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ON SERIES OF DISCRETE RANDOM VARIABLES, 1: REAL TRINOMIAL DISTRIBUTIONS WITH FIXED PROBABILITIES

by

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Abstract. — This paper begins the study of the local limit behaviour of triangular arrays of independent random variables $(\zeta_{n,k})_{1\le k\le n}$ where the law of $\zeta_{n,k}$ depends on on n. We consider the case when $\zeta_{n,1}$ takes three integral values $0 < a_1(n) < a_2(n)$ with respective probabilities p_0, p_1, p_2 which do not depend on n. We show three types of limit behaviours for the sequence of r. v. $\eta_n = \zeta_{n,1} + \cdots + \zeta_{n,n}$, according as $a_2(n)/\gcd(a_1(n), a_2(n))$ tends to infinity slower, quicker or at the same speed as $\sqrt{n}$.

These notes are a first step in the description of the local behaviour of *series* of discrete random variables, that is to say sequences $(\eta_1, \dots, \eta_n, \dots)$ of random variables such that η_n is the sum of n independent discrete random variables $(\xi_{n,k})_{1\le k\le n}$ following a same law that may depend on n. We are restricting ourselves to the case when the $\xi_{n,k}$'s take three integer values $a_0 = 0 < a_1(n) < a_2(n)$, where a_1 and a_2 are coprime, with fixed positive probabilities p_0, p_1 and p_2 respectively.

When the values $a_1(n)$ and $a_2(n)$ do not depend on n, it follows from a result of Gnedenko that we have a *local limit* result, namely

$$P\{\eta_n = N\} = \frac{1}{\sigma\sqrt{2\pi n}}\left(\exp\left(-\frac{(n\mu - N)^2}{2n\sigma^2}\right) + o(1)\right) \qquad \text{as } n \to \infty,$$

uniformly in N, where μ and σ^2 are the expectation and the variance of the $\xi_{n,k}$'s.

Our aim is to give a complete description of the case when a_1 and a_2 depend on n, showing that there exist three different behaviors according as $a_2(n)$ is bounded or tends to infinity slower than $\sqrt{n}$, tends to infinity at the same speed as $\sqrt{n}$, or tends to infinity quicker than $\sqrt{n}$. In the first case, we get a *local limit* result similar to

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the one we just quoted. The second case leads to a result of a similar structure with a limiting law which is no more normal. The third case can be seen as *isomorphic* to a two-dimensional series with a fixed law; this notion, which may be of future importance, will be presented in the last section. This notion will also be useful to explain what happens when the coprimality condition of the a_i's is removed, without having to rewrite the statement of the Theorem in a heavier form where a_1 and a_2 would be replaced by $a_1/\gcd(a_1,a_2)$ and $a_2/\gcd(a_1,a_2)$, and $\{\eta_n = N\}$ by $\{\eta_n = N\gcd(a_1,a_2)\}\ldots$

We now state our main result.

Theorem. — *Let p_0, p_1, p_2 be three positive numbers with sum 1, and $a_0 = 0 < a_1(n) < a_2(n)$ be three coprime integers. Let further*

$$\begin{aligned} \mu &= \mu_n = p_1 a_1(n) + p_2 a_2(n), \\ \sigma^2 &= \sigma_n^2 = p_1 a_1^2(n) + p_2 a_2^2(n) - \mu_n^2. \end{aligned}$$

We consider n independent random variables $(\xi_{n,k})_{1\le k\le n}$, each of which takes the values $a_0, a_1(n), a_2(n)$ with probabilities p_0, p_1, p_2 respectively, and we denote by η_n the sum $\xi_{n,1} + \cdots + \xi_{n,n}$.

When $a_2(n) = o(\sqrt{n})$ as n tends to infinity, we have, uniformly with respect to the integer N

$$P\{\eta_n = N\} = \frac{1}{\sigma_n\sqrt{2\pi n}}\left(\exp\left(-\frac{(n\mu_n - N)^2}{2n\sigma_n^2}\right) + o(1)\right) \quad \textit{as } n \to \infty.$$

When $a_2(n)/\sqrt{n}$ tends to infinity with n, we have, uniformly with respect to the integer N

$$P\{\eta_n = N\} = \frac{n!}{k_0!k_1!k_2!} p_0^{k_0} p_1^{k_1} p_2^{k_2} + o(\frac{1}{n}) \quad \textit{as } n \to \infty,$$

where the integral triple (k_0, k_1, k_2) is defined by

$$-a_2(n)/2 < k_1 - np_1 \le a_2(n)/2, \;\; k_1 a_1(n) + k_2 a_2(n) = N, \;\; k_0 + k_1 + k_2 = n.$$

When $a_2(n)/\sqrt{n}$ tends to a finite positive limit c when n tends to infinity, we have, uniformly with respect to c and to the integer N

$$P\{\eta_n = N\} = \frac{1}{2\pi n\sqrt{p_0p_1p_2}} \sum_{(k_0,k_1,k_2)} \exp(Q(k_0,k_1,k_2)) + o(\frac{1}{n}) \quad \textit{as } n \to \infty,$$

where the sum is extended to integral triples satisfying $k_0 + k_1 + k_2 = n$, $a_1(n)k_1 + a_2(n)k_2 = N$, and the quadratic form Q is defined by

$$Q(k_0,k_1,k_2) = -\frac{1}{2}\sum_{i=1}^{3} \frac{1}{np_i}(k_i - np_i)^2.$$

1. The case when $a_2(n) = o(\sqrt{n})$

Due to the arithmetical nature of our problem (the a_i's are integers), we shall use the Fourier kernel $\exp(2\pi i \cdot x)$ and define by

$$\Psi(t) = \Psi_n(t) = p_0 + p_1 \exp(2\pi i t a_1) + p_2 \exp(2\pi i t a_2)$$

the characteristic function of $\xi_{n,k}$ so that we have

$$P\{\eta_n = N\} = \int \Psi^n(t) \exp(-2\pi i t N)\, dt\ ,$$

where the integral is performed over any interval of length 1.

