamma_{\mathcal{L}}(X)\tau_{\mathcal{L}}(V_y).$$

The sum

$$\sum_{y\in Y(F)} \gamma_{\mathcal{L}}(X)\tau_{\mathcal{L}}(V_y)$$

converges for $[L_Y]$ contained in the interior of $\Lambda_{\mathrm{eff}}(Y)$. □

Let us recall the explicit formula for $\tau_{\mathcal{L}}(V_y)$ (see (2.9) in [**19**], p. 431):

$$\lim_{s\to\alpha_{\mathcal{L}}(X)} (s-\alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)} \sum_{x\in P_0(F)\backslash P(F)} H_{\mathcal{L}}(xy)^{-s} = \gamma_1 c_P H_{\mathcal{L}_Y^{-1}\otimes\mathcal{K}_Y}(y)$$

where $\gamma_1 \in \mathbf{Q}$ is an explicit constant and the constant c_P is defined in ([**19**], p. 430). It follows that

$$\Theta(\mathcal{L}) = \gamma_1 c_P \sum_{y\in Y(F)} H_{\mathcal{L}_Y^{-1}\otimes\mathcal{K}_Y}(y).$$

Next we observe that there is an explicit constant $\gamma_2 \in \mathbf{Q}$ such that we have

$$\gamma_2\tau_{\mathcal{L}}(V_y) = c_P \cdot H_{\mathcal{L}_Y^{-1}\otimes\mathcal{K}_Y}(y)$$

for all $y \in Y(F)$. To see this, we first identify $c_P = \gamma_2\tau_{\mathcal{L}}(V)$, this is done by a computation of local factors of intertwining operators ([**30**], p.160-161). The next step involves the comparison of Tamagawa measures on V_y for varying $y \in Y(F)$. Finally, we have $\gamma_{\mathcal{L}}(X) = \gamma_1\gamma_2$.

4.5. Toric varieties. — There are many equivalent ways to describe a toric variety X over a number field F together with some projective embedding (see, for example, [**12, 25, 13**]). For us, it will be useful to view $X = X_\Sigma$ as a collection of the following data ([**6**]):

1. A splitting field E of the algebraic torus T and the group $G = \mathrm{Gal}(E/F)$.
2. The lattice of E-rational characters of T, which we denote by M and its dual lattice N.
3. A G-invariant complete fan Σ in $N_{\mathbf{R}} = N \otimes \mathbf{R}$.

There is an isomorphism between the group of G-invariant integral piecewise linear functions $\varphi \in PL(\Sigma)^G$ and classes of T-linearized line bundles on X_Σ. For $\varphi \in PL(\Sigma)^G$ we denote the corresponding line bundle by $L(\varphi)$.

We define metrizations of line bundles as follows. Let $G_v \subset G$ be the decomposition group at v. We put $N_v = N^{G_v}$ for the lattice of G_v-invariants of N for non-archimedean valuations v and $N_v = N_{\mathbf{R}}^{G_v}$ for archimedean v. We have the logarithmic map

$$T(F_v)/T(\mathcal{O}_v) \to N_v$$

which is an embedding of finite index for all non-archimedean v, an isomorphism of lattices for almost all non-archimedean valuations and an isomorphism of real vector spaces for archimedean valuations. We denote by $\bar{t}_v$ the image of $t_v \in T(F_v)$ under this map.

Definition 4.5.1 **([6**, p. 607]). — For every $\varphi \in PL(\Sigma)^G$ and $t_v \in T(F_v)$ we define the local height function

$$H_{\Sigma,v}(t_v, \varphi) := e^{\varphi(\bar{t}_v) \log q_v}$$

where q_v is the cardinality of the residue field of F_v for non-archimedean valuations and $\log q_v = 1$ for archimedean valuations. For $t \in T(\mathbf{A}_F)$ we define the global height function as

$$H_\Sigma(t, \varphi) := \prod_{v \in \mathrm{Val}(F)} H_{\Sigma,v}(t_v, \varphi).$$

