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EIGENVALUES OF SUMS OF HERMITIAN MATRICES
[after A. Klyachko]

by **William FULTON**

STATEMENT OF THE RESULTS

In its simplest form, the problem solved by Klyachko [K] is to describe the eigenvalues of the sum $A + B$ of two Hermitian $n \times n$ matrices in terms of the eigenvalues of A and B. List the eigenvalues of a Hermitian matrix H in decreasing order: $\lambda(H) : \lambda_1(H) \geq \lambda_2(H) \geq \cdots \geq \lambda_n(H)$. Write $\lambda(A) : \alpha_1 \geq \cdots \geq \alpha_n$, $\lambda(B) : \beta_1 \geq \cdots \geq \beta_n$; let $C = A + B$, and $\lambda(C) : \gamma_1 \geq \cdots \geq \gamma_n$. Of course one always has the trace identity $\sum \gamma_i = \sum \alpha_i + \sum \beta_i$.

The problem arose from questions in solid mechanics, where eigenvalues of symmetric matrices determine shapes of ellipsoids. The inequality $\gamma_1 \leq \alpha_1 + \beta_1$ was known more than a century ago. In 1912 Weyl [We] proved the inequalities

$$\gamma_{i+j-1} \leq \alpha_i + \beta_j \qquad \text{for } i + j - 1 \leq n.$$

Weyl was interested in a generalization to compact selfadjoint operators on a Hilbert space. The inequalities in this case can be deduced easily from the finite-dimensional case. For this and more of the early history of the subject, see [Zw].

For $n = 2$, these inequalities $\gamma_1 \leq \alpha_1 + \beta_1$, $\gamma_2 \leq \alpha_1 + \beta_2$, and $\gamma_2 \leq \alpha_2 + \beta_1$ are easily seen to be sufficient to characterize the possible eigenvalues of the sum.

This problem was featured in Gelfand's seminar. Berezin and Gelfand [B-G] showed that $(\gamma_1, \dots, \gamma_n)$ is in the convex hull of the points $(\alpha_1 + \beta_{w(1)}, \dots \alpha_n + \beta_{w(n)})$, for $w \in \mathfrak{S}_n$. V. Lidskii [L1] gave a more elementary proof of this, and Wielandt [Wi] proved this was equivalent to the following inequalities

$$\sum_{k \in I} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \leq r} \beta_j$$

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for any subset I of $[1,n] = \{1,2,\ldots,n\}$ of cardinality r. Of course one has similar inequalities by interchanging α and β. The case when $I = [1,r]$ had been given in 1949 by K. Fan.

For $n = 3$, there are six inequalities coming from Weyl, and five from Lidskii and Wielandt. For a complete answer in this case, one more inequality is needed:

$$\gamma_2 + \gamma_3 \leq \alpha_1 + \alpha_3 + \beta_1 + \beta_3.$$

Other inequalities were found (see [AM], and [M-O] for related history and references), all having the form

$$(*_{IJK}) \qquad \sum_{k\in K} \gamma_k \leq \sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j,$$

where I, J, and K are certain subsets of $[1,n]$ of the same cardinality. A. Horn [H] gave several inequalities of this type, including the complete set for $n = 3$ and $n = 4$. Horn conjectured that necessary and sufficient conditions could be given by such inequalities, where the subsets T_r^n of triples (I,J,K) of cardinality r in $[1,n]$ that occur can be given by induction on r. For this, write $I = \{i_1 < i_2 < \cdots < i_r\}$, and similarly for J and K. When $r = 1$, (I,J,K) is in T_1^n whenever $i_1 + j_1 = k_1 + 1$. For $r > 1$, (I,J,K) is in T_r^n whenever

$$\sum_{i\in I} i + \sum_{j\in J} j = \sum_{k\in K} k + \binom{r+1}{2}$$

and, for all p with $1 \leq p \leq r-1$ and all (U,V,W) in T_p^r,

$$\sum_{u\in U} i_u + \sum_{v\in V} j_v \leq \sum_{w\in W} k_w + \binom{p+1}{2}.$$

This conjecture of Horn is still open, although we will see that something rather close to it has been proved.

The problem is to describe which triples (I,J,K) determine inequalities, and then to show that the inequalities obtained characterize the possible eigenvalues. The answer we discuss is in terms of Schubert calculus. To describe this, we need to fix some notation. To any subset J of $[1,n]$ of cardinality r, and any complete flag $F_\bullet$ in an n-dimensional vector space E, there is a Schubert variety in the Grassmann variety $\mathrm{Gr}(r,E)$ of r-dimensional subspaces of E:

$$\begin{aligned}\Omega_J(F_\bullet) &= \{\, L \in \mathrm{Gr}(r,E) \mid \dim(L\cap F_{j_t}) \geq t \text{ for } 1 \leq t \leq r \,\} \\ &= \{\, L \in \mathrm{Gr}(r,E) \mid \dim(L\cap F_k) \geq |J\cap[1,k]| \text{ for } 1 \leq k \leq n \,\}.\end{aligned}$$

(Here $|\ |$ denotes cardinality.) This is the closure of the Schubert cell $\Omega_J^\circ(F_\bullet)$, defined by replacing the inequalities in the second description by equalities. We denote the cohomology class of this Schubert variety (which is independent of the flag) by σ_J.

We also need a dual notation. For any subset I of $[1,n]$, denote by $I^\vee$ the subset $\{n+1-i \mid i \in I\}$. Set $\sigma^I = \sigma_{I^\vee}$. The relation between these notations and the more familiar notation for these Schubert classes using partitions is given by the equation

$$\sigma^I = \sigma(\lambda) = \sigma_J,$$

with $\lambda = (i_r - r, i_{r-1} - (r-1), \ldots, i_1 - 1)$, and $J = I^\vee$. The partition λ is identified with a Young diagram in an r by $n-r$ rectangle, which is outlined by a path of n steps (horizontal and vertical) between the southwest and northeast corners of the rectangle. The set I lists the vertical steps in drawing the segmented path from bottom to top, and $J = I^\vee$ lists the vertical steps in drawing it from top to bottom. For example if $n = 5$, $r = 2$, and $I = \{2,5\}$, then $\lambda = (3,1)$; the Young diagram with 3 boxes in the first row and 1 box in the second is obtained by moving a first step to the right, a second step up, third and fourth steps to the right, and the fifth step up. And $I^\vee = \{1,4\}$, which prescribes the reverse trip along this path.

