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CAYLEY TRANSFORM AND GENERALIZED WHITTAKER MODELS FOR IRREDUCIBLE HIGHEST WEIGHT MODULES

by

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Dedicated to Professor Ryoshi Hotta on his sixtieth birthday

Abstract. — We study the generalized Whittaker models for irreducible admissible highest weight modules $L(\tau)$ for a connected simple Lie group G of Hermitian type, by using certain invariant differential operators $\mathcal{D}_{\tau^*}$ of gradient type on the Hermitian symmetric space $K\backslash G$. It is shown that each $L(\tau)$ embeds, with nonzero and finite multiplicity, into the generalized Gelfand-Graev representation $\Gamma_{m(\tau)}$ attached to the unique open orbit $\mathcal{O}_{m(\tau)}$ (through the Kostant-Sekiguchi correspondence) in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. The embeddings can be intrinsically analyzed by means of the Cayley transform which carries the bounded realization of $K\backslash G$ to unbounded one. If $L(\tau)$ is unitarizable, the space $\mathcal{Y}(\tau)$ of infinitesimal homomorphisms from $L(\tau)$ into $\Gamma_{m(\tau)}$ can be described in terms of the principal symbol at the origin of the differential operator $\mathcal{D}_{\tau^*}$. For the classical groups $G = SU(p,q)$, $Sp(n,\mathbb{R})$ and $SO^*(2n)$, the space $\mathcal{Y}(\tau)$ is clearly understood through the oscillator representations of reductive dual pairs.

***Résumé* (Transformation de Cayley et modèles de Whittaker généralisés pour les modules irréductibles de plus haut poids)**

Soit G un groupe de Lie connexe simple de type hermitien. On considère les G-modules irréductibles admissibles $L(\tau)$ de plus haut poids. Dans cet article, nous étudions les modèles de Whittaker généralisés pour $L(\tau)$ en utilisant certains opérateurs différentiels de type gradient $\mathcal{D}_{\tau^*}$ sur l'espace hermitien symétrique $K\backslash G$. Il est montré que chaque $L(\tau)$ apparaît, avec une multiplicité finie et non nulle, dans la représentation de Gelfand-Graev généralisée $\Gamma_{m(\tau)}$ qui est attachée à l'unique orbite ouverte $\mathcal{O}_{m(\tau)}$ (par la correspondance de Kostant-Sekiguchi) dans la variété $\mathcal{V}(L(\tau))$ associée à $L(\tau)$. On peut analyser intrinsèquement les isomorphismes de $L(\tau)$ dans $\Gamma_{m(\tau)}$ au moyen de la transformation de Cayley qui donne un rapport entre la réalisation de $K\backslash G$ comme domaine borné et celle comme domaine non borné. Si $L(\tau)$ est unitarisable, l'espace $\mathcal{Y}(\tau)$ des homomorphismes infinitésimaux de $L(\tau)$ dans $\Gamma_{m(\tau)}$ s'exprime par le symbole principal à l'origine de l'opérateur différentiel $\mathcal{D}_{\tau^*}$. Pour les groupes classiques $G = SU(p,q)$, $Sp(n,\mathbb{R})$ et $SO^*(2n)$, on peut comprendre l'espace $\mathcal{Y}(\tau)$ en utilisant les représentations oscillateur pour les paires duales réductives.

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Introduction

For a semisimple algebraic group G, the generalized Gelfand-Graev representations introduced by Kawanaka **[14]** form a family of representations of G induced from certain one-dimensional characters of various unipotent subgroups. By construction, each of these induced G-modules is naturally attached to a nilpotent G-orbit $\mathcal{O}_G$ in the Lie algebra through the Dynkin-Kostant theory. The original (non generalized) Gelfand-Graev representations are induced from nondegenerate characters of a maximal unipotent subgroup, and they correspond to the principal nilpotent orbits. We say that an irreducible representation π of G has a generalized Whittaker model of type $\mathcal{O}_G$ if π admits an embedding into the generalized Gelfand-Graev representation attached to $\mathcal{O}_G$. The problem of describing the generalized Whittaker models is important not only in representation theory but also in connection with the theory of automorphic forms.

Generalized Whittaker models (or vectors) for irreducible representations of G have been studied by many authors (e.g., **[14]**, **[15]**, **[26]**, **[22]**, **[24]**, **[39]**, etc.). For real or complex groups, it is Kostant **[18]** who initiated a systematic study on the existence of nonzero Whittaker vectors attached to the principal nilpotent orbits of quasi-split groups, in connection with the primitive ideals of the irreducible representations in question. Later, some results of Kostant have been extended by Matumoto to those on generalized Whittaker vectors associated to arbitrary (not necessarily principal) nilpotent orbits $\mathcal{O}_G$. In fact, it is shown in **[22]** that the Harish-Chandra module of an irreducible admissible representation π has a nonzero generalized Whittaker vector of type $\mathcal{O}_G$ only if the nilpotent orbit $\mathcal{O}_G$ is contained in the associated variety of the primitive ideal $\operatorname{Ann}\pi$ in the universal enveloping algebra. For complex groups G, one of the main results in **[24]** tells us that, under certain assumptions on $\mathcal{O}_G$ and on π, the space of $C^{-\infty}$-generalized Whittaker vectors of type $\mathcal{O}_G$ is nonzero and finite-dimensional if and only if the closure of $\mathcal{O}_G$ coincides with the wave front set of π.

As to p-adic groups, Mœglin and Waldspurger have already established in 1987 a stronger result of this nature, by showing that the wave front cycle (asymptotic cycle) of an irreducible representation π of G completely controls the spaces of generalized Whittaker vectors of interest. Namely, it is proved in **[26]** that, if $\mathcal{O}_G$ is a nilpotent orbit which is maximal in the wave front set (asymptotic support) of π, the dimension of the space of generalized Whittaker vectors of type $\mathcal{O}_G$ is equal to the multiplicity of $\mathcal{O}_G$ in the wave front cycle. However, up to this time, the corresponding phenomenon is not yet fully understood in the case of real groups, except for the representations with the largest Gelfand-Kirillov dimension (see **[23]** and **[25]**).

In this article, we focus our attention on the irreducible admissible (unitary) highest weight representations of real simple Lie groups. These are representations with rather small Gelfand-Kirillov dimensions. We reveal a structure of the spaces of generalized Whittaker models in relation to the associated cycles of highest weight modules.

Now, let G be a connected simple Lie group with finite center, and let K be a maximal compact subgroup of G. Assume that $K\backslash G$ is Hermitian symmetric. The Lie algebras of G and K are denoted by $\mathfrak{g}_0$ and $\mathfrak{k}_0$ respectively. We write $K_{\mathbb{C}}$ (resp. $\mathfrak{g}, \mathfrak{k}$) for the complexifications of K (resp. $\mathfrak{g}_0, \mathfrak{k}_0$) respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a complexified Cartan decomposition of $\mathfrak{g}$, and let θ denote the corresponding Cartan involution of $\mathfrak{g}$. The G-invariant complex structure on $K\backslash G$ gives a triangular decomposition $\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{k} + \mathfrak{p}_-$ of $\mathfrak{g}$. Conventionally, the complexification in $\mathfrak{g}$ of any real vector subspace $\mathfrak{s}_0$ of $\mathfrak{g}_0$ will be denoted by $\mathfrak{s}$ by dropping the subscript 0. We write $U(\mathfrak{m})$ (resp. $S(\mathfrak{v})$) for the universal enveloping algebra of a Lie algebra $\mathfrak{m}$ (resp. the symmetric algebra of a vector space $\mathfrak{v}$).

The group G of Hermitian type has a distinguished family of irreducible admissible Hilbert representations with highest weights. The Harish-Chandra module of such a G-representation is obtained as the unique simple quotient $L(\tau)$ of generalized Verma module induced from an irreducible representation (τ, V_τ) of K. Here τ is extended to a representation of the maximal parabolic subalgebra $\mathfrak{q} := \mathfrak{k} + \mathfrak{p}_+$ of $\mathfrak{g}$ by making $\mathfrak{p}_+$ act on V_τ trivially. We call τ the extreme K-type of $L(\tau)$.

The purpose of this paper is to describe the generalized Whittaker models for irreducible highest weight $(\mathfrak{g}, K)$-modules $L(\tau)$. To be more precise, let $\{\mathcal{O}_m \mid m = 0, \ldots, r\}$ be the totality of nilpotent $K_{\mathbb{C}}$-orbits in the nilradical $\mathfrak{p}_+$ of $\mathfrak{q}$, arranged as $\dim \mathcal{O}_0 = 0 < \dim \mathcal{O}_1 < \cdots < \dim \mathcal{O}_r = \dim \mathfrak{p}_+$. We write $\mathcal{O}'_m$ for the the nilpotent G-orbit in $\mathfrak{g}_0$ corresponding to $\mathcal{O}_m$ by the Kostant-Sekiguchi bijection. Following the recipe by Kawanaka [**14**] (see also [**40**]), we can construct a generalized Gelfand-Graev representation $\Gamma_m = \mathrm{Ind}_{\mathfrak{n}(m)}^G(\eta_m)$ (GGGR for short; see Definition 4.3) of G attached to $\mathcal{O}'_m$. On the other hand, it is well-known that the associated variety $\mathcal{V}(L(\tau))$ of a highest weight module $L(\tau)$ is the closure of a single $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m(\tau)}$ in $\mathfrak{p}_+$, where $m(\tau)$ depends on τ. Then our aim is to specify the $(\mathfrak{g}, K)$-embeddings of $L(\tau)$ into these GGGRs Γ_m $(m = 0, \ldots, r)$. This is a continuation of our earlier work [**41**] on Whittaker models for the holomorphic discrete series.

In order to specify the embeddings, we use the invariant differential operator $\mathcal{D}_{\tau^*}$ on $K\backslash G$ of gradient type associated to the K-representation τ^* dual to τ (Definition 2.3). This operator $\mathcal{D}_{\tau^*}$ is due to Enright, Davidson and Stanke ([**2**], [**3**], [**4**]). The K-finite kernel of $\mathcal{D}_{\tau^*}$ realizes the dual lowest weight module $L(\tau)^*$. By virtue of the kernel theorem given as Corollary 1.8, we find that the space $\mathcal{Y}(\tau, m)$ of η_m-covariant solutions of the differential equation $\mathcal{D}_{\tau^*} F = 0$ is isomorphic to the space of $(\mathfrak{g}, K)$-homomorphisms in question, where η_m is the character of nilpotent Lie subalgebra $\mathfrak{n}(m)$ of $\mathfrak{g}$ that defines Γ_m.

The space $\mathcal{Y}(\tau, m)$ can be intrinsically analyzed by means of the unbounded realization of $K\backslash G$ via the Cayley transform (cf. [**32**], [**9**]). Some remarkable results of

Enright and Joseph **[5]**, Jakobsen **[20]** on the annihilator ideal of (unitarizable) highest weight modules are useful in the course of our study. Also, elementary properties (cf. Vogan **[33**, Section 2]) on the associated (characteristic) cycle of Harish-Chandra modules guarantee that the space $\mathcal{Y}(\tau, m)$ does not vanish for the most relevant $m = m(\tau)$. As a result, we get the following conclusions (see Theorems 4.7–4.9).

Theorem 1. — *$L(\tau)$ embeds into the GGGR Γ_m with nonzero and finite multiplicity if and only if the corresponding $\mathcal{O}_m$ is the unique open $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m(\tau)}$ in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. In this case, the space $\mathcal{Y}(\tau) := \mathcal{Y}(\tau, m(\tau))$ consists only of elementary functions on the unbounded domain $\mathcal{S}\,(\subset \mathfrak{p}_-)$ which realizes $K\backslash G$.*

Theorem 2. — *If $L(\tau)$ is unitarizable, we can specify the space $\mathcal{Y}(\tau)$ in terms of the principal symbol at the origin Ke of the differential operator $\mathcal{D}_{\tau^*}$. This reveals a natural action on $\mathcal{Y}(\tau)$ of the isotropy subgroup $K_{\mathbb{C}}(X)$ of $K_{\mathbb{C}}$ at a certain point $X \in \mathcal{O}_{m(\tau)}$. Furthermore, we find that the dimension of $\mathcal{Y}(\tau)$, that is, the multiplicity of embeddings $L(\tau) \hookrightarrow \Gamma_{m(\tau)}$, coincides with the multiplicity of the $S(\mathfrak{p}_-)$-module $L(\tau)$ at the defining ideal of $\mathcal{V}(L(\tau))$.*

For the classical groups $G = SU(p,q)$, $Sp(2n,\mathbb{R})$ and $SO^*(2n)$, the theory of reductive dual pair gives explicit realizations of unitarizable highest weight modules $L(\tau)$ (cf. **[12]**, **[7]**, **[3]**). In this setting, it is not difficult to specify the generalized Whittaker models for such $L(\tau)$'s more explicitly by using the oscillator representation of a pair (G, G') with a compact group G' dual to G. In fact, this has been done by Tagawa **[31]** for the case $SU(p,q)$, motivated by author's observation in 1997 for the case $Sp(n,\mathbb{R})$. We include this observation as well as Tagawa's result at the end of this paper (see Theorems 5.14 and 5.15 together with the isomorphism (4.15)), handling all the groups $SU(p,q)$, $Sp(2n,\mathbb{R})$ and $SO^*(2n)$ in a unified manner.

The last statement in Theorem 2 clarifies the relationship between the generalized Whittaker models and the multiplicity in the associated cycle $\mathcal{AC}(L(\tau))$ of unitarizable highest weight module $L(\tau)$. In fact, $\mathcal{Y}(\tau)$ turns to be the dual of the isotropy representation of $K_{\mathbb{C}}(X)$ attached to $\mathcal{AC}(L(\tau))$ in the sense of Vogan **[33]**. We note that the associated cycle and the Bernstein degree of $L(\tau)$ have been specified by Nishiyama, Ochiai and Taniguchi **[27]** for the above classical groups G through detailed study of K-types of $L(\tau)$, where $L(\tau)$ is assumed to be an irreducible constituent of the oscillator representations of pairs (G, G') in the stable range (with smaller G'). Recently, Kato and Ochiai **[13]** have generalized the technique in **[27]** to a large extent. They established in particular a unified formula for the degrees of nilpotent orbits $\mathcal{O}_m$, which is valid for any simple Lie group of Hermitian type.

An η_m-equivariant linear form on $L(\tau)$ is called an (algebraic) generalized Whittaker vector of type η_m. Each $(\mathfrak{g}, K)$-embedding of $L(\tau)$ into the GGGR Γ_m, composed with the evaluation at the identity $e \in G$ of functions in Γ_m, naturally gives rise to a generalized Whittaker vector of type η_m on $L(\tau)$. We can show that the converse is also true for the most relevant case $m = m(\tau)$. Namely, it turns out that every generalized Whittaker vector of type η_m comes from a function in the space $\mathcal{Y}(\tau)$ for any $L(\tau)$ (see Proposition 4.19). This allows us to interpret the main results of this article in terms of algebraic generalized Whittaker vectors associated to irreducible highest weight $(\mathfrak{g}, K)$-modules (Theorem 4.22).

We organize this paper as follows.

Section 1 gives general theory on the embeddings of irreducible $(\mathfrak{g}, K)$-modules into induced G-representations. The kernel theorem (Corollary 1.8) is our main tool for studying generalized Whittaker models. We introduce in Section 2 the differential operator $\mathcal{D}_{\tau^*}$ on $K\backslash G$ of gradient type associated to τ^*, after [**4**]. In addition, the solutions F of $\mathcal{D}_{\tau^*}F = 0$ of exponential type are specified in Proposition 2.8. Section 3 is devoted to characterizing the associated variety and multiplicity of irreducible highest weight module $L(\tau)$ by means of the principal symbol of $\mathcal{D}_{\tau^*}$ (Theorem 3.11). In Section 4 we give our main results (Theorems 4.7–4.9) that describe the generalized Whittaker models for $L(\tau)$. Relation to algebraic generalized Whittaker vectors is also investigated. Last in Section 5, we discuss the case of classical groups $SU(p,q)$, $Sp(2n,\mathbb{R})$ and $SO^*(2n)$ more explicitly.

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1. Embeddings of Harish-Chandra modules

This section prepares some generalities about the embeddings of irreducible Harish-Chandra modules into C^∞-induced representations of a semisimple Lie group, by developing our earlier observation [**42**, I, §2] for the discrete series in full generality. The results stated in this section are more or less folklore for the experts, or they are consequences of some known facts concerning the maximal globalization of Harish-Chandra modules due to Schmid and Kashiwara (cf. [**29**], [**11**]). Nevertheless we include here the detail with direct proofs in order to keep this paper more accessible and self-contained. In fact, a kernel theorem, Corollary 1.8, will be essentially used in the succeeding sections to describe generalized Whittaker models for highest weight representations.

1.1. A duality of Peter-Weyl type. — Throughout this section, let G be any connected semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. We keep the same notation and convention employed at the beginning of Introduction.

A $U(\mathfrak{g})$-module $\boldsymbol{X}$ is called a $(\mathfrak{g}, K)$-*module* if the subalgebra $U(\mathfrak{k})$ acts on $\boldsymbol{X}$ locally finitely, and if the $\mathfrak{k}_0$-action gives rise to a representation of K on $\boldsymbol{X}$ through exponential map. By a *Harish-Chandra module*, we mean a $(\mathfrak{g}, K)$-module of finite length as a $U(\mathfrak{g})$-module. By basic results of Harish-Chandra (see e.g., [**35**, Chap.3]), any admissible (i.e., K-multiplicity finite) representation of G on a Hilbert space $\boldsymbol{H}$ yields, through differentiation, a $(\mathfrak{g}, K)$-module structure on the subspace $\boldsymbol{H}_K$ of all K-finite vectors in $\boldsymbol{H}$. The continuous G-module $\boldsymbol{H}$ is irreducible if and only if the corresponding $\boldsymbol{H}_K$ is irreducible as a $(\mathfrak{g}, K)$-module. Each irreducible $(\mathfrak{g}, K)$-module $\boldsymbol{X}$ can be extended to an irreducible Hilbert G-module $\boldsymbol{H}$ with K-finite part $\boldsymbol{H}_K = \boldsymbol{X}$. Notice that the $(\mathfrak{g}, K)$-module corresponding to the irreducible G-module $\boldsymbol{H}^*$ contragredient to $\boldsymbol{H}$ is isomorphic to the K-finite part of the full dual space $\boldsymbol{X}' = \mathrm{Hom}_{\mathbb{C}}(\boldsymbol{X}, \mathbb{C})$. We denote this irreducible $(\mathfrak{g}, K)$-module by $\boldsymbol{X}^*$, and call it the dual Harish-Chandra module of $\boldsymbol{X}$.

We study in this paper the embeddings of irreducible $(\mathfrak{g}, K)$-modules $\boldsymbol{X}$ into certain smoothly induced Fréchet G-modules $\boldsymbol{F}$. Such an $\boldsymbol{F}$ has a compatible $\mathfrak{g}$ and K module structure through differentiation, and its K-finite part $\boldsymbol{F}_K$ is a $(\mathfrak{g}, K)$-module. We note that the image of $\boldsymbol{X}$ by any $\mathfrak{g}$ and K homomorphism into $\boldsymbol{F}$ is necessarily contained in $\boldsymbol{F}_K$, i.e., $\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X}, \boldsymbol{F}) = \mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X}, \boldsymbol{F}_K)$.

The group G acts on the space $C^\infty(G)$ of all smooth functions on G by left translation and by right translation as follows:

$$g^L f(x) := f(g^{-1}x), \quad g^R f(x) := f(xg) \quad (g \in G, x \in G, f \in C^\infty(G)).$$

These two actions L and R commute with each other. Through differentiation one gets two $U(\mathfrak{g})$-representations on $C^\infty(G)$ denoted again by L and R respectively. Let $C_K^\infty(G)$ be the space of functions $f \in C^\infty(G)$ which are left K-finite and also right K-finite. Then $C_K^\infty(G)$ becomes a $(\mathfrak{g}, K)$-module through L or R.

If the group G is compact, i.e., $G = K$, the regular representation $(L\hat{\otimes}R, C_G^\infty(G))$ of $G \times G$ decomposes into irreducibles as

$$C_G^\infty(G) \simeq \bigoplus_{\delta \in \hat{G}} V_\delta \otimes V_\delta^* \quad \text{as } G \times G\text{-modules}$$

by the Peter-Weyl theorem, where $\hat{G}$ denotes the set of all equivalence classes of irreducible finite-dimensional representations of G and we write V_δ for an irreducible G-module of class $\delta \in \hat{G}$. The following lemma says that we have a similar duality of Peter-Weyl type for irreducible Harish-Chandra modules of noncompact semisimple Lie groups.

Lemma 1.1. — *Let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module, and let f be in $C_K^\infty(G)$. Then the $(\mathfrak{g}, K)$-module $U(\mathfrak{g})^L f$ generated by f through the action L is isomorphic to $\boldsymbol{X}$ if and only if the corresponding $U(\mathfrak{g})^R f$ through the action R is isomorphic to $\boldsymbol{X}^*$.*

We give a proof below introducing some important notion which we use throughout this paper.

Proof of Lemma 1.1. — Let us prove the *if* part only since the converse can be proved in the same way. So, assume that $U(\mathfrak{g})^R f \simeq \boldsymbol{X}^*$ as $(\mathfrak{g}, K)$-modules.

Take a finite-dimensional K-module (τ, V_τ) which is isomorphic to $U(\mathfrak{k})^L f$. Let $i : V_\tau \xrightarrow{\sim} U(\mathfrak{k})^L f$ denote a K-isomorphism. We define a V_τ^*-valued smooth function F on G by

$$\langle F(g), v\rangle = i(v)(g) \quad (v \in V_\tau,\ g \in G),$$

where $\langle \cdot, \cdot \rangle$ denotes the natural dual pairing on $V_\tau^* \times V_\tau$. Then it is immediate to verify that F lies in the following space:

$$C^\infty_{\tau^*}(G) := \{\Phi : G \xrightarrow{C^\infty} V_\tau^* \mid \Phi(kg) = \tau^*(k)\Phi(g) \quad (g \in G,\ k \in K)\}. \tag{1.1}$$

Here (τ^*, V_τ^*) denotes the representation of K contragredient to τ. The space $C^\infty_{\tau^*}(G)$ has G- and $U(\mathfrak{g})$-module structures through right translation R. The function F is in the K-finite part, say $C^\infty_{\tau^*}(G)_K$, of $C^\infty_{\tau^*}(G)$ since $U(\mathfrak{k})^L f \subset C^\infty_K(G)$. By definition we see

$$f(g) = \langle F(g), i^{-1}(f)\rangle. \tag{1.2}$$

Now the assignment $D^R F \mapsto D^R f = \langle D^R F(\cdot), i^{-1}(f)\rangle$ $(D \in U(\mathfrak{g}))$ gives a $(\mathfrak{g}, K)$-homomorphism from $U(\mathfrak{g})^R F$ onto $U(\mathfrak{g})^R f \simeq \boldsymbol{X}^*$. We see that this homomorphism is injective. In fact, suppose $D^R f = 0$ for some $D \in U(\mathfrak{g})$. It then follows that

$$\begin{aligned} 0 = D^R f(kg) &= \langle D^R F(kg), i^{-1}(f)\rangle \\ &= \langle \tau^*(k) D^R F(g), i^{-1}(f)\rangle = \langle D^R F(g), i^{-1}((k^{-1})^L f)\rangle \end{aligned} \tag{1.3}$$

for all $g \in G$ and all $k \in K$. This implies that $D^R F = 0$ since f is a K-cyclic vector for $U(\mathfrak{k})^L f \simeq V_\tau$. Thus we have found a $(\mathfrak{g}, K)$-module embedding, say A_0, from $\boldsymbol{X}^*$ into $C^\infty_{\tau^*}(G)_K$ whose image equals $U(\mathfrak{g})^R F$.

Let $(\pi, \boldsymbol{H})$ be an irreducible admissible G-representation with Harish-Chandra module $\boldsymbol{X}$, and let $(\pi^*, \boldsymbol{H}^*)$ be the representation of G contragredient to π. We have $\boldsymbol{H}^*_K = \boldsymbol{X}^*$ as remarked before. By virtue of the Frobenius reciprocity for smoothly induced representation $\mathrm{Ind}_K^G(\tau^*)$ of G acting on $C^\infty_{\tau^*}(G)$, one obtains a linear isomorphism

$$\mathrm{Hom}_K(\boldsymbol{X}^*,\ V_\tau^*) \simeq \mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X}^*,\ C^\infty_{\tau^*}(G)_K), \tag{1.4}$$

which is given as follows. Take a K-homomorphism $T : \boldsymbol{X}^* \to V_\tau^*$. Then we can define $A(\varphi) \in C^\infty_{\tau^*}(G)$ for every $\varphi \in \boldsymbol{X}^*$ by

$$A(\varphi)(g) = \tilde{T}(\pi^*(g)\varphi) \quad (g \in G). \tag{1.5}$$

Here $\tilde{T}$ denotes the unique continuous extension of $T : \boldsymbol{X}^* \to V_\tau^*$ to $\boldsymbol{H}^*$. Then, the assignment $T \mapsto A$ gives (1.4).

We now consider our specified embedding $A_0 : \boldsymbol{X}^* \simeq U(\mathfrak{g})^R F \hookrightarrow C^\infty_{\tau^*}(G)_K$. Let T_0 denote the element of $\mathrm{Hom}_K(\boldsymbol{X}^*, V_\tau^*)$ corresponding to A_0 by (1.5). Set $\varphi_0 := A_0^{-1}(F) \in \boldsymbol{X}^*$ and $\psi_0 := i^{-1}(f) \circ \tilde{T}_0 \in \boldsymbol{X}$. Here ψ_0 is regarded as an element of $\boldsymbol{X} = ((\boldsymbol{H}^*)^*)_K$ through

$$\psi_0 : \boldsymbol{H}^* \xrightarrow{\tilde{T}_0} V_\tau^* \xrightarrow{i^{-1}(f)} \mathbb{C}$$

with $i^{-1}(f) \in V_\tau = \mathrm{Hom}_{\mathbb{C}}(V_\tau^*, \mathbb{C})$. In view of (1.2) and (1.5) we find

$$f(g) = \langle \pi^*(g)\varphi_0, \psi_0 \rangle_{\boldsymbol{H}^* \times \boldsymbol{H}} = \langle \varphi_0, \pi(g)^{-1}\psi_0 \rangle_{\boldsymbol{H}^* \times \boldsymbol{H}} \quad (g \in G). \tag{1.6}$$

Finally, (1.6) implies that the map

$$\boldsymbol{X} \ni D\psi_0 \mapsto D^L f = \langle \varphi_0, \pi(g)^{-1} D\psi_0 \rangle \in U(\mathfrak{g})^L f \quad (D \in U(\mathfrak{g}))$$

gives a $(\mathfrak{g}, K)$-isomorphism, i.e., $\boldsymbol{X} \simeq U(\mathfrak{g})^L f$ as desired. □

1.2. Maximal globalization. — Let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module. We fix once and for all an irreducible finite-dimensional representation (τ, V_τ) of K which occurs in $\boldsymbol{X}$, and fix an embedding $i_\tau : V_\tau \hookrightarrow \boldsymbol{X}$ as K-modules. Then the adjoint operator i_τ^* of i_τ gives a surjective K-homomorphism from $\boldsymbol{X}^*$ to V_τ^*. We denote by A_{τ^*} the $(\mathfrak{g}, K)$-embedding from $\boldsymbol{X}^*$ into $C^\infty_{\tau^*}(G)$ (see (1.1)) corresponding to i_τ^* through (1.4) and (1.5).

Equip $C^\infty_{\tau^*}(G)$ with a Fréchet space topology of compact uniform convergence of functions on G and each of their derivatives. The following proposition characterizes the closure $A_{\tau^*}(\boldsymbol{X}^*)^-$ of $A_{\tau^*}(\boldsymbol{X}^*)$ in $C^\infty_{\tau^*}(G)$.

