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ON AN ADDITIVE PROBLEM OF ERDŐS AND STRAUS, 2

by

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Abstract. — We denote by $s^\wedge A$ the set of integers which can be written as a sum of s pairwise distinct elements from A. The set A is called admissible if and only if $s \neq t$ implies that $s^\wedge A$ and $t^\wedge A$ have no element in common.

P. Erdős conjectured that an admissible set included in $[1, N]$ has a maximal cardinality when A consists of consecutive integers located at the upper end of the interval $[1, N]$. The object of this paper is to give a proof of Erdős' conjecture, for sufficiently large N.

Let $\mathcal{A}$ be a set of positive integers having the property that each time an integer n can be written as a sum of distinct elements of $\mathcal{A}$, the number of summands is well defined, in that the integer n cannot be written as a sum of distinct elements of $\mathcal{A}$ with a different number of summands. This notion has been introduced by P. Erdős in 1962 (cf. [**2**]) and called **admissibility** by E.G. Straus in 1966 (cf. [**5**]). In other words, if we denote by $s^\wedge\mathcal{A}$ the set of integers which can be written as a sum of s pairwise distinct elements from $\mathcal{A}$ then $\mathcal{A}$ is **admissible** if and only if $s \neq t$ implies that $s^\wedge\mathcal{A}$ and $t^\wedge\mathcal{A}$ have no element in common.

Erdős conjectured that an admissible subset $\mathcal{A}$ included in $[1, N]$ has a cardinality which is maximal when $\mathcal{A}$ consists of consecutive integers located at the upper end of the interval $[1, N]$. As it was computed by E.G. Straus, the set

$$\{N - k + 1, N - k + 2, \ldots, N\}$$

is admissible if and only if $k \leq 2\sqrt{N + 1/4} - 1$.

Straus himself proved that $\sqrt{N}$ is the right order of magnitude for the cardinality of a maximal admissible subset from $[1, N]$. More precisely, he proved the inequality $|\mathcal{A}| \leq (4/\sqrt{3} + o(1))\sqrt{N}$. The constant involved has been slightly reduced by P. Erdős, J-L. Nicolas and A. Sárközy (cf. [**3**]) and we proved (cf. [**1**]) the inequality

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$|\mathcal{A}| \leq (2+o(1))\sqrt{N}$. The object of this paper is to give a proof of Erdős conjecture, at least when N is sufficiently large.

Theorem 1. — *There exists an integer N_0, effectively computable, such that for any integer $N \geq N_0$ and any admissible subset $\mathcal{A} \subset [1, N]$ we have*

$$\mathrm{Card}\, \mathcal{A} \leq 2\sqrt{N+1/4} - 1.$$

The proof is based on the description of the structure of large admissible sets we obtained previously, namely :

Theorem 2 **(J-M. Deshouillers, G.A. Freiman [1]).** — *Let $\mathcal{A}$ be an admissible set included in $[1, N]$, such that* $\mathrm{Card}\, \mathcal{A} > 1.96\sqrt{N}$. *If N is large enough, there exist $\mathcal{C} \subset \mathcal{A}$ and an integer q having the following properties :*
(i) $\mathrm{Card}\, \mathcal{C} \leq 10^5 N^{5/12}$,
(ii) for some t the set $t^\wedge\mathcal{C}$ contains at least $3N^{5/6}$ terms in an arithmetic progression modulo q,
(iii) $\mathcal{A}\setminus\mathcal{C}$ is included in an arithmetic progression modulo q containing at most $N^{7/12}$ terms.

Although we do not develop this point, it will be clear from the proof that our arguments may be used to describe the structure of maximal admissible subsets of $[1, N]$, leading for example to the fact that when N has the shape n^2 or n^2+n (and n sufficiently large), the Erdős - Straus example is the only maximal subset of $[1, N]$.

1. We first establish a lemma expressing the fact that if a set of integers $\mathcal{D}$ is part of a finite arithmetic progression with few missing elements, then the same is locally true for $s^\wedge\mathcal{D}$.

Proposition 1. — *Let us consider integers r, s, t and a, q such that $t \geq 2s - q$, $s \geq 4r + 3 + q$ and $0 \leq a < q$.*
Let further $\mathcal{D} = \{d_1 < d_2 < \cdots < d_t\}$ be a set of t distinct integers congruent to a modulo q such that $d_t - d_1 = (t-1+r)q$, and denote by m (resp. M) the smallest (resp. largest) element in $s^\wedge\mathcal{D}$. Then, among $2r+1$ consecutive integers congruent to sa modulo q and laying in the interval $[m, M]$, at least $r+1$ belong to $s^\wedge\mathcal{D}$.

