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LOCAL MONOMIALIZATION AND FACTORIZATION OF MORPHISMS

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Abstract. — Suppose that $R \subset S$ are regular local rings of a common dimension, which are essentially of finite type over a field k of characteristic zero, such that the quotient field K of S is finite over the quotient field of R. If V is a valuation ring of K which dominates S, then we show that there are sequences of monoidal transforms (blowups of regular primes) $R \to R_1$ and $S \to S_1$ along V such that $R_1 \to S_1$ is a monomial mapping. It follows that a generically finite morphism of nonsingular varieties can be made to be a monomial mapping along a valuation, after blowups of nonsingular subvarieties. We give applications to factorization of birational morphisms and simultaneous resolution of singularities.

Résumé **(Monomialisation et factorisation locales des morphismes)**

Soient $R \subset S$ deux anneaux locaux réguliers de même dimension, essentiellement de type fini sur un corps k de caractéristique zéro, et tels que le corps des fractions K de S est fini sur celui de R. Si V est un anneau de valuation de K dominant S, nous montrons qu'il existe des suites de transformés monoïdaux (éclatements d'idéaux premiers réguliers) $R \to R_1$ et $S \to S_1$ le long de V tels que $R_1 \to S_1$ est une application monomiale. Il s'ensuit qu'un morphisme génériquement fini de variétés non singulières peut être rendu monomial le long d'une valuation après éclatement de sous-variétés non singulières. Nous donnons des applications à la factorisation des morphismes birationnels et à la résolution simultanée des singularités.

Contents

CHAPTER 1

INTRODUCTION

1.1. Statement of the main results

Suppose that we are given a system of equations

$$
\begin{aligned}
x_1 &= f_1(y_1, y_2, \ldots, y_n) \\
&\vdots \\
x_n &= f_n(y_1, y_2, \ldots, y_n)
\end{aligned}
\tag{1}
$$

which is nondegenerate, in the sense that the Jacobian determinate of the system is not (identically) zero. This system is well understood in the special case that $f_1, \ldots, f_n$ are monomials in the variables $y_1, \ldots, y_n$. For instance, by inverting the matrix A of coefficients of the monomials, we can express $y_1, \ldots, y_n$ as rational functions of d-th roots of the variables $x_1, \ldots, x_n$, where d is the determinant of A.

Our main result shows that all solutions of a system (1) can be expressed in the following simple form. There are finitely many charts obtained from a composition of monoidal tranforms in the variables x and y

$$
\begin{aligned}
x_i &= \Phi_i(\overline{x}_1, \ldots, \overline{x}_n), \quad 1 \le i \le n \\
y_i &= \Psi_i(\overline{y}_1, \ldots, \overline{y}_n), \quad 1 \le i \le n
\end{aligned}
$$

such that the transform of the system (1) becomes a system of monomial equations

$$
\begin{aligned}
\overline{x}_1 &= \overline{y}_1^{a_{11}} \cdots \overline{y}_n^{a_{1n}} \\
&\vdots \\
\overline{x}_n &= \overline{y}_1^{a_{n1}} \cdots \overline{y}_n^{a_{nn}}
\end{aligned}
$$

with $\det(a_{ij}) \neq 0$. A monoidal transform is a composition of

(1) a change of variable

(2) a transform

$$\begin{aligned} x_1 &= x_1(1)x_2(1) \\ x_i &= x_i(1) \qquad \text{if } i > 1. \end{aligned}$$

Our solution is constructive, as it consists of a series of algorithms.

This result can be interpreted geometrically as follows. Suppose that $\phi : X \to Y$ is a generically finite morphism of varieties. Then it is possible to construct a finite number of charts X_i and Y_i such that $X_i \to Y_i$ are monomial mappings, the mappings $X_i \to X$ and $Y_i \to Y$ are sequences of blowups of nonsingular subvarieties, and X_i and Y_i form complete systems, in the sense that they can be patched to obtain schemes which satisfy the existence part of the valuative criteria of properness.

Our main result is stated precisely in Theorem 1.1.

***Theorem 1.1* (Monomialization).** — *Suppose that $R \subset S$ are regular local rings, essentially of finite type over a field k of characteristic zero, such that the quotient field K of S is a finite extension of the quotient field J of R.*

Let V be a valuation ring of K which dominates S. Then there exist sequences of monoidal transforms $R \to R'$ and $S \to S'$ such that V dominates S', S' dominates R' and there are regular parameters $(x_1, \ldots, x_n)$ in R', $(y_1, \ldots, y_n)$ in S', units $\delta_1, \ldots, \delta_n \in S'$ and a matrix (a_{ij}) of nonnegative integers such that $\det(a_{ij}) \neq 0$ *and*

$$\begin{aligned} x_1 &= y_1^{a_{11}} \cdots y_n^{a_{1n}} \delta_1 \\ &\vdots \\ x_n &= y_1^{a_{n1}} \cdots y_n^{a_{nn}} \delta_n. \end{aligned} \tag{2}$$

With the assumptions of Theorem 1.1, An example of Abhyankar (Theorem 12 **[6]**) shows that it is in general not possible to perform monoidal transforms along V in R and S to obtain $R' \to S'$ such that $R' \to S'$ is (a localization of) a finite map. As such, Theorem 1.1 is the strongest possible local result for generically finite maps.

A more geometric statement of Theorem 1.1 is given in Theorem 1.2. A complete variety over a field k is an integral finite type k-scheme which satisfies the existence part of the valuative criterion for properness (cf. Chapter 8). Complete and separated is equivalent to proper. A toroidal morphism is locally a monomial mapping in uniformizing parameters on an appropriate etale extension.

Theorem 1.2. — *Let k be a field of characteristic zero, $\Phi : X \to Y$ a generically finite morphism of nonsingular proper k-varieties. Then there are birational morphisms of nonsingular complete k-varieties $\alpha : X_1 \to X$ and $\beta : Y_1 \to Y$, and a toroidal morphism $\Psi : X_1 \to Y_1$ such that the diagram*

$$\begin{array}{ccc} X_1 & \xrightarrow{\Psi} & Y_1 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Phi} & Y \end{array}$$

commutes and α and β are locally products of blowups of nonsingular subvarieties. That is, for every $z \in X_1$, there exist affine neighborhoods V_1 of z, V of $x = \alpha(z)$, such that $\alpha : V_1 \to V$ is a finite product of monoidal transforms, and there exist affine neighborhoods W_1 of $\Psi(z)$, W of $y = \alpha(\Psi(z))$, such that $\beta : W_1 \to W$ is a finite product of monoidal transforms.

Here a monoidal transform of a nonsingular k-scheme S is the map $T \to S$ induced by an open subset T of $\mathrm{Proj}(\oplus \mathcal{I}^n)$, where $\mathcal{I}$ is the ideal sheaf of a nonsingular subvariety of S. We give a proof of Theorem 1.2 in chapter 8.

In the special case of dimension two, we can strengthen the conclusions of Theorem 1.2.

Theorem 1.3. — *Let k be a field of characteristic zero, $\Phi : S \to T$ a generically finite morphism of nonsingular proper k-surfaces. Then there are products of blowups of points (quadratic transforms) $\alpha : S_1 \to S$ and $\beta : T_1 \to T$, and a morphism $\Psi : S_1 \to T_1$ such that the diagram*

$$\begin{array}{ccc} S_1 & \xrightarrow{\Psi} & T_1 \\ \downarrow & & \downarrow \\ S & \xrightarrow{\Phi} & T \end{array}$$

commutes, and Ψ is a toroidal morphism.

In the case of complex surfaces, a proof of 1.3 follows from results of Akbulut and King (Chapter 7 of [**8**]).

We also prove, as a corollary of Theorem 1.1, a local theorem on simultaneous resolution of singularities, which is valid in all dimensions. This theorem is proven in dimension 2 (and in all characteristics) by Abhyankar in Theorem 4.8 of his book "Ramification theoretic methods in algebraic geometry" [**3**].

***Theorem 1.4* (Theorem 1.1 [14])**. — *Let k be a field of characteristic zero, L/k an algebraic function field, K a finite algebraic extension of L, ν a valuation of K/k, and (R, M) a regular local ring with quotient field K, essentially of finite type over k, such that ν dominates R. Then for some sequence of monodial transforms $R \to R^*$ along ν, there exists a normal local ring S^* with quotient field L, essentially of finite type over k, such that R^* is the localization of the integral closure T of S^* in K at a maximal ideal of T.*

Stronger results hold for birational morphisms, morphisms which are an isomorphism on an open set. A birational morphism of nonsingular projective surfaces can be factored by a product of quadratic transforms. This was proved by Zariski, over an algebraically closed field of arbitrary characteristic, as a corollary to a local theorem on factorization (on page 589 of [**37**] and in section II.1 of [**38**]). The most general form of this Theorem is due to Abhyankar, in Theorem 3 of his 1956 paper [**2**]. Abhyankar proves that an inclusion $R \subset S$ of regular local rings of dimension 2 with a common quotient field can be factored by a finite sequence of quadratic transforms (blowups of points).

In higher dimensions, the simplest birational morphisms are the monoidal transforms. A monoidal transform is a blowup of a nonsingular subvariety. Sally [**30**] and Shannon [**33**] have found examples of inclusions $R \subset S$ of regular local rings of dimension 3 with a common quotient field which cannot be factored by a finite sequence of monoidal transforms (blowups of points and nonsingular curves).

In [**13**], we prove the following Theorem, which gives a positive answer to a conjecture of Abhyankar (page 237 [**5**], [**10**]), over fields of characteristic 0. In view of the counterexamples to a direct factorization, Theorem 1.5 is the best possible local factorization result in dimension three.

Theorem 1.5 **(Theorem A [13]).** — *Suppose that $R \subset S$ are excellent regular local rings such that* $\dim(R) = \dim(S) = 3$, *containing a field k of characteristic zero and with a common quotient field K. Let V be a valuation ring of K which dominates S. Then there exists a regular local ring T, with quotient field K, such that T dominates S, V dominates T, and the inclusions $R \to T$ and $S \to T$ can be factored by sequences of monoidal transforms.*

$$\begin{tikzcd} & V & \\ & T \arrow[u] & \\ R \arrow[ur] \arrow[rr] & & S \arrow[ul] \end{tikzcd}$$

It is natural to ask if the generalization of this three dimensional factorization theorem is possible in all dimensions by constructing a factorization by a sequence of blowups and blowdowns with nonsingular centers along a valuation. In this paper, we prove the following theorem which gives a positive answer to this question in all dimensions.

Theorem 1.6 **(Factorization 1).** — *Suppose that $R \subset S$ are regular local rings, essentially of finite type over a field k of characteristic zero, with a common quotient field K with* $\operatorname{trdeg}_k K = n \geq 3$. *Let V be a valuation ring of K which dominates S. Then there exist sequences of regular local rings contained in K*

$$\begin{tikzcd} & R_1 & & \cdots & & R_{n-2} & \\ R \arrow[ur] & & S_1 \arrow[ul] \arrow[ur] & & S_{n-3} \arrow[ul] \arrow[ur] & & S_{n-2} = S \arrow[ul] \end{tikzcd}$$

such that each local ring is dominated by V and each arrow is a sequence of monoidal transforms. Furthermore, we have inclusions $R \subset S_i$ for all i.

In the special case $n = 3$ of Theorem 1.6, we get the triangle of Theorem 1.5.

The proofs of the above theorems are essentially self contained in this paper. We only assume some basic results on valuation theory (as can be found in **[3]** and **[39]**) and the basic resolution theorems of Hironaka **[20]**. The Hironaka results are essentially only used in the case of a composite valuation, to establish the existence of a nonsingular center of a composite valuation.

A long standing conjecture in algebraic geometry is that one can factor a birational morphism $X \to Y$ between nonsingular projective varieties by a series of alternating blowups and blowdowns with nonsingular centers (cf. **[29]**). We will refer to this as the global factorization conjecture. In **[29]** an example is given of Hironaka, showing that it is not possible in dimension ≥ 3 to always factor birational morphisms of nonsingular varieties by blowups with nonsingular centers.

Our Theorem 1.6 shows that there is no local obstruction to the global factorization conjecture in any dimension. We prove a local form of this conjecture.

Theorem 1.7. — *Suppose that $X \to Y$ is a birational morphism of nonsingular projective n-dimensional varieties, over a field of characteristic zero, and ν is a valuation of the function field of X. Then there is a sequence of projective birational morphisms of nonsingular varieties*

$$\begin{array}{ccccccccc} & X_1 & & & & & X_{n-1} & & \\ \swarrow & & \searrow & \swarrow & \cdots \searrow & \swarrow & & \searrow & \\ X & & Y_1 & & Y_{n-1} & & & & Y_{n-2} = Y \end{array}$$

such that each morphism is a product of blowups of nonsingular subvarities in a Zariski neighbourhood of the center of ν.

Theorem 1.8. — *Let k be a field of characteristic zero, $\phi : X \to Y$ a birational morphism of nonsingular proper k-varieties of dimension n. Then there is a sequence of birational morphisms of nonsingular complete k-varieties $\alpha_i : X_{i+1} \to Y_i$ and $\beta_i : X_{i+1} \to Y_{i+1}$*

$$\begin{array}{ccccccccc} & X_1 & & & & & X_{n-1} & & \\ \swarrow & & \searrow & \swarrow & \cdots \searrow & \swarrow & & \searrow & \\ X & & Y_1 & & Y_{n-1} & & & & Y_{n-2} = Y \end{array}$$

such that each morphism is locally a product of blowups of nonsingular subvarieties. That is, for every $z \in X_{i+1}$, there exist affine neighborhoods W of z, U of $x = \alpha_i(z)$, V of $y = \beta_i(z)$ such that $\alpha_i : W \to U$ and $\beta_i : W \to V$ are finite products of monoidal transforms.

Theorem 1.8 is proved in dimension 3 by the author in **[13]**. The proof of Theorem 1.8 is exactly the same, with the use of Theorem 1.6 from this paper, which is valid in all dimensions. By Theorem 1.6, for each valuation of the function field K of X, there exist local rings for which the conclusions of Theorem 1.6 hold. These local rings can be extended to affine varieties which are related by products of monoidal transforms. By the quasi-compactness of the Zariski manifold (Theorem VI.17.40 **[39]**) all valuations of K are centered at finitely many of these affine constructions. We can then patch these affine varieties along the open sets where they are isomorphic to get complete k-varieties as desired. The proof is similar to that of Theorem 1.2 given in Chapter 8.

The Monomialization Theorem is a "resolution of singularities" type problem. Some of the difficulties which arise in it are related to those which appear in the problems of resolution of (char. 0) vector fields (cf. **[9]**, **[32]**), and in resolution of singularities in characteristic $p > 0$ (cf. [4], **[11]**, **[18]**, **[25]**). Resolution of vector fields is an open problem (locally) in dimension ≥ 4 and is open (globally) in dimension ≥ 3. Resolution of singularities in characteristic $p > 0$ is an open problem in dimension ≥ 4.

Some important papers which are directly concerned with the global factorization problem are Hironaka **[21]**, Danilov **[15]**, Crauder **[12]** and Pinkham **[29]**. An important special case where the global factorization problem has been solved is toric geometry (Danilov **[16]**, Ewald **[17]**, Wlodarczyk **[34]**, Morelli **[26]**, Abramovich, Matsuki, Rashid **[7]**).

A birational morphism of nonsingular toric varieties can be thought of as a union of monomial mappings on affine spaces. In toric geometry, the global factorization problem becomes more tractable than in the general case of arbitrary polynomial mappings, since the problem can be translated into combinatorics.

Morelli's main result **[26]** is that a birational morphism of proper nonsingular toric varieties can be factored by one sequence of blowups (with nonsingular centers) followed by one sequence of blowdowns (with nonsingular centers).

Our main result, Theorem 1.1 — Monomialization, allows us to reduce the factorization problem (locally) to monomial mappings. If we then make use of Morelli's result, which says (locally) that a birational monomial mapping can be factored by one sequence of blowups, followed by one sequence of blowdowns, we obtain an even stronger local factorization theorem than Theorem 1.6.

Abhyankar has conjectured (page 237 **[5]**, **[10]**) that in all dimensions it is possible to factor a birational mapping along a valuation by a sequence of blowups followed by a sequence of blowdowns with nonsingular centers. This is the most optimistic possible local statement.

We prove the following Theorem, which proves Abhyankar's conjecture in all dimensions (over fields of characteristic 0).

***Theorem 1.9* (Factorization 2).** — *Suppose that $R \subset S$ are regular local rings, essentially of finite type over a field k of characteristic zero, with a common quotient field K. Let V be a valuation ring of K which dominates S. Then there exists a regular local ring T, with quotient field K, such that T dominates S, V dominates T, and the* *inclusions $R \to T$ and $S \to T$ can be factored by sequences of monoidal transforms.*

$$\begin{tikzcd} & V & \\ & T \arrow[u] & \\ R \arrow[ur] \arrow[rr] & & S \arrow[ul] \end{tikzcd}$$

The solution to Abhyankar's conjecture (as stated in [**10**]) is given in Theorem 1.10.

Theorem 1.10. — *Suppose that K is a field of algebraic functions over a field k of characteristic zero, with $\mathrm{trdeg}_k K = n$, R and S are regular local rings, essentially of finite type over k, with quotient field K. Let V be a valuation ring of K which dominates R and S. Then there exists a regular local ring T, essentially of finite type over k, with quotient field K, dominated by V, containing R and S, such that $R \to T$ and $S \to T$ can be factored by products of monoidal transforms.*

In dimension 3, Theorems 1.9 and 1.10 have been proven by the author in [**13**]. Theorem 1.1, which shows that it is possible to monomialize a generically finite morphism along a valuation, is essential in this proof.

Hironaka and Abhyankar (section 6 of chapter 0 [**20**] and page 254 [**5**]) have conjectured that a birational morphism of nonsingular projective varieties can be factored by a series of blowups followed by a series of blowdowns with nonsingular centers.

Our Theorem 1.9 shows that there is no local obstruction to this global factorization conjecture in any dimension.

We prove the following global analogue of Theorem 1.9.

Theorem 1.11. — *Let k be a field of characteristic zero, $\phi : X \to Y$ a birational morphism of nonsingular proper k-varieties. Then there exists a nonsingular complete k-variety Z and birational complete morphisms $\alpha : Z \to X$ and $\beta : Z \to Y$ making the diagram*

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

commute, such that α and β are locally products of monoidal transforms. That is, for every $z \in Z$, there exist affine neighborhoods W of z, U of $x = \alpha(z)$, V of $y = \beta(z)$ such that $\alpha : W \to U$ and $\beta : W \to V$ are finite products of monoidal transforms.

Here a monoidal transform of a nonsingular k-scheme S is the map $T \to S$ induced by an open subset T of $\mathrm{Proj}(\oplus \mathcal{I}^n)$, where $\mathcal{I}$ is the ideal sheaf of a nonsingular subvariety of S.

Theorem 1.11 is proved in dimension 3 by the author in [**13**]. The proof is exactly the same, with the use of Theorem 1.9 from this paper, which is valid in all dimensions. The method of proof is similar to Theorem 1.2, which is proved in Chapter 8.

The author would like to thank the referee for a careful reading of the manuscript and for helpful comments.

1.2. Geometry and valuations

A valuation ring of a field of algebraic functions K will dominate some local ring of a projective model V of K. This leads to the "valuative criterion for properness" (cf. Theorem II.4.7 [**19**]).

The Zariski manifold M of K is a locally ringed space whose local rings are the valuations rings of K, containing the ground field k (cf. chapter VI, section 17 [**39**], [**24**], section 6 of chaper 0 [**20**]). M satisfies the universal property that for any morphism of proper k-schemes $\Psi : X \to Y$ such that X and Y have function fields (isomorphic to) K, there are projections $\pi_1 : M \to X$ and $\pi_2 : M \to Y$ making a commutative diagram

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

When K is a 1-dimensional function field, the only nontrivial valuation rings are the local rings of the points on the nonsingular model of K. As such, a projective nonsingular curve can be identified with its Zariski manifold (cf. I.6 [**19**]). If K has dimension > 1, K has many non-noetherian valuations, and M is far from being a k-scheme.

The main result of Zariski in [**36**] is his Theorem U_1, which states that for a valuation B of a field of algebraic functions K over a ground field of characteristic 0, there is a projective model V of K on which the center of B is at a nonsingular point of V.

Our Theorems 1.1, 1.6, 1.9 and 1.10 are direct analogues of Theorem U_1 for generically finite and birational morphisms of varieties.

Zariski obtained a solution to "the classical problem of local uniformization" from his Theorem U_1. In the language of schemes (cf. section 6 of chapter 0 [**20**]) Zariski's result shows that for any integral proper k-scheme X (where k is a field of characteristic 0) there exists a complete nonsingular integral k-scheme Y and a birational morphism $Y \to X$. A complete variety over a field k is an integral finite type k-scheme which satisfies the existence part of the valuative criterion for properness.

Our Theorems 1.2, 1.3, 1.8 and 1.11 are analogous to Zariski's solution of "the classical problem of local uniformization".

1.3. Overview of the proof

The main thrust of the paper is to acheive monomialization. Theorem 1.1 proves monomialization for generically finite extensions. The corollaries, Theorems 1.2 through 1.11 are then easily obtained.

Theorem 1.1 is an immediate corollary of Theorem 5.3. Theorem 5.5 is a stronger version, valid for birational extensions.

In fact, Theorems 5.3 and 5.5 prove more than monomialization. They produce a matrix of exponents $A = (a_{ij})$ which has a very special form, depending on the rational rank of the rank 1 valuations composite with V.

Theorem 5.5 reduces the proof of Theorem 1.6 (Local factorization) to the special case where $\dim R = \dim S = n$ and V has rank 1 and rational rank n. Factorization in the special case $n = 3$ and V has rational rank $n = 3$ was solved by Christensen in [**10**]. We generalize Christensen's algorithm in Theorem 6.4 to prove factorization when V has rational rank n. The proof of Theorem 6.4 uses only elementary methods of linear algebra. Theorem 1.6 then follows from Theorem 5.5.

Now we will discuss the proof of Theorem 5.3. The most difficult part of Theorem 5.3 is the case where ν has rank 1, which is proved in Theorem 5.1. Almost the entirety of the paper (chapter 4) is devoted to the proof of Theorem 5.1.

Suppose that ν has rank 1 and rational rank s. Then it is not difficult to construct sequences of monoidal transforms $R \to R(1)$ and $S \to S(1)$ such that ν dominates $S(1)$, $S(1)$ dominates $R(1)$, $R(1)$ has regular parameters $(x_1(1), \ldots, x_n(1))$, $S(1)$ has regular parameters $(y_1(1), \ldots, y_n(1))$ such that

$$\begin{aligned} x_1(1) &= y_1(1)^{c_{11}(1)} \cdots y_s(1)^{c_{1s}(1)}\delta_1 \\ &\vdots \\ x_s(1) &= y_1(1)^{c_{s1}(1)} \cdots y_s(1)^{c_{ss}(1)}\delta_s \end{aligned}$$

where $\det(c_{ij}(1)) \neq 0$ and δ_i are units in $S(1)$. This step is accomplished in the proof of Theorem 5.1.

The inductive step in the proof is Theorem 4.11, which starts with monoidal transform sequences (MTSs) $R \to R(0)$ and $S \to S(0)$ such that ν dominates $S(0)$, $S(0)$ dominates $R(0)$, $R(0)$ has regular parameters $(x_1(0), \ldots, x_n(0))$, $S(0)$ has regular parameters $(y_1(0), \ldots, y_n(0))$ such that

$$\begin{aligned} x_1(0) &= y_1(0)^{c_{11}(0)} \cdots y_s(0)^{c_{1s}(0)}\delta_1 \\ &\vdots \\ x_s(0) &= y_1(0)^{c_{s1}(0)} \cdots y_s(0)^{c_{ss}(0)}\delta_s \\ x_{s+1}(0) &= y_{s+1}(0) \\ &\vdots \\ x_l(0) &= y_l(0) \end{aligned} \tag{3}$$

where $\det(c_{ij}(0)) \neq 0$ and δ_i are units in $S(0)$, and construct MTSs $R(0) \to R(t)$, $S(0) \to S(t)$ such that ν dominates $S(t)$, $S(t)$ dominates $R(t)$, $R(t)$ has regular parameters $(x_1(t), \ldots, x_n(t))$, $S(t)$ has regular parameters $(y_1(t), \ldots, y_n(t))$ such that

$$
\begin{aligned}
x_1(t) &= y_1(t)^{c_{11}(t)} \cdots y_s(t)^{c_{1s}(t)} \delta_1 \\
&\vdots \\
x_s(t) &= y_1(t)^{c_{s1}(t)} \cdots y_s(t)^{c_{ss}(t)} \delta_s \\
x_{s+1}(t) &= y_{s+1}(t) \\
&\vdots \\
x_{l+1}(t) &= y_{l+1}(t)
\end{aligned}
\tag{4}
$$

where $\det(c_{ij}(t)) \neq 0$ and δ_i are units in $S(t)$.

To prove Theorem 4.11, we make use of special sequences of monoidal transforms which are derived from the Cremona tranformations constructed by Zariski in chapter B of **[36]**, using an algorithm of Perron. We will call such transformations Perron transforms. Our proof makes use of these transforms in local rings of etale extensions, giving the transforms the special form

$$
\begin{aligned}
x_1(i) &= x_1(i+1)^{a_{11}(i+1)} \cdots x_s(i+1)^{a_{1s}(i+1)} c_{i+1}^{a_{1,s+1}(i+1)} \\
&\vdots \\
x_s(i) &= x_1(i+1)^{a_{s1}(i+1)} \cdots x_s(i+1)^{a_{ss}(i+1)} c_{i+1}^{a_{s,s+1}(i+1)} \\
x_r(i) &= x_1(i+1)^{a_{s+1,1}(i+1)} \cdots x_s(i+1)^{a_{s+1,s}(i+1)} \\
&\quad \cdot (x_r(i+1)+1) c_{i+1}^{a_{s+1,s+1}(i+1)}
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
y_1(i) &= y_1(i+1)^{b_{11}(i+1)} \cdots y_s(i+1)^{b_{1s}(i+1)} d_{i+1}^{b_{1,s+1}(i+1)} \\
&\vdots \\
y_s(i) &= y_1(i+1)^{b_{s1}(i+1)} \cdots y_s(i+1)^{b_{ss}(i+1)} d_{i+1}^{b_{s,s+1}(i+1)} \\
y_r(i) &= y_1(i+1)^{b_{s+1,1}(i+1)} \cdots y_s(i+1)^{b_{s+1,s}(i+1)} \\
&\quad (y_r(i+1)+1) d_{i+1}^{b_{s+1,s+1}(i+1)}
\end{aligned}
$$

where $\det(a_{ij}(i+1)) = \pm 1$ and $\det(b_{ij}(i+1)) = \pm 1$, c_{i+1}, d_{i+1} are algebraic over k.

Zariski observes on page 343 of **[35]** that his Cremona transformations have "the same effect as the classical Puiseux substitution $x = x_1^\nu, y = x_1^\mu(c_1 + y_1)$ used in the determination of the branches of the curve $\phi(x,y) = 0$. The only difference—and advantage—is that our transformation does not lead to elements x_1, y_1 outside the field $k(x,y)$"

Our transforms (5) do induce a field extension. They are the direct generalization of the classical Puiseux substitution to higher dimensions. We must pay for the advantage of the simple form of the equations by introducing many difficulties arising from the need to make finite etale extensions after each transform. We call a sequence of such transforms a uniformizing transform sequence (UTS).

4.11 is proved by first constructing UTSs such that (4) holds, and then using this partial solution to construct sequences of monoidal transforms such that (4) holds. The UTSs are constructed in Theorems 4.7 and 4.11, and this is used to construct MTSs such that (4) holds in Theorems 4.8, 4.9, 4.10 and 4.11.

Underlying the whole proof is Zariski's algorithm for the reduction of the multiplicity of a polynomial along a rank 1 valuation, via Perron transforms. This algorithm is itself a a generalization of Newton's algorithm to determine the branches of a curve singularity.

We will now give an outline of the proof of Theorem 4.11 (the inductive step). We will give a formal construction, so that we need only consider UTSs, where the basic ideas are transparent. We will construct UTSs along ν (here the $U(i)$, $T(i)$ are complete local rings)

$$\begin{array}{ccccccc} U(0) & \to & U(1) & \to & \cdots & \to & U(t) \\ \uparrow & & \uparrow & & & & \uparrow \\ T(0) & \to & T(1) & \to & \cdots & \to & T(t) \end{array} \tag{6}$$

such that $T(i)$ has regular parameters $(x_1(i), \ldots, x_n(i))$, $U(i)$ has regular parameters $(y_1(i), \ldots, y_n(i))$ such that

$$\begin{aligned} x_1(i) &= y_1(i)^{c_{11}(i)} \cdots y_s(i)^{c_{1s}(i)} \\ &\vdots \\ x_s(i) &= y_1(i)^{c_{s1}(i)} \cdots y_s(i)^{c_{ss}(i)} \\ x_{s+1}(i) &= y_{s+1}(i) \\ &\vdots \\ x_l(i) &= y_l(i) \end{aligned} \tag{7}$$

where $\det(c_{ij}(i)) \neq 0$ for $0 \leq i \leq t$.

We will presume that k is algebraically closed, and isomorphic to the residue fields of R, S and V. We will also assume that various technical difficulties, such as the rank of ν increasing when ν is extended to the complete local ring $U(i)$, do not occur.

In Theorem 4.7 it is shown that

(8) Given $f \in U(0)$, there exists (6) such that $f = y_1(t)^{d_1} \cdots y_s(t)^{d_s}\gamma$ where γ is a unit in $U(t)$.

(9) Given $f \in U(0) - k[[y_1, \ldots, y_l]]$, there exists (6) such that

$$f = P(y_1(t), \ldots, y_l(t)) + y_1(t)^{d_1} \cdots y_s(t)^{d_s} y_{l+1}(t)$$

where P is a power series.

Theorem 4.11 then shows that it is possible to construct a UTS (6) such that $x_{l+1}(t) = y_{l+1}(t)$, which allows us to conclude the truth of the inductive step.

We will now give a more detailed analysis of these important steps. For simplicity, we will assume that $s = \text{rat rank}(\nu) = 1$. This is the essential case.

Suppose that R has regular parameters $(x_1, \dots, x_n)$, and $2 \le i \le n$. Zariski constructed (in **[35]**, and in a generalized form in **[36]**) a MTS $R \to R(1)$ where $R(1)$ has regular parameters $(x_1(1), x_2(1), \dots, x_n(1))$ by the following method. Since ν has rational rank 1, we can identify the value group of ν with a subgroup of $\mathbf{R}$.

$$\nu\left(\frac{x_i}{x_1}\right) = \frac{a_{i1}(1)}{a_{11}(1)}$$

where $a_{i1}(1), a_{11}(1)$ are relatively prime positive integers. We can then choose positive integers $a_{i1}(1), a_{ii}(1)$ such that $a_{11}(1)a_{ii}(1) - a_{1i}(1)a_{i1}(1) = 1$. Then

$$\begin{aligned} \nu(x_1^{a_{ii}(1)}x_i^{-a_{1i}(1)}) &> 0 \\ \nu(x_1^{-a_{i1}(1)}x_i^{a_{11}(1)}) &= 0 \end{aligned}$$

There exits then a uniquely determined, nonzero $c_1 \in k$ such that

$$\nu(x_1^{-a_{i1}(1)}x_i^{a_{11}(1)} - c_1) > 0.$$

We can define $x_j(1)$ for $1 \le j \le n$ by

$$\begin{aligned} x_1 &= x_1(1)^{a_{11}(1)}(x_i(1)+c_1)^{a_{1i}(1)} \\ x_i &= x_1(1)^{a_{i1}(1)}(x_i(1)+c_1)^{a_{ii}(1)} \\ x_j &= x_j(1) \text{ if } j \ne 1 \text{ or } i. \end{aligned} \tag{10}$$

Set $R(1) = R[x_1(1), x_i(1)]_{(x_1(1),\dots,x_n(1))}$.

Using such transformations, Zariski proves

Theorem 1.12 **(Zariski [35], [36]).** — *Given $f \in R$, there exists a MTS along ν*

$$R \to R(1) \to \cdots \to R(t)$$

such that $f = x_1(t)^d\gamma$ where gamma is a unit in $R(t)$.

In our analysis, we will consider the transformation (10) in formal coordinates. $\widehat{R}(1)$ has regular parameters $(\overline{x}_1(1), \overline{x}_2(1), \dots, \overline{x}_n(1))$ defined by

$$\begin{aligned} \overline{x}_1(1) &= x_1(1)(x_i(1)+c_1)^{a_{1i}(1)/a_{11}(1)} \\ \overline{x}_i(1) &= (x_i(1)+c_1)^{1/a_{11}(1)} - c_1^{1/a_{11}(1)} \\ \overline{x}_j(1) &= x_j(1) \text{ if } j \ne 1 \text{ or } i. \end{aligned}$$

In these coordinates, (10) becomes

$$\begin{aligned} x_1 &= \overline{x}_1(1)^{a_{11}(1)} \\ x_i &= \overline{x}_1(1)^{a_{i1}(1)}(\overline{x}_i(1) + c_1^{1/a_{11}(1)}) \\ x_j &= \overline{x}_j(1) \text{ if } j \ne 1 \text{ or } i. \end{aligned}$$

We have inclusions

$$T'(1) = R(1) \to T''(1) \to T(1) = \widehat{R(1)}$$

where

$$T''(1) = R(1)\left[(x_i(1)+c_1)^{1/a_{11}(1)}\right]_{(\overline{x}_1(1),\ldots,\overline{x}_n(1))}$$

is a localization of a finite etale extension of $R(1)$. We can extend ν to a valuation of the quotient field of $\widehat{R(1)}$ which dominates $\widehat{R(1)}$. For simplicity, we will assume that this extension still has rank 1.

We will construct sequences of UTSs

$$\begin{array}{ccccccc}
T'(1) \to T''(1) \to T(1) = \widehat{T'(1)} & & & & \\
\quad\quad\downarrow & & & & \\
\quad\quad T'(2) \to & T''(2) & \to T(2) = \widehat{T'(2)} & \\
& \downarrow & & \\
& T'(3) & \cdots & \\
& & & \vdots
\end{array}$$

where each downward arrow is of the form (10).

To prove the inductive step, we must construct UTSs (6) starting with $R \to S$. By induction, we may assume that R has regular parameters $(x_1,\ldots,x_n)$ and S has regular parameters $(y_1,\ldots,y_n)$ such that

$$\begin{aligned}
x_1 &= y_1^{t_0}\delta_1\\
x_2 &= y_2\\
&\vdots\\
x_l &= y_l
\end{aligned}$$

where δ_1 is a unit in S. (Recall that we are assuming that $s=1$.) By Hensel's lemma, δ_1^{1/t_0} is a unit in $\widehat{S}$. We then can start our sequence of UTSs by setting $U''(0) = S[\delta_1^{1/t_0}]_{(\overline{y}_1,\ldots,\overline{y}_n)}$ and $T''(0) = R$, with regular parameters $(\overline{x}_1,\ldots,\overline{x}_n)$ and $(\overline{y}_1,\ldots,\overline{y}_n)$, such that

$$\begin{aligned}
\overline{x}_1 &= \overline{y}_1^{t_0}\\
\overline{x}_2 &= \overline{y}_2\\
&\vdots\\
\overline{x}_l &= \overline{y}_l
\end{aligned}$$

We will construct 2 types of UTSs

$$\begin{array}{ccc}
U''(0) & \to & U''(1)\\
\uparrow & & \uparrow\\
T''(0) & \to & T''(1)
\end{array}$$

A transformation of Type I is defined when $2 \le i \le l$. The equations defining the horizontal maps are then

$$\begin{aligned}
\overline{x}_1 &= \overline{x}_1(1)^{a_{11}(1)}\\
\overline{x}_i &= \overline{x}_1(1)^{a_{i1}(1)}(\overline{x}_i(1)+c_1)\\
\overline{x}_j &= \overline{x}_j(1) \text{ if } j \ne 1 \text{ or } i,
\end{aligned}$$

$$\begin{aligned}
\overline{y}_1 &= \overline{y}_1(1)^{b_{11}(1)}\\
\overline{y}_i &= \overline{y}_1(1)^{b_{i1}(1)}(\overline{y}_i(1)+d_1)\\
\overline{y}_j &= \overline{y}_j(1) \text{ if } j \ne 1 \text{ or } i.
\end{aligned}$$

$$\frac{a_{i1}(1)}{a_{11}(1)} = \frac{\nu(\overline{x}_i)}{\nu(\overline{x}_1)} = \frac{\nu(\overline{y}_i)}{t_0\nu(\overline{y}_1)} = \frac{b_{i1}(1)}{t_0 b_{11}(1)}$$

$(a_{11}(1), a_{i1}(1)) = 1$ implies $a_{11}(1) \mid t_0 b_{11}(1)$.

We thus have

$$\begin{aligned}
\overline{x}_1(1) &= \overline{y}_1(1)^{t_0 b_{11}(1)/a_{11}(1)} = \overline{y}_1(1)^{t_1}\\
\overline{x}_2(1) &= \overline{y}_2(1)\\
&\vdots\\
\overline{x}_l(1) &= \overline{y}_l(1).
\end{aligned}$$

A transformation of Type II is defined when $l < i$. The equations defining the horizontal maps are then

$$\overline{x}_j = \overline{x}_j(1) \text{ for } 1 \le j \le n$$

$$\begin{aligned}
\overline{y}_1 &= \overline{y}_1(1)^{b_{11}(1)}\\
\overline{y}_i &= \overline{y}_1(1)^{b_{i1}(1)}(\overline{y}_i(1)+d_1)\\
\overline{y}_j &= \overline{y}_j(1) \text{ if } j \ne 1 \text{ or } i.
\end{aligned}$$

In this case $T''(0) = T''(1)$.

In this way, we can construct sequences of UTSs

$$\begin{array}{ccccccccc}
S & \to & U(0) & \to & U(1) & \to & \cdots & \to & U(t)\\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow\\
R & \to & T(0) & \to & T(1) & \to & \cdots & \to & T(t)
\end{array} \tag{11}$$

such that $T(k)$ has regular parameters $(\overline{x}_1(k), \dots, \overline{x}_n(k))$ and $U(k)$ has regular parameters $(\overline{y}_1(k), \dots, \overline{y}_n(k))$ related by

$$\begin{aligned}
\overline{x}_1(k) &= \overline{y}_1(k)^{t_k}\\
\overline{x}_2(k) &= \overline{y}_2(k)\\
&\vdots\\
\overline{x}_l(k) &= \overline{y}_l(k)
\end{aligned} \tag{12}$$

for $0 \leq k \leq t$. The transformations $T(k) \to T(k+1)$ and $U(k) \to U(k+1)$ are of type I or II, and we also allow changes of variables, replacing $\overline{x}_i(k)$ with $\overline{x}_i(k) - P(\overline{x}_1(k), \ldots, \overline{x}_{i-1}(k))$ and replacing $\overline{y}_i(k)$ with $\overline{y}_i(k) - P(\overline{x}_1(k), \ldots, \overline{x}_{i-1}(k))$ if $2 \leq i \leq l$, for some power series P, and we may replace $\overline{y}_i(k)$ with $\overline{y}_i(k) - P(\overline{y}_1(k), \ldots, \overline{y}_{i-1}(k))$ if $l < i$.

To prove the induction step, we must prove Theorems 1.13 and 1.14 below.

***Theorem 1.13* (Theorem 4.7 with s = rat rank $\nu = 1$)**

(13) *Given $f \in U(0)$, there exists a sequence (11) such that $f = \gamma \overline{y}_1(t)^d$ with γ a unit in $U(t)$. (If $f \in k[[\overline{y}_1, \ldots, \overline{y}_\tau]]$, the transformations of type I and II in the sequence involve only the first τ variables.)*

(14) *Suppose that $f \in k[[\overline{y}_1, \ldots, \overline{y}_m]] - k[[\overline{y}_1, \ldots, \overline{y}_l]]$. Then there exists a sequence (11) such that $f = P(\overline{y}_1(t), \ldots, \overline{y}_l(t)) + \overline{y}_1(t)^{d_1} \overline{y}_m(t)$.*

***Theorem 1.14* (Theorem 4.11).** — *there exists a sequence (11) such that $\overline{x}_{l+1}(t) = \overline{y}_{l+1}(t)$.*

Outline of proof of (13) of Theorem 1.13. — The proof is by induction on τ. Suppose that (13) is true for $\tau - 1$. We will assume that $\tau \leq l$, which is the essential case. Recall that

$$\begin{aligned} \overline{x}_1 &= \overline{y}_1^{t_0} \\ \overline{x}_2 &= \overline{y}_2 \\ &\vdots \\ \overline{x}_l &= \overline{y}_l. \end{aligned}$$

Let ω be a primitive t_0^{th} root of unity. Set

$$g(\overline{x}_1, \ldots, \overline{x}_\tau) = \prod_{i=0}^{t_0 - 1} f(\omega^i \overline{y}_1, \overline{y}_2, \ldots, \overline{y}_\tau).$$

$f \mid g$ in $U(0)$. We will perform a UTS (11) to get $g = \overline{x}_1(t)^d \Lambda$ where Λ is a unit in $T(t)$. Then $f = \overline{y}_1(t)^{d_1'} \Lambda'$ where Λ' is a unit in $U(t)$.

To transform g into the form $g = \overline{x}_1(t)^d \Lambda$, where Λ is a unit, we will make use of an algorithm of Zariski ([**35**], [**36**]) to reduce the multiplicity of g. Initially set $g = \overline{x}_1^b g_0$ where $\overline{x}_1$ does not divide g_0. Set

$$r = \mathrm{mult}(g_0(0, \ldots, 0, \overline{x}_\tau)).$$

$0 \leq r \leq \infty$. If $r = 0$, g_0 is a unit, and we are done. Suppose that $0 < r$. We can write

$$g_0 = \sum_{i=0}^{\infty} a_i(\overline{x}_1, \ldots, \overline{x}_{\tau-1}) \overline{x}_\tau^i = \sum_{i=1}^{d} a_{\alpha_i} \overline{x}_\tau^{\alpha_i} + \sum_j a_{\beta_j} \overline{x}_\tau^{\beta_j} + \overline{x}_\tau^N \Omega$$

where the terms $a_{\alpha_i}\overline{x}_\tau^{\alpha_i}$ have minimum value ρ, $N\nu(\overline{x}_\tau) > \rho$, and the $a_{\beta_j}\overline{x}_\tau^{\beta_j}$ terms are the finitely many remaining terms. We must have $\alpha_i \leq r$ for all α_i, and $\alpha_i = r$ implies α_i is a unit.

By induction, we can perform UTSs in the first $\tau - 1$ variables to reduce to the case

$$\begin{aligned} a_{\alpha_i} &= \overline{x}_1^{\gamma_i} u_{\alpha_i}(\overline{x}_1, \ldots, \overline{x}_{\tau-1}) \\ a_{\beta_j} &= \overline{x}_1^{\delta_j} u_{\beta_j}(\overline{x}_1, \ldots, \overline{x}_{\tau-1}) \end{aligned}$$

where u_{α_i} and u_{β_j} are units in $T(0)$. Now we make a UTS

$$\begin{aligned} \overline{x}_1 &= \overline{x}_1(1)^{a_{11}} \\ \overline{x}_\tau &= \overline{x}_1(1)^{a_{\tau 1}}(\overline{x}_\tau(1) + c_1). \end{aligned}$$

$$g_0 = \sum_{i=1}^{d} \overline{x}_1^{\gamma_i a_{11} + \alpha_i a_{\tau 1}} u_{\alpha_i}(\overline{x}_\tau(1) + c_1)^{\alpha_i} + \cdots = \overline{x}_1(1)^{\varepsilon}\Big(\sum_{i=1}^{d} u_{\alpha_i}(\overline{x}_\tau(1) + c_1)^{\alpha_i} + \overline{x}_1\Omega\Big).$$

Set

$$\begin{aligned} g_1 &= \sum_{i=1}^{d} u_{\alpha_i}(\overline{x}_\tau(1) + c_1)^{\alpha_i} + \overline{x}_1\Omega, \\ r_1 &= \operatorname{mult} g_1(0, \ldots, 0, \overline{x}_\tau(1)). \end{aligned}$$

$r_1 < \infty$ and $r_1 \leq r$ since all $\alpha_i \leq r$. Set

$$\zeta(t) = g_1(0, \ldots, 0, t - c_1) = \sum_{i=1}^{d} u_{\alpha_i}(0, \ldots, 0)t^{\alpha_i}.$$

If we do not have a reduction in r, so that $r_1 = r$,

$$\zeta(\overline{x}_\tau(1) + c_1) = e\overline{x}_\tau(1)^r$$

for some nonzero $e \in k$. Thus $\zeta(t) = e(t - c_1)^r$ has a nonzero t^{r-1} term. We conclude that $\alpha_d = r$, a_{α_d} is a unit, $\alpha_{d-1} = r - 1$ and

$$\rho = \nu(a_{\alpha_d}\overline{x}_\tau^r) = \nu(a_{\alpha_{d-1}}\overline{x}_\tau^{r-1}).$$

Thus

$$\nu(\overline{x}_\tau) = \nu(a_{\alpha_{d-1}}(\overline{x}_1, \ldots, \overline{x}_{\tau-1})).$$

Since $\overline{x}_\tau^r$ is a minimum value term of f,

$$\nu(\overline{x}_\tau) \leq \nu(\overline{x}_\tau^r) \leq \nu(f).$$

Now make a change of variables in $T(0)$, replacing $\overline{x}_\tau$ with $\overline{x}'_\tau = \overline{x}_\tau - \lambda a_{\alpha_{d-1}}$ where $\lambda \in k$ is chosen to make $\nu(\overline{x}_\tau) < \nu(\overline{x}'_\tau)$. Repeat the above procedure with these new variables. We eventually get a reduction in r. In fact if we didn't, we would have an infinite bounded sequence in $T(0)$

$$\nu(\overline{x}_\tau) < \nu(\overline{x}'_\tau) < \cdots < \nu(f)$$

which is impossible (by Lemma 2.3).

Outline of proof of (14) of Theorem 1.13

$$f = \sum_{i=0}^{\infty} \sigma_i(\overline{y}_1, \ldots, \overline{y}_{m-1})\overline{y}_m^i.$$

Set

$$Q = \sum_{i>0} \sigma_i(\overline{y}_1, \ldots, \overline{y}_{m-1})\overline{y}_m^i.$$

After possibly permuting the variables $\overline{y}_{l+1}, \ldots, \overline{y}_m$, we can assume that $Q \neq 0$. $Q = \overline{y}_1^{n_1} Q_0$. where $\overline{y}_1$ does not divide Q_0. Set $r = \mathrm{mult}(Q_0(0, \ldots, 0, \overline{y}_m))$. $1 \leq r \leq \infty$.

Suppose that $r > 1$. Write

$$Q_0 = \sum_{i=1}^{d} \sigma_{\alpha_i}(\overline{y}_1, \ldots, \overline{y}_{m-1})\overline{y}_m^{\alpha_i} + \cdots$$

where the $\sigma_{\alpha_i}\overline{y}_m^{\alpha_i}$ are the minimum value terms. By construction, all $\alpha_i > 0$. By (13) of 1.13, we can perform UTSs in the first $m-1$ variables to get $\sigma_i = u_{\alpha_i}\overline{y}_1^{\gamma_i}$ where u_{α_i} are units. Then we can perform a UTSs in $\overline{y}_1$ and $\overline{y}_m$ to get

$$Q_0 = \overline{y}_1(1)^{\varepsilon}(\sum_{i=1} \sigma_{\alpha_i}(\overline{y}_m(1) + c_1)^{\alpha_i} + \overline{y}_1(1)\Omega).$$

Set

$$Q_1 = \sum_{i=1}^{d} \sigma_{\alpha_i}(\overline{y}_m(1) + c_1)^{\alpha_i} + \overline{y}_1(1)\Omega - \sum_{i=1}^{d} u_{\alpha_i} c_1^{\alpha_i}.$$

Set $r_1 = \mathrm{mult}\, Q_1(0, \ldots, 0, \overline{y}_m(1))$. $0 < r_1 < \infty$ and $r_1 \leq r$. Suppose that we do not have a reduction in r, so that $r_1 = r$. Then as in the proof of (13) of Theorem 1.13, $\nu(\overline{y}_m) = \nu(\sigma_{\alpha_{d-1}}(\overline{y}_1, \ldots, \overline{y}_{m-1}))$. Now make a change of variables in $U(0)$, replacing $\overline{y}_m$ with

$$\overline{y}'_m = \overline{y}_m - \lambda\sigma_{\alpha_{d-1}}(\overline{y}_1, \ldots, \overline{y}_{m-1}),$$

where $\lambda \in k$ is chosen to make $\nu(\overline{y}_m) < \nu(\overline{y}'_m)$. We have

$$\nu(\overline{y}_m) \leq \nu(\overline{y}_m^{r-1}) \leq \nu\left(\frac{\partial Q_0}{\partial \overline{y}_m}\right) \leq \nu\left(\frac{\partial f}{\partial \overline{y}_m}\right).$$

Repeat the above algorithm with $\overline{y}_m$ replaced with $\overline{y}'_m$ in $U(0)$. Since $\partial f/\partial \overline{y}_m = \partial f/\partial \overline{y}'_m$, we get a reduction in r after finitely many iterations, by an argument similar to that at the end of the proof of (13) of Theorem 1.13.

We can then repeat this procedure to eventually get an expression

$$f = L(\overline{y}_1, \ldots, \overline{y}_{m-1}) + \overline{y}_1^{n_0}\overline{Q}$$

where $\operatorname{mult} \overline{Q}(0, \ldots, 0, \overline{y}_m) = 1$.

By induction, we can perform UTSs in the first $m-1$ variables only to get

$$L = L'(\overline{y}_1, \ldots, \overline{y}_l) + \overline{y}_1^{n_1}\overline{Q}_1$$

where $\operatorname{mult} \overline{Q}_1(0, \ldots, 0, \overline{y}_{m-1}) = 1$. Then f is in the desired form.

Outline of proof of Theorem 1.14. — By (14) of Theorem 1.13 we may assume that

$$\begin{aligned}
\overline{x}_1 &= \overline{y}_1^{t_0} \\
\overline{x}_2 &= \overline{y}_2 \\
&\vdots \\
\overline{x}_l &= \overline{y}_l \\
\overline{x}_{l+1} &= P(\overline{y}_1, \ldots, \overline{y}_l) + \overline{y}_1^{d_1}\overline{y}_{l+1}.
\end{aligned}$$

Let ω be a primitive t_0^{th} root of unity. Set

$$g(\overline{x}_1, \ldots, \overline{x}_{l+1}) = \prod_{i=0}^{t_0-1}(\overline{x}_{l+1} - P(\omega^i\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_l)).$$

$\overline{y}_{l+1}$ divides g in $U(0)$. Set $r = \operatorname{mult}(g(0, \ldots, 0, \overline{x}_{l+1}))$. $1 \leq r < \infty$.

Suppose that $r = 1$. Then in $T(0)$,

$$\begin{aligned}
g &= \operatorname{unit}(\overline{x}_{l+1} + \Phi(\overline{x}_1, \ldots, \overline{x}_l)) \\
&= \operatorname{unit}(P + \overline{y}_1^{d_1}\overline{y}_{l+1} + \Phi).
\end{aligned}$$

since $\overline{y}_{l+1}$ divides g, we must have $P = -\Phi$. We can then replace $\overline{x}_{l+1}$ with

$$\overline{x}_{l+1} + \Phi = \overline{y}_1^{d_1}\overline{y}_{l+1},$$

which can be factored to achieve the conclusions of Theorem 1.14.

Now suppose that $r > 1$. By (13) of Theorem 1.13, there exists a UTS in the first l variables so that $P = \overline{y}_1^{h_1}\overline{P}(\overline{y}_1, \ldots, \overline{y}_l)$, where $\overline{P}$ is a unit, and

$$g = \sum_{i=1}^{d} \overline{a}_{\alpha_i}\overline{x}_1^{\gamma(i)}\overline{x}_{l+1}^{\alpha_i} + \cdots$$

where the $\overline{a}_{\alpha_i}\overline{x}_{l+1}^{\alpha_i}$ are the minimum value terms and the $\overline{a}_{\alpha_i}(\overline{x}_1, \ldots, \overline{x}_l)$ are units.

Case 1. — Assume that $\nu(P) > \nu(\overline{y}_1^{d_1})$. Then $\overline{x}_{l+1} = \overline{y}_1^{d_1}G$ where $G = \overline{y}_1^{h_1-d_1}\overline{P} + \overline{y}_{l+1}$. Since $\operatorname{mult} G(0, \ldots, 0, \overline{y}_{l+1}) = 1$, we can factor this to get in the form of the conclusions of Theorem 1.14.

Case 2. — Suppose that $\nu(P) \leq \nu(\overline{y}_1^{d_1})$. Then $\overline{x}_{l+1} = \overline{y}_1^{h_1}\overline{P}_1$ where $\overline{P}_1 = \overline{P} + \overline{y}_1^{d_1-h_1}\overline{y}_{l+1}$ is a unit. Perform a UTS $T(0) \to T(1)$ defined by

$$\begin{aligned}\overline{x}_1 &= \overline{x}_1(1)^{a_{11}(1)}\\ \overline{x}_{l+1} &= \overline{x}_1(1)^{a_{l+1,1}(1)}(\overline{x}_{l+1}(1) + c_1).\end{aligned}$$

We will show that this map factors through $U(0)$.

$$\frac{\nu(\overline{x}_{l+1})}{\nu(\overline{x}_1)} = \frac{a_{l+1,1}(1)}{a_{11}(1)} = \frac{h_1\nu(\overline{y}_1)}{t_0\nu(\overline{y}_1)} = \frac{h_1}{t_0}.$$

Thus $h_1 = t_1 a_{l+1,1}(1)$ and $t_0 = t_1 a_{11}(1)$ for some positive integer t_1.

$$\overline{x}_1(1) = \overline{y}_1^{t_1}$$

$$\overline{x}_{l+1}(1) + c_1 = \frac{\overline{y}_1^{h_1}}{\overline{x}_1(1)^{a_{l+1,1}(1)}}\overline{P}_1 = \overline{P}_1$$

so that

$$\overline{x}_{l+1} = P_1(\overline{y}_1, \ldots, \overline{y}_l) + \overline{y}_1^{d}\overline{y}_{l+1}.$$

Set $g = \overline{x}_1(1)^e g_1$. $r_1 = \operatorname{mult} g_1(0, \ldots, 0, \overline{x}_{l+1}(1)) \leq r$. If $r_1 = r$, we can replace $\overline{x}_{l+1}$ with $\overline{x}_{l+1} - \sigma(\overline{x}_1, \ldots, \overline{x}_l)$ and repeat to eventually get $r_1 < r$.

We have $\overline{y}_{l+1}(1) = \overline{y}_{l+1}$ and $\overline{y}_{l+1} \mid g$, so that $\overline{y}_{l+1}(1) \mid g$. $\overline{x}_1(1) = \overline{y}_1(1)^{t_1}$ implies $r_1 > 0$. Now we can repeat the above argument to eventually either reach $r = 1$ or case 1.

CHAPTER 2

PRELIMINARIES

2.1. Valuations

Lemma 2.1. — *Suppose that R is a regular local ring, with quotient field K. Then $R = K \cap \widehat{R}$ in the quotient field of $\widehat{R}$.*

Proof. — Suppose that $f \in K \cap \widehat{R}$. Then there exist $g, h \in R$ such that $f = g/h$, with $(g,h) = 1$ in R. If $f \notin R$, there exists an irreducible $s \in R$ such that $s \mid h$ but s does not divide g. Let $s' \in \widehat{R}$ be an irreducible such that $(s') \cap R = (s)$. $hf = g$ in $\widehat{R}$. $s' \mid h$ implies $s' \mid g$ in $\widehat{R}$. Hence $s \mid g$ in R, a contradiction. □

Lemma 2.2. — *Let R be a regular local ring with quotient field K, maximal ideal m. Let v be a rank 1 valuation of K dominating R, with value group Γ, valuation ring $\mathcal{O}_\nu$. Let $\overline{K}$ be the completion of K with respect to a metric $|\cdot|$ associated to v. There exists a valuation $\overline{v}$ of $\overline{K}$ extending v, with valuation ring $\mathcal{O}_{\overline{v}}$ such that $\mathcal{O}_{\overline{v}}/m_{\overline{v}} = \mathcal{O}_v/m_v$ and the value group of $\overline{v}$ is Γ (cf. Theorems 1 and 2, Chapter 2* **[31]***).*

Then there exists a prime $p \subset \widehat{R}$ and an inclusion $\widehat{R}/p \to \overline{K}$ which extends $R \to \overline{K}$.

Proof. — Let $\{a_n\}$ be a cauchy sequence in the m-adic topology of R. Let $v(m) = \rho > 0$. Then $v(m^N) = N\rho$ implies $\{a_n\}$ is a fundamental sequence with respect to $|\cdot|$. Hence there is a natural map $\phi : \widehat{R} \to \overline{K}$ making

$$\begin{array}{ccc} \widehat{R} & \xrightarrow{\phi} & \overline{K} \\ \uparrow & \nearrow & \\ R & & \end{array}$$

commute. Let $P = \text{kernel } \phi$. □

Lemma 2.3 extracts an argument from page 345 of **[35]**.

Lemma 2.3. — *Let R be a regular local ring containing a field of characteristic zero and ν a rank 1 valuation of the quotient field of R which has nonnegative value on R.*

Suppose that $z_1, \ldots, z_n, \ldots$ is an infinite sequence of elements of R such that

$$\nu(z_1) < \nu(z_2) < \cdots < \nu(z_n) < \cdots$$

is strictly increasing. Then $\nu(z_n)$ has infinity for a limit.

Proof. — Let m be the maximal ideal of R, K the quotient field of R. Let $\{\dot{\bar{t}}_i\}$ be a transcendence basis of R/m over k. Lift $\bar{t}_i$ to $t_i \in R$. Let L be the field obtained by adjoining the t_i to k. Then $L \subset R$. Let L' be the algebraic closure of L in K. Then $L' \subset R$ since R is normal. Let $\overline{L}$ be an algebraic closure of L'. $\overline{K} = K \otimes_{L'} \overline{L}$ is a field (cf. Corollary 2, Section 15, Chapter III **[39]**). Let $\overline{\nu}$ be an extension of ν to $\overline{K}$. $\overline{\nu}$ has rank 1 since $\overline{K}$ is algebraic over K. Let $\overline{R}$ be the localization of $R \otimes_{L'} \overline{L}$ at the center of $\overline{\nu}$. Then $\overline{R}$ is a regular local ring dominating R. We can extend $\overline{\nu}$ to a valuation $\overline{\nu}$ dominating $\widehat{\overline{R}} \cong \overline{L}[[x_1, \ldots, x_n]]$, a powerseries ring.

Let ρ be a positive real number. Let $\sigma = \min(\overline{\nu}(x_i))$. Let n_ρ be the smallest integer such that $n_\rho\sigma > \rho$. Let $g(x_1, \ldots, x_n) \in \overline{L}[[x_1, \ldots, x_n]]$ be such that $\nu(g) \leq \rho$. Write $g = g' + g''$ where g' is a polynomial of degree $< n_\rho$, and g'' is a powerseries with terms of degree $\geq n_\rho$. Every form in $x_1, \ldots, x_n$ of degree m has value $\geq m\sigma$. Hence $\nu(g'') \geq n_\rho\sigma > \rho$. Since $\nu(g) \leq \rho$, $\nu(g') = \nu(g)$. Thus if a powerseries has a value $\leq \rho$, its value is the value of a polynomial of degree $< n_\rho$. Hence, among the values assumed by elements of $\widehat{\overline{R}}$, there is only a finite number of values which are less than or equal to a given fixed real number ρ. □

2.2. Birational Transforms

Suppose that R is a regular local ring, with maximal ideal m, and that $x_1, \ldots, x_n \in R$ can be extended to a system of regular parameters $(x_1, \ldots, x_d)$ in R. Let I be the ideal $I = (x_1, \ldots, x_n)$.

The blow up

$$\pi : \text{Proj}(\oplus_{n \geq 0} I^n) \to \text{spec}(R)$$

is called a monoidal transform of spec(R). Proj$(\oplus_{n \geq 0} I^n)$ is a regular scheme. let

$$p \in \pi^{-1}(m) \subset \text{Proj}(\oplus_{n \geq 0} I^n).$$

$p \in \text{spec}(R[x_1/x_i, \ldots, x_n/x_i])$ for some i. Then

$$R \to (R[x_1/x_i, \ldots, x_n/x_i])_p$$

is called a monoidal transform of R. If $n = d$, so that $I = m$,

$$R \to (R[x_1/x_i, \ldots, x_d/x_i])_p$$

is called a quadratic transform.

In this section we state results of Abhyankar and Hironaka in a form which we will use. The conclusions of Theorems 2.5 through 2.7 and Theorem 2.9 have been proved by Hironaka **[20]** in equicharacteristic zero, and have been proved by Abhyankar **[1]**, **[4]** in positive characteristic, for varieties of dimension ≤ 3.

Definition 2.4. — Let R be a regular local ring. $f \in R$ is said to have simple normal crossings (SNCs), and be a SNC divisor, if there exist regular parameters $(x_1, \ldots, x_n)$ in R such that $f = \mathrm{unit} \cdot x_1^{a_1} \cdots x_n^{a_n}$ for some non-negative integers $a_1, \ldots, a_n$.

Theorem 2.5. — *Let R be an excellent regular local ring, containing a field of characteristic zero. Let X be a nonsingular R-scheme, $f : X \to \mathrm{spec}(R)$ a projective morphism, $h \in R$. Then there exists a sequence of monoidal transforms $g : Y \to X$, such that h has SNCs at every point of Y.*

Proof. — Immediate from Main Theorem II(N) **[20]**. ☐

Theorem 2.6. — *Suppose that R, S are excellent regular local rings containing a field k of characteristic zero such that S dominates R. Let ν be a valuation of the quotient field K of S that dominates S, $R \to R_1$ a monoidal transform such that ν dominates R_1. R_1 is a local ring on $X = \mathrm{Proj}(\oplus_{n \geq 0} p^n)$ for some prime $p \subset R$. Let*

$$U = \{Q \in \mathrm{spec}(S) : pS_Q \textit{ is invertible}\}$$

an open subset of $\mathrm{spec}(S)$. *Then there exists a projective morphism $f : Y \to \mathrm{spec}(S)$ which is a product of monoidal transforms such that if S_1 is the local ring of Y dominated by ν, then S_1 dominates R_1, and $(f)^{-1}(U) \to U$ is an isomorphism.*

Proof. — Since S is a UFD, we can write $pS = gI$, where $g \in S$, $I \subset S$ has height ≥ 2. Then $U = \mathrm{spec}(S) - V(I)$. By Main Theorem II(N) **[20]**, there exists a sequence of monoidal transforms $\pi : Y \to \mathrm{spec}(S)$ such that $I\mathcal{O}_Y$ is invertible, and $\pi^{-1}(U) \to U$ is an isomorphism. Let S_1 be the local ring of the center of ν on Y. We have $pS_1 = hS_1$ for some $h \in p$. Hence $R[p/h] \subset S_1$, and since ν dominates S_1, R_1 is the localization of $R[p/h]$ which is dominated by S_1. ☐

Theorem 2.7. — *Suppose that R is an excellent local domain containing a field of characteristic zero, with quotient field K. Let ν be a valuation of K dominating R. Suppose that $f \in K$ is such that $\nu(f) \geq 0$. Then there exists a MTS along ν*

$$R \to R_1 \to \cdots \to R_n$$

such that $f \in R_n$.