We shall divide the range of summation into a *major arc*, when t is close to 0, and a *minor arc* when t is far from 0. Let ε be a positive real number (that will be specified later to be $1/(a_2(n)n^{2/5})$), and let

$$\mathfrak{M} = [-\varepsilon, \varepsilon] \qquad \text{and} \qquad \mathfrak{m} =]\varepsilon; 1-\varepsilon].$$

1.1. Contribution of the minor arc.

— The following lemma will be used to get an upper bound for Ψ on the minor arc, playing either with the term $\exp(2\pi i a_1 t)$ or with $\exp(2\pi i a_2 t)$.

Lemma 1. — *Let* $C = 8 \min\left(\frac{p_0 p_1}{p_0+p_1}, \frac{p_0 p_2}{p_0+p_2}\right)$. *We have*

$$|p_0 + p_1 \exp(2\pi i u_1) + p_2 \exp(2\pi i u_2)| \leq \exp(-C \max(||u_1||^2, ||u_2||^2))\ ,$$

where $||u||$ *denotes the distance from* u *to the nearest integer.*

Proof. — It is of course enough to prove the inequality

$$|p_0 + p_1 \exp(2\pi i u_1) + p_2 \exp(2\pi i u_2)| \leq \exp\left(\frac{-8p_0p_1}{p_0+p_1}||u_1||^2\right).$$

We have

$$\begin{aligned} |p_0 + p_1 \exp(2\pi i u_1)|^2 &= p_0^2 + p_1^2 + 2p_0p_1 \cos 2\pi u_1 \\ &= (p_0+p_1)^2 - 4p_0p_1 \sin^2 \pi u_1 \\ &\leq (p_0+p_1)^2 - 16 p_0 p_1 ||u_1||^2 \\ &\leq (p_0+p_1)^2 (1 - \frac{8p_0p_1}{(p_0+p_1)^2}||u_1||^2)^2\ . \end{aligned}$$

This implies

$$\begin{aligned} |p_0 + p_1 \exp(2\pi i u_1) + p_2(2\pi i u_2)| &\leq p_0 + p_1 + p_2 - \frac{8p_0p_1}{p_0+p_1}||u_1||^2 \\ &\leq 1 - \frac{8p_0p_1}{p_0+p_1}||u_1||^2 \\ &\leq \exp\left(-\frac{8p_0p_1}{p_0+p_1}||u_1||^2\right)\ , \end{aligned}$$

which is the inequality we looked for. □

We now present the dissection of the minor arc. For any integer r, we shall denote by $\overline{r}$ the integer in $[0, a_2/2]$ such that r is congruent either to $\overline{r}$ or to $-\overline{r}$ modulo a_2. The reader will easily check that $\mathfrak{m}$ is the disjoint union of the following intervals:

$$\begin{aligned}
\mathfrak{m}_1(r) &= \left]\frac{r}{a_2} - \frac{\overline{ra_1}}{2a_2^2}, \frac{r}{a_2} + \frac{\overline{ra_1}}{2a_2^2}\right[, \quad \text{for } r = 1, 2, \ldots, a_1 - 1, \\
\mathfrak{m}_2(0) &= \left]\varepsilon, \frac{1}{a_2} - \frac{\overline{1a_1}}{2a_2^2}\right], \\
\mathfrak{m}_2(a_2 - 1) &= \left[\frac{a_2 - 1}{a_2} + \frac{\overline{(a_2 - 1)a_1}}{2a_2^2}, 1 - \varepsilon\right[, \\
\mathfrak{m}_2(r) &= \left[\frac{r}{a_2} + \frac{\overline{ra_1}}{2a_2^2}, \frac{r+1}{a_2} - \frac{\overline{(r+1)a_2}}{2a_2^2}\right], \quad \text{for } r = 1, 2, \ldots, a_2 - 2 .
\end{aligned}$$

The intervals of type $\mathfrak{m}_2$ stay away from rationals with denominator a_2, so that $||a_2 t||$ is rather large when t is in such an interval. More precisely, if we consider $\mathfrak{m}_2^-(r) = [\frac{r}{a_2} + \frac{\overline{ra_1}}{2a_2^2}, \frac{2r+1}{2a_2}]$, in the case when $1 \le r \le a_2/2$, we have $t \in \mathfrak{m}_2^-(r) \Rightarrow ||ta_2|| = ||ta_2 - r|| = (ta_2 - r)$, so that, by Lemma 1, we have

$$\begin{aligned}
\int_{\mathfrak{m}_2^-(r)} |\Psi(t)|^n dt &\le \int_{\mathfrak{m}_2^-(r)} \exp(-Cn(ta_2 - r)^2) dt \\
&\le \frac{1}{a_2\sqrt{n}} \int_{\frac{\overline{ra_1}}{2a_2}\sqrt{n}}^{\infty} \exp(-Cu^2) du .
\end{aligned}$$

