Astérisque

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Astérisque, tome 262 (2000)

<http://www.numdam.org/item?id=AST_2000__262__R1_0>

ASTÉRISQUE 262

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Société Mathématique de France 2000
Publié avec le concours du Centre National de la Recherche Scientifique

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1991 *Mathematics Subject Classification.* — 35P25, 58G25, 81U10.
Key words and phrases. — Propagation of singularities, 3-body scattering, scattering matrix, wave front relation, broken geodesics.

PROPAGATION OF SINGULARITIES IN THREE-BODY SCATTERING

András Vasy

***Abstract.* —** In this paper we consider a compact manifold with boundary X equipped with a scattering metric g and with a collection C_i of disjoint closed embedded submanifolds of ∂X. Thus, g is a Riemannian metric in $\mathrm{int}(X)$ of the form $g = x^{-4}\,dx^2 + x^{-2}h$ near ∂X for some choice of a boundary defining function x, h being a smooth symmetric 2-cotensor on X which is non-degenerate when restricted to ∂X. We also let Δ be the (positive) Laplacian of g, suppose that $V \in \mathcal{C}^\infty([X; \cup_i C_i])$ where $[X; \cup_i C_i]$ is X blown up along the C_i, assume that V vanishes at the lift of ∂X, and consider the operator $H = \Delta + V$. Three-body scattering with smooth potentials which have an asymptotic expansion at infinity (possibly Coulomb-type) provide the standard example of this setup. We analyze the propagation of singularities of generalized eigenfunctions of H, showing that this is essentially a hyperbolic problem which has much in common with the Dirichlet and transmission problems for the wave operator, though additional features arise due to the presence of bound states of the 'two-body operators'. We also show that the wave front relation of the free-to-free part of the scattering matrix is given by the broken geodesic flow at distance π.

***Résumé* (Propagation des singularités dans la diffusion à trois corps)**

Dans cet article, nous considérons une variété X compacte à bord, munie d'une métrique de diffusion g et d'une famille C_i de sous-variétés fermées de ∂X deux à deux disjointes. Ainsi, g est une métrique riemannienne dans $\mathrm{int}(X)$ de la forme $g = x^{-4}\,dx^2 + x^{-2}h$ près de ∂X, pour un choix convenable de fonction x définissant le bord, h étant un 2-cotenseur symétrique $\mathcal{C}^\infty$ sur X qui est non dégénéré en restriction à ∂X. Nous notons aussi Δ le laplacien (positif) de g, choisissons $V \in \mathcal{C}^\infty([X; \cup_i C_i])$, où $[X; \cup_i C_i]$ est X éclaté le long des C_i, tel que V s'annule sur le relevé de ∂X, et nous considérons l'opérateur $H = \Delta + V$. Les diffusions à trois corps avec potentiels $\mathcal{C}^\infty$ qui ont un développement asymptotique à l'infini (éventuellement du type de Coulomb) sont des exemples de cette situation. Nous analysons la propagation des singularités des fonctions propres généralisées de H en montrant que c'est essentiellement un problème hyperbolique qui a beaucoup en commun avec les problèmes de Dirichlet et de transmission pour l'opérateur des ondes, avec néanmoins des propriétés supplémentaires dues à la présence d'états bornés des opérateurs à deux corps. Nous

montrons aussi que la relation de front d'onde de la partie libre-libre de la matrice de diffusion est donnée par le flot des géodésiques brisées à distance π.

CONTENTS

CHAPTER 1

INTRODUCTION

Let X be a compact manifold with boundary. In **[25]** Melrose has defined the algebra $\mathrm{Diff}_{\mathrm{sc}}(X)$ of scattering differential operators on X. In fact, let $x \in \mathcal{C}^\infty(X)$ be a boundary defining function of X, so $x \geq 0$, $dx \neq 0$ on ∂X, and $\partial X = \{x = 0\}$. The Lie algebra of b-vector fields on X, $\mathcal{V}_{\mathrm{b}}(X)$, is the set of all smooth vector fields on X which are tangent to ∂X. The Lie algebra of scattering vector fields on X, $\mathcal{V}_{\mathrm{sc}}(X)$, is simply $\mathcal{V}_{\mathrm{sc}}(X) = x\mathcal{V}_{\mathrm{b}}(X)$; this notion is independent of the choice of the boundary defining function x. Much as in the case of $\mathcal{V}_{\mathrm{b}}(X)$, $\mathcal{V}_{\mathrm{sc}}(X)$ is the set of all smooth sections of a vector bundle over X; this bundle is denoted by ${}^{\mathrm{sc}}TX$. Finally, $\mathrm{Diff}_{\mathrm{sc}}(X)$ is just the enveloping algebra of $\mathcal{V}_{\mathrm{sc}}(X)$, i.e. the ring of operators on $\mathcal{C}^\infty(X)$ generated by $\mathcal{C}^\infty(X)$ (considered as multiplication operators) and $\mathcal{V}_{\mathrm{sc}}(X)$. An example of such an operator is the Laplacian Δ associated to a scattering metric g. Thus, g is a Riemannian metric in $\mathrm{int}(X)$ of the form $g = x^{-4}\,dx^2 + x^{-2}h$ near ∂X for some choice of a boundary defining function x, h being a smooth symmetric 2-cotensor on X which is non-degenerate when restricted to ∂X. In particular, g is a metric on ${}^{\mathrm{sc}}TX$.

Let C_i, $i = 1, \ldots, k$, be disjoint closed embedded submanifolds of ∂X. Here the C_i might have different dimensions. Nevertheless, to simplify the notation, we introduce $C = \cup_i C_i$, and say that C is also a closed embedded submanifold of ∂X, although this is strictly speaking only true if the dimensions of the connected components of C are the same. Let mf ('main face') be the lift of ∂X to $[X; C]$, the blow-up of X along C (see the Appendix of **[25]** for a treatment of blow-ups, and see Figure 1 for a picture). We write ρ_{mf} for a defining function of mf. The 'three-body type' operators we are interested in are perturbations H of Δ of the form $H = \Delta + V$, where $V \in \mathcal{C}^\infty([X; C])$ is real-valued and vanishes at mf. As discussed in the following paragraphs, three-body Hamiltonians, with the center of mass removed, give an example of such operators, and explain our interest in the problem. In the degenerate case when $k = 0$, i.e. $C = \varnothing$, we arrive at the generalized 'two-body type' scattering considered in Melrose's original paper **[25]**; in this case $V \in x\mathcal{C}^\infty(X)$.

Consider the Euclidian space, $\mathbb{R}^N$, with the standard metric, and its radial compactification to the upper hemisphere $\mathbb{S}^N_+$. Embedding $\mathbb{S}^N_+$ in $\mathbb{R}^{N+1}$ as the unit upper hemisphere this is given by the map $\mathrm{SP} : \mathbb{R}^N \to \mathbb{S}^N_+$

$$\mathrm{SP}(z) = \left(\frac{1}{(1+|z|^2)^{1/2}}, \frac{z}{(1+|z|^2)^{1/2}}\right). \tag{1.1}$$

Let x be a boundary defining function of $\mathbb{S}^N_+$ such that $x = (\mathrm{SP}^{-1})^*|z|^{-1}$ near $\partial\mathbb{S}^N_+$. Then the Euclidian metric pulls back to a scattering metric on $\mathbb{S}^N_+$, with h being the standard metric on $\mathbb{S}^{N-1} = \partial\mathbb{S}^N_+$, and the Euclidian Laplacian becomes an element of $\mathrm{Diff}^2_{\mathrm{sc}}(\mathbb{S}^N_+)$.

Let X_i, $i = 1, \ldots, k$, be linear subspaces of $\mathbb{R}^N$, let X^i be the orthocomplement of X_i, $n_i = \dim X^i$, and let π^i be the orthogonal projection to X^i. By a Euclidian many-body Hamiltonian we mean an operator of the form $H = \Delta + \sum_i (\pi^i)^* V_i$ where $V_i \in \mathcal{C}^\infty(X^i; \mathbb{R})$ satisfy $(\mathrm{SP}_i^{-1})^* V_i \in \rho_i \mathcal{C}^\infty(\mathbb{S}^{n_i}_+)$ with ρ_i denoting a boundary defining function of $\mathbb{S}^{n_i}_+$, and SP_i being the radial compactification map $\mathrm{SP}_i : X^i \to \mathbb{S}^{n_i}_+$. The condition on V_i means that it is a one-step polyhomogeneous symbol on X^i of order -1. A Euclidian three-body Hamiltonian (with center of mass removed) is a many-body Hamiltonian with the additional assumption that $X_i \cap X_j = \{0\}$ for $i \neq j$. In the compactified picture, writing $\overline{X_i} = \mathrm{cl}(\mathrm{SP}(X_i)) \subset \mathbb{S}^N_+$, $C_i = \overline{X_i} \cap \mathbb{S}^{N-1}$, the condition $X_i \cap X_j = \{0\}$ for $i \neq j$ becomes $C_i \cap C_j = \varnothing$ for $i \neq j$. With the notation $C = \cup_i C_i$ as in the general case, it is straightforward to check that

$$V = (\mathrm{SP}^{-1})^* \sum_i (\pi^i)^* V_i \in \mathcal{C}^\infty([\mathbb{S}^N_+; C]), \qquad V|_{\mathrm{mf}} = 0 \tag{1.2}$$

(this will be done here in Lemma 2.1), so H is indeed a 'three-body type' operator as described above in the geometric setting. Note also that the C_i are 'subspheres' of $\mathbb{S}^{N-1}$, in particular, they are totally geodesic with respect to the standard metric. A two-body Hamiltonian corresponds to taking $k = 1$, $X_1 = \{0\}$ above, so we have $V \in x\mathcal{C}^\infty(\mathbb{S}^N_+)$, giving rise to the 'two-body type' terminology in the geometric setting. In Figure 1 below we take $N = 2$ and the X_i are lines. Hence, $X = \mathbb{S}^2_+$ is a disk, $\partial X = \mathbb{S}^1$, each C_i consists of two points. The lift of C_i to $[X; C]$ is denoted by ff_i in the figure.

Now we return to the general setting. First note that $H = \Delta + V$ is self-adjoint on $L^2_{\mathrm{sc}}(X)$, the L^2 space defined by integration with respect to the Riemannian density dg, since Δ and V are such and V is bounded. Hence, its resolvent $R(\lambda) = (H - \lambda)^{-1}$ is a bounded linear operator on $L^2_{\mathrm{sc}}(X)$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In this paper we analyze the boundary value of the resolvent at the real axis, i.e. $R(\lambda \pm i0)$. We show that $\mathrm{spec}_p(H) \cap (0, \infty) = \varnothing$ and

$$R(\lambda \pm i0) \in \mathcal{B}(x^{1/2+\epsilon} L^2_{\mathrm{sc}}(X), x^{-1/2-\epsilon} L^2_{\mathrm{sc}}(X)) \tag{1.3}$$

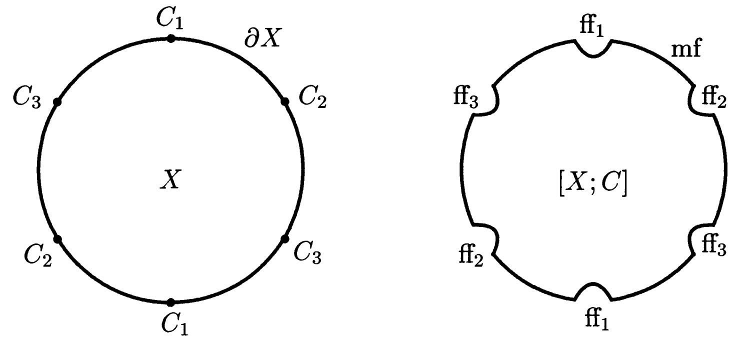

FIGURE 1. The original space X and its resolution $[X;C]$.

for all $\varepsilon > 0$. This is completely analogous to the classical result of Mourre in Euclidian three-body scattering (**[30, 31]**, see also the paper **[32]** of Perry, Sigal and Simon in which they extend Mourre's results to many-body systems), together with the absence of positive eigenvalues which was shown by Froese and Herbst **[8]** in the Euclidian case.

We also show that for $f \in \dot{\mathcal{C}}^\infty(X)$, $R(\lambda \pm i0)f$ has a complete asymptotic expansion away from C which is similar to the corresponding expansion for Euclidian two-body Hamiltonians. Here $\dot{\mathcal{C}}^\infty(X)$ is the subspace of $\mathcal{C}^\infty(X)$ consisting of functions which vanish at ∂X together with all of their derivatives. For simplicity here we only state the asymptotic expansion if $V \in \rho_{\mathrm{mf}}^2 \mathcal{C}^\infty([X;C])$ (i.e. short-range); the general case is described in Theorem 18.6. It is convenient to replace the spectral parameter λ by λ^2. Then, for $\lambda > 0$, $f \in \dot{\mathcal{C}}^\infty(X)$, the expansion can be described by

$$v_\pm = e^{\pm i\lambda/x} x^{-(N-1)/2} R(\lambda^2 \mp i0) f \in \mathcal{C}^\infty(X \setminus C). \tag{1.4}$$

The top term of such an expansion for Euclidian three-body scattering was described by Isozaki in **[21]**, assuming that the potentials were short range, by Herbst and Skibsted in **[16]** in the long-range many-body Euclidian case, and the full expansion was proved by the author in **[38]**. Moreover, we show that given any 'initial data' $a_0 \in \mathcal{C}_c^\infty(\partial X \setminus C)$ we can find $f \in \dot{\mathcal{C}}^\infty(X)$ such that with v_- as above we have $v_- \in \mathcal{C}^\infty(X)$ and $a_0 = v_-|_{\partial X}$. Then

$$u = R(\lambda^2 + i0)f - R(\lambda^2 - i0)f \in \mathcal{C}^{-\infty}(X) \tag{1.5}$$

satisfies $(H - \lambda)u = 0$, and has the form

$$u = e^{i\lambda/x} x^{(N-1)/2} v_- - e^{-i\lambda/x} x^{(N-1)/2} v_+. \tag{1.6}$$

For $\lambda > 0$ the Poisson operator corresponding to 'free initial data' is the map $P(\lambda) : \mathcal{C}_c^\infty(\partial X \setminus C) \to \mathcal{C}^{-\infty}(X)$ given by $P(\lambda)a_0 = u$. This definition is justified by the uniqueness statement of Theorem 19.1 which is again an analog of Isozaki's result **[22]**. The free-to-free part of the scattering matrix, $S(\lambda)$, relates the leading part of the expansions in (1.6) at $\partial X \setminus C$. Thus, $S(\lambda)$ is the map

$$S(\lambda) : \mathcal{C}_c^\infty(\partial X \setminus C) \to \mathcal{C}^\infty(\partial X \setminus C) \tag{1.7}$$

given by

$$S(\lambda)a_0 = -v_+|_{\partial X \setminus C}, \qquad a_0 \in \mathcal{C}_c^\infty(\partial X \setminus C). \tag{1.8}$$

The equivalence of this definition of the scattering matrix with the customary one involving the wave operators was proved by Hassell **[14]** for short-range interactions. Here, in Remark 19.12, we sketch a somewhat different proof based on Isozaki's argument in **[20]**.

Our main theorem describes the structure of $S(\lambda)$. We first introduce the broken geodesic flow of $h|_{\partial X}$ on ∂X, broken at C. For simplicity we only define this here if C is totally geodesic; for the general definition see Definition 11.6 and the remarks preceeding it. Let $I \subset \mathbb{R}$ be an interval, and let B be a discrete subset. We denote by $S\partial X$ the sphere bundle of ∂X identified as the unit-length subbundle of $T\partial X$ with respect to $h|_{\partial X}$ (we drop the restriction in the notation from now on). We say that a curve $\gamma : I \to \partial X$ is a broken geodesic of h if two conditions are satisfied. First, for all intervals $J \subset I \setminus B$, $\gamma|_J$ is a geodesic of h, such that for all $t \in J$, $\gamma'(t) \in S\partial X$. Second, if $t \in B$ then $\gamma(t) \in C$ and the limits $\gamma'(t-0)$ and $\gamma'(t+0)$ both exist and differ by a vector in $T_{\gamma(t)}\partial X$ which is orthogonal to $T_{\gamma(t)}C$ (i.e. the usual law of reflection is satisfied; see Figure 2). We say that $p, q \in S\partial X$ are related by the broken geodesic flow at time $-\pi$ if there is a broken geodesic γ defined on $[-\pi, 0]$, such that $\gamma'(0) = p$, $\gamma'(-\pi) = q$. Using the metric h to identify $S\partial X$ and $S^*\partial X$, this defines the broken geodesic flow at time $-\pi$ on $S^*\partial X$. We then have the following result:

Theorem. — *For $\lambda > 0$ the wave front relation of the free-to-free part of the scattering matrix, $S(\lambda)$, is given by the broken geodesic flow of $h|_{\partial X}$ on ∂X, broken at C, at time $-\pi$.*

This theorem was conjectured by Melrose based on his work with Zworski in the generalized 'two-body type' setting **[25, 29]**. As mentioned above, this just means that we take $C = \varnothing$. The result of Melrose and Zworski was that $S(\lambda)$ is a Fourier integral operator associated to the geodesic flow on ∂X at time $-\pi$, from which our Theorem follows when $C = \varnothing$.

In the case of Euclidian three-body scattering with rapidly decreasing two-body potentials a somewhat stronger result than the Theorem has been proved by the author in **[39]** by an explicit construction resembling Faddeev's original one **[7]**; namely the scattering matrix was shown to be a sum of Fourier integral operators associated to the broken geodesic flow. Using different methods, which are closer to those of Melrose and Zworski in **[29]**, Hassell has shown in **[13]** that the same conclusion holds. In addition, Hassell's construction states explicitly that the kernel of the Poisson operator is a sum of Legendrian distributions associated to conic Legendrian pairs.

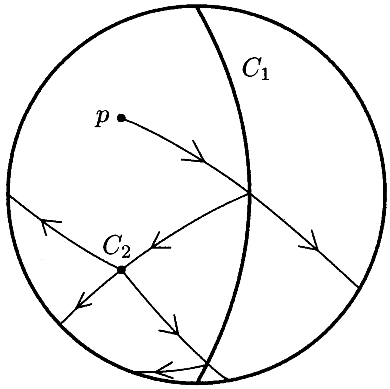

FIGURE 2. Broken geodesics on $\partial X = \mathbb{S}^2$ starting at p.

We also remark that there are other interesting operators associated to this geometry; one example is Christiansen's analysis of scattering in perturbed stratified media **[3]**.

A major difference between two-body and three-body type scattering is that in the latter case the range of $P(\lambda)$, considered as an operator on $\mathcal{C}_c^\infty(\partial X \setminus C)$, may not be dense in the nullspace of $H - \lambda$ on $\mathcal{C}^{-\infty}(X)$. Apart from those corresponding to 'free initial data', essentially characterized by restriction of their expansion to mf, there are generalized eigenfunctions of $H - \lambda$ corresponding to 'two-body bound states'; in the case of Euclidian three-body scattering these arise from eigenfunctions of $\Delta_{X^i} + V_i$ in $L^2(X^i)$. In the Euclidian setting these are easier to describe than those coming from free initial data; this was done by Isozaki **[20, 21]** and Skibsted **[37]** for short-range potentials, and by Bommier **[2]** for long-range potentials in a more general Euclidian many-body setting. Due to the lack of product structure, this task is much harder in the geometric setting, and we only prove the propagation of singularities of generalized eigenfunctions along bicharacteristics under additional assumptions. These assumptions guarantee that the spectrum of the two-body operators is constant along C, and are satisfied in the Euclidian setting. Even in these cases we do not treat the Poisson operator with initial data in a two-body bound state and the corresponding pieces of the scattering matrix. Hence, we do not consider whether every generalized eigenfunction arises from a combination of 'free' and 'two-body bound' initial data. An L^2 version of this statement is called asymptotic completeness in the Euclidian case; it was proved by Enss **[5, 6]** for both short-range and long-range three-body scattering. In the many-body setting these were proved by Sigal and Soffer **[34, 35]**, Graf **[11]**, Dereziński **[4]** and Yafaev **[41]**.

To see why a result such as the above Theorem should hold, consider first the operator $\Delta - \lambda$, and its analysis in Melrose's paper **[25]**. There is a principal symbol map

$$\sigma_{\mathrm{sc},m} : \mathrm{Diff}^m_{\mathrm{sc}}(X) \to S^m_h({}^{\mathrm{sc}}T^*X), \tag{1.9}$$

$S^m_h({}^{\mathrm{sc}}T^*X)$ denoting the space of homogeneous functions of degree m on ${}^{\mathrm{sc}}T^*X \setminus 0$; this is completely analogous to the principal symbol map on compact manifolds without boundary. We have $\sigma_{\mathrm{sc},2}(\Delta - \lambda) = |\zeta|^2$, $|\cdot|$ denoting the metric function on ${}^{\mathrm{sc}}T^*X$, the dual bundle of ${}^{\mathrm{sc}}TX$. This is independent of λ, and it is elliptic in the usual sense, i.e. it has an inverse in $S^{-2}_h({}^{\mathrm{sc}}T^*X)$. However, $\sigma_{\mathrm{sc},m}$ does not capture the behavior of $\mathrm{Diff}^m_{\mathrm{sc}}(X)$ completely, such as its compactness properties between certain Sobolev spaces. In fact, there is a symbol map, $\widehat{N}_{\mathrm{sc}}$, at ∂X as well, mapping

$$\widehat{N}_{\mathrm{sc}} : \mathrm{Diff}_{\mathrm{sc}}(X) \to \mathcal{C}^\infty({}^{\mathrm{sc}}T^*_{\partial X}X). \tag{1.10}$$

Now, $\widehat{N}_{\mathrm{sc}}(\Delta - \lambda) = |\zeta|^2 - \lambda$, i.e. λ is not lower order than Δ in this sense, meaning that it appears in $\widehat{N}_{\mathrm{sc}}(\Delta - \lambda)$. Hence, for $\lambda \geq 0$, $\widehat{N}_{\mathrm{sc}}(\Delta - \lambda)$ is not invertible in $\mathcal{C}^\infty({}^{\mathrm{sc}}T^*_{\partial X}X)$, so $\Delta - \lambda$ is not fully elliptic. *This* gives rise to scattering theory.

It is perhaps useful to remark that if $X = \mathbb{S}^N_+$ is the radial compactification of $\mathbb{R}^N$, $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^N$ (regarded as an operator acting on $\mathcal{C}^{-\infty}(\mathbb{S}^N_+)$) and $P \in \mathrm{Diff}^m_{\mathrm{sc}}(\mathbb{S}^N_+)$ then $\widetilde{P} = \mathcal{F}P\mathcal{F}^{-1} \in \Psi^0(\mathbb{R}^N)$; here $\Psi^0(\mathbb{R}^N)$ is the standard pseudo-differential calculus on $\mathbb{R}^N$. (In fact, $\widetilde{P} \in \Psi^{0,-m}_{\mathrm{sc}}(\mathbb{S}^N_+)$; $\Psi^{\infty,-\infty}_{\mathrm{sc}}(\mathbb{S}^N_+)$ is the microlocalization of $\mathrm{Diff}_{\mathrm{sc}}(\mathbb{S}^N_+)$, see **[25]**.) Moreover, $\widehat{N}_{\mathrm{sc}}(P)$ is closely related to $\sigma_0(\widetilde{P})$. Namely, with $L : {}^{\mathrm{sc}}T^*_{\mathbb{S}^{N-1}}\mathbb{S}^N_+ \to S^*\mathbb{R}^N$ being the Legendre diffeomorphism given in **[29**, Lemma 5], we have $\sigma_0(\widetilde{P}) = (L^{-1})^*\widehat{N}_{\mathrm{sc}}(P)$. If $P = \Delta - \lambda$, Δ being the standard Euclidian Laplacian, we have $\widetilde{P} = |\zeta|^2 - \lambda \in \mathrm{Diff}^0(\mathbb{R}^N)$ (i.e. a multiplication operator). We see directly here that $\widetilde{P}$ is not elliptic (in the usual sense) if $\lambda \geq 0$ since its principal symbol vanishes on the sphere $|\zeta|^2 = \lambda$. This in turn implies that $\widehat{N}_{\mathrm{sc}}(\Delta - \lambda)$ is not invertible, as discussed above. Incidentally, the argument we just described works even if g is an arbitrary scattering metric on $\mathbb{S}^N_+$ (but then, with $P = \Delta - \lambda$, $\widetilde{P} \in \Psi^0(\mathbb{R}^N)$ is not necessarily just a multiplication operator), or indeed if X is an arbitrary compact manifold with boundary; in the latter case we need to identify X with $\mathbb{S}^N_+$ locally near some point p on ∂X, and correspondingly only the restriction of $\widehat{N}_{\mathrm{sc}}(P)$ to a neighborhood of ${}^{\mathrm{sc}}T^*_pX$ is realized as a principal symbol of an element of $\Psi^0(\mathbb{R}^N)$.

Returning to the geometric setting, the lack of invertibility of $\widehat{N}_{\mathrm{sc}}(\Delta - \lambda)$ implies that generalized eigenfunctions of $\Delta - \lambda$ need not be 'trivial', i.e. they are not necessarily in $\dot{\mathcal{C}}^\infty(X)$. They are certainly smooth in the interior of X since $\sigma_{\mathrm{sc},2}(\Delta - \lambda)$ is elliptic, but their behavior at ∂X is much more complicated. Just as for interior singularities, the failure of a distribution $u \in \mathcal{C}^{-\infty}(X)$ to be in $\dot{\mathcal{C}}^\infty(X)$, i.e. its 'singularities', can be measured by a wave front set, $\mathrm{WF}_{\mathrm{sc}}(u)$. Corresponding to the symbol maps of $\mathrm{Diff}_{\mathrm{sc}}(X)$, this consists of two parts: one part is an extension of the usual wave front set from the interior to give a subset of the cosphere bundle S^*X, the other part at the boundary is a subset of ${}^{\mathrm{sc}}T^*_{\partial X}X$. The first part describes the smoothness properties of u, the second part its decay properties at ∂X. Due to the ellipticity of $\sigma_{\mathrm{sc},2}(\Delta-\lambda)$, $(\Delta-\lambda)u=0$ implies that $\mathrm{WF}_{\mathrm{sc}}(u)\subset {}^{\mathrm{sc}}T^*_{\partial X}X$.

The singularities of generalized eigenfunctions of $\Delta-\lambda$ were analyzed by Melrose in **[25]**. To facilitate this analysis, let x be the boundary defining function used in the definition of g, and let y_j be local coordinates on ∂X, extended to X. Then a covector $v\in {}^{\mathrm{sc}}T^*_pX$, p near ∂X, can be written as $v=\tau x^{-2}\,dx+\mu\cdot x^{-1}\,dy$. Hence, we have local coordinates (x,y,τ,μ) on ${}^{\mathrm{sc}}T^*X$ near ∂X. In these coordinates

$$\widehat{N}_{\mathrm{sc}}(\Delta-\lambda)=\tau^2+|\mu|^2-\lambda; \tag{1.11}$$

here $|\cdot|$ is the metric function of $h|_{\partial X}$. The characteristic set, $\Sigma_{\Delta-\lambda}\subset {}^{\mathrm{sc}}T^*_{\partial X}X$, of $\Delta-\lambda$ is the set where $\widehat{N}_{\mathrm{sc}}(\Delta-\lambda)$ vanishes. Just as in the case of operators on compact manifolds without boundary, there is a (rescaled) Hamilton vector field associated to operators $P\in\mathrm{Diff}_{\mathrm{sc}}(X)$. Its restriction to ${}^{\mathrm{sc}}T^*_{\partial X}X$ is denoted by ${}^{\mathrm{sc}}H_p$, and it only depends on $p=\widehat{N}_{\mathrm{sc}}(P)$. It is related to the commutator $[P,Q]$ for $P,Q\in\mathrm{Diff}_{\mathrm{sc}}(X)$. Indeed, $[P,Q]\in x\,\mathrm{Diff}_{\mathrm{sc}}(X)$, and $\widehat{N}_{\mathrm{sc}}(x^{-1}[P,Q])=\frac{1}{i}{}^{\mathrm{sc}}H_pq$. Correspondingly, as expected, ${}^{\mathrm{sc}}H_\lambda=0$, and with g denoting the metric function on ${}^{\mathrm{sc}}T^*_{\partial X}X$, the Hamilton vector field of $\Delta-\lambda$ is just ${}^{\mathrm{sc}}H_g$. There are two disjoint submanifolds of $\Sigma_{\Delta-\lambda}$ where ${}^{\mathrm{sc}}H_g$ vanishes, namely

$$R^{\pm}_\lambda=\{(y,\tau,\mu)\in\Sigma_{\Delta-\lambda}:\ \mu=0,\ \tau=\pm\lambda^{1/2}\}; \tag{1.12}$$

these are called 'radial surfaces'. The integral curves $\gamma(t)$ of ${}^{\mathrm{sc}}H_g$ approach $R^{\pm}_\lambda$ as $t\to\mp\infty$. The closure of the projection of each integral curve $\gamma(t)$ to ∂X gives a geodesic segment of $h|_{\partial X}$ of length π after reparametrization. Now, away from $R^{\pm}_\lambda$, where ${}^{\mathrm{sc}}H_g$ does not vanish, we have principal type propagation of singularities just as for hyperbolic operators on manifolds without boundary – in fact, we should think of $\Delta-\lambda$ as a hyperbolic operator at ∂X. Such a correspondence is made explicit by the Fourier transform if $X=\mathbb{S}^N_+$ is the radial compactification of $\mathbb{R}^N$, and by a localized version of the Fourier transform in the general case as described in the previous paragraphs. Just as in the standard case of manifolds without boundary, the propagation results can be obtained by positive commutator estimates; this is the significance of ${}^{\mathrm{sc}}H_g$. The singularities of the scattering matrix then correspond to singularities propagating from R^-_λ to R^+_λ along the bicharacteristics.

If we add a potential $V\in x\mathcal{C}^\infty(X)$ and consider $H-\lambda=\Delta+V-\lambda$, then $\widehat{N}_{\mathrm{sc}}(H-\lambda)=\widehat{N}_{\mathrm{sc}}(\Delta-\lambda)$, so in the region of principal type propagation the previous analysis applies; again, this is described in **[25]**. If, however, we consider $V\in\rho_{\mathrm{mf}}\mathcal{C}^\infty([X;C])$, then the behavior of commutators with H is radically changed. Thus, propagation of singularities for generalized eigenfunctions of H is very similar to the propagation phenomena in hyperbolic boundary and transmission problems, and the broken geodesics in the statement of the Theorem arise for similar reasons as the broken bicharacteristics in those cases. In fact, many of the proofs of those phenomena, such as those given by Hörmander in [18, Chapter XXIV], can be adapted to our setting.

Again, it may be of some use to relate certain parts of Melrose's analysis of the propagation of singularities to the standard form of Hörmander's theorem via the Fourier transform. Thus, in the Euclidian setting, $X = \mathbb{S}^N_+$, Δ the standard Laplacian, $P = \Delta - \lambda$, $\lambda > 0$, we have $\widetilde{P} = \mathcal{F}P\mathcal{F}^{-1} = |\zeta|^2 - \lambda \in \Psi^0(\mathbb{R}^N_\zeta)$ with characteristic variety $\{(\zeta, \zeta^*) : |\zeta|^2 = \lambda\}$ (ζ^* is the cotangent variable). It is easy to see that the (non-rescaled) Hamilton vector field of $\widetilde{P}$ is radial on the conormal bundle of the sphere given by $|\zeta|^2 = \lambda$, and its integral curves are straight lines in $T^*\mathbb{R}^N$ whose projection to the base variables (i.e., ζ) is constant. Indeed, projecting them to the cosphere bundle $S^*\mathbb{R}^N$ (realized, say, as the unit sphere subbundle of $T^*\mathbb{R}^N$), they become either open half great circles connecting the 'incoming' and 'outgoing' parts,

$$N^\pm_\lambda = \{(\zeta, \pm\zeta/|\zeta|) : |\zeta|^2 = \lambda\}, \tag{1.13}$$

of the cosphere bundle, or points in $N^\pm_\lambda$ (corresponding to the vector field being radial there); note that $N^\pm_\lambda = L(R^\pm_\lambda)$, L being the Legendre diffeomorphism as above. Hence, Hörmander's theorem for real principal type propagation applies in this situation: if $(\Delta - \lambda)u = 0$, then $\mathrm{WF}(\mathcal{F}u)$ either contains such a semicircle, or it is disjoint from it. Now,

$$L(\mathrm{WF}_{\mathrm{sc}}(u) \cap {}^{\mathrm{sc}}T^*_{\mathbb{S}^{N-1}}\mathbb{S}^N_+) = \mathrm{WF}(\mathcal{F}u), \tag{1.14}$$

corresponding to the analogous relationship between the indicial operator in the scattering calculus and the principal symbol map for standard pseudo-differential operators under conjugation by the Fourier transform. Hence, the propagation of singularities for $\mathcal{F}u$ in Hörmander's sense (here u is a generalized eigenfunction of $\Delta - \lambda$) implies the results on the propagation of $\mathrm{WF}_{\mathrm{sc}}(u)$ which were described in the previous paragraphs. Again, this analysis can be modified easily to accommodate two-body type scattering (by using a local Fourier transform if necessary), but the transposition of this argument to three-body scattering seems to be more complicated (see, however, [39] for a less comprehensive analysis of three-body scattering employing the Fourier transform).

We now describe the commutator constructions in somewhat more detail. First, we define a new algebra of differential operators on X which includes both $\mathrm{Diff}_{\mathrm{sc}}(X)$ and $\mathcal{C}^\infty([X; C])$. It is convenient to introduce some notation. The front face of the blown up space, $[X; C]$, is denoted by ff. Defining functions for ff and mf will be denoted by ρ_{ff} and ρ_{mf} respectively. The blow down map is written as

$$\beta : [X; C] \to X; \tag{1.15}$$

ρ_{mf} and ρ_{ff} can be chosen so that $\rho_{\mathrm{mf}}\rho_{\mathrm{ff}} = \beta^* x$. The inclusion of $\mathrm{Diff}_{\mathrm{sc}}(X)$ into the new algebra is supposed to preserve interesting analytical properties. We are thus led to define

$$\mathrm{Diff}_{3\mathrm{sc}}(X) = \mathcal{C}^\infty([X;C]) \otimes_{\mathcal{C}^\infty(X)} \mathrm{Diff}_{\mathrm{sc}}(X). \tag{1.16}$$

For reasons of brevity the notation does not include C on which $\mathrm{Diff}_{3\mathrm{sc}}(X)$ depends. Now, $\mathrm{Diff}_{3\mathrm{sc}}(X)$ is actually an algebra with respect to operator composition, since for $V \in \mathcal{V}_{\mathrm{sc}}(X)$, $f \in \mathcal{C}^\infty([X;C])$, we have $[V, f] = Vf \in \rho_{\mathrm{mf}}\mathcal{C}^\infty([X;C])$ as $\mathcal{V}_{\mathrm{sc}}(X)$ lifts to be a subset of $\rho_{\mathrm{mf}}\mathcal{V}_{\mathrm{b}}([X;C])$. In this paper we will microlocalize $\mathrm{Diff}_{3\mathrm{sc}}(X)$ by constructing the corresponding algebra of pseudo-differential operators, $\Psi^{\infty,-\infty}_{3\mathrm{sc}}(X)$.

This algebra, $\Psi^{\infty,-\infty}_{3\mathrm{sc}}(X)$, will have several properties which are similar to the fibred cusp algebras defined by Mazzeo and Melrose in **[23]**. In fact, in the interior of ff, $\mathrm{Diff}_{3\mathrm{sc}}(X)$ is a fibred cusp algebra (though on a non-compact manifold). Thus, many of the proofs are essentially adaptations of the proofs in **[23]**, although in this paper we refrain from blowing up C on many occasions (thereby hiding the similarity), and only do the blow-ups necessary to obtain the b-fibrations required for push-forward results when the need arises.

One of the main differences between $\mathrm{Diff}_{3\mathrm{sc}}(X)$ and $\mathrm{Diff}_{\mathrm{sc}}(X)$ is that the former is not commutative to 'top weight'. That is, while for $P \in \mathrm{Diff}^m_{\mathrm{sc}}(X)$, $Q \in \mathrm{Diff}^{m'}_{\mathrm{sc}}(X)$, we have $[P,Q] \in x\,\mathrm{Diff}^{m+m'-1}_{\mathrm{sc}}(X)$, this is replaced by $[P,Q] \in \rho_{\mathrm{mf}}\,\mathrm{Diff}^{m+m'-1}_{3\mathrm{sc}}(X)$ for $P \in \mathrm{Diff}^m_{3\mathrm{sc}}(X)$, $Q \in \mathrm{Diff}^{m'}_{3\mathrm{sc}}(X)$. Thus, there is no gain of a weight factor at ff.

Now consider the operator $H = \Delta + V$, $V \in \rho_{\mathrm{mf}}\mathcal{C}^\infty([X;C])$, discussed above. As indicated in the previous paragraph, for $P \in \mathrm{Diff}^m_{\mathrm{sc}}(X)$,

$$[\Delta, P] \in x\,\mathrm{Diff}^{m+1}_{\mathrm{sc}}(X) \subset \rho_{\mathrm{mf}}\rho_{\mathrm{ff}}\,\mathrm{Diff}^{m+1}_{3\mathrm{sc}}(X). \tag{1.17}$$

On the other hand,

$$[V, P] \in \rho^2_{\mathrm{mf}}\,\mathrm{Diff}^{m-1}_{3\mathrm{sc}}(X). \tag{1.18}$$

Hence, as expected, $[V,P]$ is lower order than $[\Delta, P]$ at mf. However, at ff it can actually be higher order. That is, the term $[V,P]$ can dominate $[\Delta,P]$ there! This would clearly cause very serious problems for positive commutator arguments used, for example, to prove results on the propagation of singularities. We can avoid this by choosing P carefully. Thus, we take P from the 'symbolic center', $Z\,\mathrm{Diff}_{3\mathrm{sc}}(X) \subset \mathrm{Diff}_{\mathrm{sc}}(X)$ of $\mathrm{Diff}_{3\mathrm{sc}}(X)$, i.e. we choose $P \in \mathrm{Diff}^m_{3\mathrm{sc}}(X)$ so that $[P,Q] \in \rho_{\mathrm{mf}}\rho_{\mathrm{ff}}\,\mathrm{Diff}^{m+m'-1}_{3\mathrm{sc}}(X)$ for all $Q \in \mathrm{Diff}^{m'}_{3\mathrm{sc}}(X)$. This makes $[V,P]$ the same order as $[\Delta,P]$ with additional vanishing at mf which will be sufficient for the commutator arguments. While the leading part of $[V,P]$ can be quite complicated since it does depend on 'sub-leading' terms, the standard Poisson bracket formula lets us deal with $[\Delta,P]$ easily. The additional vanishing of $[V,P]$ at mf will ensure (due to compactness arguments) that relatively simple estimates of this commutator suffice.

The commutator approach we just outlined can give global positive estimates, such as the Mourre estimate, for $H = \Delta + V$. However, we need to introduce the corresponding pseudo-differential algebra, $\Psi^{\infty,-\infty}_{3\mathrm{sc}}(X)$, for a microlocal description of the propagation of singularities at ∂X. These will propagate along broken bicharacteristics of ${}^{sc}H_g$, broken only at C, with the usual law of reflection satisfied at the 'break points'. The spreading of the singularities from a bicharacteristic to other ones when it hits C corresponds to the restriction in the choice of P in the commutator estimates mentioned above.

This paper consists of two major parts. In the first half we construct $\Psi_{3sc}^{\infty,-\infty}(X)$ and investigate its general properties. In the second half of the paper, starting with Chapter 11, we focus our attention on three-body type Hamiltonians and prove such results as the propagation of singularities for generalized eigenfunctions of these operators.

More specifically, in Chapter 2 we discuss some properties of the differential operator algebra $\mathrm{Diff}_{3sc}(X)$. Most of this chapter consists of a description of the indicial and normal operators for this algebra, since their behavior is significantly different from those for the scattering calculus. We define the microlocalization, $\Psi_{3sc}^{m,l}(X)$, of $\mathrm{Diff}_{3sc}(X)$ in Chapter 3, and in the subsequent chapters we analyze its properties, mostly following **[23]**. A more detailed plan of these chapters is given at the beginning of Chapter 3.

In Chapter 11 we describe the basic properties of the Hamiltonian, $H = \Delta + V$. We then prove the Mourre estimate in our setting in Chapter 12; our method is very similar to Froese's and Herbst's in **[9]**. This could be used to analyze spectral properties of H, just as in Mourre's work **[30]**, but we adopt instead the approach of [**18**, Chapter XXX] and **[25]**. We proceed to show in Chapters 13-16 that singularities propagate along broken bicharacteristics of the (rescaled) Hamilton vector field, ${}^{sc}H_g$, of g.

We continue in Chapter 17 by showing that H has no positive eigenvalues and in Chapter 18 we describe the boundary value of the resolvent at the real axis, $R(\lambda \pm i0)$, $\lambda > 0$, applied to $f \in \dot{C}^\infty(X)$. This is basically the many-body result of of Gérard, Isozaki and Skibsted **[10, 22]** in our setting, with the additional microlocal variables included, together with the full asymptotic expansion away from C given in **[38]**. It should be noted that the propagation estimates of **[10]** correspond to microlocalization with respect to the operator $x^2 D_x$ only. This is completely sufficient for spectral theory, uniqueness statements, and (with slightly more involved arguments) for asymptotic expansions of $R(\lambda \pm i0)f$, $f \in \dot{C}^\infty(X)$, but it cannot capture the singularities of the scattering matrix, for example.

We end the discussion by analyzing the scattering matrix in Chapter 19 using our results concerning the propagation of singularities and the plane wave construction of Melrose and Zworski **[29]** near the 'initial point' (the easy part of their construction, which we recall in Appendix A).

I am very grateful to Richard Melrose for suggesting the problem, for sharing his knowledge with me, for our frequent, fruitful and inspiring discussions affecting every part of this paper and for his encouragement. His firm belief that scattering theory can be understood in microlocal terms similarly to the well-known theory of hyperbolic operators motivated me while working on my thesis.

I am pleased to thank Rafe Mazzeo for our conversations on the subject of this paper. The construction of the pseudodifferential algebra used in this work follows closely the construction of a related algebra in a manuscript of Rafe Mazzeo and Richard Melrose **[23]** which they generously made available to me.

I am indebted to Andrew Hassell for our very helpful conversations, and for kindly providing me with preliminary versions of his manuscripts **[13]** and **[14]**. The latter paper and related discussions lead to proof of the equivalence of the two different definitions of S-matrices outlined in Remark 19.12. I am also grateful to Jared Wunsch for his comments about this paper and for giving me access to his manuscript **[40]**. I am happy to acknowledge that discussions with Arne Jensen and Erik Skibsted afforded considerable help in improving the exposition in this paper. I have also benefited from conversations with Tanya Christiansen, David Jerison and Maciej Zworski which were related to this paper.

The financial support of the Alfred P. Sloan Foundation is gratefully acknowledged.

This paper is based on the author's doctoral thesis, submitted to the Department of Mathematics of the Massachusetts Institute of Technology in partial fulfillment of the requirements for the PhD degree.

CHAPTER 2

DIFFERENTIAL OPERATORS

The aim of this chapter is to discuss basic properties of $\mathrm{Diff}_{3\mathrm{sc}}(X)$, defined in (1.16), in preparation for the constuction and analysis of the corresponding pseudo-differential calculus. Since the 'symbolic' properties (i.e., normal and indicial operators) of $\mathrm{Diff}_{3\mathrm{sc}}(X)$ differ most from the corresponding properties of $\mathrm{Diff}_{\mathrm{sc}}(X)$, we spend most of this chapter by describing these. As the main goal here is to explain the most important points, on occasion we will defer detailed proofs to the following chapters.

First, we analyze the structure of $\mathrm{Diff}_{3\mathrm{sc}}(X)$ in local coordinates. Near a point $p \in C$ we can choose coordinates

$$x,\ y_j\ (j = 1, \ldots, \operatorname{codim} C - 1),\ z_j\ (j = 1, \ldots, \dim C) \tag{2.1}$$

such that $x = 0$ defines ∂X and $x = 0$, $y = 0$ define C. Correspondingly, one can cover a neighborhood of ff $[X; C]$ by two types of coordinates. In the interior of ff we have coordinates

$$x,\ \overline{Y} = y/x,\ z. \tag{2.2}$$

Near ff $\cap$ mf in the lift of the region defined for some k by $|y_k| \geq c|y_j|$ for some $c > 0$ and all $j \neq k$

$$\widehat{x} = x/y_k,\ \widehat{Y}_j = y_j/y_k\ (j \neq k),\ y_k,\ z \tag{2.3}$$

give coordinates. In (2.2) x is the boundary defining function of ff, in (2.3) $\widehat{x}$ defines mf, and y_k defines ff.

It might be useful to discuss here how these coordinates relate to the standard coordinates on $\mathbb{R}^N$ in case of Euclidian three-body scattering. For this purpose, with the notation of the Introduction, choose an integer l, and let $w = (w', w'')$ be the standard coordinates on $\mathbb{R}^N = X_l \oplus X^l$ (so w'_j are the 'free variables' and w''_j the 'interaction variables'), $m = \dim X_l$, $n = \dim X^l$. Near $C_l = \mathrm{cl}(\mathrm{SP}(X_l)) \cap \mathbb{S}^{N-1}$ (corresponding to a conic neighborhood of X_l) we have $|w'| > c|w''|$ for some $c > 0$.

Hence, near any point $p \in C_l$ one of the coordinate functions w'_j, which we may take to be w'_m, satisfies $|w'_j| > c'|w'_k|$ for all k ($c' > 0$), $|w'_j| > c'|w''|$. We may also assume that $w'_m > 0$ for the sake of simplifying the notation. Taking into account the coordinate form of the map SP (i.e. (1.1)) we see that we can take

$$x = |w|^{-1},\ z_j = \frac{w'_j}{|w|}\ (j = 1, \dots, m-1),\ y_j = \frac{w''_j}{|w|}\ (j = 1, \dots, n) \tag{2.4}$$

as local coordinates on the compactified space $\mathbb{S}^N_+$, at least near p. Correspondingly, the coordinates (2.2)-(2.3) on the blown up space $[\mathbb{S}^N_+; C]$ become

$$x = |w|^{-1},\ z_j = \frac{w'_j}{|w|}\ (j = 1, \dots, m-1),\ \overline{Y}_j = w''_j\ (j = 1, \dots, n) \tag{2.5}$$

and

$$\widehat{x} = (w''_k)^{-1},\ z_j = \frac{w'_j}{|w|}\ (j = 1, \dots, m-1),\ \widehat{Y}_j = \frac{w''_j}{w''_k}\ (j \neq k),\ y_k = \frac{w''_k}{|w|} \tag{2.6}$$

respectively. Since $(w''_k)^{-1}$ and $\frac{w''_j}{w''_k}$ $(j \neq k)$ are coordinates in the appropriate region of the radial compactification $\mathbb{S}^n_+$ of X^l (and the w''_j, $j = 1, \dots, n$, are coordinates in the interior of $\mathbb{S}^n_+$), we have proved the following lemma.

Lemma 2.1. — *Suppose that $a \in S^r_{\mathrm{cl}}(X^l)$. Then*

$$\beta^*(\mathrm{SP}^{-1})^*(\pi^l)^* a \in \rho_{\mathrm{mf}}^{-r} \mathcal{C}^\infty([\mathbb{S}^N_+; C_l]). \tag{2.7}$$

Here $\beta : [\mathbb{S}^N_+; C_l] \to \mathbb{S}^N_+$ is the blow-down map, and the subscript cl *refers to classical (one-step polyhomogeneous) symbols.*

The scattering tangent bundle of X, ${}^{\mathrm{sc}}TX$, pulls back to the 3-body scattering tangent bundle ${}^{3\mathrm{sc}}T[X; C]$. Similarly, its dual bundle, ${}^{\mathrm{sc}}T^*X$, pulls back to give the 3-body scattering cotangent bundle ${}^{3\mathrm{sc}}T^*[X; C]$. From (1.16), 3-body scattering vector fields are just smooth sections of ${}^{3\mathrm{sc}}T[X; C]$. The Lie algebra of these vector fields is denoted by $\mathcal{V}_{3\mathrm{sc}}(X)$, i.e. just as in the case of general differential operators the underlying space X is emphasized at the expense of C. This is partially justified by the fact that ${}^{3\mathrm{sc}}T[X; C]$ is the pull back of a bundle over X.

In the local coordinates (2.1) near $p \in C$ a basis of ${}^{\mathrm{sc}}TX$ is given by

$$x^2\partial_x,\ x\partial_{y_j}\ (j = 1, \dots, \operatorname{codim} C - 1),\ x\partial_{z_j}\ (j = 1, \dots, \dim C). \tag{2.8}$$

Near the interior of $\beta^{-1}(p)$ in coordinates (2.2) these lift to a basis of ${}^{3\mathrm{sc}}T[X; C]$:

$$x^2\partial_x - \sum_j x\overline{Y}_j\partial_{\overline{Y}_j},\ \partial_{\overline{Y}_j}\ (j = 1, \dots, \operatorname{codim} C - 1),\ x\partial_{z_j}\ (j = 1, \dots, \dim C). \tag{2.9}$$

Near the corner $\mathrm{mf} \cap \beta^{-1}(p)$ in the coordinates (2.3) they give a basis

$$y_k\widehat{x}^2\partial_{\widehat{x}},\ \widehat{x}\partial_{\widehat{Y}_j}\ (j \neq k),\ y_k\widehat{x}\partial_{y_k} - \widehat{x}^2\partial_{\widehat{x}} - \sum_{j\neq k} \widehat{x}\widehat{Y}_j\partial_{\widehat{Y}_j},\ y_k\widehat{x}\partial_{z_j} \tag{2.10}$$

of ${}^{3\mathrm{sc}}T[X;C]$. Thus, over $\mathrm{int}(\beta^{-1}(p))$ sections of ${}^{3\mathrm{sc}}T[X;C]$ are spanned by

$$x^2\partial_x,\ \partial_{\overline{Y}_j},\ x\partial_{z_j} \tag{2.11}$$

over $\mathcal{C}^\infty([X;C])$ corresponding to a natural fibred cusp structure (which notion we do not use or define in this paper, see **[23]** for the definition and a discussion of its properties), but at $\partial\beta^{-1}(p)$ it does not have a simple product-type structure.

We briefly note the form taken by the lifts of the standard coordinate vector fields ∂_{w_j} in the setting of Euclidian three-body scattering. While this is somewhat complicated in the explicit coordinate expression on $\mathbb{S}^N_+$, they become slightly simpler at C. Thus, with $\mathcal{I}(C)$ denoting the ideal of functions in $\mathcal{C}^\infty(\mathbb{S}^N_+)$ which vanish at C,

$$\partial_{w'_m} = -(1-\sum z_k^2)((x^2\partial_x)+\sum_j z_j(x\partial_{z_j})) + \mathcal{I}(C)\,\mathrm{Diff}_{\mathrm{sc}}(\mathbb{S}^N_+), \tag{2.12}$$

$$\partial_{w'_j} = -z_j(x^2\partial_x)+\sum_k(\delta_{jk}-z_jz_k)(x\partial_{z_k}) + \mathcal{I}(C)\,\mathrm{Diff}_{\mathrm{sc}}(\mathbb{S}^N_+) \quad (j\neq m), \tag{2.13}$$

$$\partial_{w''_j} = x\partial_{y_j} + \mathcal{I}(C)\,\mathrm{Diff}_{\mathrm{sc}}(\mathbb{S}^N_+). \tag{2.14}$$

The significance of these expressions will be clearer soon when we discuss the normal operators in $\mathrm{Diff}_{3\mathrm{sc}}(X)$, $X=\mathbb{S}^N_+$; for this purpose we return to the general geometric framework.

First, the principal symbol map of $\mathrm{Diff}_{\mathrm{sc}}(X)$ (see (1.9)) extends by continuity to define the principal symbol map of $\mathrm{Diff}_{3\mathrm{sc}}(X)$ and to give a short exact sequence:

$$0\to \mathrm{Diff}^{m-1}_{3\mathrm{sc}}(X)\hookrightarrow \mathrm{Diff}^m_{3\mathrm{sc}}(X)\xrightarrow{\sigma_{3\mathrm{sc},m}} P^m_h({}^{3\mathrm{sc}}T[X;C])\to 0, \tag{2.15}$$

P^m_h denoting the space of mth order homogeneous polynomials.

Next, apart from the indicial operator of $\mathrm{Diff}_{\mathrm{sc}}(X)$ discussed in (1.10), we also consider the corresponding normal operator defined for $p\in\partial X$ as

$$N_{\mathrm{sc},p}:\mathrm{Diff}_{\mathrm{sc}}(X)\to \mathrm{Diff}_{\mathrm{I}}\,{}^{\mathrm{sc}}T_pX; \tag{2.16}$$

here $\mathrm{Diff}_{\mathrm{I}}\,{}^{\mathrm{sc}}T_pX$ is the algebra of translation invariant differential operators on the vector space ${}^{\mathrm{sc}}T_pX$ (see **[25**, Section 2]). In fact, for $Q\in\mathrm{Diff}_{\mathrm{sc}}(X)$, $p\in\partial X$, $N_{\mathrm{sc},p}(Q)$ is simply given by the canonical lifting of Q (obtained by 'freezing the coefficients' at p) to be a translation invariant (i.e., constant coefficient) differential operator on ${}^{\mathrm{sc}}T_pX$. This definition makes sense since $\mathrm{Diff}_{\mathrm{sc}}(X)$ is commutative to top order:

$$[\mathrm{Diff}^m_{\mathrm{sc}}(X),\mathrm{Diff}^{m'}_{\mathrm{sc}}(X)]\subset x\,\mathrm{Diff}^{m+m'-1}_{\mathrm{sc}}(X). \tag{2.17}$$

Also note that N_{sc} is multiplicative due to (2.17). Moreover, N_{sc} and $\widehat{N}_{\mathrm{sc}}$ are related via conjugation by the invariant Fourier transform on the fibers of ${}^{\mathrm{sc}}T_{\partial X}X$ (mapping functions on ${}^{\mathrm{sc}}T_pX$ to densities on its dual space ${}^{\mathrm{sc}}T^*_pX$, $p\in\partial X$). Namely, an invariant differential operator becomes a multiplication operator given by a (not necessarily homogeneous!) polynomial under conjugation.

There is also an 'oscillatory testing' definition of $\widehat{N}_{\rm sc}$ given in [**29**, Lemma 8]. The indicial operator in the three-body calculus will be constructed similarly, so we recall this definition here. First note that if $f \in \mathcal{C}^\infty(X)$ then $d(f/x) \in \mathcal{C}^\infty(X; {}^{\rm sc}T^*X)$. In addition, if $P \in \mathrm{Diff}_{\rm sc}(X)$, $u \in \mathcal{C}^\infty(X)$, then

$$v = e^{-if/x} P e^{if/x} u \in \mathcal{C}^\infty(X). \tag{2.18}$$

Moreover, if $p \in \partial X$, $\zeta \in {}^{\rm sc}T^*_pX$, $d(f/x)(p) = \zeta$, then

$$v(p) = (\widehat{N}_{\rm sc}(P)(\zeta))u(p); \tag{2.19}$$

this can be taken as the definition of $\widehat{N}_{\rm sc}$. Here note that

$$d(f/x)(p) = -f(p)(dx/x^2) + (df)(p)/x, \tag{2.20}$$

so for any $\zeta \in {}^{\rm sc}T^*_pX$ we can easily find f with $d(f/x)(p) = \zeta$. We also remark that the oscillatory testing definition is often very helpful in analyzing the 'symbolic' properties of a pseudo-differential calculus since it automatically reflects the composition properties of the operators, thus giving a multiplicative 'symbol' map.

Just like the principal symbol map, $N_{\rm sc}$ extends by continuity to define the normal operator map of $\mathrm{Diff}_{3\rm sc}(X)$ at mf, and it gives a short exact sequence:

$$0 \to \rho_{\rm mf}\,\mathrm{Diff}_{3\rm sc}(X) \hookrightarrow \mathrm{Diff}_{3\rm sc}(X) \xrightarrow{N_{\rm mf,0}} \mathrm{Diff}_{\rm I}\,{}^{3\rm sc}T_{\rm mf}[X;C] \to 0. \tag{2.21}$$

One of the main points about $\sigma_{3{\rm sc},m}$ and $N_{\rm mf,0}$ (keeping in mind that ultimately we are interested in spectral theory, hence in resolvents) is that they are multiplicative in the sense that

$$\sigma_{3{\rm sc},m}(P)\sigma_{3{\rm sc},m'}(Q) = \sigma_{3{\rm sc},m+m'}(PQ), \qquad N_{\rm mf,0}(P)N_{\rm mf,0}(Q) = N_{\rm mf,0}(PQ) \tag{2.22}$$

for $P \in \mathrm{Diff}^m_{3\rm sc}(X)$, $Q \in \mathrm{Diff}^{m'}_{3\rm sc}(X)$. We wish to define a normal operator at ff which is also multiplicative. This is somewhat complicated; we must also work out the space into which it maps. Here we just point out that the natural idea one might try, i.e. mapping into $\mathrm{Diff}_{\rm I}\,{}^{3\rm sc}T_{\rm ff}[X;C]$ does not give a multiplicative homomorphism. In fact, it cannot, since this is a commutative algebra, while $\mathrm{Diff}_{3\rm sc}(X)$ is not so even to top order as indicated in the Introduction.

Just as there is a well defined relative b-tangent bundle ${}^{\rm b}T(C;X)$ over C, we also have a relative scattering tangent bundle ${}^{\rm sc}T(C;X)$. In fact, ${}^{\rm sc}T(C;X)$ is the subbundle of ${}^{\rm sc}T_CX$ consisting of $v \in {}^{\rm sc}T_pX$, $p \in C$, for which there exists

$$V \in \mathcal{V}_{\rm sc}(X;C) \subset \mathcal{V}_{\rm sc}(X) \tag{2.23}$$

with $V(p) = v$. Here

$$\mathcal{V}_{\rm sc}(X;C) = x\mathcal{V}_{\rm b}(X;C) = x\{V \in \mathcal{V}_{\rm b}(X) :\ V \text{ is tangent to } C\}, \tag{2.24}$$

and tangency is defined using $\mathcal{V}_{\rm b}(X) \subset \mathcal{V}(X)$, i.e. the (non-injective) inclusion map ${}^{\rm b}TX \to TX$. Thus, given a boundary defining function x, ${}^{\rm b}T(C;X)$ is isomorphic to ${}^{\rm sc}T(C;X)$ (via extension and multiplication by x), but the isomorphism depends on the choice of x. It should be noted that $\dim {}^{\rm sc}T_p(C;X) = \dim C + 1$, and in the local coordinates (2.1) it is spanned by $x^2\partial_x$ and $x\partial_z$. In particular, by (2.12)-(2.14), in the Euclidian setting ${}^{\mathrm{sc}}T(C;X)$ is spanned by the lift of the 'free (coordinate) vector fields', $\partial_{w'_j}$, $j = 1,\dots,m$.

One way of seeing the importance of $\mathcal{V}_{\mathrm{sc}}(X;C)$ is to remark that for $P \in \mathcal{V}_{\mathrm{sc}}(X;C)$, $f \in \mathcal{C}^\infty([X;C])$, we have $Pf \in x\mathcal{C}^\infty([X;C])$, i.e., as operators, $[P,f] \in x\,\mathrm{Diff}_{3\mathrm{sc}}(X)$. That is, P commutes with $\mathrm{Diff}_{3\mathrm{sc}}(X)$ modulo $x\,\mathrm{Diff}_{3\mathrm{sc}}(X)$; this is not true when P is an arbitrary element of $\mathrm{Diff}_{\mathrm{sc}}(X)$.

For $p \in C$, there is a natural action of ${}^{\mathrm{sc}}T_pX/{}^{\mathrm{sc}}T_p(C;X)$ on $\mathrm{int}(\beta^{-1}(p))$ as shown in Chapter 4. In local coordinates (2.2) this is given by

$$L_v(\overline{Y},z) = (\overline{Y}+\beta,z), \qquad v = \alpha x^2\partial_x + \beta x\partial_y + \gamma x\partial_z. \tag{2.25}$$

(In the Euclidian case this is just translation in the 'interaction variables' induced by the 'interaction vector fields'.) Correspondingly, the tangent space of the fibers of the blow down map, $T_q\beta^{-1}(p)$, $q \in \mathrm{int}(\mathrm{ff})$, $\beta(q) = p$, is naturally isomorphic to ${}^{\mathrm{sc}}T_pX/{}^{\mathrm{sc}}T_p(C;X)$. This isomorphism can be realized as follows: $v \in {}^{\mathrm{sc}}T_pX$ pulls back to $\beta^*v \in {}^{3\mathrm{sc}}T_q[X;C]$. The range of the natural (non-injective) inclusion map ${}^{3\mathrm{sc}}T_q[X;C] \to T_q[X;C]$ induced by the inclusion $\mathcal{V}_{3\mathrm{sc}}(X) \hookrightarrow \mathcal{V}([X;C])$ is $T_q\beta^{-1}(p)$. The null space of the composition of the pull back with this inclusion is exactly ${}^{\mathrm{sc}}T(C;X)$, and it gives the isomorphism mentioned above. In particular, $v \in {}^{\mathrm{sc}}T_pX$ is mapped to a vector field on $T\beta^{-1}(p)$ which is invariant under the affine action.

More generally, if $V \in \mathcal{V}_{3\mathrm{sc}}(X)$ is a vector field, it can be regarded as a section of $T[X;C]$, and it can be restricted to int(ff) with the result being tangent to the fibers of β. This induces a natural map

$$\mathcal{C}^\infty(\beta^{-1}(p);{}^{3\mathrm{sc}}T_{\beta^{-1}(p)}[X;C]) \ni V \mapsto V_\partial \in \mathcal{C}^\infty(\beta^{-1}(p);T\beta^{-1}(p)); \tag{2.26}$$

this is called the boundary restriction map. The null space of this map is exactly $\mathcal{C}^\infty(\beta^{-1}(p);\beta^*_{\beta^{-1}(p)}{}^{\mathrm{sc}}T(C;X))$. In the local coordinates (2.2) this map is given by

$$\alpha x^2\partial_x + \beta\cdot\partial_{\overline{Y}} + \gamma\cdot x\partial_z \mapsto \beta\cdot\partial_{\overline{Y}}. \tag{2.27}$$

On the other hand, the basis vector fields (2.10) near $\partial\beta^{-1}(p)$ restrict to

$$0,\ \widehat{x}\partial_{\widehat{Y}_j}\ (j\neq k),\ -\widehat{x}^2\partial_{\widehat{x}} - \sum_{j\neq k}\widehat{x}\widehat{Y}_j\partial_{\widehat{Y}_j},\ 0 \tag{2.28}$$

respectively. Thus, the boundary restriction map actually maps into

$$\mathcal{V}_{\mathrm{sc}}(\beta^{-1}(p)) = \mathcal{C}^\infty(\beta^{-1}(p);{}^{\mathrm{sc}}T\beta^{-1}(p)). \tag{2.29}$$

It also extends naturally to an algebra homomorphism $\mathrm{Diff}_{3\mathrm{sc}}(X) \to \mathrm{Diff}_{\mathrm{sc}}(\beta^{-1}(p))$. Note that $\mathrm{Diff}_{\mathrm{sc}}(\beta^{-1}(p))$ is *not* a commutative algebra, so our normal operator map at ff (part of which is given by the boundary restriction map) will not be commutative, just as expected. We remark that in the Euclidian case this discussion means that $\alpha\cdot\partial_{w'} + \beta\cdot\partial_{w''}$ maps to $\beta\cdot\partial_{\overline{Y}}$, i.e. the 'free coordinate vector fields' are eliminated, and the 'interaction vector fields' preserved.

The boundary restriction map can also be analyzed using the action of $\mathrm{Diff}_{3sc}(X)$ on $\mathcal{C}^\infty([X;C])$ similarly to (2.18)-(2.19). Namely, if

$$P \in \mathrm{Diff}_{3sc}(X) \quad \text{and} \quad u \in \mathcal{C}^\infty([X;C]),$$

then $Pu \in \mathcal{C}^\infty([X;C])$ and an explicit calculation shows that for $p \in C$ we have

$$Pu|_{\beta^{-1}(p)} = P_\partial(p)(u|_{\beta^{-1}(p)}). \tag{2.30}$$

In fact, the full oscillatory testing construction can be extended to $\mathrm{Diff}_{3sc}(X)$. A straightforward calculation (or a direct consequence of (1.16)) proves that if $P \in \mathrm{Diff}_{3sc}(X)$, $f \in \mathcal{C}^\infty(X)$ (so f is smooth on the original space, X!) then $\widetilde{P}_f = e^{-if/x}Pe^{if/x} \in \mathrm{Diff}_{3sc}(X)$, so for any $p \in C$ it has a boundary restriction map $(\widetilde{P}_f)_\partial \in \mathrm{Diff}_{sc}(\beta^{-1}(p))$. Moreover, if $p \in C$, $d(f/x)(p) = \zeta \in {}^{sc}T^*_pX$ then it can be checked easily (this will be done for the pseudo-differential calculus in Chapter 6) that $(\widetilde{P}_f)_\partial$ depends on f only via ζ. Therefore, we can define the indicial operator of P at ff as the map

(2.31)
$$ {}^{sc}T^*_CX \ni \zeta \mapsto \widehat{P}_{\mathrm{ff},0}(\zeta) = (\widetilde{P}_f)_\partial(p) \in \mathrm{Diff}_{sc}(\beta^{-1}(p)), \quad \zeta \in {}^{sc}T^*_pX,\ d(f/x)(p) = \zeta;$$

this definition is independent of the choice of f satisfying (2.31).

