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LEMPERT MAPPINGS AND HOLOMORPHIC MOTIONS IN $\mathbf{C}^n$

by

Kari Astala, Zoltan Balogh & Hans Martin Reimann

Abstract. — The purpose of this note is twofold: to discuss the concept of holomorphic motions and phenomena of Mañé-Sad-Sullivan type in several complex variables and secondly, to compare the different notions of Beltrami differentials in CR-geometry which have appeared in [4] and [7].

1. Introduction

Holomorphic motions in the complex plane $\mathbf{C}$ are isotopies of subsets $A \subset \mathbf{C}$ for which the dependence on the "time" parameter is holomorphic. This simple notion has been important in explaining a number of different questions in complex analysis, in particular the rigidity phenomena in complex dynamics and the role of quasiconformal mappings in holomorphic deformations.

It was Mañé, Sad and Sullivan **[9]** who first realized that for time-holomorphic isotopies one can forget all smoothness requirements in space variables and thus produce almost automatic rigidity results in various contexts. Given a subset $A \subset \overline{\mathbf{C}}$, it is simply enough to define a holomorphic motion of A as a mapping $f : \Delta \times A \to \overline{\mathbf{C}}$, where $\Delta = \{\lambda \in \mathbf{C} : |\lambda| < 1\}$, such that

(i) for any fixed $a \in A$, the map $\lambda \to f(\lambda, a)$ is holomorphic in Δ
(ii) for any fixed $\lambda \in \Delta$, the map $a \to f(\lambda, a) = f_\lambda(a)$ is an injection and
(iii) the mapping f_0 is the identity on A.

Then f is automatically continuous in $\overline{A} \times \overline{\mathbf{C}}$ and the restrictions $f_\lambda(.)$ are quasisymmetric mappings **[9]**; in case $A = \overline{\mathbf{C}}$ they are quasiconformal with the precise

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bound on the dilatation

$$K(f_\lambda) \le \frac{1+|\lambda|}{1-|\lambda|}. \tag{1}$$

The picture was then completed by Slodkowski **[12]** who proved the so called Generalized Λ-lemma, that a holomorphic motion of any set $A \subset \overline{\mathbf{C}}$ extends to a motion of the whole $\overline{\mathbf{C}}$.

In this setting it is natural to look for similar phenomena in several complex variables, when the sets moving are of higher dimension. However, one quickly sees that the simple minded generalization does not work: Let, for example, $S(x) = x/|x|$ for $x \in \mathbf{R} \setminus \{0\}$, $S(0) = 0$ and define

$$f_\lambda(z, w) = (z + \lambda S(\mathrm{Re}\{w\}), w).$$

Then f_λ is holomorphic in λ and injective but not even continuous in $\mathbf{C}^2$.

Our first goal in this note is to introduce the proper notion or point of view to holomorphic motions in several complex variables and then show the existence of the first nontrivial examples, results of Mañé-Sad-Sullivan type. We expect that similar phenomena occur, in fact, in much larger setups.

Remark. — The generalizations to the case where the *parameter* space is higher dimensional were studied by Adrien Douady in his work **[3]**.

If there are to be holomorphic motions in $\mathbf{C}^n$, the one-dimensional theory suggests that they are connected to a notion of quasiconformality. Therefore recall that in several complex variables the appropriate concepts are the quasiconformal mappings on CR-structures **[4]**, or mappings on boundaries of pseudoconvex domains which firstly are contact transforms, *i.e.* preserve the horizontal (complex) lines of the tangent spaces

$$H_p\partial D = T_p\partial D \cap JT_p\partial D$$

where J is the complex structure as a mapping of $T_p\mathbf{C}^2$, and secondly, are there quasiconformal with respect to the corresponding Levi Form, *i.e.*

$$K(p) = \frac{\sup\{L(F_*X, F_*X) : X \in H_p\partial D,\ L(X,X) = 1\}}{\inf\{L(F_*X, F_*X) : X \in H_p\partial D,\ L(X,X) = 1\}} \le K \tag{2}$$

for all $p \in \partial D$.