Proof. — Write $f = a/b$ with $a, b \in R$. By Main Theorem II(N) **[20]** applied to the ideal $I = (a, b)$ in R, there exists a MTS along ν, $R \to R_n$ such that $IR_n = \alpha R_n$ is a principal ideal. There exist constants c, d, u_1, u_2 in R_n such that $a = c\alpha, b = d\alpha, \alpha = u_1 a + u_2 b$. Then $u_1 c + u_2 d = 1$, so that $cR_n + dR_n = R_n$, and one of c or d is a unit in R_n. If c is a unit, then $0 \leq \nu(f) = \nu(c/d) = \nu(c) - \nu(d)$ implies $\nu(d) = 0$, and since ν dominates R_n, d is a unit and $f \in R_n$. ☐

***Theorem 2.8* (Abhyankar).** — *Let R, S be two dimensional regular local rings such that R and S have the same quotient field, and S dominates R. Then there exists a unique finite sequence*

$$R_0 \to R_1 \to \cdots \to R_m$$

of quadratic transforms such that $R_m = S$.

Proof. — This is Theorem 3 of [Ab2]. □

Suppose that Y is an algebraic scheme, X, D are subschemes of Y. Suppose that $g : Y' \to Y$, $f : X' \to X$ are the monoidal transforms of Y and X with center D and $D \cap X$ respectively. Then there exists a unique isomorphism of X' to a subscheme X'' of Y' such that g induces f (cf. chapter 0, section 2 **[20]**). X'' is called the strict transform of X be the monoidal transform g.

Theorem 2.9. — *Let R be an excellent regular local ring, containing a field of characteristic zero. Let $W \subset \operatorname{spec}(R)$ be an integral subscheme, $V \subset \operatorname{spec}(R)$ be the singular locus of W. Then there exists a sequence of monoidal transforms $f : X \to \operatorname{spec}(R)$ such that the strict transform of W is nonsingular in X, and f is an isomorphism over $\operatorname{spec}(R) - V$.*

Proof. — This is immediate from Theorem $I_2^{N,n}$ **[20]**. □

Theorem 2.10. — *Suppose that $R \subset S$ are r dimensional local rings with a common quotient field K, and respective maximal ideals m and n such that S dominates R, S/mS is a finite R/m module, and R is normal and analytically irreducible. Then $R = S$.*

Proof. — This is the version of Zariski's Main Theorem proved in Theorem 37.4 **[28]**. □

***Theorem 2.11* (Theorem 1 [22]).** — *Suppose that R is an excellent regular local ring with quotient field J, K is a finite extension field of J and S is a regular local ring with quotient field K such that $R \subset S$ and $\dim(R) = \dim(S)$. Then S is essentially of finite type over R.*

Proof. — Let $(y_1, \ldots, y_n)$ be a system of regular parameters in S and suppose that K is generated by $h_1, \ldots, h_r$ over J. Let $h_i = f_i/g_i$ for $1 \le i \le r$ where $f_i, g_i \in S$. Let T be the normalization of $R[y_1, \ldots, y_n, f_1, \ldots, f_r, g_1, \ldots, g_r]$, $q = m(S) \cap T$, $U = T_q$. By Theorem 2.10 $T_q = U$. □

Theorem 2.12. — *Suppose that R is an excellent regular local ring, with maximal ideal m, S is a regular local ring with maximal ideal n, such that $R \subset S$, $\dim(R) = \dim(S)$ and the quotient field of S is a finite extension of the quotient field of R. Then there is an inclusion*

$$\widehat{R} \subset \widehat{S}$$

where $\widehat{R}$ is the m-adic completion of R, $\widehat{S}$ is the n-adic completion of S.

Proof. — By Theorem 2.11 S is essentially of finite type over R. Since S is universally catenary, the dimension formula (Theorem 15.6 **[26]**) holds.

$$\dim R + \operatorname{trdeg}_R S = \dim S + \operatorname{restrdeg}_R S$$

Since R is analytically irreducible, R is a subspace of S by Theorem 10.13 **[4]** ("A version of Zariski's Subspace Theorem"). □

We will use the notation $m(R)$ to denote the maximal ideal of a local ring R, $k(R)$ to denote the residue field $R/m(R)$. $\widehat{R}$ or $R\hat{}$ will denote the $m(R)$-adic completion of R.

Let k be a field, $0 \neq f(z_1, \ldots, z_n) \in k[[z_1, \ldots, z_n]]$. Let $m = (z_1, \ldots, z_n)$. Define $\operatorname{mult}(f) = r$ if $f \in m^r$, $f \notin m^{r+1}$.

CHAPTER 3

UNIFORMIZING TRANSFORMS

Definition 3.1. — Suppose that R is a regular local ring. A monoidal transform sequence (MTS) is a sequence of ring homomorphisms

$$R = R_0 \to R_1 \to R_2 \to \cdots \to R_n$$

such that each map $R_i \to R_{i+1}$ is a finite product of monoidal transforms.

Definition 3.2. — Suppose that R is an excellent regular local ring containing a field k of characteristic zero, with quotient field K. A uniformizing transform sequence (UTS) is a sequence of ring homomorphisms

$$\begin{tikzcd}[column sep=small, row sep=small]
R \arrow[r] & \overline{T}_0'' \arrow[r] \arrow[d] & \overline{T}_0 \arrow[dr] & & & & \\
& \overline{T}_1' \arrow[r] & \overline{T}_1'' \arrow[r] \arrow[d] & \overline{T}_1 \arrow[dr] & & & \\
& & \overline{T}_2' \arrow[r] & \overline{T}_2'' \arrow[r] \arrow[d] & \overline{T}_2 \arrow[dr] & & \\
& & & \vdots & & \vdots & \\
& & & & \arrow[d] & & \arrow[dr] & \\
& & & & \overline{T}_n' \arrow[r] & \overline{T}_n'' \arrow[r] & & \overline{T}_n
\end{tikzcd} \tag{15}$$

such that $\overline{T}_0 = \widehat{R}$, the completion of R with respect to its maximal ideal, and for all i, $\overline{T}_i$ is the completion with respect to its maximal ideal of a finite product of monoidal transforms $\overline{T}_i'$ of $\overline{T}_{i-1}''$. For all i, $\overline{T}_i''$ is a a regular local ring essentially of finite type over $\overline{T}_i'$ with quotient field K_i, such that $\overline{T}_i' \subset \overline{T}_i'' \subset \overline{T}_i$ and K_0 is a finite extension of K, K_{i+1} is a finite extension of K_i for all $i \geq 0$.

To simplify notation, we will often denote the UTS (15) by $(R, \overline{T}_n'', \overline{T}_n)$ or by

$$R \to \overline{T}_0 \to \overline{T}_1 \to \cdots \to \overline{T}_n.$$

We will denote the UTS consisting of the maps

$$\begin{array}{ccccc} \overline{T}'_{n-1} & \rightarrow & \overline{T}''_{n-1} & \rightarrow & \overline{T}_{n-1} \\ & & \downarrow & & & \searrow \\ & & \overline{T}'_n & \rightarrow & \overline{T}''_n & \rightarrow & \overline{T}_n \end{array}$$

by $\overline{T}_{n-1} \rightarrow \overline{T}_n$.

A UTS (15) is called a rational uniformizing sequence (RUTS) if there exists an associated MTS

$$R = R_0 \rightarrow R_1 \rightarrow \cdots \rightarrow R_n,$$

maps $R_i \rightarrow \overline{T}'_i$ such that $\widehat{R}_i \cong \overline{T}_i$ for $1 \leq i \leq n$, and all squares in the resulting diagram

$$\begin{array}{ccccccc} \overline{T}_0 & \rightarrow & \overline{T}_1 & \rightarrow & \cdots & \rightarrow & \overline{T}_n \\ \uparrow & & \uparrow & & & & \uparrow \\ R_0 & \rightarrow & R_1 & \rightarrow & \cdots & \rightarrow & R_n \end{array} \tag{16}$$

commute.

Suppose that v is a valuation of K which dominates R and

$$R \rightarrow \overline{T}_0 \rightarrow \overline{T}_1 \rightarrow \cdots \rightarrow \overline{T}_n \tag{17}$$

is a UTS.

Suppose that v_0 is an extension of v to the quotient field of $\overline{T}_0$ such that ν_0 dominates $\overline{T}_0$. The existence of ν_0 is shown in **[23]**. If v_0 dominates $\overline{T}'_1$ we can extend v_0 to a valuation v_1 of the quotient field of $\overline{T}_1$ which dominates $\overline{T}_1$.

Then if v_1 dominates $\overline{T}'_2$, in the same manner we can extend v_1 to a valuation v_2 of the quotient field of $\overline{T}_2$ which dominates $\overline{T}_2$. If we can inductively construct a sequence $v_1, \ldots, v_n$ of extensions of v to the quotient fields of $\overline{T}_i$ in this manner, (17) is called a UTS along ν. If there is no danger of confusion, we will denote the extensions v_i by v.

Suppose that (17) is a UTS along a rank 1 valuation ν of K. Let Γ_ν be the value group of ν. Suppose that i is such that $0 \leq i \leq n$. Let $\mathcal{O}_{\nu_i}$ be the valuation ring of ν_i and Γ_{ν_i} be the value group of ν_i. Γ_ν is a subgroup of Γ_{ν_i}. Set

$$\overline{\Gamma} = \{\beta \in \Gamma_{\nu_i} \mid -\alpha \leq \beta \leq \alpha \text{ for some } \alpha \in \Gamma_\nu\}.$$

$\overline{\Gamma}$ is an isolated subgroup of Γ_{ν_i}, since Γ_ν is a subgroup, so there is a prime a in $\mathcal{O}_{\nu_i}$ (which could be 0) such that $\overline{\Gamma}$ is the isolated subgroup Γ_a of a, (by Proposition 2.29 **[3]** or Theorem 15, chapter VI, section 10 **[39]**). Set

$$\overline{p}_i = a \cap \overline{T}_i = \{f \in \overline{T}_i \mid \nu(f) > \alpha \text{ for all } \alpha \in \Gamma_\nu\}.$$

We will say that $\nu(f) = \infty$ if $f \in \overline{p}_i$.

For the rest of this chapter we will assume that $R \subset S$ are excellent regular local rings such that $\dim(R) = \dim(S)$, containing a field k of characteristic zero, such that the quotient field K of S is a finite extension of the quotient field J of R. We will fix a valuation ν of K with valuation ring V such that ν dominates S.

Note that the restriction of ν to J has the same rank and rational rank that ν does (Lemmas 1 and 2 of section 11, chapter VI [**39**]). Observe that S is essentially of finite type over R (Theorem 2.11) and $\widehat{R} \to \widehat{S}$ is an inclusion (Theorem 2.12).

Suppose that $(R, \overline{T}''_n, \overline{T}_n)$ and $(S, \overline{U}''_n, \overline{U}_n)$ are UTSs. We will say that $(R, \overline{T}''_n, \overline{T}_n)$ and $(S, \overline{U}''_n, \overline{U}_n)$ are compatible UTSs (or a CUTS) if there are commutative diagrams of inclusions

$$\begin{array}{ccccc} \overline{U}'_i & \to & \overline{U}''_i & \to & \overline{U}_i \\ \uparrow & & \uparrow & & \uparrow \\ \overline{T}'_i & \to & \overline{T}''_i & \to & \overline{T}_i \end{array} \tag{18}$$

for $0 \le i \le n$. In particular, the quotient field of $\overline{U}''_i$ is finite over the quotient field of $\overline{T}''_i$ for all i, and $\overline{U}''_i$ is essentially of finite type over $\overline{T}''_i$ for all i.

We will say that UTSs along ν $(R, \overline{T}''_n, \overline{T}_n)$ and $(S, \overline{U}''_n, \overline{U}_n)$ are CUTS along ν if the extensions of ν are compatible in (18).

If $(R, \overline{T}''_n, \overline{T}_n)$ and $(S, \overline{U}''_n, \overline{U}_n)$ are RUTSs and CUTSs, then we will say that $(R, \overline{T}''_n, \overline{T}_n)$ and $(S, \overline{U}''_n, \overline{U}_n)$ are compatible RUTSs (or a CRUTS).

Lemma 3.3. — *Suppose that the CRUTS $(R, \overline{T}''_n, \overline{T}_n)$ and $(S, \overline{U}''_n, \overline{U}_n)$ have respective associated MTSs*

$$R = R_0 \to R_1 \to \cdots \to R_n$$

and

$$S = S_0 \to S_1 \to \cdots \to S_n.$$

Then there is a commutative diagram

$$\begin{array}{ccccccc} R & \to & R_1 & \to & \cdots & \to & R_n \\ \downarrow & & \downarrow & & & & \downarrow \\ S & \to & S_1 & \to & \cdots & \to & S_n \end{array} \tag{19}$$

Proof. — This follows from (18), (16) and Lemma 2.1, since then

$$R_i \subset \widehat{R}_i \cap J = \overline{T}_i \cap J \subset \overline{U}_i \cap K = \widehat{S}_i \cap K = S_i$$

for all i. □

CHAPTER 4

RANK 1

4.1. Perron Transforms

Throughout this chapter we will assume that $R \subset S$ are excellent regular local rings such that $\dim(R) = \dim(S)$, containing a field k of characteristic zero, such that the quotient field K of S is a finite extension of the quotient field J of R. We will fix a valuation ν of K with valuation ring V such that ν dominates S. Suppose that if m_V is the maximal ideal of V, and $p^* = m_V \cap S$, then $(S/p^*)_{p^*}$ is a finitely generated field extension of k. We will further assume that

(1) ν has rank 1 and arbitrary rational rank s ($\leq \dim(S)$).
(2) $\dim_R(\nu) = 0$ and $\mathcal{O}_\nu/m_\nu$ is algebraic over k.

Let $n = \dim(R) = \dim(S)$. We will define 2 types of UTSs. Suppose that $(R, \overline{T}'', \overline{T})$ is a UTS along ν and $\overline{T}''$ has regular parameters $(\tilde{x}'_1, \ldots, \tilde{x}'_n)$ such that

$$\nu(\tilde{x}'_1) = \tau_1, \ldots, \nu(\tilde{x}'_s) = \tau_s$$

are rationally independent. Let ν_0 be an extension of ν to the quotient field of $\overline{T}''$ which dominates $\overline{T}''$.

We first define a UTS $\overline{T} \rightarrow \overline{T}(1)$ of type I along ν. The MTS $\overline{T}'' \rightarrow \overline{T}'(1)$ is defined as follows. $\overline{T}'(1) = T_h$ where h is a positive integer and T_h is constructed as follows.

Set $\tau_i(0) = \tau_i$ for $1 \leq i \leq s$. For each positive integer h define s positive, rationally independent real numbers $\tau_1(h), \ldots, \tau_s(h)$ by the "Algorithm of Perron" (B.I of **[36]**)

$$\begin{aligned}
\tau_1(h-1) &= \tau_s(h) \\
\tau_2(h-1) &= \tau_1(h) + a_2(h-1)\tau_s(h) \\
&\vdots \\
\tau_s(h-1) &= \tau_{s-1}(h-1) + a_s(h-1)\tau_s(h)
\end{aligned}$$

where
$$a_j(h-1) = \left[\frac{\tau_j(h-1)}{\tau_1(h-1)}\right], \quad 2 \le j \le s$$
the "greatest integer" in $\tau_j(h)/\tau_1(h)$. There are then nonnegative integers $A_i(h)$ such that
$$\tau_i = A_i(h)\tau_1(h) + A_i(h+1)\tau_2(h) + \cdots + A_i(h+s-1)\tau_s(h)$$
for $1 \le i \le s$.
$$\det\begin{pmatrix} A_1(h) & \dots & A_1(h+s-1) \\ \vdots & & \vdots \\ A_s(h) & \dots & A_s(h+s-1)\end{pmatrix} = (-1)^{h(s-1)}$$
(See formula (4'), page 385 **[36]**.) These numbers have the important property that
$$\lim_{h\to\infty} \frac{A_i(h)}{A_1(h)} = \frac{\tau_i}{\tau_1} \tag{20}$$
(See formula (5), page 385 **[36]**.) Set $\tilde{x}_i(0) = \tilde{x}'_i$ for $1 \le i \le n$. Define T_h by the sequence of MTSs along ν
$$\overline{T}'' = \tilde{T}(0) \to \tilde{T}(1) \to \cdots \to \tilde{T}(h) = T_h = \overline{T}'(1)$$
Where $\tilde{T}(i+1) = \tilde{T}(i)[\tilde{x}_1(i+1), \dots, \tilde{x}_s(i+1)]_{(\tilde{x}_1(i+1),\dots,\tilde{x}_n(i+1))}$ for $0 \le i \le h-1$.
$$\begin{aligned} \tilde{x}_1(i) &= \tilde{x}_s(i+1) \\ \tilde{x}_2(i) &= \tilde{x}_1(i+1)\tilde{x}_s(i+1)^{a_2(i)} \\ &\vdots \\ \tilde{x}_s(i) &= \tilde{x}_{s-1}(i+1)\tilde{x}_s(i+1)^{a_s(i)} \end{aligned}$$
$\nu(\tilde{x}_j(i)) = \tau_j(i)$ for $1 \le j \le s$. If we set $\overline{x}_i(1) = \tilde{x}_i(h)$, we then have regular parameters $(\overline{x}_1(1), \dots, \overline{x}_n(1))$ in $\overline{T}'(1)$ satisfying
$$\begin{aligned} \tilde{x}'_1 &= \overline{x}_1(1)^{A_1(h)} \cdots \overline{x}_s(1)^{A_1(h+s-1)} \\ &\vdots \\ \tilde{x}'_s &= \overline{x}_1(1)^{A_s(h)} \cdots \overline{x}_s(1)^{A_s(h+s-1)} \\ \tilde{x}'_{s+1} &= \overline{x}_{s+1}(1) \\ &\vdots \\ \tilde{x}'_n &= \overline{x}_n(1) \end{aligned} \tag{21}$$
Then $\overline{T}'(1) = \overline{T}''[\overline{x}_1(1), \dots, \overline{x}_s(1)]_{(\overline{x}_1(1),\dots,\overline{x}_n(1))}$. Let $\overline{T}(1)$ be the completion of $\overline{T}'(1)$ at its maximal ideal. Set $\overline{T}''(1) = \overline{T}'(1)$. Then for any extension ν_1 of ν_0 to the quotient field of $\overline{T}(1)$ which dominates $\overline{T}(1)$, $\overline{T} \to \overline{T}(1)$ is a UTS and $(R, \overline{T}''(1), \overline{T}(1))$ is a UTS along ν. Note that $\nu(\overline{x}_1(1)), \dots, \nu(\overline{x}_s(1))$ are rationally independent.

Now we define a UTS $\overline{T} \to \overline{T}(1)$ of type II_r along ν (with the restriction that $s+1 \le r \le n$). The MTS $\overline{T}'' \to \overline{T}'(1)$ is constructed as follows. Set $\nu(\tilde{x}'_r) = \tau_r$. τ_r

must be rationally dependent on $\tau_1,\ldots,\tau_s$ since ν has rational rank s. There are thus integers $\lambda,\lambda_1,\ldots,\lambda_s$ such that $\lambda>0$, $(\lambda,\lambda_1,\ldots,\lambda_s)=1$ and

$$\lambda\tau_r=\lambda_1\tau_1+\cdots+\lambda_s\tau_s.$$

First perform a MTS $\overline{T}''\to\widetilde{T}(1)$ which is UTS along ν where $\widetilde{T}(1)$ has regular parameters $(\widetilde{x}_1(1),\ldots,\widetilde{x}_n(1))$ defined by $\widetilde{x}_i'=\widetilde{x}_1(1)^{A_1(h)}\cdots\widetilde{x}_s(1)^{A_s(h+s-1)}$ for $1\leq i\leq s$. Then $\nu(\widetilde{x}_i(1))=\tau_i(h)$ for $1\leq i\leq s$, $\nu(\widetilde{x}_r(1))=\tau_r$. Set

$$\lambda_i(h)=\lambda_1A_1(h+i-1)+\lambda_2A_2(h+i-1)+\cdots+\lambda_sA_s(h+i-1)$$

for $1\leq i\leq s$. Then

$$\lambda\tau_r=\lambda_1(h)\tau_1(h)+\cdots+\lambda_s(h)\tau_s(h).$$

Take h sufficiently large that all $\lambda_i(h)>0$. This is possible by (20), since $\lambda_1\tau_1+\cdots+\lambda_s\tau_s>0$. We still have $(\lambda,\lambda_1(h),\ldots,\lambda_s(h))=1$ since $\det(A_i(h+j-1))=\pm1$. After reindexing the $\widetilde{x}_i(1)$, we may suppose that $\lambda_1(h)$ is not divisable by λ. Let $\lambda_1(h)=\lambda\mu+\lambda'$, with $0<\lambda'<\lambda$. Now perform a MTS $\widetilde{T}(1)\to\widetilde{T}(2)$ along ν where $\widetilde{T}(2)$ has regular parameters $(\widetilde{x}_1(2),\ldots,\widetilde{x}_n(2))$ defined by

$$\begin{aligned}\widetilde{x}_1(1)&=\widetilde{x}_r(2)\\ \widetilde{x}_2(1)&=\widetilde{x}_2(2)\\ &\vdots\\ \widetilde{x}_s(1)&=\widetilde{x}_s(2)\\ \widetilde{x}_r(1)&=\widetilde{x}_1(2)\widetilde{x}_r(2)^{\mu}.\end{aligned}$$

Set $\tau_i'=\nu(\widetilde{x}_i(2))$ for all i. $\tau_1',\ldots,\tau_s',\tau_r'$ are positive and

$$\lambda'\tau_r'=\lambda_1'\tau_1'+\cdots+\lambda_s'\tau_s'$$

where $\lambda_1'=\lambda$, $\lambda_i'=-\lambda_i(h)$ for $2\leq i\leq s$. We have thus acheived a reduction in λ. By repeating this procedure, we get a MTS $\overline{T}''\to\widetilde{T}(\alpha)$ along ν where $\widetilde{T}(\alpha)$ has regular parameters $(\widetilde{x}_1(\alpha),\ldots,\widetilde{x}_n(\alpha))$ such that if $\overline{\tau}_i=\nu(\widetilde{x}_i(\alpha))$, $\overline{\tau}_1,\ldots,\overline{\tau}_s$ are rationally independent and

$$\overline{\tau}_r=\overline{\lambda}_1\overline{\tau}_1+\cdots+\overline{\lambda}_s\overline{\tau}_s$$

for some integers $\overline{\lambda}_i$. Now perform a MTS $\widetilde{T}(\alpha)\to\widetilde{T}(\alpha+1)$ which is a UTS of type I along ν where $T(\alpha+1)$ has regular parameters $(\widetilde{x}_1(\alpha+1),\ldots,\widetilde{x}_n(\alpha+1))$ such that if $\tau_i^*=\nu(\widetilde{x}_i(\alpha+1))$,

$$\tau_r^*=\widetilde{\lambda}_1\tau_1^*+\cdots+\widetilde{\lambda}_s\tau_s^*$$

for some positive integers $\widetilde{\lambda}_i$. Finally perform a MTS $\widetilde{T}(\alpha+1)\to\widetilde{T}(\alpha+2)$ where $\widetilde{T}(\alpha+2)=\widetilde{T}(\alpha+1)[N_r]_q$.

$$N_r=\frac{\widetilde{x}_r(\alpha+1)}{\widetilde{x}_1(\alpha+1)^{\widetilde{\lambda}_1}\cdots\widetilde{x}_s(\alpha+1)^{\widetilde{\lambda}_s}}$$

and q is the center of ν on $\widetilde{T}(\alpha+1)[N_r]$. Set $\overline{T}'(1) = \widetilde{T}(\alpha+2)$. Since $\nu(N_r) = 0$, N_r has residue $c \neq 0$ in $k(\overline{T}'(1))$. Set $N_i = \widetilde{x}_i(\alpha)$ for $1 \le i \le s$. Then there exists a matrix (a_{ij}) such that

$$\begin{aligned}
\widetilde{x}_1' &= N_1^{a_{11}} \cdots N_r^{a_{1,s+1}}\\
&\vdots\\
\widetilde{x}_s' &= N_1^{a_{s1}} \cdots N_r^{a_{s,s+1}}\\
\widetilde{x}_r' &= N_1^{a_{s+1,1}} \cdots N_r^{a_{s+1,s+1}}
\end{aligned}$$

and $\det(a_{ij}) = \pm 1$. $\overline{T}'(1)$ is a localization of $\overline{T}''[N_1, \ldots, N_s, N_r]$.

Let $\overline{T}(1)$ be the completion of $\overline{T}'(1)$ at its maximal ideal. ν_0 extends to a valuation of the quotient field of $\overline{T}(1)$ which dominates $\overline{T}(1)$. Let ν_1 be such an extension. $\overline{T}(1)$ has a regular system of parameters $(x_1^*(1), \ldots, x_n^*(1))$ defined by

$$\begin{aligned}
\widetilde{x}_1' &= x_1^*(1)^{a_{11}} \cdots x_s^*(1)^{a_{1s}}(x_r^*(1)+c)^{a_{1,s+1}}\\
&\vdots\\
\widetilde{x}_s' &= x_1^*(1)^{a_{s1}} \cdots x_s^*(1)^{a_{ss}}(x_r^*(1)+c)^{a_{s,s+1}}\\
\widetilde{x}_r' &= x_1^*(1)^{a_{s+1,1}} \cdots x_s^*(1)^{a_{s+1,s}}(x_r^*(1)+c)^{a_{s+1,s+1}}.
\end{aligned} \tag{22}$$

$\det(a_{ij}) = \pm 1$ and $\nu_1(x_1^*(1)), \ldots, \nu_1(x_s^*(1))$ are rationally independent. Set

$$\overline{c} = \det \begin{pmatrix} a_{11} & \cdots & a_{1s}\\ \vdots & & \vdots\\ a_{s1} & \cdots & a_{ss}\end{pmatrix} \det \begin{pmatrix} a_{11} & \cdots & a_{1,s+1}\\ \vdots & & \vdots\\ a_{s+1,1} & \cdots & a_{s+1,s+1}\end{pmatrix} \neq 0$$

since $\nu(\widetilde{x}_1'), \ldots, \nu(\widetilde{x}_s')$ are rationally independent. Define rational numbers $\gamma_1, \ldots, \gamma_s$ by

$$\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_s\end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1s}\\ \vdots & & \vdots\\ a_{s1} & \cdots & a_{ss}\end{pmatrix}^{-1} \begin{pmatrix} -a_{1,s+1}\\ \vdots\\ -a_{s,s+1}\end{pmatrix}.$$

$\gamma_i = m_i/\overline{c}$ for some $m_i \in \mathbf{N}$. By Cramer's rule,

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,s+1}\\ \vdots & & \vdots\\ a_{s,1} & \cdots & a_{s,s+1}\\ a_{s+1,1} & \cdots & a_{s+1,s+1}\end{pmatrix} \begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_s\\ 1\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\\ 1/\overline{c}\end{pmatrix}.$$

$\gamma_1 a_{s+1,1} + \cdots + \gamma_s a_{s_1,s} + a_{s+1,s+1} = 1/\overline{c}$.

Let $(b_{ij}) = (a_{ij})^{-1}$. Then

$$\begin{aligned}
N_1 &= x_1^*(1) = (\widetilde{x}_1')^{b_{1,1}} \cdots (\widetilde{x}_s')^{b_{1,s}} (\widetilde{x}_r')^{b_{1,s+1}}\\
&\vdots\\
N_s &= x_s^*(1) = (\widetilde{x}_1')^{b_{s,1}} \cdots (\widetilde{x}_s')^{b_{s,s}} (\widetilde{x}_r')^{b_{s,s+1}}\\
N_r &= x_r^*(1) + c = (\widetilde{x}_1')^{b_{s+1,1}} \cdots (\widetilde{x}_s')^{b_{s+1,s}} (\widetilde{x}_r')^{b_{s+1,s+1}}
\end{aligned}$$

$(x_1^*(1), \ldots, x_{r-1}^*(1), N_r', x_{r+1}^*(1), \ldots, x_n^*(1))$ are regular parameters in $\overline{T}'(1)$ where

$$N_r' = \prod (N_r - \sigma(c))$$

and the product is over all conjugates $\sigma(c)$ of c over k in an algebraic closure $\overline{k}$ of k.

$$k(\overline{T}'(1)) \cong k(\overline{T}'')(c).$$

We have $\left(\dfrac{x_r^*(1)}{c} + 1\right)^{1/\overline{c}} \in \overline{T}(1)$ where $\left(\dfrac{x_r^*(1)}{c} + 1\right)^{1/\overline{c}}$ is uniquely determined by the condition that it has residue 1 in $k(\overline{T}(1))$. Set

$$\overline{T}''(1) = \overline{T}'(1)\left[c, \left(\frac{x_r^*(1)}{c} + 1\right)^{1/\overline{c}}\right]_{(x_1^*(1), \ldots, x_n^*(1))}.$$

$\overline{T}''(1)$ has regular parameters $(\overline{x}_1(1), \ldots, \overline{x}_n(1))$ defined by

$$\overline{x}_i(1) = \begin{cases} x_i^*(1)\left(\dfrac{x_r^*(1)}{c} + 1\right)^{-\gamma_i} & 1 \le i \le s\\ \left(\dfrac{x_r^*(1)}{c} + 1\right)^{1/\overline{c}} - 1 & i = r\\ x_i^*(1) & s < i,\ i \ne r \end{cases} \tag{23}$$

We have

$$\begin{aligned}
\widetilde{x}_1' &= \overline{x}_1(1)^{a_{1,1}} \cdots \overline{x}_s(1)^{a_{1,s}} c^{a_{1,s+1}}\\
&\vdots\\
\widetilde{x}_s' &= \overline{x}_1(1)^{a_{s,1}} \cdots \overline{x}_s(1)^{a_{s,s}} c^{a_{s,s+1}}\\
\widetilde{x}_r' &= \overline{x}_1(1)^{a_{s+1,1}} \cdots \overline{x}_s(1)^{a_{s+1,s}} (\overline{x}_r(1) + 1) c^{a_{s+1,s+1}}
\end{aligned}$$

Note that

$$x_i^*(1) = \begin{cases} (\overline{x}_r(1) + 1)^{\overline{c}\gamma_i} \overline{x}_i(1) & 1 \le i \le s\\ c[(\overline{x}_r(1) + 1)^{\overline{c}} - 1] & i = r\\ \overline{x}_i(1) & s < i,\ i \ne r \end{cases} \tag{24}$$

Thus $\overline{T} \to \overline{T}(1)$ is a UTS and by our extension of ν to the quotient field of $\overline{T}(1)$, $(R, \overline{T}''(1), \overline{T}(1))$ is a UTS along ν. We will call $\overline{T} \to \overline{T}(1)$ a UTS of type II_r.

Remark 4.1. — In our constructions of UTSs of types I and II_r, $\overline{T}'' \to \overline{T}'(1)$ is a product of monoidal transforms

$$\overline{T}'' = T_0 \to T_1 \to \cdots \to T_{t-1} \to T_t = \overline{T}'(1)$$

where each $T_i \to T_{i+1}$ is a monoidal transform centered at a height 2 prime a_i and $a_i\overline{T}''(1) = (\overline{x}_1(1)^{d_1^i} \cdots \overline{x}_s(1)^{d_s^i})$ for some nonnegative integers d_j^i for all i.

Lemma 4.2. — *Suppose that $(R, \overline{T}'', \overline{T})$ is a UTS along ν, $(x_1, \ldots, x_n)$ are regular parameters in $\overline{T}''$, and $\nu(x_1), \ldots, \nu(x_s)$ are rationally independent.*

(25) *Suppose that $M_1 = x_1^{d_1^1} \cdots x_s^{d_s^1}, M_2 = x_1^{d_1^2} \cdots x_s^{d_s^2} \in \overline{T}''$ and $\nu(M_1) < \nu(M_2)$. Then there exists a UTS of type I along ν, $\overline{T} \to \overline{T}(1)$, such that $M_1 \mid M_2$ in $\overline{T}'(1)$.*

(26) *Suppose that $M = x_1^{d_1} \cdots x_s^{d_s}$ is such that the d_i are integers and $0 < \nu(M)$. Then there exists a UTS of type I along ν $\overline{T} \to \overline{T}(1)$ such that $M \in \overline{T}'(1)$.*

Proof. — The proof of (25) is from Theorem 2 **[36]**. Consider the UTS with equations (21). In $\overline{T}'(1)$,

$$M_i = \overline{x}_1(1)^{d_1^i A_1(h) + \cdots + d_s^i A_s(h)} \cdots \overline{x}_s(1)^{d_1^i A_1(h+s-1) + \cdots + d_s^i A_s(h+s-1)}$$

for $i = 1, 2$. For $h \gg 0$

$$d_1^2 A_1(h+j-1) + \cdots + d_s^2 A_s(h+j-1) > d_1^1 A_1(h+j-1) + \cdots + d_s^1 A_s(h+j-1)$$

for $1 \le j \le s$ by (20).

To prove (26) just write $M = M_2/M_1$ where M_i are monomials in $x_1, \ldots, x_s$. Since $\nu(M_2) > \nu(M_1)$, (26) follows from (25). □

Lemma 4.3. — *Suppose that $(R, \overline{T}'', \overline{T})$ and $(S, \overline{U}'', \overline{U})$ is a CUTS along ν, $\overline{T}''$ has regular parameters $(\overline{x}_1, \ldots, \overline{x}_n)$ and $\overline{U}''$ has regular parameters $(\overline{y}_1, \ldots, \overline{y}_n)$, related by*

$$\begin{aligned} \overline{x}_1 &= \overline{y}_1^{c_{11}} \cdots \overline{y}_s^{c_{1s}} \alpha_1 \\ &\vdots \\ \overline{x}_s &= \overline{y}_1^{c_{s1}} \cdots \overline{y}_s^{c_{ss}} \alpha_s \end{aligned}$$

such that $\alpha_1, \ldots, \alpha_s \in k(\overline{U})$, $\nu(\overline{x}_1), \ldots, \nu(\overline{x}_s)$ are rationally independent and $\det(c_{ij}) \neq 0$. *Suppose that $\overline{T} \to \overline{T}(1)$ is a UTS of type I along ν, such that $\overline{T}'(1) = \overline{T}''(1)$ has regular parameters $(\overline{x}_1(1), \ldots, \overline{x}_n(1))$ with*

$$\begin{aligned} \overline{x}_1 &= \overline{x}_1(1)^{a_{11}} \cdots \overline{x}_s(1)^{a_{1s}} \\ &\vdots \\ \overline{x}_s &= \overline{x}_1(1)^{a_{s1}} \cdots \overline{x}_s(1)^{a_{ss}}. \end{aligned}$$

Then there exists a UTS of type I along ν $\overline{U} \to \overline{U}(1)$ *such that* $(R, \overline{T}''(1), \overline{T}(1))$ *and* $(S, \overline{U}''(1), \overline{U}(1))$ *is a CUTS along* ν *and* $\overline{U}'(1) = \overline{U}''(1)$ *has regular parameters* $(\overline{y}_1(1), \ldots, \overline{y}_n(1))$ *with*

$$
\begin{aligned}
\overline{y}_1 &= \overline{y}_1(1)^{b_{11}} \cdots \overline{y}_s(1)^{b_{1s}} \\
&\vdots \\
\overline{y}_s &= \overline{y}_1(1)^{b_{s1}} \cdots \overline{y}_s(1)^{b_{ss}},
\end{aligned}
\tag{27}
$$

and

$$
\begin{aligned}
\overline{x}_1(1) &= \overline{y}_1(1)^{c_{11}(1)} \cdots \overline{y}_s(1)^{c_{1s}(1)} \alpha_1(1) \\
&\vdots \\
\overline{x}_s(1) &= \overline{y}_1(1)^{c_{s1}(1)} \cdots \overline{y}_s(1)^{c_{ss}(1)} \alpha_s(1)
\end{aligned}
\tag{28}
$$

where $\alpha_1(1), \ldots, \alpha_s(1) \in k(\overline{U}(1))$, $\nu(\overline{x}_1(1)), \ldots, \nu(\overline{x}_s(1))$ *are rationally independent and* $\det(c_{ij}(1)) \neq 0$.

Proof. — Let $(e_{ij}) = (a_{ij})^{-1}$, $(d_{ij}) = (e_{ij})(c_{jk})$, an integral matrix. Let $\alpha_i(1) = \alpha_1^{e_{i1}} \cdots \alpha_s^{e_{is}}$ for $1 \leq i \leq s$. Then

$$
\begin{aligned}
\overline{x}_1(1) &= \overline{y}_1^{d_{11}} \cdots \overline{y}_s^{d_{1s}} \alpha_1(1) \\
&\vdots \\
\overline{x}_s(1) &= \overline{y}_1^{d_{s1}} \cdots \overline{y}_s^{d_{ss}} \alpha_s(1)
\end{aligned}
$$

By (26) of Lemma 4.2, we can construct a UTS (27) of type I $\overline{U} \to \overline{U}(1)$ such that we have an inclusion $\overline{T}''(1) \subset \overline{U}''(1)$ and (28) holds. Then an extension of ν from the quotient field of $\overline{U}$ which dominates $\overline{U}$ to a valuation of the quotient field of $\overline{U}(1)$ which dominates $\overline{U}(1)$ restricts to an extension of ν to the quotient field of $\overline{T}(1)$ which dominates $\overline{T}(1)$ so that $(R, \overline{T}''(1), \overline{T}(1))$ and $(S, \overline{U}''(1), \overline{U}(1))$ is a CUTS along ν. ☐

Lemma 4.4. — *Suppose that* $(R, \overline{T}'', \overline{T})$, $(S, \overline{U}'', \overline{U})$ *is a CUTS along* ν, $\overline{T}''$ *has regular parameters* $(\overline{x}_1, \ldots, \overline{x}_n)$ *and* $\overline{U}''$ *has regular parameters* $(\overline{y}_1, \ldots, \overline{y}_n)$ *such that*

$$
\begin{aligned}
\overline{x}_1 &= \overline{y}_1^{c_{11}} \cdots \overline{y}_s^{c_{1s}} \alpha_1 \\
&\vdots \\
\overline{x}_s &= \overline{y}_1^{c_{s1}} \cdots \overline{y}_s^{c_{ss}} \alpha_s \\
\overline{x}_{s+1} &= \overline{y}_{s+1} \\
&\vdots \\
\overline{x}_l &= \overline{y}_l
\end{aligned}
\tag{29}
$$

with $\alpha_1, \ldots, \alpha_s \in k(\overline{U})$, $\nu(\overline{x}_1), \ldots, \nu(\overline{x}_s)$ *rationally independent,* $\det(c_{ij}) \neq 0$.

Suppose that $\overline{T} \to \overline{T}(1)$ *is a UTS of type* II_r *along* ν, *with* $s+1 \le r \le l$ *such that* $\overline{T}(1)''$ *has regular parameters* $(\overline{x}_1(1), \ldots, \overline{x}_n(1))$ *with*

$$\begin{aligned} \overline{x}_1 &= \overline{x}_1(1)^{a_{11}} \cdots \overline{x}_s(1)^{a_{1s}} c^{a_{1,s+1}} \\ &\vdots \\ \overline{x}_s &= \overline{x}_1(1)^{a_{s1}} \cdots \overline{x}_s(1)^{a_{ss}} c^{a_{s,s+1}} \\ \overline{x}_r &= \overline{x}_1(1)^{a_{s+1,1}} \cdots \overline{x}_s(1)^{a_{s+1,s}} (\overline{x}_r(1)+1) c^{a_{s+1,s+1}}. \end{aligned} \tag{30}$$

Then there exists a UTS of type II_r, *(followed by a UTS of type I)* $\overline{U} \to \overline{U}(1)$ *along* ν *such that* $\overline{U}''(1)$ *has regular parameters* $(\overline{y}_1(1), \ldots, \overline{y}_n(1))$ *satisfying*

$$\begin{aligned} \overline{y}_1 &= \overline{y}_1(1)^{b_{11}} \cdots \overline{y}_s(1)^{b_{1s}} d^{b_{1,s+1}} \\ &\vdots \\ \overline{y}_s &= \overline{y}_1(1)^{b_{s1}} \cdots \overline{y}_s(1)^{b_{ss}} d^{b_{s,s+1}} \\ \overline{y}_r &= \overline{y}_1(1)^{b_{s+1,1}} \cdots \overline{y}_s(1)^{b_{s+1,s}} (\overline{y}_r(1)+1) d^{b_{s+1,s+1}}, \end{aligned} \tag{31}$$

$\overline{T}''(1) \subset \overline{U}''(1)$, *and*

$$\begin{aligned} \overline{x}_1(1) &= \overline{y}_1(1)^{c_{11}(1)} \cdots \overline{y}_s(1)^{c_{1s}(1)} \alpha_1(1) \\ &\vdots \\ \overline{x}_s(1) &= \overline{y}_1(1)^{c_{s1}(1)} \cdots \overline{y}_s(1)^{c_{ss}(1)} \alpha_s(1) \\ \overline{x}_{s+1}(1) &= \overline{y}_{s+1}(1) \\ &\vdots \\ \overline{x}_l(1) &= \overline{y}_l(1) \end{aligned} \tag{32}$$

where $\alpha_1(1), \ldots, \alpha_s(1) \in k(\overline{U}(1))$, $\nu(\overline{x}_1(1)), \ldots, \nu(\overline{x}_s(1))$ *are rationally independent,* $\det(c_{ij}(1)) \neq 0$ *and* $(R, \overline{T}''(1), \overline{T}(1))$, $(S, \overline{U}''(1), \overline{U}(1))$ *is a CUTS along* ν.

Proof. — Identify ν with our extension of ν to the quotient field of $\overline{U}$ which dominates $\overline{U}$. Set $(g_{ij}) = (a_{ij})^{-1}$,

$$\begin{aligned} A_1 &= \overline{x}_1^{g_{11}} \cdots \overline{x}_s^{g_{1s}} \overline{x}_r^{g_{1,s+1}} \\ &\vdots \\ A_s &= \overline{x}_1^{g_{s1}} \cdots \overline{x}_s^{g_{ss}} \overline{x}_r^{g_{s,s+1}} \\ A_r &= \overline{x}_1^{g_{s+1,1}} \cdots \overline{x}_s^{g_{s+1,s}} \overline{x}_r^{g_{s+1,s+1}}. \end{aligned}$$

Then $\overline{T}'(1)$ is a localization of $\overline{T}''[A_1, \ldots, A_s, A_r]$. $\nu(A_i) > 0$ for $1 \le i \le s$ and $\nu(A_r) = 0$. We have

$$\begin{aligned} A_1 &= \overline{y}_1^{d_{11}} \cdots \overline{y}_s^{d_{1s}} \overline{y}_r^{d_{1,s+1}} \beta_1 \\ &\vdots \\ A_s &= \overline{y}_1^{d_{s1}} \cdots \overline{y}_s^{d_{ss}} \overline{y}_r^{d_{s,s+1}} \beta_s \\ A_r &= \overline{y}_1^{d_{s+1,1}} \cdots \overline{y}_s^{d_{s+1,s}} \overline{y}_r^{d_{s+1,s+1}} \beta_r \end{aligned}$$

where $\beta_i = \alpha_1^{g_{i1}} \cdots \alpha_s^{g_{is}}$ for $1 \le i \le s$, $\beta_r = \alpha_1^{g_{s+1,1}} \cdots \alpha_s^{g_{s+1,s}}$ and

$$(d_{ik}) = (a_{ij})^{-1} \begin{pmatrix} (c_{jk}) & 0 \\ 0 & 1 \end{pmatrix}. \tag{33}$$

Define B_i by

$$\begin{aligned} B_1 &= \overline{y}_1^{h_{11}} \cdots \overline{y}_s^{h_{1s}} \overline{y}_r^{h_{1,s+1}} \\ &\vdots \\ B_s &= \overline{y}_1^{h_{s1}} \cdots \overline{y}_s^{h_{ss}} \overline{y}_r^{h_{s,s+1}} \\ B_r &= \overline{y}_1^{h_{s+1,1}} \cdots \overline{y}_s^{h_{s+1,s}} \overline{y}_r^{h_{s+1,s+1}} \end{aligned}$$

where the matrix (h_{ij}) defines a UTS of type II_r $\overline{U} \to W$ along ν where $\overline{W}'$ is a localization of $\overline{U}''[B_1, \ldots, B_s, B_r]$ with $\nu(B_i) > 0$ for $1 \le i \le s$ and $\nu(B_r) = 0$. We have

$$\begin{aligned} A_1 &= B_1^{e_{11}} \cdots B_s^{e_{1s}} B_r^{e_{1,s+1}} \beta_1 \\ &\vdots \\ A_s &= B_1^{e_{s1}} \cdots B_s^{e_{ss}} B_r^{e_{s,s+1}} \beta_s \\ A_r &= B_1^{e_{s+1,1}} \cdots B_s^{e_{s+1,s}} B_r^{e_{s+1,s+1}} \beta_r \end{aligned}$$

where $(e_{ij}) = (d_{ij})(h_{ij})^{-1}$ is a matrix with integral coefficients. Since $\nu(A_r) = \nu(B_r) = 0$ and $\nu(B_1), \ldots, \nu(B_s)$ are rationally independent, we have $e_{s+1,1} = \cdots = e_{s+1,s} = 0$. Then $\det(e_{ij}) \neq 0$ implies $e_{s+1,s+1} \neq 0$. Since $\nu(A_1), \ldots, \nu(A_s) > 0$, by Lemma 4.2, we can perform a UTS of type I $\overline{W} \to \overline{U}(1)$ along ν so that $\overline{U}'(1)$ is a localization of $\overline{W}'[C_1, \ldots, C_s]$ with $\nu(C_1), \ldots, \nu(C_s)$ rationally independent, and

$$\begin{aligned} B_1 &= C_1^{b''_{11}} \cdots C_s^{b''_{1s}} \\ &\vdots \\ B_s &= C_1^{b''_{s1}} \cdots C_s^{b''_{ss}} \end{aligned}$$

to get

$$\begin{aligned} A_1 &= C_1^{f_{11}} \cdots C_s^{f_{1s}} B_r^{f_{1,s+1}} \beta_1 \\ &\vdots \\ A_s &= C_1^{f_{s1}} \cdots C_s^{f_{ss}} B_r^{f_{s,s+1}} \beta_s \\ A_r &= C_1^{f_{s+1,1}} \cdots C_s^{f_{s+1,s}} B_r^{f_{s+1,s+1}} \beta_r \end{aligned} \tag{34}$$

with $f_{ij} \ge 0$ for all i, j and

$$(f_{ij}) = (e_{ij}) \begin{pmatrix} (b''_{ij}) & 0 \\ 0 & 1 \end{pmatrix}.$$

Set

$$(b_{ij}) = (h_{ij})^{-1}\begin{pmatrix}(b''_{ij}) & 0\\ 0 & 1\end{pmatrix}.$$

(34) implies $A_1, \ldots, A_r \in \overline{U}'(1)$. Thus $\overline{U}'' \to \overline{U}'(1)$ is a MTS along ν and $\overline{T}'(1) \subset \overline{U}'(1)$. Further, we have $f_{s+1,1} = \cdots = f_{s+1,s} = 0$.

Extend ν from the quotient field of $\overline{U}$ to a valuation of the quotient field of $\overline{U}(1)$ which dominates $\overline{U}(1)$. Let d be the residue of B_r in $k(W)$. $\overline{U}(1)$ has regular parameters $(y_1^*(1), \ldots, y_n^*(1))$ such that

$$\begin{aligned} \overline{y}_1 &= y_1^*(1)^{b_{11}} \cdots y_s^*(1)^{b_{1s}}(y_r^*(1)+d)^{b_{1,s+1}}\\ &\ \vdots\\ \overline{y}_s &= y_1^*(1)^{b_{s1}} \cdots y_s^*(1)^{b_{ss}}(y_r^*(1)+d)^{b_{s,s+1}}\\ \overline{y}_r &= y_1^*(1)^{b_{s+1,1}} \cdots y_s^*(1)^{b_{s+1,s}}(y_r^*(1)+d)^{b_{s+1,s+1}}. \end{aligned} \tag{35}$$

Set

$$\overline{U}''(1) = \overline{U}'(1)\left[d, \left(\frac{y_r^*(1)}{d}+1\right)^{1/\overline{d}}\right]_{(y_1^*(1),\ldots,y_n^*(1))}.$$

We have a natural inclusion $\overline{T}(1) \subset \overline{U}(1)$.

Let $c \in k(\overline{T}'(1))$ be the residue of A_r. Then c is the residue of A_r in the residue field of our extension of ν to the quotient field of $\overline{U}(1)$, since ν dominates $\overline{U}''(1)$. Set

$$x_i^*(i) = \begin{cases} A_i & 1 \le i \le s\\ \overline{x}_i & s < i,\ i \ne r\\ A_r - c & i = r\end{cases}$$

Then $(x_1^*(1), \ldots, x_n^*(1))$ are regular parameters in $\overline{T}(1)$ such that

$$\begin{aligned} x_1^*(1) &= y_1^*(1)^{f_{11}} \cdots y_s^*(1)^{f_{1s}}(y_r^*(1)+d)^{f_{1,s+1}}\beta_1\\ &\ \vdots\\ x_s^*(1) &= y_1^*(1)^{f_{s1}} \cdots y_s^*(1)^{f_{ss}}(y_r^*(1)+d)^{f_{s,s+1}}\beta_s\\ x_r^*(1)+c &= y_1^*(1)^{f_{s+1,1}} \cdots y_s^*(1)^{f_{s+1,s}}(y_r^*(1)+d)^{f_{s+1,s+1}}\beta_r. \end{aligned} \tag{36}$$

They are related to the regular parameters $\overline{x}_i(1)$ in $\overline{T}(1)$ satisfying (30) by

$$x_i^*(1) = \begin{cases} \overline{x}_i(1)(\overline{x}_r(1)+1)^{\overline{c}\gamma_i} & 1 \le i \le s\\ c[(\overline{x}_r(1)+1)^{\overline{c}} - 1] & i = r\end{cases}$$

where

$$\overline{c} = \det\begin{pmatrix} a_{11} & \cdots & a_{1s}\\ \vdots & & \vdots\\ a_{s1} & \cdots & a_{ss}\end{pmatrix} \det\begin{pmatrix} a_{11} & \cdots & a_{1,s+1}\\ \vdots & & \vdots\\ a_{s1} & \cdots & a_{s+1,s+1}\end{pmatrix}$$

$$\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_s \\ 1 \end{pmatrix} = (a_{ij})^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1/\overline{c} \end{pmatrix}.$$

We have regular parameters $\overline{y}_i(1)$ in $\overline{U}''(1)$ satisfying (31) with

$$y_i^*(1) = \begin{cases} \overline{y}_i(1)(\overline{y}_r(1)+1)^{\overline{d}\tau_i} & 1 \le i \le s \\ d[(\overline{y}_r(1)+1)^{\overline{d}} - 1] & i = r \end{cases}$$

where

$$\overline{d} = \det \begin{pmatrix} b_{11} & \cdots & b_{1s} \\ \vdots & & \vdots \\ b_{s1} & \cdots & b_{ss} \end{pmatrix} \det \begin{pmatrix} b_{11} & \cdots & b_{1,s+1} \\ \vdots & & \vdots \\ b_{s1} & \cdots & b_{s+1,s+1} \end{pmatrix}$$

$$\begin{pmatrix} \tau_1 \\ \vdots \\ \tau_s \\ 1 \end{pmatrix} = (b_{ij})^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1/\overline{d} \end{pmatrix}.$$

$$(f_{ij}) \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_s \\ 1 \end{pmatrix} = (a_{ij})^{-1} \begin{pmatrix} (c_{ij}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1/\overline{d} \end{pmatrix} = (a_{ij})^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1/\overline{d} \end{pmatrix} = \frac{\overline{c}}{\overline{d}} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_s \\ 1 \end{pmatrix}.$$

Substitute this in (36) to get

$$\begin{aligned} \overline{x}_1(1)(\overline{x}_r(1)+1)^{\overline{c}\gamma_1} &= \overline{y}_1(1)^{f_{11}} \cdots \overline{y}_s(1)^{f_{1s}} (\overline{y}_r(1)+1)^{\overline{c}\gamma_1} d^{f_{1,s+1}} \beta_1 \\ &\vdots \\ \overline{x}_s(1)(\overline{x}_r(1)+1)^{\overline{c}\gamma_s} &= \overline{y}_1(1)^{f_{s1}} \cdots \overline{y}_s(1)^{f_{ss}} (\overline{y}_r(1)+1)^{\overline{c}\gamma_s} d^{f_{s,s+1}} \beta_s \\ c(\overline{x}_r(1)+1)^{\overline{c}} &= (\overline{y}_r(1)+1)^{\overline{c}} d^{f_{s+1,s+1}} \beta_r. \end{aligned}$$

$\nu(\overline{x}_r(1)) > 0$ and $\nu(\overline{y}_r(1)) > 0$ imply

$$c = d^{f_{s+1,s+1}} \beta_r. \tag{37}$$

Our inclusion $\overline{T}(1) \subset \overline{U}(1)$ induces

$$\overline{x}_r(1) = \omega \overline{y}_r(1) + \omega - 1$$

in $\overline{U}(1)$ for some $\overline{c}$-th root of unity ω. Since $\overline{x}_r(1) \in m(\overline{T}(1))$ and $\overline{y}_r(1) \in m(\overline{U}(1))$, we must have $\omega = 1$. We thus get (32).

$$\overline{T}''(1) = \overline{T}'(1)[c, \overline{x}_r(1)]_{(\overline{x}_1(1),\ldots,\overline{x}_n(1))} \subset \overline{U}''[d, \overline{y}_r(1)]_{(\overline{y}_1(1),\ldots,\overline{y}_n(1))} = \overline{U}''(1).$$

An extension of ν_0 to the quotient field of $\overline{U}(1)$ which dominates $\overline{U}(1)$ then makes $(R, \overline{T}''(1), \overline{T}(1))$, $(S, \overline{U}''(1), \overline{U}(1))$ a CUTS along ν. □

Lemma 4.5. — *Suppose that $(R,\overline{T}'',\overline{T})$, $(S,\overline{U}'',\overline{U})$ is a CUTS along ν, $\overline{T}''$ has regular parameters $(\overline{x}_1,\ldots,\overline{x}_n)$ and $\overline{U}''$ has regular parameters $(\overline{y}_1,\ldots,\overline{y}_n)$ such that*

$$
\begin{aligned}
\overline{x}_1 &= \overline{y}_1^{c_{11}}\cdots\overline{y}_s^{c_{1s}}\alpha_1\\
&\ \vdots\\
\overline{x}_s &= \overline{y}_1^{c_{s1}}\cdots\overline{y}_s^{c_{ss}}\alpha_s\\
\overline{x}_{s+1} &= \overline{y}_{s+1}\\
&\ \vdots\\
\overline{x}_l &= \overline{y}_l\\
\overline{x}_{l+1} &= \overline{y}_1^{d_1}\cdots\overline{y}_s^{d_s}\overline{y}_{l+1}
\end{aligned}
\tag{38}
$$

where $\alpha_1,\ldots,\alpha_s\in k(\overline{U})$, $\nu(\overline{x}_1),\ldots,\nu(\overline{x}_s)$ are rationally independent, $\det(c_{ij})\neq 0$.

Suppose that $\overline{T}\to\overline{T}(1)$ is a UTS of type II_{l+1} along ν, such that $\overline{T}''(1)$ has regular parameters $(\overline{x}_1(1),\ldots,\overline{x}_n(1))$ with

$$
\begin{aligned}
\overline{x}_1 &= \overline{x}_1(1)^{a_{11}}\cdots\overline{x}_s(1)^{a_{1s}}c^{a_{1,s+1}}\\
&\ \vdots\\
\overline{x}_s &= \overline{x}_1(1)^{a_{s1}}\cdots\overline{x}_s(1)^{a_{ss}}c^{a_{s,s+1}}\\
\overline{x}_{l+1} &= \overline{x}_1(1)^{a_{s+1,1}}\cdots\overline{x}_s(1)^{a_{s+1,s}}(\overline{x}_{l+1}(1)+1)c^{a_{s+1,s+1}}.
\end{aligned}
\tag{39}
$$

Then there exists a UTS of type II_{l+1}, (followed by a UTS of type I) $\overline{U}\to\overline{U}(1)$ along ν such that $\overline{U}''(1)$ has regular parameters $(\overline{y}_1(1),\ldots,\overline{y}_n(1))$ satisfying

$$
\begin{aligned}
\overline{y}_1 &= \overline{y}_1(1)^{b_{11}}\cdots\overline{y}_s(1)^{b_{1s}}d^{b_{1,s+1}}\\
&\ \vdots\\
\overline{y}_s &= \overline{y}_1(1)^{b_{s1}}\cdots\overline{y}_s(1)^{b_{ss}}d^{b_{s,s+1}}\\
\overline{y}_{l+1} &= \overline{y}_1(1)^{b_{s+1,1}}\cdots\overline{y}_s(1)^{b_{s+1,s}}(\overline{y}_{l+1}(1)+1)d^{b_{s+1,s+1}},
\end{aligned}
\tag{40}
$$

$\overline{T}''(1)\subset\overline{U}''(1)$, and

$$
\begin{aligned}
\overline{x}_1(1) &= \overline{y}_1(1)^{c_{11}(1)}\cdots\overline{y}_s(1)^{c_{1s}(1)}\alpha_1(1)\\
&\ \vdots\\
\overline{x}_s(1) &= \overline{y}_1(1)^{c_{s1}(1)}\cdots\overline{y}_s(1)^{c_{ss}(1)}\alpha_s(1)\\
\overline{x}_{s+1}(1) &= \overline{y}_{s+1}(1)\\
&\ \vdots\\
\overline{x}_l(1) &= \overline{y}_l(1)\\
\overline{x}_{l+1}(1) &= \overline{y}_{l+1}(1)
\end{aligned}
\tag{41}
$$

where $\alpha_1(1),\ldots,\alpha_s(1)\in k(\overline{U}(1))$, $\nu(\overline{x}_1(1)),\ldots,\nu(\overline{x}_s(1))$ are rationally independent, $\det(c_{ij}(1))\neq 0$ *and $(R,\overline{T}''(1),\overline{T}(1))$, $(S,\overline{U}''(1),\overline{U}(1))$ is a CUTS along ν.*

Proof. — Change r to $l+1$ in the proof of Lemma 4.4, and change (d_{ik}) to

$$(d_{ik}) = (a_{ij})^{-1} \begin{pmatrix} c_{11} & \cdots & c_{1s} & 0 \\ \vdots & & \vdots & \vdots \\ c_{s1} & \cdots & c_{ss} & 0 \\ d_1 & \cdots & d_s & 1 \end{pmatrix}.$$

□

Lemma 4.6. — *Suppose that $(R, \overline{T}'', \overline{T})$ and $(S, \overline{U}'', \overline{U})$ is a CUTS along ν, $\overline{T}''$ has regular parameters $(\overline{x}_1, \ldots, \overline{x}_n)$ and $\overline{U}''$ has regular parameters $(\overline{y}_1, \ldots, \overline{y}_n)$ such that*

$$\begin{aligned} \overline{x}_1 &= \overline{y}_1^{c_{11}} \cdots \overline{y}_s^{c_{1s}} \alpha_1 \\ &\vdots \\ \overline{x}_s &= \overline{y}_1^{c_{s1}} \cdots \overline{y}_s^{c_{ss}} \alpha_s \\ \overline{x}_{s+1} &= \overline{y}_{s+1} \\ &\vdots \\ \overline{x}_l &= \overline{y}_l \\ \overline{x}_{l+1} &= \overline{y}_1^{c_{s+1,1}} \cdots \overline{y}_s^{c_{s+1,s}} \delta \end{aligned}$$

where $\alpha_1, \ldots, \alpha_s \in k(\overline{U})$, $\delta \in \overline{U}''$ is a unit, $\nu(\overline{x}_1), \ldots, \nu(\overline{x}_s)$ are rationally independent and $\det(c_{ij})_{1 \le i,j \le s} \neq 0$.

Suppose that $\overline{T} \to \overline{T}(1)$ is a UTS of type II_{l+1} along ν, so that $\overline{T}''(1)$ has regular parameters $(\overline{x}_1(1), \ldots, \overline{x}_n(1))$ satisfying

$$\begin{aligned} \overline{x}_1 &= \overline{x}_1(1)^{a_{11}} \cdots \overline{x}_s(1)^{a_{1s}} c^{a_{1,s+1}} \\ &\vdots \\ \overline{x}_s &= \overline{x}_1(1)^{a_{s1}} \cdots \overline{x}_s(1)^{a_{ss}} c^{a_{s,s+1}} \\ \overline{x}_{l+1} &= \overline{x}_1(1)^{a_{s+1,1}} \cdots \overline{x}_s(1)^{a_{s+1,s}} (\overline{x}_{l+1}(1) + 1) c^{a_{s+1,s+1}}. \end{aligned}$$

Then there exists a UTS of type I along ν $\overline{U} \to \overline{U}(1)$ such that $\overline{U}'(1)$ has regular parameters $(\widehat{y}_1(1), \ldots, \widehat{y}_n(1))$ with

$$\begin{aligned} \overline{y}_1 &= \widehat{y}_1(1)^{b_{11}} \cdots \widehat{y}_s(1)^{b_{1s}} \\ &\vdots \\ \overline{y}_s &= \widehat{y}_1(1)^{b_{s1}} \cdots \widehat{y}_s(1)^{b_{ss}} \end{aligned}$$

and $\overline{U}''(1)$ *has regular parameters* $(\overline{y}_1(1),\ldots,\overline{y}_n(1))$ *such that* $\overline{y}_i(1)=\varepsilon_i\widehat{y}_i(1)$ *for* $1\le i\le s$ *for some units* $\varepsilon_i\in\overline{U}''(1)$, $\overline{T}''(1)\subset\overline{U}''(1)$,

$$\begin{aligned}
\overline{x}_1(1) &= \overline{y}_1(1)^{c_{11}(1)}\cdots\overline{y}_s(1)^{c_{1s}(1)}\alpha_1(1)\\
&\vdots\\
\overline{x}_s(1) &= \overline{y}_1(1)^{c_{s1}(1)}\cdots\overline{y}_s(1)^{c_{ss}(1)}\alpha_s(1)\\
\overline{x}_{s+1}(1) &= \overline{y}_{s+1}(1)\\
&\vdots\\
\overline{x}_l(1) &= \overline{y}_l(1)
\end{aligned}$$

where $\alpha_1(1),\ldots,\alpha_s(1)\in k(\overline{U}(1))$, $\nu(\overline{x}_1(1)),\ldots,\nu(\overline{x}_s(1))$ *are rationally independent,* $\det(c_{ij}(1))\neq 0$, *and* $(R,\overline{T}''(1),\overline{T}(1))$ *and* $(S,\overline{U}''(1),\overline{U}(1))$ *is a CUTS along* ν.