Now, allowing ζ to take any value in ${}^{sc}T^*_CX$ leads to much redundant information. To see this, simply note that if $f(x,y,z) = y_j$ (in a neighborhood of a point $p \in C$) then $d(f/x) = dy_j/x \neq 0$ but $e^{if/x} = e^{iY_j}$ in a neighborhood of int(ff), i.e. it is smooth in such a neighborhood. Hence, $(\widetilde{P}_f)_\partial = e^{-iY_j}P_\partial e^{iY_j}$, so $\widehat{P}_{\mathrm{ff},0}(dy_j/x)$ can be reconstructed from $\widehat{P}_{\mathrm{ff},0}(0)$. More generally, $\widehat{P}_{\mathrm{ff},0}(\zeta)$ can be reconstructed from $\widehat{P}_{\mathrm{ff},0}(0)$ if $\zeta \in {}^{sc}T_p(C;X)^\perp$, the annihilator of ${}^{sc}T_p(C;X)$ in ${}^{sc}T^*_pX$. This is not surprising, since some of the information about the behavior of elements of $\mathrm{Diff}_{sc}(X)$ that has to be deduced from oscillatory functions can be obtained by merely considering the boundary restriction map in $\mathrm{Diff}_{3sc}(X)$ as we allow singular test sections (i.e., we allow $u \in \mathcal{C}^\infty([X;C])$).

To eliminate this redundancy, we choose a subbundle $W \to C$ of ${}^{sc}T_CX$ which is complementary to ${}^{sc}T(C;X)$. Such a splitting arises naturally if we have a scattering metric on X, for it gives an inner product on the fibers of ${}^{sc}TX$, and we can take W to be the orthocomplement of ${}^{sc}T(C;X)$. This induces a corresponding splitting of ${}^{sc}T^*X$ over C, with $W^\perp \subset {}^{sc}T^*_CX$ being the annihilator of W. We can now choose local coordinates x, y, z near $p \in C$ such that $x = 0$ defines ∂X, $x = 0$, $y = 0$ define C, and $x\partial_{y_j}$, $j = 1, \ldots, \operatorname{codim} C - 1$, give a basis of W. This means exactly that dx/x^2 and dz_j/x, $j = 1, \ldots, \dim C$, are a basis of $W^\perp$. It follows from the discussion of the previous paragraph that we do not lose any information if we require $d(f/x) \in W^\perp$ when defining $\widehat{P}_{\mathrm{ff}}$, i.e. if we regard the indicial operator as a map

$$W^\perp \ni \zeta \mapsto \widehat{P}_{\mathrm{ff},0}(\zeta) \in \mathrm{Diff}_{sc}(\beta^{-1}(p)), \quad \zeta \in W^\perp_p. \tag{2.32}$$

Note that in the Euclidian setting W_p, $p \in C$, is spanned by the 'interaction vectors' $\partial_{w''_j}$, so $W_p^\perp$ is spanned by the 'free covectors' dw'_j. Hence, in this case, the indicial operator is constructed essentially by conjugation by the partial Fourier transform in the free variables.

It is not very easy to describe the range of the indicial operator as can be seen in the Euclidian case from the fact that partial Fourier transform maps to functions on a space where the configuration space and its dual variables are mixed up. Thus, it is useful to define a normal operator as well at ff. Taking into account the null space of the boundary restriction map, we want to define the normal operator, $N_{\mathrm{ff},0,p}(V)$, $V \in \mathcal{V}_{3\mathrm{sc}}(X)$, for each point $p \in C$ as a section of $T\beta^*_{\beta^{-1}(p)}{}^{\mathrm{sc}}T(C;X)$. In fact, $N_{\mathrm{ff},0,p}(V)$ will be a vector field which is translation invariant on the fibers of $\beta^*_{\beta^{-1}(p)}{}^{\mathrm{sc}}T(C;X) \to \beta^{-1}(p)$; the space of such vector fields is denoted by $\mathcal{V}_{\mathrm{I}}(\beta^*_{\beta^{-1}(p)}{}^{\mathrm{sc}}T(C;X))$. We remark that using β we can define horizontal sections of $T\beta^*_{\beta^{-1}(p)}{}^{\mathrm{sc}}T(C;X)$ as vector fields which are tangent to the submanifolds $\beta^*_{\beta^{-1}(p)}v$, $v \in {}^{\mathrm{sc}}T_p(C;X)$. Hence, we can canonically lift a section of $T\beta^{-1}(p)$ to a horizontal section of $T\beta^*_{\beta^{-1}(p)}{}^{\mathrm{sc}}T(C;X)$.

Now, $N_{\mathrm{ff},0,p}(V)$ can be defined using a splitting ${}^{\mathrm{sc}}T_pX = {}^{\mathrm{sc}}T_p(C;X) \oplus W_p$ discussed in the previous paragraphs. Correspondingly, with $q \in \mathrm{int(ff)}$, $\beta(q) = p$, we have a splitting ${}^{3\mathrm{sc}}T_q[X;C] = \beta_q^{*\mathrm{sc}}T(C;X) \oplus \beta_q^* W$. Let π_1 and π_2 be the projections onto the two summands. Then for $V \in \mathcal{V}_{3\mathrm{sc}}(X)$, $\pi_1(V) \in \beta_q^{*\mathrm{sc}}T(C;X)$ can be canonically regarded as a translation invariant vector field on the tangent space of this vector space, hence as a vector field on $\beta^*_{\beta^{-1}(p)}{}^{\mathrm{sc}}T(C;X)$ which is tangent to the fibers of the projection $\beta^*_{\beta^{-1}(p)}{}^{\mathrm{sc}}T(C;X) \to \beta^{-1}(p)$ (i.e., a vertical vector field) and is translation invariant on these fibers. On the other hand, $V_\partial = (\pi_2(V))_\partial \in T\beta^{-1}(p)$ can be regarded as a horizontal vector field on $\beta^*_{\beta^{-1}(p)}{}^{\mathrm{sc}}T(C;X)$. Hence, using the splitting we can define

$$N_{\mathrm{ff},0,p}(V) = \pi_1(V) + V_\partial \in \mathcal{V}_{\mathrm{I}}(\beta^*_{\beta^{-1}(p)}{}^{\mathrm{sc}}T(C;X)) \tag{2.33}$$

employing these identifications. The null space of $N_{\mathrm{ff},0,p}(V)$ is $\mathcal{I}(\beta^{-1}(p))\mathcal{V}_{3\mathrm{sc}}(X)$ where $\mathcal{I}(\beta^{-1}(p))$ is the ideal of smooth functions on $[X;C]$ vanishing at $\beta^{-1}(p)$.

The only reason for this not being surjective is the behavior of $N_{\mathrm{ff},0,p}(V)$ at $\partial\beta^{-1}(p)$. Namely, from the tensor product definition and from (2.29) it follows that $N_{\mathrm{ff},0,p}$ maps onto

$$\mathcal{V}_{\mathrm{sc,I}}(\beta^*_{\beta^{-1}(p)}{}^{\mathrm{sc}}T(C;X)). \tag{2.34}$$

It extends to an algebra homomorphism:

$$N_{\mathrm{ff},0,p} : \mathrm{Diff}_{3\mathrm{sc}}(X) \to \mathrm{Diff}_{\mathrm{sc,I}}(\beta^*_{\beta^{-1}(p)}{}^{\mathrm{sc}}T(C;X)). \tag{2.35}$$

The space

$$\mathrm{Diff}_{\mathrm{sus}({}^{\mathrm{sc}}T_p(C;X)),\mathrm{sc}}(\beta^{-1}(p)) = \mathrm{Diff}_{\mathrm{sc,I}}(\beta^*_{\beta^{-1}(p)}{}^{\mathrm{sc}}T(C;X)) \tag{2.36}$$

is analogous to the space $\mathrm{Diff}_{\mathrm{sus}({}^{\mathrm{sc}}T_p(C;X))}(\beta^{-1}(p))$ of ${}^{\mathrm{sc}}T_p(C;X)$ suspended differential operators on $\beta^{-1}(p)$ as defined by Mazzeo and Melrose **[23]**; the only difference is the appearance of the boundary $\partial\beta^{-1}(p)$. Just as in their case we can put $N_{\mathrm{ff},0,p}$, $p \in C$, together in a single operator using the fibration β of ff over C; this will be discussed in detail in Chapter 6. We thus obtain the normal homomorphism $N_{\mathrm{ff},0}$ into the algebra $\mathrm{Diff}_{\mathrm{sus}(V)-C,\mathrm{sc}}(\mathrm{ff})$ of $V = {}^{\mathrm{sc}}T(C;X)$-suspended differential operators on the fibration $\mathrm{int}(\mathrm{ff}) \to C$. It gives a short exact sequence

$$0 \to \rho_{\mathrm{ff}}\,\mathrm{Diff}_{3\mathrm{sc}}(X) \hookrightarrow \mathrm{Diff}_{3\mathrm{sc}}(X) \xrightarrow{N_{\mathrm{ff},0}} \mathrm{Diff}_{\mathrm{sus}(V)-C,\mathrm{sc}}(\mathrm{ff}) \to 0. \tag{2.37}$$

We proceed now to microlocalize $\mathrm{Diff}_{3\mathrm{sc}}(X)$ by constructing the 'small calculus', $\Psi_{3\mathrm{sc}}(X)$, of pseudo-differential operators, and to examine its properties, such as the normal operators.

CHAPTER 3

DEFINITION OF THE THREE-BODY SCATTERING CALCULUS

Following Melrose's construction of the scattering calculus in **[25]**, there are two alternatives of defining the three-body scattering calculus, $\Psi_{3\mathrm{sc}}^{\infty,-\infty}(X)$. Namely, we can either define a class of symbols which will give rise to the operators in $\Psi_{3\mathrm{sc}}^{\infty,-\infty}(X)$ by (left, right, Weyl, etc.) quantization and localization, or we can define the kernels of these operators directly on a double space $X^2_{3\mathrm{sc}}$, i.e. on an appropriate blow-up of X^2. We follow the second approach since it allows us to use the machinery of b-fibrations developed by Melrose to prove the basic properties of the calculus rather easily. However, at the end of this chapter we briefly describe how the calculus can be obtained by the first method.

The detailed plan of the chapters describing the calculus is the following. In this chapter we define $\Psi_{3\mathrm{sc}}^{\infty,-\infty}(X)$, and we analyze its action on conormal functions on $[X;C]$. In the next chapter we investigate the boundary restriction map. In Chapter 5 we show that the calculus is closed under operator composition. In Chapter 6 we define the normal and indicial operator maps. In Chapter 7 we examine the commutator of operators in the calculus more thoroughly. This is followed by a discussion of L^2 boundedness properties in Chapter 8 and by a description of the wave front set arising from the pseudo-differential algebra in Chapter 9. We finish the first part of this paper by describing the functional calculus for certain elements of $\Psi_{3\mathrm{sc}}^{m,0}(X)$, $m > 0$, in Chapter 10.

In order to define the three-body scattering calculus, we first recall the definition of the scattering double space X^2_{sc} from **[25]**. Thus, consider the b-double space and its blow-down map

$$\beta_{\mathrm{b}} : X^2_{\mathrm{b}} \to X^2, \qquad X^2_{\mathrm{b}} = [X^2; (\partial X)^2]. \tag{3.1}$$

The diagonal Δ of X^2 lifts to a p-submanifold $\Delta_{\mathrm{b}} \subset X^2_{\mathrm{b}}$ which intersects $\partial X^2_{\mathrm{b}}$ in the interior of the front face bf of the blow up (3.1). The scattering double space is then the blow up

$$\beta_{\mathrm{sc}} : X^2_{\mathrm{sc}} \to X^2_{\mathrm{b}}, \qquad X^2_{\mathrm{sc}} = [X^2_{\mathrm{b}}; \partial\Delta_{\mathrm{b}}]. \tag{3.2}$$

The lift of Δ_{b}, Δ_{sc}, is a p-submanifold of X^2_{sc} meeting $\partial X^2_{\mathrm{sc}}$ only in the front face sf of the blow up (3.2). We can also lift C from either factor of X to X^2_{b}. The lifts of C_L, C_R under β_{b} intersect bf in embedded submanifolds, and

$$C_L \cap \partial\Delta_{\mathrm{b}} = C_R \cap \partial\Delta_{\mathrm{b}} \tag{3.3}$$

is a closed p-submanifold of Δ_{b}. Hence $C_L \cap \partial\Delta_{\mathrm{b}}$ lifts to a closed p-submanifold of sf, and we can define the three-body double space:

$$X^2_{3\mathrm{sc}} = [X^2_{\mathrm{sc}}; \beta_{\mathrm{sc}}^{-1}(C_L \cap \partial\Delta_{\mathrm{b}})]. \tag{3.4}$$

We write the blow-down map as $\beta_{3\mathrm{sc}} : X^2_{3\mathrm{sc}} \to X^2_{\mathrm{sc}}$. Since $\partial\Delta_{\mathrm{b}} \cap C_L \subset \partial\Delta_{\mathrm{b}}$ are closed p-submanifolds of X^2_{b}, they can be blown up in either order, so

$$X^2_{3\mathrm{sc}} = [X^2_{\mathrm{b}}; C_L \cap \partial\Delta_{\mathrm{b}}; \partial\Delta_{\mathrm{b}}]. \tag{3.5}$$

The lift of sf to $X^2_{3\mathrm{sc}}$ is denoted by sf', while the front face of the blow up (3.4) is sf_C. Thus, we can choose boundary defining functions of sf' and sf_C so that $\beta^*_{3\mathrm{sc}}\rho_{\mathrm{sf}} = \rho_{\mathrm{sf'}}\rho_{\mathrm{sf}_C}$.

It is actually useful to construct coordinates near sf' and sf_C. Let x be a boundary defining function of X. We can choose coordinates x,y,z near some point on $C \subset X$ such that C is defined by $x = 0$, $y = 0$. Denoting the coordinates on the right factor of X by x', y', z' we then obtain coordinates in the interior of bf near $\partial\Delta_{\mathrm{b}} \cap C_L$:

$$s = x'/x,\ x,\ y,\ y',\ z,\ z'. \tag{3.6}$$

In the region of validity of these coordinates Δ_{b} is defined by $s = 1$, $y = y'$, $z = z'$, C_L is defined by $x = 0$, $y = 0$, C_R by $x = 0$, $y' = 0$. From here we can obtain coordinates in the interior of sf near $\beta_{\mathrm{sc}}^{-1}(C_L \cap \partial\Delta_{\mathrm{b}})$:

$$x,\ S = (1-s)/x,\ Y = (y-y')/x,\ Z = (z-z')/x,\ y,\ z. \tag{3.7}$$

Now Δ_{sc} is defined by $S = 0$, $Y = 0$, $Z = 0$, and $\beta_{\mathrm{sc}}^{-1}(C_L \cap \partial\Delta_{\mathrm{b}})$ is defined by $x = 0$, $y = 0$. In particular, they are p-transversal. It follows now that Δ_{sc} lifts to a p-submanifold, $\Delta_{3\mathrm{sc}}$, of $X^2_{3\mathrm{sc}}$ intersecting the boundary in $\mathrm{sf'} \cup \mathrm{sf}_C$ only. Finally, in the interior of sf_C we have coordinates

$$x,\ S,\ Y,\ Z,\ \overline{Y} = y/x,\ z, \tag{3.8}$$

while near $\mathrm{sf}_C \cap \mathrm{sf'}$ in the lift of the region $|y_k| \geq c|y_j|$ for some $c > 0$ and all $j \neq k$

$$\widehat{x} = x/y_k,\ S,\ Y,\ Z,\ \widehat{Y}_j = y_j/y_k\ (j \neq k),\ y_k,\ z. \tag{3.9}$$

In the region where (3.8) are valid $\Delta_{3\mathrm{sc}}$ is defined by $S = 0$, $Y = 0$, $Z = 0$, and similarly in the coordinates (3.9). In the coordinates (3.9) sf' is defined by $\widehat{x} = 0$ and sf_C by $y_k = 0$. Note that C_L can be replaced by C_R in the construction of $X^2_{3\mathrm{sc}}$ by

(3.3), and similarly we can swap the primed and unprimed coordinates throughout this discussion.

The scattering kernel density bundle for operators on half-densities

$$\mathrm{KD}^{1/2}_{\mathrm{sc}} = \rho_{\mathrm{sf}}^{-1/2(\dim X+1)}\,\Omega^{1/2}(X^2_{\mathrm{sc}}) \tag{3.10}$$

can be pulled back by $\beta_{3\mathrm{sc}}$ to obtain the three-body-scattering kernel density bundle

$$\mathrm{KD}^{1/2}_{3\mathrm{sc}} = (\beta^*_{3\mathrm{sc}}\rho_{\mathrm{sf}})^{-1/2(\dim X+1)}\rho_{\mathrm{sf_C}}^{\operatorname{codim} C/2}\,\Omega^{1/2}(X^2_{3\mathrm{sc}}), \tag{3.11}$$

so

$$\mathrm{KD}^{1/2}_{3\mathrm{sc}} = \rho_{\mathrm{sf_C}}^{-1/2(\dim C+1)}\rho_{\mathrm{sf'}}^{-1/2(\dim X+1)}\,\Omega^{1/2}(X^2_{3\mathrm{sc}}). \tag{3.12}$$

The space of kernels of elements of the three-body-scattering small calculus with weight $l \in \mathbb{R}$ and order $m \in \mathbb{R}$ is defined by

$$\Psi^{m,l}_{3\mathrm{scc}}(X;{}^{\mathrm{sc}}\Omega^{1/2}) = \{\kappa \in \mathcal{A}^{m,l}(X^2_{3\mathrm{sc}},\Delta_{3\mathrm{sc}};\mathrm{KD}^{1/2}_{3\mathrm{sc}}) :\ \kappa \equiv 0 \text{ at } \partial X^2_{3\mathrm{sc}} \setminus (\mathrm{sf'} \cup \mathrm{sf_C})\}. \tag{3.13}$$

We also define the corresponding one-step polyhomogeneous space:

$$\Psi^{m,l}_{3\mathrm{sc}}(X;{}^{\mathrm{sc}}\Omega^{1/2}) = \{\kappa \in \rho^l_{\mathrm{sf'}}\rho^l_{\mathrm{sf_C}} I^m_{\mathrm{os}}(X^2_{3\mathrm{sc}},\Delta_{3\mathrm{sc}};\mathrm{KD}^{1/2}_{3\mathrm{sc}}); \kappa \equiv 0 \text{ at } \partial X^2_{3\mathrm{sc}} \setminus (\mathrm{sf'} \cup \mathrm{sf_C})\}. \tag{3.14}$$

We can generalize these definitions for arbitrary vector bundles E and F over X as usual, i.e. we define $\Psi^{m,l}_{3\mathrm{sc}}(X;E,F)$ by replacing the bundle $\mathrm{KD}^{1/2}_{3\mathrm{sc}}$ in (3.14) by

$$\mathrm{KD}^{\mathrm{E,F}}_{3\mathrm{sc}} = \mathrm{KD}^{1/2}_{3\mathrm{sc}} \otimes \overline{\beta}^*_{3\mathrm{sc}} \operatorname{Hom}(\pi^*_R(E \otimes {}^{\mathrm{sc}}\Omega^{-1/2}(X)), \pi^*_L(F \otimes {}^{\mathrm{sc}}\Omega^{-1/2}(X))) \tag{3.15}$$

where $\overline{\beta}_{3\mathrm{sc}} = \beta_{\mathrm{b}}\beta_{\mathrm{sc}}\beta_{3\mathrm{sc}} : X^2_{3\mathrm{sc}} \to X^2$ is the composite blow down map, and $\pi_L, \pi_R : X^2 \to X$ are the left and right projections. We write $\Psi^{m,l}_{3\mathrm{sc}}(X;E)$ for $\Psi^{m,l}_{3\mathrm{sc}}(X;E,E)$, and if E is the trivial vector bundle, i.e. for action on functions, we simply write $\Psi^{m,l}_{3\mathrm{sc}}(X)$.

Since elements of $I^m_{\mathrm{os}}(X^2_{\mathrm{sc}},\Delta_{\mathrm{sc}};\mathrm{KD}^{\mathrm{E,F}}_{3\mathrm{sc}})$ pull back to elements of

$$I^m_{\mathrm{os}}(X^2_{3\mathrm{sc}},\Delta_{3\mathrm{sc}};\mathrm{KD}^{\mathrm{E,F}}_{3\mathrm{sc}}),$$

it follows that $\beta^*_{3\mathrm{sc}}\,\Psi^{m,l}_{\mathrm{sc}}(X;E,F) \subset \Psi^{m,l}_{3\mathrm{sc}}(X;E,F)$. Before checking that multiplication by functions in $\mathcal{C}^\infty([X;C])$ is an element of $\Psi^{0,0}_{3\mathrm{sc}}(X;E)$ we modify this definition of the double space.

The problem is that if we consider the space $[X;C]$ instead of X as the base space, then with the single blow up (3.4) the projection to either factor of $[X;C]$ is not a b-fibration (it is not even a smooth map). It would have been reasonable to define $X^2_{3\mathrm{sc}}$ so that this problem does not arise in the first place, but then the triple space (which we need for the composition of operators) would have been much more complicated. In fact, even now it is easier to define two new spaces $X^2_{3\mathrm{sc},R}$ and

$X^2_{3sc,L}$ with b-maps (actually composite blow-down maps) $\beta_{3sc,L} : X^2_{3sc,L} \to X^2_{3sc}$ and $\beta_{3sc,R} : X^2_{3sc,R} \to X^2_{3sc}$ for which the corresponding projections

$$\pi^2_{3sc,L} : X^2_{3sc,L} \to [X;C], \qquad \pi^2_{3sc,R} : X^2_{3sc,R} \to [X;C] \tag{3.16}$$

are b-fibrations.

Let lf and rf be the left and right boundary hypersurfaces of X^2_{sc}, so lf is the lift of $\partial X \times X$ under $\beta_b \circ \beta_{sc}$, and rf is defined similarly. Let bf' be lift of bf under β_{sc}. Using the stretched projections $\pi^2_{sc,L}$, $\pi^2_{sc,R}$ we define

$$X^2_{3sc,L} = [X^2_{sc}; \beta^{-1}_{sc}(C_L \cap \partial\Delta_b); (\pi^2_{sc,L})^{-1}(C) \cap \text{bf'}; (\pi^2_{sc,L})^{-1}(C) \cap \text{lf}], \tag{3.17}$$

$$X^2_{3sc,R} = [X^2_{sc}; \beta^{-1}_{sc}(C_R \cap \partial\Delta_b); (\pi^2_{sc,R})^{-1}(C) \cap \text{bf'}; (\pi^2_{sc,R})^{-1}(C) \cap \text{lf}]. \tag{3.18}$$

Lemma 3.1. — *The stretched projections $\pi^2_{3sc,L} : X^2_{3sc,L} \to [X;C]$, $\pi^2_{3sc,R} : X^2_{3sc,R} \to [X;C]$ are b-fibrations.*

Proof. — We take $\pi^2_{3sc,L}$ in this proof for definiteness; by (3.3) $\pi^2_{3sc,R}$ can be dealt with the same way. First of all, by (3.5)

$$X^2_{3sc,L} = [X^2_b; C_L \cap \partial\Delta_b; \partial\Delta_b; C_L \cap \text{bf}; C_L \cap \text{lf}]. \tag{3.19}$$

Upon blowing up $C_L \cap \partial\Delta_b$ in X^2_b, $C_L \cap \text{bf}$ and $\partial\Delta_b$ lift to be disjoint p-submanifolds, so they can be blown up in either order. Moreover, the lift of $C_L \cap \text{lf}$ to $[X^2_b; C_L \cap \partial\Delta_b]$ is disjoint from the lift of $\partial\Delta_b$, so these two can be blown up in either order too. Thus,

$$X^2_{3sc,L} = [X^2_b; C_L \cap \partial\Delta_b; C_L \cap \text{bf}; C_L \cap \text{lf}; \partial\Delta_b]. \tag{3.20}$$

Since $C_L \cap \partial\Delta_b$ is a closed p-submanifold of $C_L \cap \text{bf}$ which is disjoint from $C_L \cap \text{lf}$ we see that

$$X^2_{3sc,L} = [X^2_b; C_L \cap \text{bf}; C_L \cap \text{lf}; C_L \cap \partial\Delta_b; \partial\Delta_b] \tag{3.21}$$

In addition, $C \times \partial X$ is a closed p-submanifold of $(\partial X)^2$ in X^2, so

$$\begin{aligned} [X^2_b; C_L \cap \text{bf}; C_L \cap \text{lf}] &= [X^2; (\partial X)^2; C \times \partial X; C \times X] \\ &= [X^2; C \times \partial X; (\partial X)^2; C \times X] \\ &= [X^2; C \times \partial X; C \times X; (\partial X)^2] \end{aligned} \tag{3.22}$$

where in the last step we used that upon blowing up $C \times \partial X$, $C \times X$ and $(\partial X)^2$ become disjoint. Finally, $C \times \partial X$ is a closed p-submanifold of $C \times X$ in X^2, and $[X^2; C \times X] = [X;C] \times X$, so (ff denoting the front face of the blow up $[X;C]$)

$$[X^2; C \times \partial X; C \times X] = [[X;C] \times X; \text{ff} \times \partial X]. \tag{3.23}$$

Putting together equations (3.19)-(3.23) we see that $X^2_{3sc,L}$ can be obtained from $[X;C] \times X$ by a series of blow ups. Since the left projection $[X;C] \times X \to [X;C]$ is a fibration (hence a b-fibration), and the blow down maps are b-maps, it follows that the stretched projection $\pi^2_{3sc,L}$, defined as the composite of the blow down maps and the left projection, is also a b-map; in fact, an interior b-map.

We now check that $\pi^2_{3\mathrm{sc},L}$ is actually a b-fibration. If Y is a manifold with corners, $p \in Y$, let $\mathrm{Fa}(p)$ denote the smallest boundary face of Y which contains p. A b-fibration, f, remains a b-submersion when composed with the blow up map of a closed p-submanifold M, if for each point $p \in M$ the induced map $f : M \to \mathrm{Fa}(f(p))$ is a b-submersion [**23**, Proof of Proposition 6]. For any boundary face M this is automatically satisfied. The composite map will be a b-fibration if $f(M)$ is a boundary hypersurface of the range space.

In our case we start with a fibration $\pi : [X; C] \times X \to [X; C]$. Since π maps ff $\times \partial X$ to the boundary hypersurface ff of $[X; C]$, π lifts to a b-fibration π_1. Next, π_1 maps the lift of mf $\times \partial X$ to mf in $[X; C]$, so blowing up this lift gives another b-fibration, π_2. Note that the lift of mf $\times \partial X$ to $[[X; C] \times X; \mathrm{ff} \times \partial X]$ is just the lift of $(\partial X)^2$ to $[X^2; C \times \partial X; C \times X]$; these two spaces are the same by (3.23). Thus, the composite of the left projection and the blow down maps of (3.22), π_2, is a b-fibration.

It remains to deal with the last two blow ups of (3.21). But these can be dealt with similarly. Namely, π_2 is a diffeomorphism from the lift of $C_L \cap \partial\Delta_{\mathrm{b}}$ (to (3.22)) to ff, so we obtain a new blown up b-fibration π_3. The lift of $\partial\Delta_{\mathrm{b}}$ to this new space is mapped to mf by π_3, and π_3 is a diffeomorphism between the lift and mf, so the composite of the blow down maps of (3.21) and the left projection, i.e. $\pi^2_{3\mathrm{sc},L}$, is a b-fibration as claimed. □

The following lemmas are very useful for taking care of the behavior of functions at irrelevant boundary faces. Recall that the blow down map β of a closed boundary p-submanifold S of Y gives an isomorphism $\beta^* : \mathcal{C}^{-\infty}(Y) \to \mathcal{C}^{-\infty}([Y; S])$.

Lemma 3.2. — *Suppose that S is a closed boundary p-submanifold of Y, and let $\beta : [Y; S] \to Y$ be the blow down map. If $u \in \mathcal{C}^{-\infty}([Y; S])$ is (polyhomogeneous) conormal to $\partial[Y; S]$ in a neighborhood of* ff*, the front face of the blow up, which vanishes to infinite order at* ff*, then $v = (\beta^*)^{-1}u \in \mathcal{C}^{-\infty}(Y)$ is (polyhomogeneous) conormal to ∂Y near S and vanishes to infinite order at S.*

Lemma 3.3. — *Suppose that Z is a closed interior p-submanifold of Y, and S is a boundary hypersurface of Z. Let $\beta : [Y, S] \to Y$ be the blow down map. If $u \in \mathcal{C}^{-\infty}([Y; S])$ is (polyhomogeneous) conormal to the lift of Z and to $\partial[Y; S]$ in a neighborhood of* ff*, the front face of the blow up, and it vanishes to infinite order at* ff *then $v = (\beta^*)^{-1}u \in \mathcal{C}^{-\infty}(Y)$ is (polyhomogeneous) conormal to Z and to ∂Y in a neighborhood of S and it vanishes to infinite order at S.*

Proof. — These lemmas follow from the fact that the vector fields used in the definition of the conormal spaces on Y lift to $[Y; S]$ with finite order singularities at ff. Since u is assumed to vanish to infinite order there, it follows that the lifts of these vector fields preserve the Sobolev-regularity of u. In case u is polyhomogeneous, it even has a polyhomogeneous development in terms of these operators. This proves both lemmas. □

Corollary 3.4. — *If* $A \in \Psi^{m,l}_{3\mathrm{scc}}(X;E,F)$ *then*

$$A : \dot{\mathcal{C}}^\infty([X;C];E) \to \dot{\mathcal{C}}^\infty([X;C];F). \tag{3.24}$$

If in addition $A \in \Psi^{m,l}_{3\mathrm{sc}}(X;E,F)$ *then*

$$A : \rho^k_{\mathrm{mf}}\rho^{k'}_{\mathrm{ff}}\mathcal{C}^\infty([X;C];E) \to \rho^{k+l}_{\mathrm{mf}}\rho^{k'+l}_{\mathrm{ff}}\mathcal{C}^\infty([X;C];F) \tag{3.25}$$

for all k, $k' \in \mathbb{R}$.

Proof. — The first statement is the easiest to check since $\dot{\mathcal{C}}^\infty([X;C]) = \beta^*\dot{\mathcal{C}}^\infty(X)$. Using $\pi^2_{\mathrm{sc},R}$ we can pull back $u \in \dot{\mathcal{C}}^\infty(X;E)$ to X^2_{sc}, and then by $\beta_{3\mathrm{sc}}$ to $X^2_{3\mathrm{sc}}$ the product with the kernel of A then vanishes to infinite order at the boundary. The standard push-forward theorem now gives the result.

To check the second statement note that if $u \in \rho^k_{\mathrm{mf}}\rho^{k'}_{\mathrm{ff}}\mathcal{C}^\infty(X;E)$ then

$$\begin{aligned} A(\pi^2_{3\mathrm{sc},R})^*u \in \beta^*(\rho^{k+l}_{\mathrm{sf'}}\rho^{k'+l}_{\mathrm{sf_C}})\mathcal{C}^\infty(X^2_{3\mathrm{sc},R};\mathrm{KD}^{1/2}_{3\mathrm{sc}} \otimes \pi^*_R{}^{\mathrm{sc}}\Omega^{1/2}(X) \\ \otimes \pi^*_L(F \otimes {}^{\mathrm{sc}}\Omega^{-1/2}(X))) \end{aligned} \tag{3.26}$$

(where a few pull backs by blow down maps are dropped in the notation), and it vanishes to infinite order on all faces but the lift of $\mathrm{sf_C}$ and sf'. Thus, Lemma 3.2 implies that the blow ups of $X^2_{3\mathrm{sc}}$ in (3.18) can be undone, and

$$A(\pi^2_{3\mathrm{sc},R})^*u \in \rho^{k+l}_{\mathrm{sf'}}\rho^{k'+l}_{\mathrm{sf_C}}\mathcal{C}^\infty(X^2_{3\mathrm{sc}};\mathrm{KD}^{1/2}_{3\mathrm{sc}} \otimes(\pi_L)^{*\mathrm{sc}}\Omega^{1/2}(X) \otimes (\pi_R)^*(F \otimes {}^{\mathrm{sc}}\Omega^{-1/2}(X))) \tag{3.27}$$

with infinite order vanishing off $\mathrm{sf_C}$ and sf'. Therefore, it can be pulled back to $X^2_{3\mathrm{sc},L}$ and then pushed forward by $\pi^2_{3\mathrm{sc},L}$ to $[X;C]$ with the result being in

$$\rho^{k+l}_{\mathrm{mf}}\rho^{k'+l}_{\mathrm{ff}}\mathcal{C}^\infty([X;C];F)$$

(see **[24]**) as claimed. □

Note that this proof also shows that $\mathcal{C}^\infty([X;C]) \subset \Psi^{0,0}_{3\mathrm{sc}}(X)$ as multiplication operators, since the kernel of $u \in \mathcal{C}^\infty([X;C])$ as an operator is just $u\,\mathrm{Id}$, Id denoting the kernel of the identity operator too. Thus, it is exactly (3.26), and hence (3.27), with $A = \mathrm{Id}$, E, F trivial; and the proof is similar for

$$\mathcal{C}^\infty([X;C]) \subset \Psi^{0,0}_{3\mathrm{sc}}(X;E). \tag{3.28}$$

Finally we discuss the alternative definition of this space of operators in terms of localization and quantization of symbols, as indicated at the beginning of this chapter. Thus, we can assume that $X = \mathbb{S}^N_+$ is the radial compactification of $\mathbb{R}^N$, (w,z) are coordinates on $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$, and

$$C = \mathrm{cl}(\mathrm{SP}(\{(w,0) : \ w \in \mathbb{R}^m\})) \tag{3.29}$$

with $\mathrm{SP} : \mathbb{R}^N \to \mathbb{S}^N_+$ being the map defined in (1.1). We also take E and F to be the trivial vector bundles for simplicity. Suppose that $a \in x^l\rho^{-m}_\infty\mathcal{C}^\infty([\mathbb{S}^N_+;C] \times \mathbb{S}^N_+)$ where ρ_∞ is the boundary defining function on the second factor, and x on the first factor of X. Removing the compactification of the second factor this simply means that a is a symbol on $[\mathbb{S}^N_+; C] \times \mathbb{R}^N_\xi$; in particular

$$|PD^\alpha_\xi a(p,\xi)| \le C_\alpha x^l \langle \xi \rangle^{m-|\alpha|} \tag{3.30}$$

if $P \in \mathrm{Diff}_{\mathrm{b}}([\mathbb{S}^N_+; C])$. The Weyl quantization of this symbol is

$$A(x,\theta,x',\theta') = \int e^{i(\theta/x-\theta'/x')\cdot\xi} a\Big(\frac{\theta}{2x} + \frac{\theta'}{2x'}, \xi\Big)\, d\xi. \tag{3.31}$$

Integration by parts shows that for all $Q \in \mathrm{Diff}(\mathbb{S}^N_+ \times \mathbb{S}^N_+)$

$$|QA(x,\theta,x',\theta')| \le C_{r,Q}\Big|\frac{\theta}{x} - \frac{\theta'}{x'}\Big|^{-r} \tag{3.32}$$

for all r everywhere where the right hand side makes sense. But, just as in case of the scattering calculus, this factor gives us smoothness and infinite order vanishing near all faces but $\mathrm{sf_C}$ and sf'.

Writing $\theta = (\theta_1, \theta_2)$ near C we can take $y = \theta_1$ and z to be some components of θ_2. With this choice the phase function lifts to be smooth in the interior of $\mathrm{sf_C} \cup \mathrm{sf'}$ and it is non-degenerate in the sense of **[17]** with critical points at $\Delta_{3\mathrm{sc}}$. Hence, we deduce:

Lemma 3.5. — *The set of operators on X obtained by localization and quantization of symbols $a \in x^l \rho_\infty^{-m} \mathcal{C}^\infty([\mathbb{S}^N_+; C] \times \mathbb{S}^N_+)$, where ρ_∞ is the boundary defining function of the second factor, is exactly $\Psi^{m,l}_{3\mathrm{sc}}(X)$.*

We also note what the estimate (3.30) becomes in terms of coordinates (w,z) on $\mathbb{R}^N$. Thus, C is the closure of the inverse image of $z = 0$ under the radial compactification. Then (3.30) is replaced by

$$|D^\alpha_w D^\beta_z D^\gamma_\xi a(w,z,\xi)| \le C_{\alpha,\beta,\gamma} \langle (w,z) \rangle^{-l-|\alpha|} \langle z \rangle^{-|\beta|} \langle \xi \rangle^{m-|\gamma|}. \tag{3.33}$$

CHAPTER 4

RESTRICTION TO THE BOUNDARY

In this chapter we examine the boundary restriction map for $\Psi_{3sc}^{\infty,-\infty}(X)$ which extends the corresponding map for $\mathrm{Diff}_{3sc}(X)$ by generalizing (2.30). Namely, due to Corollary 3.4, $A \in \Psi_{3sc}^{m,0}(X; E, F)$ defines an operator

$$A_\partial : \mathcal{C}^\infty(\partial[X; C]; E) \to \mathcal{C}^\infty(\partial[X; C]; F), \tag{4.1}$$

$$A_\partial u = A\widetilde{u}|_{\partial[X;C]}, \qquad \widetilde{u}|_{\partial[X;C]} = u, \qquad \widetilde{u} \in \mathcal{C}^\infty(X; E) \tag{4.2}$$

independently of the extension $\widetilde{u}$ of u. Here we denoted the pull back of the bundles E, F to the boundary by E and F as well. In the general case $A \in \Psi_{3sc}^{m,l}(X; E, F)$ the choice of a boundary defining function x of X gives an isomorphism

$$\Psi_{3sc}^{m,l}(X; E, F) \ni A \to x^{-l}A \in \Psi_{3sc}^{m,0}(X; E, F). \tag{4.3}$$

This depends on x, but if we then restrict to the boundary, $(x^{-l}A)_\partial$ it only depends on dx restricted to the boundary. Correspondingly we can change the bundles on which $x^{-l}A$ acts to obtain a natural boundary restriction map

$$\Psi_{3sc}^{m,l}(X; E, F) \ni A \to A_{\partial,l} = (x^{-l}A)_\partial, \tag{4.4}$$

$$A_{\partial,l} : \mathcal{C}^\infty(\partial[X; C]; E) \to \mathcal{C}^\infty(\partial[X; C]; |N^*\partial X|^{-l} \otimes F). \tag{4.5}$$

However, it is often convenient to trivialize $|N^*\partial X|$ by the choice of a boundary defining function and drop the additional bundle in (4.4). For example, if we have a scattering metric g on X, then it fixes x up to $O(x^2)$, i.e. it trivializes $N^*\partial X$.

It is useful to calculate the action of $A \in \Psi_{3sc}^{m,0}(X; E, F)$ in local coordinates. We first consider the mapping properties from the coordinate chart near $\mathrm{ff} \cap \mathrm{mf}$ to itself so we use coordinates

$$\widehat{x} = x/y_k, \widehat{Y}_j = y_j/y_k \ (j \neq k), y_k, z. \tag{4.6}$$

We also assume that E and F are trivial over this patch. Pulling back the coordinates on the right factor to the region where (3.9) are valid gives

$$\widehat{x}' = \widehat{x}\frac{1-\widehat{x}y_k S}{1-\widehat{x}Y_k},\ \widehat{Y}_j' = \frac{\widehat{Y}_j - \widehat{x}Y_j}{1-\widehat{x}Y_k}\ (j \neq k),\ y_k' = y_k(1-\widehat{x}Y_k),\ z' = z - y_k\widehat{x}Z. \tag{4.7}$$

Thus, the action of A on $u \in \mathcal{C}^\infty([X;C];E)$ supported in the region of validity of these coordinates gives

(4.8)

$$Au(\widehat{x}, \widehat{Y}_j, y_k, z) = \int A(\widehat{x}, y_k, \widehat{Y}_j, z, S, Y, Z) \\ u\Big(\widehat{x}\frac{1-\widehat{x}y_kS}{1-\widehat{x}Y_k}, \frac{\widehat{Y}_j - \widehat{x}Y_j}{1-\widehat{x}Y_k}, y_k(1-\widehat{x}Y_k), z - y_k\widehat{x}Z\Big)\, dS\, dY\, dZ.$$

It is interesting to see what happens when we restrict this to ff or mf. In these coordinates ff is given by $y_k = 0$, mf by $\widehat{x} = 0$. Thus, at mf

$$Au(0, \widehat{Y}_j, y_k, z) = \Big(\int A(0, y_k, \widehat{Y}_j, z, S, Y, Z)\, dS\, dY\, dZ\Big) u(0, \widehat{Y}_j, y_k, z). \tag{4.9}$$

That is, at mf, A_∂ is simply multiplication by

$$A_{\mathrm{mf}}(y_k, \widehat{Y}_j, z) = \int A(0, y_k, \widehat{Y}_j, z, S, Y, Z)\, dS\, dY\, dZ; \tag{4.10}$$

in particular it is local. At ff

$$Au(\widehat{x}, \widehat{Y}_j, 0, z) = \int A_{\mathrm{ff}}(\widehat{x}, \widehat{Y}_j, z, Y) u\Big(\frac{\widehat{x}}{1-\widehat{x}Y_k}, \frac{\widehat{Y}_j - \widehat{x}Y_j}{1-\widehat{x}Y_k}, 0, z\Big)\, dY \tag{4.11}$$

with

$$A_{\mathrm{ff}}(\widehat{x}, \widehat{Y}_j, z, Y) = \int A(\widehat{x}, 0, \widehat{Y}_j, z, S, Y, Z)\, dS\, dZ. \tag{4.12}$$

This is only local in the z variable, that is in the fibers of the blow up.

The same result would be obtained considering the coordinate chart in the interior of ff. In fact, pulling back the coordinates from the right factor to this region (where the coordinates are x, $\overline{Y}$, z, S, Y and Z) gives

$$x' = x(1-xS),\ \overline{Y}' = (1-xS)^{-1}(\overline{Y} - Y),\ z' = z - xZ. \tag{4.13}$$

(So $x, \overline{Y}, z, S, \overline{Y}', Z$ give another coordinate system in the interior of sf_{C}! This is the coordinate system used in the fibred cusp computations in **[23]**.) Thus, for $u \in \mathcal{C}^\infty([X;C];E)$

$$A_{\mathrm{ff}} u(\overline{Y}, z) = \int A_{\mathrm{ff}}(\overline{Y}, z, Y) u(0, \overline{Y} - Y, z)\, dY, \tag{4.14}$$

$$A_{\mathrm{ff}}(\overline{Y}, z, Y) = \int A(0, \overline{Y}, z, S, Y, Z)\, dS\, dZ. \tag{4.15}$$

Of course, we must consider mapping properties between different coordinate charts, but they again give similar answers.

We put this information together to construct a space of boundary operators. First note that $A_{\mathrm{ff}} = A_\partial|_{\mathrm{ff}}$ is a smooth family of pseudodifferential operators in

$$\Psi_{\mathrm{sc}}^{m,0}(\mathbb{S}_+^n; E, F) = \Psi_{\mathrm{sc}}^{m,0}(\beta^{-1}(p); E, F)$$

on C_p; of course, E and F are trivial over $\beta^{-1}(p)$. The set of such families will be denoted by $\Psi_{\mathrm{sc}-C}^{m,0}(\mathrm{ff}, E, F)$. Also note that the boundary operator of A_{ff} at $p \in \mathrm{ff} \cap \mathrm{mf}$ is just $A_{\mathrm{mf}}(p)$ where $A_{\mathrm{mf}} \in \mathcal{C}^\infty(\mathrm{mf}; \mathrm{Hom}(E, F))$ is the restriction of A_∂ to mf identified with the smooth section by which it is a multiplication. Let $S(X; C)$ denote the subspace of

$$\mathcal{C}^\infty(\mathrm{mf}, \mathrm{Hom}(E, F)) \oplus \Psi_{\mathrm{sc}-C}^{m,0}(\mathrm{ff}; E, F) \tag{4.16}$$

consisting of pairs (a, A_0) for which the restriction of A_0 to $\partial\,\mathrm{ff}$ at $p \in \partial\,\mathrm{ff} = \mathrm{ff} \cap \mathrm{mf}$ is just $a(p)$. We thus deduce:

Lemma 4.1. — *The boundary restriction map $A \mapsto A_\partial$ gives a surjective map to $S(X; C)$.*

There is significantly more information in A_{ff} than in A_{mf}. For example, if $A \in \mathcal{V}_{3\mathrm{sc}}(X)$, then A_∂ is given by the evaluation map ${}^{3\mathrm{sc}}T[X; C] \ni A \mapsto \iota(A) \in T[X; C]$. Thus, $A_{\mathrm{mf}} = 0$ directly from the definition of A_∂, since then $A = \rho_{\mathrm{mf}} V$, $V \in \mathcal{V}_{\mathrm{b}}([X; C])$, and $\mathcal{V}_{\mathrm{b}}([X; C]) : \mathcal{C}^\infty([X; C]) \to \mathcal{C}^\infty([X; C])$, but A_{ff} does not vanish necessarily. The precise relationship between the boundary operators at the two hypersurfaces will be discussed in Chapter 6.

Since $\Psi_{\mathrm{sc}}^{m,0}(X) \subset \Psi_{3\mathrm{sc}}^{m,0}(X)$, it is important to see how the boundary restriction behaves on the smaller algebra. For $p \in C$ we have defined the fiber of the relative scattering tangent bundle ${}^{\mathrm{sc}}T_p(C; X) \subset {}^{\mathrm{sc}}T_p X$ similarly to ${}^{\mathrm{b}}T_p(C; X)$, so $v \in {}^{\mathrm{sc}}T_p(C; X)$ if and only if $v = xV|_p$ for some $V \in \mathcal{V}_{\mathrm{b}}(X)$ with V_p tangent to C. Given a boundary defining function, x, the map $\mathcal{V}_{\mathrm{b}}(X) \ni V \mapsto xV \in \mathcal{V}_{\mathrm{sc}}(X)$ restricts to an isomorphism of ${}^{\mathrm{b}}T(C; X)$ with ${}^{\mathrm{sc}}T(C; X)$, but the isomorphism depends on the choice of x. We also recall that the normal operator for $V \in \mathcal{V}_{\mathrm{sc}}(X)$ at $p \in \partial X$ is given by $V_p \in {}^{\mathrm{sc}}T_p X$ lifted to a translation invariant vector field, $N_{\mathrm{sc},p}(V)$, on ${}^{\mathrm{sc}}T_p X$ (by the natural identification of ${}^{\mathrm{sc}}T_p X$ with the fibers of its tangent bundle).

Lemma 4.2. — *There is a natural transitive free affine action of the fibers of*

$${}^{\mathrm{sc}}TX / {}^{\mathrm{sc}}T(C; X) \to C$$

on the fibers $\beta^{-1}(p) \cap \mathrm{int}(\mathrm{ff})$, $p \in C$, such that if $A \in \Psi_{\mathrm{sc}}^{m,0}(X; E, F)$ then A_{ff} is translation invariant (i.e. invariant under this action). If $A \in \mathcal{V}_{\mathrm{sc}}(X)$ then A_{ff} is given by the push-forward of $N_{\mathrm{sc}}(A)$ by the differential of this action.

Proof. — If (x, y, z) are coordinates near p, x is a defining function of ∂X, C is defined by $x = 0$, $y = 0$, then we have coordinates

$$x,\ \overline{Y} = \frac{y}{x},\ z \tag{4.17}$$

near $\beta^{-1}(p) \cap \operatorname{int}(\mathrm{ff})$. We can write $v \in {}^{\mathrm{sc}}T_pX$ as

$$v = \alpha x^2 \partial_x + \sum_j \beta_j x \partial_{y_j} + \sum_j \gamma_j x \partial_{z_j}. \tag{4.18}$$

Now define

$$L_v(\overline{Y}, z) = (\overline{Y} + \beta, z). \tag{4.19}$$

If (x', y', z') is another coordinate system near p with properties as above, then

$$x' = a(x,y,z)x, \qquad y' = b(x,y,z)x + B(x,y,z)y \tag{4.20}$$

where B is a $\operatorname{codim} C - 1$ by $\operatorname{codim} C - 1$ matrix, b is a vector in $\mathbb{R}^{\operatorname{codim} C-1}$, $a(0,y,z) > 0$, $B(0,0,z)$ is invertible. It follows that

$$v = \alpha'(x')^2 \partial_{x'} + \sum_j \beta'_j x' \partial_{y'_j} + \sum_j \gamma'_j x' \partial_{z'_j}, \tag{4.21}$$

$$\beta'_j = \sum_k a(0,0,z(p))^{-1} B_{jk}(0,0,z(p)) \beta_k. \tag{4.22}$$

In addition,

$$\begin{aligned} \overline{Y}'_j = \frac{y'_j}{x'} &= a(x,y,z)^{-1} b_j(x,y,z) + \sum_k a(x,y,z)^{-1} B_{jk}(x,y,z) \overline{Y}_k \\ &= a(x,x\overline{Y},z)^{-1} b_j(x,x\overline{Y},z) + \sum_k a(x,x\overline{Y},z)^{-1} B_{jk}(x,x\overline{Y},z)\overline{Y}_k. \end{aligned} \tag{4.23}$$

Thus, on ff

$$\overline{Y}'_j = a(0,0,z)^{-1} b_j(0,0,z) + \sum_k a(0,0,z)^{-1} B_{jk}(0,0,z) \overline{Y}_k. \tag{4.24}$$

Hence, if we define L'_v as in (4.19), i.e. by

$$L'_v(\overline{Y}', z') = (\overline{Y}' + \beta', z') \tag{4.25}$$

then, e_j denoting the jth unit vector in $\mathbb{R}^{\operatorname{codim} C-1}$

(4.26)

$$\begin{aligned} L'_v(\overline{Y}'(\overline{Y},z), z'(\overline{Y},z)) = \Big(\sum_j (a(0,0,z)^{-1} b_j(0,0,z) + \sum_k a(0,0,z)^{-1} B_{jk}(0,0,z)\overline{Y}_k \\ + \sum_k a(0,0,z(p))^{-1} B_{jk}(0,0,z(p))\beta_k) e_j, z'(0,z) \Big) \\ = \big(\overline{Y}'(L_v(\overline{Y},z)), z'(L_v(\overline{Y},z))\big). \end{aligned}$$

Therefore, L_v is well-defined independently of the coordinates on X used in the definition. Moreover, by (4.19), L_v does not depend on α and γ, so L_v is in fact the lift of an affine action by the quotient ${}^{\mathrm{sc}}TX/{}^{\mathrm{sc}}T(C;X)$ as claimed.

We can see directly from the definition (4.19) that the action is transitive and free. We can write $V \in \mathcal{V}_{\mathrm{sc}}(X)$ in local coordinates as

$$V = \alpha x^2 \partial_x + \sum_j \beta_j x \partial_{y_j} + \sum_j \gamma_j x \partial_{z_j}. \tag{4.27}$$

Its lift to $\mathcal{V}([X;C])$ near int(ff) in the coordinates (4.17) is

$$V' = \alpha\Big(x^2\partial_x - \sum_j x\overline{Y}_j \partial_{\overline{Y}_j}\Big) + \sum_j \beta_j \partial_{\overline{Y}_j} + \sum_j \gamma_j x \partial_{z_j}. \tag{4.28}$$

Thus, for $q \in \mathrm{int}(\mathrm{ff})$, $V'(q) = \sum_j \beta_j \partial_{\overline{Y}_j}$, which is exactly the push-forward of $V(\beta^{-1}(q))$ by the action. Finally, due to (4.15), $A \in \Psi_{\mathrm{sc}}^{m,0}(X;E,F)$ means exactly that A_{ff} is independent of $\overline{Y}$, so A_{ff} has a convolution kernel, i.e. it is translation invariant. □

CHAPTER 5

COMPOSITION OF OPERATORS

In this chapter we prove that $\Psi_{3sc}^{\infty,-\infty}(X)$ is an algebra with multiplication given by operator composition. We first recall Melrose's definition of the scattering triple space X_{sc}^3 from **[25]**. The b-triple space is defined by the iterated blow up

$$X_b^3 = [X^3; (\partial X)^3; (\partial X)^2 \times X; \partial X \times X \times \partial X; X \times (\partial X)^2]. \tag{5.1}$$

The three partial diagonals lifted from X_b^2 by the stretched projections are p-submanifolds and intersect in pairs only in the triple diagonal; in particular, these pairwise intersections intersect the boundary of X_b^3 in the boundary, K, of the triple diagonal. The intersection of the lifted partial diagonals with the front face of the first blow up in (5.1) is denoted by G_O, $O = F, S, C$, and the other part (which is in the front face of one of the last three blow ups in (5.1)) by J_O. The intersection of any two of the G_O is K; the J_O do not meet each other, and meet only the corresponding G_O away from K.

If we blow up K the elements of $\mathcal{G} = \{G_F, G_S, G_C\}$ become disjoint. This allows us to define the scattering triple space

$$X_{sc}^3 = [X_b^3; K; \mathcal{G}; \mathcal{J}] \tag{5.2}$$

where $\mathcal{J} = \{J_F, J_S, J_C\}$. If we denote by B_O the last three boundary faces of (5.1), and $I = (\partial X)^3$ then we also have

$$X_{sc}^3 = [X_{sc}^2 \times X; I; B_S; B_C; K; J_F; G_S; G_C; J_S; J_C]. \tag{5.3}$$

Now, using the stretched projections $\pi_{b,O}^3$, C_L can be lifted from either double space to X_b^3; these will be denoted by C_L^O. Similarly to the construction of the double space we need to blow up the intersection of the C_L^O with the boundary of the lifted partial diagonals.

$$X_{3sc}^3 = [X_b^3; K; K \cap C_L^F; \mathcal{G}; \mathcal{G}_C; \mathcal{J}; \mathcal{J}_C]. \tag{5.4}$$

Here

$$\mathcal{G}_C = \{G_F \cap C_L^F, G_S \cap C_L^S, G_C \cap C_L^C\}, \tag{5.5}$$

and similarly

$$\mathcal{J}_C = \{J_F \cap C_L^F, J_S \cap C_L^S, J_C \cap C_L^C\}. \tag{5.6}$$

Note that

$$K \cap C_L^F = K \cap C_L^S = K \cap C_L^C. \tag{5.7}$$

The problem with this definition is that we were too economical in the definition of $X^2_{3\mathrm{sc}}$ (meaning that we had only a few blow ups), so this space is too big for the streched projection to give b-fibrations. So we also construct some intermediate spaces $X^3_{3\mathrm{sc},O}$ with composite blow down maps from $\beta_O : X^3_{3\mathrm{sc}} \to X^3_{3\mathrm{sc},O}$ for which the corresponding stretched projection $\pi^3_{3\mathrm{sc},O} : X^3_{3\mathrm{sc},O} \to X^2_{3\mathrm{sc}}$ is a b-fibration. Thus, let

$$X^3_{3\mathrm{sc},O} = [X^3_{\mathrm{b}}; K; K \cap C_L^O; G_O; G_O \cap C_L^O; J_O; J_O \cap C_L^O]. \tag{5.8}$$

Since the G_O are disjoint after the blow up of K in X^3_{b} and the J_O are disjoint from each other and from all but the corresponding G_O, the blow ups in (5.4) can be rearranged so that the first blow ups are exactly those of (5.8); here we also use (5.7). Thus, there is a composite blow down map (hence an interior b-map)

$$\beta_O : X^3_{3\mathrm{sc}} \to X^3_{3\mathrm{sc},O}. \tag{5.9}$$

Now we turn our attention to the stretched projections.

Lemma 5.1. — *The stretched projections $\pi^3_{3\mathrm{sc},O} : X^3_{3\mathrm{sc},O} \to X^2_{3\mathrm{sc}}$, $O = F, S, C$, are b-fibrations.*

Proof. — It suffices to prove the claim for $\pi^3_{3\mathrm{sc},F}$, say, due to the symmetry. In (5.8) the blow ups of K and $K \cap C_L^F$ can be interchanged as $K \cap C_L^F$ is a closed p-submanifold of K. Upon blowing up $K \cap C_L^F$, K and $G_F \cap C_L^F$ become disjoint, so they can be blown up in either order. Note that in (5.8) the blow ups of G_F and $G_F \cap C_L^F$ can also be interchanged. Thus,

$$X^3_{3\mathrm{sc},F} = [X^3_{\mathrm{b}}; K \cap C_L^F; G_F \cap C_L^F; G_F; K; J_F; J_F \cap C_L^F]. \tag{5.10}$$

Here we also used that K lifts to be a closed p-submanifold of G_F, so the order of their blow up is immaterial. Commuting $K \cap C_L^F$ through $G_F \cap C_L^F$ and G_F, and commuting J_F and $J_F \cap C_L^F$ to the front (these are disjoint from K and $K \cap C_L^F$) gives

$$X^3_{3\mathrm{sc},F} = [X^3_{\mathrm{b}}; G_F; G_F \cap C_L^F; J_F; J_F \cap C_L^F; K \cap C_L^F; K]. \tag{5.11}$$

Now, as $X^3_{\mathrm{b}} = [X^2_{\mathrm{b}} \times X; I; B_S; B_C]$, we can use that B_S and B_C are disjoint from the other faces of the blow up to reorder it. Furthermore, in $X^2_{\mathrm{b}} \times X$, $G_F \cap C_L^F \subset G_F \subset I$, so these blow ups can be interchanged too. Upon blowing up G_F, I and J_F become disjoint. Using these results we see that

$$X^3_{3\mathrm{sc},F} = [X^2_{\mathrm{b}} \times X; G_F \cap C^F_L; G_F; J_F \cap C^F_L; J_F; I; K \cap C^F_L; K; B_S; B_C]. \tag{5.12}$$

Finally we use that $G_F \subset J_F$, so

(5.13)
$$[X^2_{\mathrm{b}} \times X; G_F \cap C^F_L; G_F; J_F \cap C^F_L; J_F] = [X^2_{\mathrm{b}} \times X; J_F; J_F \cap C^F_L; G_F \cap C^F_L; G_F].$$

But due to the product structure of J_F and C^F_L

$$[X^2_{\mathrm{b}} \times X; J_F; J_F \cap C^F_L] = X^2_{3\mathrm{sc}} \times X. \tag{5.14}$$

It follows now that $X^3_{3\mathrm{sc},F}$ can be obtained from $X^2_{3\mathrm{sc}} \times X$ by a series of blow ups, hence the composite of the blow down maps and the projection to the first factor, $\pi^3_{3\mathrm{sc},F}$, is an interior b-map.

We proceed as in Lemma 3.1 to show that $\pi^3_{3\mathrm{sc},F}$ is a b-fibration. The projection $\pi : X^2_{3\mathrm{sc}} \times X \to X^2_{3\mathrm{sc}}$ is certainly a b-fibration. The next two blow ups on the right hand side of (5.13) involve $\mathrm{sf}_{\mathrm{C}} \times \partial X$ and sf' $\times \partial X$ which are mapped to the boundary hypersurfaces sf_{C} and sf' respectively by π, so (see the proof or Lemma 3.1) π lifts to a b-fibration π_1 of the space in (5.13). Next, I is the lift of bf $\times \partial X$ to (5.13), and π_1 maps it to bf, hence the lifted projection, π_2 is also a b-fibration. Similarly, B_S and B_C are the lifts of rf $\times \partial X$ and lf $\times \partial X$, so the lifted projection, π_3, is a b-fibration

$$\pi_3 : [X^3_{\mathrm{b}}; G_F; G_F \cap C^F_L; J_F; J_F \cap C^F_L] \to X^2_{3\mathrm{sc}}. \tag{5.15}$$

Here we used the remarks after (5.11) to rewrite the space obtained after the blow ups.

Now, $K \cap C^F_L$ is a submanifold of the front face of the blow up of $G_F \cap C^F_L$ in (5.15), and π_3 maps it to sf_{C}, while K is a submanifold of the front face of the blow up of G_F there; it is mapped to sf' by π_3. Hence, π_3 lifts to a b-fibration even after they are blown up. But, by (5.11), the space we have constructed with these blow ups is exactly $X^3_{3\mathrm{sc},F}$, so this proves the lemma. □

Proposition 5.2. — *If $A \in \Psi^{m,l}_{3\mathrm{sc}}(X; F, G)$, $B \in \Psi^{m',l'}_{3\mathrm{sc}}(X; E, F)$ then*

$$AB \in \Psi^{m+m',l+l'}_{3\mathrm{sc}}(X; E, G) \tag{5.16}$$

and

$$(AB)_\partial = A_\partial B_\partial. \tag{5.17}$$

If we only assume that $A \in \Psi^{m,l}_{3\mathrm{scc}}(X; F, G)$, $B \in \Psi^{m',l'}_{3\mathrm{scc}}(X; E, F)$ then we still have

$$AB \in \Psi^{m+m',l+l'}_{3\mathrm{scc}}(X; E, G). \tag{5.18}$$

Proof. — Suppose that $A \in \Psi^{m,l}_{3sc}(X; F, G)$, $B \in \Psi^{m',l'}_{3sc}(X; E, F)$. The kernel of the composite operator is just

$$AB = (\pi^3_{3sc,C})_*(\beta_C^{-1})^*((\beta_F^*(\pi^3_{3sc,F})^*A)(\beta_S^*(\pi^3_{3sc,S})^*B)). \tag{5.19}$$

Since all of the $\pi^3_{3sc,O}$ are b-fibrations, and the β_O are interior b-maps, the product is polyhomogeneous conormal on X^3_{3sc}. Moreover, at each boundary hypersurface of X^3_{3sc}, except at the lift of K and $K \cap C^F_L$, one of the two factors vanishes to infinite order, hence the same holds for the product. Thus, by Lemma 3.3 $(\beta_C^{-1})^*$ applied to the product gives a polyhomogeneous conormal distribution on $X^3_{3sc,C}$. As $\pi^3_{3sc,C}$ is a b-fibration it follows that the push-forward is polyhomogeneous conormal and vanishes to infinite order at all boundary hypersurfaces of X^2_{3sc} except sf' and sf_C, proving that $AB \in \Psi^{m+m',l+l'}_{3sc}(X; E, G)$. A similar argument without the polyhomogeneity claims proves (5.18) for $A \in \Psi^{m,l}_{3scc}(X; F, G)$, $B \in \Psi^{m',l'}_{3scc}(X; E, F)$.

Now assume again that $A \in \Psi^{m,l}_{3sc}(X; F, G)$, $B \in \Psi^{m',l'}_{3sc}(X; E, F)$, and let $u \in \mathcal{C}^\infty(\partial[X; C]; E)$, and let $\tilde{u} \in \mathcal{C}^\infty([X; C]; E)$ be such that $\tilde{u}|_{\partial[X;C]} = u$. Then

$$(AB)_\partial u = AB\tilde{u}|_{\partial[X;C]}, \qquad B_\partial u = B\tilde{u}|_{\partial[X;C]}. \tag{5.20}$$

But $\tilde{v} = B\tilde{u}$ is then a smooth extension of $B_\partial u$, so

$$A_\partial(B_\partial u) = A\tilde{v}|_{\partial[X;C]} = AB\tilde{u}|_{\partial[X;C]} = (AB)_\partial u \tag{5.21}$$

indeed. □

CHAPTER 6

THE NORMAL OPERATOR

In this chapter we define the principal symbol and the normal operator for $A \in \Psi^{m,l}_{3sc}(X)$ so that the vanishing of these two together will be equivalent to $A \in \Psi^{m-1,l+1}_{3sc}(X)$. First we restrict our attention to the case $l = 0$. The principal symbol map $\sigma_{3sc,m}$ is Hörmander's symbol map **[17]** for the kernel of A which is conormal to the diagonal Δ_{3sc}. The singularity coming from the density factor in (3.15) means that

$$\sigma_{3sc,m} : \Psi^{m,0}_{3sc}(X) \to S^m_h({}^{3sc}T^*X; \pi^* \operatorname{Hom}(E,F)) \tag{6.1}$$

where S^m_h is the space of mth order homogeneous sections of $\pi^* \operatorname{Hom}(E,F)$ over ${}^{3sc}T^*[X;C]$. We radially compactify the fibers of ${}^{3sc}T^*[X;C]$ and let ${}^{3sc}S^*[X;C]$ be the new boundary face (i.e. the boundary of ${}^{3sc}T^*[X;C]$ at fiber-infinity). This allows us to write $\sigma_{3sc,m}$ as a map

$$\sigma_{3sc,m} : \Psi^{m,0}_{3sc}(X) \to \mathcal{C}^\infty({}^{3sc}S^*[X;C]; (N^{*\,3sc}S^*[X;C])^m \otimes \pi^* \operatorname{Hom}(E,F)). \tag{6.2}$$

We then have a short exact sequence:

$$\begin{aligned} 0 \to \Psi^{m-1,0}_{3sc}(X) &\to \Psi^{m,0}_{3sc}(X) \\ &\to \mathcal{C}^\infty({}^{3sc}S^*[X;C]; (N^{*\,3sc}S^*[X;C])^m \otimes \pi^* \operatorname{Hom}(E,F)) \to 0 \end{aligned} \tag{6.3}$$

as usual.

To obtain a similar short exact sequence in the boundary weighting of $\Psi^{*,*}_{3sc}(X)$ we need more information than what is given by the boundary restriction map. As in **[23]**, this is done by conjugating by 'oscillatory test functions'. Thus, suppose that $f \in \mathcal{C}^\infty(\partial X)$. Choose $\tilde{f} \in \mathcal{C}^\infty(X)$ with $\tilde{f}|_{\partial X} = f$.

