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INVERSE THEOREMS AND THE NUMBER OF SUMS AND PRODUCTS

by

Melvyn B. Nathanson & Gérald Tenenbaum

Abstract. — Let $\epsilon > 0$. Erdős and Szemerédi conjectured that if A is a set of k positive integers which large k , there must be at least $k^{2-\epsilon}$ integers that can be written as the sum or product of two elements of A . We shall prove this conjecture in the special case that the number of sums is very small.

1. A conjecture of Erdős and Szemerédi

Let A be a nonempty, finite set of positive integers, and let $|A|$ denote the cardinality of the set A . Let

$$2A = \{a + a' : a, a' \in A\}$$

denote the 2-fold *sumset* of A , and let

$$A^2 = \{aa' : a, a' \in A\}$$

denote the 2-fold *product set* of A . We let

$$E_2(A) = 2A \cup A^2$$

denote the set of all integers that can be written as the sum or product of two elements of A . If $|A| = k$, then

$$|2A| \leq \binom{k+1}{2}$$

and

$$|A^2| \leq \binom{k+1}{2},$$

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and so the number of sums and products of two elements of A is

$$|E_2(A)| \leq k^2 + k.$$

Erdős and Szemerédi [3, p. 60] made the beautiful conjecture that a finite set of positive integers cannot have simultaneously few sums and few products. More precisely, they conjectured that for every $\varepsilon > 0$ there exists an integer $k_0(\varepsilon)$ such that, if A is a finite set of positive integers and

$$|A| = k \geq k_0(\varepsilon),$$

then

$$|E_2(A)| \gg_\varepsilon k^{2-\varepsilon}.$$

Very little is known about this question. Erdős and Szemerédi [4] have shown that there exists a real number $\delta > 0$ such that

$$|E_2(A)| \gg k^{1+\delta},$$

and Nathanson [11] proved that

$$|E_2(A)| \geq ck^{32/31},$$

where $c = 0.00028\dots$

Erdős and Szemerédi [4] also remarked that, in the special case that $|2A| \leq ck$, “perhaps there are more than $k^2/(\log k)^\varepsilon$ elements in A^2 ”. This cannot be true for arbitrary finite sets of positive integers and arbitrarily small $\varepsilon > 0$. For example, if A is the set of all integers from 1 to k , then Tenenbaum [16, 17], improving a result of Erdős [2], proved that

$$(1) \quad \frac{k^2}{(\log k)^{\varepsilon_0}} e^{-c\sqrt{\log_2 k \log_3 k}} \ll |A^2| \ll \frac{k^2}{(\log k)^{\varepsilon_0} \sqrt{\log_2 k}},$$

where \log_r denotes the r -fold iterated logarithm, and

$$(2) \quad \varepsilon_0 = 1 - \left(\frac{1 + \log_2 2}{\log 2} \right) \geq 0.08607$$

(cf. Hall and Tenenbaum [8, Theorem 23]).

Using an inverse theorem of Freiman, we shall prove that if A is a set of k positive integers such that $|2A| \leq 3k - 4$, then

$$|A^2| \gg (k/\log k)^2.$$

We obtain a similar result for the sumset and product set of two possibly different sets of integers. Let A_1 and A_2 be nonempty, finite sets of positive integers, and let

$$A_1 + A_2 = \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}$$

and

$$A_1 A_2 = \{a_1 a_2 : a_1 \in A_1, a_2 \in A_2\}.$$

Let $|A_1| = |A_2| = k$. We prove that whenever $|A_1 + A_2| \leq 3k - 4$, then we have $|A_1 A_2| \gg (k/\log k)^2$.

2. Product sets of arithmetic progressions

A set Q of positive integers is an *arithmetic progression* of length ℓ and difference q if there exist positive integers r, q , and ℓ such that

$$Q = \{r + uq : 0 \leq u < \ell\}.$$

We shall always assume that

$$\ell \geq 2.$$

For any sets A and B of positive integers, let $\varrho_{A,B}(m)$ denote the number of representations of m in the form $m = ab$, where $a \in A$ and $b \in B$. Let $\varrho_A(m) = \varrho_{A,A}(m)$. Let $\tau(m)$ denote the number of positive divisors of m . Clearly, for every integer m ,

$$\varrho_{A,B}(m) \leq \tau(m).$$

If $A_1 \subseteq Q_1$ and $A_2 \subseteq Q_2$, then $\varrho_{A_1,A_2}(m) \leq \varrho_{Q_1,Q_2}(m)$.