We proved in ([**6**], p. 608) that this pairing can be extended to a pairing

$$H_\Sigma : T(\mathbf{A}_F) \times PL(\Sigma)^G_{\mathbf{C}} \to \mathbf{C}$$

and that it defines a simultaneous metrization of T-linearized line bundles on X. We will denote such metrized line bundles by $\mathcal{L} = \mathcal{L}(\varphi)$. We consider the height zeta function

$$Z(T, \mathcal{L}, s) = \sum_{t \in T(F)} H_{\mathcal{L}}(t)^{-s}.$$

Theorem 4.5.2 (**[8]**). — *Let $\mathcal{L}$ be an invertible sheaf on X (with the metrization introduced above) such that its class $[L]$ is contained in the interior of $\Lambda_{\text{eff}}(X)$. There exist an $\varepsilon(\mathcal{L}) > 0$ and a $\Theta(\mathcal{L}) > 0$ such that the height zeta function has the following representation*

$$Z(\mathcal{L}, T, S) = \frac{\Theta(\mathcal{L})}{(s - \alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)}} + \frac{f(s)}{(s - \alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)-1}}$$

where $f(s)$ is a function which is holomorphic for $\mathrm{Re}(s) > \alpha_{\mathcal{L}}(X) - \varepsilon(\mathcal{L})$.

Remark 4.5.3. — The computation of the constants $\alpha_{\mathcal{L}}(X)$ and $\beta_{\mathcal{L}}(X)$ in specific examples is a problem in linear programming. For the anticanonical line bundle on a smooth toric variety X we have $\alpha_{\mathcal{K}^{-1}}(X) = 1$ and $\beta_{\mathcal{K}^{-1}}(X) = \dim PL(\Sigma)^G_{\mathbf{R}} - \dim M^G_{\mathbf{R}}$.

Our goal is to identify the constant $\Theta(\mathcal{L})$. Let us recall some properties of toric varieties and introduce more notations (see [**6**]). The cone of effective divisors $\Lambda_{\text{eff}}(X)$ is generated by the classes of irreducible components of $X\backslash T$ which we denote by $[D_1], ..., [D_r]$. These divisors correspond to Galois orbits $\Sigma_1(1), ..., \Sigma_r(1)$ on the set of 1-dimensional cones in Σ. The line bundle $\mathcal{L}$ defines a face $\Lambda(\mathcal{L})$ of $\Lambda_{\text{eff}}(X)$. We denote by $J = J(\mathcal{L})$ the maximal set of indices $J \subset [1, ..., r]$ such that we have

$$\alpha_{\mathcal{L}}(X)[L] + [K_X] = \sum_{j \in J} r_j [D_j]$$

with $[D_j] \in \Lambda(\mathcal{L})$ and some $r_j \in \mathbf{Q}_{>0}$. We denote by $I = I(\mathcal{L})$ the set of indices $i \notin J(\mathcal{L})$. We denote by M_J the lattice given by

$$M_J := \{m \in M \mid < e, m >= 0 \quad \mathbf{R}_{\geq 0} e \in \cup_{i \in I} \Sigma_i(1)\},$$

We denote by $M_I := M/M_J$ and by N_* the corresponding dual lattices. We have an exact sequence of algebraic tori

$$1 \to T_I \to T \to T_J \to 1$$

which induces a map $\pi_{\mathcal{L}} : T(F) \to T_J(F)$ with finite cokernel and an exact sequence of lattices

$$0 \to N_I \to N \to N_J \to 0.$$

The restriction of the fan $\Sigma \subset N_{\mathbf{R}}$ to $N_{I,\mathbf{R}}$ will be denoted by Σ_I. It is again a G-invariant fan and it will define an equivariant compactification X_I of T_I. The class of the piecewise linear function $\varphi_I \in PL(\Sigma_I)^G$ with $\varphi_I(e) = 1$ for $e \in \cup_{i \in I} \Sigma_i$ corresponds to the class of the anticanonical line bundle $[-K_I] \in \mathrm{Pic}(X_I)$.

The line bundle $\mathcal{L}$ defines a fibration of varieties $\pi_{\mathcal{L}} : X_\Sigma \to Y$ with fibers isomorphic to X_I, which, when restricted to T, gives rise to the exact sequence of tori above. We denote the fiber over $y \in T_J(F)$ by $X_{I,y}$.