In a similar way, the contribution of each of $]\varepsilon, \frac{1}{2a_2}]$ and $[\frac{2a_2-1}{a_2}, 1 - \varepsilon[$ is at most

$$\frac{1}{a_2\sqrt{n}} \int_{\varepsilon a_2\sqrt{n}}^{\infty} \exp(-Cu^2) du.$$

The coprimality of a_1 and a_2 implies that $\overline{ra_1}$ is different from 0 for $r = 1, 2, \ldots, a_1 - 1$, and that any integer s in $[1, a_2/2]$ is equal to some $\overline{ra_1}$ for at most two values of r. If we denote by $\mathfrak{m}_2$ the union of the $\mathfrak{m}_2(r)$ for $r = 0, 1, \ldots, a_2 - 1$, we get

$$\int_{\mathfrak{m}_2} |\Psi(t)|^n dt \le \frac{2}{a_2\sqrt{n}} \int_{\varepsilon a_2\sqrt{n}}^{\infty} \exp(-Cu^2) du + \frac{4}{a_2\sqrt{n}} \sum_{s=1}^{\infty} \int_{\frac{s\sqrt{n}}{2a_2}}^{\infty} \exp(-Cu^2) du ,$$

so that the condition $a_2 = o(\sqrt{n})$ implies

$$\int_{\mathfrak{m}_2}^{\infty} |\Psi(t)|^n dt \le \frac{2}{a_2\sqrt{n}} \int_{\varepsilon a_2\sqrt{n}}^{\infty} \exp(-Cu^2) du + o(\frac{1}{a_2\sqrt{n}}). \tag{1}$$

We now turn our attention to the $\mathfrak{m}_1(r)$'s, the union of which is denoted by $\mathfrak{m}_1$. The length of $\mathfrak{m}_1(r)$ is $\frac{\overline{ra_1}}{2a_2^2}$, and for t in $\mathfrak{m}_1(r)$ we have

$$||a_1 t|| \geq ||\frac{a_1 r}{a_2}| - \frac{a_1 \overline{ra_1}}{2a_2^2} \geq \frac{\overline{ra_1}}{2a_2},$$

so that we get

$$\begin{aligned} \int\limits_{\mathfrak{m}_1} |\Psi(t)|^n dt &\leq \frac{2}{2a_2^2} \sum_{s=1}^{\infty} \exp(-Cn(\frac{s}{2a_2^2})^2) \\ &\leq \frac{2}{a_2^2} \exp(-\frac{Cn}{4a_2^2}) , \end{aligned}$$

as soon as $\exp(-Cn/(4a_2^2))$ is less than 1/2, which is the case when n is large enough. We thus have

$$\int\limits_{\mathfrak{m}_2}^{\infty} |\Psi(t)|^n dt \leq \frac{2}{a_2\sqrt{n}} \cdot \frac{\sqrt{n}}{a_2} \exp(-\frac{Cn}{4a_2^2}) = o(\frac{1}{a_2\sqrt{n}}).$$

Combining the contributions of the different parts of the minor arc, we get the following

Proposition 1. — *We have, for any* $\varepsilon > 0$

$$\int\limits_{\varepsilon}^{1-\varepsilon} |\Psi(t)|^n dt \leq \frac{2}{a_2\sqrt{n}} \int\limits_{\varepsilon a_2\sqrt{n}}^{\infty} \exp(-Cu^2) du + o(\frac{2}{a_2\sqrt{n}}).$$

1.2. Contribution of the major arc

Proposition 2. — *Let* $\varepsilon = O(1/(a_2 n^{1/3}))$. *We have*

$$\int\limits_{-\varepsilon}^{\varepsilon} \Psi^n(t) \exp(-2\pi i N t) dt = \frac{1}{\sigma\sqrt{2\pi n}} \exp\big(- \frac{(\mu n - N)^2}{2n\sigma^2}\big)$$

$$+ O(a_2^3 n\varepsilon^4) + O(\frac{1}{a_2\sqrt{n}} \int\limits_{\varepsilon a_2\sqrt{n}}^{\infty} \exp(-Cu^2) du),$$

uniformly in N, *where* n *tends to infinity.*

This proposition is an easy consequence of the two following lemmata

Lemma 2. — *Let* $\varepsilon = O(1/(a_2 n^{1/3}))$. *We have*

$$\int\limits_{-\varepsilon}^{\varepsilon} \Psi^n(t) \exp(-2\pi i N t) dt = \int\limits_{-\varepsilon}^{\varepsilon} \exp(-2\pi i t(n\mu - N) - 2\pi^2 n\sigma^2 t^2) dt + O(a_2^3 n\varepsilon^4),$$

uniformly in N, *when* n *tends to infinity.*

Proof. — On the interval $[-\varepsilon, +\varepsilon]$ we use the second order Taylor expansion of $\exp(2\pi i a_1 t)$ and $\exp(2\pi i a_2 t)$, noticing that $a_1 t$ and $a_2 t$ are bounded terms. We get

$$\begin{aligned} p_0 \quad + &\quad p_1 \exp(2\pi i a_1 t) + p_2 \exp(2\pi i a_2 t) \\ = &\quad 1 + 2\pi i t(a_1 p_1 + a_2 p_2) - 2\pi^2 (a_1^2 p_1 + a_2^2 p_2) t^2 + O(a_2^3 |t|^3) \\ = &\quad \exp(2\pi i \mu t - 2\pi^2 \sigma^2 t^2 + O(a_2^3 |t|^3)) \ . \end{aligned}$$