Lemma 1 (Shiu). — *Let $0 < \alpha < 1/2$ and let $0 < \beta < 1/2$. Let x and y be real numbers and let s and q be integers such that*

$$(3) \quad 0 < s \leq q \text{ and } (s, q) = 1,$$

$$(4) \quad q < y^{1-\alpha},$$

and

$$(5) \quad x^\beta < y \leq x.$$

Then

$$\sum_{\substack{w \equiv s \pmod{q} \\ x-y < w \leq x}} \tau(w) \ll_{\alpha,\beta} \frac{\varphi(q)y \log x}{q^2}.$$

Proof. This is a special case of Theorem 2 in Shiu [14] (see also Vinogradov and Linnik [18] and Barban and Vehov [1]).

Lemma 2. — *Let s, q, h , and ℓ be integers such that $h \geq 0$, $\ell \geq 2$, $0 < s \leq q$, and $(s, q) = 1$. Let Q be the arithmetic progression*

$$Q = \{s + vq : h \leq v < h + \ell\}.$$

If $(h + 1)q < \ell^5$, then

$$\sum_{w \in Q} \tau(w) \ll \ell \log \ell.$$

Proof. We apply Lemma 1 with $\alpha = \beta = 1/6$, $x = (h + \ell)q$, and $y = \ell q$. The integers s and q satisfy (3). Since $q \leq (h + 1)q < \ell^5$, we have $q^{1/6} < \ell^{5/6}$, and so

$$q = q^{1/6} q^{5/6} < (\ell q)^{5/6} = y^{1-\alpha}.$$

This shows that (4) is satisfied.

To obtain (5), we consider two cases. If $h \leq \ell$, then, since $2 \leq \ell \leq \ell q$, we have

$$x^\beta = ((h + \ell)q)^\beta \leq (2\ell q)^\beta \leq (\ell q)^{2\beta} = (\ell q)^{1/3} < \ell q = y \leq x.$$

If $h > \ell$, then, since $hq < \ell^5$, we have

$$x^\beta = \{(h + \ell)q\}^\beta < (\ell h q)^\beta < \ell^{6\beta} = \ell \leq \ell q = y \leq x.$$

This shows that (5) holds.

Applying Lemma 1, we obtain

$$\begin{aligned} \sum_{w \in Q} \tau(w) &= \sum_{\substack{w \equiv s \pmod{q} \\ hq < w \leq (h+\ell)q}} \tau(w) \ll \frac{\varphi(q)(\ell q) \log((h + \ell)q)}{q^2} \\ &\ll \ell \log(\ell(h + 1)q) \ll \ell \log \ell^6 \ll \ell \log \ell. \end{aligned}$$

This completes the proof.

Lemma 3. — *Let Q_1 and Q_2 be two arithmetic progressions of length $\ell \geq 2$, and let $m \in Q_1 Q_2$. Then*

$$(6) \quad \varrho_{Q_1, Q_2}(m) \ll_\varepsilon \ell^\varepsilon$$

for every $\varepsilon > 0$, and

$$(7) \quad \sum_{m \in Q_1 Q_2} \varrho_{Q_1, Q_2}(m)^2 \ll (\ell \log \ell)^2.$$

Proof. Let $Q_i = \{r_i + uq_i : 0 \leq u < \ell\}$ for $i = 1, 2$. We may assume without loss of generality that $(r_i, q_i) = 1$. We write $r_i = s_i + h_i q_i$, where $0 < s_i \leq q_i$ and $h_i \geq 0$. Then

$$Q_i = \{s_i + vq_i : h_i \leq v < h_i + \ell\}.$$

If $w_1 \in Q_1$ and $w_2 \in Q_2$, then, for suitable $v_1 \in [h_1, h_1 + \ell]$, $v_2 \in [h_2, h_2 + \ell]$, we have

$$(8) \quad h_1 q_1 < w_1 = s_1 + v_1 q_1 \leq (h_1 + \ell)q_1 \leq \ell(h_1 + 1)q_1$$

and

$$(9) \quad h_2 q_2 < w_2 = s_2 + v_2 q_2 \leq (h_2 + \ell)q_2 \leq \ell(h_2 + 1)q_2.$$

We can assume that

$$(h_2 + 1)q_2 \leq (h_1 + 1)q_1.$$

There are two cases. In the first case,

$$(h_1 + 1)q_1 < \ell^5.$$

By (8) and (9), we deduce that

$$w_1 \leq \ell(h_1 + 1)q_1 < \ell^6, \quad \text{and} \quad w_2 \leq \ell(h_2 + 1)q_2 \leq \ell(h_1 + 1)q_1 < \ell^6.$$

If $m \in Q_1 Q_2$, then m is of the form $m = w_1 w_2$, and so $m < \ell^{12}$. Since, by a classical estimate, $\tau(m) \ll_\varepsilon m^{\varepsilon/12}$, it follows that

$$\varrho_{Q_1, Q_2}(m) \leq \tau(m) \ll_\varepsilon m^{\varepsilon/12} \ll_\varepsilon \ell^\varepsilon.$$

This proves (6).