The same direction is, actually, suggested also by the approach of Slodkowski **[12]**. He viewed holomorphic motions (or their graphs) as disjoint analytic disks in $\mathbf{C}^2$. Namely given such a motion $f : \Delta \times A \to \mathbf{C}$ each point $a \in A$ defines a holomorphic disk $D_a \subset \mathbf{C}^2$, a holomorphic image of Δ, by

$$D_a = \{(\lambda, f(\lambda, a)) : \lambda \in \Delta\} \tag{3}$$

and these disks are clearly pointwise disjoint. Conversely, given a family of analytic disks of the form (3) with $D_a \cap D_b = \varnothing$ when $a \neq b$, they define a holomorpic motion $\Psi(\lambda, f(0,a)) = f(\lambda, a)$. (The extension of a given motion was then obtained by studying certain totally real tori whose polynomial hulls were shown to consists of disjoint families of suitable analytic disks.)

Interpreting the Mañé-Sad-Sullivan result in the language (3) of disjoint analytic disks, for a motion of the whole complex plane the disks D_a, $a \in \mathbf{C}$, fill in the domain $\Delta \times \mathbf{C}$. And when we move along the disks with λ, in the transverse direction *i.e.* on the *complex* lines of the corresponding tangent spaces of $\partial\Delta(|\lambda|) \times \mathbf{C}$ the mappings $(0, a) \mapsto (\lambda, f(\lambda, a))$ are now quasiconformal by the original λ-lemma.

This picture makes it very suggestive that similar phenomena should occur in other situations in $\mathbf{C}^2$ or $\mathbf{C}^n$ as well. That is, for suitable families of analytic disks one should expect that moving holomorphically along these disks yields automatic quasiconformality in transverse horizontal directions, quasiconformality in the Koranyi-Reimann sense, with the bound (1) on the dilatation. The philosophy of holomorphic motions in $\mathbf{C}^n$ would then be not that there is one strict definition of these motions but rather that there are several natural situations that share the common features described here.

To show that there do exist nontrivial holomorphic motions in the above sense in $\mathbf{C}^2$ (the choice $n = 2$ is made for simplicity) we make use of the theory developed by Lempert **[5]**-**[8]** and consider bounded strictly $\mathbf{R}$-convex smooth subdomains $D \subset \mathbf{C}^2$ and their generalizations the strictly linearly convex domains. The latter class consists of smooth bounded domains with the property that for each boundary point $p \in \partial D$ the horizontal space $H_p\partial D$ does not intersect $\bar{D}\backslash\{p\}$ and that $H_p\partial D$ has precisely first order contact with ∂D at p. That is, there exists $c > 0$ such that

$$\operatorname{dist}(q, H_p\partial D) \geq c \cdot \operatorname{dist}(p,q)^2, \qquad q \in D.$$

In particular, strictly convex domains are strictly linearly convex which in turn are strictly pseudoconvex.

As shown by Lempert in strictly linearly convex domains extremal Kobayashi disks are especially well behaved. For this recall that in any bounded domain containing the origin the Kobayashi indicatrix $\bar{I}$ of D is defined by

$$\bar{I} = \{f'(0) : \ f : \Delta \to D \text{ is holomorphic and } f(0) = 0\}.$$

If $v \in \partial\bar{I}$, a holomorphic mapping $f = f_v : \Delta \to D$ such that $f(0) = 0$ and $f'(0) = v$ is then called an extremal map corresponding to the vector v. In strictly linearly convex domains extremal disks are uniquely determined by v, a fact no longer true for general pseudoconvex domains. This enables us to simply define

$$\Psi : \Delta \times \partial\bar{I} \to \mathbf{C}^2, \qquad \Psi(\lambda, v) = \frac{f_v(\lambda)}{\lambda}$$

and we can describe a full counterpart of the Mañé-Sad-Sullivan result, a holomorphic motion of $\partial\bar{I}$.

Theorem 1. — *Let D be a strictly linearly convex domain containing the origin and $\Psi : \Delta \times \partial \bar{I} \to \mathbf{C}^2$ be defined by $\Psi(\lambda, v) = \lambda^{-1} f_v(\lambda)$. Then Ψ satisfies the following properties:*

(1) $\Psi(0, \cdot) = \mathrm{Id}\,|_{\partial \bar{I}}$;
(2) $\Psi(\cdot, v) : \Delta \to \mathbf{C}^2$ *is holomorphic;*
(3) $\Psi(\lambda, \cdot) : \partial \bar{I} \to A_\lambda$ *is a contact mapping where* $A_\lambda = \Psi(\lambda, \partial \bar{I})$ *is the boundary of a strictly pseudoconvex domain. In particular,* Ψ *is continuous in* $\Delta \times \partial \bar{I}$;
(4) $\Psi(\lambda, \cdot) : \partial \bar{I} \to A_\lambda$ *is* $K(\lambda)$*-quasiconformal with* $K(\lambda) \leq \frac{1+|\lambda|}{1-|\lambda|}$, $\lambda \in \Delta$.

