Astérisque

JEAN-MICHEL BISMUT
SEBASTIAN GOETTE

Families torsion and Morse functions

Astérisque, tome 275 (2001)
<http://www.numdam.org/item?id=AST_2001__275__R1_0>

ASTÉRISQUE 275

FAMILIES TORSION AND MORSE FUNCTIONS

Jean-Michel Bismut
Sebastian Goette

Société Mathématique de France 2001
Publié avec le concours du Centre National de la Recherche Scientifique

J.-M. Bismut

Département de Mathématique, Université Paris-Sud, Bâtiment 425, 91405 Orsay, France.

E-mail : Jean-Michel.Bismut@math.u-psud.fr

S. Goette

Mathematisches Institut der Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany.

E-mail : sebastian.goette@uni-tuebingen.de

2000 ***Mathematics Subject Classification.*** — 37D15, 57R20, 58G10, 58G11.

Key words and phrases. — Morse-Smale systems. Characteristic classes and numbers. Index theory and related fixed point theories. Heat and other parabolic equation methods.

Jean-Michel Bismut was supported by Institut Universitaire de France (I.U.F.). Sebastian Goette was supported by a research fellowship of the Deutsche Forschungsgemeinschaft (DFG).

FAMILIES TORSION AND MORSE FUNCTIONS

Jean-Michel Bismut, Sebastian Goette

Abstract. — To a flat vector bundle, one can associate odd real characteristic classes. Bismut and Lott have proved a Riemann-Roch-Grothendieck theorem for such classes, when taking the direct image of a flat vector bundle by a proper submersion. They have also constructed associated secondary invariants, the analytic torsion forms in de Rham theory. The component of degree 0 of these forms is the classical Ray-Singer torsion.

The present paper has five purposes:

- to extend the theory of analytic torsion forms to the equivariant setting.
- to give a proper normalization of these torsion forms.
- to prove rigidity formulas, showing that in positive degree, and up to locally computable terms, these forms are rigid under deformation of the flat connection.
- to evaluate the equivariant analytic torsion forms modulo coboundaries, under the assumption that there exists a fibrewise gradient vector field which verifies the Morse-Smale transversality conditions in every fibre.
- to compute the equivariant analytic torsion forms of sphere bundles associated to vector bundles.

Our main formula generalizes the results by Cheeger, Müller, Lott-Rothenberg and Bismut-Zhang on the relation of Ray-Singer torsion to Reidemeister torsion, and also computations by Bunke for sphere bundles.

***Résumé* (Torsion en famille et fonctions de Morse). —** À un fibré plat, on peut associer des classes caractéristiques impaires réelles. Bismut et Lott ont montré un théorème de Riemann-Roch-Grothendieck, quand on prend l'image directe d'un fibré plat par une submersion propre. Ils ont aussi construit des invariants secondaires, les formes de torsion analytique en théorie de de Rham, qui sont des formes paires sur la base de la fibration considérée. La composante de degré 0 de ces formes est la torsion analytique de Ray-Singer.

Le présent article a pour objet :

- d'étendre la théorie des formes de torsion analytique en situation équivariante.
- de normaliser les formes de torsion analytique.
- d'établir des résultats de rigidité, qui montrent qu'à des termes explicites calculables localement près, les formes de torsion ne varient pas par déformation de la connexion plate considérée, et ceci en degré positif.
- d'évaluer les formes de torsion analytique équivariantes, sous l'hypothèse qu'il existe un champ de gradient de Morse-Smale dans les fibres.
- d'évaluer les formes de torsion équivariantes des fibrés en sphères provenant de fibrés vectoriels.

Le résultat principal généralise des résultats obtenus par Cheeger, Müller, et Lott-Rothenberg et Bismut-Zhang sur le lien entre torsion analytique et torsion de Reidemeister, et aussi des calculs de Bunke pour des fibrés en sphères.

CONTENTS

INTRODUCTION

The main purpose of this paper is to give a formula for the analytic torsion forms in de Rham theory introduced by Bismut and Lott [**BLo1**]. Our formula is only valid in special cases. In this Introduction, we will describe the geometric setting, and explain our results, and also their limitations.

Let M be a smooth manifold, let E be a vector bundle on M, equipped with a flat connection ∇^E. Then the Chern classes of E vanish identically. The theory of differential characters by Cheeger-Simons [**CSi**] shows that $\left(E, \nabla^E\right)$ has secondary characteristic classes which lie in $H^{\mathrm{odd}}\left(M, \mathbf{C}/\mathbf{Z}\right)$. Since $\mathbf{C}/\mathbf{Z} = \mathbf{R} \oplus \mathbf{R}/\mathbf{Z}$, the $\mathbf{R}$ component of these classes are just ordinary cohomology classes in $H^{\mathrm{odd}}\left(M, \mathbf{R}\right)$. In [**BLo1**], Bismut and Lott obtained explicit de Rham representatives of these classes. Namely, if g^E is a Hermitian metric on E, set

$$\omega\left(\nabla^E, g^E\right) = \left(g^E\right)^{-1} \nabla^E_{\cdot} g^E. \tag{0.1}$$

Then $\omega\left(\nabla^E, g^E\right)$ is a 1-form with values in $\mathrm{End}\left(E\right)$. Let $\varphi : \Lambda^{\bullet}(T^*M) \to \Lambda^{\bullet}(T^*M)$ be given by $\varphi\alpha = (2i\pi)^{-\deg\alpha/2}\,\alpha$. If h is a holomorphic odd function, put

$$h\left(\nabla^E, g^E\right) = \sqrt{2i\pi}\,\varphi \mathrm{Tr}\left[h\left(\omega\left(\nabla^E, g^E\right)/2\right)\right]. \tag{0.2}$$

Then by [**BLo1**], the form $h\left(\nabla^E, g^E\right)$ is closed, its cohomology class $h\left(\nabla^E\right)$ does not depend on g^E, and is just one of the above odd classes.

Let $\pi : M \to S$ be a submersion of smooth manifolds with compact fibre X. Let F be a complex vector bundle on M, and let ∇^F be a flat connection on F. Let $\left(\Omega^{\bullet}\left(X, F|_X\right), d^X\right)$ be the de Rham complex of smooth forms along the fibre with coefficients in F, equipped with the fibrewise de Rham operator d^X. Let $H^{\bullet}(X, F|_X)$ be the fibrewise cohomology of X with coefficients in F. Then $H^{\bullet}(X, F|_X)$ is a $\mathbf{Z}$-graded complex vector bundle on S, equipped with the flat Gauss-Manin connection $\nabla^{H^{\bullet}(X,F|_X)}$. In [**BLo1**, Theorem 3.17], Bismut and Lott proved a Riemann-Roch-Grothendieck formula for such classes. Namely, if h is a holomorphic odd function, if

$e(TX)$ denotes the Euler class of TX, we have the identity,

$$h\left(\nabla^{H^\bullet(X,F|_X)}\right) = \int_X e(TX)\, h\left(\nabla^F\right) \text{ in } H^{\text{odd}}(X, \mathbf{C}). \tag{0.3}$$

A fundamental feature of (0.3) is that because the degree of $e(TX)$ is just $\dim(X)$, the above equality is valid 'degree by degree'. Also recall that if $\dim(X)$ is odd, $e(TX) = 0$, so that, if $\dim X$ is odd, the right-hand side of (0.3) vanishes.

Bismut and Lott [**BLo1**] refined on equality (0.3) at the level of differential forms. Namely let $T^H M \subset TM$ be a horizontal bundle on M, let g^{TX} and g^F be metrics on TX and F. Then by [**B3**, Section 1c)], the above data determine an Euclidean connection ∇^{TX} on the vector bundle TX over M, which is described in Theorem 3.5. Let $e\left(TX, \nabla^{TX}\right)$ be the differential form associated to the connection ∇^{TX}, which represents $e(TX)$ in Chern-Weil theory. By identifying $H^\bullet(X, F|_X)$ to the corresponding fibrewise harmonic forms, the vector bundle $H^\bullet(X, F|_X)$ inherits the L_2 metric $g_{L_2}^{H^\bullet(X,F|_X)}$. Set

$$h(x) = xe^{x^2}. \tag{0.4}$$

In [**BLo1**, Section 3], Bismut and Lott constructed even forms on S, the analytic torsion forms $\mathcal{T}_h\left(T^H M, g^{TX}, \nabla^F, g^F\right)$, which are such that

$$d\mathcal{T}_h\left(T^H M, g^{TX}, \nabla^F, g^F\right) = \int_X e\left(TX, \nabla^{TX}\right) h\left(\nabla^F, g^F\right) - h\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2}^{H^\bullet(X,F|_X)}\right). \tag{0.5}$$

If the fibres X are odd dimensional and F is fibrewise acyclic, so that $H^\bullet(X, F|_X) = 0$, then $\mathcal{T}_h\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ defines an even cohomology class, which does not depend on the above data.

In general the question arises of evaluating the class of $\mathcal{T}_h\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ modulo exact forms. This is in fact the main goal of this paper. First let us briefly review the present state of our knowledge on this question.

Assume temporarily that S is a point, so that we only consider the case of a single fibre X. For $0 \leqslant p \leqslant \dim X$, let $\zeta_p(s)$ be the zeta function of the Laplacian of the fibre X acting on p-forms. Put

$$\theta(s) = \sum_{p=0}^{\dim X} (-1)^{p+1}\, p\, \zeta_p(s). \tag{0.6}$$

By [**BLo1**, Theorem 3.29], if $\mathcal{T}_h\left(T^H M, g^{TX}, \nabla^F, g^F\right)^{(0)}$ is the component of degree 0 of $\mathcal{T}_h\left(T^H M, g^{TX}, \nabla^F, g^F\right)$, then

$$\mathcal{T}_h\left(T^H M, g^{TX}, \nabla^F, g^F\right)^{(0)} = \frac{1}{2}\frac{\partial \theta}{\partial s}(0). \tag{0.7}$$

The quantity $\frac{\partial \theta}{\partial s}(0)$ is called the Ray-Singer analytic torsion of the fibre. It was introduced by Ray and Singer in [**RS1**]. Quillen [**Q2**] used holomorphic analytic

torsion to define the Quillen metric on the determinant of the Dolbeault cohomology of a holomorphic vector bundle. Bismut and Zhang [**BZ1**] used its de Rham analogue to define a Ray-Singer metric on the complex line $\det(H^\bullet(X, F|_X))$. Via (0.5) and (0.7), one recovers the anomaly formulas for Ray-Singer metrics established by Bismut and Zhang in [**BZ1**, Theorem 0.1].

Assume temporarily that the metric g^F is flat. In [**RS1**], Ray and Singer conjectured that the Ray-Singer analytic torsion coincides with a combinatorial invariant, the Reidemeister torsion. To be more precise, let K be a triangulation of a fibre X. Then, as explained in Milnor [**Mi1**] and in [**BZ1**, Chapter 1], via K, one can define another metric on $\det(H^\bullet(X, F|_X))$, the Reidemeister metric. One can then show that the Reidemeister metric does not depend on the triangulation. Ray and Singer conjectured the equality of the Ray-Singer and Reidemeister metrics. This conjecture was proved independently by Cheeger [**C**] and Müller [**Mü1**] using different methods. Essentially, Cheeger studied the behaviour of these two metrics by surgery, and Müller used combinatorial invariance by taking the mesh of the triangulation to zero. This result was extended by Müller in [**Mü2**] to unimodular flat vector bundles.

In [**BZ1**], Bismut and Zhang extended the above results to the case of general flat vector bundles. Let $f : X \to \mathbf{R}$ be a Morse function, and let ∇f be the gradient field of f with respect to some metric. Assume that $Y = -\nabla f$ is Morse-Smale [**Sm1, Sm2, Th**] i.e. the corresponding stable and unstable cells intersect transversally. Then, as explained in [**Mi1**] and in [**BZ1**, Chapter I], the above data determine a finite dimensional complex $(C^\bullet(W^u, F), \partial)$, the Thom-Smale complex [**Sm1, Sm2, Th**], whose cohomology is just $H^\bullet(X, F|_X)$. If g^F is a metric on F, we construct this way a metric on $\det(H^\bullet(X, F|_X))$, the Milnor metric. If g^F is flat, this metric coincides with the Reidemeister metric. In [**BZ1**], Bismut and Zhang gave a formula comparing the Ray-Singer and the Milnor metrics. The defect is given by the integral on X of a Chern-Simons current.