Proof. — Identify ν with our extension of ν to the quotient field of $\overline{U}$ which dominates $\overline{U}$. Set $(g_{ij})=(a_{ij})^{-1}$,

$$\begin{aligned}
A_1 &= \overline{x}_1^{g_{11}}\cdots\overline{x}_s^{g_{1s}}\overline{x}_{l+1}^{g_{1,s+1}}\\
&\vdots\\
A_s &= \overline{x}_1^{g_{s1}}\cdots\overline{x}_s^{g_{ss}}\overline{x}_{l+1}^{g_{s,s+1}}\\
A_{l+1} &= \overline{x}_1^{g_{s+1,1}}\cdots\overline{x}_s^{g_{s+1,s}}\overline{x}_{l+1}^{g_{s+1,s+1}}.
\end{aligned}$$

$\overline{T}'(1)$ is a localization of $\overline{T}''[A_1,\ldots,A_s,A_{l+1}]$. $\nu(A_i)>0$ for $1\le i\le s$ and $\nu(A_{l+1})=0$.

$$\begin{aligned}
A_1 &= \overline{y}_1^{d_{11}}\cdots\overline{y}_s^{d_{1s}}\delta_1e_1\\
&\vdots\\
A_s &= \overline{y}_1^{d_{s1}}\cdots\overline{y}_s^{d_{ss}}\delta_se_s\\
A_{l+1} &= \overline{y}_1^{d_{s+1,1}}\cdots\overline{y}_s^{d_{s+1,s}}\delta_{l+1}
\end{aligned}$$

where $e_i\in k(\overline{U})$, $(d_{ik})=(a_{ij})^{-1}(c_{jk})$ and δ_i are units in $\overline{U}''$ such that δ_i has residue 1 in $k(\overline{U})$ for $1\le i\le s$. $\nu(A_{l+1})=0$ and $\nu(\overline{y}_1),\ldots,\nu(\overline{y}_s)$ rationally independent implies

$$d_{s+1,1}=\cdots=d_{s+1,s}=0.$$

Since $\nu(A_i) > 0$ for $1 \le i \le s$, by Lemma 4.2 we can perform a UTS of type I along $\nu\ \overline{U} \to \overline{U}(1)$ where $\overline{U}'(1)$ has regular parameters $(\widehat{y}_1(1), \ldots, \widehat{y}_n(1))$ satisfying

$$\begin{aligned}
\overline{y}_1 &= \widehat{y}_1(1)^{b_{11}(1)} \cdots \widehat{y}_s(1)^{b_{1s}(1)} \\
&\vdots \\
\overline{y}_s &= \widehat{y}_1(1)^{b_{s1}(1)} \cdots \widehat{y}_s(1)^{b_{ss}(1)}
\end{aligned}$$

to get

$$\begin{aligned}
A_1 &= \widehat{y}_1(1)^{c_{11}(1)} \cdots \widehat{y}_s(1)^{c_{1s}(1)} \delta_1 e_1 \\
&\vdots \\
A_s &= \widehat{y}_1(1)^{c_{s1}(1)} \cdots \widehat{y}_s(1)^{c_{ss}(1)} \delta_s e_s \\
A_{l+1} &= \delta_{l+1}
\end{aligned}$$

where all $c_{ij}(1) \ge 0$. Thus

$$\overline{T}''[A_1, \ldots, A_s, A_{l+1}] \subset \overline{U}'(1) = \overline{U}''[\widehat{y}_1(1), \ldots, \widehat{y}_s(1)]_{(\widehat{y}_1(1), \ldots, \widehat{y}_n(1))},$$

and since ν dominates $\overline{U}'(1)$ and $\overline{T}'(1)$, $\overline{U}'(1)$ dominates $\overline{T}'(1)$.

Now extend ν from the quotient field of $\overline{U}$ to a valuation of the quotient field of $\overline{U}(1)$ which dominates $\overline{U}(1)$. $\overline{T}''(1)$ has regular parameters $(x_1^*(1), \ldots, x_n^*(1))$ with

$$\begin{aligned}
\overline{x}_1 &= x_1^*(1)^{a_{11}} \cdots x_s^*(1)^{a_{1s}} (x_{l+1}^*(1) + c)^{a_{1,s+1}} \\
&\vdots \\
\overline{x}_s &= x_1^*(1)^{a_{s1}} \cdots x_s^*(1)^{a_{ss}} (x_{l+1}^*(1) + c)^{a_{s,s+1}} \\
\overline{x}_{l+1} &= x_1^*(1)^{a_{s+1,1}} \cdots x_s^*(1)^{a_{s+1,s}} (x_{l+1}^*(1) + c)^{a_{s+1,s+1}}.
\end{aligned}$$

$$\overline{T}''(1) = \overline{T}'' \left[c, \left(\frac{x_{l+1}^*(1)}{c} + 1 \right)^{1/\overline{c}} \right]_{(x_1^*(1), \ldots, x_n^*(1))}.$$

$(\overline{x}_1(1), \ldots, \overline{x}_n(1))$ are regular parameters in $\overline{T}''(1)$ which satisfy

$$\begin{aligned}
\overline{x}_1(1) &= x_1^*(1) \left(\frac{x_{l+1}^*(1)}{c} + 1 \right)^{-\gamma_1} = \widehat{y}_1(1)^{c_{11}(1)} \cdots \widehat{y}_s(1)^{c_{1s}(1)} \delta_1 e_1 \left(\frac{\delta_{l+1}}{c} \right)^{-\gamma_1} \\
&\vdots \\
\overline{x}_s(1) &= x_s^*(1) \left(\frac{x_{l+1}^*(1)}{c} + 1 \right)^{-\gamma_s} = \widehat{y}_1(1)^{c_{s1}(1)} \cdots \widehat{y}_s(1)^{c_{ss}(1)} \delta_s e_s \left(\frac{\delta_{l+1}}{c} \right)^{-\gamma_s} \\
\overline{x}_{l+1}(1) &= \left(\frac{x_{l+1}^*(1)}{c} + 1 \right)^{1/\overline{c}} - 1 = \left(\frac{\delta_{l+1}}{c} \right)^{1/\overline{c}} - 1
\end{aligned}$$

Set $(e_{ij}) = (c_{ij}(1))^{-1}$,

$$\varepsilon_i = \delta_1^{e_{i1}} \cdots \delta_s^{e_{is}} \left(\frac{\delta_{l+1}}{c} \right)^{-\gamma_1 e_{i1} - \cdots - \gamma_s e_{is}}$$

for $1 \le i \le s$. Define

$$\overline{y}_i(1) = \begin{cases} \varepsilon_i \widehat{y}_i(1) & 1 \le i \le s \\ \widehat{y}_i(1) & s+1 \le i. \end{cases}$$

Then the conclusions of Lemma 4.6 hold with

$$\overline{U}''(1) = \overline{U}'(1)\left[c, \left(\frac{\delta_{l+1}}{c}\right)^{1/\overline{c}}, \varepsilon_1, \ldots, \varepsilon_s\right]_{(\overline{y}_1(1), \ldots, \overline{y}_n(1))}.$$

□

4.2. Monomialization in rank 1

Theorem 4.7. — *Suppose that $(R, \overline{T}'', \overline{T})$ and $(S, \overline{U}'', \overline{U})$ is a CUTS along ν such that $\overline{T}''$ contains the subfield $k(c_0)$ for some $c_0 \in \overline{T}''$ and $\overline{U}''$ contains a subfield isomorphic to $k(\overline{U}'')$, $\overline{T}''$ has regular parameters $(\overline{z}_1, \ldots, \overline{z}_n)$ and $\overline{U}''$ has regular parameters $(\overline{w}_1, \ldots, \overline{w}_n)$ such that*

$$\begin{aligned} \overline{z}_1 &= \overline{w}_1^{c_{11}} \cdots \overline{w}_s^{c_{1s}} \phi_1 \\ &\vdots \\ \overline{z}_s &= \overline{w}_1^{c_{s1}} \cdots \overline{w}_s^{c_{ss}} \phi_s \\ \overline{z}_{s+1} &= \overline{w}_{s+1} \\ &\vdots \\ \overline{z}_l &= \overline{w}_l. \end{aligned}$$

where $\phi_1, \ldots, \phi_s \in k(\overline{U}'')$, $\nu(\overline{z}_1), \ldots, \nu(\overline{z}_s)$ are rationally independent, $\det(c_{ij}) \neq 0$.

Suppose that one of the following three conditions hold.

(42) *$f \in k(\overline{U})[[\overline{w}_1, \ldots, \overline{w}_m]]$ for some m such that $s \le m \le n$ with $\nu(f) < \infty$.*
(43) *$f \in k(\overline{U})[[\overline{w}_1, \ldots, \overline{w}_m]]$ for some m such that $s < m \le n$ with $\nu(f) = \infty$ and $A > 0$ is given.*
(44) *$f \in \left(k(\overline{U})[[\overline{w}_1, \ldots, \overline{w}_m]] - k(\overline{U})[[\overline{w}_1, \ldots, \overline{w}_l]]\right) \cap \overline{U}''$ for some m such that $l < m \le n$.*

Then there exists a CUTS along ν $(R, \overline{T}''(t), \overline{T}(t))$ and $(S, \overline{U}''(t), \overline{U}(t))$

$$\begin{array}{ccccccc} \overline{U} = \overline{U}(0) & \to & \overline{U}(1) & \to & \cdots & \to & \overline{U}(t) \\ \uparrow & & \uparrow & & & & \uparrow \\ \overline{T} = \overline{T}(0) & \to & \overline{T}(1) & \to & \cdots & \to & \overline{T}(t) \end{array} \tag{45}$$

such that $\overline{T}''(i)$ *has regular parameters* $(\overline{z}_1(i),\ldots,\overline{z}_n(i))$, $\overline{U}''(i)$ *has regular parameters* $(\overline{w}_1(i),\ldots,\overline{w}_n(i))$ *satisfying*

$$\begin{aligned}
\overline{z}_1(i) &= \overline{w}_1(i)^{c_{11}(i)}\cdots\overline{w}_s(i)^{c_{1s}(i)}\phi_1(i)\\
&\vdots\\
\overline{z}_s(i) &= \overline{w}_1(i)^{c_{s1}(i)}\cdots\overline{w}_s(i)^{c_{ss}(i)}\phi_s(i)\\
\overline{z}_{s+1}(i) &= \overline{w}_{s+1}(i)\\
&\vdots\\
\overline{z}_l(i) &= \overline{w}_l(i)
\end{aligned}$$

$\overline{T}''(i)$ *contains a subfield* $k(c_0,\ldots,c_i)$ *and* $\overline{U}''(i)$ *contains a subfield isomorphic to* $k(\overline{U}(i))$. $\phi_1(i),\ldots,\phi_s(i)\in k(\overline{U}(i))$, $\nu(\overline{z}_1(i)),\ldots,\nu(\overline{z}_s(i))$ *are rationally independent,* $\det(c_{ij}(i))\neq 0$ *for* $0\leq i\leq t$. *In case (42) we have*

$$f=\overline{w}_1(t)^{d_1}\cdots\overline{w}_s(t)^{d_s}u(\overline{w}_1(t),\ldots,\overline{w}_m(t))$$

where $u\in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_m(t)]]$ *is a unit power series.*

In case (43) we have

$$f=\overline{w}_1(t)^{d_1}\cdots\overline{w}_s(t)^{d_s}\Sigma(\overline{w}_1(t),\ldots,\overline{w}_m(t))$$

where $\Sigma\in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_m(t)]]$, $\nu(\overline{w}_1(t)^{d_1}\cdots\overline{w}_s(t)^{d_s})>A$.

In case (44) we have

$$f=P(\overline{w}_1(t),\ldots,\overline{w}_l(t))+\overline{w}_1(t)^{d_1}\cdots\overline{w}_s(t)^{d_s}H$$

for some powerseries $P\in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_l(t)]]$,

$$H=u(\overline{w}_m(t)+\overline{w}_1(t)^{g_1}\cdots\overline{w}_s(t)^{g_s}\Sigma)$$

where $u\in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_m(t)]]$ *is a unit,* $\Sigma\in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_{m-1}(t)]]$ *and* $\nu(\overline{w}_m(t))\leq\nu(\overline{w}_1(t)^{g_1}\cdots\overline{w}_s(t)^{g_s})$.

(45) *will be such that* $\overline{T}''(\alpha)$ *has regular parameters*

$$(\overline{z}_1(\alpha),\ldots,\overline{z}_n(\alpha)) \text{ and } (\tilde{\overline{z}}'_1(\alpha),\ldots,\tilde{\overline{z}}'_n(\alpha)),$$

$\overline{U}''(\alpha)$ *has regular parameters*

$$(\overline{w}_1(\alpha),\ldots,\overline{w}_n(\alpha)) \text{ and } (\tilde{\overline{w}}'_1(\alpha),\ldots,\tilde{\overline{w}}'_n(\alpha))$$

where $\overline{z}_i(0)=\overline{z}_i$ *and* $\overline{w}_i(0)=\overline{w}_i$ *for* $1\leq i\leq n$. (45) *will consist of three types of CUTS.*

(M1) $\overline{T}(\alpha)\to\overline{T}(\alpha+1)$ *and* $\overline{U}(\alpha)\to\overline{U}(\alpha+1)$ *are of type I.*

(M2) $\overline{T}(\alpha)\to\overline{T}(\alpha+1)$ *is of type* II_r, $s+1\leq r\leq l$, *and* $\overline{U}(\alpha)\to\overline{U}(\alpha+1)$ *is a transformation of type* II_r, *followed by a transformation of type I.*

(M3) $\overline{T}(\alpha)=\overline{T}(\alpha+1)$ *and* $\overline{U}(\alpha)\to\overline{U}(\alpha+1)$ *is of type* II_r $(l+1\leq r\leq m)$.

We will find polynomials $P_{i,\alpha}$ so that the variables will be related by:

$$\tilde{\overline{z}}'_i(\alpha) = \begin{cases} \overline{z}_i(\alpha) - P_{i,\alpha}(\overline{z}_1(\alpha), \ldots, \overline{z}_{i-1}(\alpha)) & \text{if } s+1 \leq i \leq l \\ \overline{z}_i(\alpha) & \text{otherwise} \end{cases}$$

$$\tilde{\overline{w}}'_i(\alpha) = \begin{cases} \tilde{\overline{z}}'_i(\alpha) & \text{if } s+1 \leq i \leq l \\ \overline{w}_i(\alpha) - P_{i,\alpha}(\overline{w}_1(\alpha), \ldots, \overline{w}_{i-1}(\alpha)) & \text{if } l+1 \leq i \leq m \\ \overline{w}_i(\alpha) & \text{otherwise} \end{cases}$$

The coefficients of $P_{i,\alpha}$ will be in $k(c_0, \ldots, c_\alpha)$ if $i \leq l$, and will be in $k(\overline{U}(\alpha))$ if $i > l$. For all α we will have

$$\begin{aligned} \overline{z}_1(\alpha) &= \overline{w}_1(\alpha)^{c_{11}(\alpha)} \cdots \overline{w}_s(\alpha)^{c_{1s}(\alpha)} \phi_1(\alpha) \\ &\vdots \\ \overline{z}_s(\alpha) &= \overline{w}_1(\alpha)^{c_{s1}(\alpha)} \cdots \overline{w}_s(\alpha)^{c_{ss}(\alpha)} \phi_s(\alpha) \\ \overline{z}_{s+1}(\alpha) &= \overline{w}_{s+1}(\alpha) \\ &\vdots \\ \overline{z}_l(\alpha) &= \overline{w}_l(\alpha) \end{aligned} \tag{46}$$

and

$$\begin{aligned} \tilde{\overline{z}}'_1(\alpha) &= \tilde{\overline{w}}'_1(\alpha)^{c_{11}(\alpha)} \cdots \tilde{\overline{w}}'_s(\alpha)^{c_{1s}(\alpha)} \phi_1(\alpha) \\ &\vdots \\ \tilde{\overline{z}}'_s(\alpha) &= \tilde{\overline{w}}'_1(\alpha)^{c_{s1}(\alpha)} \cdots \tilde{\overline{w}}'_s(\alpha)^{c_{ss}(\alpha)} \phi_s(\alpha) \\ \tilde{\overline{z}}'_{s+1}(\alpha) &= \tilde{\overline{w}}'_{s+1}(\alpha) \\ &\vdots \\ \tilde{\overline{z}}_l(\alpha) &= \tilde{\overline{w}}'_l(\alpha) \end{aligned} \tag{47}$$

where $\phi_1(\alpha), \ldots, \phi_s(\alpha) \in k(\overline{U}(\alpha))$. $\overline{T}''(\alpha)$ contains a subfield $k(c_0, \ldots, c_\alpha)$ and $\overline{U}''(\alpha)$ contains a subfield isomorphic to $k(\overline{U}(\alpha))$.

In a transformation $\overline{T}(\alpha) \to \overline{T}(\alpha+1)$ of type I, $\overline{T}''(\alpha+1)$ will have regular parameters $(\overline{z}_1(\alpha+1), \ldots, \overline{z}_n(\alpha+1))$

$$\begin{aligned} \tilde{\overline{z}}'_1(\alpha) &= \overline{z}_1(\alpha+1)^{a_{11}(\alpha+1)} \cdots \overline{z}_s(\alpha+1)^{a_{1s}(\alpha+1)} \\ &\vdots \\ \tilde{\overline{z}}'_s(\alpha) &= \overline{z}_1(\alpha+1)^{a_{s1}(\alpha+1)} \cdots \overline{z}_s(\alpha+1)^{a_{ss}(\alpha+1)} \end{aligned} \tag{48}$$

and $c_{\alpha+1}$ *is defined to be* 1. *In a transformation* $\overline{T}(\alpha) \to \overline{T}(\alpha+1)$ *of type* II_r $(s+1 \leq r \leq l)$ $\overline{T}''(\alpha+1)$ *will have regular parameters* $(\overline{z}_1(\alpha+1), \ldots, \overline{z}_n(\alpha+1))$

$$
(49)\qquad
\begin{aligned}
\widetilde{\overline{z}}'_1(\alpha) &= \overline{z}_1(\alpha+1)^{a_{11}(\alpha+1)} \cdots \overline{z}_s(\alpha+1)^{a_{1s}(\alpha+1)} c_{\alpha+1}^{a_{1,s+1}(\alpha+1)}\\
&\ \vdots\\
\widetilde{\overline{z}}'_s(\alpha) &= \overline{z}_1(\alpha+1)^{a_{s1}(\alpha+1)} \cdots \overline{z}_s(\alpha+1)^{a_{ss}(\alpha+1)} c_{\alpha+1}^{a_{s,s+1}(\alpha+1)}\\
\widetilde{\overline{z}}'_r(\alpha) &= \overline{z}_1(\alpha+1)^{a_{s+1,1}(\alpha+1)} \cdots \overline{z}_s(\alpha+1)^{a_{s+1,s}(\alpha+1)}\\
&\quad \cdot (\overline{z}_r(\alpha+1)+1) c_{\alpha+1}^{a_{s+1,s+1}(\alpha+1)}
\end{aligned}
$$

In a transformation $\overline{U}(\alpha) \to \overline{U}(\alpha+1)$ *of type I,* $\overline{U}''(\alpha+1)$ *will have regular parameters* $(\overline{w}_1(\alpha+1), \ldots, \overline{w}_n(\alpha+1))$

$$
(50)\qquad
\begin{aligned}
\widetilde{\overline{w}}'_1(\alpha) &= \overline{w}_1(\alpha+1)^{b_{11}(\alpha+1)} \cdots \overline{w}_s(\alpha+1)^{b_{1s}(\alpha+1)}\\
&\ \vdots\\
\widetilde{\overline{w}}'_s(\alpha) &= \overline{w}_1(\alpha+1)^{b_{s1}(\alpha+1)} \cdots \overline{w}_s(\alpha+1)^{b_{ss}(\alpha+1)}
\end{aligned}
$$

and $d_{\alpha+1}$ *is defined to be* 1. *In a transformation* $\overline{U}(\alpha) \to \overline{U}(\alpha+1)$ *of type* II_r $(s+1 \leq r \leq m)$ $\overline{U}''(\alpha+1)$ *will have regular parameters* $(\overline{w}_1(\alpha+1), \ldots, \overline{w}_n(\alpha+1))$

$$
(51)\qquad
\begin{aligned}
\widetilde{\overline{w}}'_1(\alpha) &= \overline{w}_1(\alpha+1)^{b_{11}(\alpha+1)} \cdots \overline{w}_s(\alpha+1)^{b_{1s}(\alpha+1)} d_{\alpha+1}^{b_{1,s+1}(\alpha+1)}\\
&\ \vdots\\
\widetilde{\overline{w}}'_s(\alpha) &= \overline{w}_1(\alpha+1)^{b_{s1}(\alpha+1)} \cdots \overline{w}_s(\alpha+1)^{b_{ss}(\alpha+1)} d_{\alpha+1}^{b_{s,s+1}(\alpha+1)}\\
\widetilde{\overline{w}}'_r(\alpha) &= \overline{w}_1(\alpha+1)^{b_{s+1,1}(\alpha+1)} \cdots \overline{w}_s(\alpha+1)^{b_{s+1,s}(\alpha+1)}\\
&\quad \cdot (\overline{w}_r(\alpha+1)+1) d_{\alpha+1}^{b_{s+1,s+1}(\alpha+1)}
\end{aligned}
$$

In a transformation of type (M2) $c_{\alpha+1}$ *is related to* $d_{\alpha+1}$ *by (37) of the proof of Lemma 4.4.*

We will call a UTS (CUTS) as in (45) a UTS (CUTS) in the first m *variables.*

Proof. — We will first show that it is possible to construct a UTS along ν

$$
(52)\qquad \overline{T} \to \overline{T}(1) \to \cdots \to \overline{T}(t)
$$

so that the conditions (54), (55) and (53) below hold.

(53) Suppose that $s \leq m \leq l$. Then there exists a UTS (52) in the first m variables such that

$$
p_m(i) = \{f \in k(\overline{T}(i))[[\overline{z}_1(i), \ldots, \overline{z}_m(i)]] \mid \nu(f) = \infty\}
$$

has the form

$(P(m))$ $\quad p_m(t) =$

$$
(\overline{z}_{r(1)}(t) - Q_{r(1)}(\overline{z}_1(t), \ldots, \overline{z}_{r(1)-1}), \ldots, \overline{z}_{r(\widetilde{m})}(t) - Q_{r(\widetilde{m})}(\overline{z}_1(t), \ldots, \overline{z}_{r(\widetilde{m})-1}))
$$

for some $0 \leq \widetilde{m} \leq m-s$ and $s < r(1) < r(2) < \cdots < r(\widetilde{m}) \leq m$, where $Q_{r(i)}$ are power series with coefficients in $k(c_0, \ldots, c_t)$.

(54) Suppose that $h \in k(\overline{T})[[\overline{z}_1,\dots,\overline{z}_m]]$ for some m with $s \le m \le n$ and $\nu(h) < \infty$. Then there exists a UTS (52), in the first m variables such that $P(m)$ holds in $\overline{T}(t)$ and

$$h = \overline{z}_1(t)^{d_1} \cdots \overline{z}_s(t)^{d_s} u(\overline{z}_1(t),\dots,\overline{z}_m(t))$$

where u is a unit power series with coefficients in $k(\overline{T}(t))$. If h belongs to $k(c_0)[[\overline{z}_1,\dots,\overline{z}_m]]$ then u has coefficients in $k(c_0,\dots,c_t)$.

(55) Suppose that $h \in k(\overline{T})[[\overline{z}_1,\dots,\overline{z}_m]]$ for some m with $s < m \le n$, and $\nu(h) = \infty$ and $A > 0$ is given. Then there exists a UTS (52), in the first m variables such that $P(m)$ holds in $\overline{T}(t)$ and

$$h = \overline{z}_1(t)^{d_1} \cdots \overline{z}_s(t)^{d_s} \Sigma(\overline{z}_1(t),\dots,\overline{z}_m(t))$$

where $\nu(\overline{z}_1(t)^{d_1} \cdots \overline{z}_s(t)^{d_s}) > A$, Σ is a power series with coefficients in $k(\overline{T}(t))$. If $h \in k(c_0)[[\overline{z}_1,\dots,\overline{z}_m]]$, then Σ has coefficients in $k(c_0,\dots,c_t)$.

(53) is trivial for $m = s$, since $p_s = (0)$.

Proof of (54) for $m = s$. — Set $\tau_i = \nu(\overline{z}_i)$ for $1 \le i \le s$. h has an expansion

$$h = \sum_{i \ge 1} a_i \overline{z}_1^{b_1(i)} \cdots \overline{z}_s^{b_s(i)}$$

where the $a_i \in k(\overline{T})$ (or $a_i \in k(c_0)$) and the terms have increasing value. Set

$$c = \frac{3}{\min\{\tau_i/\tau_1\}}\left(b_1(1) + b_2(1)\frac{\tau_2}{\tau_1} + \cdots + b_s(1)\frac{\tau_s}{\tau_1}\right).$$

We can perform a UTS of type I where $\overline{T} \to \overline{T}(1)$ is such that

$$\left|\frac{\tau_i}{\tau_1} - \frac{a_{ij}(1)}{a_{1j}(1)}\right| < \frac{\tau_i}{2\tau_1}$$

for $1 \le i \le s$.

$$\overline{z}_1^{b_1(i)} \cdots \overline{z}_s^{b_s(i)} = \overline{z}_1(1)^{b_1(i)a_{11}(1)+\cdots+b_s(i)a_{s1}(1)} \cdots \overline{z}_s(1)^{b_1(i)a_{1s}(1)+\cdots+b_s(i)a_{ss}(1)}.$$

Suppose that i is such that $b_1(i) + \cdots + b_s(i) > c$. Then for all $1 \le j \le s$ we have

$$\begin{aligned}
b_1(i)a_{1j}(1) + \cdots + b_s(i)a_{sj}(1) &= a_{1j}(1)\left(b_1(i) + b_2(i)\frac{a_{2j}(1)}{a_{1j}(1)} + \cdots + b_s(i)\frac{a_{sj}(1)}{a_{1j}(1)}\right)\\
&\ge \frac{a_{1j}(1)}{2}\left(b_1(i) + b_2(i)\frac{\tau_2}{\tau_1} + \cdots + b_s(i)\frac{\tau_s}{\tau_1}\right)\\
&\ge a_{1j}(1)(b_1(i) + \cdots + b_s(i))\frac{\min(\tau_i/\tau_1)}{2}\\
&> \frac{3a_{1j}(1)}{2}\left(b_1(1) + b_2(1)\frac{\tau_2}{\tau_1} + \cdots + b_s(1)\frac{\tau_s}{\tau_1}\right)\\
&> a_{1j}(1)\left(b_1(1) + b_2(1)\frac{a_{2j}(1)}{a_{1j}(1)} + \cdots + b_s(1)\frac{a_{sj}(1)}{a_{1j}(1)}\right)\\
&= b_1(1)a_{1j}(1) + \cdots + b_s(1)a_{sj}(1).
\end{aligned}$$

By Lemma 4.2 we may choose the $a_{ij}(1)$ so that the inequality

$$b_1(i)a_{1j}(1)+\cdots+b_s(i)a_{sj}(1)>b_1(1)a_{1j}(1)+\cdots+b_s(1)a_{sj}(1)$$

also holds for $1\le j\le s$ for the finitely many i such that $b_1(i)+\cdots+b_s(i)\le c$. Then h has the desired form in $\overline{T}(1)$.

We will now establish (54), (55) and (53) by proving the following inductive statements.

$A(\overline{m})$: (54), (55) and (53) for $m<\overline{m}$ imply (53) for $m=\overline{m}$.
$B(\overline{m})$: (54), (55) for $m<\overline{m}$ and (53) for $m=\overline{m}$ imply (54) and (55) for $m=\overline{m}$.

Proof of $A(\overline{m})$ ($s<\overline{m}$). — By assumption there exists a UTS $\overline{T}\to\overline{T}(t)$ satisfying (53) for $\overline{m}-1$. After replacing $\overline{T}''(0)$ with $\overline{T}''(t)$ and replacing c_0 with a primitive element of $k(c_0,\ldots,c_t)$ over k, we may assume that

$$p_{\overline{m}-1}=(\overline{z}_{r(1)}-Q_{r(1)}(\overline{z}_1,\ldots,\overline{z}_{r(1)-1}),\ldots,\overline{z}_{r(\widetilde{\overline{m}-1})}-Q_{r(\widetilde{\overline{m}-1})}(\overline{z}_1,\ldots,\overline{z}_{r(\widetilde{\overline{m}-1})-1})).$$

where $Q_{r(i)}$ are power series with coefficients in $k(c_0)$. If $p_{\overline{m}-1}k(\overline{T})[[\overline{z}_1,\ldots,\overline{z}_m]]=p_{\overline{m}}$ we are done. So suppose that there exists $\overline{f}\in p_{\overline{m}}-p_{\overline{m}-1}k(\overline{T})[[\overline{z}_1,\ldots,\overline{z}_{\overline{m}}]]$. Let L be a Galois closure of $k(\overline{T})$ over $k(c_0)$, G be the Galois group of L over $k(c_0)$. Set

$$f=\prod_{\sigma\in G}\sigma(\overline{f})\in k(c_0)[[\overline{z}_1,\ldots,\overline{z}_{\overline{m}}]].$$

$f\in p_{\overline{m}}\cap k(c_0)[[\overline{z}_1,\ldots,\overline{z}_{\overline{m}}]]$ and $\nu(f)=\infty$ since $\overline{f}|f$ in $k(\overline{T})[[\overline{z}_1,\ldots,\overline{z}_m]]$. Suppose

$$f\in p_{\overline{m}-1}k(\overline{T})[[\overline{z}_1,\ldots,\overline{z}_{\overline{m}}]].$$

Then $f\in p_{\overline{m}-1}L[[\overline{z}_1,\ldots,\overline{z}_{\overline{m}}]]$ which is a prime ideal, and $\sigma(\overline{f})\in p_{\overline{m}-1}L[[\overline{z}_1,\ldots,\overline{z}_{\overline{m}}]]$ for some $\sigma\in G$. But

$$\sigma\left(p_{\overline{m}-1}L[[\overline{z}_1,\ldots,\overline{z}_{\overline{m}}]]\right)=p_{\overline{m}-1}L[[\overline{z}_1,\ldots,\overline{z}_{\overline{m}}]]$$

for all $\sigma\in G$. Thus

$$\overline{f}\in\left(p_{\overline{m}-1}L[[\overline{z}_1,\ldots,\overline{z}_{\overline{m}}]]\right)\cap k(\overline{T})[[\overline{z}_1,\ldots,\overline{z}_{\overline{m}}]]=p_{\overline{m}-1}k(\overline{T})[[\overline{z}_1,\ldots,\overline{z}_{\overline{m}}]]$$

a contradiction. Thus $f\notin p_{\overline{m}-1}k(\overline{T})[[\overline{z}_1,\ldots,\overline{z}_m]]$.

$$f=\sum_{i=0}^{\infty}a_i(\overline{z}_1,\ldots,\overline{z}_{\overline{m}-1})\overline{z}_{\overline{m}}^i.$$

where the a_i have coefficients in $k(c_0)$. By assumption $\nu(a_i)<\infty$ for some i. Set $r=\mathrm{mult}(f(0,\ldots,0,\overline{z}_m))$.

$$f=\sum_{i=1}^{d}a_i\overline{z}_{\overline{m}}^{f_i}+\sum_j a_j\overline{z}_{\overline{m}}^{f_j}+\sum_k a_k\overline{z}_{\overline{m}}^{f_k}+\overline{z}_{\overline{m}}^{\alpha}\Omega$$

where the first sum consists of the terms of minimal value $\rho = \nu(a_i\overline{z}_{\overline{m}}^{f_i})$, $1 \le i \le d$, $\alpha\nu(\overline{z}_{\overline{m}}) > \rho$, the second sum is a finite sum of terms of finite value $\nu(a_j\overline{z}_{\overline{m}}^{f_j}) > \rho$ and the third sum is a finite sum of terms $a_k\overline{z}_{\overline{m}}^{f_k}$ of infinite value.

Set

$$\overline{R} = k(c_0)\left[\left[\overline{z}_1,\ldots,\overline{z}_{\overline{m}-1}\right]\right] / \left(p_{\overline{m}-1} \cap k(c_0)\left[\left[\overline{z}_1,\ldots,\overline{z}_{\overline{m}-1}\right]\right]\right).$$

ν induces a rank 1 valuation on the quotient field of $\overline{R}$.

Given d in the value group $\Gamma_\nu \subset \mathbf{R}$ of ν, let

$$I_d = \{f \in \overline{R} \mid \nu(f) \ge d\}.$$

By Lemma 2.3, there is a set of real numbers

$$d_1 < d_2 < \cdots < d_i < \tag{56}$$

with $\lim_{i\to\infty} d_i = \infty$ such that d_i are the possible finite values of elements of $\overline{R}$ and $\cap_{i=1}^{\infty} I_{d_i} = 0$. Thus by Theorem 13, Section 5, Chapter VIII **[39]**, there is a function $\gamma(i)$ such that $I_{d_i} \subset m(\overline{R})^{\gamma(i)}$ and $\gamma(i) \to \infty$ as $i \to \infty$.

By assumption, we can construct a UTS in the first $\overline{m}-1$ variables along ν so that for all i, j, k

$$\begin{aligned} a_i &= \overline{z}_1(t_1)^{e_1(i)} \cdots \overline{z}_s(t_1)^{e_s(i)}\overline{a}_i \\ a_j &= \overline{z}_1(t_1)^{f_1(j)} \cdots \overline{z}_s(t_1)^{f_s(j)}\overline{a}_j \\ a_k &= \overline{z}_1(t_1)^{g_1(k)} \cdots \overline{z}_s(t_1)^{g_s(k)}\Sigma_k \end{aligned}$$

in $k(c_0,\ldots,c_{t_1})\left[\left[\overline{z}_1(t_1),\ldots,\overline{z}_{\overline{m}-1}(t_1)\right]\right]$ where $\overline{a}_i$, $\overline{a}_j$ are units and

$$\nu\left(\overline{z}_1(t_1)^{g_1(k)} \cdots \overline{z}_s(t_1)^{g_s(k)}\right) > \rho.$$

Now perform a UTS of type $\mathrm{II}_{\overline{m}}$ and a UTS of type I along ν to get

$$f = \overline{z}_1(t_2)^{d_1} \cdots \overline{z}_s(t_2)^{d_s} f_1$$

where $f_1 \in k(c_0,\ldots,c_{t_2})\left[\left[\overline{z}_1(t_2),\ldots,\overline{z}_{\overline{m}}(t_2)\right]\right]$. Set

$$\lambda_i = c_{t_2}^{a_{1,s+1}(t_2)e_1(i)+\cdots+a_{s,s+1}(t_2)e_s(i)+a_{s+1,s+1}(t_2)f_i}$$

for $1 \le i \le d$. Then

$$f_1 = \sum_{i=1}^{d} \lambda_i\overline{a}_i(\overline{z}_{\overline{m}}(t_2)+1)^{f_i} + \overline{z}_1(t_2)\cdots\overline{z}_s(t_2)\Lambda,$$

for some series $\Lambda \in \overline{T}(t_2)$ by Lemma 4.2. Set $r_1 = \operatorname{mult} f_1(0,\ldots,0,\overline{z}_{\overline{m}}(t_2)) < \infty$. $r_1 \le f_d \le r$.

The residue of $\overline{a}_i$ in

$$\overline{T}(t_2)/(\overline{z}_1(t_2),\ldots,\overline{z}_{\overline{m}-1}(t_2),\overline{z}_{\overline{m}+1}(t_2),\ldots,\overline{z}_n(t_2)) \cong k(\overline{T}(t_2))\left[\left[\overline{z}_{\overline{m}}(t_2)\right]\right]$$

is a nonzero constant $\widetilde{a}_i \in k(\overline{T}(t_2))$ for $1 \leq i \leq d$. Set

$$\zeta(t) = f_1(0, \ldots, 0, t-1) = \sum_{i=1}^{d} \lambda_i \widetilde{a}_i t^{f_i}.$$

Suppose that $r_1 = r$. Then $f_d = r$ and

$$\zeta(\overline{z}_{\overline{m}}(t_2) + 1) = \sum_{i=1}^{d} \lambda_i \widetilde{a}_i (\overline{z}_{\overline{m}}(t_2) + 1)^{f_i} = \lambda_d \widetilde{a}_d \overline{z}_{\overline{m}}(t_2)^{f_d} \tag{57}$$

Thus $\zeta(t) = \lambda_d \widetilde{a}_d (t-1)^r$ has a nonzero t^{r-1} term, so that $f_{d-1} = r-1$ and $\widetilde{a}_{d-1} \neq 0$. Therefore $a_d = \overline{a}_d$ and $\nu(\overline{z}_{\overline{m}}) = \nu(a_{d-1})$. Define $\tau(0)$ by $\nu(\overline{z}_{\overline{m}}) = \nu(a_{d-1}) = d_{\tau(0)}$ in (56). Then $a_{d-1} = h + \phi$ with $h \in p_{\overline{m}-1} \cap k(c_0)[[\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1}]]$ and

$$\phi \in m(k(c_0)[[\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1}]])^{\gamma(\tau(0))}.$$

Let $\alpha = \widetilde{a}_d \in k(c_0)$ be the constant term of the power series $a_d \in k(c_0)[[\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1}]]$. Expanding out the LHS of (57), we have

$$\lambda_d r \widetilde{a}_d + \lambda_{d-1} \widetilde{a}_{d-1} = 0.$$

$$\begin{aligned}
\frac{\overline{z}_{\overline{m}}}{\phi} &= \frac{\overline{z}_{\overline{m}}^r}{(a_{d-1} - h)\overline{z}_{\overline{m}}^{r-1}} \\
&= \frac{(\overline{z}_{\overline{m}}(t_2) + 1)^r \lambda_d}{\left(\lambda_{d-1}\overline{a}_{d-1} - \dfrac{h c_{t_2}^{a_{s+1,s+1}(t_2)(r-1)}}{\overline{z}_1(t_2)^{a_{s+1,1}(t_2)} \cdots \overline{z}_s(t_2)^{a_{s+1,s}(t_2)}} \right) (\overline{z}_{\overline{m}}(t_2) + 1)^{r-1}} \\
&= \frac{(\overline{z}_{\overline{m}}(t_2) + 1)\lambda_d}{\lambda_{d-1}\overline{a}_{d-1} - \dfrac{h c_{t_2}^{a_{s+1,s+1}(t_2)(r-1)}}{\overline{z}_1(t_2)^{a_{s+1,1}(t_2)} \cdots \overline{z}_s(t_2)^{a_{s+1,s}(t_2)}}} \\
&= \frac{\dfrac{\overline{z}_{\overline{m}}(t_2)\lambda_d}{\lambda_{d-1}\overline{a}_{d-1}} + \dfrac{\lambda_d}{\lambda_{d-1}\overline{a}_{d-1}}}{1 - \dfrac{h c_{t_2}^{a_{s+1,s+1}(t_2)(r-1)}}{\lambda_{d-1}\overline{a}_{d-1}\overline{z}_1(t_2)^{a_{s+1,1}(t_2)} \cdots \overline{z}_s(t_2)^{a_{s+1,s}(t_2)}}}
\end{aligned}$$

has residue $-1/r\alpha$ in $\mathcal{O}_\nu/m_\nu$. (Recall that $\nu(h) = \infty$). Thus $\nu(\overline{z}_{\overline{m}} + \frac{1}{r\alpha}\phi) > \nu(\overline{z}_{\overline{m}})$. Since $1/r\alpha \in k(c_0)$, $\frac{1}{r\alpha}\phi \in m(k(c_0)[[\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1}]])^{\gamma(\tau(0))}$ and there exists

$$A_1 \in m(k(c_0)[\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1}])^{\gamma(\tau(0))}$$

such that $\nu(\overline{z}_{\overline{m}} - A_1) > \nu(\overline{z}_{\overline{m}})$. Set $\overline{z}_{\overline{m}}^{(1)} = \overline{z}_{\overline{m}} - A_1$.

Repeat the above algorithm, with $\overline{z}_{\overline{m}}$ replaced by $\overline{z}_{\overline{m}}^{(1)}$. If we do not achieve a reduction $r_1 < r$, we can make an infinite sequence of change of variables

$$\overline{z}_{\overline{m}}^{(i)} = \overline{z}_{\overline{m}}^{(i-1)} - A_i$$

such that $A_i \in k(c_0)[\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1}]$, $\nu(A_i) = \nu(\overline{z}_{\overline{m}}^{(i-1)})$, $\nu(A_i) = d_{\tau(i-1)}$,

$$A_i \in m(k(c_0)[\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1}])^{\gamma(\tau(i-1))}$$

and

$$\tau(0) < \tau(1) < \cdots < \tau(i) < \cdots$$

Then

$$\overline{z}_{\overline{m}}^{(i)} - \overline{z}_{\overline{m}}^{(j)} \in m(k(c_0)[\overline{z}_1, \dots, \overline{z}_{\overline{m}-1}])^{\min\{\gamma(\tau(i-1)),\gamma(\tau(j-1))\}}.$$

Thus $\{\overline{z}_{\overline{m}}^{(i)}\}$ is a Cauchy sequence, and there exists a series

$$\overline{A}(\overline{z}_1, \dots, \overline{z}_{\overline{m}-1}) \in k(c_0)[[\overline{z}_1, \dots, \overline{z}_{\overline{m}-1}]]$$

such that

$$\overline{z}_{\overline{m}}^{\infty} = \lim_{i\to\infty} \overline{z}_{\overline{m}}^{(i)} = \overline{z}_{\overline{m}} - \overline{A}$$

and $\nu(\overline{z}_{\overline{m}}^{\infty}) = \infty$. Thus $\overline{z}_{\overline{m}}^{\infty} \in p_{\overline{m}}$.

Set $\lambda(\alpha) = \gamma(\tau(\alpha))$. For all α there are series a_i, a_j, a_k and exponents f_i, f_j, f_k such that we can write

$$f = [a_1(\overline{z}_{\overline{m}}^{(\alpha)})^{f_1} + \cdots + a_r(\overline{z}_{\overline{m}}^{(\alpha)})^r] + \Sigma a_j(\overline{z}_{\overline{m}}^{(\alpha)})^{f_j} + \Sigma a_k(\overline{z}_{\overline{m}}^{(\alpha)})^{f_k} + (\overline{z}_{\overline{m}}^{(\alpha)})^{r+1}\Omega$$

where the terms in the first sum satisfy

$$\nu(a_i(\overline{z}_{\overline{m}}^{(\alpha)})^{f_i}) = r\nu(\overline{z}_{\overline{m}}^{(\alpha)}) = rd_{\tau(\alpha)},$$

a_r is a unit, the terms in the second sum satisfy $\nu(a_j(\overline{z}_{\overline{m}}^{(\alpha)})^{f_j}) > rd_{\tau(\alpha)}$, and the terms in the third sum satisfy $\nu(a_k) = \infty$. Set $m = m(k(c_0)[[\overline{z}_1, \dots, \overline{z}_{\overline{m}-1}]])$. Since for terms in the first sum, $\nu(a_i) \geq d_{\tau(\alpha)}$ implies $a_i \in p_{\overline{m}-1} + m^{\lambda(\alpha)}$, we have

$$f \equiv \widetilde{a}_r(\overline{z}_{\overline{m}}^{(\alpha)})^r \bmod (m^{\lambda(\alpha)}\overline{T} + p_{\overline{m}-1}\overline{T} + (\overline{z}_{\overline{m}}^{(\alpha)})^{r+1})$$

where $\widetilde{a}_r$ is a unit in $\overline{T}$, so that

$$f \in (\overline{z}_{\overline{m}}^{(\alpha)})^r + m^{\lambda(\alpha)}\overline{T} + p_{\overline{m}-1}\overline{T} = (\overline{z}_{\overline{m}}^{(\infty)})^r + m^{\lambda(\alpha)}\overline{T} + p_{\overline{m}-1}\overline{T}.$$

Thus

$$f \in \cap_{\alpha=1}^{\infty}((\overline{z}_{\overline{m}}^{(\infty)})^r + m^{\lambda(\alpha)}\overline{T} + p_{\overline{m}-1}\overline{T}) = (\overline{z}_{\overline{m}}^{(\infty)})^r + p_{\overline{m}-1}\overline{T}.$$

Since the $\widetilde{a}_r$ are units, we have

$$f = u(\overline{z}_m - \overline{A}(\overline{z}_1, \dots, \overline{z}_{\overline{m}-1}))^r + h \tag{58}$$

where u is a unit power series, $h \in p_{\overline{m}-1}\overline{T}$.

Suppose that we reach a reduction $r_1 < r$ after a finite number of iterations. We can repeat the whole algorithm with f replaced with f_1, r replaced with r_1, c_0 with a primitive element of $k(c_0, \dots, c_{t_2})$ over k, $\overline{T}''$ with $\overline{T}''(t_2)$. (Recall that $k(c_0, \dots, c_{t_2}) \subset \overline{T}''(t_2)$) We have $\nu(f_1) = \infty$, so that the algorithm cannot terminate with $r = 0$, and we must produce $\overline{z}_{\overline{m}}^{\infty}(t)$ such that

$$\overline{z}_{\overline{m}}^{\infty}(t) = \overline{z}_{\overline{m}}(t) - \overline{A}(\overline{z}_1(t), \dots, \overline{z}_{\overline{m}-1}(t)),$$

with $\overline{A} \in k(c_0, \dots, c_t)[[\overline{z}_1(t), \dots, \overline{z}_{\overline{m}-1}(t)]]$ and $\nu(\overline{z}_{\overline{m}}^{\infty}(t)) = \infty$. In particular, the algorithm produces $\overline{z}_{\overline{m}}(t) - Q_{\overline{m}}(\overline{z}_1(t), \dots, \overline{z}_{\overline{m}-1}(t))$ of infinite value.

By (53) for $m = \overline{m} - 1$, we can now construct a further UTS in the first $\overline{m} - 1$ variables along ν, so that

$$p_{\overline{m}-1}(t) = \left(\overline{z}_{r(1)}(t) - Q_{r(1)}(\overline{z}_1(t), \ldots, \overline{z}_{r(1)-1}(t)), \ldots, \overline{z}_{r(\widetilde{m-1})}(t) - Q_{r(\widetilde{m-1})}(\overline{z}_1(t), \ldots, \overline{z}_{r(\widetilde{m-1})-1})(t)\right).$$

Now suppose $g \in p_{\overline{m}}(t)$ and $\nu(g) = \infty$. Then there exists

$$g_0 \in k(\overline{T})[[\overline{z}_1(t), \ldots, \overline{z}_{\overline{m}-1}(t)]] \quad \text{and} \quad g_1 \in \overline{T}$$

such that $g = g_0 + (\overline{z}_{\overline{m}}(t) - Q_{\overline{m}})g_1$ and $\nu(g_0) = \infty$. Thus $g_0 \in p_{\overline{m}-1}(t)$, showing that

$$p_{\overline{m}} = (\overline{z}_{r(1)}(t) - Q_{r(1)}(\overline{z}_1(t), \ldots, \overline{z}_{r(1)-1}), \ldots, \overline{z}_{\overline{m}} - Q_{\overline{m}}(\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1})).$$

Proof of $B(\overline{m})$ ($s < \overline{m}$)

Case 1: $\nu(h) < \infty$. — There exists a UTS $\overline{T} \to \overline{T}(t)$ satisfying (53) for $m = \overline{m}$. After replacing $\overline{T}''(0)$ with $\overline{T}''(t)$ and replacing c_0 with a primitive element of $k(c_0, \ldots, c_t)$ over k, we may assume that

$$p_{\overline{m}} = (\overline{z}_{r(1)} - Q_{r(1)}(\overline{z}_1, \ldots, \overline{z}_{r(1)-1}), \ldots, \overline{z}_{r(\tilde{m})} - Q_{r(\tilde{m})}(\overline{z}_1, \ldots, \overline{z}_{r(\tilde{m})-1})).$$

where the $Q_{r(i)}$ are power series with coefficients in $k(c_0)$. Let L be a Galois closure of $k(\overline{T})$ over $k(c_0)$ and G be the Galois group of L over $k(c_0)$. Set

$$g = \prod_{\sigma \in G} \sigma(h) \in k(c_0)[[\overline{z}_1, \ldots, \overline{z}_m]].$$

$g \in k(c_0)[[\overline{z}_1, \ldots, \overline{z}_m]]$. Suppose $\nu(g) = \infty$. Then $g \in p_{\overline{m}}L[[\overline{z}_1, \ldots, \overline{z}_{\overline{m}}]]$ which is a prime ideal, invariant under G. Thus

$$h \in (p_{\overline{m}}L[[\overline{z}_1, \ldots, \overline{z}_{\overline{m}}]]) \cap k(\overline{T})[[\overline{z}_1, \ldots, \overline{z}_{\overline{m}}]] = p_{\overline{m}}$$

which implies $\nu(h) = \infty$, a contradiction. Thus $\nu(g) \neq \infty$. We will construct a UTS so that

$$g = u\overline{z}_1(t)^{e_1} \cdots \overline{z}_s(t)^{e_s}$$

where u is a unit power series in $k(c_0, \ldots, c_t)[[\overline{z}_1(t), \ldots, \overline{z}_{\overline{m}}(t)]]$ and h belongs to $k(\overline{T})[[\overline{z}_1(t), \ldots, \overline{z}_{\overline{m}}(t)]]$. Since $h \mid g$ in $k(\overline{T})[[\overline{z}_1(t), \ldots, \overline{z}_{\overline{m}}(t)]]$, we will then have h in the desired form in $\overline{T}(t)$.

Set $g = \overline{z}_1^{d_1} \cdots \overline{z}_s^{d_s} g_0$ where $\overline{z}_i$ does not divide g_0 for $1 \leq i \leq s$. Set

$$r = \operatorname{mult} g_0(0, \ldots, 0, \overline{z}_{\overline{m}}).$$

$0 \leq r \leq \infty$. We will also have an induction on r. If $r = 0$ we are done, so suppose that $r > 0$.

$$g_0 = \sum_{i=1}^{d} \sigma_i(\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1})\overline{z}_{\overline{m}}^{a_i} + \sum_j \sigma_j(\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1})\overline{z}_{\overline{m}}^{a_j} + \sum_k \sigma_k(\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1})\overline{z}_{\overline{m}}^{a_k} + \overline{z}_{\overline{m}}^{a}\Psi$$

where the coefficients of σ_i, σ_j, σ_k and Ψ are in $k(c_0)$, Ψ is a power series in $\overline{z}_1,\dots,\overline{z}_{\overline{m}}$, the first sum is over terms of minimum value ρ, a satisfies $a\nu(\overline{z}_{\overline{m}}) > \rho$, and the (finitely many) remaining terms of finite value are in the second sum, the (finitely many) remaining terms of infinite value are in the third sum.

By (54), (55) for $m < \overline{m}$ there is a UTS $\overline{T} \to \overline{T}(\alpha)$ in the first $\overline{m}-1$ variables along ν such that

$$\sigma_i = \overline{z}_1(\alpha)^{c_1(i)}\cdots\overline{z}_s(\alpha)^{c_s(i)}\overline{u}_i$$

for all i,

$$\sigma_j = \overline{z}_1(\alpha)^{c_1(j)}\cdots\overline{z}_s(\alpha)^{c_s(j)}\overline{u}_j$$

for all j and

$$\sigma_k = \overline{z}_1(\alpha)^{c_1(k)}\cdots\overline{z}_s(\alpha)^{c_s(k)}\overline{u}_k$$

for all k where $\overline{u}_i,\overline{u}_j,\overline{u}_k \in k(c_0,\dots,c_\alpha)[[\overline{z}_1(\alpha),\dots,\overline{z}_{\overline{m}-1}(\alpha)]]$, $\overline{u}_i,\overline{u}_j$ are units and

$$\nu(\overline{z}_1(\alpha)^{c_1(k)}\cdots\overline{z}_s(\alpha)^{c_s(k)}) > \rho.$$

Now perform a UTS of type $\mathrm{II}_{\overline{m}}$ $\overline{T}(\alpha) \to \overline{T}(\alpha+1)$ along ν to get

$$\begin{aligned} g_0 =& \overline{z}_1(\alpha+1)^{e_1}\cdots\overline{z}_s(\alpha+1)^{e_s}\Big(\sum_i \lambda_i\overline{u}_i(\overline{z}_{\overline{m}}(\alpha+1)+1)^{a_i}\Big) \\ &+\sum_j \overline{z}_1(\alpha+1)^{e_1^j}\cdots\overline{z}_s(\alpha+1)^{e_s^j}\lambda_j\overline{u}_j(\overline{z}_{\overline{m}}(\alpha+1)+1)^{a_j} \\ &+\sum_k \overline{z}_1(\alpha+1)^{e_1^k}\cdots\overline{z}_s(\alpha+1)^{e_s^k}\lambda_k\overline{u}_k(\overline{z}_{\overline{m}}(\alpha+1)+1)^{a_k} \\ &+(\overline{z}_1(\alpha+1)^{a_{s+1,1}}\cdots\overline{z}_s(\alpha+1)^{a_{s+1,s}})^a\Psi'. \end{aligned}$$

where

$$\lambda_i = c_{\alpha+1}^{a_{1,s+1}(\alpha+1)c_1(i)+\cdots+a_{s,s+1}(\alpha+1)c_s(i)+a_{s+1,s+1}(\alpha+1)a_i}.$$

Then perform a UTS of type I $\overline{T}(\alpha+1) \to \overline{T}(\alpha+2)$ along ν to get

$$g_0 = \overline{z}_1(\alpha+2)^{d_1(\alpha+2)}\cdots\overline{z}_s(\alpha+2)^{d_s(\alpha+2)}g_1$$

where

$$g_1 = \sum_{i=1}^{d}\lambda_i\overline{u}_i(\overline{z}_{\overline{m}}(\alpha+2)+1)^{a_i} + \overline{z}_1(\alpha+2)\cdots\overline{z}_s(\alpha+2)\Psi_1$$

is a power series with coefficients in $k(c_0,\dots,c_{\alpha+1})$, Ψ_1 a power series in $\overline{z}_1(\alpha+2),\dots,\overline{z}_{\overline{m}}(\alpha+2)$. Set $r_1 = \mathrm{mult}\, g_1(0,\dots,0,\overline{z}_{\overline{m}}(\alpha+2))$. $r_1 < \infty$ and $r_1 \le r$.

Suppose that $r_1 = r$. Then as in (57) in the proof of $A(\overline{m})$, $\overline{z}_{\overline{m}}^r$ is a minimal value term in g_0, so that $a_d = r$, $a_{d-1} = r-1$, $\sigma_{d-1} \neq 0$, and $\nu(\sigma_{d-1}) = \nu(\overline{z}_{\overline{m}})$.

As in the proof of $A(\overline{m})$, there exists $A_1 \in k(c_0)[\overline{z}_1,\dots,\overline{z}_{\overline{m}-1}]$ such that we can make a change of variable, replacing $\overline{z}_{\overline{m}}$ with $\overline{z}'_{\overline{m}} = \overline{z}_{\overline{m}} - A_1$ to get $\nu(\overline{z}_{\overline{m}} - A_1) > \nu(\overline{z}_{\overline{m}})$. We have

$$\nu(\overline{z}_{\overline{m}}) \le \nu(\overline{z}_{\overline{m}}^r) \le \nu(g_0)$$

since $\overline{z}_{\overline{m}}^{r}$ is a minimal value term in g_0. Now repeat this procedure. If we do not achieve $r_1 < r$ after a finite number of iterations, we get an infinite sequence

$$\nu(\overline{z}_{\overline{m}}) < \nu(\overline{z}'_{\overline{m}}) < \cdots < \nu(\overline{z}^{(i)}_{\overline{m}}) < \cdots$$

such that $\nu(\overline{z}^{(i)}_{\overline{m}}) \leq \nu(g_0)$ for all i. By Lemma 2.3, this is impossible.

Thus after replacing $\overline{z}_{\overline{m}}$ with

$$\widetilde{\overline{z}}'_{\overline{m}} = \overline{z}_{\overline{m}} - P_{\overline{m},0}(\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1})$$

for some appropriate polynomial $P_{\overline{m},0} \in k(c_0)[\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1}]$, we achieve a reduction $r_1 < r$ in $\overline{T}''(\alpha+2)$. By induction on r, we can construct a UTS $\overline{T} \to \overline{T}(t)$ along ν such that

$$g = \overline{z}_1(t)^{d_1} \cdots \overline{z}_s(t)^{d_s} \overline{u}(\overline{z}_1(t), \ldots, \overline{z}_{\overline{m}}(t))$$

where $\overline{u}$ is a unit power series with coefficients in $k(c_0, \ldots, c_t)$.

By (53) for $m = \overline{m}$ we can perform a further UTS to get

$$p_{\overline{m}}(t) = (\overline{z}_{r(1)}(t) - Q_{r(1)}(\overline{z}_1(t), \ldots, \overline{z}_{r(1)-1}), \ldots, \overline{z}_{r(\tilde{m})}(t) - Q_{r(\tilde{m})}(\overline{z}_1(t), \ldots, \overline{z}_{r(\tilde{m})-1}))$$

while preserving

$$g = \overline{z}_1(t)^{d_1} \cdots \overline{z}_s(t)^{d_s} \overline{u}(\overline{z}_1(t), \ldots, \overline{z}_{\overline{m}}(t))$$

where $\overline{u}$ is a unit.

Case 2: $\nu(h) = \infty$. — By (53) for $m = \overline{m}$, we can assume that

$$p_{\overline{m}} = (\overline{z}_{r(1)}(t) - Q_{r(1)}(\overline{z}_1(t), \ldots, \overline{z}_{r(1)-1}), \ldots, \overline{z}_{r(\tilde{m})} - Q_{r(\tilde{m})}(\overline{z}_1, \ldots, \overline{z}_{r(\tilde{m})-1})).$$

where the $Q_{r(i)}$ are series with coefficients in $k(c_0)$. Then

$$h = \sum_{i=1}^{\tilde{m}} \sigma_i(\overline{z}_{r(i)} - Q_{r(i)})$$

for some $\sigma_i \in k(\overline{T})[[\overline{z}_1, \ldots, \overline{z}_m]]$. Choose b so that $b\nu(m(\overline{T})) > A$. There are polynomials $P_{r(i)}(\overline{z}_1, \ldots, z_{r(i)-1})$ in $k(c_0)[\overline{z}_1, \ldots, \overline{z}_{r(i)-1}]$ such that $Q_{r(i)} - P_{r(i)} \in m(\overline{T})^b$ and

$$\nu(\overline{z}_{r(i)} - P_{r(i)}) > A.$$

Make a change of variables replacing $\overline{z}_{r(i)}$ with $\overline{z}_{r(i)} - P_{r(i)}$ for $1 \leq i \leq \tilde{m}$. Then construct the UTS $\overline{T} \to \overline{T}(t_1)$ which is a sequence of UTSs of type II_r for $s+1 \leq r \leq \overline{m}$, followed by a MTS of type I to get

$$h = \overline{z}_1(t_1)^{d_1} \cdots \overline{z}_s(t_1)^{d_s} \Sigma$$

with $\nu(\overline{z}_1(t_1)^{d_1} \cdots \overline{z}_s(t_s)^{d_s}) > A$. By (53) for $m = \overline{m}$, we can perform a UTS along ν in the first $\overline{m}$ variables to get

$$p_{\overline{m}}(t) = \big(\overline{z}_{r(1)}(t) - Q_{r(1)}(\overline{z}_1(t), \ldots, \overline{z}_{r(1)-1}(t)), \ldots, \overline{z}_{r(\tilde{m})}(t) \\ - Q_{r(\tilde{m})}(\overline{z}_1(t), \ldots, \overline{z}_{r(\tilde{m})-1}(t))\big)$$

while preserving

$$h = \overline{z}_1(t)^{d_1} \cdots \overline{z}_s(t)^{d_s}\Sigma$$

with $\nu(\overline{z}_1(t)^{d_1} \cdots \overline{z}_s(t)^{d_s}) > A$.

Proof when (42) or (43) holds

Case 1. — Suppose that $s \leq m \leq l$. After performing a CUTS in the first m variables, we may assume that

$$p_m = (\overline{z}_{r(1)} - Q_{r(1)}(\overline{z}_1, \ldots, \overline{z}_{r(1)-1}), \ldots, \overline{z}_{r(\tilde{m})} - Q_{r(\tilde{m})}(\overline{z}_1, \ldots, \overline{z}_{r(\tilde{m})-1})).$$

where the coefficients of $Q_{r(i)}$ are in $k(c_0)$. $f \in k(\overline{U})[[\overline{w}_1, \ldots, \overline{w}_m]]$

First suppose that $\nu(f) < \infty$. Let $\overline{T}_i = k(c_0)[[\overline{z}_1, \ldots, \overline{z}_i]]$, $\overline{U}_i = k(\overline{U})[[\overline{w}_1, \ldots, \overline{w}_i]]$ for $s \leq i \leq m$. Let $d = \det(c_{ij})$, (d_{ij}) be the adjoint matrix of (c_{ij}). Then

$$\begin{aligned}
\overline{w}_1 &= \overline{z}_1^{d_{11}/d} \cdots \overline{z}_s^{d_{1s}/d}\lambda_1 \\
&\vdots \\
\overline{w}_s &= \overline{z}_1^{d_{s1}/d} \cdots \overline{z}_s^{d_{ss}/d}\lambda_s \\
\overline{w}_{s+1} &= \overline{z}_{s+1} \\
&\vdots \\
\overline{w}_m &= \overline{z}_m
\end{aligned}$$

where

$$\lambda_i = \phi_1^{-d_{i1}/d} \cdots \phi_s^{-d_{is}/d} \text{ for } 1 \leq i \leq s$$

Given a CUTS (45), set $\sigma(i)$ to be the largest possible α such that after possibly permuting the parameters $\overline{z}_{s+1}(i), \ldots, \overline{z}_m(i)$, ν induces a rank 1 valuation on the quotient field of $k(\overline{T}(i))[[\overline{z}_1(i), \ldots, \overline{z}_\alpha(i)]]$. (Since $\nu(\overline{z}_1(i)), \ldots, \nu(\overline{z}_s(i)))$ are rationally independent, $\sigma(i) \geq s$.)