We should mention that statement (4) is the new result proven here; statements (1), (2) and (3), due to Lempert, are being included for the sake of completeness. To obtain the optimal dilatation bound we turn to our second goal, to compare the different notions of Beltrami differentials in contact geometry and CR-manifolds, introduced respectively, by Koranyi and Reimann **[4]** and Lempert **[7]**. This with required preliminary material will be presented in the next section.

2. Inner actions and Beltrami differentials

It will be convenient start with a version of the Riemann mapping theorem in $\mathbf{C}^n$ due to Lempert (**[5]**, **[6]**), and use the formalism introduced by Semmes **[11]**. These Riemann mappings preserve the complex structure to some extent but are flexible enough to yield general existence results. In more precise terms, Lempert considered mappings $\rho : \bar{B} \to \bar{D}$ from the unit ball in $\mathbf{C}^n$ onto domains $D \subset \mathbf{C}^n$ containing the origin which satisfy the following three requirements:

(1) $\rho : \bar{B}\backslash\{0\} \to \bar{D}\backslash\{0\}$ is a smooth diffeomorphism and $\rho : \bar{B} \to \bar{D}$ is bilipshitz;
(2) ρ restricted to any complex line through the origin is holomorphic;
(3) ρ restricted to the boundary of any ball B_r centered at the origin and of radius $0 < r \leq 1$ is contact, *i.e.*

$$\rho_* H\partial B_r = H\partial D_r \qquad\qquad (D_r = \rho B_r).$$

For the last condition recall that when a domain is strictly pseudoconvex the horizontal tangent bundle $H\partial D = T\partial D \cap JT\partial D$, where J is the complex structure, defines a contact structure on the boundary.

In what follows a mapping with the above properties (1)-(3) will be called a Lempert mapping. The basic existence result is then:

***Theorem A* (Lempert).** — *Let D be a strictly linearly convex domain. Then there exists a Lempert mapping $\rho : \bar{B} \to \bar{D}$.*

A very nice exposition of the properties of the Lempert mappings was given by Semmes in **[11]**. The statement and proof of Theorem A, for example, may be found in **[11]** in the case of strictly convex domains and it is based essentially on the results in **[5]**. The proof of Theorem A in the case of strictly linearly convex domains follows exactly as in the strictly convex case (Theorem 5.2 in **[11]**) using the results in **[6]**.

It is important for our purposes to estimate the Beltrami differential of the Lempert mapping $\rho : \bar{B} \to \bar{D}$ in the sense [4], as a quasiconformal mapping on the level surfaces $\rho : \partial B_r \to \partial D_r$. Therefore, let first $r = 1$ and use the notation

$$\partial D = M$$

for the boundary of the stricly linearly convex domain $D \subset \mathbf{C}^2$. Then on the unit sphere

$$S^3 = \{(z, w) : |z|^2 + |w|^2 = 1\}$$

we consider the $(1,0)$-vectorfield $Z = \bar{z}\dfrac{\partial}{\partial w} - \bar{w}\dfrac{\partial}{\partial z}$. It is easy to check that Z is a section of $H^{1,0}S^3$ where

$$\mathbf{C} \otimes HS^3 = H^{1,0}S^3 \oplus H^{0,1}S^3$$

is the complexified horizontal bundle of the sphere and $H_p^{1,0}$ is the space of $(1,0)$-vectors, *i.e.* vectors of the form $Y - iJY$, $Y \in H_pM$. Since ρ is a contact mapping we can write

$$\rho_* Z = Y + \bar{\nu}\bar{Y} \text{ where } Y \in H^{1,0}_{\rho(z,w)}M. \tag{4}$$

We have used here the fact that $H^{1,0}_{\rho(z,w)}M$ is complex one-dimensional and therefore Y in (4) is uniquely determined.

In fact, in an invariant formulation one should think of ν in (4) not as a number but rather a complex-antilinear map $\nu : H_p^{1,0}M \to H_p^{1,0}M$ at each point $p \in M$ so that $\rho_* Z = Y + \overline{\nu(Y)}$. Once we have chosen a basis in $H_p^{1,0}S^3$ (or Z in (4)) ν becomes a complex number.