Assume again that S is a point, and that G is a finite Lie group which acts on X, whose action lifts to F and preserves the above data. If the metric g^F is unitarily flat, Rothenberg [**Ro**] showed that one can define an equivariant version of the Reidemeister torsion. In [**LoRo**], Lott and Rothenberg showed that one can extend the Cheeger-Müller theorem to this situation, by replacing the Ray-Singer torsion by its equivariant extension. In [**BZ2**], Bismut and Zhang extended the Lott and Rothenberg formula to the case of arbitrary flat vector bundles.

From the above discussion, it should be clear that $\mathcal{T}_h\left(T^H M, g^{TX}, \nabla^F, g^F\right)^{(0)}$ and its equivariant extension are fully understood.

On the other hand, in [**BLo1**, Corollary 4.14], Bismut and Lott gave a formula for the analytic torsion forms of S^1 fibre bundles equipped with a complex Hermitian line bundle with a unitary flat connection, whose holonomy along the fibre is a root of unity. The torsion forms are power series in the first Chern class of the line bundle, the

coefficients being polylogarithms evaluated at the considered root of unity. In [**I, K**], Igusa and Klein gave a construction of higher Reidemeister torsion using algebraic K-theory. In the case of S^1 fibre bundles over S^2, they gave a formula for their torsion which coincides with the formula of Bismut-Lott [**BLo1**]. The coincidence of these two computations is unexplained. The purely algebraic character of Igusa and Klein's constructions should make a direct approach to a comparison formula very difficult. Also, in [**Bu1**], Bunke computed in particular the non equivariant analytic torsion forms for sphere bundles associated to vector bundles.

The present paper has five purposes:

- To extend the theory of analytic torsion forms to the equivariant setting.
- To give a proper normalization of the analytic torsion forms.
- To prove rigidity formulas for the equivariant analytic torsion forms in degree $\geqslant 2$, which show that up to 'local' terms, they are essentially invariant by deformation of the flat connection ∇^F.
- To evaluate the equivariant analytic torsion forms modulo coboundaries, when there is a function $f : M \to \mathbf{R}$ which is fibrewise Morse, and the fibres can be equipped with a corresponding fibrewise Morse-Smale vector field Y.
- To give a formula for the equivariant torsion forms of unit sphere bundles.

In fact, let G be a compact Lie group. Let $\left(E, \nabla^E\right)$ be a flat vector bundle as in (0.1). Assume that G acts trivially on M, and acts on E by flat automorphisms. Let g^E be a G-invariant metric. If h is a holomorphic odd function, if $g \in G$, set

$$h_g\left(\nabla^E, g^E\right) = \mathrm{Tr}\left[gh\left(\omega\left(\nabla^E, g^E\right)/2\right)\right]. \tag{0.8}$$

Then the form $h_g\left(\nabla^E, g^E\right)$ has the same properties as the form $h\left(\nabla^E, g^E\right)$. Namely $h_g\left(\nabla^E, g^E\right)$ is closed, and its cohomology class $h_g\left(\nabla^E\right)$ does not depend on g^E.

Let now $\pi : M \to S$ be a submersion taken as before. Assume that the compact Lie group G acts on M, and preserves the fibres X. Let F be a flat vector bundle on which G acts by flat automorphisms. Then $H^\bullet(X, F|_X)$ is a flat $\mathbf{Z}$-graded G-vector bundle on S. Let now $g \in G$, let $M_g \subset M$ be the submanifold of fixed points by g. which fibres on S with fibre $X_g \subset X$. Then in Theorem 3.25, if h is a holomorphic odd function, we extend (0.3) to the formula,

$$h_g\left(\nabla^{H^\bullet(X,F|_X)}\right) = \int_{X_g} e\left(TX_g\right) h_g\left(\nabla^F\right) \text{ in } H^{\mathrm{odd}}\left(X, \mathbf{C}\right). \tag{0.9}$$

Now we assume that $T^H M, g^{TX}, g^F$ are G-invariant. These data determine an Euclidean connection ∇^{TX_g} on TX_g. If $h(x)$ is still given by (0.4), in Section 3.12, we construct equivariant analytic torsion forms $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$, which are

such that

$$(0.10)\quad d\mathcal{T}_{h,g}\left(T^HM,g^{TX},\nabla^F,g^F\right)=\int_{X_g}e\left(TX_g,\nabla^{TX_g}\right)h_g\left(\nabla^F,g^F\right)\\ -h_g\left(\nabla^{H^\bullet(X,F|_X)},g_{L_2}^{H^\bullet(X,F|_X)}\right).$$

Still, as in [**BLo1**], our construction of $\mathcal{T}_{h,g}\left(T^HM,g^{TX},\nabla^F,g^F\right)$ depends explicitly on the choice of h in (0.4). In Sections 2.8 and 3.17, we show how to normalize the torsion forms $\mathcal{T}_{h,g}\left(T^HM,g^{TX},\nabla^F,g^F\right)$ into forms $\mathcal{T}_{\mathrm{ch},g}\left(T^HM,g^{TX},\nabla^F,g^F\right)$, which we call Chern analytic torsion forms. The idea is that one can normalize the classes in (0.8) to secondary Chern classes associated to the Chern character, and the forms $\mathcal{T}_{\mathrm{ch},g}\left(T^HM,g^{TX},\nabla^F,g^F\right)$ verify an equation similar to (0.10).

It is well known [**CSi**, Proposition 2.9] that the odd characteristic classes of flat vector bundles are rigid in degree $\geqslant 3$. We establish a corresponding result for the analytic torsion forms. Let $\Omega^\bullet(S)$ be the space of smooth forms on S, let $d\Omega^\bullet(S)\subset\Omega^\bullet(S)$ be the space of smooth coboundaries. In Sections 2.4 and 3.16, we show that the classes $\mathcal{T}_{h,g}\left(T^HM,g^{TX},\nabla^F,g^F\right)\in\Omega^\bullet(S)/d\Omega^\bullet(S)$ are rigid in positive degree. This result says that up to locally computable secondary characteristic classes (which are analogues of the Bott-Chern classes [**BoCh, BGS1**] in complex geometry), the class of $\mathcal{T}_{h,g}\left(T^HM,g^{TX},\nabla^F,g^F\right)$ in $\Omega^\bullet(S)/d\Omega^\bullet(S)$ is invariant under deformation of the flat connection ∇^F. We also establish corresponding results for families of finite dimensional complexes.

Assume now that $f:M\to\mathbf{R}$ is a G-invariant smooth function, and that $\nabla f\in TX$ is a G-invariant fibrewise gradient vector field for f. We assume that $Y=-\nabla f$ is Morse-Smale [**Sm1, Sm2**] in every fibre X, i.e. that the stable and unstable cells associated to Y intersect transversally. Let $\mathbf{B}$ be the zero set of Y. Then $\mathbf{B}$ is a submanifold of M, which fibres over S with finite fibre B. If $x\in\mathbf{B}$, let $\mathrm{ind}\,(x)$ be the Morse index of x, i.e. the number of negative eigenvalues of the quadratic form $d^2f(x)|_{T_xX\times T_xX}$.

Let $(C^\bullet(W^u,F),\partial)$ be the Thom-Smale complex along the fibres X, which is associated to Y and to the flat vector bundle F. Then $(C^\bullet(W^u,F),\partial)$ is a flat Hermitian complex on S, whose fibrewise cohomology is $H^\bullet(X,F|_X)$. Let $g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}$ be the metric on $H^\bullet(X,F|_X)$, which is obtained by identifying $H^\bullet(X,F|_X)$ to the corresponding harmonic objects in $C^\bullet(W^u,F)$. Then by a construction given in Section 1.10, which extends a construction in [**BLo1**, Section 2], we obtain finite dimensional torsion forms $T_{h,g}\left(A^{C^\bullet(W^u,F)\prime},g^{C^\bullet(W^u,F)}\right)$, which are such that

$$(0.11)\quad dT_{h,g}\left(A^{C^\bullet(W^u,F)\prime},g^{C^\bullet(W^u,F)}\right)=h_g\left(\nabla^{C^\bullet(W^u,F)},g^{C^\bullet(W^u,F)}\right)\\ -h_g\left(\nabla^{H^\bullet(X,F|_X)},g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}\right).$$

Let $\widetilde{h}_g\left(\nabla^{H^\bullet(X,F|X)}, g^{H^\bullet(X,F|X)}_{C^\bullet(W^u,F)}, g^{H^\bullet(X,F|X)}_{L_2}\right) \in \Omega^\bullet(S)/d\Omega^\bullet(S)$ be the class defined in [**BLo1**, Section 1] and in Section 1.5, such that

$$d\widetilde{h}_g\left(\nabla^{H^\bullet(X,F|X)}, g^{H^\bullet(X,F|X)}_{C^\bullet(W^u,F)}, g^{H^\bullet(X,F|X)}_{L_2}\right) = h_g\left(\nabla^{H^\bullet(X,F|X)}, g^{H^\bullet(X,F|X)}_{L_2}\right) - h_g\left(\nabla^{H^\bullet(X,F|X)}, g^{H^\bullet(X,F|X)}_{C^\bullet(W^u,F)}\right). \tag{0.12}$$

Let $\psi\left(TX_g, \nabla^{TX_g}\right)$ be the current defined on the total space of $p : TX_g \to M_g$ constructed by Mathai and Quillen [**MQ**, Section 7], such that if M_g is identified to the zero section of TX_g, and if δ_{M_g} is the current of integration on M_g,

$$d\psi\left(TX_g, \nabla^{TX_g}\right) = p^* e\left(TX_g, \nabla^{TX_g}\right) - \delta_{M_g}. \tag{0.13}$$

Recall that $\mathbf{B}$ is the zero set of ∇f. Put

$$\mathbf{B}_g = \mathbf{B} \cap M_g. \tag{0.14}$$

Then, as explained in Section 6.3, the current $(\nabla f)^* \psi\left(TX_g, \nabla^{TX_g}\right)$ on M_g is well defined, and is such that

$$d(\nabla f)^* \psi\left(TX_g, \nabla^{TX_g}\right) = e\left(TX_g, \nabla^{TX_g}\right) - \delta_{\mathbf{B}_g}. \tag{0.15}$$

For $y \in \mathbf{R}, s \in \mathbf{C}, \mathrm{Re}(s) > 1$, set

$$\zeta(y,s) = \sum_{n=1}^{+\infty} \frac{\cos(ny)}{n^s}, \qquad \eta(y,s) = \sum_{n=1}^{+\infty} \frac{\sin(ny)}{n^s}. \tag{0.16}$$

Then $\zeta(y,s)$ and $\eta(y,s)$ are the real and imaginary parts of the Lerch zeta function $L(y,s)$ [**Le**]. The function $\eta(y,s)$ is holomorphic in the variable $s \in \mathbf{C}$, the function $\zeta(y,s)$ is also holomorphic in $s \in \mathbf{C}$ if $y \notin 2\pi\mathbf{Z}$, and is meromorphic with a simple pole at $s = 1$ if $y \in 2\pi\mathbf{Z}$. Let $I(\theta, x)$ be the formal power series,

$$I(\theta,x) = \frac{1}{2}\left[\sum_{\substack{p\in\mathbf{N}\\ p\,\text{even}}} \frac{(2p+1)!}{(p!)^3}\frac{\partial\zeta}{\partial s}(\theta,-p)\left(\frac{x}{4}\right)^p + i\sum_{\substack{p\in\mathbf{N}\\ p\,\text{odd}}} \frac{(2p+1)!}{(p!)^3}\frac{\partial\eta}{\partial s}(\theta,-p)\left(\frac{x}{4}\right)^p\right]. \tag{0.17}$$

Put

$${}^0I(\theta,x) = I(\theta,x) - I(0,0). \tag{0.18}$$

Given $\theta \in \mathbf{R}$, we identify ${}^0I(\theta,x)$ to the corresponding additive genus. Also, if $g \in G$ acts as a parallel automorphism on the fibres of a vector bundle E, let ${}^0I_g(E)$ be the even cohomology class which is obtained by splitting E according to the angles θ of the action of g on E, and by summing the corresponding ${}^0I(\theta,x)$ genera.

