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THE DEGREES OF ORBITS OF THE MULTIPLICITY-FREE ACTIONS

by

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Abstract. — We give a formula for the degrees of orbits of the irreducible representations with multiplicity-free action. In particular, we obtain the Bernstein degree and the associated cycle of the irreducible unitary highest weight modules of the scalar type for arbitrary hermitian Lie algebras.

***Résumé* (Degrés des orbites nilpotentes des représentations irréductibles sans multiplicité)**
Nous donnons une formule pour les degrés des orbites nilpotentes des représentations irréductibles sans multiplicité. Nous obtenons les degrés de Bernstein et les cycles associés des représentations irréductibles unitaires de plus haut poids de type scalaire pour des algèbres de Lie hermitiennes.

1. Introduction

Let K be a connected reductive complex algebraic group, and V an irreducible representation of K. We assume that the action of K is multiplicity-free; that is, each irreducible representation of K occurs at most once in the polynomial ring $\mathbb{C}[V]$. We also assume that the image of K in $GL(V)$ contains all nonzero scalar matrices $\mathbb{C}^{\times}\,\mathrm{id}_V$. Such representations have been classified by Kac **[10]**. There are eight families and five exceptional representations.

In this paper, we determine the degree of each closed K-stable subset Y of V. We establish a method by which we can express some asymptotic behavior of the dimension of the filtered module in terms of a definite integral. This is a generalization of the technique presented in Ref. **[19]**. As a corollary, a formula for the degree of each K-stable closed subset can be obtained (Theorem 2.5). The multiplicity-free action contains an important family coming from the hermitian symmetric spaces. Such representations consist of four families and two exceptionals of the classification

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mentioned above. Using the detailed structure of the restricted root system, we can obtain a formula in these hermitian symmetric cases that is more concise than that obtained in the general case (Theorem 3.2). This formula unifies three kinds (i.e., homomorphism, symmetric endomorphism and skew-symmetric endomorphism) of Giambelli formulas, as well as the corresponding formula for the exceptional Lie algebras. The formula for the degree of the closure of the orbit immediately gives the Bernstein degree of the irreducible unitary highest weight module of the scalar type(Corollary 4.1). For three families of classical Lie algebras $\mathfrak{sp}(n,\mathbb{R})$, $\mathfrak{u}(p,q)$ and $\mathfrak{o}^*(2n)$, this result is obtained in Section 7 of Ref. **[19]** through case analysis. In the final section, we give two examples demonstrating the calculation of the Bernstein degree of the unitary highest weight modules of the non-scalar type. These are also derived from Theorem 2.3. In the Appendix, we list the explicit values for the degree of the closure of the orbits for all thirteen families of multiplicity-free actions, with some comment on the structure of the orbits.

A part of this paper is taken from the master thesis of the first author **[12]**.

2. Degree of the multiplicity-free action

2.1. Degree. — Let V be a finite-dimensional complex vector space, $\mathbb{C}[V]$ the ring of polynomials on V, and M a finitely-generated $\mathbb{C}[V]$-module. By a standard procedure, we can associate two additive, numerical invariants, the dimension and the multiplicity of M. This procedure is briefly summarized in Section 1 of Ref. **[19]** in this volume.

Let Y be a closed conic subvariety of V, and let $\mathbf{I}(Y)$ be the defining ideal of Y;

$$\mathbf{I}(Y) = \{p \in \mathbb{C}[V] \mid p(y) = 0 \text{ for all } y \in Y\}.$$

We define $\mathbb{C}[Y] = \mathbb{C}[V]/\mathbf{I}(Y)$. Defined in this manner, $\mathbb{C}[Y]$ is the coordinate ring of Y. Since $\mathbf{I}(Y)$ is a (reduced) graded ideal of $\mathbb{C}[V]$, $\mathbb{C}[Y]$ is naturally a graded $\mathbb{C}[V]$-module. The multiplicity of $\mathbb{C}[Y]$ is called the degree of Y, and is denoted by $\deg(Y)$. It is known that the degree of a complete intersection is elementary.

Lemma 2.1

(i) *If Y is a complete intersection, then the degree of Y is the product of the degrees of the defining equations of the irreducible components of Y.*

(ii) *If Y is a hypersurface, then the degree of Y is the degree (as a homogeneous polynomial) of the defining equation of Y.*

(iii) *If Y is a linear subspace of V, then the degree of Y is* 1.

The assertion (iii) is a special case of (ii), and (ii) is a special case of (i). The assertion (i) is found in standard textbooks, such as Ref. **[4]**. On the other hand, if the variety Y is not a complete intersection, such as a determinantal variety, its degree is non-trivial, as can be seen from the Giambelli formula.

2.2. Asymptotic behavior of some graded module. — Let K be a connected reductive complex algebraic group, and let V be a finite dimensional representation of K. Let $\mathbb{C}[V]^i$ be the set of homogeneous polynomials in $\mathbb{C}[V]$ of degree i. We assume that the image of K in $GL(V)$ contains all nonzero scalar matrices. Then, there exists an element $Z \in \mathrm{Lie}(K)$ such that $Z \cdot p = ip$ for all $p \in \mathbb{C}[V]^i$. This element is called the degree operator (or Euler operator). We denote the natural action of K on the graded algebra $\mathbb{C}[V]$ by Ad. We call M a $(\mathbb{C}[V], K)$-module if M is a $\mathbb{C}[V]$-module and is a completely reducible K-module with the compatibility condition $k \cdot (p \cdot (k^{-1} \cdot m)) = (\mathrm{Ad}\,(k)(p)) \cdot m$ for all $k \in K$, $p \in \mathbb{C}[V]$ and $m \in M$. We denote the decomposition into K-isotypic components by $M = \oplus_\mu M_\mu$. Assume that there exists some isotypic component M_λ generating M as a $\mathbb{C}[V]$-module. Such a component is unique if it exists. We define a graded component by $M^i = \mathbb{C}[V]^i M_\lambda$ for $i \in \mathbb{Z}_{\geqslant 0}$. Then $M = \oplus_i M^i$ is a graded $\mathbb{C}[V]$-module, and each graded component is given by

$$M^i = \{m \in M \mid Z \cdot m = (\lambda(Z) + i)m\}.$$

We assume, moreover, that M has a multiplicity-free decomposition

$$M = \bigoplus_{\varphi \in \Lambda(M)} F(\lambda + \varphi),$$

where $F(\mu)$ is a (finite-dimensional) irreducible K-module whose highest weight is μ, and that there exists linearly independent weights $\varphi_1, \dots, \varphi_m$ such that

$$\Lambda(M) = \{n_1\varphi_1 + \cdots + n_m\varphi_m \mid n_i \in \mathbb{Z}_{\geqslant 0}\}.$$

In this case, the graded component M^i is given by

$$M^i = \oplus F(\lambda + n_1\varphi_1 + \cdots + n_m\varphi_m),$$

where the summation is over $(n_1, \dots, n_m) \in \mathbb{Z}^m_{\geqslant 0}$ with $n_1\varphi_1(Z) + \cdots + n_m\varphi_m(Z) = i$. We will determine the asymptotic of the dimension of the graded component for large i.

Using the Weyl dimension formula, it can be shown that $\dim F(\lambda + n_1\varphi_1 + \cdots + n_m\varphi_m)$ is a polynomial in $(n_1, \dots, n_m)$. To be more explicit, let Δ_K^+ be the set of positive roots of the Lie algebra of K, and let ρ_K be the half sum of positive roots. We define

$$\Delta_M^+ = \Delta_K^+ \setminus \{\alpha \in \Delta_K^+ \mid \langle \alpha, \varphi_i \rangle = 0 \text{ for all } i = 1, \dots, m\}$$

and

$$\begin{aligned} f(x_1, \dots, x_m) = & \prod_{\alpha \in \Delta_K^+ \setminus \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle}{\langle \alpha, \rho_K \rangle} \times \\ & \times \prod_{\alpha \in \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle + x_1 \langle \alpha, \varphi_1 \rangle + \cdots + x_m \langle \alpha, \varphi_m \rangle}{\langle \alpha, \rho_K \rangle}. \end{aligned}$$

Then $\dim F(\lambda + n_1\varphi_1 + \cdots + n_m\varphi_m) = f(n_1, \ldots, n_m)$. The degree of the polynomial f is equal to the number $|\Delta_M^+|$ of roots in Δ_M^+, and the leading term, which we denote by $\bar{f}$, is

$$\bar{f}(x_1, \ldots, x_m) = \prod_{\alpha \in \Delta_K^+ \setminus \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle}{\langle \alpha, \rho_K \rangle} \times \prod_{\alpha \in \Delta_M^+} \frac{x_1\langle \alpha, \varphi_1 \rangle + \cdots + x_m\langle \alpha, \varphi_m \rangle}{\langle \alpha, \rho_K \rangle}.$$

We define a filtered module $M_l = \oplus_{i=0}^l M^i$. This $\{M_l\}_l$ gives the filtration of M; and the dimension of the filtered component is

$$\dim M_l = \sum f(n_1, \ldots, n_m),$$

where the summation is over $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_{\geqslant 0}^m$, with $n_1\varphi_1(Z) + \cdots + n_m\varphi_m(Z) \leqslant l$. We express this condition as $|\mathbf{n}| \leqslant l$ for short.