Theorem 4.5.4. — *We have*

$$\Theta(\mathcal{L}) = \gamma_{\mathcal{L}}(X)\delta_{\mathcal{L}}(X) \sum_{y \in \pi_{\mathcal{L}}(T(F))} \tau_{\mathcal{L}}(X_{I,y}).$$

Proof. — Let $\mathcal{L} = \mathcal{L}(\varphi)$ be an invertible sheaf on T with the metrization introduced above. In the domain of absolute and uniform convergence we can rearrange the order of summation and we obtain

$$Z(T, \mathcal{L}, s) = \sum_{y \in \pi_{\mathcal{L}}(T(F))} \sum_{x \in T_{I,y}(F)} H_{\Sigma}(yx, \varphi)^{-s}.$$

From the proof of our main theorem in [**8**] it follows that the sums

$$Z(T_{I,y}, \mathcal{L}, s) = \sum_{x \in T_I(F)} H_{\Sigma}(yx, \varphi)^{-s}$$

have a pole at $\alpha_{\mathcal{L}}(X)$ of order $\beta_{\mathcal{L}}(X)$. Moreover, the constant $\Theta(\mathcal{L})$ is obtained as a sum over $y \in \pi_{\mathcal{L}}(T(F))$ of residues of $Z(T_{I,y}, \mathcal{L}, s)$. Now we want to use the Tamagawa number conjecture for the anticanonical line bundle (with varying metrizations) on the toric variety X_I to conclude the proof.

In [**7**] we proved this conjecture for a specific metrization and under the assumption that the fan Σ is *regular*. We want to demonstrate that our proof goes through in the general case needed above.

Our main idea was to use the Poisson summation formula on the adelic group $T(\mathbf{A}_F)$ and to obtain an integral representations for the height zeta function. We denote by $\mathcal{A}_I = (T_I(\mathbf{A}_F)/\mathbf{K}T_I(F))^*$ the group of unitary characters of $T_I(\mathbf{A}_F)$ which are trivial on $T_I(F)$ and on the maximal compact subgroup $\mathbf{K} \subset T_I(\mathbf{A}_F)$. Using the the adelic definition of the height function we obtain

$$Z(T_{I,y}, \mathcal{L}, s) = \sum_{t \in T_I(F)} H_{\mathcal{L}}(yt)^{-s} = \int_{\mathcal{A}_I} d\chi \int_{T_I(\mathbf{A}_F)} H_{\mathcal{L}}(yt)^{-s} \chi(t) d\mu,$$

where $d\mu$ is a Haar measure on $T(\mathbf{A}_F)$ and $d\chi$ is the orthogonal Haar measure on $\mathcal{A}_I$. To apply our technical theorem in [**7**] about the analytic continuation and the residues of such integrals we need to know that

$$\int_{T_I(\mathbf{A}_F)} H_{\mathcal{L}}(yt)^{-s} \chi(t) d\mu = \prod_{i \in I} L_i(\chi_i, s) \cdot \zeta_{\Sigma_I}(\chi, s) \cdot \zeta_{\infty}(\chi, s)$$

where

$$\zeta_{\Sigma_I}(\chi, s) = \prod_{v \in \mathrm{Val}(F)} \zeta_{\Sigma_I, v}(\chi, s)$$

is an absolutely convergent Euler product for $\mathrm{Re}(s) > \alpha_{\mathcal{L}(X)} - \varepsilon(\mathcal{L})$, $L_i(\chi_i, s)$ are Hecke L-functions (with some induced characters χ_i) and $\zeta_{\infty}(\chi, s)$ satisfies certain growth conditions.

First we observe that it is unnecessary to assume that the fan Σ_I is regular. Using our definition of the height function we see that the calculation of the Fourier transform (see [**6**]) reduces to summations of the function $q_v^{\varphi(\overline{t_v})+im_v}$ ($m_v \in M_{I,\mathbf{R}}$) over the lattice $N_{I,v}$ (resp. to integrations in cones for archimedean valuation). A piecewise linear function φ induces a piecewise linear function on any subdivision of the fan. Clearly, the result of such summations and integrations does not depend on any subdivisions.

Next we see that for a fixed $y \in T_J(F)$ we have $H_{\mathcal{L},v}(yt) = H_{\mathcal{L},v}(t)$ for almost all v and all $t \in T_I(\mathbf{A}_F)$. Now we can refer to lemma 5.10 in [**8**] which proves the required statement. The local integrals for the remaining finitely many non-archimedean valuations will be absorbed into $\zeta_{\Sigma_I}(\chi, s)$. And finally, we need to check that the estimates of the Fourier transform of $H_{\mathcal{L},v}(yt)$ at archimedean valuations are still satisfied for any $y \in T_J(F)$. This is straightforward.