The codimension of σ^I is the number $|\lambda|$ of boxes in λ, which is $\sum_{i \in I} i - \binom{r+1}{2}$, where r is the cardinality of I. The classes σ^I form a basis for the cohomology ring of the Grassmann variety, as I varies over subsets of $[1,n]$ of cardinality r. (The classes σ^I and σ_I are dual by the intersection pairing.) We say that a class σ^K *occurs* in $\sigma^I \cdot \sigma^J$ if the expansion of the product $\sigma^I \cdot \sigma^J$ contains σ^K with a nonzero coefficient.

Theorem (Klyachko [K]).— *The inequalities $(*_{IJK})$, for those (I,J,K) of cardinality r for which σ^K occurs in $\sigma^I \cdot \sigma^J$, for $1 \le r \le n-1$, are necessary and sufficient for the existence of Hermitian operators A and B and $C = A + B$ having given eigenvalues α, β, and γ.*

Although it is stated in [K] that these inequalities are independent, P. Belkale has recently shown that all the inequalities for which σ^K occurs with multiplicity greater than 1 in $\sigma^I \cdot \sigma^J$ are redundant.

It is not hard to see that all the inequalities discussed at the beginning are among those in the theorem. For example, Weyl's are given by $I = \{i\}$, $J = \{j\}$, $K = \{i+j-1\}$, which occur because general linear spaces of codimensions $i-1$ and $j-1$ meet in a linear space of codimension $i+j-2$ in $\mathbb{P}^{n-1}$ if $i+j \le n-1$. Similarly, the Lidskii-Wielandt equations follow from the fact that $\sigma^{[1,r]} = 1$, so $\sigma^I \cdot \sigma^{[1,r]} = \sigma^I$. The last inequality for $n = 3$, with $I = J = \{1,3\}$ and $K = \{2,3\}$ comes from the unique nontrivial intersection in $\mathrm{Gr}(2,3) \cong \mathbb{P}^2$.

Another inequality that follows from Schubert calculus (see Freede and Thompson [T-F]) is for a triple of the form (I,J,K) of cardinality r with $k_t = i_t + j_t - t$ for $1 \le t \le r$; here I and J can be arbitrary, provided $i_r + j_r \le n + r$.

Elementary facts from Schubert calculus imply some symmetries on the triples (I,J,K) which give inequalities in the theorem, besides the obvious symmetry that (J,I,K) occurs when (I,J,K) occurs. Since (I,J,K) occurs exactly when $\sigma^I \cdot \sigma^J \cdot \sigma^{K^\vee}$ is a nonzero multiple of the class of a point, we see that (I,J,K) occurs exactly when $(K^\vee, J, I^\vee)$ occurs. The standard isomorphism of $\mathrm{Gr}(r,n)$ with $\mathrm{Gr}(n-r,n)$ takes the class of σ^I to $\sigma^{I^{c\vee}}$, where

I^c denotes the complement of I. Therefore (I, J, K) occurs exactly when $(I^{c\vee}, J^{c\vee}, K^{c\vee})$ occurs.

We write α for the sequence $\alpha_1, \ldots, \alpha_n$, and similarly for β and γ. There are some obvious properties for triples (α, β, γ) to occur as eigenvalues of Hermitian matrices A, B, and $A + B$:

(i) (α, β, γ) occurs if and only if $(\alpha + a, \beta + b, \gamma + a + b)$ occurs, for any constants a and b.

(ii) (α, β, γ) occurs if and only if (β, α, γ) occurs.

(iii) (α, β, γ) occurs if and only if $(\alpha^\vee, \beta^\vee, \gamma^\vee)$ occurs, where $\alpha^\vee = (-\alpha_n, \ldots, -\alpha_1)$.

(iv) (α, β, γ) occurs if and only if $(N\alpha, N\beta, N\gamma)$ occurs, for any positive N.

These properties are seen: (i) by adding scalar matrices to A and B; (ii) by interchanging A and B; (iii) from the fact that $\lambda(-H) = \lambda(H)^\vee$; (iv) by multiplying each of the matrices by N.

Suppose α, β, and γ are all integral, i.e., all their entries are integers. They are then highest weights for irreducible representations V^α, V^β, and V^γ of $\mathrm{GL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{C})$. It is a basic problem of representation theory to determine when V^γ is contained in the tensor product representation $V^\alpha \otimes V^\beta$. In this case the first three properties in the above list are also true (with a and b integers in (i)); (iii) follows from the fact that $V^{\lambda^\vee}$ is the dual representation to V^λ. Property (iv) on this list, for N an integer, however, is an open question:

Saturation Problem.— *If $V^{N\gamma}$ occurs in $V^{N\alpha} \otimes V^{N\beta}$ for some positive integer N, does it follow that V^γ occurs in $V^\alpha \otimes V^\beta$?*

Theorem (Klyachko [K]).— *$V^{N\gamma}$ occurs in $V^{N\alpha} \otimes V^{N\beta}$ for some positive integer N if and only if the inequality $(*_{IJK})$ is satisfied for those (I, J, K) of cardinality r for which σ^K occurs in $\sigma^I \cdot \sigma^J$, for $1 \le r \le n - 1$.*

The problem of which V^γ is contained in $V^\alpha \otimes V^\beta$ is well known to be related to Schubert calculus, but this is quite a different relation from that occurring in Klyachko's theorem. For this, using (i), one may assume all the integers occurring are nonnegative, so they correspond to partitions of lengths at most n. Then V^γ occurs in $V^\alpha \otimes V^\beta$ if and only if, for any integer m that is at least as large as the largest integer occurring in α, β, or γ, the Schubert class corresponding to the partition γ occurs in the product of the Schubert class corresponding to α with that corresponding to β, in the Grassmann variety $\mathrm{Gr}(n, n + m)$.