Proof. — We treat the special case when $a = 0, q = 1$ and $\mathcal{D}$ is included in $[1, t]$. We notice that the general case reduces to this one by writing
$d_l = d_1 + q(\delta_l - 1)$ and considering the set $\{\delta_1, \ldots, \delta_t\}$.

Let x be an integer in $s^\wedge\mathcal{D} \cap [m, (m+M)/2]$. We first show that the interval $[x, x+3r]$ contains at least $2r+1$ elements from $s^\wedge\mathcal{D}$. Since x is in $s^\wedge\mathcal{D}$, we can find $d(1) < \cdots < d(s)$, elements in $\mathcal{D}$, the sum of which is x.

Let us show that $d(1)$ is less than $t - s - 3r$. On the one hand we have

$$m + M \leq (r+1) + \cdots + (r+s) + (t+r-s+1) + \cdots + (t+r) = \frac{s}{2}(2t+4r+2),$$

and on the other hand we have

$$x \geq d(1) + (d(1)+1) + \cdots + (d(1)+s-1) = \frac{s}{2}(2d(1)+s-1).$$

The inequality $x \leq (m+M)/2$ implies that we have

$$2d(1) + s - 1 \leq t + 2r + 1,$$

whence

$$2d(1) \leq 2(t - s - 3r) - (t - s - 4r - 2),$$

and we notice that $t - s - 4r - 2$ is positive, by the assumptions of Proposition 1.

Since $d(1)$ is less than $t-s-3r$, the interval $[d(1), t+r]$ contains at least $s+4r+1$ integers. We denote by $i_1 < \cdots < i_l$ the indexes of those $d's$ such that $d(i_k+1) - d(i_k) \geq 2$, with the convention that $d(i_l+1) = 3Dt+r+1$ in the case when $d(s) < t+r$. The set

$$\bigcup_{k=1}^{l}]d(i_k)+1, d(i_k+1)-1[$$

contains at least $4r+1$ integers. We now suppress from those intervals those which contain no element from $\mathcal{D}$, and we rewrite the remaining ones as

$$]d(j_1)+1, d(j_1+1)-1[, \ldots,]d(j_h)+1, d(j_h+1)-1[.$$

They contain at least $3r+1$ integers, among which at most r are not in $\mathcal{D}$.

Let us define u_1 to be the largest integer such that $d(j_1)+u_1$ is in $\mathcal{D}$ and is less than $d(j_1+1)$, and let us define $u_2, \ldots, u_h$ in a similar way. We consider the integers

$$x = y + d(j_1) + \cdots + d(j_h) \quad \text{(which defines } y\text{)},$$

$$x + 1 = y + d(j_1) + 1 + d(j_2) + \cdots + d(j_h),$$

$$\ldots$$

$$x + u_1 = y + d(j_1) + u_1 + d(j_2) + \cdots + d(j_h),$$

$$\ldots$$

$$x + u_1 + \cdots + u_h = y + d(j_1) + u_1 + d(j_2) + u_2 + \cdots + d(j_h) + u_h.$$

One readily deduces from this construction that the interval

$$[x, x + \min(3r, u_1 + \cdots + u_h)]$$

contains at most r elements which are not in $s^\wedge\mathcal{D}$.

What we have proven so far easily implies that any interval $[z-r, z]$ with $m \leq z \leq (M+m)/2$ contains at least one element in $s^\wedge\mathcal{D}$. Let us consider an interval $[y, y+2r]$ with $m \leq y \leq (M+m)/2$. By what we have just said, the interval $[y-r, y]$ contains an element in $s^\wedge\mathcal{D}$, let us call it x. As we have shown the interval $[x, x+3r]$ contains at most r integers not in $s^\wedge\mathcal{D}$, so that $[y, y+2r]$ contains at most r integers not in $s^\wedge\mathcal{D}$, which is equivalent to say that it contains at least $r+1$ elements from $s^\wedge\mathcal{D}$.

A similar argument taking into account decreasing sequences and starting with M shows that any interval $[y-2r, y]$ with $(m+M)/2 \leq y \leq M$ contains at least $r+1$ elements from $s^\wedge\mathcal{D}$.