Lemma 2.2. — *Let $d = m + |\Delta_M^+|$. Then*

$$\lim_{l \to \infty} l^{-d} \sum_{|\mathbf{n}| \leqslant l} f(\mathbf{n}) = \int \bar{f}(x) dx_1 \cdots dx_m,$$

where the domain of integration is the simplex

$$\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_1 \geqslant 0, \ldots, x_m \geqslant 0,\ x_1\varphi_1(Z) + \cdots + x_m\varphi_m(Z) \leqslant 1\}.$$

Summarizing the above, we have the following theorem:

Theorem 2.3. — *If l is large, then*

$$\dim M_l = c \cdot l^d/d! + (\text{lower order terms}),$$

where $d = m + |\Delta_M^+|$ and

$$c = d! \prod_{\alpha \in \Delta_K^+ \setminus \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle}{\langle \alpha, \rho_K \rangle} \times \int \prod_{\alpha \in \Delta_M^+} \frac{x_1\langle \alpha, \varphi_1 \rangle + \cdots + x_m\langle \alpha, \varphi_m \rangle}{\langle \alpha, \rho_K \rangle} dx_1 \cdots dx_m,$$

with the domain of integration

$$\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_1 \geqslant 0, \ldots, x_m \geqslant 0,\ x_1\varphi_1(Z) + \cdots + x_m\varphi_m(Z) \leqslant 1\}.$$

2.3. Multiplicity-free action. — Let V and K be as in the Introduction. That is, in addition to the assumption made in the previous subsection, we assume that the representation V is irreducible and that $\mathbb{C}[V]$ is multiplicity-free. The set of highest weights of K-types arising in $\mathbb{C}[V]$ is a free semigroup. We denote the set of generators by $P\hat{A}^+(V)$.

Let Y be a closed irreducible K-stable subset of V. Since V has a finite number of K-orbits, Y is the closure of a K-orbit on V. We set $M = \mathbb{C}[Y]$. As in § 2.1, M can naturally be considered the quotient ring of $\mathbb{C}[V]$, and thus it inherits the natural grading from $\mathbb{C}[V]$. Then the $(\mathbb{C}[V], K)$-module M satisfies the first assumption in § 2.2, with the weight λ taken to be zero.

Lemma 2.4. — *Suppose Y is an irreducible closed K-stable subset of V. Then there exists a free semigroup $\Lambda(Y)$ such that $\mathbb{C}[Y] = \oplus_{\varphi\in\Lambda(Y)} F(\varphi)$ as a K-module. The generators of the free semigroup $\Lambda(Y)$ form a subset of $P\hat{A}^+(V)$. This subset is denoted by $P\hat{A}^+(Y) \subset P\hat{A}^+(V)$.*

A proof of this lemma is given in Ref. [**6**]. Also appearing there is the explicit form of the subset generating the subsemigroup, which we use in an application below.

We denote the number of elements of $P\hat{A}^+(Y)$ by m, and we set $P\hat{A}^+(Y) = \{\varphi_1, \dots, \varphi_m\}$. For a weight α, we define the vector $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ by $(\langle\alpha,\varphi_1\rangle, \dots, \langle\alpha,\varphi_m\rangle)$. We define

$$\Delta_Y^+ = \{\alpha \in \Delta^+ \mid (\alpha_1, \dots, \alpha_m) \neq 0\}$$

and $k_i = \varphi_i(Z) \in \mathbb{Z}_{>0}$. Then, the K-type $F(\varphi_i)$ appears in the homogeneous component $\mathbb{C}[Y]^{k_i}$. With this notation, we can give the degree of Y.

Theorem 2.5. — *The dimension of Y is $m + |\Delta_Y^+|$, and the degree of Y is*

$$\frac{(m+|\Delta_Y^+|)!}{\prod_{\alpha\in\Delta_Y^+}\langle\alpha,\rho_K\rangle} \times \int \prod_{\alpha\in\Delta_Y^+} (\alpha_1x_1 + \cdots + \alpha_mx_m)\, dx_1\cdots dx_m,$$

where the domain of the integration is the simplex

$$\{(x_1,\dots,x_m) \in \mathbb{R}^m \mid x_1 \geqslant 0, \dots, x_m \geqslant 0,\ k_1x_1 + \cdots + k_mx_m \leqslant 1\}.$$

Proof. — Applying Theorem 2.3 with

$$\bar{f}(x_1,\dots,x_m) = \prod_{\alpha\in\Delta_Y^+} \frac{\alpha_1x_1 + \cdots + \alpha_mx_m}{\langle\alpha,\rho_K\rangle},$$

we obtain the result. □

3. Hermitian symmetric case

In this section, we consider the subclass of the multiplicity-free actions consisting of the holomorphic tangent spaces of the hermitian symmetric spaces. In this case, we can obtain a more sophisticated formula for the degree by using the structure of the restricted root system.

3.1. Hermitian Lie algebra. — We first recall some standard notation of Lie algebras, root systems and weights.

Let $\mathfrak{g}_0$ be a non-compact real simple Lie algebra. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$. We assume that the center $\mathfrak{c}_0$ of $\mathfrak{k}_0$ is non-zero, that is, that $\mathfrak{g}_0$ is of the hermitian type. Then $\mathfrak{c}_0$ is one dimensional. Let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$. Then $\mathfrak{t}_0$ is a compact Cartan subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{t}$ denote the respective complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$ and $\mathfrak{t}_0$. We denote the Killing form by $B(\cdot,\cdot)$. The restriction of the Killing form on $\mathfrak{t}$ is a non-degenerate symmetric bilinear form.

Using this, we identify $\mathfrak{t}$ with its dual $\mathfrak{t}^*$, and introduce the non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{t}^*$. Let Δ be the root system of $(\mathfrak{g},\mathfrak{t})$, and $\mathfrak{g}_\alpha$ the root space corresponding to the root $\alpha\in\Delta$. A root α is said to be compact (resp., non-compact) if $\mathfrak{g}_\alpha\subset\mathfrak{k}$ (resp., $\mathfrak{g}_\alpha\subset\mathfrak{p}$). Let Δ_c (resp., Δ_n) denote the set of all compact (resp., non-compact) roots in Δ. We have the disjoint decomposition $\Delta=\Delta_c\cup\Delta_n$.

There exists an element $Y_0\in\sqrt{-1}\mathfrak{c}_0$ such that $\gamma(Y_0)=\pm1$ for any $\gamma\in\Delta_n$. This Y_0 is called the characteristic element. We set $\Delta_n^\pm=\{\alpha\in\Delta\mid\alpha(Y_0)=\pm1\}$. Then $\Delta_c=\{\alpha\in\Delta\mid\alpha(Y_0)=0\}$, and we have the disjoint decomposition $\Delta=\Delta_n^+\cup\Delta_c\cup\Delta_n^-$. Then $\mathfrak{k}=\mathfrak{t}\oplus(\oplus_{\alpha\in\Delta_c}\mathfrak{g}_\alpha)$ gives the root space decomposition, and if we set $\mathfrak{p}^\pm=\oplus_{\alpha\in\Delta_n^\pm}\mathfrak{g}_\alpha$, then we have the triangular decomposition $\mathfrak{g}=\mathfrak{p}^+\oplus\mathfrak{k}\oplus\mathfrak{p}^-$. We choose an ordering of Δ such that the set Δ^+ of all positive roots satisfies the condition $\Delta_n^+\subset\Delta^+$. Let $\Delta_c^\pm=\Delta^\pm\cap\Delta_c$.

As in Ref. **[1]**, we construct a maximally strongly orthogonal subset $\{\gamma_1,\dots,\gamma_r\}\subset\Delta_n^+$ such that γ_i is the smallest element of the subset of elements in Δ_n^+ orthogonal to $\gamma_1,\dots,\gamma_{i-1}$. Then γ_1 is the unique simple non-compact root. For a $\lambda\in\mathfrak{t}^*$, we define $H_\lambda\in\mathfrak{t}$ by $B(H_\lambda,h)=\lambda(h)$ for all $h\in\mathfrak{t}$, or equivalently, $\lambda'(H_\lambda)=\langle\lambda,\lambda'\rangle$ for all $\lambda'\in\mathfrak{t}^*$. Let $\mathfrak{t}^-=\sum_{i=1}^r\mathbb{C}H_{\gamma_i}$. Then $\{H_{\gamma_1},\dots,H_{\gamma_r}\}$ forms a basis of $\mathfrak{t}^-$. Then, letting $\mathfrak{t}^+=\{H\in\mathfrak{t}\mid\gamma_i(H)=0\text{ for all }i=1,\dots,r\}$, we have $\mathfrak{t}=\mathfrak{t}^+\oplus\mathfrak{t}^-$.

We summarize several facts on strongly orthogonal roots (see, e.g., **[23]**, **[24]**). Note that the strongly orthogonal roots $\{\gamma_i\}$ here are taken from the minimal γ_1, while those of **[24]** in this volume are taken from the maximal γ_r.