If $\sigma(i)$ drops during the course of the proof, we can start the corresponding algorithm again with this smaller value of $\sigma(i)$. Eventually $\sigma(i)$ must stabilize, so we may assume that $\sigma(i)$ is constant throughout the proof.

$\nu(\overline{z}_1^{d_{i1}} \cdots \overline{z}_s^{d_{is}}) > 0$ for $1 \leq i \leq s$. By Lemma 4.6, there exists a CUTS of type (M1) $\overline{T} \rightarrow \overline{T}(1)$, $\overline{U} \rightarrow \overline{U}(1)$ such that $\overline{z}_1^{d_{i1}} \cdots \overline{z}_s^{d_{is}} \in \overline{T}_m(1) = k(\overline{T})[[\overline{z}_1(1), \ldots, \overline{z}_m(1)]]$ for all i. Since $p_m\overline{T}_m(1)$ is a prime and we may assume that $\sigma(1) = \sigma(0)$, we have $p_m\overline{T}_m(1) = p_m(1)$.

Let ω be a primitive dth root of unity and L be a Galois closure of $k(\overline{U})(\omega, \lambda_1, \ldots, \lambda_s)$ over $k(c_0)$ with Galois group G. Set

$$\overline{W} = L[[\overline{z}_1(1)^{1/d}, \ldots, \overline{z}_s(1)^{1/d}, \overline{z}_{s+1}(1), \ldots, \overline{z}_m(1)]].$$

Given $i_1, \ldots, i_s \in \mathbf{N}$, Define a k-automorphism $\sigma_{i_1 \cdots i_s} : \overline{W} \rightarrow \overline{W}$ by

$$\sigma_{i_1 \cdots i_s}(\overline{z}_j(1)^{1/d}) = \omega^{i_j}\overline{z}_j(1)^{1/d} \quad \text{for } 1 \leq j \leq s.$$

Our extension of ν to the quotient field F of $\overline{U}(1)$ extends to a valuation of the finite field extension generated by L and $F(\overline{z}_1(1)^{1/d}, \ldots, \overline{z}_s(1)^{1/d})$, which induces valuations on the quotient fields of $\overline{T}_m$, $\overline{U}_m$ and $\overline{W}$ which are compatible with the inclusions $\overline{T}_m \subset \overline{U}_m \subset \overline{W}$. $\overline{T}_m(1) \to \overline{W}$ is finite, $p_m\overline{W}$ is prime implies

$$p_m\overline{W}_m = \{h \in \overline{W}_m \mid \nu(h) = \infty\}.$$

Thus

$$p_m\overline{U}_m = \{h \in \overline{U}_m \mid \nu(h) = \infty\}.$$

Set

$$\overline{g} = \prod \sigma_{i_1\cdots i_s}(f), g = \prod_{\tau \in G} \tau(\overline{g}) \in k(c_0)[[\overline{z}_1(1), \ldots, \overline{z}_m(1)]] \subset \overline{T}_m(1)$$

Suppose $\nu(g) = \infty$. Then $g \in p_m\overline{T}_m(1)$ implies $g \in p_m\overline{W}$ which implies $\tau\sigma_{i_1\cdots i_s}(f) \in p_m\overline{W}$ for some $\tau, \sigma_{i_1\cdots i_s}$ since $p_m\overline{W}$ is prime. But $\tau\sigma_{i_1\cdots i_s}(p_m\overline{W}) = p_m\overline{W}$ implies $f \in p_m\overline{W} \cap \overline{U}_m = p_m\overline{U}_m$ so that $\nu(f) = \infty$. This is a contradiction. Thus $\nu(g) < \infty$.

By (54) (and Lemmas 4.3 and 4.4) we can construct a CUTS $(R, \overline{T}''(t), \overline{T}(t))$ and $(S, \overline{U}''(t), \overline{U}(t))$ in the first m variables to transform g into the form

$$g = \overline{z}_1(t)^{\tilde{d}_1} \cdots \overline{z}_s(t)^{\tilde{d}_s}\overline{u} \tag{59}$$

in $\overline{T}(t)$ where $\overline{u}(\overline{z}_1(t), \ldots, \overline{z}_m(t))$ is a unit power series with coefficients in $k(c_0, \ldots, c_t)$. Then $f \mid g$ in $\overline{U}(t)$ implies

$$f = \overline{w}_1(t)^{d_1} \cdots \overline{w}_s(t)^{d_s} u$$

where u is a unit in $\overline{U}(t)$. But f is a series in $\overline{w}_1(t), \ldots, \overline{w}_m(t)$ with coefficients in $k(\overline{U}(t))$. Thus

$$f = \overline{w}_1(t)^{d_1} \cdots \overline{w}_s(t)^{d_s} u(\overline{w}_1(t), \ldots, \overline{w}_m(t)),$$

where the coefficients of u are in $k(\overline{U}(t))$.

Now suppose that $\nu(f) = \infty$. $p_m\overline{U}_m$ is the set of elements of $\overline{U}_m$ of infinite value. Otherwise, as argued above, we can perform a UTS $\overline{T} \to \overline{T}(1)$ to get $\sigma(1) < \sigma(0)$. Thus it suffices by (53) to prove the theorem when $f = \overline{z}_{r(i)} - Q_{r(i)}$ is a generator of p_m. This follows from (55).

Case 2. — Suppose that $m > l$. The proof is by induction on $m - l$, assuming that it is true for smaller differences $m - l$.

First suppose that $\nu(f) < \infty$. Set

$$f = \overline{w}_1^{d_1} \cdots \overline{w}_s^{d_s} f_0$$

where $\overline{w}_i$ does not divide f_0 for $1 \le i \le s$. Set $r = \operatorname{mult} f_0(0,\ldots,0,\overline{w}_m)$. $0 \le r \le \infty$. We will also have an induction on r. If $r = 0$ we are done, so suppose that $r > 0$.

$$f_0 = \sum_i \sigma_i(\overline{w}_1,\ldots,\overline{w}_{m-1})\overline{w}_m^{a_i} + \sum_j \sigma_j(\overline{w}_1,\ldots,\overline{w}_{m-1})\overline{w}_m^{a_j} + \sum_k \sigma_k(\overline{w}_1,\ldots,\overline{w}_{m-1})\overline{w}_m^{a_k} + \overline{w}_m^{a}\Psi$$

where $\sigma_i, \sigma_j, \sigma_k$ are power series with coefficients in $k(\overline{U})$, Ψ is a power series in $\overline{w}_1,\ldots,\overline{w}_m$ with coefficients in $k(\overline{U})$, the first sum is over terms of minimum value ρ, a satisfies $a\nu(\overline{w}_m) > \rho$, the (finitely many) terms in the second sum have finite value and the (finitely many) terms in the second sum have infinite value.

By induction there is a CUTS $(R,\overline{T}''(\alpha),\overline{T}(\alpha))$ and $(S,\overline{U}''(\alpha),\overline{U}(\alpha))$ in the first $m-1$ variables such that

$$\sigma_i = \overline{w}_1(\alpha)^{c_1(i)}\cdots\overline{w}_s(\alpha)^{c_s(i)}\overline{u}_i$$

for all i,

$$\sigma_j = \overline{w}_1(\alpha)^{c_1(j)}\cdots\overline{w}_s(\alpha)^{c_s(j)}\overline{u}_j$$

for all j and

$$\sigma_k = \overline{w}_1(\alpha)^{c_1(k)}\cdots\overline{w}_s(\alpha)^{c_s(k)}\overline{u}_k$$

for all k where $\overline{u}_i, \overline{u}_j, \overline{u}_k \in k(\overline{U}(\alpha))[[\overline{w}_1(\alpha),\ldots,\overline{w}_{m-1}(\alpha)]]$, $\overline{u}_i, \overline{u}_j$ are units for all i, j and

$$\nu(\overline{w}_1(\alpha)^{c_1(k)}\cdots\overline{w}_s(\alpha)^{c_s(k)}) > \rho$$

for all k. Now perform a CUTS of type (M3) where $\overline{U}(\alpha) \to \overline{U}(\alpha+1)$ is of type II_m to get

$$\begin{aligned}
f_0 &= \overline{w}_1(\alpha+1)^{e_1}\cdots\overline{w}_s(\alpha+1)^{e_s}\Big(\sum_i \lambda_i\overline{u}_i(\overline{w}_m(\alpha+1)+1)^{a_i}\Big)\\
&\quad + \sum_j \overline{w}_1(\alpha_1)^{e_1^j}\cdots\overline{w}_s(\alpha+1)^{e_s^j}\lambda_j\overline{u}_j(\overline{w}_m(\alpha+1)+1)^{a_j}\\
&\quad + \sum_k \overline{w}_1(\alpha+1)^{e_1^k}\cdots\overline{w}_s(\alpha+1)^{e_s^k}\lambda_k\overline{u}_k(\overline{w}_m(\alpha+1)+1)^{a_k}\\
&\quad + \Big(\overline{w}_1(\alpha+1)^{b_{s+1,1}(\alpha+1)}\cdots\overline{w}_s(\alpha+1)^{b_{s+1,s}(\alpha+1)}\Big)^a\Psi'\\
\lambda_i &= d_{\alpha+1}^{c_1(i)b_{1,s+1}(\alpha+1)+\cdots+c_s(i)b_{s,s+1}(\alpha+1)+b_{s+1,s+1}(\alpha+1)a_i}.
\end{aligned}$$

Now perform a CUTS of type (M1) $\overline{T}(\alpha+1) \to \overline{T}(\alpha+2)$, $\overline{U}(\alpha+1) \to \overline{U}(\alpha+2)$ to get

$$f_0 = \overline{w}_1(\alpha+2)^{d_1(\alpha+2)}\cdots\overline{w}_s(\alpha+2)^{d_s(\alpha+2)}f_1$$

where

$$f_1 = \sum_{i=1}^{d}\lambda_i\overline{u}_i(\overline{w}_m(\alpha+2)+1)^{a_i} + \overline{w}_1(\alpha+2)\cdots\overline{w}_s(\alpha+2)\Psi_1,$$

Ψ_1 a power series in $\overline{w}_1(\alpha+2),\ldots,\overline{w}_m(\alpha+2)$ with coefficients in $k(\overline{U}(\alpha+2))$. Set $r_1=\operatorname{mult} f_1(0,\ldots,0,\overline{w}_m(\alpha+2))$. $r_1<\infty$ and $r_1\le r$.

As in the proof of Case 1 of $B(\overline{m})$, there is a polynomial $P_{m,0}\in k(\overline{U})[\overline{w}_1,\ldots,\overline{w}_{m-1}]$ such that if we replace $\overline{w}_m$ with

$$\widetilde{\overline{w}}'_m=\overline{w}_m-P_{m,0}$$

we get a reduction $r_1<r$ in $\overline{U}(\alpha+2)$. By induction, we can construct a CUTS as desired.

Suppose that $\nu(f)=\infty$. Given a CUTS (45) and i such that $s\le i\le n$, set

$$a_i(t)=\{h\in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_i(t)]]\mid \nu(h)=\infty\}.$$

$$f=\sum_{i=0}^{\infty}\sigma_i(\overline{w}_1,\ldots,\overline{w}_{m-1})\overline{w}_m^i.$$

If $\nu(\sigma_i)=\infty$ for all i, we can put f in the desired form by induction on m applied to a finite set of generators of the ideal generated by the σ_i.

Suppose some $\nu(\sigma_i)<\infty$ for some i. As in the proof of $A(\overline{m})$, we can perform a UTS in the first m variables to get

$$f=\overline{w}_1(t_1)^{d_1}\cdots\overline{w}_s(t_1)^{d_s}f_1$$

such that as in (58), there is a series $\overline{A}(\overline{w}_1(t_1),\ldots,\overline{w}_{m-1}(t_1))$ with coefficients in $k(\overline{U}(t_1))$ such that $\nu(\overline{w}_m(t_1)-\overline{A})=\infty$ and

$$f_1=u(\overline{w}_m(t_1)-\overline{A})^r+h$$

where $u\in k(\overline{U}(t_1))[[\overline{w}_1(t_1),\ldots,\overline{w}_m(t_1)]]$ is a unit power series, $h\in a_{m-1}(t_1)$ and $r>0$. By induction on m, we are reduced to the case

$$f=\overline{w}_m-\overline{A}(\overline{w}_1,\ldots,\overline{w}_{m-1}).$$

We can then put f in the desired form using the argument of Case 2 of the proof of $B(\overline{m})$.

Proof when (44) holds. — Suppose that f is as in (44) of the statement of the theorem.

$$f=\sum_{i=0}^{\infty}b_i(\overline{w}_1,\ldots,\overline{w}_{m-1})\overline{w}_m^i.$$

Set $Q=\sum_{i=1}^{\infty}b_i\overline{w}_m^i$. After reindexing the $\overline{w}_i$, $l+1\le i\le m$, we may assume that $Q\ne 0$. $Q=\overline{w}_1^{n_1(0)}\cdots\overline{w}_s^{n_s(0)}Q_0$ where $\overline{w}_i$ does not divide Q_0 for $1\le i\le s$. Set $r=\operatorname{mult} Q_0(0,\ldots,0,\overline{w}_m)$. $1\le r\le\infty$. The proof will be by induction on r. Suppose that $r>1$.

$\nu(\partial f/\partial\overline{w}_m) < \infty$ since $\partial f/\partial\overline{w}_m \in \overline{U}''$ and ν restricts to a rank 1 valuation of the quotient field of $\overline{U}''$. Thus there must be some $i > 0$ such that $\nu(b_i) < \infty$.

$$Q_0 = \sum_{i=1}^{d} \sigma_i(\overline{w}_1, \ldots, \overline{w}_{m-1})\overline{w}_m^{a_i} + \sum_j \sigma_j(\overline{w}_1, \ldots, \overline{w}_{m-1})\overline{w}_m^{a_j} + \sum_k \sigma_k(\overline{w}_1, \ldots, \overline{w}_{m-1})\overline{w}_m^{a_k} + \overline{w}_m^{a}\Psi$$

where the first sum is over terms of minimum value ρ, a satisfies $a\nu(\overline{w}_m) > \rho$, the (finitely many) terms in the second sum have finite value and the (finitely many) terms in the third sum have infinite value.

By (42) of the Theorem there is a CUTS $(R, \overline{T}''(\alpha), \overline{T}(\alpha))$ and $(S, \overline{U}''(\alpha), \overline{U}(\alpha))$ in the first $m-1$ variables such that

$$\sigma_i = \overline{w}_1(\alpha)^{c_1^i(\alpha)} \cdots \overline{w}_s(\alpha)^{c_s^i(\alpha)}\overline{u}_i$$

for all i,

$$\sigma_j = \overline{w}_1(\alpha)^{c_1^j(\alpha)} \cdots \overline{w}_s(\alpha)^{c_s^j(\alpha)}\overline{u}_j$$

for all j and

$$\sigma_k = \overline{w}_1(\alpha)^{c_1^i(\alpha)} \cdots \overline{w}_s(\alpha)^{c_s^k(\alpha)}\overline{u}_k$$

for all k

where $\overline{u}_i, \overline{u}_j, \overline{u}_k \in k(\overline{U}(\alpha))[[\overline{w}_1(\alpha), \ldots, \overline{w}_{m-1}(\alpha)]]$, $\overline{u}_i, \overline{u}_j$ are units and

$$\nu(\overline{w}_1(\alpha)^{c_1^i(\alpha)} \cdots \overline{w}_s(\alpha)^{c_s^k(\alpha)}) > \rho.$$

Perform a UTS $\overline{U}(\alpha) \to \overline{U}(\alpha+1)$ of type II_m to get

$$\begin{aligned}Q_0 = {}& \overline{w}_1(\alpha+1)^{c_1^i(\alpha+1)} \cdots \overline{w}_s(\alpha+1)^{c_s^i(\alpha+1)}\Big(\sum_i \lambda_i\overline{u}_i(\overline{w}_m(\alpha+1)+1)^{a_i}\Big)\\
&+ \sum_j \overline{w}_1(\alpha+1)^{c_1^j(\alpha+1)} \cdots \overline{w}_s(\alpha+1)^{c_s^j(\alpha+1)}\lambda_j\overline{u}_j(\overline{w}_m(\alpha+1)+1)^{a_j}\\
&+ \sum_k \overline{w}_1(\alpha+1)^{c_1^k(\alpha+1)} \cdots \overline{w}_s(\alpha+1)^{c_s^k(\alpha+1)}\lambda_k\overline{u}_k(\overline{w}_m(\alpha+1)+1)^{a_k}\\
&+ \left(\overline{w}_1(\alpha+1)^{b_{s+1,1}(\alpha+1)} \cdots \overline{w}_s(\alpha+1)^{b_{s+1,s}(\alpha+1)}\right)^a \Psi'.\end{aligned}$$

$$\lambda_i = d_{\alpha+1}^{c_1^i(\alpha)b_{1,s+1}(\alpha+1)+\cdots+c_s^i(\alpha)b_{s,s+1}(\alpha+1)+b_{s+1,s+1}(\alpha+1)a_i}.$$

Now perform a CUTS of type (MI) $\overline{T}(\alpha+1) \to \overline{T}(\alpha+2)$, $\overline{U}(\alpha+1) \to \overline{U}(\alpha+2)$ to get

$$Q_0 = \overline{w}_1(\alpha+2)^{c_1^i(\alpha+2)} \cdots \overline{w}_s(\alpha+2)^{c_s^i(\alpha+2)} \cdot \Big(\sum_{i=1}^{d} \lambda_i\overline{u}_i(\overline{w}_m(\alpha+2)+1)^{a_i} + \overline{w}_1(\alpha+2)\cdots\overline{w}_s(\alpha+2)\Omega\Big).$$

Set

$$Q_1 = \sum_{i=1}^{d} \lambda_i \overline{u}_i(\overline{w}_m(\alpha+2)+1)^{a_i} + \overline{w}_1(\alpha+2)\cdots\overline{w}_s(\alpha+2)\Omega - \sum_{i=1}^{d}\lambda_i\overline{u}_i.$$

Set $r_1 = \operatorname{mult} Q_1(0,\ldots,0,\overline{w}_m(\alpha+2))$. $0 < r_1 < \infty$ and $r_1 \le r$.

Suppose $r_1 = r$. Then as in (57) in the proof of $A(\overline{m})$, $a_d = r$, $\sigma_{r-1} \neq 0$ and $\nu(\sigma_{r-1}) = \nu(\overline{w}_m)$.

As in the argument of the proof of $A(\overline{m})$, there is a polynomial

$$A_1 \in k(\overline{U})[\overline{w}_1,\ldots,\overline{w}_{m-1}] \subset \overline{U}''$$

such that we can make a change of variables, replacing $\overline{w}_m$ with $\overline{w}'_m = \overline{w}_m - A_1$, to get $\nu(\overline{w}'_m) > \nu(\overline{w}_m)$. We have

$$\nu(\overline{w}_m) \le \nu(\overline{w}_m^{r-1}) \le \nu\Big(\frac{\partial Q_0}{\partial \overline{w}_m}\Big)$$

since $\overline{w}_m^{r-1}$ is a minimum value term of $\partial Q_0/\partial\overline{w}_m$. $\partial f/\partial\overline{w}_m \in \overline{U}''$ and

$$\frac{\partial f}{\partial\overline{w}_m} = \overline{w}_1^{n_1(0)}\ldots\overline{w}_s^{n_s(0)}\frac{\partial Q_0}{\partial\overline{w}_m}$$

implies $\nu\big(\partial Q_0/\partial\overline{w}_m\big) < \infty$. Now repeat the above procedure. Since

$$\frac{\partial Q_0}{\partial\overline{w}'_m} = \frac{\partial Q_0}{\partial\overline{w}_m}$$

we will achieve a reduction in r after a finite number of iterations by Lemma 2.3.

Thus after replacing $\overline{w}_m$ with

$$\tilde{\overline{w}}'_m = \overline{w}_m - P_{m,0}$$

for some $P_{m,0} \in k(\overline{U})[\overline{w}_1,\ldots,\overline{w}_{m-1}]$, we achieve a reduction $r_1 < r$ in $\overline{U}(\alpha+2)$.

Thus we can construct a CUTS $(R,\overline{T}''(\beta),\overline{T}(\beta))$ and $(S,\overline{U}''(\beta),\overline{U}(\beta))$ such that

$$f = L(\overline{w}_1(\beta),\ldots,\overline{w}_{m-1}(\beta)) + \overline{w}_1(\beta)^{\alpha_1}\ldots,\overline{w}_s(\beta)^{\alpha_s}\overline{Q}$$

where $\operatorname{mult}\overline{Q}(0,\ldots,0,\overline{w}_m(\beta)) = 1$. Set

$$\tau = \nu(\overline{w}_1(\beta)^{\alpha_1}\ldots,\overline{w}_s(\beta)^{\alpha_s}).$$

Suppose that L is not in $k(\overline{U}(\beta))\big[\big[\overline{w}_1(\beta),\ldots,\overline{w}_l(\beta)\big]\big]$. Set

$$A = k(\overline{U}(\beta))\big[\big[\overline{w}_1(\beta),\ldots,\overline{w}_{m-1}(\beta)\big]\big].$$

We can write $L = f_1 + H$, with

$$f_1 \in k(\overline{U}(\beta))[\overline{w}_1(\beta),\ldots,\overline{w}_{m-1}(\beta)] \subset \overline{U}''(\beta),$$

$H \in m(A)^\alpha$ where $\nu(m(A)^\alpha) > \tau$. After permuting the variables

$$\{\overline{w}_{l+1}(\beta),\ldots,\overline{w}_{m-1}(\beta)\}$$

we may assume that $\partial f_1/\partial\overline{w}_{m-1}(\beta) \neq 0$. Thus

$$\nu\big(\partial f_1/\partial\overline{w}_{m-1}(\beta)\big) < \infty \quad \text{since} \quad \partial f_1/\partial\overline{w}_{m-1}(\beta) \in \overline{U}''(\beta).$$

By induction on m, we can perform a CUTS in the first $m-1$ variables to get

$$f_1 = L'(\overline{w}_1(\gamma), \ldots, \overline{w}_l(\gamma)) + \overline{w}_1(\gamma)^{\beta_1} \cdots \overline{w}_s(\gamma)^{\beta_s} \overline{Q}_1$$

so that

$$\overline{Q}_1 = \overline{u}(\overline{w}_{m-1}(\gamma) + \overline{w}_1(\gamma)^{\overline{g}_1} \cdots \overline{w}_s(\gamma)^{\overline{g}_s} \overline{\Sigma})$$

where

$$\overline{u} \in k(\overline{U}(\gamma))\left[\left[\overline{w}_1(\gamma), \ldots, \overline{w}_{m-1}(\gamma)\right]\right] \text{ is a unit,}$$
$$\Sigma \in k(\overline{U}(\gamma))\left[\left[\overline{w}_1(\gamma), \ldots, \overline{w}_{m-2}(\gamma)\right]\right],$$

and

$$\nu(\overline{w}_{m-1}(\gamma)) \leq \nu(\overline{w}_1(\gamma)^{\overline{g}_1} \cdots \overline{w}_s(\gamma)^{\overline{g}_s}).$$

Now perform a CUTS consisting of CUTSs of type (M2), $s+1 \leq r \leq m-1$ (with $P_{r,t} = 0$ for $\gamma + 1 \leq t \leq \delta - 1$) and a CUTS of type (M1) to get

$$H = \overline{w}_1(\delta)^{d_1(\delta)} \cdots \overline{w}_s(\delta)^{d_s(\delta)} \Psi$$

with

$$\nu(\overline{w}_1(\delta)^{d_1(\delta)} \cdots \overline{w}_s(\delta)^{d_s(\delta)}) > \tau.$$
$$\overline{Q}_1 = \overline{w}_1(\delta)^{e_1(\delta)} \cdots \overline{w}_s(\delta)^{e_s(\delta)} \overline{u}'(\overline{w}_{m-1}(\delta) + \phi')$$

for some $\phi' \in k(\overline{U}(\delta))\left[\left[\overline{w}_1(\delta), \ldots, \overline{w}_{m-2}(\delta)\right]\right]$, and unit

$$\overline{u}' \in k(\overline{U}(\delta))\left[\left[\overline{w}_1(\delta), \ldots, \overline{w}_{m-1}(\delta)\right]\right].$$

After possibly interchanging $\overline{w}_{m-1}(\delta)$ and $\overline{w}_m(\delta)$ and performing a CUTS of type (M1), we have f in the form

$$f = L(\overline{w}_1(\delta), \ldots, \overline{w}_l(\delta)) + \overline{w}_1(\delta)^{\alpha_1} \cdots \overline{w}_s(\delta)^{\alpha_s} Q$$

where $\operatorname{mult} Q(0, \ldots, 0, \overline{w}_m(\delta)) = 1$. Thus $Q = u(\overline{w}_m(\delta) + \Omega)$ where

$$u \in k(\overline{U}(\delta))\left[\left[\overline{w}_1(\delta), \ldots, \overline{w}_m(\delta)\right]\right]$$

is a unit and $\Omega \in k(\overline{U}(\delta))\left[\left[\overline{w}_1(\delta), \ldots, \overline{w}_{m-1}(\delta)\right]\right]$. After replacing $\overline{w}_m(\delta)$ with $\overline{w}_m(\delta)+\Psi$, where $\Psi \in k(\overline{U}(\delta))[\overline{w}_1(\delta), \ldots, \overline{w}_{m-1}(\delta)] \subset \overline{U}''(\delta)$, we can assume that

$$\Omega \in (\overline{w}_1(\delta), \ldots, \overline{w}_{m-1}(\delta))^B$$

where B is arbitrarilly large.

If $\nu(Q) < \infty$, we can choose B so large that $\nu(Q) = \nu(\overline{w}_m(\delta)) < \nu(\Omega)$. Then by the conclusions of (42) and (43) of the Theorem, we can perform a CUTS in the first $m-1$ variables to get

$$\Omega = \overline{w}_1(\varepsilon)^{g_1} \cdots \overline{w}_s(\varepsilon)^{g_s} \Sigma$$

with $\nu(\overline{w}_1(\varepsilon)^{g_1} \cdots \overline{w}_s(\varepsilon)^{g_s}) > \nu(\overline{w}_m(\varepsilon))$.

If $\nu(Q) = \infty$, we must have $\nu(\Omega) = \nu(\overline{w}_m(\delta)) < \infty$. Then by (42) of the Theorem, we can perform a CUTS in the first $m-1$ variables to get

$$\Omega = \overline{w}_1(\varepsilon)^{g_1} \cdots \overline{w}_s(\varepsilon)^{g_s} \Sigma$$

with $\nu(\overline{w}_1(\varepsilon)^{g_1}\cdots\overline{w}_s(\varepsilon)^{g_s})=\nu(\overline{w}_m(\varepsilon))$. $\square$

Theorem 4.8. — *Suppose that $T''(0)\subset\widehat{R}$ is a regular local ring essentially of finite type over R such that the quotient field of $T''(0)$ is finite over J, $U''(0)\subset\widehat{S}$ is a regular local ring essentially of finite type over S such that the quotient field of $U''(0)$ is finite over K, $T''(0)\subset U''(0)$, $T''(0)$ contains a subfield isomorphic to $k(c_0)$ for some $c_0\in k(T''(0))$ and $U''(0)$ contains a subfield isomorphic to $k(U''(0))$. Suppose that R has regular parameters $(x_1,\ldots,x_n)$, S has regular parameters $(y_1,\ldots,y_n)$, $T''(0)$ has regular parameters $(\overline{\widetilde{x}}_1,\ldots,\overline{\widetilde{x}}_n)$ and $U''(0)$ has regular parameters $(\overline{\widetilde{y}}_1,\ldots,\overline{\widetilde{y}}_n)$ such that*

$$\begin{aligned}\overline{\widetilde{x}}_1&=\overline{\widetilde{y}}_1^{c_{11}}\cdots\overline{\widetilde{y}}_s^{c_{1s}}\phi_1\\&\;\vdots\\\overline{\widetilde{x}}_s&=\overline{\widetilde{y}}_1^{c_{s1}}\cdots\overline{\widetilde{y}}_s^{c_{ss}}\phi_s\\\overline{\widetilde{x}}_{s+1}&=\overline{\widetilde{y}}_{s+1}\\&\;\vdots\\\overline{\widetilde{x}}_l&=\overline{\widetilde{y}}_l\end{aligned}$$

where $\phi_1,\ldots,\phi_s\in k(U''(0))$, $\nu(\overline{\widetilde{x}}_1),\ldots,\nu(\overline{\widetilde{x}}_s)$ are rationally independent, $\det(c_{ij})\neq 0$. Suppose that there exists a regular local ring $\widetilde{R}\subset R$ such that $(x_1,\ldots,x_l)$ are regular parameters in $\widetilde{R}$ and $k(\widetilde{R})\cong k(c_0)$. For $1\le i\le l$, there exists $\gamma_i\in k(c_0)[[x_1,\ldots,x_l]]\cap T''(0)$ such that $\gamma_i\equiv 1$ mod $(x_1,\ldots,x_l)$ and

$$x_i=\begin{cases}\gamma_i\overline{\widetilde{x}}_i & 1\le i\le l\\ \overline{\widetilde{x}}_i & l<i\le n.\end{cases}$$

In particular $k(c_0)[[x_1,\ldots,x_l]]=k(c_0)[[\overline{\widetilde{x}}_1,\ldots,\overline{\widetilde{x}}_l]]$. There exists $\gamma_i^y\in U''(0)$ such that $y_i=\gamma_i^y\overline{\widetilde{y}}_i$, $\gamma_i^y\equiv 1$ mod $m(U''(0))$ for $1\le i\le n$.

Suppose that one of the following three conditions holds.

(60) *$f\in k(U''(0))[[\overline{\widetilde{y}}_1,\ldots,\overline{\widetilde{y}}_m]]$ for some m with $l\le m\le n$ and $\nu(f)<\infty$.*
(61) *$f\in k(U''(0))[[\overline{\widetilde{y}}_1,\ldots,\overline{\widetilde{y}}_m]]$ for some m with $l<m\le n$, $\nu(f)=\infty$, and $A\in\mathbf{N}$ is given.*
(62) *$f\in U''(0)-k(U''(0))[[\overline{\widetilde{y}}_1,\ldots,\overline{\widetilde{y}}_l]]$.*

Then there exists a positive integer N_0 such that for $N\ge N_0$, we can construct a CRUTS along ν $(R,T''(t),T(t))$ and $(S,U''(t),U(t))$ with associated MTSs

$$\begin{array}{ccc}S&\to&S(t)\\\uparrow&&\uparrow\\R&\to&R(t)\end{array}$$

such that the following holds. $T''(t)$ *contains a subfield* $k(c_0,\ldots,c_t)$, $U''(t)$ *contains a subfield isomorphic to* $k(U(t))$,

$$\begin{aligned} R(t) &\ \textit{has regular parameters } (x_1(t),\ldots,x_n(t)),\\ T''(t) &\ \textit{has regular parameters } (\tilde{\overline{x}}_1(t),\ldots,\tilde{\overline{x}}_n(t)),\\ S(t) &\ \textit{has regular parameters } (y_1(1),\ldots,y_n(t)),\\ U''(t) &\ \textit{has regular parameters } (\tilde{\overline{y}}_1(t),\ldots,\tilde{\overline{y}}_n(t)) \end{aligned}$$

such that

$$x_i(t)=\begin{cases}\gamma_i(t)\tilde{\overline{x}}_i(t) & 1\le i\le l\\ \tilde{\overline{x}}_i(t) & l<i\le n\end{cases}$$

where $\gamma_i(t)\in k(c_0,\ldots,c_t)[[x_1(t),\ldots,x_l(t)]]$ *are units such that*

$$\gamma_i(t)\equiv 1 \ \textit{mod}\ (x_1(t),\ldots,x_l(t)).$$

In particular,

$$k(c_0,\ldots,c_t)[[x_1(t),\ldots,x_l(t)]]=k(c_0,\ldots,c_t)[[\tilde{\overline{x}}_1(t),\ldots,\tilde{\overline{x}}_l(t)]].$$

For $1\le i\le n$ *there exists* $\gamma_i^y(t)\in U''(t)$ *such that* $y_i(t)=\gamma_i^y(t)\tilde{\overline{y}}_i(t)$,

$$\gamma_i^y(t)\equiv 1 \ \textit{mod}\ m(U''(t)).$$

$$\begin{aligned} \tilde{\overline{x}}_1(t) &= \tilde{\overline{y}}_1(t)^{c_{11}(t)}\cdots\tilde{\overline{y}}_s(t)^{c_{1s}(t)}\phi_1(t)\\ &\ \vdots\\ \tilde{\overline{x}}_s(t) &= \tilde{\overline{y}}_1(t)^{c_{s1}(t)}\cdots\tilde{\overline{y}}_s(t)^{c_{ss}(t)}\phi_s(t)\\ \tilde{\overline{x}}_{s+1}(t) &= \tilde{\overline{y}}_{s+1}(t)\\ &\ \vdots\\ \tilde{\overline{x}}_l(t) &= \tilde{\overline{y}}_l(t) \end{aligned} \tag{63}$$

$\phi_1(t),\ldots,\phi_s(t)\in k(U(t))$, $\nu(\tilde{\overline{x}}_1(t)),\ldots,\nu(\tilde{\overline{x}}_s(t))$ *are rationally independent,* $\det(c_{ij}(t))$ $\neq 0$ *and there exists a regular local ring* $\tilde{R}(t)\subset R(t)$ *such that* $(x_1(t),\ldots,x_l(t))$ *are regular parameters in* $\tilde{R}(t)$ *and* $k(\tilde{R}(t))\cong k(c_0,\ldots,c_t)$. *Furthermore,* $x_i(t)=x_i$ *for* $l+1\le i\le n$, $y_i(t)=y_i$ *for* $m+1\le i\le n$, *so that the CRUTS is in the first* m *variables where* $m=n$ *in case (62). Set* $n_{t,l}=m\left(k(U(t))[[\tilde{\overline{y}}_1(t),\ldots,\tilde{\overline{y}}_l(t)]]\right)$.

In case (60) we have

$$f\equiv\tilde{\overline{y}}_1(t)^{d_1}\cdots\tilde{\overline{y}}_s(t)^{d_s}u(\tilde{\overline{y}}_1(t),\ldots,\tilde{\overline{y}}_m(t)) \ \textit{mod}\ m(U(t))^N \tag{64}$$

where u *is a unit power series. Further if* $f\in k(\overline{U})[[\tilde{\overline{y}}_1,\ldots,\tilde{\overline{y}}_l]]$,

$$f\equiv\tilde{\overline{y}}_1(t)^{d_1}\cdots\tilde{\overline{y}}_s(t)^{d_s}u(\tilde{\overline{y}}_1(t),\ldots,\tilde{\overline{y}}_l(t)) \ \textit{mod}\ n_{t,l}^N.$$

In case (61) we have

$$f\equiv\tilde{\overline{y}}_1(t)^{d_1}\cdots\tilde{\overline{y}}_s(t)^{d_s}\Sigma(\tilde{\overline{y}}_1(t),\ldots,\tilde{\overline{y}}_m(t)) \ \textit{mod}\ m(U(t))^N \tag{65}$$

with $\nu(\widetilde{\overline{y}}_1(t)^{d_1}\cdots\widetilde{\overline{y}}_s(t)^{d_s}) > A$. *Further if* $f\in k(\overline{U})[[\widetilde{\overline{y}}_1,\ldots,\widetilde{\overline{y}}_l]]$,

$$f\equiv\widetilde{\overline{y}}_1(t)^{d_1}\cdots\widetilde{\overline{y}}_s(t)^{d_s}u(\widetilde{\overline{y}}_1(t),\ldots,\widetilde{\overline{y}}_l(t))\ mod\ n_{t,l}^N.$$

In case (62) we have

$$f\equiv P(\widetilde{\overline{y}}_1(t),\ldots,\widetilde{\overline{y}}_l(t))+\widetilde{\overline{y}}_1(t)^{d_1}\cdots\widetilde{\overline{y}}_s(t)^{d_s}H\ mod\ m(U(t))^N \tag{66}$$

where P *is a series with coefficients in* $k(U(t))$ *and*

$$H=u(\widetilde{\overline{y}}_{l+1}(t)+\widetilde{\overline{y}}_1(t)^{g_1}\cdots\widetilde{\overline{y}}_s(t)^{g_s}\Sigma)$$

where $u\in U(t)$ *is a unit,* $\Sigma\in k(U(t))[[\widetilde{\overline{y}}_1(t),\ldots,\widetilde{\overline{y}}_l(t),\widetilde{\overline{y}}_{1+2}(t),\ldots,\widetilde{\overline{y}}_n(t)]]$ *and*

$$\nu(\widetilde{\overline{y}}_{l+1}(t))\le\nu(\widetilde{\overline{y}}_1(t)^{g_1}\cdots\widetilde{\overline{y}}_s(t)^{g_s}).$$

Proof. — Set $\overline{T}=\widehat{R}$, $\overline{U}=\widehat{S}$, $\overline{T}''=T''(0)$, $\overline{U}''=U''(0)$. Set $\overline{z}_i=\widetilde{\overline{x}}_i$, $\overline{w}_i=\widetilde{\overline{y}}_i$ for $1\le i\le n$. In case (62) set $m=n$. By Theorem 4.7 there is a CUTS along ν $(R,\overline{T}''(t),\overline{T}(t))$ and $(S,\overline{U}''(t),\overline{U}(t))$

$$\begin{array}{ccc}\overline{U}(0)&\to&\overline{U}(t)\\ \uparrow&&\uparrow\\ \overline{T}(0)&\to&\overline{T}(t)\end{array}$$

so that in the notation of Theorem 4.7 and its proof, for $0\le\alpha\le t$, $\overline{T}''(\alpha)$ has regular parameters

$$(\overline{z}_1(\alpha),\ldots,\overline{z}_n(\alpha))\text{ and }(\widetilde{\overline{z}}'_1(\alpha),\ldots,\widetilde{\overline{z}}'_n(\alpha)),$$

$\overline{U}''(\alpha)$ has parameters

$$(\overline{w}_1(\alpha),\ldots,\overline{w}_n(\alpha))\text{ and }(\widetilde{\overline{w}}'_1(\alpha),\ldots,\widetilde{\overline{w}}'_n(\alpha))$$

such that in case (60) we have

$$f=\overline{w}_1(t)^{d_1}\cdots\overline{w}_s(t)^{d_s}u(\overline{w}_1(t),\ldots,\overline{w}_m(t))$$

where u is a unit power series. In case (61) we have

$$f=\overline{w}_1(t)^{d_1}\cdots\overline{w}_s(t)^{d_s}\Sigma(\overline{w}_1(t),\ldots,\overline{w}_m(t))$$

where $\nu(\overline{w}_1(t)^{d_1}\cdots\overline{w}_s(t)^{d_s})>A$. In case (62) we have

$$f=P(\overline{w}_1(t),\ldots,\overline{w}_l(t))+\overline{w}_1(t)^{d_1}\cdots\overline{w}_s(t)^{d_s}H.$$

for some powerseries $P\in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_l(t)]]$,

$$H=u(\overline{w}_n(t)+\overline{w}_1(t)^{g_1}\cdots\overline{w}_s(t)^{g_s}\Sigma)$$

where $u\in\overline{U}(t)$ is a unit, $\Sigma\in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_{n-1}(t)]]$ and

$$\nu(\overline{w}_n(t))\le\nu(\overline{w}_1(t)^{g_1}\cdots\overline{w}_s(t)^{g_s}).$$

Step 1. — Fix $N > 0$. To begin with, we will construct commutative diagrams of inclusions of regular local rings

$$\begin{array}{ccccc} U'(\alpha) & \to & U''(\alpha) & \to & U(\alpha) \\ \uparrow & & \uparrow & & \uparrow \\ T'(\alpha) & \to & T''(\alpha) & \to & T(\alpha) \end{array} \tag{67}$$

for $1 \le \alpha \le t$ such that $T(\alpha) = T'(\alpha)^\frown$, $U(\alpha) = U'(\alpha)^\frown$ for all α, $T''(\alpha)$ has regular parameters

$$(\overline{x}_1(\alpha), \ldots, \overline{x}_n(\alpha)), (\tilde{\overline{x}}_1(\alpha), \ldots, \tilde{\overline{x}}_n(\alpha)), (\tilde{\overline{x}}'_1(\alpha), \ldots, \tilde{\overline{x}}'_n(\alpha)).$$

$U''(\alpha)$ has regular parameters

$$(\overline{y}_1(\alpha), \ldots, \overline{y}_n(\alpha)), (\tilde{\overline{y}}_1(\alpha), \ldots, \tilde{\overline{y}}_n(\alpha)), (\tilde{\overline{y}}'_1(\alpha), \ldots, \tilde{\overline{y}}'_n(\alpha)).$$

where $\overline{x}_i(0) = \tilde{\overline{x}}_i$ and $\overline{y}_i(0) = \tilde{\overline{y}}_i$ for $1 \le i \le n$. We will have isomorphisms

$$\begin{array}{l} \eta_T^\alpha : k(T(\alpha)) \to k(\overline{T}(\alpha)) \quad \text{and} \\ \eta_U^\alpha : k(U(\alpha)) \to k(\overline{U}(\alpha)) \end{array} \tag{68}$$

such that the diagrams

$$\begin{array}{ccc} k(\overline{T}(\alpha)) & \to & k(\overline{T}(\alpha+1)) \\ \uparrow & & \uparrow \\ k(T(\alpha)) & \to & k(T(\alpha+1)) \end{array} \tag{69}$$

and

$$\begin{array}{ccc} k(\overline{U}(\alpha)) & \to & k(\overline{U}(\alpha+1)) \\ \uparrow & & \uparrow \\ k(U(\alpha)) & \to & k(U(\alpha+1)) \end{array}$$

commute for $0 \le \alpha \le t-1$. For all α we will have

$$\begin{aligned} \overline{x}_1(\alpha) &= \overline{y}_1(\alpha)^{c_{11}(\alpha)} \cdots \overline{y}_s(\alpha)^{c_{1s}(\alpha)} \phi_1(\alpha) \\ &\vdots \\ \overline{x}_s(\alpha) &= \overline{y}_1(\alpha)^{c_{s1}(\alpha)} \cdots \overline{y}_s(\alpha)^{c_{ss}(\alpha)} \phi_s(\alpha) \\ \overline{x}_{s+1}(\alpha) &= \overline{y}_{s+1}(\alpha) \\ &\vdots \\ \overline{x}_l(\alpha) &= \overline{y}_l(\alpha) \end{aligned} \tag{70}$$

with $\phi_1(\alpha), \ldots, \phi_s(\alpha) \in k(U(\alpha))$ the coefficients of (46) of Theorem 4.7, $(c_{ij}(\alpha))$ the exponents of (46) of Theorem 4.7.

We will construct (67) inductively. Suppose that (67) has been constructed out to $T(\alpha) \to U(\alpha)$ and regular parameters

$$(\overline{x}_1(\alpha), \ldots, \overline{x}_n(\alpha)) \text{ in } T''(\alpha) \quad \text{and} \quad (\overline{y}_1(\alpha), \ldots, \overline{y}_n(\alpha)) \text{ in } U''(\alpha)$$

have been defined so that (68) and (70) hold.

If we identity $k(T(\alpha))$ with $k(\overline{T}(\alpha))$ and $k(U(\alpha))$ with $k(\overline{U}(\alpha))$ we have isomorphisms $T(\alpha) \cong k(\overline{T}(\alpha))[[\overline{x}_1(\alpha), \ldots, \overline{x}_n(\alpha)]]$ and $U(\alpha) \cong k(\overline{U}(\alpha))[[\overline{y}_1(\alpha), \ldots, \overline{y}_n(\alpha)]]$.

We can choose Λ_α and $\Omega_{i,\alpha}$ arbitrarily subject to the following conditions, to define regular parameters in $T''(\alpha)$ by

$$(71)\quad \widetilde{\overline{x}}_i(\alpha) = \begin{cases} \overline{x}_r(\alpha) + \Lambda_\alpha(\overline{x}_1(\alpha), \ldots, \overline{x}_l(\alpha)) & \text{if } \overline{T}(\alpha-1) \to \overline{T}(\alpha) \text{ is of type II}_r \\ & \text{and } i = r \\ \overline{x}_i(\alpha) & \text{otherwise} \end{cases}$$

with $\Lambda_\alpha \in k(c_0, \ldots, c_\alpha)[[\overline{x}_1(\alpha), \ldots, \overline{x}_l(\alpha)]] \cap T''(\alpha)$ and $\mathrm{mult}(\Lambda_\alpha) \geq N$. We will take $\Lambda_0 = 0$.

Recall that the $P_{i,\alpha}$ constructed in Theorem 4.7 are polynomials with coefficients in $k(c_0, \ldots, c_\alpha)$ if $i \leq l$. Define

$$(72)\quad \widetilde{\overline{x}}'_i(\alpha) = \begin{cases} \widetilde{\overline{x}}_i(\alpha) - P_{i,\alpha}(\widetilde{\overline{x}}_1(\alpha), \ldots, \widetilde{\overline{x}}_{i-1}(\alpha)) & \\ \qquad + \Omega_{i,\alpha}(\widetilde{\overline{x}}_1(\alpha), \ldots, \widetilde{\overline{x}}_l(\alpha)), & \text{if } s+1 \leq i \leq l \\ \widetilde{\overline{x}}_i(\alpha) & \text{otherwise} \end{cases}$$

with $\Omega_{i,\alpha} \in k(c_0, \ldots, c_\alpha)[[\overline{x}_1(\alpha), \ldots, \overline{x}_l(\alpha)]] \cap T''(\alpha)$ and $\mathrm{mult}(\Omega_{i,\alpha}) \geq N$. (If $P_{i,\alpha} = 0$, or if $1 \leq i \leq s$ we will have $\overline{x}'_i(\alpha) = \widetilde{\overline{x}}_i(\alpha)$.) We then have

$$\begin{aligned} k(c_0, \ldots, c_\alpha)[[\overline{x}_1(\alpha), \ldots, \overline{x}_l(\alpha)]] &= k(c_0, \ldots, c_\alpha)[[\widetilde{\overline{x}}_1(\alpha), \ldots, \widetilde{\overline{x}}_l(\alpha)]] \\ &= k(c_0, \ldots, c_\alpha)[[\widetilde{\overline{x}}'_1(\alpha), \ldots, \widetilde{\overline{x}}'_l(\alpha)]] \end{aligned}$$

and

$$\overline{x}_i(\alpha) = \widetilde{\overline{x}}_i(\alpha) = \widetilde{\overline{x}}'_i(\alpha) = x_i$$

for $l < i \leq n$. Define

$$(73)\quad \widetilde{\overline{y}}_i(\alpha) = \begin{cases} \widetilde{\overline{x}}_i(\alpha) & \text{if } s+1 \leq i \leq l \\ \overline{y}_i(\alpha) + \Lambda_\alpha(\overline{y}_1(\alpha), \ldots, \overline{y}_n(\alpha)), & \text{if } S(\alpha-1) \to S(\alpha) \text{ is of type II}_r \\ & \text{and } i = r \geq l+1 \\ \overline{y}_i(\alpha) & \text{otherwise} \end{cases}$$

with $\Lambda_\alpha \in U''(\alpha)$ and $\mathrm{mult}(\Lambda_\alpha) \geq N$. We will take $\Lambda_0 = 0$.

Recall that the $P_{i,\alpha}$ constructed in Theorem 4.7 are polynomials with coefficents in $k(\overline{U}(\alpha))$ for $l+1 \leq i$. Define

$$(74)\quad \widetilde{\overline{y}}'_i(\alpha) = \begin{cases} \widetilde{\overline{x}}'_i(\alpha) & \text{if } s+1 \leq i \leq l \\ \widetilde{\overline{y}}_i(\alpha) - P_{i,\alpha}(\widetilde{\overline{y}}_1(\alpha), \ldots, \widetilde{\overline{y}}_{i-1}(\alpha)) & \\ \qquad + \Omega_{i,\alpha}(\widetilde{\overline{y}}_1(\alpha), \ldots, \widetilde{\overline{y}}_n(\alpha)) & \text{if } l+1 \leq i \leq m \\ \widetilde{\overline{y}}_i(\alpha) & \text{otherwise} \end{cases}$$

with $\Omega_{i,\alpha} \in U''(\alpha)$ and $\mathrm{mult}(\Omega_{i,\alpha}) \geq N$.

These variables are such that for all α,

$$\begin{aligned} k(U(\alpha))[[\overline{y}_1(\alpha),\dots,\overline{y}_l(\alpha)]] &= k(U(\alpha))[[\widetilde{\overline{y}}_1(\alpha),\dots,\widetilde{\overline{y}}_l(\alpha)]] \\ &= k(U(\alpha))[[\widetilde{\overline{y}}'_1(\alpha),\dots,\widetilde{\overline{y}}'_l(\alpha)]] \end{aligned}$$

and $\overline{y}_i(\alpha) = \widetilde{\overline{y}}_i(\alpha) = \widetilde{\overline{y}}'_i(\alpha) = y_i$ for $m < i \le n$.

If $\overline{T}(\alpha) \to \overline{T}(\alpha+1)$ is of type I, defined by (48) of Theorem 4.7, $T(\alpha) \to T(\alpha+1)$ will be the UTS of type I such that $T''(\alpha+1)$ has regular parameters $(\overline{x}_1(\alpha+1),\dots,\overline{x}_n(\alpha+1))$ defined by

$$(75)\qquad \begin{aligned} \widetilde{\overline{x}}'_1(\alpha) &= \overline{x}_1(\alpha+1)^{a_{11}(\alpha+1)}\cdots\overline{x}_s(\alpha+1)^{a_{1s}(\alpha+1)} \\ &\;\vdots \\ \widetilde{\overline{x}}'_s(\alpha) &= \overline{x}_1(\alpha+1)^{a_{s1}(\alpha+1)}\cdots\overline{x}_s(\alpha+1)^{a_{ss}(\alpha+1)}. \end{aligned}$$

Suppose that $\overline{T}(\alpha) \to \overline{T}(\alpha+1)$ is of type II_r, defined by (49) of Theorem 4.7. Set $(e_{ij}) = (a_{ij}(\alpha+1))^{-1}$,

$$(76)\qquad \begin{aligned} M_1 &= \widetilde{\overline{x}}'_1(\alpha)^{e_{11}}\cdots\widetilde{\overline{x}}'_s(\alpha)^{e_{1s}}\widetilde{\overline{x}}'_r(\alpha)^{e_{1,s+1}} \\ &\;\vdots \\ M_s &= \widetilde{\overline{x}}'_1(\alpha)^{e_{s1}}\cdots\widetilde{\overline{x}}'_s(\alpha)^{e_{ss}}\widetilde{\overline{x}}'_r(\alpha)^{e_{s,s+1}} \\ M_r &= \widetilde{\overline{x}}'_1(\alpha)^{e_{s+1,1}}\cdots\widetilde{\overline{x}}'_s(\alpha)^{e_{s+1,s}}\widetilde{\overline{x}}'_r(\alpha)^{e_{s+1,s+1}}. \end{aligned}$$

Let k_1 be the integral closure of k in $T(\alpha)$. Set

$$A = \left(T''(\alpha)[M_1,\dots,M_s,M_r]\otimes_{k_1} k(\overline{T}(\alpha+1))\right)_a$$

where

$$a = (M_1,\dots,M_s,\widetilde{\overline{x}}'_{s+1}(\alpha),\dots,\widetilde{\overline{x}}'_{r-1}(\alpha), M_r - c_{\alpha+1},\widetilde{\overline{x}}'_{r+1}(\alpha),\dots,\widetilde{\overline{x}}'_n(\alpha)).$$

Set $q_1 = T''(\alpha)[M_1,\dots,M_s,M_r]\cap a$ where the intersection is in A. Define

$$T'(\alpha+1) = T''(\alpha)[M_1,\dots,M_s,M_r]_{q_1}.$$

Define $T(\alpha+1) = T'(\alpha+1)\hat{}$. Our inclusion $T'(\alpha+1)\subset A$ induces an isomorphism $\eta_T^{\alpha+1} : k(T(\alpha+1)) \to k(\overline{T}(\alpha+1))$. We can thus identify $c_{\alpha+1}$ with $(\eta_T^{\alpha+1})^{-1}(c_{\alpha+1})$. $T(\alpha+1)$ has regular parameters $(\widehat{\overline{x}}_1(\alpha+1),\dots,\widehat{\overline{x}}_n(\alpha+1))$ defined by

$$(77)\qquad \widehat{\overline{x}}_i(\alpha+1) = \begin{cases} M_i & 1\le i\le s \\ M_r - c_{\alpha+1} & i = r \\ \widetilde{\overline{x}}'_i(\alpha) & s<i,\ i\ne r \end{cases}$$

Set

$$(78)\qquad T''(\alpha+1) = T''(\alpha)\left[c_{\alpha+1},\left(\frac{\widehat{\overline{x}}_r(\alpha+1)}{c_{\alpha+1}}+1\right)^{1/\overline{c}_{\alpha+1}}\right]_{(\widehat{\overline{x}}_1(\alpha+1),\dots,\widehat{\overline{x}}_n(\alpha+1))}$$

where $\left(\frac{\widehat{\overline{x}}_r(\alpha+1)}{c_{\alpha+1}}+1\right)^{1/\overline{c}_{\alpha+1}}$ has residue 1 in $k(T(\alpha+1))$. $T''(\alpha+1)$ has regular parameters $(\overline{x}_1(\alpha+1),\ldots,\overline{x}_n(\alpha+1))$ defined by

$$\overline{x}_i(\alpha+1)=\begin{cases}\widehat{\overline{x}}_i(\alpha+1)\left(\frac{\widehat{\overline{x}}_r(\alpha+1)}{c_{\alpha+1}}+1\right)^{-\gamma_i(\alpha+1)} & 1\le i\le s\\ \left(\frac{\widehat{\overline{x}}_r(\alpha+1)}{c_{\alpha+1}}+1\right)^{1/\overline{c}_{\alpha+1}}-1 & i=r\\ \widehat{\overline{x}}_i(\alpha+1) & s<i,\ i\ne r\end{cases}\tag{79}$$

Then $T(\alpha)\to T(\alpha+1)$ is a UTS of type II_r with

$$\begin{aligned}\widetilde{\overline{x}}'_1(\alpha)&=\overline{x}_1(\alpha+1)^{a_{11}(\alpha+1)}\cdots\overline{x}_s(\alpha+1)^{a_{1s}(\alpha+1)}c_{\alpha+1}^{a_{1,s+1}(\alpha+1)}\\&\ \ \vdots\\ \widetilde{\overline{x}}'_s(\alpha)&=\overline{x}_1(\alpha+1)^{a_{s1}(\alpha+1)}\cdots\overline{x}_s(\alpha+1)^{a_{ss}(\alpha+1)}c_{\alpha+1}^{a_{s,s+1}(\alpha+1)}\\ \widetilde{\overline{x}}'_r(\alpha)&=\overline{x}_1(\alpha+1)^{a_{s+1,1}(\alpha+1)}\cdots\overline{x}_s(\alpha+1)^{a_{s+1,s}(\alpha+1)}\\&\qquad\cdot(\overline{x}_r(\alpha+1)+1)c_{\alpha+1}^{a_{s+1,s+1}(\alpha+1)}\end{aligned}\tag{80}$$

If $\overline{U}(\alpha)\to\overline{U}(\alpha+1)$ is of type I, defined by (50) of Theorem 4.7, $U(\alpha)\to U(\alpha+1)$ will be the UTS of type I such that $U''(\alpha+1)$ has regular parameters $(\overline{y}_1(\alpha+1),\ldots,\overline{y}_s(\alpha+1))$ defined by

$$\begin{aligned}\widetilde{\overline{y}}'_1(\alpha)&=\overline{y}_1(\alpha+1)^{b_{11}(\alpha+1)}\cdots\overline{y}_s(\alpha+1)^{b_{1s}(\alpha+1)}\\&\ \ \vdots\\ \widetilde{\overline{y}}'_s(\alpha)&=\overline{y}_1(\alpha+1)^{b_{s1}(\alpha+1)}\cdots\overline{y}_s(\alpha+1)^{b_{ss}(\alpha+1)}.\end{aligned}\tag{81}$$

Suppose that $\overline{U}(\alpha)\to\overline{U}(\alpha+1)$ is of type II_r, defined by (51) of Theorem 4.7. Set $(f_{ij})=(b_{ij}(\alpha+1))^{-1}$,

$$\begin{aligned}N_1&=\widetilde{\overline{y}}'_1(\alpha)^{f_{11}}\cdots\widetilde{\overline{y}}'_s(\alpha)^{f_{1s}}\widetilde{\overline{y}}'_r(\alpha)^{f_{1,s+1}}\\&\ \ \vdots\\ N_s&=\widetilde{\overline{y}}'_1(\alpha)^{f_{s1}}\cdots\widetilde{\overline{y}}'_s(\alpha)^{e_{ss}}\widetilde{\overline{y}}'_r(\alpha)^{f_{s,s+1}}\\ N_r&=\widetilde{\overline{y}}'_1(\alpha)^{f_{s+1,1}}\cdots\widetilde{\overline{y}}'_s(\alpha)^{f_{s+1,s}}\widetilde{\overline{y}}'_r(\alpha)^{f_{s+1,s+1}}.\end{aligned}\tag{82}$$

Let k_2 be the integral closure of k in $U(\alpha)$. Set

$$B=\left(U''(\alpha)[N_1,\ldots,N_s,N_r]\otimes_{k_2}k(\overline{U}(\alpha+1))\right)_b\tag{83}$$

where

$$b=(N_1,\ldots,N_s,\widetilde{\overline{y}}'_{s+1}(\alpha),\ldots,\widetilde{\overline{y}}'_{r-1}(\alpha),N_r-d_{\alpha+1},\widetilde{\overline{y}}'_{r+1}(\alpha),\ldots,\widetilde{\overline{y}}'_{s+1}(\alpha)).$$

Set $q_2=U''(\alpha)[N_1,\ldots,N_s,N_r]\cap b$ where the intersection is in B. Define

$$U'(\alpha+1)=U''(\alpha)[N_1,\ldots,N_s,N_r]_{q_2}.\tag{84}$$

Define $U(\alpha+1)=U'(\alpha+1)\hat{}$. Our inclusion $U'(\alpha+1)\subset B$ induces an isomorphism $\eta_U^{\alpha+1}:k(U(\alpha+1))\to k(\overline{U}(\alpha+1))$. We can thus identify $d_{\alpha+1}$ with $(\eta_U^{\alpha+1})^{-1}(d_{\alpha+1})$.

$U(\alpha+1)$ has regular parameters $(\widehat{\overline{y}}_1(\alpha+1),\ldots,\widehat{\overline{y}}_n(\alpha+1))$ defined by

$$\widehat{\overline{y}}_i(\alpha+1)=\begin{cases} N_i & 1\le i\le s\\ N_r-d_{\alpha+1} & i=r\\ \widetilde{\overline{y}}'_i(\alpha) & s<i,\ i\ne r\end{cases} \tag{85}$$

Set

$$U''(\alpha+1)=U''(\alpha)\left[d_{\alpha+1},\left(\frac{\widehat{\overline{y}}_r(\alpha+1)}{d_{\alpha+1}}+1\right)^{1/\overline{d}_{\alpha+1}}\right]_{(\widehat{\overline{y}}_1(\alpha+1),\ldots,\widehat{\overline{y}}_n(\alpha+1))} \tag{86}$$

where $\left(\frac{\widehat{\overline{y}}_r(\alpha+1)}{d_{\alpha+1}}+1\right)^{1/\overline{d}_{\alpha+1}}$ has residue 1 in $k(U(\alpha+1))$. $U''(\alpha+1)$ has regular parameters $(\overline{y}_1(\alpha+1),\ldots,\overline{y}_n(\alpha+1))$ defined by

$$\overline{y}_i(\alpha+1)=\begin{cases} \widehat{\overline{y}}_i(\alpha+1)\left(\frac{\widehat{\overline{y}}_r(\alpha+1)}{d_{\alpha+1}}+1\right)^{-r_i(\alpha+1)} & 1\le i\le s\\ \left(\frac{\widehat{\overline{y}}_r(\alpha+1)}{d_{\alpha+1}}+1\right)^{1/\overline{d}_{\alpha+1}}-1 & i=r\\ \widehat{\overline{y}}_i(\alpha+1) & s<i,\ i\ne r\end{cases} \tag{87}$$

Then $U(\alpha)\to U(\alpha+1)$ is a UTS of type II_r with

$$\begin{aligned} \widetilde{\overline{y}}'_1(\alpha) &= \overline{y}_1(\alpha+1)^{b_{11}(\alpha+1)}\cdots\overline{y}_s(\alpha+1)^{b_{1s}(\alpha+1)}d_{\alpha+1}^{b_{1,s+1}(\alpha+1)}\\ &\ \ \vdots\\ \widetilde{\overline{y}}'_s(\alpha) &= \overline{y}_1(\alpha+1)^{b_{s1}(\alpha+1)}\cdots\overline{y}_s(\alpha+1)^{b_{ss}(\alpha+1)}d_{\alpha+1}^{b_{s,s+1}(\alpha+1)}\\ \widetilde{\overline{y}}'_r(\alpha) &= \overline{y}_1(\alpha+1)^{b_{s+1,1}(\alpha+1)}\cdots\overline{y}_s(\alpha+1)^{b_{s+1,s}(\alpha+1)}\\ &\quad\cdot(\overline{y}_r(\alpha+1)+1)d_{\alpha+1}^{b_{s+1,s+1}(\alpha+1)}. \end{aligned} \tag{88}$$

We will now prove that (67), (69) and (70) hold for $\alpha+1$. The essential case is when $\overline{T}(\alpha)\to\overline{T}(\alpha+1)$ is of type II_r with $s+1\le r\le l$.