Let $TX|^s_{\mathbf{B}}, TX|^u_{\mathbf{B}}$ be the stable and unstable subbundles of $TX|_{\mathbf{B}}$ with respect to Y. Then we have the splitting,

$$TX|_{\mathbf{B}} = TX|^s_{\mathbf{B}} \oplus TX|^u_{\mathbf{B}}. \tag{0.19}$$

Let o^u be the orientation bundle of $TX|^u_{\mathbf{B}}$.

Clearly g acts on $TX|_{\mathbf{B}_g}$ and preserves the splitting (0.19). Let $I_g\left(TX|_{\mathbf{B}_g}\right)$ be the genus ${}^0I_g$ evaluated on the $\mathbf{Z}_2$-graded vector bundle $TX|_{\mathbf{B}_g}$, i.e.

$$ {}^0I_g\left(TX|_{\mathbf{B}_g}\right) = {}^0I_g\left(TX|^s_{\mathbf{B}_g}\right) - {}^0I_g\left(TX|^u_{\mathbf{B}_g}\right). \tag{0.20}$$

Finally, if $x \in \mathbf{B}_g$, let $\mathrm{Tr}^{F_x\otimes o^u_x}[g]$ be the trace of the action of g on $F_x \otimes o^u_x$.

The main result of this paper, proved in Theorem 7.2, is as follows.

Theorem 0.1. — *For any $g \in G$, the following identity holds,*

$$\begin{aligned} &\mathcal{T}_{h,g}\left(T^HM, g^{TX}, \nabla^F, g^F\right) - T_{h,g}\left(A^{C^\bullet(W^u,F)\prime}, g^{C^\bullet(W^u,F)}\right) \\ &\qquad + \widetilde{h}_g\left(\nabla^{H^\bullet(X,F|_X)}, g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F)}, g^{H^\bullet(X,F|_X)}_{L_2}\right) \\ &= -\int_{X_g} h_g\left(\nabla^F, g^F\right)(\nabla f)^*\,\psi\left(TX_g, \nabla^{TX_g}\right) \\ &\qquad + \sum_{x\in B_g} (-1)^{\mathrm{ind}(x)}\,\mathrm{Tr}^{F_x\otimes o^u_x}[g]\,{}^0I_g\left(T_xX|_{\mathbf{B}_g}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \end{aligned} \tag{0.21}$$

In Chapter 7, we show that Theorem 0.1 is compatible with all the known properties of the equivariant analytic torsion forms, including anomaly formulas, rigidity and products. Using Theorem 0.1 and Poincaré duality, we also derive certain properties of $TX|_{\mathbf{B}}$. Also, as explained in Chapter 7, in degree 0, Theorem 0.1 is equivalent to the main result of [**BZ2**], which in turn extends results by Cheeger [**C**], Müller [**Mü1, Mü2**], Lott and Rothenberg [**LoRo**] and Bismut-Zhang [**BZ1, BZ2**].

Put

$$J(\theta,x) = \frac{1}{2}\left[\sum_{\substack{p\in\mathbf{N}\\ p\,\text{even}}} \frac{\partial\zeta}{\partial s}(\theta,-p)\frac{x^p}{p!} + i\sum_{\substack{p\in\mathbf{N}\\ p\,\text{odd}}} \frac{\partial\eta}{\partial s}(\theta,-p)\frac{x^p}{p!}\right]. \tag{0.22}$$

Set

$${}^0J(\theta,x) = J(\theta,x) - J(0,0)). \tag{0.23}$$

In Theorem 7.4, we show that if the forms $\mathcal{T}_{h,g}\left(T^HM, g^{TX}, \nabla^F, g^F\right)$ are replaced by the Chern analytic torsion forms $\mathcal{T}_{\mathrm{ch},g}\left(T^HM, g^{TX}, \nabla^F, g^F\right)$, then the obvious analogue of Theorem 0.1 remains true, with ${}^0I(\theta,x)$ replaced by ${}^0J(\theta,x)$.

Theorem 0.1 is only a first step to the evaluation of the analytic torsion forms in full generality. As shown in Chapter 5, our fibrations are such that the fundamental group of the base S acts as a finite group on the cohomology $H^\bullet(X,\mathbf{Z})$. By passing to a finite normal cover of S, this action can then be made trivial. This puts a severe restriction of the fibrations to which our formula applies. Fibrations by torus bundles associated to general $\mathrm{SL}(n,\mathbf{Z})$ vector bundles do not verify our assumptions. In particular the evaluation by Bismut and Lott [**BLo2**] of the analytic torsion forms of such torus fibrations cannot be obtained from our main result.

However, as we shall see in Chapter 16, there is a variant of Theorem 0.1, in which f is only assumed to be a fibrewise Morse-Bott function. In Chapter 16, we establish this more general version of Theorem 0.1 in the context of unit sphere bundles. This formula is used to evaluate the torsion forms of these sphere bundles. We now explain in more detail the main formula of Chapter 16.

Let E be a real vector bundle of dimension $n+1 \geqslant 2$ on a manifold S. Let g^E be an Euclidean metric on E, and let ∇^E be an Euclidean connection. Assume that G fixes S and acts on E by unitary automorphisms which preserve ∇^E. Let S^E be the unit sphere bundle of $\left(E, g^E\right)$. Let $\mathcal{E}$ be the total space of S^E. If $g \in G$, we can then define analytic torsion forms associated to the projection $\pi : \mathcal{E} \to S$, which are closed and whose cohomology class $\mathcal{T}_{h,g}(\mathcal{E})$ does not depend on the choices of g^E and ∇^E. If $g \in G$, let $\det(g) = \pm 1$ be the determinant of g acting on the fibres of E. The following result is established in Theorem 16.1, in part as a consequence of Theorem 0.1.

Theorem 0.2. — *For any $g \in G$, the following identity holds,*

$$\mathcal{T}_{h,g}(\mathcal{E}) = \left(1 - (-1)^n \det(g)\right)\left({}^0I_g(E) - \frac{1}{2}\log\left(\frac{2\pi^{(n+1)/2}}{\Gamma\left((n+1)/2\right)}\right)\right) \quad \text{in } H^{\text{even}}(S, \mathbf{C}). \tag{0.24}$$

Theorem 0.2 was already obtained by Bunke [**Bu1**] in the case where $g = 1$. As explained in Remark 16.3, from Theorem (0.2), we recover the evaluation by Bismut and Lott [**BLo1**, Corollary 4.14] of the analytic torsion forms associated associated with a circle bundle equipped with a complex line bundle, with holonomy around the circle given by a root of unity.

The analogue of Theorem 0.1 for the Chern analytic torsion forms suggests a possible link with Arakelov theory. In fact let us recall that in [**B8**], Bismut introduced a genus $R(\theta, x)$, extending the $R(x)$ genus of Gillet and Soulé [**GS1**], given by the formula,

$$\begin{aligned} R(\theta,x) = \sum_{\substack{p \geqslant 0 \\ p \text{ even}}} i\left\{\sum_{j=1}^{p}\frac{1}{j}\eta(\theta,-p) + 2\frac{\partial\eta}{\partial s}(\theta,-p)\right\}\frac{x^p}{p!} \\ + \sum_{\substack{p \geqslant 0 \\ p \text{ odd}}}\left\{\sum_{j=1}^{p}\frac{1}{j}\zeta(\theta,-p) + 2\frac{\partial\zeta}{\partial s}(\theta,-p)\right\}\frac{x^p}{p!}, \end{aligned} \tag{0.25}$$

so that $R(x) = R(x, 0)$. In [**B9**], it was shown that the genus $R(\theta, x)$ appears as a defect in an immersion formula for equivariant Quillen metrics, extending the main result of [**BL**]. In [**KöRoe**], Köhler and Roessler extended the Riemann-Roch formula in Arakelov geometry of Gillet and Soulé [**GS2**] to an equivariant situation. Recall

that $L(y,s)$ is the Lerch zeta function. In Proposition 4.40, we give the obvious identity,

$$R(\theta,x)+4J(\theta,x)=\sum_{p\in\mathbf{N}}\Big(\sum_{j=1}^{p}\frac{1}{j}L(\theta,-p)+2\frac{\partial L}{\partial s}(\theta,-p)\Big)\frac{x^p}{p!}. \tag{0.26}$$

As explained in Proposition 4.41, the factor 4 in the left-hand side of (0.26) is 'natural'. This result suggests a possible mysterious connection of analytic torsion forms in de Rham theory with their holomorphic counterpart [**BGS1, BK**] in Arakelov theory.

Now we briefly describe the techniques which are used in the proofs of the above results. The idea is to combine the methods used by Bismut-Zhang [**BZ1, BZ2**] in their proof of Theorem 0.1 in degree 0 with the formalism of Bismut-Lott [**BLo1**]. The proofs also bear some similarity with the proofs of Bismut in [**B10**], where an analogue of the above problem was considered in the context of the holomorphic analytic torsion forms of [**BGS1, BK**].

Let us also indicate that in [**BGo4, BGo5**], we have constructed an infinitesimal version of the equivariant torsion, which is essentially a version of the Chern analytic torsion forms $\mathcal{T}_{\mathrm{ch},g}\left(T^HM,g^{TX},\nabla^F,g^F\right)$ in the case where the fibration $\pi:M\to S$ comes from a G-principal fibre bundle, and we have also given a local formula relating the classical equivariant analytic torsion to its infinitesimal version. The results of [**BGo4, BGo5**] demonstrate that $\mathcal{T}_{\mathrm{ch},g}\left(T^HM,g^{TX},\nabla^F,g^F\right)$ gives indeed the 'right' normalization of the analytic torsion forms. Besides, we show that these results are compatible to the results we obtain in the present paper, and also with results of Bunke [**Bu2**].

Let us now describe in more detail the techniques used in the present paper.

1. *Superconnections and Chern-Simons theory.*— In [**BLo1**], Quillen's superconnections [**Q1**] are the key tool to the proof of the Riemann-Roch-Grothendieck formula (0.3) and to the construction of the analytic torsion forms. In Chapters 1 and 2 of this paper, we make the link with Chern-Simons theory more explicit than in [**BLo1**]. By making this link, we obtain a natural construction of the Chern analytic torsion forms, and also a direct understanding of the rigidity of the analytic torsion forms.

2. *An extended de Rham map.*— In [**La**], Laudenbach proved that under standard assumption on $Y=-\nabla f$, the stable and unstable cells of Y can be compactified into submanifolds with C^1 conical singularities. In particular, it is possible to integrate smooth differential forms on these cells. Laudenbach proved that one gets this way a de Rham map $P^\infty:\Omega^\bullet\big(X,F|_X\big)\to C^\bullet\left(W^u,F\right)$, which is a quasi-isomorphism. In Section 5.6, we extend this result to the situation considered above. Namely, we show that there is an integration along the fibre map $\mathbf{P}^\infty:\Omega^\bullet(M,F)\to\Omega^\bullet\left(S,C^\bullet\left(W^u,F\right)\right)$, which maps the de Rham operator d^M on M into the canonical flat superconnection $A^{C^\bullet(W^u,F)\prime}$ of $C^\bullet\left(W^u,F\right)$.

3. *The Witten complex and the instanton calculus.*— In [**BZ1, BZ2**], a key idea is to use the Witten deformation of the de Rham operator [**W**], i.e. for $T \in \mathbf{R}$, to replace d^X by $e^{-Tf} d^X e^{Tf}$, and make $T \to +\infty$. Besides the 0 eigenvalue, which corresponds to the cohomology $H^\bullet(X, F|_X)$, a finite number of nonzero eigenvalues of the Laplacian decay exponentially as $T \to +\infty$. In [**BZ1**], results of Helffer-Sjöstrand [**HSj**] were needed to obtain a precise estimate of the 'small' eigenvalues. In [**BZ2**, Section 6], the results of [**BZ1**] were given a much simpler proof by using the results of Laudenbach [**La**] on the Thom-Smale complex $C^\bullet(W^u, F)$.