Lemma 3.1

(1) *For $1\leqslant i<j\leqslant r$, γ_i and γ_j are strongly orthogonal: $\gamma_i\pm\gamma_j\notin\Delta$.*
(2) *The number r of maximally strongly orthogonal roots is equal to the split rank of $\mathfrak{g}_0$.*
(3) *If $\alpha\in\Delta_c^+$, then the restriction $\alpha|_{\mathfrak{t}^-}$ takes one of the following possible forms:*
   - $-\bar\gamma_i/2$ *for some $i=1,\dots,r$.*
   - $-(\bar\gamma_k-\bar\gamma_l)/2$ *for some $1\leqslant k<l\leqslant r$.*
   - 0.
(4) *If $\alpha\in\Delta_n^+$, then the restriction $\alpha|_{\mathfrak{t}^-}$ takes one of the following possible forms:*
   - *$\bar\gamma_i/2$, $\bar\gamma_i$ for some $i=1,\dots,r$.*
   - *$(\bar\gamma_k+\bar\gamma_l)/2$ for some $1\leqslant k<l\leqslant r$.*
(5) *The set of non-zero restrictions of $\Delta(\mathfrak{g},\mathfrak{t})$ to $\mathfrak{t}^-$ is one of the following two:*
   - *$\Delta(\mathfrak{g},\mathfrak{t}^-)=\{\pm\bar\gamma_i,\pm(\bar\gamma_k\pm\bar\gamma_l)/2\mid 1\leqslant i\leqslant r,1\leqslant k<l\leqslant r\}$: type C_r,*
   - *$\Delta(\mathfrak{g},\mathfrak{t}^-)=\{\pm\bar\gamma_i/2,\pm\bar\gamma_i,(\bar\gamma_k\pm\bar\gamma_l)/2\mid 1\leqslant i\leqslant r,1\leqslant k<l\leqslant r\}$: type BC_r.*

   The root system is of type C_r if and only if the hermitian Lie algebra $\mathfrak{g}_0$ is of the tube type.
(6) *By the Cayley transformation, the toral subalgebra $\mathfrak{t}^-$ is isomorphic to the complexification of a split Cartan subalgebra of $\mathfrak{g}_0$. This implies that the root system $\Delta(\mathfrak{g},\mathfrak{t}^-)$ coincides with the restricted root system of $\mathfrak{g}_0$.*

(7) *The dimension of root spaces has the following properties:*
– $\dim \mathfrak{g}(\mathfrak{t}^-, \pm\bar\gamma_i) = 1$.
– *The dimension* $\dim \mathfrak{g}(\mathfrak{t}^-, \pm(\bar\gamma_k \pm \bar\gamma_l)/2)$ *does not depend on* k *or* l. *This dimension is called the multiplicity of middle roots.*
– *The dimension* $\dim \mathfrak{g}(\mathfrak{t}^-, \pm\bar\gamma_i/2)$ *does not depend on* i. *This dimension is called the multiplicity of short roots. The multiplicity of short roots is zero if and only if the Lie algebra* $\mathfrak{g}_0$ *is of the tube type.*

(8) *The number of compact roots has the following properties:*
– *The cardinality of the set* $\{\alpha \in \Delta_c^+ \mid \alpha|_{\mathfrak{t}^-} = -(\bar\gamma_k - \bar\gamma_l)/2\}$ *is equal to the multiplicity of middle roots.*
– *The cardinality* $\#\{\alpha \in \Delta_c^+ \mid \alpha|_{\mathfrak{t}^-} = -\bar\gamma_i/2\} = \#\{\alpha \in \Delta_n^+ \mid \alpha|_{\mathfrak{t}^-} = -\bar\gamma_i/2\}$ *is equal to half of the multiplicity of short roots.*

(9) *Let* $\Delta_0 = \{\alpha \in \Delta \mid \alpha|_{\mathfrak{t}^-} = 0\}$. *Then* Δ_0 *is a subset of* Δ_c *and is the root system corresponding to the reductive subalgebra* $Z_{\mathfrak{k}}(\mathfrak{t}^-) = \{X \in \mathfrak{k} \mid [X, H] = 0,$ *for all* $H \in \mathfrak{t}^-\}$.

(10) *The strongly orthogonal roots* $\gamma_1, \ldots, \gamma_r$ *are long roots and have the same length.*

We recall the classification of the hermitian Lie algebra $\mathfrak{g}_0$ and some relevant information which we will use later.

	CI	AIII	DIII	BI, DI	EIII	EVII
$\mathfrak{g}_0$	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{su}(p,q)$	$\mathfrak{so}^*(2n)$	$\mathfrak{so}(2,n)$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{7(-25)}$
r	n	$\min(p,q)$	$[n/2]$	2	2	3
c	$1/2$	1	2	$(n-2)/2$	3	4
middle	1	2	4	$n-2$	6	8
short	0	$2\|p-q\|$	0 or 4	0	8	0

Here, we follow the notation of Ref. [**1**]. The split rank r of $\mathfrak{g}_0$ is denoted by t in Table 1 of Ref. [**2**]. The length of the interval c of the Wallach set is given in Table 2.9 of Ref. [**1**]. It is denoted by $\varepsilon = \varepsilon_{\mathfrak{g},\alpha}$ in Table 1 of Ref. [**2**]. Then c is equal to half of the multiplicity of the middle roots. The entries in the row labeled 'middle' (resp. 'short') are the root multiplicities of the restricted root system of $\mathfrak{g}_0$. These values are quoted from [**5**](Table VI, Ch.X). The multiplicity of short roots for type DIII is zero (resp., four) if n is even (resp., odd).

3.2. Degree of the orbit. — Let $G_{\mathbb{C}}$ be a connected linear Lie group with Lie algebra $\mathfrak{g}$, and let $G_{\mathbb{R}}$, K and $K_{\mathbb{R}}$ be the connected analytic subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{g}_0$, $\mathfrak{k}$ and $\mathfrak{k}_0$, respectively. The restriction of the adjoint action of $G_{\mathbb{C}}$ on $\mathfrak{g}$ to the subgroup K preserves the subspaces $\mathfrak{k}$ and $\mathfrak{p}^{\pm}$. We now recall the orbit decomposition of the action of K on $\mathfrak{p}^+$. (See Section 3.1 of Ref. **[24]**.) In this decomposition, the closure relation of the orbits is a linear ordering, and the number of K-orbits on $\mathfrak{p}^+$ is $r+1$. Then we can enumerate orbits $\mathcal{O}_m$ with $m = 0, 1, \ldots, r$ so that the closure is given by $\overline{\mathcal{O}_m} = \mathcal{O}_m \cup \cdots \cup \mathcal{O}_1 \cup \mathcal{O}_0$. Any K-stable closed subset of $\mathfrak{p}^+$ is irreducible and of the form $\overline{\mathcal{O}_m}$.

We define $\varphi_i = -(\gamma_1 + \cdots + \gamma_i)$. Then, with the notation of Lemma 2.4, $P\hat{A}^+(V) = \{\varphi_1, \ldots, \varphi_r\}$ and $P\hat{A}^+(\overline{\mathcal{O}_m}) = \{\varphi_1, \ldots, \varphi_m\}$.

We define the following definite integral:

$$I^\alpha(s,m) = \int_{D_m} (x_1 x_2 \cdots x_m)^s \prod_{1 \leqslant i < j \leqslant m} |x_i - x_j|^\alpha \, dx_1 \cdots dx_m.$$

Here the parameters α and s are positive real numbers, and the domain of the integration D_m is the simplex

$$D_m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \geqslant 0,\ x_1 + \cdots + x_m \leqslant 1\}.$$

This integral is evaluated in Ref. **[11]** (see also Example VI.10.7(c) of Ref. **[16]** and Theorem 2.2 of Ref. **[18]**). The result is

$$I^\alpha(s,m) = \frac{m! \prod_{i=1}^m \Gamma(i\alpha/2)}{\Gamma(\alpha/2)^m} \frac{\prod_{i=1}^m \Gamma((s+1) + (i-1)\alpha/2)}{\Gamma(1 + m(s+1) + (m-1)m\alpha/2)}.$$

Let s_r (resp., c) be equal to half of the root multiplicity of the short (resp., middle) roots of the restricted root system. In particular, $s_r = 0$ for the tube type. For $m = 0, \ldots, r$, we define $s_m = s_r + 2c(r-m)$, $d_m = m(s_m + 1) + (m-1)mc$, and

$$\Delta_{c,m}^+ = \{\alpha \in \Delta_c^+ \mid \langle \alpha, \gamma_i \rangle = 0 \text{ for all } i = 1, \ldots, m\}$$

Theorem 3.2

(1) *The dimension of the orbit $\mathcal{O}_m$ is d_m. This is the homogeneous degree of the integrand of the integral $I^{2c}(s_m, m)$.*

(2) *The degree of $\overline{\mathcal{O}_m}$ is given by*

$$\deg(\overline{\mathcal{O}_m}) = d_m! \times \frac{(\langle \gamma_1, \gamma_1 \rangle / 2)^{d_m - m}}{\prod_{\alpha \in \Delta_c^+ \setminus \Delta_{c,m}^+} \langle \alpha, \rho_c \rangle} \times \frac{1}{m!} I^{2c}(s_m, m).$$

The explicit values of degrees are given in the Appendix. We remark that the degrees of almost all orbits in the present case can be obtained without using the above theorem, as they can be obtained from the previously obtained results appearing in many detailed works. This theorem, however, gives a unified formula for the degree in terms of K-types corresponding to the orbits.

3.3. Proof of Theorem 3.2. — We first apply Theorem 2.5. Let $V = \mathfrak{p}^+$ and $Y = \overline{\mathcal{O}_m}$ with $0 \leqslant m \leqslant r$. Then $\Delta_K^+ = \Delta_c^+$ and $\Delta_Y^+ = \Delta_c^+ \setminus \Delta_{c,m}^+$. Using the Killing form, we can identify the dual of $\mathfrak{p}^+$ with $\mathfrak{p}^-$. Then $\mathbb{C}[V]$ is isomorphic to the symmetric algebra $S(\mathfrak{p}^-)$. The degree operator Z is $-Y_0$, where Y_0 is the characteristic element. Since $\gamma_i \in \Delta_n^+$, we have $\varphi_i'(Z) = 1$ for all i.