To study the variation of ν on M let us fix a point $(z, w) \in S^3$. Then $\zeta(z, w) \in S^3$ for $\zeta \in S^1$ and (4) gives us a function $\nu : S^1 \to \mathbf{C}$, $\nu = \nu(\zeta) = \nu_p(\zeta)$, $p = \rho(z, w)$. Moreover using property 3 of Lempert mappings we can do this consideration for any $\zeta \in \bar{\Delta} \setminus 0$ and we obtain a function ν on $\bar{\Delta}\setminus 0$.

The following simple statement (see also **[4]**) will be useful later in our note:

***Lemma* 2.** — *Let us consider the Lempert mapping $\rho : \bar{B} \to \bar{D}$ of a strictly linearly convex domain D and the Beltrami differential ν as constructed above. Then we have the inequality: $|\nu| < 1$.*

Proof. — Let us denote by $u_0(z, w) = |z|^2 + |w|^2 - 1$ the defining function of B and $u_1(p) = u_0(\rho^{-1}(p))$, $p \in \bar{D}$. Then u_1 is a defining function of D that is smooth on $\bar{D}\setminus\{0\}$ (see **[11]**, lemma 2.9). Moreover if we consider the $(1,1)$-forms $\Omega_0 = \frac{1}{2}\partial\bar{\partial}u_0$ and $\Omega_1 = \frac{1}{2}\partial\bar{\partial}u_1$ by Lemma 2.6 in **[11]** we have

$$\Omega_0 = \rho^*\Omega_1. \tag{5}$$

We insert in the forms Ω_0 and Ω_1 vectors from the complexified horizontal spaces $\mathbf{C}\otimes HS^3$ and, respectively, $\mathbf{C}\otimes HM$. Relation (5) gives

$$\Omega_0(Z,\bar{Z}) = \Omega_1(Y+\bar{\nu}\bar{Y},\bar{Y}+\nu Y). \tag{6}$$

Since $\Omega_1(Y,Y) = \Omega_1(\bar{Y},\bar{Y}) = 0$ we have

$$0 < \Omega_0(Z,\bar{Z}) = (1-|\nu|^2)\Omega_1(Y,\bar{Y}). \tag{7}$$

By strict pseudoconvexity of D we have $\Omega_1(Y,\bar{Y}) > 0$ and hence $|\nu| < 1$. □

Furthermore, note that like in the classical case of planar quasiconformal mappings there is a relationship between the norm of the Beltrami coefficient and the quasiconformal dilatation of (2), given by

$$K(\rho) := \sup_{p\in M} K(p) = \frac{1+||\nu||_\infty}{1-||\nu||_\infty}, \tag{8}$$

see [4].

In his work [7] on embeddability of CR-structures Lempert studied inner actions on the boundary M of the strictly linearly convex domain D and defined for them another, apparently quite different Beltrami differential. Our next goal is to find a relationship between these two notions.

We begin by basic notation. Let $\{g_t\}, t\in\mathbf{R}$, be a smooth $\mathbf{R}$-action on M. It is called transverse if its infinitesimal generator $\{dg_t/dt\}_{t=0}$ is everywhere transverse to the contact plane field HM and a contact action $\{g_t\}$ in case the tangent maps g_{t*} preserve the contact distribution HM for all t. A consistent orientation of M is obtained by declaring the frame $X, JX, [JX,X]$ to be positive for a nonvanishing local section of HM. Then a transverse contact action is called positive if for one (or any) local section X of HM the frame X, JX and $\{dg_t/dt\}_{t=0}$ is positively oriented.

Let us now consider a positive contact action $\{g_\zeta\}, \zeta\in S^1$ of the unit circle. In general this action does not respect the complex structure J and one can measure this fact as follows. Let $p\in M$ and $X\in\mathbf{C}\otimes H_pM$ be such that X and $\bar{X}$ are linearly independent. Since the complex dimension of $\mathbf{C}\otimes H_pM$ is 2 this means that X and $\bar{X}$ span $\mathbf{C}\otimes H_pM$, so that there are complex numbers (not unique of course) a, b not both zero such that

$$a\bar{X} + bX \in H_p^{0,1}M. \tag{9}$$

Clearly $|a| \neq |b|$ for otherwise the $(0,1)$-vector in (9) and its conjugate would be dependent. We shall assume that X is chosen so that $|a| > |b|$ holds. In this case we say that X is a $(1,0)$-like vector.