To prove (7), we use the submultiplicativity of the divisor function, that is, $\tau(uv) \leq \tau(u)\tau(v)$ for all positive integers u, v . Then

$$\begin{aligned} \sum_{m \in Q_1 Q_2} \varrho_{Q_1, Q_2}(m)^2 &= \sum_{w_1 \in Q_1} \sum_{w_2 \in Q_2} \varrho_{Q_1, Q_2}(w_1 w_2) \\ &\leq \sum_{w_1 \in Q_1} \sum_{w_2 \in Q_2} \tau(w_1 w_2) \\ &\leq \sum_{w_1 \in Q_1} \tau(w_1) \sum_{w_2 \in Q_2} \tau(w_2) \ll \ell^2 (\log \ell)^2, \end{aligned}$$

where the last upper bound follows from Lemma 2.

Consider now the second case

$$(h_1 + 1)q_1 \geq \ell^5.$$

We shall prove that

$$(10) \quad \varrho_{Q_1, Q_2}(m) \leq 3$$

for all $m \geq 1$. Suppose that $w_1 = r_1 + uq_1 \in Q_1$ and $w'_1 = r_1 + u'q_1 \in Q_1$ are distinct divisors of m , and that $w_1 < w'_1$. Then $(r_1, q_1) = 1$ implies that $(w_1, q_1) = (w'_1, q_1) = 1$, and so $((w_1, w'_1), q_1) = 1$. Since (w_1, w'_1) divides

$$w'_1 - w_1 = (u' - u)q_1,$$

it follows that (w_1, w'_1) divides $u' - u$, and so

$$1 \leq (w_1, w'_1) \leq u' - u < \ell.$$

Suppose that $\varrho_{Q_1, Q_2}(m) \geq 4$. Then m has at least four distinct representations in the form $m = w_1 w_2$ with $w_1 \in Q_1$ and $w_2 \in Q_2$, and so m has at least four different divisors in Q_1 , that is, at least four divisors of the form

$$r_1 + uq_1 = s_1 + (h_1 + u)q_1$$

with $0 \leq u < \ell$. At most one of these divisors is $s_1 + h_1 q_1$, and so m has at least three different divisors, which we shall denote by w_1, w'_1 , and w''_1 , such that

$$\min\{w_1, w'_1, w''_1\} \geq s_1 + (h_1 + 1)q_1 > (h_1 + 1)q_1 \geq \ell^5.$$

Let $[w_1, w'_1, w''_1]$ denote the least common multiple of w_1, w'_1 , and w''_1 . Since each of these three numbers is a divisor of m , we have

$$\begin{aligned} m &\geq [w_1, w'_1, w''_1] \geq \frac{w_1 w'_1 w''_1}{(w_1, w'_1)(w_1, w''_1)(w'_1, w''_1)} \\ &> \left(\frac{(h_1 + 1)q_1}{\ell}\right)^3 = \frac{(h_1 + 1)q_1}{\ell^3} (h_1 + 1)^2 q_1^2 \\ &\geq \ell^2 \left((h_1 + 1)q_1\right)^2 \geq \ell(h_1 + 1)q_1 \cdot \ell(h_1 + 1)q_1 \geq w_1 w_2 = m, \end{aligned}$$

which is impossible. This proves (10), and inequalities (6) and (7) follow immediately.

Lemma 4. — *Let Q be an arithmetic progression of length $\ell \geq 2$, and let $m \in Q^2$. Then*

$$(11) \quad \varrho_Q(m) \ll_\varepsilon \ell^\varepsilon$$

for every $\varepsilon > 0$, and

$$(12) \quad \sum_{m \in Q^2} \varrho_Q(m)^2 \ll (\ell \log \ell)^2.$$

Proof. This follows immediately from Lemma 3 with $Q_1 = Q_2 = Q$.