We employ a new set of coordinates y_i, defined in terms of the original coordinates by $y_i = x_i + \cdots + x_m$, in the integral in Theorem 2.5. We define $\varphi'_1 = \varphi_1$ and $\varphi'_i = \varphi_i - \varphi_{i-1}$ for $i \geqslant 2$. Since $\varphi_i = -(\gamma_1 + \cdots + \gamma_i)$, $\varphi'_i = -\gamma_i$. Then the semigroup $\Lambda(M)$ can be expressed as

$$\Lambda(M) = \{n'_1\varphi'_1 + \cdots + n'_m\varphi'_m \mid n'_i \in \mathbb{Z}_{\geqslant 0},\ n'_1 \geqslant n'_2 \geqslant \cdots \geqslant n'_m\}.$$

Next, we define $(\alpha'_1, \ldots, \alpha'_m) = (\langle\alpha, \varphi'_1\rangle, \ldots, \langle\alpha, \varphi'_m\rangle)$ for $\alpha \in \Delta_M^+$. Then $\alpha_1 x_1 + \cdots + \alpha_m x_m = \alpha'_1 y_1 + \cdots + \alpha'_m y_m$. Clearly, the integral

$$\int \prod_{\alpha \in \Delta_M^+} (\alpha_1 x_1 + \cdots + \alpha_m x_m) dx_1 \cdots dx_m$$

over the domain

$$\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant m,\ x_1\varphi_1(Z) + \cdots + x_m\varphi_m(Z) \leqslant 1\}$$

is equal to

$$\int \prod_{\alpha \in \Delta_M^+} (\alpha'_1 y_1 + \cdots + \alpha'_m y_m) dy_1 \cdots dy_m \tag{1}$$

over the domain

$$\{(y_1, \ldots, y_m) \in \mathbb{R}^m \mid y_1 \geqslant y_2 \geqslant \cdots \geqslant y_m \geqslant 0,\ y_1\varphi'_1(Z) + \cdots + y_m\varphi'_m(Z) \leqslant 1\}.$$

We next determine Δ_Y^+ and $(\alpha'_1, \ldots, \alpha'_m)$ for each $\alpha \in \Delta_Y^+$.

Lemma 3.3. — *Let e_i be the i-th unit vector in $\mathbb{R}^m$.*

(1) *For any $\alpha \in \Delta_c^+$, $(\alpha'_1, \ldots, \alpha'_m) = (\langle\alpha, -\gamma_1\rangle, \ldots, \langle\alpha, -\gamma_m\rangle)$ takes one of the following forms:*
- *$(\langle\gamma_1, \gamma_1\rangle/2)(e_k - e_l)$ with some $1 \leqslant k < l \leqslant m$.*
- *$(\langle\gamma_1, \gamma_1\rangle/2)e_i$ with some $1 \leqslant i \leqslant m$.*
- *0.*

(2) *For each $1 \leqslant k < l \leqslant m$, the number of $\alpha \in \Delta_c^+$ satisfying the condition $(\alpha'_1, \ldots, \alpha'_m) = (\langle\gamma_1, \gamma_1\rangle/2)(e_k - e_l)$ is equal to the root multiplicity $2c$ of the middle roots.*

(3) *For each i, the number of $\alpha \in \Delta_c^+$ satisfying $(\alpha'_1, \ldots, \alpha'_m) = (\langle\gamma_1, \gamma_1\rangle/2)e_i$ is equal to s_m.*

Proof. — For $m = r$, the assertion follows from the identity $\mathfrak{t}_r = \mathfrak{t}^-$. We next consider the case for general m.

We define $a = (\langle\gamma_1, \gamma_1\rangle/2)$ for convenience. For $1 \leqslant k < l \leqslant m$, we have

$$\{\alpha \in \Delta_c^+ \mid (\alpha'_1, \ldots, \alpha'_m) = a(e_k - e_l)\} = \{\alpha \in \Delta_c^+ \mid (\alpha'_1, \ldots, \alpha'_r) = a(e_k - e_l)\}.$$

This demonstrate the assertion for the middle roots. For $1 \leqslant i \leqslant m$, the set

$$\{\alpha \in \Delta_c^+ \mid (\alpha'_1, \ldots, \alpha'_m) = ae_i\}$$

is the disjoint union of

$$\{\alpha \in \Delta_c^+ \mid (\alpha_1', \dots, \alpha_r') = ae_i\}$$

and

$$\bigsqcup_{j=m+1}^{r} \{\alpha \in \Delta_c^+ \mid (\alpha_1', \dots, \alpha_r') = a(e_i - e_j)\}.$$

This shows that $s_m = s_r + 2c(r - m)$. □

We now complete the proof of Theorem 3.2. By Lemma 3.3, the number of elements in Δ_Y^+ is $2c \times m(m-1)/2 + ms_m$. This implies the formula $d_m = m(s_m+1)+(m-1)mc$. Also by this lemma, we have the following formula for $\bar{f}'$:

$$\begin{aligned} \bar{f}'(y_1, \dots, y_m) &= C \prod_{i=1}^{m} (ay_i)^{\dim \mathfrak{k}(\mathfrak{t}_m, e_i)} \prod_{1 \leqslant k < l \leqslant m} (ay_k - ay_l)^{\dim \mathfrak{k}(\mathfrak{t}_m, e_k - e_l)} \\ &= Ca^{d_m - m}(y_1 \cdots y_m)^{s_m} \prod_{1 \leqslant k < l \leqslant m} (y_k - y_l)^{2c}, \end{aligned}$$

where we denote

$$C = \frac{1}{\prod_{\alpha \in \Delta_c^+ \setminus \Delta_{c,m}^+} \langle \alpha, \rho_c \rangle}.$$

The domain of integral in (1) is

$$D_m' = \{(y_1, \dots, y_m) \in \mathbb{R}^m \mid y_1 \geqslant y_2 \geqslant \cdots \geqslant y_m \geqslant 0,\ y_1 + \cdots + y_m \leqslant 1\}. \tag{2}$$

Since the integrand of $I^\alpha(s, m)$ is symmetric with respect to permutations of the variables $(y_1, \dots, y_m)$, the integral $\int_{D_m'} \bar{f}'(y)dy$ is equal to $\frac{1}{m!}\int_{D_m} \bar{f}'(y)dy$. Hence the degree of $\overline{\mathcal{O}_m}$ is equal to $d_m! C a^{d_m - m} I^{2c}(s_m, m)/m!$. This completes the proof of Theorem 3.2. □

4. Unitary highest weight modules of the scalar type

4.1. Highest weight modules. — We keep the notation of Section 3. We define $\mathfrak{p}^+ = \oplus_{\alpha \in \Delta_n^+} \mathfrak{g}_\alpha$ and $\mathfrak{p}^- = \oplus_{\alpha \in \Delta_n^-} \mathfrak{g}_\alpha$. Then $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ is a graded Lie algebra with a characteristic element Y_0. We next define $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$. Then $\mathfrak{q}$ is a maximal parabolic subalgebra of $\mathfrak{g}$ with the commutative nilpotent radical $\mathfrak{p}^+$. Every maximal parabolic subalgebra with a commutative nilpotent radical arises in this way.

A weight $\lambda \in \mathfrak{t}^*$ is said to be a Δ_c^+-dominant integral weight if $2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}_{\geqslant 0}$ for all $\alpha \in \Delta_c^+$. We denote the set of all Δ_c^+-dominant integral weights of $\mathfrak{t}^*$ by P_c^+. Also, we denote the fundamental weight corresponding to the non-compact simple root γ_1 by ζ. In other words, the element $\zeta \in \mathfrak{t}^*$ is characterized by the conditions

$$\langle \zeta, \alpha \rangle = 0 \text{ for all } \alpha \in \Delta_c, \qquad \text{and } \langle \zeta, \gamma_1 \rangle = \langle \gamma_1, \gamma_1 \rangle / 2.$$

Let ρ_c be equal to half of the sum of roots in Δ_c^+ and ρ that of Δ^+.

We denote the irreducible finite dimensional representation of $\mathfrak{k}$ with the highest weight $\lambda \in P_c^+$ by $F(\lambda)$. Through the Levi decomposition $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$, a $\mathfrak{k}$-module is considered as a $\mathfrak{q}$-module on which $\mathfrak{p}^+$ acts trivially. We define the generalized Verma module (or induced module) by

$$N(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} F(\lambda),$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. By definition, $N(\lambda)$ is a highest weight $\mathfrak{g}$-module. It is well known that $N(\lambda)$ has a unique simple quotient $L(\lambda)$. Note that, as in the definition in Ref. **[1]**, we employ no rho-shift in our definition of an irreducible highest weight module $L(\lambda)$. The infinitesimal character of $N(\lambda)$ and $L(\lambda)$ is $\lambda + \rho \in \mathfrak{t}^*$.

The Poincaré-Birkhoff-Witt theorem implies the isomorphism $N(\lambda) \cong U(\mathfrak{p}^+) \otimes_{\mathbb{C}} F(\lambda)$ as a $(U(\mathfrak{p}^+), K)$-module. Note that $\mathfrak{p}^+$ is commutive and that the enveloping algebra $U(\mathfrak{p}^+)$ is canonically isomorphic to the symmetric algebra $S(\mathfrak{p}^+)$. It is significant that the module $N(\lambda)$ together with $L(\lambda)$ is not only filtered by $U(\mathfrak{g})$ but also is graded by the action of the characteristic element.