2. We now prove the following result concerning the structure of a large admissible finite set.

Theorem 3. — *Let $\mathcal{A} = \{a_1 < \cdots < a_A\}$ be an admissible subset of $[1, N]$ with cardinality $A = 2N^{1/2} + O(N^{5/12})$, and let us define q to be the largest integer such that $\mathcal{A}$ is contained in an arithmetic progression modulo q. We have $q = O(N^{5/12})$ and there exists an integer u in $[N^{11/24}, 2N^{11/24}]$ such that*

$$a_{A-u} - a_{u+1} = q(2N^{1/2} + O(N^{11/24})).$$

Proof. — The proof is based on the structure result we quoted in the introduction as Theorem 2. We keep its notation and first show that an integer q satisfying (ii) and (iii) is indeed the largest integer such that $\mathcal{A}$ is contained in an arithmetic progression modulo q. We let $\mathcal{B}$ denote $\mathcal{A} \setminus \mathcal{C}$.

A simple counting argument will show that $\mathcal{A}$ is included in the same arithmetic progression as $\mathcal{B}$. Otherwise, let us consider an element $a \in \mathcal{A}$ which is not in the same arithmetic progression as $\mathcal{B}$ modulo q. The set $s^\wedge\mathcal{A}$ contains the disjoint sets $s^\wedge\mathcal{B}$ and $a + (s-1)^\wedge\mathcal{B}$. We thus have $|s^\wedge\mathcal{A}| \geq |s^\wedge\mathcal{B}| + |(s-1)^\wedge\mathcal{B}|$. It is well-known (cf. [4] for example) that $|s^\wedge\mathcal{B}| \geq s(|\mathcal{B}| - s)$ for $s \leq |\mathcal{B}|$, and since $\mathcal{A} \subset [1, N]$ is admissible we have

$$\begin{array}{rcl} N(|\mathcal{B}|+1) & \geq & \text{Card } (\bigcup_s (s^\wedge\mathcal{B} \cup (a + (s-1)^\wedge\mathcal{B})) \\ & \geq & 2\sum_s |s^\wedge\mathcal{B}| \geq 2\sum_s s = 20(|\mathcal{B}| - s) = \frac{1}{3}|\mathcal{B}|^3 + O(N), \end{array}$$

which implies $|\mathcal{B}| \leq (\sqrt{3} + o(1))\sqrt{N}$, so that we have $|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| \leq (\sqrt{3} + o(1))\sqrt{N}$, a contradiction.

We have so far proven that q divides $g := gcd(a_2 - a_1, \ldots, a_A - a_1)$. Property (ii) implies that q is a multiple of g, so that we have $q = g$, as we wished to show.

The second step in the proof consists in showing that for $0 < k \leq |\mathcal{B}| - q$, any element in $k^\wedge\mathcal{B}$ is less than any element in $(k+q)^\wedge\mathcal{B}$. Let us call J the $3N^{5/6}$ consecutive terms of the arithmetic progression modulo q, the existence of which is asserted in (ii). Since $\mathcal{B}$ is included in an arithmetic progression modulo q with less that $3N^{5/6}$ terms, the sets $k^\wedge\mathcal{B} + J$ and $(k+q)^\wedge\mathcal{B} + J$ consists of consecutive terms of arithmetic progressions modulo q, and moreover, they are in the *same* class modulo q. Since $\mathcal{A}$ is admissible, the sets $k^\wedge\mathcal{B} + J$ (included in $(k+t)^\wedge\mathcal{A}$) and $(k+q)^\wedge\mathcal{B} + J$ (included in $(k+q+t)^\wedge\mathcal{A}$) do not intersect. To prove that any element of $k^\wedge\mathcal{B}$ is less that any element of $(k+q)^\wedge\mathcal{B}$, it is now sufficient to notice that $k^\wedge\mathcal{B}$ contains an element (we can consider the smallest element of $k^\wedge\mathcal{B}$), which is smaller than some element of $(k+q)^\wedge\mathcal{B}$.

We now prove that $q = O(N^{5/12})$. The cardinality of $\mathcal{A}$ and Theorem 2 imply that $|\mathcal{B}| = 2N^{1/2} + O(N^{5/12})$. We choose k so that $2k + q$ is $|\mathcal{B}|$ or $|\mathcal{B}| - 1$. (We notice that this is always possible since $\mathcal{A}$ contains at least $N^{1/2}$ integers from $[1, N]$ in an arithmetic progression modulo q, so that $q \leq N^{1/2}$). By the second step, the largest element in $k^\wedge\mathcal{B}$ is smaller than the largest element in $(k+q)^\wedge\mathcal{B}$. Let z be $(k+q)$-th element from $\mathcal{B}$, in the increasing order. We have