In the present paper, we follow essentially the same strategy as in [**BZ1, BZ2**]. However, we need not only to estimate the 'size' of the small eigenvalues, but also subtle properties connected in particular with the variation of the corresponding eigenspaces. This is done by two distinct methods developed in Chapters 10 and 11, which both lead to the same results. In Chapter 10, we use a theory of 'generalized' metrics, and in Chapter 11, we show how to use the eigenvalue estimates or [**BZ2**] to obtain the required result, when the function f is fibrewise nice.

4. *Local families index theory and Berezin integrals.*— In [**BLo1**], the main results were proved using the local families index Theorem of [**B3**]. Here, we use the local families index techniques developed in Berline-Getzler-Vergne [**BeGeV**]. In [**BZ1, BZ2**], two copies of the bundle of Clifford algebras of (TX, g^{TX}) appeared naturally, and the considered operators exhibited a symmetry property when these two copies were interchanged, which made the local index computations rather easy. Ultimately, the Berezin integrals of Mathai-Quillen appeared in the local index computations. Here, this symmetry property is broken. A more sophisticated calculus to handle Berezin integrals is needed, which is developed in Chapter 6.

5. *Finite propagation speed and localization.* — As in [**BL, B9, BGo1**], finite propagation speed of solutions of hyperbolic equations [**ChP, T**] plays a key role in the proofs that certain estimates can be made local. In fact in our estimates, the question often arises of showing that these estimates can be proved 'locally'. Finite propagation speed is one of the tools one can use to prove that such a localization of the estimates is indeed possible.

This paper is organized as follows. In Chapter 1, we recall the formalism of [**BLo1**] on flat superconnections, and we construct the equivariant generalization of the torsion forms of finite dimensional complexes of [**BLo1**]. In Chapter 2, we prove rigidity results in positive degree for such torsion forms, and we construct their proper normalization, the Chern torsion forms. In Chapter 3, we prove the equivariant extension of the Riemann-Roch-Grothendieck theorem of [**BLo1**] for flat vector bundles, and we construct corresponding equivariant analytic torsion forms. We prove that in a suitable sense, these forms are rigid under deformation of the flat connection on the given vector bundle F, and we also construct their Chern normalization.

In Chapter 4, we construct the analytic torsion forms of a $\mathbf{Z}_2$-graded vector bundle E. The explicit computation of these torsion forms is of fundamental interest, since it produces the genus ${}^0I(\theta, x)$. In Chapter 5, we describe the main properties of the family of complexes $C^\bullet(W^u, F)$, and we describe the de Rham map of Laudenbach [**La**]. In Chapter 6, we prove various properties of Berezin integrals in our geometric setting, and we recall the construction of the Mathai-Quillen currents.

In Chapter 7, we check that Theorem 0.1 is compatible with known results on analytic torsion forms. Chapters 8-15 are devoted to the proof of Theorem 0.1. The general organization of the proof is closely related to the proof of corresponding results in [**BZ1, BZ2**]. In Chapter 8, using a contour integral, we prove a basic identity, depending on three parameters ε, A, T_0. Theorem 0.1 will be obtained by taking adequate limits in this identity. In Chapter 9, we state, without proof, a number of intermediate results, from which Theorem 0.1 is then derived. Chapters 10 and 11 provide two different proofs of one of these results, Chapters 12–15 give proofs of the other intermediate results. The proofs involve various kinds of localization on the fixed point fibre X_g or the fixed critical points B_g. Finally, in Chapter 16, we establish Theorem 0.2.

The results contained in this paper were announced in [**BGo2, BGo3**].

Acknowledgments. — The authors are very much indebted to François Laudenbach for useful discussions. They are also grateful to a referee, for his detailed observations and comments.

CHAPTER 1

FLAT SUPERCONNECTIONS AND EQUIVARIANT TORSION FORMS

The purpose of this Chapter is to explain the superconnection formalism, and also to state the main results of Bismut-Lott [**BLo1**] on flat superconnections and torsion forms. The only minor difference is that we work in an equivariant context. However, as explained later, the objects which we consider are just linear combinations of objects considered by Bismut and Lott.

This Chapter is organized as follows. In Section 1.1, we recall the definition by Quillen [**Q1**] of superconnections on a $\mathbf{Z}_2$-graded vector bundle E. In Section 1.2, we construct the transpose of a superconnection, and in Section 1.3, the adjoint of a superconnection with respect to a metric . In Section 1.4, we define the action of a Lie group G on superconnections. In Section 1.5, we consider flat superconnections, and we construct the associated odd closed forms of [**BLo1**] in an equivariant context. In Section 1.6, we consider superconnections which have total degree 1 on a $\mathbf{Z}$-graded vector bundle. In Section 1.7, we define a canonical rescaling of the given Hermitian metric with respect to a parameter $t > 0$. In Section 1.8, we evaluate the limit as $t \to +\infty$ of the odd forms associated to the rescaled metric. Finally, in Sections 1.9 and 1.10, we introduce two versions of equivariant torsion forms, and we establish corresponding anomaly formulas.

1.1. The superconnection formalism

Here we follow Quillen [**Q1**]. Let M be a smooth manifold. Let $E = E_+ \oplus E_-$ be a $\mathbf{Z}_2$-graded complex vector bundle on M. Let $\tau = \pm 1$ on $E_\pm$ be the involution of E which defines the $\mathbf{Z}_2$-grading. Then $\mathrm{End}\,(E)$ is a $\mathbf{Z}_2$-graded bundle of algebras, whose even (resp. odd) elements commute (resp. anticommute) with τ. If $A \in \mathrm{End}\,(E)$, we define its supertrace $\mathrm{Tr_s}\,[A]$ by the formula,

$$\mathrm{Tr_s}\,[A] = \mathrm{Tr}\,[\tau A]\,. \tag{1.1}$$

Let $\Lambda^\bullet(T^*M)$ be the complexified exterior algebra of T^*M. We extend $\mathrm{Tr_s}$ to a linear map $\Lambda^\bullet(T^*M) \widehat{\otimes} \mathrm{End}\,(E) \to \mathbf{C}$, so that, if $\omega \in \Lambda^\bullet(T^*M), A \in \mathrm{End}\,(E)$,

$$\mathrm{Tr_s}\,[\omega A] = \omega \mathrm{Tr_s}\,[A]\,. \tag{1.2}$$

If $\mathcal{A}$ is any $\mathbf{Z}_2$-graded algebra, if $\alpha, \alpha' \in \mathcal{A}$, the supercommutator $[\alpha, \alpha']$ is defined by the formula,

$$[\alpha, \alpha'] = \alpha\alpha' - (-1)^{\deg(\alpha)\deg(\alpha')}\alpha'\alpha. \tag{1.3}$$

In the whole paper, [ ] is our notation for the supercommutator. Then by [**Q1**], the supertrace of supercommutators in $\Lambda^\bullet(T^*M) \widehat{\otimes} \operatorname{End}(E)$ vanishes.

Observe that $\Lambda^\bullet(T^*M)\widehat{\otimes}E$ is a $\Lambda^\bullet(T^*M)$ module. By definition, a superconnection A is an odd differential operator acting on $C^\infty\left(M, \Lambda^\bullet(T^*M) \widehat{\otimes} E\right)$, which verifies a Leibnitz rule. Namely, if $\omega \in C^\infty(M, \Lambda^\bullet(T^*M))$, $s \in C^\infty\left(M, \Lambda^\bullet(T^*M) \widehat{\otimes} E\right)$, then

$$A(\omega s) = d\omega s + (-1)^{\deg(\omega)}\omega A s. \tag{1.4}$$

If $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ is any connection on E which preserves E_+ and E_-, then there is $S \in C^\infty\left(M, \left(\Lambda^\bullet(T^*M) \widehat{\otimes} \operatorname{End}(E)\right)^{\text{odd}}\right)$ such that

$$A = \nabla^E + S, \tag{1.5}$$

and conversely any object of the form (1.5) is a superconnection.

The curvature of a superconnection A is its square A^2. The curvature A^2 is a smooth section of $\left(\Lambda^\bullet(T^*M) \widehat{\otimes} \operatorname{End}(E)\right)^{\text{even}}$.

A superconnection is said to be flat if $A^2 = 0$.

1.2. The transpose of a superconnection

Here, we follow [**BLo1**, Section 1 (c)]. Let $\overline{E}^* = \overline{E}^*_+ \oplus \overline{E}^*_-$ be the antidual bundle of $E = E_+ \oplus E_-$. Let * be the even antilinear map from $\Lambda^\bullet(T^*M) \widehat{\otimes} \operatorname{End}(E)$ into $\Lambda^\bullet(T^*M) \widehat{\otimes} \operatorname{End}\left(\overline{E}^*\right)$ which is defined by the following relations:

- If $\alpha, \alpha' \in \Lambda^\bullet(T^*M) \widehat{\otimes} \operatorname{End}(E)$, then
$$\overline{(\alpha\alpha')}^* = \overline{\alpha}'^*\overline{\alpha}^*.$$
- If $\omega \in T^*M \otimes_{\mathbf{R}} \mathbf{C}$,
$$\overline{\omega}^* = -\overline{\omega}.$$
- If $B \in \operatorname{End}(E)$, $\overline{B}^*$ is the obvious transpose of B.

Given a superconnection A on E, we write A as in (1.5). Let $\nabla^{\overline{E}^*}$ be the connection on $\overline{E}^*$ induced by ∇^E. Then $\nabla^{\overline{E}^*}$ preserves the splitting $\overline{E}^* = \overline{E}^*_+ \oplus \overline{E}^*_-$.

Definition 1.1. — The transpose of the superconnection A is the superconnection $\overline{A}^*$ given by

$$\overline{A}^* = \nabla^{\overline{E}^*} + \overline{S}^*. \tag{1.6}$$

One verifies easily that (1.6) does not depend on the splitting (1.5).

1.3. The adjoint of a superconnection

We follow [**BLo1**, Section 1 (d)]. Let now $g^E = g^{E_+} \oplus g^{E_-}$ be a Hermitian metric on $E = E_+ \oplus E_-$, such that E_+ and E_- are mutually orthogonal in E. Then g^E induces a linear even isomorphism $E \to \overline{E}^*$.

Definition 1.2. — The adjoint A^* of the superconnection A is the superconnection on E given by

$$A^* = \left(g^E\right)^{-1} \overline{A}^* g^E. \tag{1.7}$$

Also note that $S \in \Lambda^\bullet(T^*M) \widehat{\otimes} \operatorname{End}(E)$, we define S^* by the formula,

$$S^* = \left(g^E\right)^{-1} \overline{S}^* g^E. \tag{1.8}$$

Remark 1.3. — An important example of the above situation is the case where $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ is a connection on $E = E_+ \oplus E_-$. Let $\omega\left(\nabla^E, g^E\right)$ be the 1-form on M with values in even self-adjoint sections of $\operatorname{End}(E)$,

$$\omega\left(\nabla^E, g^E\right) = \left(g^E\right)^{-1} \nabla^E g^E. \tag{1.9}$$

Then if $\nabla^{E,*}$ is the connection on E which is the adjoint of ∇^E,

$$\nabla^{E,*} = \nabla^E + \omega\left(\nabla^E, g^E\right). \tag{1.10}$$

Set

$$\nabla^{E,u} = \frac{1}{2}\left(\nabla^E + \nabla^{E,*}\right). \tag{1.11}$$

By (1.10) and (1.11),

$$\nabla^{E,u} = \nabla^E + \frac{1}{2}\omega\left(\nabla^E, g^E\right). \tag{1.12}$$

Then $\nabla^{E,u}$ is a unitary connection on E.