By (34) of Lemma 4.4 in the construction of $\overline{T}(\alpha)\to\overline{T}(\alpha+1)$ and $\overline{U}(\alpha)\to\overline{U}(\alpha+1)$,

$$\begin{aligned} M_1 &= N_1^{g_{11}}\cdots N_s^{g_{1s}}N_r^{g_{1,s+1}}\beta_1\\ &\ \ \vdots\\ M_s &= N_1^{g_{s1}}\cdots N_s^{g_{ss}}N_r^{g_{s,s+1}}\beta_s\\ M_r &= N_r^{g_{s,s+1}}\beta_r \end{aligned} \tag{89}$$

$\beta_i=\phi_1(\alpha)^{e_{i1}}\cdots\phi_s(\alpha)^{e_{is}}\in k(U(\alpha))\subset U''(\alpha)$ for $1\le i\le s$,

$$\beta_r=\phi_1(\alpha)^{e_{s+1,1}}\cdots\phi_s(\alpha)^{e_{s+1,s}}.$$

$$(g_{ij})=(a_{ij}(\alpha+1))^{-1}\begin{pmatrix}(c_{ij}(\alpha)) & 0\\ 0 & 1\end{pmatrix}(b_{ij}(\alpha+1))$$

$g_{s+1,1} = \cdots = g_{s+1,s} = 0$ and $g_{ij} \geq 0$ for all i, j. Thus $T''(\alpha)[M_1, \ldots, M_s, M_r] \subset U''(\alpha)[N_1, \ldots, N_s, N_r]$. Our inclusion $k(\overline{T}(\alpha+1)) \to k(\overline{U}(\alpha+1))$ induces an identification $c_{\alpha+1} = d_{\alpha+1}^{g_{s+1,s+1}} \beta_r$. Then by (37) of Lemma 4.4,

$$\begin{aligned} M_r - c_{\alpha+1} &= N_r^{g_{s+1,s+1}} \beta_r - c_{\alpha+1} = (N_r^{g_{s+1,s+1}} - d_{\alpha+1}^{g_{s+1,s+1}})\beta_r \\ &= \prod_{i=1}^{g_{s+1,s+1}} (N_r - \omega^i d_{\alpha+1})\beta_r \end{aligned}$$

where ω is a primitive $g_{s+1,s+1}$-th root of unity (in an algebraic closure of $k(U(\alpha+1))$). Thus $N_r - d_{\alpha+1}$ divides $M_r - c_{\alpha+1}$ in $U''(\alpha)[N_1, \ldots, N_s, N_r]$ and we have an inclusion $A \subset B$ which induces $T'(\alpha+1) \subset U'(\alpha+1)$ and $T(\alpha+1) \subset U(\alpha+1)$. Thus (69) holds for $\alpha+1$.

By the argument of Lemma 4.4 in the construction of $\overline{T}(\alpha+1) \to \overline{U}(\alpha+1)$ and (77)-(79), (85)-(87), we have that $T''(\alpha+1) \subset U''(\alpha+1)$ and (70) holds for $\alpha+1$.

Step 2. — Suppose that $T(t) \to U(t)$ is constructed as in Step 1, and f satisfies (60), (61) or (62) in the statement of Theorem 4.8. We will show that f satisfies the respective equation (64), (65) or (66) in $U(t)$. It suffices to prove the following statement.

Suppose that $0 \leq j \leq t$ and $\overline{f}_j$ is defined by $f(\overline{w}_1, \ldots, \overline{w}_n) = \overline{f}_j(\overline{w}_1(j), \ldots, \overline{w}_n(j))$ in $\overline{U}(j)$. Then

$$f(\widetilde{\overline{y}}_1, \ldots, \widetilde{\overline{y}}_n) \equiv \overline{f}_j(\overline{y}_1(j), \ldots, \overline{y}_n(j)) \bmod m(U(j))^N. \tag{90}$$

The statement (90) will be proved by induction on j. By induction, suppose that

$$f(\widetilde{\overline{y}}_1, \ldots, \widetilde{\overline{y}}_n) \equiv \overline{f}_j(\overline{y}_1(j), \ldots, \overline{y}_n(j)) \bmod m(U(j))^N.$$

We have $\overline{f}_j(\overline{w}_1(j), \ldots, \overline{w}_n(j)) = \overline{f}_{j+1}(\overline{w}_1(j+1), \ldots, \overline{w}_n(j+1))$ in $\overline{U}(j+1)$.

There are series $\overline{P}_{i,j}$ with coefficients in $k(\overline{U}(j))$ such that

$$\overline{P}_{i,j}(\widetilde{\overline{w}}'_1(j), \ldots, \widetilde{\overline{w}}'_{i-1}(j)) = \begin{cases} P_{i,j}(\overline{z}_1(j), \ldots, \overline{z}_{i-1}(j)) & s+1 \leq i \leq l \\ P_{i,j}(\overline{w}_1(j), \ldots, \overline{w}_{i-1}(j)) & l+1 < i \end{cases}$$

We have

$$\begin{aligned} f(\widetilde{\overline{y}}_1, \ldots, \widetilde{\overline{y}}_n) &\equiv \overline{f}_j(\overline{y}_1(j), \ldots, \overline{y}_n(j)) \bmod m(U(j+1))^N \\ &\equiv \overline{f}_j(\widetilde{\overline{y}}'_1(j), \ldots, \widetilde{\overline{y}}'_s(j), \widetilde{\overline{y}}'_{s+1}(j) + \overline{P}_{s+1,j}(\widetilde{\overline{y}}'_1(j), \ldots, \widetilde{\overline{y}}'_s(j)), \ldots, \\ &\qquad \widetilde{\overline{y}}'_n(j) + \overline{P}_{n,j}(\widetilde{\overline{y}}'_1(j), \ldots, \widetilde{\overline{y}}'_{n-1}(j))) \bmod m(U(j+1))^N \\ &\equiv \overline{f}_{j+1}(\overline{y}_1(j+1), \ldots, \overline{y}_n(j+1)) \bmod m(U(j+1))^N \end{aligned}$$

Set $n_{\alpha,l} = m\left(k(U(\alpha))[[\overline{y}_1(\alpha), \ldots, \overline{y}_l(\alpha)]]\right)$ for $1 \leq \alpha \leq t$. In the case of $f(\widetilde{\overline{y}}_1, \ldots, \widetilde{\overline{y}}_l) \in k(U)[[\widetilde{\overline{y}}_1, \ldots, \widetilde{\overline{y}}_l]]$, the above argument is valid with n replaced by l and $m(U(j+1))$ replaced by $n_{j+1,l}$, since $\overline{U}(0) \to \overline{U}(t)$ is then a UTS in the first l variables.

Step 3. — Now we will construct, with suitable choice of the series Λ_α and $\Omega_{i,\alpha}$, in (71)-(74) of Step 1, and our fixed N, a CRUTS $(R, T''(t), T(t))$ and $(S, U''(t), U(t))$ with associated MTS (91)

$$\begin{array}{ccccccc} S & = S(0) & \to & S(1) & \to \cdots \to & S(t) \\ & \uparrow & & \uparrow & & \uparrow \\ R & = R(0) & \to & R(1) & \to \cdots \to & R(t) \end{array} \tag{91}$$

such that $R(\alpha)$ has regular parameters

$$(x_1(\alpha), \ldots, x_n(\alpha)), (\widetilde{x}_1(\alpha), \ldots, \widetilde{x}_n(\alpha)), (\widetilde{x}'_1(\alpha), \ldots, \widetilde{x}'_n(\alpha)).$$

$S(\alpha)$ has regular parameters

$$(y_1(\alpha), \ldots, y_n(\alpha)), (\widetilde{y}_1(\alpha), \ldots, \widetilde{y}_n(\alpha)), (\widetilde{y}'_1(\alpha), \ldots, \widetilde{y}'_n(\alpha)).$$

(91) will consist of three types of MTSs.

(M1) $R(\alpha) \to R(\alpha+1)$ and $S(\alpha) \to S(\alpha+1)$ are of type I.

(M2) $R(\alpha) \to R(\alpha+1)$ is of type II_r, $s+1 \le r \le l$, and $S(\alpha) \to S(\alpha+1)$ is a MTS of type II_r, followed by a MTS of type I.

(M3) $R(\alpha) = R(\alpha+1)$ and $S(\alpha) \to S(\alpha+1)$ is of type II_r $(l+1 \le r \le m)$.

There exists for all α a regular local ring $\widetilde{R}(\alpha) \subset R(\alpha)$ such that $\widetilde{R}(\alpha)$ has regular parameters $(x_1(\alpha), \ldots, x_l(\alpha))$ and $\widetilde{R}(\alpha)\hat{} \cong k(c_0, \ldots, c_\alpha)[[x_1(\alpha), \ldots, x_l(\alpha)]]$.

The series Λ_α in (71) is chosen so that

$$x_i(\alpha) = \gamma_i(\alpha)\overline{\overline{x}}_i(\alpha)$$

for $1 \le i \le l$ where $\gamma_i(\alpha) \in k(c_0, \ldots, c_\alpha)[[x_1(\alpha), \ldots x_l(\alpha)]] \cap T''(\alpha)$ are units such that $\gamma_i(\alpha) \equiv 1 \mod (x_1(\alpha), \ldots, x_l(\alpha))$. In fact, in conjunction with an appropriate choice of $\lambda_{\alpha-1}$ in (93) below, we will have $x_r(\alpha) = \gamma_r(\alpha)\overline{x}_r(\alpha) + \psi_\alpha$ where

$$\psi_\alpha \in k(c_0, \ldots, c_\alpha)[[\overline{x}_1(\alpha), \ldots, \overline{x}_l(\alpha)]],$$

and $\mathrm{mult}(\psi_\alpha) \ge N$ if $R(\alpha-1) \to R(\alpha)$ is of type II_r.

The series $\Omega_{i,\alpha}$ in (72) is chosen so that we can define regular parameters $\widetilde{x}_i(\alpha)$ in $R(\alpha)$ by

$$\widetilde{x}_i(\alpha) = \begin{cases} \gamma_i(\alpha)\widetilde{\overline{x}}'_i(\alpha) & 1 \le i \le l \\ \widetilde{\overline{x}}'_i(\alpha) & l < i \end{cases} \tag{92}$$

(If $P_{i,\alpha} = 0$, or if $1 \le i \le s$ we will have $\widetilde{x}_i(\alpha) = x_i(\alpha)$.) Define regular parameters $\widetilde{x}'_i(\alpha)$ in $R(\alpha)$ by

$$\widetilde{x}'_i(\alpha) = \begin{cases} \lambda_\alpha \widetilde{x}_r(\alpha), & \text{if } R(\alpha) \to R(\alpha+1) \text{ is of type } \mathrm{II}_r \text{ and } i = r \\ \widetilde{x}_i(\alpha) & \text{otherwise} \end{cases} \tag{93}$$

We will have $\lambda_\alpha \in \widetilde{R}(\alpha) \subset k(c_0, \ldots, c_\alpha)[[x_1(\alpha), \ldots, x_l(\alpha)]]$, $\lambda_\alpha \equiv 1 \mod m(\widetilde{R}(\alpha))$.

These variables are such that for all α,

$$\begin{aligned}k(c_0,\ldots,c_\alpha)[[x_1(\alpha),\ldots,x_l(\alpha)]] &= k(c_0,\ldots,c_\alpha)[[\widetilde{x}_1(\alpha),\ldots,\widetilde{x}_l(\alpha)]]\\ &= k(c_0,\ldots,c_\alpha)[[\widetilde{x}'_1(\alpha),\ldots,\widetilde{x}'_l(\alpha)]]\\ &= k(c_0,\ldots,c_\alpha)[[\overline{x}_1(\alpha),\ldots,\overline{x}_l(\alpha)]]\\ &= k(c_0,\ldots,c_\alpha)[[\widetilde{\overline{x}}_1(\alpha),\ldots,\widetilde{\overline{x}}_l(\alpha)]]\\ &= k(c_0,\ldots,c_\alpha)[[\widetilde{\overline{x}}'_1(\alpha),\ldots,\widetilde{\overline{x}}'_l(\alpha)]]\end{aligned}$$

and

$$x_i(\alpha) = \widetilde{x}_i(\alpha) = \widetilde{x}'_i(\alpha) = \overline{x}_i(\alpha) = \widetilde{\overline{x}}_i(\alpha) = \widetilde{\overline{x}}'_i(\alpha)$$

for $l < i \le n$.

$k(c_0,\ldots,c_\alpha) \subset k(R(\alpha))$ and $k(\widetilde{R}(\alpha)) \cong k(c_0,\ldots,c_\alpha)$ for all i.

The series Λ_α in (73) is chosen so that

$$y_i(\alpha) = \gamma_i^y(\alpha)\widetilde{\overline{y}}_i(\alpha)$$

where $\gamma_i^y(\alpha) \in U''(\alpha)$ is a unit such that $\gamma_i^y(\alpha) \equiv 1 \bmod m(U''(\alpha))$ for $1 \le i \le m$. In fact, we will have $y_r(\alpha) = \gamma_r^y(\alpha)\overline{y}_r(\alpha) + \psi_\alpha$ where $\psi_\alpha \in S(\alpha)^\frown$, and $\mathrm{mult}(\psi_\alpha) \ge N$ if $S(\alpha-1) \to S(\alpha)$ is of type II_r, with $l+1 \le r$.

The series $\Omega_{i,\alpha}$ in (74) is chosen so that we can define regular parameters $\widetilde{y}_i(\alpha)$ in $S(\alpha)$ by

$$\widetilde{y}_i(\alpha) = \gamma_i^y(\alpha)\widetilde{\overline{y}}'_i(\alpha). \tag{94}$$

which satisfy

$$\widetilde{y}_i(\alpha) = \begin{cases} y_i(\alpha) & \text{if } 1 \le i \le s \text{ or } m < i \le n\\ \widetilde{x}_i(\alpha) & \text{if } s+1 \le i \le l\\ \gamma_i^y(\alpha)\widetilde{\overline{y}}'_i(\alpha), & \text{if } l+1 \le i \le m.\end{cases}$$

We will have $\gamma_i^y(\alpha) = \gamma_i(\alpha)$ if $s+1 \le i \le l$. Define regular parameters $\widetilde{y}'_i(\alpha)$ in $S(\alpha)$ by

$$\widetilde{y}'_i(\alpha) = \begin{cases} \widetilde{x}'_i(\alpha) & \text{if } s+1 \le i \le l\\ \lambda_\alpha \widetilde{y}_r(\alpha) & \text{if } S(\alpha) \to S(\alpha+1) \text{ is of type } \mathrm{II}_r \text{ with } r \ge l+1, i = r\\ \widetilde{y}_i(\alpha) & \text{otherwise}\end{cases} \tag{95}$$

We will have $\lambda_\alpha \in S(\alpha), \lambda_\alpha \equiv 1 \bmod m(S(\alpha))$.

These variables are such that for all α, $y_i(\alpha) = \widetilde{y}_i(\alpha) = y_i$ for $m < i \le n$.

Suppose that we have constructed the CRUTS out to $(R, T''(\alpha), T(\alpha))$ and $(S, U''(\alpha), U(\alpha))$, the MTS out to

$$\begin{array}{ccc} S & \to & S(\alpha)\\ \uparrow & & \uparrow\\ R & \to & R(\alpha)\end{array},$$

we have constructed $\widetilde{R}(\alpha) \subset R(\alpha)$ and have defined regular parameters

$$\begin{aligned} (x_1(\alpha),\ldots,x_n(\alpha)) &\text{ in } R(\alpha),\\ (\overline{x}_1(\alpha),\ldots,\overline{x}_n(\alpha)),\ (\widetilde{\overline{x}}_1(\alpha),\ldots,\widetilde{\overline{x}}_n(\alpha)) &\text{ in } T''(\alpha),\\ (y_1(\alpha),\ldots,y_n(\alpha)) &\text{ in } S(\alpha),\\ (\overline{y}_1(\alpha),\ldots,\overline{y}_n(\alpha)) \text{ and } (\widetilde{\overline{y}}_1(\alpha),\ldots,\widetilde{\overline{y}}_n(\alpha)) &\text{ in } U''(\alpha). \end{aligned}$$

Case 1. — Suppose that both $\overline{T}(\alpha) \to \overline{T}(\alpha+1)$ and $\overline{U}(\alpha) \to \overline{U}(\alpha+1)$ are of type I. By assumption $x_i(\alpha) = \gamma_i(\alpha)\widetilde{\overline{x}}_i(\alpha)$ for $1 \le i \le l$.

$$-\gamma_i(\alpha)P_{i,\alpha}(\widetilde{\overline{x}}_1(\alpha),\ldots,\widetilde{\overline{x}}_{i-1}(\alpha)) \in k(c_0,\ldots,c_\alpha)[[\widetilde{\overline{x}}_1(\alpha),\ldots,\widetilde{\overline{x}}_l(\alpha)]] = \widetilde{R}(\alpha)\hat{},$$

the completion of $\widetilde{R}(\alpha)$ for $s+1 \le i \le l$. Thus there exists $A \in \widetilde{R}(\alpha) \subset R(\alpha)$, $\overline{\Omega}_{i,\alpha} \in m\left(k(c_0,\ldots,c_\alpha)[[\widetilde{\overline{x}}_1(\alpha),\ldots,\widetilde{\overline{x}}_l(\alpha)]]\right)^N$ such that $A - \overline{\Omega}_{i,\alpha} = -\gamma_i(\alpha)P_{i,\alpha}$. Set $\Omega_{i,\alpha} = \gamma_i(\alpha)^{-1}\overline{\Omega}_{i,\alpha}$.

$$\begin{aligned} \gamma_i(\alpha)[\widetilde{\overline{x}}_i(\alpha) - P_{i,\alpha} + \Omega_{i,\alpha}] &= \gamma_i(\alpha)\widetilde{\overline{x}}_i(\alpha) - \gamma_i(\alpha)P_{i,\alpha} + \overline{\Omega}_{i,\alpha}\\ &= x_i(\alpha) + A \in \widetilde{R}(\alpha). \end{aligned}$$

Thus by suitable choice of the $\Omega_{i,\alpha}$, we have regular parameters $\widetilde{x}'_i(\alpha)$ in $R(\alpha)$ and regular parameters $\widetilde{\overline{x}}'_i(\alpha)$ in $T''(\alpha)$ satisfying (92) and (93). We can also define $\Omega_{i,\alpha}$ for $l+1 \le i \le m$ to get regular parameters $\widetilde{y}'_i(\alpha)$ in $S(\alpha)$, and regular parameters $\widetilde{\overline{y}}'_i(\alpha)$ in $U''(\alpha)$ satisfying (94) and (95). Define $R(\alpha) \to R(\alpha+1)$ and $S(\alpha) \to S(\alpha+1)$ by

$$\begin{aligned} \widetilde{x}'_1(\alpha) &= x_1(\alpha+1)^{a_{11}(\alpha+1)}\cdots x_s(\alpha+1)^{a_{1s}(\alpha+1)}\\ &\vdots\\ \widetilde{x}'_s(\alpha) &= x_1(\alpha+1)^{a_{s1}(\alpha+1)}\cdots x_s(\alpha+1)^{a_{ss}(\alpha+1)} \end{aligned}$$

and

$$\begin{aligned} \widetilde{y}'_1(\alpha) &= y_1(\alpha+1)^{b_{11}(\alpha+1)}\cdots y_s(\alpha+1)^{b_{1s}(\alpha+1)}\\ &\vdots\\ \widetilde{y}'_s(\alpha) &= y_1(\alpha+1)^{b_{s1}(\alpha+1)}\cdots y_s(\alpha+1)^{b_{ss}(\alpha+1)}. \end{aligned}$$

Then $T(\alpha+1) = R(\alpha+1)\hat{}$ and $U(\alpha+1) = S(\alpha+1)\hat{}$. Set $\widetilde{\overline{x}}_i(\alpha+1) = \overline{x}_i(\alpha+1)$, $\widetilde{\overline{y}}_i(\alpha+1) = \overline{y}_i(\alpha+1)$ for all i.

Set $(e_{ij}) = (a_{ij}(\alpha+1))^{-1}$, set

$$\gamma_i(\alpha+1) = \begin{cases} \gamma_1(\alpha)^{e_{i1}}\cdots\gamma_s(\alpha)^{e_{is}} & 1 \le i \le s\\ \gamma_i(\alpha) & s < i \le l \end{cases}$$

$\gamma_i(\alpha+1) \in k(c_0, \ldots, c_{\alpha+1})[[x_1(\alpha+1), \ldots, x_l(\alpha+1)]] \cap T''(\alpha+1)$ for $1 \le i \le l$.

$$x_i(\alpha+1) = \begin{cases} \gamma_i(\alpha+1)\widetilde{\overline{x}}_i(\alpha+1) & 1 \le i \le l \\ \widetilde{\overline{x}}_i(\alpha+1) & l+1 < i \le n \end{cases}$$

Set $(f_{ij}) = (b_{ij}(\alpha+1))^{-1}$, set

$$\gamma_i^y(\alpha+1) = \begin{cases} \gamma_1^y(\alpha)^{f_{i1}} \cdots \gamma_s^y(\alpha)^{f_{is}} & 1 \le i \le s \\ \gamma_i^y(\alpha) & s < i \le n \end{cases}$$

Then $y_i(\alpha+1) = \gamma_i^y(\alpha+1)\widetilde{\overline{y}}_i(\alpha+1)$ for $1 \le i \le n$. Set

$$\widetilde{R}(\alpha+1) = \widetilde{R}(\alpha)[x_1(\alpha+1), \ldots, x_s(\alpha+1)]_{(x_1(\alpha+1), \ldots, x_l(\alpha+1))}.$$

Case 2. — Suppose that both $\overline{T}(\alpha) \to \overline{T}(\alpha+1)$ and $\overline{U}(\alpha) \to \overline{U}(\alpha+1)$ are of type II_r with $s+1 \le r \le l$. By suitable choice of the $\Omega_{i,\alpha}$ as in Case 1, we have regular parameters $\widetilde{x}_i(\alpha)$ in $R(\alpha)$ and regular parameters $\widetilde{y}_i(\alpha)$ in $S(\alpha)$ satisfying (92) and (94). Set $(e_{ij}) = (a_{ij}(\alpha+1))^{-1}$. Choose $\lambda_\alpha \in \widetilde{R}(\alpha) \subset R(\alpha)$ in (93) so that

$$\begin{aligned}(96)\quad & \gamma_1(\alpha)^{e_{s+1,1}} \cdots \gamma_s(\alpha)^{e_{s+1,s}} \gamma_r(\alpha)^{e_{s+1,s+1}} \lambda_\alpha^{e_{s+1,s+1}} \equiv 1 \\ & \mod (x_1(\alpha), \ldots, x_l(\alpha))^N k(c_0, \ldots, c_\alpha)[[x_1(\alpha), \ldots, x_l(\alpha)]].\end{aligned}$$

Set

$$(97)\qquad \begin{aligned} A_1 &= \widetilde{x}_1'(\alpha)^{e_{11}} \cdots \widetilde{x}_s'(\alpha)^{e_{1s}} \widetilde{x}_r'(\alpha)^{e_{1,s+1}} \\ &\vdots \\ A_s &= \widetilde{x}_1'(\alpha)^{e_{s1}} \cdots \widetilde{x}_s'(\alpha)^{e_{ss}} \widetilde{x}_r'(\alpha)^{e_{s,s+1}} \\ A_r &= \widetilde{x}_1'(\alpha)^{e_{s+1,1}} \cdots \widetilde{x}_s'(\alpha)^{e_{s+1,s}} \widetilde{x}_r'(\alpha)^{e_{s+1,s+1}} \end{aligned}$$

where $(e_{ij}) = (a_{ij}(\alpha+1))^{-1}$.

Let k_3 be the integral closure of k in $R(\alpha)$. We have

$$k_3 \to k(R(\alpha)) \to k(T(\alpha)) \to k(T(\alpha+1)).$$

Set

$$C = (R(\alpha)[A_1, \ldots, A_s, A_r] \otimes_{k_3} k(T(\alpha+1)))_c$$

where $c = (A_1, \ldots, A_s, \widetilde{x}_{s+1}'(\alpha), \ldots, \widetilde{x}_{r-1}'(\alpha), A_r - c_{\alpha+1}, \widetilde{x}_{r+1}'(\alpha), \ldots, \widetilde{x}_n'(\alpha))$. Set $q_3 = R(\alpha)[A_1, \ldots, A_s, A_r] \cap c$ where the intersection is in C. Define

$$R(\alpha+1) = R(\alpha)[A_1, \ldots, A_s, A_r]_{q_3}.$$

Our construction gives an isomorphism $k(R(\alpha+1)) \cong k(T(\alpha+1))$. Define regular parameters $(x_1^*(\alpha+1), \ldots, x_n^*(\alpha+1))$ in $R(\alpha+1)\hat{}$ by

$$(98)\qquad x_i^*(\alpha+1) = \begin{cases} A_i & 1 \le i \le s \\ A_r - c_{\alpha+1} & i = r \\ \widetilde{x}_i'(\alpha) & s < i,\ i \ne r \end{cases}$$

$$\begin{aligned}
\widetilde{x}_1'(\alpha) &= x_1^*(\alpha+1)^{a_{11}(\alpha+1)}\cdots x_s^*(\alpha+1)^{a_{1s}(\alpha+1)}(x_r^*(\alpha+1)+c_{\alpha+1})^{a_{1,s+1}(\alpha+1)}\\
&\vdots\\
\widetilde{x}_s'(\alpha) &= x_1^*(\alpha+1)^{a_{s1}(\alpha+1)}\cdots x_s^*(\alpha+1)^{a_{ss}(\alpha+1)}(x_r^*(\alpha+1)+c_{\alpha+1})^{a_{s,s+1}(\alpha+1)}\\
\widetilde{x}_r'(\alpha) &= x_1^*(\alpha+1)^{a_{s+1,1}(\alpha+1)}\cdots x_s^*(\alpha+1)^{a_{s+1,s}(\alpha+1)}(x_r^*(\alpha+1)+c_{\alpha+1})^{a_{s+1,s+1}(\alpha+1)}
\end{aligned}$$

Set $\widetilde{R}(\alpha+1)=\widetilde{R}(\alpha)[A_1,\ldots,A_s,A_r]_q$ where $q=\widetilde{R}(\alpha)[A_1,\ldots,A_s,A_r]\cap c$.

$$k(\widetilde{R}(\alpha+1))\cong k(\widetilde{R}(\alpha))(c_{\alpha+1})\cong k(c_0,\ldots,c_{\alpha+1})$$

and

$$\widetilde{R}(\alpha+1)\hat{}\cong k(c_0,\ldots,c_{\alpha+1})\,[[x_1^*(\alpha+1),\ldots,x_l^*(\alpha+1)]].$$

Set

$$\begin{aligned}
B_1 &= \widetilde{y}_1'(\alpha)^{f_{11}}\cdots\widetilde{y}_s'(\alpha)^{f_{1s}}\widetilde{y}_r'(\alpha)^{f_{1,s+1}}\\
&\vdots\\
B_s &= \widetilde{y}_1'(\alpha)^{f_{s1}}\cdots\widetilde{y}_s'(\alpha)^{f_{ss}}\widetilde{y}_r'(\alpha)^{f_{s+1,s+1}}\\
B_r &= \widetilde{y}_1'(\alpha)^{f_{s+1,1}}\cdots\widetilde{y}_s'(\alpha)^{f_{s+1,s}}\widetilde{y}_r'(\alpha)^{f_{s+1,s+1}}
\end{aligned}\tag{99}$$

where $(f_{ij})=(b_{ij}(\alpha+1))^{-1}$. Let k_4 be the integral closure of k in $S(\alpha)$. We have

$$k_4\to k(S(\alpha))\to k(U(\alpha))\to k(U(\alpha+1)).$$

Set

$$D=(S(\alpha)[B_1,\ldots,B_s,B_r]\otimes_{k_4}k(U(\alpha+1)))_d$$

where $d=(B_1,\ldots,B_s,\widetilde{y}_{s+1}'(\alpha),\ldots,\widetilde{y}_{r-1}'(\alpha),B_r-d_{\alpha+1},\widetilde{y}_{r+1}'(\alpha),\ldots,\widetilde{y}_n'(\alpha))$. Set $q_4=S(\alpha)[B_1,\ldots,B_s,B_r]\cap d$ where the intersection is in D. Define

$$S(\alpha+1)=S(\alpha)[B_1,\ldots,B_s,B_r]_{q_4}.$$

Our construction gives an isomorphism $k(S(\alpha+1))\cong k(U(\alpha+1))$. Define regular parameters $(y_1^*(\alpha+1),\ldots,y_n^*(\alpha+1))$ in $S(\alpha+1)\hat{}$ by

$$y_i^*(\alpha+1)=\begin{cases}B_i & 1\le i\le s\\ B_r-d_{\alpha+1} & i=r\\ \widetilde{y}_i'(\alpha) & s<i,\ i\ne r\end{cases}\tag{100}$$

$$\begin{aligned}
\widetilde{y}_1'(\alpha) &= y_1^*(\alpha+1)^{b_{11}(\alpha+1)}\cdots y_s^*(\alpha+1)^{b_{1s}(\alpha+1)}(y_r^*(\alpha+1)+d_{\alpha+1})^{b_{1,s+1}(\alpha+1)}\\
&\vdots\\
\widetilde{y}_s'(\alpha) &= y_1^*(\alpha+1)^{b_{s1}(\alpha+1)}\cdots y_s^*(\alpha+1)^{b_{ss}(\alpha+1)}(y_r^*(\alpha+1)+d_{\alpha+1})^{b_{s,s+1}(\alpha+1)}\\
\widetilde{y}_r'(\alpha) &= y_1^*(\alpha+1)^{b_{s+1,1}(\alpha+1)}\cdots y_s^*(\alpha+1)^{b_{s+1,s}(\alpha+1)}(y_r^*(\alpha+1)+d_{\alpha+1})^{b_{s+1,s+1}(\alpha+1)}.
\end{aligned}$$

Set $\sigma_i=\gamma_1(\alpha)^{e_{i1}}\cdots\gamma_s(\alpha)^{e_{is}}\gamma_r(\alpha)^{e_{i,s+1}}\lambda_\alpha^{e_{i,s+1}}$ for $1\le i\le s$, and set

$$\sigma_r=\gamma_1(\alpha)^{e_{s+1,1}}\cdots\gamma_s(\alpha)^{e_{s+1,s}}\gamma_r(\alpha)^{e_{s+1,s+1}}\lambda_\alpha^{e_{s+1,s+1}}.$$

By (92), (93), (76) and (97)

$$
\begin{aligned}
A_1 &= \sigma_1 M_1 \\
&\vdots \\
A_s &= \sigma_s M_s \\
A_r &= \sigma_r M_r.
\end{aligned}
\tag{101}
$$

Thus $R(\alpha)[A_1, \ldots, A_s, A_r] \subset T''(\alpha)[M_1, \ldots, M_s, M_r]$. We then have an inclusion $C \to A$ which induces an inclusion $R(\alpha+1) \to T'(\alpha+1)$. By (98), (101), (77) and (79),

$$x_i^*(\alpha+1) = \sigma_i(\overline{x}_r(\alpha+1)+1)^{\overline{c}_{\alpha+1}\gamma_i(\alpha+1)}\overline{x}_i(\alpha+1)$$

for $1 \le i \le s$ and

$$
\begin{aligned}
x_r^*(\alpha+1) + c_{\alpha+1} &= \sigma_r(\widehat{\overline{x}}_r(\alpha+1) + c_{\alpha+1}) \\
&= \sigma_r\left(c_{\alpha+1}(\overline{x}_r(\alpha+1)+1)^{\overline{c}_{\alpha+1}}\right) \\
&= \sigma_r(\overline{c}_{\alpha+1}c_{\alpha+1}u\overline{x}_r(\alpha+1) + c_{\alpha+1})
\end{aligned}
$$

where $u \in \mathbf{Q}[\overline{x}_r(\alpha+1)]$ is a polynomial with constant term 1.

$$\frac{x_r^*(\alpha+1)}{\overline{c}_{\alpha+1}c_{\alpha+1}} = \sigma_r u\overline{x}_r(\alpha+1) + \frac{\sigma_r - 1}{\overline{c}_{\alpha+1}}$$

By (96), $T(\alpha+1) = R(\alpha+1)\hat{}$ since $k(R(\alpha+1)) \cong k(T(\alpha+1))$, $(x_1^*(\alpha+1), \ldots, x_n^*(\alpha+1))$ are regular parameters in $T(\alpha+1)$, by our construction of σ_r.

Thus there exists

$$\Omega \in (\overline{x}_1(\alpha+1), \ldots, \overline{x}_l(\alpha+1))^N k(c_0, \ldots, c_{\alpha+1})[[\overline{x}_1(\alpha+1), \ldots, \overline{x}_l(\alpha+1)]]$$

such that

$$\sigma_r u\overline{x}_r(\alpha+1) + \Omega \in \widetilde{R}(\alpha+1) \subset R(\alpha+1).$$

Set

$$x_i(\alpha+1) = \begin{cases} x_i^*(\alpha+1) & i \ne r \\ \sigma_r u\overline{x}_r(\alpha+1) + \Omega & i = r \end{cases}$$

$$\gamma_i(\alpha+1) = \begin{cases} \sigma_i(\overline{x}_r(\alpha+1)+1)^{\overline{c}_{\alpha+1}\gamma_i(\alpha+1)} & 1 \le i \le s \\ \sigma_r u & i = r \\ \gamma_i(\alpha) & s < i \le l, i \ne r \end{cases}$$

Set $\Lambda_{\alpha+1} = \sigma_r^{-1}u^{-1}\Omega$. By definition

$$\widetilde{\overline{x}}_i(\alpha+1) = \begin{cases} \overline{x}_r(\alpha+1) + \Lambda_{\alpha+1} & i = r \\ \overline{x}_i(\alpha+1) & i \ne r \end{cases}$$

Then $(x_1(\alpha+1), \ldots, x_n(\alpha+1))$ are regular parameters in $R(\alpha+1)$ and

$$x_i(\alpha+1) = \gamma_i(\alpha+1)\widetilde{\overline{x}}_i(\alpha+1)$$

for $1 \le i \le l$. Since $\sigma_i \in T''(\alpha)$ for all i, $c_{\alpha+1} \in T''(\alpha+1)$, $\overline{c}_{\alpha+1}\gamma_i(\alpha+1)$ are integers and $(\overline{x}_1(\alpha+1), \ldots, \overline{x}_n(\alpha+1))$ are regular parameters in $T''(\alpha+1)$, $\gamma_i(\alpha+1) \in T''(\alpha+1)$ for all i and $(\overline{\overline{x}}_1(\alpha+1), \ldots, \overline{\overline{x}}_n(\alpha+1))$ are regular parameters in $T''(\alpha+1)$.

Set $\sigma_i^y = \gamma_1^y(\alpha)^{f_{i1}} \cdots \gamma_s^y(\alpha)^{f_{is}} \gamma_r^y(\alpha)^{f_{i,s+1}} \lambda_\alpha^{f_{i,s+1}}$ for $1 \le i \le s$ and set

$$\sigma_r^y = \gamma_1^y(\alpha)^{f_{s+1,1}} \cdots \gamma_s^y(\alpha)^{f_{s+1,s}} \gamma_r^y(\alpha)^{f_{s+1,s+1}} \lambda_\alpha^{f_{s+1,s+1}}.$$

By (94), (95), (82) and (99)

$$\begin{aligned} B_1 &= \sigma_1^y N_1 \\ &\vdots \\ B_s &= \sigma_s^y N_s \\ B_r &= \sigma_r^y N_r. \end{aligned} \tag{102}$$

Thus $S(\alpha)[B_1, \ldots, B_s, B_r] \subset U''(\alpha)[N_1, \ldots, N_s, N_r]$. We then have an inclusion $D \subset B$ which induces an inclusion $S(\alpha+1) \to U'(\alpha+1)$. By (100), (102), (85) and (87)

$$\begin{aligned} y_i^*(\alpha+1) &= \sigma_i^y \widehat{\overline{y}}_i(\alpha+1) \\ &= \sigma_i^y (\overline{y}_r(\alpha+1)+1)^{\overline{d}_{\alpha+1}\tau_i(\alpha+1)} \overline{y}_i(\alpha+1) \end{aligned}$$

for $1 \le i \le s$ so that $y_i^*(\alpha+1) = \text{unit } \overline{y}_i(\alpha+1)$ in $U(\alpha+1)$ for $1 \le i \le s$, and

$$\begin{aligned} y_r^*(\alpha+1) + d_{\alpha+1} &= \sigma_r^y (\widehat{\overline{y}}_r(\alpha+1) + d_{\alpha+1}) \\ &= \sigma_r^y d_{\alpha+1} (\overline{y}_r(\alpha+1)+1)^{\overline{d}_{\alpha+1}} \\ &= \sigma_r^y [\overline{d}_{\alpha+1} d_{\alpha+1} u \overline{y}_r(\alpha+1) + d_{\alpha+1}] \end{aligned}$$

where $u \in \mathbf{Q}[\overline{y}_r(\alpha+1)]$ is a polynomial with constant term 1. $\sigma_r^y \equiv 1 \bmod m(U(\alpha))$. Thus

$$\sigma_r^y \equiv 1 \bmod (\overline{y}_1(\alpha+1), \ldots, \overline{y}_{r-1}(\alpha+1), \overline{y}_{r+1}(\alpha+1), \ldots, \overline{y}_n(\alpha+1)).$$

$$\begin{aligned} \frac{y_r^*(\alpha+1)}{\overline{d}_{\alpha+1} d_{\alpha+1}} &= \sigma_r^y u \overline{y}_r(\alpha+1) + \frac{\sigma_r^y - 1}{\overline{d}_{\alpha+1}} \\ &\equiv u \overline{y}_r(\alpha+1) \\ &\qquad \bmod (\overline{y}_1(\alpha+1), \ldots, \overline{y}_{r-1}(\alpha+1), \overline{y}_{r+1}(\alpha+1), \ldots, \overline{y}_n(\alpha+1)). \end{aligned}$$

Thus $U(\alpha+1) = S(\alpha+1)\hat{}$ since $(y_1^*(\alpha+1), \ldots, y_n^*(\alpha+1))$ are regular parameters in $U(\alpha+1)$ and $k(S(\alpha+1)) \cong k(U(\alpha+1))$. By Lemma 3.3, $T(\alpha+1) \to U(\alpha+1)$ induces a map $R(\alpha+1) \to S(\alpha+1)$. Set

$$y_i(\alpha+1) = \begin{cases} y_i^*(\alpha+1) & i \ne r \\ x_r(\alpha+1) & i = r \end{cases}$$

$$\gamma_i^y(\alpha+1) = \begin{cases} \sigma_i^y(\overline{y}_r(\alpha+1)+1)^{\overline{d}_{\alpha+1}\tau_i(\alpha+1)} & 1 \le i \le s \\ \gamma_i(\alpha+1) & i = r \\ \gamma_i^y(\alpha) & s < i \le n, i \ne r \end{cases}$$

By definition

$$\widetilde{\overline{y}}_i(\alpha+1) = \begin{cases} \overline{y}_i(\alpha+1) & i \neq r \\ \widetilde{\overline{x}}_i(\alpha+1) & i = r \end{cases}$$

Then $(y_1(\alpha+1), \ldots, y_n(\alpha+1))$ are regular parameters in $S(\alpha+1)$ and

$$y_i(\alpha+1) = \gamma_i^y(\alpha+1)\widetilde{\overline{y}}_i(\alpha+1)$$

for $1 \leq i \leq n$, $\gamma_i^y(\alpha+1) \in U''(\alpha+1)$ and $(\widetilde{\overline{y}}_1(\alpha+1), \ldots, \widetilde{\overline{y}}_n(\alpha+1))$ are regular parameters in $U''(\alpha+1)$.

Case 3. — Suppose that $\overline{T}(\alpha) = \overline{T}(\alpha+1)$ and $\overline{U}(\alpha) \rightarrow \overline{U}(\alpha+1)$ is of type II_r with $l+1 \leq r \leq m$. By suitable choice of the $\Omega_{i,\alpha}$ as in Case 1, we have regular parameters $\widetilde{y}_i(\alpha)$ in $S(\alpha)$ satisfying (94). Set $(f_{ij}) = (b_{ij}(\alpha+1))^{-1}$. Choose λ_α in (95) so that

$$\gamma_1^y(\alpha)^{f_{s+1,1}} \cdots \gamma_s^y(\alpha)^{f_{s+1,s}} \gamma_r^y(\alpha)^{f_{s+1,s+1}} \lambda_\alpha^{f_{s+1,s+1}} \equiv 1 \bmod m(U(\alpha))^N.$$

As in the argument of Case 2, we can define, by (99),

$$S(\alpha+1) = S(\alpha)[B_1, \ldots, B_s, B_r]_{q_4}$$

so that $S(\alpha+1)^\frown$ has regular parameters $(y_1^*(\alpha+1), \ldots, y_n^*(\alpha+1))$ defined by (100). Set $\sigma_i^y = \gamma_1^y(\alpha)^{f_{i1}} \cdots \gamma_s^y(\alpha)^{f_{is}} \gamma_r^y(\alpha)^{f_{i,s+1}} \lambda_\alpha^{f_{i,s+1}}$ for $1 \leq i \leq s$ and set

$$\sigma_r = \gamma_1^y(\alpha)^{f_{s+1,1}} \cdots \gamma_s^y(\alpha)^{f_{s+1,s}} \gamma_r^y(\alpha)^{f_{s+1,s+1}} \lambda_\alpha^{f_{s+1,s+1}}.$$

Then equations (102) hold, and we have an inclusion $S(\alpha+1) \rightarrow U'(\alpha+1)$ as in the argument of Case 2.

By (100), (102), (85) and (87),

$$\begin{aligned} y_i^*(\alpha+1) &= \sigma_i \widehat{\overline{y}}_i(\alpha+1) \\ &= \sigma_i(\overline{y}_r(\alpha+1)+1)^{\overline{d}_{\alpha+1}\tau_i}\overline{y}_i(\alpha+1) \end{aligned}$$

for $1 \leq i \leq s$ and

$$\begin{aligned} y_r^*(\alpha+1) + d_{\alpha+1} &= \sigma_r(\widehat{\overline{y}}_r(\alpha+1) + d_{\alpha+1}) \\ &= \sigma_r^y d_{\alpha+1}(\overline{y}_r(\alpha+1)+1)^{\overline{d}_{\alpha+1}} \\ &= \sigma_r^y[\overline{d}_{\alpha+1} d_{\alpha+1} u \overline{y}_r(\alpha+1) + d_{\alpha+1}] \end{aligned}$$

where $u \in \mathbf{Q}[\overline{y}_r(\alpha+1)]$ is a polynomial with constant term 1.

$$\frac{y_r^*(\alpha+1)}{\overline{d}_{\alpha+1} d_{\alpha+1}} = \sigma_r^y u \overline{y}_r(\alpha+1) + \frac{\sigma_r^y - 1}{\overline{d}_{\alpha+1}}$$

Recall that $\sigma_r^y \equiv 1 \bmod m(U(\alpha))^N$. Thus $U(\alpha+1) = S(\alpha+1)^\frown$ since $(y_1^*(\alpha+1), \ldots, y_n^*(\alpha+1))$ are regular parameters in $U(\alpha+1)$ and $k(S(\alpha+1)) \cong k(U(\alpha+1))$.

Thus there exists

$$\Omega \in (\overline{y}_1(\alpha+1), \ldots, \overline{y}_n(\alpha+1))^N U(\alpha+1) = (\overline{y}_1(\alpha+1), \ldots, \overline{y}_n(\alpha+1))^N S(\alpha+1)^\frown$$

such that

$$\sigma_r u \overline{y}_r(\alpha+1) + \Omega \in S(\alpha+1).$$

Set

$$y_i(\alpha+1) = \begin{cases} y_i^*(\alpha+1) & i \neq r \\ \sigma_r u \overline{y}_r(\alpha+1) + \Omega & i = r \end{cases}$$

$$\gamma_i^y(\alpha+1) = \begin{cases} \sigma_i(\overline{y}_r(\alpha+1)+1)^{\overline{d}_{\alpha+1}\tau_i} & 1 \leq i \leq s \\ \sigma_r u & i = r \\ \gamma_i^y(\alpha) & s < i \leq n, i \neq r \end{cases}$$

Set $\Lambda_{\alpha+1} = \sigma_r^{-1} u^{-1} \Omega$. By definition

$$\widetilde{\overline{y}}_i(\alpha+1) = \begin{cases} \overline{y}_i(\alpha+1) & i \neq r \\ \overline{y}_r(\alpha+1) + \Lambda_{\alpha+1} & i = r \end{cases}$$

Then $(y_1(\alpha+1), \ldots, y_n(\alpha+1))$ are regular parameters in $S(\alpha+1)$ and

$$y_i(\alpha+1) = \gamma_i^y(\alpha+1)\widetilde{\overline{y}}_i(\alpha+1)$$

for $1 \leq i \leq n$. $\gamma_i^y(\alpha+1) \in U''(\alpha+1)$ for all i and $(\widetilde{\overline{y}}_1(\alpha+1), \ldots, \widetilde{\overline{y}}_n(\alpha+1))$ are regular parameters in $U''(\alpha+1)$.

Step 4. — It remains to show that the CRUTS (67), (91); $(R, T''(t), T(t))$ and $(S, U''(t), U(t))$, constructed in step 3 is a CRUTS along ν if N is sufficiently large.

We have an extension of ν to the quotient field of $\overline{U}(t)$ which dominates $\overline{U}(t)$. Define $\widetilde{U}(t) = \overline{U}(t)/B(t)$ where $B(t)$ is the prime ideal of elements of $\overline{U}(t)$ of infinite value. Let $G(t)$ be the quotient field of $\widetilde{U}(t)$. Let $\overline{K}$ be the completion of K with respect to a metric associated to ν (cf. Lemma 2.2), $\overline{G}(t)$ be the completion of $G(t)$ with respect to a metric associated to ν. We have a natural inclusion of complete fields $\overline{K} \to \overline{G}(t)$. Suppose that for some $0 \leq \beta \leq t$

$$\begin{array}{ccccccc} U(0) & \to & U(1) & \to & \cdots & \to & U(\beta) \\ \uparrow & & \uparrow & & & & \uparrow \\ T(0) & \to & T(1) & \to & \cdots & \to & T(\beta) \end{array}$$

is a CRUTS along ν. Then by Lemma 2.2, we have natural maps:

$$U(i) = \widehat{S}(i) \to \overline{K} \quad \text{for } 0 \leq i \leq \beta.$$

Let $\mathcal{O}_w$ be the valuation ring of the natural extension w of ν to $\overline{G}(t)$, m_w be the maximal ideal of $\mathcal{O}_w$ and Γ_w be the value group of w. We have an inclusion $k(\overline{U}(t)) \to \mathcal{O}_w/m_w$.

We will prove the following inductive statement on α with $0 \leq \alpha \leq t$. Given a positive element $\lambda'_\alpha \in \Gamma_w$, such that

$$\lambda'_\alpha > \max\{w(\overline{w}_1(\alpha)), \ldots, w(\overline{w}_n(\alpha))\}, \tag{103}$$

there exists a positive element N_α such that if $N \geq N_\alpha$, and

$$\begin{array}{ccccccc} U(0) & \to & U(1) & \to & \cdots & \to & U(\alpha) \\ \uparrow & & \uparrow & & & & \uparrow \\ T(0) & \to & T(1) & \to & \cdots & \to & T(\alpha) \end{array} \tag{104}$$

is a CRUTS (67) as constructed in Step 3, then

(A1) $U(\alpha) \to \mathcal{O}_w$ and there is a commutative diagram

$$\begin{array}{ccc} k(\overline{U}(\alpha)) & \to & \mathcal{O}_w/m_w \\ \uparrow \eta_U^\alpha & \nearrow & \\ k(U(\alpha)) & & \end{array}$$

(A2) $w(\overline{y}_i(\alpha)) > 0$ for $1 \leq i \leq n$.

(A3) $\overline{y}_i(\alpha) = \overline{w}_i(\alpha)$ for $1 \leq i \leq s$, $w(\overline{w}_i(\alpha) - \overline{y}_i(\alpha)) > \lambda'_\alpha$ for $s+1 \leq i \leq n$.

Since $\overline{y}_i(0) = \overline{w}_i(0)$, for $1 \leq i \leq n$, and $U''(0) = \overline{U}''(0)$, the statement is true for $m = 0$.

Suppose the inductive statement is true for CRUTS of length α, and for any given $\lambda'_\alpha \in \Gamma_w$ satisfying (103). We will prove it for sequences of length $\alpha+1$, and any given $\lambda'_{\alpha+1}$ such that

$$\lambda'_{\alpha+1} > \max\{w(\overline{w}_1(\alpha+1)), \ldots, w(\overline{w}_n(\alpha+1))\}.$$

Choose $\lambda' = \lambda'_{\alpha+1}$ if $\overline{U}(\alpha) \to \overline{U}(\alpha+1)$ is of type I,

$$\lambda' > \lambda'_{\alpha+1} + b_{s+1,1}(\alpha+1)w(\overline{w}_1(\alpha+1)) + \cdots + b_{s+1,s}(\alpha+1)w(\overline{w}_s(\alpha+1))$$

if $\overline{U}(\alpha) \to \overline{U}(\alpha+1)$ is of type II_r. By induction, there exists N'' such that $N \geq N''$ implies $w(\overline{w}_i(\alpha) - \overline{y}_i(\alpha)) > \lambda'$ for all i, and we can further choose N'' so large that $Nw(m(\overline{U}(\alpha))) > \lambda'$. Then $\nu(\overline{y}_i(\alpha)) = \nu(\overline{w}_i(\alpha))$ for all i and $Nw(m(U(\alpha))) > \lambda'$.

$\nu(\Lambda_\alpha) > \lambda'$ implies $w(\overline{w}_i(\alpha) - \widetilde{\overline{y}}_i(\alpha)) > \lambda'$ for $s+1 \leq i \leq n$. $\nu(\Omega_{i,\alpha}) > \lambda'$ implies $w(\widetilde{\overline{w}}'_i(\alpha) - \widetilde{\overline{y}}'_i(\alpha)) > \lambda'$ for $s+1 \leq i \leq n$. Thus there exists $\sigma_i \in \mathcal{O}_w$ with $w(\sigma_i) > \lambda'$ such that $\widetilde{\overline{y}}'_i(\alpha) = \widetilde{\overline{w}}'_i(\alpha) + \sigma_i$ for $s+1 \leq i \leq n$ and $\widetilde{\overline{y}}'_i(\alpha) = \widetilde{\overline{w}}'_i(\alpha)$ for $1 \leq i \leq s$.

Suppose that $\overline{U}(\alpha) \to \overline{U}(\alpha+1)$ is of type I.

$$\widetilde{\overline{y}}'_i(\alpha) = \prod_{j=1}^{s} \overline{y}_j(\alpha+1)^{b_{ij}(\alpha+1)}$$

and

$$\widetilde{\overline{w}}'_i(\alpha) = \prod_{j=1}^{s} \overline{w}_j(\alpha+1)^{b_{ij}(\alpha+1)}$$

for $1 \leq i \leq s$. Thus $\overline{y}_j(\alpha+1) = \overline{w}_{j+1}(\alpha+1)$ for $1 \leq j \leq s$ and (A1), (A2) and (A3) hold for $\alpha+1$.

Suppose that $\overline{U}(\alpha) \to \overline{U}(\alpha+1)$ is of type II_r. Set $b_{ij} = b_{ij}(\alpha+1)$. Set $(f_{ij}) = (b_{ij}(\alpha+1))^{-1}$. $\overline{U}(\alpha+1)$ has regular parameters $(\widehat{\overline{w}}_1(\alpha+1), \ldots, \widehat{\overline{w}}_n(\alpha+1))$ such that

$$\begin{aligned}
\widehat{\overline{w}}_1(\alpha+1) &= \widetilde{\overline{w}}_1'(\alpha)^{f_{11}} \cdots \widetilde{\overline{w}}_s'(\alpha)^{f_{1s}} \widetilde{\overline{w}}_r'(\alpha)^{f_{1,s+1}} \\
&\ \vdots \\
\widehat{\overline{w}}_s(\alpha+1) &= \widetilde{\overline{w}}_1'(\alpha)^{f_{s1}} \cdots \widetilde{\overline{w}}_s'(\alpha)^{f_{ss}} \widetilde{\overline{w}}_r'(\alpha)^{f_{s,s+1}} \\
\widehat{\overline{w}}_r(\alpha+1) + d_{\alpha+1} &= \widetilde{\overline{w}}_1'(\alpha)^{f_{s+1,1}} \cdots \widetilde{\overline{w}}_s'(\alpha)^{f_{s+1,s}} \widetilde{\overline{w}}_r'(\alpha)^{f_{s+1,s+1}}
\end{aligned} \tag{105}$$

Recall equation (82).

$$\begin{aligned}
N_1 &= \widetilde{\overline{y}}_1'(\alpha)^{f_{11}} \cdots \widetilde{\overline{y}}_s'(\alpha)^{f_{1s}} \widetilde{\overline{y}}_r'(\alpha)^{f_{1,s+1}} \\
&\ \vdots \\
N_s &= \widetilde{\overline{y}}_1'(\alpha)^{f_{s1}} \cdots \widetilde{\overline{y}}_s'(\alpha)^{f_{ss}} \widetilde{\overline{y}}_r'(\alpha)^{f_{s,s+1}} \\
N_r &= \widetilde{\overline{y}}_1'(\alpha)^{f_{s+1,1}} \cdots \widetilde{\overline{y}}_s'(\alpha)^{f_{s+1,s}} \widetilde{\overline{y}}_r'(\alpha)^{f_{s+1,s+1}}
\end{aligned} \tag{106}$$

There is a natural map

$$U''(\alpha)[N_1, \ldots, N_s, N_r] \to \overline{K} \to G(t).$$

From (105) and (106) we have

$$\frac{N_i}{\widehat{\overline{w}}_i(\alpha+1)} = \left(\frac{\widetilde{\overline{y}}_r'(\alpha)}{\widetilde{\overline{w}}_r'(\alpha)}\right)^{f_{i,s+1}} \in \mathcal{O}_w \tag{107}$$

for $1 \le i \le s$ and

$$\frac{N_r}{\widehat{\overline{w}}_r(\alpha+1) + d_{\alpha+1}} = \left(\frac{\widetilde{\overline{y}}_r'(\alpha)}{\widetilde{\overline{w}}_r'(\alpha)}\right)^{f_{s+1,s+1}} \in \mathcal{O}_w \tag{108}$$

All of these ratios have residue 1 in $\mathcal{O}_w/m_w$. Thus $U''(\alpha)[N_1, \ldots, N_s, N_r] \to \mathcal{O}_w$. Then since $k(\overline{U}(\alpha+1)) \subset \mathcal{O}_w$, and we have an inclusion

$$k(U(\alpha)) \to k(\overline{U}(\alpha)) \to k(\overline{U}(\alpha+1)) \to \mathcal{O}_w/m_w.$$

There is a natural map

$$U''(\alpha)[N_1, \ldots, N_s, N_r] \otimes_{k_2} k(\overline{U}(\alpha+1)) \to \mathcal{O}_w$$

where k_2 is the integral closure of k in $U(\alpha)$.

By (107) and (108), $\nu(N_i) = \nu(\widehat{\overline{w}}_i(\alpha+1)) > 0$ for $1 \le i \le s$ and

$$N_r \longmapsto u(\widehat{\overline{w}}_r(\alpha+1) + d_{\alpha+1})$$

where $u \in \mathcal{O}_w$ is a unit, $u \equiv 1 \bmod m_w$. Thus the residue of N_r in $\mathcal{O}_w/m_w$ is $d_{\alpha+1}$. Thus $w(N_r - d_{\alpha+1}) > 0$ and we have a map $U'(\alpha+1) \to \mathcal{O}_w$ by (83) and (84), which induces a map $U(\alpha+1) \to \mathcal{O}_w$ such that

$$\begin{array}{ccc}
k(\overline{U}(\alpha+1)) & \to & k(\mathcal{O}_w) \\
\uparrow \eta_{\alpha+1}^U & \nearrow & \\
k(U(\alpha+1)) & &
\end{array}$$

commutes, verifying (A1) for $\alpha+1$.

The regular parameters $(\widehat{\overline{y}}_1(\alpha+1),\ldots,\widehat{\overline{y}}_n(\alpha+1))$ in $U(\alpha+1)$ defined by (85) satisfy

$$\widehat{\overline{y}}_i(\alpha+1)=\begin{cases} N_i & 1\le i\le s\\ N_r-d_{\alpha+1} & i=r\\ \widetilde{\overline{y}}'_i(\alpha) & s<i,\ i\ne r\end{cases} \tag{109}$$

$\overline{U}(\alpha+1)$ has regular parameters $(\overline{w}_1(\alpha+1),\ldots,\overline{w}_n(\alpha+1))$ defined by

$$\overline{w}_i(\alpha+1)=\begin{cases} \widehat{\overline{w}}_i(\alpha+1)\left(\frac{\widehat{\overline{w}}_r(\alpha+1)}{d_{\alpha+1}}+1\right)^{-\tau_i(\alpha+1)} & 1\le i\le s\\ \left(\frac{\widehat{\overline{w}}_r(\alpha+1)}{d_{\alpha+1}}+1\right)^{1/\overline{d}_{\alpha+1}}-1 & i=r\\ \widehat{\overline{w}}_i(\alpha+1) & s<i,\ i\ne r\end{cases} \tag{110}$$

The regular parameters $(\overline{y}_1(\alpha+1),\ldots,\overline{y}_n(\alpha+1))$ of $U(\alpha+1)$ are defined by (87),

$$\overline{y}_i(\alpha+1)=\begin{cases} \widehat{\overline{y}}_i(\alpha+1)\left(\frac{\widehat{\overline{y}}_r(\alpha+1)}{d_{\alpha+1}}+1\right)^{-\tau_i(\alpha+1)} & 1\le i\le s\\ \left(\frac{\widehat{\overline{y}}_r(\alpha+1)}{d_{\alpha+1}}+1\right)^{1/\overline{d}_{\alpha+1}}-1 & i=r\\ \widehat{\overline{y}}_i(\alpha+1) & s<i,\ i\ne r\end{cases} \tag{111}$$

$$\begin{aligned} \widetilde{\overline{y}}'_1(\alpha) &= \overline{y}_1(\alpha+1)^{b_{11}(\alpha+1)}\cdots\overline{y}_s(\alpha+1)^{b_{1s}(\alpha+1)}d_{\alpha+1}^{b_{1,s+1}(\alpha+1)}\\ &\ \vdots\\ \widetilde{\overline{y}}'_s(\alpha) &= \overline{y}_1(\alpha+1)^{b_{s1}(\alpha+1)}\cdots\overline{y}_s(\alpha+1)^{b_{ss}(\alpha+1)}d_{\alpha+1}^{b_{s,s+1}(\alpha+1)}\\ \widetilde{\overline{y}}'_r(\alpha) &= \overline{y}_1(\alpha+1)^{b_{s+1,1}(\alpha+1)}\cdots\overline{y}_s(\alpha+1)^{b_{s+1,s}(\alpha+1)}\\ &\qquad\cdot(\overline{y}_r(\alpha+1)+1)d_{\alpha+1}^{b_{s+1,s+1}(\alpha+1)} \end{aligned} \tag{112}$$

$$\begin{aligned} \widetilde{\overline{w}}'_1(\alpha) &= \overline{w}_1(\alpha+1)^{b_{11}(\alpha+1)}\cdots\overline{w}_s(\alpha+1)^{b_{1s}(\alpha+1)}d_{\alpha+1}^{b_{1,s+1}(\alpha+1)}\\ &\ \vdots\\ \widetilde{\overline{w}}'_s(\alpha) &= \overline{w}_1(\alpha+1)^{b_{s1}(\alpha+1)}\cdots\overline{w}_s(\alpha+1)^{b_{ss}(\alpha+1)}d_{\alpha+1}^{b_{s,s+1}(\alpha+1)}\\ \widetilde{\overline{w}}'_r(\alpha) &= \overline{w}_1(\alpha+1)^{b_{s+1,1}(\alpha+1)}\cdots\overline{w}_s(\alpha+1)^{b_{s+1,s}(\alpha+1)}\\ &\qquad\cdot(\overline{w}_r(\alpha+1)+1)d_{\alpha+1}^{b_{s+1,s+1}(\alpha+1)} \end{aligned} \tag{113}$$

Comparing (112) and (113), we see that for $1\le i\le s$ $\overline{y}_i(\alpha+1)=\lambda_i\overline{w}_i(\alpha+1)$ for some $\overline{d}_{\alpha+1}$-th roots of unity λ_i. By our construction of UTSs (at the beginning of Chapter (3))

$$\left(\frac{\widehat{\overline{y}}_r(\alpha+1)}{d_{\alpha+1}}+1\right)^{1/\overline{d}_{\alpha+1}} \quad\text{and}\quad \left(\frac{\widehat{\overline{w}}_r(\alpha+1)}{d_{\alpha+1}}+1\right)^{1/\overline{d}_{\alpha+1}}$$

have residue 1 in $\mathcal{O}_w/m_w$. By (107), (109), (110) and (111) we see that

$$\frac{\overline{y}_i(\alpha+1)}{\overline{w}_i(\alpha+1)}$$

has residue 1 in $\mathcal{O}_w/m_w$ for $1 \leq i \leq s$. Thus we have $\overline{y}_i(\alpha+1) = \overline{w}_i(\alpha+1)$ for $1 \leq i \leq s$, proving the first half of (A3) for $\alpha+1$.

$$\begin{aligned}\overline{w}_r(\alpha+1) - \overline{y}_r(\alpha+1) &= \frac{\widetilde{\overline{w}}'_r(\alpha)}{d_{\alpha+1}^{b_{s+1,s+1}}\overline{w}_1(\alpha+1)^{b_{s+1,1}}\cdots\overline{w}_s(\alpha+1)^{b_{s+1,s}}} \\ &\quad - \frac{\widetilde{\overline{y}}'_r(\alpha)}{d_{\alpha+1}^{b_{s+1,s+1}}\overline{y}_1(\alpha+1)^{b_{s+1,1}}\cdots\overline{y}_s(\alpha+1)^{b_{s+1,s}}} \\ &= \frac{\widetilde{\overline{w}}'_r(\alpha) - \widetilde{\overline{y}}'_r(\alpha)}{d_{\alpha+1}^{b_{s+1,s+1}}\overline{w}_1(\alpha+1)^{b_{s+1,1}}\cdots\overline{w}_s(\alpha+1)^{b_{s+1,s}}}\end{aligned}$$

Thus

$$\begin{aligned}w(\overline{w}_r(\alpha+1) - \overline{y}_r(\alpha+1)) &= w(\widetilde{\overline{w}}'_r(\alpha) - \widetilde{\overline{y}}'_r(\alpha)) \\ &\quad - b_{s+1,1}w(\overline{w}_1(\alpha+1)) - \cdots - b_{s+1,s}w(\overline{w}_s(\alpha+1)) \\ &> \lambda' - b_{s+1,1}w(\overline{w}_1(\alpha+1)) - \cdots - b_{s+1,s}w(\overline{w}_s(\alpha+1)) \\ &> \lambda'_{\alpha+1}\end{aligned}$$

verifying the second half of (A3), and (A2) for $\alpha+1$. □

Theorem 4.9. — *Suppose that $T''(0) \subset \widehat{R}$ is a regular local ring essentially of finite type over R such that the quotient field of $T''(0)$ is finite over J, $U''(0) \subset \widehat{S}$ is a regular local ring essentially of finite type over S such that the quotient field of $U''(0)$ is finite over K, $T''(0) \subset U''(0)$, $T''(0)$ contains a subfield isomorphic to $k(c_0)$ for some $c_0 \in k(T''(0))$ and $U''(0)$ contains a subfield isomorphic to $k(U''(0))$. Suppose that R has regular parameters $(x_1,\ldots,x_n)$, S has regular parameters $(y_1,\ldots,y_n)$, $T''(0)$ has regular parameters $(\overline{x}_1,\ldots,\overline{x}_n)$ and $U''(0)$ has regular parameters $(\overline{y}_1,\ldots,\overline{y}_n)$ such that*

$$\begin{aligned}\overline{x}_1 &= \overline{y}_1^{c_{11}}\cdots\overline{y}_s^{c_{1s}}\phi_1 \\ &\vdots \\ \overline{x}_s &= \overline{y}_1^{c_{s1}}\cdots\overline{y}_s^{c_{ss}}\phi_s \\ \overline{x}_{s+1} &= \overline{y}_{s+1} \\ &\vdots \\ \overline{x}_l &= \overline{y}_l\end{aligned}$$

where $\phi_1,\ldots,\phi_s \in k(U''(0))$, $\nu(\overline{x}_1),\ldots,\nu(\overline{x}_s)$ are rationally independent, $\det(c_{ij}) \neq 0$. *Suppose that there exists a regular local ring $\widetilde{R} \subset R$ such that $(x_1,\ldots,x_l)$ are regular parameters in $\widetilde{R}$, $k(\widetilde{R}) \cong k(c_0)$ and*

$$x_i = \begin{cases}\gamma_i\overline{x}_i & 1 \leq i \leq l \\ \overline{x}_i & l < i \leq n\end{cases}$$

with $\gamma_i \in k(c_0)[[x_1,\ldots,x_l]] \cap T''(0)$ *for* $1 \le i \le l$ *and* $\gamma_i \equiv 1$ *mod* $(x_1,\ldots,x_l)$*, there exist* $\gamma_i^y \in U''(0)$ *such that* $y_i = \gamma_i^y \overline{y}_i$, $\gamma_i^y \equiv 1$ *mod* $m(U''(0))$ *for* $1 \le i \le n$*. Further suppose that*

$$x_{l+1} = \overline{P}(\overline{y}_1,\ldots,\overline{y}_l) + \overline{y}_1^{d_1}\cdots\overline{y}_s^{d_s}\overline{H} + \Omega$$

where $\overline{P}$ *is a power series with coefficients in* $k(U''(0))$,

$$\overline{H} = \overline{u}(\overline{y}_{l+1} + \overline{y}_1^{\overline{g}_1}\cdots\overline{y}_s^{\overline{g}_s}\overline{\Sigma})$$

where $\overline{u} \in U''(0)\hat{}$ *is a unit,*

$$\overline{\Sigma} \in k(U''(0))[[\overline{y}_1,\ldots,\overline{y}_l,\overline{y}_{l+2},\ldots,\overline{y}_n]],$$
$$\nu(\overline{y}_{l+1}) \le \nu(\overline{y}_1^{\overline{g}_1}\cdots\overline{y}_s^{\overline{g}_s}) \quad \textit{and}$$
$$\Omega \in m(U(0))^N \quad \textit{with } N\nu(m(U(0))) > \nu(\overline{y}_1^{d_1}\cdots\overline{y}_s^{d_s}\overline{y}_{l+1}).$$

Then there exists a CRUTS along ν $(R, T''(t'), T(t'))$ *and* $(S, U''(t'), U(t'))$ *with associated MTS*

$$\begin{array}{ccc} S & \to & S(t') \\ \uparrow & & \uparrow \\ R & \to & R(t') \end{array}$$

such that $T''(t')$ *contains a subfield isomorphic to* $k(c_0,\ldots,c_{t'})$, $U''(t')$ *contains a subfield isomorphic to* $k(U(t'))$,

$$\begin{aligned} R(t') &\textit{ has regular parameters } (x_1(t'),\ldots,x_n(t')),\\ T''(t') &\textit{ has regular parameters } (\tilde{\overline{x}}_1(t'),\ldots\tilde{\overline{x}}_n(t')),\\ S(t') &\textit{ has regular parameters } (y_1(t'),\ldots,y_n(t')),\\ U''(t') &\textit{ has regular parameters } (\tilde{\overline{y}}_1(t'),\ldots,\tilde{\overline{y}}_n(t')) \end{aligned}$$

where

$$\begin{aligned} \tilde{\overline{x}}_1(t') &= \tilde{\overline{y}}_1(t')^{c_{11}(t')}\cdots\tilde{\overline{y}}_s(t')^{c_{1s}(t')}\phi_1(t')\\ &\vdots\\ \tilde{\overline{x}}_s(t') &= \tilde{\overline{y}}_1(t')^{c_{s1}(t')}\cdots\tilde{\overline{y}}_s(t')^{c_{ss}(t')}\phi_s(t')\\ \tilde{\overline{x}}_{s+1}(t') &= \tilde{\overline{y}}_{s+1}(t')\\ &\vdots\\ \tilde{\overline{x}}_l(t') &= \tilde{\overline{y}}_l(t')\\ x_{l+1}(t') = \tilde{\overline{x}}_{l+1}(t') &= P(\tilde{\overline{y}}_1(t'),\ldots,\tilde{\overline{y}}_l(t')) + \tilde{\overline{y}}_1(t')^{d_1(t')}\cdots\tilde{\overline{y}}_s(t')^{d_s(t')}H \end{aligned}$$

where P, H *are power series with coefficients in* $k(U(t'))$*, with*

$$\operatorname{mult} H(0,\ldots,0,\tilde{\overline{y}}_{l+1}(t'),0,\ldots,0) = 1,$$

$\phi_1(t'),\ldots,\phi_s(t') \in k(U(t'))$, $\nu(\widetilde{\overline{x}}_1(t')),\ldots,\nu(\widetilde{\overline{x}}_s(t'))$ are rationally independent, $\det(c_{ij}(t')) \neq 0$. There exists a regular local ring $\widetilde{R}(t') \subset R(t')$ such that

$$(x_1(t'),\ldots,x_l(t')) \text{ are regular parameters in } \widetilde{R}(t')$$

and $k(\widetilde{R}(t')) \cong k(c_0,\ldots,c_{t'})$.

$$x_i(t') = \begin{cases} \gamma_i(t')\widetilde{\overline{x}}_i(t') & 1 \leq i \leq l \\ \widetilde{\overline{x}}_i(t') & l < i \leq n \end{cases}$$

with $\gamma_i(t') \in k(c_0,\ldots,c_{t'})[[x_1(t'),\ldots,x_l(t')]] \cap T''(t')$ units for $1 \leq i \leq l$, such that $\gamma_i(t') \equiv 1 \mod (x_1(t'),\ldots,x_l(t'))$ and for $1 \leq i \leq n$ there exists $\gamma_i^y(t') \in U''(t')$ such that $y_i(t') = \gamma_i^y(t')\widetilde{\overline{y}}_i(t')$, $\gamma_i^y(t') \equiv 1 \mod m(U''(t'))$.