The multiplicity with which this occurs is the Littlewood-Richardson coefficient $c_{\alpha\beta}^\gamma$, for which there are many combinatorial formulas. The Saturation Problem asks if $c_{N\alpha\, N\beta}^{N\gamma} \neq 0$ for some positive N implies that $c_{\alpha\beta}^\gamma \neq 0$. A. Buch has written a computer program which has verified that there is no counterexample to the Saturation Problem whenever $N \cdot |\gamma| \le 68$. (The converse, that $c_{\alpha\beta}^\gamma \neq 0$ implies $c_{N\alpha\, N\beta}^{N\gamma} \neq 0$ for all positive N, is an easy consequence of the original Littlewood-Richardson rule for computing these coefficients; in fact, the triples (α,β,γ) for which $c^{\gamma}_{\alpha\beta} \neq 0$ form a semigroup [Z].) By Klyachko's theorem, the Saturation Problem is equivalent to the assertion that the nonvanishing of $c^{\gamma}_{\alpha\beta}$, when the three partitions have length at most n, is controlled by Schubert calculus in smaller Grassmannians $\mathrm{Gr}(r,n)$. Note that Horn's conjecture and an affirmative solution to the Saturation Problem imply an inductive rule for determining when a Littlewood-Richardson coefficient is nonzero. For an interesting discussion of this Saturation Problem, including its relation to Klyachko's work, and its failure for other classical groups, see [Z].

In fact, Klyachko's theorems, together with a positive solution to the Saturation Problem, imply a strong version of Horn's conjecture: a triple (I,J,K) of subsets of cardinality r of $[1,n]$ is in his set T^n_r if and only if σ^K occurs in $\sigma^I \cdot \sigma^J$.

The theorems generalize to sums of more than two Hermitian matrices, and to tensor products of more than two representations. At the same time we will rephrase the theorems, which will make the proofs easier. Fix positive integers n and $m \geq 3$. For $1 \leq s \leq m$, let $\lambda^s = (\lambda^s_1 \geq \cdots \geq \lambda^s_n)$ be a decreasing sequence of n real numbers. For any m-tuple $I^1,\ldots,I^m$ of subsets of the same cardinality r of $[1,n]$, with $1 \leq r \leq n-1$, consider the following inequality:

$$(*_{I^1\ldots I^m}) \qquad \frac{1}{r}\sum_{s=1}^{m}\sum_{i\in I^s}\lambda^s_i \leq \frac{1}{n}\sum_{s=1}^{m}\sum_{i=1}^{n}\lambda^s_i.$$

We say that conditions $(*)$ are satisfied if the inequality $(*_{I^1\ldots I^m})$ is valid whenever $\sigma_{I^1}\cdot\ldots\cdot\sigma_{I^m}$ is a nonzero multiple of the class $\sigma_{[1,r]}$ (the class of a point) in the cohomology of $\mathrm{Gr}(r,n)$. In fact, it is an easy consequence of Pieri's formula for multiplying a Schubert class by a hyperplane, that conditions $(*)$ imply the inequality $(*_{I^1\ldots I^m})$ whenever $\sigma_{I^1}\cdot\ldots\cdot\sigma_{I^m}$ is not zero.

Theorem 1 (Klyachko).— *There are Hermitian $n\times n$ matrices $H^1,\ldots,H^m$ with $\lambda(H^s) = \lambda^s$ for all s and $\sum_s H^s = c\,1$ a scalar if and only if conditions $(*)$ are satisfied.*

Theorem 2 (Klyachko).— *Suppose each λ^s_i is an integer (or a rational number). There is a positive integer N such that the representation $V^{N\lambda^1}\otimes\cdots\otimes V^{N\lambda^m}$ contains the trivial representation of $\mathrm{SL}_n(\mathbb{C})$ if and only if conditions $(*)$ are satisfied.*

Theorem 3 (Belkale).— *The conditions $(*)$ follow from the subset of inequalities $(*_{I^1\ldots I^m})$ for which $\sigma_{I^1}\cdot\ldots\cdot\sigma_{I^m}$ is equal to the class $\sigma_{[1,r]}$.*

Theorem 1 is equivalent to the following assertion: given weakly decreasing n-tuples μ and λ^s, $1 \leq s \leq m$, with $\sum \mu_i = \sum_s\sum_i \lambda^s_i$, there are Hermitian matrices H^s with $\lambda(H^s) = \lambda^s$ and $H = \sum H^s$ having $\lambda(H) = \mu$, if and only if

$$\sum_{k\in K}\mu_k \leq \sum_{s=1}^{m}\sum_{i\in I^s}\lambda^s_i$$

for every $1 \le r \le n-1$, and subsets $I^1, \dots, I^m$ and K of $[1,n]$ of cardinality r such that σ^K occurs in $\prod \sigma^{I^s}$. To see this, apply Theorem 1 to the $(m+1)$-tuple $(-H^1, \dots, -H^m, H)$, and the subsets $I^{1\vee}, \dots, I^{m\vee}, K$. Conversely, if $\sum_{s=1}^m H^s = c1$, apply this to $(-H^1, \dots, -H^{m-1})$ and $H = H^m - c1$ to recover Theorem 1.

Similarly, Theorem 2 is equivalent to the claim that, given integral (or rational) weakly decreasing n-tuples μ and λ^s, $1 \le s \le m$, with $\sum \mu_i = \sum_s \sum_i \lambda_i^s$, the representation $V^{N\mu}$ is contained in $V^{N\lambda^1} \otimes \cdots \otimes V^{N\lambda^m}$ for some positive integer N if and only if the same conditions hold. When $m = 2$, we recover the earlier theorems. Note that these versions of the theorems are true for $m \ge 1$.

