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SUBSET SUMS OF SETS OF RESIDUES

by

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Dedicated to Grisha Freiman, with respect and affection

Abstract. — The number m is called the critical number of a finite abelian group G , if it is the minimal natural number with the property: for every subset A of G with $|A| \geq m, 0 \notin A$, the set of subset sums A^* of A is equal to G . In this paper, we prove the conjecture of G. Diderrich about the value of the critical number of the group G , in the case $G = \mathbb{Z}_q$, for sufficiently large q .

Let G be a finite Abelian group, $A \subset G$ such that $0 \notin A$. Let $A = \{a_1, a_2, \dots, a_{|A|}\}$, where $|A| = \text{card}A$.

Let

$$A^* := \{x \mid x = a_1\varepsilon_1 + a_2\varepsilon_2 + \dots + \varepsilon_{|A|}a_{|A|}, \varepsilon_j \in \{0, 1\}, 1 \leq j \leq |A|, \sum_{j=1}^{|A|} \varepsilon_j > 0\}$$

and

$$X := \{m \in \mathbb{N} \mid \forall A \subset G, |A| \geq m \Rightarrow A^* = G\}.$$

Since $|G| - 1 \in X$, then $X \neq \emptyset$ if $|G| > 2$. The number

$$c(G) = \min_{m \in X} m$$

was introduced by George T. Diderrich in [1] and called the critical number of the group G .

In this note we study the magnitude of $c(G)$ in the case $G = \mathbb{Z}_q$, where \mathbb{Z}_q is a group of residue classes modulo q . We set $c(q) := c(\mathbb{Z}_q)$. A survey of the problem was given by G.T. Diderrich and H.B. Mann in [2].

In the case when q is a prime number John Olson [3] proved that

$$c(q) \leq \sqrt{4q - 3} + 1.$$

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Recently J.A. Dias da Silva and Y.O. Hamidoune [4] have found the exact value of $c(q)$ for which an estimate

$$2q^{1/2} - 2 < c(q) < 2q^{1/2}$$

is valid.

If $q = p_1 p_2$, $p_1 \geq p_2$, p_1, p_2 - prime numbers, then

$$p_1 + p_2 - 2 \leq c(G) \leq p_1 + p_2 - 1$$

as was proved by Diderrich [1].

It was proved in [2] that for $q = 2\ell$, $\ell > 1$

$$c(G) = \ell \text{ if } \ell \geq 5 \text{ or } q = 8$$

$$c(G) = \ell + 1 \text{ in all other cases.}$$

Thus, to give thorough solution for $G = \mathbb{Z}_q$ we have to find $c(q)$ when q is a product of no less than three prime odd numbers.

G. Diderrich in [1] has formulated the following conjecture:

Let G be an Abelian group of odd order $|G| = ph$ where p is the least prime divisor of $|G|$ and h is a composite number. Then

$$c(G) = p + h - 2.$$

We prove here this conjecture for the case $G = \mathbb{Z}_q$ for sufficiently large q .

Theorem 1. — *There exists a positive integer q_0 that if $q > q_0$ and $q = ph$, $p > 2$, where p is the least prime divisor of q and h is a composite number, we have*

$$c(q) = p + h - 2.$$

To prove Theorem 1 we need the following results.

Lemma 1. — *Let $A = \{a_1, a_2, \dots, a_{|A|}\} \subset N, N = \{1, 2, \dots, \ell\}, S(A) = \sum_{i=1}^{|A|} a_i$,*

$A(g) = \{x \in A | x \equiv 0 \pmod{g}\}, B(A) = \frac{1}{2} \left(\sum_{i=1}^{|A|} a_i^2 \right)^{1/2}$. *Suppose that for some $\varepsilon > 0$*

and $\ell > \ell_1(\varepsilon)$ we have $|A| \geq \ell^{2/3+\varepsilon}$ and

$$(1) \quad |A(g)| \leq |A| - \ell^{\frac{2}{3}+\frac{\varepsilon}{2}},$$

for every $g \geq 2$. Then for every M for which

$$|M - \frac{1}{2}S(A)| \leq B(A)$$

we have $M \subset A^$.*

Lemma 2. — *Let ε be a constant, $0 < \varepsilon \leq 1/3$. There exists $\ell_0 = \ell_0(\varepsilon)$ such that for every $\ell \geq \ell_0$ and every set of integers $A \subset [1, \ell]$, for which*

$$(2) \quad |A| \geq \ell^{\frac{2}{3}+\varepsilon},$$

the set A^* contains an arithmetic progression of ℓ elements and difference d satisfying the condition

$$(3) \quad d < \frac{2\ell}{|A|}.$$

We cited as Lemma 1 the Proposition 1.3 on page 298 of [5].