1.4. Superconnections and group actions

We make the same assumptions as in Sections 1.1-1.3. Let G be a compact Lie group. We assume that G acts fibrewise on the vector bundle E over M by even automorphisms of E. We define a right action of G on superconnections, so that if A is a superconnection and $g \in G$,

$$A \cdot g = g^{-1} A g. \tag{1.13}$$

Then

$$(A \cdot g)^2 = g^{-1} A^2 g. \tag{1.14}$$

In particular, G preserves flat superconnections.

Clearly, if A is a superconnection, we can write A in the form,

$$A = \sum_{j=0}^{\dim M} A^{(j)}, \tag{1.15}$$

where $A^{(0)}$ is a smooth section of $\mathrm{End}(E)^{\mathrm{odd}}$, $A^{(1)}$ is a connection on E which preserves the splitting $E = E_+ \oplus E_-$, and for $j \geqslant 2$, $A^{(j)}$ is a smooth section of $\left(\Lambda^j(T^*M) \widehat{\otimes} \mathrm{End}(E)\right)^{\mathrm{odd}}$. By (1.14), (1.15), we find that if A is G-invariant, the components $A^{(j)}$ are themselves G-invariant. In particular, the connection $A^{(1)}$ is G-invariant.

Remark 1.4. — Take $g \in G$. Let $e^{i\theta_j}, 0 \leqslant \theta_j < 2\pi$ be the distinct eigenvalues of the action of g on a given fibre E_x. The above shows that these eigenvalues are locally constant with respect to $x \in M$, so that the vector bundle E splits into a direct sum of $\mathbf{Z}_2$-graded eigenbundles,

$$E = \bigoplus_{j=1}^{q} E^{e^{i\theta_j}}. \tag{1.16}$$

The superconnection A then splits as a direct sum of superconnections $A^{e^{i\theta_j}}$ on the $E^{e^{i\theta_j}}$'s.

1.5. Flat superconnections, Hermitian metrics and odd closed forms

Let $g^E = g^{E_+} \oplus g^{E_-}$ be a G-invariant Hermitian metric on $E = E_+ \oplus E_-$. Let A' be a G-invariant flat superconnection on E. Let A'' be the adjoint of A' with respect to g^E. Then A'' is also a G-invariant flat superconnection on E. Put

$$A = \frac{1}{2}(A'' + A'), \qquad B = \frac{1}{2}(A'' - A'). \tag{1.17}$$

Then A is a G-invariant superconnection on E, and B is a smooth G-invariant section of $\left(\Lambda^\bullet(T^*M) \widehat{\otimes} \mathrm{End}(E)\right)^{\mathrm{odd}}$, such that

$$B^* = -B. \tag{1.18}$$

The following trivial relations are taken from [**BLo1**, Proposition 1.2].

Proposition 1.5. — *The following identities hold,*

$$\begin{aligned} &B^2 = -A^2, \quad [A, B] = 0, \\ &\left[A', B^2\right] = 0, \quad \left[A'', B^2\right] = 0, \quad \left[A, B^2\right] = 0. \end{aligned} \tag{1.19}$$

Now we have the result of [**BLo1**, Proposition 1.3].

Proposition 1.6. — *Let f be a holomorphic function. For any $g \in G$,*

$$\mathrm{Tr}_s\left[gf\left(B^2\right)\right] = \mathrm{Tr}_s[g] f(0). \tag{1.20}$$

Proof. — Take $t \in \mathbf{R}$. Using the fact that $\mathrm{Tr_s}$ vanishes on supercommutators [**BeGeV**, page 40], we get

$$\frac{\partial}{\partial t}\mathrm{Tr_s}\left[gf\left(tB^2\right)\right] = \mathrm{Tr_s}\left[gB^2 f'\left(tB^2\right)\right] = \frac{1}{2}\mathrm{Tr_s}\left[\left[B, gBf'\left(tB^2\right)\right]\right] = 0. \tag{1.21}$$

By (1.21), we get (1.20). □

We will say that a holomorphic function $h : \mathbf{C} \to \mathbf{C}$ is real if for any $x \in \mathbf{C}$, $h\left(\overline{x}\right) = \overline{h\left(x\right)}$. We fix a square root $i^{1/2}$ of i. Our formulas will not depend on the choice of the square root. Let $\varphi : \Lambda^\bullet(T^*M) \to \Lambda^\bullet(T^*M)$ be given by

$$\varphi\omega = \left(2i\pi\right)^{-\deg(\omega)/2}\omega. \tag{1.22}$$

In the sequel, we fix $g \in G$.

Definition 1.7. — If $h\left(x\right)$ is a holomorphic odd function, put

$$h_g\left(A', g^E\right) = \left(2i\pi\right)^{1/2}\varphi\mathrm{Tr_s}\left[gh\left(B\right)\right]. \tag{1.23}$$

If we use the notation in (1.16), we get

$$h_g\left(A', g^E\right) = \sum_{j=1}^{q} e^{i\theta_j} h_1\left(A'^{e^{i\theta_j}}, g^{E^{e^{i\theta_j}}}\right). \tag{1.24}$$

Therefore, as explained in the Introduction, $h_g\left(A', g^E\right)$ is a linear combination of objects already considered in [**BLo1**].

The following result was established in [**BLo1**, Theorems 1.8 and 1.11].

Theorem 1.8. — *The form $h_g\left(A', g^E\right)$ is odd and closed, and it is real if h is real and $g = 1$. Its cohomology class, denoted by $h_g\left(A'\right)$, does not depend on g^E.*

Proof. — Using Proposition 1.5, and the G-invariance of A, we find that

$$[A, B] = 0, \qquad [A, g] = 0. \tag{1.25}$$

Using (1.25) and the fact that supertraces vanish on supercommutators, we find that the form $h_g\left(A', g^E\right)$ is closed. By functoriality, it follows that its cohomology class does not depend on g^E. If h is real and $g = 1$, by [**BLo1**, Theorem 1.8], the form $h_g\left(A', g^E\right)$ is real. The proof of our Theorem is completed. □

Definition 1.9. — Let $\Omega^\bullet(M)$ be the space of smooth complex differential forms on M, let $d\Omega^\bullet(M) \subset \Omega^\bullet(M)$ be the subspace of exact smooth differential forms.

Let $\ell \in [0,1] \to g_\ell^E$ be a smooth family of Hermitian metrics on E taken as before. We denote by A_ℓ, B_ℓ the objects associated to g_ℓ^E which we defined in (1.17) .

Definition 1.10. — Put

$$\widetilde{h}_g\left(A', g_\ell^E\right) = \int_0^1 \varphi\mathrm{Tr_s}\left[g\frac{1}{2}\left(g_\ell^E\right)^{-1}\frac{\partial g_\ell^E}{\partial \ell}h'\left(B_\ell\right)\right]d\ell. \tag{1.26}$$

Now we state a result established in [**BLo1**, Theorems 1.9 and 1.11].

Theorem 1.11. — *The class of the form* $\widetilde{h}_g\left(A', g_\ell^E\right)$ *in* $\Omega^\bullet(M)/d\Omega^\bullet(M)$ *only depends on* g_0^E *and* g_1^E. *Moreover,*

$$d\widetilde{h}_g\left(A', g_\ell^E\right) = h_g\left(A', g_1^E\right) - h_g\left(A', g_0^E\right). \tag{1.27}$$

If h *is real and* $g = 1$, *the form* $\widetilde{h}_g\left(A', g_\ell^E\right)$ *is real.*

Proof. — We lift E to $M \times [0,1]$. Then the flat superconnection A' lifts to a flat superconnection $\widetilde{A}'$. On $M \times \{\ell\}$, we equip E with the metric g_ℓ^E. Therefore, on $M \times [0,1]$, E is equipped with a metric $\widetilde{g}^E$. One verifies easily that

$$h_g\left(\widetilde{A}', \widetilde{g}^E\right) = h_g\left(A', g_\ell^E\right) + d\ell\varphi \mathrm{Tr}_{\mathrm{s}}\left[g\frac{1}{2}\left(g_\ell^E\right)^{-1}\frac{\partial g_\ell^E}{\partial \ell}h'\left(B_\ell\right)\right]. \tag{1.28}$$

Our Theorem is a consequence of Theorem 1.8 and of (1.28). □

We will denote by $\widetilde{h}_g\left(A', g_0^E, g_1^E\right)$ the class of $\widetilde{h}_g\left(A', g_\ell^E\right)$ in $\Omega^\bullet(M)/d\Omega^\bullet(M)$.

Remark 1.12. — As in Remark 1.3, take a connection $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$, and assume that ∇^E is flat. We use the notation in (1.9)-(1.12). Then $\nabla^{E,u}$ is a unitary connection on E, which by (1.10), is exactly the connection A in (1.17) associated to $A' = \nabla^E$. Moreover if B is defined as in (1.17),

$$B = \frac{1}{2}\omega\left(\nabla^E, g^E\right). \tag{1.29}$$

Then from (1.19), we find in particular that

$$\nabla^{E,u,2} = -\frac{1}{4}\omega^2\left(\nabla^E, g^E\right), \qquad \nabla^{E,u}\omega\left(\nabla^F, g^F\right) = 0. \tag{1.30}$$

1.6. Superconnections of total degree 1

Let $E = \bigoplus_{i=0}^m E^i$ be a $\mathbf{Z}$-graded complex vector bundle on M. Set

$$E_+ = \bigoplus_{i \text{ even}} E^i, \qquad E_- = \bigoplus_{i \text{ odd}} E^i. \tag{1.31}$$

Then $E = E_+ \oplus E_-$ is a $\mathbf{Z}_2$-graded vector bundle.

Let A' be a superconnection on $E = E_+ \oplus E_-$. As in (1.15), we write A' in the form,

$$A' = \sum_{j=0}^{\dim M} A'^{(j)}, \tag{1.32}$$

where $A'^{(j)}$ is of partial degree j in the Grassmann variables in $\Lambda^\bullet(T^*M)$.

Definition 1.13. — We say that A' is of total degree 1 (resp. -1) if $A'^{(1)}$ is a connection on E which preserves the grading, and if for $j \neq 1$, $A'^{(j)}$ is a section of $\Lambda^j\left(T^*M\right) \widehat{\otimes} \mathrm{Hom}\left(E^\bullet, E^{\bullet+1-j}\right)$ (resp. $\Lambda^j\left(T^*M\right) \widehat{\otimes} \mathrm{Hom}\left(E^\bullet, E^{\bullet-1+j}\right)$).

In what follows we assume that A' is a flat superconnection of total degree 1. Put

$$v = A'^{(0)}, \qquad \nabla^E = A'^{(1)}. \tag{1.33}$$

Then v is a section of $\mathrm{Hom}\,(E^\bullet, E^{\bullet+1})$, and ∇^E is a connection on E which preserves the grading.

The following statement was established in [**BLo1**, Proposition 2.2].

Proposition 1.14. — *We have the identities,*

$$v^2 = 0, \qquad \left[\nabla^E, v\right] = 0, \qquad \nabla^{E,2} + \left[v, A'^{(2)}\right] = 0. \tag{1.34}$$

Proof. — This follows from the identity,

$$A'^2 = 0. \tag{1.35}$$

□

Definition 1.15. — Given $x \in M$, let $H^\bullet(E,v)_x = \bigoplus_{i=0}^m H^i(E,v)_x$ be the cohomology of the complex $(E,v)_x$.

By Proposition 1.14, since v is parallel with respect to ∇^E, there is a complex $\mathbf{Z}$-graded vector bundle $H^\bullet(E,v)$ whose fibres are the $H^\bullet(E,v)_x$. Also by [**BLo1**, Definition 2.4 and Proposition 2.5], the connection ∇^E induces on $H^\bullet(E,v)$ a connection $\nabla^{H^\bullet(E,v)}$, which is flat. This result is in fact a trivial consequence of Proposition 1.14.

Let G be a compact Lie group acting fibrewise on E and preserving the $\mathbf{Z}$-grading. Assume that A' is G-invariant. Then the $A'^{(j)}$ are also G-invariant. It follows that G acts naturally on $H^\bullet(E,v)$. One verifies easily that this action is parallel with respect to $\nabla^{H^\bullet(E,v)}$.