$$z \leq N - (k-1)q$$

and

$$(z+q)+\cdots+(z+qk) \le z+(z-q)+\cdots+(z-(k+q-1)q) \quad ;$$

by an easy computation, we get

$$(q+2k)^2 \le 2N+2k^2+3q,$$

but $2k+q=|\mathcal{B}|+O(1)=|\mathcal{A}|+O(N^{5/12})$, which implies

$$2k^2 \ge 2N(1+O(N^{-1/12})),$$

so that we have

$$k=N^{1/2}+O(N^{5/12}).$$

We now use again the same argument, being more precise. Let us write $\mathcal{B}=\{b_1 < \cdots < b_{k+q} < b_{k+q+1} < \cdots < b_{2k+q} \le b_B\}$. We have

$$b_{k+q+1}+\cdots+b_{2k+q} < b_1 \cdots + b_k + b_{k+1} + b_{k+q}.$$

Let t be any integer in $[1,k]$. We have

$$b_{k+1}+\cdots+b_{k+q} > (b_{2k+q}-b_1)+\cdots+(b_{2k+q-t+1}-b_t)+\cdots+(b_{k+q+1}-b_k).$$

We clearly have the inequalities

$$\begin{aligned}
&b_{k+q+1}-b_k \ge (q+1)q,\\
&b_{k+q+2}-b_{k-1} \ge (q+3)q,\\
&\cdots\\
&b_{2k+q-t-1}-b_{t+2} \ge (q+1+2(k-t-2))q,\\
&b_{2k+q-t}-b_{t+1} \ge b_{2k+q-t}-b_{t+1},\\
&b_{2k+q-t+1}-b_t \ge (b_{2k+q-t}-b_{t+1})+2q,\\
&\cdots\\
&(b_{2k+q}-b_1) \ge (b_{2k+q-t}-b_{t+1})+2tq.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
b_{k+1}+\cdots+b_{k+q} \quad &> (t+1)(b_{2k+q-t}-b_{t+1})\\
&+q\textstyle\sum_{l=0}^{k-t-2}(q+1+2l)+q\sum_{h=0}^{t}2h.
\end{aligned}$$

Taking into account that $b_{k+q} \le N-kq$, a dull computation leads to

$$(t+1)(b_{2k+q-t}-b_{t+1}) \le q(N-k^2+2kt+O(N^{11/12})),$$

when $t=O(N^{11/24})$. This in turn leads to

$$b_{2k+q-t}-b_{t+1} \le q(2k+O(N^{11/24})),$$

when $t=\frac{3}{2}N^{11/24}+O(1)$.
Let C the cardinality of $\mathcal{C}$. Since $\mathcal{A}=\mathcal{B}\cup\mathcal{C}$, we have

$$b_{t+1} \le a_{t+C} \le a_{A+t+C-2k-q+1} \le a_{2k+q-C-t} \le b_{2k+q-t} \quad ;$$

we choose $u=A+t+C-2k-q$ and recall that $A-2k-q \le C+1=O(N^{5/12})$, so that Theorem 3 is proven.

3. We now embark on the proof of Theorem 1 which will follow from Theorem 3 and Proposition 1. Let $\mathcal{A}$ be an admissible subset of $[1, N]$ with maximal cardinality. By [**1**], we know that $A = 2\sqrt{N} + O(N^{5/12})$, so we can apply Theorem 3 : there exists integers u and r such that

$$a_{A-u} - a_{u+1} = q(A - 2u + r),$$

with $u \in [N^{11/24}, 2N^{11/24}]$ and $r = O(N^{11/24})$.

We let

$$\mathcal{D} := \mathcal{A} \cap [a_{u+1}, a_{A-u}], \quad t := A - 2u, \quad \sigma := [(t-q)/2],$$

and we shall apply Proposition 1 with $s = \sigma$ and $s = \sigma + q$ (one readily checks that the conditions of application of Proposition 1 are fulfilled). Let us further denote by $m(s)$ (resp. $M(s)$) the smallest (resp. largest) element in $s^\wedge\mathcal{D}$.