***Theorem 1.2* (cf. [29], [11]).** — *Under the above notation, $A_{\tau^*}(\boldsymbol{X}^*)^-$ is a G-submodule of $C^\infty_{\tau^*}(G)$, and one gets an isomorphism of G-modules*

$$\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G)) \ni W \overset{\sim}{\longmapsto} F \in A_{\tau^*}(\boldsymbol{X}^*)^-$$

through

$$\langle F(g), v \rangle = ((W \circ i_\tau)(v))(g) \quad (g \in G,\ v \in V_\tau). \tag{1.7}$$

Here $C^\infty(G)$ is viewed as a smooth G-module by left translation L, and the right action R on $C^\infty(G)$ naturally gives a G-module structure on $\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G))$.

It follows essentially from [**29**, page 316] that the G-module $A_{\tau^*}(\boldsymbol{X}^*)^-$ gives a *maximal globalization* of the Harish-Chandra module $\boldsymbol{X}^*$. Namely, if a complete, locally convex Hausdorff topological vector space $\boldsymbol{F}$ admits a continuous G-action with underlying Harish-Chandra module $\boldsymbol{X}^*$, then the identity map on $\boldsymbol{X}^*$ extends uniquely to a continuous embedding $\boldsymbol{F} \hookrightarrow A_{\tau^*}(\boldsymbol{X}^*)^-$ as G-modules. One can get the above theorem from the first statement of Theorem 2.8, or equivalently (2.9), in [**11**].

In what follows, we give a direct proof of the above theorem to keep this article self-contained. This is done by generalizing our argument in [**42**, I, §2].

Proof of Theorem 1.2. — Let W be a $(\mathfrak{g}, K)$-embedding of $\boldsymbol{X}$ into $C^\infty(G)$. Since $W \circ i_\tau : V_\tau \xrightarrow{\sim} W(i_\tau(V_\tau)) \subset C^\infty(G)$ is a K-isomorphism, we can see just as in the beginning of the proof of Lemma 1.1 that there exists a unique $F \in C^\infty_{\tau^*}(G)$ satisfying (1.7). It is then easy to observe that the map $W \mapsto F$ sets up a G-homomorphism, say Υ, from $\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G))$ to $C^\infty_{\tau^*}(G)$ and that Υ is injective because of the irreducibility of $\boldsymbol{X}$. Hence we will get the theorem if we can show

$$\mathrm{Im}\,\Upsilon = A_{\tau^*}(\boldsymbol{X}^*)^-, \tag{1.8}$$

where $\mathrm{Im}\,\Upsilon$ denotes the image of Υ.

To prove (1.8) we use the projection to K-isotypic component. Let $\boldsymbol{M}$ be any smooth Fréchet K-module. For each $\delta \in \hat{K}$, the unitary dual of K, the integral operator Q_δ defined by

$$Q_\delta(v) = (\dim\delta)\cdot\int_K \overline{\mathrm{tr}(\delta(k))}\cdot kv\,dk \quad (v \in \boldsymbol{M}),$$

gives a continuous K-equivariant projection of $\boldsymbol{M}$ onto its δ-isotypic component $\boldsymbol{M}_\delta$. Here dk denotes the normalized Haar measure on K. By Harish-Chandra, the Fourier series $\sum_{\delta\in\hat{K}} Q_\delta(v)$ converges absolutely to v. (cf. [**36**, Th.4.4.2.1]).

Now the right hand side of (1.8) is described in terms of the projections Q_δ as

$$A_{\tau^*}(\boldsymbol{X}^*)^- = \{F \in C^\infty_{\tau^*}(G) \mid F_\delta := Q_\delta(F) \in A_{\tau^*}(\boldsymbol{X}^*) \quad \text{for all}\ \ \delta \in \hat{K}\}. \tag{1.9}$$

In reality, the inclusion $\supset$ is evident since the sum $F = \sum_{\delta\in\hat{K}} F_\delta$ converges in $C^\infty_{\tau^*}(G)$. Conversely assume F be in the closure $A_{\tau^*}(\boldsymbol{X}^*)^-$. Then there exists a sequence $\{F_j\}_{j=1,2,..}$ in $A_{\tau^*}(\boldsymbol{X}^*)$ such that $F = \lim_{j\to\infty} F_j$. Since the projection Q_δ is continuous, one obtains $F_\delta = \lim_{j\to\infty}(F_j)_\delta$ for every $\delta \in \hat{K}$. Noting that $(F_j)_\delta$ lies in a finite-dimensional (and hence closed) subspace $A_{\tau^*}(\boldsymbol{X}^*)_\delta \simeq \boldsymbol{X}^*_\delta$, we find that $F_\delta \in A_{\tau^*}(\boldsymbol{X}^*)_\delta$.

We are going to show just as in the proof of [**42**, I, Th.2.4] that $\mathrm{Im}\,\Upsilon$ coincides with the right hand side of (1.9) by using Lemma 1.1 instead of [**42**, I, Lemma 2.5].

Let F be a nonzero function in $C^\infty_{\tau^*}(G)$ such that $F_\delta \in A_{\tau^*}(\boldsymbol{X}^*)$ for every $\delta \in \hat{K}$. We write Ξ for the totality of finite subsets S of $\hat{K}$ consisting of elements δ such that $F_\delta \neq 0$. Define $F_S \in A_{\tau^*}(\boldsymbol{X}^*)$ and $f_{S,v} \in C^\infty(G)$ by

$$F_S = \sum_{\delta\in S} F_\delta, \quad f_{S,v} = \langle F_S(\cdot), v\rangle$$

for every $S \in \Xi$ and $v \in V_\tau\backslash\{0\}$. Then, $U(\mathfrak{g})^R F_S = A_{\tau^*}(\boldsymbol{X}^*) \simeq \boldsymbol{X}^*$ as $(\mathfrak{g}, K)$-modules. This implies that $U(\mathfrak{g})^R f_{S,v} \simeq \boldsymbol{X}^*$ for all S and v (cf. (1.3)). Set $Q_S := \sum_{\delta\in S} Q_\delta$ and $f_v := \langle F(\cdot), v\rangle$. We now use Lemma 1.1 to deduce

$$Q_S(U(\mathfrak{g})^L f_v) = U(\mathfrak{g})^L f_{S,v} \simeq \boldsymbol{X} \quad (S \in \Xi,\ v \in V_\tau\backslash\{0\}), \tag{1.10}$$

by noting that Q_S commutes with $U(\mathfrak{g})$-action L. It then follows from the irreducibility of $\boldsymbol{X}$ that the kernel of projection Q_S restricted to $U(\mathfrak{g})^L f_v$ is independent of a choice of $S \in \Xi$. Indeed, let S_1 and S_2 be in Ξ, and set $S' := S_1 \cup S_2$. Note that $Q_{S_i} = Q_{S_i}Q_{S'}$ $(i = 1, 2)$. One sees from (1.10) that Q_{S_i} on $U(\mathfrak{g})^L f_{S',v}$ gives a $U(\mathfrak{g})$-isomorphism from $U(\mathfrak{g})^L f_{S',v}$ onto $U(\mathfrak{g})^L f_{S_i,v}$ by Schur's lemma. This implies that

$$\operatorname{Ker}(Q_{S_1}|U(\mathfrak{g})^L f_v) = \operatorname{Ker}(Q_{S'}|U(\mathfrak{g})^L f_v) = \operatorname{Ker}(Q_{S_2}|U(\mathfrak{g})^L f_v).$$

The above kernel space must be $\{0\}$ since

$$\bigcap_{S\in\Xi} \operatorname{Ker}(Q_S|U(\mathfrak{g})^L f_v) \subset \bigcap_{S} \operatorname{Ker} Q_S = \{0\},$$

where S in the middle term runs over all finite subsets of $\hat{K}$. We thus find an embedding

$$\boldsymbol{X} \simeq U(\mathfrak{g})^L f_v \hookrightarrow C^\infty(G)$$

corresponding to F through Υ.

Conversely, let $W : \boldsymbol{X} \hookrightarrow C^\infty(G)$ be any $(\mathfrak{g}, K)$-embedding. Set $F := \Upsilon(W)$. We want to prove $F_\delta = Q_\delta(F) \in A_{\tau^*}(\boldsymbol{X}^*)$ for every $\delta \in \hat{K}$. To do this, define an element $\xi \in \boldsymbol{X}^*$ by

$$\langle \xi, a\rangle = ((Q_\delta \circ W)(a))(e) \quad (a \in \boldsymbol{X}),$$

where e denotes the identity element of G. It then follows for any $D \in U(\mathfrak{g})$ and $v \in V_\tau$ that

$$\begin{aligned}\langle D^L F_\delta(e), v\rangle &= D^L((Q_\delta \circ W \circ i_\tau)(v))(e)\\ &= ((Q_\delta \circ W)(D i_\tau(v)))(e)\\ &= \langle \xi, D i_\tau(v)\rangle = \langle i_\tau^* {}^T D\xi, v\rangle,\end{aligned}$$

since D^L commutes with Q_δ and with W. Here $U(\mathfrak{g}) \ni D \mapsto {}^T D \in U(\mathfrak{g})$ denotes the principal anti-automorphism of $U(\mathfrak{g})$ such that ${}^T X = -X$ if $X \in \mathfrak{g}$. We thus deduce

$$D^L F_\delta(e) = i_\tau^*({}^T D\xi) \quad \text{for all } D \in U(\mathfrak{g}). \tag{1.11}$$

This yields that

$$F_\delta(g) = \tilde{i}_\tau^*(\pi^*(g)\xi) = A_{\tau^*}(\xi)(g) \quad (g \in G)$$

as desired, because the both functions F_δ and $A_{\tau^*}(\xi)$ are real analytic on the connected Lie group G, and because they have the same Taylor series expansion at e by (1.11).

Thus the theorem has been proved completely. □

1.3. Kernel theorem. — To study the embeddings of $\boldsymbol{X}$ into various induced G-modules, it is useful to characterize the G-module $A_{\tau^*}(\boldsymbol{X}^*)^-$ as the full kernel space of a continuous G-homomorphism $\mathcal{D}$ defined on $C^\infty_{\tau^*}(G)$ in the following way.

Theorem 1.3. — *Let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module, and let (τ, V_τ) be a K-type of $\boldsymbol{X}$. Fix an embedding $i_\tau : V_\tau \hookrightarrow \boldsymbol{X}$ as K-modules, and write A_{τ^*} for the $(\mathfrak{g}, K)$-embedding $\boldsymbol{X}^* \hookrightarrow C^\infty_{\tau^*}(G)$ associated with the adjoint operator i_τ^* by* (1.4) *and* (1.5).

If $\mathcal{D}$ is any continuous G-homomorphism from the $C^{\infty}_{\tau^*}(G)$ to a smooth Fréchet G-module $\boldsymbol{M}$ such that

$$A_{\tau^*}(\boldsymbol{X}^*) = \{F \in C^{\infty}_{\tau^*}(G) \mid F \textit{ is right } K\textit{-finite and } \mathcal{D}F = 0\}, \tag{1.12}$$

then the full kernel space $\operatorname{Ker}\mathcal{D}$ *of* $\mathcal{D}$ *in* $C^{\infty}_{\tau^*}(G)$ *equals the G-module* $A_{\tau^*}(\boldsymbol{X}^*)^-$, *the closure of* $A_{\tau^*}(\boldsymbol{X}^*)$ *in* $C^{\infty}_{\tau^*}(G)$. *Hence one gets from Theorem 1.2*

$$\operatorname{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^{\infty}(G)) \simeq \operatorname{Ker}\mathcal{D} = A_{\tau^*}(\boldsymbol{X}^*)^- \quad \textit{as } G\textit{-modules.} \tag{1.13}$$

Proof. — We show that $\operatorname{Ker}\mathcal{D} = A_{\tau^*}(\boldsymbol{X}^*)^-$. The inclusion $\supset$ is obvious because $\operatorname{Ker}\mathcal{D}$ is a closed subspace of $C^{\infty}_{\tau^*}(G)$ by the continuity of $\mathcal{D}$ and because $A_{\tau^*}(\boldsymbol{X}^*) \subset \operatorname{Ker}\mathcal{D}$ by (1.12). Conversely if $F \in \operatorname{Ker}\mathcal{D}$, then it follows from (1.12) that $F_\delta = Q_\delta(F) \in A_{\tau^*}(\boldsymbol{X}^*)$ for every $\delta \in \hat{K}$, because $\mathcal{D}F_\delta = Q_\delta(\mathcal{D}F) = 0$. Hence we get $F = \sum_{\delta \in \hat{K}} F_\delta \in A_{\tau^*}(\boldsymbol{X}^*)^-$. Now the assertion follows from Theorem 1.2. □

Remark 1.4. — The above proof tells us that the assumption on $\mathcal{D}$ can be weakened. Namely, the theorem is still true for any K-homomorphism $\mathcal{D}$ from $C^{\infty}_{\tau^*}(G)$ to a smooth K-module $\boldsymbol{M}$ satisfying (1.12).

Example 1.5. — We mention that an operator $\mathcal{D}$ satisfying the requirement in Theorem 1.3 has been constructed when $\boldsymbol{X}^*$ is the $(\mathfrak{g}, K)$-module associated with: (a) discrete series (**[28]**, **[10]**) more generally Zuckerman cohomologically induced module (**[38]**, **[1]**), with parameter "far from the walls", or (b) highest weight representation (**[2]**, **[4]**; see also Theorem 2.6). In each of these cases, $\mathcal{D}$ is given as a *G-invariant differential operator of gradient type* acting on $C^{\infty}_{\tau^*}(G)$, where τ^* is the unique extreme K-type of $\boldsymbol{X}^*$ which occurs in $\boldsymbol{X}^*$ with multiplicity one.

We conclude this section by giving an application of Theorem 1.3. For this we need

Definition 1.6. — Let $\mathfrak{n}$ be a complex Lie subalgebra of $\mathfrak{g}$, and $(\eta, \boldsymbol{E})$ be a representation of $\mathfrak{n}$ on a Fréchet space $\boldsymbol{E}$ such that the linear endomorphism $\eta(Z)$ is continuous on $\boldsymbol{E}$ for every $Z \in \mathfrak{n}$. Then the space

$$C^{\infty}(G;\eta) := \{f : G \overset{C^{\infty}}{\longrightarrow} \boldsymbol{E} \mid Z^R f = -\eta(Z) f \quad (Z \in \mathfrak{n})\},$$

endowed with the natural Fréchet space topology, has a structure of smooth G-module by L. We write Γ_η for the resulting G-representation on $C^{\infty}(G;\eta)$, and call it the *representation of G induced from η in C^{∞}-context.*

Remark 1.7. — If $\mathfrak{n}$ is the complexification of real Lie subalgebra $\mathfrak{n}_0$ of $\mathfrak{g}_0$ corresponding to a simply connected analytic subgroup N of G, then $C^{\infty}(G;\eta)$ coincides with the space of $\boldsymbol{E}$-valued smooth functions f on G such that

$$f(gn) = \eta(n)^{-1} f(g) \quad (g \in G,\ n \in N),$$

at least when $\boldsymbol{E}$ is finite-dimensional. Here η denotes the well-defined representation of the group N defined by $\eta : \mathfrak{n}_0 \to \boldsymbol{E}$ through exponential map.

Let the notation and assumption be as in Theorem 1.3 and in Definition 1.6. We write $C^\infty_{\tau^*}(G;\eta)$ for the space of C^∞-functions on G with values in $V_\tau^* \otimes \boldsymbol{E}$ satisfying the following conditions.

$$\begin{cases} Z^R F = -(\mathrm{id}_{V_\tau^*} \otimes \eta(Z))F & (Z \in \mathfrak{n}), \\ k^L F = (\tau^*(k^{-1}) \otimes \mathrm{id}_{\boldsymbol{E}})F & (k \in K), \end{cases}$$

where id_V denotes the identity map on a set V. Let $\boldsymbol{E}'$ be the space of continuous linear functionals on $\boldsymbol{E}$ equipped with dual $U(\mathfrak{n})$-action. We define a linear map

$$\mathcal{D}(\eta) : C^\infty_{\tau^*}(G;\eta) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\boldsymbol{E}', \boldsymbol{M})$$

through $\mathcal{D}$ by

$$(\mathcal{D}(\eta)F)(\zeta) = \mathcal{D}(\langle F(\cdot), \zeta\rangle) \quad (F \in C^\infty_{\tau^*}(G;\eta),\ \zeta \in \boldsymbol{E}').$$

Here $\langle \cdot\,, \cdot \rangle$ stands for the canonical dual pairing on $(V_\tau^* \otimes \boldsymbol{E}) \times \boldsymbol{E}'$ with values in V_τ^*. If η is a one-dimensional $\mathfrak{n}$-representation, the above $\mathcal{D}(\eta)$ is naturally identified with the restriction of $\mathcal{D}$ to the subspace $C^\infty_{\tau^*}(G;\eta)$ of $C^\infty_{\tau^*}(G)$.

By using (1.13), we can now deduce the following

***Corollary 1.8* (Kernel Theorem).** — *Under the above notation, assume that the representation $(\eta, \boldsymbol{E})$ of $\mathfrak{n}$ is weakly cyclic in the following sense: there exists a $\zeta_0 \in \boldsymbol{E}'$ such that $U(\mathfrak{n})\zeta_0$ is dense in $\boldsymbol{E}'$ with respect to the weak $*$-topology. Then the embeddings of irreducible $(\mathfrak{g}, K)$-module $\boldsymbol{X}$ into induced module $C^\infty(G;\eta)$ are characterized as*

$$\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X}, C^\infty(G;\eta)) \simeq \mathrm{Ker}\,\mathcal{D}(\eta) \quad \textit{as vector spaces.}$$

Here the isomorphism is given as in (1.7).

Remark 1.9. — The above kernel thoerem has been proved in our earlier work [**42**, I, Th.2.4] in case that $\boldsymbol{X}$ is the $(\mathfrak{g}, K)$-module of discrete series and that $\mathcal{D}$ is a differential operator of gradient type (Schmid operator).

Proof of Corollary 1.8. — First, we observe just as in the proof of Theorem 1.2 that the map

$$\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G;\eta)) \ni W \overset{\Upsilon_\eta}{\longmapsto} F \in C^\infty_{\tau^*}(G;\eta)$$

defined as in (1.7) yields an injective linear map. For a nonzero element $F \in C^\infty_{\tau^*}(G;\eta)$ and a nonzero vector $v \in V_\tau$, we put $f_v := \langle F(\cdot), v\rangle_{(V_\tau^* \otimes \boldsymbol{E}) \times V_\tau} \in C^\infty(G;\eta)$. Then F lies in the image of Υ_η if and only if

$$U(\mathfrak{g})^L f_v \simeq \boldsymbol{X} \quad \text{as } (\mathfrak{g}, K)\text{-modules.} \tag{1.14}$$

It follows from the Hahn-Banach extension theorem that the G-homomorphism

$$C^\infty(G;\eta) \ni f \longmapsto \langle f(\cdot), \zeta_0\rangle_{\boldsymbol{E}\times\boldsymbol{E}'} \in C^\infty(G) \tag{1.15}$$

is injective because $U(\mathfrak{n})\zeta_0$ is weak $*$-dense in $\boldsymbol{E}'$. Then (1.14) and (1.15) together with (1.13) imply that $F \in \mathrm{Im}\,\Upsilon_\eta$ if and only if $(\mathcal{D}(\eta)F)(\zeta_0) = 0$. Since the function

F is η-covariant, the latter condition is equivalent to $(\mathcal{D}(\eta)F)(U(\mathfrak{n})\zeta_0) = 0$. This implies $\mathcal{D}(\eta)F = 0$ (and vice versa) by virtue of the Banach-Steinhaus theorem. $\square$

We will apply the above kernel theorem later in this paper to describe generalized Whittaker models for irreducible highest weight representations.

2. Differential operators, and lowest or highest weight modules

From now on, we assume that $K\backslash G$ is an irreducible Hermitian symmetric space with G-invariant complex structure. We consider the irreducible highest weight $(\mathfrak{g}, K)$-modules $L(\tau)$ with extreme K-types τ. In this section we describe, following [4], the differential operators $\mathcal{D}_{\tau^*}$ of gradient type on $K\backslash G$ whose K-finite kernels realize the dual lowest weight $(\mathfrak{g}, K)$-modules $L(\tau)^*$ (Theorem 2.6). This combined with Theorem 1.3 enables us to identify the maximal globalization of $L(\tau)^*$ with the full (not necessarily K-finite) kernel space of $\mathcal{D}_{\tau^*}$ (Proposition 2.7). We also specify for later use the solutions of differential equation $\mathcal{D}_{\tau^*}F = 0$ of exponential type.

2.1. Simple Lie group of Hermitian type. — We begin with summarizing some basic facts on fine structure for simple Lie groups of Hermitian type, following the notation in [**41**, Part I, §5] and [**8**, 3.3]. It is known that there exists a unique (up to sign) central element Z_0 of $\mathfrak{k}_0$ such that $\operatorname{ad} Z_0$ restricted to $\mathfrak{p}_0$ gives an $\operatorname{Ad}(K)$-invariant complex structure on $\mathfrak{p}_0$. One gets a triangular decomposition of $\mathfrak{g}$ as follows:

$$\begin{aligned} &\mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+ \quad \text{such that} \\ &[\mathfrak{k}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm, \quad [\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k}, \quad [\mathfrak{p}_+, \mathfrak{p}_+] = [\mathfrak{p}_-, \mathfrak{p}_-] = \{0\}, \end{aligned} \tag{2.1}$$

where $\mathfrak{p}_\pm$ denotes the eigenspace of $\operatorname{ad} Z_0$ on $\mathfrak{g}$ with eigenvalue $\pm\sqrt{-1}$ respectively. We extend $\operatorname{ad} Z_0$ on $\mathfrak{p}_0$ to a G-invariant complex structure on the Hermitian symmetric space $K\backslash G$ canonically through the identification $\mathfrak{p}_0 = T(K\backslash G)_{Ke}$, the tangent space of $K\backslash G$ at the origin Ke.

Let $\mathfrak{t}_0$ be a compact Cartan subalgebra of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$. We write Δ for the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$, and for each $\gamma \in \Delta$ the corresponding root subspace of $\mathfrak{g}$ will be denoted by $\mathfrak{g}(\mathfrak{t}; \gamma)$:

$$\mathfrak{g}(\mathfrak{t}; \gamma) = \{X \in \mathfrak{g} \mid (\operatorname{ad} H)X = \gamma(H)X \ \text{ for all } H \in \mathfrak{t}\}.$$

We can choose root vectors $X_\gamma \in \mathfrak{g}(\mathfrak{t}; \gamma)$ $(\gamma \in \Delta)$ such that

$$X_\gamma - X_{-\gamma},\ \sqrt{-1}(X_\gamma + X_{-\gamma}) \in \mathfrak{k}_0 + \sqrt{-1}\mathfrak{p}_0, \quad [X_\gamma, X_{-\gamma}] = H_\gamma, \tag{2.2}$$

where H_γ is the element of $\sqrt{-1}\mathfrak{t}_0$ corresponding the coroot $\gamma^\vee := 2\gamma/(\gamma, \gamma)$ through the identification $\mathfrak{t}^* = \mathfrak{t}$ by the Killing form B of $\mathfrak{g}$. Let Δ_c (resp. Δ_n) denote the subset of all compact (resp. noncompact) roots in Δ.

Take a positive system Δ^+ of Δ compatible with the decomposition (2.1):

$$\mathfrak{p}_\pm = \bigoplus_{\gamma\in\Delta_n^+} \mathfrak{g}(\mathfrak{t}; \pm\gamma) \quad \text{with} \quad \Delta_n^+ := \Delta^+ \cap \Delta_n,$$

and fix a lexicographic order on $\sqrt{-1}\mathfrak{t}_0^*$ which yields Δ^+. Using this order we define a fundamental sequence $(\gamma_1, \gamma_2, \ldots, \gamma_r)$ of strongly orthogonal (i.e., $\gamma_i \pm \gamma_j \notin \Delta \cup \{0\}$ for $i \neq j$) noncompact positive roots in such a way that γ_k is the maximal element of Δ^+, which is strongly orthogonal to $\gamma_{k+1}, \ldots, \gamma_r$.

Now, put $\mathfrak{t}^- := \sum_{k=1}^r \mathbb{C}H_{\gamma_k} \subset \mathfrak{t}$, and denote by $\gamma^- \in (\mathfrak{t}^-)^*$ the restriction to $\mathfrak{t}^-$ of a linear form $\gamma \in \mathfrak{t}^*$. For integers k, l with $1 \leqslant l < k \leqslant r$, we define subsets P_{kl}, P_k, P_0 of Δ_n^+ and subsets C_{kl}, C_k, C_0 of Δ_c^+ respectively by

$$P_{kl} := \left\{ \gamma \in \Delta_n^+ \,\middle|\, \gamma^- = \left(\frac{\gamma_k + \gamma_l}{2}\right)^- \right\}, \tag{2.3}$$

$$C_{kl} := \left\{ \gamma \in \Delta_c^+ \,\middle|\, \gamma^- = \left(\frac{\gamma_k - \gamma_l}{2}\right)^- \right\}, \tag{2.4}$$

$$P_k := \left\{ \gamma \in \Delta_n^+ \,\middle|\, \gamma^- = \left(\frac{\gamma_k}{2}\right)^- \right\}, \quad C_k := \left\{ \gamma \in \Delta_c^+ \,\middle|\, \gamma^- = \left(\frac{\gamma_k}{2}\right)^- \right\}, \tag{2.5}$$

$$P_0 := \{\gamma_1, \gamma_2, \ldots, \gamma_r\}, \quad C_0 := \{\gamma \in \Delta_c^+ \mid \gamma^- = 0\}. \tag{2.6}$$

By Harish-Chandra the subsets Δ_n^+ and Δ_c^+ are decomposed as

$$\Delta_n^+ = \Big(\bigcup_{1\leqslant k\leqslant r} P_k\Big) \bigcup P_0 \bigcup \Big(\bigcup_{1\leqslant l<k\leqslant r} P_{kl}\Big),$$

$$\Delta_c^+ = C_0 \bigcup \Big(\bigcup_{1\leqslant k\leqslant r} C_k\Big) \bigcup \Big(\bigcup_{1\leqslant l<k\leqslant r} C_{kl}\Big),$$

where the unions are disjoint. Moreover the maps

$$C_{kl} \ni \gamma \longmapsto \gamma + \gamma_l \in P_{kl} \quad \text{and} \quad C_k \ni \gamma \longmapsto -\gamma + \gamma_k \in P_k \tag{2.7}$$

give rise to bijections from C_{kl} to P_{kl} and from C_k to P_k respectively. Note that the subsets P_{kl} and C_{kl} are always non-empty, and that P_k and C_k $(1 \leqslant k \leqslant r)$ are empty if and only if the Hermitian symmetric space $K\backslash G$ is analytically equivalent to a tube domain.

We now introduce a *Cayley transform* $\boldsymbol{c} = \mathrm{Ad}(c)$ on $\mathfrak{g}$ defined by the following element of $G_{\mathbb{C}}^{\circ}$:

$$c = \exp\Big(\frac{\pi}{4}\sum_{k=1}^r (X_{\gamma_k} - X_{-\gamma_k})\Big), \tag{2.8}$$

where $G_{\mathbb{C}}^{\circ}$ denotes a connected Lie group with Lie algebra $\mathfrak{g}$. Note that $-\boldsymbol{c}^2$ gives the identity map on $\mathfrak{t}^-$. It follows that

$$\mathfrak{a}_{\mathfrak{p},0} := \boldsymbol{c}^{-1}(\mathfrak{t}^- \cap \sqrt{-1}\mathfrak{t}_0) = \boldsymbol{c}(\mathfrak{t}^- \cap \sqrt{-1}\mathfrak{t}_0) \tag{2.9}$$

is a maximal abelian subspace of $\mathfrak{p}_0$, and that the elements

$$H_k := c^{-1}(H_{\gamma_k}) = -c(H_{\gamma_k}) = X_{\gamma_k} + X_{-\gamma_k} \quad (k = 1, 2, \ldots, r) \tag{2.10}$$

form an orthogonal basis of vector space $\mathfrak{a}_{\mathfrak{p},0}$ with respect to the inner product defined by the Killing form B. This implies in particular that r equals the real rank of G. The restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}_{\mathfrak{p}}$ has been described by Moore in terms of linear forms $\psi_k := \gamma_k \circ (c|\mathfrak{a}_{\mathfrak{p}})$ on $\mathfrak{a}_{\mathfrak{p}}$ (see e.g., [**8**, Th.3.5] for the description).