We can now apply the main technical theorem of [**7**] and obtain

$$\lim_{s\to\alpha_{\mathcal{L}}} (s-\alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)} Z(T_{I,y}, \mathcal{L}, s) = \gamma_{\mathcal{L}}(X)\delta_{\mathcal{L}}(X)\tau_{\mathcal{L}}(X_{I,y}).$$

□

Remark 4.5.5. — It is possible to compute $\tau_{\mathcal{L}}(X_{I,y})$ and to observe that it is related to the height of the point $y \in T_J(F)$.

Remark 4.5.6. — Similar statements hold for equivariant compactifications of homogeneous spaces G/U where G is a split reductive group and U is its maximal unipotent subgroup [**33**]. We hope that these results can be extended to equivariant compactifications of other homogeneous spaces, in particular, to equivariant compactifications of reductive and non-reductive groups.

5. Singular Fano varieties

5.1. Weighted projective spaces. — Let $W := \mathbf{P}(w) = \mathbf{P}(w_0, \ldots, w_n)$ be a weighted projective space of dimension n with weights $w = (w_0, \ldots, w_n)$. We remark that W is a rational variety over $\mathbf{Q}$ with $\mathrm{Pic}(W) \cong \mathbf{Z}$. Moreover, the anticanonical class K_W^{-1} is an ample $\mathbf{Q}$-Cartier divisor. So W is a (singular) Fano variety of index

$$r = \frac{w_0 + \cdots + w_n}{\text{l.c.m.}\{w_0, \ldots, w_n\}}.$$

One could try to generalize the method of Schanuel [**31**] for counting $\mathbf{Q}$-rational points of bounded height on usual projective spaces to the case of weighted projective spaces. Let $z_0, z_1, \ldots, z_n$ be homogeneous coordinates on Y. Then a first approximation to counting points of bounded height would be a counting of all $(n+1)$-tuples $(x_0, x_1, \ldots, x_n) \in \mathbf{Z}^n \setminus \{0\}$ satisfying the conditions

$$|x_i| \leqslant B^{\frac{w_i}{w_0+w_1+\cdots+w_n}} \quad i = 0, \ldots, n.$$

Since the volume of the domain restricted by these inequalities is

$$B = \prod_{i=0}^{n} B^{\frac{w_i}{w_0+w_1+\cdots+w_n}},$$

one could expect that the asymptotic number of solutions of these inequalities agrees with "expected" linear growth for the anticanonical height. However, this "intuition" turns out to be wrong, in general, because the singularities of Y could be even worse than canonical. A typical class of singularities that appear on Y are so called log-terminal singularities introduced by Kawamata [**28**]. We give below a simple example of a Del Pezzo surface with a log-terminal singularity and we show that for every dense Zariski open subsets $U \subset W$ the number $N(U, B)$ of F-rational points of anticanonical height $\leqslant B$ in U has more than linear growth:

$$N(U, B) = c(U) B^{2-\frac{4}{m+2}}(1 + o(B)).$$

Moreover, there are no dense Zariski open subsets $U' \subset X$ such that the adelic term in the constant $c(U)$ in the asymptotic formula for $N(U, B)$ would be independent of U for $U \subset U'$.

Example 5.1.1 **(Del-Pezzo surface with a log-terminal singularity)**
Let $W = \mathbf{P}(1, 1, m)$ be a singular weighted projective plane with weights $(1, 1, m)$, $m \geqslant 2$. Then the anticanonical class of W is an ample $\mathbf{Q}$-Cartier divisor (i.e., W is Del Pezzo surface) and $p = (0 : 0 : 1)$ is the unique singularity of W. Let $X \to W$ be the minimal resolution of the singularity at $p \in W$. Then X is isomorphic to a ruled surface $\mathbf{F}_m = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(m))$ and the exceptional divisor $E = f^{-1}(p)$ is a smooth rational curve which is a section of the $\mathbf{P}^1$-bundle over $\mathbf{P}^1$ and $\langle E, E \rangle = -m$. Then we have

$$K_X = f^* K_W + \frac{2-m}{m} E.$$

Therefore, p is canonical $\Leftrightarrow m = 2$ and p is log-terminal $\Leftrightarrow m \geqslant 3$.