The integrand in the LHS of the formula in Lemma 2 can thus be written as

$$\exp(2\pi i (n\mu - N)t - 2\pi^2 n\sigma^2 t^2 + O(na_2^3|t|^3)).$$

which is also equal to

$$\exp(2\pi i (n\mu - N)t - 2\pi^2 n\sigma^2 t^2) + O(na_2^3 \varepsilon^3),$$

since $na_2^3\varepsilon^3$ is bounded. The lemma follows by integrating over $[-\varepsilon, \varepsilon]$. □

Lemma 3. — *We have*

$$\int_{|t|>\varepsilon} |\exp(2\pi t i(n\mu - N) - 2\pi^2 n\sigma^2 t^2)| dt \le \frac{2}{a_2\sqrt{n}} \int_{\varepsilon a_2\sqrt{n}}^{\infty} \exp(-Cu^2) du.$$

Proof. — We first notice that $\sigma^2 \ge a_2^2 \frac{p_0 p_2}{p_0+p_2}$. We have

$$\begin{aligned} \sigma^2 &= p_1 a_1^2 + p_2 a_2^2 - (p_1 a_1 + p_2 a_2)^2 \\ &= p_1 q_1 (a_1^2 - 2\frac{p_2}{q_1} a_1 a_2 + \frac{p_2 q_2}{p_1 q_1} a_2^2) \\ &= p_1 q_1 ((a_1 - \frac{p_2}{q_1} a_2)^2 + (\frac{p_2 q_2}{p_1 q_1} - \frac{p_2^2}{q_1^2}) a_2^2) \\ &\ge a_2^2 \left(\frac{p_1 q_2 q_1 - p_1 p_2^2}{q_1} \right) \\ &= a_2^2 \frac{(q_2 q_1 - p_1 p_2) p_2}{q_1} \\ &= a_2^2 \frac{p_0 p_2}{p_0 + p_2} \ . \end{aligned}$$

This inequality on σ^2 implies $2\pi^2\sigma^2 \ge Ca_2^2$.
We thus obtain

$$\begin{aligned} \int_{|t|>\varepsilon} |\exp(2\pi i t(n\mu - N) \quad - \quad 2\pi^2 n\sigma^2 t^2| dt &= \int_{|t|>\varepsilon} \exp(-2\pi^2 n\sigma^2 t^2) dt \\ &\le 2 \int_{\varepsilon}^{\infty} \exp(-Cna_2^2 t^2) dt \\ &= \frac{2}{a_2\sqrt{n}} \int_{\varepsilon a_2\sqrt{n}}^{\infty} \exp(-Cu^2) du \ . \end{aligned}$$

□

The proof of Proposition 2 follows from Lemma 2, Lemma 3, and the well-known relation

$$\int_{-\infty}^{\infty} \exp(2\pi it(n\mu - N) - 2\pi^2 n\sigma^2 t^2)dt = \frac{1}{\sigma\sqrt{2\pi n}} \exp(-\frac{(\mu n - N)^2}{2n\sigma^2}).$$

1.3. Proof of the Theorem when $a_2(n) = o(\sqrt{n})$. — According to Proposition 1 and Proposition 2, we merely have to notice that σ and a_2 have the same order of magnitude (we already showed that $\sigma^2 \geq a_2^2 \frac{p_0 p_2}{p_0+p_2}$, and the relation $\sigma^2 \leq a_2^2$ is trivial), and to show that one can find a function ε such that:

(i) $\varepsilon a_2\sqrt{n} \to \infty$,
(ii) $a_2^3 n\varepsilon^4 = o(\frac{1}{a_2\sqrt{n}})$,
(iii) $a_2 n^{1/3}\varepsilon = O(1)$.

As we already mentioned it, the quantity $\varepsilon = \frac{1}{a_2(n)n^{2/5}}$ satisfies these three conditions.

2. The case when $a_2(n)/\sqrt{n}$ tends to infinity.

For non-negative integers k_0, k_1, k_2 we define

$$P(k_0, k_1, k_2) = \frac{(k_0 + k_1 + k_2)!}{k_0!k_1!k_2!} p_0^{k_0} p_1^{k_1} p_2^{k_2},$$

and we extend this definition by letting the RHS be 0 when at least one of the k's is negative.

The second case of the theorem easily follows from the following lemma.

Lemma 4. — *Let $a_1(n)$ and $a_2(n)$ be positive coprime integers. For an integer N, we define*

$$\mathcal{E}(N) = \{(k_0, k_1, k_2), k_0 + k_1 + k_2 = n \quad \text{and} \quad k_1 a_1(n) + k_2 a_2(n) = N\}.$$

Let φ be a function that tends to infinity with its argument. Uniformly in N, $a_1(n)$ and $a_2(n)$ we have

$$\sum_{\substack{(k_0,k_1,k_2)\in\mathcal{E}(N)\\ |k_1-p_1 n|>\varphi(n)\sqrt{n}}} P(k_0, k_1, k_2) = o\left(\frac{1}{n}\right).$$

Proof. — We consider the fundamental triple (k_0^*, k_1^*, k_2^*) in $\mathcal{E}(N)$ satisfying $-a_2/2 < k_1^* - p_1 n \leq a_2/2$. We have

$$\mathcal{E}(N) = \{(k_0^* - s(a_2 - a_1), k_1^* + sa_2^*, k_2^* - sa_1), s \in \mathbb{Z}\}.$$

We remember for later use that the second components of two triples in $\mathcal{E}(N)$ differ by a multiple of a_2, and that for any given k_1, there exists in $\mathcal{E}(N)$ at most one triple the second element of which is k_1.