Let now $g^E = \bigoplus_{i=0}^m g^{E^i}$ be a G-invariant Hermitian metric on $E = \bigoplus_{i=0}^m E^i$, such that the E^i's are mutually orthogonal in E. Let v^* be the adjoint of v with respect to g^E. Put

$$V = \frac{1}{2}\left(v^* - v\right). \tag{1.36}$$

It follows from finite dimensional Hodge theory that for any $x \in M$, we have a canonical isomorphism,

$$H^\bullet(E,v)_x \simeq \ker V_x. \tag{1.37}$$

As a $\mathbf{Z}$-graded subbundle of E, $\ker V$ inherits a G-invariant Hermitian metric from the metric g^E. Let $g^{H^\bullet(E,v)}$ be the corresponding G-invariant metric on $H^\bullet(E,v)$. Let $P^{\ker V}$ be the orthogonal projection operator from E on $\ker V$.

Recall that $\omega\left(\nabla^E, g^E\right)$ was defined in (1.9). Also $\omega\left(\nabla^{H^\bullet(E,v)}, g^{H^\bullet(E,v)}\right)$ is given by a similar formula.

Then by [**BLo1**, Proposition 2.6],

$$\begin{aligned}
&\nabla^{H^\bullet(E,v)} = P^{\ker V}\nabla^E,\\
&\nabla^{H^\bullet(E,v),*} = P^{\ker V}\nabla^{E,*},\\
&\omega\left(\nabla^{H^\bullet(E,v)}, g^{H^\bullet(E,v)}\right) = P^{\ker V}\omega\left(\nabla^E, g^E\right)P^{\ker V}.
\end{aligned} \tag{1.38}$$

The unitary connections $\nabla^{E,u}$ and $\nabla^{H^\bullet(E,v),u}$ are defined as in (1.9). It follows from (1.38) that

$$\nabla^{H^\bullet(E,v),u} = P^{\ker V}\nabla^{E,u}. \tag{1.39}$$

1.7. A rescaled metric

We make the same assumptions as in Section 1.6. Let N be the number operator acting on E, i.e. N acts on E^k by multiplication by k.

Definition 1.16. — For $t > 0$, let g_t^E be the metric on E,

$$g_t^E = t^N g^E. \tag{1.40}$$

Let A_t'' be the adjoint of A' with respect to g_t^E. Clearly $A'' = A_1''$. Also,

$$A_t'' = t^{-N}A''t^N. \tag{1.41}$$

We define A_t, B_t as in (1.17), i.e.

$$A_t = \frac{1}{2}\left(A_t'' + A'\right), \qquad B_t = \frac{1}{2}\left(A_t'' - A'\right). \tag{1.42}$$

Take $g \in G$. As in Definition 1.7, set

$$h_g\left(A', g_t^E\right) = (2i\pi)^{1/2}\varphi\mathrm{Tr_s}\left[gh\left(B_t\right)\right]. \tag{1.43}$$

Definition 1.17. — Set

$$h_g^\wedge\left(A', g_t^E\right) = \varphi\mathrm{Tr_s}\left[\frac{N}{2}gh'\left(B_t\right)\right]. \tag{1.44}$$

The following result was established in [**BLo1**, Theorem 2.9].

Theorem 1.18. — *The form $h_g^\wedge\left(A', g_t^E\right)$ is even. It is real if h is real and $g = 1$. Moreover,*

$$\frac{\partial}{\partial t}h_g\left(A', g_t^E\right) = d\frac{h_g^\wedge\left(A', g_t^E\right)}{t}. \tag{1.45}$$

Definition 1.19. — For $t > 0$, set

$$C_t' = t^{N/2}A't^{-N/2}, \qquad C_t'' = t^{-N/2}A''t^{N/2}. \tag{1.46}$$

Then C'_t is a flat superconnection of total degree 1, and C''_t is its adjoint with respect to the metric g^E. Set

$$C_t = \frac{1}{2}\left(C''_t + C'_t\right), \qquad D_t = \frac{1}{2}\left(C''_t - C'_t\right). \tag{1.47}$$

By (1.41), (1.46), we get

$$C_t = t^{N/2} A_t t^{-N/2}, \qquad D_t = t^{N/2} B_t t^{-N/2}. \tag{1.48}$$

From (1.48), we deduce that

$$h_g\left(A', g_t^E\right) = h_g\left(C'_t, g^E\right), \qquad h_g^\wedge\left(A', g_t^E\right) = h_g^\wedge\left(C'_t, g^E\right). \tag{1.49}$$

If $a \in \mathbf{R}_+$, let $\psi_a : \Lambda^\bullet(T^*M) \to \Lambda^\bullet(T^*M)$ be given by

$$\psi_a \omega = a^{\deg\omega/2}\omega. \tag{1.50}$$

Recall that $A = A_1, B = B_1$.

Proposition 1.20. — *For $t > 0$, the following identities hold,*

$$C_t = \psi_t^{-1}\sqrt{t}\, A \psi_t, \qquad D_t = \psi_t^{-1}\sqrt{t}\, B \psi_t. \tag{1.51}$$

Proof. — We use the notation in (1.32). By (1.46), since A' is of total degree 1, and A'' is of total degree -1,

$$C'_t = \sum_{j=0}^{\dim M} t^{(1-j)/2} A'^{(j)}, \qquad C''_t = \sum_{j=0}^{\dim M} t^{(1-j)/2} A''^{(j)}. \tag{1.52}$$

From (1.52), we get (1.51). □

Proposition 1.21. — *For $t > 0$, the following identities hold,*

$$\begin{aligned} h_g\left(A', g_t^E\right) &= (2i\pi)^{1/2}\varphi\psi_t^{-1}\mathrm{Tr_s}\left[gh\left(\sqrt{t}\, B\right)\right], \\ h_g^\wedge\left(A', g_t^E\right) &= \varphi\psi_t^{-1}\mathrm{Tr_s}\left[\frac{N}{2} g h'\left(\sqrt{t}\, B\right)\right]. \end{aligned} \tag{1.53}$$

Proof. — This follows from (1.49) and from Proposition 1.20. □

1.8. The limit of $h_g\left(A', g_t^E\right)$ and of $h^\wedge\left(A', g_t^E\right)$ as $t \to +\infty$

We make the same assumptions as in Sections 1.6 and 1.7. Let $h(x)$ be a holomorphic odd function. We assume there is $c > 0$ such that for any $k \in \mathbf{N}$, there exists $C_k > 0$ such that

$$\sup_{\substack{x \in \mathbf{C} \\ |\mathrm{Re}\, x| \leqslant c}} (1 + |x|)^k \, |h(x)| \leqslant C_k. \tag{1.54}$$

Let $(\alpha_t)_{t \in \mathbf{R}_+^* \cup \{+\infty\}}$ be smooth forms on M. We will write that as $t \to +\infty$,

$$\alpha_t = \alpha_{+\infty} + \mathcal{O}\left(1/\sqrt{t}\right), \tag{1.55}$$

if for any compact set $K \subset M$ and any $k \in \mathbf{N}$, there is $C > 0$ such that the supremum of the norms of $\alpha_t - \alpha_{+\infty}$ and its derivatives of order $\leqslant k$ over K is bounded by $C/\sqrt{t}$.

If $x \in M, g \in G$, set

$$\begin{aligned} \chi_g(E_x) &= \sum_{j=0}^{m} (-1)^j \operatorname{Tr}^{E_x^j}[g], \\ \chi'_g(E_x) &= \sum_{j=0}^{m} (-1)^j j \operatorname{Tr}^{H^j(E,v)_x}[g], \\ \widetilde{\chi}'_g(E_x) &= \sum_{j=0}^{m} (-1)^j j \operatorname{Tr}^{E^j}[g]. \end{aligned} \tag{1.56}$$

Since the action of g on E is parallel, the functions of $x \in M$ in (1.56) are locally constant. Classically, $\chi_g(E)$ is the equivariant Euler characteristic of the complex (E, v), i.e.

$$\chi_g(E) = \sum_{j=0}^{m} (-1)^j \operatorname{Tr}^{H^j(E,v)}[g]. \tag{1.57}$$

By definition,

(1.58)

$$h_g\left(\nabla^{H^\bullet(E,v)}, g^{H^\bullet(E,v)}\right) = \sum_{j=1}^{m} (-1)^j (2i\pi)^{1/2} \varphi \operatorname{Tr}\left[gh\left(\frac{1}{2}\omega\left(\nabla^{H^j(E,v)}, g^{H^j(E,v)}\right)\right)\right].$$

The following result was established in [**BLo1**, Theorem 2.13].

Theorem 1.22. — *As* $t \to +\infty$,

$$\begin{aligned} h_g\left(A', g_t^E\right) &= h_g\left(\nabla^{H^\bullet(E,v)}, g^{H^\bullet(E,v)}\right) + \mathcal{O}\left(1/\sqrt{t}\right), \\ h_g^{\wedge}\left(A', g_t^E\right) &= \frac{1}{2} h'(0) \chi'_g(E) + \mathcal{O}\left(1/\sqrt{t}\right). \end{aligned} \tag{1.59}$$

Remark 1.23. — From Theorems 1.8 and 1.22, we deduce that

$$h_g(A') = h_g\left(\nabla^{H^\bullet(E,v)}\right) \text{ in } H^{\mathrm{odd}}(M, \mathbf{C}), \tag{1.60}$$

which is just [**BLo1**, Theorem 2.14].

1.9. The form $S_{h,g}\left(A', g^E\right)$

We make the same assumptions as in Sections 1.6-1.8. We still assume that the holomorphic odd function $h(x)$ is such that (1.54) holds. Also, we use the notation in (1.44).

Definition 1.24. — Set

$$S_{h,g}\left(A', g^E\right) = -\int_1^{+\infty} \left(h_g^{\wedge}\left(A', g_t^E\right) - \frac{1}{2} h'(0) \chi'_g(E)\right) \frac{dt}{t}. \tag{1.61}$$

The following result was established in [**BLo1**, Theorem 2.16].

Theorem 1.25. — *The form $S_{h,g}\left(A', g^E\right)$ is even, and real if h is real and $g = 1$. The following identity holds,*

$$dS_{h,g}\left(A', g^E\right) = h_g\left(A', g^E\right) - h_g\left(\nabla^{H^\bullet(E,v)}, g^{H^\bullet(E,v)}\right). \tag{1.62}$$

Proof. — This follows from Theorems 1.18 and 1.22. □

Let g_0^E and g_1^E be two G-invariant Hermitian metrics which are taken as before. Let $g_0^{H^\bullet(E,v)}, g_1^{H^\bullet(E,v)}$ be the corresponding metrics on $H^\bullet(E, v)$. The following result was proved in [**BLo1**, Theorem 2.17].

Theorem 1.26. — *The following identity holds,*

$$\begin{aligned} &S_{h,g}\left(A', g_1^E\right) - S_{h,g}\left(A', g_0^E\right) \\ &= \widetilde{h}_g\left(A', g_0^E, g_1^E\right) - \widetilde{h}_g\left(\nabla^{H^\bullet(E,v)}, g_0^{H^\bullet(E,v)}, g_1^{H^\bullet(E,v)}\right) \text{ in } \Omega^\bullet(M)/d\Omega^\bullet(M). \end{aligned} \tag{1.63}$$

Proof. — This is an easy consequence of Theorem 1.25 and of the functoriality of the forms $S_{h,g}\left(A', g^E\right)$. □

1.10. Flat complexes of vector bundles and their torsion forms

Let

$$(E, v) : 0 \to E^0 \xrightarrow{v} E^1 \xrightarrow{v} \cdots \xrightarrow{v} E^m \to 0 \tag{1.64}$$

be a flat complex of vector bundles. By definition, $E = \bigoplus E^i$ is equipped with a flat connection $\nabla^E = \bigoplus_{i=0}^m \nabla^{E^i}$, and v is parallel with respect to ∇^E, i.e.

$$\nabla^E v = 0. \tag{1.65}$$

Put

$$A' = v + \nabla^E. \tag{1.66}$$

Then A' is a flat superconnection on E of total degree 1.