4.2. Unitary highest weight modules. — An irreducible highest weight $\mathfrak{g}$-module $L(\lambda)$ is called *unitarizable* if it has a $\mathfrak{g}_0$-invariant positive definite sesqui-linear form. The set of irreducible unitary highest weight modules consists of two classes; one is the set of induced modules (irreducible generalized Verma modules), and the other is the set of irreducible unitary highest weight modules which is not induced. In particular, the latter class with one-dimensional lowest K-types is called the Wallach set. It is easy to see (e.g., Section 2.2 of Ref. **[19]**) that the associated cycle of the generalized Verma module $N(\lambda)$ is $(\dim F(\lambda)) \cdot [\mathfrak{p}^+]$. In what follows, we consider the representation which is not induced.

Let us recall the number c introduced in Section 3.1. For unitary highest weight modules of the *scalar* type $L(z\zeta)$, the Wallach set corresponds to the set of parameters $z = 0, -c, \ldots, -(r-1)c$. It is shown in Ref. **[2]** that the annihilator is

$$\mathrm{Ann}_{U(\mathfrak{p}^-)} L(-mc\zeta) = \mathbf{I}(\overline{\mathcal{O}_m})$$

for $m = 0, \ldots, r$. Since for $m = r$ the Verma module $N(-rc\zeta)$ is irreducible, the unitarizable $L(-rc\zeta)$ does not belong to the Wallach set. However, since the situation is the same for the case $m = r$, we do not exclude the case $m = r$. As a $(U(\mathfrak{p}^-), K)$-module, we have the isomorphism

$$L(-mc\zeta) = U(\mathfrak{p}^-)/\mathbf{I}(\overline{\mathcal{O}_m}) \cong \mathbb{C}[\overline{\mathcal{O}_m}].$$

Thus, the associated variety of $L(-mc\zeta)$ is $\overline{\mathcal{O}_m}$, and the associated cycle of $L(-mc\zeta)$ is $[\overline{\mathcal{O}_m}]$. The Gelfand-Kirillov dimension of $L(-mc\zeta)$ is the dimension of the variety $\overline{\mathcal{O}_m}$, and the Bernstein degree of $L(-mc\zeta)$ is the degree of $\overline{\mathcal{O}_m}$.

As a direct consequence of Theorem 3.2, we can determine the Gelfand-Kirillov dimension and the Bernstein degree of the unitary highest weight module $L(-mc\zeta)$ of the scalar K-type.

Corollary 4.1. — *Let s_m, d_m and $\Delta_{c,m}^+$ be the same as in Theorem 3.2. We consider the representation $L(-mc\zeta)$ with $m = 0, 1, \dots, r$.*

(1) *The Gelfand-Kirillov dimension of $L(-mc\zeta)$ is d_m.*

(2) *The Bernstein degree of $L(-mc\zeta)$ is*

$$d_m! \frac{(\langle \gamma_1, \gamma_1 \rangle / 2)^{d_m - m}}{\prod_{\alpha \in \Delta_c^+ \setminus \Delta_{c,m}^+} \langle \alpha, \rho_c \rangle} \times \frac{1}{m!} I^{2c}(s_m, m).$$

We note that K-type decompositions like that in Lemma 2.4 are given in Ref. **[21]** for the generalized Verma module and in Theorem 5.10 of Ref. **[23]** for the module $L(-mc\zeta)$ in the Wallach set.

5. Further example of the degree of unitary highest weight modules

In the previous section we saw the method introduced in Section 2 is effective for modules of the scalar type. We now consider its application to modules of non-scalar type. In this section, we give calculations of the degrees of some unitary highest weight modules of non-scalar type. These examples are based on the examples in Ref. **[1]**, and we follow the notation used there for the root system.

Let $\{\alpha_1, \dots, \alpha_l\} \subset \Delta^+$ be the set of simple roots and $\{\omega_1, \dots, \omega_l\}$ the set of the corresponding fundamental weights.

5.1. EIII, case II, "the last unitarizable place". — Let $\mathfrak{g}_0$ be of type EIII. The corresponding multiplicity-free action is of the type (xi) in the Appendix. The compact root system Δ_c is of type D_5. Let us consider the Δ_c^+-dominant integral weight of the form

$$\lambda = a\omega_6 + (-a - 4)\omega_1,$$

with positive integer $a \in \mathbb{Z}_{>0}$. Here, the simple root α_1 is taken to be non-compact, and the fundamental weight ω_1 is perpendicular to Δ_c. The weight of this form is referred to in Ref. **[1]** as "the last unitarizable place of Case II". The set $\Delta_\lambda = \{\alpha \in \Delta_c \mid \langle \lambda, \alpha \rangle = 0\}$ is the root system of type D_4 whose simple system is $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

We consider the unitary highest weight representation $L(\lambda)$. This is the only unitary highest weight module $L(\lambda)$ of non-scalar type which is neither induced nor at 'the first reduction point'.

Proposition 5.1. — *For $L(a\omega_6 + (-a-4)\omega_1)$, the Gelfand-Kirillov dimension is* 16 *and the Bernstein degree is* 1.

Proof. — From the K-type decomposition

$$L(\lambda) = \bigoplus_{n_1 \geqslant n_2 \geqslant 0, n_i \in \mathbb{Z}} F(\lambda - n_1\gamma_1 - n_2\gamma_2)$$

given in Proposition 12.5 of Ref. [1], we have $m = 2$. Using the realization of the root system in the standard Euclidean space [1], we have $\Delta_c^+ = \{\pm e_i + e_j \mid 1 \leqslant i < j \leqslant 5\}$. We calculate

$$\Delta_{c,m}^+ = \{e_1 + e_i \mid i = 2, 3, 4\} \cup \{e_j - e_k \mid 2 \leqslant k < j \leqslant 4\},$$

which is the root system of type A_3 with the simple system $\{\alpha_2, \alpha_4, \alpha_5\}$. This implies that $\langle \alpha, \lambda \rangle = 0$ for all $\alpha \in \Delta_{c,m}^+$. Then

$$\prod_{\alpha \in \Delta_{c,m}^+} \frac{\langle \alpha, \lambda + \rho_c \rangle}{\langle \alpha, \rho_c \rangle} = 1.$$

Hence, by Theorem 3.2, the asymptotic of the dimension of the filtered pieces of $L(\lambda)$ is identical to that of the scalar case with $m = 2$. The proposition thus follows from (xi) in the Appendix or (iii) of Lemma 2.1. □

5.2. EVII, case II, the last unitarizable place. — Let $\mathfrak{g}_0$ be a Lie algebra of type EVII. The corresponding multiplicity-free action is given in (xiii) in the Appendix. The root system Δ_c is of type E_6. Let us consider the weight

$$\lambda = k\omega_6 + (-2k - 8)\omega_7,$$

with positive integer k. The fundamental weight ω_7 corresponds to the non-compact simple root α_7. The weight of this form is called "the last unitarizable place of Case II". The subset $\Delta_\lambda = \{\alpha \in \Delta_c \mid \langle \lambda, \alpha \rangle = 0\}$ is the root system of type D_5 whose simple system is $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

We consider the representation $L(\lambda)$. This is the only unitary highest weight module of non-scalar type which is neither induced nor at the first reduction point.

Proposition 5.2. — *For $L(k\omega_6 + (-2k - 8)\omega_7)$, the Gelfand-Kirillov dimension is* 26 *and the Bernstein degree is*

$$\frac{3(2k + 7) \prod_{i=1}^{6} (k + i)}{7!}.$$

Proof. — The K-type decomposition is given in Proposition 13.10 of Ref. [1]. We have

$$L(\lambda) = \bigoplus_{\substack{n_i \in \mathbb{Z} \\ n_1 \geqslant n_2 \geqslant 0 \\ 0 \leqslant n_3 \leqslant k}} F(\lambda - n_1\gamma_1 - n_2\gamma_2 - n_3\delta),$$

where $\delta = \alpha_6 + \alpha_7$. We apply Theorem 2.3 with $m = 2$ and with λ replaced by $\lambda - n_3\delta$ for each $n_3 = 0, \dots, k$. For $m = 2$, $\Delta_{c,m}^+ = \{\pm e_i + e_j \mid 1 \leqslant i < j \leqslant 4\}$, which is the root system of type D_4 with the simple roots $\{\alpha_5, \alpha_4, \alpha_3, \alpha_2\}$. This implies that $\langle \alpha, \lambda \rangle = 0$ for all $\alpha \in \Delta_{c,m}^+$. Then, for each n_3, the contribution to the degree is

$$\deg(\overline{\mathcal{O}_2}) \times \prod_{\alpha \in \Delta_{c2}^+} \frac{\langle \alpha, \lambda - n_3\delta + \rho_c \rangle}{\langle \alpha, \rho_c \rangle} = \deg(\overline{\mathcal{O}_2}) \times \prod_{\alpha \in \Delta_{c2}^+} \frac{\langle \alpha, -n_3\delta + \rho_c \rangle}{\langle \alpha, \rho_c \rangle}, \tag{3}$$

Here, the root systems Δ_c^+ and Δ_{c2}^+ have the following significant relation.