We do not know if the inequalities that appear in Theorem 3 are independent. It may also be remarked that the saturation problem is open for more than three representations, i.e., if the power N can be replaced by $N = 1$ in the preceding assertion. This does not seem to follow from the Saturation Problem (the case where $m = 2$).

It may be worth pointing out that there are no similar theorems for sums of arbitrary matrices. For example, if $A = \left(\begin{smallmatrix} \epsilon & a \\ 0 & -\epsilon \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} -\epsilon & 0 \\ a & \epsilon \end{smallmatrix}\right)$, then A and B have eigenvalues $\pm\epsilon$, while $C = A + B$ has eigenvalues $\pm a$, unrelated to the eigenvalues of A and B. On the other hand, the theorems can be applied to the *singular values* of a general matrix A, which are the eigenvalues of the matrix $\sqrt{AA^*}$; this is because these values, taken with positive and negative sign, are the eigenvalues of the matrix $\left(\begin{smallmatrix} 0 & A \\ A^* & 0 \end{smallmatrix}\right)$.

In the next section we present complete proofs of Theorems 1–3. The final section contains some applications, and a brief discussion of some more recent work on eigenvalues of unitary matrices.

PROOF OF THE THEOREMS

The necessity of the inequalities in Theorem 1 is relatively elementary, and has been known for some time, cf. [H-Z], [T-F], [J], [H-R]. Let $E \cong \mathbb{C}^n$ be an n-dimensional complex vector space with a Hermitian inner product $\langle\ ,\ \rangle$. For any subspace L of dimension r of E, and any Hermitian operator H on E, set (after Rayleigh)

$$R_H(L) = \sum_{i=1}^{r} \langle Hu_i, u_i \rangle,$$

where $u_1, \dots, u_r$ is an orthonormal basis of L. Equivalently, $R_H(L)$ is the trace of the composite $L \subset E \xrightarrow{H} E \to L$, where the map from E to L is orthogonal projection. Note that $R_{A+B}(L) = R_A(L) + R_B(L)$.

Let $F_\bullet(H)$ be a complete flag in E with F_p spanned by eigenvectors $v_1, \dots, v_p$, where $Hv_i = \lambda_i(H)v_i$ for $1 \le i \le n$. (This flag is unique if the eigenvalues are distinct; otherwise an arbitrary choice is made.)

Lemma 1 (Hersch-Zwahlen [H-Z]).— *For any subset I of $[1,n]$, we have*

$$\sum_{i\in I}\lambda_i(H) = \operatorname*{Min}_{L\in\Omega_I(F_\bullet(H))} R_H(L).$$

Proof. Suppose L is in $\Omega_I(F_\bullet(H))$. Choose a unit vector u_1 in $L\cap F_{i_1}$, and note that $\langle Hu_1,u_1\rangle \geq \lambda_{i_1}$. Choose a unit vector u_2 in $L\cap F_{i_2}$ that is perpendicular to u_1, and note that $\langle Hu_2,u_2\rangle \geq \lambda_{i_2}$. Continuing in this way, we see that $R_H(L)\geq \sum_{i\in I}\lambda_i$. The minimum occurs by taking L to be the span of $v_{i_1}, v_{i_2},\ldots,v_{i_r}$.

Now if $\sum H^s$ is a scalar $c1$, and $\prod\sigma_{I(s)}\neq 0$, then the intersection $\bigcap\Omega_{I^s}(F_\bullet(H^s))$ must contain some point L. Therefore

$$\sum_{s=1}^{m}\sum_{i\in I^s}\lambda_i(H^s) \leq \sum_{s=1}^{m} R_{H^s}(L) = R_{c1}(L) = c\,r,$$

where $c=\frac{1}{n}\sum_s\sum_i\lambda_i^s$. This proves that all the inequalities $(*_{I^1\ldots I^m})$ are satisfied. This part of the theorem was also proved by S. Johnson [J] and Helmke and Rosenthal [H-R].

To prove the converse, we start by showing that the polyhedral cone defined by the conditions $(*)$ has a nonempty interior.

Lemma 2.— *There are $\lambda^1,\ldots,\lambda^m$ such that each inequality $(*_{I^1\ldots I^m})$ holds with strict inequality, whenever the sum of the codimensions of the σ_{I^s} is at most the dimension $r(n-r)$ of the Grassmannian, and with $\lambda_i^s>\lambda_{i+1}^s$ for $1\leq s\leq m$ and $1\leq i\leq n-1$.*

Proof. Let $\lambda^s=(n-1,n-3,\ldots,-(n-3),-(n-1))$ for $1\leq s\leq m$. If $I^s=[n-r+1,n]$ for all s, so $\sigma_{I^s}=1$, then $\sum_{i\in I^s}\lambda_i^s=-r(n-r)$. Each change in an I^s by replacing one of its entries by the next smallest integer increases this sum by 2, while it increases the codimension of σ_{I^s} by 1. So if $\sum_s \operatorname{codim}\sigma_{I^s}\leq r(n-r)$, then

$$\frac{1}{r}\sum_{s=1}^{m}\sum_{i\in I^s}\lambda_i^s \leq \frac{-mr(n-r)+2r(n-r)}{r} = -(m-2)(n-r),$$

which is negative, since $m>2$, so it is strictly less than $\frac{1}{n}\sum_s\sum_i\lambda_i^s=0$.