Lemma 5. — *Let Q_1 and Q_2 be arithmetic progressions of length $\ell \geq 2$. Then*

$$|Q_1 Q_2| \gg \left(\frac{\ell}{\log \ell} \right)^2.$$

Proof. Let $\varrho_{Q_1, Q_2}(m)$ denote the number of representations of m in the form $m = q_1 q_2$, where $q_1 \in Q_1$ and $q_2 \in Q_2$. By the Cauchy-Schwarz inequality and inequality (7) of Lemma 3,

$$\begin{aligned} \ell^2 &= \sum_{m \in Q_1 Q_2} \varrho_{Q_1, Q_2}(m) \leq |Q_1 Q_2|^{1/2} \left(\sum_{m \in Q_1 Q_2} \varrho_{Q_1, Q_2}(m)^2 \right)^{1/2} \\ &\ll |Q_1 Q_2|^{1/2} \ell \log \ell. \end{aligned}$$

Therefore,

$$|Q_1 Q_2| \gg \left(\frac{\ell}{\log \ell} \right)^2.$$

This completes the proof.

Lemma 6. — *Let Q be an arithmetic progression of length $\ell \geq 2$. Then*

$$|Q^2| \gg \left(\frac{\ell}{\log \ell} \right)^2.$$

Proof. This follows immediately from Lemma 5 with $Q_1 = Q_2 = Q$.

3. Application of some inverse theorems

We shall use the following two inverse theorems of Freiman.

Lemma 7 (Freiman). — *Let A be a nonempty set of k positive integers. If*

$$|2A| \leq 3k - 4,$$

then A is a subset of an arithmetic progression of length $\ell < 2k$.

Proof. See [5, 7, 10, 12].

Lemma 8 (Freiman). — *Let A_1 and A_2 be nonempty finite sets of positive integers, and let $|A_i| = k_i$ for $i = 1, 2$. If*

$$|A_1 + A_2| \leq k_1 + k_2 + \min\{k_1, k_2\} - 4,$$

then A_1 and A_2 are subsets of arithmetic progressions Q_1 and Q_2 , respectively, where Q_1 and Q_2 have the same difference and the same length $\ell < k_1 + k_2$.

Proof. See [6, 9, 12, 15].

Theorem 1. — *Let A be a finite set of positive integers, and let $|A| = k \geq 2$. If*

$$|2A| \leq 3k - 4,$$

then

$$|A^2| \gg \left(\frac{k}{\log k}\right)^2.$$

Proof. By Lemma 7, if $|2A| \leq 3k - 4$, then there exists an arithmetic progression Q of length $\ell < 2k$ such that $A \subseteq Q$. Since

$$\varrho_A(m) \leq \varrho_Q(m),$$

it follows from (12) that

$$\begin{aligned} k^2 &= \sum_{m \in A^2} \varrho_A(m) \leq |A^2|^{1/2} \left(\sum_{m \in A^2} \varrho_A(m)^2 \right)^{1/2} \\ &\leq |A^2|^{1/2} \left(\sum_{m \in Q^2} \varrho_Q(m)^2 \right)^{1/2} \\ &\ll |A^2|^{1/2} \ell \log \ell \ll |A^2|^{1/2} k \log k. \end{aligned}$$

Therefore,

$$(13) \quad |A^2| \gg \left(\frac{k}{\log k}\right)^2.$$

This completes the proof.

Theorem 2. — *Let $\lambda \geq 1$. Let A_1 and A_2 be finite sets of positive integers such that $|A_i| = k_i \geq 2$ for $i = 1, 2$ and*

$$(14) \quad \frac{1}{\lambda} \leq \frac{k_2}{k_1} \leq \lambda.$$

If

$$|A_1 + A_2| \leq k_1 + k_2 + \min\{k_1, k_2\} - 4,$$

then

$$|A_1 A_2| \gg_{\lambda} \frac{k_1 k_2}{\left(\log(k_1 k_2)\right)^2}.$$

Proof. It follows from (14) that

$$(k_1 + k_2)^2 \leq (1 + \lambda)^2 k_1^2 = (1 + \lambda)^2 \lambda k_1 (k_1/\lambda) \leq (1 + \lambda)^2 \lambda k_1 k_2,$$

and so

$$k_1 + k_2 \ll_\lambda (k_1 k_2)^{1/2}.$$

By Lemma 8, if $|A_1 + A_2| \leq k_1 + k_2 + \min\{k_1, k_2\} - 4$, there exist arithmetic progressions Q_1 and Q_2 , each of length $\ell < k_1 + k_2$, such that $A_1 \subseteq Q_1$ and $A_2 \subseteq Q_2$. Since

$$\varrho_{A_1, A_2}(m) \leq \varrho_{Q_1, Q_2}(m),$$

it follows from (7) that

$$\begin{aligned} k_1 k_2 &= \sum_{m \in A_1 A_2} \varrho_{A_1, A_2}(m) \\ &\leq |A_1 A_2|^{1/2} \left(\sum_{m \in A_1 A_2} \varrho_{A_1, A_2}(m)^2 \right)^{1/2} \\ &\leq |A_1 A_2|^{1/2} \left(\sum_{m \in Q_1 Q_2} \varrho_{Q_1, Q_2}(m)^2 \right)^{1/2} \\ &\ll |A_1 A_2|^{1/2} \ell \log \ell \ll |A_1 A_2|^{1/2} (k_1 + k_2) \log(k_1 + k_2) \\ &\ll_\lambda |A_1 A_2|^{1/2} (k_1 k_2)^{1/2} \log(k_1 k_2). \end{aligned}$$