Proof. — Perform the following sequence of CUTSs of type (M2) for $0 \leq r \leq l-s-1$

$$\begin{aligned}
\overline{x}_1(r) &= \overline{x}_1(r+1)^{a_{11}(r+1)}\cdots\overline{x}_s(r+1)^{a_{1s}(r+1)}c_{r+1}^{a_{1,s+1}(r+1)}\\
&\;\;\vdots\\
\overline{x}_s(r) &= \overline{x}_1(r+1)^{a_{s1}(r+1)}\cdots\overline{x}_s(r+1)^{a_{ss}(r+1)}c_{r+1}^{a_{s,s+1}(r+1)}\\
\overline{x}_{s+r+1}(r) &= \overline{x}_1(r+1)^{a_{s+1,1}(r+1)}\cdots\overline{x}_s(r+1)^{a_{s+1,s}(r+1)}\\
&\qquad\cdot(\overline{x}_{s+r+1}(r+1)+1)c_{r+1}^{a_{s+1,s+1}(r+1)}
\end{aligned}$$

$$\begin{aligned}
\overline{y}_1(r) &= \overline{y}_1(r+1)^{b_{11}(r+1)}\cdots\overline{y}_s(r+1)^{b_{1s}(r+1)}d_{r+1}^{b_{1,s+1}(r+1)}\\
&\;\;\vdots\\
\overline{y}_s(r) &= \overline{y}_1(r+1)^{b_{s1}(r+1)}\cdots\overline{y}_s(r+1)^{b_{ss}(r+1)}d_{r+1}^{b_{s,s+1}(r+1)}\\
\overline{y}_{s+r+1}(r) &= \overline{y}_1(r+1)^{b_{s+1,1}(r+1)}\cdots\overline{y}_s(r+1)^{b_{s+1,s}(r+1)}\\
&\qquad\cdot(\overline{y}_{s+r+1}(r+1)+1)d_{r+1}^{b_{s+1,s+1}(r+1)},
\end{aligned}$$

the sequence of CUTSs of type (M3) for $l-s \leq r \leq n-s-1$

$$\begin{aligned}
\overline{y}_1(r) &= \overline{y}_1(r+1)^{b_{11}(r+1)}\cdots\overline{y}_s(r+1)^{b_{1s}(r+1)}d_{r+1}^{b_{1,s+1}(r+1)}\\
&\;\;\vdots\\
\overline{y}_s(r) &= \overline{y}_1(r+1)^{b_{s1}(r+1)}\cdots\overline{y}_s(r+1)^{b_{ss}(r+1)}d_{r+1}^{b_{s,s+1}(r+1)}\\
\overline{y}_{s+r+1}(r) &= \overline{y}_1(r+1)^{b_{s+1,1}(r+1)}\cdots\overline{y}_s(r+1)^{b_{s+1,s}(r+1)}\\
&\qquad\cdot(\overline{y}_{s+r+1}(r+1)+1)d_{r+1}^{b_{s+1,s+1}(r+1)},
\end{aligned}$$

followed by a CUTS of type (M1), with $t' = n - s + 1$

$$\begin{aligned}
\overline{x}_1(t'-1) &= \overline{x}_1(t')^{a_{11}(t')} \cdots \overline{x}_s(t')^{a_{1s}(t')}\\
&\vdots\\
\overline{x}_s(t'-1) &= \overline{x}_1(t')^{a_{s1}(t')} \cdots \overline{x}_s(t')^{a_{ss}(t')}\\
\overline{y}_1(t'-1) &= \overline{y}_1(t')^{b_{11}(t')} \cdots \overline{y}_s(t')^{b_{1s}(t')}\\
&\vdots\\
\overline{y}_s(t'-1) &= \overline{y}_1(t')^{b_{s1}(t')} \cdots \overline{y}_s(t')^{b_{ss}(t')},
\end{aligned}$$

so that

$$\begin{aligned}
x_{l+1} = P(\overline{y}_1(t'), \ldots, \overline{y}_l(t')) + \overline{y}_1(t')^{d_1(t')} \cdots \overline{y}_s(t')^{d_s(t')} \overline{u}\left((\overline{y}_{l+1}(t') + 1)\lambda + \Sigma\right)\\
+ \overline{y}_1(t')^{d_1(t')+1} \cdots \overline{y}_s(t')^{d_s(t')+1}\Psi
\end{aligned}$$

where

$$\Sigma \in k(U(t'))[[\overline{y}_1(t'), \ldots, \overline{y}_l(t'), \overline{y}_{l+2}(t'), \ldots, \overline{y}_n(t')]],$$

$P(\overline{y}_1(t'), \ldots, \overline{y}_l(t')) = \overline{P}(\overline{y}_1, \ldots, \overline{y}_l)$, $\overline{y}_1(t') \cdots \overline{y}_s(t') \mid (\overline{u} - \overline{u}(0, \ldots, 0))$, and

$$\Omega = \overline{y}_1(t')^{d_1(t')+1} \cdots \overline{y}_s(t')^{d_s(t')+1}\Psi$$

with $\lambda = d_{l-s+1}^{b_{s+1,s+1}(l-s+1)} \in k(U(t'))$, $\Psi \in U(t')$. Then after replacing P with

$$P + \lambda\overline{u}(0, \ldots, 0)\overline{y}_1(t')^{d_1(t')} \cdots \overline{y}_s(t')^{d_s(t')},$$

we can put $\overline{x}_{l+1}(t') = x_{l+1}$ in the desired form with

$$H = \overline{u}\,\overline{y}_{l+1}(t')\lambda + (\overline{u} - \overline{u}(0, \ldots, 0))\lambda + \overline{u}\Sigma + \overline{y}_1(t') \cdots \overline{y}_s(t')\Psi.$$

The proof that

$$\begin{array}{ccc}
U(0) & \rightarrow & U(t')\\
\uparrow & & \uparrow\\
T(0) & \rightarrow & T(t')
\end{array}$$

is a CRUTS is a simplification of the argument of step 3 in the proof of Theorem 4.8. We will give an outline of the proof.

We can define MTSs $R \to R(t')$ and $S \to S(t')$ such that $R(r)\widehat{}$, $S(r)\widehat{}$ have respective regular parameters $(x_1^*(r), \ldots, x_n^*(r))$ and $(y_1^*(r), \ldots, y_n^*(r))$ for $0 \le r \le n$.

For $0 \le r \le l-s-1$

$$\begin{aligned}
x_1^*(r) &= x_1^*(r+1)^{a_{11}(r+1)} \cdots x_s^*(r+1)^{a_{1s}(r+1)} \\
&\quad \cdot (x_{s+r+1}^*(r+1) + c_{r+1})^{a_{1,s+1}(r+1)} \\
&\vdots \\
x_s^*(r) &= x_1^*(r+1)^{a_{s1}(r+1)} \cdots x_s^*(r+1)^{a_{ss}(r+1)} \\
&\quad \cdot (x_{s+r+1}^*(r+1) + c_{r+1})^{a_{s,s+1}(r+1)} \\
x_{s+r+1}^*(r) &= x_1^*(r+1)^{a_{s+1,1}(r+1)} \cdots x_s^*(r+1)^{a_{s+1,s}(r+1)} \\
&\quad \cdot (x_{s+r+1}^*(r+1) + c_{r+1})^{a_{s+1,s+1}(r+1)}
\end{aligned}$$

$$\begin{aligned}
y_1^*(r) &= y_1^*(r+1)^{b_{11}(r+1)} \cdots y_s^*(r+1)^{b_{1s}(r+1)} \\
&\quad \cdot (y_{s+r+1}^*(r+1) + d_{r+1})^{b_{1,s+1}(r+1)} \\
&\vdots \\
y_s^*(r) &= y_1^*(r+1)^{b_{s1}(r+1)} \cdots y_s^*(r+1)^{b_{ss}(r+1)} \\
&\quad \cdot (y_{s+r+1}^*(r+1) + d_{r+1})^{b_{s,s+1}(r+1)} \\
y_{s+r+1}^*(r) &= y_1^*(r+1)^{b_{s+1,1}(r+1)} \cdots y_s^*(r+1)^{b_{s+1,s}(r+1)} \\
&\quad \cdot (y_{s+r+1}^*(r+1) + d_{r+1})^{b_{s+1,s+1}(r+1)},
\end{aligned}$$

For $l-s \le r \le n-s-1$

$$\begin{aligned}
y_1^*(r) &= y_1^*(r+1)^{b_{11}(r+1)} \cdots y_s^*(r+1)^{b_{1s}(r+1)} \\
&\quad \cdot (y_{s+r+1}^*(r+1) + d_{r+1})^{b_{1,s+1}(r+1)} \\
&\vdots \\
y_s^*(r) &= y_1^*(r+1)^{b_{s1}(r+1)} \cdots y_s^*(r+1)^{b_{ss}(r+1)} \\
&\quad \cdot (y_{s+r+1}^*(r+1) + d_{r+1})^{b_{s,s+1}(r+1)} \\
y_{s+r+1}^*(r) &= y_1^*(r+1)^{b_{s+1,1}(r+1)} \cdots y_s^*(r+1)^{b_{s+1,s}(r+1)} \\
&\quad \cdot (y_{s+r+1}^*(r+1) + d_{r+1})^{b_{s+1,s+1}(r+1)},
\end{aligned}$$

followed by a MTS of type (M1), with $t' = n-s+1$

$$\begin{aligned}
x_1^*(t'-1) &= x_1^*(t')^{a_{11}(t')} \cdots x_s^*(t')^{a_{1s}(t')} \\
&\vdots \\
x_s^*(t'-1) &= x_1^*(t')^{a_{s1}(t')} \cdots x_s^*(t')^{a_{ss}(t')}
\end{aligned}$$

$$
\begin{aligned}
y_1^*(t'-1) &= y_1^*(t')^{b_{11}(t')}\cdots y_s^*(t')^{b_{1s}(t')}\\
&\vdots\\
y_s^*(t'-1) &= y_1^*(t')^{b_{s1}(t')}\cdots y_s^*(t')^{b_{ss}(t')}.
\end{aligned}
$$

For $1 \le r \le l-s$ we have

$$
x_i^*(r) = \begin{cases} \lambda_i(r)\overline{x}_i(r) & 1 \le i \le s\\ \lambda_{s+r}(r)\overline{x}_{s+r}(r) + \Phi_{s+r} & i = s+r\\ x_i^*(r-1) & s < i,\ i \ne s+r \end{cases}
$$

where $\lambda_i(r), \Phi_{s+r} \in k(c_0, \ldots, c_r)[[\overline{x}_1(r), \ldots, \overline{x}_l(r)]]$, the $\lambda_i(r)$ are units and

$$
\Phi_{s+r} \in (\overline{x}_1(r), \ldots, \overline{x}_{s+r-1}(r), \overline{x}_{s+r+1}(r), \ldots, \overline{x}_l(r)).
$$

For $1 \le r \le n-s$ we have

$$
y_i^*(r) = \begin{cases} \lambda_i^y(r)\overline{y}_i(r) & 1 \le i \le s\\ \lambda_{s+r}^y(r)\overline{y}_{s+r}(r) + \Phi_{s+r}^y & i = s+r\\ y_i^*(r-1) & s < i,\ i \ne s+r \end{cases}
$$

where $\lambda_i^y(r), \Phi_{s+r}^y \in k(U(r))[[\overline{y}_1(r), \ldots, \overline{y}_{s+r-1}(1), \overline{y}_{s+r+1}(r), \ldots, \overline{y}_n(r)]]$, the $\lambda_i^y(r)$ are units.

$R(t')$ has regular parameters $(x_1(t'), \ldots, x_n(t'))$ where

$$
x_i(t') = \begin{cases} x_i^*(t') & 1 \le i \le s, \quad l+1 \le i \le n\\ \prod(x_i^*(t') + c_{i-s} - \sigma(c_{i-s})) & s+1 \le i \le l \end{cases}
$$

where the product is over the distinct conjugates $\sigma(c_{i-s})$ of c_{i-s} in an algebraic closure $\overline{k}$ of k over k. For $0 \le r \le l-s-1$ define

$$
\widetilde{R}(r+1) = \widetilde{R}(r)[x_1^*(r+1), \ldots, x_s^*(r+1), \tau(r+1)]_{q_{r+1}}
$$

where

$$
\tau(r+1) = \prod(x_{s+r+1}^*(r+1) + c_{r+1} - \sigma(c_{r+1}))
$$

is the product over the distinct conjugates $\sigma(c_{r+1})$ of c_{r+1} over k,

$$
q_{r+1} = m(R(r+1)) \cap \widetilde{R}(r)[x_1^*(r+1), \ldots, x_s^*(r+1), \tau(r+1)].
$$

For $l-s \le r \le n-s+1$ define $\widetilde{R}(r+1) = \widetilde{R}(r)$. Define

$$
\widetilde{R}(t') = \widetilde{R}(n-s)[x_1^*(t'), \ldots, x_s^*(t')]_{q_{t'}}
$$

where

$$
q_{t'} = m(R(t')) \cap \widetilde{R}(n-s)[x_1^*(t'), \ldots, x_s^*(t')].
$$

$S(t')$ has regular parameters $(y_1(t'),\ldots,y_n(t'))$ where

$$y_i(t')=\begin{cases} y_i^*(t') & 1\le i\le s\\ x_i(t') & s+1\le i\le l\\ \prod(y_i^*(t')+d_{i-s}-\sigma(d_{i-s})) & l+1\le i\le n\end{cases}$$

where the product is over the distinct conjugates $\sigma(d_{i-s})$ of d_{i-s} in an algebraic closure $\overline{k}$ of k over k.

Now set

$$\widetilde{\overline{x}}_i(t')=\begin{cases} x_i(t') & \text{for } s+1\le i\le l\\ \overline{x}_i(t') & \text{otherwise}\end{cases}$$

$$\widetilde{\overline{y}}_i(t')=\begin{cases} \widetilde{\overline{x}}_i(t') & \text{for } s+1\le i\le l\\ \overline{y}_i(t') & \text{otherwise.}\end{cases}$$

We are then in the form of the conclusions of the Theorem. □

The proof of Theorem 4.9 also proves the following Theorem.

Theorem 4.10. — *Let $n_{0,l}=m\left(k(U''(0))[[\overline{y}_1,\ldots,\overline{y}_l]]\right)$ in the assumptions of Theorem 4.9.*

(1) *If $\Omega\in n_{0,l}^N$ in the assumptions of Theorem 4.9, then a sequence of MTSs of type (M2) and a MTS of type (M1) (so that the CRUTS along ν is in the first l variables) are sufficient to transform x_{l+1} into the form of the conclusions of Theorem 4.9.*

(2) *Suppose that*

$$g=\overline{y}_1^{d_1}\cdots\overline{y}_s^{d_s}u(\overline{y}_1,\ldots,\overline{y}_l)+\Omega$$

where u is a unit power series with coefficients in $k(U''(0))$ and $\Omega\in n_{0,l}^N$ with $N\nu(n_{0,l})>\nu(\overline{y}_1^{d_1}\cdots\overline{y}_s^{d_s})$. Then a sequence of MTSs of type (M2) and a MTS of type (M1) (so that the CRUTS along ν is in the first l variables) are sufficient to transform g into the form

$$g=\overline{y}_1(t')^{d_1(t')}\cdots\overline{y}_s(t')^{d_s(t')}\overline{u}(\overline{y}_1(t'),\ldots,\overline{y}_l(t'))$$

where $\overline{u}$ is a unit power series.

(3) *Suppose that*

$$g=\overline{y}_1^{d_1}\cdots\overline{y}_s^{d_s}\Sigma(\overline{y}_1,\ldots,\overline{y}_l)+\Omega$$

where $\nu(\overline{y}_1^{d_1}\cdots\overline{y}_s^{d_s})>A$ and $\Omega\in n_{0,l}^N$ with $N\nu(n_{0,l})>\nu(\overline{y}_1^{d_1}\cdots\overline{y}_s^{d_s})$. Then a sequence of MTSs of type (M2) and a MTS of type (M1) (so that the CRUTS along ν is in the first l variables) are sufficient to transform g into the form

$$g=\overline{y}_1(t')^{d_1(t')}\cdots\overline{y}_s(t')^{d_s(t')}\overline{\Sigma}(\overline{y}_1(t'),\ldots,\overline{y}_l(t'))$$

where $\nu(\overline{y}_1(t')^{d_1(t')}\cdots\overline{y}_s(t')^{d_s(t')})>A$.

(4) *Suppose that*

$$g = \overline{y}_1^{d_1} \cdots \overline{y}_s^{d_s} u(\overline{y}_1, \ldots, \overline{y}_l) + \Omega$$

where u is a unit power series with coefficients in $k(U''(0))$ and $\Omega \in m(U''(0))^N$ with $N\nu(m(U''(0))) > \nu(\overline{y}_1^{d_1} \cdots \overline{y}_s^{d_s})$. Then there exists A CRUTS along ν as in the conclusions of Theorem 4.9 such that

$$g = \overline{y}_1(t')^{d_1(t')} \cdots \overline{y}_s(t')^{d_s(t')} \overline{u}(\overline{y}_1(t'), \ldots, \overline{y}_l(t'))$$

where $\overline{u}$ is a unit power series.

Theorem 4.11. — *Suppose that $T''(0) \subset \widehat{R}$ is a regular local ring essentially of finite type over R such that the quotient field of $T''(0)$ is finite over J, $U''(0) \subset \widehat{S}$ is a regular local ring essentially of finite type over S such that the quotient field of $U''(0)$ is finite over K, $T''(0) \subset U''(0)$, $T''(0)$ contains a subfield isomorphic to $k(c_0)$ for some $c_0 \in k(T''(0))$ and $U''(0)$ contains a subfield isomorphic to $k(U''(0))$. Suppose that R has regular parameters $(x_1, \ldots, x_n)$, S has regular parameters $(y_1, \ldots, y_n)$, $T''(0)$ has regular parameters $(\overline{x}_1, \ldots, \overline{x}_n)$ and $U''(0)$ has regular parameters $(\overline{y}_1, \ldots, \overline{y}_n)$ such that*

$$\begin{aligned}
\overline{x}_1 &= \overline{y}_1^{c_{11}} \cdots \overline{y}_s^{c_{1s}} \phi_1 \\
&\vdots \\
\overline{x}_s &= \overline{y}_1^{c_{s1}} \cdots \overline{y}_s^{c_{ss}} \phi_s \\
\overline{x}_{s+1} &= \overline{y}_{s+1} \\
&\vdots \\
\overline{x}_l &= \overline{y}_l
\end{aligned}$$

where $\phi_1, \ldots, \phi_s \in k(U''(0))$, $\nu(\overline{x}_1), \ldots, \nu(\overline{x}_s)$ are rationally independent, $\det(c_{ij}) \neq 0$. *Suppose that there exists a regular local ring $\widetilde{R} \subset R$ such that $(x_1, \ldots, x_l)$ are regular parameters in $\widetilde{R}$, $k(\widetilde{R}) \cong k(c_0)$ and*

$$x_i = \begin{cases} \gamma_i \overline{x}_i & 1 \leq i \leq l \\ \overline{x}_i & l < i \leq n \end{cases}$$

with $\gamma_i \in k(c_0)[[x_1, \ldots, x_l]] \cap T''(0)$ for $1 \leq i \leq l$ and $\gamma_i \equiv 1 \bmod (x_1, \ldots, x_l)$, there exist $\gamma_i^y \in U''(0)$ such that $y_i = \gamma_i^y \overline{y}_i$, $\gamma_i^y \equiv 1 \bmod m(U''(0))$ for $1 \leq i \leq n$.

Then there exists a CRUTS along ν $(R, T''(t), T(t))$ and $(S, U''(t), U(t))$ with associated MTSs

$$\begin{array}{ccc} S & \rightarrow & S(t) \\ \uparrow & & \uparrow \\ R & \rightarrow & R(t) \end{array}$$

such that $T''(t)$ contains a subfield isomorphic to $k(c_0, \ldots, c_t)$, $U''(t)$ contains a subfield isomorphic to $k(U(t))$, $R(t')$ has regular parameters $(x_1(t), \ldots, x_n(t))$, $S(t')$ has *regular parameters* $(y_1(t'), \ldots, y_n(t'))$, $T''(t)$ *has regular parameters* $(\tilde{\overline{x}}_1(t), \ldots \tilde{\overline{x}}_n(t))$, $U''(t)$ *has regular parameters* $(\tilde{\overline{y}}_1(t), \ldots, \tilde{\overline{y}}_n(t))$ *where*

$$\begin{aligned}
\tilde{\overline{x}}_1(t) &= \tilde{\overline{y}}_1(t)^{c_{11}(t)} \cdots \tilde{\overline{y}}_s(t)^{c_{1s}(t)} \phi_1(t) \\
&\vdots \\
\tilde{\overline{x}}_s(t) &= \tilde{\overline{y}}_1(t)^{c_{s1}(t)} \cdots \tilde{\overline{y}}_s(t)^{c_{ss}(t)} \phi_s(t) \\
\tilde{\overline{x}}_{s+1}(t) &= \tilde{\overline{y}}_{s+1}(t) \\
&\vdots \\
\tilde{\overline{x}}_{l+1}(t) &= \tilde{\overline{y}}_{l+1}(t)
\end{aligned}$$

such that $\phi_1(t), \ldots, \phi_s(t) \in k(U(t))$, $\nu(\tilde{\overline{x}}_1(t)), \ldots, \nu(\tilde{\overline{x}}_s(t))$ *are rationally independent,* $\det(c_{ij}(t)) \neq 0$.

$$x_i(t) = \tilde{\overline{x}}_i(t) \textit{ for } 1 \leq i \leq n.$$

For $1 \leq i \leq n$ *there exist* $\gamma_i^y(t) \in U''(t)$ *such that* $y_i(t) = \gamma_i^y(t)\tilde{\overline{y}}_i(t)$,

$$\gamma_i^y(t) \equiv 1 \textit{ mod } m(U''(t)).$$

Proof. — We will construct a CRUTS $(R, T''(t), T(t))$ and $(S, U''(t), U(t))$ along ν with associated MTS

$$\begin{array}{ccccc}
S = & S(0) & \to & S(t) \\
& \uparrow & & \uparrow \\
R = & R(0) & \to & R(t).
\end{array}$$

We will say that $\mathrm{CN}(\beta)$ holds (with $0 \leq \beta \leq t$) if $T''(\beta)$ contains a subfield isomorphic to $k(c_0, \ldots, c_\beta)$, $U''(\beta)$ contains a subfield isomorphic to $k(U(\beta))$, $R(\beta)$ has regular parameters $(x_1(\beta), \ldots, x_n(\beta))$, $T''(\beta)$ has regular parameters $(\tilde{\overline{x}}_1(\beta), \ldots, \tilde{\overline{x}}_n(\beta))$, $U''(\beta)$ has regular parameters $(\tilde{\overline{y}}_1(\beta), \ldots, \tilde{\overline{y}}_n(\beta))$, such that

$$x_i(\beta) = \begin{cases} \gamma_i(\beta)\tilde{\overline{x}}_i(\beta) & 1 \leq i \leq l \\ \tilde{\overline{x}}_i(\beta) & l < i \leq n \end{cases}$$

with

$$\gamma_i(\beta) \in k(c_0, \ldots, c_\beta)[[x_1(\beta), \ldots, x_l(\beta)]] \cap T''(\beta),$$

$\gamma_i(\beta) \equiv 1 \bmod (x_1(\beta), \ldots, x_l(\beta))$ for $1 \leq i \leq l$ and for $1 \leq i \leq n$ there exist $\gamma_i^y(\beta) \in U''(\beta)$ such that $y_i(\beta) = \gamma_i^y(\beta)\tilde{\overline{y}}_i(\beta)$, $\gamma_i^y(\beta) \equiv 1 \bmod m(U''(\beta))$. We must further have

$$\begin{aligned}
\widetilde{\overline{x}}_1(\beta) &= \widetilde{\overline{y}}_1(\beta)^{c_{11}(\beta)}\cdots\widetilde{\overline{y}}_s(\beta)^{c_{1s}(\beta)}\phi_1(\beta)\\
&\vdots\\
\widetilde{\overline{x}}_s(\beta) &= \widetilde{\overline{y}}_1(\beta)^{c_{s1}(\beta)}\cdots\widetilde{\overline{y}}_s(\beta)^{c_{ss}(\beta)}\phi_s(\beta)\\
\widetilde{\overline{x}}_{s+1}(\beta) &= \widetilde{\overline{y}}_{s+1}(\beta)\\
&\vdots\\
\widetilde{\overline{x}}_l(\beta) &= \widetilde{\overline{y}}_l(\beta)
\end{aligned}$$

with $\phi_i(\beta)\in k(S(\beta))$, $\nu(\widetilde{\overline{x}}_1(\beta)),\ldots,\nu(\widetilde{\overline{x}}_s(\beta))$ are rationally independent, $\det(c_{ij}(\beta))\neq 0$, and there exists a regular local ring $\widetilde{R}(\beta)\subset R(\beta)$ such that $(x_1(\beta),\ldots,x_l(\beta))$ are regular parameters in $\widetilde{R}(\beta)$ and $k(\widetilde{R}(\beta))\cong k(c_0,\ldots,c_\beta)$.

By Theorem 2.12 and Theorems 4.8 (with $f=x_{l+1}$) and 4.9 we may assume that

$$x_{l+1}=\overline{x}_{l+1}=P(\overline{y}_1,\ldots,\overline{y}_l)+\overline{y}_1^{d_1}\cdots\overline{y}_s^{d_s}\Sigma_0.$$

where Σ_0 is a series with $\operatorname{mult}\Sigma_0(0,\ldots,0,\overline{y}_{l+1},0,\ldots,0)=1$. Suppose that $\nu(P)=\infty$ (this includes the case $P=0$). Then by Theorems 4.8 and 4.10, we have a CRUTS along ν in the first l variables such that

$$x_{l+1}=\widetilde{\overline{y}}_1(t)^{d_1(t)}\cdots\widetilde{\overline{y}}_s(t)^{d_s(t)}\Sigma_t(\widetilde{\overline{y}}_1(t),\ldots,\widetilde{\overline{y}}_n(t))\tag{114}$$

$\operatorname{mult}\Sigma_t(0,\ldots,0,\widetilde{\overline{y}}_{l+1}(t),0\ldots,0)=1$, and CN($t$) holds. We can thus make changes of variables, replacing $\widetilde{\overline{x}}_i(t)$ with $x_i(t)$ (for $1\le i\le n$) and $\widetilde{\overline{y}}_i(t)$ with $\widetilde{\overline{x}}_i(t)$ (for $s+1\le i\le l$) so that (114) holds and CN(t) holds with $\gamma_i(t)=1$ for $1\le i\le l$, and $y_i(t)=x_i(t)$ for $s+1\le i\le l$.

Suppose that $\nu(P)<\infty$. Set $d=\det(c_{ij})$. Let (f_{ij}) be the adjoint matrix of (c_{ij}), so that (f_{ij}/d) is the inverse of (c_{ij}). Let ω be a primitive d^{th} root of unity in an algebraic closure $\overline{k}$ of k. Set

$$g'=\prod_{i_1,\ldots,i_s=1}^{d}\left(x_{l+1}-P(\omega^{i_1}\overline{y}_1,\ldots,\omega^{i_s}\overline{y}_s,\overline{y}_{s+1},\ldots,\overline{y}_l)\right).$$

$$g'\in k(U''(0))\left[\left[\overline{y}_1^d,\ldots,\overline{y}_s^d,\overline{y}_{s+1},\ldots,\overline{y}_l\right]\right][x_{l+1}].$$

Let G be the Galois group of a Galois closure of $k(U''(0))$ over $k(c_0)$. Since x_{l+1} is analytically independent of $\overline{y}_1^d,\ldots,\overline{y}_s^d,\overline{y}_{s+1},\ldots,\overline{y}_l$ (by Theorem 2.12) we can define $g=\prod_{\tau\in G}\tau(g')$ where τ acts on the coefficients of g'.

$$g\in k(c_0)\left[\left[\overline{y}_1^d,\ldots,\overline{y}_s^d,\overline{y}_{s+1},\ldots,\overline{y}_l\right]\right][x_{l+1}].$$

Since $\overline{y}_i^d=\overline{x}_1^{f_{i1}}\cdots\overline{x}_s^{f_{is}}\phi_1^{-f_{i1}}\cdots\phi_s^{-f_{is}}$ for $1\le i\le s$, by Lemma 4.2 we can perform a MTS of type (M1) to get $g\in k(c_0)[[\overline{x}_1(1),\ldots,\overline{x}_l(1)]][x_{l+1}]$. Σ_0 is irreducible in $U(1)$

(since $\operatorname{mult}(\Sigma_0(0,\ldots,0,\overline{y}_{l+1},0,\ldots,0)) = 1$) and $\Sigma_0 \mid g$ in $U(1)$.

$$g = \sum_{i=1}^{d} a_i \overline{x}_{l+1}(1)^{f_i} + \sum_j a_j \overline{x}_{l+1}(1)^{f_j} + \sum_k a_k \overline{x}_{l+1}(1)^{f_k} + \overline{x}_{l+1}(1)^{\tau}\Gamma \tag{115}$$

where a_i, a_j, a_k are series in $\overline{x}_1(1),\ldots,\overline{x}_l(1)$ with coefficients in $k(c_0)$, $m > 0$ and the first sum consists of the terms of (finite) minimum value ρ.

$$\rho < \infty \text{ since } \operatorname{mult} g(0,\ldots,\overline{x}_{l+1}(1)) < \infty.$$

$\tau\nu(\overline{x}_{l+1}(1)) > \rho$, the second sum consists of (finitely many) remaining terms of finite value, the third sum consists of (finitely many) terms of infinite value. Let

$$r = \operatorname{mult} g(0,\ldots,0,\overline{x}_{l+1}(1)), \quad 1 \le r < \infty.$$

Suppose that $r > 1$. By Theorems 4.8 and 4.10, we can construct a CRUTS in the first l variables, with associated MTS

$$\begin{array}{ccc} S(1) & \rightarrow & S(\alpha) \\ \uparrow & & \uparrow \\ R(1) & \rightarrow & R(\alpha) \end{array}$$

such that $\mathrm{CN}(\alpha)$ holds, to get

$$P = \tilde{\overline{y}}_1(\alpha)^{g_1(\alpha)} \cdots \tilde{\overline{y}}_s(\alpha)^{g_s(\alpha)} \overline{P}(\tilde{\overline{y}}_1(\alpha),\ldots,\tilde{\overline{y}}_l(\alpha))$$

where $\overline{P}$ is a unit, and

$$a_\zeta = \tilde{\overline{x}}_1(\alpha)^{e_\zeta^1} \cdots \tilde{\overline{x}}_s(\alpha)^{e_\zeta^s} \overline{a}_\zeta(\overline{x}_1(\alpha),\ldots,\overline{x}_l(\alpha)) \tag{116}$$

for $\zeta = i,j,k$ where $\overline{a}_i, \overline{a}_j$ are units and $\nu(\tilde{\overline{x}}_1(\alpha)^{e_k^1} \cdots \tilde{\overline{x}}_s(\alpha)^{e_k^s}) > \rho$ for all i,j,k. We have an expression

$$x_{l+1}(\alpha) = x_{l+1} = P + \tilde{\overline{y}}_1(\alpha)^{d_1(\alpha)} \cdots \tilde{\overline{y}}_s(\alpha)^{d_s(\alpha)} \Sigma_\alpha(\tilde{\overline{y}}_1(\alpha),\ldots,\tilde{\overline{y}}_n(\alpha)).$$

where $\operatorname{mult}\Sigma_\alpha(0,\ldots,0,\tilde{\overline{y}}_{l+1}(\alpha),0\ldots,0) = 1$. If $\nu(P) > \nu(\tilde{\overline{y}}_1(\alpha)^{d_1(\alpha)} \cdots \tilde{\overline{y}}_1(\alpha)^{d_s(\alpha)})$, then after possibly performing a CRUTS of type (M1), so that

$$\tilde{\overline{y}}_1(\alpha)^{d_1(\alpha)} \cdots \tilde{\overline{y}}_s(\alpha)^{d_s(\alpha)} \mid \tilde{\overline{y}}_1(\alpha)^{g_1(\alpha)} \cdots \tilde{\overline{y}}_s(\alpha)^{g_s(\alpha)}$$

in $U(\alpha+1)$, we can set

$$\Sigma_{\alpha+1} = \Sigma_\alpha + \tilde{\overline{y}}_1(\alpha)^{g_1(\alpha)-d_1(\alpha)} \cdots \tilde{\overline{y}}_s(\alpha)^{g_s(\alpha)-d_s(\alpha)} \overline{P}$$

to get

$$x_{l+1} = \tilde{\overline{y}}_1(\alpha+1)^{d_1(\alpha+1)} \cdots \tilde{\overline{y}}_s(\alpha+1)^{d_s(\alpha+1)} \Sigma_{\alpha+1}.$$

Now suppose that $\nu(P) \le \nu(\tilde{\overline{y}}_1(\alpha)^{d_1(\alpha)} \cdots \tilde{\overline{y}}_s(\alpha)^{d_s(\alpha)})$. After possibly performing a CRUTS of type (M1), so that

$$\tilde{\overline{y}}_1(\alpha)^{g_1(\alpha)} \cdots \tilde{\overline{y}}_s(\alpha)^{g_s(\alpha)} \mid \tilde{\overline{y}}_1(\alpha)^{d_1(\alpha)} \cdots \tilde{\overline{y}}_s(\alpha)^{d_s(\alpha)}$$

in $U(\alpha+1)$, we have

$$\begin{aligned}
&x_{l+1}(\alpha+1) = x_{l+1}\\
&= \tilde{\overline{y}}_1(\alpha+1)^{d_1(\alpha+1)}\cdots\tilde{\overline{y}}_s(\alpha+1)^{d_s(\alpha+1)}\\
&\qquad(\overline{P}+\tilde{\overline{y}}_1(\alpha+1)^{\varepsilon_1(\alpha+1)}\cdots\tilde{\overline{y}}_s(\alpha+1)^{\varepsilon_s(\alpha+1)}\overline{\Sigma}_{\alpha+1}(\tilde{\overline{y}}_1(\alpha+1),\ldots,\tilde{\overline{y}}_n(\alpha+1))).
\end{aligned}$$

where

$$\mathrm{mult}(\overline{\Sigma}_{\alpha+1}(0,\ldots,0,\tilde{\overline{y}}_{l+1}(\alpha+1),0,\ldots,0))=1,$$

$\overline{P}+\tilde{\overline{y}}_1(\alpha+1)^{\varepsilon_1(\alpha+1)}\cdots\tilde{\overline{y}}_s(\alpha+1)^{\varepsilon_s(\alpha+1)}\Sigma_{\alpha+1}$ is a unit and $\mathrm{CN}(\alpha+1)$ holds.
Set $(e_{ij})=(c_{ij}(\alpha+1))^{-1}$, $d=\det(c_{ij}(\alpha+1))$. We can replace

$$\begin{aligned}
&\tilde{\overline{y}}_i(\alpha+1) \text{ with } \tilde{\overline{y}}_i(\alpha+1)\gamma_1(\alpha+1)^{e_{i1}}\cdots\gamma_s(\alpha+1)^{e_{is}} \quad \text{for } 1\le i\le s,\\
&\tilde{\overline{y}}_i(\alpha+1) \text{ with } \tilde{\overline{y}}_i(\alpha+1)\gamma_i(\alpha+1) \quad \text{for } s+1\le i\le l \quad \text{and}\\
&U''(\alpha+1) \text{ with } U''(\alpha+1)[\gamma_1(\alpha+1)^{1/d},\ldots,\gamma_s(\alpha+1)^{1/d}]_q
\end{aligned}$$

where

$$q=m(U(\alpha+1))\cap\Big(U''(\alpha+1)[\gamma_1(\alpha+1)^{1/d},\ldots,\gamma_s(\alpha+1)^{1/d}]\Big)$$

to get

$$\begin{aligned}
x_1(\alpha+1) &= \tilde{\overline{y}}_1(\alpha+1)^{c_{11}(\alpha+1)}\cdots\tilde{\overline{y}}_s(\alpha+1)^{c_{1s}(\alpha+1)}\phi_1(\alpha+1)\\
&\ \vdots\\
x_s(\alpha+1) &= \tilde{\overline{y}}_1(\alpha+1)^{c_{s1}(\alpha+1)}\cdots\tilde{\overline{y}}_s(\alpha+1)^{c_{ss}(\alpha+1)}\phi_s(\alpha+1)\\
x_{s+1}(\alpha+1) &= \tilde{\overline{y}}_{s+1}(\alpha+1)\\
&\ \vdots\\
x_l(\alpha+1) &= \tilde{\overline{y}}_l(\alpha+1)
\end{aligned}$$

and have an expression in $U(\alpha+1)\cong k(U(\alpha+1))[[\tilde{\overline{y}}_1(\alpha+1),\ldots,\tilde{\overline{y}}_n(\alpha+1)]]$

$$\begin{aligned}
x_{l+1} = {}&\tilde{\overline{y}}_1(\alpha+1)^{d_1(\alpha+1)}\cdots\tilde{\overline{y}}_s(\alpha+1)^{d_s(\alpha+1)}\tau[(\overline{P}_{\alpha+1}(\tilde{\overline{y}}_1(\alpha+1),\ldots,\tilde{\overline{y}}_l(\alpha+1))\\
&+\tilde{\overline{y}}_1(\alpha+1)^{\varepsilon_1(\alpha+1)}\cdots\tilde{\overline{y}}_s(\alpha+1)^{\varepsilon_s(\alpha+1)}\Sigma_{\alpha+1}(\tilde{\overline{y}}_1(\alpha+1),\ldots,\tilde{\overline{y}}_n(\alpha+1))]).
\end{aligned}$$

where mult $\Sigma_{\alpha+1}(0,\ldots,0,\tilde{\overline{y}}_{l+1}(\alpha+1),0\ldots,0)=1$, $\overline{P}_{\alpha+1}$ is a power series with constant term 1 and $\tau\in k(U(\alpha+1))$. We have $\overline{x}_{l+1}(\alpha+1)=x_{l+1}$ and $\overline{y}_{l+1}(\alpha+1)=\overline{y}_{l+1}$. By Lemma 4.6 we can construct MTSs $R(\alpha+1)\to R(\alpha+2)$ of type II_{l+1} and $S(\alpha+1)\to S(\alpha+2)$ of type I such that $R(\alpha+2)\hat{}$ has regular parameters

$(x_1^*(\alpha+2),\ldots,x_n^*(\alpha+2))$

$$
\begin{aligned}
x_1(\alpha+1) &= x_1^*(\alpha+2)^{a_{11}(\alpha+2)}\cdots x_s^*(\alpha+2)^{a_{1s}(\alpha+2)}\\
&\quad\cdot(x_{l+1}^*(\alpha+2)+c_{\alpha+2})^{a_{1,s+1}(\alpha+2)}\\
&\vdots\\
x_s(\alpha+1) &= x_1^*(\alpha+2)^{a_{s1}(\alpha+2)}\cdots x_s^*(\alpha+2)^{a_{ss}(\alpha+2)}\\
&\quad\cdot(x_{l+1}^*(\alpha+2)+c_{\alpha+2})^{a_{s,s+1}(\alpha+2)}\\
x_{l+1}(\alpha+1) &= x_1^*(\alpha+2)^{a_{s+1,1}(\alpha+2)}\cdots x_s^*(\alpha+2)^{a_{s+1,s}(\alpha+2)}\\
&\quad\cdot(x_{l+1}^*(\alpha+2)+c_{\alpha+2})^{a_{s+1,s+1}(\alpha+2)}
\end{aligned}
\tag{117}
$$

$S(\alpha+2)\hat{}$ has regular parameters $(\overline{y}_1(\alpha+2),\ldots,\overline{y}_n(\alpha+2))$

$$
\begin{aligned}
\widetilde{\overline{y}}_1(\alpha+1) &= \overline{y}_1(\alpha+2)^{b_{11}(\alpha+2)}\cdots\overline{y}_s(\alpha+2)^{b_{1s}(\alpha+2)}\\
&\vdots\\
\widetilde{\overline{y}}_s(\alpha+1) &= \overline{y}_1(\alpha+2)^{b_{s1}(\alpha+2)}\cdots\overline{y}_s(\alpha+2)^{b_{ss}(\alpha+2)}
\end{aligned}
$$

such that $R(\alpha+2)\subset S(\alpha+2)$.

Set

$$\gamma=\overline{P}_{\alpha+1}+\widetilde{\overline{y}}_1(\alpha+1)^{\varepsilon_1(\alpha+1)}\cdots\widetilde{\overline{y}}_s(\alpha+1)^{\varepsilon_s(\alpha+1)}\Sigma_{\alpha+1}\tag{118}$$

so that

$$x_{l+1}=\widetilde{\overline{y}}_1(\alpha+1)^{d_1(\alpha+1)}\cdots\widetilde{\overline{y}}_s(\alpha+1)^{d_s(\alpha+1)}\tau\gamma.$$

This shows that $\gamma\in U''(\alpha+1)$. Set $(e_{ij})=(a_{ij}(\alpha+2))^{-1}$. By construction there are positive integers f_{ij} such that

$$
\begin{aligned}
x_1^*(\alpha+2) &= x_1(\alpha+1)^{e_{11}}\cdots x_s(\alpha+1)^{e_{1s}}x_{l+1}(\alpha+1)^{e_{1,s+1}}\\
&= \overline{y}_1(\alpha+2)^{f_{11}}\cdots\overline{y}_s(\alpha+2)^{f_{1s}}\gamma^{e_{1,s+1}}\tau^{e_{1,s+1}}\\
&\quad\cdot\phi_1(\alpha+1)^{e_{11}}\cdots\phi_s(\alpha+1)^{e_{1s}}\\
&\vdots\\
x_s^*(\alpha+2) &= x_1(\alpha+1)^{e_{s1}}\cdots x_s(\alpha+1)^{e_{ss}}x_{l+1}(\alpha+1)^{e_{s,s+1}}\\
&= \overline{y}_1(\alpha+2)^{f_{s1}}\cdots\overline{y}_s(\alpha+2)^{f_{ss}}\gamma^{e_{s,s+1}}\tau^{e_{s,s+1}}\\
&\quad\cdot\phi_1(\alpha+1)^{e_{s1}}\cdots\phi_s(\alpha+1)^{e_{ss}}\\
x_{l+1}^*(\alpha+2)+c_{\alpha+2} &= x_1(\alpha+1)^{e_{s+1,1}}\cdots x_s(\alpha+1)^{e_{s+1,s}}x_{l+1}(\alpha+1)^{e_{s+1,s+1}}\\
&= \overline{y}_1(\alpha+2)^{f_{s+1,1}}\cdots\overline{y}_s(\alpha+2)^{f_{s+1,s}}\gamma^{e_{s+1,s+1}}\tau^{e_{s+1,s+1}}\\
&\quad\cdot\phi_1(\alpha+1)^{e_{s+1,1}}\cdots\phi_s(\alpha+1)^{e_{s+1,s}}
\end{aligned}
\tag{119}
$$

in $S(\alpha+2)\hat{}$. $\nu(x_{s+1}^*(\alpha+2)+c_{\alpha+2})=0$ implies $f_{s+1,1}=\cdots=f_{s+1,s}=0$. Set

$$\widetilde{\omega}=\phi_1(\alpha+1)^{e_{s+1,1}}\cdots\phi_s(\alpha+1)^{e_{s+1,s}}\tau^{e_{s+1,s+1}}\in k(U(\alpha+1)).$$

Substituting (118), we have

$$
\begin{aligned}
x_{l+1}^*(\alpha+2)+c_{\alpha+2} &= \widetilde{\omega}(\overline{P}_{\alpha+1}+\widetilde{\overline{y}}_1(\alpha+1)^{\varepsilon_1(\alpha+1)}\cdots\widetilde{\overline{y}}_s(\alpha+1)^{\varepsilon_s(\alpha+1)}\Sigma_{\alpha+1})^{e_{s+1,s+1}}\\
&= Q_0(\overline{y}_1(\alpha+2),\ldots,\overline{y}_l(\alpha+2))\\
&\quad+\overline{y}_1(\alpha+2)^{\varepsilon_1(\alpha+2)}\cdots\overline{y}_s(\alpha+2)^{\varepsilon_s(\alpha+2)}\Lambda_0(\overline{y}_1(\alpha+2),\ldots,\overline{y}_n(\alpha+2))
\end{aligned}
$$

where Q_0 is a unit and $\operatorname{mult} \Lambda_0(0,\ldots,0,\overline{y}_{l+1}(\alpha+2),0\ldots,0)=1$. Define $\alpha_i \in \mathbf{Q}$ by

$$\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_s\end{pmatrix} = \begin{pmatrix} f_{11} & \cdots & f_{1s}\\ \vdots & & \vdots\\ f_{s1} & \cdots & f_{ss}\end{pmatrix}^{-1} \begin{pmatrix}-e_{1,s+1}\\ \vdots\\ -e_{s,s+1}\end{pmatrix}$$

and set

$$\widehat{y}_i(\alpha+2) = \begin{cases} \gamma^{-\alpha_i}\overline{y}_i(\alpha+2) & 1\le i\le s\\ \overline{y}_i(\alpha+2) & s<i\end{cases}$$

to get

$$\begin{aligned} x_1^*(\alpha+2) &= \widehat{y}_1(\alpha+2)^{c_{11}(\alpha+2)}\ldots\widehat{y}_s(\alpha+2)^{c_{1s}(\alpha+2)}\psi_1\\ &\vdots \\ x_s^*(\alpha+2) &= \widehat{y}_1(\alpha+2)^{c_{s1}(\alpha+2)}\ldots\widehat{y}_s(\alpha+2)^{c_{ss}(\alpha+2)}\psi_s.\end{aligned} \tag{120}$$

$(\widehat{y}_1(\alpha+2),\ldots,\widehat{y}_n(\alpha+2))$ are regular parameters in $S(\alpha+2)^\frown$, $\psi_1,\ldots,\psi_s \in k(S(\alpha+2))$.

There are unit power series Q_i and power series Λ_i such that

$$\begin{aligned}\gamma^{\alpha_i} &= Q_i(\overline{y}_1(\alpha+2),\ldots,\overline{y}_l(\alpha+2))\\ &\quad + \overline{y}_1(\alpha+2)^{\varepsilon_1(\alpha+2)}\cdots\overline{y}_s(\alpha+2)^{\varepsilon_s(\alpha+2)}\Lambda_i(\overline{y}_1(\alpha+2),\ldots,\overline{y}_n(\alpha+2))\end{aligned}$$

in $S(\alpha+2)^\frown$ for $1\le i\le s$, where $\operatorname{mult}(\Lambda_i(0,\ldots,0,\overline{y}_{l+1}(\alpha+2),0,\ldots,0))=1$,

$$Q_i(0,\ldots,0)=1.$$

$$\begin{aligned}\overline{y}_i(\alpha+2) &\equiv Q_i(\overline{y}_1(\alpha+2),\ldots,\overline{y}_l(\alpha+2))\widehat{y}_i(\alpha+2)\\ &\qquad \bmod (\widehat{y}_1(\alpha+2)^{\varepsilon_1(\alpha+2)}\cdots\widehat{y}_s(\alpha+2)^{\varepsilon_s(\alpha+2)}\widehat{y}_i(\alpha+2))\end{aligned}$$

for $1\le i\le s$. We will show that there exist unit power series Ω_i such that

$$\begin{aligned}\gamma^{\alpha_i} &\equiv \Omega_i(\widehat{y}_1(\alpha+2),\ldots,\widehat{y}_l(\alpha+2))\\ &\quad + \overline{y}_1(\alpha+2)^{\varepsilon_1(\alpha+2)}\cdots\overline{y}_s(\alpha+2)^{\varepsilon_s(\alpha+2)}\Lambda_i(\overline{y}_1(\alpha+2),\ldots,\overline{y}_n(\alpha+2))\\ &\quad \bmod \widehat{y}_1(\alpha+2)^{\varepsilon_1(\alpha+2)}\cdots\widehat{y}_s(\alpha+2)^{\varepsilon_s(\alpha+2)}(\widehat{y}_1(\alpha+2),\ldots,\widehat{y}_l(\alpha+2)).\end{aligned}$$

This follows from induction, since for any series $A(\overline{y}_1(\alpha+2),\ldots,\overline{y}_l(\alpha+2))$, there exist series $A_{i_1\cdots i_l}$ such that $\operatorname{mult}(A_{i_1,\ldots,i_l})>0$ and

$$\begin{aligned}A(\overline{y}_1(\alpha+2),\ldots,\overline{y}_l(\alpha+2)) &\equiv A(\widehat{y}_1(\alpha+2),\ldots,\widehat{y}_l(\alpha+2))\\ &\quad + \sum_{i_1+\cdots+i_l>0} A_{i_1\cdots i_l}(\overline{y}_1(\alpha+2),\ldots,\overline{y}_l(\alpha+2))\widehat{y}_1(\alpha+2)^{i_1}\cdots\widehat{y}_l(\alpha+2)^{i_l}\\ &\quad \bmod \widehat{y}_1(\alpha+2)^{\varepsilon_1(\alpha+2)}\cdots\widehat{y}_l(\alpha+2)^{\varepsilon_s(\alpha+2)}(\widehat{y}_1(\alpha+2),\ldots,\widehat{y}_l(\alpha+2)).\end{aligned}$$

Thus

$$\begin{aligned}\gamma^{\alpha_i} \equiv\ & \Omega_i(\widehat{y}_1(\alpha+2),\ldots,\widehat{y}_l(\alpha+2)) \\ & + \widehat{y}_1(\alpha+2)^{\varepsilon_1(\alpha+2)}\cdots\widehat{y}_s(\alpha+2)^{\varepsilon_s(\alpha+2)}\Phi_i(\widehat{y}_{l+1}(\alpha+2),\ldots,\widehat{y}_n(\alpha+2)) \\ & \bmod \widehat{y}_1(\alpha+2)^{\varepsilon_1(\alpha+2)}\cdots\widehat{y}_s(\alpha+2)^{\varepsilon_s(\alpha+2)}(\widehat{y}_1(\alpha+2),\ldots,\widehat{y}_l(\alpha+2))\end{aligned}$$

with

$$\operatorname{mult}\Phi_i(\widehat{y}_{l+1}(\alpha+2),0,\ldots,0)=1.$$

$R(\alpha+2)$ has regular parameters $(x_1(\alpha+2),\ldots,x_n(\alpha+2))$ defined by

$$x_i(\alpha+2)=\begin{cases} x_i^*(\alpha+2) & 1\neq l+1 \\ \prod(x_{l+1}^*(\alpha+2)+c_{\alpha+2}-\sigma(c_{\alpha+2})) & i=l+1\end{cases} \tag{121}$$

where the product is over the distinct conjugates $\sigma(c_{\alpha+2})\in\overline{k}$ of $c_{\alpha+2}$ over k.