If we fix $p\in M$ and X then the images $g_{\zeta*}X$ and $g_{\zeta*}\bar{X}$ span $\mathbf{C}\otimes H_{g_\zeta(p)}M$ as $g_{\zeta*}$ preserves the contact bundle HM. Therefore there are again complex numbers $a(\zeta)$,

$b(\zeta)$ not both zero, such that

$$a(\zeta)g_{\zeta*}\bar{X} + b(\zeta)g_{\zeta*}X \in H^{0,1}_{g_\zeta(p)}M. \tag{10}$$

Although $a(\zeta)$ and $b(\zeta)$ are not unique their quotient $\mu(\zeta) = b(\zeta)/a(\zeta) \in \mathbf{C} \cup \{\infty\}$ is uniquely determined and depends smoothly on ζ. Since $|\mu(\zeta)| \neq 1$ as before and $\mu(1) = b/a$ it follows that $|\mu(\zeta)| < 1$ for all $\zeta \in S^1$.

In this setting, Lempert calls μ the Beltrami differential associated to the trajectory of p. In fact μ depends on p itself and X but this dependence is just up to Moebius transformations of the unit disc ([**7**], proposition 3.1).

A positive contact action is now called *inner* if its Lempert-Beltrami differential μ, for each trajectory, extends to a smooth function on the closed unit disc $\bar{\Delta}$ in such a way that this extension is holomorphic in Δ.

The existence of the inner actions on boundaries of strictly linearly convex domains was proved by Lempert in [**7**]. We present here a different approach (in Theorem 3) that gives us the possibility to draw some precise consequences in the proof of Theorem 1. In fact, we are going to consider the action $g_\zeta : M \to M$ given by $g_\zeta(p) = \rho(\zeta(\rho^{-1}(p)))$ for $p \in M$, $\zeta \in S^1$. In other words, $g_\zeta : M \to M$ is the conjugate of the standard action $h_\zeta : S^3 \to S^3$, $h_\zeta(z,w) = \zeta(z,w)$ from the sphere S^3 to M by the Lempert mapping ρ.

It is in this setup that we can observe a connection between the two Beltrami differentials μ and ν. The observation is based on the fact that since the definition of μ depends on the choice of $p \in M$ and a vector $X \in \mathbf{C} \otimes H_pM$ one must do this selection carefully. After fixing $(z,w) \in S^3$ and $p = \rho(z,w) \in M$ (and hence also the function $\nu_p(\zeta)$) a consistent way is to take $X = \rho_* Z_{(z,w)} = Y + \bar{\nu}\bar{Y}$. We are allowed to do that since $|\nu| < 1$ by Lemma 2 and therefore X and $\bar{X}$ are independent. With these particular choices we then consider the Beltrami differential μ of g_ζ as given by (9) and (10).

Theorem 3. — *Let $D \subseteq \mathbf{C}^2$ be a strictly linearly convex domain with boundary M and let $\rho : \bar{B} \to \bar{D}$ be the Lempert mapping whose Beltrami coefficient, as a contact quasiconformal transformation, is ν. Then the action $g_\zeta : M \to M$ given by $g_\zeta(q) = \rho(\zeta(\rho^{-1}(q)))$ is an inner action. If the Lempert-Beltrami differential μ of this action g_ζ is defined as above we have then the relation*

$$\nu(\zeta) = -\left(\frac{\zeta}{\bar{\zeta}}\right)^2 \mu(\zeta) \quad \textit{for} \quad \zeta \in \bar{\Delta}\backslash\{0\}. \tag{11}$$