Lemma 5.3. — *For any $\alpha \in \Delta_{c2}^+$, we have*

$$\langle \alpha, \delta \rangle = \langle \alpha, \omega_{1,D_4} \rangle \text{ and } \langle \alpha, \rho_c \rangle = \langle \alpha, \rho_{c,D_4} \rangle.$$

Here, $\rho_{c,D_4} = e_2 + 2e_3 + 3e_4$ is equal to half of the sum of the roots in Δ_{c2}^+. The fundamental weight $\omega_{1,D_4} = e_4$ of the natural representation of $\mathfrak{so}(8)$ corresponds to the simple root α_5.

This lemma implies that the quantity (3) is equal to the dimension of the irreducible finite-dimensional representation $F(\mathfrak{so}(8), n_3\omega_{1,D_4})$ of the Lie algebra $\mathfrak{so}(8)$ with the highest weight $n_3\omega_{1,D_4}$. Its value is

$$(n_3+1)(n_3+2)(n_3+3)^2(n_3+4)(n_3+5)/(3\cdot 5!) = \binom{n_3+6}{6} + \binom{n_3+5}{6}.$$

Then, the degree of the representation $L(\lambda)$ is $d = \deg(\overline{\mathcal{O}_2})$ multiplied by the quantity

$$\sum_{n_3=0}^{k} \prod_{\alpha \in \Delta_{c2}^+} \frac{\langle \alpha, -n_3\delta + \rho_c \rangle}{\langle \alpha, \rho_c \rangle} = \binom{k+7}{7} + \binom{k+6}{7} = \frac{(2k+7)\prod_{i=1}^{6}(k+i)}{7!},$$

as is required in the proposition. □

Corollary 5.4. — *The associated cycle of $L(k\omega_6 + (-2k-8)\omega_7)$ is*

$$\frac{(2k+7)\prod_{i=1}^{6}(k+i)}{7!} \times [\overline{\mathcal{O}_2}].$$

Remark 5.5. — Vogan **[22]** has introduced the *isotropy representation* of the isotropy subgroup of the generic point of the associated variety on the space of the multiplicity of a given $(\mathfrak{g}, K)$-module. In our case, the Lie algebra of the Levi part of the isotropy subgroup of a point of the nilpotent orbit $\mathcal{O}_2$ in K is isomorphic to $\mathfrak{so}(9)$. Let ω_{1,B_4} be the fundamental weight corresponding to the natural (vector) representation of $\mathfrak{so}(9)$, and $F(\mathfrak{so}(9), k\omega_{1,B_4})$ the irreducible finite-dimensional representation of $\mathfrak{so}(9)$ with highest weight $k\omega_{1,B_4}$. It is easy to see, by the Weyl dimension formula, that

$$\frac{(2k+7)\prod_{i=1}^{6}(k+i)}{7!} = \dim F(\mathfrak{so}(9), k\omega_{1,B_4}).$$

Since the restriction of the irreducible representation $F(\mathfrak{so}(9), k\omega_{1,B_4})$ to the subalgebra $\mathfrak{so}(8)$ is decomposed as $\oplus_{n_3=0}^{k} F(\mathfrak{so}(8), n_3\omega_{1,D_4})$, the proof above may suggest interpreting the number as the dimension of the representation as above. Hence, it is suggested that the isotropy representation attached to the representation $L(k\omega_6 + (-2k-8)\omega_7)$ is precisely $F(\mathfrak{so}(9), k\omega_{1,B_4})$.

6. Appendix : List of degrees of orbits

We define the orbit $\mathcal{O}_0 = \{0\}$. The orbit $\mathcal{O}_{\max}$ is open dense.

6.1. Hermitian symmetric case. — The following case (i), (ii), (iii), (iv), (xi), or (xiii) corresponds to the case with Cartan label AIII, CI, DIII, (BI and DI), EIII, or EVII, respectively. (c.f. Table in § 3.1.) We use Theorem 3.2. In the following, we normalize the inner product $\langle \cdot, \cdot \rangle$ so that the restriction on Δ_c is induced from the Killing form on $\mathfrak{k}$. For example, $\langle \gamma_i, \gamma_i \rangle = 4$ for the case (i), while $\langle \gamma_i, \gamma_i \rangle = 2$ for other five cases.

(i) $GL_p \times GL_q$ with $p \geqslant q$: Here the orbits are parametrized by $\{0, 1, \ldots, q\}$. We apply the following identifications to Theorem 2.5: Δ_c is of type $A_{p-1} \times A_{q-1}$, $\Delta_c(\mathfrak{t}_m)$ is of type $A_{p-1-m} \times A_{q-1-m}$, the denominator of the formula is $\prod_{\alpha \in \Delta_c^+ \setminus \Delta_{c,m}^+} \langle \alpha, \rho_c \rangle = \prod_{i=1}^m ((p-i)!(q-i)!)$, and $s_m = p+q-2m$. In this case,

$$\dim(\overline{\mathcal{O}_m}) = m(p+q) - m^2$$

$$\deg(\overline{\mathcal{O}_m}) = \frac{0!1! \cdots (m-1)! \times (p+q-2m)! \cdots (p+q-m-2)!(p+q-m-1)!}{(p-m)!(p-m+1)! \cdots (p-1)! \times (q-m)!(q-m+1)! \cdots (q-1)!}.$$

This coincides with the Giambelli formula.

(ii) S^2GL_n: Here the orbits are parametrized by $\{0, 1, \ldots, n\}$. In this case, Δ_c is of type A_{n-1}, $\Delta_c(\mathfrak{t}_m)$ is of type A_{n-1-m}, $\prod_{\alpha \in \Delta_c^+ \setminus \Delta_{c,m}^+} \langle \alpha, \rho_c \rangle = \prod_{i=1}^m (n-i)!$, and $s_m = n - m$. We then obtain

$$\dim(\overline{\mathcal{O}_m}) = mn - (m-1)m/2$$

$$\deg(\overline{\mathcal{O}_m}) = \frac{0!1! \cdots (m-1)!}{0!!1!! \cdots (m-1)!!} \frac{(2n-2m)!!(2n-2m+1)!! \cdots (2n-m+1)!!}{(n-m)!(n-m+1)! \cdots (n-1)!},$$

where $l!! = l(l-2) \cdots 4 \cdot 2$ for an even integer l, and $l!! = l(l-2) \cdots 3 \cdot 1$ for odd l. This coincides with the Giambelli formula.

(iii) $\Lambda^2 GL_n$: In this case, we parameterize the orbits by $\{0, 1, 2, \ldots, [n/2]\}$, not by $\{0, 2, 4, \ldots, 2[n/2]\}$, since our numbering should be compatible with the enumeration of the Wallach set for the unitary highest weight module of the scalar K-type. Here, Δ_c is of type A_{n-1}, $\Delta_c(\mathfrak{t}_m)$ is of type $A_{n-1-2m} \times A_1^m$, $\prod_{\alpha \in \Delta_c^+ \setminus \Delta_{c,m}^+} \langle \alpha, \rho_c \rangle = \prod_{i=1}^{2m} (n-i)!$, and $s_m = 2n - 4m$. We have

$$\dim(\overline{\mathcal{O}_m}) = 2mn - (2m+1)m$$

$$\deg(\overline{\mathcal{O}_m}) = \frac{1!3! \cdots (2m-1)! \times (2n-4m)!(2n-4m+2)! \cdots (2n-2m-2)!}{(n-2m)! \cdots (n-m+1)!(n-m)! \cdots \cdots (n-1)!}$$

This coincides with the Giambelli formula.

(iv) $O_n \times GL_1$: In this case, the orbits are parametrized by $\{0, 1, 2\}$. Then, Δ_c is the root system of $\mathfrak{so}(n)$, $\Delta_c(\mathfrak{t}_2) = \Delta_c(\mathfrak{t}_1)$ is the root system of $\mathfrak{so}(n-2)$, and for $m = 1$ the denominator of the formula is $\prod_{\alpha \in \Delta_c^+ \setminus \Delta_{c,m}^+} \langle \alpha, \rho_c \rangle = (n-3)!(\frac{1}{2}n-1) =$

$(n-2)!/2$. Here we have

$$\begin{aligned}\dim(\overline{\mathcal{O}_m}) &= 0, n-1, n \\ \deg(\overline{\mathcal{O}_m}) &= 1, 2, 1 \qquad \text{for } m = 0, 1, 2, \text{ respectively.}\end{aligned}$$

Since the closure of the orbit $\overline{\mathcal{O}_1}$ is a quadratic hypersurface, the formula above follows from Lemma 2.1.