2.2. Generalized Verma module and its maximal submodule. — Let (τ, V_τ) be any irreducible finite-dimensional representation of K with Δ_c^+-highest weight $\lambda = \lambda(\tau)$. We consider the *generalized Verma* $U(\mathfrak{g})$*-module* induced from τ:

$$M(\tau) := U(\mathfrak{g}) \otimes_{U(\mathfrak{k}+\mathfrak{p}_+)} V_\tau.$$

Here τ is extended to a representation of the maximal parabolic subalgebra $\mathfrak{k} + \mathfrak{p}_+$ by letting $\mathfrak{p}_+$ act on V_τ trivially: $\mathfrak{p}_+ V_\tau = \{0\}$. $M(\tau)$ has a structure of $(\mathfrak{g}, K)$-module through

$$D' \cdot (D \otimes v) := D'D \otimes v, \quad k \cdot (D \otimes v) := \mathrm{Ad}(k)D \otimes \tau(k)v$$

for $D' \in U(\mathfrak{g})$, $k \in K$ and $D \otimes v \in M(\tau)$ with $D \in U(\mathfrak{g})$, $v \in V_\tau$. Let $N(\tau)$ be the unique maximal proper $(\mathfrak{g}, K)$-submodule of $M(\tau)$. Then the quotient $L(\tau) := M(\tau)/N(\tau)$ gives an irreducible $(\mathfrak{g}, K)$-module with Δ^+-highest weight λ.

We now summarize for later use some basic known results concerning the structure of $N(\tau)$. One finds from the decomposition (2.1) that $M(\tau) = U(\mathfrak{p}_-)V_\tau$ is canonically isomorphic to the tensor product $S(\mathfrak{p}_-) \otimes V_\tau = S(\mathfrak{p}_-) \otimes_{\mathbb{C}} V_\tau$ as a K-module, where $S(\mathfrak{p}_-)$ ($\simeq U(\mathfrak{p}_-)$) denotes the symmetric algebra of $\mathfrak{p}_-$ looked upon as a K-module by the adjoint action. This isomorphism yields a gradation of the K-module $M(\tau)$:

$$M(\tau) = \bigoplus_{j=0}^{\infty} M_j(\tau) \quad \text{with} \quad M_j(\tau) := S^j(\mathfrak{p}_-)V_\tau \simeq S^j(\mathfrak{p}_-) \otimes V_\tau. \tag{2.11}$$

Here we write $S^j(\mathfrak{p}_-)$ for the K-submodule of $S(\mathfrak{p}_-)$ consisting of all homogeneous elements of $S(\mathfrak{p}_-)$ of degree j. Note that the submodule $N(\tau)$ is graded:

$$N(\tau) = \bigoplus_{j=0}^{\infty} N_j(\tau) \quad \text{with} \quad N_j(\tau) := N(\tau) \cap M_j(\tau), \tag{2.12}$$

because $N(\tau)$ is stable under the action of the central element $\sqrt{-1}Z_0 \in \mathfrak{t}$ which gives the gradation $S(\mathfrak{p}_-) = \oplus_{j=0}^{\infty} S^j(\mathfrak{p}_-)$.

Since $M(\tau) = S(\mathfrak{p}_-)V_\tau$ is finitely generated over the Noetherian ring $S(\mathfrak{p}_-)$, so is the submodule $N(\tau)$, too. This implies that, if $N(\tau) \neq \{0\}$, there exist finitely many irreducible K-submodules $W_1, \ldots, W_q$ of $N(\tau)$ such that

$$N(\tau) = \sum_{u=1}^{q} S(\mathfrak{p}_-)W_u \quad \text{with} \quad W_u \subset S^{i_u}(\mathfrak{p}_-)V_\tau \simeq S^{i_u}(\mathfrak{p}_-) \otimes V_\tau \tag{2.13}$$

for some positive integers i_u $(u = 1, \ldots, q)$ arranged as

$$i(\tau) := i_1 = \min\{j |\ N_j(\tau) \neq \{0\}\}. \tag{2.14}$$

We call $i(\tau)$ *the level of reduction* of $M(\tau)$.

An irreducible $(\mathfrak{g}, K)$-module $\boldsymbol{X}$ is called *unitarizable* if $\boldsymbol{X}$ is isomorphic to the Harish-Chandra module $\boldsymbol{H}_K$ of an irreducible unitary representation of G on a Hilbert space $\boldsymbol{H}$. The unitarizable highest weight $(\mathfrak{g}, K)$-modules have been completely classified by Enright, Howe and Wallach [**7**]. Note that their work contains case-by-case analysis, and that it uses some results of former contributors such as [**12**], [**6**], etc. Later, Enright and Joseph [**5**], and also Jakobsen [**20**] gave a more intrinsic classification.

For unitarizable $L(\tau)$'s, [**5**] gives a simple description of the maximal submodule $N(\tau)$ as follows. Assume that $L(\tau)$ is unitarizable and that $N(\tau) \neq \{0\}$. Then the level $i(\tau)$ of reduction of $M(\tau)$ is an integer such that $1 \leqslant i(\tau) \leqslant r$, where r is the real rank of G as in 2.1. Let $Q_{i(\tau)}$ be the irreducible K-submodule of $S^{i(\tau)}(\mathfrak{p}_-)$ with lowest weight $-\gamma_r - \cdots - \gamma_{r-i(\tau)+1}$. Then the tensor product $Q_{i(\tau)} \otimes V_\tau$ has a unique irreducible K-submodule W_1, called the Parthasarathy, Rao and Varadarajan component (the PRV-component for short), with extreme weight $\lambda - \gamma_r - \cdots - \gamma_{r-i(\tau)+1}$. Noting that

$$Q_{i(\tau)} \otimes V_\tau \subset S^{i(\tau)}(\mathfrak{p}_-) \otimes V_\tau \simeq M_{i(\tau)}(\tau), \tag{2.15}$$

we regard W_1 as a K-submodule of $M_{i(\tau)}(\tau)$.

Theorem 2.1 **([5,** 5.2, 6.5 and 8.3**], see also** [**3**, 3.1]**).** — *Keep the above notation. If $L(\tau)$ is unitarizable and if the maximal submodule $N(\tau)$ of $M(\tau)$ does not vanish, $N(\tau)$ is a highest weight $(\mathfrak{g}, K)$-module generated over $S(\mathfrak{p}_-)$ by the PRV-component W_1:*

$$N(\tau) = S(\mathfrak{p}_-)W_1.$$

2.3. A realization of lowest weight module $L(\tau)^*$. — For each irreducible representation (τ, V_τ) of K, let $L(\tau)^*$ be the irreducible lowest weight $(\mathfrak{g}, K)$-module which is dual to $L(\tau)$. This subsection gives after [**4**] a realization of $L(\tau)^*$ as the K-finite kernel of a certain G-invariant differential operator of gradient type defined on the symmetric space $K\backslash G$. This together with the kernel theorem (Corollary 1.8) will tell us how to describe the $(\mathfrak{g}, K)$-embeddings of highest weight module $L(\tau)$ into various induced G-representations.

Now, let $O^*_{\tau^*}(G)$ denote the space of functions F in $C^\infty_{\tau^*}(G)$ (see (1.1)) satisfying

$$X^L F = 0 \quad \text{for all} \quad X \in \mathfrak{p}_+. \tag{2.16}$$

Then we see that $O^*_{\tau^*}(G)$ is a closed G-submodule of $C^\infty_{\tau^*}(G)$ through right translation R, and that it is canonically isomorphic to the space of anti-holomorphic sections of the G-homogeneous vector bundle on $K\backslash G$ associated to the K-module V^*_τ.

It is useful to employ another realization of the G-module $O^*_{\tau^*}(G)$ as a space of holomorphic V^*_τ-valued functions on a bounded domain $\mathcal{B}$ of $\mathfrak{p}_-$. To be more precise, we take a connected linear Lie group G° with a covering homomorphism

$$\varpi : G \to G^\circ.$$

Such a G° always exists (we can take $G^\circ = \mathrm{Ad}(G)$ for example). Let $G^\circ_{\mathbb{C}}$ denote the connected complexification of G°. We write K°, $K^\circ_{\mathbb{C}}$ and $P_\pm = \exp \mathfrak{p}_\pm$ for the connected Lie subgroups of $G^\circ_{\mathbb{C}}$ with Lie algebras $\mathfrak{k}_0$, $\mathfrak{k}$ and $\mathfrak{p}_\pm$, respectively. Note that the exponential map gives holomorphic diffeomorphisms from $\mathfrak{p}_\pm$ onto $P_\pm$. Consider an open dense subset $P_+K^\circ_{\mathbb{C}}P_-$ of $G^\circ_{\mathbb{C}}$, which is holomorphically diffeomorphic to the direct product $P_+ \times K^\circ_{\mathbb{C}} \times P_-$ through multiplication. For each $x \in P_+K^\circ_{\mathbb{C}}P_-$, let $p_+(x)$, $k_{\mathbb{C}}(x)$, and $p_-(x)$ denote respectively the elements of P_+, $K^\circ_{\mathbb{C}}$, and P_- such that $x = p_+(x)k_{\mathbb{C}}(x)p_-(x)$. We set $\xi(x) := \log p_-(x) \in \mathfrak{p}_-$. Note that $G^\circ \subset P_+K^\circ_{\mathbb{C}}P_-$. We extend the assignment $x \mapsto \xi(x)$ $(x \in G^\circ)$ to a map, denoted again by $\xi(x)$, from G to $\mathfrak{p}_-$ through ϖ. This (extended) ξ naturally induces an anti-holomorphic diffeomorphism, say $\tilde{\xi}$, from the symmetric space $K\backslash G$ onto a bounded domain

$$\mathcal{B} := \{\xi(x) \in \mathfrak{p}_- \mid x \in G\} \tag{2.17}$$

of $\mathfrak{p}_-$, where $\tilde{\xi}(Kx) := \xi(x)$. (See for example [**16**, 7.129].) Let $K_{\mathbb{C}}$ denote the complexification of K. Then, ϖ restricted to K yields a covering homomorphism from $K_{\mathbb{C}}$ to $K^\circ_{\mathbb{C}}$, and the map $x \mapsto k_{\mathbb{C}}(x)$ $(x \in G^\circ)$ lifts to a map from G to $K_{\mathbb{C}}$ which we denote again by $k_{\mathbb{C}}(x)$ $(x \in G)$.

Let $O(\mathcal{B}, V^*_\tau)$ be the space of all V^*_τ-valued holomorphic functions on $\mathcal{B}$. It is easily verified that the above $\tilde{\xi}$ gives rise to a linear isomorphism Θ from $O^*_{\tau^*}(G)$ onto $O(\mathcal{B}, V^*_\tau)$ by

$$(\Theta F)(\tilde{\xi}(Kx)) := \tau^*(k_{\mathbb{C}}(x))^{-1}F(x) \quad (x \in G) \tag{2.18}$$

for $F \in O^*_{\tau^*}(G)$. Then $O(\mathcal{B}, V^*_\tau)$ has a G-module structure inherited from $(R, O^*_{\tau^*}(G))$ through Θ:

$$(g \cdot f)(\xi(x)) = \tau^*(k_{\mathbb{C}}(\exp \xi(x)\, g))f(\xi(xg)) \quad (x \in G) \tag{2.19}$$

for $g \in G$ and $f \in O(\mathcal{B}, V^*_\tau)$. Here one should notice that

$$\exp \xi(x)\, g = (p_+(x)k_{\mathbb{C}}(x))^{-1}xg \in P_+K^\circ_{\mathbb{C}}G^\circ \subset P_+K^\circ_{\mathbb{C}}P_-$$

for $x, g \in G^\circ$, and that the map

$$\mathcal{B} \times G^\circ \ni (z, g) \longmapsto k_{\mathbb{C}}(\exp z\, g) \in K^\circ_{\mathbb{C}}$$

lifts to a map from $\mathcal{B} \times G$ to $K_{\mathbb{C}}$ in the canonical way (cf. [**4**, Prop.4.7]).

By differentiating the G-action (2.19) one obtains a $\mathfrak{g}$-module $O(\mathcal{B}, V^*_\tau)$. We remark that the action of each element Y in $\mathfrak{p}_-$ is described simply as

$$(Y \cdot f)(z) = \frac{d}{dt}f(z + tY)|_{t=0} \quad (z \in \mathcal{B}). \tag{2.20}$$

An $f \in O(\mathcal{B}, V_\tau^*)$ is K-finite if and only if f is a polynomial, because one sees from (2.19) that

$$(h_t \cdot f)(z) = \tau^*(h_t) f(e^{\sqrt{-1}t} z)$$

for $h_t := \exp tZ_0 \in K$ $(t \in \mathbb{R})$, where Z_0 is the central element of $\mathfrak{k}_0$ defined in 2.1. Hence the K-finite part $O_{\tau^*}^*(G)_K$ of $O_{\tau^*}^*(G)$ is isomorphic, through Θ, to the space $\mathcal{P}(\mathfrak{p}_-, V_\tau^*) = S(\mathfrak{p}_+) \otimes V_\tau^*$ of V_τ^*-valued polynomial functions on $\mathfrak{p}_-$. Here we identify the symmetric algebra $S(\mathfrak{p}_+)$ of $\mathfrak{p}_+$ with the ring of polynomial functions on $\mathfrak{p}_-$ through the Killing form B restricted to $\mathfrak{p}_+ \times \mathfrak{p}_-$.

We now define a bilinear form $\langle \cdot, \cdot \rangle_\tau$ on $O_{\tau^*}^*(G) \times (U(\mathfrak{g}) \otimes_{\mathbb{C}} V_\tau)$ by

$$\langle F, D \otimes v \rangle_\tau := \langle D^L F(e), v \rangle = \langle ({}^T D)^R F(e), v \rangle \tag{2.21}$$

for $F \in O_{\tau^*}^*(G), D \in U(\mathfrak{g})$, and $v \in V_\tau$. Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing on $V_\tau^* \times V_\tau$, and $D \mapsto {}^T D$ the principal anti-automorphism of $U(\mathfrak{g})$, respectively. Then it is a routine task to verify that $\langle \cdot, \cdot \rangle_\tau$ naturally gives rise to a $(\mathfrak{g}, K)$-invariant bilinear form on $O_{\tau^*}^*(G) \times M(\tau)$, which we denote again by $\langle \cdot, \cdot \rangle_\tau$. Note that this pairing is described through the above isomorphism Θ as

$$\langle F, D \otimes v \rangle_\tau = \langle ({}^T D \cdot f)(0), v \rangle \quad \text{with} \quad f := \Theta F \in O(\mathcal{B}, V_\tau^*),$$

where $D \in U(\mathfrak{p}_-) = S(\mathfrak{p}_-)$, $v \in V_\tau$, and ${}^T D \cdot f$ is defined through the directional derivative action (2.20). This implies the following

Lemma 2.2 **(cf. [3, §2])**

(1) *The $(\mathfrak{g}, K)$-invariant pairing $\langle \cdot, \cdot \rangle_\tau$ is nondegenerate on $O_{\tau^*}^*(G)_K \times M(\tau)$.*

(2) *Let $R(\tau^*)$ be the orthogonal of the maximal submodule $N(\tau)$ in $O_{\tau^*}^*(G)_K \simeq \mathcal{P}(\mathfrak{p}_-, V_\tau^*)$ with respect to $\langle \cdot, \cdot \rangle_\tau$. Then $R(\tau^*)$ is the unique, nonzero irreducible $(\mathfrak{g}, K)$-submodule of $O_{\tau^*}^*(G)_K$, and it is isomorphic to the lowest weight module $L(\tau)^*$ dual to $L(\tau) = M(\tau)/N(\tau)$. The $(\mathfrak{g}, K)$-isomorphism A_{τ^*} from $L(\tau)^*$ onto $R(\tau^*)$ is given by*

$$\langle A_{\tau^*}(\varphi), w \rangle_\tau = \langle \varphi, \ w + N(\tau) \rangle_{L(\tau)^* \times L(\tau)} \quad (w \in M(\tau))$$

for $\varphi \in L(\tau)^*$.

We are now going to introduce a differential operator of gradient type whose K-finite kernel characterizes the $(\mathfrak{g}, K)$-module $R(\tau^*) = A_{\tau^*}(L(\tau)^*)$. For this, we take a basis $X_1, \dots, X_s$ of the $\mathbb{C}$-vector space $\mathfrak{p}_+$ such that $B(X_j, \overline{X}_k) = \delta_{jk}$ (Kronecker's δ), where $\overline{X}_i \in \mathfrak{p}_-$ denotes the complex conjugate of $X_i \in \mathfrak{p}_+$ with respect to the real form $\mathfrak{g}_0$. Set

$$X^\alpha := X_1^{\alpha_1} \cdots X_s^{\alpha_s} \in U(\mathfrak{p}_+) \quad \text{and} \quad \overline{X}^\alpha := \overline{X}_1^{\alpha_1} \cdots \overline{X}_s^{\alpha_s} \in U(\mathfrak{p}_-)$$

for every multi-index $\alpha = (\alpha_1, \dots, \alpha_s)$ of nonnegative integers $\alpha_1, \dots, \alpha_s$. We denote by $|\alpha| := \alpha_1 + \cdots + \alpha_s$ the length of α. For each positive integer n we define the gradients ∇^n and $\overline{\nabla}^n$ of order n on $C^\infty_{\tau^*}(G)$ as follows.

$$\nabla^n F(x) := \sum_{|\alpha|=n} \frac{1}{\alpha!}\overline{X}^\alpha \otimes (X^\alpha)^L F(x),$$

$$\overline{\nabla}^n F(x) := \sum_{|\alpha|=n} \frac{1}{\alpha!}X^\alpha \otimes (\overline{X}^\alpha)^L F(x),$$

for $x \in G$ and $F \in C^\infty_{\tau^*}(G)$, where $\alpha! := \alpha_1!\cdots\alpha_n!$. It is then easy to see that $\nabla^n F$ and $\overline{\nabla}^n F$ are independent of the choice of a basis $X_1, \ldots, X_s$, and that the operators ∇^n and $\overline{\nabla}^n$ give continuous G-homomorphisms

$$\nabla^n : C^\infty_{\tau^*}(G) \to C^\infty_{\tau^*(-n)}(G), \quad \overline{\nabla}^n : C^\infty_{\tau^*}(G) \to C^\infty_{\tau^*(+n)}(G).$$

Here $\tau^*(\pm n)$ denotes the K-representation on the tensor product $S^n(\mathfrak{p}_\pm) \otimes V_\tau^*$ respectively.

Let W_u $(u = 1, \ldots, q)$ be, as in (2.13), the irreducible K-submodules of $S^{i_u}(\mathfrak{p}_-)V_\tau \subset N(\tau)$ which generate $N(\tau)$ over $S(\mathfrak{p}_-)$ when $N(\tau) \neq \{0\}$. For each u, the adjoint operator P_u of the embedding

$$W_u \hookrightarrow S^{i_u}(\mathfrak{p}_-)V_\tau \simeq S^{i_u}(\mathfrak{p}_-) \otimes V_\tau \tag{2.22}$$

gives a surjective K-homomorphism:

$$P_u : S^{i_u}(\mathfrak{p}_+) \otimes V_\tau^* \simeq (S^{i_u}(\mathfrak{p}_-) \otimes V_\tau)^* \longrightarrow W_u^*. \tag{2.23}$$

Definition 2.3. — Under the above notation, let $\mathcal{D}_{\tau^*}$ be a G-invariant differential operator from $C^\infty_{\tau^*}(G)$ to $C^\infty_\rho(G)$ defined by

$$\mathcal{D}_{\tau^*}F(x) := \nabla^1 F(x) \oplus (\oplus_{u=1}^q P_u(\overline{\nabla}^{i_u} F(x)))$$

for $x \in G$ and $F \in C^\infty_{\tau^*}(G)$. Here we write $\rho = \rho(\tau^*)$ for the representation of K on

$$(\mathfrak{p}_- \otimes V_\tau^*) \oplus (\oplus_{u=1}^q W_u^*),$$

and $\mathcal{D}_{\tau^*}$ should be understood as $\mathcal{D}_{\tau^*} = \nabla^1$ if $N(\tau) = \{0\}$, or equivalently $M(\tau) = L(\tau)$. We call $\mathcal{D}_{\tau^*}$ the *differential operator of gradient type* associated to τ^*.

Remark 2.4. — A function $F \in C^\infty_{\tau^*}(G)$ lies in the G-submodule $O^*_{\tau^*}(G)$ defined by (2.16) if and only if $\nabla^1 F = 0$. Hence we have $\operatorname{Ker}\mathcal{D}_{\tau^*} \subset O^*_{\tau^*}(G)$ for every τ^*, and the equality holds if and only if $N(\tau) = \{0\}$.

Remark 2.5. — If $L(\tau)$ is unitarizable, one sees from Theorem 2.1 that

$$\mathcal{D}_{\tau^*} = \nabla^1 \oplus (P_1 \circ \overline{\nabla}^{i(\tau)}).$$

Here $i(\tau)$ is the level of reduction of $M(\tau)$, and the K-homomorphism P_1 is defined through the PRV-component $W_1 \subset S^{i(\tau)}(\mathfrak{p}_-) \otimes V_\tau$.

The following theorem, equivalent to [**4**, Prop.7.6] due to Davidson and Stanke, realizes the lowest weight module $L(\tau)^*$ by means of $\mathcal{D}_{\tau^*}$.

***Theorem 2.6* (cf. [4]).** — *The image $R(\tau^*)$ of the $(\mathfrak{g}, K)$-embedding A_{τ^*} from $L(\tau)^*$ into $\mathcal{O}^*_{\tau^*}(G)_K$ defined in Lemma 2.2 coincides with the K-finite kernel of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type:*

$$R(\tau^*) = \{F \in C^{\infty}_{\tau^*}(G) \mid F \text{ is right } K\text{-finite and } \mathcal{D}_{\tau^*}F = 0\}.$$

2.4. Maximal globalization of $L(\tau)^*$. — The above theorem together with Theorem 1.3 implies that the full kernel space $\operatorname{Ker}\mathcal{D}_{\tau^*}$ gives a maximal globalization of the lowest weight module $L(\tau)^*$, as follows.

Proposition 2.7. — (1) *The closure $R(\tau^*)^-$ of $R(\tau^*) = A_{\tau^*}(L(\tau)^*)$ in $C^{\infty}_{\tau^*}(G)$ coincides with* $\operatorname{Ker}\mathcal{D}_{\tau^*}$. *It coincides also with the orthogonal, say $R'(\tau^*)$, of $N(\tau)$ in the whole (not necessarily K-finite) space $\mathcal{O}^*_{\tau^*}(G)$ with respect to the paring $\langle\ \cdot\ ,\ \cdot\ \rangle_\tau$ in* (2.21).

(2) *One has an isomorphism of G-modules*

$$\operatorname{Hom}_{\mathfrak{g},K}(L(\tau),\ C^{\infty}(G)) \simeq \operatorname{Ker}\mathcal{D}_{\tau^*}(= R(\tau^*)^- = R'(\tau^*))$$

by the correspondence given in Theorem 1.2 through the canonical K-embedding i_τ of V_τ into $L(\tau)$.

Proof. — The statements except $R(\tau^*)^- = R'(\tau^*)$ follow immediately from Theorems 1.3 and 2.6. The equality $R(\tau^*)^- = R'(\tau^*)$ can be shown just as in the proof of Theorem 1.3, by bearing in mind that $R'(\tau^*)$ is K-stable. □

We end this section by specifying for later use the solutions $F \in \mathcal{O}^*_{\tau^*}(G)$ of exponential type of the differential equation $\mathcal{D}_{\tau^*}F = 0$.

For each $X \in \mathfrak{p}_+$ and each $v^* \in V^*_\tau$, let $f_{X,v^*} = \exp X \otimes v^*$ denote the V^*_τ-valued holomorphic function on $\mathfrak{p}_-$ defined by

$$f_{X,v^*}(z) := \exp B(X, z) \cdot v^* \quad (z \in \mathfrak{p}_-).$$

We set $F_{X,v^*} := \Theta^{-1} f_{X,v^*} \in \mathcal{O}^*_{\tau^*}(G)$. Then the function F_{X,v^*} is described as

$$F_{X,v^*}(x) = \exp B(X, \xi(x)) \cdot \tau^*(k_{\mathbb{C}}(x))v^* \quad (x \in G) \tag{2.24}$$

by the definition of Θ (see (2.18)).

Proposition 2.8. — *The function F_{X,v^*} satisfies the differential equation $\mathcal{D}_{\tau^*}F = 0$ if and only if*

$$P_u(X^{i_u} \otimes v^*) = 0 \quad \textit{for} \quad u = 1, \ldots, q. \tag{2.25}$$

Here P_u is a K-homomorphism from $S^{i_u}(\mathfrak{p}_+) \otimes V^*_\tau$ onto W^*_u defined in (2.22) *and in* (2.23).

Proof. — Let $D \in S(\mathfrak{p}_-)$. In view of (2.20) we observe that

$$D^R F_{X,v^*} = \Theta^{-1}(D \cdot f_{X,v^*}) = D(X) F_{X,v^*},$$

because f_{X,v^*} is an exponential function defined by X, where $D \in S(\mathfrak{p}_-)$ in the right hand side is looked upon as a polynomial on $\mathfrak{p}_+ = \mathfrak{p}_-^*$. It then follows that F_{X,v^*} is orthogonal to $N(\tau) = \sum_{u=1}^{q} S(\mathfrak{p}_-)W_u$ with respect to the $\mathfrak{g}$-invariant pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\tau$ in (2.21) if and only if

$$\langle F_{X,v^*}, w\rangle_\tau = 0 \quad \text{for all } w \in W_u \ (u = 1,\dots,q). \tag{2.26}$$

We now express $w \in W_u$ as $w = \sum_{j=1}^{N} D_j v_j$ with $D_j \in S^{i_u}(\mathfrak{p}_-)$ and $v_j \in V_\tau$. Then the left hand side of (2.26) is calculated as

$$\langle F_{X,v^*}, w\rangle_\tau = (-1)^{i_u} \sum_{j=1}^{N} D_j(X)\langle v^*, v_j\rangle = (-1)^{i_u}(X^{i_u} \otimes v^*, w),$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the dual pairing between $S^{i_u}(\mathfrak{p}_+)\otimes V_\tau^*$ and $S^{i_u}(\mathfrak{p}_-)\otimes V_\tau$. Hence, the element F_{X,v^*} is orthogonal to W_u with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_\tau$ if and only if the linear form $X^{i_u} \otimes v^*$ on $S^{i_u}(\mathfrak{p}_-) \otimes V_\tau$ vanishes on the subspace W_u, or equivalently $P_u(X^{i_u} \otimes v^*) = 0$ by the definition of P_u. We thus conclude that (2.25) gives a necessary and sufficient condition for $\mathcal{D}_{\tau^*} F_{X,v^*} = 0$ by Proposition 2.7 (1). □

In the next section we will study the condition (2.25) in connection with the associated variety and the multiplicity of highest weight module $L(\tau)$.

3. Associated variety and multiplicity of highest weight modules

The purpose of this section is to understand the associated variety and multiplicity for each irreducible highest weight module $L(\tau)$ by means of the principal symbol $\boldsymbol{\sigma}$ of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type. The harvest of our discussion is summarized as Theorem 3.11. The symbol $\boldsymbol{\sigma}$ yields a $K_\mathbb{C}$-homogeneous vector bundle on the unique open orbit $\mathcal{O}_{m(\tau)}$ in $\mathcal{V}(L(\tau))$. The dimension of fibers can be understood as the multiplicity of $S(\mathfrak{p}_-)$-module $L(\tau)/I_{m(\tau)}L(\tau)$ at the prime ideal $I_{m(\tau)}$ which defines the variety $\mathcal{V}(L(\tau))$, and a result of Joseph (cf. Theorem 3.7) tells us $I_{m(\tau)}L(\tau) = \{0\}$, i.e., $L(\tau)/I_{m(\tau)}L(\tau) = L(\tau)$, for unitarizable $L(\tau)$'s.