Lemma 6.1. — *For any $A \in \Psi^{m,l}_{3sc}(X;E,F)$*

$$\tilde{A} = e^{-i\tilde{f}/x} A e^{i\tilde{f}/x} \in \Psi^{m,l}_{3sc}(X;E,F). \tag{6.4}$$

Proof. — The kernel of $\widetilde{A}$ is $\widetilde{A} = e^{-i\widetilde{f}(x,y,z)/x+i\widetilde{f}(x',y',z')/x'}A$. The exponential factor is

$$\exp\big(i((1-xS)^{-1}\widetilde{f}(x(1-xS),y-xY,z-xZ)-\widetilde{f}(x,y,z))/x\big) \tag{6.5}$$

near sf. Now, $(1-xS)^{-1}\widetilde{f}(x(1-xS),y-xY,z-xZ)-\widetilde{f}(x,y,z)$ vanishes at $x=0$, so it is of the form

$$x\big(Sf(y,z)-Y\cdot\partial_y f(y,z)-Z\cdot\partial_z f(y,z)\big)+x^2 g(x,y,z,S,Y,Z) \tag{6.6}$$

with g smooth. It follows that (6.5) is smooth up to sf and its restriction to sf is

$$\exp\big(i(Sf(y,z)-Y\cdot\partial_y f(y,z)-Z\cdot\partial_z f(y,z))\big). \tag{6.7}$$

Although this exponential is not smooth up to the other faces of X^2_{sc}, it only has a finite order singularity there. Since the kernel of A vanishes to infinite order at the lift of these faces to to $X^2_{3\mathrm{sc}}$, it follows that $\widetilde{A}\in\Psi^{m,l}_{3\mathrm{sc}}(X;E,F)$. □

If $A\in\Psi^{m,l}_{3\mathrm{sc}}(X;E,F)$ then by (4.10) $\widetilde{A}_{\mathrm{mf},l}(y,z)$ only depends on $f(y,z)$ and $df(y,z)$, and similarly, by (4.12), $\widetilde{A}_{\mathrm{ff},l}(\widehat{x},\widehat{Y}_j,z,Y)$ only depends on $f(0,z)$ and $df(0,z)$. Moreover, the dependence of $\widetilde{A}_{\mathrm{ff},l}$ on $d_y f(0,z)$ is only via conjugation by a nonvanishing smooth function. At the operator level (as in (6.4)) this can be seen from the fact that if $f(0,z)$ and $d_z f(0,z)$ vanish, then $e^{i\widetilde{f}/x}$ extends from $\mathrm{int}([X;C])$ to be a smooth function on int(ff), since $\overline{Y}_j=y_j/x$ is a smooth function on the interior of ff. Hence, denoting $\widetilde{A}$ obtained from f via (6.4) by A^f, if $f_1(0,z)=f_2(0,z)$ and $d_z f_1(0,z)=d_z f_2(0,z)$ then $A^{f_1}_{\mathrm{ff},l}(z)$ and $A^{f_2}_{\mathrm{ff},l}(z)$ are unitarily equivalent on $L^2(\mathbb{S}^n_+;E_z)=L^2(\beta^{-1}(p);E)$; L^2 is taken with respect to any translation invariant metric (in the sense of Lemma 4.2).

A convenient way of incorporating the information about both $f(0,z)$ and $df(0,z)$ is to consider

$$d\Big(\frac{\widetilde{f}}{x}\Big)=-\widetilde{f}\,\frac{dx}{x^2}+\frac{d\widetilde{f}}{x}\in\mathcal{C}^\infty(X;{}^{\mathrm{sc}}T^*X). \tag{6.8}$$

Then

$$d\Big(\frac{\widetilde{f}}{x}\Big)(0,z)=-f(0,z)\,\frac{dx}{x^2}+\frac{d_{(y,z)}f(0,z)}{x}. \tag{6.9}$$

Hence, the statements of the previous paragraph show that $\widetilde{A}_{\mathrm{ff},l}(p)$ only depends on $d(f/x)(0,z)\in{}^{\mathrm{sc}}T^*_pX$, $p=(0,z)\in C$, and its dependence on $x^{-1}\,d_y f(0,z)$ is somewhat redundant.

To eliminate the ambiguity we choose a subbundle $W\to C$ of ${}^{\mathrm{sc}}TX$ which is complementary to ${}^{\mathrm{sc}}T(C;X)$. Such a splitting arises naturally if we have a scattering metric on X, for it gives an inner product on ${}^{\mathrm{sc}}TX$, and we can take W to be the orthocomplement of ${}^{\mathrm{sc}}T(C;X)$. This induces a corresponding splitting of ${}^{\mathrm{sc}}T^*X$ over C, with $W^\perp\subset{}^{\mathrm{sc}}T^*_CX$ being the annihilator of W. We can now choose local coordinates x, y, z near $p\in C$ such that $x=0$ defines ∂X, $x=0$, $y=0$ define C, and $x\partial_{y_j}$, $j=1,\dots,\operatorname{codim}C-1$, give a basis of W. This means exactly that dx/x^2

and dz_j/x, $j = 1, \dots, \dim C$, are a basis of $W^\perp$. It follows from the discussion of the previous paragraph that we do not lose any information if we require $d(f/x) \in W^\perp$ when defining $\widehat{A}_{\mathrm{ff}}$. Note that the choice of a boundary defining function x, modulo $x^2\mathcal{C}^\infty(X)$, fixes dx/x^2 as an element of ${}^{\mathrm{sc}}T^*_{\partial X}X$, so in this case W induces a splitting of $T_p\partial X$ by defining a complementary bundle $\widetilde{W}$ of TC. In particular, this is the case if we are given a scattering metric g on X.

Definition 6.2. — For $A \in \Psi^{m,l}_{3\mathrm{sc}}(X; E, F)$ the indicial operator

$$\widehat{A}_{\mathrm{ff},l}(p, \tau, \nu) \in \Psi^{m,0}_{\mathrm{sc}}(\beta^{-1}(p); E, F \otimes |N^*\partial X|^{-l}), \tag{6.10}$$

$p \in C$, $(p, \tau, \nu) \in W^\perp$ is the restriction $A^f_{\mathrm{ff},l}$ with A^f given by (6.4) with $f(p) = -\tau$, $df(p) = \nu$ (i.e. $d(f/x) = \tau(dx/x^2) + \nu/x$). Similarly,

$$\widehat{A}_{\mathrm{mf}}(p, \tau, \overline{\xi}) \in \mathrm{Hom}(E, F) \otimes |N^*\partial X|^{-l}, \tag{6.11}$$

$(p, \overline{\xi}) \in T^*_p\,\mathrm{mf}$, is $A^f_{\mathrm{mf},l}$ with $f(p) = -\tau$, $df(p) = \overline{\xi}$. We often write $\widehat{A}_{\mathrm{mf},0} = \widehat{A}_{\mathrm{mf}}$, $\widehat{A}_{\mathrm{ff},0} = \widehat{A}_{\mathrm{ff}}$.

Lemma 6.3. — *For each $p \in C$, $(\tau, \nu) \in W^\perp_p$, the indicial operators at* ff *and* mf *give multiplicative homomorphisms*

$$\Psi^{m,l}_{3\mathrm{sc}}(X; E, F) \to \Psi^{m,0}_{\mathrm{sc}}(\beta^{-1}(p); E, F \otimes |N^*\partial X|^{-l}), \tag{6.12}$$

$$\Psi^{m,l}_{3\mathrm{sc}}(X; E, F) \to \mathcal{C}^\infty(\mathrm{mf}; \mathrm{Hom}(E, F \otimes |N^*\partial X|^{-l})) \tag{6.13}$$

respectively. If $A \in \Psi^{m,l}_{3\mathrm{sc}}(X; E, F)$ and $\widehat{A}_{\mathrm{ff},l}$, $\widehat{A}_{\mathrm{mf},l}$ vanish identically then $A \in \Psi^{m,l+1}_{3\mathrm{sc}}(X; E, F)$.

Proof. — The multiplicative property follows from

$$e^{-i\tilde{f}/x} A B e^{i\tilde{f}/x} = (e^{-i\tilde{f}/x} A e^{i\tilde{f}/x})(e^{-i\tilde{f}/x} B e^{i\tilde{f}/x}). \tag{6.14}$$

Since for $A \in \Psi^{m,0}_{3\mathrm{sc}}(X; E, F)$

$$\widehat{A}_{\mathrm{ff}}(\overline{Y}, z, Y, \tau, \nu) = \int e^{i(-S\tau - Z\cdot\nu)} A(0, \overline{Y}, z, S, Y, Z)\, dS\, dZ, \tag{6.15}$$

the vanishing of $\widehat{A}_{\mathrm{ff}}(z, \tau, \nu)$ for all z, τ and ν means that the partial Fourier transform (6.15) vanishes identically, so (by taking the inverse Fourier transform) we see that the kernel of A vanishes identically when restricted to sf_C. In the case of $\widehat{A}_{\mathrm{mf}}$ vanishing means that

$$\widehat{A}_{\mathrm{mf}}(y_k, \widehat{Y}_j, z, \tau, \overline{\xi}) = \int e^{i(-S\tau - (Y,Z)\cdot\overline{\xi})} A(0, y_k, \widehat{Y}_j, z, S, Y, Z)\, dS\, dY\, dZ \equiv 0, \tag{6.16}$$

so by taking the inverse Fourier transform we deduce that A vanishes when restricted to sf'. But the vanishing of A at these two boundary hypersurfaces means that $A = xA'$, $A' \in \Psi^{m,0}_{3\mathrm{sc}}(X; E, F)$, i.e. that $A \in \Psi^{m,1}_{3\mathrm{sc}}(X; E, F)$. In the case $A \in \Psi^{m,l}_{3\mathrm{sc}}(X; E, F)$ we only have to note that $A \to x^{-l}A$ is a bijection. □

One of the main differences between the indicial operators at mf and at ff (and hence between $\Psi_{sc}^{m,l}(X)$ and $\Psi_{3sc}^{m,l}(X)$) is that the former maps into a commutative algebra while the latter does not. Thus, for $A, B \in \Psi_{3sc}^{m,0}(X)$, $\widehat{[A,B]}_{\mathrm{mf}} = 0$, but $\widehat{[A,B]}_{\mathrm{ff}}$ does not necessarily vanish. Since commutation properties are very important in spectral theory, we are interested in finding the pseudo-differential operators which commute with all others to 'top order'. We thus make the following definition:

Definition 6.4. — We say that $A \in Z\,\Psi_{3sc}^{m,l}(X;E)$, if for all $B \in \Psi_{3sc}^{m',l'}(X;E)$

$$[A,B] \in \Psi_{3sc}^{m+m'-1,l+l'+1}(X;E). \tag{6.17}$$

Lemma 6.5. — *Let $A \in \Psi_{3sc}^{m,l}(X;E)$. Then $A \in Z\,\Psi_{3sc}^{m,l}(X;E)$ if and only if*

$$\widehat{A}_{\mathrm{ff},l}(p,\tau,\nu) = a(p,\tau,\nu)\,\mathrm{Id}, \qquad a \in \mathcal{C}^\infty(W^\perp). \tag{6.18}$$

Proof. — Since multiples of the identity operators commute with all operators and the indicial operator is multiplicative, if (6.18) holds then $\widehat{[A,B]}_{\mathrm{ff},l+l'} = 0$. Thus, by Lemma 6.3 (and the commutativity of the indicial operator at mf and of the principal symbol map) (6.17) holds.

On the other hand, for each $p \in C$, $(\tau,\nu) \in W_p^\perp$, the indicial operator gives a surjective map

$$\Psi_{3sc}^{m',0}(X;E) \ni B \mapsto \widehat{B}_{\mathrm{ff}}(p,\tau,\nu) \in \Psi_{sc}^{m',0}(\beta^{-1}(p);E). \tag{6.19}$$

Since the center of $\Psi_{sc}^{\infty,-\infty}(\beta^{-1}(p))$ consists of multiples of the identity map (see e.g. [**18**, Lemma 7.1.4]), (6.18) follows. □

Remark 6.6. — The subalgebra of $\Psi_{sc}^{\infty,-\infty}(X)$ generated by $\mathcal{V}_{sc}(X;C)$ (over $\mathcal{C}^\infty(X)$) certainly lies in $Z\,\Psi_{3sc}^{\infty,-\infty}(X)$. In fact, for $g \in \mathcal{C}^\infty(X)$,

$$\widehat{g}_{\mathrm{ff},0}(p,\tau,\nu) = g(p), \tag{6.20}$$

and for $V \in \mathcal{V}_{sc}(X;C)$,

$$\widehat{V}_{\mathrm{ff},0}(p,\tau,\nu) = \alpha(p)\tau + \sum_j \gamma_j(p)\nu_j \tag{6.21}$$

if $V(p) = \alpha(p)x^2\partial_x + \sum_j \gamma_j(p)x\partial_{z_j}$.

We now define the normal operators which contain the same information as the indicial operators but which are sometimes more convenient. First let M be a compact manifold with boundary and let V be a real vector space. Generalizing the results of [**23**] we define the V-suspended algebra of scattering pseudo-differential operators on M, denoted $\Psi_{\mathrm{sus}(V),\mathrm{sc}}^{m,l}(M)$. We do so by defining their kernels as convolution operators in V, i.e. we demand that

$$A \in \mathcal{C}^{-\infty}(M_{sc}^2 \times V; {}^{sc}\Omega_R) \tag{6.22}$$

is polyhomogeneous conormal to $\Delta_{\mathrm{sc}} \times \{0\}$ and $\mathrm{sf} \times V$ of order m and l respectively, decays rapidly at ∞ (in V) with all derivatives, vanishes to infinite order on all other boundary faces, and acts on $\mathcal{S}(\mathrm{int}(M) \times V)$ as

$$Au(m,v) = \int A(m,m',v-v')u(v')\,dm'\,dv'. \tag{6.23}$$

We could rephrase the definition by radially compactifying V to $\overline{V}$, and demanding that $A \in \mathcal{C}^{-\infty}(M^2_{\mathrm{sc}} \times \overline{V}; {}^{\mathrm{sc}}\Omega_R)$ should be conormal to $\Delta_{\mathrm{sc}} \times \{0\}$ and $\mathrm{sf} \times \overline{V}$ and vanish to infinite order on all other boundary faces. Here ${}^{\mathrm{sc}}\Omega_R$ is the pull back of ${}^{\mathrm{sc}}\Omega(M) \otimes {}^{\mathrm{sc}}\Omega(\overline{V})$ by the right projection $\pi_R : M^2_{\mathrm{sc}} \times V \to M \times V$.

We also need to define a corresponding algebra associated to operators mapping sections of a vector bundle $E' \to M \times V$ to another one $F' \to M \times V$. For the V-convolution structure (i.e. the related translation invariance) to make sense, we require that E' and F' are pull backs of vector bundles $E \to M$ and $V \to M$. Then $\Psi^{m,l}_{\mathrm{sus}(V),\mathrm{sc}}(M; E, F)$ is defined as was $\Psi^{m,l}_{\mathrm{sus}(V),\mathrm{sc}}(M)$, except that (6.22) must be replaced by

$$A \in \mathcal{C}^{-\infty}(M^2_{\mathrm{sc}} \times V; {}^{\mathrm{sc}}\Omega_R \otimes \pi^* \operatorname{Hom}(E,F)). \tag{6.24}$$

Here $\pi : M^2_{\mathrm{sc}} \times V \to M^2$ is the projection.

Since $\Psi^{m,l}_{\mathrm{sus}(V),\mathrm{sc}}(M; E, F)$ is invariant under diffeomorphisms of M, linear transformations of V, and bundle transformations of E and F over M, we can define the analogous object for vector bundles over a manifold C.

Definition 6.7. — Suppose that $V \to C$ is a real vector bundle over a compact manifold, Y a compact manifold with boundary and $\beta : Y \to C$ is a fibration. The V-suspended scattering double space is

$$Y^2_{\mathrm{sus}(V)-C,\mathrm{sc}} = [Y \times_C Y \times_C V; \partial Y \times_C \partial Y \times_C V; \Delta_{\mathrm{b},Y}] \tag{6.25}$$

where $\Delta_{\mathrm{b},Y}$ is the lift of the Y-diagonal,

$$\{(y,y',v) : \ y = y',\ \beta(y) = \beta(v)\} \subset Y \times_C Y \times_C V, \tag{6.26}$$

to the first blow up. The front face of the last blow up is denoted by $\mathrm{sf}_{\mathrm{sus}(V)}$. The V-suspended scattering diagonal, $\Delta_{\mathrm{sus-sc}}$, is the lift of

$$\{(y,y',o) : \ y = y',\ \beta(y) = \beta(o)\}, \tag{6.27}$$

o denoting elements of the zero section of V.

We now define the generalization of $\Psi^{m,l}_{\mathrm{sus}(V),\mathrm{sc}}(M; E, F)$ when M is a fiber of a fibration $\beta : Y \to C$ over a compact manifold C. Here we also need to generalize V to a real vector bundle over C. Thus, we want elements of the new algebra to be a smooth family of operators on C_p with values in $\Psi^{m,l}_{\mathrm{sus}(V_p),\mathrm{sc}}(\beta^{-1}(p); E, F)$. More precisely, we make the following definition.

Definition 6.8. — Suppose that $V \to C$ is a real vector bundle over a compact manifold, $E \to Y$, $F \to Y$ are vector bundles, Y a compact manifold with boundary and $Y \to C$ is a fibration. The algebra of V-suspended scattering pseudo-differential operators, $\Psi^{m,l}_{\mathrm{sus}(V)-C,\mathrm{sc}}(Y; E, F)$, is the space of operators with V-convolution kernel $A \in \mathcal{C}^{-\infty}(Y^2_{\mathrm{sus}(V)-C,\mathrm{sc}}; {}^{\mathrm{sc}}\Omega_R \otimes \pi^* \operatorname{Hom}(E, F))$ which are conormal to $\Delta_{\mathrm{sus-sc}}$ and to $\mathrm{sf}_{\mathrm{sus}(V)}$, vanish to infinite order on all other boundary faces, and decay rapidly with all derivatives at infinity in V. Here $\pi : Y^2_{\mathrm{sus}(V)-C,\mathrm{sc}} \to Y \times_C Y$ is the projection.

We can finally define the normal operator at the front face essentially as the restriction of the kernel to sf_C; or as the inverse Fourier transform of the indicial operators (cf. (6.15)). We actually have slightly more structure than this (after all, we want to realize the normal operator as an operator). First we note the lift of the basis vector fields on the left factor of X^2 (as in (2.8)) from X^2 to $X^2_{3\mathrm{sc}}$. Namely, as calculated in **[25]**, they lift to

$$\partial_S + x\mathcal{V}_{\mathrm{b}}(X^2_{3\mathrm{sc}}),\ \partial_Y + x\mathcal{V}_{\mathrm{b}}(X^2_{3\mathrm{sc}}),\ \partial_Z + x\mathcal{V}_{\mathrm{b}}(X^2_{3\mathrm{sc}}) \tag{6.28}$$

in both coordinate systems (3.8) and (3.9). In particular, restricted to sf_C they become ∂_S, ∂_Y and ∂_Z respectively. This means that we can naturally identify ${}^{\mathrm{sc}}T_pX$ with the fibers $z = z(p)$, $\overline{Y} = \mathit{const}$ of $\mathrm{int}(\mathrm{sf}_C)$ since the lift from the left factor gives translation invariant vector fields on the these fibers which can be identified with points of the fibers. Thus, we have a natural identification of ${}^{3\mathrm{sc}}T_{\mathrm{ff}}[X; C]$ with sf_C. Hence, the subspace $\beta^{*}_{\mathrm{ff}}{}^{\mathrm{sc}}T(C; X)$ is also identified with a submanifold of sf_C, namely with $Y = 0$. More generally, the lift of ${}^{\mathrm{sc}}T(C; X)$ gives a 'distribution' on sf_C whose integral submanifolds correspond to elements of ${}^{3\mathrm{sc}}T_{\mathrm{ff}}[X; C]$ with the same image in the tangent space $T\beta^{-1}(p)$. These are the submanifolds $Y = \mathit{const}$. Now the splitting ${}^{\mathrm{sc}}TX = W \oplus {}^{\mathrm{sc}}T(C; X)$ over C means that we have a splitting ${}^{3\mathrm{sc}}T_q[X; C] = T_q\beta^{-1}(p) \oplus \beta^{*}_q{}^{\mathrm{sc}}T(C; X)$. We can identify $T\beta^{-1}(p)$ with $(\beta^{-1}(p))^2$ by the exponential map, that is by $(\overline{Y}, \beta) \mapsto (\overline{Y}, \overline{Y} - \beta)$, which gives us an identification of sf_C with $\mathrm{ff} \times_C \mathrm{ff} \times_C {}^{\mathrm{sc}}T(C; X)$. Thus, we are exactly in the setting of Definition 6.8 with $\beta : \mathrm{ff} \to C$ a fibration, and $V = {}^{\mathrm{sc}}T(C; X) \to C$ a vector bundle. We can then regard the restriction of the kernel of $A \in \Psi^{m,0}_{3\mathrm{sc}}(X)$ to sf_C as a distribution on this space, and directly from the definition of $\Psi^{m,0}_{3\mathrm{sc}}(X)$, we obtain an element, $N_{\mathrm{ff},0}(A)$, of $\Psi^{m,0}_{\mathrm{sus}(V)-C,\mathrm{sc}}(\mathrm{ff})$.

There is a better way of thinking about this which is more analogous to **[23]**. As noted in Chapter 4, Y can be replaced by $\overline{Y}'$ as a coordinate on sf_C: $Y = \overline{Y} - \overline{Y}'$. In these coordinates the identification of $(\beta^{-1}(p))^2 \times {}^{\mathrm{sc}}T_p(C; X)$ with the submanifolds of sf_C given by $z = \mathit{const}$ is more natural, but it still depends on the choice of W since there is no natural origin $S = 0$, $Z = 0$ to correspond to the fibers of ${}^{\mathrm{sc}}T(C; X)$.

Since the convolution kernel of

$$N_{\mathrm{ff},0,p} : \Psi^{m,0}_{3\mathrm{sc}}(X) \to \Psi^{m,0}_{\mathrm{sus}(V),\mathrm{sc}}(\beta^{-1}(p)), \tag{6.29}$$

as in (6.22), is just the inverse Fourier transform of the indicial operator over $\beta^{-1}(p)$ (giving a distributional density as required) as follows from (6.15), multiplicativity of $N_{\mathrm{ff},0}$ follows from the corresponding property of indicial operators. For general $l \in \mathbb{R}$, $A \in \Psi_{3\mathrm{sc}}^{m,l}(X)$, we define $N_{\mathrm{ff},l}(A) = N_{\mathrm{ff},0}(x^{-l}A)$. We thus conclude:

Proposition 6.9. — *The normal operator at the front face, $N_{\mathrm{ff},l}$, gives a multiplicative short exact sequence*

$$\begin{aligned} 0 \to & \rho_{\mathrm{sf}_C}\, \Psi_{3\mathrm{sc}}^{m,l}(X;E,F) \hookrightarrow \\ & \to \Psi_{3\mathrm{sc}}^{m,l}(X;E,F) \xrightarrow{N_{\mathrm{ff},l}} \Psi_{\mathrm{sus}(V)-C,\mathrm{sc}}^{m,0}(\mathrm{ff};E,F \otimes |N^*\partial X|^{-l}) \to 0 \end{aligned} \tag{6.30}$$

with $V = {}^{\mathrm{sc}}T(C;X)$, the relative scattering tangent bundle of C in X.

CHAPTER 7

COMMUTATORS

The proof of the propagation of singularities used in this paper is based on a positive commutator estimate. We thus proceed to compute the commutator of $A \in \Psi_{3sc}^{m,0}(X)$, $B \in \Psi_{3sc}^{m',0}(X)$. As we saw in Chapter 6, in general we only have $[A,B] \in \rho_{mf}\,\Psi_{3sc}^{m+m'-1,0}(X)$, but if $[\widehat{A}_{ff}(\xi), \widehat{B}_{ff}(\xi)] = 0$ for all $\xi \in W^\perp$, then $[A,B] \in \Psi_{3sc}^{m+m'-1,1}(X)$. Since this happens in many interesting cases, we need to compute $[A,B]$ modulo $\Psi_{3sc}^{m+m'-2,2}(X)$. In fact, we are interested in $\widehat{[A,B]}_{ff,1}$ (under the assumption that $[A,B] \in \Psi_{3sc}^{m+m'-1,1}(X)$), so it suffices to compute $[A,B]u$ for $u \in \dot{\mathcal{C}}_{ff}^\infty([X;C])$.

Lemma 7.1. — *If $u \in \dot{\mathcal{C}}_{ff}^\infty([X;C])$, $A \in \Psi_{3sc}^{m,0}(X)$, then $Au \in \dot{\mathcal{C}}_{ff}^\infty([X;C])$ and*

(7.1)
$$Au = A_{ff}u + x((\partial_x A)u + A_{ff}(\partial_x u) - (D_\tau \widehat{A}_{ff}(0))(\overline{Y}\partial_{\overline{Y}}u) + (D_\nu \widehat{A}_{ff}(0))(\partial_z u)) \quad \mathit{mod}\ x^2\dot{\mathcal{C}}_{ff}^\infty([X;C]).$$

Proof. — Since in the local coordinates valid in the interior of ff (which suffice as $u \in \dot{\mathcal{C}}_{ff}^\infty([X;C])$)

$$Au(x,\overline{Y},z) = \int A(x,\overline{Y},z,S,Y,Z)u\Big(x(1-xS), \frac{\overline{Y}-Y}{1-xS}, z-xZ\Big)\,dS\,dY\,dZ, \tag{7.2}$$

differentiation with respect to x gives

(7.3)
$$\begin{aligned}\partial_x Au(x,\overline{Y},z) = \int (\partial_x A)(x,\overline{Y},z,S,Y,Z)u\Big(x(1-xS), \frac{\overline{Y}-Y}{1-xS}, z-xZ\Big)\,dS\,dY\,dZ \\ + \int A(x,\overline{Y},z,S,Y,Z)\Big((1-2Sx)\partial_x + S\frac{\overline{Y}-Y}{(1-xS)^2}\partial_{\overline{Y}} - Z\partial_z\Big) \\ \cdot u\Big(x(1-xS), \frac{\overline{Y}-Y}{1-xS}, z-xZ\Big)\,dS\,dY\,dZ.\end{aligned}$$

Restricting this to ff, i.e. letting $x = 0$, gives

(7.4)
$$\partial_x Au(0,\overline{Y},z) = \int (\partial_x A)(0,\overline{Y},z,S,Y,Z)u(0,\overline{Y}-Y,z)\,dS\,dY\,dZ + \int A(0,\overline{Y},z,S,Y,Z)(\partial_x + S(\overline{Y}-Y)\partial_{\overline{Y}} - Z\partial_z)u(0,\overline{Y}-Y,z)\,dS\,dY\,dZ.$$

Now, the first term is just $(\partial_x A)_{\mathrm{ff}}u$, ∂_x denoting the derivative of the kernel of A; here $\partial_x A \in \Psi^{m,0}_{3\mathrm{sc}}(X)$ since the kernel is in the appropriate space. The second term is $A_{\mathrm{ff}}(\partial_x u)$, while the last term is

(7.5)
$$-\int\Big(\int ZA(0,\overline{Y},z,S,Y,Z)\,dS\,dZ\Big)(\partial_z u)(0,\overline{Y}-Y,z)\,dY = \int D_\nu \mathcal{F}_{S,Z}A(0,\overline{Y},z,0,Y,0)(\partial_z u)(0,\overline{Y}-Y,z)\,dY = (D_\nu \widehat{A}_{\mathrm{ff}}(0))(\partial_z u),$$

and the third is of similar form taking into account that

$$(\overline{Y}-Y)\partial_{\overline{Y}}u(0,\overline{Y}-Y,z) = (\overline{Y}\partial_{\overline{Y}}u)(0,\overline{Y}-Y,z). \tag{7.6}$$

Since $Au - Au|_{\mathrm{ff}} - x(\partial_x Au)|_{\mathrm{ff}} \in x^2\mathcal{C}^\infty(X)$, this proves the lemma. □

We can now discuss commutators.

Lemma 7.2. — *If $A \in \Psi^{m,0}_{3\mathrm{sc}}$, $B \in \Psi^{m',0}_{3\mathrm{sc}}(X)$ and $u \in \dot{\mathcal{C}}^\infty_{\mathrm{ff}}([X;C])$, then*

$$\begin{aligned}[B,A]u = {} & [B_{\mathrm{ff}},A_{\mathrm{ff}}]u + x[B_{\mathrm{ff}},A_{\mathrm{ff}}]\partial_x u + x[(\partial_x B)_{\mathrm{ff}},A_{\mathrm{ff}}]u + x[B_{\mathrm{ff}},(\partial_x A)_{\mathrm{ff}}]u \\ & + x([D_\nu \widehat{A}_{\mathrm{ff}}(0),B_{\mathrm{ff}}] + [A_{\mathrm{ff}},D_\nu \widehat{B}_{\mathrm{ff}}(0)])\partial_z u + xD_\tau \widehat{A}_{\mathrm{ff}}(0)[\overline{Y},B]\partial_{\overline{Y}}u \\ & - xD_\tau \widehat{B}_{\mathrm{ff}}(0)[\overline{Y},A]\partial_{\overline{Y}}u - x([B_{\mathrm{ff}},D_\tau \widehat{A}_{\mathrm{ff}}(0)] - [D_\tau \widehat{B}_{\mathrm{ff}},A_{\mathrm{ff}}])\overline{Y}\partial_{\overline{Y}}u \\ & - x(D_\tau \widehat{B}_{\mathrm{ff}}(0)(\overline{Y}\partial_{\overline{Y}}A) - D_\tau \widehat{A}_{\mathrm{ff}}(0)(\overline{Y}\partial_{\overline{Y}}B))u \\ & + x(D_\nu \widehat{B}_{\mathrm{ff}}(0)(\partial_z A_{\mathrm{ff}}) - D_\nu \widehat{A}_{\mathrm{ff}}(0)(\partial_z B_{\mathrm{ff}}))u \\ & \qquad (\mathit{mod}\ x^2\dot{\mathcal{C}}^\infty_{\mathrm{ff}}([X;C])).\end{aligned} \tag{7.7}$$

Proof. — This is just an application of the previous lemma; first one calculates Au modulo $x^2\mathcal{C}^\infty([X;C])$, then $B(Au)$ the same way, and one deals with $A(Bu)$ similarly. In addition we write

$$\partial_z(A_{\mathrm{ff}}u) = (\partial_z A_{\mathrm{ff}})u + A_{\mathrm{ff}}\partial_z u, \tag{7.8}$$

and $\partial_{\overline{Y}}(A_{\mathrm{ff}}u)$ similarly. □

It is easy to extend this result to the indicial operators since for $\tilde{f} \in \mathcal{C}^\infty(X)$

$$[A,B]^{\tilde{f}} = e^{-i\tilde{f}/x}[A,B]e^{i\tilde{f}/x} = [e^{-i\tilde{f}/x}Ae^{i\tilde{f}/x}, e^{-i\tilde{f}/x}Be^{i\tilde{f}/x}]. \tag{7.9}$$

However, in general $[A,B]^{\tilde f}u$, regarded as an element of $\dot{\mathcal{C}}^\infty_{\mathrm{ff}}([X;C])/x^2\dot{\mathcal{C}}^\infty_{\mathrm{ff}}([X;C])$, depends on $\tilde f$ in a more complicated way than in the case of the indicial operators where we quotiented out by $x\dot{\mathcal{C}}^\infty_{\mathrm{ff}}([X;C])$. The situation is much simpler if $[\widehat{B}_{\mathrm{ff}},\widehat{A}_{\mathrm{ff}}] = 0$ on $W^\perp$. Then $[B,A] \in \Psi^{m+m'-1,1}_{3\mathrm{sc}}(X)$, and $[B,A]^{\tilde f}$ gives the indicial operator of $[B,A]$, which hence depends only on $f(0,z_0)$ and $df(0,z_0)$ where $f = \tilde f|_{\partial X}$. In this case we can simply center our coordinate system at $(0,z_0)$, i.e. we may assume that $z_0 = 0$, assume that u is supported near $z=0$, and we can take $f(y,z) = -\tau + \nu z$ to calculate $\widehat{[A,B]}(0,\tau,\nu)$, since the result of the computation is independent of all such choices.

Proposition 7.3. — *If $A \in \Psi^{m,0}_{3\mathrm{sc}}(X)$, $B \in \Psi^{m',0}_{3\mathrm{sc}}(X)$ and $[\widehat{B}_{\mathrm{ff}},\widehat{A}_{\mathrm{ff}}] = 0$ on $W^\perp$, then $[B,A] \in \Psi^{m+m'-1,1}_{3\mathrm{sc}}(X)$ and*

$$
\begin{aligned}
\widehat{[B,A]}_{\mathrm{ff},1} &= [(\partial_x\widehat{B})_{\mathrm{ff}},\widehat{A}_{\mathrm{ff}}] + [\widehat{B}_{\mathrm{ff}},(\partial_x\widehat{A})_{\mathrm{ff}}] + (D_\tau\widehat{A}_{\mathrm{ff}})[\overline{Y},\widehat{B}_{\mathrm{ff}}]\partial_{\overline{Y}} \\
&\quad - (D_\tau\widehat{B}_{\mathrm{ff}})[\overline{Y},\widehat{A}_{\mathrm{ff}}]\partial_{\overline{Y}} + \big((D_\tau\widehat{A}_{\mathrm{ff}})(\overline{Y}\partial_{\overline{Y}}\widehat{B}_{\mathrm{ff}}) - (D_\tau\widehat{B}_{\mathrm{ff}})(\overline{Y}\partial_{\overline{Y}}\widehat{A}_{\mathrm{ff}})\big) \\
&\quad + \big((D_\nu\widehat{B}_{\mathrm{ff}})(\partial_z\widehat{A}_{\mathrm{ff}}) - (D_\nu\widehat{A}_{\mathrm{ff}})(\partial_z\widehat{B}_{\mathrm{ff}})\big) \\
&\quad + \big((\nu\cdot D_\nu\widehat{A}_{\mathrm{ff}})(\partial_\tau\widehat{B}_{\mathrm{ff}}) - (\nu\cdot D_\nu\widehat{B}_{\mathrm{ff}})(\partial_\tau\widehat{A}_{\mathrm{ff}})\big).
\end{aligned}
\tag{7.10}
$$

Proof. — The additional ingredient to Lemma 7.2 is the understanding of operators such as $(\partial_x A^{\tilde f})_{\mathrm{ff}}$ and $(\partial_z A^{\tilde f})_{\mathrm{ff}}$. Now, with our choice of $f = -\tau + \nu z$,

$$
A^{\tilde f}|_{\mathrm{ff}}(\overline{Y},z,Y) = \int e^{i(S(-\tau+z\nu)-Z\nu)}A(0,\overline{Y},z,S,Y,Z)\,dS\,dZ,
\tag{7.11}
$$

so

$$
\begin{aligned}
\partial_{z_j}A^{\tilde f}|_{\mathrm{ff}}(\overline{Y},z,Y) &= \nu_j\int iSe^{i(S(-\tau+z\nu)-Z\nu)}A(0,\overline{Y},z,S,Y,Z)\,dS\,dZ \\
&\qquad + \int e^{i(S(-\tau+z\nu)-Z\nu)}\partial_{z_j}A(0,\overline{Y},z,S,Y,Z)\,dS\,dZ.
\end{aligned}
\tag{7.12}
$$

Thus, restricting to $z=0$ gives

$$
\partial_{z_j}A^{\tilde f}|_{\mathrm{ff}}(\overline{Y},0,Y) = -\nu_j\partial_\tau\widehat{A}_{\mathrm{ff}}(\overline{Y},0,Y) + \partial_{z_j}\widehat{A}_{\mathrm{ff}}(\overline{Y},0,Y).
\tag{7.13}
$$

Substituting this into Lemma 7.2 and noting that

$$
D_\nu[\widehat{A}_{\mathrm{ff}},\widehat{B}_{\mathrm{ff}}] = [D_\nu\widehat{A}_{\mathrm{ff}},\widehat{B}_{\mathrm{ff}}] + [\widehat{A}_{\mathrm{ff}},D_\nu\widehat{B}_{\mathrm{ff}}],
\tag{7.14}
$$

with a similar result for D_τ, proves the proposition. □

It is interesting to see how this proposition gives the usual commutator formula if $A \in \Psi^{m,0}_{\mathrm{sc}}(X)$, $B \in \Psi^{m',0}_{\mathrm{sc}}(X)$. In that case the kernel of A is the pull back of a distribution A' on X^2_{sc}, so

$$
A(x,\overline{Y},z,S,Y,Z) = A'(x,x\overline{Y},z,S,Y,Z).
\tag{7.15}
$$

Let a' be the Fourier transform of A' in S, Y and Z, i.e. it gives $j_{\mathrm{sc},m,0}(A')$ when restricted to $C_{\mathrm{sc}}X$, and define b' similarly. Thus,

$$\widehat{A}_{\mathrm{ff}}(z,\tau,\nu,\overline{Y},Y) = \mathcal{F}_\mu^{-1} a'(0,0,z,\tau,Y,\nu), \tag{7.16}$$

$$(\partial_x \widehat{A})_{\mathrm{ff}}(z,\tau,\nu,\overline{Y},Y) = \partial_x \mathcal{F}_\mu^{-1} a'(0,0,z,\tau,Y,\nu) + \sum_j \overline{Y}_j \partial_{y_j} \mathcal{F}_\mu^{-1} a'(0,0,z,\tau,Y,\nu). \tag{7.17}$$

Thus, the only dependence on $\overline{Y}$ in (7.10) comes from the multiplication by $\overline{Y}_j$ in expressions such as the last term of (7.17). Thus, we can explicitly compute the operator commutators in (7.10). Moreover,

$$\begin{aligned}[\widehat{B}_{\mathrm{ff}}, (\partial_x \widehat{A})_{\mathrm{ff}}] &= \sum_j [\widehat{B}_{\mathrm{ff}}, \overline{Y}_j] \partial_{y_j} \mathcal{F}_\mu^{-1} a'(0,0,z,\tau,Y,\nu) \\ &= \sum_j \mathcal{F}_\mu^{-1} D_{\mu_j} b'(0,0,z,\tau,Y,\nu) \partial_{y_j} \mathcal{F}_\mu^{-1} a'(0,0,z,\tau,Y,\nu).\end{aligned} \tag{7.18}$$

Similarly,

$$[\overline{Y}_j, \widehat{B}_{\mathrm{ff}}] = -\mathcal{F}_\mu^{-1} D_{\mu_j} b'. \tag{7.19}$$

The other terms can be computed similarly giving

$$\begin{aligned}\widehat{[B,A]}_{\mathrm{ff},1} &= \sum_j \mathcal{F}_\mu^{-1} \big((D_{\mu_j} b')(\partial_{y_j} a') - (D_{\mu_j} a')(\partial_{y_j} b') \big) \\ &+ \sum_j \mathcal{F}_\mu^{-1} \big((D_{\nu_j} b')(\partial_{z_j} a') - (D_{\nu_j} a')(\partial_{z_j} b') \big) \\ &+ \mathcal{F}_\mu^{-1} \big((\nu \cdot D_\nu a')(\partial_\tau b') - (\nu \cdot D_\nu b')(\partial_\tau a') \big) \\ &+ \mathcal{F}_\mu^{-1} \big((\mu \cdot D_\mu a')(\partial_\tau b') - (\mu \cdot D_\mu b')(\partial_\tau a') \big).\end{aligned} \tag{7.20}$$

Here the right hand side is just the inverse Fourier transform of the standard (rescaled) Poisson bracket formula [**25**, Equation 5.23] of the scattering calculus, as expected.

CHAPTER 8

MAPPING PROPERTIES

Proposition 8.1. — *If $A \in \Psi^{0,0}_{3\mathrm{scc}}(X)$ and $L^2_{\mathrm{sc}}(X)$ is defined with respect to a scattering density, $\nu \in \mathcal{C}^\infty(X; {}^{\mathrm{sc}}\Omega(X))$, then A defines a bounded linear operator on $L^2_{\mathrm{sc}}(X)$.*

Proof. — This can be proved by the construction of an approximate square root as usual, at least in the case of $A \in \Psi^{0,0}_{3\mathrm{sc}}(X)$ where we have discussed the symbol and indicial maps in detail, or simply using the local description of kernels which implies that with $X = \mathbb{S}^N_+$, $\Psi^{0,0}_{3\mathrm{scc}}(X) \subset \Psi^0_\infty(\mathbb{R}^N)$ (this is the algebra corresponding to symbols in [**18**, Definition 18.1.1]), so we can apply Hörmander's theorem [**18**, Theorem 18.1.11]. □

Corollary 8.2. — *If $A \in \Psi^{m,l}_{3\mathrm{scc}}(X)$ then for all m', $l' \in \mathbb{R}$, A defines a continuous operator from $H^{m',l'}_{\mathrm{sc}}(X)$ to $H^{m'-m,l'+l}_{\mathrm{sc}}(X)$. In particular, if $m < 0$, $l > 0$, then A is a compact operator on $L^2_{\mathrm{sc}}(X)$.*

Proof. — Suppose $m' > 0$, $l' > 0$. Let $P_0 \in \Psi^{|m'|/2,0}_{\mathrm{sc}}(X)$ be fully elliptic (i.e. $j_{\mathrm{sc},|m'|/2,0}(P)$ is invertible). Then $Q_0 = \mathrm{Id} + P_0^* P_0 \in \Psi^{|m'|,0}_{\mathrm{sc}}(X)$ is invertible with $Q_0^{-1} \in \Psi^{-|m'|,0}_{\mathrm{sc}}(X)$. If $m' \geq 0$ let $Q = Q_0 x^{-l'}$, while if $m' < 0$ let $Q = Q_0^{-1} x^{-l'}$. Thus, $Q \in \Psi^{m',-l'}_{\mathrm{sc}}(X)$ is invertible with inverse $Q^{-1} \in \Psi^{-m',l'}_{\mathrm{sc}}(X)$. Similarly, we can construct $Q' \in \Psi^{m'-m,-l'-l}_{\mathrm{sc}}(X)$ with inverse in $\Psi^{-m'+m,l'+l}_{\mathrm{sc}}(X)$. Now, $Q'AQ^{-1} \in \Psi^{0,0}_{3\mathrm{scc}}(X)$, so by the proposition $Q'AQ^{-1}$ is bounded on $L^2_{\mathrm{sc}}(X)$. Since $Q \in \mathcal{B}(H^{m',l'}_{\mathrm{sc}}(X), L^2_{\mathrm{sc}}(X))$ and $(Q')^{-1} \in \mathcal{B}(L^2_{\mathrm{sc}}(X), H^{m'-m,l'+l}_{\mathrm{sc}}(X))$, the composite operator is

$$A = (Q')^{-1}(Q'AQ^{-1})Q \in \mathcal{B}(H^{m',l'}_{\mathrm{sc}}(X), H^{m'-m,l'+l}_{\mathrm{sc}}(X)). \tag{8.1}$$

□

Proposition 8.3. — *If $A \in \Psi^{m,l}_{3\mathrm{sc}}(X)$, and $\sigma_m(A)$, $N_{\mathrm{mf},l}(A)$ and $N_{\mathrm{ff},l}(A)$ are invertible, then there exists $P \in \Psi^{-m,-l}_{3\mathrm{sc}}(X)$ such that*

$$PA - \mathrm{Id} \in \Psi^{-\infty,\infty}_{3\mathrm{sc}}(X), \qquad AP - \mathrm{Id} \in \Psi^{-\infty,\infty}_{3\mathrm{sc}}(X). \tag{8.2}$$

Proof. — This is just the standard proof using the symbol calculus. Thus, using the full ellipticity and the exactness of the symbol mappings we can find $P_0 \in \Psi_{3sc}^{-m,-l}(X)$ such that

(8.3)
$$\sigma_{-m}(P_0) = \sigma_m(A)^{-1}, \qquad N_{\mathrm{mf},-l}(P_0) = N_{\mathrm{mf},l}(A)^{-1}, \qquad N_{\mathrm{ff},-l}(P_0) = N_{\mathrm{ff},l}(A)^{-1}.$$

Thus,

$$R_L = P_0 A - \mathrm{Id} \in \Psi_{3sc}^{-1,1}(X), \qquad R_R = AP_0 - \mathrm{Id} \in \Psi_{3sc}^{-1,1}(X). \tag{8.4}$$

Then we asymptotically sum the Neumann series

$$P_R \sim P_0(\mathrm{Id} + \sum_{j=1}^{\infty}(-1)^j R_R^j), \tag{8.5}$$

and define P_L similarly. The standard argument shows then that

$$P_L - P_R \in \Psi_{3sc}^{-\infty,\infty}(X),$$

so we can take P to be either one of these two. □

CHAPTER 9

WAVEFRONT SET

Just as Melrose has defined the scattering wave front set, $\mathrm{WF}_{\mathrm{sc}}$, arising from $\Psi_{\mathrm{sc}}^{\infty,-\infty}(X)$, on the boundary $C_{\mathrm{sc}}X$ of ${}^{\mathrm{sc}}\overline{T}^*X$, we can define an analogous notion of wave front set for $\Psi_{3\mathrm{sc}}^{\infty,-\infty}(X)$. In fact, since the operators for which we want do develop a scattering theory are elliptic in the usual sense, i.e. $\sigma_{3\mathrm{sc},m}(H)$ is invertible for these H, we will only consider the part of the wave front set which captures the behavior of distributions at ∂X. First, however, we define the simpler notion of operator wave front set.

The operator wave front set, $\mathrm{WF}'_{3\mathrm{sc}}$, of elements of $\Psi_{3\mathrm{sc}}^{m,l}(X)$ is closely related to the indicial operators. Namely, for $A \in \Psi_{3\mathrm{sc}}^{m,l}(X)$ we could first consider the set of points $\alpha \in {}^{3\mathrm{sc}}T^*_{\mathrm{mf}}[X;C]$ which have a neighborhood in ${}^{3\mathrm{sc}}T^*_{\mathrm{mf}}[X;C]$ on which $\widehat{A}_{\mathrm{mf},l}$ vanishes and call it the complement of the 'top order operator wave front set of A at mf'. Similarly we could define 'the top order operator wave front set of A at ff' by saying that $\xi \in W^\perp$ is not in it if ξ has a neighborhood on which $\widehat{A}_{\mathrm{ff},l}$ vanishes. Although the 'full indicial operator' of A is not well defined, the following statement has an invariant meaning: the amplitude of A, $a(A) \in x^l\rho_\infty^{-m}\mathcal{C}^\infty({}^{3\mathrm{sc}}\overline{T}^*[X;C])$, defined as the Fourier transform of the kernel of A in S, Y and Z given by some local coordinates on X (as in Chapter 2), vanishes to infinite order on a neighborhood of α in ${}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C]$ or on a neighborhood of $\beta^{-1}(\pi^\perp)^{-1}(\{\xi\}) \subset {}^{3\mathrm{sc}}\overline{T}^*_{\partial[X;C]}[X;C]$. Here $\pi^\perp : {}^{\mathrm{sc}}T^*_C X \to W^\perp$ is the projection. As a general principle we should work on compact spaces. Hence we consider the radial compactification $\overline{W}^\perp$ of $W^\perp$. If $K \subset \overline{W}^\perp$ is closed and $K = \mathrm{cl}(\mathrm{int}(K))$, by $\beta^{-1}(\pi^\perp)^{-1}(K)$ we mean the set $\mathrm{cl}(\beta^{-1}(\pi^\perp)^{-1}(K \cap \mathrm{int}(\overline{W}^\perp)))$. It is also useful to have the notion of operator wave front set at ${}^{3\mathrm{sc}}S^*X$ corresponding to the symbol map. Thus, the operator wave front set will be defined on the disjoint union of three compact manifolds with corners:

$$C_{3\mathrm{sc}}[X;C] = {}^{3\mathrm{sc}}S^*[X;C] \cup {}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C] \cup \overline{W}^\perp. \tag{9.1}$$

We thus make the following definition.

Definition 9.1. — The operator wave front set,

$$\mathrm{WF}'_{3\mathrm{sc}}(A) = \mathrm{WF}'_{3\mathrm{sc},\sigma}(A) \cup \mathrm{WF}'_{3\mathrm{sc},\mathrm{mf}}(A) \cup \mathrm{WF}'_{3\mathrm{sc},\mathrm{ff}}(A) \subset C_{3\mathrm{sc}}[X;C], \tag{9.2}$$

of $A \in \Psi^{m,l}_{3\mathrm{sc}}(X)$ is given by

$$\begin{aligned} {}^{\mathrm{sc}}T^*_{\partial X}X \setminus \mathrm{WF}'_{3\mathrm{sc},\mathrm{mf}}(A) = \{\alpha \in {}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}X : \exists U \subset {}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C] \text{ open} \\ \text{such that } \alpha \in U,\ a(A) \text{ vanishes to infinite order on } \mathrm{cl}(U)\} \end{aligned} \tag{9.3}$$

at mf, and at ff by

$$\begin{aligned} \overline{W}^{\perp} \setminus \mathrm{WF}'_{3\mathrm{sc},\mathrm{ff}}(A) = \{\xi \in \overline{W}^{\perp} : \exists U \subset \overline{W}^{\perp} \text{ open} \\ \text{such that } \xi \in U,\ a(A) \text{ vanishes to infinite order on } \beta^{-1}(\pi^{\perp})^{-1}(\mathrm{cl}(U))\}. \end{aligned} \tag{9.4}$$

Finally, at ${}^{3\mathrm{sc}}S^*[X;C]$ it is given by

$$\begin{aligned} {}^{3\mathrm{sc}}S^*[X;C] \setminus \mathrm{WF}'_{3\mathrm{sc},\sigma}(A) = \{\alpha \in {}^{3\mathrm{sc}}S^*[X;C] : \exists U \subset {}^{3\mathrm{sc}}S^*[X;C] \text{ open} \\ \text{such that } \alpha \in U,\ a(A) \text{ vanishes to infinite order on } \mathrm{cl}(U)\}. \end{aligned} \tag{9.5}$$

It follows immediately from the definition that $\mathrm{WF}'_{3\mathrm{sc}}(A) = \varnothing$ implies that $a(A)$ vanishes to infinite order at $\partial\, {}^{3\mathrm{sc}}\overline{T}^*[X;C]$, so $A \in \Psi^{-\infty,\infty}_{3\mathrm{sc}}(X)$. We also have the corresponding 'partial residual' results, i.e. for $A \in \Psi^{m,l}_{3\mathrm{sc}}(X)$

$$\mathrm{WF}'_{3\mathrm{sc},\sigma}(A) = \varnothing \Longrightarrow A \in \Psi^{-\infty,l}_{3\mathrm{sc}}(X), \tag{9.6}$$

$$\mathrm{WF}'_{3\mathrm{sc},\mathrm{mf}}(A) = \varnothing \text{ and } \mathrm{WF}'_{3\mathrm{sc},\mathrm{ff}}(A) = \varnothing \Longrightarrow A \in \Psi^{m,\infty}_{3\mathrm{sc}}(X). \tag{9.7}$$

The operator wave front set behaves under composition just as expected.

Lemma 9.2. — *If $A \in \Psi^{m,l}_{3\mathrm{sc}}(X)$, $B \in \Psi^{m',l'}_{3\mathrm{sc}}(X)$, then*

$$\mathrm{WF}'_{3\mathrm{sc}}(AB) \subset \mathrm{WF}'_{3\mathrm{sc}}(A) \cap \mathrm{WF}'_{3\mathrm{sc}}(B). \tag{9.8}$$

Proof. — This follows easily from the microlocality of the composition formula. In particular, at the top level at ff, $\widehat{AB}_{\mathrm{ff},l+l'} = \widehat{A}_{\mathrm{ff},l}\widehat{B}_{\mathrm{ff},l'}$, which vanishes if either factor on the right hand side vanishes. This argument extends to the full amplitude and to the other faces. □

We now show the existence of a microlocal parametrix of operators $A \in \Psi^{m,l}_{3\mathrm{sc}}(X)$ whose normal operator is microlocally invertible. Such a result holds in the scattering calculus, so we only need to concern ourselves with the behavior at C.

Lemma 9.3. — *If $A \in \Psi^{m,l}_{3\mathrm{sc}}(X)$, $\xi_0 \in W^{\perp}$, $\widehat{A}_{\mathrm{ff}}(\xi_0)$ is invertible, then there exists $B \in \Psi^{-m,-l}_{3\mathrm{sc}}(X)$ such that*

$$\xi_0 \notin \mathrm{WF}'_{3\mathrm{sc},\mathrm{ff}}(AB - \mathrm{Id}), \qquad \xi_0 \notin \mathrm{WF}'_{3\mathrm{sc},\mathrm{ff}}(BA - \mathrm{Id}). \tag{9.9}$$

Proof. — We only consider $m = l = 0$; the extension to other values of m and l is straightforward. Let $a \in \mathcal{C}^\infty([X;C] \times \mathbb{S}^N_+)$ be the left symbol of A, and let

$$b_0 \in \mathcal{C}^\infty(U \times \mathbb{S}^n_+ \times \mathbb{S}^n_+)$$

be the symbol of $\widehat{A}_{\mathrm{ff}}(\xi)^{-1}$ when $\xi \in U$, a sufficiently small neighborhood of ξ_0 in $W^\perp$. We wish to show that there exists a symbol $b \in \mathcal{C}^\infty([X;C] \times \mathbb{S}^N_+)$ which restricts to b_0 in $U' \times \mathbb{S}^n_+ \times \mathbb{S}^n_+$ for some U' open containing ξ_0. But the invertibility of $\widehat{A}_{\mathrm{ff}}(\xi_0)$ implies that $\widehat{A}_{\mathrm{mf}}(\alpha) \neq 0$ if $\beta(\alpha) = \xi_0$, i.e. $a(0, \widehat{Y}_j, z_0, \tau_0, \mu, \nu_0)$ is invertible for all μ and $\widehat{Y}_j$. In fact, more is true. Compactifying the fibers of ${}^{3\mathrm{sc}}T^*[X;C]$ to obtain ${}^{3\mathrm{sc}}\overline{T}^*[X;C]$, we see that $a(p, (0, \widehat{\mu}, 0)) \neq 0$ for any $\widehat{\mu} \in \mathbb{S}^{n-1}$, $p \in \mathrm{ff}$. This allows us to define b' on a neighborhood of U' by $b = a^{-1}$ away from the interior of $\mathrm{ff} \times \mathbb{S}^N_+$, and b_0 on U. Let $\phi \in \mathcal{C}^\infty({}^{3\mathrm{sc}}\overline{T}^*[X;C])$ be identically 1 at $(0, \overline{Y}, z_0, \tau_0, \mu, \nu_0)$ and be supported in a small neighborhood V of this set in ${}^{3\mathrm{sc}}\overline{T}^*[X;C]$, and let $b_1 = \phi b'$, $B_1 = q_L(b)$. Then $B_1 A - \mathrm{Id} = R_1 + R_2$ where $R_1 \in \Psi^{-1,1}_{3\mathrm{sc}}(X)$ and the left symbol of R_2 vanishes in an open subset V' of V containing $S = \beta^{-1}(\pi^\perp)^{-1}(\xi_0)$. We can now follow the usual argument (asymptotic summation of the Neumann series) to remove R_1 and obtain B such that $BA - \mathrm{Id} = R_2'$, R_2' vanishing in $V'' \subset V'$ open still containing S. Thus, $\xi_0 \notin \mathrm{WF}'_{3\mathrm{sc}}(R_2')$, and B satisfies the second equation in (9.9). Now, we could have constructed similarly B' satisfying the first equation there, and the standard argument shows that $\xi_0 \notin \mathrm{WF}'_{3\mathrm{sc}}(B - B')$, so B also satisfies the first equation. □

Remark 9.4. — If $K \subset C_{3\mathrm{sc}}[X;C]$ is compact and A is elliptic on K, then essentially the same proof shows that we can pick $B \in \Psi^{-m,-l}_{3\mathrm{sc}}(X)$ such that

$$K \cap \mathrm{WF}'_{3\mathrm{sc}}(AB - \mathrm{Id}) = \varnothing, \qquad K \cap \mathrm{WF}'_{3\mathrm{sc}}(BA - \mathrm{Id}) = \varnothing. \tag{9.10}$$

We define the wave front set $\mathrm{WF}_{3\mathrm{sc}}$, consisting of two pieces: one at mf and one at ff. For $u \in \mathcal{C}^{-\infty}(X)$ we want

$$\mathrm{WF}_{3\mathrm{sc,mf}}(u) \subset {}^{\mathrm{sc}}T^*_{\partial X}X, \qquad \mathrm{WF}_{3\mathrm{sc,ff}}(u) \subset W^\perp. \tag{9.11}$$

Definition 9.5. — The relative 3-body scattering wave front set, $\mathrm{WF}^{m,l}_{3\mathrm{sc}}(u)$, of $u \in \mathcal{C}^{-\infty}(X)$ ('relative' to $H^{m,l}_{\mathrm{sc}}(X)$), is given by

(9.12)
$$\begin{aligned}{}^{\mathrm{sc}}T^*_{\partial X}X \setminus \mathrm{WF}^{m,l}_{3\mathrm{sc,mf}}(u) = \{p \in {}^{\mathrm{sc}}T^*_{\partial X}X : \exists\, A \in \Psi^{0,0}_{3\mathrm{sc}}(X) \text{ such that } Au \in H^{m,l}_{\mathrm{sc}}(X) \\ \text{and } \widehat{A}_{\mathrm{mf}}(q) \neq 0\ \forall\, q \in {}^{3\mathrm{sc}}T^*_{\mathrm{mf}}[X;C] \text{ with } \beta(q) = p\}\end{aligned}$$

at mf, while at ff by

(9.13)
$$\begin{aligned}W^\perp \setminus \mathrm{WF}^{m,l}_{3\mathrm{sc,ff}}(u) = \{p \in W^\perp : \exists\, A \in \Psi^{0,0}_{3\mathrm{sc}}(X) \text{ such that } \widehat{A}_{\mathrm{ff}}(p) \text{ is invertible,} \\ Au \in H^{m,l}_{\mathrm{sc}}(X)\}.\end{aligned}$$

The absolute 3-body scattering wave front set, $\mathrm{WF}_{3\mathrm{sc}}(u)$, is defined by replacing $H^{m,l}_{\mathrm{sc}}(X)$ by $\dot{\mathcal{C}}^\infty(X)$ in (9.12) and (9.13).

First note that for $u \in \mathcal{C}^{-\infty}(X)$

$$\mathrm{WF}_{3\mathrm{sc,mf}}(u) \cap {}^{\mathrm{sc}}T^*_{\partial X \setminus C} X = \mathrm{WF}_{\mathrm{sc}}(u) \cap {}^{\mathrm{sc}}T^*_{\partial X \setminus C} X. \tag{9.14}$$

In fact, it is clear from the definition that the left hand side is a subset of the right hand side. On the other hand, if $p \notin \mathrm{WF}_{3\mathrm{sc,mf}}(u)$, $p \in {}^{\mathrm{sc}}T^*_{\partial X \setminus C} X$ and $A \in \Psi^{0,0}_{3\mathrm{sc}}(X)$ is such that $\widehat{A}(p) \neq 0$, $Au \in \dot{\mathcal{C}}^\infty(X)$, then let $\rho \in \mathcal{C}^\infty(X)$ be supported in $X \setminus C$, identically 1 near $\pi(p)$, $\pi : {}^{\mathrm{sc}}T^*X \to X$ being the projection. Then we have $\rho A \in \Psi^{0,0}_{\mathrm{sc}}(X)$, $j_{\mathrm{sc},0,0}(\rho A)(p) \neq 0$, $\rho A u \in \dot{\mathcal{C}}^\infty(X)$, so $p \notin \mathrm{WF}_{\mathrm{sc}}(u)$.

There is a natural map from ${}^{\mathrm{sc}}T^*_C X$ to $W^\perp$ given by $\pi^\perp$, the orthogonal projection to $W^\perp$. Now, if $A \in \Psi^{0,0}_{3\mathrm{sc}}(X)$ and $\widehat{A}_{\mathrm{ff}}(\xi) \in \Psi^{0,0}_{\mathrm{sc}}(\mathbb{S}^n_+)$ is invertible for some $\xi \in W^\perp$, then certainly $j_{\mathrm{sc},0,0}(\widehat{A}_{\mathrm{ff}}(\xi))$ is invertible. But this means that for all $q \in {}^{3\mathrm{sc}}T^*_{\mathrm{mf}}[X;C]$ with $\pi^\perp\beta(q) = \xi$, $\widehat{A}_{\mathrm{mf}}(q) \neq 0$. Thus, if $p \in {}^{\mathrm{sc}}T^*_C X$, $\pi^\perp(p) = \xi$, $\xi \notin \mathrm{WF}^{m,l}_{3\mathrm{sc,ff}}(u)$ then $p \notin \mathrm{WF}^{m,l}_{3\mathrm{sc,mf}}(u)$. This means that $\mathrm{WF}_{3\mathrm{sc,mf}}$ restricted to ${}^{\mathrm{sc}}T^*_C X$ is somewhat redundant.

Note that (9.12) and (9.13) can be replaced by uniform statements over compact sets disjoint from $\mathrm{WF}_{3\mathrm{sc}}(u)$. Namely if $\xi \notin \mathrm{WF}_{3\mathrm{sc}}(u)$ then by definition $A_\xi u \in \dot{\mathcal{C}}^\infty(X)$ for some $A_\xi \in \Psi^{0,0}_{3\mathrm{sc}}(X)$ with $\widehat{(A_\xi)}_{\mathrm{ff}}(\xi)$ invertible. But then $\widehat{(A_\xi)}_{\mathrm{ff}}$ is invertible on a neighborhood U_ξ of ξ in $W^\perp$. Let U'_ξ be a neighborhood of ξ such that $\mathrm{cl}(U'_\xi) \subset U_\xi$. Now, if $K_{\mathrm{ff}} \subset W^\perp \setminus \mathrm{WF}_{3\mathrm{sc,ff}}(u)$ is compact, then $\{U'_\xi : \xi \in K_{\mathrm{ff}}\}$ is an open cover of K_{ff}, so it has a finite subcover $\{U'_{\xi_j} : j = 1, \ldots, k\}$. Now let $A = \sum_j A^*_{\xi_j} A_{\xi_j} \in \Psi^{0,0}_{3\mathrm{sc}}(X)$. Then $Au \in \dot{\mathcal{C}}^\infty(X)$ by construction. Moreover, if $\xi \in K_{\mathrm{ff}}$, then $\xi \in U'_{\xi_j}$ for some j, and on U_{ξ_j}, $\widehat{(A_{\xi_j})}_{\mathrm{ff}}$ is invertible, so on U'_{ξ_j}, $A^*_{\xi_j} A_{\xi_j} \geq \delta > 0$ for some δ. Hence, $\widehat{A}_{\mathrm{ff}}(\xi)$ is invertible. Since mf can be dealt with similarly, we conclude:

Lemma 9.6. — *Suppose that $K_{\mathrm{mf}} \subset \mathrm{WF}_{3\mathrm{sc,mf}}(u)^c$ and $K_{\mathrm{ff}} \subset \mathrm{WF}_{3\mathrm{sc,ff}}(u)^c$ are compact. Then there exists $A \in \Psi^{0,0}_{3\mathrm{sc}}(X)$ with $Au \in \dot{\mathcal{C}}^\infty(X)$ and $\widehat{A}_{\mathrm{ff}}$ invertible on K_{ff}, $\widehat{A}_{\mathrm{mf}} \neq 0$ on K_{mf}.*

Remark 9.7. — We can easily prove the analogous result for $\mathrm{WF}^{m,l}_{3\mathrm{sc}}(u)$.

In the next chapters we show that the wave front set of approximate generalized eigenfunctions u of the operators we are interested in stays in a compact subset of ${}^{\mathrm{sc}}T^*_{\partial X} X$ and $W^\perp$. Correspondingly, we are interested in applying operators $A \in \Psi^{-\infty,l}_{3\mathrm{sc}}(X)$ to u where $\mathrm{WF}'_{3\mathrm{sc}}(A) \subset {}^{3\mathrm{sc}}T^*_{\mathrm{mf}}[X;C] \cup W^\perp$, i.e. it stays away from $\partial\overline{W}^\perp$ and ${}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C] \cap {}^{3\mathrm{sc}}S^*[X;C]$. Although it is not true in general that $\mathrm{WF}_{3\mathrm{sc}}(Au) \subset \mathrm{WF}_{3\mathrm{sc}}(u)$, we can prove the following weaker result.