(xi) $Spin_{10} \times GL_1$: Here the orbits are parametrized by $\{0, 1, 2\}$, and in this case $\Delta_c = \{\alpha_i \mid 2 \leqslant i \leqslant 6\}$ is of type D_5, $\Delta_c(\mathfrak{t}_1) = \{\alpha_i \mid i = 2, 4, 5, 6\}$ is of type A_4, and $\Delta_c(\mathfrak{t}_2) = \{\alpha_i \mid i = 2, 4, 5\}$ is of type A_3. The denominator $\prod_{\alpha \in \Delta_c^+ \setminus \Delta_{c,m}^+} \langle \alpha, \rho_c \rangle$ for $m = 1$ is $7!5!/2$, and that for $m = 2$ is $7!5!4!/2$. We have

$$\begin{aligned}\dim(\overline{\mathcal{O}_m}) &= 0, 11, 16, \\ \deg(\overline{\mathcal{O}_m}) &= 1, 12, 1 \qquad \text{for } m = 0, 1, 2, \text{ respectively.}\end{aligned}$$

(xiii) $E_6 \times GL_1$: Here the orbits are parametrized by $\{0, 1, 2, 3\}$ and in this case $\Delta_c = \{\alpha_i \mid 1 \leqslant i \leqslant 6\}$ is of type E_6, $\Delta_c(\mathfrak{t}_1) = \{\alpha_i \mid 1 \leqslant i \leqslant 5\}$ is of type D_5, and $\Delta_c(\mathfrak{t}_2) = \Delta_c(\mathfrak{t}_3) = \{\alpha_i \mid 2 \leqslant i \leqslant 5\}$ is of type D_4. The denominator $\prod_{\alpha \in \Delta_c^+ \setminus \Delta_{c,m}^+} \langle \alpha, \rho_c \rangle$ for $m = 1$ is $11!8!/6$, and that for $m = 2, 3$ is $2 \cdot 11!8!7!/3$.

$$\begin{aligned}\dim(\overline{\mathcal{O}_m}) &= 0, 17, 26, 27, \\ \deg(\overline{\mathcal{O}_m}) &= 1, 78, 3, 1 \qquad \text{for } m = 0, 1, 2, 3, \text{ respectively.}\end{aligned}$$

Since the hermitian symmetric space of type EVII is of the tube type, it is known that the orbit $\overline{\mathcal{O}_2}$ is a hypersurface, and that the defining equation, which is the basic relative invariant of the corresponding prehomogeneous vector space, is cubic. The degree here was known previously, except for the case $m = 1$.

6.2. Non-hermitian case

(v) $Sp_{2n} \times GL_1$: In this case, the orbit structure is the same as that for $GL_{2n} \times GL_1$, which is a special case of case (i).

(ix) $Spin_7 \times GL_1$: In this case, the orbit structure is the same as that for $O(7) \times GL_1$, which is a special case of case (iv).

(xii) $G_2 \times GL_1$: Here, the orbit structure is the same as that for $O(7) \times GL_1$, which is a special case of case (iv).

(vi) $Sp_{2n} \times GL_2$: In this case, the orbits are parametrized by $\{(0,0), (1,0), (2,0), (2,2)\}$. Comparing with the orbits $\{\mathcal{O}_0^{(\mathrm{i})}, \mathcal{O}_1^{(\mathrm{i})}, \mathcal{O}_2^{(\mathrm{i})}\}$ of case (i) $GL_{2n} \times GL_2$, we have

$$\mathcal{O}_0^{(\mathrm{i})} = \mathcal{O}_{(0,0)}^{(\mathrm{vi})}, \mathcal{O}_1^{(\mathrm{i})} = \mathcal{O}_{(1,0)}^{(\mathrm{vi})}, \mathcal{O}_2^{(\mathrm{i})} = \mathcal{O}_{(2,0)}^{(\mathrm{vi})} \cup \mathcal{O}_{(2,2)}^{(\mathrm{vi})}.$$

We also know that $\overline{\mathcal{O}_{(2,0)}}$ is a quadratic hypersurface. Then the formula for the degrees of the closure of orbits can be reduced to that in known cases.

$$\dim(\overline{\mathcal{O}_m}) = 0, 2n+1, 4n-1, 4n$$
$$\deg(\overline{\mathcal{O}_m}) = 1, 2n, 2, 1.$$

(vii) $Sp_{2n} \times GL_3$: In this case, the orbits are parametrized by $\{(0,0), (1,0), (2,0), (2,2), (3,0), (3,2)\}$. Comparing with the orbits $\{\mathcal{O}_0^{(\mathrm{i})}, \mathcal{O}_1^{(\mathrm{i})}, \mathcal{O}_2^{(\mathrm{i})}, \mathcal{O}_3^{(\mathrm{i})}\}$ of the case (i) $GL_{2n} \times GL_3$, we have

$$\mathcal{O}_0^{(\mathrm{i})} = \mathcal{O}_{(0,0)}^{(\mathrm{vii})}, \mathcal{O}_1^{(\mathrm{i})} = \mathcal{O}_{(1,0)}^{(\mathrm{vii})}, \mathcal{O}_2^{(\mathrm{i})} = \mathcal{O}_{(2,0)}^{(\mathrm{vii})} \cup \mathcal{O}_{(2,2)}^{(\mathrm{vii})}, \mathcal{O}_3^{(\mathrm{i})} = \mathcal{O}_{(3,0)}^{(\mathrm{vii})} \cup \mathcal{O}_{(3,2)}^{(\mathrm{vii})}.$$

It is not difficult to see that the variety $\overline{\mathcal{O}_{(3,0)}}$ is the complete intersection of three quadratic hypersurfaces. Thus the degree of the variety for this case, except for $\overline{\mathcal{O}_{(2,0)}}$, was known previously. We have

$$\dim(\overline{\mathcal{O}_m}) = 0, 2n+2, 4n+1, 4n+2, 6n-3, 6n$$
$$\deg(\overline{\mathcal{O}_m}) = 1, n(2n+1), 4n(n-1), n(2n-1), 8, 1$$

(viii) $Sp_4 \times GL_n$: Here, the orbits are parametrized by $\{(0,0), (1,0), (2,0), (2,2), (3,2), (4,4)\}$. Comparing with the orbits $\{\mathcal{O}_0^{(\mathrm{i})}, \mathcal{O}_1^{(\mathrm{i})}, \mathcal{O}_2^{(\mathrm{i})}, \mathcal{O}_3^{(\mathrm{i})}, \mathcal{O}_4^{(\mathrm{i})}\}$ of the case (i) $GL_4 \times GL_n$, we have

$$\mathcal{O}_0^{(\mathrm{i})} = \mathcal{O}_{(0,0)}^{(\mathrm{viii})}, \mathcal{O}_1^{(\mathrm{i})} = \mathcal{O}_{(1,0)}^{(\mathrm{viii})}, \mathcal{O}_2^{(\mathrm{i})} = \mathcal{O}_{(2,0)}^{(\mathrm{viii})} \cup \mathcal{O}_{(2,2)}^{(\mathrm{viii})}, \mathcal{O}_3^{(\mathrm{i})} = \mathcal{O}_{(3,2)}^{(\mathrm{viii})}, \mathcal{O}_4^{(\mathrm{i})} = \mathcal{O}_{(4,4)}^{(\mathrm{viii})}.$$

In this case the degree of the variety, except for $\overline{\mathcal{O}_{(2,0)}}$, was known previously. Here we have

$$\dim(\overline{\mathcal{O}_m}) = 0, n+3, 2n+3, 2n+4, 3n+1, 4n$$
$$\deg(\overline{\mathcal{O}_m}) = 1, n(n+1)(n+2)/6, (n-1)n(n+1)/3, (n-1)n^2(n+1)/12,$$
$$(n-2)(n-1)n/6, 1, \text{ respectively.}$$

(x) $Spin_9 \times GL_1$: Here, the orbits are parametrized by $\{0, 1, 2, 2'\}$. This representation is equivalent to the isotropy representation on the tangent space of the Riemannian symmetric space $F_4/Spin_9$ of rank one. Thus it has an invariant quadratic form. Comparing the orbits $\{\mathcal{O}_0^{(\mathrm{iv})}, \mathcal{O}_1^{(\mathrm{iv})}, \mathcal{O}_2^{(\mathrm{iv})}\}$ of the case (iv) $O_{16} \times GL_1$, we find

$$\mathcal{O}_0^{(\mathrm{iv})} = \mathcal{O}_0^{(\mathrm{x})}, \mathcal{O}_1^{(\mathrm{iv})} = \mathcal{O}_1^{(\mathrm{x})} \cup \mathcal{O}_2^{(\mathrm{x})}, \mathcal{O}_2^{(\mathrm{iv})} = \mathcal{O}_{2'}^{(\mathrm{x})}.$$

On the other hand, the representation (x) is the restriction of the representation (xi). The correspondence between orbits is

$$\mathcal{O}_0^{(\mathrm{xi})} = \mathcal{O}_0^{(\mathrm{x})}, \mathcal{O}_1^{(\mathrm{xi})} = \mathcal{O}_1^{(\mathrm{x})}, \mathcal{O}_2^{(\mathrm{xi})} = \mathcal{O}_2^{(\mathrm{x})} \cup \mathcal{O}_{2'}^{(\mathrm{x})}.$$

Then the formula for the degrees of the closure of orbits can be reduced to that in cases (iv) and (xi). We obtain

$$\dim(\overline{\mathcal{O}_m}) = 0, 11, 15, 16$$
$$\deg(\overline{\mathcal{O}_m}) = 1, 12, 2, 1, \text{ respectively.}$$

6.3. Examples. — Let us illustrate the calculations necessary to obtain the above results by considering the spaces (vii) and (viii). In these cases, the group K is a direct product, say, $K = {}^lK \times {}^rK$ with ${}^lK = Sp(n, \mathbb{C}) = Sp_{2n}$ and ${}^rK = GL(n', \mathbb{C})$. We consider the case in which $(n, n') = (n, 3)$ or $(2, n')$. We use the superscript l or r to indicate an object corresponding to ${}^lK$ or ${}^rK$. The root system ${}^l\Delta_K^+$ is of type C_n, and ${}^r\Delta_K^+$ is of type $A_{n'-1}$. In the standard realization,

$$\begin{aligned} {}^l\Delta_K^+ &= \{e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\} \cup \{2e_i \mid 1 \leqslant i \leqslant n\}, \\ \text{and} \qquad {}^r\Delta_K^+ &= \{e_i - e_j \mid 1 \leqslant i < j \leqslant n'\}. \end{aligned}$$

We denote the weight $e_1 + \cdots + e_i = (1, \ldots, 1, 0, \ldots, 0)$ of K_l (resp., K_r) by l_i (resp., r_i).