Proof of Lemma 2. — Let us first assume that A fulfills the condition (1) in Lemma 1. Since we have

$$B(A) \geq \frac{1}{2} \sqrt{\sum_{i=1}^{|A|} i^2} > \frac{1}{2} \sqrt{\frac{|A|^3}{3}} > \frac{1}{2\sqrt{3}} \ell^{1+\frac{3}{2}\varepsilon}$$

and every M from the interval $(\frac{1}{2}S(A) - B(A), \frac{1}{2}S(A) + B(A))$ belong to A^* , there exists an arithmetic progression in A^* of the length $2B(A) > \ell$, if $\ell > \ell_0 = \ell_1(\varepsilon)$.

Now we study the case when A does not satisfy (1). We can then find an integer $g_1 \geq 2$ such that $B_1 \subset A = A_0$ and B_1 contains those elements of A_0 which are divisible by g_1 and for the set $A_1 = \{x/g_1 | x \in B_1 \text{ and } x \equiv 0(\text{mod } g_1)\}$ we have

$$|A_1| > |A_0| - \ell^{\frac{2}{3}+\frac{\varepsilon}{2}}.$$

Suppose that this process was repeated s times and numbers g_1, g_2, \dots, g_s were found and sets A_1, A_2, \dots, A_s defined inductively, B_j being a subset of A_{j-1} containing those elements of A_{j-1} which are divisible by g_j and

$$A_j = \{x/g_j | x \in B_j \text{ and } x \equiv 0(\text{mod } g_j)\}$$

so that we have

$$|A_j| > |A_{j-1}| - \ell^{\frac{2}{3}+\frac{\varepsilon}{2}}, \quad j = 1, 2, \dots, s.$$

From

$$|A_s| \geq |A_{s-1}| - \ell^{\frac{2}{3}+\frac{\varepsilon}{2}} > |A| - s\ell^{\frac{2}{3}+\frac{\varepsilon}{2}}$$

and

$$\ell_s = \left\lfloor \frac{\ell_{s-1}}{q_s} \right\rfloor \leq \frac{\ell}{2^s}$$

it follows that

$$(4) \quad |A_s| \geq \frac{1}{2}|A| \geq \frac{1}{2}\ell^{\frac{2}{3}+\frac{\varepsilon}{2}} > \ell_s^{2+\varepsilon}.$$

The condition (2) of Lemma 2 for A_s is verified, for some sufficiently large s the condition (3) is fulfilled and thus A_s^* contains an interval

$$\left(\frac{1}{2}S(A_s) - B(A_s), \frac{1}{2}S(A_s) + B(A_s) \right).$$

We have, in view of (4),

$$(5) \quad \begin{aligned} B(A_s) &\geq \frac{1}{2} \sqrt{\sum_{i=1}^{|A_s|} i^2} > \frac{1}{2} \sqrt{\frac{|A_s|^3}{3}} \\ &\geq \frac{1}{4\sqrt{6}} \ell^{1+\frac{3}{2}\varepsilon} > \ell. \end{aligned}$$

We have shown that A_s^* contains an arithmetic progression of length ℓ and difference $d = g_1 g_2 \cdots g_s$, and thus A^* has the same property.

We now prove (2). From

$$\ell_s = \left\lceil \frac{\ell}{d} \right\rceil, \quad \ell_s \geq |A_s| \geq \frac{1}{2}|A|$$

we have

$$\left\lceil \frac{\ell}{d} \right\rceil \geq \frac{1}{2}|A|$$

or

$$d \leq \frac{2\ell}{|A|}.$$

Lemma 2 is proved.