The group $\mathrm{Pic}(X)$ is isomorphic to $\mathbf{Z}^2$ where $\{[E], [C]\}$ is a $\mathbf{Z}$-basis. Moreover, $[E], [C]$ are generators for the cone $\Lambda_{\mathrm{eff}}(X)$ of effective divisors in $\mathrm{Pic}(X)_{\mathbf{R}}$. We have

$$K_X = -2E - (m+2)C,$$

$$L := f^*(-K_X) = \frac{m+2}{m} E + (m+2) C$$

and

$$\alpha_L(W) = \inf\{t \in \mathbf{Q} \ : \ t[L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)\} = \frac{2m}{m+2}.$$

Since X is a smooth toric variety, we can apply our main result in [**8**] and obtain the following:

Theorem 5.1.2. — *Let $\pi : W \setminus p \to \mathbf{P}^1$ be the natural projection, C_x the fiber of π over $x \in \mathbf{P}^1(\mathbf{Q})$. Then for any dense Zariski open subset in $W \setminus p$, one has*

$$N(U,B) = c(U)B^{2-\frac{4}{m+2}}(1+o(B))$$

Moreover,

$$c(U) = \sum_{x \in \mathbf{P}^1(\mathbf{Q}) \cap \pi(U)} c(C_x).$$

5.2. Vaughan-Wooley cubic

Example 5.2.1. — Let $Y \subset \mathbf{P}^5$ be a singular cubic defined by the equation $z_0z_1z_2 - z_3z_4z_5 = 0$, X the intersection of Y with the linear subspace in $\mathbf{P}^5$ with the equation:

$$z_0 + z_1 + z_2 - z_3 - z_4 - z_5 = 0.$$

Vaughan and Wooley proved [**34**]:

Theorem 5.2.2. — *Let $U \subset X$ be the complement in X to the following* 15 *divisors $D_{i_1i_2i_3}, D_{ij} \subset X$ ($\{i_1,i_2,i_3\} = \{0,1,2\}$, $i \in \{0,1,2\}$, $j \in \{3,4,5\}$), where*

$$D_{i_1i_2i_3} = \{(z_0 : \ldots : z_5) \in \mathbf{P}^5 \ : \ z_{i_1} = z_3,\, z_{i_2} = z_4,\, z_{i_3} = z_5\}$$
$$D_{ij} = \{(z_0 : \ldots : z_5) \in X \ : \ z_i = z_j = 0\}.$$

If $N(U,B)$ is the number of $\mathbf{Q}$-rational points in U of the anticanonical height $\leqslant B$, then there exist some constants $c_1 > c_2 > 0$ such that

$$c_2B^2(\log B)^5 \leqslant N(U,B) \leqslant c_1B^2(\log B)^5.$$

We want to show that this result is compatible with predictions in [**4**]. First of all we note that Y is a 4-dimensional toric Fano variety: an equivariant compactification of a 4-dimensional algebraic torus T with respect to a 4-dimensional polyhedron Δ with 6 lattice vertices

$$v_0 = (0,0,0,0),\ v_1 = (1,0,0,0),\ v_2 = (0,1,0,0),$$
$$v_3 = (0,0,1,0),\ v_4 = (0,0,0,1),\ v_5 = (1,1,-1,-1)$$

(Δ is the support of global sections of a very ample divisor Y corresponding to the embedding $Y \hookrightarrow \mathbf{P}^4$). The polyhedron Δ has 9 faces Θ_{ij} of codimension 1:

$$\Theta_{ij} = \mathrm{Conv}(\{v_0,v_1,v_2,v_3,v_4,v_5\} \setminus \{v_i,v_j\}).$$

Each face Θ_{ij} defines an torus invariant divisor $Y_{ij} \subset Y \setminus T$ such that $D_{ij} = Y_{ij} \cap X$. It is easy to check that all singularities of Y are at worst terminal and that the hypersurface $X \subset Y$ intersects all strata Y_{ij} transversally. From these facts we obtain that the only exceptional divisors with the discrepancy

0 that appear in a resolution of singularities of X come from singularities in $X \cap T$. We write down the affine equation of $X \cap T$ as

$$1 + x + y - z - t - \frac{xy}{zt},$$

where $T = \operatorname{Spec} \mathbf{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}, t^{\pm 1}]$. From this equation one immediately sees that the only singularities in $X \cap T$ are the A_1-double points lying on the curve $C \ : \ x = y = z = t$. Therefore, we obtain $\operatorname{rk} \operatorname{Cl}(X) = \operatorname{rk} \operatorname{Cl}(Y) = 9 - \dim T = 5$. Moreover, there exists exactly one crepant divisor (over C) in a resolution of singularities of X. So the predicted power of $\log B$ in the asymptotic formula for $N(U, B)$ is $(\operatorname{rk} \operatorname{Cl}(X) - 1) + 1 = 5$.