Lemma 9.8. — *If $A \in \Psi^{-\infty,l}_{3\mathrm{sc}}(X)$, $u \in \mathcal{C}^{-\infty}(X)$, and*

$$\mathrm{WF}'_{3\mathrm{sc}}(A) \subset {}^{3\mathrm{sc}}T^*_{\mathrm{mf}}[X;C] \cup W^\perp, \tag{9.15}$$

then for m', $l' \in \mathbb{R}$

$$\mathrm{WF}'_{3\mathrm{sc}}(A) \cap \mathrm{WF}^{m',l'}_{3\mathrm{sc}}(u) = \varnothing \Rightarrow Au \in H^{\infty,l+l'}_{\mathrm{sc}}(X). \tag{9.16}$$

Proof. — Suppose that $\mathrm{WF}'_{3\mathrm{sc}}(A) \cap \mathrm{WF}^{m',l'}_{3\mathrm{sc}}(u) = \varnothing$. Thus, $\mathrm{WF}'_{3\mathrm{sc}}(A)$ is a compact subset of $\mathrm{WF}^{m',l'}_{3\mathrm{sc}}(u)^c$, so by Lemma 9.6 and the remark following it there exists $P \in \Psi^{0,0}_{3\mathrm{sc}}(X)$ with P elliptic on $\mathrm{WF}'_{3\mathrm{sc}}(A)$ and $Pu \in H^{m',l'}_{\mathrm{sc}}(X)$. Moreover, by Lemma 9.3 and the remark following it, there exist $Q, R \in \Psi^{0,0}_{3\mathrm{sc}}(X)$ such that $\mathrm{Id} = QP + R$, and $\mathrm{WF}'_{3\mathrm{sc}}(R) \cap \mathrm{WF}'_{3\mathrm{sc}}(A) = \varnothing$. Now,

$$Au = A(QP + R)u = AQ(Pu) + (AR)u. \tag{9.17}$$

Since $Pu \in H^{m',l'}_{\mathrm{sc}}(X)$ and $AQ \in \Psi^{-\infty,l}_{3\mathrm{sc}}(X)$, the first term is in $H^{\infty,l+l'}_{\mathrm{sc}}(X)$. Moreover, $\mathrm{WF}'_{3\mathrm{sc}}(A) \cap \mathrm{WF}'_{3\mathrm{sc}}(R) = \varnothing$, so $AR \in \Psi^{-\infty,\infty}_{3\mathrm{sc}}(X)$. Thus, the second term is in $\dot{C}^\infty(X)$, proving that $Au \in H^{\infty,l+l'}_{\mathrm{sc}}(X)$. □

CHAPTER 10

FUNCTIONAL CALCULUS

In [**33**] Seeley used integration along a contour avoiding the spectrum to define complex powers of pseudo-differential operators with real symbols and to show that they were also pseudo-differential operators. He also showed that holomorphic functions of a zeroth order pseudo-differential operator on a compact manifold are also pseudo-differential operators. This method does not work directly for non-holomorphic functions of an operator, but Stokes' theorem can be used in certain cases. In [**15**] Helffer and Sjöstrand applied this to compactly supported smooth functions of self-adjoint operators by using almost analytic extension. We now show that compactly supported smooth functions of pseudodifferential operators in $\Psi_{3\text{sc}}^{m,0}(X)$ with $\sigma_{3\text{sc},m}$ elliptic, are in $\Psi_{3\text{sc}}^{-\infty,0}(X)$. Here we need an L^2 inner product defined by a smooth positive scattering density, $\nu \in \mathcal{C}^\infty(X; {}^{\text{sc}}\Omega(X))$. First, however, we state a uniform version of the parametrix construction in the scattering calculus.

Lemma 10.1. — *If $P \in \Psi_{\text{sc}}^{m,0}(X)$ is self-adjoint, $\sigma_{3\text{sc},m}(P)$ is elliptic, $m > 0$, and $k > 0$ is an integer then there exists a family of order k parametrices $B_z = B_z^k \in \Psi_{\text{sc}}^{-m,0}(X)$, $z \in \mathbb{C} \setminus \mathbb{R}$ such that*

$$(P - z)B_z - \text{Id},\ B_z(P - z) - \text{Id} \in \Psi_{\text{sc}}^{-k,k}(X), \tag{10.1}$$

and the seminorm of order k of B_z, as well as that of the error terms in (10.1) *are bounded by $C_k\,|\text{Im}\,z|^{-c(k)}$.*

Proof. — Let $p \in \rho_\infty^{-m}\mathcal{C}^\infty(X; {}^{\text{sc}}\overline{T}^* X)$ be a smooth extension of $\sigma_{\text{sc},m}(P)$; here ρ_∞ is the boundary defining function of ${}^{\text{sc}}\overline{T}^* X$ at 'fiber-infinity'. In the uncompactified notation this just means that p is a symbol of order m on ${}^{\text{sc}}T^*X$. Since $|p - z| \geq |\text{Im}\,z|$, it follows from the chain rule that for $T \in \text{Diff}_{\text{b}}^r(X)$

$$|TD_\xi^\beta (p - z)^{-1}| \leq C_{r,\beta}\,|\text{Im}\,z|^{-r-|\beta|}\,\langle\xi\rangle^{m-|\beta|} \tag{10.2}$$

with $C_{\alpha,\beta}$ independent of z (in fact, it depends only on the $r + |\beta|$ seminorm of p). Let Q_z be a Weyl quantization of $(p - z)^{-1}$ (constructed by some cutoffs). Then

$$E_{z,L} = \mathrm{Id} - Q_z(P - z),\ E_{z,R} = \mathrm{Id} - (P - z)Q_z \in \Psi_{\mathrm{sc}}^{-1,1}(X), \tag{10.3}$$

and the seminorm of order j of Q_z, $E_{z,L}$, $E_{z,R}$ are bounded by $C'_j\,|\mathrm{Im}\,z|^{-c'(j)}$. Using the standard Neumann series argument we define

$$B_{z,L} = \Big(\mathrm{Id} + \sum_{j=1}^{k-1} E_{z,L}^j\Big)Q_z, \qquad B_{z,R} = Q_z\Big(\mathrm{Id} + \sum_{j=1}^{k-1} E_{z,R}^j\Big). \tag{10.4}$$

It follows from the continuity of the composition that the kth seminorms of $B_{z,L}$ and $B_{z,R}$, as well as those of the error terms

$$F_{z,L} = B_{z,L}(P - z) - \mathrm{Id},\ F_{z,R} = (P - z)B_{z,R} - \mathrm{Id} \in \Psi_{\mathrm{sc}}^{-k,k}(X) \tag{10.5}$$

are bounded by $C''_k\,|\mathrm{Im}\,z|^{-k}$. In addition,

(10.6)

$$B_{z,L} = B_{z,L}((P - z)B_{z,R} - F_{z,R}) = (\mathrm{Id} + F_{z,L})B_{z,R} - B_{z,L}F_{z,R} \in \Psi_{\mathrm{sc}}^{-m-k,k}(X)$$

with kth seminorm bounded by $\widetilde{C}_k\,|\mathrm{Im}\,z|^{-c(k)}$, so we can take, say, $B_z = B_{z,L}$ in (10.1) above. This completes the proof of the lemma. □

Proposition 10.2. — *Suppose that $\phi \in C_c^\infty(\mathbb{R})$, and $P \in \Psi_{\mathrm{sc}}^{m,0}(X)$ is self-adjoint, $\sigma_{m,\mathrm{sc}}$ is elliptic, $m > 0$. Then $\phi(P) \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$ and*

$$j_{\mathrm{sc},0,0}(\phi(P))|_{{}^{\mathrm{sc}}T^*_{\partial X}X} = \phi(j_{\mathrm{sc},m,0}(P)). \tag{10.7}$$

Proof. — Let $\widetilde{\phi}$ be a compactly supported almost analytic extension of ϕ. Then, as shown in **[15]**

$$\phi(P) = \frac{1}{2\pi i}\int \overline{\partial}_z\widetilde{\phi}(z)(z - P)^{-1}\,dz \wedge d\overline{z}. \tag{10.8}$$

Let B_z be a family of order k parametrices as in the previous lemma. Define $\widetilde{P}^k_{\widetilde{\phi}}$ by replacing $(P - z)^{-1}$ by B_z^k in (10.8). Interpreting the integral as that of the kernels it follows that $\widetilde{P}^k_{\widetilde{\phi}} \in \Psi_{\mathrm{sc}}^{-m,0}(X)$ for all k. Moreover, using the error estimate of the previous lemma,

$$\phi(P) - \widetilde{P}^k_{\widetilde{\phi}} = -\frac{1}{2\pi i}\int \overline{\partial}_z\widetilde{\phi}(z)F_{z,L}\,dz \wedge d\overline{z}, \tag{10.9}$$

and $|\mathrm{Im}\,z|^k\,F_{z,L}$ is a bounded family of linear operators in $\mathcal{B}(H_{\mathrm{sc}}^{r,s}(X), H_{\mathrm{sc}}^{r+k,s+k}(X))$ for any r and s. Hence, directly from (10.9),

$$\phi(P) - \widetilde{P}^k_{\widetilde{\phi}} \in \mathcal{B}(H_{\mathrm{sc}}^{r,s}(X), H_{\mathrm{sc}}^{r+k,s+k}(X)). \tag{10.10}$$

Since $B_z^{k+1} - B_z^k \in \Psi_{\mathrm{sc}}^{-m-k-1,k+1}(X)$, we also have that

$$\widetilde{P}^{k+1}_{\widetilde{\phi}} - \widetilde{P}^k_{\widetilde{\phi}} \in \Psi_{\mathrm{sc}}^{-m-k-1,k+1}(X). \tag{10.11}$$

Thus, we can asymptotically sum

$$\widetilde{P}_{\widetilde{\phi}} \sim P^1_{\widetilde{\phi}} + \sum_{k=1}^{\infty}(\widetilde{P}^{k+1}_{\widetilde{\phi}} - \widetilde{P}^k_{\widetilde{\phi}}) \in \Psi_{\mathrm{sc}}^{-m,0}(X). \tag{10.12}$$

By (10.10)

$$\phi(P) - \widetilde{P}_{\widetilde{\phi}} : \mathcal{C}^{-\infty}(X) \to \dot{\mathcal{C}}^{\infty}(X) \tag{10.13}$$

is continuous, so it is in $\Psi_{\mathrm{sc}}^{-\infty,\infty}(X)$, proving that

$$\phi(P) \in \Psi_{\mathrm{sc}}^{-m,0}(X). \tag{10.14}$$

Noting that $\phi \in \mathcal{C}^\infty_c(\mathbb{R}_t)$, we can write it as $\phi = (t^2+1)^{-k}\psi_k$ with $\psi_k \in \mathcal{C}^\infty_c(\mathbb{R}_t)$ for all $k > 0$. Thus, applying (10.14) for $\psi_k(P)$, it follows that $\phi(P) = (P^2+1)^{-k}\psi_k(P)$ is in $\Psi^{k,0}_{\mathrm{sc}}(X)$ for all k, i.e. in $\Psi^{-\infty,0}_{\mathrm{sc}}(X)$.

Finally, (10.7) follows from

$$\begin{aligned} j_{\mathrm{sc},0,0}(\phi(P)) &= \frac{1}{2\pi i}\int \overline{\partial}_z\widetilde{\phi}(z) j_{\mathrm{sc},0,0}((z-P)^{-1})\,dz\wedge d\overline{z} \\ &= \frac{1}{2\pi i}\int \overline{\partial}_z\widetilde{\phi}(z)(z - j_{\mathrm{sc},0,0}(P))^{-1}\,dz\wedge d\overline{z} = \phi(j_{\mathrm{sc},0,0}(P)). \end{aligned} \tag{10.15}$$

□

The corresponding statement in $\Psi^{m,0}_{3\mathrm{sc}}(X)$ can be proved similarly.

Proposition 10.3. — *Suppose that $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$, and $P \in \Psi^{m,0}_{3\mathrm{sc}}(X)$ is self-adjoint with $\sigma_{3\mathrm{sc},m}(P)$ is elliptic and $m > 0$. Then $\phi(P) \in \Psi^{-\infty,0}_{3\mathrm{sc}}(X)$. Moreover,*

$$N_{\mathrm{ff},0}(\phi(P)) = \phi(N_{\mathrm{ff},0}(P)), \qquad N_{\mathrm{mf},0}(\phi(P)) = \phi(N_{\mathrm{mf},0}(P)). \tag{10.16}$$

Proof. — We proceed as in the case of scattering differential operators to prove $\phi(P) \in \Psi^{-\infty,0}_{3\mathrm{sc}}(X)$. Thus, we first prove an analogue of Lemma 10.1. The main difference is that now we need to use that the seminorms of $(\widehat{P}_{\mathrm{ff}}(\xi) - z)^{-1}$ are bounded by powers of $|\mathrm{Im}\, z|^{-1}$. We finish the argument as in the previous proposition.

Finally, (10.16) holds since restricting the integral (10.8) (considered as an integral of the kernels) to sf_C and sf' gives the analogous definition of $\phi(N_{\mathrm{ff},0}(P))$ and $\phi(N_{\mathrm{mf},0}(P))$. □

We can also show that $(\mathrm{Id} - \phi(P))(P-\lambda)^{-1} \in \Psi^{-m,0}_{3\mathrm{sc}}(X)$ under the same assumptions as above if $\lambda \notin \mathrm{supp}\,\phi$.

Proposition 10.4. — *Suppose that $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$, $\lambda \notin \mathrm{supp}\,\phi$, and $P \in \Psi^{m,0}_{3\mathrm{sc}}(X)$ is self-adjoint with $\sigma_{3\mathrm{sc},m}(P)$ is elliptic and $m > 0$. Then $(Id - \phi(P))(P-\lambda)^{-1} \in \Psi^{-m,0}_{3\mathrm{sc}}(X)$.*

Proof. — Let $\widetilde{\phi}$ be a compactly supported almost analytic extension of ϕ. Then $f(z) = (1-\widetilde{\phi}(z))(z-\lambda)^{-1}$ is an almost analytic extension of $(1-\phi(t))(t-\lambda)^{-1}$ which is analytic outside a compact set. Let $\Gamma = \Gamma(t)$ be a curve such that near Γ $\widetilde{\phi}(z)$

vanishes, $\Gamma(t) = |t| \pm i|t|$ when $|t|$ is sufficiently large and $\mp t > 0$, and Γ is disjoint from $\operatorname{spec}(P)$. By the Cauchy-Stokes formula we need to replace (10.8) by

$$\phi(P) = \frac{1}{2\pi i} \int \overline{\partial}_z f(z)(z-P)^{-1}\, dz \wedge d\overline{z} + \int_\Gamma f(z)(z-P)^{-1}\, dz. \tag{10.17}$$

We can see that the first term is an element of $\Psi_{3sc}^{-\infty,0}(X)$ exactly as above. On Γ we have $f(z) = (z-\lambda)^{-1}$, so we can apply a construction analogous to Seeley's **[33]** to conclude that the second term is indeed in $\Psi_{3sc}^{-m,0}(X)$. □

CHAPTER 11

THE HAMILTONIAN

In this chapter we start using the machinery developed in the first half of the paper to analyze the three-body type problems described in the Introduction. First, we remark that the Laplacian, Δ, of a scattering metric

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2} \tag{11.1}$$

where h is a smooth symmetric 2-tensor when restricted to ∂X is in $\mathrm{Diff}^2_{\mathrm{sc}}(X)$ (see Melrose's paper **[25]**). Its normal operator is the flat Laplacian on ${}^{\mathrm{sc}}T_pX$, $p \in \partial X$, of the metric induced by g.

From now on we choose the bundle W used in the construction of the indicial operator, Definition 6.2, to be the orthocomplement of ${}^{\mathrm{sc}}T(C;X)$. Thus, if $p \in C$, we can choose coordinates (y,z) on a neighborhood U of p in ∂X, such that $x\partial_{y_j}$ give an orthonormal basis of W_q at each $q \in U \cap C$. Hence in these coordinates, henceforth called coordinates adapted to $W^\perp$, the dual boundary metric h (restricted to $T^*\partial X$) becomes

$$h = \sum h^{ij}_{nn}(y,z)\partial_{y_i}\partial_{y_j} + \sum h^{ij}_{nt}(y,z)(\partial_{y_i}\partial_{z_j} + \partial_{z_j}\partial_{y_i}) + \sum h^{ij}_{tt}(y,z)\partial_{z_i}\partial_{z_j} \tag{11.2}$$

with

$$h^{ij}_{nn}(0,z) = \delta_{ij}, \qquad h^{ij}_{nt}(0,z) = 0. \tag{11.3}$$

Note that g fixes x modulo $x^2\mathcal{C}^\infty(X)$ (to make g of the form in (11.1)), so W induces a splitting of $T_C\partial X$ and $T^*_C\partial X$. Namely, $T^*_C\partial X = N^*C \oplus \widetilde{W}^\perp$, $\widetilde{W}^\perp$ being the orthocomplement of $N^*C \subset T^*_C\partial X$ with respect to $h|_{\partial X}$. In particular, we can identify T^*C with $\widetilde{W}^\perp$. We define

$$\widetilde{h} = h|_{\widetilde{W}^\perp} = \sum_{ij} h^{ij}_{tt}(0,z)\partial_{z_i}\partial_{z_j} \tag{11.4}$$

which is a metric on $\widetilde{W}^\perp$, i.e. it can be thought of as a metric on T^*C. We denote the metric functions by g, h, $\widetilde{h}$ as well, so $\widetilde{h} \in \mathcal{C}^\infty(\widetilde{W}^\perp)$ is given in these local coordinates by

$$\widetilde{h}(z,\nu) = \sum_{ij} h_{tt}^{ij}(0,z)\nu_i\nu_j. \tag{11.5}$$

We will also use the following notation:

$$\widetilde{h}(z,\nu) = \widetilde{h}_z(\nu) = |\nu|_z^2, \tag{11.6}$$

and similarly for h and g.

Since in the scattering calculus $\widehat{N}_{\mathrm{sc}}(\Delta) = g$, $N_{\mathrm{sc},p}(\Delta) = \Delta_{g(p)}$, Lemma 4.2 implies that the restriction of Δ to ff, now considering Δ as an element of $\Psi_{3\mathrm{sc}}^{2,0}(X)$, is $\Delta_{g|W}$. This can also seen very explicitly from the local coordinate expression for Δ. Namely, the standard formula for the Laplacian of a metric in local coordinates x_j, i.e.

$$\Delta = \sum_{i,j} \frac{1}{\sqrt{\det(g_{kl})}} D_i\, g^{ij} \sqrt{\det(g_{kl})}\, D_k, \tag{11.7}$$

gives

$$\Delta = (x^2 D_x)^2 + x^2 \Delta_{h|\partial X} \quad \mathrm{mod}\ x\,\mathrm{Diff}_{\mathrm{sc}}^2(X), \tag{11.8}$$

(see [**25**, Lemma 3]), and (11.3) implies then (by the same formula)

$$\Delta_{h|\partial X} = \sum_j D_{y_j}^2 + \sum h_{tt}^{ij}(0,z) D_{z_i} D_{z_j} \quad \mathrm{mod}\ \mathcal{I}(C)\,\mathrm{Diff}^2(\partial X), \tag{11.9}$$

$\mathcal{I}(C) \subset \mathcal{C}^\infty(\partial X)$ denoting the ideal of smooth functions on ∂X which vanish at C. Thus by (2.9), the coordinate expression for Δ in the interior of ff, in the coordinates (2.2) which are valid there, becomes

$$\Delta = (x^2 D_x)^2 + \sum_j D_{Y_j}^2 + \sum_{ij} h_{tt}^{ij}(0,z)(xD_{z_i})(xD_{z_j}) \quad \mathrm{mod}\ \rho_{\mathrm{ff}}\,\mathrm{Diff}_{3\mathrm{sc}}^2(X). \tag{11.10}$$

The last term of (11.10) is just $x^2 \Delta_{\widetilde{h}}$ modulo $x^3\,\mathrm{Diff}^1(C)$. Hence, we have proved the following lemma.

Lemma 11.1. — *For the Laplacian Δ of a scattering metric* (11.1), *Δ_{ff} is the Laplacian of the translation invariant metric on the fibers $\beta^{-1}(p)$ of ff $(p \in C)$ given by the push-forward in Lemma 4.2. The indicial operator is $\Delta_{\mathrm{ff}} + \tau^2 + |\nu|_z^2$ if we choose W in Definition 6.2 to be the orthocomplement of ${}^{\mathrm{sc}}T(C;X)$ with respect to g. Correspondingly, the normal operator is $\Delta_{\mathrm{ff}} + \Delta_{g|{}^{\mathrm{sc}}T(C;X)}$, $\Delta_{g|{}^{\mathrm{sc}}T(C;X)}$ denoting the Laplacian of the lift of $g|_{{}^{\mathrm{sc}}T(C;X)}$ to $\beta^{*\mathrm{sc}}T(C;X)$.*

In this paper we shall consider operators of the following type:

$$H = \Delta + V, \quad V \in \rho_{\mathrm{mf}}\mathcal{C}^\infty([X;C];\mathbb{R}),\ \Delta = \Delta_g,\ g \text{ as in } (11.1). \tag{11.11}$$

We proceed to analyze the characteristic set of H to conclude a regularity result outside it.

The characteristic set $\Sigma_{\Delta-\lambda}$ of $\Delta - \lambda$, $\lambda \in \mathbb{R}$, on X is the submanifold of ${}^{\mathrm{sc}}T^*_{\partial X}X$ where $j_{\mathrm{sc},2,0}(\Delta - \lambda) = 0$. Thus, using the product decomposition of ${}^{\mathrm{sc}}T^*_{\partial X}X$ induced by the choice of x to bring the metric to the form (11.1) we have

$$\Sigma_{\Delta-\lambda} = \{(\tau, q) \in \mathbb{R} \times T^*\partial X : \ \tau^2 + h(q) = \lambda\}. \tag{11.12}$$

In the local coordinates (y, z) discussed above, we have

$$\Sigma_{\Delta-\lambda} = \{(y, z, \tau, \mu, \nu) : \ \tau^2 + h_{(y,z)}(\mu, \nu) = \lambda\}, \tag{11.13}$$

so

$$\Sigma_{\Delta-\lambda} \cap {}^{\mathrm{sc}}T^*_C X = \{(0, z, \tau, \mu, \nu) : \ \tau^2 + |\mu|^2 + |\nu|_z^2 = \lambda\}. \tag{11.14}$$

Thus, with $\pi^\perp : {}^{\mathrm{sc}}T^*_C X \to W^\perp$ the orthogonal projection,

$$\pi^\perp(\Sigma_{\Delta-\lambda} \cap {}^{\mathrm{sc}}T^*_C X) = \{(z, \tau, \nu) : \ \tau^2 + |\nu|_z^2 \leq \lambda\}. \tag{11.15}$$

This set splits into two parts. In fact, with

$$\Sigma_n(\lambda) = \{(z, \tau, \nu) : \ \tau^2 + |\nu|_z^2 < \lambda\}, \tag{11.16}$$

$$\Sigma_t(\lambda) = \{(z, \tau, \nu) : \ \tau^2 + |\nu|_z^2 = \lambda\}, \tag{11.17}$$

we see that on $(\pi^\perp)^{-1}(\Sigma_n(\lambda)) \cap \Sigma_{\Delta-\lambda}$, $\mu \neq 0$, while on $(\pi^\perp)^{-1}(\Sigma_t(\lambda)) \cap \Sigma_{\Delta-\lambda}$, μ vanishes. As we shall see this corresponds to the (rescaled) Hamiltonian vector field ${}^{\mathrm{sc}}H_g$ of g being normal or tangent to ${}^{\mathrm{sc}}T^*_C X$ at points on $\Sigma_{\Delta-\lambda}$, which in turn will affect the propagation results considerably. Note that $\Sigma_n(\lambda) = \varnothing$ if $\lambda \leq 0$, and $\Sigma_t(\lambda) = \varnothing$ if $\lambda < 0$.

By our assumption on V it follows that $\sigma_{3\mathrm{sc},2}(H - \lambda)$ is elliptic, $\widehat{(H - \lambda)}_{\mathrm{mf}} = \Delta_{\mathrm{mf}} - \lambda$, so the characteristic set of $H - \lambda$ on mf is exactly $\Sigma_{\Delta-\lambda}$. Now, the indicial operator of H at ff is

$$\widehat{H}_{\mathrm{ff}} = \Delta_{\mathrm{ff}} + V_{\mathrm{ff}} + \tau^2 + |\nu|_z^2 = H_{\mathrm{ff}}(z) + \tau^2 + |\nu|_z^2. \tag{11.18}$$

Now $H_{\mathrm{ff}}(z) - \sigma$ is invertible with the inverse in $\Psi_{\mathrm{sc}}^{-2,0}(\mathbb{S}^n_+)$ if and only if $\sigma < 0$ and $\sigma \notin \mathrm{spec}_p(H_{\mathrm{ff}}(z))$. Note that

$$\mathrm{spec}_p(H_{\mathrm{ff}}(z)) \setminus \{0\} \subset (a, 0), \qquad a < 0, \tag{11.19}$$

is discrete, and each eigenspace is finite dimensional by analytic Fredholm theory, applied in the scattering calculus [**25**, Theorem 1], as $H_{\mathrm{ff}}(z)$ is bounded below, and by the absence of positive eigenvalues [**25**, Theorem 2]. We thus conclude that $\widehat{(H - \lambda)}_{\mathrm{ff}}(z, \tau, \nu)$ is invertible with inverse in $\Psi_{\mathrm{sc}}^{-2,0}(\mathbb{S}^n_+)$ if an only if

$$\lambda - \tau^2 - |\nu|_z^2 \notin [0, \infty) \cup \mathrm{spec}_p(H_{\mathrm{ff}}(z)). \tag{11.20}$$

It is convenient to define

$$\Sigma_b(\lambda) = \{(z, \tau, \nu) : \ \lambda - \tau^2 - |\nu|_z^2 \in \mathrm{spec}_p(H_{\mathrm{ff}}) \setminus \{0\}\}. \tag{11.21}$$

With this notation we have thus proved the following proposition:

Proposition 11.2. — *Let H be as in* (11.11), $\lambda \in \mathbb{R}$. *Then the characteristic set of* $H - \lambda$ *is given by*

$$\Sigma_{\mathrm{mf}}(H-\lambda) = \Sigma_{\Delta-\lambda}, \tag{11.22}$$

$$\Sigma_{\mathrm{ff}}(H-\lambda) = \Sigma_n(\lambda) \cup \Sigma_t(\lambda) \cup \Sigma_b(\lambda). \tag{11.23}$$

Thus, for $u \in \mathcal{C}^{-\infty}(X)$,

$$\alpha \in {}^{\mathrm{sc}}T^*_{\partial X}X \setminus \Sigma_{\Delta-\lambda} \text{ and } \alpha \notin \mathrm{WF}_{3\mathrm{sc,mf}}((H-\lambda)u) \Longrightarrow \alpha \notin \mathrm{WF}_{3\mathrm{sc,mf}}(u), \tag{11.24}$$

and similarly

$$\xi \in W^{\perp} \setminus \Sigma_{\mathrm{ff}}(H-\lambda) \text{ and } \xi \notin \mathrm{WF}_{3\mathrm{sc,ff}}((H-\lambda)u) \Longrightarrow \xi \notin \mathrm{WF}_{3\mathrm{sc,ff}}(u). \tag{11.25}$$

Remark 11.3. — Note that $\Sigma_{\mathrm{ff}}(H-\lambda)$ is a compact subset of $W^{\perp}$ due to (11.19).

We now discuss the basic properties of the (rescaled) Hamilton vector field, ${}^{\mathrm{sc}}H_g$ of Δ. Let $R_{\overline{\mu}} = \overline{\mu} \cdot \partial_{\overline{\mu}}$ be the $\overline{\mu}$-radial vector field in coordinates $(x, \overline{y}, \tau, \overline{\mu})$ on ${}^{\mathrm{sc}}T^*X$ above a neighborhood of $p \in \partial X$; this is well-defined at ${}^{\mathrm{sc}}T^*_{\partial X}X$ independently of the coordinates. Then the Hamilton vector field of g becomes

$${}^{\mathrm{sc}}H_g = 2\tau(x\partial_x + \overline{\mu} \cdot \partial_{\overline{\mu}}) - 2h\partial_\tau + H_h + xW', \qquad W' \in \mathcal{V}_{\mathrm{b}}({}^{\mathrm{sc}}T^*X), \tag{11.26}$$

where H_h is the Hamilton vector field of $h \in \mathcal{C}^\infty(T^*\partial X)$; see [**25**, Equation (8.17)]. Noting that h is positive definite, ${}^{\mathrm{sc}}H_g$ vanishes at ∂X if and only if $\overline{\mu} = 0$ since $x\partial_x$ restricts to ∂X as 0. For $\lambda > 0$ we define the 'radial surfaces'

$$R^{\pm}_{\lambda} = \{(\overline{y}, \tau, \overline{\mu}) : \ \tau = \pm\lambda^{1/2}, \ \overline{\mu} = 0\}. \tag{11.27}$$

Thus, for $\lambda > 0$, $H - \lambda$ gives rise to real principal type propagation of singularities on $\Sigma_{\Delta-\lambda} \setminus (R^{\pm}_{\lambda} \cup {}^{\mathrm{sc}}T^*_C X)$ as in Hörmander's theorem; in this setting it was proved by Melrose in [**25**, Proposition 7]. All integral curves $\gamma(t)$ of ${}^{\mathrm{sc}}H_g$ in $\Sigma_{\Delta-\lambda}$ tend to R^-_λ as $t \to \infty$ and to R^+_λ as $t \to -\infty$; the signs correspond to the negative sign of the ∂_τ component of ${}^{\mathrm{sc}}H_g$.

In the local coordinates used above near $p \in C$ we compute H_h:

$$\begin{aligned} H_h = {} & 2\sum_{i,j} h^{ij}_{nn}\mu_j\partial_{y_i} + 2\sum_{i,j} h^{ij}_{nt}\mu_i\partial_{z_j} + 2\sum_{ij} h^{ij}_{nt}\nu_j\partial_{y_i} + 2\sum_{i,j} h^{ij}_{tt}\nu_j\partial_{z_i} \\ & + \sum_{i,j,k}(\partial_{z_k}h^{ij}_{nn})\mu_i\mu_j\partial_{\nu_k} + 2\sum_{i,j,k}(\partial_{z_k}h^{ij}_{nt})\mu_i\nu_j\partial_{\nu_k} + \sum_{i,j,k}(\partial_{z_k}h^{ij}_{tt})\nu_i\nu_j\partial_{\nu_k} + W' \end{aligned} \tag{11.28}$$

with $W' = \sum \alpha_j \partial_{\mu_j}$ for some $\alpha_j \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*\partial X)$. By (11.3)

$$H_h(0, z, \tau, \mu, \nu) - 2\mu \cdot \partial_y \in T({}^{\mathrm{sc}}T^*_C X) \tag{11.29}$$

so we see that ${}^{\mathrm{sc}}H_g$ is normal to ${}^{\mathrm{sc}}T^*_C X$ on $(\pi^{\perp})^{-1}(\Sigma_n(\lambda)) \cap \Sigma_{\Delta-\lambda}$, but it is tangent to it on $(\pi^{\perp})^{-1}(\Sigma_t(\lambda)) \cap \Sigma_{\Delta-\lambda}$ as claimed. Hence singularities can be expected to leave C normally in the former case, while in the latter case more complicated phenomena could occur. Since $(\pi^\perp)^{-1}(\Sigma_b(\lambda))$ is disjoint from $\Sigma_{\Delta-\lambda}$, singularities at Σ_b can be expected to remain at C.

We shall see that if H_{ff} is independent of z in some local coordinates, as in the actual three-body problem, the propagation of singularities at $\Sigma_b(\lambda)$ is governed by

$$W = 2\tau(\nu \cdot \partial_\nu) - 2\widetilde{h}\partial_\tau + H_{\widetilde{h}} \in \mathcal{V}(W^\perp). \tag{11.30}$$

Thus, $W(z,\tau,\nu) = \pi_*^\perp|_{(0,z,\tau,0,\nu)}{}^{\mathrm{sc}}H_g$. Note that W vanishes if and only if $\widetilde{h} = 0$, so we will see propagation outside of $\widetilde{h} = 0$. Along the integral curves $\gamma(t)$ of W, $\tau^2 + \widetilde{h}$ is constant, and it is greater than λ on $\Sigma_b(\lambda)$. In addition, the integral curves tend to $R^\pm$ as $t \to \mp\infty$; here

$$R^\pm = \{(z,\tau,\nu) : \ \widetilde{h} = 0, \ \pm\tau > 0\}. \tag{11.31}$$

This follows from the formula for the ∂_τ component of W; we provide a more detailed analysis of these integral curves in the following paragraphs along the lines of the description of the bicharacteristics of g by Melrose and Zworski [**29**, Lemma 2]. Recall also that $W^\perp$ is a subbundle of ${}^{\mathrm{sc}}T^*_C X$ over C, and it is given by $\mu = 0$ in our local coordinates. Correspondingly, we can think of $\Sigma_t(\lambda) \cup \Sigma_b(\lambda)$ as a subset of ${}^{\mathrm{sc}}T^*_C X$.

In fact, we see that under certain assumptions singularities of (approximate) eigenfunctions of H propagate along broken bicharacterstics. The definition we take is analogous to Hörmander's in [**18**, Definition 24.2.2].

Definition 11.4. — A broken bicharacteristic of $H-\lambda$, H as in (11.11), is a continuous map

$$\gamma : I \setminus B \to \Sigma_{\Delta-\lambda} \cup \Sigma_b(\lambda) \subset {}^{\mathrm{sc}}T^*_{\partial X} X \tag{11.32}$$

where $I \subset \mathbb{R}$ is an interval and B is a discrete subset such that

(i) if J is an interval, $J \subset I \setminus B$, then $\gamma|_J$ is an integral curve of either ${}^{\mathrm{sc}}H_g$ or W,
(ii) if $t \in B$ then the limits $\gamma(t-0)$ and $\gamma(t+0)$ both exist, belong to ${}^{\mathrm{sc}}T^*_C X$, and $\pi^\perp(\gamma(t-0)) = \pi^\perp(\gamma(t+0))$.

Broken bicharacteristics will be sufficient for describing the propagation of singularities if no bicharacteristic of ${}^{\mathrm{sc}}H_g$ which does not lie completely in $W^\perp$ is tangent to $W^\perp$ to infinite order. For example, this is satisfied if C is totally geodesic. If this condition is not satisfied, we need to generalize this notion similarly to [**18**, Definition 24.3.7] which comes from the original definition by Melrose and Sjöstrand [**28**, Definition 3.1]. We need to make some modifications however. Since the glancing set of order precisely 2 does not break up into the disjoint union of a diffractive set and a gliding set (even if C has codimension 1 in ∂X, there is no natural notion of 'diffractive' and 'gliding'), the above mentioned definition has to be changed so that the diffractive set is treated on equal footing with the rest of the glancing set. This means that the generalized broken bicharacteristics are just like the analytic rays defined by Sjöstrand in **[36]**; except that we are in a higher codimensional setting, and even in the codimension 1 case C has 'two sides'.

Definition 11.5. — A generalized broken bicharacteristic of $H-\lambda$, H as in (11.11), is a continuous map

$$\gamma : I \setminus B \to \Sigma_{\Delta-\lambda} \cup \Sigma_b(\lambda) \subset {}^{\mathrm{sc}}T^*_{\partial X}X \tag{11.33}$$

where $I \subset \mathbb{R}$ is an interval and B is a subset of I such that

(i) if $t \in I \setminus B$ then γ is differentiable at t, and $\gamma'(t) = {}^{\mathrm{sc}}H_g(\gamma(t))$ or $\gamma'(t) = W(\gamma(t))$,
(ii) if $t \in B$ then t is an isolated point of B, the limits $\gamma(t-0)$ and $\gamma(t+0)$ both exist, belong to ${}^{\mathrm{sc}}T^*_C X$, and $\pi^\perp(\gamma(t-0)) = \pi^\perp(\gamma(t+0))$.

We often say 'broken bicharacteristics' instead of 'generalized broken bicharacteristics' when it is clear from the context what is meant.

Finally we define (generalized) broken geodesics. We actually only state the definition of broken geodesics (which is very similar to Definition 11.4), Definition 11.5 can be modified similarly to yield a definition of generalized broken geodesics. In the following definition we regard $S^*\partial X \subset T^*\partial X$ as the unit cosphere bundle with respect to the metric $h|_{\partial X}$. Also, let $\widetilde{\pi^\perp} : T^*_C\partial X \to \widetilde{W} = (N^*C)^\perp$ be the orthogonal projection to the orthocomplement of N^*C with respect to $h|_{\partial X}$.

Definition 11.6. — A broken geodesic of $h|_{\partial X}$, h as in (11.1), is a continuous map

$$\tilde{\gamma} : I \setminus B \to S^*\partial X \subset T^*\partial X \tag{11.34}$$

where $I \subset \mathbb{R}$ is an interval and B is a discrete subset such that

(i) if J is an interval, $J \subset I \setminus B$, then $\tilde{\gamma}|_J$ is an integral curve of either $H_{h/2}$ or $H_{\tilde{h}/2}$,
(ii) if $t \in B$ then the limits $\tilde{\gamma}(t-0)$ and $\tilde{\gamma}(t+0)$ both exist, belong to $S^*\partial X$, and $\widetilde{\pi^\perp}(\tilde{\gamma}(t-0)) = \widetilde{\pi^\perp}(\tilde{\gamma}(t+0))$.

The factor $1/2$ in $H_{h/2}$ and $H_{\tilde{h}/2}$ only appears to make sure that the tangent vector to a broken geodesic, when it is defined, has unit length. There is a close connection between (generalized) broken bicharacteristics and broken geodesics. Namely, the projection of a broken bicharacteristic to $T^*\partial X$ first, and then rescaled to $S^*\partial X$ using the $\mathbb{R}^+$ action on $T^*\partial X$, is a reparametrized, non-maximally extended, broken geodesic whose projection to ∂X has length π (with respect to $h|_{\partial X}$). To see this, first recall Melrose's and Zworski's discussion **[29**, Lemma 2] of the corresponding relationship between bicharacteristics of g in $\Sigma_{\Delta-\lambda} \setminus (R^-_\lambda \cup R^+_\lambda)$ and geodesics of $h|_{\partial X}$.

Thus, Melrose and Zworski showed that after rescaling the parameter along the bicharacteristics of g in $\Sigma_{\Delta-\lambda} \setminus (R^-_\lambda \cup R^+_\lambda)$ to $s \in (0,\pi)$, with the rescaling given by

$ds/dt = \frac{1}{2}h^{1/2}$, they are curves of the form

$$\tau = \lambda^{1/2} \cos s, \tag{11.35}$$

$$\overline{\mu} = \lambda^{1/2} (\sin s)\widehat{\overline{\mu}}, \tag{11.36}$$

$$(\overline{y}, \widehat{\overline{\mu}}) = \exp(sH_{h/2})(\overline{y}', \overline{\mu}') \tag{11.37}$$

where $(\overline{y}', \overline{\mu}') \in T^*\partial X$ and $h(\overline{y}', \overline{\mu}') = 1$, i.e. $(\overline{y}', \overline{\mu}') \in S^*\partial X$ by our identification of $S^*\partial X$. Equivalently, they are curves of the form

$$\tau = \lambda^{1/2} \cos s, \tag{11.38}$$

$$\overline{\mu} = \lambda^{1/2} (\sin s)\widehat{\overline{\mu}}, \tag{11.39}$$

$$(\overline{y}, \widehat{\overline{\mu}}) = \exp((s-\pi)H_{h/2})(\overline{y}', \overline{\mu}'), \tag{11.40}$$

$s \in (0, \pi)$. In particular, as s varies, $(\overline{y}, \widehat{\overline{\mu}})$ moves along the geodesic with initial point $(\overline{y}', \overline{\mu}')$. Given $(\overline{y}', \overline{\mu}') \in S^*\partial X$, we let $\gamma_-(t; \overline{y}', \overline{\mu}')$ be the unique bicharacteristic of g which is of the form (11.35)-(11.37) after reparametrization and which satisfies $\tau(\gamma_-(0; \overline{y}', \overline{\mu}')) = 0$. Similarly, let $\gamma_+(t; \overline{y}', \overline{\mu}')$ be the unique bicharacteristic of g which is of the form (11.38)-(11.40) after reparametrization and which satisfies $\tau(\gamma_+(0; \overline{y}', \overline{\mu}')) = 0$. It is useful to introduce some notation for the corresponding relation between points of $\Sigma_{\Delta-\lambda} \setminus (R^-_\lambda \cup R^+_\lambda)$ and points of $S^*\partial X$.

Definition 11.7. — Suppose $\alpha \in \Sigma_{\Delta-\lambda}$, $\zeta \in S^*\partial X$. We say that $\alpha \sim'_\pm \zeta$ if there is a time $t \in \mathbb{R}$ such that $\gamma_\pm(t; \zeta) = \alpha$.

Now, the (generalized) broken bicharacteristics of $H - \lambda$, $\lambda > 0$, can be described similarly. Namely, after reparametrizing them, letting $ds/dt = \frac{1}{2}h^{1/2}$, they become curves of the form

$$\tau = \widetilde{\lambda}^{1/2} \cos s, \tag{11.41}$$

$$\overline{\mu} = \widetilde{\lambda}^{1/2} (\sin s)\widehat{\overline{\mu}}, \tag{11.42}$$

$$(\overline{y}, \widehat{\overline{\mu}}) = \widetilde{\gamma}(s - \pi, \overline{y}', \overline{\mu}'), \tag{11.43}$$

where $s \in (0, \pi)$, $\widetilde{\lambda} = \lambda$ or $\lambda - \widetilde{\lambda} \in \operatorname{spec}_p(H_{\mathrm{ff}})$, and $\widetilde{\gamma}$ is a (generalized) broken geodesic satisfying $\widetilde{\gamma}(\pi, \overline{y}', \overline{\mu}') = (\overline{y}', \overline{\mu}')$. Moreover, if $\widetilde{\lambda} \neq \lambda$, then $\widetilde{\gamma}$ must be an integral curve of $H_{\widetilde{h}/2}$. Here we only stated the second parametrization; the first one can be stated similarly. This parametization can be deduced similarly to the way (11.38)-(11.40) is proved in [**29**, Lemma 2]. Namely, changing into $\overline{\mu}$ polar coordinates, i.e.

$$\widehat{\overline{\mu}} = h(\overline{y}, \overline{\mu})^{-1/2} \overline{\mu}, \quad |\overline{\mu}| = h(\overline{y}, \overline{\mu})^{1/2}, \tag{11.44}$$

and changing the parametrization by $ds/dt = \frac{1}{2}|\overline{\mu}|$, yields

$$\frac{d\tau}{ds} = -|\overline{\mu}|, \quad \frac{d}{ds}|\overline{\mu}| = \tau, \tag{11.45}$$

$$\frac{d}{ds}(\overline{y}, \overline{\mu}) = H_{h/2} \text{ if } \frac{d}{dt}\gamma|_{t(s)} = {}^{\mathrm{sc}}H_g, \tag{11.46}$$

$$\frac{d}{ds}(\overline{y}, \overline{\mu}) = H_{\widetilde{h}/2} \text{ if } \frac{d}{dt}\gamma|_{t(s)} = W. \tag{11.47}$$

This proves that the projection of a (generalized) broken bicharacteristic γ to ∂X is a (generalized) broken geodesic of length π as claimed. Note that this also shows that if γ is a broken bicharacteristic through $\alpha = (\overline{y}, \tau, \overline{\mu})$, $\overline{y} \notin C$, $\gamma(t_0) = \alpha$, and $\lambda^{1/2} \cos \operatorname{dist}(\overline{y}, C) < \tau$, then in (11.41), $s < \operatorname{dist}(\overline{y}, C)$, so by (11.43), γ does not intersect ${}^{\mathrm{sc}}T^*_C X$ for $t \leq t_0$ (since the broken geodesics have tangent vectors of unit length), i.e. γ is actually a bicharacteristic for $t \leq t_0$. Similarly, if $\lambda^{1/2} \cos \operatorname{dist}(\overline{y}, C) < -\tau$, $\gamma(t_0) = \alpha = (\overline{y}, \tau, \overline{\mu})$, then γ is a bicharacteristic for $t \geq t_0$. Note also that by (11.42) $\overline{\mu} \to 0$ as $t \to \pm\infty$, and $\tau \to \mp\widetilde{\lambda}^{1/2}$ as $t \to \pm\infty$.

We also introduce the relation corresponding to $\sim'_{\pm}$ for (generalized) broken bicharacteristics.

Definition 11.8. — Suppose $\alpha \in \Sigma_{\Delta - \lambda}$, $\xi \in \Sigma_{\mathrm{ff}}(H - \lambda)$, $\zeta \in S^*_{\partial X \setminus C} \partial X$. We say that $\alpha \sim_{\pm} \zeta$ if there is a (generalized) broken bicharacteristic γ through α and a constant C which satisfies $\gamma(t) = \gamma_{\pm}(t; \zeta)$ for $\pm t > C$. We also say that $\xi \sim_{\pm} \zeta$ if for some, and hence for all, $\alpha' \in \Sigma_{\Delta - \lambda^2}$ with $\pi^{\perp}(\alpha') = \xi$, $\alpha' \sim_{\pm} \zeta$.

In the propagation theorems we shall see that if for some $\xi_0 \in W^{\perp}$, ξ_0 does not belong to $\mathrm{WF}_{3\mathrm{sc,ff}}((H - \lambda)u)$ and certain additional conditions hold then ξ_0 does not belong to $\mathrm{WF}^{m,l}_{3\mathrm{sc,ff}}(u)$ for any m and l. We now prove that m does not play a role at all since $\sigma_{3\mathrm{sc},2}(H)$ is elliptic.

Lemma 11.9. — *Suppose that $\lambda \in \mathbb{R}$, $\xi_0 \in W^{\perp}$ and $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc,ff}}((H - \lambda)u)$. If in addition there exist m and l such that $\xi_0 \notin \mathrm{WF}^{m,l}_{3\mathrm{sc,ff}}(u)$ then for any m' we have $\xi_0 \notin \mathrm{WF}^{m',l}_{3\mathrm{sc,ff}}(u)$.*

Proof. — It is convenient to use that H is self-adjoint, so $(H + i)^{-1} \in \Psi^{-2,0}_{3\mathrm{sc}}(X)$. By our assumptions we have some $Q \in \Psi^{0,0}_{3\mathrm{sc}}(X)$ with $\widehat{Q}(\xi_0)$ invertible in $\Psi^{0,0}_{\mathrm{sc}}(\mathbb{S}^n_+)$ for which $Qu \in H^{m,l}_{\mathrm{sc}}(X)$. Since $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc,ff}}((H - \lambda)u)$ we can also arrange that

$$Q(H - \lambda)u \in \dot{C}^{\infty}(X)$$

by reducing $\mathrm{WF}'_{3\mathrm{sc}}(Q)$ if necessary. Writing $\lambda + i = (H + i) - (H - \lambda)$ we see that

$$Qu = (\lambda + i)^{-1} Q(H + i)u - (\lambda + i)^{-1} Q(H - \lambda)u. \tag{11.48}$$

By our assumption Qu belongs to $H^{m,l}_{\mathrm{sc}}(X)$, and we have seen that the same holds for $Q(H-\lambda)u$. Thus, $Q(H+i)u \in H^{m,l}_{\mathrm{sc}}(X)$, so

$$Q'u \in H^{m+2,l}_{\mathrm{sc}}(X), \qquad Q' = (H+i)^{-1}Q(H+i) \in \Psi^{0,0}_{3\mathrm{sc}}(X). \tag{11.49}$$

But $\widehat{Q}'(\xi_0)$ is invertible in $\Psi^{0,0}_{\mathrm{sc}}(\mathbb{S}^n_+)$, so we conclude that $\xi_0 \notin \mathrm{WF}^{m+2,l}_{3\mathrm{sc,ff}}(X)$. We can repeat this argument if necessary, thus completing the proof of the lemma. □

Since $H \in \Psi^{2,0}_{3\mathrm{sc}}(X)$ is self-adjoint and $\sigma_{3\mathrm{sc},2}(H)$ is elliptic, we have for all $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$ that $\psi(H) \in \Psi^{-\infty,0}_{3\mathrm{sc}}(X)$. Moreover, if $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ and $\phi \equiv 1$ on $\operatorname{supp}\psi$ then $(\mathrm{Id}-\phi(H))\psi(H) = 0$, $\psi(H)(\mathrm{Id}-\phi(H)) = 0$. Now,

$$\widehat{\phi(H)}_{\mathrm{ff}} = \phi(\widehat{H}_{\mathrm{ff}}) = \phi(H_{\mathrm{ff}}(z) + \tau^2 + |\nu|^2_z), \tag{11.50}$$

and $H_{\mathrm{ff}}(z) = \Delta_{\mathrm{ff}} + V_{\mathrm{ff}}(z) \geq c$ for some $c \in \mathbb{R}$, so for a sufficiently large C, $\tau^2 + |\nu|^2_z \geq C$ implies that $\widehat{H}_{\mathrm{ff}} \geq 1 + \sup\operatorname{supp}\phi$, so $\phi(\widehat{H}_{\mathrm{ff}}(\xi)) = 0$ when $\tau^2 + |\nu|^2_z$ is large, $\xi = (z,\tau,\nu)$. In particular, $(\widehat{\mathrm{Id}-\phi(H)})_{\mathrm{ff}}(\xi) = \mathrm{Id}$ outside a compact subset of $W^\perp$. Taking a microlocal parametrix P of $\mathrm{Id}-\phi(H)$ at such a ξ, so $\mathrm{Id} = P(\mathrm{Id}-\phi(H)) + R$, $\xi \notin \mathrm{WF}'_{3\mathrm{sc,ff}}(R)$, shows that

$$\psi(H) = P(\mathrm{Id}-\phi(H))\psi(H) + R\psi(H) = R\psi(H). \tag{11.51}$$

Since $\xi \notin \mathrm{WF}'_{3\mathrm{sc}}(R)$, we conclude that $\xi \notin \mathrm{WF}'_{3\mathrm{sc}}(\psi(H))$. Since a similar argument works at mf, we deduce that $\mathrm{WF}'_{3\mathrm{sc}}(\psi(H)) \subset {}^{3\mathrm{sc}}T^*_{\mathrm{mf}}[X;C] \cup W^\perp$. Correspondingly we can drop the compactifications ${}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C]$, $\overline{W}^\perp$ in our arguments.

CHAPTER 12

THE MOURRE ESTIMATE

The Mourre estimate is a global positive commutator estimate for perturbations of the Laplacian which has been used widely in the analysis of many-body scattering. Before discussing it we make a definition.

Definition 12.1. — Suppose that H satisfies (11.11). The set of the thresholds of H is defined as

$$\Lambda(H) = \{0\} \cup_{p \in C} \operatorname{spec}_p(H_{\mathrm{ff}}(p)). \tag{12.1}$$

To prove the Mourre estimate (which is the statement of the following Theorem) in this generalized 3-body type setting we first reduce the problem to obtaining the estimate for the normal operators. Then we can use the proof of Froese and Herbst **[9]** for unreduced two-body operators (i.e. two-body operators from which the center of mass motion is not removed).

Theorem 12.2. — *Suppose that H satisfies* (11.11). *Let $A \in x^{-1}\operatorname{Diff}^1_{\mathrm{sc}}(X)$ be self-adjoint with*

$$\widehat{N}_{\mathrm{sc},-1}(A) = \widehat{N}_{\mathrm{sc},-1}(xD_x) \tag{12.2}$$

and let $H = \Delta + V$. For $\lambda \geq \inf \Lambda(H)$ let

$$s(\lambda) = \sup(\Lambda(H) \cap (-\infty, \lambda]), \tag{12.3}$$

otherwise define $s(\lambda) < \lambda$ arbitrarily. Then for $\lambda \in \mathbb{R}$ and $\varepsilon > 0$ there exists an open interval $I \subset (\lambda - \varepsilon, \lambda + \varepsilon)$ such that for all $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ supported in I

$$i\phi(H)[A,H]\phi(H) \geq 2(\lambda - s(\lambda) - \varepsilon)\phi(H)^2 + K \tag{12.4}$$

where $K \in \Psi^{-\infty,1}_{3\mathrm{sc}}(X)$.

Proof. — First, $A \in Z\,\Psi^{1,-1}_{3\mathrm{sc}}(X)$ by Lemma 6.5, so $[A,V] \in \Psi^{0,0}_{3\mathrm{sc}}(X)$, and actually in $\rho_{\mathrm{mf}}\,\Psi^{0,0}_{3\mathrm{sc}}(X)$ due to the additional vanishing of V at mf. Of course, $[A,\Delta] \in \Psi^{2,0}_{\mathrm{sc}}(X)$

already, since the scattering calculus is commutative at the level of normal operators. Now it suffices to prove (12.4) for the normal operators, i.e. that

$$iN_{\mathrm{ff},0}(\phi(H))N_{\mathrm{ff},0}([A,H])N_{\mathrm{ff},0}(\phi(H)) \geq 2(\lambda - s(\lambda) - \varepsilon)\phi(H)^2, \tag{12.5}$$

$$iN_{\mathrm{mf},0}(\phi(H))N_{\mathrm{mf},0}([A,H])N_{\mathrm{mf},0}(\phi(H)) \geq 2(\lambda - s(\lambda) - \varepsilon)\phi(H)^2. \tag{12.6}$$

In fact, if these hold, then Proposition C.2, applied with $C = 2(\lambda - s(\lambda) - \varepsilon)\mathrm{Id}$, shows (12.4) with 2ε in place of ε.

Note that (12.6) is just the standard estimate of the scattering calculus since $N_{\mathrm{mf},0}(H) = N_{\mathrm{mf},0}(\Delta)$, so as

$$N_{\mathrm{mf},0}(i[A,H]) = N_{\mathrm{mf},0}(i[A,\Delta]) = \{j_{\mathrm{sc}}(A), j_{\mathrm{sc}}(\Delta)\} = 2j_{\mathrm{sc}}(\Delta).$$

Also, by Proposition 10.3, $N_{\mathrm{mf},0}(\phi(H)) = \phi(j_{\mathrm{sc}}(\Delta))$. Thus, (12.6) follows from $\operatorname{supp}\phi \subset (\lambda - \varepsilon, \lambda + \varepsilon)$ and $\operatorname{spec}(\Delta) = [0,\infty)$.

On the other hand, the normal operator estimate on the front face can be replaced by its Fourier transform, i.e. the corresponding indicial operator estimate. Recall that Δ_{ff} is the fiber Laplacian of β as in Lemma 11.1, and note that

$$H_{\mathrm{ff}} = \Delta_{\mathrm{ff}} + V_{\mathrm{ff}} = \Delta_{\mathrm{ff}} + V|_{\mathrm{ff}}. \tag{12.7}$$

Now, using the local coordinate expression of A in the interior of ff

$$i\widehat{[A,H]}_{\mathrm{ff},0} = 2(\Delta_{\mathrm{ff}} + \tau^2 + |\nu|_z^2) + \widehat{[-\overline{Y}\partial_{\overline{Y}}, V]}_{\mathrm{ff},0} = \widehat{[-\overline{Y}\partial_{\overline{Y}}}, H_{\mathrm{ff}}] + 2(\tau^2 + |\nu|_z^2). \tag{12.8}$$

Thus, it suffices to prove that for $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ with sufficiently small support

$$\begin{aligned}\phi(H_{\mathrm{ff}} + \eta)([-\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}] + 2\eta)\phi(H_{\mathrm{ff}} + \eta)\\ \geq 2(\lambda - s(\lambda) - \varepsilon)\phi(H_{\mathrm{ff}} + \eta)^2\end{aligned} \tag{12.9}$$

where we have written $\eta = \tau^2 + |\nu|_z^2$ for simplicity. As the notation indicates ϕ is not allowed to depend on τ, ν and $p \in C$. This is exactly the 2-body estimate of the Theorem of Froese and Herbst in [9] if $N_{\mathrm{ff},0}(V)$ is the same on each fiber up to translations and metric preserving transformations of ${}^{\mathrm{sc}}T_pX$ with ${}^{\mathrm{sc}}T_qX$, $p,q \in C$ (this statement makes sense due to 4.2). The general case requires only minor modifications.

Namely, the point is to reduce the estimate to first a similar one but with ϕ possibly depending on $\xi = (p,\tau,\nu) \in W^\perp$, and then further to an estimate analogous to (12.4) for the two body operators. In fact, a weaker estimate than (12.4) suffices for two-body operators. More precisely, suppose that $\lambda \in \mathbb{R}$, and $\varepsilon > 0$. If $\lambda \geq 0$ let $\delta = \varepsilon$, if $\lambda < 0$ let $\delta = \min\{-\lambda, \varepsilon\}$. Then for $\sigma \in (-\infty, \lambda]$ and for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$ supported in the open interval $I = (\sigma - \delta, \sigma + \delta)$ we have that

$$\phi(H_{\mathrm{ff}}(z))[-\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}(z)]\phi(H_{\mathrm{ff}}(z)) \geq 2(\sigma - s(\lambda) - \varepsilon)\phi(H_{\mathrm{ff}}(z))^2 + R(z) \tag{12.10}$$

where $R(z)$ is a continuous function on C with values in $\Psi_{sc}^{-\infty,1}(\beta^{-1}(p))$ if we fix ϕ. The analog of (12.4) would have $s_2(\sigma)$ instead of $s(\lambda)$ on the right hand side where $s_2(\sigma) = 0$ if $\sigma \geq 0$ and it can be taken arbitrary if $\sigma < 0$. Thus, (12.10) is weaker than the two-body Mourre estimate since $s_2(\sigma) \leq s(\lambda)$ if $\sigma \leq \lambda$. Now, since

$$\widehat{[A,V]}_{\mathrm{ff},0} \in \Psi_{sc}^{0,1}(\beta^{-1}(p)), \tag{12.11}$$

and, by Proposition 10.2,

$$\phi(H_{\mathrm{ff}}(z)) - \phi(\Delta) \in \Psi_{sc}^{-\infty,1}(\beta^{-1}(p)), \tag{12.12}$$

so taking into account $[-\overline{Y}\partial_{\overline{Y}}, \Delta_{\mathrm{ff}}] = 2\Delta_{\mathrm{ff}}$, (12.10) is a consequence of

$$\phi(\Delta)\Delta\phi(\Delta) \geq (\sigma - \varepsilon)\phi(\Delta)^2 \tag{12.13}$$

for $\lambda \geq 0$ and the vanishing of both sides of (12.13) if $\lambda < 0$. These in turn follow from $\operatorname{supp}\phi \subset I$, $\operatorname{spec}(\Delta) = [0,\infty)$, and from the fact that if $\lambda < 0$ then $I \subset (-\infty, 0)$.

Now suppose that $\psi \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$ is supported in $(\lambda-\delta, \lambda+\delta)$. Let $\sigma(\xi) = \lambda - \tau^2 - |\nu|_z^2$. Then with $\phi_\xi(t) = \psi(t + \tau^2 + |\nu|_z^2)$ we have $\operatorname{supp}\phi_\xi \subset (\sigma(\xi)-\delta, \sigma(\xi)+\delta)$ and $\phi_\xi(H_{\mathrm{ff}}(z)) = \psi(\widehat{H}_{\mathrm{ff}}(\xi))$. Thus, by (12.10)

$$\psi(\widehat{H}_{\mathrm{ff}}(\xi))[-\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}(z)]\psi(\widehat{H}_{\mathrm{ff}}(\xi)) \geq 2(\sigma(\xi) - s(\lambda) - \varepsilon)\psi(\widehat{H}_{\mathrm{ff}}(\xi))^2 + R(\xi) \tag{12.14}$$

where now $R(\xi)$ is a continuous function on $W^\perp$ with values in $\Psi_{sc}^{-\infty,1}(\mathbb{S}_+^n)$ if we fix ψ.

Choose ψ, $\widetilde{\psi}$, $\phi \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$ such that ψ is identically 1 near $\operatorname{supp}\widetilde{\psi}$, $\operatorname{supp}\psi \subset I = (\lambda-\delta, \lambda+\delta)$, and $\widetilde{\psi} \equiv 1$ near $\operatorname{supp}\phi$. Thus, multiplying (12.14) by $\widetilde{\psi}(\widehat{H}_{\mathrm{ff}})$ from both left and right,

$$\begin{aligned}\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}(\xi))[-\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}]\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}(\xi)) \geq 2(\sigma(\xi) - s(\lambda) - \varepsilon)\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}(\xi))^2 \\ + \widetilde{\psi}(\widehat{H}_{\mathrm{ff}})R(\xi)\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}).\end{aligned} \tag{12.15}$$

Suppose that $\sigma(\xi)$ is not an eigenvalue of $H_{\mathrm{ff}}(z)$, i.e. λ is not an eigenvalue of $\widehat{H}_{\mathrm{ff}}(\xi)$. Then $\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}(\xi)) \to 0$ strongly as $\operatorname{supp}\widetilde{\psi} \to \{\lambda\}$. Thus,

$$\|\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}(\xi))R(\xi)\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}(\xi))\| < \varepsilon \tag{12.16}$$

if we assume that $\widetilde{\psi}$ is supported in a sufficiently small open interval

$$I'_\xi = (\lambda - \delta'_\xi, \lambda + \delta'_\xi).$$

Hence,

$$\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}(\xi))[-\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}(z)]\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}(\xi)) - 2(\sigma(\xi) - s(\lambda) - \varepsilon)\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}(\xi))^2 \geq -\varepsilon \tag{12.17}$$

whenever $\operatorname{supp}\widetilde{\psi} \subset I'_\xi$. The left hand side is a continuous function of ξ with values in $\mathcal{B}(L^2_{sc}(\mathbb{S}_+^n), L^2_{sc}(\mathbb{S}_+^n))$ if we keep $\widetilde{\psi}$ fixed, so there is a neighborhood U_ξ of ξ such that for $\xi' \in U_\xi$

$$\widetilde{\psi}(\widehat{H}_{\rm ff}(\xi'))[-\overline{Y}\partial_{\overline{Y}}, H_{\rm ff}(z')]\widetilde{\psi}(\widehat{H}_{\rm ff}(\xi')) - 2(\sigma(\xi') - s(\lambda) - \varepsilon)\widetilde{\psi}(\widehat{H}_{\rm ff}(\xi'))^2 \geq -2\varepsilon \tag{12.18}$$

if $\operatorname{supp}\widetilde{\psi} \subset I'_\xi$. Multiplying (12.18) by $\phi(\widehat{H}_{\rm ff}(\xi))$ from both left and right and rearranging the equation, we deduce that for all $\xi' \in U_\xi$

$$\phi(\widehat{H}_{\rm ff}(\xi))[-\overline{Y}\partial_{\overline{Y}}, H_{\rm ff}(z)]\phi(\widehat{H}_{\rm ff}(\xi)) \geq 2(\sigma - s(\lambda) - 2\varepsilon)\phi(\widehat{H}_{\rm ff}(\xi))^2 \tag{12.19}$$

whenever $\operatorname{supp}\phi \subset (\lambda - \delta'_\xi/2, \lambda + \delta'_\xi/2)$.

If $\sigma(\xi)$ is an eigenvalue of $H_{\rm ff}(z)$, then $\sigma \leq s(\lambda)$ by the definition of $s(\lambda)$. We want to prove that even in this case there exists a neighborhood U_ξ of ξ and $\delta'_\xi > 0$ such that (12.19) holds whenever $\xi' \in U_\xi$, $\phi \in \mathcal{C}_c^\infty(\mathbb{R}; [0,1])$, $\operatorname{supp}\phi \subset (\lambda - \delta'_\xi/2, \lambda + \delta'_\xi/2)$. We again follow the proof of Froese and Herbst, though in the particular case of three-body scattering the estimate of Lemma 15.1, which we use in the microlocal propagation theorems, would make the proof slightly simpler. So let $E = E_{H_{\rm ff}(z)}(\{\sigma(\xi)\})$, $E_{H_{\rm ff}}$ denoting the spectral projection. We proceed to show that there exists $R_1(\xi)$ compact such that

$$\begin{aligned}\widetilde{\psi}(\widehat{H}_{\rm ff}(\xi))&[-\overline{Y}\partial_{\overline{Y}}, H_{\rm ff}(z)]\widetilde{\psi}(\widehat{H}_{\rm ff}(\xi)) \\ &\geq 2(\sigma(\xi) - s(\lambda) - \varepsilon)\widetilde{\psi}(\widehat{H}_{\rm ff})^2 + \widetilde{\psi}(\widehat{H}_{\rm ff})(\operatorname{Id} - E)R_1(\xi)(\operatorname{Id} - E)\widetilde{\psi}(\widehat{H}_{\rm ff}),\end{aligned} \tag{12.20}$$

from which (12.17) follows as in the previous case since $\widetilde{\psi}(\widehat{H}_{\rm ff}(\xi))(\operatorname{Id} - E) \to 0$ strongly as $\operatorname{supp}\widetilde{\psi} \to \{\lambda\}$. To prove (12.20), choose a finite dimensional orthogonal projection with $\operatorname{Ran} F \subset \operatorname{Ran} E$ and

$$\|(\operatorname{Id} - E)R(\xi)(\operatorname{Id} - E) - (\operatorname{Id} - F)R(\xi)(\operatorname{Id} - F)\| < \varepsilon/2. \tag{12.21}$$

This implies that

$$\begin{aligned}0 \geq &-(\varepsilon/2)(\widetilde{\psi}(\widehat{H}_{\rm ff}(\xi))^2 - F) \\ &+ \widetilde{\psi}(\widehat{H}_{\rm ff})((\operatorname{Id} - E)R(\xi)(\operatorname{Id} - E) - (\operatorname{Id} - F)R(\xi)(\operatorname{Id} - F))\widetilde{\psi}(\widehat{H}_{\rm ff})\end{aligned} \tag{12.22}$$

since $\widetilde{\psi} \equiv 1$ near λ, so $\widetilde{\psi}(\widehat{H}_{\rm ff}(\xi))F = F$. We now use (12.14) with ε replaced by $\varepsilon/4$. Multiplying it by $\widetilde{\psi}(\widehat{H}_{\rm ff})(\operatorname{Id} - F)$ from left and right (noting that the two factors commute) and adding (12.22) to it gives

$$\begin{aligned}(\widetilde{\psi}(\widehat{H}_{\rm ff}(\xi)) - F)&[-\overline{Y}\partial_{\overline{Y}}, H_{\rm ff}(z)](\widetilde{\psi}(\widehat{H}_{\rm ff}(\xi)) - F) \\ &\geq 2(\sigma(\xi) - s(\lambda) - \varepsilon/2)(\widetilde{\psi}(\widehat{H}_{\rm ff})^2 - F) + \widetilde{\psi}(\widehat{H}_{\rm ff})(\operatorname{Id} - E)R(\xi)(\operatorname{Id} - E)\widetilde{\psi}(\widehat{H}_{\rm ff}).\end{aligned} \tag{12.23}$$

Following the proof in **[9]**, we note that now it suffices to show that for some $R_2(\xi)$ compact we have

$$\begin{aligned}F[-\overline{Y}\partial_{\overline{Y}}, H_{\rm ff}(z)]&\widetilde{\psi}(\widehat{H}_{\rm ff}(\xi))(\operatorname{Id} - E) + (\operatorname{Id} - E)\widetilde{\psi}(\widehat{H}_{\rm ff}(\xi))[-\overline{Y}\partial_{\overline{Y}}, H_{\rm ff}(z)]F \\ &\geq 2(\sigma(\xi) - s(\lambda) - \varepsilon/2)F + \widetilde{\psi}(\widehat{H}_{\rm ff})(\operatorname{Id} - E)R_2(\xi)(\operatorname{Id} - E)\widetilde{\psi}(\widehat{H}_{\rm ff}).\end{aligned} \tag{12.24}$$

Indeed, adding (12.23) and (12.24) proves (12.20) since by the virial theorem

$$E[\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}(\xi)]E = 0.$$

Now, we simply let

$$C = C(\xi) = F[-\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}(z)]\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}(\xi))(\mathrm{Id} - E), \tag{12.25}$$

$R_2(\xi) = -\widetilde{\varepsilon}C(\xi)^*C(\xi)$, $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\xi) = \frac{1}{2}(s(\lambda) - \sigma(\xi) + \varepsilon/2)^{-1} > 0$. In this notation (12.24) becomes

$$C^*F + FC \geq -(\widetilde{\varepsilon}^{-1}C^*C + \widetilde{\varepsilon}F), \tag{12.26}$$

and to prove (12.26) it suffices to note that

$$(\widetilde{\varepsilon}^{-1/2}C + \widetilde{\varepsilon}^{1/2}F)^*(\widetilde{\varepsilon}^{-1/2}C + \widetilde{\varepsilon}^{1/2}F) \geq 0. \tag{12.27}$$

Now if $\delta'_\xi > 0$ is sufficiently small and $\widetilde{\psi} = \widetilde{\psi}_\xi$ is supported in $I'_\xi = (\lambda - \delta'_\xi, \lambda + \delta'_\xi)$ then

$$\|\widetilde{\psi}_\xi(\widehat{H}_{\mathrm{ff}}(\xi))(\mathrm{Id} - E)R_1(\xi)(\mathrm{Id} - E)\widetilde{\psi}_\xi(\widehat{H}_{\mathrm{ff}}(\xi))\| < \varepsilon. \tag{12.28}$$

So from (12.20) we conclude that (12.17) holds. Then the very same continuity argument as after (12.17) proves (12.19).