3.1. $K_\mathbb{C}$-orbits $\mathcal{O}_m$ in $\mathfrak{p}_+$. — We keep the notation in 2.1. Let us begin by describing the $K_\mathbb{C}$-orbit decomposition of the vector space $\mathfrak{p}_+$ under the adjoint action. For every integer m such that $0 \leqslant m \leqslant r = \mathbb{R}\text{-rank}\, G$, we set

$$\mathcal{O}_m := \mathrm{Ad}(K_\mathbb{C})X(m) \quad \text{with} \quad X(m) := \sum_{k=r-m+1}^{r} X_{\gamma_k}. \tag{3.1}$$

Here $X_{\gamma_k} \in \mathfrak{g}(\mathfrak{t};\gamma_k)$ (see (2.2)) is a root vector for noncompact positive root γ_k, and $X(0)$ should be understood as 0. It then follows that every $X \in \mathfrak{p}_+$ is conjugate to some $X(m)$ under $K_\mathbb{C}$:

$$\mathfrak{p}_+ = \mathcal{O}_0 \cup \cdots \cup \mathcal{O}_r.$$

In reality, there exists an element $k \in K$ such that

$$\mathrm{Ad}(k)(X+\overline{X}) \in \mathfrak{a}_{\mathfrak{p},0} = \sum_{k=1}^{r} \mathbb{R}(X_{\gamma_k} + X_{-\gamma_k})$$

(see (2.9) and (2.10)), since $X + \overline{X} \in \mathfrak{p}_0 = \mathrm{Ad}(K)\mathfrak{a}_{\mathfrak{p},0}$. This shows that

$$\mathrm{Ad}(k)X = \sum_{k \in I} c_k X_{\gamma_k} \quad \text{with} \quad c_k \in \mathbb{R}\backslash\{0\},$$

for some subset I of $\{1, \ldots, r\}$. Note that the complex torus

$$\exp\Big(\sum_{k\in I} \mathbb{C} H_{\gamma_k}\Big) \subset K_{\mathbb{C}}$$

acts on the set $\sum_{k\in I} \mathbb{C}^{\times} X_{\gamma_k}$ transitively, and that $\sum_{k\in I} X_{\gamma_k}$ is conjugate to $X(|I|)$ under the action of the Weyl group $N_K(\mathfrak{a}_{\mathfrak{p},0})/Z_K(\mathfrak{a}_{\mathfrak{p},0})$ of the pair $(\mathfrak{g}_0, \mathfrak{a}_{\mathfrak{p},0})$ (see e.g., [**41**, Prop.5.1(3)]), where $|J|$ denotes the cardinal number of any set J. We thus find that $X \in \mathcal{O}_m$ with $m = |I|$, and that the elements $X(0), \ldots, X(m-1)$ are in the closure of the orbit $\mathcal{O}_m$ with respect to the usual topology (or the Zariski topology) on $\mathfrak{p}_+$.

One can compute the the dimension of each $K_{\mathbb{C}}$-orbit $\mathcal{O}_m$ as follows. In view of Harish-Chandra's result (2.3)–(2.7) on the restricted roots, we easily find that the tangent space $T_{X(m)}(\mathcal{O}_m) = [\mathfrak{k}, X(m)]$ of $\mathcal{O}_m$ at the point $X(m) \in \mathcal{O}_m$ is described as

$$[\mathfrak{k}, X(m)] = \bigoplus_{\gamma \in \Delta^+(m)} \mathfrak{g}(\mathfrak{t}; \gamma) \tag{3.2}$$

with

$$\Delta^+(m) := \{\gamma_r, \ldots, \gamma_{r-m+1}\} \bigcup \Big(\bigcup_{\substack{k>l \\ k>r-m}} P_{kl} \Big) \bigcup \Big(\bigcup_{k>r-m} P_k \Big). \tag{3.3}$$

Hence one obtains

$$\dim \mathcal{O}_m = m + \sum_{\substack{k>l \\ k>r-m}} |P_{kl}| + \sum_{k>r-m} |P_k|.$$

This implies in particular that

$$\dim \mathcal{O}_r > \dim \mathcal{O}_{r-1} > \cdots > \dim \mathcal{O}_0 = 0,$$

and that

$$\dim \mathcal{O}_m - \dim \mathcal{O}_{m-1} = 1 + |P_{r-m+1}| + \sum_{l<r-m+1} |P_{r-m+1,l}|.$$

Note that the right hand side of the above equality is at least two if either $P_{r-m+1} \neq \varnothing$ (namely, $K\backslash G$ is not of tube-type) or $m < r$.

Thus we have proved the following well-known result.

Proposition 3.1. — *The subspace $\mathfrak{p}_+$ splits into a disjoint union of $r+1$ number of $K_{\mathbb{C}}$-orbits $\mathcal{O}_m$ $(0 \leqslant m \leqslant r)$: $\mathfrak{p}_+ = \coprod_{0\leqslant m\leqslant r} \mathcal{O}_m$, and the closure $\overline{\mathcal{O}_m}$ of each orbit $\mathcal{O}_m$ is equal to $\cup_{k\leqslant m}\mathcal{O}_k$ for every m.*

3.2. Associated variety $\mathcal{V}(L(\tau))$. — Let $L(\tau) = M(\tau)/N(\tau)$ be, as in 2.2, the irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type (τ, V_τ). Consider the annihilator ideal

$$\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau) := \{D \in S(\mathfrak{p}_-) \mid Dw = 0 \quad \text{for all } w \in L(\tau)\}.$$

of $L(\tau)$ in $S(\mathfrak{p}_-) = U(\mathfrak{p}_-)$. It should be remarked that an element $D \in S(\mathfrak{p}_-)$ belongs to $\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)$ if and only if $Dv = 0$ for all $v \in V_\tau$, since $L(\tau) = S(\mathfrak{p}_-)V_\tau$ with commutative algebra $S(\mathfrak{p}_-)$.

Definition 3.2. — The affine algebraic variety

$$\mathcal{V}(L(\tau)) := \{X \in \mathfrak{p}_+ \mid D(X) = 0 \quad \text{for all } D \in \mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)\} \subset \mathfrak{p}_+$$

defined by the ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)$ is called the *associated variety* of the $(\mathfrak{g}, K)$-module $L(\tau)$. Here $S(\mathfrak{p}_-)$ is identified with the ring of polynomial functions on $\mathfrak{p}_+$ through the Killing form B of $\mathfrak{g}$.

Remark 3.3. — The notion of the associated variety has been introduced by Vogan **[33]** for arbitrary Harish-Chandra modules (see also **[44]**,**[8]**). As for the highest weight modules $L(\tau)$, the above definition of $\mathcal{V}(L(\tau))$ coincides with Vogan's original one. Indeed, let

$$\mathrm{gr}\, L(\tau) := \bigoplus_{n=0}^{\infty} U_n(\mathfrak{g})V_\tau / U_{n-1}(\mathfrak{g})V_\tau$$

be the graded $(S(\mathfrak{g}), K)$-module defined through the filtration

$$\{0\} := U_{-1}(\mathfrak{g})V_\tau \subset V_\tau = U_0(\mathfrak{g})V_\tau \subset \cdots \subset U_{n-1}(\mathfrak{g})V_\tau \subset U_n(\mathfrak{g})V_\tau \subset \dots,$$

of $L(\tau)$. Here $U_n(\mathfrak{g})$ $(n = 0, 1, \dots)$ denotes the natural increasing filtration of $U(\mathfrak{g})$, and $S(\mathfrak{g}) \simeq \oplus_{n=0}^{\infty} U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$. Then one easily sees that $\mathfrak{k} + \mathfrak{p}_+$ annihilates $\mathrm{gr}\, L(\tau)$, and that

$$\mathrm{gr}\, L(\tau) \simeq L(\tau) \quad \text{as } (S(\mathfrak{p}_-), K)\text{-modules}$$

by (2.11) and (2.12). Hence the algebraic variety in $\mathfrak{g}^* = \mathfrak{g}$ (the identification through B) defined by the annihilator of $\mathrm{gr}\, L(\tau)$ in $S(\mathfrak{g})$, which is the associated variety by Vogan, is nothing but $\mathcal{V}(L(\tau))$.

Since the ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)$ is stable under $\mathrm{Ad}(K_{\mathbb{C}})$, so is the variety $\mathcal{V}(L(\tau))$. In view of Proposition 3.1, we see that there exists a unique integer $m = m(\tau)$ $(0 \leqslant m \leqslant r)$ such that

$$(3.4) \qquad \mathcal{V}(L(\tau)) = \overline{\mathcal{O}_m} \quad \text{with} \quad \mathcal{O}_m = \mathrm{Ad}(K_{\mathbb{C}})X(m) \quad \text{and} \quad m = m(\tau).$$

In particular, the variety $\mathcal{V}(L(\tau))$ is irreducible.

Now let I_m be the prime ideal of $S(\mathfrak{p}_-)$ associated to the irreducible variety $\overline{\mathcal{O}_m}$ $(m = 0, \ldots, r)$:

$$(3.5) \qquad I_m := \{D \in S(\mathfrak{p}_-) \mid D(X) = 0 \quad \text{for all } X \in \overline{\mathcal{O}_m}\}.$$

It holds that $I_r = \{0\}$ since $\overline{\mathcal{O}_r} = \mathfrak{p}_+$. If $m < r$, one knows that

$$(3.6) \qquad I_m = S(\mathfrak{p}_-)Q_{m+1}$$

by [**5**, 8.1] and [**21**, Prop.2.3], where Q_{m+1} denotes as in (2.15) the irreducible K-submodule of $S^{m+1}(\mathfrak{p}_-) \subset S(\mathfrak{p}_-)$ with lowest weight $-\gamma_r - \cdots - \gamma_{r-m}$.

By Hilbert's Nullstellensatz, $I_{m(\tau)}$ coincides with the radical of the annihilator ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)$ for every τ. This allows us to deduce the following

Lemma 3.4. — *The annihilator in $S(\mathfrak{p}_-)$ of $(S(\mathfrak{p}_-), K)$-module $L(\tau)/I_{m(\tau)}L(\tau)$ is equal to $I_{m(\tau)}$.*

Proof. — Since $\sqrt{\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)} = I_{m(\tau)}$, there exists an integer $n_0 > 0$ such that $B^{n_0} \in \mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)$ for every $B \in Q_{m(\tau)+1}$, the finite-dimensional generating subspace of $I_{m(\tau)}$. This implies that

$$(3.7) \qquad (I_{m(\tau)})^{n_0} \subset \mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau).$$

If $D \in \mathrm{Ann}_{S(\mathfrak{p}_-)}(L(\tau)/I_{m(\tau)}L(\tau))$, then $DL(\tau) \subset I_{m(\tau)}L(\tau)$. Inductively, one gets

$$(3.8) \qquad D^n L(\tau) \subset (I_{m(\tau)})^n L(\tau) \quad (n = 1, 2, \ldots).$$

We thus find from (3.7) and (3.8) that $D^{n_0} \in \mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)$, and so $D \in I_{m(\tau)}$. This proves the inclusion $\mathrm{Ann}_{S(\mathfrak{p}_-)}(L(\tau)/I_{m(\tau)}L(\tau)) \subset I_{m(\tau)}$. The converse inclusion is obvious. □

For each $X \in \mathfrak{p}_+$, let $\mathfrak{m}(X)$ be the maximal ideal of $S(\mathfrak{p}_-)$ which defines the variety $\{X\}$ of one element X:

$$(3.9) \qquad \mathfrak{m}(X) := \sum_{Y \in \mathfrak{p}_-} (Y - B(X, Y))S(\mathfrak{p}_-).$$

We set

$$(3.10) \qquad \mathcal{W}(X, \tau) := L(\tau)/\mathfrak{m}(X)L(\tau).$$

Then we see that $\dim \mathcal{W}(X, \tau) < \infty$, and that the isotropy subgroup $K_{\mathbb{C}}(X)$ of $K_{\mathbb{C}}$ at X acts on $\mathcal{W}(X, \tau)$ naturally. Note that, if J is an ideal of $S(\mathfrak{p}_-)$ that defines the variety $\mathcal{V}(L(\tau))$, then

$$(3.11) \qquad \mathfrak{m}(X) \supset J \iff X \in \mathcal{V}(L(\tau)) = \overline{\mathcal{O}_{m(\tau)}}.$$

By applying [**33**, Cor.2.10 and Def.2.12] in view of Lemma 3.4, we immediately deduce

Proposition 3.5. — *Assume that $X \in \mathcal{O}_{m(\tau)}$. Then the dimension of $K_{\mathbb{C}}(X)$-module $\mathcal{W}(X,\tau)$ coincides with the multiplicity of the $S(\mathfrak{p}_-)$-module $L(\tau)/I_{m(\tau)}L(\tau)$ at the unique minimal associated prime $I_{m(\tau)}$:*

$$\dim \mathcal{W}(X,\tau) = \mathrm{mult}_{I_{m(\tau)}}(L(\tau)/I_{m(\tau)}L(\tau)) \quad (X \in \mathcal{O}_{m(\tau)}).$$

So in particular, one has $\mathcal{W}(X,\tau) \neq \{0\}$.

Remark 3.6. — See [**33**, Section 2] and also [**27**, 1.1] for the definition and elementary properties of the multiplicities of finitely generated modules over a commutative Noetherian ring (in connection with Harish-Chandra modules). The multiplicity $\mathrm{mult}_{I_{m(\tau)}}(L(\tau))$ of the whole $L(\tau)$ at $I_{m(\tau)}$ is described as

$$\text{(3.12)} \qquad \sum_{j=0}^{n_0-1} \dim\{(I_{m(\tau)})^j L(\tau)/\mathfrak{m}(X)(I_{m(\tau)})^j L(\tau)\} \qquad (X \in \mathcal{O}_{m(\tau)}),$$

through the filtration

$$L(\tau) = (I_{m(\tau)})^0 L(\tau) \supset (I_{m(\tau)})^1 L(\tau) \supset \cdots \supset (I_{m(\tau)})^{n_0} L(\tau) = \{0\}$$

of the $(S(\mathfrak{p}_-), K)$-module $L(\tau)$. Here n_0 is as in (3.7), and the summand at $j = 0$ in (3.12) is equal to the above $\dim \mathcal{W}(X,\tau)$.

The above proposition will be used in the next subsection to study the associated variety $\mathcal{V}(L(\tau))$ in connection with the principal symbol of differential operator $\mathcal{D}_{\tau^*}$ of gradient type.

As for the unitarizable highest weight modules, the following remarkable result of Joseph (due to Davidson, Enright and Stanke [**3**] for $\mathfrak{g}$ classical) gives a clearer understanding of the above proposition.

Theorem 3.7 ([**21**, Lem.2.4 and Th.5.6]). — *If $L(\tau)$ is unitarizable, the annihilator $\mathrm{Ann}_{S(\mathfrak{p}_-)}w$ in $S(\mathfrak{p}_-)$ of any nonzero vector $w \in L(\tau)$ coincides with the prime ideal $I_{m(\tau)}$. Especially, one has $\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau) = I_{m(\tau)}$.*

Remark 3.8. — For unitarizable $L(\tau) = M(\tau)/N(\tau)$ with nonzero $N(\tau)$, the above theorem together with (3.6) implies the inequality:

$$i(\tau) \leqslant m(\tau) + 1,$$

where $i(\tau)$ is as in (2.14) the level of reduction of the generalized Verma module $M(\tau)$. A description of the number $m(\tau)$ in terms of $i(\tau)$ has been given in [**21**].

Corollary 3.9 **(to Prop.3.5 and Th.3.7).** — *One has*

$$\dim \mathcal{W}(X,\tau) = \mathrm{mult}_{I_{m(\tau)}}(L(\tau)) \quad (X \in \mathcal{O}_{m(\tau)})$$

for every irreducible unitarizable highest weight module $L(\tau)$.

For the classical groups $Sp(2n,\mathbb{R})$, $U(p,q)$ and $O^*(2p)$, Nishiyama, Ochiai and Taniguchi [**27**, Th.7.18 and Th.9.1] have described the associated cycle:

$$\mathcal{AC}(L(\tau)) = \mathrm{mult}_{I_{m(\tau)}}(L(\tau)) \cdot [\overline{\mathcal{O}_{m(\tau)}}], \tag{3.13}$$

and also the Bernstein degree

$$\operatorname{Deg} L(\tau) = \mathrm{mult}_{I_{m(\tau)}}(L(\tau)) \cdot \deg(\overline{\mathcal{O}_{m(\tau)}}) \tag{3.14}$$

of the unitarizable highest weight module $L(\tau)$ (our $\mathfrak{p}_+$ is replaced by $\mathfrak{p}_-$ in [**27**]) by using the theory of reductive dual pairs (G, G') with compact G'. They treat the case where the dual pair (G, G') is in the stable range with smaller G', and then the multiplicity $\mathrm{mult}_{I_{m(\tau)}}(L(\tau))$ is specified as the dimension of the corresponding irreducible representation of G', through detailed study of K-types of $L(\tau)$. On the other hand, the above corollary allows us to give another simple proof of this description of the multiplicity by investigating the $K_{\mathbb{C}}(X)$-module $\mathcal{W}(X,\tau)$, where the dual pairs (G, G') need not be in the stable range. We will do it later in Section 5 (see Theorems 5.14 and 5.15).

3.3. Principal symbol $\boldsymbol{\sigma}$ and associated cycle. — Let

$$\mathcal{D}_{\tau^*} = \nabla^1 \oplus (\oplus_{u=1}^q P_u \circ \overline{\nabla}^{i_u})$$

be, as in Definition 2.3, the differential operator of gradient type whose kernel realizes the maximal globalization of dual lowest weight module $L(\tau)^*$ (see Proposition 2.7). We put

$$\boldsymbol{\sigma}(X, v^*) := \sum_{u=1}^{q} P_u(X^{i_u} \otimes v^*) \in W^* := \oplus_{u=1}^q W_u^* \tag{3.15}$$

for $X \in \mathfrak{p}_+$ and $v^* \in V_\tau^*$, where $P_u : S^{i_u}(\mathfrak{p}_+) \otimes V_\tau^* \longrightarrow W_u^*$ is the K-homomorphism in (2.23). Here $\boldsymbol{\sigma}$ should be understood as $\boldsymbol{\sigma}(X, v^*) = 0$ for every $X \in \mathfrak{p}_+$ and every $v^* \in V_\tau^*$, when $\mathcal{D}_{\tau^*} = \nabla^1$, or equivalently $N(\tau) = \{0\}$. Note that $\boldsymbol{\sigma}$ is naturally identified with the principal symbol at the origin Ke of differential operator $\mathcal{D}_{\tau^*}$, where the symbol is considered only on $\mathfrak{p}_+ \times V_\tau^*$ with the anti-holomorphic cotangent space $\mathfrak{p}_+ = \mathfrak{p}_-^*$ of $K\backslash G$ at Ke. By abuse of language, we call $\boldsymbol{\sigma}$ the principal symbol of $\mathcal{D}_{\tau^*}$ at the origin, since we are concerned mainly with the anti-holomorphic sections of G-homogeneous vector bundle $V_\tau^* {}_K\!\times G$.

We are now going to describe the associated variety $\mathcal{V}(L(\tau))$ by means of $\boldsymbol{\sigma}$. To do this, fix any $X \in \mathfrak{p}_+$ for a while. Then the map $v^* \mapsto \boldsymbol{\sigma}(X, v^*)$ gives a $K_{\mathbb{C}}(X)$-homomorphism $\boldsymbol{\sigma}(X, \cdot\,)$ from V_τ^* to W^*. Hence $\operatorname{Ker}\boldsymbol{\sigma}(X, \cdot\,)$ is a $K_{\mathbb{C}}(X)$-submodule of V_τ^*. By Proposition 2.8 we can describe $\operatorname{Ker}\boldsymbol{\sigma}(X, \cdot\,)$ as

$$\operatorname{Ker}\boldsymbol{\sigma}(X, \cdot\,) = \{v^* \in V_\tau^* \mid \mathcal{D}_{\tau^*} F_{X,v^*} = 0\},$$

where $F_{X,v^*} \in C^\infty_{\tau^*}(G)$ is the function of exponential type defined by (2.24).

The following lemma relates the above kernel with the $K_{\mathbb{C}}(X)$-module $\mathcal{W}(X,\tau)$ in (3.10).

Lemma 3.10. — *For each $X \in \mathfrak{p}_+$, the natural map*

$$V_\tau \hookrightarrow M(\tau) \to L(\tau) = M(\tau)/N(\tau) \to \mathcal{W}(X,\tau) = L(\tau)/\mathfrak{m}(X)L(\tau) \tag{3.16}$$

from V_τ onto $\mathcal{W}(X,\tau)$ induces a $K_{\mathbb{C}}(X)$-isomorphism

$$\mathcal{W}(X,\tau)^* \simeq \operatorname{Ker} \boldsymbol{\sigma}(X,\,\cdot\,) \subset V_\tau^* \tag{3.17}$$

through the contravariant functor $\operatorname{Hom}_{\mathbb{C}}(\,\cdot\,,\mathbb{C})$.

Proof. — First, the natural map from $M(\tau)$ to $\mathcal{W}(X,\tau)$ in (3.16) induces a linear isomorphism from $\mathcal{W}(X,\tau)^*$ onto the space $\mathcal{U}$ of all linear forms ψ on $M(\tau)$ satisfying

$$\psi \circ D = D(X)\psi \quad \text{for} \quad D \in S(\mathfrak{p}_-) \tag{3.18}$$

and

$$\psi | N(\tau) = 0 \quad \text{with} \quad N(\tau) = \sum_{u=1}^{q} S(\mathfrak{p}_-) W_u \quad \text{as in (2.13).} \tag{3.19}$$

In view of (3.18), one sees that the second condition (3.19) is equivalent to

$$\psi | W_u = 0 \quad \text{for } u = 1, \ldots, q.$$

Second, pull back each $\psi \in \mathcal{U}$ to an element of V_τ^* through the embedding $V_\tau \hookrightarrow M(\tau)$:

$$\mathcal{U} \ni \psi \mapsto v^* := \psi | V_\tau \in V_\tau^*. \tag{3.20}$$

By (3.18), this map is injective. We can show just as in the proof of Proposition 2.8 that an element $v^* \in V_\tau^*$ lies in the image of the map (3.20) if and only if $P_u(X^{i_u} \otimes v^*) = 0$ for $u = 1, \ldots, q$, or equivalently, $v^* \in \operatorname{Ker} \boldsymbol{\sigma}(X,\,\cdot\,)$. One thus gets the linear isomorphism (3.17), which is in fact a $K_{\mathbb{C}}(X)$-homomorphism since so is the map (3.16). □

We are now in a position to give a characterization of the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$ and the multiplicity $\operatorname{mult}_{I_{m(\tau)}}(L(\tau)/I_{m(\tau)}L(\tau))$ in terms of the principal symbol $\boldsymbol{\sigma}$, as follows.

Theorem 3.11. — *Let $L(\tau)$ be any irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type τ, and let $\boldsymbol{\sigma} : \mathfrak{p}_+ \times V_\tau^* \to W^*$ be the principal symbol of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type associated to τ^*. Then it holds that*

$$\mathcal{V}(L(\tau)) = \{X \in \mathfrak{p}_+ \mid \operatorname{Ker} \boldsymbol{\sigma}(X,\,\cdot\,) \neq \{0\}\}. \tag{3.21}$$

Moreover, if X is an element of the unique open $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m(\tau)}$ of $\mathcal{V}(L(\tau))$, the dimension of vector space $\operatorname{Ker} \boldsymbol{\sigma}(X,\,\cdot\,)$ coincides with the multiplicity of $S(\mathfrak{p}_-)$-module $L(\tau)/I_{m(\tau)}L(\tau)$ at the prime ideal $I_{m(\tau)}$ of $S(\mathfrak{p}_-)$ corresponding to the variety $\mathcal{V}(L(\tau)) = \overline{\mathcal{O}_{m(\tau)}}$.

Remark 3.12. — We can get the same kind of characterization of the associated variety and the multiplicity also for irreducible $(\mathfrak{g}, K)$-modules of discrete series G-representations, by using the results of **[10]** and **[44]**. We will discuss it elsewhere.

Proof of Theorem 3.11. — We write $\mathcal{V}'$ for the set in the right hand side of (3.21). First, we immediately find that $\mathcal{V}'$ is an affine algebraic variety of $\mathfrak{p}_+$, by noting that

$$\operatorname{Ker} \boldsymbol{\sigma}(X, \cdot\,) \neq \{0\} \quad \Longleftrightarrow \quad \operatorname{rank} \boldsymbol{\sigma}(X, \cdot\,) \leqslant \dim V_\tau^* - 1.$$

Moreover $\mathcal{V}'$ is $K_{\mathbb{C}}$-stable, because one has

$$\operatorname{Ker} \boldsymbol{\sigma}(\operatorname{Ad}(k)X, \cdot\,) = \tau^*(k) \operatorname{Ker} \boldsymbol{\sigma}(X, \cdot\,) \quad \text{for all } k \in K_{\mathbb{C}}$$

by the definition of $\boldsymbol{\sigma}$.

Second, the inclusion $\mathcal{O}_{m(\tau)} \subset \mathcal{V}'$ and the second assertion of the theorem are direct consequences of Proposition 3.5 and Lemma 3.10. If $X \notin \overline{\mathcal{O}_{m(\tau)}} = \mathcal{V}(L(\tau))$, we get $\mathfrak{m}(X) + \operatorname{Ann}_{S(\mathfrak{p}_-)} L(\tau) = S(\mathfrak{p}_-)$ by (3.11). This implies that

$$\mathfrak{m}(X)L(\tau) = (\mathfrak{m}(X) + \operatorname{Ann}_{S(\mathfrak{p}_-)} L(\tau))L(\tau) = L(\tau).$$

So one gets $\operatorname{Ker} \boldsymbol{\sigma}(X, \cdot\,) \simeq \mathcal{W}(X, \tau)^* = \{0\}$ again by Lemma 3.10. We thus find $\mathcal{O}_{m(\tau)} \subset \mathcal{V}' \subset \overline{\mathcal{O}_{m(\tau)}}$, and so $\mathcal{V}' = \mathcal{V}(L(\tau))$ as desired. □

4. Generalized Whittaker models for highest weight modules

In this section we describe the generalized Whittaker models for irreducible highest weight modules $L(\tau)$. The main results are summarized as Theorems 4.7–4.9. We find that each $L(\tau)$ embeds, with nonzero and finite multiplicity, into the generalized Gelfand-Graev representation $\Gamma_{m(\tau)}$ attached to the Cayley transform of the open $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m(\tau)}$ in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. It is shown that, if $L(\tau)$ is unitarizable, the multiplicity of $(\mathfrak{g}, K)$-embeddings $L(\tau) \hookrightarrow \Gamma_{m(\tau)}$ coincides with the multiplicity of $L(\tau)$ at the defining prime ideal of $\mathcal{V}(L(\tau))$.

4.1. Generalized Gelfand-Graev representations. — We keep the notation in 2.1 and 3.1. We begin with introducing in this subsection the generalized Gelfand-Graev representations of G attached to the Cayley transforms of nilpotent $K_{\mathbb{C}}$-orbits $\mathcal{O}_m = \operatorname{Ad}(K_{\mathbb{C}})X(m)$ in $\mathfrak{p}_+$, where m ranges over the integers such that $0 \leqslant m \leqslant r = \mathbb{R}\text{-rank}\, G$.