Lemma 3 (M. Chaimovich [6]). — *Let $B = \{b_i\}$ be a multiset, $B \subset \mathbb{Z}_q$. Suppose that for every $s \geq 2$, s dividing q , we have*

$$(6) \quad |B \setminus B(s)| \geq s - 1.$$

There exists $F \subset B$ for which

$$\begin{aligned} |F| &\leq q - 1, \\ F^* &= \mathbb{Z}_q. \end{aligned}$$

Proof of Theorem 1. — Let $q = p_1 p_2 \cdots p_k$, $k \geq 4$, $p = p_1 \leq p_2 \leq \cdots \leq p_k$. We have

$$(7) \quad p^k \leq q \Rightarrow p \leq q^{1/4}.$$

Let $A \subset \mathbb{Z}_q$ be such that $0 \notin A$ and

$$(8) \quad |A| \geq \frac{q}{p} + p - 2;$$

we have to prove that $A^* = \mathbb{Z}_p$.

From (7) and (8) we get

$$(9) \quad |A| > \frac{q}{p} \geq q^{3/4}.$$

Let us consider some divisor d of q , and denote by A_d a multiset A viewed as a multiset of residues mod d . Let us show that for every δ dividing d the number of residues in A_d which are not divisible by δ satisfies the condition of Lemma 3.

The number of residues in \mathbb{Z}_q which are divisible by δ is equal to q/δ . Therefore the number of such residues in A (which are all different) is not larger than $q/\delta - 1$, because $0 \notin A$.

From this reasoning and from (7) we get the estimate

$$(10) \quad \begin{aligned} |A_d \setminus A(\delta)| &\geq |A| - \left(\frac{q}{\delta} - 1 \right) \geq \\ \frac{q}{p} + p - 2 - \frac{q}{\delta} + 1 &= \frac{q}{p} + p - \left(\frac{q}{\delta} + \delta \right) + \delta - 1. \end{aligned}$$

The function $x + q/x$ is decreasing on the segment $[1, \sqrt{q}]$.

The least divisor of q is equal to p , and the maximal one to q/p . Therefore

$$p \leq \delta \leq \frac{q}{p} .$$

If $p \leq \delta \leq \sqrt{q}$, we have

$$(11) \quad \frac{q}{p} + p \geq \frac{q}{\delta} + \delta .$$

In the case $\sqrt{q} \leq \delta \leq \frac{q}{p}$, let $\rho = \frac{q}{\delta}$. Then $\delta = \frac{q}{\rho}$, $\sqrt{q} \leq \frac{q}{\rho} \leq \frac{q}{p}$ and $p \leq \rho \leq \sqrt{q}$ and we have

$$(12) \quad \frac{q}{p} + p \geq \frac{q}{\rho} + \rho = \delta + \frac{q}{\delta} .$$

From (11) and (12) it follows from (10) that we have

$$(13) \quad |A_d \setminus A(\delta)| \geq \delta - 1 .$$

Let us apply the Lemma 3 to A_d . Condition (13) is condition (6) of Lemma 3. Therefore there exists $F_d \subset A_d$ such that $|F_d| \leq d - 1$ and $F_d^* = \mathbb{Z}_d$.

Viewing F_d as a set of residues mod q , let

$$A' = \bigcup_{\substack{d/q \\ p \leq d < q^{1/3}}} F_d .$$

It is well known that the number of divisors $d(q) = O(q^\varepsilon)$ for every $\varepsilon > 0$ so that

$$|A'| < q^{\frac{1}{3} + \varepsilon}$$

for sufficiently large q .

Take now $A'' = A \setminus A'$. Take the least positive integer from each class of residues of the set A'' and denote this set by \widehat{A}'' . We have $\widehat{A}'' \subset [1, q - 1]$. We set $\ell = q$ and see that all conditions of Lemma 1 are valid for \widehat{A}'' . Thus, $(\widehat{A}'')^*$ contains an arithmetic progression \mathcal{L} with a length q and a difference Δ such that

$$(14) \quad \Delta < \frac{2q}{q^{\frac{3}{4}}} = 2q^{1/4} .$$

If $(\Delta, q) = 1$ then $(A'')^* = \mathbb{Z}_q$. Suppose that $D = (\Delta, q) > 1$. Then \mathcal{L} (and therefore $(\widehat{A}'')^*$ which contains \mathcal{L}) contains the residues of \mathbb{Z}_q which are divisible by D . If \mathbb{Z}_D is a system of residues mod q representing a system of all residues mod D/q , then $(\widehat{A}'')^* + \mathbb{Z}_D = \mathbb{Z}_q$. But $F_D \subset A'$ and $F_D^* = \mathbb{Z}_D$. Thus

$$A^* \supset (\widehat{A}'')^* + (A')^* = \mathbb{Z}_q .$$

Theorem 1 is proved in the case $k \geq 4$.