Let now k_1 be a given non-negative integers less than n. We define

$$\widetilde{p_0} = \frac{p_0}{p_0 + p_2}, \quad \widetilde{p_2} = \frac{p_2}{p_0 + p_2}, \quad \widetilde{k_0} = \lfloor (n - k_1)\widetilde{p_0} \rfloor, \quad \widetilde{k_2} = \lfloor (n - k_1)\widetilde{p_2} \rfloor.$$

By a simple property of the binomial distribution $\mathcal{B}(n-k_1,\widetilde{p_0})$, we have

$$\frac{(n-k_1)!}{k_0!k_2!}\widetilde{p_0}^{k_0}\widetilde{p_2}^{k_2}=O\Big(\frac{(n-k_1)!}{\widetilde{k_0}!\widetilde{k_2}!}\widetilde{p_0}^{\widetilde{k_0}}\widetilde{p_2}^{\widetilde{k_2}}\Big)=O\Big(\frac{1}{\sqrt{n-k_1}}\Big).$$

From this follows that for (k_0,k_1,k_2) in $\mathcal{E}(N)$ we have

$$\begin{aligned}\frac{n!}{k_0!k_1!k_2!}p_0^{k_0}p_1^{k_1}p_2^{k_2} &= \frac{n!}{k_1!(n-k_1)!}p_1^{k_1}\frac{(n-k_1)!}{k_0!k_2!}\widetilde{p_0}^{k_0}\widetilde{p_2}^{k_2}\Big(\frac{p_0^{k_0}p_2^{k_2}}{\widetilde{p_0}^{k_0}\widetilde{p_2}^{k_2}}\Big)\\ &= O\Big(\frac{n!}{k_1!(n-k_1)!}p_1^{k_1}\frac{p_0^{k_0}p_2^{k_2}}{\widetilde{p_0}^{k_0}\widetilde{p_2}^{k_2}}\frac{1}{\sqrt{n-k_1}}\Big)\\ &= O\Big(\frac{n!}{k_1!(n-k_1)!}p_1^{k_1}(p_0+p_2)^{k_0+k_2}\frac{1}{\sqrt{n-k_1}}\Big)\\ &= O\Big(\frac{1}{\sqrt{n-k_1}}\frac{n!}{k_1!(n-k_1)!}p_1^{k_1}q_1^{n-k_1}\Big)\ .\end{aligned}$$

Using the properties of the terms k_1 in the triple (k_0,k_1,k_2) belonging to $\mathcal{E}(N)$, the series

$$\sum_{\substack{(k_0,k_1,k_2)\in\mathcal{E}(N)\\ |k_1-p_1n|>\varphi(n)\sqrt{n}}} P(k_0,k_1k_2)$$

is – up to a constant factor – at most equal to its largest term, which is $o(\frac{1}{n})$. □

3. The case when $a_2(n)/\sqrt{n}$ tends to a positive limit.

We retain the definition of $P(k_0,k_1,k_2)$ and $\mathcal{E}(N)$ introduced in the previous section, and notice that Lemma 4 may still be applied in our present case, as well as in the description of $\mathcal{E}(N)$ in terms of the fundamental triple (k_0^*,k_1^*,k_2^*) satisfying $-a_2/2<k_1^*-np_1\le a_2/2$. We further write $\mathbf{k}=(k_0,k_1,k_2)$. Let $\varphi(n)=n^{1/10}$; we have

$$\begin{aligned}&\Big|\sum_{\mathbf{k}\in\mathcal{E}(N)}P(\mathbf{k})-\sum_{\mathbf{k}\in\mathcal{E}(N)}\frac{\exp(Q(\mathbf{k}))}{2\pi n\sqrt{p_0p_1p_2}}\Big|\\ \le\ & \sum_{\substack{\mathbf{k}\in\mathcal{E}(N)\\ |k_1-np_1|\le\varphi(n)\sqrt{n}\\ |k_2-np_2|\le\varphi(n)\sqrt{n}}}\Big|P(\mathbf{k})-\frac{\exp(Q(\mathbf{k}))}{2\pi n\sqrt{p_0p_1p_2}}\Big|\\ +\ & \sum_{\substack{\mathbf{k}\in\mathcal{E}(N)\\ |k_1-np_1|>\varphi(n)\sqrt{n}}}P(\mathbf{k})+\sum_{\substack{\mathbf{k}\in\mathcal{E}(N)\\ |k_2-np_2|\le\varphi(n)\sqrt{n}}}P(\mathbf{k})\\ +\ & O\Big(\frac{1}{n}\sum_{\substack{\mathbf{k}\in\mathcal{E}(N)\\ |k_1-np_1|>\varphi(n)\sqrt{n}}}\exp(Q(\mathbf{k})\Big)+O\Big(\frac{1}{n}\sum_{\substack{\mathbf{k}\in\mathcal{E}(N)\\ |k_1-np_1|\le\varphi(n)\sqrt{n}\\ |k_2-np_2|>\varphi(n)\sqrt{n}}}\exp(Q(\mathbf{k}))\Big)\ .\end{aligned}$$