For this, we consider an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$:

$$(4.1) \quad X(m) = \sum_{k=r-m+1}^{r} X_{\gamma_k}, \quad H(m) := \sum_{k=r-m+1}^{r} H_{\gamma_k}, \quad Y(m) := \sum_{k=r-m+1}^{r} X_{-\gamma_k},$$

with commutation relation

$$\begin{cases} [H(m), X(m)] = 2X(m), \quad [H(m), Y(m)] = -2Y(m), \\ [X(m), Y(m)] = H(m). \end{cases}$$

We put

$$
(4.2)\qquad \begin{cases} X'(m) := -\sqrt{-1}\boldsymbol{c}^{-1}(X(m)) = \dfrac{\sqrt{-1}}{2}(H(m) - X(m) + Y(m)), \\ H'(m) := \boldsymbol{c}^{-1}(H(m)) = X(m) + Y(m) = \displaystyle\sum_{k=r-m+1}^{r} H_k \quad \text{(cf. (2.10))}, \\ Y'(m) := \sqrt{-1}\boldsymbol{c}^{-1}(Y(m)) = -\dfrac{\sqrt{-1}}{2}(H(m) + X(m) - Y(m)), \end{cases}
$$

where $\boldsymbol{c} = \mathrm{Ad}(c)$, with c as in (2.8), is the Cayley transform on $\mathfrak{g}$. Then $(X'(m), H'(m), Y'(m))$ forms an $\mathfrak{sl}_2$-triple in the real form $\mathfrak{g}_0$ of $\mathfrak{g}$, since $\overline{H(m)} = -H(m)$, $\overline{X(m)} = Y(m)$ by (2.2). Set $\mathcal{O}'_m := \mathrm{Ad}(G)X'(m)$. We note that the nilpotent G-orbit $\mathcal{O}'_m$ in $\mathfrak{g}_0$ corresponds to the $K_{\mathbb{C}}$-orbit $\mathcal{O}_m$ in $\mathfrak{p}_+ \subset \mathfrak{p}$ through the Kostant-Sekiguchi correspondence (cf. [**8**, Th.3.1]).

Lemma 4.1 ([**8**, Lemma 3.2])

(1) *The Lie algebra* $\mathfrak{g}$ *decomposes into a direct sum of the* j*-eigensubspaces* $\mathfrak{g}_j(m)$ *for* $\mathrm{ad}\, H'(m)$ *as*

$$\mathfrak{g} = \mathfrak{g}_{-2}(m) \oplus \mathfrak{g}_{-1}(m) \oplus \mathfrak{g}_0(m) \oplus \mathfrak{g}_1(m) \oplus \mathfrak{g}_2(m).$$

(2) *Let* $\Delta(m, j)$ $(j = 0, \pm 1, \pm 2)$ *be the subsets of the root system* Δ *of* $(\mathfrak{g}, \mathfrak{t})$ *defined by*

$$\Delta(m,2) := \{\gamma_{r-m+1}, \ldots, \gamma_r\} \bigcup \Big(\bigcup_{r-m<l<k} P_{kl} \Big),$$

$$(4.3)\qquad \Delta(m,1) := \Big(\bigcup_{l \leqslant r-m<k} (P_{kl} \cup C_{kl}) \Big) \bigcup \Big(\bigcup_{r-m<k} (P_k \cup C_k) \Big),$$

$$
(4.4)\qquad \begin{aligned} \Delta^+(m,0) := C_0 \bigcup \{\gamma_1, \ldots, \gamma_{r-m}\} \bigcup \Big(\bigcup_{r-m<l<k} C_{kl} \Big) \\ \bigcup \Big(\bigcup_{l<k\leqslant r-m} (P_{kl} \cup C_{kl}) \Big) \bigcup \Big(\bigcup_{k \leqslant r-m} (P_k \cup C_k) \Big), \end{aligned}
$$

$$(4.5)\qquad \Delta(m,0) := \Delta^+(m,0) \cup (-\Delta^+(m,0)), \quad \Delta(m,-j) := -\Delta(m,j) \quad (j = 1,2).$$

Then each subspace $\boldsymbol{c}(\mathfrak{g}_j(m)) = \mathrm{Ad}(c)\mathfrak{g}_j(m)$ *is described in terms of the root subspaces* $\mathfrak{g}(\mathfrak{t}; \gamma)$ *as*

$$\boldsymbol{c}(\mathfrak{g}_j(m)) = \begin{cases} \oplus_{\gamma \in \Delta(m,j)} \mathfrak{g}(\mathfrak{t}; \gamma) & \text{if } j \neq 0, \\ \mathfrak{t} \oplus \left(\oplus_{\gamma \in \Delta(m,0)} \mathfrak{g}(\mathfrak{t}; \gamma) \right) & \text{if } j = 0. \end{cases}$$

Now we set

$$\Delta^-(m) := (\Delta(m,-2) \cup \Delta(m,-1)) \cap \Delta_n = -\Delta^+(m) \text{ (cf. (3.3))}.$$

Let $\mathfrak{p}_-(m)$ and $\mathfrak{n}(m)$ be nilpotent, abelian Lie subalgebras of $\mathfrak{g}$ defined respectively by

$$\mathfrak{p}_-(m) := \bigoplus_{\gamma\in\Delta^-(m)} \mathfrak{g}(\mathfrak{t};\gamma) \quad \text{and} \quad \mathfrak{n}(m) := \boldsymbol{c}(\mathfrak{p}_-(m)). \tag{4.6}$$

Note that $\mathfrak{p}_-(r) = \mathfrak{p}_-$. If $K\backslash G$ is of tube type, the Lie subalgebra $\mathfrak{n}(m)$ is stable under the complex conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

We get the following lemma on the structure of these subalgebras $\mathfrak{p}_-(m)$ and $\mathfrak{n}(m)$.

Lemma 4.2

(1) *One has the equality*

$$\mathfrak{p}_-(m) = [\mathfrak{k}, Y(m)]. \tag{4.7}$$

Namely, $\mathfrak{p}_-(m)$ is canonically isomorphic to the tangent space at the point $Y(m)$ of $K_{\mathbb{C}}$-orbit $\mathrm{Ad}(K_{\mathbb{C}})Y(m)$ in $\mathfrak{p}_-$.

(2) *Let $\mathfrak{v}(m)$ be the subspace of $\mathfrak{g}_1(m)$ such that*

$$\mathfrak{v}(m) := \boldsymbol{c}^{-1}(\oplus_{\gamma\in\Xi(m)}\mathfrak{g}(\mathfrak{t};\gamma))$$

with

$$\Xi(m) := \Big(\bigcup_{l\leqslant r-m<k} P_{kl}\Big) \bigcup \Big(\bigcup_{k>r-m} C_k\Big) \subset \Delta(m,1). \tag{4.8}$$

Then it holds that

$$\mathfrak{n}(m) = \mathfrak{v}(m) \oplus \mathfrak{g}_2(m) \quad \textit{and} \quad \dim \mathfrak{v}(m) = \frac{1}{2}\dim \mathfrak{g}_1(m). \tag{4.9}$$

Proof. — First, (4.7) is a direct consequence of (3.2). To prove (2), we note that

$$\boldsymbol{c}^2 = \mathrm{Ad}(c)^2 = \prod_{k=1}^{r} s_{\gamma_k} \quad \text{with} \quad s_{\gamma_k} := \mathrm{Ad}\big(\exp\frac{\pi}{2}(X_{\gamma_k} - X_{-\gamma_k})\big),$$

gives rise to an element of the Weyl group of $(\mathfrak{g},\mathfrak{t})$ such that

$$\boldsymbol{c}^2\gamma_k = -\gamma_k, \quad \boldsymbol{c}^2 C_k = -P_k, \quad \boldsymbol{c}^2 P_k = -C_k$$

for $k = 1,\dots,r$. In fact, s_{γ_k} gives the orthogonal reflection with respect to γ_k, and (2.7) implies $\boldsymbol{c}^2 C_k = -P_k$ and so $\boldsymbol{c}^2 P_k = -C_k$. We thus find that

$$\boldsymbol{c}^2\Delta^-(m) = \Delta(m,2) \cup \Xi(m) \quad \text{(disjoint union)},$$

and correspondingly

$$\mathfrak{n}(m) = \mathfrak{v}(m) \oplus \mathfrak{g}_2(m)$$

by Lemma 4.1(2). In view of (4.3) and (4.8), one gets the second equality in (4.9). □

Let η_m be the one-dimensional representation (i.e., character) of abelian Lie subalgebra $\mathfrak{n}(m) = \mathfrak{v}(m) \oplus \mathfrak{g}_2(m)$ defined by

$$\eta_m(U) := \sqrt{-1}B(U, \theta X'(m)) = -\sqrt{-1}B(U, Y'(m)) \quad \text{for} \quad U \in \mathfrak{n}(m). \tag{4.10}$$

Here θ denotes the complexified Cartan involution of $\mathfrak{g}$, and B the Killing form of $\mathfrak{g}$. Then, just as in Definition 1.6 we get a C^∞-induced G- and $(\mathfrak{g}, K)$-representation $\Gamma_m := \Gamma_{\eta_m}$ acting on the space

$$(4.11) \qquad C^\infty(G;\eta_m) = \{f \in C^\infty(G) \mid U^R f = -\eta_m(U) f \quad (U \in \mathfrak{n}(m))\}$$

through left translation L. Note that

$$(4.12) \qquad C^\infty(G;\eta_r) \subset C^\infty(G;\eta_{r-1}) \subset \cdots \subset C^\infty(G;\eta_0) = C^\infty(G),$$

since one sees $\mathfrak{n}(m) \subset \mathfrak{n}(m')$ and $\eta_{m'}|\mathfrak{n}(m) = \eta_m$ for $m \leqslant m'$.

Definition 4.3. — We call $(\Gamma_m, C^\infty(G;\eta_m))$ the *generalized Gelfand-Graev representation* (GGGR for short) of G attached to the nilpotent orbit $\mathcal{O}'_m = \mathrm{Ad}(G)X'(m)$ in $\mathfrak{g}_0$.

Remark 4.4. — The GGGRs attached to arbitrary nilpotent orbits have been constructed in full generality by Kawanaka [**14**] for reductive algebraic groups. See also [**40**] for the GGGRs of real semisimple Lie groups.

Remark 4.5. — It should be noticed that the above Γ_m's are slightly different from the C^∞-induced GGGRs discussed in [**40**]. In fact, we extend η_m to a linear form on the Lie subalgebra $\mathfrak{g}_1(m) \oplus \mathfrak{g}_2(m)$ by (4.10). Let ζ_m be the irreducible unitary representation of the nilpotent Lie subgroup

$$\tilde{N}(m) := \exp((\mathfrak{g}_1(m) \oplus \mathfrak{g}_2(m)) \cap \mathfrak{g}_0)$$

of G which corresponds to the coadjoint orbit $\mathrm{Ad}^*(\tilde{N}(m))(-\sqrt{-1}\eta_m)$ by the Kirillov orbit method. In [**40**, Def.1.11], the C^∞-GGGR attached to $\mathcal{O}'_m$ is defined to be the representation $C^\infty\text{-}\mathrm{Ind}_{\tilde{N}(m)}^G(\zeta_m)$ of G induced from ζ_m in C^∞-context.

Nevertheless, we can show just as in [**40**, Prop.4.10] that $\mathfrak{n}(m)$ is a totally complex, positive polarization of the linear form $-\sqrt{-1}\eta_m$ on the Lie algebra of $\tilde{N}(m)$. This implies that

$$C^\infty\text{-}\mathrm{Ind}_{\tilde{N}(m)}^G(\zeta_m) \hookrightarrow \Gamma_m \quad \text{as } G\text{-modules},$$

and the image of this embedding is always dense in Γ_m. So we treat Γ_m in this paper instead of $C^\infty\text{-}\mathrm{Ind}_{\tilde{N}(m)}^G(\zeta_m)$.

4.2. Generalized Whittaker models. — For any irreducible finite-dimensional K-module (τ, V_τ), let $L(\tau) = M(\tau)/N(\tau)$ (see 2.2) be the irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type τ. Consider the GGGRs $(\Gamma_m, C^\infty(G;\eta_m))$ $(m = 0, \ldots, r)$ induced from the characters $\eta_m : \mathfrak{n}(m) \to \mathbb{C}$. We say that $L(\tau)$ has a *generalized Whittaker model* of type η_m if $L(\tau)$ is isomorphic to a $(\mathfrak{g}, K)$-submodule of $C^\infty(G;\eta_m)$.

We are going to describe the generalized Whittaker models for $L(\tau)$ by specifying the vector space $\mathrm{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G;\eta_m))$ of $(\mathfrak{g}, K)$-homomorphisms from $L(\tau)$ into $C^\infty(G;\,\eta_m)$. To do this, let $\mathcal{D}_{\tau^*} : C^\infty_{\tau^*}(G) \to C^\infty_\rho(G)$ be, as in Definition 2.3, the

G-invariant differential operator of gradient type whose kernel realizes the maximal globalization of lowest weight module $L(\tau)^*$ (see Proposition 2.7). We set

$$\begin{aligned}(4.13)\qquad \mathcal{Y}(\tau,m) &:= \operatorname{Ker}\mathcal{D}_{\tau^*}(\eta_m)\\ &= \{F\in C^\infty_{\tau^*}(G)\mid \mathcal{D}_{\tau^*}F=0,\quad U^RF=-\eta_m(U)F\ (U\in\mathfrak{n}(m))\}.\end{aligned}$$

Then the kernel theorem (Corollary 1.8) gives a linear isomorphism

$$(4.14)\qquad \operatorname{Hom}_{\mathfrak{g},K}(L(\tau),C^\infty(G;\eta_m))\simeq\mathcal{Y}(\tau,m)$$

through the correspondence (1.7).

Now our aim is to describe the space $\mathcal{Y}(\tau,m)$ for each τ and m. For this purpose, we essentially utilize the following unbounded realization of Hermitian symmetric space $K\backslash G$.

Proposition 4.6 **(cf. [16, page 455], [9], [32]).** — *Retain the notation at the beginning of 2.3, and consider the open dense subset $P_+K^\circ_{\mathbb{C}}P_-$ of $G^\circ_{\mathbb{C}}$ with $P_\pm=\exp\mathfrak{p}_\pm$.*

(1) *One has $G^\circ c\subset P_+K^\circ_{\mathbb{C}}P_-$, where c is the Cayley element of $G^\circ_{\mathbb{C}}$ in (2.8).*

(2) *Set $\xi'(x):=\log p_-(xc)\in\mathfrak{p}_-$ $(x\in G^\circ)$, where $xc=p_+(xc)k_{\mathbb{C}}(xc)p_-(xc)$ with $k_{\mathbb{C}}(xc)\in K^\circ_{\mathbb{C}}$ and $p_\pm(xc)\in P_\pm$. Extend the assignment $x\mapsto\xi'(x)$ $(x\in G^\circ)$ to a map from G to $\mathfrak{p}_-$ through the covering homomorphism $\varpi: G\to G^\circ$. Then, the extended $\xi'(x)$ $(x\in G)$ sets up an anti-holomorphic diffeomorphism, say $\tilde{\xi}'$, from $K\backslash G$ onto an unbounded domain*

$$\mathcal{S}:=\{\xi'(x)\mid x\in G\}\subset\mathfrak{p}_-.$$

Note that the map $x\mapsto k_{\mathbb{C}}(xc)$ $(x\in G^\circ)$ lifts to a map from G to $K_{\mathbb{C}}$ (cf. [32]). We write $k_{\mathbb{C}}(x\cdot c)$ $(x\in G)$ for this lift.

We are now in a position to state the principal results of this article. Let $\mathcal{O}_{m(\tau)}$ be, as in (3.4), the unique open $K_{\mathbb{C}}$-orbit in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. Among the generalized Whittaker models for $L(\tau)$, those of type $\eta_{m(\tau)}$ are most important. We obtain the following theorem on the corresponding linear space $\mathcal{Y}(\tau,m)$ with $m=m(\tau)$.

Theorem 4.7. — *Let (τ,V_τ) be an irreducible finite-dimensional representation of K. Set $m=m(\tau)$ and $\mathcal{Y}(\tau):=\mathcal{Y}(\tau,m)$ for short. Then,*

(1) *$\mathcal{Y}(\tau)$ is a nonzero, finite-dimensional vector space.*

(2) *For any $F\in\mathcal{Y}(\tau)$, there exists a unique polynomial function φ on $\mathfrak{p}_-$ with values in V^*_τ such that*

$$F(x)=\exp B(X(m),\xi'(x))\tau^*(k_{\mathbb{C}}(x\cdot c))\varphi(\xi'(x))\quad(x\in G).$$

(3) *Let $\boldsymbol{\sigma}:\mathfrak{p}_+\times V^*_\tau\to W^*$ be the principal symbol of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type, defined by (3.15). Consider the functions $F_{X(m),v^*}\in C^\infty_{\tau^*}(G)$ of exponential type in Proposition 2.8. Then the assignment*

$$v^*\longmapsto c^RF_{X(m),v^*}:=F_{X(m),v^*}(\cdot\, c)\quad(v^*\in\operatorname{Ker}\boldsymbol{\sigma}(X(m),\ \cdot\ ))$$

yields an injective linear map

$$\chi_\tau : \operatorname{Ker} \boldsymbol{\sigma}(X(m), \ \cdot\ ) \hookrightarrow \mathcal{Y}(\tau).$$

The linear map χ_τ is not surjective in general. In fact, if $L(\tau)$ is finite-dimensional, one has $\operatorname{Ker} \boldsymbol{\sigma}(X(m), \ \cdot\ ) = V_\tau^*$ since $X(m) = X(0) = 0$ in this case. However Lemma 1.1 implies that $\mathcal{Y}(\tau) \simeq L(\tau)^*$.

Nevertheless, we can show the surjectivity of χ_τ for relevant $L(\tau)$'s.

Theorem 4.8. — *Assume that $L(\tau)$ is unitarizable. Then the linear embedding χ_τ in Theorem 4.7 is surjective. Hence one gets*

$$\operatorname{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G;\eta_m)) \simeq \mathcal{Y}(\tau) \simeq \operatorname{Ker} \boldsymbol{\sigma}(X(m), \ \cdot\ ) \simeq \mathcal{W}(X(m),\tau)^* \tag{4.15}$$

as vector spaces, where $m = m(\tau)$, and $\mathcal{W}(X(m),\tau) = L(\tau)/\mathfrak{m}(X(m))L(\tau)$ is as in (3.10)*. Moreover, the dimension of the vector spaces in* (4.15) *equals the multiplicity* $\operatorname{mult}_{I_m}(L(\tau))$ *of the $S(\mathfrak{p}_-)$-module $L(\tau)$ at the unique associated prime $I_m \subset S(\mathfrak{p}_-)$ by Corollary 3.9.*

Theorem 4.7 for $m = m(\tau)$ allows us to deduce the following result on the structure of $\mathcal{Y}(\tau, m')$ for $m' \neq m(\tau)$.

Theorem 4.9. — *The linear space $\mathcal{Y}(\tau, m')$ vanishes* (*resp. is infinite-dimensional*) *if $m' > m(\tau)$* (*resp. $m' < m(\tau)$*)*.*

Remark 4.10. — Theorem 4.8 recovers, to a great extent, our earlier work [**41**, Part II] on the generalized Whittaker models for the holomorphic discrete series $L(\tau) = M(\tau) = U(\mathfrak{g}) \otimes_{U(\mathfrak{k}+\mathfrak{p}_+)} V_\tau$:

$$\operatorname{Hom}_{\mathfrak{g},K}(M(\tau), C^\infty(G;\eta_r)) \simeq V_\tau^*.$$

Moreover, the above three theorems applied to the special case $m(\tau) = r$ gives an answer to Problem 12.7 (*for $i = 0$*) posed in [**41**]. But this answer does not seem to be new. In fact, D. H. Collingwood kindly informed me in 1992 that he had settled Problem 12.7.

Remark 4.11. — The vanishing of $\mathcal{Y}(\tau, m')$ $(m' > m(\tau))$ in Theorem 4.9 follows also from a general result of Matumoto [**22**, Th.2].

The following three subsections 4.3–4.5 will be devoted to proving the above three theorems.

4.3. Key lemmas. — In this subsection we prepare two lemmas which are crucially important to prove Theorems 4.7 and 4.8.

The first lemma is the following somewhat surprising result of Jakobsen.

Lemma 4.12 ([**20**, Prop.2.9]). — *Let g be any element of $G_{\mathbb{C}}^{\circ}$, and let $L(\tau)$ be an irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type τ. Then one has the equality*

$$\mathrm{Ann}_{U(\mathrm{Ad}(g)\mathfrak{p}_-)} L(\tau) = \mathrm{Ad}(g)(\mathrm{Ann}_{U(\mathfrak{p}_-)} L(\tau))$$

on the annihilator of $L(\tau)$ in $U(\mathrm{Ad}(g)\mathfrak{p}_-)$ and that in $U(\mathfrak{p}_-) = S(\mathfrak{p}_-)$.

Second, for each integer $m = 0, \dots, r$, let J_m denote the ideal of $S(\mathfrak{p}_-)$ generated by the elements $Y - B(X(m), Y)$ $(Y \in \mathfrak{p}_-(m))$:

$$J_m = \sum_{Y \in \mathfrak{p}_-(m)} (Y - B(X(m), Y)) S(\mathfrak{p}_-). \tag{4.16}$$

A method of Joseph (cf. [**5**, 2.4]) for describing the lowest weight vector of irreducible K-module $Q_{m+1} = I_m \cap S^{m+1}(\mathfrak{p}_-)$ (see (3.6)) can be applied to deduce the following

Lemma 4.13. — *It holds that $I_m + J_m = \mathfrak{m}(X(m))$. Here I_m (see (3.5)) is the prime ideal of $S(\mathfrak{p}_-)$ corresponding to the irreducible algebraic variety $\overline{\mathcal{O}_m}$, and $\mathfrak{m}(X(m))$ (see (3.9)) is the maximal ideal of $S(\mathfrak{p}_-)$ corresponding to $X(m) \in \mathfrak{p}_+$.*

Proof. — The inclusion $I_m + J_m \subset \mathfrak{m}(X(m))$ is obvious since any polynomial in I_m or in J_m vanishes at $X(m)$ by definition. If $m = r$, the equality $I_r + J_r = \mathfrak{m}(X(r))$ holds since $I_r = \{0\}$ and $J_r = \mathfrak{m}(X(r))$.

Now we assume that $m < r$. In order to prove the sum $I_m + J_m$ exhausts the whole $\mathfrak{m}(X(m))$, we consider the subspace

$$\mathfrak{q}_m := \sum_{\gamma \in \Delta_n \cap \Delta^+(m,0)} \mathfrak{g}(\mathfrak{t}; -\gamma) \subset \mathfrak{p}_-.$$

(See (4.4) for the definition of $\Delta^+(m, 0)$.) Then one gets $\mathfrak{p}_- = \mathfrak{p}_-(m) \oplus \mathfrak{q}_m$ as vector spaces, and hence

$$\mathfrak{m}(X(m)) = J_m + \mathfrak{q}_m S(\mathfrak{p}_-) \tag{4.17}$$

by the definitions of $\mathfrak{m}(X(m))$ and J_m.

We set

$$\Omega(m) := (\Delta_c \cap \Delta(m, 0)) \setminus \Big(\bigcup_{r-m<l<k} (C_{kl} \cup -C_{kl}) \Big) \quad \text{(cf. (4.5))},$$

and let $\mathfrak{k}_m$ be the Lie subalgebra of $\mathfrak{k}$ defined by

$$\mathfrak{k}_m := \mathfrak{t} \oplus (\oplus_{\gamma \in \Omega(m)} \mathfrak{g}(\mathfrak{t}; \gamma)).$$

We write $(K_{\mathbb{C}})_m$ for the analytic subgroup of $K_{\mathbb{C}}$ with Lie algebra $\mathfrak{k}_m$. Note that $\mathfrak{q}_m, J_m$ and I_m are all stable under the adjoint action of $(K_{\mathbb{C}})_m$. Further, by using (2.7) one easily checks that $\mathfrak{q}_m$ is an irreducible $(K_{\mathbb{C}})_m$-module with lowest weight vector $X_{-\gamma_{r-m}} \in \mathfrak{q}_m$. This together with (4.17) reduces our task to showing

$$X_{-\gamma_{r-m}} \in I_m + J_m, \tag{4.18}$$

which can be done as follows.

Let $Q_{m+1} = I_m \cap S^{m+1}(\mathfrak{p}_-)$ be the irreducible K-submodule of I_m with lowest weight $-\gamma_r - \cdots - \gamma_{r-m}$. Take a nonzero lowest weight vector $D_{m+1} \in Q_{m+1}$. By virtue of [**5**, 2.4(*)], we find that

$$D_{m+1} \equiv cX_{-\gamma_r} \cdots X_{-\gamma_{r-m}} \mod J_m$$

for some nonzero constant $c \in \mathbb{C}$. This implies that

$$D_{m+1} \equiv \{c \prod_{k>r-m} B(X_{\gamma_k}, X_{-\gamma_k})\} \cdot X_{-\gamma_{r-m}} \mod J_m.$$

Thus we have obtained (4.18) as desired. □

4.4. A role of the Cayley transform. — Keep the notation at the beginning of 2.3. We recall that the bounded realization $\mathcal{B} = \{\xi(x) \mid x \in G\} \subset \mathfrak{p}_-$ of $K\backslash G$ gives a linear isomorphism

$$\Theta : O^*_{\tau^*}(G) \xrightarrow{\sim} O(\mathcal{B}, V^*_\tau)$$

by (2.18). Let $O(\mathfrak{p}_-, V^*_\tau)$ be the space of all holomorphic functions on the whole $\mathfrak{p}_-$ with values in V^*_τ. Naturally, we regard $O(\mathfrak{p}_-, V^*_\tau)$ as a subspace of $O(\mathcal{B}, V^*_\tau)$. Set

$$O^*_{\tau^*}(G)_0 := \Theta^{-1} O(\mathfrak{p}_-, V^*_\tau).$$

Just in the same way, the unbounded realization

$$\mathcal{S} = \{\xi'(x) = \xi(\varpi(x)c) \mid x \in G\} \subset \mathfrak{p}_-$$

of $K\backslash G$ in Proposition 4.6 gives a linear isomorphism

$$\Theta^c : O^*_{\tau^*}(G) \xrightarrow{\sim} O(\mathcal{S}, V^*_\tau),$$

by

$$\Theta^c F(\xi'(x)) := \tau^*(k_{\mathbb{C}}(x \cdot c))^{-1} F(x) \quad (x \in G; F \in O^*_{\tau^*}(G)). \tag{4.19}$$

See also [**32**, 2.4]. Similarly we put

$$O^*_{\tau^*}(G)^c_0 := (\Theta^c)^{-1} O(\mathfrak{p}_-, V^*_\tau).$$

Then the composite $(\Theta^c)^{-1} \circ \Theta$ induces an isomorphism from $O^*_{\tau^*}(G)_0$ onto $O^*_{\tau^*}(G)^c_0$ as vector spaces. This is exactly the (well-defined) right translation of functions on G by the Cayley element $c \in G^\circ_{\mathbb{C}}$:

$$O^*_{\tau^*}(G)_0 \ni F \longmapsto c^R F \in O^*_{\tau^*}(G)^c_0, \tag{4.20}$$

where

$$c^R F(x) = \tau^*(k_{\mathbb{C}}(x \cdot c))(\Theta F)(\xi'(x)) \quad \text{with} \quad \xi'(x) = \log p_-(\varpi(x)c)$$

for $x \in G$.

The function $c^R F = ((\Theta^c)^{-1} \circ \Theta)(F)$ can be interpreted as follows. First, take an open neighbourhood U of e (the identity element) in G such that $\varpi : G \to G^\circ$

restricted to U gives a diffeomorphism from U onto $U^\circ := \varpi(U)$ $(\subset G^\circ)$. Define a V_τ^*-valued function F° on U° by setting

$$F^\circ(\varpi(x)) = F(x) \quad (x \in U). \tag{4.21}$$

One sees that F° extends, in a unique way, to a (multi-valued) complex analytic function $\tilde{F}^\circ$ on the open dense subset $P_+K_{\mathbb{C}}^\circ P_-$ of $G_{\mathbb{C}}^\circ$, and that $\tilde{F}^\circ$ comes from the function

$$P_+ \times K_{\mathbb{C}} \times P_- \ni (p_+, k_{\mathbb{C}}, p_-) \longmapsto \tau^*(k_{\mathbb{C}})(\Theta F)(\log p_-) \in V_\tau^* \tag{4.22}$$

through the covering map from $K_{\mathbb{C}}$ to $K_{\mathbb{C}}^\circ$. Second, we consider the right translation $c^R\tilde{F}^\circ$ of $\tilde{F}^\circ$ by the Cayley element $c \in G_{\mathbb{C}}^\circ$. This is a complex analytic function defined on $P_+K_{\mathbb{C}}^\circ P_-c^{-1}$. Noting that $G^\circ \subset P_+K_{\mathbb{C}}^\circ P_-c^{-1}$ by Proposition 4.6 (1), we write c^RF° for the restriction to G° of $c^R\tilde{F}^\circ$. Then our $c^RF = ((\Theta^c)^{-1} \circ \Theta)(F)$ gives a (single-valued) lift of c^RF° to G.