5.3. Cubic $xyz = u^3$. — We consider the singular cubic surface $X \subset \mathbf{P}^3$ over $\mathbf{Q}$ given by the homogeneous equation $xyz = u^3$. This is a toric variety, an equivariant compactification of the torus

$$T = \operatorname{Spec} \mathbf{Q}[x, y, z]/(xyz - 1)$$

given by the condition $u \neq 0$. We can fix an isomorphism $T \cong \mathbf{G}_m^2$ by choosing $\{x, y\}$ as a basis of the group of algebraic characters of T. Consider the problem of the computation of the asymptotic of

$$N(T, B) = \operatorname{Card}\{(x, y) \in (\mathbf{Q}^*)^2 \ : \ H(x, y) \leq B \ \}$$

for $B \to \infty$, where

$$H(x, y) = \prod_{v \in \operatorname{Val}(\mathbf{Q})} \max\{\|x\|_v, \|y\|_v, \|(xy)^{-1}\|_v, \|1\|_v\}$$

This problem is addressed in [**18**]. We would like to use this problem as a down-to-earth illustration of our general theory of height zeta functions of toric varieties. First of all we note that the relation $\|x\|_v \|y\|_v \|(xy)^{-1}\|_v = \|1\|_v = 1$ implies that

$$H_v(x, y) := \max\{\|x\|_v, \|y\|_v, \|(xy)^{-1}\|_v\}.$$

Since $l_1 = \log \|x\|_v$, $l_2 = \log \|y\|_v$, $l_3 = \log \|(xy)^{-1}\|_v$ are linear functions on the logarithmic space $N_{\mathbf{R},v} \cong \mathbf{R}^2$:

$$N_{\mathbf{R},v} = \begin{cases} T(\mathbf{Q}_v)/T(\mathcal{O}_v) \otimes_{\mathbf{Z}} \mathbf{R}, & \text{if } v = p \in \operatorname{Spec} \mathbf{Z} \\ T(\mathbf{Q}_v)/T(\mathcal{O}_v), & \text{if } v = \infty, \end{cases}$$

we can consider $\log h_v(x, y)$ as a piecewise linear function on $N_{\mathbf{R},v}$. Let $e_1 = (-2, 1)$, $e_2 = (1, -2)$ and $e_3 = (1, 1)$ be lattice vectors in $\mathbf{Z}^2 \subset \mathbf{R}^2$. We define the following 3 convex cones in $\mathbf{R}^2$:

$$\sigma_1 = \mathbf{R}_{\geqslant 0} e_2 + \mathbf{R}_{\geqslant 0} e_3,$$
$$\sigma_2 = \mathbf{R}_{\geqslant 0} e_1 + \mathbf{R}_{\geqslant 0} e_3,$$
$$\sigma_3 = \mathbf{R}_{\geqslant 0} e_1 + \mathbf{R}_{\geqslant 0} e_2.$$

Then $N_{\mathbf{R},v} = \bigcup_{i=1}^{3} \sigma_i$ and the restriction of $\log H_v(x,y)$ to σ_i coincides with the linear function l_i. Let

$$A := \bigoplus_{v \in \mathrm{Val}(\mathbf{Q})} T(\mathbf{Q}_v)/T(\mathcal{O}_v)$$

be the logarithmic adelic group. In order to compute the height zeta function

$$Z(s) = \sum_{(x,y)\in(\mathbf{Q}^*)^2 = T(\mathbf{Q})} H(x,y)^{-s}$$

we use the natural homomorphism Log of $T(\mathbf{Q})$ to A. Denote by B the subgroup $\mathrm{Log}(T(\mathbf{Q})) \subset A$. We remark that the kernel of Log consists of 4 elements of finite order in $(\mathbf{Q}^*)^2$ and the quotient A/B is isomorphic to $\mathbf{R}^2$. Moreover the functions $H_v(x,y)^{-s}$ on each $T(\mathbf{Q}_v)/T(\mathcal{O}_v)$ define a natural extension of $h(x,y)^{-s}$ to a function on A. So we obtain:

$$Z(s) = 4 \sum_{b \in B} \prod_{v \in \mathrm{Val}(\mathbf{Q})} H_v(b_v)^{-s} \tag{4}$$