By Lemma 4, the second and the third terms in the RHS are $o(1/n)$. The fourth term is easily dealt with since we have

$$\sum_{\substack{\mathbf{k}\in\mathcal{E}(N)\\ |k_1-np_1|>\varphi(n)\sqrt{n}}} \exp(Q(\mathbf{k})) \le \sum_{\substack{\mathbf{k}\in\mathcal{E}(N)\\ |k_1-np_1|>\varphi(n)\sqrt{n}}} \exp\Big(-\frac{1}{2np_1}(k_1-np_1)^2\Big)$$

$$\le 2 \sum_{\substack{s\in N\\ s>(\varphi(n)-1)(\sqrt{n}/a_2)}} \exp\Big(-\frac{1}{2p_1}(s-1)^2\frac{a_2^2}{n}\Big) = o(1).$$

Concerning the fifth term in the RHS, we first notice that the growth condition on a_2 implies that there are $O(\varphi(n))$ elements $\mathbf{k}$ in $\mathcal{E}(N)$ such that $|k_1-np_1| \le \varphi(n)\sqrt{n}$. We thus have

$$\sum_{\substack{\mathbf{k}\in\mathcal{E}(N)\\ |k_1-np_1|\le\varphi(n)\sqrt{n}\\ |k_2-np_2|>\varphi(n)\sqrt{n}}} \exp(Q(\mathbf{k})) = O\Big(\varphi(n)\exp\Big(-\frac{1}{2np_2}\varphi^2(n)n\Big)\Big) = o(1).$$

We now turn our attention toward the first and main term. We are going to prove that the conditions $\mathbf{k}\in\mathcal{E}(N)$, $|k_1-np_1|\le\varphi(n)\sqrt{n}$, $|k_2-np_2|\le\varphi(n)\sqrt{n}$ imply

$$P(\mathbf{k}) = \frac{1}{2\pi n\sqrt{p_0p_1p_2}}\exp(Q(\mathbf{k})) + o(\frac{1}{n\varphi(n)}),$$

which is sufficient to induce

$$\sum_{\mathbf{k}\in\mathcal{E}(N)} P(\mathbf{k}) = \sum_{\mathbf{k}\in\mathcal{E}(N)} \frac{\exp(Q(\mathbf{k}))}{2\pi n\sqrt{p_0p_1p_2}} + o(\frac{1}{n}),$$

which is the last part of the Theorem. We notice that the conditions we have stated imply $|k_i-np_i|\le 2\varphi(n)\sqrt{n}$ for $i=0,1$ and 2.

We use Stirling's formula in the shape

$$s! = \sqrt{2\pi s}\Big(\frac{s}{e}\Big)^s e^{\Theta_s}, \quad \text{with } |\Theta_s| \le \frac{1}{12s}.$$

We have

$$P(\mathbf{k}) = \frac{1}{2\pi}\sqrt{\frac{n}{k_0k_1k_2}}(\frac{np_0}{k_0})^{k_0}(\frac{np_1}{k_1})^{k_1}(\frac{np_2}{k_2})^{k_2}\exp(O(\frac{1}{n}))$$

$$= \frac{1}{2\pi n\sqrt{p_0p_1p_2}}(\frac{np_0}{k_0})^{k_0+1/2}(\frac{np_1}{k_1})^{k_1+1/2}(\frac{np_2}{k_2})^{k_2+1/2}\exp(O(\frac{1}{n})).$$

A second order expansion easily leads to

$$\begin{aligned}\log(\frac{np}{k})^{k+1/2} &= (k+1/2)\log(1+\frac{np-k}{k})\\ &= np-k-\frac{(np-k)^2}{2np}+O(\frac{\varphi(n)^3}{\sqrt{n}})\ ;\end{aligned}$$

since the sum $(np_1-k_1)+(np_2-k_2)+(np_3-k_3)$ is 0, we have

$$P(\mathbf{k}) = \frac{1}{2\pi n\sqrt{p_0p_1p_2}}\exp(Q(\mathbf{k})) + O(\frac{\varphi(n)^3}{n^{3/2}})$$

which is stronger that what we need as soon as $\varphi(n) = o(n^{1/8})$. The Theorem is now completely proved. We can even notice that the error term in the last case can be further reduced.

4. Isomorphism between series of discrete random variables

In the case when $a_2(n)/\sqrt{n}$ tends to infinity with n, the Theorem tells us that at most one term (the fundamental one, if any) is meaningful in the series

$$P\{\eta_n = N\} = \sum_{\mathbf{k}\in\mathcal{E}(N)} P(\mathbf{k}).$$

This situation is similar to a two-dimensional one: let $(\zeta_{n,k})$ be a family of independent bivariate random variables taking the values $(0,0), (0,1)$ and $(1,0)$ with respective probabilities p_0, p_1 and p_2. Letting $\varphi_n = \zeta_{n,1} + \cdots + \zeta_{n,n}$, we have

$$P\{\varphi_n = (k_1, k_2)\} = P(n - k_1 - k_2, k_1, k_2).$$

We thus would like to consider that the sequence of linear maps $\mathbb{R}^2 \overset{\mathbf{a}(n)}{\to} \mathbb{R}$ defined by $\mathbf{a}(n).\mathbf{x} = a_1(n)x_1 + a_2(n)x_2$ induces an isomorphism between the series $(\xi_{n,k})$ and $(\zeta_{n,k})$; this notion, for which we now suggest a precise definition, bears some similarity with the one that has been introduced by the second named author when dealing with additive problems (cf. [**1**]). Let us first recall this notion.