The following proposition assures that the above right translation c^R preserves the kernel of differential operator $\mathcal{D}_{\tau^*}$.

Proposition 4.14. — *Let $\mathcal{D}_{\tau^*} : C_{\tau^*}^\infty(G) \to C_\rho^\infty(G)$ be the differential operator of gradient type associated to τ^*. Then* (4.20) *yields a linear isomorphism*

$$\operatorname{Ker}\mathcal{D}_{\tau^*} \cap O_{\tau^*}^*(G)_0 \simeq \operatorname{Ker}\mathcal{D}_{\tau^*} \cap O_{\tau^*}^*(G)_0^c.$$

Namely, a function F in $O_{\tau^*}^*(G)_0$ satisfies the differential equation $\mathcal{D}_{\tau^*}F = 0$ if and only if the corresponding c^RF in $O_{\tau^*}^*(G)_0^c$ satisfies the same equation.

As shown in the next subsection, this proposition together with two key lemmas in 4.3 allows us to describe the space $\mathcal{Y}(\tau) = \mathcal{Y}(\tau, m(\tau))$ of generalized Whittaker functions on G associated to the highest weight module $L(\tau)$.

Proof of Proposition 4.14. — Let $F \in O_{\tau^*}^*(G)_0$. We employ the interpretation of c^RF and also the notation given just before the proposition. Note that $\mathcal{D}_{\tau^*}$ naturally gives rise to a right $G_{\mathbb{C}}^\circ$-invariant, holomorphic differential operator, say $\tilde{\mathcal{D}}_{\tau^*}^\circ$ defined on the complex group $G_{\mathbb{C}}^\circ$.

Now assume that $\mathcal{D}_{\tau^*}F = 0$. Then one finds that $\tilde{\mathcal{D}}_{\tau^*}^\circ\tilde{F}^\circ = 0$ on $P_+K_{\mathbb{C}}^\circ P_-$. In reality, $\tilde{\mathcal{D}}_{\tau^*}^\circ\tilde{F}^\circ$ is the complex analytic extension of $\tilde{\mathcal{D}}_{\tau^*}^\circ F^\circ$ on U°, and the latter $\tilde{\mathcal{D}}_{\tau^*}^\circ F^\circ$ equals zero by assumption (cf. (4.21)). We thus get

$$\tilde{\mathcal{D}}_{\tau^*}^\circ(c^R\tilde{F}^\circ) = c^R(\tilde{\mathcal{D}}_{\tau^*}^\circ\tilde{F}^\circ) = 0 \quad \text{on } P_+K_{\mathbb{C}}P_-c^{-1}.$$

This implies that $\mathcal{D}_{\tau^*}(c^RF) = 0$, because $\mathcal{D}_{\tau^*}(c^RF)$ is a lift to G of the restriction $(\tilde{\mathcal{D}}_{\tau^*}^\circ(c^R\tilde{F}^\circ))|G^\circ$.

The reverse implication can be proved in the same way by using the inverse Cayley transform. □

4.5. Proof of the main theorems. — We are now ready to prove our main theorems given in 4.2.

Proof of Theorem 4.7. — Let F be any function in $\mathcal{Y}(\tau) = \mathcal{Y}(\tau, m)$ (see (4.13)) with $m = m(\tau)$. Set $f^c := \Theta^c F \in O(\mathcal{S}, V_\tau^*)$.

STEP 1. We first see that the requirement $U^R F = -\eta_m(U)F$ ($U \in \mathfrak{n}(m) = \boldsymbol{c}(\mathfrak{p}_-(m))$) for F is equivalent to

$$(4.23)\qquad D_1 \cdot f^c = 0 \quad \text{for} \quad D_1 \in J_m \quad (\text{cf. } (4.16)),$$

for the corresponding f^c, by noting that

$$\eta_m(\boldsymbol{c}Y) = -\sqrt{-1}B(Y, \boldsymbol{c}^{-1}(Y'(m))) = -B(Y, X(m)) \quad (Y \in \mathfrak{p}_-(m)).$$

Here the action of $S(\mathfrak{p}_-)$ on $O(\mathcal{S}, V_\tau^*)$ is defined by the directional derivative (2.20).

STEP 2. Consider the point $Y_0 := \xi'(e) \in \mathcal{S}$ (e the identity element of G), which is expressed as

$$Y_0 = \log p_-(c) = -\sum_{k=1}^{r} X_{-\gamma_k}.$$

Let D_2 be any element of the annihilator ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)$ of $L(\tau)$ in $S(\mathfrak{p}_-) = U(\mathfrak{p}_-)$. Then it is standard to verify that

$$(D_2 \cdot f^c)(Y_0) = (\boldsymbol{c}(D_2))^R F(e) = (\boldsymbol{c}({}^T D_2))^L F(e).$$

Here $D \mapsto {}^T D$ denotes the principal anti-automorphism of $U(\mathfrak{g})$ as in 1.2.

Noting that the ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)$ is homogeneous, we can apply Lemma 4.12 to deduce that $\boldsymbol{c}({}^T D_2)$ lies in the annihilator of $L(\tau)$ in $U(\boldsymbol{c}(\mathfrak{p}_-))$. This implies that

$$(\boldsymbol{c}({}^T D_2))^L F = 0,$$

because $U(\mathfrak{g})^L \langle v^*, F(\cdot) \rangle \simeq L(\tau)$ for every nonzero vector $v^* \in V_\tau^*$. We thus conclude

$$(4.24)\qquad (D_2 \cdot f^c)(Y_0) = 0 \quad (D_2 \in \mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)).$$

STEP 3. We are going to specify the function $f^c \in O(\mathcal{S}, V_\tau^*)$. It follows from Hilbert's Nullstellensatz that

$$\sqrt{\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau) + J_m} = \mathfrak{m}(X(m)),$$

since $\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau) + J_m$ defines the variety $\{X(m)\}$ of one point $X(m)$ in $\mathfrak{p}_+$ by virtue of Lemma 4.13. Hence (4.23) and (4.24) imply that there exists a nonnegative integer N such that

$$(4.25)\qquad ((Y - B(X(m), Y))^{N+1} \cdot f^c)(Y_0) = 0 \quad \text{for all } Y \in \mathfrak{p}_-.$$

This means that the function f^c is of the form

$$(4.26)\qquad f^c(Y) = \exp B(X(m), Y)\varphi(Y),$$

where φ is a V_τ^*-valued polynomial function on $\mathfrak{p}_-$ of degree at most N. In particular, F lies in $O^*_{\tau^*}(G)^c_0$ by (4.26), and so one finds that

$$\mathcal{Y}(\tau) \subset O^*_{\tau^*}(G)^c_0.$$

Thus we have proved the claim (2) of Theorem 4.7 as well as the finite-dimensionality of $\mathcal{Y}(\tau)$ in the assertion (1).

STEP 4. Let $v^* \in V_\tau^*$. By Proposition 2.8, the function $F_{X(m),v^*} \in O^*_{\tau^*}(G)_0$ of exponential type (see (2.24)) satisfies $\mathcal{D}_{\tau^*} F_{X(m),v^*} = 0$ if and only if the vector v^* lies in $\operatorname{Ker}\boldsymbol{\sigma}(X(m),\ \cdot\ )$. Proposition 4.14 says that the former condition is equivalent to $\mathcal{D}_{\tau^*}(c^R F_{X(m),v^*}) = 0$. Noting that $\Theta^c(c^R F_{X(m),v^*})$ is of the form (4.26) with constant function $\varphi(Y) = v^*$ ($Y \in \mathfrak{p}_-$), we deduce that $c^R F_{X(m),v^*} \in \mathcal{Y}(\tau)$ for every $v^* \in \operatorname{Ker}\boldsymbol{\sigma}(X(m),\ \cdot\ )$. This proves the assertion (3). Finally, the vector space $\mathcal{Y}(\tau)$ does not vanish because

$$\{0\} \neq \mathcal{W}(X(m),\tau)^* \simeq \operatorname{Ker}\boldsymbol{\sigma}(X(m),\ \cdot\ ) \hookrightarrow \mathcal{Y}(\tau),$$

thanks to Proposition 3.5 and Lemma 3.10. □

Proof of Theorem 4.8. — Suppose that $L(\tau)$ is unitarizable. We set $m = m(\tau)$. Then one knows that $I_m = \operatorname{Ann}_{S(\mathfrak{p}_-)} L(\tau)$ by Theorem 3.7. This combined with Lemma 4.12 allows us to refine the discussion in Step 3 of the proof of Theorem 4.7. As a result, we find that, for any $F \in \mathcal{Y}(\tau)$, the corresponding function $f^c = \Theta^c F$ in (4.26) is necessarily of exponential type, i.e., $f^c(Y) = \exp B(X(m),Y)v^*$ ($Y \in \mathfrak{p}_-$) for some $v^* \in V_\tau^*$. This proves the surjectivity of χ_τ in Theorem 4.7. Now the remainder of the theorem is a consequence of Corollary 3.9 and Lemma 3.10. □

Proof of Theorem 4.9. — First, assume that $m' > m := m(\tau)$. Let F be any function in the space $\mathcal{Y}(\tau,m')$. By (4.12), F belongs to $\mathcal{Y}(\tau) = \mathcal{Y}(\tau,m(\tau))$ also. Hence the corresponding $f^c := \Theta^c F \in O(\mathfrak{p}_-, V_\tau^*)$ is of the form (4.26). It follows in particular that

$$(X_{-\gamma_{r-m'+1}})^n \cdot f^c = \exp B(X(m),\ \cdot\ )((X_{-\gamma_{r-m'+1}})^n \cdot \varphi) = 0$$

for sufficiently large integers n, because $B(X(m), X_{-\gamma_{r-m'+1}}) = 0$ and because φ in (4.26) is a polynomial on $\mathfrak{p}_-$. On the other hand, since F is in $C^\infty(G;\eta_{m'})$, we see just as in Step 1 of the proof of Theorem 4.7 that

$$(X_{-\gamma_{r-m'+1}}) \cdot f^c = B(X(m'), X_{-\gamma_{r-m'+1}}) f^c = B(X_{\gamma_{r-m'+1}}, X_{-\gamma_{r-m'+1}}) f^c.$$

Thus one gets $f^c = 0$ since $B(X_{\gamma_{r-m'+1}}, X_{-\gamma_{r-m'+1}}) \neq 0$. This shows that $\mathcal{Y}(\tau,m') = \{0\}$.

Second, assume that $m' < m = m(\tau)$. Take a nonzero function F in $\mathcal{Y}(\tau)$ by Theorem 4.7 (1). Note that $\mathcal{Y}(\tau) \subset \mathcal{Y}(\tau,m')$. For each $t \in \mathbb{R}$, we define an element $a_t \in G$ by

$$a_t := \exp\{-t(X_{\gamma_{r-m+1}} + X_{-\gamma_{r-m+1}})\} = \exp tc(H_{\gamma_{r-m+1}}) \text{ (cf. (4.2))}.$$

Then it is easily checked that

$$\mathrm{Ad}(a_t)\mathfrak{n}(m') = \mathfrak{n}(m') \quad \text{and} \quad \eta_{m'} \circ \mathrm{Ad}(a_t) = \eta_{m'}.$$

This implies that the functions $(a_t)^R F$ still lie in $\mathcal{Y}(\tau, m')$ for all $t \in \mathbb{R}$, by noting that the differential operator $\mathcal{D}_{\tau^*}$ is right G-invariant. These vectors $(a_t)^R F$ $(t \in \mathbb{R})$ in $\mathcal{Y}(\tau, m')$ turn to be linearly independent, because one gets

$$(\boldsymbol{c}(X_{-\gamma_{r-m+1}}))^R((a_t)^R F) = e^{2t} \cdot B(X_{\gamma_{r-m+1}}, X_{-\gamma_{r-m+1}}) \cdot ((a_t)^R F).$$

by direct computation. Hence the vector space $\mathcal{Y}(\tau, m')$ in question is infinite-dimensional if $m' < m(\tau)$. □

Now we have completely proved the main theorems, Theorems 4.7–4.9.

4.6. Relation to generalized Whittaker vectors. — We end this section by interpreting our results (Theorems 4.7–4.9) in terms of generalized Whittaker vectors in the algebraic dual of an irreducible highest weight $(\mathfrak{g}, K)$-module. To do this we prepare the following lemma.

Lemma 4.15. — *Set* $\mathfrak{n} := \mathfrak{n}(r) = \boldsymbol{c}(\mathfrak{p}_-)$ *(cf.* (4.6)*). Let b be the linear map from $\mathfrak{n}$ to $\mathfrak{p}_-$ defined by*

$$b(Z) \equiv Z \qquad \mathrm{mod}\ \mathfrak{k} + \mathfrak{p}_+$$

for $Z \in \mathfrak{n}$. Then b is a surjective linear isomorphism.

Proof. — Write an element $Y \in \mathfrak{p}_-$ as a linear combination of root vectors:

$$Y = \sum_{\gamma \in \Delta_n^+} c_\gamma X_{-\gamma} \quad \text{with} \quad c_\gamma \in \mathbb{C}.$$

Then it is easy to compute the Cayley transform $\boldsymbol{c}(X_{-\gamma})$ of $X_{-\gamma}$ for each noncompact positive root γ (see [**32**, 2.1 and 2.2] and also [**41**, 9.1]). As a result, one finds that

$$\boldsymbol{c}(X_{-\gamma}) \equiv \kappa_\gamma X_{-\gamma} \qquad \mathrm{mod}\ \ \mathfrak{p}_+ + \mathfrak{k},$$

where $\kappa_\gamma = 1/2$ or $1/\sqrt{2}$ depending on $\gamma \in \Delta_n^+$. This implies that

$$b(\boldsymbol{c}(Y)) = \sum_{\gamma \in \Delta_n^+} \kappa_\gamma c_\gamma X_{-\gamma}.$$

We thus get the lemma. □

Let $L(\tau)$ be the irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type τ. Let us look upon $L(\tau)$, by restriction, as a module over $U(\mathfrak{n}) = S(\mathfrak{n})$. (Note that $\mathfrak{n}$ is an abelian subalgebra of $\mathfrak{g}$.) Then Lemma 4.15 immediately implies the following

Proposition 4.16. — *$L(\tau)$ is finitely generated as a $U(\mathfrak{n})$-module. Moreover, the Gelfand-Kirillov dimension* $\mathrm{Dim}(\mathfrak{n}; L(\tau))$ *and the Bernstein degree* $\mathrm{Deg}(\mathfrak{n}; L(\tau))$ *of $L(\tau)$ as a $U(\mathfrak{n})$-module coincide with those* $\mathrm{Dim}\, L(\tau) = \mathrm{Dim}(\mathfrak{g}; L(\tau))$ *and* $\mathrm{Deg}\, L(\tau) = \mathrm{Deg}(\mathfrak{g}; L(\tau))$ *as a $U(\mathfrak{g})$-module, respectively.*

Remark 4.17. — The argument in the proof of Lemma 4.15 allows us to show

$$\mathfrak{n}^{\perp} \cap \mathcal{V}(L(\tau)) \subset \mathfrak{n}^{\perp} \cap \mathfrak{p}_+ = \{0\},$$

where $\mathfrak{n}^{\perp}$ denotes the orthogonal of $\mathfrak{n}$ in $\mathfrak{g}$ with respect to the Killing form. In view of this property, we can apply a criterion [**43**, Th.2.2] for the finiteness of restriction of $U(\mathfrak{g})$-modules to subalgebras. This gives another proof of the above proposition.

In view of Lemma 4.12, the annihilator ideal of $L(\tau)$ in $U(\mathfrak{n})$ turns to be

$$\operatorname{Ann}_{U(\mathfrak{n})} L(\tau) = \boldsymbol{c}(\operatorname{Ann}_{U(\mathfrak{p}_-)} L(\tau)), \tag{4.27}$$

and it defines the associated variety

$$\mathcal{V}(\mathfrak{n}; L(\tau)) = \overline{\boldsymbol{c}(\mathcal{O}_{\mathrm{m}(\tau)})}$$

of $U(\mathfrak{n})$-module $L(\tau)$, which is an irreducible affine algebraic variety in $\mathfrak{n}^* = \boldsymbol{c}(\mathfrak{p}_+)$. Thus, the associated cycle $\mathcal{AC}(\mathfrak{n}; L(\tau))$ of $U(\mathfrak{n})$-module $L(\tau)$ is of the form

$$\mathcal{AC}(\mathfrak{n}; L(\tau)) = \operatorname{mult}_{\boldsymbol{c}(I_{m(\tau)})}(\mathfrak{n}; L(\tau)) \cdot [\overline{\boldsymbol{c}(\mathcal{O}_{\mathrm{m}(\tau)})}],$$

where $\operatorname{mult}_{\boldsymbol{c}(I_{m(\tau)})}(\mathfrak{n}; L(\tau))$ denotes the multiplicity of $U(\mathfrak{n})$-module $L(\tau)$ at the unique associated prime $\boldsymbol{c}(I_{m(\tau)})$. Further, the Bernstein degree of $U(\mathfrak{n})$-module $L(\tau)$ is described as

$$\operatorname{Deg}(\mathfrak{n}; L(\tau)) = \operatorname{mult}_{\boldsymbol{c}(I_{m(\tau)})}(\mathfrak{n}; L(\tau)) \cdot \deg(\overline{\boldsymbol{c}(\mathcal{O}_{\mathrm{m}(\tau)})}), \tag{4.28}$$

where $\deg(\overline{\boldsymbol{c}(\mathcal{O}_{\mathrm{m}(\tau)})})$ denotes the degree of the nilpotent cone $\overline{\boldsymbol{c}(\mathcal{O}_{\mathrm{m}(\tau)})}$ (cf. [**27**, Lemma 1.1]).

The above discussion tells us the following coincidence of two types of multiplicities of $L(\tau)$.

Proposition 4.18. — *One has the equality*

$$\operatorname{mult}_{I_{m(\tau)}}(L(\tau)) = \operatorname{mult}_{\boldsymbol{c}(I_{m(\tau)})}(\mathfrak{n}; L(\tau)),$$

where $\operatorname{mult}_{I_{m(\tau)}}(L(\tau))$ *is the multiplicity in the associated cycle of* $(\mathfrak{g}, K)$*-module* $L(\tau)$ (*cf.* (3.13)).

Proof. — The assertion follows from Proposition 4.16 together with the equalities (3.14) and (4.28), by noting that the degrees of orbits $\overline{\mathcal{O}_{m(\tau)}}$ and $\overline{\boldsymbol{c}(\mathcal{O}_{\mathrm{m}(\tau)})}$ coincide with each other. □

Now, for each $m = 0, \ldots, r = \mathbb{R}\text{-rank}(G)$, let η_m be the one-dimensional representation of $\mathfrak{n}(m) = \boldsymbol{c}(\mathfrak{p}_-(m))$ which induces the GGGR $(\Gamma_m, C^{\infty}(G; \eta_m))$ (cf. (4.10) and (4.11)). A linear form ψ on $L(\tau)$ is called an (algebraic) *generalized Whittaker vector* of type η_m if

$$\psi(Uw) = \eta_m(U)\psi(w) \quad \text{for all} \quad U \in \mathfrak{n}(m) \text{ and } w \in L(\tau).$$

We write $\mathrm{Wh}^*_{\eta_m}(L(\tau))$ for the space of such generalized Whittaker vectors. By definition, one observes that

$$\mathrm{Wh}^*_{\eta_m}(L(\tau)) \simeq \left(L(\tau)/\boldsymbol{c}({}^T J_m)L(\tau)\right)^* := \mathrm{Hom}_{\mathbb{C}}(L(\tau)/\boldsymbol{c}({}^T J_m)L(\tau),\ \mathbb{C}), \tag{4.29}$$

as vector spaces, where J_m is the ideal of $S(\mathfrak{p}_-) = U(\mathfrak{p}_-)$ defined by (4.16), and T denotes the automorphism of $S(\mathfrak{p}_-)$ such that ${}^T Y = -Y$ for $Y \in \mathfrak{p}_-$. Further, every $(\mathfrak{g}, K)$-embedding T from $L(\tau)$ into $C^\infty(G; \eta_m)$ yields a generalized Whittaker vector $\psi \in \mathrm{Wh}^*_{\eta_m}(L(\tau))$ by

$$\psi(w) = (Tw)(e) \quad (w \in L(\tau)).$$

This assignment $T \mapsto \psi$ sets up a linear embedding

$$\mathrm{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G; \eta_m)) \hookrightarrow \mathrm{Wh}^*_{\eta_m}(L(\tau)). \tag{4.30}$$

We can show that this embedding is actually surjective for the most relevant case, as follows.

Proposition 4.19. — *If $m = m(\tau)$, the map* (4.30) *is surjective. Namely, every nonzero generalized Whittaker vector in $\mathrm{Wh}^*_{\eta_{m(\tau)}}(L(\tau))$ gives an embedding of $L(\tau)$ into the GGGR $C^\infty(G; \eta_{m(\tau)})$.*

Remark 4.20. — Let $L(\tau)^\infty$ denote the smooth G-module consisting of all C^∞-vectors for an irreducible admissible representation of G corresponding to $L(\tau)$. In view of the discussion in [**41**, 12.5], one finds that, if $L(\tau)$ is a member of holomorphic discrete series, any vector in $\mathrm{Wh}^*_{\eta_{m(\tau)}}(L(\tau))$ extends also to a continuous G-isomorphism from $L(\tau)^\infty$ into $C^\infty(G; \eta_{m(\tau)})$. This appears to be true for any $L(\tau)$ not necessarily in the discrete series, but we do not discuss it here.

Proof of Proposition 4.19. — First, we set $\tilde{\mathfrak{m}} := \mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau) + J_{m(\tau)}$. By virtue of Lemma 4.13, $\tilde{\mathfrak{m}}$ is an ideal of $S(\mathfrak{p}_-)$ that defines the one point variety $\{X(m(\tau))\}$, and in particular, the codimension of $\tilde{\mathfrak{m}}$ in $S(\mathfrak{p}_-)$ is finite. By (4.27), the isomorphism (4.29) turns out to be

$$\mathrm{Wh}^*_{\eta_{m(\tau)}}(L(\tau)) \simeq \left(L(\tau)/\boldsymbol{c}({}^T\tilde{\mathfrak{m}})L(\tau)\right)^*. \tag{4.31}$$

Second, we consider the generalized Verma module $M(\tau) = U(\mathfrak{g}) \otimes_{U(\mathfrak{k}+\mathfrak{p}_+)} V_\tau$ and its unique maximal submodule $N(\tau)$. The natural quotient map $M(\tau) \to L(\tau) = M(\tau)/N(\tau)$ induces a linear isomorphism

$$L(\tau)/\boldsymbol{c}({}^T\tilde{\mathfrak{m}})L(\tau) \simeq M(\tau)/(N(\tau) + \boldsymbol{c}({}^T\tilde{\mathfrak{m}})M(\tau)) \tag{4.32}$$

in the canonical way. Now let $\langle\ \cdot, \cdot\ \rangle_\tau$ be the $(\mathfrak{g}, K)$-invariant bilinear form on $O^*_{\tau^*}(G) \times M(\tau)$ constructed in 2.3. We write $\mathcal{E}$ for the orthogonal of $\boldsymbol{c}({}^T\tilde{\mathfrak{m}})M(\tau)$ in $O^*_{\tau^*}(G)$. Then, the bilinear form $\langle\ \cdot, \cdot\ \rangle_\tau$ naturally induces a linear embedding

$$\mathcal{E} \hookrightarrow (M(\tau)/\boldsymbol{c}({}^T\tilde{\mathfrak{m}})M(\tau))^*. \tag{4.33}$$

Third, just as in the proof of Theorem 4.7 (see 4.5), one finds that an element $F \in O^*_{\tau^*}(G)$ belongs to $\mathcal{E}$ if and only if

$$(4.34) \qquad D \cdot f^c(Y_0) = 0 \quad \text{for all} \quad D \in \tilde{\mathfrak{m}},$$

where $Y_0 = \xi'(e)$, and $f^c = (\Theta^c)^{-1}F \in O(\mathcal{S}, V^*_\tau)$ as in (4.19). It then follows from (4.34) that

$$(4.35) \qquad \dim \mathcal{E} = \dim V_\tau \times \dim\, S(\mathfrak{p}_-)/\tilde{\mathfrak{m}}.$$

Since we have

$$M(\tau) = U(\mathfrak{n})V_\tau \simeq U(\mathfrak{n}) \otimes V_\tau$$

by Lemma 4.15, the dimension of the quotient space $M(\tau)/\boldsymbol{c}({}^T\tilde{\mathfrak{m}})M(\tau)$ is equal to the right hand side of (4.35). This shows that the linear isomorphism (4.33) is surjective:

$$(4.36) \qquad \mathcal{E} \simeq (M(\tau)/\boldsymbol{c}({}^T\tilde{\mathfrak{m}})M(\tau))^*.$$

In view of Proposition 2.7 (1), (4.31), (4.32) and (4.36) give rise to isomorphisms

$$\mathcal{E} \cap \mathrm{Ker}\mathcal{D}_{\tau^*} \simeq (M(\tau)/(N(\tau) + \boldsymbol{c}({}^T\tilde{\mathfrak{m}})M(\tau)))^* \simeq \mathrm{Wh}^*_{\eta_{m(\tau)}}(L(\tau))$$

as vector spaces, where $\mathcal{D}_{\tau^*}$ is the differential operator of gradient-type associated to τ^*. This proves the proposition, because every function in $\mathcal{E} \cap \mathrm{Ker}\mathcal{D}_{\tau^*}$ gives a $(\mathfrak{g}, K)$-embedding of $L(\tau)$ into $C^\infty(G; \eta_{m(\tau)})$ by virtue of (4.14). □

Proposition 4.21. — *If $L(\tau)$ is unitarizable, one gets*

$$\dim \mathrm{Wh}^*_{\eta_{m(\tau)}}(L(\tau)) = \dim L(\tau)/\mathfrak{m}(X(m(\tau)))L(\tau) = \dim L(\tau)/\boldsymbol{c}(\mathfrak{m}(-X(m(\tau))))L(\tau),$$

where $\mathfrak{m}(X)$ is the maximal ideal of $S(\mathfrak{p}_-)$ definining a point $X \in \mathcal{O}_{m(\tau)}$ (cf. (4.1)). Moreover, the above dimension is equal to the multiplicity in the associated cycle $\mathcal{AC}(L(\tau))$ of $L(\tau)$.

Proof. — The assertions follow from Theorem 4.8 and Proposition 4.19 by noting the isomorphism (4.31), where ${}^T\tilde{\mathfrak{m}} = \mathfrak{m}(-X(m(\tau)))$ in this case. □

Concerning the spaces of algebraic generalized Whittaker vectors, we are now in a position to give the following consequence of the main results of this article.

Theorem 4.22. — *The dimension of the vector space $\mathrm{Wh}^*_{\eta_m}(L(\tau))$ is given as*

$$\dim \mathrm{Wh}^*_{\eta_m}(L(\tau)) = \begin{cases} 0 & \text{if } m > m(\tau), \\ \text{finite } (\neq 0) & \text{if } m = m(\tau), \\ \infty & \text{if } m < m(\tau). \end{cases}$$

Here $\mathcal{O}_{m(\tau)}$ is the unique open $K_{\mathbb{C}}$-orbit in the associated variety of $L(\tau)$. Moreover, if $L(\tau)$ is unitarizable, the dimension of $\mathrm{Wh}^*_{\eta_{m(\tau)}}(L(\tau))$ coincides with the multiplicity $\mathrm{mult}_{I_{m(\tau)}}(L(\tau))$.