Therefore,

$$(15) \quad |A_1 A_2| \gg_\lambda \frac{k_1 k_2}{(\log(k_1 k_2))^2}.$$

This completes the proof.

Theorem 3. — *Let A_1 and A_2 be finite sets of positive integers such that $|A_1| = |A_2| = k \geq 2$. If*

$$|A_1 + A_2| \leq 3k - 4,$$

then

$$|A_1 A_2| \gg \left(\frac{k}{\log k} \right)^2.$$

Proof. This follows immediately from Theorem 2 with $k_1 = k_2 = k$ and $\lambda = 1$.

4. Open problems

By Theorem 1, if $|A| = k$ and $|2A| \leq 3k - 4$, then $|A^2| \gg k^{2-\epsilon}$. This gives the first general case in which we know that the conjecture of Erdős and Szemerédi is true. It would be nice to prove that if $c \geq 3$ and if A is a finite set of k positive integers such that

$$(16) \quad |2A| \leq ck,$$

then

$$|A^2| \gg_{c,\varepsilon} k^{2-\varepsilon}.$$

By a general inverse theorem of Freiman [7, 12, 13], a finite set of integers whose sumset satisfies inequality (16) is a "large" subset of what is called an n -dimensional arithmetic progression. This is a set Q with the following structure: For $n \geq 1$, there exist positive integers $r, q_1, \dots, q_n, \ell_1, \dots, \ell_n$ such that

$$(17) \quad Q = \{r + u_1q_1 + \dots + u_nq_n : 0 \leq u_i < \ell_i \text{ for } i = 1, \dots, n\}.$$

The length of Q is defined as $\ell(Q) = \ell_1 \cdots \ell_n$. Clearly,

$$|Q| \leq \ell(Q)$$

for every n -dimensional arithmetic progression. Freiman's inverse theorem should be applicable to the Erdős-Szemerédi conjecture for sets satisfying the additive condition (16).

Let Q be an n -dimensional arithmetic progression of the form (17). If j is such that $\ell_j = \max\{\ell_i : i = 1, \dots, n\}$ in (17), then

$$Q \supseteq Q_j = \{r + u_jq_j : 0 \leq u_j < \ell_j\}.$$

It follows from Lemma 6 that

$$(18) \quad |Q^2| \geq |Q_j^2| \gg \left(\frac{\ell_j}{\log \ell_j}\right)^2.$$

The following example shows that this inequality is almost best possible. Fix $n \geq 2$. For $\ell \geq 2$, consider the n -dimensional arithmetic progression Q with $r = 1, q_i = i$ and $\ell_i = \ell$ for $i = 1, \dots, n$. Then

$$Q = \{1 + \sum_{i=1}^n iu_i : 0 \leq u_i < \ell\} \subseteq \left[1, 1 + \frac{1}{2}n(n+1)(\ell-1)\right] \subseteq [1, n^2\ell].$$

We apply the lower bound (18) with $\ell = \max\{\ell_i : i = 1, \dots, n\}$, and we apply the upper bound (1) with $k = n^2\ell$. For sufficiently large ℓ we obtain

$$\left(\frac{\ell}{\log \ell}\right)^2 \ll |Q^2| \ll \frac{k^2}{(\log k)^{\varepsilon_0}} \ll_n \frac{\ell^2}{(\log \ell)^{\varepsilon_0}},$$

where ε_0 is defined by (2). Since $\ell(Q) = \ell^n$, it is clearly not true that

$$|Q^2| \gg_{n,\varepsilon} \ell(Q)^{2-\varepsilon}.$$

It would be interesting to obtain sufficient conditions for an n -dimensional arithmetic progression Q to satisfy

$$|Q^2| \gg_{n,\varepsilon} |Q|^{2-\varepsilon}.$$

Let A be a set of k positive integers. For $h \geq 3$, let $E_h(A)$ denote the set of all numbers that can be written as the sum or product of h elements of A . Erdős and Szemerédi [4] also conjectured that

$$|E_h(A)| \gg_\varepsilon k^{h-\varepsilon}$$

for all $\varepsilon > 0$. Nothing is known about this.

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