Set $\widetilde{\overline{x}}_i(\alpha+2)=x_i(\alpha+2)$, $\widetilde{\overline{y}}_i(\alpha+2)=\widehat{y}_i(\alpha+2)$ for $1\leq i\leq n$. Set $U''(\alpha+2)=U'(\alpha+2)[\gamma^{\alpha_1},\ldots,\gamma^{\alpha_s}]_q$ where

$$q=m(U(\alpha+2))\cap U'(\alpha+2)[\gamma^{\alpha_1},\ldots,\gamma^{\alpha_s}].$$

Then $\mathrm{CN}(\alpha+2)$ holds. We have $x_{l+1}^*(\alpha+2)+c_{\alpha+2}=\gamma^{e_{s+1,s+1}}\lambda$ for some $\lambda\in k(S(\alpha+2))$ by (119). $e_{s+1,s+1}\neq 0$ by Theorem 2.12.

$$\begin{aligned}x_{l+1}^*(\alpha+2) =\ & \widetilde{P}_{\alpha+2}(\widetilde{\overline{y}}_1(\alpha+2),\ldots,\widetilde{\overline{y}}_l(\alpha+2)) \\ & + \widetilde{\overline{y}}_1(\alpha+2)^{\varepsilon_1(\alpha+2)}\cdots\widetilde{\overline{y}}_s(\alpha+2)^{\varepsilon_s(\alpha+2)}\widetilde{\Sigma}_{\alpha+2}(\widetilde{\overline{y}}_1(\alpha+2),\ldots,\widetilde{\overline{y}}_n(\alpha+2))\end{aligned}$$

where $\operatorname{mult}(\widetilde{\Sigma}_{\alpha+2}(0,\ldots,0,\widetilde{\overline{y}}_{l+1}(\alpha+2),0\ldots,0))=1$.

$$\begin{aligned}x_{l+1}(\alpha+2) =\ & \prod_\sigma\left(\widetilde{P}_{\alpha+2}(\widetilde{\overline{y}}_1(\alpha+2),\ldots,\widetilde{\overline{y}}_l(\alpha+2))+(c_{\alpha+2}-\sigma(c_{\alpha+2}))\right) \\ & + \widetilde{\overline{y}}_1(\alpha+2)^{\varepsilon_1(\alpha+2)}\cdots\widetilde{\overline{y}}_s(\alpha+2)^{\varepsilon_s(\alpha+2)} \\ & \cdot\Big[\sum_\sigma\left(\widetilde{P}_{\alpha+2}(\widetilde{\overline{y}}_1(\alpha+2),\ldots,\widetilde{\overline{y}}_l(\alpha+2))+(c_{\alpha+2}-\sigma(c_{\alpha+2}))\right)\widetilde{\Sigma}_{\alpha+2} \\ & \qquad + \widetilde{\overline{y}}_1(\alpha+2)^{\varepsilon_1(\alpha+2)}\cdots\widetilde{\overline{y}}_s(\alpha+2)^{\varepsilon_s(\alpha+2)}\Omega\Big]\end{aligned}$$

in $S(\alpha+2)\hat{}\cong k(S(\alpha+2))[[\widetilde{\overline{y}}_1(\alpha+2),\ldots,\widetilde{\overline{y}}_n(\alpha+2)]]$ and the product and sum are over the distinct conjugates $\sigma(c_{\alpha+2})$ of $c_{\alpha+2}$ in $\overline{k}$ over k. If $c_{\alpha+2}\not\in k$, we have

$$\sum_\sigma(c_{\alpha+2}-\sigma(c_{\alpha+2}))\neq 0$$

since if this sum were 0, we would have $c_{\alpha+2}$ invariant under the automorphism group of a Galois closure of $k(c_{\alpha+2})$ over k which is impossible. We have an expression

$$\begin{aligned}x_{l+1}(\alpha+2) =\ & P_{\alpha+2}(\widetilde{\overline{y}}_1(\alpha+2),\ldots,\widetilde{\overline{y}}_l(\alpha+2)) \\ & + \widetilde{\overline{y}}_1(\alpha+2)^{\varepsilon_1(\alpha+2)}\cdots\widetilde{\overline{y}}_s(\alpha+2)^{\varepsilon_s(\alpha+2)}\Sigma_{\alpha+2}(\widetilde{\overline{y}}_1(\alpha+2),\ldots,\widetilde{\overline{y}}_n(\alpha+2))\end{aligned} \tag{122}$$

where $P_{\alpha+2}$, $\Sigma_{\alpha+2}$ are power series with coefficients in $k(S(\alpha+2))$ and

$$\mathrm{mult}(\Sigma_{\alpha+2}(0,\ldots,0,\widetilde{\overline{y}}_{l+1},0,\ldots,0))=1.$$

If $c_{\alpha+2}\in k$, $x_{l+1}(\alpha+2)=x^*_{l+1}(\alpha+2)$ and this expression is immediate. By (115) and (116),

$$\begin{aligned}
g=&\sum_{i=1}^{d} a'_i\widetilde{\overline{x}}_1(\alpha+1)^{e_i^1(\alpha+1)}\cdots\widetilde{\overline{x}}_s(\alpha+1)^{e_i^s(\alpha+1)}\widetilde{\overline{x}}_{l+1}(\alpha+1)^{f_i}\\
&+\sum_{j} a'_j\widetilde{\overline{x}}_1(\alpha+1)^{e_j^1(\alpha+1)}\cdots\widetilde{\overline{x}}_s(\alpha+1)^{e_j^s(\alpha+1)}\widetilde{\overline{x}}_{l+1}(\alpha+1)^{f_j}\\
&+\sum_{k} a'_k\widetilde{\overline{x}}_1(\alpha+1)^{e_k^1(\alpha+1)}\cdots\widetilde{\overline{x}}_s(\alpha+1)^{e_k^s(\alpha+1)}\widetilde{\overline{x}}_{l+1}(\alpha+1)^{f_k}\\
&+\widetilde{\overline{x}}_{l+1}(\alpha+1)^{\tau}\Gamma
\end{aligned}$$

with $a'_i,a'_j,a'_k\in k(c_0,\ldots,c_{\alpha+1})[[\widetilde{\overline{x}}_1(\alpha+1),\ldots,\widetilde{\overline{x}}_l(\alpha+1)]]$. Since

$$\widetilde{\overline{x}}_{l+1}(\alpha+1)=x_{l+1}(\alpha+1)=x_{l+1}$$

and $\mathrm{CN}(\alpha+1)$ holds, so that

$$k(c_0,\ldots,c_{\alpha+1})[[x_1(\alpha+1),\ldots,x_l(\alpha+1)]]=k(c_0,\ldots,c_{\alpha+1})[[\widetilde{\overline{x}}_1(\alpha+1),\ldots,\widetilde{\overline{x}}_l(\alpha+1)]],$$

we have an expansion

$$\begin{aligned}
g=&\sum_{i=1}^{d} \widetilde{a}_i x_1(\alpha+1)^{e_i^1(\alpha+1)}\cdots x_s(\alpha+1)^{e_i^s(\alpha+1)}x_{l+1}(\alpha+1)^{f_i}\\
&+\sum_{j} \widetilde{a}_j x_1(\alpha+1)^{e_j^1(\alpha+1)}\cdots x_s(\alpha+1)^{e_j^s(\alpha+1)}x_{l+1}(\alpha+1)^{f_j}\\
&+\sum_{k} \widetilde{a}_k x_1(\alpha+1)^{e_k^1(\alpha+1)}\cdots x_s(\alpha+1)^{e_k^s(\alpha+1)}x_{l+1}(\alpha+1)^{f_k}\\
&+x_{l+1}(\alpha+1)^{\tau}\Gamma
\end{aligned}$$

with $\widetilde{a}_i,\widetilde{a}_j,\widetilde{a}_k\in k(c_0,\ldots,c_{\alpha+1})[[x_1(\alpha+1),\ldots,x_l(\alpha+1)]]$, $\widetilde{a}_i,\widetilde{a}_j$ units for all i,j and

$$\nu(x_1(\alpha+1)^{e_k^1(\alpha+1)}\cdots x_s(\alpha+1)^{e_k^s(\alpha+1)}x_{l+1}(\alpha+1)^{f_k})>\rho$$

for all k. In $R(\alpha+2)\widehat{\ }$, by (117)

$$g=\sum_{i=1}^{d}\widetilde{a}_i\prod_{a=1}^{s}x_a^*(\alpha+2)^{a_{1a}(\alpha+2)e_i^1(\alpha+1)+\cdots+a_{sa}(\alpha+2)e_i^s(\alpha+1)+a_{s+1,a}(\alpha+2)f_i}$$
$$(x_{l+1}^*(\alpha+2)+c_{\alpha+2})^{a_{1,s+1}(\alpha+2)e_i^1(\alpha+1)+\cdots+a_{s,s+1}(\alpha+2)e_i^s(\alpha+1)+a_{s+1,s+1}(\alpha+2)f_i}$$
$$+\sum_j\widetilde{a}_j\prod_{a=1}^{s}x_a^*(\alpha+2)^{a_{1a}(\alpha+2)e_j^1(\alpha+1)+\cdots+a_{sa}(\alpha+2)e_j^s(\alpha+1)+a_{s+1,a}(\alpha+2)f_j}$$
$$(x_{l+1}^*(\alpha+2)+c_{\alpha+2})^{a_{1,s+1}(\alpha+2)e_j^1(\alpha+1)+\cdots+a_{s,s+1}(\alpha+2)e_j^s(\alpha+1)+a_{s+1,s+1}(\alpha+2)f_j}$$
$$+\sum_k\widetilde{a}_k\prod_{a=1}^{s}x_a^*(\alpha+2)^{a_{1a}(\alpha+2)e_k^1(\alpha+1)+\cdots+a_{sa}(\alpha+2)e_k^s(\alpha+1)+a_{s+1,a}(\alpha+2)f_k}$$
$$(x_{l+1}^*(\alpha+2)+c_{\alpha+2})^{a_{1,s+1}(\alpha+2)e_k^1(\alpha+1)+\cdots+a_{s,s+1}(\alpha+2)e_k^s(\alpha+1)+a_{s+1,s+1}(\alpha+2)f_k}$$
$$+\left[\left(\prod_{a=1}^{s}x_a^*(\alpha+2)^{a_{s+1,a}(\alpha+2)}\right)(x_{l+1}^*(\alpha+2)+c_{\alpha+2})^{a_{s+1,s+1}(\alpha+2)}\right]^{\tau}\Gamma$$

with $\widetilde{a}_i,\widetilde{a}_j,\widetilde{a}_k\in k(c_0,\ldots,c_{\alpha+2})[[x_1^*(\alpha+2),\ldots,x_{l+1}^*(\alpha+2)]]$, $\widetilde{a}_i,\widetilde{a}_j$ units and

$$\nu\Big(\prod_{a=1}^{s}x_a^*(\alpha+2)^{a_{1a}(\alpha+2)e_k^1(\alpha+1)+\cdots+a_{sa}(\alpha+2)e_k^s(\alpha+1)+a_{s+1,a}(\alpha+2)f_k}\Big)>\rho.$$

We have that

$$\sum_{a=1}^{s}(a_{1a}(\alpha+2)e_i^1(\alpha+1)+\cdots+a_{sa}(\alpha+2)e_i^s(\alpha+1)+a_{s+1,a}(\alpha+2)f_i)\nu(x_a^*(\alpha+2))$$

are equal for $1\le i\le d$ since the corresponding terms in g have equal value ρ. Set

$$\alpha_\zeta=a_{1\zeta}(\alpha+2)e_1^1(\alpha+1)+\cdots+a_{s\zeta}(\alpha+2)e_1^s(\alpha+1)+a_{s+1,\zeta}(\alpha+2)f_1$$

for $1\le\zeta\le s$. Since $\nu(x_1^*(\alpha+2)),\ldots,\nu(x_s^*(\alpha+2))$ are rationally independent, we have

$$\alpha_\zeta=a_{1\zeta}(\alpha+2)e_i^1(\alpha+1)+\cdots+a_{s\zeta}(\alpha+2)e_i^s(\alpha+1)+a_{s+1,\zeta}(\alpha+2)f_i$$

for $1\le i\le d$ and $1\le\zeta\le s$. Set

$$m_{f_i}=a_{1,s+1}(\alpha+2)e_i^1(\alpha+1)+\cdots+a_{s,s+1}(\alpha+2)e_i^s(\alpha+1)+a_{s+1,s+1}(\alpha+2)f_i$$

for $1\le i\le d$. Set

$$\varepsilon=\frac{\det\begin{pmatrix}a_{11}(\alpha+2)&\cdots&a_{1s}(\alpha+2)\\ \vdots&&\vdots\\ a_{s1}(\alpha+2)&\cdots&a_{ss}(\alpha+2)\end{pmatrix}}{\det\begin{pmatrix}a_{11}(\alpha+2)&\cdots&a_{1,s+1}(\alpha+2)\\ \vdots&&\vdots\\ a_{s+1,1}(\alpha+2)&\cdots&a_{s+1,s+1}(\alpha+2)\end{pmatrix}}.$$

ε is a nonzero integer. By Cramer's rule,

$$f_i - f_1 = \varepsilon(m_{f_i} - m_{f_1})$$

for $1 \le i \le d$. Assume that $\varepsilon > 0$. We can perform a CRUTS of type (M1) with associated MTSs $R(\alpha+2) \to R(\alpha+3)$ and $S(\alpha+2) \to S(\alpha+3)$ where $R(\alpha+3)$ has regular parameters $(x_1(\alpha+3), \ldots, x_n(\alpha+3))$ and $T''(\alpha+3)$ has regular parameters $(\overline{x}_1(\alpha+3), \ldots, \overline{x}_n(\alpha+3))$ such that

$$\begin{aligned} x_1(\alpha+2) = x_1^*(\alpha+2) &= x_1(\alpha+3)^{a_{11}(\alpha+3)} \cdots x_s(\alpha+3)^{a_{1s}(\alpha+3)} \\ &\vdots \\ x_s(\alpha+2) = x_s^*(\alpha+2) &= x_1(\alpha+3)^{a_{s1}(\alpha+3)} \cdots x_s(\alpha+3)^{a_{ss}(\alpha+3)} \end{aligned} \tag{123}$$

and

$$\overline{x}_i(\alpha+3) = \begin{cases} x_i(\alpha+3) & i \le s \\ x_i^*(\alpha+2) & s < i \end{cases}$$

$x_i(\alpha+2) = x_i(\alpha+3)$ for $s < i$ to get

$$g = \overline{x}_1(\alpha+3)^{a'_1} \cdots \overline{x}_s(\alpha+3)^{a'_s}((\overline{x}_{l+1}(\alpha+3) + c_{\alpha+2})^{m_{f_1}}\Phi + \overline{x}_1(\alpha+3)\cdots\overline{x}_s(\alpha+3)G)$$

in $k(c_0, \ldots, c_{\alpha+3})[[\overline{x}_1(\alpha+3), \ldots, \overline{x}_{l+1}(\alpha+3)]]$, with

$$\Phi = \widetilde{a}_1 + \widetilde{a}_2(\overline{x}_{l+1}(\alpha+3) + c_{\alpha+2})^{(f_2-f_1)/\varepsilon} + \cdots + \widetilde{a}_d(\overline{x}_{l+1}(\alpha+3) + c_{\alpha+2})^{(f_d-f_1)/\varepsilon}.$$

In the case $\varepsilon < 0$, we must consider an expression

$$g = \overline{x}_1(\alpha+3)^{a'_1} \cdots \overline{x}_s(\alpha+3)^{a'_s}((\overline{x}_{l+1}(\alpha+3) + c_{\alpha+2})^{m_{f_d}}\Phi' + \overline{x}_1(\alpha+3)\cdots\overline{x}_s(\alpha+3)G')$$

with

$$\Phi' = \widetilde{a}_1(\overline{x}_{l+1}(\alpha+3) + c_{\alpha+2})^{(f_1-f_d)/\varepsilon} + \cdots + \widetilde{a}_d.$$

Again assume that $\varepsilon > 0$. The proof when $\varepsilon < 0$ is similar. Let

$$r' = \mathrm{mult}(\Phi(0, \ldots, 0, \overline{x}_{l+1}(\alpha+3))).$$

$\rho = \nu(a_i) + f_i\nu(\overline{x}_{l+1}(1)) \le r\nu(\overline{x}_{l+1}(1))$ implies $f_d \le r$. Set $\eta_i = (f_i - f_1)/\varepsilon$. The residue of $\widetilde{a}_i$ in

$$T(\alpha+3)/(\overline{x}_1(\alpha+3), \ldots, \overline{x}_l(\alpha+3), \overline{x}_{l+2}(\alpha+3), \ldots, \overline{x}_n(\alpha+3)) \cong k(T(\alpha+3))[[\overline{x}_{l+1}(\alpha+3)]]$$

is a nonzero constant $\widehat{a}_i \in k(c_0, \ldots, c_{\alpha+2})$ for $1 \le i \le d$. Set

$$\zeta(t) = \widehat{a}_1 + \widehat{a}_2 t^{\eta_2} + \cdots + \widehat{a}_d t^{\eta_d},$$

$r' = \mathrm{mult}(\zeta(\overline{x}_{l+1}(\alpha+3) + c_{\alpha+2}))$. $r' \le \eta_d = (f_d - f_1)/\varepsilon \le r$. Suppose that $r' = r$. Then $f_1 = 0$, $f_d = r$, $\varepsilon = 1$ and $\zeta(t) = \widehat{a}_d(t - c_{\alpha+2})^r$. Thus there exist nonzero $(\overline{x}_{l+1}(\alpha+3) + c_{\alpha+2})^r$ and $(\overline{x}_{l+1}(\alpha+3) + c_{\alpha+2})^{r-1}$ terms in $\Phi(0, \ldots, 0, \overline{x}_{l+1}(\alpha+3))$

and $f_{d-1} = f_d - 1 = r - 1$. Thus a_d is a unit so that $a_d = \widetilde{a}_d$ and $\nu(a_{d-1}\overline{x}_{l+1}(1)^{r-1}) = \nu(a_d\overline{x}_{l+1}(1)^r)$ implies $\nu(\overline{x}_{l+1}(1)) = \nu(a_{d-1})$.

$$\sum_{i=1}^{d} \widehat{a}_i(\overline{x}_{l+1}(\alpha+3) + c_{\alpha+2})^{(f_i-f_1)/\varepsilon} = \widehat{a}_d\overline{x}_{l+1}(\alpha+3)^r. \tag{124}$$

Expanding out the LHS of (124), we have

$$r\widehat{a}_d c_{\alpha+2} + \widehat{a}_{d-1} = 0$$

which implies

$$\frac{c_{\alpha+2}}{\widehat{a}_{d-1}} = -\frac{1}{r\widehat{a}_d}.$$

Let $\alpha = \widehat{a}_d \in k(c_0)$ be the constant term of the power series

$$a_d \in k(c_0)[[x_1(1), \ldots, x_l(1)]].$$

$$\begin{aligned}
\frac{\overline{x}_{l+1}(1)}{a_{d-1}} = \frac{a_d\overline{x}_{l+1}(1)^r}{a_d a_{d-1}\overline{x}_{l+1}(1)^{r-1}} &= \frac{\widetilde{a}_d(\overline{x}_{l+1}(\alpha+3)+c_{\alpha+2})^{m_{f_d}}}{\widetilde{a}_d\widetilde{a}_{d-1}(\overline{x}_{l+1}(\alpha+3)+c_{\alpha+2})^{m_{f_{d-1}}}} \\
&= \frac{\widetilde{a}_d(\overline{x}_{l+1}(\alpha+3)+c_{\alpha+2})^{(f_d-f_1)/\varepsilon}}{\widetilde{a}_d\widetilde{a}_{d-1}(\overline{x}_{l+1}(\alpha+3)+c_{\alpha+2})^{(f_{d-1}-f_1)/\varepsilon}} \\
&= \frac{\widetilde{a}_d(\overline{x}_{l+1}(\alpha+3)+c_{\alpha+2})}{\widetilde{a}_d\widetilde{a}_{d-1}}
\end{aligned}$$

which has residue $-1/r\alpha$ in $k(R(\alpha+3)) \subset \mathcal{O}_\nu/m_\nu$. (Here $\mathcal{O}_v$ is the valuation ring of our extension of ν to the quotient field of $S(\alpha+3)\widehat{\ }$.) There exists $Q \in \widetilde{R}(1)$ such that Q is equivalent to $-\frac{1}{r\alpha}a_{d-1}$ modulo a sufficiently high power of the maximal ideal of $k(c_0)[[x_1(1), \ldots, x_l(1)]]$ (recall that $c_1 = 1$) so that $\nu(x_{l+1}(1) - Q) > \nu(x_{l+1}(1))$ (Recall that $\overline{x}_{l+1}(1) = x_{l+1}(1)$).

Since $a_d\overline{x}_{l+1}(1)^r$ is a minimum value term of g, we have $\nu(\overline{x}_{l+1}(1)) \leq \nu(g)$. Make a change of variables in $R(1)$ and $T''(1)$, replacing $x_{l+1}(1)$ and $\overline{x}_{l+1}(1)$ with

$$\overline{x}_{l+1}^{(1)}(1) = x_{l+1}^{(1)}(1) = x_{l+1}(1) - Q$$

CN(1) holds for these new variables. Further, in $S(1)\widehat{\ }$, we have

$$\overline{x}_{l+1}^{(1)}(1) = P^{(1)}(\overline{y}_1(1), \ldots, \overline{y}_l(1)) + \overline{y}_1(1)^{d_1(1)} \cdots \overline{y}_s(1)^{d_s(1)}\Sigma_0$$

where $\mathrm{mult}(\Sigma_0(0, \ldots, 0, \overline{y}_{l+1}(1), 0, \ldots, 0)) = 1$. Then repeat the above procedure with this change of variable and our previous g. If $\nu(P^{(1)}) > \nu(\widetilde{\overline{y}}_1(\alpha)^{d_1(\alpha)} \cdots \widetilde{\overline{y}}_s(\alpha)^{d_s(\alpha)})$ the above algorithm produces an expression

$$x_{l+1}^{(1)}(1) = \widetilde{\overline{y}}_1(\alpha+1)^{d_1(\alpha+1)} \cdots \widetilde{\overline{y}}_s(\alpha+1)^{d_s(\alpha+1)}\Sigma_\alpha^{(1)}$$

where $\mathrm{mult}(\Sigma_\alpha^{(1)}(0, \ldots, 0, \widetilde{\overline{y}}_{l+1}(\alpha+1), 0, \ldots, 0)) = 1$. So suppose that

$$\nu(P^{(1)}) \leq \nu(\widetilde{\overline{y}}_1(\alpha)^{d_1(\alpha)} \cdots \widetilde{\overline{y}}_s(\alpha)^{d_s(\alpha)}).$$

If we do not get a reduction $r_1 < r$, we have

$$\nu(\overline{x}_{l+1}(1)) < \nu(\overline{x}_{l+1}^{(1)}(1)) \leq \nu(g).$$

We can repeat this process. By Lemma 2.3, We eventually get a reduction $r' < r$, or $\nu(g) = \infty$ and we can construct (as in (58) in the proof of $A(\overline{m})$ in Theorem 4.7)

$$\phi = \lim_{i\to\infty} Q_i(\overline{x}_1(1), \ldots, \overline{x}_l(1)) \in k(c_0)[[\overline{x}_1(1), \ldots, \overline{x}_l(1)]]$$

such that

$$g = u(\overline{x}_{l+1}(1) - \phi)^r + h. \tag{125}$$

where $h \in a_l k(c_0)[[\overline{x}_1(1), \ldots, \overline{x}_{l+1}]]$ with

$$a_l = \{f \in k(c_0)[[\overline{x}_1(1), \ldots, \overline{x}_l(1)]] \mid \nu(f) > \infty\}$$

and $u(\overline{x}_1(1), \ldots, \overline{x}_{l+1})$ is a unit power series.

Suppose that $r' < r$. In our construction, $\overline{y}_{l+1}(\alpha+3) = \overline{y}_{l+1}$, so that if

$$\Sigma_0 = \Sigma_0^{\alpha+3}(\overline{y}_1(\alpha+3), \ldots, \overline{y}_n(\alpha+3)),$$

$\mathrm{mult}(\Sigma_0^{\alpha+3}(0, \ldots, 0, \overline{y}_{l+1}(\alpha+3), 0, \ldots, 0)) = 1$. Set

$$g_1 = (\overline{x}_{l+1}(\alpha+3) + c_{\alpha+2})^{m_{f_1}}\Phi + \overline{x}_1(\alpha+3)\cdots\overline{x}_s(\alpha+3)G.$$

$x_{l+1}(\alpha+2) = \eta x^*_{l+1}(\alpha+2)$ where $\eta \in k(c_{\alpha+2})[x^*_{l+1}(\alpha+2)]$ is a unit which implies

$$x_{l+1}(\alpha+3) = \eta\overline{x}_{l+1}(\alpha+3).$$

Thus

$$\begin{aligned} g_1 &\in k(c_0, \ldots, c_{\alpha+3})[[\overline{x}_1(\alpha+3), \ldots, \overline{x}_{l+1}(\alpha+3)]] \\ &= k(c_0, \ldots, c_{\alpha+3})[[x_1(\alpha+3), \ldots, x_{l+1}(\alpha+3)]] \subset R(\alpha+3)\widehat{\ } \end{aligned}$$

$\Sigma_0 \mid g_1$ so that

$$1 \le \mathrm{mult}\, g_1(0, \ldots, 0, x_{l+1}(\alpha+3)) = \mathrm{mult}\, g_1(0, \ldots, 0, \overline{x}_{l+1}(\alpha+3)) = r_1 < r.$$

By (122), there is an expression in $S(\alpha+3)\widehat{\ }$

$$\begin{aligned} x_{l+1}(\alpha+3) &= x_{l+1}(\alpha+2) \\ &= P_{\alpha+3}(\widetilde{\overline{y}}_1(\alpha+3), \ldots, \widetilde{\overline{y}}_l(\alpha+3)) \\ &\quad + \widetilde{\overline{y}}_1(\alpha+3)^{d_1(\alpha+3)}\cdots\widetilde{\overline{y}}_s(\alpha+3)^{d_s(\alpha+3)}\Sigma_{\alpha+3}(\widetilde{\overline{y}}_1(\alpha+3), \ldots, \widetilde{\overline{y}}_n(\alpha+3)) \end{aligned}$$

where $\mathrm{mult}(\Sigma_{\alpha+3}(0, \ldots, 0, \widetilde{\overline{y}}_{l+1}(\alpha+3), 0, \ldots, 0)) = 1$. Now set $\widetilde{\overline{x}}_i(\alpha+3) = x_i(\alpha+3)$ for $1 \le i \le n$. By (120), (121) and (123) $\mathrm{CN}(\alpha+3)$ holds (with $\gamma_i(\alpha+3) = 1$ for $1 \le i \le l$).

By induction on r we can now repeat the procedure following (115), with $R(1)$, $\widetilde{R}(1)$, $S(1)$ replaced with $R(\alpha+3)$, $\widetilde{R}(\alpha+3)$, $S(\alpha+3)$ respectively, c_0 with a primitive element of $k(c_0, \ldots, c_{\alpha+3})$ over k, g with g_1, to eventually get t such that $\mathrm{CN}(t)$ holds with $\widetilde{\overline{x}}_i(t) = x_i(t)$ for $1 \le i \le n$ and

$$x_{l+1}(t) = \widetilde{\overline{y}}_1(t)^{d_1(t)}\cdots\widetilde{\overline{y}}_s(t)^{d_s(t)}\Sigma_t(\widetilde{\overline{y}}_1(t), \ldots, \widetilde{\overline{y}}_n(t))$$

where $\operatorname{mult}\Sigma_t(0,\ldots,0,\widetilde{\overline{y}}_{l+1}(t),0,\ldots,0)=1$ or we have

$$x_{l+1}(t)=P_t(\widetilde{\overline{y}}_1(t),\ldots,\widetilde{\overline{y}}_l(t))+\widetilde{\overline{y}}_1(t)^{d_1(t)}\cdots\widetilde{\overline{y}}_s(t)^{d_s(t)}\Sigma_t(\widetilde{\overline{y}}_1(t),\ldots,\widetilde{\overline{y}}_n(t))$$

where $\operatorname{mult}\Sigma_t(0,\ldots,0,\widetilde{\overline{y}}_{l+1}(t),0\ldots,0)=1$, P_t,Σ_t are series with coefficients in $k(S(t))$ and

$$g=u(x_1(t),\ldots,x_{l+1}(t))x_1(t)^{d_1}\cdots x_s(t)^{d_s}[x_{l+1}(t)+\Phi(x_1(t),\ldots,x_l(t))]^a \tag{126}$$

for some $a>0$ where u,Φ are series with coefficients in $k(c_0,\ldots,c_t)$, u is a unit and $\Sigma_0\mid g$. Suppose that (126) holds. Define

$$\Sigma_0^t(\widetilde{\overline{y}}_1(t),\ldots,\widetilde{\overline{y}}_n(t))=\Sigma_0.$$

$\operatorname{mult}\Sigma_0^t(0,\ldots,0,\widetilde{\overline{y}}_{l+1}(t),0,\ldots,0)=1$. We have regular parameters $(\widehat{y}_1(t),\ldots,\widehat{y}_n(t))$ in $S(t)^\frown$ defined by

$$\widehat{y}_i(t)=\begin{cases}\widetilde{\overline{y}}_i(t) & i\neq l+1\\ \Sigma_0 & i=l+1\end{cases}$$

There is an expression

$$x_{l+1}(t)=\overline{P}_t(\widehat{y}_1(t),\ldots,\widehat{y}_l(t))+\widehat{y}_1(t)^{d_1(t)}\cdots\widehat{y}_s^{d_s(t)}\widehat{\Sigma}_t(\widehat{y}_1(t),\ldots,\widehat{y}_n(t))$$

with $\operatorname{mult}\widehat{\Sigma}_t(0,\ldots,0,\widehat{y}_{l+1}(t),0,\ldots,0)=1$. Thus

$$\widehat{y}_{l+1}(t)\mid x_{l+1}(t)+\Phi(x_1(t),\ldots,x_l(t))$$

in $S(t)^\frown$. Since $P_t+\Phi\in k(S(t))[[\widetilde{\overline{y}}_1(t),\ldots,\widetilde{\overline{y}}_l(t)]]$,

$$P_t+\Phi=\Omega\widetilde{\overline{y}}_1(t)^{d_1(t)}\cdots\widetilde{\overline{y}}_s(t)^{d_s(t)}$$

with $\Omega\in k(S(t))[[\widetilde{\overline{y}}_1(t),\ldots,\widetilde{\overline{y}}_l(t)]]$ and

$$\Sigma_0\mid\Omega+\Sigma_t.$$

Set $m_{t,l}=m\left(k(c_0,\ldots,c_t)[[x_1(t),\ldots,x_l(t)]]\right)$. Choose N so that

$$N\nu(m_{t,l})>\nu(\widetilde{\overline{y}}_1(t)^{d_1(t)}\cdots\widetilde{\overline{y}}_s(t)^{d_s(t)}).$$

There exists $\Phi'\in k(c_0,\ldots,c_t)[x_1(t),\ldots,x_l(t)]$ such that $\Phi'\equiv\Phi \bmod m_{t,l}^N$. Make a change of variables, replacing $x_{l+1}(t)$ with $x_{l+1}(t)+\Phi'$ to get

$$x_{l+1}(t)=\widetilde{\overline{y}}_1(t)^{d_1(t)}\cdots\widetilde{\overline{y}}_s(t)^{d_s(t)}(\Omega+\Sigma_t)+(\Phi'-\Phi)$$

By Theorem 4.10, we can perform a CRUTS along ν in the first l variables, with associated MTSs $R(t)\to R(t')$, $S(t)\to S(t')$ to get

$$\widetilde{\overline{x}}_{l+1}(t')=x_{l+1}(t')=\widetilde{\overline{y}}_1(t')^{d_1(t')}\cdots\widetilde{\overline{y}}_s(t')^{d_s(t')}\Sigma_{t'},$$

where $\operatorname{mult}\Sigma_{t'}(0,\ldots,0,\widetilde{\overline{y}}_{l+1}(t'),0,\ldots,0))=1$ and such that $\mathrm{CN}(t')$ holds.

We thus have regular parameters $(x_1(t'),\ldots,x_n(t'))$ in $R(t')$ and $(y_1(t'),\ldots,y_n(t'))$ in $S(t')$, units $\tau_1(t'),\ldots,\tau_s(t')\in S(t')$ such that

$$\begin{aligned}
x_1(t') &= y_1(t')^{c_{11}(t')}\cdots y_s(t')^{c_{1s}(t')}\tau_1(t')\\
&\vdots\\
x_s(t') &= y_1(t')^{c_{s1}(t')}\cdots y_s(t')^{c_{ss}(t')}\tau_s(t')\\
x_{s+1}(t') &= y_{s+1}(t')\\
&\vdots\\
x_l(t') &= y_l(t')\\
x_{l+1}(t') &= y_1(t')^{d_1(t')}\cdots y_s(t')^{d_s(t')}y_{l+1}(t').
\end{aligned}$$

Let $\phi_i(t')$ be the residue of $\tau_i(t')$ in $k(S(t'))$, $\overline{\tau}_i=\tau_i(t')/\phi_i(t')$. Let $(e_{ij})=(c_{ij}(t'))^{-1}$. Define

$$\overline{y}_i(t')=\begin{cases}\overline{\tau}_1^{e_{i1}}\cdots\overline{\tau}_s^{e_{is}}y_i(t') & 1\le i\le s\\ y_i(t') & s<i, i\neq l+1\\ \overline{\tau}_1^{-e_{11}d_1(t')-\cdots-e_{s1}d_s(t')}\cdots\overline{\tau}_s^{-e_{1s}d_1(t')-\cdots-e_{ss}d_s(t')}y_{l+1}(t') & i=l+1\end{cases}$$

We have

$$\begin{aligned}
x_1(t') &= \overline{y}_1(t')^{c_{11}(t')}\cdots\overline{y}_s(t')^{c_{1s}(t')}\phi_1(t')\\
&\vdots\\
x_s(t') &= \overline{y}_1(t')^{c_{s1}(t')}\cdots\overline{y}_s(t')^{c_{ss}(t')}\phi_s(t')\\
x_{s+1}(t') &= \overline{y}_{s+1}(t')\\
&\vdots\\
x_l(t') &= \overline{y}_l(t')\\
x_{l+1}(t') &= \overline{y}_1(t')^{d_1(t')}\cdots\overline{y}_s(t')^{d_s(t')}\overline{y}_{l+1}(t')
\end{aligned}$$

in

$$U''(t')=S(t')[\overline{\tau}_1^{e_{11}},\ldots,\overline{\tau}_s^{e_{ss}}]_{(\overline{y}_1(t'),\ldots,\overline{y}_n(t'))}.$$

By Lemma 4.5, we can perform a MTS of type II_{l+1} $R(t')\to R(t'+1)$

$$\begin{aligned}
\overline{x}_1(t') &= \overline{x}_1(t'+1)^{a_{11}(t'+1)}\cdots\overline{x}_s(t'+1)^{a_{1s}(t'+1)}c_{t'+1}^{a_{1,s+1}(t'+1)}\\
&\vdots\\
\overline{x}_s(t') &= \overline{x}_1(t'+1)^{a_{s1}(t'+1)}\cdots\overline{x}_s(t'+1)^{a_{ss}(t'+1)}c_{t'+1}^{a_{s,s+1}(t'+1)}\\
x_{l+1}(t') &= \overline{x}_1(t'+1)^{a_{s+1,1}(t'+1)}\cdots\overline{x}_s(t'+1)^{a_{s+1,s}(t'+1)}\\
&\quad\cdot(\overline{x}_{l+1}(t'+1)+1)c_{t'+1}^{a_{s+1,s+1}(t'+1)}
\end{aligned}$$

and a MTS of type II_{l+1} (possibly followed by a transformation of type I) $S(t') \to S(t'+1)$

$$\begin{aligned}
\overline{y}_1(t') &= \overline{y}_1(t'+1)^{b_{11}(t'+1)} \cdots \overline{y}_s(t'+1)^{b_{1s}(t'+1)} d_{t'+1}^{b_{1,s+1}(t'+1)} \\
&\vdots \\
\overline{y}_s(t') &= \overline{y}_1(t'+1)^{b_{s1}(t'+1)} \cdots \overline{y}_s(t'+1)^{b_{ss}(t'+1)} d_{t'+1}^{b_{s,s+1}(t'+1)} \\
\overline{y}_{l+1}(t') &= \overline{y}_1(t'+1)^{b_{s+1,1}(t'+1)} \cdots \overline{y}_s(t'+1)^{b_{s+1,s}(t'+1)} \\
&\qquad \cdot (\overline{y}_{l+1}(t'+1)+1) d_{t'+1}^{b_{s+1,s+1}(t'+1)}
\end{aligned}$$

such that $R(t'+1) \subset S(t'+1)$, and $\overline{x}_{l+1}(t'+1) = \overline{y}_{l+1}(t'+1)$. By adding an appropriate series Ω to $\overline{x}_{l+1}(t'+1)$, we will have regular parameters in $R(t'+1) \to S(t'+1)$ as desired. $\square$

CHAPTER 5

MONOMIALIZATION

Theorem 5.1. — *Suppose that $R \subset S$ are excellent regular local rings such that* $\dim(R) = \dim(S)$*, containing a field k of characteristic* 0 *such that the quotient field K of S is a finite extension of the quotient field J of R. Suppose that ν is a valuation of K with valuation ring V such that V dominates S. Suppose that ν has rank* 1 *and rational rank s. Suppose that if m_V is the maximal ideal of V, and $p^* = m_V \cap S$, then $(S/p^*)_{p^*}$ is a finitely generated field extension of k. Then there exist sequences of monodial transforms $R \to R'$ and $S \to S'$ along ν such that* $\dim(R') = \dim(S')$*, S' dominates R', ν dominates S' and there are regular parameters $(x_1', \ldots, x_n')$ in R', $(y_1', \ldots, y_n')$ in S', units $\delta_1, \ldots, \delta_s \in S'$ and a matrix (c_{ij}) of nonnegative integers such that* $\det(c_{ij}) \neq 0$ *and*

$$\begin{aligned}
x_1' &= (y_1')^{c_{11}} \cdots (y_s')^{c_{1s}} \delta_1 \\
&\ \vdots \\
x_s' &= (y_1')^{c_{s1}} \cdots (y_s')^{c_{ss}} \delta_s \\
x_{s+1}' &= y_{s+1}' \\
&\ \vdots \\
x_n' &= y_n'.
\end{aligned}$$

Proof. — By Theorem 2.7, applied to the lift to V of a transcendence basis of V/m_ν over R/m (which is finite by Theorem 1 [**2**] or Appendix 2 [**39**]), there exists a MTS along ν, $R \to R_1$, such that $\dim_{R_1}(\nu) = 0$. Let m_1 be the maximal ideal of R_1. $\mathrm{trdeg}_{R/m}(R_1/m_1) = \dim_R(\nu)$ and $\dim(R_1) = \dim(R) - \dim_R(\nu)$ by the dimension formula (Theorem 15.6 [**26**]). By Theorem 2.6, there exists a MTS $S \to S_1$ along ν such that S_1 dominates R_1. Let n_1 be the maximal ideal of S_1. S is essentially of finite type (a spot) over R by Theorem 2.11, since $\dim(R) = \dim(S)$. Hence S_1 is a spot over R_1. By the dimension formula,

$$\dim(R_1) = \dim(S_1) + \mathrm{trdeg}_{R_1/m_1}(S_1/n_1) = \dim(S_1),$$

since $\mathrm{trdeg}_{R_1/m_1}(S_1/n_1) \leq \dim_{R_1}(\nu) = 0$. We may thus assume that $\dim_R(\nu) = 0$.

Let $\{\bar{t}_i\}$ be a transcendence basis of R/m over k. Let t_i be lifts of $\bar{t}_i$ to R. Then the field L obtained by adjoining all of the t_i to k is contained in R, and ν is trivial on $L - \{0\}$. hence we can replace k by L. We may thus assume that assumptions (1) and (2) of Chapter 4 hold.

There exist $f_1, \ldots, f_s \in J$ such that $\nu(f_1), \ldots, \nu(f_s)$ are positive and rationally independent. By Theorem 2.7, there exists a MTS $R \rightarrow R'$ along ν, such that $f_1, \ldots, f_s \in R'$. By Theorem 2.5, there exists a MTS $R' \rightarrow R''$ along ν such that $f_1 \cdots f_s$ is a SNC divisor in R''. Then R'' has regular parameters $(x_1'', \ldots, x_n'')$ such that $\nu(x_1''), \ldots, \nu(x_s'')$ are rationally independent. By Theorem 2.6, there exists a MTS $S \rightarrow S'$ along ν, such that $R'' \subset S'$. We may thus assume that there exist regular parameters $(x_1, \ldots, x_n)$ in R such that $\nu(x_1), \ldots, \nu(x_s)$ are rationally independent.

By Theorem 2.5, after replacing S with a MTS along ν we may assume that $x_1 \cdots x_n$ has SNCs in S. Thus there are regular parameters $(y_1, \ldots, y_n)$ in S and units ψ_i such that

$$x_i = y_1^{c_{i1}} \cdots y_n^{c_{in}} \psi_i$$

for $1 \leq i \leq s$. Thus $\nu(y_1), \ldots, \nu(y_n)$ span a rational vector space of dimension s. After possibly reindexing the y_i, we may assume that $\nu(y_1), \ldots, \nu(y_s)$ are rationally independent. By (60) of Theorem 4.8 with $R = S$, $f = x_1 \ldots x_s$ and Theorem 4.10, we can replace S with a MTS along ν to get

$$x_i = y_1^{c_{i1}} \cdots y_s^{c_{is}} \psi_i$$

for $1 \leq i \leq s$, where ψ_i are units and $\det(c_{ij}) \neq 0$.

Let ϕ_i be the residue of ψ_i in S/n. For $1 \leq j \leq s$ set

$$\varepsilon_i = \prod_{i=1}^{s} \left(\frac{\psi_j}{\phi_j}\right)^{e_{ij}}$$

where $(e_{ij}) = (c_{ij})^{-1}$, a matrix with rational coefficients. $\varepsilon_j \in \widehat{S}$ for $1 \leq j \leq s$.

Set $T''(0) = R$, $\overline{x}_i = x_i$ for $1 \leq i \leq n$. Set $U''(0) = S[d_0, \varepsilon_1, \ldots, \varepsilon_s]_q$ where $d_0 \in \widehat{S}$ is such that $k(d_0) \cong k(S)$, $q = m(\widehat{S}) \cap S[d_0, \varepsilon_1, \ldots, \varepsilon_s]$. $U''(0)$ has regular parameters

$$\overline{y}_j = \begin{cases} \varepsilon_j y_j & 1 \leq j \leq s \\ y_j & s < j. \end{cases}$$

$$\overline{x}_i = \overline{y}_1^{c_{i1}} \cdots \overline{y}_s^{c_{is}} \phi_i$$

for $1 \leq i \leq s$. Set $\widetilde{R}(0) = k[x_1, \ldots, x_s]_q$, $q = m(R) \cap k[x_1, \ldots, x_s]$, $c_0 = 1$. Thus the assumptions of Theorem 4.11 are satisfied with $l = s$ and by the conclusions of Theorem 4.11 applied $n - s$ times consecutively, we can construct MTSs $R \rightarrow R'$, $S \rightarrow S'$ such that V dominates S', S' dominates R' and R' has regular parameters

$(x_1', \ldots, x_n')$, S' has regular parameters $(y_1', \ldots, y_n')$ satisfying the conclusions of the Theorem. □

Corollary 5.2. — *Suppose that $R \subset S$ are excellent regular local rings such that $\dim(R) = \dim(S)$, containing a field k of characteristic 0 and with a common quotient field K. Suppose that ν is a valuation of K with valuation ring V such that V dominates S. Suppose that ν has rank 1 and rational rank s. Suppose that if m_V is the maximal ideal of V, and $p^* = m_V \cap S$, then $(S/p^*)_{p^*}$ is a finitely generated field extension of k. Then there exist sequences of monodial transforms $R \to R'$ and $S \to S'$ along ν such that $\dim(R') = \dim(S')$, S' dominates R', ν dominates S' and there are regular parameters $(x_1', \ldots, x_n')$ in R', $(\widetilde{y}_1, \ldots, \widetilde{y}_n)$ in S' such that*

$$
\begin{aligned}
x_1' &= \widetilde{y}_1^{c_{11}} \cdots \widetilde{y}_s^{c_{1s}} \\
&\vdots \\
x_s' &= \widetilde{y}_1^{c_{s1}} \cdots \widetilde{y}_s^{c_{ss}} \\
x_{s+1}' &= \widetilde{y}_{s+1} \\
&\vdots \\
x_n' &= \widetilde{y}_n
\end{aligned}
$$

where $\det(c_{ij}) = \pm 1$ and $k(R') \cong k(S')$.

Proof. — We can construct MTSs along ν $R \to R'$, $S \to S'$ such that the conclusions of Theorem 5.1 hold. To finish the proof, we must show that $\det(c_{ij}) = \pm 1$ and $k(R') \cong k(S')$. We will analyze (c_{ij}) by constructing MTSs which may not be dominated by ν. Since interchanging the variables x_i' will only change the sign of $\det(c_{ij})$, we may assume that $c_{11} \neq 0$.

Case 1. — Suppose that $c_{11} < c_{21}$. Then we can perform a MTS $S' \to S(1)$ where $S(1)$ has regular parameters $(y_1(1), \ldots, y_n(1))$ such that

$$
y_i' = \begin{cases} y_1(1) y_2(1)^m \cdots y_s(1)^m & i = 1 \\ y_i(1) & i \neq 1 \end{cases}
$$

Then for $m \gg 0$ the monoidal transform $R' \to R(1)$ factors through $S(1)$, where $R(1)$ has regular parameters $(x_1(1), \ldots, x_n(1))$ defined by

$$
x_i' = \begin{cases} x_1(1) x_2(1) & i = 2 \\ x_i(1) & i \neq 2 \end{cases}
$$

Then

$$\begin{aligned} x_1(1) &= y_1(1)^{c_{11}} \cdots \\ x_2(1) &= y_1(1)^{c_{21}-c_{11}} \ldots \\ x_3(1) &= y_1(1)^{c_{31}} \cdots \\ &\vdots \end{aligned}$$

Case 2. — Suppose that $c_{21} < c_{11}$. As in Case 1, we can perform MTSs to get

$$\begin{aligned} x_1(1) &= y_1(1)^{c_{11}-c_{21}} \ldots \\ x_2(1) &= y_1(1)^{c_{21}} \cdots \\ x_3(1) &= y_1(1)^{c_{31}} \cdots \\ &\vdots \end{aligned}$$

Case 3. — Suppose that $c_{11} = c_{21}$ and $c_{1j} < c_{2j}$ for some j. Perform a MTS $S' \to S(1)$ where $S(1)$ has regular parameters $(y_1(1), \ldots, y_n(1))$ such that

$$y_i' = \begin{cases} y_j(1)y_2(1)^m \cdots y_{j-1}(1)^m y_{j+1}(1)^m \cdots y_s(1)^m & i = j \\ y_i(1) & i \neq j \end{cases}$$

Then for $m \gg 0$ the monoidal tranform $R' \to R(1)$ factors through $S(1)$, where $R(1)$ has regular parameters $(x_1(1), \ldots, x_n(1))$ defined by

$$x_i' = \begin{cases} x_1(1)x_2(1) & i = 2 \\ x_i(1) & i \neq 2 \end{cases}$$

Then

$$\begin{aligned} x_1(1) &= y_1(1)^{c_{11}} \cdots \\ x_2(1) &= y_2(1)^{c_{22}-c_{12}+m(c_{2j}-c_{1j})} \ldots \\ x_3(1) &= y_1(1)^{c_{31}} \cdots \\ &\vdots \end{aligned}$$

Case 4. — In the remaining case $c_{11} = c_{21}$ and $c_{1j} \geq c_{2j}$ for all j. Then the monoidal tranform $R' \to R(1)$ factors through S', where $R(1)$ has regular paramaters $(x_1(1), \ldots, x_n(1))$ defined by

$$x_i' = \begin{cases} x_1(1)x_2(1) & i = 1 \\ x_i(1) & i \neq 1 \end{cases}$$

Then

$$\begin{aligned} x_1(1) &= (y_2')^{c_{12}-c_{22}}\cdots \\ x_2(1) &= (y_1')^{c_{21}}(y_2')^{c_{22}}\cdots \\ x_3(1) &= (y_1')^{c_{31}}\cdots \\ &\vdots \end{aligned}$$

By continuing to apply these four cases, we can construct $R' \to R(\alpha)$ and $S' \to S(\alpha)$ such that $S(\alpha)$ dominates $R(\alpha)$,

$$\begin{aligned} x_1(\alpha) &= y_1(\alpha)^{c_{11}(\alpha)}\cdots y_s^{c_{1s}(\alpha)}\phi_1 \\ &\vdots \\ x_s(\alpha) &= y_1(\alpha)^{c_{s1}(\alpha)}\cdots y_s^{c_{ss}(\alpha)}\phi_s \\ x_{s+1}(\alpha) &= y_{s+1}(\alpha) \\ &\vdots \\ x_n(\alpha) &= y_n(\alpha) \end{aligned} \tag{127}$$

with ϕ_i units in $S(\alpha)$ and $c_{21}(\alpha) = 0$. By repeating the above procedure on successive rows we can construct a MTS (127) with

$$c_{21}(\alpha) = \cdots = c_{s1}(\alpha) = 0.$$

Then the algorithm can be applied to the matrix obtained by removing the first row and column from (c_{ij}) to construct (127) such that $(c_{ij}(\alpha))$ is a upper triangular matrix.

Set $q = (y_1(\alpha))$, $p = R(\alpha) \cap (y_1(\alpha))$. Our assumption that (c_{ij}) is upper triangular implies

$$qS(\alpha)\hat{} \cap R(\alpha)\hat{} = x_1(\alpha)R(\alpha)\hat{},$$

so that $p = (x_1(\alpha))$ and $\dim R(\alpha)_p = \dim S(\alpha)_q$. By the dimension formula, $A = (S(\alpha)/q)_q$ is finite over $(R(\alpha)/p)_p$. $S(\alpha)_q/pS(\alpha)_q \cong A[y_1(\alpha)]/(y_1(\alpha)^{c_{11}(\alpha)})$ is then finite over $(R(\alpha)/p)_p$, so that $R(\alpha)_p = S(\alpha)_q$ and $c_{11}(\alpha) = 1$ by Theorem 2.10.

Now perform the MTS $S(\alpha) \to S(\alpha+1)$ where $S(\alpha+1)$ has regular parameters $(y_1(\alpha+1), \ldots, y_n(\alpha+1))$ such that

$$y_i(\alpha) = \begin{cases} y_1(\alpha+1)y_2(\alpha+1)^{m_1}y_3(\alpha+1)^{m_2}\cdots y_s(\alpha+1)^{m_2} & i = 1 \\ y_i(\alpha+1) & i > 1 \end{cases}$$

where m_1 is chosen so that

$$m_1 + c_{12}(\alpha) = rc_{22}(\alpha)$$

for some $r > 0$ and m_2 is sufficiently large that

$$m_2 + c_{1j}(\alpha) > rc_{2j}(\alpha)$$

for $3 \leq j \leq s$. Then the MTS $R(\alpha) \to R(\alpha+1)$ factors through $S(\alpha+1)$, where

$$x_i(\alpha) = \begin{cases} x_1(\alpha+1)x_2(\alpha+1)^r & i=1 \\ x_i(\alpha+1) & i>1 \end{cases}$$

and

$$\begin{aligned} x_1(\alpha+1) &= y_1(1)y_3(1)^{c_{31}(\alpha+1)}\cdots \\ x_2(\alpha+1) &= y_2(1)^{c_{22}(\alpha+1)}\cdots \\ x_3(\alpha+1) &= y_3(1)^{c_{33}(\alpha+1)}\cdots \\ &\vdots \end{aligned}$$

Now perform a series of similar MTSs to get (127) with $(c_{ij}(\alpha))$ an upper triangular matrix with

$$c_{1j}(\alpha) = \begin{cases} 1 & j=1 \\ 0 & j>1. \end{cases}$$

Set $q = (y_1(\alpha), y_2(\alpha))$, $p = R(\alpha) \cap q = (x_1(\alpha), x_2(\alpha))$.

$$S(\alpha)_q/pS(\alpha)_q \cong (S(\alpha)/q)_q[y_2(\alpha)]/(y_2(\alpha)^{c_{22}(\alpha)})$$

is finite over $(R(\alpha)/p)_p$. By Theorem 2.10, $R(\alpha)_p = S(\alpha)_q$ and $c_{22}(\alpha) = 1$. We can repeat the above procedure to get (127) where $(c_{ij}(\alpha))$ is the identity matrix and $R(\alpha) = S(\alpha)$.

Thus $\det(c_{ij}) = \pm 1$. Furthermore,

$$k(R') \cong k(R(\alpha)) \cong k(S(\alpha)) \cong k(S').$$

Set $(e_{ij}) = (c_{ij})^{-1}$, a matrix with integral coefficients. Set

$$\widetilde{y}_i = \begin{cases} \delta_1^{e_{i1}} \cdots \delta_s^{e_{is}} y_i' & 1 \leq i \leq s \\ y_i' & s < i \end{cases}$$

then $(\widetilde{y}_1, \ldots, \widetilde{y}_n)$ are regular parameters in S' satisfying the conclusions of the Theorem. $\square$

Suppose that $R \subset S$ are excellent regular local rings such that $\dim(R) = \dim(S) = n$, containing a field k of characteristic 0, such that the quotient field K of S is a finite extension of the quotient field J of R. Suppose that ν is a valuation of K with valuation ring V such that V dominates S and ν has rank r. The primes of V are a chain

$$0 = p_0 \subset \cdots \subset p_r \subset V.$$

We will begin by reviewing basic facts on specialization and composition of valuations (cf. sections 8,9,10 of **[3]** and chapter VI, section 10 of **[39]**). Suppose that Γ_ν is the value group of ν. The isolated subgroups of Γ_ν are a chain

$$0 = \Gamma_r \subset \cdots \subset \Gamma_0 = \Gamma_\nu.$$

Set

$$U_i = \{\nu(a) \mid a \in p_i\}$$

Then the isolated subgroup Γ_i of p_i is defined to be the complement of U_i and $-U_i$ in Γ_ν.

For $i < j$ ν induces a valuation on the field $(V/p_i)_{p_i}$ with valuation ring $(V/p_i)_{p_j}$ and value group Γ_i/Γ_j. If $j = i+1$, Γ_i/Γ_j has rank 1.

For all i, V_{p_i} is a valuation ring of K dominating $R_{p_i\cap R}$. Thus

$$\operatorname{trdeg}_{(R/P_i\cap R)_{p_i\cap R}}(V/p_i)_{p_i} < \infty$$

by Theorem 1 **[2]** or Appendix 2 **[39]**. We can lift transcendence bases of $(V/p_i)_{p_i}$ over $(R/p_i\cap R)_{p_i\cap R}$ for $1 \le i \le r$ to $t_1, \ldots, t_a \in V$. After possibly replacing the t_i with $1/t_i$, we have $\nu(t_i) \ge 0$ for all t_i. By Theorem 2.7, there exists a MTS $R \to R'$ along ν such that $t_i \in R'$ for all i. Let $p_i' = R' \cap p_i$. Then

$$\operatorname{trdeg}_{(R'/p_i')_{p_i'}}(V/p_i)_{p_i} = 0$$

for $1 \le i \le r$. By Theorem 2.6, there exists a MTS $S \to S'$ along ν such that S' dominates R'. Replacing R by R' and S by S', we may thus assume that

$$\operatorname{trdeg}_{(R/p_i\cap R)_{p_i\cap R}}(V/p_i)_{p_i} = 0$$

for $1 \le i \le r$. Then

$$\operatorname{trdeg}_{(R/p_i\cap R)_{p_i\cap R}}(S/p_i\cap S)_{p_i\cap S} = 0$$

for all i. By the dimension formula (cf. Theorem 15.6 **[26]**)

$$\dim R/p_i\cap R = \dim S/p_i\cap S$$

for $0 \le i \le r$.

Theorem 5.3. — *Suppose that $R \subset S$ are excellent regular local rings such that $\dim(R) = \dim(S) = n$, containing a field k of characteristic 0 such that the quotient field K of S is a finite extension of the quotient field J of R. Suppose that ν is a valuation of K with valuation ring V such that V dominates S and ν has rank r. Suppose that if m_V is the maximal ideal of V, and $p^* = m_V \cap S$, then $(S/p^*)_{p^*}$ is a finitely generated field extension of k. Suppose that the segments of Γ_ν are*

$$0 = \Gamma_r \subset \cdots \subset \Gamma_0 = \Gamma_\nu$$

with associated primes

$$0 = p_0 \subset \cdots \subset p_r \subset V.$$

Suppose that Γ_{i-1}/Γ_i has rational rank s_i for $1 \le i \le r$ and

$$\operatorname{trdeg}_{(R/p_i\cap R)_{p_i\cap R}}(V/p_i)_{p_i} = 0$$

for $1 \le i \le r$. *Set* $t_i = \dim(R/p_{i-1} \cap R)_{p_i \cap R}$ *for* $1 \le i \le r$, *so that* $n = t_1 + \cdots + t_r$. *Then there exist MTSs* $R \to R'$ *and* $S \to S'$ *along* ν *such that* S' *dominates* R', R' *has regular parameters* $(z_1, \ldots, z_n)$, S' *has regular parameters* $(w_1, \ldots, w_n)$ *such that*

$$p_i \cap R' = (z_1, \ldots, z_{t_1+\cdots+t_i})$$
$$p_i \cap S' = (w_1, \ldots, w_{t_1+\cdots+t_i})$$

for $1 \le i \le r$ *and*

$$\begin{aligned}
z_1 &= w_1^{g_{11}(1)} \cdots w_{s_1}^{g_{1s_1}(1)} w_{t_1+1}^{h_{1,t_1+1}(1)} \cdots w_n^{h_{1n}(1)} \delta_{11} \\
&\vdots \\
z_{s_1} &= w_1^{g_{s_11}(1)} \cdots w_{s_1}^{g_{s_1s_1}(1)} w_{t_1+1}^{h_{s_1,t_1+1}(1)} \cdots w_n^{h_{s_1n}(1)} \delta_{1s_1} \\
z_{s_1+1} &= w_{s_1+1} w_{t_1+1}^{h_{s_1+1,t_1+1}(1)} \cdots w_n^{h_{s_1+1,n}(1)} \delta_{1s_1+1} \\
&\vdots \\
z_{t_1} &= w_{t_1} w_{t_1+1}^{h_{t_1,t_1+1}(1)} \cdots w_n^{h_{t_1n}(1)} \delta_{1t_1} \\
z_{t_1+1} &= w_{t_1+1}^{g_{11}(2)} \cdots w_{t_1+s_2}^{g_{1s_2}(2)} w_{t_1+t_2+1}^{h_{1,t_1+t_2+1}(2)} \cdots w_n^{h_{1n}(2)} \delta_{21} \\
&\vdots \\
z_{t_1+s_2} &= w_{t_1+1}^{g_{s_21}(2)} \cdots w_{t_1+s_2}^{g_{s_2s_2}(2)} w_{t_1+t_2+1}^{h_{s_2,t_1+t_2+1}(2)} \cdots w_n^{h_{s_2n}(2)} \delta_{2s_2} \\
z_{t_1+s_2+1} &= w_{t_1+s_2+1} w_{t_1+t_2+1}^{h_{s_2+1,t_1+t_2+1}(2)} \cdots w_n^{h_{s_2+1,n}(2)} \delta_{2s_2+1} \\
&\vdots \\
z_{t_1+t_2} &= w_{t_1+t_2} w_{t_1+t_2+1}^{h_{t_2,t_1+t_2+1}(2)} \cdots w_n^{h_{t_2n}(2)} \delta_{2t_2} \\
&\vdots \\
z_{t_1+\cdots+t_{r-1}+1} &= w_{t_1+\cdots+t_{r-1}+1}^{g_{11}(r)} \cdots w_{t_1+\cdots+t_{r-1}+s_r}^{g_{1s_r}(r)} \delta_{r1} \\
&\vdots \\
z_{t_1+\cdots+t_{r-1}+s_r} &= w_{t_1+\cdots+t_{r-1}+1}^{g_{s_r1}(r)} \cdots w_{t_1+\cdots+t_{r-1}+s_r}^{g_{s_rs_r}(r)} \delta_{rs_r} \\
z_{t_1+\cdots+t_{r-1}+s_r+1} &= w_{t_1+\cdots+t_{r-1}+s_r+1} \delta_{rs_r+1} \\
&\vdots \\
z_{t_1+\cdots+t_r} &= w_{t_1+\cdots+t_r} \delta_{rt_r}
\end{aligned}$$

where

$$\det \begin{pmatrix} g_{11}(i) & \cdots & g_{1s_i}(i) \\ \vdots & & \vdots \\ g_{s_i1}(i) & \cdots & g_{s_is_i}(i) \end{pmatrix} \ne 0,$$

δ_{ij} *are units in* S' *for* $1 \le i \le r$, *and* $h_{jk}(i)$ *are nonnegative integers.*

Proof. — The proof is by induction on r. $r = 1$ is immediate from Theorem 5.1. Suppose that the Theorem is true for rank $r-1$. Set $p_i(0) = p_i \cap R$, $q_i(0) = p_i \cap S$. Then there exist MTSs $R_{p_{r-1}(0)} \to T_1$ and $S_{q_{r-1}(0)} \to U_1$ such that $V_{p_{r-1}}$ dominates U_1, U_1 dominates T_1 and the conclusions of the Theorem hold for $T_1 \subset U_1$. By Theorems 2.9 and 2.6 there exist MTSs along ν $R \to R(1)$ and $S \to S(1)$ such that V dominates $S(1)$, $S(1)$ dominates $R(1)$ and if $p_i(1) = p_i \cap R(1)$, $q_i(1) = p_i \cap S(1)$, $R(1)_{p_{r-1}(1)} \cong T_1$, $S(1)_{q_{r-1}(1)} \cong U_1$ and $R(1)/p_i(1)$, $S(1)/q_i(1)$ are regular local rings for $1 \leq i \leq r$.

By assumption, $R(1)_{p_{r-1}(1)}$ has regular parameters $(\widetilde{x}_1, \ldots, \widetilde{x}_{t_1+\cdots+t_{r-1}})$ and $S(1)_{q_{r-1}(1)}$ has regular parameters $(\widetilde{y}_1, \ldots, \widetilde{y}_{t_1+\cdots+t_{r-1}})$ satisfying the conclusions of the Theorem. Set $\lambda = t_1 + \cdots + t_{r-1}$. $R(1)$ has regular parameters $(x_1(1), \ldots, x_n(1))$ such that $p_{r-1}(1) = (x_1(1), \ldots, x_\lambda(1))$. Let $\pi(1) : R(1) \to R(1)/p_{r-1}(1)$. There exist $\overline{a}^i_j \in R(1)_{p_{r-1}(1)}$, $1 \leq i \leq \lambda$, $1 \leq j \leq \lambda$ such that

$$\widetilde{x}_i = \overline{a}^i_1 x_1(1) + \cdots + \overline{a}^i_\lambda x_\lambda(1)$$

and $\det(\overline{a}^i_j) \notin p_{r-1}(1)_{p_{r-1}(1)}$. There exists $u(1) \in R(1) - p_{r-1}(1)$ such that $u(1)\widetilde{x}_i \in R(1)$ for $1 \leq i \leq \lambda$ and if we define $\widetilde{x}_i(1) = u(1)\widetilde{x}_i$

$$\widetilde{x}_i(1) = a^i_1 x_1(1) + \cdots + a^i_\lambda x_\lambda(1)$$

for $1 \leq i \leq \lambda$ where $a^i_j \in R(1)$ for all i, j and $\det(a^i_j) \notin p_{r-1}(1)$. After reindexing the $\widetilde{x}_i(1)$, we may assume that $a^1_1 \notin p_{r-1}(1)$. Let $b^1_1 = \pi(1)(a^1_1)$.

V/p_{r-1} is a rank 1, rational rank s_r valuation ring. The quotient field of V/p_{r-1} is algebraic over the quotient field of $R(1)/p_{r-1}(1)$ so that if L is the quotient field of $R(1)/p_{r-1}(1)$, then $L \cap V/p_{r-1}$ is a rank 1, rational rank s_r valuation ring. Let $\overline{\nu}$ denote the valuation induced by ν on L.