Let G be a compact Lie group action fibrewise on E by automorphisms which preserve the $\mathbf{Z}$-grading, the chain map v and the connection ∇^E. Then the superconnection A' is G-invariant. Let $g^E = \bigoplus_{i=0}^m g^{E^i}$ be a G-invariant Hermitian metric on $E = \bigoplus_{i=0}^m E^i$. Then, by using the notation in Sections 1.6-1.8,

$$A''_t = tv^* + \nabla^{E,*}, \qquad C'_t = \sqrt{t}\, v + \nabla^E, \qquad C''_t = \sqrt{t}\, v^* + \nabla^{E,*}. \tag{1.67}$$

Let h be a holomorphic odd function such that 1.54 holds. Now we have the easy result established in [**BLo1**, Proposition 2.18].

Proposition 1.27. — *As* $t \to 0$,

$$h_g\left(A', g_t^E\right) = h_g\left(\nabla^E, g^E\right) + \mathcal{O}(t), \tag{1.68}$$
$$h_g^{\wedge}\left(A', g_t^E\right) = \frac{1}{2}\widetilde{\chi}_g'(E) + \mathcal{O}(t).$$

Remark 1.28. — From Theorem 1.8 and from Proposition 1.27, we find that

$$h_g\left(\nabla^E\right) = h_g\left(\nabla^{H^\bullet(E,v)}\right) \text{ in } \Omega^\bullet(M)/d\Omega^\bullet(M). \tag{1.69}$$

Definition 1.29. — Put

$$T_{h,g}\left(A', g^E\right) = -\int_0^{+\infty}\left[h_g^{\wedge}\left(A', g_t^E\right) - \frac{1}{2}\chi_g'(E)\, h'(0)\right. \tag{1.70}$$
$$\left. - \frac{1}{2}\left(\widetilde{\chi}_g'(E) - \chi_g'(E)\right) h'\left(i\sqrt{t}/2\right)\right]\frac{dt}{t}.$$

The form $T_{h,g}\left(A', g^E\right)$ is called an equivariant torsion form. The following result was established in [**BLo1**, Theorem 2.22].

Theorem 1.30. — *The form* $T_{h,g}\left(A', g^E\right)$ *is even, and real if* h *is real and* $g = 1$. *The following identity holds,*

$$dT_{h,g}\left(A', g^E\right) = h_g\left(\nabla^E, g^E\right) - h_g\left(\nabla^{H^\bullet(E,v)}, g^{H^\bullet(E,v)}\right). \tag{1.71}$$

Proof. — Equation (1.71) follows from Theorems 1.18, 1.22 and from Proposition 1.27. □

Let g_0^E, g_1^E be two G-invariant Hermitian metrics taken as before. Recall that the classes of forms $\widetilde{h}_g\left(\nabla^E, g_0^E, g_1^E\right), \widetilde{h}_g\left(\nabla^{H^\bullet(E,v)}, g_0^{H^\bullet(E,v)}, g_1^{H^\bullet(E,v)}\right) \in \Omega^\bullet(M)/d\Omega^\bullet(M)$ were defined in Definition 1.10. The following result was established in [**BLo1**, Theorem 2.24].

Theorem 1.31. — *The following identity holds,*

$$T_{h,g}\left(A', g_1^E\right) - T_{h,g}\left(A', g_0^E\right) = \tag{1.72}$$
$$\widetilde{h}_g\left(\nabla^E, g_0^E, g_1^E\right) - \widetilde{h}_g\left(\nabla^{H^\bullet(E,v)}, g_0^{H^\bullet(E,v)}, g_1^{H^\bullet(E,v)}\right) \text{ in } \Omega^\bullet(M)/d\Omega^\bullet(M).$$

Proof. — Our Theorem is an easy consequence of Theorem 1.30 and of the functoriality of $T_{h,g}\left(A', g^E\right)$. □

1.11. Functorial characterization of the torsion forms

We use the same notation as in Section 1.10.

Assume now that the complex (E, v) is acyclic, i.e. $H^\bullet(E, v) = \{0\}$. By following [**BLo1**, Appendix I], we will say that the complex $\left((E, v), g^E\right)$ splits if there are flat vector bundles $F^0, \ldots, F^{m-1}$ and corresponding Hermitian metrics $g^{F^0}, \ldots, g^{F^{m-1}}$ such that we have the identification of Hermitian flat vector bundles,

$$E^i = F^{i-1} \oplus F^i, \ 0 \leqslant i \leqslant m, \tag{1.73}$$

and moreover $v : E^i \to E^{i+1}$ is just the identity map $F^i \to F^i$ and vanishes on F^{i-1}. Then we state [**BLo1**, Theorem A1.1].

Theorem 1.32. — *The following identity holds,*

$$dT_{h,g}\left(A', g^E\right) = h_g\left(\nabla^E, g^E\right). \tag{1.74}$$

If M' is another manifold and $\alpha : M' \to M$ is a smooth map, then

$$T_{h,g}\left(\alpha^* A', \alpha^* g^E\right) = \alpha^* T_{h,g}\left(A', g^E\right). \tag{1.75}$$

If $\left((E, v), g^E\right)$ splits, then

$$T_{h,g}\left(A', g^E\right) = 0. \tag{1.76}$$

Finally $T_{h,g}\left(A', g^E\right)$ depends smoothly on A', g^E.

Proof. — Our Theorem follows in particular from Theorem 1.30. □

Now we state a result established in [**BLo1**, Theorem A1.2].

Theorem 1.33. — *Given a manifold M, let $T'_{h,g}\left(A', g^E\right)$ be an even form on M verifying the four conditions in Theorem 1.32. Then*

$$T'_{h,g}\left(A', g^E\right) = T_{h,g}\left(A', g^E\right) \text{ in } \Omega^\bullet(M)/d\Omega^\bullet(M). \tag{1.77}$$

Remark 1.34. — One can easily extend the above characterization of $T_{h,g}\left(A', g^E\right)$ to the case where (E, v) is not acyclic.

CHAPTER 2

RIGIDITY OF TORSION FORMS AND THEIR CHERN NORMALIZATION

One purpose of this Chapter is to prove rigidity of the torsion forms $T_{h,g}\left(A', g^E\right)$ which were introduced in [**BLo1**] and in Section 1.1. In fact in [**BLo1**, Theorem 2.24] and in Theorem 1.31, we showed that these forms verify anomaly formulas when we deform the metric g^E. Here we will show that in degree $\geqslant 2$, there are analogous anomaly formulas when we deform the flat superconnection A'.

The second purpose of this Chapter is to produce a 'natural' normalization of the torsion forms, the Chern torsion forms.

This Chapter is organized as follows. In Section 2.1, we prove that in degree $\geqslant 3$, the class $h_g\left(A'\right)$ is rigid under deformation of the superconnection A', and we produce explicit transgression formulas for the corresponding forms $h_g\left(A', g^E\right)$. In Section 2.2, when k is a holomorphic odd function, we give a residue formula for $k\left(D_t\right)$. In Section 2.3, we establish a convergence result on certain forms as $t \to +\infty$. In Section 2.4, we prove that the forms $S_{h,g}\left(A', g^E\right)$ verify anomaly formulas in degree $\geqslant 2$ when A' varies, and in Section 2.5 we prove the corresponding result for the torsion forms $T_{h,g}\left(A', g^E\right)$. In Section 2.6, by following [**BLo1**, Section 1 (g)] we construct other odd forms in the Chern-Simons formalism. In Section 2.7, we relate these Chern-Simons forms to the forms $h_g\left(A', g^E\right)$. In particular, we show how the transgression formulas can be obtained in the Chern-Simons formalism. Also we obtain associated odd Chern character forms $\mathrm{ch}_g^{\circ}\left(A', g^E\right)$. In Section 2.8, we construct the Chern torsion forms. In Section 2.9, we extend the construction of the torsion forms, when we replace standard metrics g^E by so called generalized metrics $\mathbf{g}^E$. Such a construction will be needed in Chapter 10. Finally, in Section 2.10, we consider generalized metrics on flat complexes.

2.1. Rigidity properties of the superconnection odd classes

Here, we use the notation of Sections 1.1-1.5. In particular, M is a smooth manifold, and $E = E_+ \oplus E_-$ is a complex $\mathbf{Z}_2$-graded vector bundle on M. Also G is a compact

Lie group acting fibrewise on E by even automorphisms. Let $g^E = g^{E_+} \oplus g^{E_-}$ be a G-invariant Hermitian metric on $E = E_+ \oplus E_-$, which is such that that E_+ and E_- are orthogonal in E.

Let $\mathcal{M}$ be a smooth manifold of G-invariant flat superconnections A' on the $\mathbf{Z}_2$-graded vector bundle E. We will consider the manifold $M \times \mathcal{M}$. We denote by $d^M, d^{\mathcal{M}}$ the de Rham operators on $M \times \mathcal{M}$, so that the total de Rham operator is d is given by $d^M + d^{\mathcal{M}}$.

We still denote by $\left(E, g^E\right)$ the pull-back of $\left(E, g^E\right)$ to $M \times \mathcal{M}$.

If $A' \in \mathcal{M}$, let A'' be the adjoint of A' with respect to the metric g^E. As in (1.17), set

$$A = \frac{1}{2}\left(A'' + A'\right), \qquad B = \frac{1}{2}\left(A'' - A'\right). \tag{2.1}$$

Let $h(x)$ be a holomorphic odd function. We assume that $\deg h \geqslant 3$. Put

$$k(x) = \frac{h'(x)}{2x}. \tag{2.2}$$

Recall that $\varphi : \Lambda^\bullet(T^*M) \to \Lambda^\bullet(T^*M)$ was defined in (1.22). We define $\varphi : \Lambda\left(T^*\left(M \times \mathcal{M}\right)\right) \to \Lambda\left(T^*\left(M \times \mathcal{M}\right)\right)$ as in (1.22). More generally, in the sequel, we will use the same notation φ on any manifold.

Also the forms $h_g\left(A', g^E\right)$ on M were defined in Definition 1.7.

Theorem 2.1. — *The form $\sqrt{2i\pi}\,\varphi \mathrm{Tr}_{\mathrm{s}}\left[gk(B)d^{\mathcal{M}}A\right]$ on $M \times \mathcal{M}$ is odd. It is real if h is real and $g = 1$. Moreover we have the identity of forms on $M \times \mathcal{M}$,*

$$d^{\mathcal{M}} h_g\left(A', g^E\right) = d^M \sqrt{2i\pi}\,\varphi\ \mathrm{Tr}_{\mathrm{s}}\left[gk(B)d^{\mathcal{M}}A\right]. \tag{2.3}$$

Proof. — Clearly,

$$\left(d^{\mathcal{M}}A\right)^* = d^{\mathcal{M}}A. \tag{2.4}$$

Since $B^* = -B$, and k is an odd function,

$$\left(k(B)d^{\mathcal{M}}A\right)^* = -d^{\mathcal{M}}Ak(B). \tag{2.5}$$

From (2.4)-(2.5), we conclude that if h is real, the form $\sqrt{2i\pi}\,\varphi \mathrm{Tr}_{\mathrm{s}}\left[k(B)d^{\mathcal{M}}A\right]$ is real as in [**BLo1**, Theorem 1.8]. Using the fact that supertraces vanish on supercommutators, we get

$$\begin{aligned} d^{\mathcal{M}}\mathrm{Tr}_{\mathrm{s}}\left[gh(B)\right] = \mathrm{Tr}_{\mathrm{s}}\left[gh'(B)d^{\mathcal{M}}B\right] &= 2\mathrm{Tr}_{\mathrm{s}}\left[gk(B)Bd^{\mathcal{M}}B\right] \\ &= \mathrm{Tr}_{\mathrm{s}}\left[gk(B)[B, d^{\mathcal{M}}B]\right]. \end{aligned} \tag{2.6}$$

Since $A^2 = -B^2$,

$$[A, d^{\mathcal{M}}A] = -[B, d^{\mathcal{M}}B]. \tag{2.7}$$

Since by Proposition 1.5, $[A, B] = 0$, using (2.7), we get

$$d^M \mathrm{Tr}_{\mathrm{s}}\left[gk(B)d^{\mathcal{M}}A\right] = -\mathrm{Tr}_{\mathrm{s}}\left[gk(B)[A, d^{\mathcal{M}}A]\right] = \mathrm{Tr}_{\mathrm{s}}\left[gk(B)[B, d^{\mathcal{M}}B]\right]. \tag{2.8}$$

By (2.6), (2.8), we get (2.3). The proof of our Theorem is completed. □

Take $m \in \mathbf{N}$, m odd, such that $m \geqslant 3$.