Proof. — We begin by proving the second statement for $\zeta \in S^1$. If we denote by $h_\zeta : S^3 \to S^3$ the standard action $h_\zeta(z,w) = \zeta \cdot (z,w)$ then it is easy to see that

$$h_{\zeta*}Z = \frac{\zeta}{\bar{\zeta}}Z, \ \zeta \in \bar{\Delta}. \tag{12}$$

Fix a point $(z,w) \in S^3$ and $p = \rho(z,w)$. For the value $\zeta = 1$ we have that $X = Y + \bar{\nu}\bar{Y}$ and $\bar{X} = \bar{Y} + \nu Y$ are linearly independent and there are numbers $a, b \in \mathbf{C}$ such that $a\bar{X} + bX \in H_p^{0,1}M$ or $(a\nu + b)Y + (a + b\bar{\nu})\bar{Y} \in H_p^{0,1}M$. Therefore $a\nu + b = 0$ and thus $\nu(1) = -b/a = -\mu(1)$. Furthermore $g_{\zeta *}X(p)$ and $g_{\zeta *}\bar{X}(p)$ are again independent and there are numbers $a(\zeta), b(\zeta) \in \mathbf{C}$ such that

$$a(\zeta)g_{\zeta *}\bar{X}(p) + b(\zeta)g_{\zeta *}X(p) \in H^{0,1}_{g_\zeta(p)}, \tag{13}$$

i.e. $\mu(\zeta) = b(\zeta)/a(\zeta)$.

On the other hand using (12)

$$\begin{aligned} g_{\zeta *}X(p) &= (\rho_* \circ h_{\zeta *} \circ \rho_*^{-1})X(p) = (\rho_* \circ h_{\zeta *})Z(z,w) = \rho_*\Big(\frac{\zeta}{\bar{\zeta}}Z(\zeta(z,w))\Big) \\ &= \frac{\zeta}{\bar{\zeta}}X(\rho(\zeta(z,w))) = \frac{\zeta}{\bar{\zeta}}(Y + \bar{\nu}(\zeta)\bar{Y}). \end{aligned}$$

Consequently (13) becomes

$$\left(a(\zeta)\frac{\bar{\zeta}}{\zeta}\nu(\zeta) + b(\zeta)\frac{\zeta}{\bar{\zeta}}\right)Y + \left(a(\zeta)\frac{\bar{\zeta}}{\zeta} + b(\zeta)\frac{\zeta}{\bar{\zeta}}\bar{\nu}(\zeta)\right)\bar{Y} \in H^{0,1}_{g_\zeta(p)}M, \tag{14}$$

which gives

$$a(\zeta)\frac{\bar{\zeta}}{\zeta}\nu(\zeta) + b(\zeta)\frac{\zeta}{\bar{\zeta}} = 0, \tag{15}$$

and therefore (11) follows for the points ζ on the unit circle S^1.

For the remaining part, the action $g_\zeta : M \to M$ was given by

$$g_\zeta = \rho \circ h_\zeta \circ \rho^{-1},\ \zeta \in S^1. \tag{16}$$

It is clear that g_ζ is a contact action because ρ is a contact mapping by property 3 and h_ζ is a contact action on S^3. Furthermore g_ζ is transverse since $\rho|_{S^3} : S^3 \to M$ is a diffeomorphism and h_ζ is transverse on S^3. For the the first statement it therefore suffices to show that $\mu(\zeta)$ extends holomorphically to Δ. This follows along the same lines as in **[7]** but we present the argument for the convenience of the reader.

Let us recall that $\mu(\zeta)$ is given by

$$g_{\zeta *}\bar{X} + \mu(\zeta)g_{\zeta *}X \in H^{0,1}_{g_\zeta(p)}M,\ \zeta \in S^1 \tag{17}$$

where $X = Y + \bar{\nu}\bar{Y} \in \mathbf{C} \otimes H_pM$. Now $X = Y_1 + iY_2$ for some $Y_1, Y_2 \in H_pM$ where Y_1 and Y_2 are independent over $\mathbf{R}$.

Applying the projection map $\pi^{1,0} : \mathbf{C} \otimes H_{g_\zeta(p)}M \to H^{1,0}_{g_\zeta(p)}M$ to (17) we obtain

$$\pi^{1,0}g_{\zeta *}Y_1 - i\pi^{1,0}g_{\zeta *}Y_2 + \mu(\zeta)[\pi^{1,0}g_{\zeta *}Y_1 + i\pi^{1,0}g_{\zeta *}Y_2] = 0, \tag{18}$$

and thus (18) determines $\mu(\zeta)$ for $\zeta \in S^1$. To obtain a holomorphic extension we extend (18) to $\zeta \in \Delta$. Namely, we denote by $\xi(\zeta) = \pi^{1,0}g_{\zeta *}(Y_1)$ and $\eta(\zeta) = \pi^{1,0}g_{\zeta *}(Y_2)$ and we see as in **[7]** that ξ and η are holomorphic sections of the pullback bundle

$g_\zeta^* H^{1,0}M$. Our purpose is to show that (18) has then a solution for $\mu(\zeta)$ for $\zeta \in \Delta$ and the function $\mu : \Delta \to \mathbf{C}$ is holomorphic.