Now we have to study the case when q is a product of three primes. Let $q = p_1 p_2 p_3$, $p = p_1 \leq p_2 \leq p_3$. Suppose that for some positive ε we have $p < p_3^{\frac{1}{3+\varepsilon}}$. The proof may be completed in a similar way to what was done.

In the general case we can use a stronger result than Lemma 2. Namely, the formulation of Lemma 2 is valid if in (2) we replace the number $2/3$ in the exponent by $1/2$ (see G. Freiman [7] and A. Sárközy [8]). So, in the case of q being a product of three primes, we can use this stronger version and prove Theorem 1.

As we have seen, the version of Lemma 1 with the exponent $2/3$ was sufficient in the majority of cases. It is preferable to use this version, for its proof is much simpler than the case $1/2$. Secondly, in the case $2/3$ estimates of error terms have been obtained explicitly by M. Chaimovich. It provides us with the possibility to get an explicit range of validity for Theorem 1.

Lemma 4. — Define a function of ℓ in the following manner:

$$(15) \quad m_0(\ell) = \left(\frac{12}{\pi^2}\right)^{1/3} \ell^{2/3} (\log \ell + 1/6)^{1/3} \left(2 - \frac{4\gamma}{3}\right)^{1/3}$$

where $\gamma = \left(\frac{12}{\pi^2} \frac{\log \ell + 1/6}{\ell}\right)^{1/3}$.

Then for $\ell > 155$ a subset sum of each subset $A \subset \{1, 2, \dots, \ell\}$ with $|A| = m > m_0(\ell)$ contains an arithmetic progression of cardinality ℓ .

Simplifying (15) we can take

$$m_0(\ell) = 1.3 \ell^{2/3} (\log \ell + 1/6)^{1/3}.$$

In the case of four or more primes in a representation of q we have to verify an inequality

$$(16) \quad \ell^{3/4} > 1.3 \ell^{2/3} (\log \ell + 1/6)^{1/3}$$

which is fulfilled for

$$\ell \geq 3000.$$

In some special cases we can give better estimates. For example, if $p = 3$ we have $m > q/3$ and instead of (16) we have

$$\begin{aligned} \ell/3 &> 1.3 \ell^{2/3} (\log \ell + 1/6)^{1/3}, \\ \ell &> 64(\log \ell + 1/6) \end{aligned}$$

which is valid for

$$\ell \geq 500.$$

References

- [1] Diderrich G. T., *An addition theorem for Abelian groups of order pq* , Journal of Number Theory, **7**, 1975, 33–48.
- [2] Diderrich G. T., Mann H. B., *Combinatorial problems in finite Abelian groups*, J.N.Srivastava et al., eds., A Survey of Combinatorial Theory. North-Holland Publishing Company, 1973, 95–100.
- [3] Olson J. E., *An addition theorem modulo p* , Journal of Combinatorial Theory, **5**, 1968, 45–52.
- [4] Dias da Silva J.A., Hamidoune Y.O., *Cyclic spaces for Grassman derivatives and additive theory*, Bull London. Math. Soc., **26**, 1994, 140–146.
- [5] Alon N., Freiman G. A., *On sums of subsets of a set of integers*, Combinatorica, **8**(4), 1988, 297–306.
- [6] Chaimovich M., *Solving a value-independent knapsack problem with the use of methods of additive number theory*, Congressus Numerantium, **72**, 1990, 115–123.

- [7] Freiman G.A., *New analytical results in subset-sum problem*, Discrete Mathematics, **114**, 1993, 205–218.
- [8] Sárközy A., *Finite addition theorems*, II. J. Number Theory, **48(2)**, 1994, 197–218.

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