Let A and B be two finite subsets of two monoids $(E, +)$ and $(F, +)$, respectively, and let s be a positive integer. We denote by sum_s the map from E^s to E defined by $\mathrm{sum}_s(n_1, \ldots, n_s) = n_1 + \cdots + n_s$, and, since there is no fear of confusion, we use the same symbol to denote the sum of s elements of F. We let sA denote the set $\mathrm{sum}_s(A^S)$. The sets A and B are said to be *s-isomorphic* if there exist a bijection $\varphi : A \to B$ and a bijection $\varphi^{(s)} : sA \to sB$ such that

$$\forall x \in sA : \mathrm{sum}_s \circ \varphi^{\otimes s} \circ \mathrm{sum}_s^{-1}(\{x\}) = \{\varphi^{(s)}(x)\}\ ,$$
$$\forall y \in sB : \mathrm{sum}_s \circ (\varphi^{-1})^{\otimes s} \circ \mathrm{sum}_s^{-1}(\{y\}) = \{(\varphi^{(s)})^{-1}(y)\}\ .$$

A straightforward way to extend this definition to the s-isomorphism of probability measures P and Q, supported respectively by A and B, is to request that A and B are s-isomorphic and that the image of P by φ is Q and the image of P^{*s} by $\varphi^{(s)}$ is Q^{*s}.

For practical purposes, this definition is however too strong, as we shall show after stating a convenient definition of a weaker concept. We may use to our benefit the fact that we measure sets, by allowing the loss of some points in sA and sB with a total measure not exceeding some given δ, and allowing also the preservation of the measure by $\varphi^{(s)}$ to be approximate, up to some given ε. This leads to the following definition.

Definition. — *Let P and Q be two probability measures with finite supports A and B in monoids $(E, +)$ and $(F, +)$, respectively. Let s be a positive integer and δ and*

ε be two non-negative real numbers. The measures P and Q are said to be (s,δ,ε)-isomorphic if there exists a bijection $\varphi : A \to B$, two subsets $A^{(s)} \subset sA$, $B^{(s)} \subset sB$ and a bijection $\varphi^{(s)} : A^{(s)} \to B^{(s)}$ such that:

(i) $P^{*s}(A^{(s)}) \geq 1-\delta, \quad Q^{*s}(B^{(s)}) \geq 1-\delta,$
(ii) $\forall x \in A^{(s)} : \mathrm{sum}_s \circ \varphi^{\otimes s} \circ \mathrm{sum}_s^{-1}(\{x\}) = \{\varphi^{(s)}(x)\}, \quad \forall y \in B^{(s)} : \mathrm{sum}_s \circ (\varphi^{-s})^{\otimes s} \circ \mathrm{sum}_s^{-1}(\{y\}) = \{(\varphi^{(s)})^{-1}(y)\},$
(iii) $\forall x \in A^{(s)} : |P^{*s}(\{x\}) - Q^{*s}(\{\varphi^{(s)}(x)\})| \leq \varepsilon.$

We first notice that when (ii) is satisfied, (iii) is equivalent to

(iii′) $\forall y \in B^{(s)} : |Q^{*s}(\{y\}) - P^{*s}(\{(\varphi^{(s)})^{-1}(y)\})| \leq \varepsilon,$
which implies that when P and Q are (s,δ,ε)-isomorphic, so are Q and P.

When A and B are exactly the support of P and Q, the $(s,0,0)$-isomorphism is exactly the strong isomorphism we introduced at first.

As an example, we consider the case where $E = \mathbb{R}$, $F = \mathbb{R}^2$, $A = \{0,1,n\}, B = \{(0,0),(0,1),(1,0)\}$ with uniform measures, which we denote by P and Q, respectively. The map $\varphi : A \to B$ defined by $\varphi(0) = (0,0), \varphi(1) = (0,1), \varphi(n) = (1,0)$ naturally extends to sA with $2 \leq s < n$ and defines an $(s,0,0)$-isomorphism from A to B. For $s = n$, one difficulty comes from the fact that n has $n+1$ representations in nA (namely, $1+1+\cdots+1$ and the n permutations of $n+0+\cdots+0$), which correspond to the unique representation of $(0,n)$ in nB and the n representations of (1,0) in nB. Of course, from the beginning we knew that there is non $(n,0,0)$-isomorphism between A and B since nA and nB do not have the same cardinality; but we were not far from succeeding.

There are indeed ways to build satisfactory approximate isomorphisms between A and B. One way is to consider $A^{(n)} = nA \backslash \{n\}$ and $B^{(n)} = nB \backslash \{(0,n),(1,0)\}$; in this way, we get a $\Big(n,(n+1)3^{-n},0\Big)$-isomorphism.

Another way to proceed is to keep the n representations of (1,0) in nB, and take $B^{(n)} = nB \backslash \{(0,n)\}$; this implies that we have to keep n in nA, with its $(n+1)$ representations; we thus take $A^{(n)} = nA$. We are losing one representation on the ε-side (local side), but winning n of them on the δ-side (global side), and finally obtain a $\Big(n,3^{-n},3^{-n}\Big)$-isomorphism.

In general, the problem will be to find a trade between the δ and ε-sides, which play in opposite directions. For example, when dealing with sequences of probability measures (P_n) and (Q_n), a natural choice will be to obtain $(n,\delta_n,\varepsilon_n)$-isomorphisms such that $\delta_n \to 0$ and $\varepsilon_n = o\Big(\max_x \sup\big(P_n^{*n}(\{x\}),\, Q_n^{*n}(\{x\})\big)\Big)$.