We next show that it suffices to construct Hermitian operators H^s with $\lambda(H^s)=\lambda^s$, whose sum is scalar, provided the given λ^s satisfy the conditions $(*)$, with each inequality strict, each λ^s integral, and the n integers in each λ^s distinct. The reason for this is that a Hermitian matrix can be written in the form $H=UDU^*$, where D is diagonal and U is unitary. Using the compactness of the unitary group, it suffices to find Hermitian operators with eigenvalues arbitrarily close to the given eigenvalues λ^s, whose sum is a scalar. By Lemma 2 we may assume that each of the inequalities in $(*)$ is strict, and that the n numbers in each λ^s are rational and distinct. Finally, by multiplying each by a large integer N, we may assume all of the eigenvalues are integers.

Because we want the argument to apply also to the setting of Theorem 2, we assume to start only that each λ^s is integral. (Those interested only in Theorem 1 may make the extra assumptions of the previous paragraph.) We can also translate each λ^s by a constant, so we assume that $\lambda_n^s = 0$ for all s. Set $a_i^s = \lambda_i^s - \lambda_{i+1}^s$ for $1 \le i \le n-1$. Let X be the product of partial flag manifolds, so a point x of X is an m-tuple $(E_\bullet^1, \ldots, E_\bullet^m)$, where each $E_\bullet^s$ is a partial flag in the fixed vector space E, where the $E_\bullet^s$ includes a vector space E_k^s of dimension k exactly when a_k^s is positive. The partial flag variety has a canonical Plücker embedding in the product of projective spaces $\mathbb{P}(\wedge^k(E))$. Using the Veronese and Segre embeddings, we have a canonical embedding of X in $\mathbb{P}(V)$, where V is the tensor product of all the spaces $\mathrm{Sym}^{a_k^s}(\wedge^k(E))$. The bundle $\mathcal{L}$ which is the restriction of $\mathcal{O}(1)$ from $\mathbb{P}(V)$ to X, is a very ample bundle whose sections are canonically isomorphic to the dual of the space $V^{\lambda^1} \otimes \cdots \otimes V^{\lambda^m}$ (cf. [F]).

Take m general (complete) flags $E_\bullet^s$. Here general will mean that for any $1 \le r \le n-1$, and subsets $I^1, \ldots, I^m$ of $[1,n]$ of cardinality r, the intersection $\bigcap \Omega_{I^s}^\circ(E_\bullet^s)$ is not empty if and only if the class $\prod \sigma_{I^s}$ is not zero.

Lemma 3.— *Conditions* $(*)$ *are satisfied if and only if the following conditions* $(**)$ *are satisfied: for general complete flags* $E_\bullet^s$ *in* E, *and any nonzero subspace* F *of* E,

$$\frac{1}{\dim(F)} \sum_{s=1}^{m} \sum_{k=1}^{n-1} a_k^s \cdot \dim\,(F \cap E_k^s) \le \frac{1}{\dim(E)} \sum_{s=1}^{m} \sum_{k=1}^{n-1} a_k^s \cdot \dim\,(E_k^s)\,.$$

Proof. Note first that for any subset I^s of $[1,n]$,

$$\sum_{i \in I^s} \lambda_i^s = \sum_{i \in I^s} a_i^s + a_{i+1}^s + \cdots + a_{n-1}^s = \sum_{k=1}^{n-1} a_k^s \cdot |I^s \cap [1,k]|.$$

Therefore $(*)$ is equivalent to the following conditions $(*')$: for any subsets $I^1, \ldots, I^m$ of $[1,n]$ of cardinality r with $\prod \sigma_{I^s} \ne 0$,

$$\frac{1}{r} \sum_{s=1}^{m} \sum_{k=1}^{n-1} a_k^s \cdot |I^s \cap [1,k]| \le \frac{1}{n} \sum_{s=1}^{m} \sum_{k=1}^{n-1} a_k^s \cdot k.$$

Now let $E_\bullet^1, \ldots, E_\bullet^m$ be general flags. For subsets $I^1, \ldots, I^m$ of $[1,n]$ of cardinality r, we have $\prod \sigma_{I^s} \ne 0 \iff \bigcap \Omega_{I^s}^\circ(E_\bullet^s) \ne \emptyset \iff$ there is an r-dimensional subspace F of E with $\dim(F \cap E_k^s) = |I^s \cap [1,k]|$ for all s and k. Note that any r-dimensional subspace F determines subsets I^s so that $\dim(F \cap E_k^s) = |I^s \cap [1,k]|$ for $1 \le k \le n-1$. From this the equivalence of $(*')$ and $(**)$ follows.

Note that conditions $(**)$ involve only partial flags $E_\bullet^s$ that include subspaces E_k^s of dimension k for those k with $a_k^s > 0$. These general flags determine a point x in X. The group $G = \mathrm{SL}(E)$ acts on X, compatibly with its natural action on $\mathbb{P}(V)$.

Proposition.— *A general point* $x = (E_\bullet^1, \ldots, E_\bullet^m)$ *in* X *is semistable with respect to* G *and* $\mathcal{L}$ *if and only if conditions* $(*)$ *are satisfied.*

Proof. We use the Hilbert-Mumford criterion for stability [M-F-K]. For any 1- parameter subgroup $\rho : \mathbb{C}^* \to G$, there is a number $\mu^{\mathcal{L}}(x, \rho)$, which is nonnegative (resp. positive) for all ρ if and only if the point x is semistable (resp. stable). To compute this, choose a basis $e_1, \ldots, e_n$ of E such that $\rho(t) \cdot e_i = t^{r_i} e_i$, with $r_1 \geq \cdots \geq r_n$ integers, not all zero, and $\sum r_i = 0$. Let $F_\bullet$ be the complete flag with F_p is spanned by $e_1, \ldots, e_p$. For a point $y = [y_1 : \cdots : y_n]$ in $\mathbb{P}(E)$, $\mu^{\mathcal{O}(1)}(y, \rho) = -r_i$, where i is maximal with $y_i \neq 0$. By looking at the Plücker, Veronese, and Segre mappings that embed X in $\mathbb{P}(V)$, one finds that

$$\mu^{\mathcal{L}}(x, \rho) = \sum_{s=1}^{m} \sum_{k=1}^{n-1} a_k^s \Big(- \sum_{i \in J_k^s} r_i \Big),$$

where J_k^s is the subset of cardinality k of $[1, n]$ such that E_k^s is in $\Omega^\circ_{J_k^s}(F_\bullet)$. Since this is a linear function of $(r_1, \ldots, r_n)$, it will be nonnegative for all $(r_1, \ldots, r_n)$ as above if and only if it is nonnegative for $(r_1, \ldots, r_n) = (n-p, \ldots, n-p, -p, \ldots, -p)$, where the $n-p$ is repeated p times, and the $-p$ is repeated $n-p$ times. In this case

$$\sum_{i \in J} r_i = (n-p)\,|J \cap [1,p]| - p\,|J \cap [p+1, n]| = n\,|J \cap [1,p]| - p\,|J|.$$