First, we consider the space $Sp_{2n} \times GL_3$. Here, the set of primitive weights arising in V is $P\hat{A}^+(V) = \{\psi_1, \psi_2, \psi_3, \psi_2', \psi_3', \psi_4'\}$, where $\psi_1 = (l_1; r_1)$, $\psi_2 = (l_2; r_2)$, $\psi_3 = (l_3; r_3)$, $\psi_2' = (0; r_2)$, $\psi_3' = (l_1; r_3)$ and $\psi_4' = (l_2; r_1 + r_3)$. The generators of the subsemigroups corresponding to orbits are known:

$$P\hat{A}^+(\overline{\mathcal{O}_{(0,0)}}) = \varnothing; \quad P\hat{A}^+(\overline{\mathcal{O}_{(1,0)}}) = \{\psi_1\}; \quad P\hat{A}^+(\overline{\mathcal{O}_{(2,0)}}) = \{\psi_1, \psi_2\};$$
$$P\hat{A}^+(\overline{\mathcal{O}_{(2,2)}}) = \{\psi_1, \psi_2, \psi_2'\}; \quad P\hat{A}^+(\overline{\mathcal{O}_{(3,0)}}) = \{\psi_1, \psi_2, \psi_3\}.$$

We consider the degree of the closure of the orbit $Y = \overline{\mathcal{O}_{(3,0)}}$. First, we note that the lattice $\hat{A}^+(\overline{\mathcal{O}_{(3,0)}})$ is

$$\begin{aligned} &\{n_1\psi_1 + n_2\psi_2 + n_3\psi_3 \mid n_i \in \mathbb{Z}_{\geqslant 0}\} \\ &\qquad = \{n_1'\psi_1 + n_2'(\psi_2 - \psi_1) + n_3'(\psi_3 - \psi_2) \mid n_1' \geqslant n_2' \geqslant n_3' \geqslant 0\}. \end{aligned}$$

Next, for each α, we set $(\alpha_1', \alpha_2', \alpha_3') = (\langle \alpha, \psi_1 \rangle, \langle \alpha, \psi_2 - \psi_1 \rangle, \langle \alpha, \psi_3 - \psi_2 \rangle)$. Then Theorem 2.5 implies that the degree of Y is

$$\frac{d!}{\prod\limits_{\alpha \in {}^l\Delta_Y^+} \langle \alpha, {}^l\rho_K \rangle \prod\limits_{\alpha \in {}^r\Delta_Y^+} \langle \alpha, {}^r\rho_K \rangle} \int_{D_m'} {}^l\bar{f}(x_1, \ldots, x_m) \cdot {}^r\bar{f}(x_1, \ldots, x_m) dx_1 \cdots dx_m, \tag{4}$$

where $m = 3$, $d = m + |{}^l\Delta_Y^+| + |{}^r\Delta_Y^+|$, ${}^l\bar{f} = \prod_{\alpha \in {}^l\Delta_Y^+} (\alpha_1' x_1 + \cdots + \alpha_m' x_m)$, and ${}^r\bar{f}$ is similar to ${}^l\bar{f}$. The domain of integration is D_3' of (2), since the degree of $\psi_1, \psi_2 - \psi_1$ and $\psi_3 - \psi_2$ is 1. From the explicit form of ψ_i, we know that ${}^l\Delta_K^+ \setminus {}^l\Delta_Y^+$ is the positive root system of type C_{n-3} and ${}^l\Delta_Y^+ = \{e_1 \pm e_j, e_2 \pm e_j \mid 3 \leqslant j \leqslant n\} \cup \{e_1 \pm e_2, 2e_1, 2e_2\}$. We also have ${}^r\Delta_Y^+ = {}^r\Delta_K^+$. Then, with some calculation, we obtain the denominator of the formula as

$$\prod_{\alpha\in{}^l\Delta_Y^+} \langle \alpha, {}^l\rho_K\rangle = 8(2n-1)!(2n-3)!(2n-5)!, \quad \text{and} \quad \prod_{\alpha\in{}^r\Delta_Y^+} \langle \alpha, {}^r\rho_K\rangle = 2.$$

The leading polynomials here are

$${}^l\bar{f}(x_1,x_2,x_3) = (x_1x_2x_3)^{2n-6}(x_1^2-x_2^2)(x_1^2-x_3^2)(x_2^2-x_3^2)(2x_1)(2x_2)(2x_3),$$
$${}^r\bar{f}(x_1,x_2,x_3) = (x_1-x_2)(x_1-x_3)(x_2-x_3).$$

Finally, we recall the evaluation of the integral

$$(5)\quad d!\int_{D'_m}(x_1\cdots x_m)^r\Big(\prod_{1\leqslant i<j\leqslant m}(x_i-x_j)\Big)^2 s(x)dx_1\cdots dx_m$$
$$= 2^{m(m-1)/2}(m-1)!\cdots 1!\cdot r!(r+2)!\cdots(r+2m-2)!,$$

where $d = m(r+1)+3m(m-1)/2$, and $s(x) = \prod_{1\leqslant i<j\leqslant m}(x_i+x_j)$ is the Schur function attached to the staircase partition $(m-1,\dots,1,0)$. This is a special case of the formula in Example VI.10.7(c) of Ref. **[16]**. The right-hand side of (5) equals $2r!(r+2)!$ for $m=2$, and $16r!(r+2)!(r+4)!$ for $m=3$. We use the formula (5) for $m=3$ and $r=2n-5$. With this information, we conclude that the degree of $\overline{\mathcal{O}_{(3,0)}}$ is 8.

We now consider another orbit, $Y=\overline{\mathcal{O}_{(2,0)}}$. The lattice $\hat{A}^+(\overline{\mathcal{O}_{(2,0)}})$ is $\{n_1\psi_1+n_2\psi_2 \mid n_i\in\mathbb{Z}_{\geqslant 0}\} = \{n'_1\psi_1+n'_2(\psi_2-\psi_1)\mid n'_1\geqslant n'_2\geqslant 0\}$. Here, we can again use (4), with $m=2$. The denominators and the leading polynomials in this case are

$$\prod_{\alpha\in{}^l\Delta_Y^+} \langle \alpha, {}^l\rho_K\rangle = 4(2n-1)!(2n-3)!, \qquad \prod_{\alpha\in{}^r\Delta_Y^+} \langle \alpha, {}^r\rho_K\rangle = 2.$$
$${}^l\bar{f} = (x_1x_2)^{2n-4}(x_1-x_2)(x_1+x_2)(2x_1)(2x_2), \qquad {}^r\bar{f} = x_1x_2(x_1-x_2).$$

Then, the integral is of the form (5) with $m=2$ and $r=2n-2$. Hence, we conclude that the degree of $\overline{\mathcal{O}_{(2,0)}}$ is $4n(n-1)$.

Finally, we consider the orbit $Y=\overline{\mathcal{O}_{(2,0)}}$ of the space $Sp_4\times GL_{n'}$. In this case, the primitive weights are known to be $\psi_1=(l_1;r_1)$, $\psi_2=(l_2;r_2)$, $\psi_3=(l_1;r_3)$, $\psi_4=(0;r_4)$, $\psi'_2=(0;r_2)$, and $\psi'_4=(l_2;r_1+r_3)$. We also know that $P\hat{A}^+(V)=\{\psi_1,\psi_2,\psi_3,\psi_4,\psi'_2,\psi'_4\}$, $P\hat{A}^+(\mathcal{O}_{(0,0)})=\varnothing$, $P\hat{A}^+(\mathcal{O}_{(1,0)})=\{\psi_1\}$, $P\hat{A}^+(\mathcal{O}_{(2,0)})=\{\psi_1,\psi_2\}$, $P\hat{A}^+(\mathcal{O}_{(2,2)})=\{\psi_1,\psi_2,\psi'_2\}$, and $P\hat{A}^+(\mathcal{O}_{(3,2)})=\{\psi_1,\psi_2,\psi_3,\psi'_2,\psi'_4\}$. Then, we see that ${}^l\Delta_Y^+={}^l\Delta_K^+$ and that ${}^r\Delta_K^+\setminus{}^r\Delta_Y^+$ is a positive system of type $A_{n'-3}$. We then find that the denominators and the leading polynomials in the formula giving the degree are

$$\prod_{\alpha\in{}^l\Delta_Y^+} \langle \alpha, {}^l\rho_K\rangle = 4, \qquad \prod_{\alpha\in{}^r\Delta_Y^+} \langle \alpha, {}^r\rho_K\rangle = (n'-1)!(n'-2)!,$$
$${}^l\bar{f} = (2x_1)(2x_2)(x_1-x_2)(x_1+x_2), \qquad {}^r\bar{f} = (x_1x_2)^{n'-2}(x_1-x_2).$$

Thus we can apply the integral formula (5) with $m=2$ and $r=n'-1$. Hence, we conclude that the degree of $\overline{\mathcal{O}_{(2,0)}}$ is $(n'-1)n'(n'+1)/3$.

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