It only remains to show that for a fixed $\lambda \in \mathbb{R}$ we can choose δ' independently of $\xi \in W^\perp$. We have already shown this for a neighborhood U_ξ of each ξ. Note that $H_{\mathrm{ff}}(z)$ is bounded below uniformly, so there exists $c > 0$ such that $\phi(\widehat{H}_{\mathrm{ff}})$ vanishes if ϕ is supported in $(\lambda - 1, \lambda + 1)$, and $\tau^2 + |\nu|_z^2 \geq c$. Thus, (12.19) is automatically satisfied outside a compact subset K of $W^\perp$. Now, $\{U_\xi : \xi \in K\}$ is an open cover of K, so it has a finite subcover $\{U_{\xi_j} : j = 1, \ldots, J\}$. Let $\delta' = \min\{\delta'_{\xi_j} : j = 1, \ldots, J\}/2$. Then for $\phi \in \mathcal{C}_c^\infty(\mathbb{R}; [0,1])$ supported in $I'' = (\lambda - \delta', \lambda + \delta')$, (12.19) shows that

$$\phi(\widehat{H}_{\mathrm{ff}}(\xi))[-\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}]\phi(\widehat{H}_{\mathrm{ff}}(\xi)) \geq 2(\sigma(\xi) - s(\lambda) - 2\varepsilon)\phi(\widehat{H}_{\mathrm{ff}}(\xi))^2 \tag{12.29}$$

since $\xi \in U_{\xi_j}$ for some j. Adding $2(\tau^2 + |\nu|_z^2)\phi(\widehat{H}_{\mathrm{ff}})^2$ to both sides and noting that $\sigma(\xi) + \tau^2 + |\nu|_z^2 = \lambda$ proves (12.9), and hence the theorem. □

One of the main uses of the Mourre estimate is to construct the weak limit of the resolvent of H at the real axis. Thus, note that $s(\lambda) \leq \lambda$ for all λ, and away from the thresholds $\lambda - s(\lambda) - \varepsilon > 0$ for $\varepsilon > 0$ sufficiently small. Here it should be noted that by the absence of positive eigenvalues of the two-body Hamiltonians there are no positive thresholds, so for $\lambda > 0$, and more generally for $\lambda \notin \Lambda(H)$ the Mourre estimate is a positive commutator estimate.

CHAPTER 13

THE BASIC COMMUTATOR ESTIMATE

In the following chapters we analyze the propagation of singularities of generalized eigenfunctions u of H (so $u \in \mathcal{C}^{-\infty}(X)$, $(H-\lambda)u = 0$) by constructing $\widetilde{Q} \in \Psi_{3\mathrm{sc}}^{-\infty,0}(X)$ such that $[\widehat{\widetilde{Q}\phi(H), H}]_{\mathrm{ff},1}$ is positive where $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ is supported near λ. Here $\widehat{\widetilde{Q}}_{\mathrm{ff}}$ will have the form $f\widehat{\psi(H)}_{\mathrm{ff}}$ with $f \in \mathcal{C}^\infty(W^\perp)$, $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$. In fact, f will arise as the restriction of $q \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*X)$ to $W^\perp$, where $q|_{{}^{\mathrm{sc}}T^*_C X}$ is independent of μ, i.e. it is just the extension of a function on $W^\perp$ by the orthogonal projection. Unfortunately, we will have $q \notin \mathcal{C}^\infty({}^{\mathrm{sc}}\overline{T}^* X)$, meaning that the behavior of q at fiber-infinity on ${}^{\mathrm{sc}}T^*X$ is not sufficiently nice (${}^{\mathrm{sc}}\overline{T}^*X$ is the (fiber-)radial compactification of ${}^{\mathrm{sc}}T^*X$). Hence q does not give rise to an element of $\Psi_{\mathrm{sc}}^{\infty,0}(X)$. However, this is a rather irrelevant difficulty since we wish to mutliply the quantization of q by $\psi(H)$, which is in $\Psi_{3\mathrm{sc}}^{-\infty,0}(X)$, i.e. it is trivial at fiber-infinity ('smoothing up to ∂X').

We deal with this difficulty by realizing that we can write down the full symbol $\widetilde{q}$ of $Q\psi(H)$ explicitly, where Q would be defined by a quantization of a symbol with complicated behavior at fiber-infinity, and the quantization of $\widetilde{q}$ gives rise to an element of $\Psi_{3\mathrm{sc}}^{-\infty,0}(X)$ with all the desired properties. Although it is straightforward to compute the indicial operators of $Q\psi(H)$ and $[Q\psi(H), H]$ directly from this point of view, the arguments are much simpler (and more transparent) if we also consider Q as an operator acting on oscillatory functions. On such functions Q behaves essentially as an element of the scattering calculus, thereby simplifying the discussion (indeed, this motivates the choice of q in the following chapters). In particular, we can use $[\psi(H), H] = 0$ explicitly in such an argument.

Since we work locally in what follows, we may replace X by $\widetilde{U} \subset \mathbb{S}^N_+$ open, $\mathbb{S}^N_+$ being the radial compactification of $\mathbb{R}^N$. We have coordinates $(x, \overline{y})$ on $\mathbb{S}^N_+$, $\overline{y}_j$ being local coordinates on $\mathbb{S}^{N-1} = \partial\mathbb{S}^N_+$. Thus, the standard polar coordinates on $\mathbb{R}^N_w$ are $(x^{-1}, \overline{y})$, so $w = x^{-1}\overline{y}$. The canonical coordinates on $T^*\mathbb{R}^N$ induced by w are denoted (w, ξ); the canonical coordinates induced by $(x, \overline{y})$ are denoted $(x, \overline{y}, \tau, \overline{\mu})$ as usual. Note that embedding $\mathbb{S}^{N-1}$ into $\mathbb{R}^N$ as the unit sphere and using the standard metric on both $\mathbb{S}^{N-1}$ and $\mathbb{R}^N$, a covector $\overline{\mu}\cdot d\overline{y} \in T^*_{\overline{y}}\mathbb{S}^{N-1}$ can be regarded first as a vector in $T_{\overline{y}}\mathbb{S}^{N-1}$, hence as a vector in $T_{\overline{y}}\mathbb{R}^N$, which is orthogonal to the radial vector $\overline{y}$. (See also Appendix A, in particular the discussion in the proof of Proposition A.1.) Thus, $\tau = -\xi\cdot\overline{y}$, and $\overline{\mu} = \xi - (\xi\cdot\overline{y})\overline{y}$ with $\overline{\mu}$ regarded as a vector orthogonal to $\overline{y}$. We also use the notation $\langle\xi\rangle^2 = 1+|\xi|^2$, $\langle(\tau,\overline{\mu})\rangle^2 = 1+\tau^2+|\overline{\mu}|^2$ (here $|\cdot|$ is the Euclidian metric in our coordinates). Thus, $\langle\xi\rangle = \langle(\tau,\overline{\mu})\rangle$.

As discussed in Chapter 3, locally in X we can write $\psi(H)$ as the right quantization of a symbol p, i.e. if $\rho \in \mathcal{C}^\infty(X)$ is supported in $U \subset \mathbb{S}^N_+$ open, $\mathrm{cl}(U) \subset \widetilde{U}$, identically 1 on a smaller open set $U' \subset U$, $\widetilde{\rho} \in \mathcal{C}^\infty_c(\widetilde{U})$, $\widetilde{\rho} \equiv 1$ on U, then

$$P = \rho\psi(H)\widetilde{\rho} = (2\pi)^{-N}\int e^{i(\overline{y}/x-\overline{y}'/x')\cdot\xi}p(x',\overline{y}',\xi)\,d\xi. \tag{13.1}$$

Here p is smooth in the blown-up coordinates at C, i.e. it is in $\mathcal{C}^\infty([\mathbb{S}^N_+;C]\times\mathbb{R}^N_\xi)$. Since $\psi(H) \in \Psi^{-\infty,0}_{3\mathrm{sc}}(X)$, i.e. it has smooth kernel, p and its derivatives are actually rapidly decreasing in ξ.

Now suppose that

$$q \in \mathcal{C}^\infty(\mathbb{R}^{N,1}_{x,y,z}\times\mathbb{R}^N_{\tau,\mu,\nu}),\qquad \mathbb{R}^{N,1}_{x,y,z} = [0,\infty)_x\times\mathbb{R}^n_y\times\mathbb{R}^{m-1}_z,\ N = n+m, \tag{13.2}$$

q is supported in $K\times\mathbb{R}^N$ for some compact set $K\subset U'$, and it satisfies the estimates

$$|D^\alpha_{x,y,z}D^\beta_{\tau,\mu,\nu}q| \le C_{\alpha,\beta}\langle(\tau,\mu,\nu)\rangle^{m_{\alpha,\beta}} \tag{13.3}$$

for some $C_{\alpha,\beta}$ and $m_{\alpha,\beta}$ independent of (x,y,z,τ,μ,ν). Changing to the dual coordinates ξ of w, (13.3) becomes

$$|D^\alpha_{x,y,z}D^\beta_\xi q| \le C'_{\alpha,\beta}\langle\xi\rangle^{m'_{\alpha,\beta}} \tag{13.4}$$

Thus, $q \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*X)$, but typically $q \notin \mathcal{C}^\infty({}^{\mathrm{sc}}\overline{T}^*X)$, so q is not the symbol of an element of $\Psi^{0,0}_{\mathrm{sc}}(X)$ (under left, right, Weyl, or other 'reasonable' quantizations). We are mainly interested in q with much better properties; in our positive commutator estimates we take q whose support projects to a compact set in the (τ,ν) coordinates, and behaves as a (classical) symbol in μ. However, it is actually convenient to treat this more general class of q in this chapter.

Although in general $q \notin \mathcal{C}^\infty({}^{\mathrm{sc}}\overline{T}^*X)$, it is easy to see that q defines an operator acting on oscillatory functions $u = e^{i\widetilde{f}/x}v$, $\widetilde{f}\in\mathcal{C}^\infty(X)$, $v\in\mathcal{C}^\infty_c(\widetilde{U})$ by, say, left quantization. Namely,

$$\begin{aligned} Qu &= (2\pi)^{-N}\int e^{i(\overline{y}/x-\overline{y}'/x')\cdot\xi}q(x,\overline{y},\xi)u(x',\overline{y}')(x')^{-N-1}\,d\xi\,dx'\,d\overline{y}' \\ &= (2\pi)^{-N}\int e^{i\overline{y}\cdot\xi/x}q(x,\overline{y},\xi)\mathcal{F}u(\xi)\,d\xi \end{aligned} \tag{13.5}$$

where the integral makes sense as a distributional pairing since $\mathcal{F}u$ is a Lagrangian distribution with compact singular support and it is Schwartz outside a compact subset of $\mathbb{R}^N_\xi$ (e.g. if $\widetilde{f}=0$, then $\mathcal{F}u$ is conormal to the origin); see also Appendix A, in particular the proof of Proposition A.1 for more details. In fact, we can prove more generally that Q defines an operator acting on singular oscillatory functions $u = e^{i\tilde{f}/x}v$, $\tilde{f} \in \mathcal{C}^\infty(X)$, $v \in \mathcal{C}_c^\infty([\tilde{U}; C])$ since such u lies in $H_{\mathrm{sc}}^{\infty,l}(\mathbb{S}_+^N)$ for some l, and correspondingly $\mathcal{F}u \in H_{\mathrm{sc}}^{l,\infty}(\mathbb{S}_+^N)$, so $\mathcal{F}u$ vanishes to infinite order at infinity in a L^2 sense. To prove the existence of this action, we 'regularize q to finite order', i.e. write $q\mathcal{F}u = (\langle\xi\rangle^{-k}q)(\langle\xi\rangle^k\mathcal{F}u)$ in the integrand above, note that $\langle\xi\rangle^k\mathcal{F}u \in H_{\mathrm{sc}}^{l,\infty}(\mathbb{S}_+^N)$ for all k, and $\langle\xi\rangle^{-k}q$ satisfies an arbitrarily large number of the scattering symbol estimates (arbitrarily many seminorms of $\langle\xi\rangle^{-k}$ in $\mathcal{C}^\infty(\mathbb{S}_+^N \times \mathbb{S}_+^N)$ are bounded) provided that we chose k sufficiently large, so we can apply the corresponding results in the scattering calculus (discussed in Chapter 6 here).

Now, choosing a cutoff,

$$\rho' \in \mathcal{C}_c^\infty(\tilde{U}), \qquad \mathrm{supp}(1-\rho') \cap K = \varnothing, \tag{13.6}$$

allows us to extend Q to an operator acting on singular oscillatory sections $u = e^{i\tilde{f}/x}v$, $\tilde{f} \in \mathcal{C}^\infty(X)$, $v \in \mathcal{C}^\infty([X; C])$, on the original manifold by

$$Q_{\rho'}u = Q(\rho' u). \tag{13.7}$$

Based on this, choosing $\rho' \equiv 1$ on U, we can consider the composite operator

$$\tilde{Q} = (Q\rho')(\rho\psi(H)) = Q\rho\psi(H), \tag{13.8}$$

a priori acting on oscillatory functions $u = e^{i\tilde{f}/x}v$, $\tilde{f} \in \mathcal{C}^\infty(X)$, $v \in \mathcal{C}^\infty([X; C])$; $\tilde{Q}$ is independent of the choice of ρ'. Since we have written Q as a left, and $\rho\psi(H)$ as a right quantization, we conclude that the kernel of the composite operator is

$$(Q\rho\psi(H)\tilde{\rho})(x,\overline{y},x',\overline{y}') = (2\pi)^{-N} \int e^{i(\overline{y}/x-\overline{y}'/x')\cdot\xi} q(x,\overline{y},\xi)p(x',\overline{y}',\xi)\,d\xi. \tag{13.9}$$

Note that

$$\rho\psi(H)(1-\tilde{\rho}) \in \Psi_{3\mathrm{sc}}^{-\infty,\infty}(X) = \Psi_{\mathrm{sc}}^{-\infty,\infty}(X), \tag{13.10}$$

so $\rho\psi(H)(1-\tilde{\rho}) : \mathcal{C}^{-\infty}(X) \to \dot{\mathcal{C}}^\infty(X)$ is continuous. Since $Q_{\rho'} : \dot{\mathcal{C}}^\infty(X) \to \dot{\mathcal{C}}^\infty(X)$ is also continuous, we conclude that

$$Q\rho\psi(H)(1-\tilde{\rho}) \in \Psi_{\mathrm{sc}}^{-\infty,\infty}(X). \tag{13.11}$$

Since this is a 'trivial' term, we sometimes write that $\tilde{Q}$ is given by (13.9) (i.e. neglect $\tilde{\rho}$) to simplify the notation.

Motivated by (13.9), we now consider the symbol

$$\tilde{q}(x,\overline{y},x',\overline{y}',\xi) = q(x,\overline{y},\xi)p(x',\overline{y}',\xi). \tag{13.12}$$

Due to the rapid decay of p in ξ, and using the radially compactified notation in the ξ variable, we can deduce that

$$\tilde{q} \in \mathcal{C}^\infty(\mathbb{S}_+^N \times [\mathbb{S}_+^N; C] \times \mathbb{S}_+^N) \tag{13.13}$$

vanishes to infinite order at $\mathbb{S}^N_+ \times \mathbb{S}^N_+ \times \mathbb{S}^{N-1}$; here $\mathbb{S}^{N-1} = \partial\mathbb{S}^N_+$. It follows that the operator $\widetilde{Q}$ obtained by quantizing this 'double-symbol' as

$$\widetilde{Q} = (2\pi)^{-N} \int e^{i(\overline{y}/x - \overline{y}'/x')\cdot\xi} \widetilde{q}(x, \overline{y}, x', \overline{y}', \xi)\, d\xi \tag{13.14}$$

(this is really the kernel of $\widetilde{q}$) is in $\Psi^{-\infty,0}_{3\mathrm{sc}}(X)$ since the integral converges absolutely and away from sf_C and sf' the exponential factor gives infinite order vanishing (cf. Chapter 3).

The simplest way to analyze the symbolic properties of $\widetilde{Q}$ is via the oscillatory testing definition of the indicial operator and recalling that (13.8) holds with the right hand side considered as the composition of operators acting on oscillatory sections. Thus, we only need to compute the leading part of $Q\rho' u$ for $u = e^{i\widetilde{f}/x}v$, $\widetilde{f} \in \mathcal{C}^\infty(X)$, $v \in \mathcal{C}^\infty([X;C])$. But 'regularizing q to finite order' as above shows that this is given by the same formula as for scattering pseudo-differential operators. First, with $f = \widetilde{f}|_{\partial X}$,

$$e^{-i\widetilde{f}/x} Q\rho' e^{i\widetilde{f}/x} v(0,y,z) = q(0,y,z,-f(y,z), \partial_y f(y,z), \partial_z f(y,z)) v(0,y,z), \tag{13.15}$$

so $\widehat{Q}_{\mathrm{mf}} = q|_{{}^{\mathrm{sc}}T^*_{\partial X}X}$. Moreover, with $a = \langle\xi\rangle^{-k} q$ (which can be regarded as a scattering symbol satisfying a sufficient number of symbolic estimates), A (say) the left-quantization of a, using the formula of the scattering calculus (see also Chapter 4 and Chapter 6), we see that

$$A(0,y,z,S,Y,Z) = (2\pi)^{-N} \int e^{i(S\tau + Y\cdot\mu + Z\cdot\nu)} a(0,y,z,\tau,\mu,\nu)\, d\tau\, d\mu\, d\nu. \tag{13.16}$$

Thus,

$$\widehat{A}_{\mathrm{ff}}(z,\tau,\nu; \overline{Y}, Y) = (2\pi)^{-n} \int e^{iY\cdot\mu} a(0,0,z,\tau,\mu,\nu)\, d\mu. \tag{13.17}$$

Correspondingly,

$$Q\rho' u(0,\overline{Y},z) = (2\pi)^{-n} \int e^{iY\cdot\mu} q(0,0,z,\tau,\mu,\nu) v(0, \overline{Y} - Y, z)\, d\mu, \tag{13.18}$$

so

$$\widehat{Q}_{\mathrm{ff}}(z,\tau,\nu; \overline{Y}, Y) = (2\pi)^{-n} \int e^{iY\cdot\mu} q(0,0,z,\tau,\mu,\nu)\, d\mu. \tag{13.19}$$

This operator becomes particularly simple if q satisfies

$$q(0,0,z,\tau,\mu,\nu) = f(z,\tau,\nu), \qquad f \in \mathcal{C}^\infty_c(W^\perp), \tag{13.20}$$

i.e. q is independent of μ at C, since then f can be factored out of the integral giving

$$\widehat{Q}_{\mathrm{ff}}(z,\tau,\nu) = f(z,\tau,\nu)\, \mathrm{Id} \in \Psi^{0,0}_{\mathrm{sc}}(\mathbb{S}^n_+). \tag{13.21}$$

Finally, from (13.8) (using that $\mathrm{supp}(1-\rho) \times \mathbb{R}^N$ and $\mathrm{supp}\, \widetilde{q}$ are disjoint) we deduce that

$$\widehat{\widetilde{Q}}_{\mathrm{mf}} = \widehat{Q}_{\mathrm{mf}} \widehat{\psi(H)}_{\mathrm{mf}}, \qquad \widehat{\widetilde{Q}}_{\mathrm{ff}} = \widehat{Q}_{\mathrm{ff}} \widehat{\psi(H)}_{\mathrm{ff}}. \tag{13.22}$$

The same discussion can be carried out more directly from (13.12); we briefly outline the argument. Namely, it is straightforward to check that

$$x^{-1}(\overline{y}-\overline{y}')\cdot\xi = S\tau + Y\cdot\mu + Z\cdot\nu + xr(x,y,z,S,Y,Z,\tau,\mu,\nu) \tag{13.23}$$

where r and its derivatives is polynomially bounded in (S,Y,Z,τ,μ,ν). Using (4.13), (13.14) gives

(13.24)

$$\widetilde{Q}(0,y,z,S,Y,Z) = (2\pi)^{-N}\int e^{i(S\tau+Y\cdot\mu+Z\cdot\nu)}\widetilde{q}(0,\overline{Y},z,0,\overline{Y}-Y,z,\tau,\mu,\nu)\,d\tau\,d\mu\,d\nu.$$

The indicial operator at mf is given by the (S,Y,Z) Fourier transform of the restriction of the kernel $\widetilde{Q}$ to sf', so it is simply

$$\widehat{\widetilde{Q}}_{\mathrm{mf}}(y,z,S,Y,Z) = \widetilde{q}(0,y,z,0,y,z,\tau,\mu,\nu). \tag{13.25}$$

The indicial operator at ff is given by the (S,Z) Fourier transform of the restriction of the kernel to sf_C, so it is

$$\widehat{\widetilde{Q}}_{\mathrm{ff}}(z,\tau,\nu;\overline{Y},\overline{Y}') = (2\pi)^{-n}\int e^{i(\overline{Y}-\overline{Y}')\cdot\mu}\widetilde{q}(0,\overline{Y},z,0,\overline{Y}',z,\tau,\mu,\nu)\,d\mu. \tag{13.26}$$

If q satisfies (13.20), then we can substitute $\widetilde{q}=qp$ in the above formula, and pull out the factor q as f to conclude that

$$\begin{aligned}\widehat{\widetilde{Q}}_{\mathrm{ff}}(z,\tau,\nu;\overline{Y},Y) &= (2\pi)^{-n}f(z,\tau,\nu)\int e^{iY\cdot\mu}p(0,\overline{Y}-Y,z,\tau,\mu,\nu)\,d\mu\\ &= f(z,\tau,\nu)\widehat{\psi(H)}_{\mathrm{ff}}(z,\tau,\nu)\end{aligned} \tag{13.27}$$

in agreement with the previous results.

Summarizing the previous two paragraphs, we have proved the following proposition.

Proposition 13.1. — *Suppose that q is as in* (13.2) *and $\psi\in C_c^\infty(\mathbb{R})$. Then $\widetilde{q}$ given by* (13.12) *and* (13.1) *defines $\widetilde{Q}\in\Psi_{3\mathrm{sc}}^{-\infty,0}(X)$ via* (13.14). *We also have*

$$\widehat{\widetilde{Q}}_{\mathrm{mf},0}(y,z,\tau,\mu,\nu) = q(0,y,z,\tau,\mu,\nu)\widehat{\psi(H)}_{\mathrm{mf}}(y,z,\tau,\mu,\nu), \tag{13.28}$$

and for $\xi=(z,\tau,\nu)\in W^\perp$, $\widehat{\widetilde{Q}}_{\mathrm{ff},0}$ is given by (13.26). *If in addition* (13.20) *holds, then*

$$\widehat{\widetilde{Q}}_{\mathrm{ff},0}(\xi) = f(\xi)\widehat{\psi(H)}_{\mathrm{ff},0}(\xi). \tag{13.29}$$

Suppose that H is as in (11.11). The condition $[\widetilde{Q},H]\in\Psi_{3\mathrm{sc}}^{-\infty,1}(X)$ is equivalent $[\widehat{\widetilde{Q}}_{\mathrm{ff}},\widehat{H}_{\mathrm{ff}}]\equiv 0$ on $W^\perp$. If it is satisfied, we can compute the indicial operator $\widehat{[\widetilde{Q},H]}_{\mathrm{ff},1}$. Namely, it is just defined by the action of the commutator on oscillatory test functions:

$$\widehat{[\widetilde{Q},H]}_{\mathrm{ff},1}v = (x^{-1}e^{-i\bar{f}/x}[\widetilde{Q},H]e^{i\bar{f}/x}v)|_{\mathrm{ff}}. \tag{13.30}$$

Since the action of Q on such oscillatory sections u has been defined above, we can write $\widetilde{Q} = Q\psi(H)$, $[\widetilde{Q}, H] = [Q, H]\psi(H)$, expand the commutator on the right hand side, and apply the discussion of Chapter 7 even though Q is not in $\Psi_{3sc}^{\infty,0}(X)$. Again, this can be justified by 'finite order regularization of q'. Thus, we have the following proposition:

Proposition 13.2. — *If H satisfies* (11.11), $\psi \in C_c^\infty(\mathbb{R})$, *and q is as in* (13.2) *satisfying*

$$q(0,0,z,\tau,\mu,\nu) = f(z,\tau,\nu), \qquad f \in C_c^\infty(W^\perp) \tag{13.31}$$

then $[\widetilde{Q}, H] \in \Psi_{3sc}^{-\infty,1}(X)$. *Moreover, for each* $\xi \in W^\perp$

$$\widehat{[\widetilde{Q}, H]}_{\mathrm{ff},1}(\xi) = \widehat{[Q, H]}_{\mathrm{ff},1}(\xi)\psi(H)_{\mathrm{ff}}(\xi) \in \Psi_{sc}^{-\infty,0}(\mathbb{S}_+^n) \tag{13.32}$$

where $\widehat{[Q, H]}_{\mathrm{ff},1}$ *is given by the Proposition 7.3 with* $\widehat{\partial_x Q_{\mathrm{ff}}}$ *the operator obtained by replacing* $q(0,0,z,\tau,\mu,\nu)$ *by* $\partial_x q(x, x\overline{Y}, z, \tau, \mu, \nu)|_{x=0}$ *in* (13.26).

Proof. — By the previous proposition $\widetilde{Q} \in \Psi_{3sc}^{-\infty,0}(X)$, and using

$$\widehat{\widetilde{Q}}_{\mathrm{ff},0}(\xi) = f(\xi)\psi(\widehat{H}_{\mathrm{ff},0}(\xi)), \tag{13.33}$$

so

$$\widehat{[\widetilde{Q}, H]}_{\mathrm{ff},0}(\xi) = [\widehat{\widetilde{Q}}_{\mathrm{ff},0}(\xi), \widehat{H}_{\mathrm{ff},0}(\xi)] = f(\xi)[\psi(\widehat{H}_{\mathrm{ff},0}(\xi)), \widehat{H}_{\mathrm{ff},0}(\xi)] = 0. \tag{13.34}$$

We can then use the discussion preceeding this proposition to compute the indicial operator $\widehat{[Q\psi(H), H]}_{\mathrm{ff},1}$, giving the claimed result. □

Remark 13.3. — An alternative proof of the proposition is to calculate $\widehat{\partial_x \widetilde{Q}_{\mathrm{ff}}}$ from (13.14). It is not hard to see that it gives the same result; the main point is to realize that the terms arising from differentiating either the exponential or p are exactly the same as the terms that would arise if we dropped q (i.e. assumed that it was 1), multiplied by $f(z,\tau,\nu) = q(0,0,z,\tau,0,\nu)$. Since $\psi(H)$ commutes with H, such terms must cancel against others in the commutator formula of Proposition 7.3.

The following corollary of the preceeding discussion is the basic commutator estimate for the propagation results.

Corollary 13.4. — *Let H, ψ, q and f be as in Proposition 13.2, and let $l \in \mathbb{R}$. For $\xi \in W^\perp$ let*

$$R(\xi) = \widehat{[x^l\widetilde{Q}, H]}_{\mathrm{ff},l+1}(\xi) - \widehat{[x^l Q, \Delta]}_{\mathrm{ff},l+1}(\xi)\widehat{\psi(\Delta)}_{\mathrm{ff},0}(\xi). \tag{13.35}$$

Then

$$\widehat{[x^l\widetilde{Q}, H]}_{\mathrm{mf},l+1} = \widehat{[x^l Q, \Delta]}_{\mathrm{mf},l+1}\widehat{\psi(\Delta)}_{\mathrm{mf},0}, \tag{13.36}$$

and $R(\xi) \in \Psi_{sc}^{-\infty,1}(\mathbb{S}^n_+)$. *Moreover, there exist* C *and* k *independent of* ξ *and* q *such that*

$$\begin{aligned}(13.37)\quad &\|R(\xi)\|_{\mathcal{B}(L^2_{sc}(\mathbb{S}^n_+),H^{1,1}_{sc}(\mathbb{S}^n_+))}\\ &\leq C\sup\{|D^{\beta}_{x,y,z,\tau,\nu}D^{\alpha}_{\mu}q(0,0,z,\tau,\mu,\nu)|:\ |\alpha|\leq k,\ |\beta|\leq 1,\ \mu\in\mathbb{R}^n\}.\end{aligned}$$

Proof. — First, (13.36) follows from $\widehat{\psi(H)}_{\mathrm{mf}} = \widehat{\psi(\Delta)}_{\mathrm{mf}}$, $\widehat{H}_{\mathrm{mf}} = \widehat{\Delta}_{\mathrm{mf}}$, and the commutativity of the indicial operators at mf.

At ff we use the formula in Proposition 7.3 together with (13.33) and $\widehat{\partial_x Q}_{\mathrm{ff}}$ given in Proposition 13.2. Thus,

$$\begin{aligned}[\widehat{Q\psi(H),H}]_{\mathrm{ff},1} = ([\widehat{\partial_x Q}_{\mathrm{ff}},\widehat{H}_{\mathrm{ff}}] &- (D_\tau f)[\overline{Y},\widehat{H}_{\mathrm{ff}}]\partial_{\overline{Y}} - (D_\tau f)(\overline{Y}\partial_{\overline{Y}}\widehat{H}_{\mathrm{ff}})\\ (13.38)\qquad &+ (D_\nu f)(\partial_z\widehat{H}_{\mathrm{ff}}) - (D_\nu\widehat{H}_{\mathrm{ff}})(\partial_z f)\\ &+ (\nu\cdot D_\nu\widehat{H}_{\mathrm{ff}})(\partial_\tau f) - (\nu\cdot D_\nu f)(\partial_\tau\widehat{H}_{\mathrm{ff}}))\widehat{\psi(H)}_{\mathrm{ff}}.\end{aligned}$$

Here we can write $\widehat{H}_{\mathrm{ff}} = \widehat{\Delta}_{\mathrm{ff}} + V_{\mathrm{ff}}$. Since V vanishes at mf, $V_{\mathrm{ff}} \in \Psi^{0,1}_{sc}(\mathbb{S}^n_+)$. As $\overline{Y}_j \in \Psi^{0,-1}_{sc}(\mathbb{S}^n_+)$, it follows that all terms of (13.38) arising from V are in $\Psi^{0,1}_{sc}(\mathbb{S}^n_+)$, and the q dependence of all but the first one is simply via multiplication by a derivative of f. It is particularly easy to deal with the first term, $[\widehat{\partial_x Q}_{\mathrm{ff}}, V_{\mathrm{ff}}]\widehat{\psi(H)}_{\mathrm{ff}}$, if the full 'amplitude',

$$(13.39)\qquad (\partial_x + \overline{Y}\partial_y)q(0,0,z,\tau,\mu,\nu),$$

of $\widehat{\partial_x Q}_{\mathrm{ff}}$ is actually a symbol in μ of, say, order 0 (which is the case we will be using in the following chapters). Namely, then $\widehat{\partial_x Q}_{\mathrm{ff}} \in \Psi^{0,-1}_{sc}(\mathbb{S}^n_+)$, and we only need that the norm in $\mathcal{B}(L^2_{sc}(\mathbb{S}^n_+), H^{1,1}_{sc}(\mathbb{S}^n_+))$ of its commutator with $V_{\mathrm{ff}} \in \Psi^{0,1}_{sc}(\mathbb{S}^n_+)$ is bounded by a seminorm of the full symbol, (13.39), of $\widehat{\partial_x Q}_{\mathrm{ff}}$; this is a standard result in the scattering calculus. In general, when (13.39) is not a symbol in μ, we can use a regularization argument, i.e. we multiply (13.39) by $\langle\mu\rangle^{-m}\langle\mu\rangle^{m}$, and use that $\langle\mu\rangle^{-m}(\partial_x + \overline{Y}\partial_y)q(0,0,z,\tau,\mu,\nu)$ satisfies an arbitrary large number of symbolic estimates if m is sufficiently large, and note that $\widehat{\psi(H)}_{\mathrm{ff}} \in \Psi^{-\infty,0}_{sc}(\mathbb{S}^n_+)$. This shows that $[Q,\widehat{V]\psi(H)}_{\mathrm{ff},1}$ satisfies the estimate of (13.37). We also have an additional term in $[x^l Q, V]\psi(H)$, namely $[x^l, V]\widetilde{Q}$, but here the commutator actually vanishes.

It remains to deal with $[x^l Q, \Delta](\psi(H) - \psi(\Delta))$. Since under our assumption on V we have

$$(13.40)\qquad \widehat{\psi(H)}_{\mathrm{ff}} - \widehat{\psi(\Delta)}_{\mathrm{ff}} \in \Psi^{-\infty,1}_{sc}(\mathbb{S}^n_+),$$

it suffices to show that for some m the norms of $\widehat{Q}_{\mathrm{ff}}[\widehat{x^l,\Delta}]_{\mathrm{ff},l+1}$ and $[\widehat{Q,\Delta}]_{\mathrm{ff},1}$ as elements of $\mathcal{B}(H^{m,1}_{sc}(\mathbb{S}^n_+), H^{1,1}_{sc}(\mathbb{S}^n_+))$ have bounds as in (13.37). If (13.39) is a (classical) symbol of order 0 in μ, these follow from Proposition 13.1 and (13.38) respectively where now $\widehat{\Delta}_{\mathrm{ff}} \in \Psi^{2,0}_{sc}(\mathbb{S}^n_+)$ only ensures that the commutator is in $\Psi^{1,0}_{sc}(\mathbb{S}^n_+)$ as opposed to the case of $[Q,V]$. Even in general we do not need to use a regularization argument since Δ is a scattering differential operator, so the commutator $[\widehat{\partial_x Q_{\mathrm{ff}}}, \widehat{\Delta}_{\mathrm{ff}}]$ is just the product of $[\overline{Y}, \widehat{\Delta}_{\mathrm{ff}}]$ and the quantization of $\partial_y q(0,0,z,\tau,\mu,\nu)$, and the required estimate follows directly.

Since $\widehat{\psi(H)}_{\mathrm{ff}}$ has compact support in $W^\perp$, we see that C can be chosen to be independent of ξ. □

We also need to show that the operator wave front set of $\widetilde{Q}$ is indeed where we expect it to be. For $q \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*X)$ we let

$$\text{ess supp}(q) = \{\alpha \in {}^{\mathrm{sc}}T^*_{\partial X}X : \ q \text{ vanishes with all derivatives near } \alpha\}^c. \tag{13.41}$$

At ff we need a uniform version of this in the μ variable:

$$\begin{aligned}\text{ess supp}_{\mathrm{ff}}(q) = \{\xi = (z,\tau,\nu) \in W^\perp : \ \exists\, \rho \in \mathcal{C}^\infty(W^\perp),\ \chi \in \mathcal{C}^\infty(X),\\ \rho(\xi) \neq 0,\ \chi(0,0,z) \neq 0,\ \chi\rho q \in \dot{\mathcal{C}}^\infty({}^{\mathrm{sc}}\overline{T}^*_{\partial X}X)\}^c.\end{aligned} \tag{13.42}$$

Lemma 13.5. — *Suppose that q, $\widetilde{Q}$ are as in Proposition 13.1, $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$. Then*

$$\mathrm{WF}'_{\mathrm{3sc,mf}}(\widetilde{Q}) \subset \beta^{-1}(\text{ess supp}(q\psi(g))) \subset {}^{\mathrm{3sc}}T^*_{\mathrm{mf}}[X;C], \tag{13.43}$$

$$\mathrm{WF}'_{\mathrm{3sc,ff}}(\widetilde{Q}) \subset \text{ess supp}_{\mathrm{ff}}(q) \cap \mathrm{WF}'_{\mathrm{3sc,ff}}(\psi(H)) \subset W^\perp. \tag{13.44}$$

Proof. — This follows from the definition of $\widetilde{Q}$ via the quantization map, taking into account that composition is 3sc-microlocal. □

CHAPTER 14

PROPAGATION OF SINGULARITIES IN NORMAL DIRECTIONS

We are now ready to prove that singularities of generalized eigenfunctions of H which are incident along integral curves of ${}^{\mathrm{sc}}H_g$ which are not tangent to C propagate along broken bicharacteristics. Recall that $\pi^\perp : {}^{\mathrm{sc}}T^*_C X \to W^\perp$ is the orthogonal projection to the orthocomplement (with respect to the metric g) of the annihilator of ${}^{\mathrm{sc}}T(C;X)$, $g \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*X)$ is the metric function on X, $h \in \mathcal{C}^\infty(T^*\partial X)$ the metric function on ∂X, and $\widetilde{h} = h|_{\widetilde{W}^\perp}$. We only state the result for propagation in the forward direction of ${}^{\mathrm{sc}}H_g$ flow, but it is equally true for the flow in the opposite direction as a minor modification of the arguments shows.

Proposition 14.1. — *Let H be as in (11.11), $\lambda > 0$. Let $\xi_0 = (z_0, \tau_0, \nu_0) \in \Sigma_n(\lambda)$. Let $\varepsilon > 0$ be such that $\exp(s\,{}^{\mathrm{sc}}H_g)(\alpha) \notin {}^{\mathrm{sc}}T^*_C X$ if $\alpha \in {}^{\mathrm{sc}}T^*_C X \cap \Sigma_{\Delta-\lambda}$, $\pi^\perp\alpha = \xi_0$, $s \in (-\varepsilon, \varepsilon) \setminus \{0\}$. Suppose that $u \in \mathcal{C}^{-\infty}(X)$, $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc}}((H-\lambda)u)$, and for all $\alpha \in {}^{\mathrm{sc}}T^*_C X \cap \Sigma_{\Delta-\lambda}$ with $\pi^\perp\alpha = \xi_0$, we have $\exp(s\,{}^{\mathrm{sc}}H_g)(\alpha) \notin \mathrm{WF}_{3\mathrm{sc}}((H-\lambda)u)$ for all $s \in (-\varepsilon, \varepsilon)$. If in addition for each such α there exists $s \in (-\varepsilon, 0)$ such that $\exp(s\,{}^{\mathrm{sc}}H_g)(\alpha) \notin \mathrm{WF}_{\mathrm{sc}}(u)$, then $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc},\mathrm{ff}}(u)$. Hence, for all such α and for all $s \in (-\varepsilon, \varepsilon) \setminus \{0\}$, $\exp(s\,{}^{\mathrm{sc}}H_g)(\alpha) \notin \mathrm{WF}_{\mathrm{sc}}(u)$.*

Proof. — Notice first that Melrose's form of Hörmander's propagation theorem [**25**, Proposition 7] implies that under our assumptions $\exp(s\,{}^{\mathrm{sc}}H_g)(\alpha) \notin \mathrm{WF}_{\mathrm{sc}}(u)$ for all $s \in (-\varepsilon, 0)$ and $\alpha \in {}^{\mathrm{sc}}T^*_C X \cap \Sigma_{\Delta-\lambda}$ satisfying $\pi^\perp\alpha = \xi_0$. Similarly, if we just prove $\exp(s\,{}^{\mathrm{sc}}H_g)(\alpha) \notin \mathrm{WF}_{\mathrm{sc}}(u)$ for sufficiently small $s > 0$, it follows for all $s \in (0, \varepsilon)$. This in turn will follow from $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc},\mathrm{ff}}(u)$ since the wave front set is closed. Thus, we can work above a coordinate neighborhood U of $(0, z_0)$, and hence we can use local coordinates (y, z, τ, μ, ν) adapted to $W^\perp$ in this proof.

The proof is by induction on microlocal regularity, i.e. we prove that

$$\xi_0 \notin \mathrm{WF}^{m,l}_{3\mathrm{sc},\mathrm{ff}}(u) \tag{14.1}$$

for all m, l. Here m is irrelevant by standard elliptic regularity, i.e. by Lemma 11.9, which shows that if (14.1) holds for one m, then it holds for all m. So assume that

(14.1) holds for some m and l, and we proceed to show that it also holds if we replace l by $l+1/2$.

We first construct a symbol q which has a positive commutator with H microlocally away from $\exp(s^{\mathrm{sc}}H_g)(\alpha)$, $s \in (-\varepsilon, 0)$, and which is elliptic at $\exp(s^{\mathrm{sc}}H_g)(\alpha)$ for sufficiently small $s \in (0, \varepsilon)$. Note that the our commutator construction will be similar to, though much simpler than, the one used in the proof of [**18**, Proposition 24.5.1]; that proof will be more closely followed when we investigate the propagation of singularities at $\Sigma_t(\lambda)$ in the next chapter. Let

$$\Sigma = \{(y, z, \tau, \mu, \nu) : \mu \cdot y = 0,\ \mu \neq 0\}. \tag{14.2}$$

Thus, $\Sigma \subset {}^{\mathrm{sc}}T^*_U X$ is a smooth hypersurface. Moreover, in these local coordinates (11.26) states that

$$^{\mathrm{sc}}H_g = 2\tau(x\partial_x + \mu \cdot \partial_\mu + \nu \cdot \partial_\nu) - 2h\partial_\tau + H_h + xW',\ W' \in \mathcal{V}_{\mathrm{b}}({}^{\mathrm{sc}}T^*X), \tag{14.3}$$

and by (11.29)

$$H_h - 2\mu \cdot \partial_y \in T({}^{\mathrm{sc}}T^*_C X). \tag{14.4}$$

As $\mu \cdot y \equiv 0$ on ${}^{\mathrm{sc}}T^*_C X$, this proves that ${}^{\mathrm{sc}}H_g(\mu \cdot y)|_{y=0} = -2|\mu|^2$, so ${}^{\mathrm{sc}}H_g$ is transversal to $\Sigma \cap {}^{\mathrm{sc}}T^*_{\widetilde{U}} X$ if $\widetilde{U}$ is a sufficiently small neighborhood of $(0, z_0)$ in ∂X.

Let $\rho \in \mathcal{C}^\infty_c(\mathbb{R}; [0,1])$ be supported near λ, and it is identically 1 in a smaller neighborhood of λ. Now, on a neighborhood $U' \subset {}^{\mathrm{sc}}T^*_{\partial X} X$ of $\operatorname{supp} \rho(g) \cap {}^{\mathrm{sc}}T^*_C X$ we can solve the Cauchy problem

$$^{\mathrm{sc}}H_g \omega = 0, \qquad \omega|_\Sigma = |y|^2 + |z - z_0|^2 + |\tau - \tau_0|^2 + |\nu - \nu_0|^2 \tag{14.5}$$

where $|\cdot|$ denotes the Euclidian metric in these local coordinates. Since $\omega|_\Sigma \geq 0$, we have $\omega \geq 0$ on U'. Also, $\omega|_\Sigma$ vanishes exactly at

$$S = \{\alpha \in {}^{\mathrm{sc}}T^*_C X \cap U' :\ \pi^\perp \alpha = \xi_0\}, \tag{14.6}$$

so ω will vanish exactly at the flow-out

$$\widetilde{S} = \{\exp(s^{\mathrm{sc}}H_g)(S) \cap U' :\ s \in (-\varepsilon, \varepsilon)\} \tag{14.7}$$

of this set under ${}^{\mathrm{sc}}H_g$. Moreover, $d\omega$ will also vanish on $\widetilde{S}$, since it does at S, but for the same reason the Hessian is positive in directions transversal to ${}^{\mathrm{sc}}H_g$. In particular, on compact subsets K of U' we have for some C_1, C_2, $C_3 > 0$ depending on K

$$C_1 \omega^{1/2} \leq \operatorname{dist}(p, \widetilde{S}) \leq C_2 \omega^{1/2}, \tag{14.8}$$

where dist is the Euclidian distance. By reducing the size of U' (while keeping it a neighborhood of $\operatorname{supp} \rho(g) \cap {}^{\mathrm{sc}}T^*_C X$) we may assume that this holds everywhere on U'.

Propagation along the integral curves of ${}^{\mathrm{sc}}H_g$ can be measured by $\mu \cdot y$ since it vanishes on Σ and ${}^{\mathrm{sc}}H_g(\mu \cdot y) \geq c_0$ on U'. It will be, however, convenient to introduce a new propagation variable N so that ${}^{\mathrm{sc}}H_g N = 1$, $N|_\Sigma = 0$ (i.e. parametrize the integral curves by the time it takes to flow from Σ to the given point). Thus, for some c_1, $c_2 > 0$

$$c_1(\mu \cdot y) \leq N \leq c_2(\mu \cdot y). \tag{14.9}$$

Let $\chi_0 \in \mathcal{C}^\infty(\mathbb{R})$ be $\chi_0(t) = \exp(-1/t)$ for $t > 0$, identically 0 for $t \leq 0$, and also let $\chi_1 \in \mathcal{C}^\infty(\mathbb{R}; [0,1])$ be 0 on $(-\infty, 0]$, 1 on $[1, \infty)$, and satisfy $0 \leq \chi_1' \in \mathcal{C}_c^\infty((0,1))$. We now define for $\widetilde{\varepsilon} > 0$, $\delta > 0$, $A > 0$

$$\phi = N + \widetilde{\varepsilon}^{-1}\omega, \tag{14.10}$$

$$\widetilde{q}_0(y, z, \tau, \mu, \nu) = \chi_0(A^{-1}(2 - \phi/\delta))\chi_1(N/\delta + 2). \tag{14.11}$$

Note that on the support of the first factor $\phi \leq 2\delta$, and on the support of the second one $N \geq -2\delta$. Thus,

$$\text{on } \operatorname{supp} \widetilde{q}_0,\ \omega \leq 4\delta\widetilde{\varepsilon},\ \text{ and } |N| \leq 2\delta, \tag{14.12}$$

so if we choose $\widetilde{\varepsilon}$, $\delta > 0$ sufficiently small then for some $K \subset U$ compact $\operatorname{supp} \widetilde{q}_0 \subset {}^{\mathrm{sc}}T^*_K X$, i.e. $\widetilde{q}_0$ can be regarded as a function on ${}^{\mathrm{sc}}T^*X$. Next, ${}^{\mathrm{sc}}H_g \phi = {}^{\mathrm{sc}}H_g N = 1$ since ${}^{\mathrm{sc}}H_g \omega = 0$, so

$${}^{\mathrm{sc}}H_g \widetilde{q}_0 = -g_0^2 + e_0 \tag{14.13}$$

with

$$g_0^2 = A^{-1}\delta^{-1}\chi_0'(A^{-1}(2 - \phi/\delta))\chi_1(N/\delta + 2), \tag{14.14}$$

$$e_0 = 2\delta^{-1}\chi_0(A^{-1}(2 - \phi/\delta))\chi_1'(N/\delta + 2). \tag{14.15}$$

Noting that on ${}^{\mathrm{sc}}T^*_C X \cap U'$, g_0 is independent of μ, let

$$f^\flat = A^{-1}\delta^{-1}\chi_0'\chi_1 \in \mathcal{C}^\infty(W^\perp), \tag{14.16}$$

so $f^\flat|_{U'} = g_0^2|_{{}^{\mathrm{sc}}T^*_C X \cap U'}$, and in particular $g_0^2|_S = f^\flat|_S = 2A^{-1}\delta^{-1}\chi_0'(2/A) > 0$. On the other hand,

$$\text{on } \operatorname{supp} e_0,\ -2\delta \leq N \leq -\delta,\ \omega \leq 4\delta\widetilde{\varepsilon}. \tag{14.17}$$

Now, $\chi_1(N/\delta + 2)|_{{}^{\mathrm{sc}}T^*_C X} = 1$, and $\widetilde{q}_0|_{{}^{\mathrm{sc}}T^*_C X \cap U'}$ is independent of μ, namely it is

$$f(z, \tau, \nu) = \widetilde{q}_0(0, z, \tau, \mu, \nu) = \chi_0(A^{-1}(2 - \omega_0/(\widetilde{\varepsilon}\delta))) \tag{14.18}$$

with

$$\omega_0 = |y|^2 + |z - z_0|^2 + |\tau - \tau_0|^2 + |\nu - \nu_0|^2 \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*_U X), \tag{14.19}$$

so we define $\widetilde{q} \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*_U X)$ by

$$\widetilde{q} = \rho(g)\widetilde{q}_0 + (1 - \rho(g))\chi_0(A^{-1}(2 - \omega_0/(\widetilde{\varepsilon}\delta))). \tag{14.20}$$

On the support of the second term $\omega_0 \leq 2\widetilde{\varepsilon}\delta$, so $|y|^2 \leq 2\widetilde{\varepsilon}\delta$, i.e. $\operatorname{supp} \widetilde{q} \in {}^{\mathrm{sc}}T^*_K X$ (with $K \subset U$ compact) as well. Now (14.20) implies that $\widetilde{q}(0, z, \tau, \mu, \nu)$ is independent of μ, and taking into account that $\chi_0'(s) = s^{-2}\chi_0(s)$ and that $A^{-1}(2 - \omega_0/(\widetilde{\varepsilon}\delta)) \leq 2A^{-1}$, we conclude that

$$\widetilde{q}|_{{}^{\mathrm{sc}}T^*_C X} \leq 4A^{-2}\chi_0'(A^{-1}(2 - \phi/\delta))\chi_1 \leq C' A^{-1}\delta f^\flat. \tag{14.21}$$

In addition, $d\phi = dN + \widetilde{\varepsilon}^{-1}\, d\omega$, and $\operatorname{supp} \rho(g)$ is compact. Furthermore, ω_0 is independent of μ and the set $\{(y, z, \tau, \nu) : \omega_0 \leq 2\}$ is compact. Since on the support of the second term of (14.20), $\omega_0 \leq 2\widetilde{\varepsilon}\delta \leq 2$ if we make sure that $\widetilde{\varepsilon} < 1$, $\delta \leq 1$, we conclude that

$$|d\widetilde{q}|_{{}^{\mathrm{sc}}T^*_C X}| \leq A^{-1}\delta^{-1}C'(1 + \widetilde{\varepsilon}^{-1})\chi_0'\chi_1 \leq C''(1 + \widetilde{\varepsilon}^{-1})f^\flat. \tag{14.22}$$

More generally, taking into account that on ${}^{\mathrm{sc}}T^*_C X$, ω_0, ω and N are independent of μ, so when differentiating $d\widetilde{q}|_{{}^{\mathrm{sc}}T^*_C X}$ with respect to μ no additional derivative may fall on χ_0, we obtain that for all multiindices α

$$|\partial^\alpha_\mu d\widetilde{q}|_{{}^{\mathrm{sc}}T^*_C X}| \leq C''(1 + \widetilde{\varepsilon}^{-1})f^\flat. \tag{14.23}$$

Finally, we estimate $\widetilde{q}$ on $\operatorname{supp}(1 - \rho(g))^c$, i.e. near $\Sigma_{\Delta-\lambda}$. As on $\operatorname{supp}\widetilde{q}_0$, $|N| \leq 2\delta$, we have $\phi \geq -2\delta$, hence $A^{-1}(2 - \phi/\delta) \leq 4A^{-1}$, and correspondingly

$$\widetilde{q}|_{\operatorname{supp}(1-\rho(g))^c} \leq CA^{-1}\delta g_0^2. \tag{14.24}$$

In particular, given any $M > 0$, $\varepsilon' > 0$ and keeping $\delta \leq 1$, we can make sure (by choosing A sufficiently large, only depending on M and ε') that

$${}^{\mathrm{sc}}H_g\widetilde{q} + M\widetilde{q} = -(1 - r)g_0^2 + e_0 \text{ on } \operatorname{supp}(1 - \rho(g))^c, \tag{14.25}$$

here

$$r = MA^{-1}(2 - \phi/\delta)^2\delta, \qquad |r| \leq \varepsilon'/2. \tag{14.26}$$

We now fix $\widetilde{\varepsilon}$ and δ, but will leave M to be determined later. For small $\delta' > 0$

$$\widetilde{K}_{\delta'} = \{\alpha \in {}^{\mathrm{sc}}T^*_U X : \ \omega_0 \leq \delta'\} \subset {}^{\mathrm{sc}}T^*_U X, \tag{14.27}$$

and choose $\delta' \in (0, 1)$ such that $p(\widetilde{K}_{\delta'})$ is compact (p being the projection ${}^{\mathrm{sc}}T^*_U X \to U$), and

$$\mathrm{WF}_{3\mathrm{sc,mf}}((H - \lambda)u) \cap \widetilde{K}_{\delta'} = \varnothing, \ \mathrm{WF}_{3\mathrm{sc,ff}}((H - \lambda)u) \cap \pi^\perp(\widetilde{K}_{\delta'}) = \varnothing, \tag{14.28}$$

$$\begin{aligned} F \in \Psi^{0,0}_{3\mathrm{sc}}(X),\ \mathrm{WF}'_{3\mathrm{sc,mf}}(F) \subset \widetilde{K}_{\delta'},\ \mathrm{WF}'_{3\mathrm{sc,ff}}(F) \subset \pi^\perp(\widetilde{K}_{\delta'} \cap {}^{\mathrm{sc}}T^*_C X) \\ \Longrightarrow Fu \in H^{m,l}_{\mathrm{sc}}(X). \end{aligned} \tag{14.29}$$

This can be arranged as $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc,ff}}((H - \lambda)u)$ and as (14.1) holds, since by making δ' small we can make sure that $\widetilde{K}_{\delta'}$ is included in any fixed neighborhood of $(\pi^\perp)^{-1}(\{\xi_0\})$. Then, corresponding to (14.17), let

(14.30)

$$K = K_{\delta,\widetilde{\varepsilon}} = \{\alpha \in {}^{\mathrm{sc}}T^*_U X : \ -2\delta \leq N \leq -\delta,\ |g - \lambda| \leq \delta\widetilde{\varepsilon},\ \omega \leq 4\delta\widetilde{\varepsilon}\} \subset {}^{\mathrm{sc}}T^*_U X,$$

and choose $\widetilde{\varepsilon} \in (0,1)$ and $\delta \in (0,1)$ such that $K_{\delta,0} \subset \widetilde{K}_{\delta'}$ and

$$E \in \Psi_{\mathrm{sc}}^{\infty,-\infty}(X),\ \mathrm{WF}'_{\mathrm{sc}}(E) \subset K \Longrightarrow Eu \in \dot{\mathcal{C}}^\infty(X). \tag{14.31}$$

Note that this can also be arranged since we know that for $\alpha \in {}^{\mathrm{sc}}T^*_U X$ with $N(\alpha) \in (-\varepsilon, 0)$, $\omega(\alpha) = 0$, $g(\alpha) = \lambda$ we have $\alpha = \exp(N(\alpha)^{\mathrm{sc}}H_g)(\alpha_0)$ for some $\alpha_0 \in {}^{\mathrm{sc}}T^*_C X \cap \Sigma_{\Delta-\lambda}$ with $\pi^\perp \alpha_0 = \xi_0$, so $\alpha \notin \mathrm{WF}_{\mathrm{sc}}(u)$. Hence fixing any $\delta > 0$ so that the flow stays inside U' for time $|N| \leq 4\delta$, we have that $K_{\delta,0}$ is compact and is disjoint from $\mathrm{WF}_{\mathrm{sc}}(u)$ so for an appropriate neighborhood of $K_{\delta,0}$, and hence for some $\widetilde{\varepsilon} > 0$ (14.31) holds.

Let $\psi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ be identically 1 near 0 and supported sufficiently close to 0 so that the product decomposition of X near ∂X is valid on $\operatorname{supp}\psi_0$. We also define

$$q = \psi_0(x)\widetilde{q}. \tag{14.32}$$

Now note that q satisfies the estimates in (13.2) and let Q be the left quantization $q_L(q)$ of q as in (13.12) We intend to compute the commutator $i[Q,H]$. Corollary 13.4 reduces our task to computing

$$[Q,\Delta]\psi(\Delta) = [Q\psi(\Delta), \Delta]. \tag{14.33}$$

Since $Q\psi(\Delta) \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$ we can use the commutator formula in the scattering calculus to give

$$j_{\mathrm{sc},0,1}(i[Q\psi(\Delta),\Delta]) = -({}^{\mathrm{sc}}H_g q)\psi(g), \tag{14.34}$$

i.e. with q_L denoting left quantization

$$i[Q,\Delta]\psi(\Delta) - xq_L(-{}^{\mathrm{sc}}H_g q)\psi(\Delta) \in \Psi_{\mathrm{sc}}^{-\infty,2}(X). \tag{14.35}$$

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$ be supported sufficiently close to λ so that $\rho \equiv 1$ near $\operatorname{supp}\psi$. Now, $f^\flat$ is independent of μ, so

$$i\widehat{[Q,\Delta]}_{\mathrm{ff},1}\widehat{\psi(\Delta)}_{\mathrm{ff}} = f^\flat \widehat{\psi(\Delta)}_{\mathrm{ff}}. \tag{14.36}$$

Since $\psi(H) - \psi(\Delta) \in \rho_{\mathrm{mf}}\,\Psi_{3\mathrm{sc}}^{-\infty,0}(X)$, we have

$$\widehat{\psi(H)}_{\mathrm{ff}} - \widehat{\psi(\Delta)}_{\mathrm{ff}} \in \Psi_{\mathrm{sc}}^{-\infty,1}(\mathbb{S}^n_+), \tag{14.37}$$

so

$$i\widehat{[Q,\Delta]}_{\mathrm{ff},1}\widehat{\psi(\Delta)}_{\mathrm{ff}} - f^\flat\,\widehat{\psi(H)}_{\mathrm{ff}} = f^\flat(\widehat{\psi(\Delta)}_{\mathrm{ff}} - \widehat{\psi(H)}_{\mathrm{ff}}) \in \Psi_{\mathrm{sc}}^{-\infty,1}(\mathbb{S}^n_+) \tag{14.38}$$

and its norm in $\mathcal{B}(L^2_{\mathrm{sc}}(\mathbb{S}^n_+), H^{1,1}_{\mathrm{sc}}(\mathbb{S}^n_+))$ is bounded by a constant multiple of $f^\flat(\xi)$. Combining this with Corollary 13.4, (14.21) and (14.23) shows that

$$R_1(\xi) = i\widehat{[Q,H]}_{\mathrm{ff},1}\widehat{\psi(H)}_{\mathrm{ff}} - f^\flat\,\widehat{\psi(H)}_{\mathrm{ff}} \in \Psi_{\mathrm{sc}}^{-\infty,1}(\mathbb{S}^n_+), \tag{14.39}$$

$$\|R_1(\xi)\|_{\mathcal{B}(L^2_{\mathrm{sc}}(\mathbb{S}^n_+),H^{1,1}_{\mathrm{sc}}(\mathbb{S}^n_+))} \leq C' f^\flat(z,\tau,\nu) \tag{14.40}$$

with C' independent of A, hence of M, if we keep $A \geq 1$.

It is useful to replace Q by a self-adjoint operator, so we consider $\psi(H)Q^*Q\psi(H)$ in place of $Q\psi(H)$. Thus, from (14.40) and Proposition 13.1 (employing (14.37)) we deduce that for some $C > 0$

$$\|i(\psi(H)\widehat{[Q^*Q, H]}\psi(H))_{\mathrm{ff},1}(\xi) - 2f^{\flat} f\widehat{\psi(H)}_{\mathrm{ff}}^2\|_{\mathcal{B}(L^2_{\mathrm{sc}}, H^{1,1}_{\mathrm{sc}})} \leq Cf^{\flat}(\xi)f(\xi) \tag{14.41}$$

for all $\xi \in W^{\perp}$.

Now we can follow the proof of Theorem 12.2. Thus, choose $\widetilde{\psi}, \phi \in \mathcal{C}_c^{\infty}(\mathbb{R})$ identically 1 near λ, $\widetilde{\psi} \equiv 1$ on $\operatorname{supp}\phi$, $\psi \equiv 1$ on $\operatorname{supp}\widetilde{\psi}$. Let $\varepsilon' \in (0, 1)$. On $\operatorname{supp} f$, $\lambda - \tau^2 - |\nu|_z^2$ is not an eigenvalue of H_{ff} (since it is positive). Thus,

$$\widehat{\widetilde{\psi}(H)}_{\mathrm{ff}}(\xi) = \widetilde{\psi}(H_{\mathrm{ff}}(z) + \tau^2 + |\nu|_z^2) \to 0 \tag{14.42}$$

strongly as $\operatorname{supp}\widetilde{\psi} \to \{\lambda\}$. Since $\operatorname{supp} f$ is compact, and the inclusion map $T \in \mathcal{B}(H^{1,1}_{\mathrm{sc}}(\mathbb{S}^n_+), L^2_{\mathrm{sc}}(\mathbb{S}^n_+))$ is compact, for $\widetilde{\psi}$ with sufficiently small support we have

$$\|(\widehat{\widetilde{\psi}(H)T})_{\mathrm{ff}}(\xi)\|_{\mathcal{B}(H^{1,1}_{\mathrm{sc}}(\mathbb{S}^n_+), L^2_{\mathrm{sc}}(\mathbb{S}^n_+))} \leq \varepsilon' C^{-1} \tag{14.43}$$

for all $\xi \in \operatorname{supp} f$. Thus, on $\operatorname{supp} f$

$$\|i(\widetilde{\psi}(H)\widehat{[Q^*Q, H]}\widetilde{\psi}(H))_{\mathrm{ff},1}(\xi) - 2f^{\flat} f\widehat{\widetilde{\psi}(H)}_{\mathrm{ff}}^2\|_{\mathcal{B}(L^2, L^2)} \leq \varepsilon' f^{\flat}(\xi)f(\xi). \tag{14.44}$$

Note that by (14.41), (14.44) holds automatically outside $\operatorname{supp} f$, so it holds for all $\xi \in W^{\perp}$. Thus,

$$i(\widetilde{\psi}(H)\widehat{[Q^*Q, H]}\widetilde{\psi}(H))_{\mathrm{ff},1} \geq 2f^{\flat} f\widehat{\widetilde{\psi}(H)}_{\mathrm{ff}}^2 - \varepsilon' f^{\flat} f. \tag{14.45}$$

Multiplying by $\phi(H)$ from both left and right we finally conclude that

$$i\phi(H)\widehat{[Q^*Q, H]}\phi(H)_{\mathrm{ff},1} \geq (2 - \varepsilon')f^{\flat} f\widehat{\phi(H)}_{\mathrm{ff}}^2. \tag{14.46}$$

The other face, mf, is much easier to deal with. In fact, from (13.36) we deduce at once that

$$i(\psi(H)\widehat{[Q^*Q, H]}\psi(H))_{\mathrm{mf},1} = -2\psi(g)^2\widetilde{q}({}^{\mathrm{sc}}H_g\widetilde{q}). \tag{14.47}$$

This also holds if we replace ψ by ϕ.