Now $|J_k^s \cap [1,p]| = \dim(F_p \cap E_k^s)$. Therefore

$$\mu^{\mathcal{L}}(x, \rho) = -n \sum_{s=1}^{m} \sum_{k=1}^{n-1} a_k^s \cdot \dim\left(F_p \cap E_k^s\right) + p \sum_{s=1}^{m} \sum_{k=1}^{n-1} a_k^s \cdot k.$$

Condition $(**)$ therefore says precisely that $\mu^{\mathcal{L}}(x, \rho) \geq 0$ for all such ρ. Note that any F_p occurs for some such ρ, so the semistability of a general x implies $(**)$.

We now prove Theorem 1. As we have seen, we may also assume that each of the inequalities in $(*)$ is strict, and that the n integers in each λ^s are distinct. In this case all of the inequalities in $(**)$ are also strict. This means that a general point x in X is actually a stable point for the action of G and the line bundle $\mathcal{L}$. If $v \in V$ is a point representing x in $\mathbb{P}(V)$, stability implies that the orbit $G \cdot v$ is closed in V.

Choose an arbitrary Hermitian metric on E. This determines a metric on exterior and symmetric powers, so on $V = \bigotimes \mathrm{Sym}^{a_k^s}(\wedge^k E)$. Since $G \cdot v$ is closed, there is a point $g \cdot v$ in $G \cdot v$ that is closest to the origin. Replacing the given flags $E_\bullet^s$ by the flags $g(E_\bullet^s)$, we may assume that v is the closest point to the origin in the orbit, i.e., that $\|g \cdot v\| \geq \|v\|$ for all $g \in G$.

For each s take an orthonormal basis $e_1^s, \ldots, e_n^s$ of E such that $e_1^s, \ldots, e_k^s$ is a basis for E_k^s for all k. Define a Hermitian operator H^s on E by the rule $H^s(e_i^s) = \lambda_i^s \, e_i^s$ for all i.

Lemma 4.— *If $\|g \cdot v\| \geq \|v\|$ for all $g \in G$, then $\sum_s H^s$ is a scalar.*

Proof. We may take $v = v^1 \otimes \cdots \otimes v^m$, where

$$v^s = (e_1^s)^{a_1^s} \otimes (e_1^s \wedge e_2^s)^{a_2^s} \otimes \cdots \otimes (e_1^s \wedge \ldots \wedge e_{n-1}^s)^{a_{n-1}^s} \in \bigotimes_{k=1}^{n-1} \mathrm{Sym}^{a_k^s}(\wedge^k E).$$

The fact that $g \mapsto \|g \cdot v\|$ has a critical point at $g = 1$ is equivalent to the assertion that $\sum_s \ell^s(g) = 0$ for all $g \in \mathfrak{sl}(E)$, where $\ell^s(g) = \langle g \cdot v^s, v^s \rangle + \langle v^s, g \cdot v^s \rangle$. We claim that

$$\ell^s(g) = \mathrm{Trace}((g + g^*) \cdot H^s)$$

for all $g \in \mathfrak{sl}(E)$. From this it follows that $\mathrm{Trace}((g + g^*) \cdot \sum_s H^s) = 0$ for all $g \in \mathfrak{sl}(E)$, and this implies easily that $\sum_s H^s$ is a scalar. The claim is proved by a straightforward calculation. Indeed, using the basis $e_1^s, \ldots, e_n^s$ for E, note that both sides vanish when g is a strictly upper triangular or lower triangular matrix. Since both sides are additive in g, it therefore suffices to verify the claim with g diagonal. And if $g \cdot e_i^s = c_i e_i^s$, with $\sum c_i = 0$, then $g \cdot v^s = \sum_{k=1}^{n-1} a_k^s (c_1 + \cdots + c_k) v^s$, so

$$\ell^s(g) = \sum_{k=1}^{n-1} (c_k + \overline{c_k})(a_k^s + \cdots + a_{n-1}^s) = \sum_{k=1}^{n} (c_k + \overline{c_k}) \lambda_k^s = \mathrm{Trace}((g + g^*) \cdot H^s).$$

For the proof of Theorem 2, we assume the λ_i^s are integral and the conditions $(**)$ are satisfied, but only weakly. We have seen that this makes the general point x in X semistable. This means that there is a section s of $\mathcal{L}^{\otimes N}$, for some positive N, that is invariant by $\mathrm{SL}(E)$, with $s(x) \neq 0$. Since $\Gamma(X, \mathcal{L}^{\otimes N}) = (\bigotimes V^{N\lambda^s})^\vee$, this means that $\bigotimes V^{N\lambda^s}$ contains the trivial representation of $\mathrm{SL}(E)$. Conversely, if $\bigotimes V^{N\lambda^s}$ contains the trivial representation, then there is a nonzero invariant section s of $\mathcal{L}^{\otimes N}$, and then a general point x with $s(x) \neq 0$ will be semistable; we conclude by the proposition that $(*)$ is satisfied.