The main idea of our proof in [**6**] is to apply the Poisson summation formula on the group A and to express the height zeta function $Z(s)$ as an integral

$$Z(s) = \frac{4}{(2\pi)^2} \int_{\mathbf{R}^2} \left(\prod_p Q_p(s, i\mathbf{m}) \cdot Q_\infty(s, i\mathbf{m}) \right) \mathbf{dm},$$

where $\mathbf{m} = (m_1, m_2) \in \mathbf{R}^2$, $\mathbf{dm} = dm_1 dm_2$,

$$Q_p(s, i\mathbf{m}) = \sum_{(b_{1,p}, b_{2,p}) \in \mathbf{Z}^2} h_p(b_p)^{-s} p^{i<b,\mathbf{m}>},$$

and

$$Q_\infty(s, i\mathbf{m}) = \int_{\mathbf{R}^2} H_\infty(b)^{-s} \exp(i < b, \mathbf{m} >).$$

An exact computation of $Q_p(s, i\mathbf{m})$ and $Q_\infty(s, i\mathbf{m})$ can be obtained by a subdivision of each of the cones $\sigma_1, \sigma_2, \sigma_3$ into a union of 3 subcones generated by a basis of the lattice $\mathbf{Z}^2 \subset \mathbf{R}^2$. (From the viewpoint of toric geometry this means that we reduce the counting problem for rational points in a torus with respect to a singular compactification to a counting problem for rational points in a torus with respect to the minimal resolution of singularities of this compactification). This calculation is done in [**6**], Section 2 for arbitrary smooth toric varieties. What remains is the analytic continuation of the integral and the identification of the constant at the leading pole. For this it is necessary to work on the whole complexified space $PL(\Sigma)_{\mathbf{C}}$ and to invoke the technical theorems in Section 6 in [**7**]. Applying the main theorem of [**7**], we obtain

$$N(T,B) = \frac{\gamma_{\mathcal{K}^{-1}}(X)\delta_{\mathcal{K}^{-1}}(X)\tau_{\mathcal{K}^{-1}}(X)}{6!} B(\log B)^6(1+o(1))$$

for $B \to \infty$. The constants are as follows: $\gamma_{\mathcal{K}^{-1}}(X) = 1/36$, $\delta(X) = 1$ and $\tau_{\mathcal{K}^{-1}}(X) = \tau_{\mathcal{K}^{-1}}(X)_\infty \prod_p \tau_{\mathcal{K}^{-1}}(X)_p$ where

$$\tau_{\mathcal{K}^{-1}}(X)_p = \left(1 + \frac{7}{p} + \frac{1}{p^2}\right) \cdot (1 - 1/p)^7$$

for all primes p and $\tau_{\mathcal{K}^{-1}}(X)_\infty = 9{\cdot}4$. Similar statements hold over any number field. One can compute the constant $\gamma_{\mathcal{K}^{-1}}(X)$ by observing that $\Lambda^*_{\mathrm{eff}}(X)$ (the dual cone to the cone of effective divisors) is a union of two simplicial cones.

References

[1] D. Abramovich, J. Wang, *Equivariant resolution of singularities in characteristic 0*, alg-geom/9609013.

[2] V. Alekseev, *Fractional indices of log del Pezzo surfaces*, Math. USSR, Izvestia, **33** (1989), 613-629.

[3] S. J. Arakelov, *Theory of intersections on the arithmetic surface*, in Proceedings of the International Congress of Math., Vol. 1, Canad. Math. Congress, Montreal, (1975), 405-408.

[4] V. Batyrev, Yu.I. Manin, *Sur le nombre des points rationnels de hauteur bornée des variétés algébriques*, Math. Ann., **286**, (1990), 27-43.

[5] V. Batyrev, *The cone of effective divisors of threefolds*, Proc. Mal'cev Conference, Contemporary Mathematics, **131**, Part 3 (1992), 337-352.

[6] V. Batyrev, Yu. Tschinkel, *Rational points of bounded height on compactifications of anisotropic tori*, Intern. Math. Research Notices, **12**, (1995), 591-635.

[7] V. Batyrev, Yu. Tschinkel, *Manin's conjecture for toric varieties*, Journal of Alg. Geometry, **7** (1998), 15-53.