We now turn our attention to our main result.

First, this notion of isomorphism helps us to understand that the conditions $a_1 = 0$ and (a_1,a_2) coprime induce no real restriction, since two random variables ξ and ζ, taking respectively the integral values

$$(a_0,a_1,a_2) \quad \text{and} \quad \Big(0, \frac{a_1-a_0}{gcd(a_1-a_0,a_2-a_0)}, \frac{a_2-a_0}{gcd(a_1-a_0,a_2-a_0)}\Big)$$

with the same weights (p_0, p_1, p_2), have laws which are $(h, 0, 0)$-isomorphic for any $h \geq 1$.

Second, in the case when $a_2(n)/\sqrt{n}$ tends to infinity with n, we can indeed show that $\xi_{n,1}$ is $(n, \delta_n, \varepsilon_n)$-isomorphic to the two dimensional integral valued random variable η taking the *fixed* values (0,0), (1,0), (0,1) with respective probabilities p_0, p_1, p_2, for some $\delta_n = o(1)$ and $\varepsilon_n = o(\frac{1}{n})$. This requires some explanation. We let

$$A = \{(0,0), (1,0), (0,1)\}, \quad B = \{0, a_1(n), a_2(n)\} ,$$

P_n be the law of $\xi_{n,1}$, $\quad Q$ be the law of η ,

$\varphi : A \to B$ such that $\varphi(0,0) = 0, \quad \varphi(1,0) = a_1, \quad \varphi(0,1) = a_2$,

$$A^{(n)} = \{(k_1, k_2) : -a_2/2 < k_1 - p_1 n \leq a_2/2, \quad k_1 + k_2 \leq n\} ,$$

$\varphi^{(n)} : A^{(n)} \to nB$ such that $\varphi(k_1, k_2) = k_1 a_1 + k_2 a_2$,

$$B^{(n)} = \varphi^{(n)}(A^{(n)}) ,$$

and we check that we have an isomorphism.

(i) The set $A^{(n)}$ contains

$$\{(k_1, k_2)/ - a_2/2 < k_1 - p_1 n \leq a_2/2, \ -a_2/2 < k_2 - p_2 n \leq a_2/2\} ;$$

since $a_2(n)/\sqrt{n}$ tends to infinity, we have $P^{*n}(A^{(n)}) \to 1$ when n tends to infinity. Now let N be in $B^{(n)}$. By the definition of $B^{(n)}$, there exists a triple (k_0, k_1, k_2) such that (k_1, k_2) is in $A^{(n)}$, $k_1 a_1 + k_2 a_2 = N$ and $k_0 + k_1 + k_2 = n$. The measure $Q^{*n}(\{N\})$ is the sum over all the triples (k_0, k_1, k_2) such that $k_1 a_1 + k_2 a_2 = N$ and $k_0 + k_1 + k_2 = n$ of the expression $\frac{n!}{k_0! k_1! k_2!} p_0^{k_0} p_1^{k_1} p_2^{k_2}$; but this last expression is $P^{*n}(\{(k_1, k_2)\})$. We thus have

$$Q^{*n}(B^{(n)}) = \sum_{N \in B^{(n)}} Q^{*n}(\{N\}) \geq \sum_{(k_1,k_2) \in A^{(n)}} P^{*n}(\{(k_1, k_2)\}) = P^{*n}(A^{(n)}) ,$$

and so we have $Q^{*n}(B^{(n)}) \to 1$ when $n \to \infty$.

(ii) Let (k_1, k_2) be in $A^{(n)}$. The only way to write (k_1, k_2) as the sum of n elements of A is to take k_1-times (1,0), k_2-times (0,1) and $(n - k_1 - k_2)$-times (0,0). This leads to $\mathrm{sum}_n \circ \varphi^{\otimes s} \circ \mathrm{sum}_n^{-1}(\{(k_1, k_2)\}) = \{k_1 a_1 + k_2 a_2\}$.

Now let N be in $B^{(n)}$. By the definition of $B^{(n)}$, there exists (k_1, k_2) in $A^{(n)}$ such that $N = k_1 a_1 + k_2 a_2$. But there exists at most one pair (k_1, k_2) with $N = k_1 a_1 + k_2 a_2$ and $-a_2/2 < k_1 - p_1 n \leq a_2/2$ (since a_1, a_2 are coprime). This implies that $\mathrm{sum}_n \circ (\varphi^{-1})^{\otimes s} \circ \mathrm{sum}_n^{-1}(\{N\})$ consists of a single pair (k_1, k_2).

(iii) The result of the theorem states precisely that for N in $B^{(n)}$, written as $k_1 a_1 + k_2 a_2$ with $(k_1, k_2) \in A^{(n)}$, we have

$$Q^{*n}(\{N\}) = P^{*n}(\{(k_1, k_2)\}) + o\Big(\frac{1}{n}\Big) .$$

We may further notice, which is not asked in the definition, that when N is not in $B^{(n)}$ we have $Q^{*n}(\{N\}) = o(\frac{1}{n})$ and when (k_1, k_2) is not in $A^{(n)}$, we have $P^{*n}(\{(k_1, k_2)\}) = o(\frac{1}{n})$, so that $nA \backslash A^{(n)}$ as well as $nB \backslash B^{(n)}$ is not only globally small (condition (i)) but also locally small.

References

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