Belkale [B] proves Theorem 3 as follows. Assume the inequalities specified in Theorem 3. If Theorem 3 were false, there would be a subspace F of E violating the inequality of Lemma 3, for general complete flags $E_\bullet^s$. Let $\mu(F)$ be the left side of the inequality in Lemma 3. He shows that if μ is the maximum of $\mu(F)$ as F varies over all subspaces, and F is a subspace of maximum dimension with $\mu(F) = \mu$, then F is unique. (This follows a standard argument for slopes of destabilizing subbundles.) But this means that F is the unique point in the intersection of the corresponding Schubert varieties, so the product of the corresponding Schubert classes is the class of a point. Since we are assuming the inequalities in this case, this is a contradiction.

For example, if $m = 3$, $n = 6$, and $r = 3$, and $I(s) = \{2, 4, 6\}$ for all s, then the product of the Schubert classes is twice the class of a point. The corresponding inequality is equivalent to the inequality $\sum_s \lambda_2^s + \lambda_4^s + \lambda_6^s \leq \sum_s \lambda_1^s + \lambda_3^s + \lambda_5^s$. But this follows from the inequalities $\lambda_2^s \leq \lambda_1^s$, $\lambda_4^s \leq \lambda_3^s$, and $\lambda_6^s \leq \lambda_5^s$.

RELATED RESULTS

An examination of the proof shows that the same theorem is true for real symmetric matrices, although this result is not stated in [K]. The point is that one can take the space E to be the complexification of a real vector space $E_{\mathbb{R}}$, with its metric coming from an inner product on $E_{\mathbb{R}}$. The real flags are Zariski dense in X, so one can take the general flags $E^s_\bullet$ to be complexification of real flags. The resulting Hermitian operators H^s will then come from real symmetric operators on $E_{\mathbb{R}}$. This proves that Theorem 1, in all its variations, is valid as stated for real symmetric matrices. In particular:

Theorem 4.— *Given weakly decreasing n-tuples μ and λ^s, $1 \le s \le m$, with $\sum \mu_i = \sum_s \sum_i \lambda_i^s$, there are real symmetric matrices A^s with $\lambda(A^s) = \lambda^s$ for each s, with $A = \sum A^s$ having $\lambda(A) = \mu$, if and only if*

$$\sum_{k \in K} \mu_k \le \sum_{s=1}^{m} \sum_{i \in I^s} \lambda_i^s$$

for every $1 \le r \le n-1$, and subsets $I^1, \ldots, I^m$ and K of $[1,n]$ of cardinality r such that σ^K occurs in $\prod_s \sigma^{I^s}$ in the cohomology of $\mathrm{Gr}(r,n)$.

Inequalities of the form $(*)$ have also appeared in the literature for problems about principal ideal domains, and Klyachko's theorem gives some insight into these problems. If R is a principal ideal domain, any finitely generated torsion module is isomorphic to $R/d_1 \oplus R/d_2 \oplus \cdots \oplus R/d_n$ for nonzero ideals d_i with $d_1 \subset d_2 \subset \cdots \subset d_n$; these invariant factors are unique if only proper ideals are included, but we allow some unit ideals at the end. Suppose we have an exact sequence

$$0 \to \mathcal{A} \to \mathcal{C} \to \mathcal{B} \to 0$$

of such modules, where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ have invariant factors a_i, b_i, and c_i, $1 \le i \le n$. One is interested in the possibilities for such invariants. Equivalently, one can consider $n \times n$ matrices A, B, and C with entries in R and nonzero determinants, such that $C = B \cdot A$. One wants to relate the invariants of C to those of A and B. The invariants of A are the same as the invariants of the cokernel of the map from R^n to R^n given by A. The equivalence of the two versions of the problem is seen by applying the snake lemma to the diagram:

$$\begin{tikzcd} R^n \arrow[r, "A"] \arrow[d, "C"'] & R^n \arrow[d, "B"] \\ R^n \arrow[r, equal] & R^n \end{tikzcd} \tag{1}$$

Various divisibility conditions have been given for these invariants, all of the form

$$\prod_{k \in K} c_k \;\Big|\; \prod_{i \in I} a_i \cdot \prod_{j \in J} b_j \tag{$***_{IJK}$}$$

for certain subsets I, J, and K of the same cardinality r in $[1,n]$. There are many instances of these inequalities in the literature (cf. [T] and [Thi]), some for the case when $R = \mathbb{C}\{z\}$. The following result was proved combinatorially by Thompson [T]:

Theorem 5.— *The divisibility conditions* $(***_{IJK})$ *are valid whenever* σ^K *occurs in* $\sigma^I \cdot \sigma^J$ *in the cohomology of* $\mathrm{Gr}(r,n)$.

Proof. By localizing one may assume R is a discrete valuation ring, with maximal ideal $\mathfrak{p}$. Write $a_i = \mathfrak{p}^{\alpha_i}$, $b_i = \mathfrak{p}^{\beta_i}$, $c_i = \mathfrak{p}^{\gamma_i}$. It is a theorem of T. Klein (see [Mac], Chapter II) that the existence of such an exact sequence is equivalent to the nonvanishing of the Littlewood-Richardson coefficient $c_{\alpha\beta}^{\gamma}$. By Theorem 2, this nonvanishing implies the inequalities $(*_{IJK})$, which is the same as $(***_{IJK})$.

It has been conjectured (see [T]) that inequalities of this type should suffice to determine the existence of matrices $C = BA$ with given invariants. In fact, we see that the converse to Theorem 5 (assuming that $\prod_{i=1}^n c_i = \prod_{i=1}^n a_i \prod_{i=1}^n b_i$) would follow from a positive solution to the Saturation Problem.