Proof. — The claims for the cases $m < m(\tau)$ and $m = m(\tau)$ follow from Theorems 4.7 and 4.8 coupled with (4.30) and Proposition 4.19. The property $\mathrm{Wh}^*_{\eta_m}(L(\tau)) = \{0\}$ for $m > m(\tau)$ can be proved by an argument similar to the one given in the proof of Theorem 4.9, or alternatively, one can apply a general result [**22**, Corollary 4] of Matumoto. □

5. Case of the classical groups

Hereafter, we assume that G is one of the classical groups $SU(p,q)$ $(p \geqslant q)$, $Sp(n,\mathbb{R})$ or $SO^*(2n)$. The theory of reductive dual pairs gives concrete realizations of unitarizable highest weight modules $L[\sigma] = L(\tau[\sigma])$ for these groups G (see Theorem 5.1), by decomposing the oscillator representation of the pair (G, G'), where $G' = U(k)$, $O(k)$, or $Sp(k)$ respectively, and $\sigma \in \hat{G}'$.

For such $L[\sigma]$'s, we specify in this section the $K_{\mathbb{C}}(X(m))$-modules $\mathcal{W}(X(m), \tau[\sigma]) = L[\sigma]/\mathfrak{m}(X(m))L[\sigma]$ (cf. (3.10)) with $m = m(\tau[\sigma])$ explicitly by using the Fock model of the oscillator module (see Theorems 5.14 and 5.15). In view of Theorem 4.8, this leads us to a clearer understanding of the generalized Whittaker models for $L[\sigma]$.

5.1. Oscillator representation. — We start with constructing the oscillator representation of the pair (G, G'), following [**3**, §7]. First, realize our classical groups G as

$$\begin{cases} SU(p,q) = \left\{ g \in SL(p+q,\mathbb{C}) \;\middle|\; g \begin{pmatrix} I_p & O \\ O & -I_q \end{pmatrix} {}^t\bar{g} = \begin{pmatrix} I_p & O \\ O & -I_q \end{pmatrix} \right\} & (p \geqslant q), \\ Sp(n,\mathbb{R}) = \left\{ g \in SU(n,n) \;\middle|\; {}^t g J_n g = J_n \right\} \text{ with } J_n := \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}, & \\ SO^*(2n) = \left\{ g \in SU(n,n) \;\middle|\; {}^t g \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} g = \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} \right\}, & \end{cases}$$

where I_n denotes the identity matrix of size n. The totality of unitary matrices in G forms a maximal compact subgroup K.

Let $M_{p,q}$ denote the space of all complex matrices of size $p \times q$. We write Sym_n (resp. Alt_n) for the set of all symmetric (resp. alternating) complex matrices of size n. Then, the real rank $r = \mathbb{R}\text{-rank}\, G$, the complexification $K_{\mathbb{C}}$ of K, and the irreducible $K_{\mathbb{C}}$-module $\mathfrak{p}_+$ under Ad, can be described for each G respectively as in the following table.

(5.1)

G	r	$K_{\mathbb{C}}$	$\mathfrak{p}_+$
$SU(p,q)$	q	$S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$	$M_{p,q}$
$Sp(n,\mathbb{R})$	n	$GL(n,\mathbb{C})$	Sym_n
$SO^*(2n)$	$[\frac{n}{2}]$	$GL(n,\mathbb{C})$	Alt_n

Here the $K_{\mathbb{C}}$-action on $\mathfrak{p}_+$ is given as

$$g \cdot X = g_1 X g_2^{-1}, \quad g = (g_1, g_2) \in S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C})), \; X \in M_{p,q} \tag{5.2}$$

for $G = SU(p,q)$, and

$$g \cdot X = gX^t g, \quad g \in GL(n,\mathbb{C}), \; X \in \mathrm{Sym}_n \text{ or } \mathrm{Alt}_n$$

for $G = Sp(n,\mathbb{R})$ or $SO^*(2n)$. For this, see also [**27**, 7.1].

For every positive integer k, we realize the compact group G' as

$$\begin{cases} U(k) = \{g \in GL(k,\mathbb{C}) \mid g^t\overline{g} = I_k\} & \text{for } G = SU(p,q), \\ O(k) = U(k) \cap GL(k,\mathbb{R}) & \text{for } G = Sp(n,\mathbb{R}), \\ Sp(k) = \{g \in U(2k) \mid {}^t g J_k g = J_k\} & \text{for } G = SO^*(2n). \end{cases}$$

The complexification of G' will be denoted by $G'_{\mathbb{C}}$, i.e., $G'_{\mathbb{C}} = GL(k,\mathbb{C})$, $O(k,\mathbb{C})$, $Sp(k,\mathbb{C})$ respectively. Define a space M of complex matrices by

$$M := \begin{cases} M_{n,k} \; (n := p+q) & \text{for } G = SU(p,q), \\ M_{n,k} & \text{for } G = Sp(n,\mathbb{R}), \\ M_{n,2k} & \text{for } G = SO^*(2n). \end{cases}$$

For $G = SU(p,q)$, the elements $Z \in M$ will be written as

$$Z = \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{with} \quad A \in M_{p,k}, B \in M_{q,k}.$$

The group $K_{\mathbb{C}} \times G'_{\mathbb{C}}$ acts on M by

$$(g, g') \cdot Z := \begin{pmatrix} g_1 A g'^{-1} \\ {}^t g_2^{-1} B \, {}^t g' \end{pmatrix} \quad \text{with } g = (g_1, g_2), \tag{5.3}$$

for $G = SU(p,q)$, and by

$$(g, g') \cdot Z := gZg'^{-1} \tag{5.4}$$

for $G = Sp(n,\mathbb{R})$ or $SO^*(2n)$, where $(g,g') \in K_{\mathbb{C}} \times G'_{\mathbb{C}}$ and $Z \in M$.

We now prepare some notation to describe the oscillator representation. Let ψ be a map from M to $\mathfrak{p}_+$ such that

$$\psi(Z) := \begin{cases} A\,{}^t B & \text{for } G = SU(p,q), \\ \frac{1}{2} Z\,{}^t Z & \text{for } G = Sp(n,\mathbb{R}), \\ \frac{1}{2} Z J_k \,{}^t Z & \text{for } G = SO^*(2n). \end{cases} \tag{5.5}$$

Note that $\psi : M \to \mathfrak{p}_+$ is a $K_{\mathbb{C}} \times G'_{\mathbb{C}}$-equivariant polynomial map of degree two, where we let $G'_{\mathbb{C}}$ act on $\mathfrak{p}_+$ trivially. For each $Y \in \mathfrak{p}_-$, let h_Y be a polynomial on M defined by

$$h_Y(Z) := B(\psi(Z), Y) \quad (B \text{ the Killing form of } \mathfrak{g}).$$

We set for $g \in K_{\mathbb{C}}$,

$$\delta_k(g) := \begin{cases} (\det g_1)^{-k} & \text{for } G = SU(p,q), \\ (\det g)^{-k/2} & \text{for } G = Sp(n,\mathbb{R}), \\ (\det g)^{-k} & \text{for } G = SO^*(2n), \end{cases} \tag{5.6}$$

where $g = (g_1, g_2)$ as in (5.2) for $G = SU(p,q)$. If $G = Sp(n,\mathbb{R})$, the function δ_k is two-valued on $K_{\mathbb{C}} = GL(n,\mathbb{C})$. We need to go up to the two fold cover of $K_{\mathbb{C}}$ in order that δ_k determines a genuine character of the group. Hereafter, we replace K and $K_{\mathbb{C}}$ by their two fold covering groups when $G = Sp(n,\mathbb{R})$. By abuse of notation, the latter covering groups will be denoted by K and $K_{\mathbb{C}}$ again.

Let $\mathbb{C}[M]$ denote the ring of polynomial functions on the complex vector space M. One can define a $(\mathfrak{g}, K)$-representation ω on $\mathbb{C}[M]$ in the following fashion. First, the $\mathfrak{p}_-$ action on $\mathbb{C}[M]$ is given by multiplication:

$$\omega(Y)f(Z) := h_Y(Z)f(Z), \quad Y \in \mathfrak{p}_-, \tag{5.7}$$

for $f \in \mathbb{C}[M]$. Second, $\mathfrak{p}_+$ acts by differentiation:

$$\omega(X)f(Z) := \kappa(h_{\overline{X}}(\partial)f)(Z), \quad X \in \mathfrak{p}_+.$$

Here $h_{\overline{X}}(\partial)$ stands for the constant coefficient differential operator on M defined by the polynomial $h_{\overline{X}}$, and the constant κ depends only on the Lie algebra $\mathfrak{g}_0$ of G. Third, the complexification $K_{\mathbb{C}}$ acts on $\mathbb{C}[M]$ holomorphically as

$$\omega(g)f(Z) := \delta_k(g)f((g^{-1}, e)\cdot Z), \quad g \in K_{\mathbb{C}}.$$

On the other hand, $\mathbb{C}[M]$ has a natural $G'_{\mathbb{C}}$-module structure through

$$R(g')f(Z) := f((e, {g'}^{-1})\cdot Z), \quad g' \in G'_{\mathbb{C}}.$$

Then it is easily seen that these two representations ω and R commute with each other. The resulting $(\mathfrak{g}, K) \times G'_{\mathbb{C}}$-representation (ω, R) on $\mathbb{C}[M]$ will be called the Fock model of the (infinitesimal) *oscillator representation* of the pair (G, G').

It should be mentioned that the above oscillator representation ω of the pair (G, G') comes from the *Weil representation* of a metaplectic group. In fact, $G \times G'$ forms a reductive dual pair in a real symplectic group $Sp(N,\mathbb{R})$. Consider the Weil representation Ω (cf. **[12]**) of the metaplectic group $Mp(N, R)$, which is the two fold cover of $Sp(N,\mathbb{R})$. Restrict Ω to the metaplectic cover $\tilde{G} \times \tilde{G}'$ of $G \times G'$, and then twist it by a certain one-dimensional character of the compact group $\tilde{G}'$. One thus gets ω.

5.2. Unitarizable highest weight modules $L[\sigma]$. — Let (σ, V_σ) be an irreducible (finite-dimensional) unitary representation of the compact group G'. Extend σ to a holomorphic representation of $G'_{\mathbb{C}}$ in the canonical way. We set

$$L[\sigma] := \mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[M]), \tag{5.8}$$

which turns to be a $(\mathfrak{g}, K)$-module through the representation ω on $\mathbb{C}[M]$. Let $\Sigma(k)$ denote the totality of equivalence classes of irreducible unitary representations σ of G' such that $L[\sigma] \neq \{0\}$. Note that the $G'_{\mathbb{C}}$-action on $\mathbb{C}[M]$ is locally finite since $G'_{\mathbb{C}}$ preserves each subspace of homogeneous polynomials of any fixed degree. Then one gets

$$\mathbb{C}[M] \simeq \bigoplus_{\sigma \in \Sigma(k)} L[\sigma] \otimes V_\sigma \quad \text{as } (\mathfrak{g}, K) \times G'_{\mathbb{C}}\text{-modules.} \tag{5.9}$$

The isomorphism is given by

$$L[\sigma] \otimes V_\sigma \ni T \otimes v \longmapsto T(v) \in \mathbb{C}[M],$$

on each $G'_{\mathbb{C}}$-isotypic component $L[\sigma] \otimes V_\sigma$.

The following theorem states the celebrated Howe duality correspondence associated to (G, G').

Theorem 5.1 **([12], [6], [7]; cf. [3, §7])**

(1) *$L[\sigma]$ is an irreducible unitarizable highest weight $(\mathfrak{g}, K)$-module for every $\sigma \in \Sigma(k)$. In particular,* (5.9) *gives the irreducible decomposition of the $(\mathfrak{g}, K) \times G'_{\mathbb{C}}$-module $\mathbb{C}[M]$.*

(2) *Let $\sigma_1, \sigma_2 \in \Sigma(k)$. Then, $V_{\sigma_1} \simeq V_{\sigma_2}$ as $G'_{\mathbb{C}}$-modules if and only if $L[\sigma_1] \simeq L[\sigma_2]$ as $(\mathfrak{g}, K)$-modules.*

(3) *If $G = SU(p, q)$ or $Sp(n, \mathbb{R})$, any irreducible unitarizable highest weight $(\mathfrak{g}, K)$-module is isomorphic to an $L[\sigma]$, where $\sigma \in \Sigma(k)$ for some positive integer k.*

Let $\tau[\sigma]$ denote the extreme K-type of highest weight $(\mathfrak{g}, K)$-module $L[\sigma]$, i.e., $L[\sigma] = L(\tau[\sigma])$. We note that the correspondence $\sigma \leftrightarrow \tau[\sigma]$ can be explicitly described in terms of their highest weights. For this, see the articles cited in the above theorem.

It follows from the standard argument in linear algebra that each $K_{\mathbb{C}}$-orbit $\mathcal{O}_m$ in $\mathfrak{p}_+$ (see 3.1) consists of all the matrices in $\mathfrak{p}_+ = M_{p,q}, \mathrm{Sym}_n$ (resp. Alt_n) of rank m (resp. $2m$) for $G = SU(p, q), Sp(n, \mathbb{R})$ (resp. $SO^*(2n)$). Let $E_{s,t}(i, j)$ denote the (i, j)-matrix unit of size $s \times t$ whose (k, l)-matrix entry e_{kl} is equal to 1 if $(k, l) = (i, j)$; $e_{kl} = 0$ otherwise. We put

$$I_{s,t}(m) := \sum_{i=1}^{m} E_{s,t}(i, i) \in M_{s,t} \quad (m = 0, \ldots, \min(s, t)), \tag{5.10}$$

where $I_{s,t}(0) := 0$. Then, we take an element $X(m) \in \mathcal{O}_m$ explicitly as

$$X(m) := \begin{cases} I_{p,q}(m) & \text{for } G = SU(p, q), \\ I_{n,n}(m)/2 & \text{for } G = Sp(n, \mathbb{R}), \\ \sum_{i=1}^{m} (E_{n,n}(i, m+i) - E_{n,n}(m+i, i))/2 & \text{for } G = SO^*(2n). \end{cases} \tag{5.11}$$

Now, it is easily seen that the image $\psi(M)$ of the $K_{\mathbb{C}} \times G'_{\mathbb{C}}$-equivariant map $\psi : M \to \mathfrak{p}_+$ in (5.5) is a $K_{\mathbb{C}}$-stable, irreducible algebraic variety described as

$$\psi(M) = \overline{\mathcal{O}_{m_k}} \quad \text{with} \quad m_k := \min(k, r), \tag{5.12}$$

where M and ψ depend on k. By (5.7) and (5.9), the annihilator ideal in $S(\mathfrak{p}_-)$ of $L[\sigma]$ ($\sigma \in \Sigma(k)$) consists exactly of all the elements $D \in S(\mathfrak{p}_-) = \mathbb{C}[\mathfrak{p}_+]$ vanishing on $\psi(M)$. In this way we have shown the following well-known fact.

***Proposition 5.2* (cf. [3, §12]).** — *For any $\sigma \in \Sigma(k)$, the associated variety of unitarizable highest weight module $L[\sigma]$ is equal to the closure of the $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m_k} = \mathrm{Ad}(K_{\mathbb{C}})X(m_k)$. More precisely, $\mathrm{Ann}_{S(\mathfrak{p}_-)}L[\sigma]$ coincides to the prime ideal I_{m_k} defining $\overline{\mathcal{O}_{m_k}}$ (cf. Theorem 3.7).*

5.3. Variety $\mathcal{V}_k$ and ideal $\omega(\mathfrak{m})\mathbb{C}[M]$. — Now we consider the maximal ideal:

$$\mathfrak{m} := \mathfrak{m}(X(m_k)) = \sum_{Y \in \mathfrak{p}_-} (Y - B(X(m_k), Y))S(\mathfrak{p}_-) \subset S(\mathfrak{p}_-) \quad (\text{cf. } (3.9)),$$

for each positive integer k. For $m = 0, \dots, r$, let $K_{\mathbb{C}}(m) := K_{\mathbb{C}}(X(m))$ be the isotropy subgroup of $K_{\mathbb{C}}$ at $X(m) \in \mathcal{O}_m$. We want to describe the $K_{\mathbb{C}}(m_k)$-modules

$$\mathcal{W}[\sigma] := \mathcal{W}(X(m_k), \tau[\sigma]) = L[\sigma]/\mathfrak{m}L[\sigma] \quad (\sigma \in \Sigma(k)).$$

In view of (5.8) and (5.9), one gets an isomorphism

$$\mathcal{W}[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[M]/\omega(\mathfrak{m})\mathbb{C}[M]) \quad \text{as } K_{\mathbb{C}}(m_k)\text{-modules}. \tag{5.13}$$

So, our task is to decompose the quotient $\mathbb{C}[M]/\omega(\mathfrak{m})\mathbb{C}[M]$ as $K_{\mathbb{C}}(m_k) \times G'_{\mathbb{C}}$-modules.

To do this, we note that, by virtue of (5.7), $\omega(\mathfrak{m})\mathbb{C}[M]$ is equal to the ideal of $\mathbb{C}[M]$ generated by all matrix entries of the following polynomial function of degree two:

$$M \ni Z \longmapsto \psi(Z) - X(m_k) \in \mathfrak{p}_+. \tag{5.14}$$

We consider the corresponding affine algebraic variety $\mathcal{V}_k$ of M:

$$\mathcal{V}_k := \{Z \in M \mid \psi(Z) = X(m_k)\} = \psi^{-1}(X(m_k)),$$

which is the inverse image of $X(m_k)$ by ψ. Clearly, the variety $\mathcal{V}_k$ is stable under the action of $K_{\mathbb{C}}(m_k) \times G'_{\mathbb{C}}$. Note that the codimension of $\mathcal{V}_k$ is given as

$$\dim M - \dim \mathcal{V}_k = \dim \mathcal{O}_{m_k} = \dim \mathfrak{p}_-(m_k) \quad (\text{cf. Lemma 4.2})$$

by virtue of (5.12).

Now, let us give the $G'_{\mathbb{C}}$-orbit decomposition of $\mathcal{V}_k$ for each group G separately, where $G'_{\mathbb{C}}$ is identified with the subgroup $\{e\} \times G'_{\mathbb{C}}$ of $K_{\mathbb{C}} \times G'_{\mathbb{C}}$. We define a subgroup $G'_{\mathbb{C}}(k-r)$ ($r = \mathbb{R}\text{-rank}\, G$) of $G'_{\mathbb{C}}$ by

$$G'_{\mathbb{C}}(k-r) := \begin{cases} \{I_k\} \text{ (the unit group)} & \text{if } k \leqslant r, \\ \left\{ \begin{pmatrix} I_r & O \\ O & h \end{pmatrix} \in G'_{\mathbb{C}} \right\} & \text{if } k > r, \end{cases} \tag{5.15}$$

for $G = SU(p,q)$, $Sp(n,\mathbb{R})$, and by

(5.16)

$$G'_{\mathbb{C}}(k-r) := \begin{cases} \{I_{2k}\} \text{ (the unit group)} & \text{if } k \leqslant r, \\ \left\{ \begin{pmatrix} I_k & O & O & O \\ O & h_{11} & O & h_{12} \\ O & O & I_k & O \\ O & h_{21} & O & h_{22} \end{pmatrix} \in G'_{\mathbb{C}} \;\middle|\; h_{ij} \in M_{k-r,k-r} \right\} & \text{if } k > r, \end{cases}$$

for $G = SO^*(2n)$. Note that if $k > r$, the group $G'_{\mathbb{C}}(k-r)$ is naturally isomorphic to $GL(k-r,\mathbb{C})$, $O(k-r,\mathbb{C})$ or $Sp(k-r,\mathbb{C})$ according as $G = SU(p,q)$, $Sp(n,\mathbb{R})$ or $SO^*(2n)$ respectively.

First, the following lemma for the case $SU(p,q)$ is due to Tagawa.

Lemma 5.3 ([**31**, 3.5 and 3.8]). — *Assume that $G = SU(p,q)$ $(r = q,\ G'_{\mathbb{C}} = GL(k,\mathbb{C}))$.*

(1) *If $k \leqslant q$, the group $G'_{\mathbb{C}} = GL(k,\mathbb{C})$ acts on $\mathcal{V}_k$ simply transitively. One gets*

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \begin{pmatrix} I_{p,k}(k) \\ I_{q,k}(k) \end{pmatrix} \simeq G'_{\mathbb{C}} \quad \text{as } G'_{\mathbb{C}}\text{-sets}, \tag{5.17}$$

where the matrices $I_{p,k}(k), I_{q,k}(k)$ are as in (5.10).

(2) *If $k > q$ and $p = q$, then the $G'_{\mathbb{C}}$-action on $\mathcal{V}_k$ is still transitive, and it holds that*

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \begin{pmatrix} I_{q,k}(q) \\ I_{q,k}(q) \end{pmatrix} \simeq G'_{\mathbb{C}}/G'_{\mathbb{C}}(k-q) \quad \text{as } G'_{\mathbb{C}}\text{-sets.}$$

Here $G'_{\mathbb{C}}(k-q)$ coincides with the isotropy subgroup of $G'_{\mathbb{C}}$ at $\begin{pmatrix} I_{q,k}(q) \\ I_{q,k}(q) \end{pmatrix} \in \mathcal{V}_k$.

(3) *If $k > q$ and $p > q$, $\mathcal{V}_k$ is no longer $G'_{\mathbb{C}}$-homogeneous. In fact, let $\tilde{M}_{p-q,k-q}$ be the subspace of M defined by*

$$\tilde{M}_{p-q,k-q} := \left\{ \tilde{U} = \begin{pmatrix} I_q & O \\ O & U \\ I_q & O \end{pmatrix} \;\middle|\; U \in M_{p-q,k-q} \right\}.$$

Then $\mathcal{V}_k$ is decomposed as

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \tilde{M}_{p-q,k-q} = \coprod_{\tilde{U} \in \Lambda} G'_{\mathbb{C}} \cdot \tilde{U}, \tag{5.18}$$

where Λ denotes a complete system of representatives in $\tilde{M}_{p-q,k-q}$ of the $G'_{\mathbb{C}}(k-q)$-orbit space

$$\tilde{M}_{p-q,k-q}/G'_{\mathbb{C}}(k-q) \simeq M_{p-q,k-q}/GL(k-q,\mathbb{C}).$$

Second, the structure of $G'_{\mathbb{C}}$-variety $\mathcal{V}_k$ is much simpler for $Sp(n,\mathbb{R})$. This is because the corresponding Hermitian symmetric space is always of tube type.

Lemma 5.4. — *Assume that $G = Sp(n,\mathbb{R})$ ($r = n$ and $G'_{\mathbb{C}} = O(k,\mathbb{C})$). Then it holds that*

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot I_{n,k}(m_k) \simeq G'_{\mathbb{C}}/G'_{\mathbb{C}}(k-n) \quad \textit{as } G'_{\mathbb{C}}\textit{-sets,}$$

Here $m_k = \min(k,n)$, and the isotropy subgroup of $G'_{\mathbb{C}}$ at $I_{n,k}(m_k)$ is equal to the group $G'_{\mathbb{C}}(k-n)$ in (5.15).

Third, one obtains the following lemma for $SO^*(2n)$.

Lemma 5.5. — *Assume that $G = SO^*(2n)$ ($r = [n/2]$ and $G'_{\mathbb{C}} = Sp(k,\mathbb{C})$).*

(1) *If $k \leqslant r$, one has*

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot I_{n,2k}(2k) \simeq G'_{\mathbb{C}} \quad \textit{as } G'_{\mathbb{C}}\textit{-sets.} \tag{5.19}$$

(2) *If $k > r = n/2$ with even integer n, the variety $\mathcal{V}_k$ is described as*

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix} \simeq G'_{\mathbb{C}}/G'_{\mathbb{C}}(k-r), \tag{5.20}$$

where $G'_{\mathbb{C}}(k-r) \simeq Sp(k-r,\mathbb{C})$ (cf. (5.16)) coincides with the isotropy subgroup of $G'_{\mathbb{C}}$ at the matrix $\begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix}$ in $M = M_{2r,2k}$.

(3) *If $k > r = (n-1)/2$ with odd integer n, $\mathcal{V}_k$ consists of two $G'_{\mathbb{C}}$-orbits. In fact, we set*

$$(z_1,z_2)^{\sim} := \begin{pmatrix} I_r & O & O & O \\ O & O & I_r & O \\ o & z_1 & o & z_2 \end{pmatrix}$$

for $(z_1,z_2) \in M_{1,2(k-r)} = M_{1,k-r} \times M_{1,k-r}$. Then $\mathcal{V}_k$ decomposes as

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \tilde{M}_{1,2(k-r)} = G'_{\mathbb{C}} \cdot (0\ldots0,0\ldots0)^{\sim} \coprod G'_{\mathbb{C}} \cdot (1\;0\ldots0,0\ldots0)^{\sim}, \tag{5.21}$$

where $\tilde{M}_{1,2(k-r)} := \{(z_1,z_2)^{\sim} \mid z_1,z_2 \in M_{1,k-r}\}$.

We give below a proof of Lemma 5.5 for $G = SO^*(2n)$. Lemmas 5.3 and 5.4 can be shown in the same way (so we omit the proofs of these two lemmas).

Proof of Lemma 5.5. — (1) Suppose $k \leqslant r = [n/2]$. In view of (5.5) and (5.11), one observes that an element

$$Z = \begin{pmatrix} C \\ D \end{pmatrix} \in M \quad \text{with} \quad C \in M_{2k,2k},\; D \in M_{n-2k,2k}$$

belongs to $\mathcal{V}_k$ if and only if

$$CJ_k\,{}^tC = J_k, \quad CJ_k\,{}^tD = O, \quad \text{and} \quad DJ_k\,{}^tD = O,$$

which means that $C \in Sp(k,\mathbb{C})$ and $D = O$. We thus get (5.19).

(2) Consider the case $k > r = n/2$ with even integer n. Take any matrix Z in $\mathcal{V}_k$. Let $c_i \in M_{1,2k} = \mathbb{C}^{2k}$ ($i = 1,\ldots,n$) denote the i-th row vector of Z. Set $d_i := c_{r+i}$ ($i = 1,\ldots,r$). By the condition $ZJ_k\,{}^tZ = J_r$ ($\Leftrightarrow \psi(Z) = X(r)$), we can extend $\{c_1,\ldots,c_r,d_1,\ldots,d_r\}$ to a symplectic basis $\{c_1,\ldots,c_k,d_1,\ldots,d_k\}$ of $\mathbb{C}^{2k}$ with

respect to the nondegenerate alternating form defined by J_k. Then, there exists an element $g' \in G'_{\mathbb{C}} = Sp(k,\mathbb{C})$ such that $e_i g' = c_i, e_{k+i}g' = d_i$ $(i = 1,\dots,k)$, where $e_1,\dots,e_{2k}$ denotes the standard basis of $\mathbb{C}^{2k}$. This implies that

$$Z = g'^{-1} \cdot \begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix}.$$

Furthermore, $g' \in G'_{\mathbb{C}}$ fixes the above matrix if and only if $e_i g' = e_i$ and $e_{k+i}g' = e_{k+i}$ for all $i = 1,\dots,r$, or equivalently, $g' \in G'_{\mathbb{C}}(k-r)$.

(3) Suppose that $k > r = (n-1)/2$ with odd integer n. Just as in (2), one can show that any element in $\mathcal{V}_k$ lies in the $G'_{\mathbb{C}}$-orbit through a matrix Z of the form

$$Z = \begin{pmatrix} e_1 \\ \vdots \\ e_r \\ e_{k+1} \\ \vdots \\ e_{k+r} \\ z \end{pmatrix} \quad \text{for some } z \in \mathbb{C}^{2k}.$$

Then the condition $\psi(Z) = X(r)$ imposes

$$e_i J_k {}^t z = e_{k+i} J_k {}^t z = 0 \quad \text{for} \quad i = 1,\dots,r.$$

Hence one finds that $Z = (z_1, z_2)^{\tilde{}}$ for some $(z_1,z_2) \in M_{1,2(k-r)}$, i.e., $\mathcal{V}_k = G'_{\mathbb{C}} \cdot \tilde{M}_{1,2(k-r)}$.