By Theorems 5.1, 4.8 and 4.10 (with $R = S = R(1)/p_{r-1}(1)$) there exists a MTS

$$\widetilde{R}(1) = R(1)/p_{r-1}(1) \to \widetilde{R}(2) \to \cdots \to \widetilde{R}(m)$$

where each $\widetilde{R}(i) \to \widetilde{R}(i+1)$ is a monoidal transform and $\widetilde{R}(m)$ has regular parameters $(\overline{y}_{\lambda+1}(m), \ldots, \overline{y}_n(m))$ such that $\overline{\nu}(\overline{y}_{\lambda+1}(m)), \ldots, \overline{\nu}(\overline{y}_{\lambda+s_r}(m))$ are rationally independent and $b^1_1 = \overline{y}_{\lambda+1}(m)^{a_{\lambda+1}} \cdots \overline{y}_{\lambda+s_r}(m)^{a_{\lambda+s_r}} \overline{u}$ where $\overline{u} \in \widetilde{R}(m)$ is a unit. There exist regular parameters $(\overline{y}_{\lambda+1}(1), \ldots, \overline{y}_n(1))$ in $\widetilde{R}(1)$ and $a \leq n$ such that

$$\widetilde{R}(2) = \widetilde{R}(1)\left[\frac{\overline{y}_{\lambda+2}(1)}{\overline{y}_{\lambda+1}(1)}, \ldots, \frac{\overline{y}_a(1)}{\overline{y}_{\lambda+1}(1)}\right]_{\overline{Q}(2)}$$

where $\overline{Q}(2)$ is a maximal ideal. Let $y_i(1)$ be lifts of $\overline{y}_i(1)$ to $R(1)$ for $\lambda + 1 \leq i \leq n$. Then $(x_1(1), \ldots, x_\lambda(1), y_{\lambda+1}(1), \ldots, y_n(1))$ are regular parameters in $R(1)$. We have a surjection

$$\Phi_1 : R(1)\left[\frac{y_{\lambda+2}(1)}{y_{\lambda+1}(1)}, \ldots, \frac{y_a(1)}{y_{\lambda+1}(1)}\right] \to \widetilde{R}(1)\left[\frac{\overline{y}_{\lambda+2}(1)}{\overline{y}_{\lambda+1}(1)}, \ldots, \frac{\overline{y}_a(1)}{\overline{y}_{\lambda+1}(1)}\right].$$

Let $Q_2 = \Phi_1^{-1}(\overline{Q}(2))$. Set
$$R_2 = R(1)\left[\frac{y_{\lambda+2}(1)}{y_{\lambda+1}(1)}, \ldots, \frac{y_a(1)}{y_{\lambda+1}(1)}\right]_{Q_2}.$$
$R(1) \rightarrow R_2$ is a monoidal transform along ν and $p_{r-1} \cap R_2 = (x_1(1), \ldots, x_\lambda(1))$,
$$(R_2)_{p_{r-1}\cap R_2} \cong R(1)_{p_{r-1}(1)} \text{ and } R_2/p_{r-1} \cap R_2 \cong \widetilde{R}(2).$$
We can inductively construct a MTS along ν

$$R(1) \rightarrow R_2 \rightarrow \cdots \rightarrow R_m = R(2) \tag{128}$$

such that $R(2)_{p_{r-1}(2)} \cong R(1)_{p_{r-1}(1)}$, $R(2)/p_{r-1}(2) \cong \widetilde{R}(m)$ with $p_{r-1}(2) = p_{r-1} \cap R(2)$ and $R(2)$ has regular parameters $(x_1(2), \ldots, x_n(2))$ such that
$$x_i(2) = \begin{cases} x_i(1) & 1 \le i \le \lambda \\ y_i(2) & \lambda+1 \le \lambda \le n \end{cases}$$
where $y_i(2)$ are lifts of $\overline{y}_i(m)$ to $R(2)$. Thus
$$a_1^1 = x_{\lambda+1}(2)^{a_{\lambda+1}} \cdots x_{\lambda+s_r}(2)^{a_{\lambda+s_r}} u + b_1 x_1(2) + \cdots + b_\lambda x_\lambda(2)$$
where $u, b_1, \ldots, b_\lambda \in R(2)$ and u is a unit. Thus
$$\widetilde{x}_1(1) = x_{\lambda+1}(2)^{a_{\lambda+1}} \cdots x_{\lambda+s_r}(2)^{a_{\lambda+s_r}} u x_1(2) + a x_1(2)^2 + \sum_{i=2}^{\lambda} a_i x_i(2)$$
with $a_i, a \in R(2)$. Now perform a MTS $R(2) \rightarrow R(3)$ along ν
$$x_i(2) = \begin{cases} x_{\lambda+1}(3)^{a_{\lambda+1}+1} \cdots x_{\lambda+s_r}(3)^{a_{\lambda+s_r}+1} x_1(3) & i = 1 \\ x_{\lambda+1}(3)^{2a_{\lambda+1}+2} \cdots x_{\lambda+s_r}(3)^{2a_{\lambda+s_r}+2} x_i(3) & 2 \le i \le \lambda \\ x_i(3) & \lambda+1 \le i \le n \end{cases}$$
Thus $x_i(3) \in R(2)_{p_{r-1}(2)}$ for $1 \le i \le n$. Set $p_{r-1}(3) = p_{r-1} \cap R(3)$. Then $R(2)_{p_{r-1}(2)} = R(3)_{p_{r-1}(3)}$.
$$\widetilde{x}_1(1) = x_{\lambda+1}(3)^{2a_{\lambda+1}+1} \cdots x_{\lambda+s_r}(3)^{2a_{\lambda+s_r}+1}(x_1(3)u + x_{\lambda+1}(3) \cdots x_{\lambda+s_r}(3)c)$$
for some $c \in p_{r-1}(3)$. Set
$$\widehat{x}_i(3) = \begin{cases} x_1(3)u + x_{\lambda+1}(3) \cdots x_{\lambda+s_r}(3)c & i = 1 \\ x_i(3) & 2 \le i \le n \end{cases}$$
Then $(\widehat{x}_1(3), \ldots, \widehat{x}_n(3))$ are regular parameters in $R(3)$ with
$$p_{r-1}(3) = (\widehat{x}_1(3), \ldots, \widehat{x}_\lambda(3)).$$
$p_{r-1}(3)R(3)_{p_{r-1}(3)} = (\widetilde{x}_1, \ldots, \widetilde{x}_\lambda)R(3)_{p_{r-1}(3)}$ implies there exists $\overline{a}_j^i(3) \in R(3)_{p_{r-1}(3)}$ such that
$$\widetilde{x}_i = \begin{cases} \overline{a}_1^1(3)\widehat{x}_1(3) & i = 1 \\ \overline{a}_1^i(3)\widehat{x}_1(3) + \cdots + \overline{a}_\lambda^i(3)\widehat{x}_\lambda(3) & 2 \le i \le \lambda \end{cases}$$

and

$$\det\begin{pmatrix}\overline{a}_2^2(3) & \cdots & \overline{a}_\lambda^2(3)\\ \vdots & & \vdots\\ \overline{a}_2^\lambda(3) & \cdots & \overline{a}_\lambda^\lambda(3)\end{pmatrix}\not\in p_{r-1}(3)_{p_{r-1}(3)}.$$

We can repeat the above argument to construct a MTS $R(1)\rightarrow R''$ along ν such that if $p''_{r-1}=p_{r-1}\cap R''$, $R(1)_{p_{r-1}(1)}=R''_{p''_{r-1}}$ and there exists a regular system of parameters $(x''_1,\ldots,x''_n)$ in R'' and $u_1,\ldots,u_\lambda\in R''_{p''_{r-1}}-(p''_{r-1})_{p''_{r-1}}$ such that $\widetilde{x}_i=u_ix''_i$ for $1\le i\le\lambda$. By Theorem 2.6 and the above argument, there exists a MTS $S(1)\rightarrow S''$ along ν such that if $q''_{r-1}=p_{r-1}\cap S''$, $S(1)_{q_{r-1}(1)}=S''_{q''_{r-1}}$, S'' dominates R'' and there are regular parameters $(y''_1,\ldots,y''_n)$ in S'' and $v_1,\ldots,v_\lambda\in S''_{q''_{r-1}}-(q''_{r-1})_{q''_{r-1}}$ such that $\widetilde{y}_i=v_iy''_i$ for $1\le i\le\lambda$. Thus we have

$$\begin{aligned}x''_1&=\psi_1(y''_1)^{g_{11}(1)}\cdots(y''_{s_1})^{g_{1s_1}(1)}(y''_{t_1+1})^{h_{1t_1+1}(1)}\cdots(y''_\lambda)^{h_{1\lambda}(1)}\\ &\ \ \vdots\\ x''_\lambda&=\psi_\lambda y''_\lambda\end{aligned}\tag{129}$$

with $\psi_1,\ldots,\psi_\lambda\in S''_{q''_{r-1}}-(q''_{r-1})_{q''_{r-1}}$. $\psi_i=f_i/g_i$ with $f_i,g_i\in S''-q''_{r-1}$, f_i,g_i relatively prime in S''. There are nonnegative integers d_1^i such that $g_ix''_i=f_i(y''_1)^{d_1^i}\cdots(y''_\lambda)^{d_s^i}$ for $1\le i\le\lambda$ so that $g_i\mid f_i$ in S'' and $\psi_i\in S''-q''_{r-1}$ for $1\le i\le\lambda$.

Let $\pi':R''\rightarrow R''/p''_{r-1}$ and $\pi'':S''\rightarrow S''/q''_{r-1}$. Let $\overline{x}''_i=\pi'(x''_i)$ and $\overline{y}''_i=\pi''(y''_i)$. ν induces a rank 1 rational rank s_r valuation on $\overline{K}=(S''/q''_{r-1})_{q''_{r-1}}$. By Theorem 5.1, there exist MTSs

$$R''/p''_{r-1}=\widetilde{R}\rightarrow\widetilde{R}(1)\rightarrow\cdots\rightarrow\widetilde{R}(m)=T$$

and

$$S''/q''_{r-1}=\widetilde{S}\rightarrow\widetilde{S}(1)\rightarrow\cdots\rightarrow\widetilde{S}(m)=U$$

such that the valuation ring $(V/p_{r-1})\cap\overline{K}$ dominates U, U dominates T, T has regular parameters $(\overline{x}_{\lambda+1},\ldots,\overline{x}_n)$, U has regular parameters $(\overline{y}_{\lambda+1},\ldots,\overline{y}_n)$ such that $\overline{\nu}(\overline{y}_{\lambda+1}),\ldots,\overline{\nu}(\overline{y}_{\lambda+s_r})$ are rationally independent, where $\overline{\nu}$ is the valuation induced by ν on the quotient field of U and

$$\begin{aligned}\overline{x}_{\lambda+1}&=\overline{y}_{\lambda+1}^{g_{11}(r)}\cdots\overline{y}_{\lambda+s_r}^{g_{rs_r}(r)}\overline{\delta}_{\lambda+1}\\ &\ \ \vdots\\ \overline{x}_{\lambda+s_r}&=\overline{y}_{\lambda+1}^{g_{s_r1}(r)}\cdots\overline{y}_{\lambda+s_r}^{g_{s_rs_r}(r)}\overline{\delta}_{\lambda+s_r}\\ \overline{x}_{\lambda+s_r+1}&=\overline{y}_{\lambda+s_r+1}\overline{\delta}_{\lambda+s_r+1}\\ &\ \ \vdots\\ \overline{x}_n&=\overline{y}_n\overline{\delta}_n\end{aligned}\tag{130}$$

where $\overline{\delta}_i$ are units in U.

Each $\widetilde{R}(i) \to \widetilde{R}(i+1)$ is a monoidal transform centered at a prime $\overline{a}_i$. By Theorems 4.8 and 4.10 and Lemma 4.2, there exist MTSs along ν

$$T = \widetilde{R}(m) \to \cdots \to \widetilde{R}(m') = T'$$

and

$$U = \widetilde{S}(m) \to \cdots \to \widetilde{S}(m') = U'$$

such that U' dominates T', T' has regular parameters $(\overline{x}_{\lambda+1}, \ldots, \overline{x}_n)$, U' has regular parameters $(\overline{y}_{\lambda+1}, \ldots, \overline{y}_n)$ such that (130) holds, and

$$\overline{a}_i U' = (\overline{y}_{\lambda+1}^{d_1^i} \cdots \overline{y}_{\lambda+s_r}^{d_{s_r}^i})$$

for some nonnegative integers $d_1^i, \ldots, d_{s_r}^i$ for $1 \le i \le m$, and

$$\pi''(\psi_i) = \overline{y}_{\lambda+1}^{a_{i,\lambda+1}} \cdots \overline{y}_{\lambda+s_r}^{a_i \lambda+s_r} \overline{u}_i$$

where $\overline{u}_i$ are units, a_{ij} are positive integers for $1 \le i \le \lambda$.

For $m \le i \le m' - 1$ each $\widetilde{R}(i) \to \widetilde{R}(i+1)$ is a monoidal transform centered at a height 2 prime $\overline{a}_i$ (cf. Remark 4.1) such that $\overline{a}_i U' = (\overline{y}_{\lambda+1}^{d_1^i} \cdots \overline{y}_{\lambda+s_r}^{d_{s_r}^i})$ for some nonnegative integers $d_1^i, \ldots, d_{s_r}^i$. Consider the MTSs along ν

$$R'' \to R(1) \to \cdots \to R(m')$$

and

$$S'' \to S(1) \to \cdots \to S(m') = \widetilde{S}$$

constructed as in (128), so that for $1 \le i \le m'$, $R(i)_{p_{r-1}(i)} \cong R''_{p''_{r-1}}$, $R(i)/p_{r-1}(i) \cong \widetilde{R}(i)$, $S(i)_{q_{r-1}(i)} \cong S''_{q''_{r-1}}$, $S(i)/q_{r-1}(i) \cong \widetilde{S}(i)$ where $p_{r-1}(i) = p_{r-1} \cap R(i)$, $q_{r-1} = p_{r-1} \cap S(i)$ and $\widetilde{S}$ has regular parameters $(\widetilde{y}_1, \ldots, \widetilde{y}_n)$ such that $\widetilde{y}_i$ has residue $\overline{y}_i$ in U' for $\lambda+1 \le i \le n$ and $\widetilde{y}_i = y''_i$ for $1 \le i \le \lambda$. For $0 \le i \le m'-1$, $R(i) \to R(i+1)$ is the blowup of $a_i \subset R(i)$ such that $a_i \widetilde{R}(i) = \overline{a}_i$. Thus $a_i \widetilde{S}/\overline{q}_{r-1} = (\overline{y}_{\lambda+1}^{d_1^i} \cdots \overline{y}_{\lambda+s_r}^{d_{s_r}^i})$ where $\overline{q}_{r-1} = p_{r-1} \cap \widetilde{S}$. Set $\Phi_i = \widetilde{y}_{\lambda+1}^{d_1^i} \cdots \widetilde{y}_{\lambda+s_r}^{d_{s_r}^i}$. Then

$$a_i \widetilde{S} = (\Phi_i + \widetilde{y}_1 b_1^1(i) + \cdots + \widetilde{y}_\lambda b_\lambda^1(i), \ldots, \Phi_i + \widetilde{y}_1 b_1^t(i) + \cdots + \widetilde{y}_\lambda b_\lambda^t(i))$$

for some t, $b_j^k(i) \in \widetilde{S}$, $1 \le i \le m'-1$.

Perform a MTS along ν

$$\widetilde{S} = \overline{S}(0) \to \overline{S}(1) \to \cdots \overline{S}(m')$$

where $\overline{S}(j)$ has regular parameters $(\overline{y}_1(j), \ldots, \overline{y}_n(j))$ defined by $\overline{y}_i(0) = \widetilde{y}_i$ for $1 \le i \le n$,

$$\overline{y}_i(j) = \begin{cases} \Phi_{j+1} \overline{y}_i(j+1) & 1 \le i \le \lambda \\ \overline{y}_i(j+1) & i < \lambda \le n \end{cases}$$

for $0 \le j \le m'-1$. Then we have $a_i \overline{S}(m') = (\Phi_i)$ for $1 \le i \le m'-1$. $R(m') \subset \overline{S}(m')$ (by Theorem 2.6) and $\overline{S}(m')/\overline{q}(m')_{r-1} \cong U'$.

Let z_i be lifts of $\overline{x}_i$ to $R(m')$ for $\lambda+1 \le i \le n$. Define regular parameters $(x_1(m'),\ldots,x_n(m'))$ in $R(m')$ by

$$x_i(m') = \begin{cases} x_i'' & 1 \le i \le \lambda \\ z_i & \lambda+1 \le i \le n \end{cases}$$

There exists a matrix of nonnegative integers (e_{ij}) such that

$$\begin{aligned}
x_1(m') &= \overline{y}_1(m')^{g_{11}(1)}\cdots\overline{y}_{s1}(m')^{g_{1s_1}(1)}\overline{y}_{t_1+1}(m')^{h_{1,t_1+1}(1)}\cdots\overline{y}_\lambda(m')^{h_{1\lambda}(1)}\\
&\quad\cdot\overline{y}_{\lambda+1}(m')^{e_{1,\lambda+1}}\cdots\overline{y}_n(m')^{e_{1n}}\psi_1\\
&\ \ \vdots\\
x_\lambda(m') &= \overline{y}_\lambda(m')\overline{y}_{\lambda+1}(m')^{e_{\lambda,\lambda+1}}\cdots\overline{y}_n(m')^{e_{\lambda,n}}\psi_\lambda\\
x_{\lambda+1}(m') &= \overline{y}_{\lambda+1}(m')^{g_{11}(r)}\cdots\overline{y}_{\lambda+s_r}(m')^{g_{s_rs_r}(r)}\delta_{\lambda+1}\\
&\quad+f_1^{\lambda+1}\overline{y}_1(m')+\cdots+f_\lambda^{\lambda+1}\overline{y}_\lambda(m')\\
&\ \ \vdots\\
x_n(m') &= \overline{y}_n(m')\delta_n+f_1^n\overline{y}_1(m')+\cdots+f_\lambda^n\overline{y}_\lambda(m')
\end{aligned}\tag{131}$$

where δ_i are lifts of $\overline{\delta}_i$ to $\overline{S}(m')$, $f_i^j \in \overline{S}(m')$. For $1 \le i \le \lambda$,

$$\psi_i = u_i'\overline{y}_{\lambda+1}(m')^{a_{i,\lambda+1}}\cdots\overline{y}_{\lambda+s_r}^{a_i\lambda+s_r}+h_1^i\overline{y}_1(m')+\cdots+h_\lambda^i\overline{y}_\lambda(m')$$

where u_i' are lifts of $\overline{u}_i$ to $\overline{S}(m')$, the u_i' and δ_i are units in $\overline{S}(m')$. Choose

$$t > \max\{a_{ij}, g_{ij}(1)\}.$$

Now perform a MTS $\overline{S}(m') \to \overline{S}(m'+1)$ along ν where $\overline{S}(m'+1)$ has regular parameters $(\overline{y}_1(m'+1),\ldots,\overline{y}_n(m'+1))$ defined by

$$\overline{y}_i(m') = \begin{cases} \overline{y}_{\lambda+1}(m'+1)^t\cdots\overline{y}_n(m'+1)^t\overline{y}_i(m'+1) & 1 \le i \le \lambda \\ \overline{y}_i(m+1) & \lambda+1 \le i \le n \end{cases}$$

to get

$$\psi_i = u_i\overline{y}_{\lambda+1}(m'+1)^{a_{i\lambda+1}}\cdots\overline{y}_{\lambda+s_r}(m'+1)^{a_{\lambda+s_r}}$$

for some units $u_i \in \overline{S}(m'+1)$, $1 \le i \le \lambda$. $\overline{S}(m'+1)/\overline{q}(m'+1)_{r-1} \cong U'$ and there is a matrix of nonnegative integers (b_{ij}), units $u_{\lambda+1},\ldots,u_n \in \overline{S}(m+1)$ such that

$$\begin{aligned}
x_1(m') &= \overline{y}_1(m'+1)^{g_{11}(1)}\cdots\overline{y}_{s_1}(m'+1)^{g_{1s_1}(1)}\overline{y}_{t'+1}(m'+1)^{h_{1,t_1+1}(1)}\cdots\\
&\quad\cdot\overline{y}_\lambda(m'+1)^{h_{1\lambda}(1)}\overline{y}_{\lambda+1}(m'+1)^{b_{1,\lambda+1}}\cdots\overline{y}_n(m'+1)^{b_{1n}}u_1\\
&\ \ \vdots\\
x_\lambda(m') &= \overline{y}_\lambda(m'+1)\overline{y}_{\lambda+1}(m'+1)^{b_{\lambda,\lambda+1}}\cdots\overline{y}_n(m'+1)^{b_{\lambda n}}u_\lambda\\
x_{\lambda+1}(m') &= \overline{y}_{\lambda+1}(m'+1)^{g_{11}(r)}\cdots\overline{y}_{\lambda+s_r}(m'+1)^{g_{1s_r}(r)}u_{\lambda+1}\\
&\ \ \vdots\\
x_n(m') &= \overline{y}_n(m')u_n
\end{aligned}\tag{132}$$

□

Theorem 5.4 is immediate from Theorem 5.3.

***Theorem 5.4* (Monomialization).** — *Suppose that $R \subset S$ are excellent regular local rings such that* $\dim(R) = \dim(S)$, *containing a field k of characteristic zero, such that the quotient field K of S is a finite extension of the quotient field J of R.*

Let V be a valuation ring of K which dominates S. Suppose that if m_V is the maximal ideal of V, and $p^* = m_V \cap S$, then $(S/p^*)_{p^*}$ is a finitely generated field extension of k. Then there exist sequences of monoidal transforms $R \to R'$ and $S \to S'$ such that V dominates S', S' dominates R' and there are regular parameters $(x_1, \ldots, x_n)$ in R', $(y_1, \ldots, y_n)$ in S', units $\delta_1, \ldots, \delta_n \in S'$ and a matrix (a_{ij}) of nonnegative integers such that $\det(a_{ij}) \neq 0$ *and*

$$\begin{aligned} x_1 &= y_1^{a_{11}} \cdots y_n^{a_{1n}} \delta_1 \\ &\vdots \\ x_n &= y_1^{a_{n1}} \cdots y_n^{a_{nn}} \delta_n. \end{aligned}$$

Theorem 1.1 now follows from Theorem 5.4, since we can perform monoidal transforms along V to reduce to $\dim(R) = \dim(S)$ as in the proof of Theorem 1.10 (Chapter 7).

Theorem 5.5. — *Suppose that $R \subset S$ are excellent regular local rings such that* $\dim(R) = \dim(S) = n$, *containing a field k of characteristic* 0 *and with a common quotient field K. Suppose that ν is a valuation of K with valuation ring V such that V dominates S and ν has rank r. Suppose that if m_V is the maximal ideal of V, and $p^* = m_V \cap S$, then $(S/p^*)_{p^*}$ is a finitely generated field extension of k. Suppose that the segments of Γ_ν are*

$$0 = \Gamma_r \subset \cdots \subset \Gamma_0 = \Gamma_\nu$$

with associated primes

$$0 = p_0 \subset \cdots \subset p_r \subset V.$$

Suppose that Γ_{i-1}/Γ_i has rational rank s_i for $1 \leq i \leq r$ and

$$\mathrm{trdeg}_{(R/p_i \cap R)_{p_i \cap R}} (V/p_i)_{p_i} = 0$$

for $1 \leq i \leq r$. Set $t_i = \dim(R/p_{i-1} \cap R)_{p_i \cap R}$ for $1 \leq i \leq r$, so that $n = t_1 + \cdots + t_r$. Then there exist MTSs $R \to R'$ and $S \to S'$ along ν such that S' dominates R', R' has regular parameters $(z_1, \ldots, z_n)$, S' has regular parameters $(w_1, \ldots, w_n)$ such that

$$p_i \cap R' = (z_1, \ldots, z_{t_1 + \cdots + t_i})$$

$$p_i \cap S' = (w_1, \ldots, w_{t_1 + \cdots + t_i})$$

for $1 \le i \le r$ *and*

$$
\begin{aligned}
z_1 &= w_1^{g_{11}(1)} \cdots w_{s_1}^{g_{1s_1}(1)}\\
&\vdots\\
z_{s_1} &= w_1^{g_{s_11}(1)} \cdots w_{s_1}^{g_{s_1s_1}(1)}\\
z_{s_1+1} &= w_{s_1+1}\\
&\vdots\\
z_{t_1} &= w_{t_1}\\
z_{t_1+1} &= w_{t_1+1}^{g_{11}(2)} \cdots w_{t_1+s_2}^{g_{1s_2}(2)}\\
&\vdots\\
z_{t_1+s_2} &= w_{t_1+1}^{g_{s_21}(2)} \cdots w_{t_1+s_2}^{g_{s_2s_2}(2)}\\
z_{t_1+s_2+1} &= w_{t_1+s_2+1}\\
&\vdots\\
z_{t_1+t_2} &= w_{t_1+t_2}\\
&\vdots\\
z_{t_1+\cdots+t_{r-1}+1} &= w_{t_1+\cdots+t_{r-1}+1}^{g_{11}(r)} \cdots w_{t_1+\cdots+t_{r-1}+s_r}^{g_{1s_r}(r)}\\
&\vdots\\
z_{t_1+\cdots+t_{r-1}+s_r} &= w_{t_1+\cdots+t_{r-1}+1}^{g_{s_r1}(r)} \cdots w_{t_1+\cdots+t_{r-1}+s_r}^{g_{s_rs_r}(r)}\\
z_{t_1+\cdots+t_{r-1}+s_r+1} &= w_{t_1+\cdots+t_{r-1}+s_r+1}\\
&\vdots\\
z_{t_1+\cdots+t_r} &= w_{t_1+\cdots+t_r}
\end{aligned}
$$

where

$$
\det\begin{pmatrix} g_{11}(i) & \cdots & g_{1s_i}(i)\\ \vdots & & \vdots\\ g_{s_i1}(i) & \cdots & g_{s_is_i}(i)\end{pmatrix} = \pm 1
$$

and $(R'/p_i \cap R')_{p_i\cap R'} \cong (S'/p_i \cap S')_{p_i\cap S'}$ *for* $1 \le i \le r$.

Proof. — The proof is a refinement of that of Theorem 5.3. The stronger Corollary 5.2 is used instead of Theorem 5.1. Formula (129) then becomes

$$
\begin{aligned}
x_1'' &= \psi_1 (y_1'')^{g_{11}(1)} \cdots (y_{s_1}'')^{g_{1s_1}(1)}\\
&\vdots\\
x_\lambda'' &= \psi_\lambda y_\lambda''
\end{aligned}
\tag{133}
$$

(131) becomes

$$
\begin{aligned}
x_1(m') &= \overline{y}_1(m')^{g_{11}(1)} \cdots \overline{y}_{s1}(m')^{g_{1s_1}(1)} \overline{y}_{\lambda+1}(m')^{e_{1,\lambda+1}} \cdots \overline{y}_n(m')^{e_{1n}} \psi_1 \\
&\ \vdots \\
x_\lambda(m') &= \overline{y}_\lambda(m') \overline{y}_{\lambda+1}(m')^{e_{\lambda,\lambda+1}} \cdots \overline{y}_n(m')^{e_{\lambda,n}} \psi_\lambda \\
x_{\lambda+1}(m') &= \overline{y}_{\lambda+1}(m')^{g_{11}(r)} \cdots \overline{y}_{\lambda+s_r}(m')^{g_{s_r s_r}(r)} \delta_{\lambda+1} \\
&\quad + f_1^{\lambda+1} \overline{y}_1(m') + \cdots + f_\lambda^{\lambda+1} \overline{y}_\lambda(m') \\
&\ \vdots \\
x_n(m') &= \overline{y}_n(m') \delta_n + f_1^n \overline{y}_1(m') + \cdots + f_\lambda^n \overline{y}_\lambda(m')
\end{aligned}
\tag{134}
$$

(132) becomes

$$
\begin{aligned}
x_1(m') &= \overline{y}_1(m'+1)^{g_{11}(1)} \cdots \overline{y}_{s_1}(m'+1)^{g_{1s_1}(1)} \overline{y}_{\lambda+1}(m'+1)^{b_{1,\lambda+1}} \cdots \\
&\quad \cdot \overline{y}_n(m'+1)^{b_{1n}} u_1 \\
&\ \vdots \\
x_\lambda(m') &= \overline{y}_\lambda(m'+1) \overline{y}_{\lambda+1}(m'+1)^{b_{\lambda,\lambda+1}} \cdots \overline{y}_n(m'+1)^{b_{\lambda n}} u_\lambda \\
x_{\lambda+1}(m') &= \overline{y}_{\lambda+1}(m'+1)^{g_{11}(r)} \cdots \overline{y}_{\lambda+s_r}(m'+1)^{g_{1s_r}(r)} u_{\lambda+1} \\
&\ \vdots \\
x_n(m') &= \overline{y}_n(m') u_n.
\end{aligned}
\tag{135}
$$

The MTS $R(m') \to R(m'+1)$, where $R(m'+1)$ has regular parameters

$$(x_1(m'+1), \ldots, x_n(m'+1))$$

defined by

$$
x_i(m') = \begin{cases} x_i(m'+1) x_{\lambda+s_r+1}(m'+1)^{b_{i,\lambda+s_r+1}} \cdots x_n(m'+1)^{b_{in}} & 1 \le i \le \lambda \\ x_i(m'+1) & \lambda+1 \le i \le n \end{cases}
$$

factors through $\overline{S}(m'+1)$, and

$$
\begin{aligned}
x_1(m'+1) &= \overline{y}_1(m'+1)^{g_{11}(1)} \cdots \overline{y}_{s_1}(m'+1)^{g_{1s_1}(1)} \\
&\quad \cdot \overline{y}_{\lambda+1}(m'+1)^{b_{1,\lambda+1}} \cdots \overline{y}_{\lambda+s_r}(m'+1)^{b_{1,\lambda+s_r}} u_1' \\
&\ \vdots \\
x_\lambda(m'+1) &= \overline{y}_\lambda(m'+1) \overline{y}_{\lambda+1}(m'+1)^{b_{\lambda,\lambda+1}} \cdots \overline{y}_{\lambda+s_r}(m'+1)^{b_{\lambda,\lambda+s_r}} u_\lambda' \\
x_{\lambda+1}(m'+1) &= \overline{y}_{\lambda+1}(m'+1)^{g_{11}(r)} \cdots \overline{y}_{\lambda+s_r}(m'+1)^{g_{1s_r}(r)} u_{\lambda+1} \\
&\ \vdots \\
x_n(m') &= \overline{y}_n(m') u_n
\end{aligned}
\tag{136}
$$

for some units $u_i' \in \overline{S}(m'+1)$. Since $\det(g_{ij}(l)) = \pm 1$ for $1 \le l \le r$, we can make a change of variables in $\overline{S}(m'+1)$, replacing $\overline{y}_i(m'+1)$ with a unit times $\overline{y}_i(m'+1)$ for all i to get that the u_i and u'_j in (136) are 1 for all i,j. Let

$$(h_{ij}) = \begin{pmatrix} g_{11}(r) & \cdots & g_{s_1 1}(r) \\ \vdots & & \vdots \\ g_{s_r 1}(r) & \cdots & g_{s_r s_r}(r) \end{pmatrix}^{-1},$$

an integral matrix.

$$\begin{aligned} \overline{y}_{\lambda+1}(m'+1) &= x_{\lambda+1}(m'+1)^{h_{11}} \cdots x_{\lambda+s_r}(m'+1)^{h_{1,s_r}} \\ &\vdots \\ \overline{y}_{\lambda+s_r}(m'+1) &= x_{\lambda+1}(m'+1)^{h_{s_r,1}} \cdots x_{\lambda+s_r}(m'+1)^{h_{s_r,s_r}}. \end{aligned}$$

$\nu(\overline{y}_i(m'+1)) > 0$ for $\lambda+1 \le i \le \lambda+s_r$, so by Lemmas 4.2 and 4.3 there exist MTSs $R(m'+1) \to R(m'+2)$ and $\overline{S}(m'+1) \to \overline{S}(m'+2)$ along ν such that $R(m'+2)$ has regular parameters $(x_1(m'+2), \ldots, x_n(m'+2))$, $S(m'+2)$ has regular parameters $(\overline{y}_1(m'+2), \ldots, \overline{y}_n(m'+2))$ defined by

$$x_i(m'+1) = \begin{cases} x_i(m'+2) & 1 \le i \le \lambda, \\ & \lambda+s_r < i \le n \\ x_{\lambda+1}(m'+2)^{a_{i1}(m'+2)} \cdots & \\ \qquad \cdot x_{\lambda+s_r}(m'+2)^{a_{i,\lambda+s_r}(m'+2)} & \lambda+1 \le i \le \lambda+s_r \end{cases}$$

$$\overline{y}_i(m'+1) = \begin{cases} \overline{y}_i(m'+2) & 1 \le i \le \lambda, \\ & \lambda+s_r < i \le n \\ \overline{y}_{\lambda+1}(m'+2)^{b_{i1}(m'+2)} \cdots & \\ \qquad \cdot \overline{y}_{\lambda+s_r}(m'+2)^{b_{i,\lambda+s_r}(m+2)} & \lambda+1 \le i \le \lambda+s_r. \end{cases}$$

such that $R(m'+2) \subset \overline{S}(m'+2)$ and

$$\overline{y}_{\lambda+i}(m'+1) = x_{\lambda+1}(m'+2)^{e_{i1}} \cdots x_{\lambda+s_r}(m'+2)^{e_{is_r}}$$

for $1 \le i \le s_r$, where $e_{ij} \ge 0$ for all i,j. Set

$$d_{ij} = e_{1j} b_{i,\lambda+1} + \cdots + e_{s_r j} b_{i,\lambda+s_r}$$

for $1 \le i \le \lambda$, $1 \le j \le s_r$. Then the MTS $R(m'+2) \to R(m'+3)$ where $R(m'+3)$ has regular parameters $(x_1(m'+3), \ldots, x_n(m'+3))$ defined by

$$x_i(m'+2) = \begin{cases} x_{\lambda+1}(m'+3)^{d_{i1}} \cdots x_{\lambda+s_r}(m'+3)^{d_{is_r}} x_i(m'+3) & 1 \le i \le \lambda \\ x_i(m'+3) & \lambda < i \le n \end{cases}$$

factors through $\overline{S}(m'+2)$ and the conclusions of the Theorem hold for the variables $x_i(m'+3)$ and $\overline{y}_i(m'+2)$. □

CHAPTER 6

FACTORIZATION 1

In this chapter we prove Theorem 1.6, which shows that it is possible to factor a birational map along a valuation by alternating sequences of blowing ups and blowing downs. Theorem 5.5 reduces this to a question of monomial morphisms and valuations of maximal rational rank. This reduces the problem to a question in combinatorics. Christensen, in **[10]**, using elementary linear algebra, gives a proof, that in dimension 3, factorization holds along a rational rank 3 valuation. His algorithm produces a factorization with one series of blowups and one series of blowdowns. We generalize his methods to give a proof of factorization of monomial mappings in the special case of valuations of maximal rational rank. Then Theorem 1.6 follows from Theorem 5.5.

Lemma 6.1. — *Suppose that $M = (a_{ij})$ is an $n \times n$ matrix such that the $a_{ij} \geq 0$ for all i, j and $\det(a_{ij}) = \pm 1$. Suppose that R is a regular local ring with regular parameters $(x_1, \ldots, x_n)$. Then there exists a regular local ring S in the quotient field of R such that S has regular parameters $(y_1, \ldots, y_n)$ satisfying (137).*

$$x_1 = y_1^{a_{11}} y_2^{a_{12}} \cdots y_n^{a_{1n}} \tag{137}$$
$$\vdots$$
$$x_n = y_1^{a_{n1}} y_2^{a_{n2}} \cdots y_n^{a_{nn}}$$

Proof. — Set $(b_{ij}) = M^{-1}$. There exists monomials f_i in $x_1, \ldots, x_n$ for $0 \leq i \leq n$ such that $x_1^{b_{i1}} \cdots x_n^{b_{in}} = f_i/f_0$ for $1 \leq i \leq n$. In $R[f_1/f_0, \ldots, f_n/f_0]$ we have

$$x_i = \left(\frac{f_1}{f_0}\right)^{a_{i1}} \cdots \left(\frac{f_n}{f_0}\right)^{a_{in}}$$

for $1 \leq i \leq n$ so that the maximal ideal $m = (x_1, \ldots, x_n, f_1/f_0, \ldots, f_n/f_0)$ is generated by $f_1/f_0, \ldots, f_n/f_0$. Set $S = R[f_1/f_0, \ldots, f_n/f_0]_m$ and $y_i = f_i/f_0$ for $1 \leq i \leq n$. Then S is a regular local ring and (137) holds. □

Suppose $R \to S$ is as in (137). An inverse monoidal transform (IMT) $R \to S(1) \to S$ consists of a regular local ring $S(1)$ such that $R \subset S(1) \subset S$ which has regular parameters $(y_1(1),\ldots,y_n(1))$ such that $y_r(1) = y_r y_s$ for some $r \neq s$ and $y_i(1) = y_i$ for $i \neq r$.

Lemma 6.2. — *Suppose that (137) holds for $R \to S$ and the coefficients of the s^{th} column of M minus the r^{th} column of M are nonnegative ($a_{is} - a_{ir} \geq 0$ for all i). Then there exists an IMT $R \to S(1) \to S$ such that*

$$
\begin{aligned}
x_1 &= y_1(1)^{a_{11}(1)} \cdots y_n(1)^{a_{1n}(1)}\\
&\vdots\\
x_n &= y_1(1)^{a_{n1}(1)} \cdots y_n(1)^{a_{nn}(1)},
\end{aligned}
\tag{138}
$$

$M(1) = (a_{ij}(1))$ is M with the r^{th} column subtracted from the s^{th} column. The adjoint matrix $A(1)$ of $M(1)$ is obtained from the adjoint matrix A of M by adding the s^{th} row of A to the r^{th} row of A.

Proof. — This follows from Lemma 6.1. □

Let $A = (A_{ij})$ be the adjoint matrix of M in (137). Consider a monoidal transform along ν $S \to S'$, where S' has regular parameters $(y_1',\ldots,y_n')$ defined by

$$
y_i = \begin{cases} y_s' y_r' & i = r \\ y_i' & i \neq r \end{cases}
$$

Of course, this means that $\nu(y_r) > \nu(y_s)$. Then the matrix $M' = (a_{ij}')$ where $x_i = (y_1')^{a_{i1}'} \cdots (y_n')^{a_{in}'}$ for $1 \leq i \leq n$ is obtained from M by adding the r^{th} column to the s^{th} column. The adjoint matrix of M', $A' = (A_{ij}')$ is obtained from A by subtracting the s^{th} row from the r^{th} row.

Theorem 6.3. — *Suppose that $R \subset S$ are excellent regular local rings of dimension n, containing a field k of chracteristic 0, with a common quotient field K. Suppose that ν is a valuation of K which dominates S, with valuation ring V. Suppose that*

1. *V has rational rank n*
2. *R has regular parameters $(x_1,\ldots,x_n)$, S has regular parameters $(y_1,\ldots,y_n)$ such that*

$$
\begin{aligned}
x_1 &= y_1^{a_{11}} y_2^{a_{12}} \cdots y_n^{a_{1n}}\\
&\vdots\\
x_n &= y_1^{a_{n1}} y_2^{a_{n2}} \cdots y_n^{a_{nn}}
\end{aligned}
$$

where $\det(a_{ij}) = \pm 1$.

Then there exists a MTS along ν

$$
S \to S(1) \to \cdots \to S(k) \tag{139}
$$

where $S(i)$ has regular parameters $(y_1(i),\ldots,y_n(i))$ for $0\le i\le k$ with

$$
\begin{aligned}
x_1 &= y_1(i)^{a_{11}(i)}y_2(i)^{a_{12}(i)}\cdots y_n(i)^{a_{1n}(i)}\\
&\ \vdots\\
x_n &= y_1(i)^{a_{n1}(i)}y_2(i)^{a_{n2}(i)}\cdots y_n(i)^{a_{nn}(i)}.
\end{aligned}
\tag{140}
$$

such that if $M(k)=(a_{ij}(k))$ is the coefficient matrix of $R\to S(k)$, with adjoint matrix $A(k)$, then all but at most two of $A_{11}(k),A_{12}(k),\ldots,A_{1n}(k)$ are zero.

Proof. — Set $M=(a_{ij})$. Let A be the adjoint matrix of M. In a sequence such as (139), define $M(i)=(a_{jk}(i))$ and $A(i)=(A_{jk}(i))$ to be the adjoint matrix of $M(i)$.

We will call a monoidal transform $S(l)\to S(l+1)$ along ν allowable if it is centered at $P(l)=P_{ij}=(y_i(l),y_j(l))$ where $A_{1i}(l),A_{1j}(l)$ are nonzero and have the same sign. If $T\subset\{1,2,\ldots,n\}$ is a subset containing i and j, and $P(l)$ is allowable, then

$$\max\{|A_{1k}(l+1)|:k\in T\}\le\max\{|A_{1k}(l)|:k\in T\}.$$

Suppose that there exists an infinite sequence of allowable monoidal transforms

$$S\to S(1)\to\cdots\to S(l)\to \tag{141}$$

where $S(l)\to S(l+1)$ is centered at $P(l)$. We will derive a contradiction. The Theorem will then follow since at least three $A_{1i}(l)$ nonzero imply two of them must have the same sign, which implies that there exists an allowable monoidal transform.

Set

$$
\begin{aligned}
U(l)&=\{i:A_{1i}(l)\ne 0\}\\
\alpha(l)&=|U(l)|\\
T(l)&=\{i:i\text{ occurs as an index in a }P(k)\text{ for some }k\ge l\}\\
\gamma(l)&=|T(l)|\\
\beta(l)&=\max\{|A_{1i}(l)|:i\in T(l)\}\\
W(l)&=\{j\in T(l):|A_{1j}(l)|=\beta(j)\}\\
\delta(l)&=|W(l)|
\end{aligned}
$$

We have $\alpha(l+1)\le\alpha(l)$, $\beta(l+1)\le\beta(l)$, $\gamma(l+1)\le\gamma(l)$ and if $\beta(l+1)=\beta(l)$ then $\delta(l+1)\le\delta(l)$. Hence in the lexicographic ordering,

$$(\alpha(l+1),\beta(l+1),\gamma(l+1),\delta(l+1))\le(\alpha(l),\beta(l),\gamma(l),\delta(l))$$

for all l.

It suffices to show that this invariant decreases after a finite number of steps, so we may assume that

$$(\alpha(l),\beta(l),\gamma(l),\delta(l))=(\alpha,\beta,\gamma,\delta)$$

in (141) for all l, and derive a contradiction. Set $U=U(l)$, $T=T(l)$, $W=W(l)$.

If there is some l such that $P(l)=P_{rs}$ with $r,s\in W$ and $\nu(y_r(l))>\nu(y_s(l))$, then

$$A_{1r}(l+1)=A_{1r}(l)-A_{1s}(l)=0,$$

and $\alpha(l+1) < \alpha(l)$. This kind of monoidal transform can thus not occur in (141).

If some $P(l) = P_{ir}$ with $i \in T-W$, $r \in W$ and $\nu(y_r(l)) > \nu(y_i(l))$, then $A_{1r}(l+1) = A_{1r}(l) - A_{1i}(l)$. Hence $\beta(l+1) < \beta(l)$ or $\beta(l+1) = \beta(l)$ and $\delta(l+1) < \delta(l)$. Thus such a monoidal transform cannot occur in (141).

Since $\gamma(l)$ cannot decrease, we must have infinitely many l such that $P(l) = P_{ir}$ with $r \in W$, $i \in T-W$ and $P(l) = P_{is}$ with $i, s \in T-W$ for all other l.

We must thus have $y_j(l) = y_j$ for $j \in W$ and for all l. Furthermore, $\nu(y_i(l)) \leq \nu(y_i)$ for all i and l.

At each step where $P(l) = P_{ir}$ with $r \in W$ and $i \in T-W$ we have

$$y_i(l+1) = \frac{y_i(l)}{y_r(l)} = \frac{y_i(l)}{y_r}$$

and $\nu(y_i(l+1)) = \nu(y_i(l)) - \nu(y_r)$. After a finite number of steps we must have $\nu(y_i(l)) < 0$ for some $i \in T-W$, a contradiction. □

When $n = 3$, Theorem 6.4 is proved by Christensen [Ch].

Theorem 6.4. — *Suppose that $R \subset S$ are excellent regular local rings of dimension $n \geq 3$, containing a field k of chracteristic 0, with a common quotient field K. Suppose that ν is a valuation of K which dominates S, with valuation ring V. Suppose that*

(1) *V has rational rank n*
(2) *R has regular parameters $(x_1, \ldots, x_n)$, S has regular parameters $(y_1, \ldots, y_n)$ such that*

$$\begin{aligned} x_1 &= y_1^{a_{11}} y_2^{a_{12}} \cdots y_n^{a_{1n}} \\ &\vdots \\ x_n &= y_1^{a_{n1}} y_2^{a_{n2}} \cdots y_n^{a_{nn}} \end{aligned}$$

where $\det(a_{ij}) = \pm 1$.

Then there is a sequence of regular local rings contained in K

$$\begin{array}{ccccccccc} & R_1 & & & & & R_{n-2} & & \\ \nearrow & & \nwarrow & \nearrow \cdots & \nwarrow & \nearrow & & \nwarrow & \\ R & & S_1 & & S_{n-3} & & & & S_{n-2} = S \end{array}$$

such that each local ring is dominated by V and each arrow is a sequence of monoidal transforms (blow ups of regular primes). Furthermore, we have inclusions $R \subset S_i$ for all i.

Proof. — The proof is by induction on n. For $n = 2$ there is a direct factorization by a MTS. Suppose that $n \geq 3$ and the theorem is true for smaller values of n. We will show that there is a MTS $S \to S'$ along ν and a sequence of IMTs $R \to S'' \to S'$ such that a column of the matrix M'' of $R \to S''$ consists of a single 1 and zeros in the remaining entries. Without loss of generality, the first column of M'' has this form. By Lemma 6.2, there is then a sequence of IMTs $R \to S_{n-3} \to S''$ such that the matrix $\overline{M}$ of $R \to S_{n-3}$ has the form

$$\overline{M} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & \overline{a}_{n2} & \cdots & \overline{a}_{nn} \end{pmatrix}.$$

By induction on n, there will then exist a factorization of the desired form.

By Theorem 6.3, there exists a MTS $S \to S'$ along ν such that, after possibly interchanging variables, $A_{1j} = 0$ for $j > 2$ and

$$a_{11}A_{11} + a_{12}A_{12} = 1 \tag{142}$$

Case 1. — Suppose that $A_{11} < 0$ and $A_{12} > 0$. (The case $A_{12} < 0$ and $A_{11} > 0$ is similar.) Then $1 = -a_{11}(-A_{11}) + a_{12}A_{12}$. Set $m = [-A_{11}/A_{12}]$, $n = [-A_{12}/A_{11}]$.

Suppose that $m > 0$. Note that $m = 0$ implies $n > 0$.

$$a_{i1}A_{11} + a_{i2}A_{12} + \cdots + a_{in}A_{1n} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

$$\begin{aligned} a_{i2} - a_{i1}m &\geq a_{i2} - a_{i1}\left(\frac{-A_{11}}{A_{12}}\right) \\ &= \frac{1}{A_{12}}(a_{i1}A_{11} + a_{i2}A_{12} + \cdots + a_{in}A_{1n}). \end{aligned}$$

Hence $a_{12} - a_{11}m \geq 1$ and $a_{i2} - a_{i1}m \geq 0$ for $2 \leq i \leq n$. Let M' be the matrix obtained from M by performing the column operation of subtracting m times the first column from the second column. All of the coefficients of M' are positive, so by Lemma 6.2 there is an IMT $R \to S' \to S$ such that M' is the matrix of $R \to S'$. We have $A'_{1j} = A_{1j}$ if $j \neq 1$ and $A'_{11} = A_{11} + mA_{12}$ so that $A_{11} < A'_{11} \leq 0$. If $A'_{11} \neq 0$, then

$$m' = \left[\frac{-A'_{11}}{A'_{12}}\right] = \left[\frac{-A_{11} - mA_{12}}{A_{12}}\right] = \left[\frac{-A_{11}}{A_{12}} - m\right] = 0$$

so that $n' > 0$.

Now suppose that $n > 0$.

$$\begin{aligned} a_{i1} - a_{i2}n &\geq a_{i1} - a_{i2}\left(\frac{A_{12}}{-A_{11}}\right) \\ &= \frac{1}{A_{11}}(a_{i1}A_{11} + a_{i2}A_{12} + \cdots + a_{in}A_{1n}). \end{aligned}$$

Thus we have $a_{i1} - a_{i2}n \geq 0$ for $2 \leq i \leq n$. Suppose that $A_{11} \neq -1$. Then $a_{11} - a_{12}n > -1$, and since this is an integer, $a_{11} - a_{12}n \geq 0$. We can then construct an IMT $R \to S' \to S$ such that the matrix M' of $R \to S'$ is obtained from M by subtracting n times the second column from the first column. We have $A'_{1j} = A_{1j}$ if

$j \neq 2$ and $A'_{12} = A_{12} + nA_{11}$ so that $A_{12} > A'_{12} \geq 0$. If $A'_{12} \neq 0$, then $n' = 0$ and $m' > 0$ so that we can repeat Case 1.

Suppose that $A_{11} = -1$. $1 = -a_{11} + a_{12}A_{12}$ implies $a_{12} > 0$.

$$a_{i1} - (A_{12} - 1)a_{i2} = -(a_{i1}A_{11} + a_{i2}A_{12} + \cdots + a_{in}A_{1n}) + a_{i2}$$

so that $a_{i1} - (A_{12} - 1)a_{i2} = a_{i2} \geq 0$ if $i > 1$, $a_{11} - (A_{12} - 1)a_{12} = -1 + a_{12} \geq 0$. We can then construct an IMT $R \to S' \to S$ such that the matrix M' of $R \to S'$ is obtained from M by subtracting $(A_{12} - 1)$ times the second column from the first column. Now construct the IMT $R \to S'' \to S'$ where the matrix M'' of $R \to S''$ is obtained from M' by subtracting the first column from the second column. The second column of M'' consists of a 1 in the first row, and the remaining rows are 0.

After a finite number of iterations of Case 1 we either prove the induction step, or reach the case $A_{12} = 0$ or $A_{11} = 0$.

Case 2. — Suppose that $A_{11} = 0$ or $A_{12} = 0$ (and $A_{13} = \cdots = A_{1n} = 0$). Without loss of generality we may assume that $A_{12} = 0$. $1 = a_{11}A_{11}$ implies $a_{11} = A_{11} = 1$. for $i > 1$ we have $a_{i1} = a_{i1}A_{11} + a_{i2}A_{12} + \cdots + a_{in}A_{1n} = 0$ so that the first column of M consists of a 1 in the first row, and the remaining rows are 0.

Case 3. — Suppose that $A_{11} > 0$ and $A_{12} > 0$. Then $a_{11} = A_{11} = 1$ and $a_{12} = 0$, or $a_{11} = 0$ and $a_{12} = A_{12} = 1$. Without loss of generality we have the first case. For $i > 0$ we have $0 \leq a_{i1} \leq a_{i1}A_{11} + a_{i2}A_{12} + \cdots a_{in}A_{1n} = 0$ Hence the first column of M consists of a 1 in the first row, and the remaining rows are 0.

This completes the induction step for the proof of the Theorem, since the case $A_{11} < 0, A_{12} < 0$ is not possible. □

Proof of Theorem 1.6. — We can perform MTSs $R \to R'$ and $S \to S'$ so that the conclusions of Theorem 5.5 hold. We can further replace R' by a MTS $R' \to R''$ such that S' dominates R', the conclusions of Theorem 5.5 hold, and if $s_i = 2$ for some i, then

$$\begin{aligned} z_{t_1+\cdots+t_{i-1}+1} &= w_{t_1+\cdots+t_{i-1}+1} \\ z_{t_1+\cdots+t_{i-1}+2} &= w_{t_1+\cdots+t_{i-1}+2} \end{aligned}$$

since factorization is possible if $n = 2$. Let $\lambda_1, \ldots, \lambda_a$ be the λ_i such that $1 \leq \lambda_i \leq r$ and $s_{\lambda_i} > 2$. Set

$$\begin{aligned} x_1 &= z_{t_1+\cdots+t_{\lambda_i-1}+1} \\ &\ \vdots \\ x_{s_{\lambda_i}} &= z_{t_1+\cdots+t_{\lambda_i-1}+s_{\lambda_i}} \end{aligned}$$

$$\begin{aligned} y_1 &= w_{t_1+\cdots+t_{\lambda_i-1}+1} \\ &\vdots \\ y_{s_{\lambda_i}} &= w_{t_1+\cdots+t_{\lambda_i-1}+s_{\lambda_i}}. \end{aligned}$$

Set $\overline{R}_{\lambda_i} = k[x_1, \ldots, x_{s_{\lambda_i}}]_{(x_1,\ldots,x_{s_{\lambda_i}})}$, $\overline{S}_{\lambda_i} = k[y_1, \ldots, y_{s_{\lambda_i}}]_{(y_1,\ldots,y_{s_{\lambda_i}})}$. Let $\overline{K}_{\lambda_i}$ be the quotient field of $\overline{S}_{\lambda_i}$. Then $\overline{R}_{\lambda_i} \subset \overline{S}_{\lambda_i}$ and $\overline{V}_{\lambda_i} = V \cap \overline{K}_{\lambda_i}$ is a rank 1, rational rank s_{λ_i} valuation ring dominating $\overline{S}_{\lambda_i}$. By Theorem 6.4, for all λ_i, there exist MTSs of regular local rings contained in $\overline{K}_{\lambda_i}$,

$$\begin{array}{ccccccc} & (\overline{R}_{\lambda_i})_1 & & & (\overline{R}_{\lambda_i})_{s_{\lambda_i}-2} & & \\ \nearrow & & \nwarrow & \nearrow \cdots \nearrow & & \nwarrow & \\ \overline{R}_{\lambda_i} & & & (\overline{S}_{\lambda_i})_1 & & & (\overline{S}_{\lambda_i})_{s_{\lambda_i}-2} = \overline{S}_{\lambda_i} \end{array}$$

such that each local ring is dominated by $\overline{V}_{\lambda_i}$ and $\overline{R}_{\lambda_i} \subset (\overline{S}_{\lambda_i})_j$ for all j.

We can perform the corresponding sequences of MTSs along ν on R to construct a sequence of MTSs

$$\begin{array}{ccccccc} & (R_{\lambda_1})_1 & & & (R_{\lambda_1})_{s_{\lambda_1}-2} & & \\ \nearrow & & \nwarrow & \nearrow \cdots \nearrow & & \nwarrow & \\ R & & & (S_{\lambda_1})_1 & & & (S_{\lambda_1})_{s_{\lambda_1}-2} = S_{\lambda_1} = R_{\lambda_2} \\ & & & \vdots & & & \\ & (R_{\lambda_a})_1 & & & (R_{\lambda_a})_{s_{\lambda_a}-2} & & \\ \nearrow & & \nwarrow & \nearrow \cdots \nearrow & & \nwarrow & \\ R_{\lambda_a} & & & (S_{\lambda_a})_1 & & & (S_{\lambda_a})_{s_{\lambda_a}-2} = S \end{array}$$

$(s_{\lambda_1} - 2) + (s_{\lambda_2} - 2) + \cdots + (s_{\lambda_a} - 2) \le n - 2$ since $s_{\lambda_1} + \cdots + s_{\lambda_a} \le n$. Thus the conclusions of the Theorem hold.

CHAPTER 7

FACTORIZATION 2

In the special case of a monomial mapping, local factorization by one sequence of blowups followed by one sequence of blowdowns follows from Morelli's Theorem on factorization of birational morphisms of toric varieties **[27]**, **[7]**. Theorem 7.1 states this result precisely.

Theorem 7.1. — *Suppose that R, S are excellent regular local rings of dimension n, containing a field k of characteristic zero, with a common quotient field K, such that S dominates R. Suppose that R has regular parameters $(x_1, \ldots, x_n)$, S has regular parameters $(y_1, \ldots, y_n)$ and there exists a matrix (a_{ij}) of natural numbers such that* $\det(a_{ij}) = \pm 1$ *and*

$$\begin{aligned} x_1 &= y_1^{a_{11}} \cdots y_n^{a_{1n}} \\ &\vdots \\ x_n &= y_1^{a_{n1}} \cdots y_n^{a_{1n}}. \end{aligned} \tag{143}$$

Let V be a valuation ring of K which dominates S. Then there exists a regular local ring T, with quotient field K, such that T dominates S, V dominates T, and the inclusions $R \to T$ and $S \to T$ can be factored by sequences of monoidal transforms (blowups of regular primes).

$$\begin{tikzcd} & V & \\ & T \arrow[u] & \\ R \arrow[ur] \arrow[rr] & & S \arrow[ul] \end{tikzcd}$$

Proof. — With the given assumptions

$$\operatorname{Spec}(k[y_1, \ldots, y_n]) \to \operatorname{Spec}(k[x_1, \ldots, x_n]) \tag{144}$$

is a toric birational morphism of toric varieties. There exist projective toric varieties X and Y and a birational projective toric morphism $f : X \to Y$ extending (144). By the main result of **[27]**, **[7]** (Strong factorization of birational toric morphisms) there exists a factorization

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & \longrightarrow & Y \end{array}$$

where Z is a projective toric variety, $Z \to X$ and $Z \to Y$ are composities of blowups of orbit closures. $Z \to X$ and $Z \to Y$ induce MTSs along ν $R \to T$ and $S \to T$. □

Proof of Theorem 1.9. — By Theorem 1.1, we can perform sequences of monoidal transforms $R \to R_1$ and $S \to S_1$ so that V dominates S_1, S_1 dominates R_1, and R_1 and S_1 have regular parameters satisfying (143). The proof of Theorem 1.9 now follows from Theorem 7.1.

Proof of Theorem 1.10. — If K is a field containing a ground field k, and v is a valuation of K, trivial on k, then the transcendence degree of $\mathcal{O}_v/m_v$ over k is called the dimension of v ($\dim(v)$). We have

$$\mathrm{rank}(v) \leq \mathrm{rrank}(v) \leq \mathrm{trdeg}_k K$$

(cf. the Corollary and note at the end of Chapter VI, Section 10 **[39]**).

Suppose that ν is a valuation associated to V. By Theorem 2.7, applied to the lift to V of a transcendence basis of V/m_ν, there exists a MTS along ν, $R \to R_1$, such that $\dim_{R_1}(\nu) = 0$. By assumption, R_1 is a localization of $k[f_1, \ldots, f_m]$ for some $f_1, \ldots, f_m \in K$, such that $\nu(f_i) \geq 0$ for all i. By Theorem 2.7, there exists a MTS $S \to S_1$ along ν such that $f_1, \ldots, f_n$ are in S_1. Hence S_1 dominates R_1.

$$\dim(R_1) = \mathrm{trdeg}_k(K) - \mathrm{trdeg}_k(R_1/m_1) = n - \dim(\nu)$$

and $\dim(S_1) = n - \dim(\nu)$. Now the Theorem follows from Theorem 1.9.

CHAPTER 8

THE ZARISKI MANIFOLD

Let k be a field, X be an integral proper k-scheme. Define $\mathcal{M}(X)$ to be the set of pairs (X_1, f_1) of proper birational morphisms $f_1 : X_1 \to X$.

Theorem 8.1 **(Zariski).** — *There exists a locally ringed space $Z(X)$ with morphisms*

$$h(X', f) : Z(X) \to X'$$

for $(X', f) \in \mathcal{M}(X)$ such that

(1) *If $(X_i, f_i) \in \mathcal{M}(X)$ for $i = 1, 2$ such that $f_1^{-1} \circ f_2$ is a morphism, then $h(X_1, f_1) = (f_1^{-1} \circ f_2) \circ h(X_2, f_2)$ and*
(2) *If $Z'(X)$ with maps $h'(X', f) : Z'(X) \to X'$ for $X' \in \mathcal{M}(X)$ has the property (1), then there exists a unique morphism $g : Z'(X) \to Z(X)$ such that $h'(X', f) = h(X', f) \circ g$ for all $(X', f) \in \mathcal{M}(X)$.*

$Z(X)$ is called the Zariski manifold of X (cf. section 17 **[39]**, **[24]**, section 6 of chapter 0 **[20]**). The formulation of Theorem 8.1 follows **[20]**.

$Z(X)$ can be constructed explicitly as follows (cf. **[39]**, **[24]**). Let ζ be the generic point of X, $K = \mathcal{O}_{X,\zeta}$. Define $Z(X)$ to be the set of valuation rings V of K such that $\mathcal{O}_{X,p} \subset V$ for some $p \in X$. The basic open sets U of a topology on X can be defined as follows. Suppose that $f_1 : X_1 \to X$ is a birational morphism of finite type. Let ζ_1 be the generic point of X_1. f_1^* induces an identification of $\mathcal{O}_{X_1,\zeta_1}$ with K. Set U to be the set of valuation rings V of $Z(X)$ such that $\mathcal{O}_{X_1,q} \subset V$ for some $q \in X_1$. $Z(X)$ has the structure of a locally ringed space, by defining

$$\Gamma(U, \mathcal{O}_{Z(X)}) = \cap_{V \in U} V$$

for open sets U of $Z(X)$.

Given a proper birational morphism $f : X' \to X$, we can define $h(X', f) : Z(X) \to X'$ by $h(X', f)(V) = p$ if V dominates p. p exists by the valuative criteria for properness (cf. Theorem II.4.7 **[19]**).

Theorem 8.2 **(Zariski).** — *$Z(X)$ is quasi-compact.*

This is proved in chapter VI, section 17, Theorem 40 **[39]**.

***Definition 8.3* (Hironaka, chapter 0, section 6 [20]).** — Let $f : X' \to X$ be a finite type morphism of integral k-schemes. f is complete (or X' is complete over X) if

(1) The morphism f is surjective.
(2) For every point $x' \in X'$, there exists a 4-tuple $(U, \overline{X}', \overline{f}, j)$ consisting of an open dense subset U of the underlying topological space of X' which contains x', an integral finite type k-scheme $\overline{X}'$, a proper morphism $\overline{f} : \overline{X}' \to X$ and a morphism $j : X'|U \to \overline{X}'$ which induces an isomorphism of the same to the restriction of $\overline{X}'$ to an open dense subset of its underlying topological space and such that $\overline{f} \circ j = f|U$ and
(3) Every point $x \in X$ admits an open neighbourhood V in the underlying topological space of X such that $X|V$ is a finite type k-scheme, and if we identify in a canonical way the Zariski spaces $Z(\overline{X}', \overline{f}^{-1}(V))$ for all 4-tuples $(U, \overline{X}', \overline{f}, j)$ of (2) and call it $Z(X'|V)$, then the underlying topological space of $Z(X'|V)$ is equal to the union of $h(\overline{X}'|\overline{f}^{-1}(V))^{-1}(j(U) \cap \overline{f}^{-1}(V))$ for all $(U, \overline{X}', \overline{f}, j)$.

Lemma 8.4. — *A complete separated morphism $f : X \to Y$ of integral finite type k-schemes with X nonsingular is proper.*

Proof. — This is Corollary 9.5 **[13]**. □

Proof of Theorem 1.2. — Let $Z(X)$ be the Zariski manifold of X with projection $\pi_X : Z(X) \to X$. Suppose that $V \in Z(X)$. Let α be the center of V on X, β the center of V on Y, $R = \mathcal{O}_{Y,\beta}$, $S = \mathcal{O}_{X,\alpha}$. By Theorem 1.1, there exist sequences of monoidal transforms $R \to R'$, $S \to S'$ along V such that R', S' have regular parameters satisfying (2) of Theorem 1.1. There exist affine neighborhoods U_V of $\alpha \in X$, W_V of β in Y such that $\Phi(U_V) \subset W_V$, projective morphisms $a_V : \overline{U}_V \to U_V$ and $b_V : \overline{W}_V \to W_V$ which are products of monoidal transforms, and affine neighborhoods U'_V of the center α' of V on $\overline{U}_V$, W'_V of the center β' of V on $\overline{W}_V$ such that Φ induces a morphism $\Phi_V : U'_V \to W'_V$, $R' = \mathcal{O}_{W'_V,\beta'}$, $S' = \mathcal{O}_{U'_V,\alpha'}$, $(x_1, x_2, \ldots, x_n)$ are uniformizing parameters in W'_V, $(y_1, y_2, \ldots, y_n)$ are uniformizing parameters in U'_V, and $\delta_1, \ldots, \delta_n$ are units on U'_V.

U'_V is an open subset of a proper k-scheme $\overline{U}'_V$ with a birational morphism onto X. Hence there exists a canonical map $\pi_{\overline{U}'_V} : Z(X) \to \overline{U}'_V$. Let $Z_V = \pi^{-1}_{\overline{U}'_V}(U'_V)$. Z_V is an open neighborhood of V in $Z(X)$. The $\{Z_V\}$ indexed over $V \in Z(X)$ are an open cover of $Z(X)$. There exists a finite subcover, which can be indexed as $\{Z_1, \ldots, Z_n\}$, since $Z(X)$ is quasi-compact (Theorem 8.2). Let $\{U'_1, \ldots, U'_n\}$ be the corresponding U'_V.

Let A_i be the largest open subset of U_i such that $a_i : a_i^{-1}(A_i) \to A_i$ is an isomorphism. For all i, j, we have isomorphisms

$$a_j^{-1} \circ a_i : a_i^{-1}(A_i \cap A_j) \to a_j^{-1}(A_i \cap A_j).$$

Let X_1 be the scheme obtained by glueing the U_i' along the open sets $a_i^{-1}(A_i \cap A_j)$.

Let B_i be the largest open subset of W_i such that $b_i : b_i^{-1}(B_i) \to B_i$ is an isomorphism. For all i, j, we have isomorphisms

$$b_j^{-1} \circ b_i : b_i^{-1}(B_i \cap B_j) \to b_j^{-1}(B_i \cap B_j).$$

Let Y_1 be the scheme obtained by glueing the W_i' along the open sets $b_i^{-1}(B_i \cap B_j)$.

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