We can rewrite (18) in the form

$$\xi(\zeta) = i\frac{1-\mu(\zeta)}{1+\mu(\zeta)}\eta(\zeta), \quad \zeta \in \Delta. \tag{19}$$

If we denote $\alpha = i\frac{1-\mu}{1+\mu}$ we obtain $\xi(\zeta) = \alpha(\zeta)\eta(\zeta)$, $\zeta \in \Delta\backslash\{0\}$. Because ξ and η are holomorphic sections of $g_\zeta^* H^{1,0}M$ and the fibers of $H^{1,0}M$ have complex dimension 1 we conclude that (19) has a holomorphic solution $\alpha : \Delta\backslash\{0\} \to \mathbf{C}$. The point $\zeta = 0$ is troublesome as ξ and η vanish there.

On the other hand $\mu(\zeta) = \frac{i-\alpha(\zeta)}{i+\alpha(\zeta)}$ and thus we obtain that μ is a holomorphic function on $\Delta\backslash\{0\}$ if we show that $\alpha(\zeta) \neq -i$ for $\zeta \in \Delta\backslash\{0\}$. To see this let us assume $\alpha(\zeta_0) = -i$ for some $\zeta_0 \in \Delta\backslash\{0\}$. This gives

$$\pi^{1,0} g_{\zeta_0 *} X(p) = \pi^{1,0} g_{\zeta_0 *}(Y_1 + iY_2) = 0. \tag{20}$$

But then we use again (12) and write $g_{\zeta *}(X(p)) = \rho_*((\zeta/\bar{\zeta})Z(\zeta(z,w)))$. By definition $\rho : B_r \to D_r$ is a contact mapping for any $0 < r \leq 1$. Therefore

$$g_{\zeta *}(X(p)) = \frac{\zeta}{\bar{\zeta}}[Y(\rho(\zeta(z,w))) + \bar{\nu}(\zeta)\bar{Y}(\rho(\zeta(z,w)))] \tag{21}$$

with $Y(\rho(\zeta(z,w))) \in H^{1,0}_{\rho(\zeta(z,w))}M_{|\zeta|}$. Consequently (20) and (21) would imply

$$g_{\zeta_0 *} X(p) = 0$$

which is impossible, g_{ζ_0} being a diffeomorphism.

If we show that $|\mu| < 1$ on $\Delta\backslash\{0\}$ we can remove the isolated singularity $\xi = 0$ and conclude that μ is holomorphic on Δ. It is enough to show that $|\mu(\zeta)| \neq 1$ for $\zeta \in \Delta\backslash\{0\}$. Assuming $|\mu(\zeta_0)| = 1$ for some $\zeta_0 \in \Delta\backslash\{0\}$ we get $\alpha(\zeta_0) \in \mathbf{R}$ and $\xi(\zeta_0)$, $\eta(\zeta_0)$ are dependent over $\mathbf{R}$. But this is again impossible since $Y_1, Y_2 \in H_pM$ are independent and g_{ζ_0} is a diffeomorphism.

Finally, we note that the above arguments give now (11) for general ζ since μ satisfies (16) (and therefore (17)) in $\Delta\backslash\{0\}$. We can therefore use (21) and the same consideration as for $\zeta \in S^1$ to obtain

$$\nu(\zeta)\frac{\bar{\zeta}}{\zeta} + \mu(\zeta)\frac{\zeta}{\bar{\zeta}} = 0, \quad \zeta \in \bar{\Delta}\backslash\{0\}$$

which proves (11). □

Corollary 4. — *Consider $Y = \pi^{1,0}\rho_* Z$ as a vector field on M and define the action $g_\zeta(p) = \rho(\zeta\rho^{-1}(p))$ as above. Then*

$$g_{\zeta *} Y = Y' + \bar{\eta}(\zeta)\bar{Y}' \tag{22}$$

where η is holomorphic in Δ.