Now we can follow the usual proof of the principal-type propagation theorem [**25**, Proposition 7]. Let

$$b = \psi_0(x)\widetilde{q}^{1/2}(1 - r)^{1/2}g_0, \tag{14.48}$$

and let $B = q_L(b)\phi(H)$. Also, let

$$E = \phi(H)q_L((\widetilde{q}e_0)^{1/2})^* q_L((\widetilde{q}e_0)^{1/2})\phi(H). \tag{14.49}$$

Thus, by (14.46), (14.47), (14.25) and Proposition C.3

$$\begin{aligned} ix^{-1/2}\phi(H)[Q^*Q, H]\phi(H)x^{-1/2} &- Mx^{1/2}\phi(H)Q^*Q\phi(H)x^{-1/2} \\ &- Mx^{-1/2}\phi(H)Q^*Q\phi(H)x^{1/2} \geq (2 - 2\varepsilon')B^*B + E + F \end{aligned} \tag{14.50}$$

where $B \in \Psi_{3sc}^{-\infty,0}(X)$, $E \in \Psi_{sc}^{-\infty,0}(X)$, $F \in \Psi_{3sc}^{-\infty,1}(X)$, and

$$\mathrm{WF}'_{sc}(E) \subset K = K_{\widetilde{\varepsilon},\delta} \tag{14.51}$$

$$\mathrm{WF}'_{3sc,mf}(F) \subset \operatorname{supp} \widetilde{q} \subset \{\alpha \in {}^{sc}T^*_U X : \ -2\delta \leq N \leq 2\delta, \ |g-\lambda| \leq \delta\widetilde{\varepsilon}, \ \omega \leq 4\delta\widetilde{\varepsilon}\}, \tag{14.52}$$

$$\mathrm{WF}'_{3sc,ff}(F) \subset \operatorname{supp} f. \tag{14.53}$$

Let

$$\Lambda_r = x^{-l-1/2}(1+r/x)^{-1}, \qquad r \in (0,1). \tag{14.54}$$

Also define

$$Q_r = Q\phi(H)\Lambda_r x^{-1/2}, \ B_r = B\Lambda_r, \ E_r = \Lambda_r E \Lambda_r, \ F_r = \Lambda_r F \Lambda_r. \tag{14.55}$$

Then multiplying (14.50) by $(1+r/x)^{-1}$ from left and right and rearranging the terms we obtain the following estimate of self-adjoint bounded operators on $L^2_{sc}(X)$:

$$\begin{aligned} ix^{l+1/2}[Q_r^*Q_r, H]x^{l+1/2} - x^{l+1/2}((Mx^{1/2}\Lambda_r + G_r^*)\phi(H)Q^*Q_r \\ + Q_r^*Q\phi(H)(G_r + Mx^{1/2}\Lambda_r))x^{l+1/2} \\ \geq x^{l+1/2}((2-\varepsilon')B_r^*B_r + E_r + F_r)x^{l+1/2} \end{aligned} \tag{14.56}$$

where $G_r = i[\Lambda_r x^{-1/2}, H]$. Now, $G_r \in \mathcal{B}(H_{sc}^{-m+1,-1/2}(X), H_{sc}^{-m,-l-1/2}(X))$ remains bounded when we let $r \to 0$. Hence, $\|x^l G_r\| \leq M$ if we chose M sufficiently large. The point of the commutator calculation is that in $L^2_{sc}(X)$

$$\langle u, [Q_r^*Q_r, H]u\rangle = 2i \operatorname{Im}\langle u, Q_r^*Q_r(H-\lambda)u\rangle; \tag{14.57}$$

the pairing makes sense for $r > 0$ since $Q_r \in \Psi_{3sc}^{-\infty,-l}(X)$. Now apply (14.56) to $x^{-l-1/2}u$ and pair it with $x^{-l-1/2}u$ in $L^2_{sc}(X)$. Then for $r > 0$

$$\|B_r u\|^2 \leq |\langle u, E_r u\rangle| + |\langle u, F_r u\rangle| + 2|\langle u, Q_r^*Q_r(H-\lambda)u\rangle|. \tag{14.58}$$

Letting $r \to 0$ now keeps the right hand side of (14.58) bounded since $(1+r/x)^{-1} \to \mathrm{Id}$ strongly on $\mathcal{B}(H_{sc}^{m',l'}(X), H_{sc}^{m',l'}(X))$. In fact, by (14.28) $Q_r(H-\lambda)u \in \dot{\mathcal{C}}^\infty(X)$ remains bounded in $\dot{\mathcal{C}}^\infty(X)$ as $r \to 0$. Similarly, by (14.31) $E_r u$ remains bounded in $\dot{\mathcal{C}}^\infty(X)$ as $r \to 0$. Also, F_r is bounded in $\mathcal{B}(H_{sc}^{m,l}(X), H_{sc}^{-m,-l}(X))$, so $\langle u, F_r u\rangle$ stays bounded by (14.29). These show that $B_r u$ is uniformly bounded in $L^2_{sc}(X)$ which implies that $Bx^{-l-1/2}u \in L^2_{sc}(X)$.

Let

$$B' = Bx^{-l-1/2} + P(1-\phi(H)) \tag{14.59}$$

with $P \in \Psi_{3sc}^{0,-l-1/2}(X)$ with $\widehat{P}_{ff,-l-1/2}(\xi_0) = \mathrm{Id}$. Although $\widehat{B}(\xi_0)$ is not invertible, $\widehat{B}'(\xi_0)$ is by (14.48). If P is chosen with $\mathrm{WF}'_{3sc}(P)$ sufficiently small, then $\xi_0 \notin \mathrm{WF}_{3sc}((H-\lambda)u)$ implies that $P(1-\phi(H))u \in \dot{\mathcal{C}}^\infty(X)$ too, so we conclude that

$$B'u \in L^2_{sc}(X). \tag{14.60}$$

As $\widehat{B}'(\xi_0)$ is invertible, this implies that

$$\xi_0 \notin \mathrm{WF}^{m,l+1/2}_{3\mathrm{sc},\mathrm{ff}}(u). \tag{14.61}$$

This is exactly the iterative step we wanted to prove. Hence, we deduce that (14.1) holds for all m and l, proving the proposition. □

An immediate corollary is a complete description of propagation of singularities away from C if C is totally geodesic.

Corollary 14.2. — *Suppose that H satisfies* (11.11), *C is totally geodesic, and $\lambda > 0$. If $u \in \mathcal{C}^{-\infty}(X)$, $(H - \lambda)u \in \dot{\mathcal{C}}^{\infty}(X)$, $\alpha \in {}^{\mathrm{sc}}T^*_{\partial X \setminus C}X$, $\alpha \in \Sigma_{\Delta - \lambda}$, and for every broken bicharacteristic γ of $H - \lambda$ satisfying $\gamma(0) = \alpha$, there exists $t < 0$ such that $\gamma(t) \notin \mathrm{WF}_{3\mathrm{sc}}(u)$, then $\alpha \notin \mathrm{WF}_{3\mathrm{sc}}(u)$.*

CHAPTER 15

PROPAGATION OF SINGULARITIES IN TANGENTIAL DIRECTIONS

We proceed to analyze the propagation of singularities along the tangential directions to C. First we prove a result showing that if either one of two spectral conditions on H_{ff}, given below in (15.1) and (15.2), is satisfied, then for (microlocal) solutions of $(H - \lambda)u \in \dot{\mathcal{C}}^\infty(X)$ the absence of $\mathrm{WF}_{3\mathrm{sc}}(u)$ in a ball implies the absence $\mathrm{WF}_{3\mathrm{sc}}(u)$ in a corresponding parabolic region. This is completely analogous to Theorem 2.50 of Melrose and Sjöstrand [**27**]. The other main ingredient of proving that singularities propagate along generalized broken geodesics is the understanding of the generalized broken geodesic flow. Since the geometry is essentially the same as in [**27**] and [**28**], we can make this conclusion. As we are primarily interested in the actual three-body problem where C is totally geodesic, we will provide a simpler proof in this special case.

If H_{ff} has eigenvalues, propagation can be much more complicated. However, in the case when in some local coordinates adapted to $W^\perp$, $H_{\mathrm{ff}}(z)$ is independent of z, it can be described just as in the eigenvalueless case. It is convenient to state our assumptions here. From now on in this chapter we assume that H is as in (11.11), and either

$$H_{\mathrm{ff}}(z) \text{ does not have any eigenvalues in } L^2_{\mathrm{sc}}(\mathbb{S}^n_+) \text{ for any } z, \tag{15.1}$$

or

$$\text{in some local coordinates } H_{\mathrm{ff}}(z) \text{ is independent of } z. \tag{15.2}$$

Note that (15.2) does not give any conditions for $\widetilde{h}_z(\nu)$, and it is satisfied for the actual three-body operators. We first prove a commutator estimate which will be useful if (15.2) holds.

Lemma 15.1. — *Suppose that $\lambda \leq 0$ and H satisfies (11.11). Then given $\varepsilon > 0$ there exists $\delta > 0$ such that for $\widetilde{\psi} \in \mathcal{C}^\infty_c(\mathbb{R}; [0,1])$ supported in $(\lambda - \delta, \lambda + \delta)$*

$$\|\widetilde{\psi}(H_{\mathrm{ff}})[\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}]\widetilde{\psi}(H_{\mathrm{ff}})\| < \varepsilon. \tag{15.3}$$

Proof. — Let $E = E_{H_{\mathrm{ff}}}(\{\lambda\})$, $E_{H_{\mathrm{ff}}}$ denoting the spectral projection. First choose $\phi \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$ which is identically 1 on $\operatorname{supp}\widetilde{\psi}$. Then $\widetilde{\psi}(H_{\mathrm{ff}}) = \phi(H_{\mathrm{ff}})\widetilde{\psi}(H_{\mathrm{ff}})$. Let

$$\begin{aligned} R &= [\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}]\phi(H_{\mathrm{ff}}) - [\overline{Y}\partial_{\overline{Y}}, \Delta_{\mathrm{ff}}]\phi(\Delta_{\mathrm{ff}}) \\ &= [\overline{Y}\partial_{\overline{Y}}, \Delta_{\mathrm{ff}}](\phi(H_{\mathrm{ff}}) - \phi(\Delta_{\mathrm{ff}})) + [\overline{Y}\partial_{\overline{Y}}, V_{\mathrm{ff}}]\phi(H_{\mathrm{ff}}). \end{aligned} \tag{15.4}$$

Now, $[\overline{Y}\partial_{\overline{Y}}, \Delta_{\mathrm{ff}}] = -2\Delta_{\mathrm{ff}}$, and

$$\phi(H_{\mathrm{ff}}) - \phi(\Delta_{\mathrm{ff}}) \in \Psi_{\mathrm{sc}}^{-\infty,1}(\mathbb{S}_+^n) \tag{15.5}$$

by Proposition 10.2. Moreover, $V_{\mathrm{ff}} \in \Psi_{\mathrm{sc}}^{0,1}(\mathbb{S}_+^n)$, and $\overline{Y}\partial_{\overline{Y}} \in \Psi_{\mathrm{sc}}^{1,-1}(X)$, so their commutator is in $\Psi_{\mathrm{sc}}^{0,1}(\mathbb{S}_+^n)$. Hence, $R \in \Psi_{\mathrm{sc}}^{-\infty,1}(\mathbb{S}_+^n)$, and thus it is compact on $L^2_{\mathrm{sc}}(\mathbb{S}_+^n)$. Now write

$$\begin{aligned} \widetilde{\psi}(H_{\mathrm{ff}})[\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}]\widetilde{\psi}(H_{\mathrm{ff}}) =& (\widetilde{\psi}(H_{\mathrm{ff}}) - E)[\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}]\phi(H_{\mathrm{ff}})(\widetilde{\psi}(H_{\mathrm{ff}}) - E) \\ &+ E[\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}]\phi(H_{\mathrm{ff}})(\widetilde{\psi}(H_{\mathrm{ff}}) - E) \\ &+ (\widetilde{\psi}(H_{\mathrm{ff}}) - E)[\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}]\phi(H_{\mathrm{ff}})E + E[\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}]E. \end{aligned} \tag{15.6}$$

Here the last term vanishes by the virial theorem. Also, $\widetilde{\psi}(H_{\mathrm{ff}}) - E$ goes to 0 strongly as $\operatorname{supp}\widetilde{\psi} \to \{\lambda\}$, so in particular $\widetilde{\psi}$ supported sufficiently close to λ

$$\|(\widetilde{\psi}(H_{\mathrm{ff}}) - E)R\| < \varepsilon/8, \qquad \|R(\widetilde{\psi}(H_{\mathrm{ff}}) - E)\| < \varepsilon/8. \tag{15.7}$$

In addition, $\lambda \leq 0$ and $\Delta_{\mathrm{ff}} \geq 0$, so if ϕ is supported in $(\lambda - \varepsilon/32, \lambda + \varepsilon/32)$ then

$$\|2\Delta_{\mathrm{ff}}\phi(\Delta_{\mathrm{ff}})\| < \varepsilon/16. \tag{15.8}$$

Since $\|\widetilde{\psi}(H_{\mathrm{ff}})\| \leq 1$, and the same holds for E we see that if $\widetilde{\psi}$ is supported sufficiently close to λ then the first three terms on the right hand side of (15.6) are bounded in norm by $\varepsilon/2$, $\varepsilon/4$ and $\varepsilon/4$ respectively. This proves the lemma. □

Recall that

$$\widetilde{h}(z,\nu) = \widetilde{h}_z(\nu) = h_{tt}|_{y=0}(z,\nu) \tag{15.9}$$

is the restriction of the boundary metric h to $\widetilde{W}^\perp$, and we have defined

$$W = 2\tau(\nu\cdot\partial_\nu) - 2\widetilde{h}_z(\nu)\partial_\tau + H_{\widetilde{h}} \in \mathcal{V}(W^\perp). \tag{15.10}$$

This definition ensures that $W - {}^{\mathrm{sc}}H_g|_{W^\perp\cap\Sigma_{\Delta-\lambda}} = \sum \alpha_i\partial_{\mu_i}$ in the local coordinates adapted to $W^\perp$. We also assume in what follows that we have chosen some

$$K \subset \Sigma_t(\lambda) \setminus (R^- \cup R^+) \tag{15.11}$$

which is compact. Since the propagation result is local, we can work in local coordinates. In particular, it will be useful to extend the projection $\pi^\perp : {}^{\mathrm{sc}}T^*_C X \to W^\perp$ using a product decomposition given by local coordinates to a projection (also denoted by $\pi^\perp$) from ${}^{\mathrm{sc}}T^*_U X$ to $W^\perp$, where $U \subset \partial X$ is a neighborhood of C. We also write $|\cdot|$ for the Euclidian metric on ${}^{\mathrm{sc}}T^*_U X$ in the local coordinates.

Proposition 15.2. — *Suppose that H satisfies* (11.11) *and $\lambda > 0$. Suppose also that either* (15.1) *or* (15.2) *holds. Given K as in* (15.11) *there exist constants $C_0 > 0$, $\delta_0 > 0$ such that the following holds. If $\xi_0 = (z_0, \tau_0, \nu_0) \in K$, $u \in \mathcal{C}^{-\infty}(X)$, $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc}}((H-\lambda)u)$ and in addition for some $0 < \varepsilon < 1$, $0 < \delta < \min\{C_0\varepsilon, \delta_0\}$ and for all $\alpha = (y, z, \tau, \mu, \nu) \in {}^{\mathrm{sc}}T^*_{\partial X}X$*

$$|y| \leq \varepsilon\delta,\ |\pi^{\perp}(\alpha) - \exp(-\delta W)(\xi_0)| \leq \varepsilon\delta \Longrightarrow \alpha \notin \mathrm{WF}_{3\mathrm{sc,mf}}(u) \tag{15.12}$$

and

$$y = 0,\ |\pi^{\perp}(\alpha) - \exp(-\delta W)(\xi_0)| \leq \varepsilon\delta \Longrightarrow \pi^{\perp}\alpha \notin \mathrm{WF}_{3\mathrm{sc,ff}}(u) \tag{15.13}$$

then $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc,ff}}(u)$.

Proof. — The proof is essentially a combination of the proofs of Proposition 14.1 and of the propagation along generalized bicharecteristics found in [**18**, Proposition 24.5.1] which in turn is based on Melrose's and Sjöstrand's paper [**27**]. Thus, we have to change the construction of q; the point being that now ${}^{\mathrm{sc}}H_g$ is tangent to $W^{\perp}$ at some points of the broken geodesics, so we cannot use the flow-out of ${}^{\mathrm{sc}}H_g$ from some hypersurface including ${}^{\mathrm{sc}}T^*_C X$ as in the normal case to define ω, and hence $\widetilde{q}$. Of course, we still want to arrange Q to have a positive commutator with H in the region which we wish to exclude from the wave front set. The main difference from the proof of Proposition 14.1 will be that we define ω by using the flow-out of W from some hypersurface; in particular ω will be completely independent of μ. Naturally, we cannot expect ${}^{\mathrm{sc}}H_g\omega$ to vanish, but it will be small in the region of interest, and we will have to do careful estimates to make sure that it is actually sufficiently small. In the first part of the argument we follow the proof of [**18**, Proposition 24.5.1] closely with a few necessary changes.

We have

(15.14)

$$\begin{aligned} H_h = {} & 2\sum_{i,j} h^{ij}_{nn}\mu_j\partial_{y_i} + 2\sum_{i,j} h^{ij}_{nt}\mu_i\partial_{z_j} + 2\sum_{ij} h^{ij}_{nt}\nu_j\partial_{y_i} + 2\sum_{i,j} h^{ij}_{tt}\nu_j\partial_{z_i} \\ & + \sum_{i,j,k}(\partial_{z_k}h^{ij}_{nn})\mu_i\mu_j\partial_{\nu_k} + 2\sum_{i,j,k}(\partial_{z_k}h^{ij}_{nt})\mu_i\nu_j\partial_{\nu_k} + \sum_{i,j,k}(\partial_{z_k}h^{ij}_{tt})\nu_i\nu_j\partial_{\nu_k} + W' \end{aligned}$$

with $W' = \sum \alpha_j\partial_{\mu_j}$. Hence, if $\omega \in \mathcal{C}^{\infty}(\mathbb{R}^{m-1}_z \times \mathbb{R}^m_{\tau,\nu})$ then

$$H_h\omega|_{y=0} = H_{\widetilde{h}}\omega, \tag{15.15}$$

so we have

$${}^{\mathrm{sc}}H_g\omega|_{y=0} = W\omega - 2(h - \widetilde{h})\partial_\tau\omega. \tag{15.16}$$

We now define ω such that the second term is small near $\alpha_0 = (0, z_0, \tau_0, 0, \nu_0) \in {}^{\mathrm{sc}}T^*_C X$, the unique point on $\Sigma_{\Delta-\lambda}$ such that $\pi^{\perp}\alpha_0 = \xi_0$. Now, $W\tau = -2\widetilde{h}$, and

$\widetilde{h}_{z_0}(\nu_0) \neq 0$, so near ξ_0, $W\tau \neq 0$, i.e. W is transversal to the hypersurface $\tau = \tau_0$. Thus, near ξ_0 in $W^\perp$ we can solve the Cauchy problem

$$W\omega = 0, \qquad \omega|_{\tau=\tau_0} = (z - z_0)^2 + (\nu - \nu_0)^2. \tag{15.17}$$

Since ω and $d\omega$ vanish at ξ_0, the same holds on the bicharacteristic of W through ξ_0, but $\omega \geq 0$ and the Hessian is still positive in directions transversal to the bicharacteristics as these hold at ξ_0. Moreover, by [**18**, Lemma 7.7.2],

$$|d\omega| \leq C\omega^{1/2}. \tag{15.18}$$

Let

$$r_0 = \tau^2 + \widetilde{h}_z(\nu) - \lambda, \tag{15.19}$$

so $Wr_0 = 0$. Now at $\tau = \tau_0$ we have $r_0 = \widetilde{h}_z(\nu) - \widetilde{h}_{z_0}(\nu_0)$, so

$$|r_0| \leq C'|d\omega| \leq C''\omega^{1/2} \tag{15.20}$$

when $\tau = \tau_0$, and then $W\omega = 0 = Wr_0$ implies that this inequality holds everywhere. Therefore,

$$|\widetilde{h} - h| \leq |\lambda - \tau^2 - h| + |\lambda - \tau^2 - \widetilde{h}| \leq |\lambda - \tau^2 - h| + C\omega^{1/2}. \tag{15.21}$$

Note that $h^{ij}_{nn}(0,y) = \delta_{ij}$, $h^{ij}_{nt}(0,y) = 0$, and

(15.22)
$$\begin{aligned}
{}^{\mathrm{sc}}H_g\omega = {}^{\mathrm{sc}}H_g\omega - W\omega = &-2(h - \widetilde{h})\partial_\tau\omega \\
&+ 2\sum_{i,j} h^{ij}_{nt}(y,z)\mu_i\partial_{z_j}\omega + 2\sum_{i,j}(h^{ij}_{tt}(y,z) - h^{ij}_{tt}(0,z))\nu_j\partial_{z_i}\omega \\
&+ \sum_{i,j,k}\partial_{z_k}h^{ij}_{nn}(y,z)\mu_i\mu_j\partial_{\nu_k}\omega + 2\sum_{i,j,k}\partial_{z_k}h^{ij}_{nt}(y,z)\mu_i\nu_j\partial_{\nu_k}\omega \\
&+ \sum_{i,j,k}\partial_{z_k}(h^{ij}_{tt}(y,z) - h^{ij}_{tt}(0,z))\nu_i\nu_j\partial_{\nu_k}\omega,
\end{aligned}$$

so for some $C, C' > 0$

$$\begin{aligned}
|{}^{\mathrm{sc}}H_g\omega - W\omega| &\leq C'(|y| + |\tau^2 + h - \lambda| + \omega^{1/2})|d\omega| \\
&\leq C(|y| + |\tau^2 + h - \lambda| + \omega^{1/2})\omega^{1/2}.
\end{aligned} \tag{15.23}$$

Now we define for δ, $\varepsilon > 0$

$$\phi = \tau_0 - \tau + \frac{1}{\varepsilon^2\delta}(|y|^2 + \omega). \tag{15.24}$$

Note that now $|y|^2 + \omega$ plays the role of ω in (14.5) and (14.10), and our propagation variable is $\tau_0 - \tau$ since ${}^{\mathrm{sc}}H_g(\tau_0 - \tau) = 2h$ is positive near α_0. Thus,

$$ {}^{\mathrm{sc}}H_g\phi = 2h + \frac{1}{\varepsilon^2\delta}\Big(4\sum_{i,j} h^{ij}_{nn}\mu_j y_i + 4\sum_{ij} h^{ij}_{nt}\nu_j y_i + {}^{\mathrm{sc}}H_g\omega\Big). \tag{15.25}$$

We have already estimated ${}^{\mathrm{sc}}H_g\omega$. On the other hand,

$$|h_{nt}^{ij}(y,z)\nu_j y_i| \leq C|y|^2, \qquad |h_{nn}^{ij}(y,z)\mu_j y_i| \leq C|y||\mu|. \tag{15.26}$$

We can also estimate $|\mu|$ near $\Sigma_{\Delta-\lambda}$. In fact, $|\mu_i\nu_j| \leq |\mu|^2 + |\nu|^2$, so

$$\Big|\sum_{i,j} h_{nt}^{ij}(y,z)\mu_i\nu_j\Big| \leq C|y|(|\mu|^2 + |\nu|^2). \tag{15.27}$$

Also,

$$\Big|\sum_{i,j}(h_{nn}^{ij}(y,z) - \delta_{ij})\mu_i\mu_j\Big| \leq C|y||\mu|^2, \tag{15.28}$$

$$\Big|\sum_{i,j}(h_{tt}^{ij}(y,z) - h_{tt}^{ij}(0,z))\nu_i\nu_j\Big| \leq C|y||\nu|^2, \tag{15.29}$$

so

$$|h - |\mu|^2 - \widetilde{h}_z(\nu)| \leq C|y|(|\mu|^2 + |\nu|^2). \tag{15.30}$$

By the triangle inequality

$$|\mu|^2 \leq |(h - \widetilde{h}_z(\nu)) - |\mu|^2| + |h - \widetilde{h}|. \tag{15.31}$$

Hence, by (15.21)

$$|\mu|^2 \leq C(|y| + \omega^{1/2} + |\tau^2 + h - \lambda|) \tag{15.32}$$

Summarizing these estimates we see that

$$\begin{aligned}|{}^{\mathrm{sc}}H_g\phi - 2h| \leq \frac{C}{\varepsilon^2\delta}(|y|(|y| + \omega^{1/2} + |\tau^2 + h - \lambda|)^{1/2} + |y|^2 \\ + (|y| + |\tau^2 + h - \lambda| + \omega^{1/2})\omega^{1/2}).\end{aligned} \tag{15.33}$$

Now,

$$\phi \leq 2\delta \text{ and } \tau - \tau_0 \leq 2\delta \implies |\tau - \tau_0| < 2\delta,\ |y| \leq 2\varepsilon\delta,\ \omega \leq (2\varepsilon\delta)^2. \tag{15.34}$$

Thus, under the additional assumption that $|\tau^2 + h - \lambda| \leq \varepsilon\delta$,

$$|{}^{\mathrm{sc}}H_g\phi - 2h| \leq C_1((\delta/\varepsilon)^{1/2} + \delta). \tag{15.35}$$

Note that C_1 and $\delta_1 > 0$ can be chosen so that (15.35) is valid for all $\xi_0 \in K$ if $\delta < \delta_1$. Thus, there exist $C_0 > 0$ and $\delta_0 > 0$ such that if $\xi_0 \in K$, $\delta < \delta_0$, $\varepsilon < 1$, $\delta/\varepsilon < C_0$ then

$${}^{\mathrm{sc}}H_g\phi \geq c = \inf\{|\nu_0|^2_{z_0} : \xi_0 \in K\}, \tag{15.36}$$

when the assumptions of (15.34) are satisfied and $|\tau^2 + h - \lambda| \leq \varepsilon\delta$.

Still following the proof of [**18**, Proposition 24.5.1] we let $\chi_0(t) = \exp(-1/t)$ for $t > 0$, 0 for $t \leq 0$, and we let $\chi_1 \in \mathcal{C}^\infty(\mathbb{R})$ to be identically 1 on $[1,\infty)$, 0 on $(-\infty, 0]$, and to have $0 \leq \chi_1' \in \mathcal{C}_c^\infty((0,1))$. For $t \in [0,1]$, $\varepsilon \in (0,1)$, $0 < \delta \ll \varepsilon$, $A > 0$ we define

$$\widetilde{q}_t(y,z,\tau,\nu) = \chi_0(A^{-1}(1 + t - \phi/\delta))\chi_1((\tau_0 - \tau + \delta)/(\varepsilon\delta) + t). \tag{15.37}$$

On the support of the first factor $\phi \leq 2\delta$, and on the support of the second factor $\tau - \tau_0 \leq \delta + \varepsilon\delta t \leq 2\delta$. Now,

(15.38) $$^{\mathrm{sc}}H_g \widetilde{q}_t = -g_0^2 + e_0$$

where

(15.39) $$g_0^2 = A^{-1}\delta^{-1}(^{\mathrm{sc}}H_g\phi)\chi_0'(A^{-1}(1+t-\phi/\delta))\chi_1((\tau_0-\tau+\delta)/(\varepsilon\delta)+t),$$

(15.40) $$e_0 = 2h(\varepsilon\delta)^{-1}\chi_0(A^{-1}(1+t-\phi/\delta))\chi_1'((\tau_0-\tau+\delta)/(\varepsilon\delta)+t).$$

Note that $\chi_0'(s) = s^{-2}\chi_0(s)$, and on $\operatorname{supp} q_t$, $1+t-\phi/\delta \leq 4$, so

(15.41) $$A^{-1}\chi_0'(A^{-1}(1+t-\phi/\delta))\chi_1 \geq (A/16)\widetilde{q}_t.$$

By (15.35) we see that when $|\tau^2+h-\lambda| \leq \varepsilon\delta$, we have similarly to (14.24)

(15.42) $$\widetilde{q}_t \leq C'A^{-2}\chi_0'\chi_1 \leq CA^{-1}\delta g_0^2.$$

On the other hand, e_t is supported where

(15.43) $$-t\varepsilon\delta \leq \tau_0 - \tau + \delta \leq (1-t)\varepsilon\delta$$

in addition to (15.34). With $\xi = \exp(-\delta' W)\xi_0$, $\delta' = \delta/(2|\nu_0|_{z_0}^2)$, this implies that $|\tau - \tau(\xi)| \leq \varepsilon\delta + C\delta^2$. From (15.34) we also conclude that

(15.44) $$|y|^2 + |z - z(\xi)|^2 + |\nu - \nu(\xi)|^2 \leq C\varepsilon^2\delta^2.$$

We drop the index t for the time being. We now let Q be the quantization of

(15.45) $$q = \psi_0(x)\widetilde{q}$$

as in Proposition 14.1, and we consider the commutator $[Q\psi(H), H]$ where we still have $Q\psi(H) \in \Psi_{3\mathrm{sc}}^{-\infty,1}(X)$ since $\psi \in C_c^\infty(\mathbb{R})$. If $\lambda - \tau^2 - |\nu|_z^2$ is not an eigenvalue of $H_{\mathrm{ff}}(z)$ then we can employ Corollary 13.4 as in the normal case to reduce the computation to that of $[Q\psi(\Delta), \Delta]$. Since $Q\psi(\Delta) \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$, the joint symbol of the commutator is given by the Poisson bracket of the symbols:

(15.46) $$j_{\mathrm{sc},0,1}(i[Q\psi(\Delta),\Delta]) = -(^{\mathrm{sc}}H_g\widetilde{q})\psi(g).$$

We have already estimated $^{\mathrm{sc}}H_g q$ near $\Sigma_{\Delta-\lambda}$, so if we arrange the inclusion $\operatorname{supp}\psi \subset (\lambda-\varepsilon\delta, \lambda+\varepsilon\delta)$, and $\delta > 0$ is sufficiently small, we can conclude that away from $\operatorname{supp} e$

(15.47) $$j_{\mathrm{sc},0,1}(i[Q\psi(\Delta),\Delta]) \leq -A^{-1}\delta^{-1}c\chi_0'\chi_1\psi(g).$$

We can also estimate $dq|_{y=0}$ since on $\operatorname{supp} q$

(15.48) $$|d\phi|_{y=0}| \leq C' + \frac{1}{\varepsilon^2\delta}|d\omega| \leq C''\Big(1 + \frac{1}{\varepsilon^2\delta}\omega^{1/2}\Big) \leq C(1+\varepsilon^{-1}).$$

Thus, we see that away from $\operatorname{supp} e$

(15.49) $$|d\widetilde{q}(0,z,\tau,\nu)| \leq C(1+\varepsilon^{-1})f^\flat(z,\tau,\nu)$$

where, in accordance with (14.16) and (15.47) we let

(15.50) $$f^\flat = A^{-1}\delta^{-1}c\chi_0'|_{y=0}\chi_1.$$

Since q is independent of μ, this proves (14.21) and (14.23) in our setting.

If (15.1) holds, then we can apply the argument in the proof of Proposition 14.1 after (14.40) verbatim, taking into account the support properties of e in (15.43), and reducing the size of t in the iterative steps (of improving regularity by order $\frac{1}{2}$) as in the proof of [**18**, Proposition 24.5.1], to deduce the conclusion of this proposition. Note that the presence of ε^{-1} in (15.49) will not cause any problems since in the compactness argument after (14.42) we will just choose a spectral cutoff function $\widetilde{\psi} \in \mathcal{C}_c^\infty(\mathbb{R})$ supported sufficiently close to λ, with the size of support depending on ε.

Suppose now that (15.2) holds. Note that

$$\widehat{H}_{\mathrm{ff}} = \Delta_{\mathrm{ff}} + V_{\mathrm{ff}} + \widetilde{h} + \tau^2 \tag{15.51}$$

By (13.38) we have with $f = \widetilde{q}|_{y=0}$

$$i[Q\widehat{\psi(H)}, H]_{\mathrm{ff},1} = (-(\partial_\tau f)[\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}] - Wf)\psi(\widehat{H}_{\mathrm{ff}}) \tag{15.52}$$

since

$$W = 2\tau(\nu\cdot\partial_\nu) - 2\widetilde{h}\partial_\tau + (\partial_\nu\widetilde{h})\partial_z - (\partial_z\widetilde{h})\partial_\nu. \tag{15.53}$$

Now, if $\tau^2 + \widetilde{h} \geq \lambda$, then by Lemma 15.1 we can arrange for any $\xi \in W^\perp$ and for any $\varepsilon' > 0$ that

$$\|\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}(\xi))[\overline{Y}\partial_{\overline{Y}}, H_{\mathrm{ff}}(z)]\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}(\xi))\| < \varepsilon' \tag{15.54}$$

if $\widetilde{\psi} \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$ satisfies $\operatorname{supp}\widetilde{\psi} \subset (\lambda-\delta, \lambda+\delta)$ where $\delta = \delta_{\xi,\varepsilon'}$, since

$$\widetilde{\psi}(\widehat{H}_{\mathrm{ff}}) = \widetilde{\psi}(H_{\mathrm{ff}} + \tau^2 + \widetilde{h}). \tag{15.55}$$

Since $W\phi = 2\widetilde{h}$, we have

$$W\widetilde{q} = -2A^{-1}\delta^{-1}\widetilde{h}\chi_0'(A^{-1}(1+t-\phi/\delta))\chi_1 + 2\widetilde{h}\chi_0\chi_1'((\tau_0-\tau+\delta)/(\varepsilon\delta)+t). \tag{15.56}$$

We multiply both sides of (15.52) by $\psi(\widehat{H}_{\mathrm{ff}})\widehat{Q}_{\mathrm{ff}}^*$, note that $\widehat{Q}_{\mathrm{ff}}^* = f$, so it commutes with $\psi(\widehat{H}_{\mathrm{ff}})$, so we see that

$$i\psi(H)[\widehat{Q^*Q\psi(H)}, H]_{\mathrm{ff},1} \geq 2ff^\flat\psi(\widehat{H}_{\mathrm{ff}})^2 \tag{15.57}$$

if $\xi \notin \operatorname{supp} e$. This is completely analogous to (14.46). Taking into account (15.42), we actually conclude that for any M we can choose $A > 0$ sufficiently large so that

$$i\psi(H)[\widehat{Q^*Q\psi(H)}, H]_{\mathrm{ff},1} + 2Mf^2\psi(\widehat{H}_{\mathrm{ff}})^2 \geq (2-2\varepsilon')ff^\flat\psi(\widehat{H}_{\mathrm{ff}})^2 \tag{15.58}$$

Although we have assumed that $\tau^2 + \widetilde{h} \geq \lambda$, (15.57) also holds if this is not satisfied, since in that case $\lambda - (\tau^2 + \widetilde{h})$ cannot be an eigenvalue of H_{ff}, so we can use the eigenvalueless argument from above. Since the right hand side of (15.57) is a continuous function of ξ with values in $\mathcal{B}(L^2_{\mathrm{sc}}(\mathbb{S}^n_+), L^2_{\mathrm{sc}}(\mathbb{S}^n_+))$, if (15.57) holds for some ψ at ξ, it also holds in some neighborhood of ξ with ε' replaced by $2\varepsilon'$. Since $\operatorname{supp} f$ is compact, (15.57) holds on $\operatorname{supp} f$ if we choose $\operatorname{supp}\psi$ sufficiently small, and hence it holds everywhere in this case. Combining this with the argument for on supp q at mf proves the proposition when (15.2) is satisfied. □

As mentioned above, this result is completely analogous to Theorem 2.50 of Melrose's and Sjöstrand's first paper **[27]**. The argument of their second paper **[28]**, see also Sjöstrand's paper on analytic singularities **[36]** and the arguments of **[18**, Section 24.3], can be repeated to prove that our proposition implies that $\mathrm{WF}_{3\mathrm{sc}}$ propagates along generalized broken bicharacteristics. Namely, we conclude:

Proposition 15.3. — *Suppose that H satisfies* (11.11) *and $\lambda > 0$. Suppose also that either* (15.1) *or* (15.2) *holds. Let $\xi_0 \in \Sigma_t(\lambda)$. Assume that $u \in \mathcal{C}^{-\infty}(X)$ and $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc,ff}}((H-\lambda)u)$. If in addition $\xi_0 \in \mathrm{WF}_{3\mathrm{sc,ff}}(u)$, then there exist $\varepsilon > 0$ and a generalized bicharacteristic γ of H with $\gamma(0) = \xi_0$ such that $\gamma|_{(-\varepsilon,\varepsilon)} \subset \mathrm{WF}_{3\mathrm{sc,ff}}(u)$.*

We are particularly interested in the case when C is totally geodesic. Then the argument of the previous proposition can be strengthened to give an analog of Proposition 14.1 immediately, without the additional analysis of the generalized bicharacteristics. Namely, in this case the bicharacteristic γ of g going through $\alpha_0 \in W^\perp \subset {}^{\mathrm{sc}}T^*_{\partial X}X$ stays in $W^\perp$, and $\pi^\perp(\gamma)$ is a bicharacteristic of W. We now show that for microlocal solutions of $(H-\lambda)u \in \dot{\mathcal{C}}^\infty(X)$, $\mathrm{WF}_{3\mathrm{sc,ff}}(u)$ either includes the whole of γ or is disjoint from it.

Proposition 15.4. — *Suppose that C is totally geodesic, H as in* (11.11), *$\lambda > 0$ and either* (15.1) *or* (15.2) *holds. Let $\xi_0 = (z_0, \tau_0, \nu_0) \in \Sigma_t(\lambda) \setminus (R^- \cup R^+)$. Suppose also that $u \in \mathcal{C}^{-\infty}(X)$, $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc}}((H-\lambda)u)$. Then there exists $\varepsilon' > 0$ such that if in addition for the unique α_0 with $\pi^\perp\alpha_0 = \xi_0$, $g(\alpha_0) = \lambda$, and for some $s \in (-\varepsilon', 0)$ we have $\exp(s\,{}^{\mathrm{sc}}H_g)(\alpha_0) \notin \mathrm{WF}_{3\mathrm{sc}}(u)$, then $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc,ff}}(u)$.*

Proof. — Note first that $\varepsilon' > 0$ appears in the statement only to ensure that for $s \in (-\varepsilon', 0)$, $\exp(s\,{}^{\mathrm{sc}}H_g)(\alpha_0) \notin \mathrm{WF}_{3\mathrm{sc}}((H-\lambda)u)$. As usual, it suffices to prove that the set

$$\{s \in (-\varepsilon, \varepsilon) : \exp(s\,{}^{\mathrm{sc}}H_g)(\alpha_0) \notin \mathrm{WF}_{3\mathrm{sc}}(u)\} \tag{15.59}$$

is closed. We again work in local coordinates and note that C totally geodesic means that

$$\partial_y h^{ij}_{tt}(0, z) = 0 \tag{15.60}$$

for all z. It is useful to introduce geodesic normal coordinates (y', z') with respect to C. In these coordinates $h^{ij}_{nn}(y', z') - \delta_{ij}$ vanishes with its first derivative at $y' = 0$, and the same holds for $h^{ij}_{nt}(y', z')$. Moreover, (15.60) is still satisfied when the variables are replaced by the primed ones. From now on we assume that our coordinates are geodesic normal coordinates and we drop the primes.

The additional vanishing of the coefficients allows strong improvements in the arguments of the previous proposition. First, in (15.22) every term but the first one,

$-2(h-\widetilde{h})\partial_\tau\omega$, has an additional order of vanishing in $|y|$, so (15.23) can be replaced by

$$\begin{aligned}|{}^{\mathrm{sc}}H_g\omega - W\omega| &\leq C'(|y|^2+|\tau^2+h-\lambda|+\omega^{1/2})|d\omega| \\ &\leq C(|y|^2+|\tau^2+h-\lambda|+\omega^{1/2})\omega^{1/2}.\end{aligned} \tag{15.61}$$

Similarly, in (15.27)-(15.30) we gain an extra factor of $|y|$ in the estimates, so (15.32) can be replaced by

$$|\mu|^2 \leq C(|y|^2+\omega^{1/2}+|\tau^2+h-\lambda|). \tag{15.62}$$

Moreover, the first equation of (15.26) can be replaced by $|h^{ij}_{nt}(y,z)\nu_j y_i| \leq C|y|^3$.

For $\varepsilon>0$ let

$$\phi = \tau_0 - \tau + \varepsilon^{-1}|y|^2 + \varepsilon^{-2}\omega. \tag{15.63}$$

Thus, (15.33) is replaced by

$$\begin{aligned}|{}^{\mathrm{sc}}H_g\phi - 2h| \leq C(\varepsilon^{-1}&|y|(|y|^2+\omega^{1/2}+|\tau^2+h-\lambda|)^{1/2}+|y|^3 \\ &+\varepsilon^{-2}(|y|^2+|\tau^2+h-\lambda|+\omega^{1/2})\omega^{1/2}).\end{aligned} \tag{15.64}$$

Therefore,

$$\phi\leq 2\delta \text{ and } \tau-\tau_0\leq 2\delta \implies |\tau-\tau_0|\leq 2\delta,\ |y|\leq (4\varepsilon\delta)^{1/2},\ \omega\leq 4\varepsilon^2\delta. \tag{15.65}$$

Hence, under the additional assumption that $|\tau^2+h-\lambda|\leq\varepsilon\delta$,

$$|{}^{\mathrm{sc}}H_g\phi - 2h| \leq C_1(\delta+\delta^{3/4}+\delta^{3/2}\varepsilon^{1/2}+\delta^{3/2}). \tag{15.66}$$

Thus, there exists $\delta_0>0$ such that if $\delta<\delta_0$ and $\varepsilon<1$ then

$${}^{\mathrm{sc}}H_g\phi \geq c = \inf\{|\nu_0|^2_{z_0} : \xi_0\in K\}, \tag{15.67}$$

when the assumptions of (15.65) are satisfied and $|\tau^2+h-\lambda|\leq\varepsilon\delta$. This has the tremendous advantage over the non-totally geodesic case that we can fix $\delta>0$ first, and then choose $\varepsilon>0$ as small as we wish.

We can repeat the arguments of the previous proposition. Since we altered the definition of ϕ, (15.48) is replaced by

$$|d\phi|_{y=0}| \leq C' + \varepsilon^{-2}|d\omega| \leq C''(1+\varepsilon^{-2}\omega^{1/2}) \leq C(1+\varepsilon^{-1}\delta^{1/2}). \tag{15.68}$$

Again, the presence of ε^{-1} will not cause any problems since we will simply choose our spectral cutoff, $\widetilde{\psi}$ to have sufficiently small support (depending on ε) near λ. The rest of the proof can be followed verbatim to conclude that statement of Proposition 15.2 can be replaced by the following assertion. There exists a constant $\delta_0>0$ such that if $\xi_0=(z_0,\tau_0,\nu_0)\in K$, $u\in\mathcal{C}^{-\infty}(X)$, $\xi_0\notin \mathrm{WF}_{3\mathrm{sc}}((H-\lambda)u)$ and in addition for some $0<\varepsilon<1$, $0<\delta<\delta_0$ and for all $\alpha\in {}^{\mathrm{sc}}T^*_{\partial X}X$

$$|y|\leq\varepsilon\delta,\ |\alpha-\exp(-\delta W)(\xi_0)|\leq\varepsilon\delta \implies \alpha\notin \mathrm{WF}_{3\mathrm{sc,mf}}(u) \tag{15.69}$$

and

$$y = 0, \; |\alpha - \exp(-\delta W)(\xi_0)| \le \varepsilon\delta \Rightarrow \pi^\perp \alpha \notin \mathrm{WF}_{3\mathrm{sc},\mathrm{ff}}(u) \tag{15.70}$$

then $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc},\mathrm{ff}}(u)$.

It is very easy to interpret these conditions geometrically. First, $W - {}^{\mathrm{sc}}H_g$ vanishes when $y = 0$ and $\mu = 0$ by the assumption of total geodesity, so

$$\exp(-\delta W)(\xi_0) = \pi^\perp \exp(-\delta^{\mathrm{sc}} H_g)\alpha_0 \tag{15.71}$$

where α_0 is the unique element of $\Sigma_{\Delta-\lambda}$ with $\pi^\perp\alpha_0 = \xi_0$. Next, suppose that for some $\delta < \varepsilon'$, $\delta < \delta_0$, $\pi^\perp \exp(-\delta^{\mathrm{sc}} H_g)(\alpha_0) \notin \mathrm{WF}_{3\mathrm{sc}}(u)$. Then for sufficiently small $\varepsilon > 0$ (15.69) and (15.70) are satisfied, so we conclude that $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc},\mathrm{ff}}(u)$. This shows that (15.59) is closed, hence we have proved the proposition. □

CHAPTER 16

BOUND STATES WITH STRICTLY NEGATIVE ENERGY

We now analyze the propagation of singularities along bound states with strictly negative energy, i.e. at points in $\Sigma_b(\lambda)$. We assume that (15.2) holds. On the other hand, since $(\pi^{\perp})^{-1}(\Sigma_b(\lambda))$ is disjoint from $\Sigma_{\Delta-\lambda}$, the singularities at the bound states will be unable to leave C, and correspondingly we can implement the argument of Proposition 15.4 without the assumption that C is totally geodesic.

Proposition 16.1. — *Suppose that* (15.2) *holds and* $\lambda > 0$. *Let* $\xi_0 = (z_0, \tau_0, \nu_0) \in \Sigma_b(\lambda) \setminus (R^+ \cup R^-)$. *Suppose that* $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc},\mathrm{ff}}((H-\lambda)u)$. *Then there exists* $\varepsilon' > 0$ *such that if in addition* $\exp(sW)(\xi_0) \notin \mathrm{WF}_{3\mathrm{sc},\mathrm{ff}}(u)$ *for some* $s \in (0, -\varepsilon')$ *then* $\xi_0 \notin \mathrm{WF}_{3\mathrm{sc},\mathrm{ff}}(u)$.

Proof. — We just follow the proof of Proposition 15.2. We define ω, $\widetilde{q}$, etc., exactly the same way, but now we will not make use of the estimates on ${}^{\mathrm{sc}}H_g\widetilde{q}$. Now if we choose $\operatorname{supp}\psi$ close to λ then $\operatorname{supp}\psi(g)$ and $\operatorname{supp}\widetilde{q}$ are disjoint, so

$$\widehat{[Q,\Delta]}_{\mathrm{mf},1}\widehat{\psi(H)}_{\mathrm{mf}} = 0. \tag{16.1}$$

On ff we can follow the calculations following (15.51). Since it only involves estimates on $W\omega$ and the use of Lemma 15.1, the arguments given there can be followed without a change. □

CHAPTER 17

RADIAL SETS

In this chapter we study the wave front set near the radial sets

$$R^{\pm}_{\lambda} \quad \text{and} \quad R^{\pm} \cap (\Sigma_t(\lambda) \cup \Sigma_b(\lambda)).$$

As mentioned in Chapter 11, the significance of these sets is that they are the part of the characteristic set of H where the Hamilton vector field vanishes, and hence where the real principal type propagation of singularities does not hold. Indeed, singularities of generalized eigenfunctions of H can appear here without affecting the rest of the characteristic set of H, as is the case of $u = R(\lambda \pm i0)f$, $f \in \dot{C}^{\infty}(X)$, $\lambda > 0$ (see the following chapter). It might be confusing that this set is (the intersection of the characteristic set with) the 'propagation set' of Sigal and Soffer **[34]**; they use the physical meaning of the word 'propagation' (i.e. understanding it as the location of the particles).

We shall also show that any $L^2_{\text{sc}}(X)$ eigenfunction of $H - \lambda$ with $\lambda > 0$ is actually in $\dot{C}^{\infty}(X)$. A theorem of Froese and Herbst **[8]** implies that there are no such eigenfunctions in Euclidian many-body scattering. We extend this result to the geometric setting for the three-body problem, largely following their proof, in Appendix B.

In general, when we do not assume either of (15.1) and (15.2), we do not have a complete picture of propagation of singularities. Namely, the propagation is understood in normal directions to C, but tangential directions and bound states are more troublesome. However, even in these cases we can prove resolvent estimates and uniqueness results which are analogous to those of Gérard, Isozaki and Skibsted **[10, 22]**. In fact, these results only require propagation estimates in the τ variable, i.e. no complete microlocalization. If either of the above mentioned assumptions holds, so in particular for the actual three-body problem, we can obtain sharp uniqueness statements in the sense that the wave front set assumptions are minimal. We first prove the standard commutator identity.

Lemma 17.1. — *Suppose that H satisfies* (11.11), $Q \in \Psi_{3sc}^{-\infty,-2l-1}(X)$, $Q = Q^*$, $[Q, H] \in \Psi_{3sc}^{-\infty,-2l}(X)$, *and* $v \in \mathcal{C}^{-\infty}(X)$ *satisfies*

$$\mathrm{WF}_{3sc}^{0,l}(v) \cap \mathrm{WF}'_{3sc}(Q) = \varnothing, \ \mathrm{WF}_{3sc}^{0,l+1}((H-\lambda)v) \cap \mathrm{WF}'_{3sc}(Q) = \varnothing. \tag{17.1}$$

Then

$$\langle v, [Q, H]v\rangle = 2i \operatorname{Im}\langle v, Q(H-\lambda)v\rangle. \tag{17.2}$$

Proof. — Let $m', l' \in \mathbb{R}$ be such that $v \in H_{sc}^{m',l'}(X)$. Also let $P \in \Psi_{3sc}^{0,0}(X)$ with $\mathrm{WF}'_{3sc}(\mathrm{Id} - P) \cap \mathrm{WF}'_{3sc}(Q) = \varnothing$ such that $Pv \in H_{sc}^{0,l}(X)$, $P(H-\lambda)v \in H_{sc}^{0,l+1}(X)$; this can be arranged by (17.1). For the same reason note that both sides of (17.2) are indeed defined. First note that (17.2) holds under the slightly stronger assumption $Pv \in H_{sc}^{0,l+1}(X)$. In fact,

$$(H-\lambda)Q \in \mathcal{B}(H_{sc}^{0,l+1}(X), H_{sc}^{0,-l}(X)), \tag{17.3}$$

so we can write $[Q, H] = Q(H-\lambda) - (H-\lambda)Q$ and expand the left hand side of (17.2). We also write $v = Pv + (\mathrm{Id} - P)v$, and manipulate the arising terms of $\langle v, [Q, H]v\rangle$ separately using that

$$(H-\lambda)Q(\mathrm{Id} - P), \ Q(H-\lambda)(\mathrm{Id} - P) \in \Psi_{3sc}^{-\infty,\infty}(X); \tag{17.4}$$

then the standard argument gives (17.2). Moreover, again writing

$$\langle v, [Q, H]v\rangle = \langle Pv, [Q, H]Pv\rangle + \langle v, (\mathrm{Id} - P^*)[Q, H]Pv\rangle + \langle v, [Q, H](\mathrm{Id} - P)v\rangle, \tag{17.5}$$

and similarly with the right hand side of (17.2), we have

$$|\langle v, [Q, H]v\rangle| \le C\big(\|Pv\|_{H_{sc}^{0,l}(X)} + \|v\|_{H_{sc}^{m',l'}(X)}\big)^2, \tag{17.6}$$

$$\begin{aligned}|\langle v, Q(H-\lambda)v\rangle| \le C\big(&\|Pv\|_{H_{sc}^{0,l}(X)} + \|v\|_{H_{sc}^{m',l'}(X)}\big)\\ &\cdot \big(\|P(H-\lambda)v\|_{H_{sc}^{-1,l+1}(X)} + \|v\|_{H_{sc}^{m',l'}(X)}\big).\end{aligned} \tag{17.7}$$

Thus, by continuity, it suffices to show that there exists a sequence v_s in $H_{sc}^{0,l+1}(X)$ such that $v_s \to v$ in $H_{sc}^{m',l'}(X)$, $Pv_s \to Pv$ in $H_{sc}^{0,l}(X)$ and $P(H-\lambda)v_s \to P(H-\lambda)v$ in $H_{sc}^{-1,l+1}(X)$. But now consider $\Lambda_s = (1 + sx^{-1})^{-1}$, and let $v_s = \Lambda_s v$ for $s \in [0,1]$. For $s > 0$, $Pv_s \in H_{sc}^{0,l+1}(X)$, and $Pv_s \to Pv$ in $H_{sc}^{0,l}(X)$. Moreover, we can also choose P' such that $P'v \in H_{sc}^{0,l}(X)$ and $\mathrm{WF}'_{3sc}(P' - \mathrm{Id}) \cap \mathrm{WF}'_{3sc}(P) = \varnothing$. Hence,

$$P(H-\lambda)v_s = \Lambda_s P(H-\lambda)v + [P(H-\lambda), \Lambda_s]P'v + [P(H-\lambda), \Lambda_s](\mathrm{Id} - P')v. \tag{17.8}$$

Now, $\Lambda_s \to \mathrm{Id}$ strongly on $H_{sc}^{0,l+1}(X)$, so the first term converges to $P(H-\lambda)v$ in $H_{sc}^{0,l+1}(X)$ as $s \to 0$. Also, $[P(H-\lambda), \Lambda_s] \to 0$ strongly in

$$\mathcal{B}(H_{sc}^{0,l}(X), H_{sc}^{-1,l+1}(X)),$$

so the second term converges to 0 in $H_{\mathrm{sc}}^{-1,l+1}(X)$. Finally,

$$[P(H-\lambda),\Lambda_s](\mathrm{Id}-P') \to 0 \text{ strongly in } \mathcal{B}(H_{\mathrm{sc}}^{0,l'}(X), H_{\mathrm{sc}}^{-1,l+1}(X)),$$

so the last term also converges to 0 in $H_{\mathrm{sc}}^{-1,l+1}(X)$. This shows that (17.2) indeed holds if we just assume that $Pv \in H_{\mathrm{sc}}^{0,l}(X)$, $P(H-\lambda)v \in H_{\mathrm{sc}}^{0,l+1}(X)$. □

First we deal with the general case; the improved statements under the additional assumptions, (15.1) or (15.2), follow at once from the propagation results of the previous chapters. For $\tau_0 \in \mathbb{R}$ let

$$T_{\mathrm{ff}}^{\pm}(\tau_0) = \{(z,\tau,\nu) \in \Sigma_{\mathrm{ff}}(H-\lambda) :\ \pm\tau \geq \pm\tau_0\}. \tag{17.9}$$

If the additional assumptions hold, then we can use $R^- \cap (\Sigma_t(\lambda) \cup \Sigma_b(\lambda))$ instead of $T_{\mathrm{ff}}^-(-\lambda^{1/2})$ in (17.10) and (17.11) in the statement of the following lemma.

Lemma 17.2. — *Suppose that H is as in* (11.11), $\lambda > 0$. *Suppose also that*

$$\mathrm{WF}_{3\mathrm{sc,mf}}^{m,l}(u) \cap R_\lambda^- = \varnothing, \qquad \mathrm{WF}_{3\mathrm{sc,ff}}^{m,l}(u) \cap T_{\mathrm{ff}}^-(-\lambda^{1/2}) = \varnothing \tag{17.10}$$

for some $m \in \mathbb{R}$ and $l > -1/2$, and $(H-\lambda)u \in \dot{\mathcal{C}}^\infty(X)$. Then

$$\mathrm{WF}_{3\mathrm{sc,mf}}(u) \cap R_\lambda^- = \varnothing, \qquad \mathrm{WF}_{3\mathrm{sc,ff}}(u) \cap T_{\mathrm{ff}}^-(-\lambda^{1/2}) = \varnothing. \tag{17.11}$$

The same result holds with R_λ^- and $T_{\mathrm{ff}}^-(-\lambda^{1/2})$ replaced by R_λ^+ and $T_{\mathrm{ff}}^+(\lambda^{1/2})$ respectively.

Proof. — Assume iteratively that (17.11) holds for $\mathrm{WF}_{3\mathrm{sc}}^{0,l}(u)$ where $l > -1/2$; we want to show that it holds when l is replaced by $l+1/2$. Note that by our initial assumption the claim holds for some $l > -1/2$.

With $\varepsilon \in (0, \lambda^{1/2}/3)$ small, let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ be supported in $(-\infty, -\lambda^{1/2}+2\varepsilon)$, identically 1 in $(-\infty, -\lambda^{1/2}+\varepsilon)$ and $\chi' \leq 0$. Let $\psi_0 \in \mathcal{C}^\infty(X)$ be supported in a product neighborhood of ∂X, identically 1 near ∂X. Define

$$q = x^{-l-1/2}\chi(\tau)\psi_0 \geq 0; \tag{17.12}$$

q is a globally defined function on ${}^{\mathrm{sc}}T^*X$ and on $\operatorname{supp} q$, $\tau \leq -\lambda^{1/2}+2\varepsilon < 0$. Thus, $Q\psi(H) \in \Psi_{3\mathrm{sc}}^{-\infty,-l-1/2}(X)$. Near ${}^{\mathrm{sc}}T^*_{\partial X}X$ we have

$${}^{\mathrm{sc}}H_g q = 2(-(l+1/2)\tau\chi(\tau) - h\chi'(\tau))x^{-l-1/2} \geq 0. \tag{17.13}$$

Let $f = x^{l+1}q|_{{}^{\mathrm{sc}}T^*_C X}$; since f is independent of μ, it can be regarded as a function on $W^\perp$. Now, let $\psi \in \mathcal{C}_c^\infty((\lambda/2, 2\lambda))$ supported near λ, so that in a neighborhood of $\operatorname{supp}\psi(g) \cap \operatorname{supp}\chi'(\tau)$ we have $h > \delta$; here $\delta > 0$ is just some fixed constant. This can be arranged since on $\Sigma_{\Delta-\lambda}$, $h = \lambda - \tau^2$ and on $\operatorname{supp}\chi'(\tau)$, $\tau \in [-\lambda^{1/2}+\varepsilon, -\lambda^{1/2}+2\varepsilon]$. Thus, with

$$c_2 = 2\inf h|_{\operatorname{supp}\psi(g)\cap\operatorname{supp}\chi'(\tau)} > 0, \tag{17.14}$$

and

$$c_1 = 2(l+1/2)(\lambda^{1/2}-2\varepsilon) > 0, \tag{17.15}$$

we have

$$x^{l+1/2}\big({}^{\mathrm{sc}}H_g q|_{\operatorname{supp}\psi(g)}\big) \geq f^\flat = c_1\chi(\tau) - c_2\chi'(\tau) > 0. \tag{17.16}$$

Hence,

$$j_{\mathrm{sc},1,-l+1/2}(-i[Q\psi(\Delta),\Delta]) = x^{l+1/2}\big({}^{\mathrm{sc}}H_g q\big)\psi(g) \geq f^\flat\psi(g). \tag{17.17}$$

Note in particular that q is independent of $\overline{\mu} = (\mu, \nu)$,

$$x^{l+1/2} q \leq C_1 f^\flat \tag{17.18}$$

and in the standard local coordinates near C

$$|d(x^{l+1/2}q)| \leq C_2 f^\flat \tag{17.19}$$

corresponding to (14.24), (14.21) and (14.23). Using Corollary 13.4 and the argument of Proposition 14.1 we see that with $Q = q_L(q)$

$$R_1(\xi) = -i\widehat{[Q,H]}_{\mathrm{ff},-l+1/2}\widehat{\psi(H)}_{\mathrm{ff}} - f^\flat\,\widehat{\psi(H)}_{\mathrm{ff}} \in \Psi_{\mathrm{sc}}^{-\infty,1}(\mathbb{S}^n_+), \tag{17.20}$$

$$\|R_1(\xi)\|_{\mathcal{B}(L^2_{\mathrm{sc}}(\mathbb{S}^n_+),H^{1,1}_{\mathrm{sc}}(\mathbb{S}^n_+))} \leq C' f^\flat(z,\tau,\nu). \tag{17.21}$$

When $\lambda - (\tau^2 + |\nu|_z^2)$ is not an eigenvalue of H_{ff}, we can follow the proof of Proposition 14.1 after (14.40) to conclude that for $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ supported sufficiently close to λ we have

$$-i\big(\phi(H)\widehat{[Q^*Q,H]}\phi(H)\big)_{\mathrm{ff},-2l} \geq (2-\varepsilon')ff^\flat\phi(H)^2. \tag{17.22}$$

If $\lambda - (\tau^2 + |\nu|_z^2)$ is an eigenvalue of H_{ff}, we can follow the proof of Proposition 15.2 starting from (15.51). We need not make use of (15.2) since q is independent of ν, and so the term $(D_\nu f)(\partial_z \widehat{H}_{\mathrm{ff}})$ automatically vanishes in (13.38). This proves (17.22) in this setting too. We can then apply the standard compactness argument to show that ϕ can be chosen to be independent of $\xi \in W^\perp$. Of course, at mf the analog of (17.22) holds automatically. Now note that

$$-i[(x+r)^{-1/2}, H] - q_L\big(x\,{}^{\mathrm{sc}}H_g(x+r)^{-1/2}\big) \in \Psi^{1,3/2}_{3\mathrm{scc}}(X) \tag{17.23}$$

uniformly for $r \in (0,1)$, and

$${}^{\mathrm{sc}}H_g(x+r)^{-1/2} = -\tau x(x+r)^{-3/2} \tag{17.24}$$

which is positive on $\operatorname{supp} q$. Hence we have shown that with $Q_r = Q\psi(H)(x+r)^{-1/2}$ where $\phi \equiv 1$ on $\operatorname{supp}\psi$,

$$-i[Q_r^*Q_r, H] \geq \psi(H)(x+r)^{-1/2}B^2(x+r)^{-1/2}\psi(H) + E_r^2 + F_r \tag{17.25}$$

where $B \in \Psi^{-\infty,-l}_{3\mathrm{sc}}(X)$ is self-adjoint, $F_r \in \Psi^{-\infty,-2l}_{3\mathrm{scc}}(X)$ uniformly bounded, $E_r \in \Psi^{-\infty,-l}_{3\mathrm{scc}}(X)$ for $r > 0$ and it is uniformly bounded in $\Psi^{-\infty,-l-1/2}_{3\mathrm{scc}}(X)$ (and it is self-adjoint). Note also that $Q_r \in \Psi^{-\infty,-l-1/2}_{3\mathrm{scc}}(X)$ for $r > 0$ and it is uniformly bounded in $\Psi^{-\infty,-l-1}_{3\mathrm{scc}}(X)$.

Apply now (17.2) with u in place of v, $Q_r^* Q_r$ in place of Q, and use (17.25). Thus, we see that for $r > 0$

$$\|B(x+r)^{-1/2}\psi(H)u\|^2 \leq |\langle u, F_r u\rangle| + 2\,|\mathrm{Im}\langle u, Q_r^* Q_r(H-\lambda)u\rangle|\,. \tag{17.26}$$

Letting $r \to 0$ keeps the right hand side bounded, and $B(x+r)^{-1/2}\psi(H)u \to Bx^{-1/2}\psi(H)u$ in $H_{\mathrm{sc}}^{0,-1/2}(X)$, so it follows that $Bx^{-1/2}\psi(H)u \in L^2_{\mathrm{sc}}(X)$. Noting that $x^{-l-1/2}(\mathrm{Id}-\psi(H)) + Bx^{-1/2}\psi(H)$ has an invertible indicial operator where $f > 0$ by (17.22) and $(\mathrm{Id}-\psi(H))u \in \dot{\mathcal{C}}^\infty(X)$ shows that the set where $f > 0$ is disjoint from $\mathrm{WF}_{3\mathrm{sc,ff}}^{0,l+1/2}(u)$, which provides the iterative step in the proof. □

We can also analyze propagation in τ in the region $\tau \in (-\lambda^{1/2}, \lambda^{1/2})$. Of course, we have the detailed picture at $\Sigma_n(\lambda)$ in general, but at $\Sigma_t(\lambda) \cup \Sigma_b(\lambda)$ only if either (15.1) or (15.2) is satisfied. For $\tau_0 \in \mathbb{R}$ we now introduce similarly to (17.9)

$$T^{\pm}(\tau_0) = \{(\overline{y}, \tau, \overline{\mu}) \in \Sigma_{\Delta-\lambda} :\ \pm\tau \geq \pm\tau_0\}. \tag{17.27}$$

Lemma 17.3. — *Suppose that H satisfies* (11.11), $\lambda > 0$. *Suppose also that for some* $\tau_0 \in (-\lambda^{1/2}, \lambda^{1/2})$

$$\mathrm{WF}_{3\mathrm{sc,mf}}(u) \cap T^-(\tau_0) = \varnothing, \qquad \mathrm{WF}_{3\mathrm{sc,ff}}(u) \cap T^-_{\mathrm{ff}}(\tau_0) = \varnothing \tag{17.28}$$

and $(H-\lambda)u \in \dot{\mathcal{C}}^\infty(X)$. *Then for any* $\tau_0' \in (-\lambda^{1/2}, \lambda^{1/2})$ *we have*

$$\mathrm{WF}_{3\mathrm{sc,mf}}(u) \cap T^-(\tau_0') = \varnothing, \qquad \mathrm{WF}_{3\mathrm{sc,ff}}(u) \cap T^-_{\mathrm{ff}}(\tau_0') = \varnothing. \tag{17.29}$$

The same result holds with T^- and T^-_{ff} replaced by T^+ and T^+_{ff} respectively.

Proof. — This is a simple one-variable version of the propagation theorems. Thus, we only sketch the proof. We let $\chi_0 \in \mathcal{C}^\infty(\mathbb{R})$ be $\chi_0(t) = \exp(-1/t)$ for $t > 0$, $\chi_0(t) = 0$ for $t \leq 0$, and also choose $\chi_1 \in \mathcal{C}^\infty(\mathbb{R}; [0,1])$ be 0 on $(-\infty, 0]$, 1 on $[1, \infty)$. For $\delta > 0$ small, $A > 0$ large, define

$$\widetilde{q} = \chi_0(A^{-1}(\tau_0' + \delta - \tau))\chi_1((\tau - \tau_0)/\delta + 2). \tag{17.30}$$

Then we proceed just as in the proof of Proposition 14.1 to obtain a positive commutator estimate and prove this lemma. □

Remark 17.4. — This one-variable propagation result follows easily from the methods of Gérard, Isozaki and Skibsted in **[10]** in the setting of Euclidian many-body scattering, with an appropriate notion of wavefront set.

Corollary 17.5. — *Suppose that H satisfies* (11.11), $\lambda > 0$, $u \in H_{\mathrm{sc}}^{m,l}(X)$, $l > -1/2$, *and* $(H-\lambda)u \in \dot{\mathcal{C}}^\infty(X)$. *Then* $u \in \dot{\mathcal{C}}^\infty(X)$.