Finally, observe that two matrices $(z_1,z_2)^{\tilde{}}$ and $(z'_1,z'_2)^{\tilde{}}$ in $\mathcal{V}_k$ belong to the same $G'_{\mathbb{C}}$-orbit if and only if the corresponding vectors (z_1,z_2) and (z'_1,z'_2) in $M_{1,2(k-r)}$ are conjugate under the action of $Sp(k-r,\mathbb{C})$. This yields the second equality in (5.21), by noting that $Sp(k-r,\mathbb{C})$ acts on $M_{1,2(k-r)}\backslash\{0\}$ transitively. □

The above three lemmas imply in particular the following

Proposition 5.6. — *The affine algebraic variety $\mathcal{V}_k$ is irreducible except the case $G = Sp(n,\mathbb{R})$ with $k \leqslant n$.*

Remark 5.7. — If $G = Sp(n,\mathbb{R})$ with $k \leqslant n$, then $\mathcal{V}_k \simeq O(k,\mathbb{C})$ has two irreducible components according as the coset decomposition $O(k,\mathbb{C}) = SO(k,\mathbb{C}) \cup g'SO(k,\mathbb{C})$ with $g' \in O(k,\mathbb{C})\backslash SO(k,\mathbb{C})$.

Proof of Proposition 5.6. — Let $(G'_{\mathbb{C}})_o = GL(k,\mathbb{C})$, $SO(k,\mathbb{C})$ or $Sp(k,\mathbb{C})$ be the identity component of the complex classical group $G'_{\mathbb{C}} = GL(k,\mathbb{C})$, $O(k,\mathbb{C})$ or $Sp(k,\mathbb{C})$ respectively. Under the hypothesis of the proposition, we find from Lemmas 5.3–5.5 that $\mathcal{V}_k$ is the image of an irreducible variety $(G'_{\mathbb{C}})_o$ or $(G'_{\mathbb{C}})_o \times \mathbb{C}^p$ (for some $p > 0$) by a continuous map (with respect to the Zariski topology) between two affine spaces over $\mathbb{C}$. This proves the proposition. □

The next proposition is important to specify our $K_{\mathbb{C}}(m_k)$-modules $\mathcal{W}[\sigma]$.

Proposition 5.8. — *The ideal $\omega(\mathfrak{m})\mathbb{C}[M]$ of $\mathbb{C}[M]$ coincides with the defining ideal of $\mathcal{V}_k$ in $\mathbb{C}[M]$:*

$$\omega(\mathfrak{m})\mathbb{C}[M] = \{f \in \mathbb{C}[M] \mid f(Z) = 0 \quad \text{for all } Z \in \mathcal{V}_k\}. \tag{5.22}$$

Hence one gets a natural isomorphism

$$\mathbb{C}[M]/\omega(\mathfrak{m})\mathbb{C}[M] \simeq \mathbb{C}[\mathcal{V}_k] \quad \text{as } K_{\mathbb{C}}(m_k) \times G'_{\mathbb{C}}\text{-modules,} \tag{5.23}$$

where $\mathbb{C}[\mathcal{V}_k]$ denotes the affine coordinate ring of $\mathcal{V}_k$ consisting of all functions on $\mathcal{V}_k$ given by restricting polynomials on M to $\mathcal{V}_k$.

Proof. — We write $\mathcal{I}_k$ for the defining ideal of $\mathcal{V}_k$, the right hand side of (5.22). By definition one has $\omega(\mathfrak{m})\mathbb{C}[M] \subset \mathcal{I}_k$. So we want to show the converse inclusion $\mathcal{I}_k \subset \omega(\mathfrak{m})\mathbb{C}[M]$.

First, we prove the inclusion in question when the variety $\mathcal{V}_k$ is irreducible. Namely, we exclude the case $G = Sp(n,\mathbb{R})$ with $k \leqslant n$ exactly (see Proposition 5.6). Take any basis $Y_1, \ldots, Y_t$ of the vector space $\mathfrak{p}_-(m_k) = [\mathfrak{k}, Y(m_k)]$ (cf. Lemma 4.2). We define $f_1, \ldots, f_t \in \omega(\mathfrak{m})\mathbb{C}[M]$ by

$$f_i(Z) := B(\psi(Z) - X(m_k), Y_i) \quad \text{for} \quad Z \in M.$$

Lemma 4.13 together with (5.9) yields

$$\mathcal{V}_k = \{Z \in M \mid f_i(Z) = 0 \quad (i = 1, \ldots, t)\}.$$

By case-by-case examination, we can find an element $Z_0 \in \mathcal{V}_k$ on which the differentials $(df_i)_{Z_0}$ $(i = 1, \ldots, t)$ are linearly independent. In fact, the "identitylike" matrices given in Lemmas 5.3–5.5 satisfy this requirement if $\mathcal{V}_k$ is a single $G'_{\mathbb{C}}$-orbit. Otherwise, one can choose Z_0 as

$$\begin{cases} \tilde{O} \in \tilde{M}_{p-q,k-q} & (SU(p,q),\ k > q,\ p > q), \\ (0\ldots0,0\ldots0)^{\sim} \in \tilde{M}_{1,2(k-r)} & (SO^*(2n), k > (n-1)/2 \text{ with odd } n). \end{cases}$$

Thus we get $(f_1, \ldots, f_t) = \mathcal{I}_k$, by applying Lemma 4 of [**17**, page 345]. This shows $\mathcal{I}_k \subset \omega(\mathfrak{m})\mathbb{C}[M]$ as desired.

Second, consider the case $G = Sp(n,\mathbb{R})$ with $k \leqslant n$. Then we know $\mathcal{V}_k \simeq O(k,\mathbb{C})$ by Lemma 5.4, and hence the equality $\omega(\mathfrak{m})\mathbb{C}[M] = \mathcal{I}_k$ is an easy consequence of a classical theorem of Weyl [**37**, Theorem (5.2.C)].

Now the equality (5.22) and so the isomorphism (5.23) have been proved completely. □

5.4. $K_{\mathbb{C}}(m_k)$-modules $\mathcal{W}[\sigma]$. — We are now in a position to specify the $K_{\mathbb{C}}(m_k)$-modules $\mathcal{W}[\sigma]$ for every $\sigma \in \Sigma(k)$ $(k = 1, 2, \dots)$. First, we prepare some notation to state the results in a unified form. Let $G'_{\mathbb{C}}(k-r)$ $(r = \mathbb{R}\text{-rank}\, G)$ be the subgroup of $G'_{\mathbb{C}}$ in (5.15) and (5.16). With Lemmas 5.3–5.5 in mind, we introduce a $G'_{\mathbb{C}}(k-r)$-stable subvariety $\mathcal{U}_k$ of $\mathcal{V}_k$ as follows. We set

$$\mathcal{U}_k := \begin{cases} \left\{ \begin{pmatrix} I_{p,k}(m_k) \\ I_{q,k}(m_k) \end{pmatrix} \right\} & (k \leqslant q \ \text{ or } \ k > q = p), \\ \tilde{M}_{p-q,k-q} & (k > q \ \text{ and } \ p \neq q) \end{cases}$$

for $G = SU(p,q)$, and

$$\mathcal{U}_k := \{I_{n,k}(m_k)\} \quad (k = 1, 2, \dots) \quad \text{for } G = Sp(n, \mathbb{R}),$$

where $m_k = \min(k, r)$ as before. The variety $\mathcal{U}_k$ for $G = SO^*(2n)$ is defined to be

$$\mathcal{U}_k := \begin{cases} \{I_{n,2k}(2k)\} & (k \leqslant r = [n/2]) \\ \left\{ \begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix} \right\} & (k > r = n/2 \ \text{ with } \ n \text{ even}), \\ \tilde{M}_{1,2(k-r)} & (k > r = (n-1)/2 \ \text{ with } \ n \text{ odd}). \end{cases}$$

Then, Lemmas 5.3–5.5 imply that

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \mathcal{U}_k, \tag{5.24}$$

and that the $G'_{\mathbb{C}}$-orbits $\mathcal{X}$ in $\mathcal{V}_k$ are in one-one correspondence with the $G'_{\mathbb{C}}(k-r)$-orbits $\mathcal{X} \cap \mathcal{U}_k$ in $\mathcal{U}_k$.

Definition 5.9. — We say that the pair (G, G') is of type (S$\vee$T) if the pair (G, G') is in the stable range with smaller member G' (i.e., $k \leqslant r$), or the symmetric space $K\backslash G$ is of tube type (i.e., $G = SU(p,q)$ with $p = q$, $Sp(n, \mathbb{R})$, or $SO^*(2n)$ with n even). This happens exactly when $\mathcal{U}_k$ consists of a single $G'_{\mathbb{C}}(k-r)$-fixed point, say Z_0. We call it the case (S$\vee$T), too.

Now Proposition 5.8 allows us to deduce the following

Proposition 5.10. — *Under the above notation, let $\mathbb{C}[\mathcal{U}_k]$ be the coordinate ring of $G'_{\mathbb{C}}(k-r)$-stable variety $\mathcal{U}_k$ viewed as a $G'_{\mathbb{C}}(k-r)$-module in the canonical way. Then one has a linear isomorphism*

$$\mathcal{W}[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbb{C}}(k-r)}(V_\sigma, \, \mathbb{C}[\mathcal{U}_k]) \simeq (V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k])^{G'_{\mathbb{C}}(k-r)} \quad (\sigma \in \Sigma(k)). \tag{5.25}$$

In particular, it holds that

$$\mathcal{W}[\sigma] \simeq (V_\sigma^*)^{G'_{\mathbb{C}}(k-r)} \quad \textit{for the case } (\mathrm{S}\vee\mathrm{T}). \tag{5.26}$$

Here $(V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k])^{G'_{\mathbb{C}}(k-r)}$ denotes the subspace of $V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k]$ of $G'_{\mathbb{C}}(k-r)$-fixed vectors, and the right hand side of (5.26) *turns to be V_σ^* if $k \leqslant r$.*

Proof. — We know the $K_{\mathbb{C}}(m_k)$-isomorphism $\mathcal{W}[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[\mathcal{V}_k])$ thanks to (5.13) and (5.23). Let T be a $G'_{\mathbb{C}}$-homomorphism from V_σ to $\mathbb{C}[\mathcal{V}_k]$. Set $T_0(v) := T(v)|\mathcal{U}_k$, the restriction of $T(v) \in \mathbb{C}[\mathcal{V}_k]$ to $\mathcal{U}_k$, for each $v \in V_\sigma$. Then T_0 gives a homomorphism of $G'_{\mathbb{C}}(k-r)$-modules from V_σ to $\mathbb{C}[\mathcal{U}_k]$. By using Lemmas 5.3–5.5 (see also (5.24)), it is standard to verify that the assignment $T \mapsto T_0$ sets up a linear isomorphism

$$\mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[\mathcal{V}_k]) \simeq \mathrm{Hom}_{G'_{\mathbb{C}}(k-r)}(V_\sigma, \mathbb{C}[\mathcal{U}_k]), \tag{5.27}$$

which is a variant of the Frobenius reciprocity. We thus obtain (5.25) (the second isomorphism is a natural one). (5.26) follows from (5.25) immediately, since $\mathbb{C}[\mathcal{U}_k]$ is the one-dimensional trivial $G'_{\mathbb{C}}(k-r)$-module for the case (S$\vee$T). $\square$

Remark 5.11. — For the case $G = SU(p,q)$, the above proposition is due to Tagawa [**31**, Th.3.10.1].

Remark 5.12. — The irreducible decomposition of $G'_{\mathbb{C}}(k-r)$-module $\mathbb{C}[\mathcal{U}_k]$ is well-known even if $\mathcal{U}_k$ is not a variety of single point. Indeed, $\mathbb{C}[\mathcal{U}_k]$ is isomorphic to the natural $GL(k-q,\mathbb{C})$-module $\mathbb{C}[M_{p-q,k-q}]$ (resp. such $Sp(k-r,\mathbb{C})$-module $\mathbb{C}[M_{1,2(k-r)}]$) when $G = SU(p,q)$ with $p > q$ and $k > q$ (resp. $G = SO^*(2n)$ with $k > (n-1)/2$ and n odd). On one hand, the $GL_{p-q} \times GL_{k-q}$ duality can be used to decompose $\mathbb{C}[M_{p-q,k-q}]$ into irreducibles. On the other hand, the space $S^l(M_{1,2(k-r)})$ of homogeneous polynomials on $M_{1,2(k-r)}$ of any fixed degree l turns to be an irreducible $Sp(k-r,\mathbb{C})$-module with highest weight $(l,0,\ldots,0)$. This yields the irreducible decomposition

$$\mathbb{C}[M_{1,2(k-r)}] = \oplus_{l\geqslant 0} S^l(M_{1,2(k-r)})$$

as $Sp(k-r,\mathbb{C})$-modules.

Hence the right hand side of (5.25) can be described concretely by a combinatorial method, once one knows the branching rule of irreducible representations of $G'_{\mathbb{C}}$ restricted to the subgroup $G'_{\mathbb{C}}(k-r)$ (cf. [**19**], [**30**]). Although we do not discuss it in this paper, the author would like to thank K. Koike for kind communication on the branching rule of finite-dimensional representations of complex classical groups.

In view of Corollary 3.9, we get a direct consequence of Proposition 5.10 as follows.

Corollary 5.13. — *Let σ be in $\Sigma(k)$. Then, the multiplicity* $\mathrm{mult}_{I_{m_k}}(L[\sigma])$ *of irreducible highest weight module $L[\sigma]$ at the defining ideal I_{m_k} of the associated variety $\mathcal{V}(L[\sigma])$ coincides with the dimension of vector space $(V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k])^{G'_{\mathbb{C}}(k-r)}$. Especially, one gets* $\mathrm{mult}_{I_{m_k}}(L[\sigma]) = \dim\sigma$ *if $k \leqslant r$ (cf.* [**27**, Th.9.1]).

At the end, we are going to clarify how the isotropy subgroup $K_{\mathbb{C}}(m_k)$ acts on the space $\mathcal{W}[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbb{C}}(k-r)}(V_\sigma, \mathbb{C}[\mathcal{U}_k])$. To do this, we note that the elements g of the subgroup $K_{\mathbb{C}}(m)$ $(0 \leqslant m \leqslant r)$ of $K_{\mathbb{C}}$ (see Table (5.1)) are written for each group $G = SU(p,q)$, $Sp(n,\mathbb{R})$ and $SO^*(2n)$ respectively as follows.

$$g = \begin{cases} \left(\begin{pmatrix} g_{11} & g_{12} \\ O & g_{22} \end{pmatrix}, \begin{pmatrix} g_{11} & O \\ g_{43} & g_{44} \end{pmatrix}\right) \in K_{\mathbb{C}} \ \text{ with } \ g_{11} \in GL(m,\mathbb{C}) & (SU(p,q)), \\ \begin{pmatrix} g_{11} & g_{12} \\ O & g_{22} \end{pmatrix} \in K_{\mathbb{C}} \ \text{ with } \ g_{11} \in O(m,\mathbb{C}) & (Sp(n,\mathbb{R})), \\ \begin{pmatrix} g_{11} & g_{12} \\ O & g_{22} \end{pmatrix} \in K_{\mathbb{C}} \ \text{ with } \ g_{11} \in Sp(m,\mathbb{C}) & (SO^*(2n)). \end{cases}$$

This enables us to define a group homomorphism

$$\alpha : K_{\mathbb{C}}(m_k) \to G'_{\mathbb{C}}, \quad g \mapsto \alpha(g)$$

by

$$\alpha(g) := \begin{pmatrix} g_{11} & O \\ O & I_{k-r} \end{pmatrix} \quad \text{for} \quad SU(p,q) \text{ or } Sp(n,\mathbb{R}),$$

and by

$$\alpha(g) := \begin{pmatrix} p_{11} & O & p_{12} & O \\ O & I_{k-r} & O & O \\ p_{21} & O & p_{22} & O \\ O & O & O & I_{k-r} \end{pmatrix} \quad \text{with } g_{11} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \quad \text{for } SO^*(2n).$$

Here p_{ij} is a matrix of size k, and $\alpha(g)$ should be understood as g_{11} if $k \leqslant r$. Note that the elements of $\alpha(K_{\mathbb{C}}(m_k))$ commute with those of the subgroup $G'_{\mathbb{C}}(k-r)$.

Now we can deduce

Theorem 5.14 **(Case (S$\vee$T))**. — *Assume that the pair (G, G') is of type* (S$\vee$T) *in Definition 5.9. Then it holds that*

$$\mathcal{W}[\sigma] \simeq (\delta_k \otimes (\sigma^* \circ \alpha), \ (V_\sigma^*)^{G'_{\mathbb{C}}(k-r)}) \quad \text{as } K_{\mathbb{C}}(m_k)\text{-modules,} \tag{5.28}$$

where δ_k is the character of $K_{\mathbb{C}}$ in (5.6). *In particular, $\mathcal{W}[\sigma]$ is an irreducible $K_{\mathbb{C}}(m_k)$-module if $k \leqslant r$.*

Proof. — Let Z_0 be the unique element of $\mathcal{U}_k$. By noting that

$$g \cdot Z_0 = \alpha(g)^{-1} \cdot Z_0 \quad (g \in K_{\mathbb{C}}(m_k)),$$

it is a routine task to transfer the $K_{\mathbb{C}}(m_k)$-action on $\mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[\mathcal{V}_k])$ to that on $(V_\sigma^*)^{G'_{\mathbb{C}}(k-r)} \simeq \mathrm{Hom}_{G'_{\mathbb{C}}(k-r)}(V_\sigma, \mathbb{C}[\mathcal{U}_k])$ through the isomorphism (5.27). We thus get (5.28). If $k \leqslant r$, the homomorphism α is surjective. Hence (5.28) implies the irreducibility of $\mathcal{W}[\sigma]$ for $k \leqslant r$. $\square$

Next we consider the remaining case, and assume that (G, G') is not of type (S$\vee$T). Then one has $k > r$ and so $m_k = r$. Set $l := p - q$ for $G = SU(p,q)$ $(p > q)$, and $l = 1$ for $G = SO^*(2n)$ (n odd). Then, $\beta(g) := g_{22}$ $(g \in K_{\mathbb{C}}(r))$ defines a group homomorphism β from $K_{\mathbb{C}}(r)$ to $GL(l, \mathbb{C})$. The group $K_{\mathbb{C}}(r)$ acts on

$$\mathbb{C}[\mathcal{U}_k] \simeq \mathbb{C}[M_{l,\epsilon(k-r)}]$$

naturally through the left multiplication composed with β, where $\epsilon := 1$ for $G = SU(p,q)$, and $\epsilon := 2$ for $G = SO^*(2n)$. We denote by ν the resulting representation of $K_{\mathbb{C}}(r)$ on $\mathbb{C}[\mathcal{U}_k]$. Note that ν as well as $\sigma^* \circ \alpha$ commutes with the $G'_{\mathbb{C}}(k-r)$-action.

Theorem 5.15 **(Non (S$\vee$T) case).** — *Under the above assumption and notation, the reductive part of $K_{\mathbb{C}}(r)$ acts on $\mathcal{W}[\sigma] \simeq (V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k])^{G'_{\mathbb{C}}(k-r)}$ by the representation $\delta_k \otimes (\sigma^* \circ \alpha) \otimes \nu$.*

Proof. — This theorem can be proved just as in the proof of Theorem 5.14 by noting that

$$g \cdot \tilde{U} = \alpha(g)^{-1} \cdot (\beta(g)U)^{\sim} \quad (U \in M_{l,\epsilon(k-r)})$$

holds if $g \in K_{\mathbb{C}}(r)$ lies in the reductive part of $K_{\mathbb{C}}(r)$, i.e., $g_{12} = 0$. We omit the detail of the proof. □

References

[1] L. Barchini, Szegö kernels associated with Zuckerman modules, J. Funct. Anal., **131** (1995), 145–182.

[2] M.G. Davidson, T.J. Enright and R.J. Stanke, Covariant differential operators, Math. Ann., **288** (1990), 731–739.

[3] M.G. Davidson, T.J. Enright and R.J. Stanke, Differential operators and highest weight representations, Mem. Amer. Math. Soc. No. **455**, American Mathematical Society, Rrovidence, R.I., 1991.

[4] M.G. Davidson and T.J. Stanke, Szegö maps and highest weight representations, Pacific J. Math., **158** (1993), 67–91.

[5] T.J. Enright and A. Joseph, An intrinsic analysis of unitarizable highest weight modules, Math. Ann., **288** (1990), 571–594.

[6] T.J Enright and R. Parthasarathy, A proof of a conjecture of Kashiwara and Vergne. *in* "Noncommutative harmonic analysis and Lie groups (Marseille, 1980)", Lecture Notes in Math., **880**, Springer, Berlin-New York, 1981, pp. 74–90.

[7] T.J. Enright, R. Howe, and N.R. Wallach, A classification of unitary highest weight modules, *in* "Representation theory of reductive groups (Park City, Utah, 1982; P.C.Trombi ed.)", Progress in Math., Vol. **40**, Birkhäuser, 1983, pp.97–143.

[8] A. Gyoja and H. Yamashita, Associated variety, Kostant-Sekiguchi correspondence, and locally free $U(\mathfrak{n})$-action on Harish-Chandra modules, J. Math. Soc. Japan, **51** (1999), 129–149.

[9] M. Harris and H.P. Jakobsen, Singular holomorphic representations and singular modular forms, Math. Ann., **259** (1982), 227–244.

[10] R. Hotta and R. Parthasarathy, Multiplicity formulae for discrete series, Invent. Math., **26** (1974), 133–178.

[11] M. Kashiwara and W. Schmid, Quasi-equivariant $\mathcal{D}$-modules, equivariant derived category, and representations of reductive Lie groups, *in* "Lie theory and geometry (J.L.Brylinski et al. eds.)", Birkhäuser, 1994, pp.457–488.

[12] M. Kashiwara and M. Vergne, On the Segal-Shale-Weil representations and harmonic polynomials, Invent. Math., **44** (1978), 1–47.

[13] S. Kato and H. Ochiai, The degrees of orbits of the multiplicity actions, in this volume.

[14] N. Kawanaka, Generalized Gelfand-Graev representation and Ennola duality, *in* "Algebraic groups and related topics", Advanced Studies in Pure Math., **6** (1985), 175–206.

[15] N. Kawanaka, Generalized Gelfand-Graev representations of exceptional simple algebraic groups over a finite field. I, Invent. Math., **84** (1986), 575–616.

[16] A.W. Knapp, *Lie groups beyond an introduction*, Progress in Mathematics Vol. **140**, Birkhäuser, Boston-Besel-Stuttgart, 1996.

[17] B. Kostant, Lie group representations on polynomial rings, Amer. J. Math., **85** (1963), 327–404.

[18] B. Kostant, On Whittaker vectors and representation theory, Invent. Math., **48** (1978), 101–184.

[19] K. Koike and I. Terada, Young diagrammatic methods for the restriction of representations of complex classical Lie groups to reductive subgroups of maximal rank, Adv. Math., **79** (1990), 104–135.

[20] H.P. Jakobsen, An intrinsic classification of the unitarizable highest weight modules as well as their associated varieties, Compositio Math. **101** (1996), 313–352.

[21] A. Joseph, Annihilators and associated varieties of unitary highest weight modules, Ann. Sci. Éc. Norm. Sup., **25** (1992), 1–45.

[22] H. Matumoto, Whittaker vectors and associated varieties, Invent. Math., **89** (1987), 219–224.

[23] H. Matumoto, Whittaker vectors and the Goodman-Wallach operators, Acta Math., **161** (1988), 183–241.

[24] H. Matumoto, $C^{-\infty}$-Whittaker vectors for complex semisimple Lie groups, wave front sets, and Goldie rank polynomial representations, Ann. Sci. Éc. Norm. Sup., **23** (1990), 311–367.

[25] H. Matumoto, $C^{-\infty}$-Whittaker vectors corresponding to a principal nilpotent orbit of a real reductive linear Lie group, and wave front sets. Compositio Math. **82** (1992), 189–244.

[26] C. Mœglin and J.-L. Waldspurger, Modèles de Whittaker dégénérés pour des groupes p-adiques, Math. Z. **196** (1987), 427–452.

[27] K. Nishiyama, H. Ochiai and K. Taniguchi, Bernstein degree and associated cycles of Harish-Chandra modules – Hermitian symmetric case, in this volume.

[28] W. Schmid, Homogeneous complex manifolds and representations of semisimple Lie groups, Dissertation, University of California, Berkeley, 1967; reprinted *in* "Representation theory and harmonic analysis on semisimple Lie groups (P.L Sally and D.A Vogan eds.)", Mathematical Surveys and Monograph Vol. **31**, Amer. Math. Soc., 1989, pp.223–286.

[29] W. Schmid, Boundary value problems for group invariant differential equations, *in* "Elie Cartan et les Mathématiques d'Aujourd'hui", Astérisque, Numéro hors série, 1985, pp.311–321.

[30] R.S. Strichartz, The explicit Fourier decomposition of $L^2(SO(n)/SO(n-m))$, Canad. J. Math., **27** (1975), 294–310.

[31] M. Tagawa, Generalized Whittaker models for unitarizable highest weight representations (in Japanese), Master Thesis, Kyoto University, 1998.

[32] M. Vergne and H. Rossi, Analytic continuation of the holomorphic discrete series of a semi-simple Lie group, Acta Math., **136** (1976), 1–59.

[33] D.A. Vogan, Associated varieties and unipotent representations, *in* "Harmonic Analysis on Reductive Groups (W.Barker and P.Sally eds.)," Progress in Math., Vol. **101**, Birkhäuser, 1991, pp.315–388.

[34] N.R. Wallach, The analytic continuation of the discrete series. II, Trans. Amer. Math. Soc., **251** (1979), 19–37.

[35] N.R. Wallach, *Real reductive groups I*, Pure and Applied Mathematics Vol.**132**, Academic Press, San Diego-London, 1988.

[36] G. Warner, *Harmonic analysis on semi-simple Lie groups I,* Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[37] H. Weyl, *The classical groups. Their invariants and representations*, Eighth printing, Princeton University Press, Princeton, NJ, 1973.

[38] H.W. Wong, Dolbeault cohomological realization of Zuckerman modules associated with finite rank representations, J. Funct. Anal., **129** (1995), 428–454.

[39] H. Yamashita, On Whittaker vectors for generalized Gelfand-Graev representations of semisimple Lie groups, J. Math. Kyoto Univ., **26** (1986), 263–298.

[40] H. Yamashita, Finite multiplicity theorems for induced representations of semisimple Lie groups II: Applications to generalized Gelfand-Graev representations, J. Math. Kyoto. Univ., **28** (1988), 383–444.

[41] H. Yamashita, Multiplicity one theorems for generalized Gelfand-Graev representations of semisimple Lie groups and Whittaker models for the discrete series, Advanced Studies in Pure Math. **14** (1988), 31–121.

[42] H. Yamashita, Embeddings of discrete series into induced representations of semisimple Lie groups, I: General theory and the case of $SU(2,2)$, Japan. J. Math.,**16** (1990), 31–95; II: generalized Whittaker models for $SU(2,2)$, J. Math. Kyoto Univ., **31** (1991), 543–571.

[43] H. Yamashita, Criteria for the finiteness of restriction of $U(\mathfrak{g})$-modules to subalgebras and applications to Harish-Chandra modules: a study in relation to the associated varieties, J. Funct. Anal., **121** (1994), 296–329.

[44] H. Yamashita, Description of the associated varieties for the discrete series representations of a semisimple Lie group: An elementary proof by means of differential operators of gradient type, Comment. Math. Univ. St. Paul., **47** (1998), 35–52.

[45] T. Yoshinaga and H. Yamashita, The embeddings of discrete series into principal series for an exceptional real simple Lie group of type G_2, J. Math. Kyoto Univ., **36** (1996), 557–595.