As a first step, we show that $a_1 + a_2 + \cdots + a_q$ cannot be too small. We have

$$M(\sigma) - m(\sigma) \geq (a_{A-u-\sigma+1} - a_{u+\sigma}) + \cdots + (a_{A-u} - a_{u+1})$$

$$\geq q(2 + 4 + \cdots + 2(\sigma - 1) = q\sigma(\sigma - 1)$$

$$= qN + O(qN^{23/24}).$$

If $\alpha_q := a_1 + \cdots + a_q$ were less than $M(\sigma) - m\sigma - (2r+1)q$, the intersection of $[m(\sigma), M(\sigma)]$ and $[m(\sigma) + \alpha_q, M(\sigma) + \alpha_q]$ would be an interval containing at least $(2r+1)$ integers in each class modulo q. By the property of $\sigma^\wedge\mathcal{D}$ established in Proposition 1, property obviously shared by $\alpha_q + \sigma^\wedge\mathcal{D}$, the pigeon-hole principle would imply that $\sigma^\wedge\mathcal{D}$ and $\alpha_q + \sigma^\wedge\mathcal{D}$ have an element in common, and this would contradict the admissibility of $\mathcal{A}$. (We may notice that this implies that a_1 itself is not too small, but we shall not use this fact).

By using the same pigeon-hole argument, we see that the admissibility of $\mathcal{A}$ implies

$$M(\sigma) + a_{A-u+1} + \cdots + a_A \leq m(\sigma + q) + a_1 + \cdots + a_u + (2r-1)q,$$

that is to say

$$a_{A-u-\sigma+1} + \cdots + a_{A-u} + \cdots a_A \leq a_1 + \cdots + a_u + a_{u+1} + \cdots + a_{u+\sigma+q} + (2r-1)q,$$

whence we deduce

$$\begin{aligned}(a_A - a_1) + (a_{A-1} - a_2) \quad &+ \cdots + (a_{A-u-\sigma+1} - a_{u+\sigma}) \leq \\ &a_{u+\sigma+1} + \cdots + a_{u+\sigma+q} + (2r-1)q.\end{aligned}$$

We have $a_{A-u-\sigma+1} - a_{u+\sigma} \geq q(A - u - \sigma + 1 - u - \sigma) = q(A - 2u - 2\sigma + 1)$ and, by the definition of σ, we can write

$$A - 2u - 2\sigma = q + \theta,$$

where $\theta = 0$ if $A - q$ is even and $\theta = 1$ if $A - q$ is odd. We thus have

$$\begin{aligned}urq + q(1 + q + \theta) + q(3 + q + \theta) \quad &+ \cdots + q(2(u+\sigma) - 1 + q + \theta) \leq \\ &a_{u+\sigma+1} + \cdots + a_{u+\sigma+q} + (2r-1)q.\end{aligned}$$

Since $u \geq 2$ and $r \geq 0$, we have

$$\begin{aligned} q(u+\sigma)(u+\sigma+q+\theta) &\leq a_{u+\sigma+1}+\cdots+a_{u+\sigma+q}-q \\ &\leq N-(A-u-\sigma-1)q+\cdots+ \\ &\quad N-(A-u-\sigma-q)q-q \\ &\leq Nq-Aq^2+uq^2+\sigma q^2+\tfrac{q^2(q+1)}{2}-q. \end{aligned}$$

We now replace $u+\sigma$ by $\frac{A-q-\theta}{2}$, which leads to

$$q\left(\frac{A-q-\theta}{2}\right)\left(\frac{A+q+\theta}{2}\right) \leq Nq-q^2\left(\frac{A+q+\theta}{2}\right)+\frac{q^2(q-1)}{2}.$$

If $A-q$ is even, we get

$$A^2-q^2 \leq 4N-2Aq-2q^2+2q^2-2q,$$

whence

$$A^2+2Aq+q^2 \leq 4N+2q^2-2q,$$

or

$$(A+q)^2 \leq 4N+2q^2-2q.$$

if $q=1$, this is $(A+1)^2 \leq 4N$;
if $q \geq 2$, we have

$$\begin{aligned} (A+1)^2 &\leq (A+q)^2-(A+q)^2+(A+1)^2 \\ &\leq 4N+2q^2-2q-A^2-2Aq-q^2+A^2+2A+1 \\ &\leq 4N+2A(1-q)+(q-1)^2 \\ &\leq 4N-(q-1)(2A-q+1) \leq 4N. \end{aligned}$$

If $A-q$ is odd, we get

$$A^2-(1+q)^2 \leq 4N-2Aq-2q^2-2q+2q^2-2q.$$

if $q=1$, this is $A^2+2A+1 \leq 4N+1$;
if $q \geq 2$, we have

$$\begin{aligned} (A+1)^2 &\leq A^2-(1+q)^2+2q+q^2+2+2A \\ &\leq 4N-2A(q-1)+q^2-2q+2 \\ &\leq 4N-(q-1)(2A-q+1)+1 \\ &\leq 4N+1. \end{aligned}$$

In all cases, we thus have $(A+1)^2 \leq 4N+1$, which ends the proof of our main result.

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