Proof. — By Lemma 17.2

$$\mathrm{WF}_{3\mathrm{sc,mf}}(u) \cap \left(T^-(-\lambda^{1/2}) \cup T^+(\lambda^{1/2})\right) = \varnothing, \tag{17.31}$$

$$\mathrm{WF}_{3\mathrm{sc,ff}}(u) \cap \left(T^-_{\mathrm{ff}}(-\lambda^{1/2}) \cup T^+_{\mathrm{ff}}(\lambda^{1/2})\right) = \varnothing. \tag{17.32}$$

By Lemma 17.3 and the closedness of the wave front set we conclude that

$$\mathrm{WF}_{3\mathrm{sc,mf}}(u) \cap \Sigma_{\Delta-\lambda} = \varnothing, \ \mathrm{WF}_{3\mathrm{sc,ff}}(u) \cap \Sigma_{\mathrm{ff}}(H-\lambda) = \varnothing. \tag{17.33}$$

Combining this with Proposition 11.2 shows that $\mathrm{WF}_{3\mathrm{sc,mf}}(u)$ and $\mathrm{WF}_{3\mathrm{sc,ff}}(u)$ vanish, hence $\psi(H)u \in \dot{\mathcal{C}}^\infty(X)$ if $\psi \in \mathcal{C}_c^\infty(X)$. Taking $\psi \equiv 1$ near λ we also have $(1-\psi(H))u \in \dot{\mathcal{C}}^\infty(X)$ since $(H-\lambda)u \in \dot{\mathcal{C}}^\infty(X)$, so we conclude that $u \in \dot{\mathcal{C}}^\infty(X)$. □

As mentioned at the beginning of this chapter, we can extend the result of Froese and Herbst on the absence of positive eigenvalues to the general geometric setting. This is done in Appendix B; here we only state the result.

***Theorem 17.6* (cf. Froese and Herbst [8, Corollary 1.4]).** — *Let H be as in (11.11) and let $\lambda > 0$. Then $(H-\lambda)u = 0$, $u \in H_{\mathrm{sc}}^{m,l}(X)$ for some $m \in \mathbb{R}$ and for some $l > -1/2$ implies that $u = 0$. In particular, H has no positive eigenvalues.*

We now prove a 'rough' regularity theorem near the radial sets.

Lemma 17.7. — *Let H be as in (11.11) and let $\lambda > 0$. Suppose that*

$$\mathrm{WF}_{3\mathrm{sc,mf}}(u) \subset R_\lambda^+, \qquad \mathrm{WF}_{3\mathrm{sc,ff}}(u) \subset T_{\mathrm{ff}}^+(\lambda^{1/2}), \tag{17.34}$$

and $(H-\lambda)u \in \dot{\mathcal{C}}^\infty(X)$. Then $u \in H_{\mathrm{sc}}^{m,l}(X)$ for all $m \in \mathbb{R}$ and $l < -1/2$. The same result holds with R_λ^+ and $T_{\mathrm{ff}}^+(\lambda^{1/2})$ replaced by R_λ^- and $T_{\mathrm{ff}}^-(-\lambda^{1/2})$ respectively.

Proof. — The proof proceeds similarly to that of Lemma 17.2. Thus, we assume that $u \in H_{\mathrm{sc}}^{m,l}(X)$, $l < -1$, and we proceed to show it when l is replaced by $l + 1/2$. Let $\varepsilon \in (0, \lambda^{1/2}/3)$, $\chi \in \mathcal{C}^\infty(\mathbb{R})$ supported in $(\lambda^{1/2} - 2\varepsilon, \infty)$, identically 1 in $(\lambda - \varepsilon, \infty)$, $\chi' \geq 0$. Also let $\psi_0 \in \mathcal{C}^\infty(X)$ be supported in a product neighborhood of ∂X, identically 1 near ∂X. We define

$$q = x^{-l-1}\chi(\tau)\psi_0 \geq 0. \tag{17.35}$$

Now, however, near ${}^{\mathrm{sc}}T^*_{\partial X}X$ we have

$${}^{\mathrm{sc}}H_g q = -2(\tau(l+1)\chi(\tau) + h\chi'(\tau))x^{-l-1}, \tag{17.36}$$

so the two terms have opposite signs. However, $\chi'(\tau)$ is supported in

$$\tau \in (\lambda^{1/2} - 2\varepsilon, \lambda^{1/2} - \varepsilon],$$

i.e. in the region where u does not have wave front set by (17.34). We have

$$x^{l+1}\, {}^{\mathrm{sc}}H_g q|_{\tau \geq \lambda^{1/2} - 3/4\varepsilon} \geq f^\flat = -2(l+1)(\lambda^{1/2} - \varepsilon). \tag{17.37}$$

On the set $\{\tau \geq \lambda^{1/2} - 3/4\varepsilon\}$, χ' vanishes, so we have $x^{l+1}q \leq C_1 f^\flat$, $d(x^{l+1}q) = 0$ in this region. In addition, for $r > 0$

$${}^{\mathrm{sc}}H_g(1 + r/x)^{-1} = 2\tau x r(x+r)^{-2} \tag{17.38}$$

is positive on supp q. Correspondingly, using the arguments of Lemma 17.2 following (17.20) we see that with $Q_r = q_L(q)(1+r/x)^{-1}\phi(H)$, $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$

$$-i[Q_r^* Q_r, H] = B_r^2 + E_r + F_r \tag{17.39}$$

where now $B_r \in \Psi_{3\mathrm{sc}}^{-\infty,-l+1/2}(X)$ for $r > 0$, bounded in $\Psi_{3\mathrm{scc}}^{-\infty,-l-1/2}(X)$, $E_r \in \Psi_{3\mathrm{sc}}^{-\infty,-2l+1}(X)$ has $\mathrm{WF}'_{3\mathrm{sc}}$ in $\tau \le \lambda - 3/4\varepsilon$ and is bounded in $\Psi_{3\mathrm{scc}}^{-\infty,-2l+1}(X)$, and $F_r \in \Psi_{3\mathrm{scc}}^{-\infty,-2l}(X)$ uniformly. Thus, we conclude that for $r > 0$

$$\|B_r u\|^2 \le |\langle u, (E_r + F_r)u\rangle| + 2|\langle u, Q_r^* Q_r (H-\lambda)u\rangle|. \tag{17.40}$$

Now the right hand side is bounded as $r \to 0$ as we have noted, so we have proved this lemma. □

We now prove that the conclusion of Theorem 17.6 also holds if $(H-\lambda)u = 0$ and one of the radial sets is missing from the wave front set of u. This only requires a simple additional commutator estimate which is very similar to Isozaki's proof in [**22**, Lemma 4.5].

Proposition 17.8. — *Suppose that H satisfies* (11.11), $\lambda > 0$. *Suppose also that $u \in \mathcal{C}^{-\infty}(X)$,*

$$\mathrm{WF}^{m,l}_{3\mathrm{sc,mf}}(u) \cap R_\lambda^- = \varnothing, \qquad \mathrm{WF}^{m,l}_{3\mathrm{sc,ff}}(u) \cap T_{\mathrm{ff}}^-(-\lambda^{1/2}) = \varnothing \tag{17.41}$$

for some $m \in \mathbb{R}$ and $l > -1/2$, and $(H-\lambda)u = 0$. Then $u = 0$. The same result holds with R_λ^- and $T_{\mathrm{ff}}^-(-\lambda^{1/2})$ replaced by R_λ^+ and $T_{\mathrm{ff}}^+(\lambda^{1/2})$ respectively.

Proof. — By Lemma 17.2 and 17.3 we see at once that

$$\mathrm{WF}_{3\mathrm{sc,mf}}(u) \subset R_\lambda^+, \qquad \mathrm{WF}_{3\mathrm{sc,ff}}(u) \subset T_{\mathrm{ff}}^+(\lambda^{1/2}). \tag{17.42}$$

By Lemma 17.7, $u \in H_{\mathrm{sc}}^{m',l'}(X)$ for any $m' \in \mathbb{R}$, $l' < -1/2$. Now let $l \in (-1/2, 0)$, and let $\phi \in \mathcal{C}^\infty(\mathbb{R})$ be 0 on $(-\infty, 1]$, 1 on $[2,\infty)$. For $r > 0$ let

$$\chi_r(x) = r^{-2l-1}\int_0^{x/r} \phi^2(s) s^{-2l-2}\, ds. \tag{17.43}$$

Thus, $\chi_r \in \mathcal{C}_c^\infty(\mathrm{int}(X))$ and

$$x^2 \partial_x \chi_r(x) = x^{-2l}\phi^2(x/r). \tag{17.44}$$

Now, by [**25**, Equation 3.7]

$$\Delta = (x^2 D_x)^2 + i(N-1)x^3 D_x + x^2 \Delta_h + x^3 \,\mathrm{Diff}^2_{\mathrm{b}}(X), \tag{17.45}$$

so

$$-i[\chi_r(x), H] = 2x^{-2l}\phi^2(x/r)(x^2 D_x) + F'_r \tag{17.46}$$

where F'_r is bounded in $\Psi^{1,-2l+1}_{3scc}(X)$. Let $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$ supported close to λ, identically 1 near λ. Let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be 0 on $(-\infty, \lambda^{1/2}/3)$, 1 on $(2\lambda^{1/2}/3, \infty)$, and let $b = \rho(\tau)$, $B = q_L(b)\psi(H)$, $E = q_L(1-\rho^2(\tau))\psi(H)$. Thus, we see that

$$-i[\chi_r(x)\psi(H), H] = 2x^{-l}\phi(x/r)(B^2+E)\phi(x/r)x^{-l} + F_r \tag{17.47}$$

with $B \in \Psi^{-\infty,0}_{3sc}(X)$, $E \in \Psi^{-\infty,0}_{3sc}(X)$, $\mathrm{WF}'_{3sc}(E)$ disjoint from R^+_λ and T^+_{ff}, F_r bounded in $\Psi^{-\infty,-2l+1}_{3scc}(X)$. Now, for $r > 0$ we have

$$\langle u, [\chi_r(x), H]u\rangle = 2i \operatorname{Im}\langle u, \chi_r(x)(H-\lambda)u\rangle = 0. \tag{17.48}$$

Hence,

$$\|x^{-l}\phi(x/r)Bu\|^2 \le |\langle x^{-l}\phi(x/r)u, Ex^{-l}\phi(x/r)u\rangle| + |\langle u, F_r u\rangle|. \tag{17.49}$$

Taking into account (17.42) and $u \in H^{m',l'}_{sc}(X)$ for all $l' < -1/2$ we see that the right hand side stays bounded as $r \to 0$, so we conclude that $x^{-l}Bu \in L^2_{sc}(X)$, so by (17.42) we have $u \in H^{\infty,l}_{sc}(X)$. Since $l \in (-1/2, 0)$, we can apply Theorem 17.6 to conclude that $u \in \dot{\mathcal{C}}^\infty(X)$. Note that $\chi_r(x)$ is not bounded in $\Psi^{m',l'}_{3sc}(X)$ for any m' and l', so the place where we really used the assumption $(H-\lambda)u = 0$ was to eliminate the term on the right hand side of (17.48) from the right hand side of (17.49). □

We only state the improved version of this proposition; the preceding lemmas can be strenghtened similarly.

Corollary 17.9. — *Suppose that H is as in* (11.11), *$\lambda > 0$ and either one of* (15.1) *and* (15.2) *holds. Suppose also that $u \in \mathcal{C}^{-\infty}(X)$,*

$$\mathrm{WF}^{m,l}_{3sc,mf}(u) \cap R^-_\lambda = \varnothing, \qquad \mathrm{WF}^{m,l}_{3sc,\mathrm{ff}}(u) \cap R^- \cap (\Sigma_t(\lambda) \cup \Sigma_b(\lambda)) = \varnothing \tag{17.50}$$

for some $m \in \mathbb{R}$ and $l > -1/2$, and $(H-\lambda)u = 0$. Then $u = 0$. The same result holds with R^-_λ and R^- replaced by R^+_λ and R^+ respectively.

Proof. — We only have to prove that the second assumption of (17.50) implies the second assumption of (17.41). Since $\mathrm{WF}_{3sc,\mathrm{ff}}$ is closed, $R^- \cap (\Sigma_t(\lambda) \cup \Sigma_b(\lambda))$ has a neighborhood in $W^\perp$ which is disjoint from $\mathrm{WF}^{m,l}_{3sc,\mathrm{ff}}(u)$. But all integral curves of the vector field W in $\Sigma_t \cup \Sigma_b$ go to $R^- \cap (\Sigma_t(\lambda) \cup \Sigma_b(\lambda))$ as $t \to \infty$, so by Propositions 15.2 and 16.1 they are disjoint from $\mathrm{WF}^{m,l}_{3sc,\mathrm{ff}}(u)$. Hence, (17.41) is satisfied and we can apply Proposition 17.8. □

CHAPTER 18

THE RESOLVENT

In this chapter we examine the behavior of the resolvent applied to elements of $\dot{\mathcal{C}}^\infty(X)$ as the spectral parameter approaches the real axis. First we prove a simple global result on the wave front set of $u = (H - (\lambda \pm i0))^{-1} f$, $f \in \dot{\mathcal{C}}^\infty(X)$, assuming that

$$(H - (\lambda \pm it))^{-1} f \in L^\infty((0,1)_t; H^{0,s}_{\mathrm{sc}}(X)), \qquad s \in (-1,-1/2). \tag{18.1}$$

It is completely analogous to the theorem proved by Gérard, Isozaki and Skibsted in [**10**], and it is really just a version of the results of the previous chapter. Note that if one uses the Mourre estimate and the corresponding argument to estimate the resolvent, see [**32**], (18.1) is automatically satisfied. However, we do not need this; we prove the limiting absorption principle here similarly to Hörmander's proof in [**18**, Theorem 30.2.10]. For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we let

$$R(\lambda) = (H - \lambda)^{-1} \in \Psi^{-2,0}_{3\mathrm{sc}}(X). \tag{18.2}$$

Lemma 18.1. — *Suppose that H satisfies* (11.11) *and $\lambda > 0$. Let $f \in \dot{\mathcal{C}}^\infty(X)$ and $u_t = R(\lambda \pm it) f$, and assume that* (18.1) *holds. Then there exist $C > 0$ and $B \in \Psi^{0,0}_{3\mathrm{sc}}(X)$ such that $\widehat{B}_{\mathrm{mf}}$, $\widehat{B}_{\mathrm{ff}}$ are invertible for $\pm\tau \geq \lambda$, $\sigma_{3\mathrm{sc},0}(B)$ is invertible everywhere, and Bu_t is bounded in $\dot{\mathcal{C}}^\infty(X)$.*

Proof. — This is just a variation of the proof of Lemma 17.2, but we do not need the additional factor $(x+r)^{-1/2}$ since for $t > 0$ all pairings are defined. For the sake of definiteness we consider $u_t = R(\lambda - it) f$, $t \in (0,1)$. Let q be as in (17.12). Thus, (17.22) implies that with $l > -1/2$, $\widetilde{Q} = Q\psi(H)$

$$-i[\widetilde{Q}^* \widetilde{Q}, H] \geq B^2 + E^2 + F \tag{18.3}$$

where $B \in \Psi^{0,-l}_{3\mathrm{sc}}(X)$ is self-adjoint, $F \in \Psi^{0,-2l+1}_{3\mathrm{sc}}(X)$, $E \in \Psi^{0,-l}_{3\mathrm{sc}}(X)$ is self-adjoint, $\widetilde{Q} \in \Psi^{-\infty,-l-1/2}_{3\mathrm{sc}}(X)$.

Now, for $t > 0$, $u_t \in \dot{\mathcal{C}}^\infty(X)$, so

$$\langle u_t, [\widetilde{Q}^*\widetilde{Q}, H]u_t\rangle = 2i\,\mathrm{Im}\langle u_t, \widetilde{Q}^*\widetilde{Q}(H-(\lambda-it))u_t\rangle - 2it\|\widetilde{Q}u_t\|^2. \tag{18.4}$$

Hence,

$$\|Bu_t\|^2 \le 2\left|\mathrm{Im}\langle u_t, \widetilde{Q}^*\widetilde{Q}(H-(\lambda-it))u_t\rangle\right| + |\langle u_t, Fu_t\rangle| - 2t\|\widetilde{Q}u_t\|^2. \tag{18.5}$$

As $2t\|\widetilde{Q}u_t\|^2$ is nonnegative, it can be dropped. If $l = s + 1/2 > -1/2$, then the right hand side remains bounded as $t \to 0$, and we can conclude that $Bu_t \in L^\infty((0,1)_t; H^{0,l}_{\mathrm{sc}}(X))$, i.e. we improved the regularity given in (18.1) by 1/2. Note that we only needed $s > -1$ in (18.1). Thus, we can proceed iteratively as usual, thereby proving the proposition. □

We can also analyze the singularities of $R(\lambda \pm i0)$ at the opposite radial regions, i.e. where $\mp\tau \ge \lambda$. Of course, we expect that wave front set appears there, and correspondingly we prove a 'rougher' regularity result. This will only improve (18.1) by a little: it shows that s can be replaced by any $l < -1/2$.

Lemma 18.2. — *Suppose that H satisfies* (11.11) *and $\lambda > 0$. Let $f \in \dot{\mathcal{C}}^\infty(X)$ and $u_t = R(\lambda \pm it)f$ and suppose that* (18.1) *holds. Given $l < -1/2$, $m \in \mathbb{R}$, there exist $C > 0$ and $B \in \Psi^{0,0}_{3\mathrm{sc}}(X)$ such that $\widehat{B}_{\mathrm{mf}}$, $\widehat{B}_{\mathrm{ff}}$ are invertible for $\mp\tau \ge \lambda$, $\sigma_{3\mathrm{sc},0}(B)$ is invertible everywhere, and Bu_t is bounded in $H^{m,l}_{\mathrm{sc}}(X)$.*

Proof. — We again consider $u_t = R(\lambda - it)f$. The proof is very similar to that of Lemma 17.7. Thus, we let $\varepsilon \in (0, \lambda^{1/2}/3)$, $\chi \in \mathcal{C}^\infty(\mathbb{R})$ supported in $(\lambda^{1/2} - 2\varepsilon, \infty)$, identically 1 in $(\lambda - \varepsilon, \infty)$, $\chi' \ge 0$. We define q as in Lemma 17.7 as well, so with $\psi_0 \in \mathcal{C}^\infty(X)$ supported near ∂X, identically 1 in a smaller neighborhood of ∂X,

$$q = x^{-l-1/2}\chi(\tau)\psi_0 \ge 0. \tag{18.6}$$

Just as in the parameterless case, near ${}^{\mathrm{sc}}T^*_{\partial X}X$, we have

$${}^{\mathrm{sc}}H_g q = -2(\tau(l+1/2)\chi(\tau) + h\chi'(\tau))x^{-l-1/2}, \tag{18.7}$$

so the two terms have opposite signs. Again, $\chi'(\tau)$ is supported in

$$\tau \in (\lambda^{1/2} - 2\varepsilon, \lambda^{1/2} - \varepsilon].$$

By the previous lemma and the propagation results, which can be modified similarly to include the parameter t, we know that Pu_t is bounded in $\dot{\mathcal{C}}^\infty(X)$ if $\mathrm{WF}'_{3\mathrm{sc}}(P)$ does not meet $\tau > \lambda^{1/2} - \varepsilon/2$, so the second term in (18.7), applied to u_t, is automatically bounded in $\dot{\mathcal{C}}^\infty(X)$ as $t \to 0$. We have

$$x^{l+1/2}\,{}^{\mathrm{sc}}H_g q|_{\tau \ge \lambda^{1/2} - 3/4\varepsilon} \ge f^\flat = -(2l+1)(\lambda^{1/2} - \varepsilon). \tag{18.8}$$

On the set $\{\tau \ge \lambda^{1/2} - 3/4\varepsilon\}$, χ' vanishes, so we have $x^{l+1/2}q \le C_1 f^\flat$, $d(x^{l+1/2}q) = 0$ in this region. Correspondingly, using the arguments of Lemma 17.2 following (17.20) we see that with $Q_0 = q_L(q)\phi(H)$, $\phi \in C_c^\infty(\mathbb{R})$

$$-i[Q_0^* Q_0, H] = B_0^2 + E_0 + F_0 \tag{18.9}$$

where now $B_0 \in \Psi_{3sc}^{-\infty,-l}(X)$, $E_0 \in \Psi_{3sc}^{-\infty,-2l}(X)$ has WF'_{3sc} in $\tau \leq \lambda - 3/4\varepsilon$, and $F_0 \in \Psi_{3sc}^{-\infty,-2l+1}(X)$. Thus, we conclude that for $t > 0$

$$\|Bu_t\|^2 \leq |\langle u_t, (E_0 + F_0)u_t\rangle| + 2|\langle u_t, Q_0^* Q_0(H - (\lambda - it))u_t\rangle| - 2t\|Q_0 u_t\|^2. \tag{18.10}$$

Now the right hand side is bounded as $t \to 0$ as we have noted (the last term can be dropped again), so we have proved this lemma. □

We can now state the weak form of the limiting absorption principle, namely that $R(\lambda \pm it)$, $t > 0$, has a limit as $t \to 0$. We again state this in the general case, but just as in Corollary 17.9 we can replace $T_{\mathrm{ff}}^{\pm}(\pm\lambda^{1/2})$ by $R^{\pm} \cap (\Sigma_t(\lambda) \cup \Sigma_b(\lambda))$ in (18.11) if either (15.1) or (15.2) is satisfied.

Theorem 18.3. — *Suppose that H satisfies* (11.11), *$\lambda > 0$. Let $f \in \dot{C}^\infty(X)$, $u_t^{\pm} = R(\lambda \mp it)f$, $t > 0$. Then $u_t^{\pm}$ has a limit $u_{\pm} = R(\lambda \mp i0)f$ in $H_{sc}^{m,l}(X)$, $l < -1/2$, as $t \to 0$. In addition,*

$$\mathrm{WF}_{3sc,mf}(u_{\pm}) \subset R_{\lambda}^{\pm}, \qquad \mathrm{WF}_{3sc,ff}(u_{\pm}) \subset T_{\mathrm{ff}}^{\pm}(\pm\lambda^{1/2}). \tag{18.11}$$

Proof. — We consider $u_t = R(\lambda - it)f$ only and we follow the proof of [**25**, Proposition 14]. So suppose that $\delta > 0$, and u_t is not bounded in $H_{sc}^{0,-1/2-\delta}(X)$ as $t \to 0$. Hence we can take a sequence t_j, $j \in \mathbb{N}$, $t_j \to 0$, such that $\|u_{t_j}\|_{H_{sc}^{0,-1/2-\delta}(X)} \to \infty$. Now consider the sequence

$$v_j = \frac{u_{t_j}}{\|u_{t_j}\|_{H_{sc}^{0,-1/2-\delta}(X)}}. \tag{18.12}$$

Thus, v_j is bounded in $H_{sc}^{0,-1/2-\delta}(X)$. Taking some $m < 0$, $l < -1/2-\delta$, we can pick a subsequence v'_j of v_j which converges in $H_{sc}^{m,l}(X)$, since the inclusion of $H_{sc}^{0,-1/2-\delta}(X)$ to $H_{sc}^{m,l}(X)$ is compact; we let v be the limit. Note that $(H-\lambda)v'_j \to 0$ in distributions, so $(H - \lambda)v = 0$. We know by the previous lemmas (together with the propagation theorems) that Bv'_j is bounded in $\dot{C}^\infty(X)$ if $\mathrm{WF}'_{3sc}(B)$ is in $\tau \leq \lambda^{1/2} - \varepsilon$. Consequently, v satisfies the assumptions of Proposition 17.8, i.e. $v = 0$. This, however, contradicts $v'_j \to v$, $\|v'_j\|_{H_{sc}^{0,-1/2-\delta}(X)} = 1$. Thus, u_t is bounded in $H_{sc}^{0,-1/2-\delta}(X)$ for any $\delta > 0$ as $t \to 0$. Again, we can take a convergent subsequence in $H_{sc}^{m,l}(X)$, $m < 0$, $l < -1/2$, and argue as above that the difference of the limit of two such convergent subsequences must vanish. This argument also proves (18.11). □

Remark 18.4. — A slight modification of Lemmas 18.1 and 18.2 which allows f to depend on t as long as it stays bounded in $H_{sc}^{0,s}(X)$, $s > 1/2$, can be used as in Hörmander's proof of [**18**, Theorem 30.2.10] to prove that $R(\lambda \pm i0)$ is a bounded operator from $H_{sc}^{0,1/2+\varepsilon}(X)$ to $H_{sc}^{0,-1/2-\varepsilon}(X)$ for any $\varepsilon > 0$.

As a corollary of this theorem we note that $R(\lambda \pm i0)v$ also exists for distributions v which satisfy a wave front set condition. Again, if either (15.1) or (15.2) holds then $T^{\pm}_{\mathrm{ff}}(\pm\lambda^{1/2})$ can be replaced by $R^{\pm} \cap (\Sigma_t(\lambda) \cup \Sigma_b(\lambda))$.

Corollary 18.5. — *Suppose that H satisfies (11.11), $\lambda > 0$. Suppose also that $v \in \mathcal{C}^{-\infty}(X)$, and let $u_t^{\pm} = R(\lambda \mp it)v$, $t > 0$. If in addition v satisfies*

$$\mathrm{WF}_{3\mathrm{sc,mf}}(v) \cap R^{\mp}_{\lambda} = \varnothing, \qquad \mathrm{WF}_{3\mathrm{sc,ff}}(v) \cap T^{\mp}_{\mathrm{ff}}(\mp\lambda^{1/2}) = \varnothing, \tag{18.13}$$

then u_t has a limit $u_{\pm} = R(\lambda \mp i0)v$ in $\mathcal{C}^{-\infty}(X)$, as $t \to 0$. In addition,

$$\mathrm{WF}_{3\mathrm{sc,mf}}(u_{\pm}) \cap R^{\mp}_{\lambda} = \varnothing, \qquad \mathrm{WF}_{3\mathrm{sc,ff}}(u_{\pm}) \cap T^{\mp}_{\mathrm{ff}}(\mp\lambda^{1/2}) = \varnothing. \tag{18.14}$$

Furthermore, $u_{\pm}$ are the unique elements of $\dot{\mathcal{C}}^{\infty}(X)$ satisfying $(H - \lambda)u_{\pm} = v$ and (18.14).

Proof. — For $t > 0$ we have $R(\lambda \pm it)^{\dagger} = R(\lambda \pm it)$, $\dagger$ denoting transpose, so for $f \in \dot{\mathcal{C}}^{\infty}(X)$

$$v(R(\lambda \pm it)f) = (R(\lambda \pm it)v)(f). \tag{18.15}$$

(Recall that the distributional pairing is the real pairing, not the complex (i.e. L^2) one.) Since under our assumptions the left hand side converges as $t \to 0$ due to (18.11), so we can define the limit $R(\lambda \pm i0)v$ in $\mathcal{C}^{-\infty}(X)$ using this equation. Here we need to know the continuity implied by Remark 18.4. In fact, using this remark and the analog of Lemma 18.1 stated for $f \in H^{m,s}_{\mathrm{sc}}$, $s > 1/2$, we can easily conclude that

$$\mathrm{WF}^{\infty,r}_{3\mathrm{sc,mf}}(u_{\pm}) \cap R^{\mp}_{\lambda} = \varnothing, \qquad \mathrm{WF}^{\infty,r}_{3\mathrm{sc,ff}}(u_{\pm}) \cap T^{\mp}_{\mathrm{ff}}(\mp\lambda^{1/2}) = \varnothing. \tag{18.16}$$

for $r < -1/2$. Once we know the existence of such a limit satisfying (18.16), we can use a slightly stronger version of the uniform propagation estimates (in so far as only microlocal assumptions on v are used) to conclude (18.14). In fact, we can use the commutator formula in Lemma 18.1 and simply note that we only need the regularity of f on $\mathrm{WF}'_{3\mathrm{sc}}(\widetilde{Q})$. Finally, the uniqueness follows from taking the difference of two such distributions and using Proposition 17.8. □

We can also discuss the asymptotic expansion of $R(\lambda \pm i0)f$, $f \in \dot{\mathcal{C}}^{\infty}(X)$ away from C. This result was obtained in **[38]** in the case of Euclidian scattering covering the same class of potentials as in this paper, and it used the paper **[10]** of Gérard, Isozaki and Skibsted to show that

$$\mathrm{WF}_{\mathrm{sc}}(R(\lambda \mp i0)f) \cap {}^{\mathrm{sc}}T^*_{\partial X \setminus C}X \subset R^{\pm}_{\lambda}, \tag{18.17}$$

after which a local version of Melrose's original argument **[25**, Proposition 12**]** implied the existence of the asymptotic expansions. Since the necessary fact from **[10]** has been proved above in Theorem 18.3, the proof from **[38]** applies verbatim. For the statement of the result it is convenient to renormalize the resolvent. Thus, we let

$$\widetilde{R}(\pm\lambda) = R(\lambda^2 \mp i0), \qquad \lambda > 0. \tag{18.18}$$

To deal with the case of long-range interactions we make two definitions. If $V \in \rho_{\mathrm{mf}}\mathcal{C}^\infty([X;C])$, then we can can write $V = xV'$, $V' \in \mathcal{C}^\infty(X \setminus C)$. We let

$$\alpha_\lambda = (2\lambda)^{-1}V'|_{\partial X \setminus C} \in \mathcal{C}^\infty(\partial X \setminus C), \qquad \lambda \in \mathbb{R} \setminus \{0\}. \tag{18.19}$$

We also introduce an index set

$$\mathcal{K} = \{(m,p) : \ m, p \in \mathbb{N}, \ p \leq 2m\}. \tag{18.20}$$

For a description of the space $\mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X \setminus C)$ of polyhomogeneous conormal distributions to the boundary, $\partial X \setminus C$, see **[24]**. Essentially, $u \in \mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X \setminus C)$ means that u has a full asymptotic expansion in $x^m(\log x)^p$, $p \leq 2m$, $m \to \infty$, with smooth coefficients on $\partial X \setminus C$. We hence conclude:

Theorem 18.6. — *Suppose that $f \in \dot{\mathcal{C}}^\infty(X)$, H as in* (11.11), *$\lambda \in \mathbb{R} \setminus \{0\}$. Then $u = \widetilde{R}(\lambda)f$ has a full asymptotic expansion away from C as follows. If $V \in \rho^2_{\mathrm{mf}}\mathcal{C}^\infty([X;C])$ (short-range interaction) then*

$$e^{i\lambda/x}x^{-(N-1)/2}u \in \mathcal{C}^\infty(X \setminus C). \tag{18.21}$$

If $V \in \rho_{\mathrm{mf}}\mathcal{C}^\infty([X;C])$ (long-range interaction) then

$$e^{i\lambda/x}x^{i\alpha_\lambda-(N-1)/2}u \in \mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X \setminus C). \tag{18.22}$$

CHAPTER 19

THE SCATTERING MATRIX

In the concluding chapter of this paper we translate our results on the propagation of singularities of generalized eigenfunctions of H to describe the free-to-free (i.e. three-cluster to three-cluster) (part of the) scattering matrix of H. This part of the scattering matrix geometrically using the asymptotic expansion of Theorem 18.6 exactly in the same way as it was discussed in [**38**, Theorem 4.1]. The proof of that theorem involves the resolvent estimates of Gérard, Isozaki and Skibsted [**10**], Isozaki's uniqueness theorem [**22**, Theorem 1.2], and the construction of generalized eigenfunctions with arbitrary expansion, supported away from C, at one of the radial surfaces, which is again Melrose's construction [**25**, Proposition 12]. Since these have been proved in our context, in particular the uniqueness theorem is just Proposition 17.8, [**38**, Theorem 4.1] is also valid in this more general context. Namely, we have the following:

Theorem 19.1. — *Suppose that* H *is as in* (11.11), $\lambda \in \mathbb{R} \setminus \{0\}$, *and let* α_λ *and* $\mathcal{K}$ *be as in* (18.19) *and* (18.20). *Suppose also that either* (15.1) *or* (15.2) *holds. Then for* $a_0 \in \mathcal{C}_c^\infty(\partial X \setminus C)$ *there exists a unique* $u \in \mathcal{C}^{-\infty}(X)$ *such that*

$$(H - \lambda^2)u = 0, \qquad u = u_+ + u_-, \tag{19.1}$$

$$v_- = e^{-i\lambda/x} x^{-i\alpha_\lambda - (N-1)/2} u_- \in \mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X), \qquad v_-|_{\partial X} = a_0, \tag{19.2}$$

(19.3)
$$\mathrm{WF}_{\mathrm{3sc,mf}}(u_+) \cap R^{-\operatorname{sign}\lambda}_{\lambda^2} = \varnothing,\ \ \mathrm{WF}_{\mathrm{3sc,ff}}(u_+) \cap R^{-\operatorname{sign}\lambda} \cap (\Sigma_t(\lambda^2) \cup \Sigma_b(\lambda^2)) = \varnothing.$$

Moreover, there exists $f \in \dot{\mathcal{C}}^\infty(X)$ *such that* $u_\pm = \mp \widetilde{R}(\pm\lambda) f$. *In particular,* u_+ *has an asymptotic expansion as in Theorem 18.6. If* $V \in \rho_{\mathrm{mf}}^2 \mathcal{C}^\infty([X; C])$, *then* $\alpha_\lambda = 0$ *and* $\mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X)$ *can be replaced by* $\mathcal{C}^\infty(X)$.

Remark 19.2. — If neither (15.1) nor (15.2) holds, then this theorem is still true if we replace $R^{-\operatorname{sign}\lambda} \cap (\Sigma_t(\lambda^2) \cup \Sigma_b(\lambda^2))$ by $T_{\mathrm{ff}}^{-\operatorname{sign}\lambda}(-\lambda)$. This can be proved by the very same argument.

We can now define the free-to-free (three-cluster to three-cluster) scattering matrix as the operator relating the leading terms of $u_\pm$ on $\partial X \setminus C$.

Definition 19.3. — With the notation of Theorem 19.1, the free-to-free scattering matrix $S(\lambda)$, $\lambda \in \mathbb{R} \setminus \{0\}$, is defined as

$$S(\lambda) : \mathcal{C}_c^\infty(\partial X \setminus C) \to \mathcal{C}^\infty(\partial X \setminus C), \tag{19.4}$$

$$S(\lambda)a_0 = v_+|_{\partial X \setminus C}, \qquad v_+ = e^{i\lambda/x} x^{i\alpha_\lambda - (N-1)/2} u_+. \tag{19.5}$$

We also define the Poisson operator:

Definition 19.4. — With the notation of Theorem 19.1, $\lambda \in \mathbb{R} \setminus \{0\}$, the Poisson operator corresponding to free incoming data is the map

$$P(\lambda) : \mathcal{C}_c^\infty(\partial X \setminus C) \to \mathcal{C}^{-\infty}(X), \qquad P(\lambda)a_0 = u. \tag{19.6}$$

Thus, the Poisson operator associates to incoming data the unique generalized eigenfunction of H with eigenvalue λ^2 which has this 'λ-incoming part', and the scattering matrix maps the incoming data to the outgoing data. The Poisson operators $P(\lambda)$ and $P(-\lambda)$ are closely related.

Lemma 19.5. — *If $a_0 \in \mathcal{C}_c^\infty(\partial X \setminus C)$ then $P(-\lambda)\overline{a_0} = \overline{P(\lambda)a_0}$.*

Proof. — We can assume that $\lambda > 0$. Let $u = P(-\lambda)\overline{a_0}$. Thus, $(H - \lambda^2)u = 0$, $u = u_+ + u_-$,

$$\mathrm{WF}_{3\mathrm{sc,mf}}(u_+) \cap R^+_{\lambda^2} = \varnothing, \qquad \mathrm{WF}_{3\mathrm{sc,ff}}(u_+) \cap R^+ \cap (\Sigma_t(\lambda^2) \cup \Sigma_b(\lambda^2)) = \varnothing, \tag{19.7}$$

and u_- has an asymptotic expansion

$$v_- = e^{i\lambda/x} x^{i\alpha_\lambda - (N-1)/2} u_- \in \mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X), \qquad v_-|_{\partial X} = \overline{a_0}. \tag{19.8}$$

Now, taking the complex conjugate of u gives another generalized eigenfunction of H: $(H - \lambda^2)\overline{u} = 0$. Moreover, $\overline{u} = \overline{u_+} + \overline{u_-}$. Since $\overline{e^{if/x}} = e^{-if/x}$ if f is real valued, we see that

$$\mathrm{WF}_{3\mathrm{sc,mf}}(\overline{u_+}) \cap R^-_{\lambda^2} = \varnothing, \qquad \mathrm{WF}_{3\mathrm{sc,ff}}(\overline{u_+}) \cap R^- \cap \big(\Sigma_t(\lambda^2) \cup \Sigma_b(\lambda^2)\big) = \varnothing. \tag{19.9}$$

Moreover, the asymptotic expansion of $\overline{u_-}$ becomes

$$\overline{v_-} = e^{-i\lambda/x} x^{-i\alpha_\lambda - (N-1)/2} \overline{u_-} \in \mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X), \qquad \overline{v_-}|_{\partial X} = a_0. \tag{19.10}$$

By Theorem 19.1, the unique generalized eigenfunction of H with these properties is $P(\lambda)a_0$, so $P(\lambda)a_0 = \overline{P(-\lambda)\overline{a_0}}$, completing the proof of the lemma. □

In the case of two-body type scattering on X (i.e. $V \in x\mathcal{C}^\infty(X)$) the Poisson operator $P_0(\lambda)$ has been analyzed in detail by Melrose and Zworski in **[29]**, and they used it to conclude that the scattering matrix is a Fourier integral operator associated to the geodesic flow on ∂X at distance π. In this paper we have only proved simpler wave front set propagation estimates, so we cannot expect that we can draw such strong conclusions. Nevertheless, we are able to analyze the wave front set of the scattering matrix. First, however, we recall how the Poisson operator is constructed in **[29]**.

Thus, one constructs 'plane waves' starting at $\overline{y} = \overline{y}' \in \partial X$, and does so uniformly in $\overline{y}'$. For this note that $X \times \partial X$ is a manifold with boundary and we write the product coordinates on it as $(x, \overline{y}, \overline{y}')$. We can also use the product coordinates on ${}^{sc}T^*X$ near $\partial X \times \partial X$, namely they are just $(x, \overline{y}, \overline{y}', \tau, \overline{\mu}, \overline{\mu}')$. The construction microlocally near the initial point $\overline{y} = \overline{y}'$, i.e. near $(x, \overline{y}', \overline{y}', -\lambda, 0, \overline{\mu}') \in R_{\lambda^2}^{-\operatorname{sign}\lambda}$, is rather explicit. It is based on solving the eikonal equation and then the corresponding transport equations near $\overline{y} = \overline{y}'$. The simplicity is due to the fact that we are just dealing with a smooth Legendre submanifold of ${}^{sc}T^*(X \times \partial X)$ which has a simple parametrization. To proceed with the construction farther from $\overline{y}'$, Melrose and Zworski discuss Legendrian distibutions, and they use Legendre distributions associated to a pair of Legendre submanifolds with conic points to finish the construction near the outgoing radial surface, $R_{\lambda^2}^{\operatorname{sign}\lambda}$.

It would be harder to carry out the same program in our setting, though in the case of V vanishing to infinite order at mf this has been done by Hassell in **[13]**. Instead, we can use the initial part of the Melrose-Zworski construction to start plane waves at $\overline{y} = \overline{y}' \in \partial X \setminus C$, but we cut them off away from $R_{\lambda^2}^{-\operatorname{sign}\lambda}$ but before they hit ${}^{sc}T^*_C X$. This construction is described in Appendix A with the slight modification that we allow long-range potentials (V simply vanishing at mf). It is convenient to take $\lambda > 0$ in what follows; in general we just need to switch some signs.

Since in Appendix A we describe the global two-body type construction, we now indicate the modifications necessary to accommodate three-body scattering. So we fix a compact set $K \subset \partial X \setminus C$, and use the plane waves constructed in the Appendix for initial points near K, cut off before they hit ${}^{sc}T^*_C X$. Thus, let $\widetilde{V} \in x\mathcal{C}^\infty(X)$ be such that $\widetilde{V} = V$ in a neighborhood of K in X. Let $P_0(\lambda) : \mathcal{C}_c^{-\infty}(K) \to \mathcal{C}^{-\infty}(X)$, with kernel $K^\flat \in \mathcal{C}^{-\infty}(X \times \partial X; \pi_R^*\Omega)$, be the operator constructed in the Appendix for $\Delta + \widetilde{V}$ instead of H. Recall that $\sim'_+$ is the relation given by broken bicharacteristics between points in $\Sigma_{\Delta-\lambda^2}$ and $S^*\partial X$, defined in Definition 11.7. Thus, by Proposition A.1, and the remarks preceeding it about the cutoff ψ having support close to $\partial X \times \partial X$, we have for $u \in \mathcal{C}_c^{-\infty}(K)$

$$\begin{aligned}\mathrm{WF}_{sc}(P_0(\lambda)u) \subset &\{(\overline{y}, -\lambda, 0) : \ \overline{y} \in \operatorname{supp} u\} \\ &\cup \{\alpha \in \Sigma_{\Delta-\lambda^2} \setminus {}^{sc}T^*_C X : \ \exists \zeta \in \mathrm{WF}(u), \ \alpha \sim'_+ \zeta\},\end{aligned} \tag{19.11}$$

and correspondingly

$$\begin{aligned}(19.12)\quad &\mathrm{WF_{sc}}((\Delta+\widetilde{V}-\lambda^2)P_0(\lambda)u)\\ &\subset\{\alpha\in\Sigma_{\Delta-\lambda^2}\setminus(R^-_{\lambda^2}\cup {}^{\mathrm{sc}}T^*_C X):\ \exists\zeta\in\mathrm{WF}(u),\ \alpha\sim'_+\zeta\}.\end{aligned}$$

Moreover, if $u\in\mathcal{C}^\infty_c(K)$ then

$$v=e^{-i\lambda/x}x^{-i\alpha_\lambda-(N-1)/2}P_0(\lambda)u\in\mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X),\qquad v|_{\partial X}=u. \tag{19.13}$$

Now, as $V-\widetilde{V}\in\mathcal{C}^\infty([X;C])$, with $\mathrm{WF}'_{3\mathrm{sc}}(V-\widetilde{V})\cap{}^{\mathrm{sc}}T^*_K X=\varnothing$, (19.11) and (19.12) show that for $u\in\mathcal{C}^{-\infty}_c(K)$

(19.14)

$$\mathrm{WF_{sc}}((H-\lambda^2)P_0(\lambda)u)\subset\{\alpha\in\Sigma_{\Delta-\lambda^2}\setminus(R^-_{\lambda^2}\cup{}^{\mathrm{sc}}T^*_C X):\ \exists\zeta\in\mathrm{WF}(u),\ \alpha\sim'_+\zeta\}.$$

We can thus apply the outgoing resolvent, $\widetilde{R}(\lambda)$, to the error, $(H-\lambda^2)P_0(\lambda)u$; this is justified by Corollary 18.5.

Thus, for $u\in\mathcal{C}^\infty_c(\partial X\setminus C)$, $\operatorname{supp}u\subset K$, consider $P_0(\lambda)u$. We define

$$v=P_0(\lambda)u-\widetilde{R}(\lambda)(H-\lambda^2)P_0(\lambda)u. \tag{19.15}$$

Note first that by (19.14)

$$(H-\lambda^2)P_0(\lambda)u\in\dot{\mathcal{C}}^\infty(X). \tag{19.16}$$

Hence, the right hand side of (19.15) makes sense, and $(H-\lambda^2)v=0$,

$$\mathrm{WF_{3sc}}(\widetilde{R}(\lambda)(H-\lambda^2)P_0(\lambda)u)\cap\big(R^-_{\lambda^2}\cup(R^-\cap(\Sigma_b(\lambda^2)\cup\Sigma_t(\lambda^2)))\big)=\varnothing. \tag{19.17}$$

Therefore, we conclude that $P(\lambda)u-v$ is a generalized eigenfunction of H with no incoming wave front set, so by Corollary 17.9 it vanishes, i.e.

$$P(\lambda)u=P_0(\lambda)u-\widetilde{R}(\lambda)(H-\lambda^2)P_0(\lambda)u. \tag{19.18}$$

Since we have analyzed the propagation of singularities in terms of wave front sets, we can at once deduce the wave front relation of the Poisson operator.

Proposition 19.6. — *Suppose that H satisfies* (11.11), $\lambda\in\mathbb{R}\setminus\{0\}$. *Assume in addition that either* (15.1) *or* (15.2) *holds. Then the Poisson operator extends to a continuous linear map*

$$P(\lambda):\mathcal{C}^{-\infty}_c(\partial X\setminus C)\to\mathcal{C}^{-\infty}(X). \tag{19.19}$$

In addition, for $u\in\mathcal{C}^{-\infty}_c(\partial X\setminus C)$, $\lambda>0$,

$$\begin{aligned}(19.20)\quad \mathrm{WF_{3sc}}(P(\lambda)u)\subset&\{(\overline{y},-\lambda,0):\overline{y}\in\operatorname{supp}u\}\cup R^+_{\lambda^2}\cup(R^+\cap(\Sigma_b(\lambda^2)\cup\Sigma_t(\lambda^2)))\\ &\cup\{\alpha\in\Sigma_{\Delta-\lambda^2}:\ \exists\zeta\in\mathrm{WF}(u),\ \alpha\sim_+\zeta\}\\ &\cup\{\xi\in\Sigma_{\mathrm{ff}}(H-\lambda^2):\ \exists\zeta\in\mathrm{WF}(u),\ \xi\sim_+\zeta\}.\end{aligned}$$

If $\lambda<0$ this still holds with $R^+_{\lambda^2}$ and $R^-_{\lambda^2}$, R^+ and R^-, $\sim_+$ and $\sim_-$ interchanged.

Proof. — If $u \in \mathcal{C}_c^{-\infty}(\partial X \setminus C)$, then $P_0(\lambda)u$ is still defined, and it satisfies (19.11) and (19.14). Hence, $\widetilde{R}(\lambda)(H-\lambda^2)P_0(\lambda)u$ is defined by Corollary 18.5, and the right-hand side of (19.18) extends by continuity from $\mathcal{C}_c^\infty(\partial X \setminus C)$ to define $P(\lambda)u$.

Since $\mathrm{WF}_{3\mathrm{sc}}(P_0(\lambda)u)$ satisfies the statement of the proposition by (19.11), it suffices to consider $v = \widetilde{R}(\lambda)(H-\lambda^2)P_0(\lambda)u$. Thus, with $f = (H-\lambda^2)v$,

$$f = (H-\lambda^2)P_0(\lambda)u, \tag{19.21}$$

so $\mathrm{WF}_{\mathrm{sc}}(f)$ is estimated by (19.14). We can thus apply our propagation results, namely Propositions 14.1, 15.3 and 16.1, see also Corollary 14.2 and Proposition 15.4, to deduce bounds for $\mathrm{WF}_{3\mathrm{sc}}(v)$ which prove the proposition. □

Remark 19.7. — If C is totally geodesic but neither (15.1) nor (15.2) hold necessarily, then for $u \in \mathcal{C}_c^{-\infty}(\partial X \setminus C)$ we still have

$$\begin{aligned}\mathrm{WF}_{3\mathrm{sc}}(P(\lambda)u) \cap {}^{\mathrm{sc}}T^*_{\partial X\setminus C}X \subset &\{(\overline{y}, -\lambda, 0) : \overline{y} \in \operatorname{supp} u\} \cup R^+_{\lambda^2}\\ &\cup \{\alpha \in \Sigma_{\Delta-\lambda^2} : \exists \zeta \in \mathrm{WF}(u),\ \alpha \sim_+ \zeta\},\end{aligned} \tag{19.22}$$

since then the broken bicharacteristics through $\alpha \in \Sigma_{\Delta-\lambda^2} \cap {}^{\mathrm{sc}}T^*_{\partial X\setminus C}X$ can only hit ${}^{\mathrm{sc}}T^*_C X$ normally, so Corollary 14.2 suffices to prove (19.22). We also note that if the assumptions (15.1) and (15.2) are removed, then in (19.17), $R^- \cap (\Sigma_b(\lambda^2) \cup \Sigma_t(\lambda^2))$ must be replaced by $T^-_{\mathrm{ff}}(-\lambda)$ just as in Theorem 19.1; see the remark following the statement of the Theorem.

We can also analyze the wave front set of the scattering matrix. For this purpose consider the usual boundary pairing. Its statement is slightly complicated, since now we do not have such simple asymptotic expansions globally as in two-body (i.e. $V \in x\mathcal{C}^\infty(X)$) case.

Lemma 19.8. — *Suppose that* $u^{(j)} \in \mathcal{C}^{-\infty}(X)$, $j = 1, 2$,

$$u^{(j)} = u_+^{(j)} + u_-^{(j)},\ f^{(j)} = (H-\lambda^2)u_j \in \dot{\mathcal{C}}^\infty(X), \tag{19.23}$$

$u_+^{(1)} = \widetilde{R}(\lambda)g^{(1)}$, $u_-^{(2)} = \widetilde{R}(-\lambda)g^{(2)}$, $g^{(j)} \in \dot{\mathcal{C}}^\infty(X)$, $j = 1, 2$, *and*

$$v_\pm^{(j)} = e^{\pm i\lambda/x}x^{\pm i\alpha_\lambda - (N-1)/2}u_\pm^{(j)} \tag{19.24}$$

satisfy

$$v_-^{(1)} \in \mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X),\ v_-^{(1)}|_{\partial X} \in \mathcal{C}_c^\infty(\partial X \setminus C), \tag{19.25}$$

$$v_+^{(2)} \in \mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X),\ v_+^{(2)}|_{\partial X} \in \mathcal{C}_c^\infty(\partial X \setminus C). \tag{19.26}$$

Then with $w_\pm^{(j)} = v_\pm^{(j)}|_{\partial X\setminus C}$,

$$-2i\lambda \int_{\partial X} (w_+^{(1)}\overline{w_+^{(2)}} - w_-^{(1)}\overline{w_-^{(2)}})\,dh = \int_X (u^{(1)}\overline{f^{(2)}} - f^{(1)}\overline{u^{(2)}})\,dg. \tag{19.27}$$

Proof. — Since $w_-^{(1)}$ and $w_+^{(2)}$ (and hence both terms on the left hand side of (19.27)) are supported away from C, the two-body proof [**25**, Proposition 13] applies. □

Corollary 19.9. — *Let $a_0, a_0' \in \mathcal{C}_c^\infty(\partial X \setminus C)$ be supported in $K \subset \partial X \setminus C$ compact. Then*

$$\int_K S(\lambda)a_0\,\overline{a_0'}\,dh = \int_K a_0\,\overline{S(-\lambda)a_0'}\,dh. \tag{19.28}$$

Proof. — Take $u^{(1)} = P(\lambda)a_0$, $u^{(2)} = P(-\lambda)a_0'$ and apply Lemma 19.8. The right hand side of (19.27) vanishes and $w_-^{(1)} = a_0$, $w_+^{(1)} = S(\lambda)a_0$, $w_+^{(2)} = a_0'$, $w_-^{(2)} = S(-\lambda)a_0'$, so (19.28) follows. □

For the sake of definiteness we assume that $\lambda > 0$ in the following argument. Changing the sign of λ will only change some signs. Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$, identically 1 near λ^2, and let $Q \in \Psi_{3\mathrm{sc}}^{-\infty,0}(X)$ satisfy

$$\mathrm{WF}'_{3\mathrm{sc}}(\psi(H) - Q) \cap \big(R^-_{\lambda^2} \cup (R^- \cap (\Sigma_t(\lambda^2) \cup \Sigma_b(\lambda^2)))\big) = \varnothing, \tag{19.29}$$

$$\mathrm{WF}'_{3\mathrm{sc}}(Q) \cap \big(R^+_{\lambda^2} \cup (R^+ \cap (\Sigma_t(\lambda^2) \cup \Sigma_b(\lambda^2)))\big) = \varnothing. \tag{19.30}$$

For example, we can take Q' corresponding to the symbol $q(\tau)$, $q \in \mathcal{C}^\infty(\mathbb{R})$, $q \equiv 1$ near $(-\infty, -\lambda]$, $q \equiv 0$ near $[\lambda, \infty)$, and then let $Q = \psi(H)Q'$. Now given $a_0, a_0' \in \mathcal{C}_c^\infty(\partial X \setminus C)$, let $u = QP(\lambda)a_0$. Note that

$$\mathrm{WF}_{3\mathrm{sc}}(P(\lambda)a_0) \subset R^-_{\lambda^2} \cup R^+_{\lambda^2} \cup \big((R^- \cup R^+) \cap (\Sigma_t(\lambda^2) \cup \Sigma_b(\lambda^2))\big). \tag{19.31}$$

Thus, with $f = (H - \lambda^2)u$, we have $f \in \dot{\mathcal{C}}^\infty(X)$. In fact, $f = [H,Q]P(\lambda)a_0$, and

$$\mathrm{WF}'_{3\mathrm{sc}}([H,Q]) \subset \mathrm{WF}'_{3\mathrm{sc}}(Q) \cap \mathrm{WF}'_{3\mathrm{sc}}(\psi(H) - Q), \tag{19.32}$$

hence $\mathrm{WF}'_{3\mathrm{sc}}([H,Q]) \cap \mathrm{WF}_{3\mathrm{sc}}(P(\lambda)a_0) = \varnothing$, so by Lemma 9.8 we deduce that $f \in \dot{\mathcal{C}}^\infty(X)$. Lemma 19.8 implies then that for $a_0, a_0' \in \mathcal{C}_c^\infty(\partial X \setminus C)$ we have

$$2i\lambda \int_{\partial X} a_0 \overline{S(-\lambda)a_0'}\,dh = -\int_X (H - \lambda^2)QP(\lambda)a_0\overline{P(-\lambda)a_0'}\,dg. \tag{19.33}$$

Therefore, by Corollary 19.9

$$2i\lambda \int_{\partial X} S(\lambda)a_0\,\overline{a_0'}\,dh = -\int_X (H - \lambda^2)QP(\lambda)a_0\overline{P(-\lambda)a_0'}\,dg, \tag{19.34}$$

so

$$S(\lambda) = \frac{i}{2\lambda}P(-\lambda)^*(H - \lambda^2)QP(\lambda). \tag{19.35}$$

We choose Q so that on $\mathrm{WF}'_{3\mathrm{sc}}(Q) \cap \mathrm{WF}'_{3\mathrm{sc}}(\psi(H) - Q)$, $\tau \in (-\lambda + \varepsilon, -\lambda + 2\varepsilon)$, $\varepsilon > 0$ small. Fix $a_0 \in \mathcal{C}_c^{-\infty}(\partial X \setminus C)$. Now, by Proposition 19.6,

$$\begin{aligned}\mathrm{WF}_{3\mathrm{sc}}(P(\lambda)a_0) \cap \mathrm{WF}'_{3\mathrm{sc}}([H,Q]) \subset \{&\alpha \in \Sigma_{\Delta-\lambda^2} : \exists \zeta \in \mathrm{WF}(a_0),\ \alpha \sim_+ \zeta\} \\ &\cup \{\xi \in \Sigma_{\mathrm{ff}}(H - \lambda^2) : \exists \zeta \in \mathrm{WF}(a_0),\ \xi \sim_+ \zeta\}.\end{aligned} \tag{19.36}$$

Since $\varepsilon > 0$ is small, we have $\pi - s$ small in the parametrization of the bicharacteristic through α in the set on the right hand side of (19.36) due to (11.38), so the projection of $\mathrm{WF}_{3\mathrm{sc}}(P(\lambda)a_0) \cap \mathrm{WF}'_{3\mathrm{sc}}([H,Q])$ to ∂X is close to sing supp a_0, and hence it is away from C. Correspondingly, the second term of (19.36) can be dropped. This also shows that $\sim_+$ in (19.36) is actually given by the (unbroken) bicharacteristics of g in $\Sigma_{\Delta-\lambda^2}$. Thus, by (a local version of) Lemma 9.8

$$\begin{aligned}\mathrm{WF}_{3\mathrm{sc}}((H-\lambda^2)QP(\lambda)a_0) \subset \{\alpha \in \Sigma_{\Delta-\lambda^2} : \exists \zeta \in \mathrm{WF}(a_0),\ \alpha \sim_+ \zeta\} \\ \cap \mathrm{WF}'_{3\mathrm{sc}}(Q) \cap \mathrm{WF}'_{3\mathrm{sc}}(\psi(H)-Q).\end{aligned} \tag{19.37}$$

Now recall that the complex pairing

$$\langle u, u'\rangle_X = \int_X u\,\overline{u'}\,dg \tag{19.38}$$

extends by continuity from $u, u' \in \dot{\mathcal{C}}^\infty(X)$ to $u, u' \in \mathcal{C}^{-\infty}(X)$ satisfying

$$\mathrm{WF}_{\mathrm{sc}}(u) \cap \mathrm{WF}_{\mathrm{sc}}(u') = \varnothing.$$

To see this just let $A \in \Psi_{\mathrm{sc}}^{0,0}(X)$ with

$$\mathrm{WF}'_{\mathrm{sc}}(A) \cap \mathrm{WF}_{\mathrm{sc}}(u) = \varnothing, \quad \mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id}-A^*) \cap \mathrm{WF}_{\mathrm{sc}}(u') = \varnothing,$$

and note that

$$\langle u, u'\rangle_X = \langle Au, u'\rangle_X + \langle u, (\mathrm{Id}-A^*)u'\rangle_X \tag{19.39}$$

extends as claimed. Since for $a'_0 \in \mathcal{C}_c^\infty(\partial X \setminus C)$,

$$\mathrm{WF}_{3\mathrm{sc}}(P(-\lambda)a'_0) \subset R^+_{\lambda^2} \cup R^-_{\lambda^2} \cup \big(R^- \cap (\Sigma_b(\lambda^2) \cup \Sigma_t(\lambda^2))\big), \tag{19.40}$$

$\mathrm{WF}_{\mathrm{sc}}(P(-\lambda)a'_0)$ is disjoint from $\mathrm{WF}_{\mathrm{sc}}((H-\lambda^2)QP(\lambda)a_0)$ (which is away from C), so the pairing on the right hand side of (19.34) is certainly defined if $a'_0 \in \mathcal{C}_c^\infty(\partial X \setminus C)$. Note that (19.40) uses that either (15.1) or (15.2) holds. However, it is easy to see that we can still draw the desired conclusion from the results of Chapter 18 using $T^-_{\mathrm{ff}}(-\lambda)$ instead of $R^- \cap (\Sigma_b(\lambda^2) \cup \Sigma_t(\lambda^2))$; see Remark 19.2. This will also be true for some similar equations in what follows.

We now show that the pairing on the right hand side of (19.34) extends by continuity from $a'_0 \in \mathcal{C}_c^\infty(\partial X \setminus C)$ to $a'_0 \in \mathcal{C}_c^{-\infty}(\partial X \setminus C)$ with $\mathrm{WF}(a'_0)$ in a fixed compact subset of $S^*(\partial X \setminus C)$ which is disjoint from the image of $\mathrm{WF}(a_0)$ under the (generalized) broken geodesic flow at distance $-\pi$. As we saw above, the complex pairing used in (19.34) is defined by continuity whenever

$$\mathrm{WF}_{\mathrm{sc}}((H-\lambda^2)QP(\lambda)a_0) \cap \mathrm{WF}_{\mathrm{sc}}(P(-\lambda)a'_0) = \varnothing. \tag{19.41}$$

Since the first term in the intersection has wave front set away from C, the part of $\mathrm{WF}_{3\mathrm{sc}}(P(-\lambda)a'_0)$ at (in fact, near) C does not cause any problems. By Proposition 19.6 and the remark following it,

$$\begin{aligned}\mathrm{WF}_{\mathrm{sc}}(P(-\lambda)a'_0) \cap {}^{\mathrm{sc}}T^*_{\partial X\setminus C}X \cap \mathrm{WF}'_{3\mathrm{sc}}(Q) \cap \mathrm{WF}'_{3\mathrm{sc}}(\psi(H)-Q) \\ \subset \{\alpha \in \Sigma_{\Delta-\lambda^2} : \exists \zeta \in \mathrm{WF}(a'_0),\ \alpha \sim_- \zeta\}.\end{aligned} \tag{19.42}$$

Using (19.37) we conclude that

$$\begin{aligned}(19.43)\quad &\mathrm{WF}_{\mathrm{sc}}((H-\lambda^2)QP(\lambda)a_0)\cap \mathrm{WF}_{\mathrm{sc}}(P(-\lambda)a_0')\\ &\subset \{\alpha\in\Sigma_{\Delta-\lambda^2}\cap {}^{\mathrm{sc}}T^*_{\partial X\setminus C}X:\ \exists\zeta\in \mathrm{WF}(a_0),\ \zeta'\in\mathrm{WF}(a_0'),\ \alpha\sim_+\zeta,\ \alpha\sim_-\zeta'\}.\end{aligned}$$

Since there is a unique bicharacteristic of Δ through $\alpha\in\Sigma_{\Delta-\lambda^2}$, we see that if there are no $\zeta\in\mathrm{WF}(a_0)$, $\zeta'\in\mathrm{WF}(a_0')$ such that ζ' is related to ζ by the (generalized) broken geodesic flow on $S^*\partial X$ at time $-\pi$ then (19.41) holds. Thus, under this assumption the left hand side of (19.34) is also defined by continuity from $\mathcal{C}_c^\infty(\partial X\setminus C)$. This shows that $\mathrm{WF}(S(\lambda)a_0)$ is given the broken geodesic flow at distance $-\pi$. In fact, this statement simply means that taking $A\in\Psi^0(\partial X)$ with $\mathrm{WF}'(A)$ disjoint from the image of $\mathrm{WF}(a_0)$ under the broken geodesic flow at time $-\pi$ we need to show that $AS(\lambda)a_0\in\mathcal{C}^\infty(\partial X\setminus C)$. For this it suffices to show that

$$(19.44)\qquad \int_{\partial X} AS(\lambda)a_0\,\overline{a_0'}\,dh=\int_{\partial X} S(\lambda)a_0\,\overline{A^*a_0'}\,dh$$

is defined for all $a_0'\in\mathcal{C}_c^{-\infty}(\partial X\setminus C)$ by continuity from $\mathcal{C}_c^\infty(\partial X\setminus C)$. But, due to the assumption on $\mathrm{WF}'(A)$, this is exactly what we proved above. Hence, we deduce our main theorem:

Theorem 19.10. — *Suppose that H is as in* (11.11) *and $\lambda\in\mathbb{R}\setminus\{0\}$. Suppose also that either C is totally geodesic, or* (15.1), *or* (15.2) *holds. Then the free-to-free scattering matrix, $S(\lambda)$, extends to a continuous linear map $\mathcal{C}_c^{-\infty}(\partial X\setminus C)\to\mathcal{C}^{-\infty}(\partial X\setminus C)$. The wave front relation of $S(\lambda)$ is given by the (generalized) broken geodesic flow at time $-(\operatorname{sign}\lambda)\pi$.*

Remark 19.11. — This can be proved using (19.35) and Wunsch's push forward theorem **[40]** as well. Namely, the kernel of $P(-\lambda)$ is given by Melrose's and Zworski's plane wave construction near $\tau=\lambda$ as discussed above, hence we can write down the kernel $P_{-\lambda}\in\mathcal{C}^{-\infty}(\partial X\times X)$ of $P(-\lambda)^*$ explicitly as well. We take Q such that on $\mathrm{WF}'_{3\mathrm{sc}}(Q)\cap\mathrm{WF}'_{3\mathrm{sc}}(\psi(H)-Q)$, $\tau\in(\lambda-2\varepsilon,\lambda-\varepsilon)$, $\varepsilon>0$ small. Thus, the application of $P(-\lambda)^*$ to $v=(H-\lambda^2)QP(\lambda)a_0$, $a_0\in\mathcal{C}_c^{-\infty}(\partial X\setminus C)$, can be written as a push forward:

$$(19.45)\qquad (S(\lambda)a_0)(\overline{y})=\frac{i}{2\lambda}\int_X P_{-\lambda}(\overline{y},.)v\,dg.$$

It is then completely straightforward to check that Wunsch's push forward result in the scattering calculus **[40]** proves Theorem 19.10.

Remark 19.12. — Hassell has proved in **[14]** that the usual scattering matrix in Euclidian three-body scattering defined by the wave operators coincides with the one obtained by asymptotic expansions (i.e. ours) up to normalization. Here we present a somewhat different proof under the assumptions that the potentials are short range (so $V\in\rho_{\mathrm{mf}}^2\mathcal{C}^\infty([X;C])$). Namely, we relate our formula (19.35) to an analog of Isozaki's expression **[20**, Equation (3.10)**]** for the 2-cluster to 3-cluster S-matrix, with incoming channel α replaced by 0. In fact, we show that the wave-operator scattering matrix, $\widehat{S}_{00}(\lambda^2)$ (as in [20]), considered as a map $\mathcal{C}_c^\infty(\mathbb{S}^{N-1}\setminus C)\to\mathcal{C}^{-\infty}(\mathbb{S}^{N-1}\setminus C)$, satisfies

$$\widehat{S}_{00}(\lambda^2)=i^{N-1}S(-\lambda)R,\qquad \lambda>0, \tag{19.46}$$

where R is pull back by the antipodal map. Note that we write λ^2 rather than λ for the argument of $\widehat{S}_{00}(\lambda^2)$ to conform to the usual notation, and in Hassell's notation our $S(-\lambda)$ is $\widetilde{S}(\lambda)$ (see [14]).