In addition to the two theorems of Klyachko discussed here, there are other interesting ideas in [K] that we can mention only briefly. One of these relates the problem to equivariant vector bundles on the plane $\mathbb{P}^2$. (Earlier work of Klyachko described equivariant vector bundles on toric varieties, so one may surmise that this was the origin of his work on this problem.) The group is the torus T of diagonal 3×3 matrices of determinant 1. Indeed, Klyachko had previously shown that to give a T-equivariant vector bundle of rank n on the plane is equivalent to giving three flags $E_\bullet^s$ on a vector space E and three weakly decreasing sequences λ^s of integers. (The space E is the fiber of the bundle at the central point $p = [1:1:1]$, and the filtrations and sequences are determined by looking at the behavior of limits of points $t \cdot e$, for e in E, as $t \cdot p$ approaches one of the three fixed points of the torus.) When the flags are general, condition $(*)$ is equivalent to the semistability of this vector bundle: the ratio of the first Chern class to the rank is no larger for any subbundle than it is for the bundle. For m bundles, there is a similar relation on $\mathbb{P}^{m-1}$, but here one must consider equivariant coherent sheaves and not just vector bundles.

For $n = 2$, a traceless Hermitian matrix has the form

$$H = \begin{pmatrix} a & b+ic \\ b-ic & -a \end{pmatrix},$$

and the positive eigenvalue of H is the length of the vector $v = (a,b,c)$ in $\mathbb{R}^3$. Changing H to UHU^* for U unitary corresponds to a rotation of the vector v. An equation $H^1 + \cdots + H^m = 0$ corresponds to an equation $v^1 + \cdots + v^m = 0$. The quotient moduli space $X/\!/\,\mathrm{GL}(E)$, constructed via GIT with respect to the ample bundle $\mathcal{L}$, can be identified with the space of m-tuples of vectors in 3-space, with given integral lengths, whose sum is zero, modulo rotations. In particular, this gives the structure of an algebraic variety to this moduli space of polygons. One has similar moduli spaces for all n. Theorem 1 gives a criterion for them to be nonempty. For recent work on these spaces see [H-K] and [K-M].

There is some recent work of Agnihotri and Woodward [A-W], and, independently, Belkale [B], that characterizes the possible eigenvalues of unitary matrices $A_1, \ldots, A_m$, each of determinant 1, whose product $A_1 \cdot \ldots \cdot A_m$ is the identity matrix. For $A \in \mathrm{SU}(n)$, its eigenvalues can be written uniquely in the form $e^{2\pi i \lambda_1}, \ldots, e^{2\pi i \lambda_n}$, with $\sum \lambda_i = 0$ and

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_1 - 1.$$

We set $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$. The problem is to characterize the m-tuples $\lambda^1, \ldots, \lambda^m$ that can occur, i.e. so that $\lambda^s = \lambda(A_s)$ for some $A_s \in \mathrm{SU}(n)$ with $A_1 \cdot \ldots A_m = 1$. Their result is that the answer is controlled, not by Schubert calculus, but by *quantum* Schubert calculus. Necessary and sufficient conditions are that inequalities

$$\sum_{s=1}^{m} \sum_{i \in I^s} \lambda_i^s \leq d$$

are valid whenever the Gromov-Witten number $\langle \sigma_{I^1}, \ldots, \sigma_{I^m} \rangle_d \neq 0$, for subsets $I^1, \ldots I^m$ of $[1, n]$ of the same cardinality r. For this Gromov-Witten number to be nonzero, it is necessary that $\sum_s \operatorname{codim} \sigma_{I^s} = dn + r(n-r)$, and then it is the number of maps from $\mathbb{P}^1$ to $\mathrm{Gr}(r, n)$ of degree d such that, for fixed points $p_1, \ldots, p_m$ in $\mathbb{P}^1$, the image of p_s maps lies in a given general Schubert variety in the class σ_{I^s}. The nonvanishing of this Gromov-Witten number is equivalent to the class $q^d \sigma^{I^m} = q^d \sigma_{I^{m\vee}}$ occurring in the quantum product $\sigma_{I^1} * \cdots * \sigma_{I^{m-1}}$, in the (small) quantum cohomology ring of the Grassmannian $\mathrm{Gr}(r, n)$. There are algorithms for computing in this ring (cf. [B-C-F]), but one does not yet have a quantum generalization of the Littlewood-Richardson rule; nor are there simple criteria — proved or conjectured (cf. Horn's conjecture) — for the nonvanishing of a Gromov-Witten number.

The reason that one can expect such a relation with quantum Schubert calculus can be seen roughly as follows. The matrices A_s correspond to a map from the fundamental group of $\mathbb{P}^1 \smallsetminus \{p_1, \ldots p_m\}$ to $\mathrm{SU}(n)$. By the work of Mehta and Seshadri [M-S], one knows that this amounts to giving a vector bundle $\mathcal{E}$ of rank n and degree 0 on $\mathbb{P}^1$ with filtrations at each of the points p_s and weights λ^s, each satisfying the conditions $\lambda_1^s \geq \cdots \geq \lambda_n^s \geq \lambda_1^s - 1$. In fact, if there are such bundles, they can be found with a trivial underlying holomorphic structure, and where the filtrations are generic. A subbundle $\mathcal{F}$ of rank r of $\mathcal{E}$ is then given by a mapping ϕ from $\mathbb{P}^1$ to $\mathrm{Gr}(r, n)$. The parabolic slope of $\mathcal{F}$ is $-\deg(\phi) + \sum_s \sum_{i \in I^s} \lambda_i^s$, where I^s describes the position of the r-plane $\phi(p_i)$ with respect to the filtration at p_i. The bundle $\mathcal{E}$ will be parabolically semistable, which means that all the parabolic slope of all such $\mathcal{F}$ will be nonnegative.

Both of these theorems about eigenvalues imply that the possible eigenvalues that arise form a convex polyhedron, which is explicitly described. Even the fact that it is a convex polyhedron is not immediately obvious, but this follows from general theorems about moment mappings in symplectic geometry (see [A-W]).

Note added in proofs: The saturation problem has been solved affirmatively by A. Knutson and T. Tao, in "The honeycomb model of the Berenstein-Zelevinsky polytope I. Klyachko's saturation conjecture" math.RT/9807160.

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