Proof. — Determining the Lempert-Beltrami coefficient μ of the trajectory of p by the vector $X_p = Y_p + \bar{\nu}(1)Y_p$ as above, we have from the argument in (17), (18) that

$$\pi^{1,0}[\mu(\zeta)g_{\zeta *}X + g_{\zeta *}\bar{X}] = 0$$

for all points $\zeta \in \Delta$. Since by assumption (22)

$$\pi^{1,0}[\mu(\zeta)g_{\zeta *}X + g_{\zeta *}\bar{X}] = (\mu(\zeta) + \mu(\zeta)\eta(\zeta)\bar{\nu}(1) + \eta(\zeta) + \nu(1))Y'_{g_\zeta(p)},$$

we must therefore have that

$$\eta(\zeta) = -\frac{\mu(\zeta) + \nu(1)}{1 + \bar{\nu}(1)\mu(\zeta)}.$$

As g_ζ is an inner action, it follows that $\eta(\zeta)$ is holomorphic. □

3. Holomorphic motions and Kobayashi indicatrix

The proof of Theorem 1 can now be obtained quickly from the results of the previous section. Indeed, the Lempert mapping gives a natural homeomorphism R from the unit ball B to the Kobayashi indicatrix $\bar{I}$ of the domain D,

$$R(w) = \frac{d}{d\zeta}\rho(\zeta w)_{|\zeta=0}.$$

The mapping R is **C**-homogenous of degree 1 on complex lines through the origin and as shown in **[5]**, **[11]** it is a contact transformation on ∂B. Therefore our holomorphic motion of the boundary $\partial\bar{I}$ of the indicatrix,

$$\Psi(\lambda, v) = \frac{f_v(\lambda)}{\lambda}, \qquad (\lambda, v) \in \Delta \times \partial\bar{I},$$

admits a natural factorization

$$\Psi(\lambda, v) = \frac{1}{\lambda} g_\lambda \circ \sigma(v),$$

where we have used the abreviation

$$\sigma = \rho|_{\partial B} \circ R^{-1}|_{\partial\bar{I}}.$$

Let $Y = \pi^{1,0}\rho_* Z$ be the vector field as above in Corollary 4. Since $\sigma : \partial\bar{I} \to \partial D$ is a smooth contact transformation we find a vector field W, $W(x) \in H_x^{1,0}\partial\bar{I}$ for all $x \in \partial\bar{I}$, such that

$$\sigma_* W = Y + \bar{\kappa}\bar{Y}$$

on $\partial\bar{I}$.

With these tools we can now estimate the quasiconformal dilatation of the holomorphic motions $\Psi(\lambda, \cdot)$. Namely by (8) one has to bound their Beltrami differentials and for this we have from Corollary 4 that

$$\Psi_*(\lambda, \cdot)W = \frac{1}{\lambda}(g_\lambda)_*(Y + \bar{\kappa}\bar{Y}) \tag{23}$$

$$= \frac{1}{\lambda}(Y' + \bar{\eta}(\lambda)\bar{Y}' + \bar{\kappa}\bar{Y}' + \eta(\lambda)\bar{\kappa}Y') \tag{24}$$

$$= Y'' + \frac{\bar{\lambda}}{\lambda} \cdot \frac{\bar{\eta}(\lambda) + \bar{\kappa}}{1 + \bar{\eta}(\lambda)\kappa} Y'',$$

where $Y'' = \lambda^{-1}(1+\eta(\lambda)\bar{\kappa})Y'$. Consequently, the absolute values satisfy $K(\Psi_*(\lambda, \cdot)) = (1 + |\delta(\lambda)|)/(1 - |\delta(\lambda)|)^{-1}$, where

$$\delta(\lambda) = \frac{\eta(\lambda) + \kappa}{1 + \eta(\lambda)\bar{\kappa}}. \tag{25}$$

Since by Corollary 4, η is holomorphic in the unit disk with $||\eta||_\infty \leq 1$, the same holds true for δ as well. On the other hand, as $\Psi_*(0, \cdot)$ is the identity mapping, $\delta(0) = 0$. By Schwartz lemma we then get $|\delta(\lambda)| \leq |\lambda|$ which completes the proof of Theorem 1.

Remark. — We would like to mention at the end that the results of Theorems 1 and 3 are true in more general situations than for strictly linearly convex domains. Whenever we have a Lempert mapping with the properties stated at the beginning the proofs carry over. In particular, the new work **[2]** finds the Lempert mappings for the larger class named circular-like domains; these are characterized by the fact that their Lempert invariants are bounded by 1.

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