We mostly use the notation of this paper rather than that of [20] in what follows. We let $H_0=\Delta$ (where Δ is the standard positive Laplacian on $\mathbb{R}^N_z$), $H=H_0+V$, choose $\chi\in\mathcal{C}^\infty(\mathbb{R})$ identically 0 on $(\lambda-\varepsilon,\infty)$, identically 1 on $(-\infty,-\lambda+\varepsilon)$ for some $\varepsilon>0$, and let $Q=q_L(\chi(\tau))\phi(\Delta)$ where $\phi\in\mathcal{C}_c^\infty(\mathbb{R})$ is identically 1 near λ^2. We also let $\mathcal{F}$ be the (standard non-unitary) Fourier transform on $\mathbb{R}^N$, and define

$$\mathcal{F}(\lambda):\mathcal{S}(\mathbb{R}^N)\to\mathcal{C}^\infty(\mathbb{S}^{N-1}) \tag{19.47}$$

to be the trace of $\mathcal{F}$:

$$\mathcal{F}(\lambda)f(\omega)=(\mathcal{F}f)(\lambda\omega). \tag{19.48}$$

Now Isozaki's proof of [20, Lemma 3.1] can be repeated nearly verbatim to conclude that for $f,g\in\mathcal{C}_c^\infty(\mathbb{S}^{N-1}\setminus C)$

$$\begin{aligned}&\langle(\widehat{S}_{00}(\lambda^2)-\mathrm{Id})f,g\rangle\\&\qquad=\langle C_0(-\mathcal{F}(\lambda)Q^*V+\mathcal{F}(\lambda)(HQ-QH_0)^*R(\lambda^2+i0)V)\mathcal{F}^*(\lambda)f,g\rangle\end{aligned} \tag{19.49}$$

with $C_0=\frac{1}{2}i(2\pi)^{-(N-1)}\lambda^{N-2}$. Thus, as a map $\mathcal{C}_c^\infty(\mathbb{S}^{N-1}\setminus C)\to\mathcal{C}^{-\infty}(\mathbb{S}^{N-1}\setminus C)$ we have

$$\widehat{S}_{00}(\lambda^2)-\mathrm{Id}=C_0(-\mathcal{F}(\lambda)Q^*V+\mathcal{F}(\lambda)(HQ-QH_0)^*R(\lambda^2+i0)V)\mathcal{F}^*(\lambda). \tag{19.50}$$

Here, as remarked in [20], we need to check that the second term makes sense and we can interchange a limit with an integral, but that is easy to see under our assumptions (e.g. $V\mathcal{F}^*(\lambda)f\in H^{0,1/2+\varepsilon}(\mathbb{R}^N)$ automatically for some $\varepsilon>0$, as follows from the asymptotic expansion using stationary phase).

Let $\widetilde{P}_0(\pm\lambda)$ be the free Poisson operator (i.e. that of Δ), so

$$(\widetilde{P}_0(\pm\lambda)f)(z)=\lambda^{(N-1)/2}e^{\pm\pi(N-1)i/4}(2\pi)^{-(N-1)/2}\int e^{\pm i\lambda z\cdot\omega}f(\omega)\,d\omega. \tag{19.51}$$

Then we have

$$\mathcal{F}^*(\lambda)=C\widetilde{P}_0(\lambda)=C'\widetilde{P}_0(-\lambda)R, \tag{19.52}$$

$$C=\lambda^{-(N-1)/2}e^{-\pi(N-1)i/4}(2\pi)^{(N-1)/2},\qquad C'=C^*. \tag{19.53}$$

Thus, (19.50) becomes

(19.54)
$$\widehat{S}_{00}(\lambda^2) - \mathrm{Id} = C''(-(VQ\widetilde{P}_0(\lambda))^* + ((HQ - QH_0)\widetilde{P}_0(\lambda))^* R(\lambda^2 + i0)V)\widetilde{P}_0(-\lambda)R,$$

$C'' = i^N/2\lambda$. But now note that

$$\frac{1}{2i\lambda}([\Delta, Q]\widetilde{P}_0(\lambda))^* \widetilde{P}_0(-\lambda) = i^{-(N-1)}R, \tag{19.55}$$

since the left hand side is the asymptotic expansion scattering matrix of Δ at $-\lambda$ (this is just a special case of (19.33)), which is $i^{-(N-1)}R$ (see **[26]**; this simply comes from stationary phase methods), so

$$C''([\Delta, Q]\widetilde{P}_0(\lambda))^* \widetilde{P}_0(-\lambda)R = -\mathrm{Id}, \tag{19.56}$$

Thus, from (19.54)

$$\widehat{S}_{00}(\lambda^2) = -C''((HQ - QH_0)\widetilde{P}_0(\lambda))^*(\mathrm{Id} - R(\lambda^2 + i0)V)\widetilde{P}_0(-\lambda)R. \tag{19.57}$$

But the Poisson operator for H is given by

$$P(\pm\lambda) = (\mathrm{Id} - R(\lambda^2 \mp i0)V)\widetilde{P}_0(\pm\lambda) \tag{19.58}$$

as we have seen, and as $(H_0 - \lambda^2)\widetilde{P}_0(\lambda) = 0$, we can conclude that

$$\widehat{S}_{00}(\lambda^2) = -C''((H - \lambda^2)Q\widetilde{P}_0(\lambda))^* P(-\lambda)R = i^{N-1}S(-\lambda)R, \tag{19.59}$$

as claimed. Here we used the analog of (19.33) and we also replaced $P(\lambda)$ by $\widetilde{P}_0(\lambda)$. It is easy to see from Lemma 19.8 that this replacement is legitimate since (19.33) also holds after this replacement. Thus, up to normalization, our scattering matrix indeed coincides with the wave-operator one in Euclidian scattering.

APPENDIX A

CONSTRUCTION OF PLANE WAVES NEAR THE INITIAL POINT

This chapter is essentially taken from Sections 1 and 15 of Melrose's and Zworski's paper [29] with the minor modification that we allow long-range potentials. We thus construct the kernel of the Poisson operator for $\Delta + V - \lambda^2$, $V \in x\mathcal{C}^\infty(X)$, on $X \times \partial X$ microlocally near the incoming set

(A.1)
$$G^\sharp(-\lambda) = \operatorname{graph}\left\{\lambda \frac{dx}{x^2}\right\} = \{(y, y', -\lambda, 0, 0) : \ y, y' \in \partial X\} \subset {}^{\mathrm{sc}}T^*_{\partial X \times \partial X}(X \times \partial X).$$

Note that $X \times \partial X$ is also a manifold with boundary, hence with a natural scattering structure. In particular, if x is a boundary defining function of X so that g is a scattering metric on X, y are local coordinates on ∂X near a point q, then near the point $p = (q, q) \in \partial X_y \times \partial X_{y'} \subset X \times \partial X$ we have coordinates (x, y, y'). Correspondingly, on ${}^{\mathrm{sc}}T^*_{\partial X \times \partial X} X \times \partial X$ we obtain coordinates (y, y', τ, μ, μ'). The Legendre submanifold associated to the plane waves is

$$\begin{aligned} G(-\lambda) =& \{(y, y'; \tau, \mu, \mu') : \ (y, \widehat{\mu}) = \exp((s - \pi)H_{h/2})(y', \widehat{\mu}'), \ \tau = \lambda \cos s, \\ & \mu = \lambda(\sin s)\widehat{\mu}, \ \mu' = -\lambda(\sin s)\widehat{\mu}', \ s \in (0, \pi)\} \subset {}^{\mathrm{sc}}T^*_{\partial X \times \partial X} X \times \partial X; \end{aligned} \tag{A.2}$$

see [29, Proposition 4]. Note that the incoming and outgoing sets are defined the opposite way in [29]; we follow the notation of [25]. In particular, this is the reason for some sign changes above.

Near $G^\sharp(-\lambda)$, $G(-\lambda)$ is parametrized the function $\lambda\phi(y, y')$ where

$$\phi(y, y') = \cos d(y, y'),$$

and d denotes the distance on ∂X with respect to $h|_{\partial X}$. Thus, if u is a Legendre distribution of order m associated to $G(-\lambda)$, i.e. $u \in I^m_{sc}(X \times \partial X, G(-\lambda))$, and $A \in \Psi^0_{\mathrm{sc}}(X \times \partial X)$, $\mathrm{WF}'_{\mathrm{sc}}(A)$ is near $G^\sharp(-\lambda)$, then Au has the form

$$\begin{aligned} Au = (2\pi)^{-(2N-1)/4} x^{m+(2N-1)/4} e^{i\lambda\phi(y,y')/x} a(x, y, y') + u_0, \\ a \in \mathcal{C}^\infty(X \times \partial X), \ u_0 \in \dot{\mathcal{C}}^\infty(X \times \partial X) \end{aligned} \tag{A.3}$$

(see [29, Definition 2]).

We will need to consider slightly more general distributions, namely ones of the form $v = x^{i\alpha(y')}Au$, with Au as above, $\alpha \in \mathcal{C}^\infty(\partial X)$. These are Legendre distributions in the non-polyhomogeneous sense, and they can be thought of as polyhomogeneous distributions with variable order. By the stationary phase lemma we also have the pushforward result that for $f \in \mathcal{C}^\infty(\partial X_{y'})$,

$$\int_{\partial X} x^{i\alpha(y')} Au(x,y,y') f(y')\, dh = e^{i\lambda/x} x^{(N-1)/2+i\alpha(y)} Q(u,f) \tag{A.4}$$

where $Q(u,f)$ is a polyhomogeneous distribution on X with index set as in (18.20), i.e.

$$\mathcal{K} = \{(m,p) : \ m,p \in \mathbb{N},\ p \le 2m\}. \tag{A.5}$$

In particular, there exists $w \in \mathcal{C}^\infty(\partial X)$ so that $|Q(u,f) - w| \le Cx(\log x)^2$ for some constant $C > 0$. Define $Q^0_{-\lambda}(u) : \mathcal{C}^\infty(\partial X) \to \mathcal{C}^\infty(\partial X)$ by $Q^0_{-\lambda}(u)f = Q(u,f)|_{x=0} = w$. The stationary phase lemma also gives that $Q^0_{-\lambda}(u)f(y) = q(y)f(y)$ where $q \in \mathcal{C}^\infty(\partial X)$, i.e. $Q^0_{-\lambda}(u)$ is just multiplication by a smooth function.

The only modification that we need to make in Melrose's and Zworski's construction is that at the initial points, i.e. at $G(-\lambda) \cap G^\sharp(-\lambda)$, an additional factor must be introduced (which then 'propagates' along $G(-\lambda)$). Thus, we seek a Legendre distribution $K^\flat$ satisfying

$$\mathrm{WF}_{\mathrm{sc}}((\Delta_X + V_X - \lambda^2)K^\flat) \cap G^\sharp(-\lambda) = \varnothing, \tag{A.6}$$

$$Q^0_{-\lambda}(K^\flat) = \mathrm{Id}\,. \tag{A.7}$$

Here 'Legendre distribution' is understood in the sense discussed above, so $K^\flat = x^{i\alpha(y')}\widetilde{K^\flat}$, $\alpha \in \mathcal{C}^\infty(\partial X)$ and $\widetilde{K^\flat}$ is Legendre in the sense of **[29]**. It is easy to specify $\alpha = \alpha_\lambda$; it is the function in (18.19) that appears in the asymptotic expansion of $\widetilde{R}(\pm\lambda)f$, $f \in \dot{\mathcal{C}}^\infty(X)$, i.e. with $V = xV'$, $V' \in \mathcal{C}^\infty(X)$, $\alpha = (2\lambda)^{-1}V'|_{\partial X}$. We construct $K^\flat$ as an asymptotic sum

$$K^\flat \sim \sum_{j=0}^\infty K_j, \qquad x^{-i\alpha(y')}K_j \in I_{sc}^{-(2N-1)/4+j}(X \times \partial X, G(-\lambda), \Omega_R). \tag{A.8}$$

Hence, microlocally near $G^\sharp(\lambda)$, K_0 must satisfy

$$(\Delta_X + V_X - \lambda^2)K_0 \in x^{i\alpha(y')} I_{sc}^{-(2N-1)/4+2}(X \times \partial X, G(-\lambda), \Omega_R), \tag{A.9}$$

$$\sigma_0(Q^0_{-\lambda}(K_0)) = \sigma_0(\mathrm{Id}), \tag{A.10}$$

and for $j \ge 1$ we need

$$\begin{aligned}(\Delta_X + V_X - \lambda^2)K_j + (\Delta_X + V_X - \lambda^2)\Big(\sum_{l=0}^{j-1} K_l\Big)& \\ \in x^{i\alpha(y')} I_{sc}^{-(2N-1)/4+j+2}(X \times \partial X, G(-\lambda), \Omega_R)&.\end{aligned} \tag{A.11}$$

The kernels K_j take the form of oscillatory functions

(A.12)
$$K_j = x^{j+i\alpha(y')}e^{i\lambda\phi(y,y')/x}a_j(x,y,y')\pi_R^*\nu, \qquad a_j \in \mathcal{C}^\infty(X \times \partial X),\ \nu \in \mathcal{C}^\infty(\partial X, \Omega),$$

(A.13)
$$\phi(y,y') = \cos d(y,y'),$$

$d(y,y')$ still being the metric distance between y and y' with respect to $h|_{\partial X}$. Regarding y' as a parameter and introducing Riemannian normal coordinates in y centered at y' we obtain transport equations for $a_j' = a_j|_{x=0}$

(A.14)
$$(y \cdot \partial_y + j)a_j' + (i(\alpha(0) - (2\lambda)^{-1}V'(y)) + b_j)a_j' = c_j \in \mathcal{C}^\infty(X \times \partial X)$$

near $y = 0$ with b_j vanishing at $y = 0$ and $c_0 \equiv 0$. Since

$$2\lambda\alpha(0) - V'(y)$$

vanishes at $y = 0$, the transport equation for a_0' has a unique smooth solution with $a_0'(y,y) \in \mathcal{C}^\infty(\partial X)$ specified, and the equations for a_j', $j \geq 1$ have unique smooth solutions. This is true for the same reasons as in Hadamard's construction, see e.g. [**18**, Lemma 17.4.1].

Hence, the K_j exist microlocally near $G^\sharp(-\lambda)$, and if ψ is supported near the diagonal in $\partial X \times \partial X \subset X \times \partial X$, identically 1 in a smaller neighborhood of the diagonal, then the ψK_j can be considered distributions on $X \times \partial X$. They can be summed by Borel's lemma, to obtain $K^\flat \in \mathcal{C}^{-\infty}(X \times \partial X; \Omega_R)$ with the desired properties. By choosing ψ to have sufficiently small support with sufficiently small support we can arrange that the projection of $\mathrm{WF}_{\mathrm{sc}}(K^\flat)$ to $\partial X \times \partial X$ is close to the diagonal at the expense of making $\mathrm{WF}_{\mathrm{sc}}((\Delta_X + V_X - \lambda^2)K^\flat)$ close to (but disjoint from) $G^\sharp(-\lambda)$. Now recall that $\sim'_+$ is the relation induced by the bicharacteristics of g between points in $\Sigma_{\Delta-\lambda^2}$ and points in $S^*\partial X$; see Definition 11.7. We can finally deduce the following result.

Proposition A.1. — *$K^\flat \in \mathcal{C}^{-\infty}(X \times \partial X; \Omega_R)$, constructed above, is the kernel of an operator $P_0(\lambda) : \mathcal{C}^\infty(\partial X) \to \mathcal{C}^{-\infty}(X)$, which extends to an operator $P_0(\lambda) : \mathcal{C}^{-\infty}(\partial X) \to \mathcal{C}^{-\infty}(X)$, and for $u \in \mathcal{C}^{-\infty}(\partial X)$*

(A.15)
$$\begin{aligned}\mathrm{WF}_{\mathrm{sc}}(P_0(\lambda)u) \subset &\{(y,-\lambda,0) :\ y \in \operatorname{supp} u\} \\ &\cup \{\alpha \in \Sigma_{\Delta-\lambda^2} \setminus R^-_{\lambda^2} :\ \exists\zeta \in \mathrm{WF}(u),\ \alpha \sim'_+ \zeta\},\end{aligned}$$

(A.16)
$$\begin{aligned}&\mathrm{WF}_{\mathrm{sc}}((\Delta + V - \lambda^2)P_0(\lambda)u) \\ &\qquad\subset \{\alpha \in \Sigma_{\Delta-\lambda^2} \setminus R^-_{\lambda^2} :\ \exists\zeta \in \mathrm{WF}(u),\ \alpha \sim'_+ \zeta\}.\end{aligned}$$

Proof. — Since $K^\flat$ is supported near the diagonal of $\partial X_y \times \partial X_{y'}$, we can work in local coordinates. Thus, we may assume that $u \in \mathcal{C}^{-\infty}(\partial X)$ is supported in a small open set $U \subset \mathbb{R}^{N-1}_{y'}$, and we can replace X by $\mathbb{S}^N_+$, i.e. the radial compactification of $\mathbb{R}^N$, which is $[0,1)_x \times \mathbb{S}^{N-1}_y$ near $\mathbb{S}^{N-1} = \partial\mathbb{S}^N_+$ (so (x^{-1}, y) are the standard polar coordinates on $\mathbb{R}^N$). We take the partial Fourier transform of $K^\flat$ with respect to (x, y), i.e. consider

(A.17)
$$\begin{aligned}\widehat{K^\flat} &= \int_X e^{-i\xi\cdot y/x} K^\flat(x,y,y')\,dx\,dy\\ &= \int_X e^{i(-\xi\cdot y+\lambda\phi(y,y'))/x} a(x,y,y')\,dx\,dy \in \mathcal{C}^{-\infty}(\mathbb{R}^N_\xi \times \mathbb{R}^{N-1}_{y'});\end{aligned}$$

$a \in x^{i\alpha(y')}\mathcal{C}^\infty(X \times \partial X; \Omega_R) \subset S^\varepsilon(X \times \partial X)$ for all $\varepsilon > 0$. Here we are using the compactified notation for symbol spaces, i.e. the non-trivial behavior of the symbols is at $x = 0$. It follows that $\widehat{K^\flat}$ is a Lagrangian distribution associated to a conic Lagrangian submanifold Λ of $T^*(\mathbb{R}^N \times \mathbb{R}^{N-1})$ with compact projection to the base, since

(A.18)
$$\psi(\xi, y', x, y) = (-\xi \cdot y + \lambda\phi(y, y'))/x$$

is a non-degenerate phase function (again, we are using a compactified notation). Namely, Λ is given by

(A.19)
$$C \ni (\xi, y', x, y) \mapsto (\xi, y', d_\xi\psi, d_{y'}\psi) \in \Lambda \subset T^*(\mathbb{R}^N \times \partial X),$$

where C is the critical set

(A.20)
$$C = \{(\xi, y', x, y) : \; d_{(x,y)}\psi(\xi, y', x, y) = 0\}.$$

It is convenient to think of $\mathbb{S}^{N-1}_y$ as the unit sphere in $\mathbb{R}^N_{\tilde{y}}$; correspondingly we can identify $\alpha \in T^*_y\mathbb{S}^{N-1}$ with a covector in $T^*_y\mathbb{R}^N$ using the standard metric on both $\mathbb{S}^{N-1}$ and $\mathbb{R}^N$. Then

(A.21)
$$\partial_y(\xi \cdot y)\,dy = (\xi - (\xi \cdot y)y)\,d\tilde{y}.$$

Hence, (A.20) becomes

(A.22)
$$C = \{(\xi, y', x, y) : \; \xi \cdot y = \lambda\phi(y, y'), \; \xi - (\xi \cdot y)y = \lambda\,\partial_y\phi(y, y')\}.$$

Moreover,

(A.23)
$$d_{(\xi,y')}\psi = -\frac{1}{x}\,y \cdot d\xi + \frac{\lambda}{x}\,\partial_{y'}\phi(y, y')\,dy',$$

so

(A.24)
$$\Lambda = \left\{\left(\xi, y', -\frac{y}{x}, \frac{\lambda}{x}\,\partial_{y'}\phi(y, y')\right) : \; \xi \cdot y = \lambda\phi(y, y'), \; \xi - (\xi \cdot y)y = \lambda\,\partial_y\phi(y, y')\right\}.$$

Since

(A.25)
$$\phi(y, y')^2 + |\partial_y\phi(y, y')|^2_h \equiv 1$$

(this being the eikonal equation satisfied by ϕ; here $|\partial_y\phi(y, y')|_h$ is the metric length of $\partial_y\phi(y, y')\,dy$ with respect to $h|_{\partial X}$), this proves that Λ indeed has compact projection to $\mathbb{R}^N \times \partial X$. Moreover, as $\widehat{K^\flat}$ is a Lagrangian distribution associated to Λ, $\mathrm{WF}(\widehat{K^\flat}) \subset$

Λ. It is also easy to see that $(1-\rho)\widehat{K^\flat} \in \mathcal{S}(\mathbb{R}^N \times \partial X)$ if $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^N \times \partial X)$ is identically 1 in a neighborhood of the projection of Λ to the base, so

$$\mathrm{WF}_{\mathrm{sc}}(\widehat{K^\flat}) = \mathrm{WF}(\widehat{K^\flat}) \subset \Lambda. \tag{A.26}$$

Now,

$$\begin{aligned} P_0(\lambda)u &= \int_{\partial X_{y'}} K^\flat(x,y,y')u(y') \\ &= (2\pi)^{-N} \int_{\mathbb{R}^N} e^{i\xi\cdot y/x}\Big(\int_{\partial X_{y'}} \widehat{K^\flat}(\xi,y')u(y')\Big)\, d\xi. \end{aligned} \tag{A.27}$$

We write $((\xi,y'),(\xi^*,\eta))$ for the canonical coordinates induced on $T^*(\mathbb{R}^N \times \partial X)$ by the coordinates (ξ,y'). We also write (ξ,ξ^*) for the coordinates on $T^*\mathbb{R}^N$, identify $S^*\mathbb{R}^N$ as the set $\{(\xi,\xi^*) : \ |\xi^*|=1\}$, and write $\widehat{\xi^*} = \xi^*/|\xi^*|$. Similarly, if $(y,\eta) \in T^*\partial X$, we let $\widehat{\eta} = \eta/|\eta|_h$. As usual, we regard the wave front set of a distribution on, say, $\mathbb{R}^N$, both as a conic subset of $T^*\mathbb{R}^N \setminus 0$ and as a subset of $S^*\mathbb{R}^N$.

The standard wave front set calculus [**18**, Theorem 8.2.13] allows us to estimate the wave front set of

$$v = \int_{\partial X_{y'}} \widehat{K^\flat}(\xi,y')u(y'). \tag{A.28}$$

Namely, we have

$$\begin{aligned} \mathrm{WF}(v) \subset &\{(\xi,\xi^*) : \ \exists y', \ (\xi,y',\xi^*,0) \in \mathrm{WF}(\widehat{K^\flat}), \ y' \in \operatorname{supp} u\} \\ &\cup \{(\xi,\xi^*) : \ \exists (y',-\eta) \in \mathrm{WF}(u), \ (\xi,y',\xi^*,\eta) \in \mathrm{WF}(\widehat{K^\flat})\}. \end{aligned} \tag{A.29}$$

In the first set on the right hand side we have $\partial_{y'}\phi(y,y') = 0$ by (A.24), so (using that $\phi(y,y') = \cos d(y,y')$), $y = y'$. Then (A.24) also gives $\xi\cdot y = \lambda$, and $\partial_y(\xi\cdot y) = 0$, so $\xi = \lambda y = \lambda y'$. Moreover, by the same equation, $\xi^* = -y/x$, i.e. $\widehat{\xi^*} = -y$. Thus, the first set on the right hand side of (A.29) is

$$\{(\lambda y, -y) \in S^*\mathbb{R}^N \ : \ y \in \operatorname{supp} u\}. \tag{A.30}$$

Equation (A.24) also shows that the second set on the right hand side of (A.29) is

$$\begin{aligned} \{(\xi,-y) \in S^*\mathbb{R}^N \ : \ (y, -\partial_{y'}\phi(y,y')) \in \mathrm{WF}(u), \ \xi\cdot y = \lambda\phi(y,y'), \\ \xi - (\xi\cdot y)y = \lambda\partial_y\phi(y,y')\}. \end{aligned} \tag{A.31}$$

Now, $\mathrm{WF}_{\mathrm{sc}}(P_0(\lambda)u)$ and $\mathrm{WF}(v) = \mathrm{WF}_{\mathrm{sc}}(\mathcal{F}P_0(\lambda)u)$ are related by the Legendre diffeomorphism [**29**, Lemma 5 and Proposition 8]. This is the map $L^{-1} : S^*\mathbb{R}^N \to {}^{\mathrm{sc}}T^*_{\mathbb{S}^{N-1}}\mathbb{S}^N_+$ which in coordinates (y,τ,μ) on ${}^{\mathrm{sc}}T^*_{\mathbb{S}^{N-1}}\mathbb{S}^N_+$ is given by

$$L^{-1}(\xi,\widehat{\xi^*}) = (-\widehat{\xi^*}, \xi\cdot\widehat{\xi^*}, \xi - (\xi\cdot\widehat{\xi^*})\widehat{\xi^*}). \tag{A.32}$$

Hence, the set in (A.30) corresponds to

$$\{(y,-\lambda,0) : \ y \in \operatorname{supp} u\}, \tag{A.33}$$

while the set in (A.31) corresponds to

$$\begin{aligned}\{(y,\tau,\mu):\ \exists\ (y',\eta')\in \mathrm{WF}(u),\ \tau=-\lambda\cos d(y,y'),\ \mu=\lambda\,\partial_y\phi(y,y'),\\ \eta'=-\partial_{y'}\phi(y,y')\}.\end{aligned} \tag{A.34}$$

Now, by (A.25) we have $\tau^2+|\mu|_h^2=\lambda^2$ in (A.34). Since $\phi(y,y')=\cos d(y,y')$, so

$$d_{y'}\phi=-(\sin d(y,y'))\partial_{y'}d(y,y')\,dy', \tag{A.35}$$

we see that $\mu=-\lambda\partial_y\phi(y,y')$, $\eta'=-\partial_{y'}\phi(y,y')$ mean that $(y,\mu/|\mu|)$ lies on the 'backward' geodesic starting at $(y',\widehat{\eta}')$. Thus, we conclude that (A.34) can be written as

(A.36)

$$\begin{aligned}\{(y,\tau,\mu):\ \exists\ (y',\widehat{\eta}')\in \mathrm{WF}(u)\subset S^*\partial X,\ \tau=-\lambda\cos d(y,y'),\ \tau^2+|\mu|_h^2=\lambda^2\\ \exp(-d(y,y')H_{h/2})(y',\widehat{\eta}')=(y,\mu/|\mu|)\}.\end{aligned}$$

This proves (A.15). In view of (A.6) the proof of (A.16) is similar. □

APPENDIX B

ABSENCE OF POSITIVE EIGENVALUES

This chapter follows the paper [8] of Froese and Herbst, and we only emphasize the modifications necessary to accommodate the more general setting. The main point is that we have to estimate the error terms introduced by the general geometry carefully. On the other hand, we do not have any of the complications arising due to the lack of smoothness of the potential. In the proof of super-exponential decay of eigenfunctions with positive energy, the analogue of [8, Theorem 2.1], the error terms arising from the general geometry are similar to those in the Euclidian setting, so the proof of Froese and Herbst requires only minor modifications. On the other hand, they use the exact form of the metric very strongly in their proof of the unique continuation theorem at infinity [8, Theorem 3.1], so there will be many error terms in our case which we have to control and which do not arise in Euclidian scattering.

Fix a boundary defining function x on X such that $g = x^{-4}\,dx^2 + x^{-2}h$ is a scattering metric, and choose a product decomposition of a neighborhood U_0 of ∂X: $U_0 = [0, \varepsilon_0)_x \times \partial X_y$. It is convenient to eliminate cross terms $dx \otimes dy$ by adjusting the product decomposition. This is not necessary for the first proposition (super-exponential decay), but it will be important in the proof of unique continuation at ∂X.

First note that the coefficients of the dual metric

$$g^{-1} = g^{00}\,\partial_x \otimes \partial_x + \sum g^{0j}\,\partial_x \otimes \partial_{y_j} + \sum g^{j0}\,\partial_{y_j} \otimes \partial_x + \sum g^{ij}\,\partial_{y_i} \otimes \partial_{y_j} \tag{B.1}$$

satisfy

$$g^{00} = x^4(1 + O(x^2)),\ g^{0j} = O(x^4),\ g^{ij} = x^2(\widetilde{h}^{ij} + O(x)) \tag{B.2}$$

where $\widetilde{h}$ is the pull back of h to ∂X (see [25, Lemma 3]). Thus,

$$g^{-1}(dx) = x^4(\alpha_0'\partial_x + \sum \alpha_j'\partial_{y_j}), \qquad \alpha_0' = 1 + O(x^2) \tag{B.3}$$

so in a neighborhood of ∂X

$$W = (\alpha_0')^{-1}x^{-4}g^{-1}(dx) = \partial_x + \sum \alpha_j \partial_{y_j}, \tag{B.4}$$

is a smooth vector field on a neighborhood of ∂X which is transversal to the hypersurfaces $x = const$ in a smaller neighborhood of ∂X. Let $\gamma(t,y)$ be the integral curve of W satisfying $\gamma(0,y) = (0,y) \in \partial X$; so $x(\gamma(t,y)) = t$. If $p = \gamma(t,y)$, let $y'(p) = y$, $t(p) = t \equiv x(p)$. This introduces a product decomposition of a neighborhood U of X with $U = [0,\varepsilon)_x \times \partial X_{y'}$. Moreover, by our definition of γ, ∂_x and $T\{x = const\}$ are orthogonal with respect to g, so the coefficients of the cross terms $dx \otimes dy_j'$ in g vanish with respect to this product decomposition. Thus, we can assume, as we will in what follows, that

$$\begin{aligned} g = a^{-1}x^{-4}\, dx \otimes dx + x^{-2}h, \qquad & a \in \mathcal{C}^\infty(U),\ a = 1 + O(x^2), \\ h \in \mathcal{C}^\infty(U; T^*\partial X \otimes T^*\partial X),\ & h_0 = h|_{\partial X} \text{ is a metric on } \partial X \end{aligned} \tag{B.5}$$

(here we really mean the pull back of the cotensor bundle). The Laplacian of g becomes

$$\Delta = a(x^2D_x)^2 + i(N-1)x(x^2D_x) + x^2\Delta_0 + x^3P' + x^2Q' \tag{B.6}$$

where Δ_0 is the Laplacian of $h|_{\partial X}$, $P \in \mathrm{Diff}^2(\partial X)$ (lifted by the product decomposition), $Q \in \mathrm{Diff}^1_{\mathrm{sc}}(X)$, and $N = \dim X$. In fact, it is convenient to replace this by the more general expression

$$\Delta = a(x^2D_x)^2 + x^2\Delta_0 + x^3P + xQ \tag{B.7}$$

with $Q \in \mathrm{Diff}^1_{\mathrm{sc}}(X)$, $P \in \mathrm{Diff}^2(\partial X)$.

Let $\widetilde{W}$ be a vector field supported near ∂X such that on a smaller neighborhood of ∂X, $\widetilde{W} = xD_x$, and let $A = \frac{1}{2}(\widetilde{W} + \widetilde{W}^*)$. It is easy to check that if $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ is identically 1 near 0 and has sufficiently small support then

$$\phi(x)(A - (xD_x + iN/2)) \in x\mathcal{C}^\infty(X). \tag{B.8}$$

We next state the analog of Lemma 2.2 of Froese and Herbst. We let $S^m([0,1)_x)$ be the space of all symbols a of order m on $[0,1)$, which satisfy $a \in \mathcal{C}^\infty((0,1))$, vanish on $(1/2,1)$, and for which $\sup|x^{m+k}\partial_x^k a| < \infty$ for all k. The topology of $S^m([0,1))$ is given by the seminorms $\sup|x^{m+k}\partial_x^k a|$. Also, the space $S^m(X)$ of symbols is defined similarly, i.e. it is given by seminorms $\sup|x^m Pa|$, $P \in \mathrm{Diff}^k_{\mathrm{b}}(X)$. In the following lemma $\mathrm{Diff}_{\mathrm{scc}}(X)$, as usual, stands for non-classical (non-polyhomogeneous) scattering differential operators (i.e. scattering differential operators with non-polyhomogeneous coefficients), corresponding to the lack of polyhomogeneity of F. In particular, $\mathrm{Diff}^0_{\mathrm{scc}}(X) = S^0(X)$ (considered as multiplication operators).

Lemma B.1. — *Suppose that H is as in (11.11), $\lambda > 0$, $H\psi = \lambda\psi$, $\psi \in L^2_{\mathrm{sc}}(X)$. Suppose also that $\alpha \geq 0$, and for all β we have $x^{-\beta}\exp(\alpha/x)\psi \in L^2_{\mathrm{sc}}(X)$. Then* *with* $F \in S^1([0,1))$, $F \leq \alpha/x + \beta|\log x|$ *for some* β, $\operatorname{supp} F \subset U$, $\psi_F = e^F\psi$, $H(F) = H + e^F[H, e^{-F}]$ *we have* $\psi_F \in \dot{\mathcal{C}}^\infty(X)$,

$$H(F)\psi_F = \lambda\psi_F, \tag{B.9}$$

$$H(F) = H - 2a(x^2D_xF)(x^2D_x) + a(x^2D_xF)^2 + R_1, \qquad R_1 \in xS^0(X), \tag{B.10}$$

$$(\psi_F, H\psi_F) = (\psi_F, (\lambda - a(x^2D_xF)^2)\psi_F). \tag{B.11}$$

If in addition $\partial_x F \leq 0$ *then*

(B.12)
$$\begin{aligned}(\psi_F, i[A,H]\psi_F) = -4\|(ax)^{1/2}(-x^2\partial_xF)^{1/2}A\psi_F\|^2 + (\psi_F, (x\partial_x(x^2\partial_xF)^2)\psi_F)\\ + (\psi_F, R_2\psi_F) + (R_3\psi_F, A\psi_F) + (A\psi_F, R_4\psi_F),\\ R_2, R_3, R_4 \in xS^0(X).\end{aligned}$$

Here R_1, R_3 *and* R_4 *are bounded by some seminorms of* F, *and* R_2 *is bounded by a quadratic polynomial in some seminorms of* F.

Proof. — Formally this is just an explicit computation, carefully taking into account the error terms. It can be justified exactly as in the setting of the paper of Froese and Herbst. Here we just note that

$$[x^2D_x, e^F] = (x^2D_xF)e^F, \qquad x^2D_xF \in S^0([0,1)), \tag{B.13}$$

so $e^F[H, e^{-F}] \in \operatorname{Diff}^1_{\mathrm{scc}}(X)$ ($V \in \mathcal{C}^\infty([X;C])$ commutes with e^F). Hence (B.9), which a priori holds in a distributional sense, and the ellipticity of $\sigma_{3\mathrm{sc},2}(H)$ show that $\psi_F \in \dot{\mathcal{C}}^\infty(X)$. Moreover, we use

(B.14)
$$\lambda\|\psi_F\|^2 = (\psi_F, H(F)\psi_F) = \operatorname{Re}(\psi_F, H(F)\psi_F) = (\psi_F, (H + \frac{1}{2}[e^F, [H, e^{-F}]])\psi_F)$$

to prove (B.11), and

$$0 = (\psi, [H, e^FAe^F]\psi) = (\psi_F, (e^{-F}[H, e^F]A + [H,A] + A[H, e^F]e^{-F})\psi_F) \tag{B.15}$$

to prove (B.12). The estimates of the error terms are facilitated by (B.13). In particular, the dependence of R_j on seminorms of F arises by commuting e^F through x^2D_x. Each such commutation gives a factor bounded in $S^0([0,1))$ by seminorms of F, but it also eliminates the vectorfield, i.e. reduces the degree of the differential operator by 1. Since H is second order, and the only non-tangential second order part is $a(x^2D_x)^2$, the previous formulas give the claimed bounds. □

Using this lemma and the Mourre estimate (Theorem 12.2) we can follow Froese and Herbst very closely in the proof of the following result:

Proposition B.2 **(Froese and Herbst, [8, Theorem 2.1]).** — *Let H be as in* (11.11), $\lambda > 0$, *and suppose that* $\psi \in L^2_{\mathrm{sc}}(X)$ *satisfies* $H\psi = \lambda\psi$. *Then* $e^{\alpha/x}\psi \in L^2_{\mathrm{sc}}(X)$ *for all* $\alpha \in \mathbb{R}$.

Proof. — The proof is by contradiction. First note that $\psi \in \dot{\mathcal{C}}^\infty(X)$ by Corollary 17.5. Let

$$\alpha_1 = \sup\{\alpha \in [0,\infty) : \ \exp(\alpha/x)\psi \in L^2_{\mathrm{sc}}(X)\}, \tag{B.16}$$

and suppose that $\alpha_1 < \infty$. If $\alpha_1 = 0$, then let $\alpha = 0$, otherwise suppose that $\alpha < \alpha_1$, and $\alpha + \gamma > \alpha_1$. We show that for sufficiently small γ (depending only on α_1) $\exp((\alpha+\gamma)/x)\psi \in L^2_{\mathrm{sc}}(X)$, which contradicts our assumption on α_1 if α is close enough to α_1. In what follows we assume that $\gamma \in (0,1]$.

Note first that we certainly have for all $\beta \in \mathbb{R}$, $\exp(\alpha/x)x^\beta\psi \in L^2_{\mathrm{sc}}(X)$, due to our choice of α. We apply the previous lemma with

$$F = \phi(x)\Big(\frac{\alpha}{x} + \beta\log\big(1 + \frac{\gamma}{\beta x}\big)\Big), \tag{B.17}$$

$\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ identically 1 near 0 (it is convenient to take it to be identically 1 on $\operatorname{supp} A$), and let $\psi_\beta = e^F\psi$, $\Psi_\beta = \psi_\beta/\|\psi_\beta\|$. Here $F = F_\beta \in S^1([0,1))$, and F_β is uniformly bounded in $S^1([0,1))$ for $\beta \in [1,\infty)$, $\alpha \in [0,\alpha_1)$ (or $\alpha = \alpha_1$ if $\alpha_1 = 0$), $\gamma \in (0,1]$.

Now, F is an increasing function of β, and $F(x)$ converges to $\phi(x)(\alpha+\gamma)/x$ as $\beta \to \infty$. Thus, by the monotone convergence theorem

$$\|\psi_\beta\|^2 \to \|\exp(\phi(x)(\alpha+\gamma)/x)\psi\| = \infty \tag{B.18}$$

since $\alpha + \gamma > \alpha_1$. On the other hand, for any compact subset B of $\operatorname{int}(X)$, e^F is uniformly bounded, and so are its derivatives, so for any $Q \in \operatorname{Diff}^k(X)$

$$\lim_{\beta\to\infty} \|Q\Psi_\beta\|_{L^2(B)} = 0. \tag{B.19}$$

In what follows, we write b_j, $j \in \mathbb{N}$, for positive constants which are independent of α, β and γ. Now,

$$-x^2\partial_x F = (\alpha + \gamma(1+\gamma/\beta x)^{-1})\phi(x) + F_1 \le b_1, \qquad F_1 \in \mathcal{C}^\infty_c(\operatorname{int}(X)). \tag{B.20}$$

Hence, by (B.9), (B.10) and the ellipticity of $\sigma_{3\mathrm{sc},2}(H)$,

$$\|\Psi_\beta\|_{H^k_{\mathrm{sc}}(X)} \le b_{2,k} \tag{B.21}$$

for all k. Note that (B.19) and (B.21) prove that for any $Q \in \operatorname{Diff}^k_{3\mathrm{scc}}(X)$, $Q\Psi_\beta$ converges weakly to 0.

Still following [8] we next show that

$$\lim_{\beta\to\infty} \|(H - \lambda - (x^2\partial_x F)^2)\Psi_\beta\| = 0. \tag{B.22}$$

In fact, by (B.10) we have

$$\limsup_{\beta\to\infty} \|(H-\lambda-(x^2\partial_x F)^2)\Psi_\beta\| = \limsup_{\beta\to\infty} \|(2((x^2\partial_x)F)x^2D_x + i\widetilde{R}_1)\Psi_\beta\|. \tag{B.23}$$

Now, $\widetilde{R}_1 = xR_1'$, $R_1' \in \mathrm{Diff}^1_{\mathrm{scc}}(X)$ with uniformly bounded coefficients. Thus, by (B.21), $\|R_1'\Psi_\beta\|_{L^2_{\mathrm{sc}}(X)} \le b_3$. Hence, for any $\delta > 0$

$$\|\widetilde{R}_1\Psi_\beta\|^2 \le \|\widetilde{R}_1\Psi_\beta\|^2_{L^2(B_\delta)} + \delta^2 b_3^2 \tag{B.24}$$

where $B_\delta = \{p \in X : x(p) \ge \delta\}$. Since $\widetilde{R}_1$ has uniformly bounded coefficients, (B.19) proves that

$$\limsup_{\beta\to\infty} \|(2((x^2\partial_x)F)x^2D_x + i\widetilde{R}_1)\Psi_\beta\| = \limsup_{\beta\to\infty} \|(2((x^2\partial_x)F)x^2D_x\Psi_\beta\|. \tag{B.25}$$

In fact, x^2D_x can be replaced by xA since the additional term also vanishes as $\beta \to \infty$. An explicit calculation shows that

$$x\partial_x(x^2\partial_x F)^2 - 2\gamma(\alpha+\gamma) \le b_4 x, \tag{B.26}$$

so from (B.12)

$$(\Psi_\beta, i[A,H]\Psi_\beta) \le -4\|(-ax^3\partial_x F)^{1/2}A\Psi_\beta\|^2 + 2\gamma(\alpha+\gamma) + (\Psi_\beta, R_5\Psi_\beta) \tag{B.27}$$

with R_5 uniformly bounded in $\mathrm{Diff}^1_{\mathrm{scc}}(X)$. In addition, $[H,A] \in \mathrm{Diff}^2_{3\mathrm{sc}}(X)$, so the left hand side, as well as the last term on the right hand side, are bounded as $\beta \to \infty$. This proves that

$$\|x^{1/2}(-x^2\partial_x F)^{1/2}A\Psi_\beta\| \le b_5. \tag{B.28}$$

Since $|x^2\partial_x F| \le b_6$, we conclude as above that

$$\lim_{\beta\to\infty} \|(x^2\partial_x F)xA\Psi_\beta\| = 0, \tag{B.29}$$

which proves (B.22). Since $|x^2\partial_x F| \le \alpha + \gamma$, we deduce that

$$\limsup_{\beta\to\infty} \|(H-\lambda-\alpha^2)\Psi_\beta\| \le 2\gamma\alpha + \gamma^2. \tag{B.30}$$

Hence, for $\widetilde{\phi} \in \mathcal{C}_c^\infty(\mathbb{R})$ supported in $(\Lambda-\varepsilon, \Lambda+\varepsilon)$, identically 1 on $(\Lambda-\varepsilon/2, \Lambda+\varepsilon/2)$, $\Lambda = \lambda + \alpha^2$, $\varepsilon < \lambda/2$ fixed, we see that

$$\limsup_{\beta\to\infty} \|(\mathrm{Id} - \widetilde{\phi}(H))\Psi_\beta\| \le \limsup \|(H-\Lambda)(2/\varepsilon)(\mathrm{Id}-\widetilde{\phi}(H))\Psi_\beta\| \le b_7\gamma, \tag{B.31}$$

and hence

$$\limsup_{\beta\to\infty} \|(H+i)(\mathrm{Id}-\widetilde{\phi}(H))\Psi_\beta\| \le b_8\gamma. \tag{B.32}$$

Now, from (B.29), writing $R_3 = xR_3'$ in (B.12), R_3' bounded in $S^0(X)$, and noting that $(x^2\partial_x F)^{-1}$ is bounded in $S^0(X)$ near $\mathrm{supp}\, A$,

$$\lim_{\beta\to\infty} (R_3\Psi_\beta, A\Psi_\beta) = \lim_{\beta\to\infty} ((x^2\partial_x F)^{-1}R_3'\Psi_\beta, x(x^2\partial_x F)A\Psi_\beta) = 0, \tag{B.33}$$

and similarly for the R_4 term in(B.12). Thus, from (B.27), taking into account the form of the R_5 term from (B.12) and (B.26),

$$\limsup_{\beta\to\infty}(\Psi_\beta, i[A,H]\Psi_\beta) \leq 2\gamma(\alpha+\gamma) \leq b_9\gamma, \tag{B.34}$$

and by (B.31) and (B.32) (using that $[A,H] \in \mathrm{Diff}^2_{3\mathrm{sc}}(X)$)

$$\limsup_{\beta\to\infty} \|[A,H](\mathrm{Id}-\widetilde{\phi}(H))\Psi_\beta\| \leq b_{10}\gamma. \tag{B.35}$$

Hence,

$$\limsup_{\beta\to\infty}(\Psi_\beta, \widetilde{\phi}(H)i[A,H]\widetilde{\phi}(H)\Psi_\beta) \leq b_{11}\gamma. \tag{B.36}$$

For small γ, however, this contradicts the Mourre estimate of Theorem 12.2 which, together with the weak convergence of Ψ_β to 0, implies that

$$\liminf_{\beta\to\infty}(\Psi_\beta, \widetilde{\phi}(H)i[A,H]\widetilde{\phi}(H)\Psi_\beta) \geq b_{12}\liminf_{\beta\to\infty}\|\widetilde{\phi}(H)\Psi_\beta\|^2 \geq b_{12}(1-b_{13}\gamma). \tag{B.37}$$

This contradiction proves the proposition. □

We next prove, following Froese and Herbst, that faster than exponential decay of an eigenfunction of H implies that it vanishes. As mentioned in the introduction, this requires more substantial modifications than the previous proof.

Proposition B.3 **(cf. Froese and Herbst, [8,** Theorem 3.1]). — *Let H be as in* (11.11), $\lambda \in \mathbb{R}$. *Suppose that $H\psi = \lambda\psi$, $\exp(\alpha/x)\psi \in L^2_{\mathrm{sc}}(X)$ for all α. Then $\psi = 0$.*

Proof. — Let $F = F_\alpha = \phi(x)\frac{\alpha}{x}$ where $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ is supported near 0, identically 1 in a smaller neighborhood of 0, and let $\psi_\alpha = e^F\psi$, $\Psi_\alpha = \psi_\alpha/\|\psi_\alpha\|$. Then (B.11) and (B.12) give

$$(\Psi_\alpha, H\Psi_\alpha) = \lambda + \alpha^2 + \alpha^2(\Psi_\alpha, xf_1\Psi_\alpha), \tag{B.38}$$

with $f_1 \in S^0(X)$ independent of α,

$$\begin{aligned}(\Psi_\alpha, i[A,H]\Psi_\alpha) = &-4\|(\alpha a x)^{1/2}A\Psi_\alpha\|^2 + (\Psi_\alpha, x(\alpha f_2 + \alpha^2 f_3)\Psi_\alpha)\\ &+\alpha(xA\Psi_\alpha, f_4\Psi_\alpha) + \alpha(f_5\Psi_\alpha, xA\Psi_\alpha),\end{aligned} \tag{B.39}$$

$f_j \in S^0(X)$, $j = 2,\dots,5$, independent of α. In addition we have

$$i[A,H] = 2\Delta + i[A,V] + xP \tag{B.40}$$

where $P \in \mathrm{Diff}^2_{\mathrm{sc}}(X)$. Also note that $[A,V] \in \mathcal{C}^\infty([X;C]) \subset L^\infty(X)$.

Since V is bounded and $\|\Psi_\alpha\| = 1$, it follows from (B.38) that $(\Psi_\alpha, \Delta\Psi_\alpha) \leq C(1+\alpha^2)$, so

$$\|d\Psi_\alpha\|_{L^2_{\mathrm{sc}}(X;{}^{\mathrm{sc}}\Lambda^1)} \leq C'(1+\alpha). \tag{B.41}$$

In particular, for $Q \in \mathrm{Diff}^1_{\mathrm{sc}}(X)$ we see that

$$\|Q\Psi_\alpha\| \leq C_1(1+\alpha). \tag{B.42}$$

Write $xP = P_1 x P_2$, $P_1, P_2 \in \mathrm{Diff}^1_{\mathrm{sc}}(X)$, and let C_2 be such that

$$\|P_1^* \Psi_\alpha\| \leq C_2(1+\alpha),\ \|P_2\Psi_\alpha\| \leq C_2(1+\alpha),\ \|xA\Psi_\alpha\| \leq C_2(1+\alpha). \tag{B.43}$$

In fact, we can improve the last statement by noting that in (B.39), by virtue of (B.43) and (B.40), every term but the first one on the right hand side is bounded by $C_2''(1+\alpha^2)$, so

$$\|(\alpha x)^{1/2} A\Psi_\alpha\| \leq C_2'(1+\alpha). \tag{B.44}$$

Let $\Omega_\delta = \{p \in X :\ x(p) \geq \delta\}$. Thus,

$$\|\psi_\alpha\|_{L^2(\Omega_\delta)} \leq C_3 e^{\alpha/\delta} \|\psi\|_{L^2_{\mathrm{sc}}(X)}. \tag{B.45}$$

Similarly, we can estimate the derivatives of ψ_α as well in $L^2(\Omega_\delta)$, taking into account that $|x^2\partial_x e^F| \leq C_4\alpha$, so

$$\|Q\psi_\alpha\|_{L^2(\Omega_\delta)} \leq C_5(1+\alpha)e^{\alpha/\delta}, \qquad Q \in \mathrm{Diff}^1_{\mathrm{sc}}(X) \tag{B.46}$$

and more generally

$$\|Q\psi_\alpha\|_{L^2(\Omega_\delta)} \leq C_6(1+\alpha^k)e^{\alpha/\delta}, \qquad Q \in \mathrm{Diff}^k_{\mathrm{sc}}(X). \tag{B.47}$$

Here C_6 is independent of α and δ; it only depends on Q. Let C_7 be such that

$$\|\psi_\alpha\|_{L^2(\Omega_\delta)} \leq C_7 e^{\alpha/\delta} \tag{B.48}$$

and for each of $Q = xA, P_1^*, P_2$

$$\|\psi_\alpha\|_{L^2(\Omega_\delta)} \leq C_7(1+\alpha)e^{\alpha/\delta}. \tag{B.49}$$

Let $C_8 = \max_j(\sup|xf_j| + \sup|x^{1/2}f_j|) + \sup x$, and choose $\delta \in (0, \varepsilon/2)$ so that

$$\delta(1 + C_2 + C_2^2)(1 + \sup|f_j|) < \frac{1}{8} \tag{B.50}$$

for all j. Then

$$|(\Psi_\alpha, xf_j\Psi_\alpha)| \leq C_8\|\Psi_\alpha\|^2_{L^2(\Omega_\delta)} + \delta\sup|f_j|\|\Psi_\alpha\|^2, \quad j = 1,2,3, \tag{B.51}$$

so

$$|(\Psi_\alpha, xf_j\Psi_\alpha)| \leq C_8 C_7^2 e^{2\alpha/\delta}\|\psi_\alpha\|^{-2} + \frac{1}{8}, \quad j = 1,2,3, \tag{B.52}$$

Similarly

$$|(P_1^*\Psi_\alpha, xP_2\Psi_\alpha)| \leq C_8 C_7^2 (1+\alpha)^2 e^{2\alpha/\delta}\|\psi_\alpha\|^{-2} + \frac{1}{8}(1+\alpha)^2. \tag{B.53}$$

Finally,

$$|(x^{1/2}A\Psi_\alpha, x^{1/2}f_4\Psi_\alpha)| \leq C_2' C_8 (1+\alpha)^{1/2} \tag{B.54}$$

and analogously for f_5.

We now assume that $\mathrm{supp}\,\psi \cap \{p :\ x(p) \leq \delta/4\}$ is not empty; soon we obtain a contradiction. Under this assumption

$$\|\psi_\alpha\|_{L^2_{\mathrm{sc}}(X)} \geq e^{2\alpha/\delta}\|\psi\|_{L^2_{\mathrm{sc}}(\{x \leq \delta/2\})} \geq C_9 e^{2\alpha/\delta} \tag{B.55}$$

with $C_9 > 0$. Hence, our estimates above and (B.38), together with $|(\Psi_\alpha, V\Psi_\alpha)| \leq \sup|V|$ show that

$$(\Psi_\alpha, \Delta\Psi_\alpha) \geq \alpha^2 - C_{10} - \alpha^2\Big(C_{11}e^{-2\alpha/\delta} + \frac{1}{8}\Big). \tag{B.56}$$

Similarly, from (B.39), using (B.40), $[A, V] \in L^\infty(X)$, and that the first term on the right hand side of (B.39) is negative, we have

$$(\Psi_\alpha, 2\Delta\Psi_\alpha) \leq C_{12}(1+\alpha)^{3/2} + ((\alpha+\alpha^2) + (1+\alpha)^2)\Big(C_{12}e^{-2\alpha/\delta} + \frac{1}{8}\Big). \tag{B.57}$$

Thus, for sufficiently large α, (B.56) shows that

$$(\Psi_\alpha, \Delta\Psi_\alpha) \geq \frac{3}{4}\alpha^2, \tag{B.58}$$

while (B.57) implies for large α that

$$(\Psi_\alpha, \Delta\Psi_\alpha) \leq \frac{1}{2}\alpha^2, \tag{B.59}$$

providing the contradiction. Hence, $\operatorname{supp}\psi$ is a compact subset of the interior of X. Then the standard Carleman-type unique continuation theorem [**18**, Theorem 17.2.1] implies that ψ vanishes identically as claimed. □

The absence of positive eigenvalues is just a combination of the previous two propositions. Thus, we have proved Theorem 17.6.

APPENDIX C

POSITIVE OPERATORS

In this chapter we show that, roughly speaking, the positivity of the indicial operators of $A \in \Psi_{3sc}^{-\infty,0}(X)$ implies the positivity of A modulo compact operators. We prove this by constructing an approximate square root of A.

Throughout this section we assume that H is a three-body Hamiltonian. We start with the basic square root construction.

Lemma C.1. — *Suppose that H is a three-body Hamiltonian and $\lambda \in \mathbb{R}$. Suppose also that $A \in \Psi_{3sc}^{-\infty,0}(X)$ is self-adjoint, and for some $c > 0$ and $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ which is identically 1 near λ,*

$$\psi(H)A\psi(H) \geq c\psi(H)^2. \tag{C.1}$$

Then for any $c' \in (0, c)$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that

$$\operatorname{supp}\phi \cap \operatorname{supp}(1-\psi) = \varnothing, \tag{C.2}$$

there exists $B \in \Psi_{3sc}^{-\infty,0}(X)$ such that

$$\phi(H)(A - c')\phi(H) = \phi(H)B^*B\phi(H). \tag{C.3}$$

Proof. — Let

$$P = \psi(H)A\psi(H) + c(\mathrm{Id} - \psi(H)^2) \in \Psi_{3sc}^{0,0}(X). \tag{C.4}$$

Thus, $P \geq c$, so $P - c' \geq c - c' > 0$. Since the spectrum of $P - c'$ is a subset of $[c - c', \infty)$ and $c - c' > 0$, we have $(P - c')^{1/2} = f(P - c')$ where $f \in \mathcal{C}_c^\infty(\mathbb{R})$ and $f(t) = \sqrt{t}$ if t is in the spectrum of $P - c'$. By Proposition 10.3, or rather its modification for $m = 0$, in which case ellipticity of $\sigma_{3sc,m}(P)$ is not required just as the indicial operators of P need not to invertible,

$$Q = (P - c')^{1/2} = f(P - c') \in \Psi_{3sc}^{0,0}(X). \tag{C.5}$$

Let ψ_1 be identically 1 near $\operatorname{supp}\phi$ and vanish near $\operatorname{supp}(1-\psi)$. Then

$$\psi_1(H)Q^2\psi_1(H) = \psi_1(H)(P - c')\psi_1(H) = \psi_1(H)(A - c')\psi_1(H). \tag{C.6}$$

Let

$$B = Q\psi_1(H) \in \Psi_{3sc}^{-\infty,0}(X). \tag{C.7}$$

Multiplying (C.6) from both sides by $\phi(H)$ then proves (C.3). □

We now show that under certain additional assumptions, the positivity of the indicial operators implies positivity of the operator modulo lower order (hence compact) terms in the calculus. We start by assuming strict positivity of the indicial operators when localized in the spectrum of H.

Proposition C.2. — *Suppose that H is a three-body Hamiltonian and $\lambda \in \mathbb{R}$. Suppose also that $A, C \in \Psi_{3sc}^{-\infty,0}(X)$ are self-adjoint and $\widehat{C}_{a,0}(\xi) = c_a(\xi)\psi_0(\widehat{H}_a(\xi))^2$ for $a = \mathrm{mf}, \mathrm{ff}$ and $\xi \in {}^{3sc}T^*_{\mathrm{mf}}[X;C]$, $\xi \in W^\perp$ respectively, where $c_a(\xi)$ is a function with $c_a(\xi) > 0$, $\psi_0 \equiv 1$ near $\lambda \in \mathbb{R}$, $\psi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$. Assume in addition that there exists $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ which is identically 1 near λ, $\operatorname{supp}\psi \cap \operatorname{supp}(1-\psi_0) = \varnothing$, such that*

$$\psi(\widehat{H}_a(\xi))\widehat{A}_a(\xi)\psi(\widehat{H}_a(\xi)) \geq \psi(\widehat{H}_a(\xi))c_a(\xi)\psi(\widehat{H}_a(\xi)) \tag{C.8}$$

for $a = \mathrm{mf}, \mathrm{ff}$ and $\xi \in {}^{3sc}T^*_{\mathrm{mf}}[X;C]$, $\xi \in W^\perp$ respectively. Then for any $\varepsilon \in (0,1)$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ with

$$\operatorname{supp}\phi \cap \operatorname{supp}(1-\psi) = \varnothing, \tag{C.9}$$

there exists $R \in \Psi_{3sc}^{-\infty,1}(X)$ such that

$$\phi(H)A\phi(H) \geq (1-\varepsilon)\phi(H)C\phi(H) + R. \tag{C.10}$$

Proof. — We apply a parameter dependent version of the previous lemma to the indicial operators to conclude that for each ξ there exists $\widehat{B}_a(\xi)$ with

$$\phi(\widehat{H}_a(\xi))(\widehat{A}_a(\xi) - (1-\varepsilon)\widehat{C}_a(\xi))\phi(\widehat{H}_a(\xi)) = \phi(\widehat{H}_a(\xi))\widehat{B}_a(\xi)^*\widehat{B}_a(\xi)\phi(\widehat{H}_a(\xi)). \tag{C.11}$$

It follows from the Cauchy integral formula construction of the square root in the calculus and the explicit formulae (C.4), (C.5) and (C.7) that the indicial operators $\widehat{B}_a(\xi)$ match up so that there exists $B \in \Psi_{3sc}^{-\infty,0}(X)$ with indicial operators $\widehat{B}_a(\xi)$. Here note that the set where $\psi(\widehat{H}_a(\xi))$ does not vanish has compact closure, hence c is bounded below on it by a positive constant. Thus, we can take the same smooth function f in the expression (C.5) for the square root for every a and ξ. By (C.11),

$$\phi(H)(A - (1-\varepsilon)C)\phi(H) = \phi(H)B^*B\phi(H) + R \tag{C.12}$$

with $R \in \Psi_{3sc}^{-\infty,1}(X)$. Since $\phi(H)B^*B\phi(H) \geq 0$, rearranging this proves the proposition. □

Similar conclusions hold if we allow the vanishing of the indicial operators of A, but compensate by requiring matching upper bounds on A.

Proposition C.3. — *Suppose that H is a three-body Hamiltonian and $\lambda \in \mathbb{R}$. Suppose also that $A, C \in \Psi_{3sc}^{-\infty,0}(X)$ are self-adjoint and $\widehat{C}_{a,0}(\xi) = c_a(\xi)\psi_0(\widehat{H}_a(\xi))^2$ for $a = \mathrm{mf}, \mathrm{ff}$ and $\xi \in {}^{3sc}T^*_{\mathrm{mf}}[X;C]$, $\xi \in W^\perp$ respectively, where $c_a(\xi)$ is a function with $c_a(\xi) \geq 0$ with $\sqrt{c_a} \in \mathcal{C}^\infty({}^{3sc}T^*_{\mathrm{mf}}[X;C])$ resp. $\sqrt{c_a} \in \mathcal{C}^\infty(W^\perp)$ vanishing to infinite order at points ξ with $c_a(\xi) = 0$, $\psi_0 \equiv 1$ near $\lambda \in \mathbb{R}$, $\psi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$, $\widehat{A}_a(\xi) = 0$ if $c_a(\xi) = 0$, and for any differential operator $Q \in \mathrm{Diff}({}^{3sc}T^*_{\mathrm{mf}}[X;C])$ or $Q \in \mathrm{Diff}(W^\perp)$ (as $a = \mathrm{mf}$ or $a = \mathrm{ff}$), $Q(c_a(\xi)^{-1}\widehat{A}_a(\xi))$, resp. all seminorms of $Q(c_a(\xi)^{-1}\widehat{A}_a(\xi))$ in $\Psi_{sc}^{-\infty,0}(\beta^{-1}(p))$, $\xi \in W_p^\perp$, are uniformly bounded on the set of ξ's with $c_a(\xi) > 0$.*

Assume in addition that there exists $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ which is identically 1 near λ, $\operatorname{supp}\psi \cap \operatorname{supp}(1-\psi_0) = \varnothing$, such that

$$\psi(\widehat{H}_a(\xi))\widehat{A}_a(\xi)\psi(\widehat{H}_a(\xi)) \geq \psi(\widehat{H}_a(\xi))c_a(\xi)\psi(\widehat{H}_a(\xi)) \tag{C.13}$$

for $a = \mathrm{mf}, \mathrm{ff}$ and $\xi \in {}^{3sc}T^*_{\mathrm{mf}}[X;C]$, $\xi \in W^\perp$ respectively. Then the conclusion of the previous proposition holds, i.e. for any $\varepsilon \in (0,1)$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ with

$$\operatorname{supp}\phi \cap \operatorname{supp}(1-\psi) = \varnothing, \tag{C.14}$$

there exists $R \in \Psi_{3sc}^{-\infty,1}(X)$, with seminorms bounded by those of A and C in $\Psi_{3sc}^{-\infty,0}(X)$, and with $\mathrm{WF}'_{3sc}(R) \subset \mathrm{WF}'_{3sc}(A) \cup \mathrm{WF}'_{3sc}(C)$ such that

$$\phi(H)A\phi(H) \geq (1-\varepsilon)\phi(H)C\phi(H) + R. \tag{C.15}$$

Proof. — We define $\widehat{B}_a(\xi) = 0$ if $c_a(\xi) = 0$, otherwise we define $\widehat{B}_a(\xi)$ as in the previous proposition. The only additional ingredient is the analysis of $\widehat{B}_a(\xi)$ near ξ with $c_a(\xi) = 0$. To do this analysis, we follow the construction of $\widehat{B}_a(\xi)$ in detail. So let

$$\widehat{P}_a(\xi) = \psi(\widehat{H}_a(\xi))\widehat{A}_a(\xi)\psi(\widehat{H}_a(\xi)) + c_a(\xi)(\mathrm{Id} - \psi(\widehat{H}_a(\xi))^2), \tag{C.16}$$

and let

$$c'_a(\xi) = (1-\varepsilon)c_a(\xi). \tag{C.17}$$

Thus, $\widehat{P}_a(\xi) - c'_a(\xi) \geq \varepsilon c_a(\xi)$. Let

$$\widehat{Q}_a(\xi) = (\widehat{P}_a(\xi) - c'_a(\xi))^{1/2} = c_a(\xi)^{1/2}(c_a(\xi)^{-1}\widehat{P}_a(\xi) - (1-\varepsilon))^{1/2}. \tag{C.18}$$

By our assumption, there exists $M > 0$ such that the norm of $\widehat{P}_a(\xi)$ in $\mathcal{B}(L^2, L^2)$ is bounded by $Mc_a(\xi)$. Now choose $f \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $f(t) = \sqrt{t}$ on $[1-\varepsilon, M]$. Then $M \geq c_a(\xi)^{-1}\widehat{P}_a(\xi) - 1 + \varepsilon \geq \varepsilon$, so

$$\widehat{Q}_a(\xi) = c_a(\xi)^{1/2} f(c_a(\xi)^{-1}\widehat{P}_a(\xi) - (1-\varepsilon)). \tag{C.19}$$

By our assumptions, for $a = \mathrm{mf}$, $c_a(\xi)^{-1}\widehat{P}_a(\xi)$, $\xi \in {}^{3sc}T^*_{\mathrm{mf}}[X;C]$, resp. for $a = \mathrm{ff}$, the seminorms of $c_a(\xi)^{-1}\widehat{P}_a(\xi)$ in $\Psi_{sc}^{0,0}(\beta^{-1}(p))$, $\xi \in W_p^\perp$, remain uniformly bounded as $c_a(\xi) \to 0$, so the Cauchy integral representation of f, via an almost analytic extension, shows that $f(c_a(\xi)^{-1}\widehat{P}_a(\xi) - (1-\varepsilon))$ remains uniformly bounded. Thus, $\widehat{Q}_a(\xi)$ is continuous as a function on ${}^{3sc}T^*_{\mathrm{mf}}[X;C]$ or $W^\perp$ (again as $a = \mathrm{mf}$ or $a = \mathrm{ff}$)

with values in reals and Ψ_{sc} respectively. A similar argument also holds for the derivatives of $\widehat{Q}_a(\xi)$. Let ψ_1 be identically 1 near $\operatorname{supp}\phi$ and vanish near $\operatorname{supp}(1-\psi)$, and let

$$\widehat{B}_a(\xi) = \widehat{Q}_a(\xi)\psi_1(H). \tag{C.20}$$

Again, the $\widehat{B}_a(\xi)$ match up so there exists $B \in \Psi_{3\mathrm{sc}}^{-\infty,0}(X)$ with these indicial operators. We can also make sure that the lower order terms also vanish where c does, i.e. that $\mathrm{WF}'_{3\mathrm{sc}}(B) \subset \operatorname{supp} c$. Then the indicial operators of $\phi(H)(A-(1-\varepsilon)C)\phi(H)$ and $\phi(H)B^*B\phi(H)$ are the same, so

$$\phi(H)(A-(1-\varepsilon)C)\phi(H) = \phi(H)B^*B\phi(H) + R \tag{C.21}$$

with $R \in \Psi_{3\mathrm{sc}}^{-\infty,1}(X)$, proving the proposition. □

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