[8] V. Batyrev, Yu. Tschinkel, *Height zeta functions of toric varieties*, Algebraic geometry, 5, (Manin's Festschrift), J. Math. Sci., **82**, no. 1, (1996), 3220-3239.

[9] V. Batyrev, Yu. Tschinkel, *Rational points on some Fano cubic bundles*, C. R. Acad. Sci., Paris, **323**, Ser. I, (1996), 41-46.

[10] A. Borel, *Linear algebraic groups*, New York-Amsterdam, (1969).

[11] J. L. Brylinski, *Décomposition simpliciale d'un réseau, invariante par un groupe fini d'automorphismes*, C. R. Acad. Sci., Paris, Sér. A-B, **288**, (1979), A137-A139.

[12] D. Cox, *Recent developments in toric geometry*, alg-geom 9606016.

[13] V. I. Danilov, *The geometry of toric varieties*, Russ. Math. Surveys, **33**, (1978), n.2, 97-154.

[14] H. Delange, *Généralisation du théorème de Ikehara*, Ann. Sci. Ecole Norm. Sup., (4), **71**, (1954), 213-242.

[15] P. Deligne, *La conjecture de Weil I*, Inst. Hautes Études Sci. Publ. Math., **43**, (1973), 273-307.

[16] J. Denef, *On the degree of Igusa's local zeta function*, Amer. J. Math., **109**, (1987), 991-1008.

[17] G. Faltings, *Calculus on arithmetic surfaces*, Ann. of Math., (2), **119**, (1984), 387-424.

[18] E. Fouvry, *Sur la hauteur de points d'une certaine surface cubique singulière*, this volume, (1996).

[19] J. Franke, Yu. I. Manin, Yu. Tschinkel, *Rational points of bounded height on Fano varieties*, Inventiones Math., **95**, (1989), p. 421-435.

[20] T. Fujita, *On Kodaira energy and adjoint reduction of polarized manifolds*, Manuscripta Math., **76**, (1992), 59–84.

[21] T. Fujita, *Notes on Kodaira energies of polarized varieties*, Preprint, alg-geom/9205004.

[22] T. Fujita, *Kodaira energies of polarized log varieties*, Preprint, alg-geom/9305004.

[23] T. Fujita, *On Kodaira energy of polarized log varieties*, J. Math. Soc. Japan, **48-1**, (1996), 1-12.

[24] T. Fujita, *On Kodaira energy and adjoint reduction of polarized threefolds*, alg-geom/9611031.

[25] W. Fulton, *Introduction to toric varieties*, Princeton U. Press, Princeton, (1993).

[26] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math., **79**, (1964), 109-326.

[27] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the Minimal Model Program*, Adv. Studies in Pure Math., **10**, (1987), 283-360.

[28] Y. Kawamata, *Boundedness of* $\mathbf{Q}$*-Fano* 3*-folds*, Contemp. Math., **131**, (1992), 439-445.

[29] S. Lang, *Fundamentals of Diophantine Geometry*, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, (1983).

[30] E. Peyre, *Hauteurs et nombres de Tamagawa sur les variétés de Fano*, Duke Math. J., **79**, (1995), 101-218.

[31] S. Schanuel, *Heights in number fields*, Bull. Soc. Math. France, **107** (1979), 433-449.

[32] K.-H. Shin, 3*-dimensional Fano varieties with canonical singularities*, Tokyo J. Math., **12**, (1989), 375-385.

[33] M. Strauch, Yu. Tschinkel, *Height zeta functions of twisted products*, Math. Res. Letters, **4**, (1997), 273-282.

[34] R.C. Vaugan, T.D. Wooley, *On a certain nonary cubic form and related equations*, Duke Math. J., **80**, (1995), 669-735.

[35] A. Weil, *Adèles and algebraic groups*, Progr. Math., **23**, Birkhäuser, Boston (1982).

Victor V. Batyrev, Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany • *E-mail :* `batyrev@bastau.mathematik.uni-tuebingen.de`

Yuri Tschinkel, Dept. of Mathematics, U.I.C., Chicago (IL) 60607-7045, U.S.A. *E-mail :* `yuri@math.uic.edu`

smf_ast_251_08
Astérisque
251, 1998, p. 299–341

TAMAGAWA NUMBERS OF POLARIZED ALGEBRAIC VARIETIES

Victor V. Batyrev & Yuri Tschinkel

Veuillez compiler ce fichier séparément et l'insérer ici

Please compile this file separately and insert it here