Definition 2.2. — Put

$$\alpha_{m,g} = \sqrt{2i\pi}\,\varphi \mathrm{Tr}_s\left[gB^m d^{\mathcal{M}} A\right],$$

$$\beta_{m,g} = \frac{1}{2}\sqrt{2i\pi}\,\varphi \sum_{0\leqslant j\leqslant (m-3)/2} \mathrm{Tr}_s\Big[gB^j d^{\mathcal{M}} A B^{m-2-j} d^{\mathcal{M}} A$$

$$+ (-1)^j\, gB^j d^{\mathcal{M}} B B^{m-2-j} d^{\mathcal{M}} B\Big] \tag{2.9}$$

$$+ \frac{1}{4}\sqrt{2i\pi}\,\varphi \mathrm{Tr}_s\Big[gB^{(m-1)/2} d^{\mathcal{M}} A B^{(m-3)/2} d^{\mathcal{M}} A$$

$$+ (-1)^{(m-1)/2}\, gB^{(m-1)/2} d^{\mathcal{M}} B B^{(m-3)/2} d^{\mathcal{M}} B\Big].$$

Theorem 2.3. — *The forms $\alpha_{m,g}$ and $\beta_{m,g}$ on $M \times \mathcal{M}$ are odd. They are real if $g = 1$. Moreover,*

$$d^{\mathcal{M}} \alpha_{m,g} = d^M \beta_{m,g}. \tag{2.10}$$

Proof. — The proof that, if $g = 1$, $\alpha_{m,g}$ and $\beta_{m,g}$ are real is the same as in Theorem 2.1. Clearly,

$$d^{\mathcal{M}} \mathrm{Tr}_s\left[gB^m d^{\mathcal{M}} A\right] = \sum_{0\leqslant j\leqslant m-1} (-1)^j\, \mathrm{Tr}_s\left[gB^j d^{\mathcal{M}} B B^{m-1-j} d^{\mathcal{M}} A\right]. \tag{2.11}$$

Using (1.19), (2.7), we get

$$\begin{aligned} &d^M \mathrm{Tr}_s\left[gB^j d^{\mathcal{M}} A B^{m-2-j} d^{\mathcal{M}} A\right] \\ &= (-1)^{j+1}\, \mathrm{Tr}_s\left[gB^j [B, d^{\mathcal{M}} B] B^{m-2-j} d^{\mathcal{M}} A\right] + \mathrm{Tr}_s\left[gB^j d^{\mathcal{M}} A B^{m-2-j} [B, d^{\mathcal{M}} B]\right] \\ &= (-1)^j \Big\{ \mathrm{Tr}_s\left[gB^j d^{\mathcal{M}} B B^{m-1-j} d^{\mathcal{M}} A\right] - \mathrm{Tr}_s\left[gB^{j+1} d^{\mathcal{M}} B B^{m-2-j} d^{\mathcal{M}} A\right] \\ &\qquad + \mathrm{Tr}_s\left[gB^{m-1-j} d^{\mathcal{M}} B B^j d^{\mathcal{M}} A\right] - \mathrm{Tr}_s\left[gB^{m-2-j} d^{\mathcal{M}} B B^{j+1} d^{\mathcal{M}} A\right] \Big\}. \end{aligned} \tag{2.12}$$

Since $[A, B] = 0$,

$$[d^{\mathcal{M}} A, B] = [A, d^{\mathcal{M}} B]. \tag{2.13}$$

Using (2.13), we get

$$\begin{aligned} &d^M \mathrm{Tr}_s\left[gB^j d^{\mathcal{M}} B B^{m-2-j} d^{\mathcal{M}} B\right] \\ &= (-1)^j\, \mathrm{Tr}_s\left[gB^j [d^{\mathcal{M}} A, B] B^{m-2-j} d^{\mathcal{M}} B\right] - \mathrm{Tr}_s\left[gB^j d^{\mathcal{M}} B B^{m-2-j} [d^{\mathcal{M}} A, B]\right] \\ &= \mathrm{Tr}_s\left[gB^j d^{\mathcal{M}} B B^{m-1-j} d^{\mathcal{M}} A\right] + \mathrm{Tr}_s\left[gB^{j+1} d^{\mathcal{M}} B B^{m-2-j} d^{\mathcal{M}} A\right] \\ &\qquad + \mathrm{Tr}_s\left[gB^{m-1-j} d^{\mathcal{M}} B B^j d^{\mathcal{M}} A\right] + \mathrm{Tr}_s\left[gB^{m-2-j} d^{\mathcal{M}} B B^{j+1} d^{\mathcal{M}} A\right]. \end{aligned} \tag{2.14}$$

From (2.11)-(2.14), we get (2.10). The proof of our Theorem is completed. □

The operator $\varphi : \Lambda^\bullet(T^*M) \to \Lambda^\bullet(T^*M)$ is now defined as in (1.22). Let k be a holomorphic odd function. Let $\ell \in [0,1] \to A'_\ell$ be a smooth one parameter family of G-invariant flat superconnections.

Definition 2.4. — Let $L_{k,g}\left(A'_\ell, g^E\right)$ be the form on M,

$$L_{k,g}\left(A'_\ell, g^E\right) = \int_0^1 \varphi \mathrm{Tr_s}\left[gk(B_\ell)\frac{\partial A_\ell}{\partial \ell}\right] d\ell. \tag{2.15}$$

Observe that the form $L_{k,g}\left(A'_\ell, g^E\right)$ depends explicitly on the path $\ell \in [0,1] \to A'_\ell$.

Theorem 2.5. — *The form $L_{k,g}(A'_\ell, g^E)$ is even, and real if k is real and $g = 1$. If $\deg k \geqslant 3$, given A'_0, A'_1, the class of the form $L_{k,g}\left(A'_\ell, g^E\right)$ in $\Omega^\bullet(M)/d\Omega^\bullet(M)$ depends only on the homotopy class of the path $\ell \to A'_\ell$. If h is a holomorphic odd function such that $\deg h \geqslant 3$, if $k(x) = h'(x)/2x$, then*

$$d^M L_{k,g}\left(A'_\ell, g^E\right) = h_g\left(A'_1, g^E\right) - h_g(A'_0, g^E)\,. \tag{2.16}$$

Proof. — The first part of our Theorem is a trivial consequence of Theorem 2.3. Equation (2.16) follows from Theorem 2.1. □

Remark 2.6. — Theorem 2.5 indicates that if $\deg h \geqslant 3$, the cohomology classes $h_g(A')$ are rigid, i.e. they are invariant under deformations of A'. This result is well-known [**CSi**, Proposition 2.9] for the odd Chern classes associated to flat vector bundles. This rigidity result is not true in degree 1. Also under the assumptions of the second part of Theorem 2.5, equation (2.16) refines on the rigidity result at the level of differential forms. If $\deg h \geqslant 5$, the first part of the Theorem indicates that the class of the transgression form $L_{k,g}\left(A'_\ell, g^E\right)$ in $\Omega^\bullet(M)/d\Omega^\bullet(M)$ is itself canonical, i.e. it is rigid under deformation of the path $\ell \to A'_\ell$. This class should be thought of as an analogue of the Bott-Chern classes in complex geometry [**BGS1**], since it is obtained by a double transgression formula.

Remark 2.7. — One verifies easily that in degree $\geqslant 3$, (1.27) follows from (2.16).

2.2. An expression for $k(D_t)$

Now, we make the same assumptions and we use the same notation as in Sections 1.6-1.9. In particular $E = \bigoplus_{i=0}^m E^i$ is a $\mathbf{Z}$-graded vector bundle on M, G is a compact Lie group which acts fibrewise by $\mathbf{Z}$-graded automorphisms of E, A' is a flat G-invariant superconnection on E of total degree 1, and $g^E = \bigoplus_{i=0}^m g^{E^i}$ is a Hermitian metric on E such that the E^i's are mutually orthogonal in E.

With the notation in (1.36),

$$B^{(0)} = V. \tag{2.17}$$

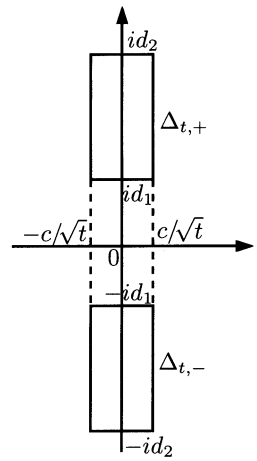

FIGURE 2.1

If $x \in M, A \in \left(\Lambda^\bullet(T^*M) \widehat{\otimes} \operatorname{End}(E)\right)_x$, let $\operatorname{Sp}(A_x) \subset \mathbf{C}$ be the spectrum of A. Clearly,

$$\operatorname{Sp}(B) = \operatorname{Sp}\left(B^{(0)}\right). \tag{2.18}$$

Also the spectrum of $B^{(0)}$ is purely imaginary. Moreover, by (1.37),

$$\ker B^{(0)} \simeq H^\bullet(E, v). \tag{2.19}$$

Let $k(x)$ be a holomorphic odd function. We assume there is $c > 0$ such that for any $p \in \mathbf{N}$, there exists $C_p > 0$ such that

$$\sup_{\substack{x \in \mathbf{C} \\ |\operatorname{Re} x| \leqslant c}} (1 + |x|)^p \, |k(x)| \leqslant C_p. \tag{2.20}$$

To make our arguments simpler, we will temporarily assume that M is compact.

Since $\ker B^{(0)}$ has constant rank, there exists $d_1, d_2 \in \mathbf{R}_+^*$, with $d_1 < d_2/4$, such that

$$\left|\operatorname{Sp}\left(B^{(0)}\right)\right| \subset \{0\} \cup [2d_1, d_2/2]. \tag{2.21}$$

Let $\delta \subset \mathbf{C}$ be the circle of centre 0 and radius 1. For $t > 0$, let $\Delta_t = \Delta_{t,+} \cup \Delta_{t,-}$ be the contour in $\mathbf{C}$ indicated in Figure 2.1.

Definition 2.8. — For $t > 0$, put

$$G_t = \psi_t^{-1} \frac{1}{2i\pi} \int_{\Delta_t} \frac{k(\sqrt{t}\,\lambda)}{\lambda - B} d\lambda \psi_t, \qquad H_t = \psi_t^{-1} \frac{1}{2i\pi} \int_{\frac{d_1}{2}\delta} \frac{k(\sqrt{t}\,\lambda)}{\lambda - B} d\lambda \psi_t. \tag{2.22}$$

Proposition 2.9. — *For any* $t > 0$,

$$k(D_t) = G_t + H_t. \tag{2.23}$$

Proof. — By (2.18), (2.21), the spectrum of B is included in the domain bounded by $\Delta_t \cup \frac{d_1}{2}\delta$. Using Proposition 1.20, (2.20) and the theorem of residues, we get (2.23). □

Proposition 2.10. — *Given* $p \in \mathbf{N}$, *there exists* $C > 0$ *such that for* $t \geqslant 1$,

$$|G_t| \leqslant \frac{C}{t^p}. \tag{2.24}$$

Proof. — Using (2.20), we find that given $p' \in \mathbf{N}$ there exists $C > 0$ such that if $t \geqslant 1, \lambda \in \Delta_t$,

$$\left|k\left(\sqrt{t}\,\lambda\right)\right| \leqslant \frac{C}{t^{p'}\,|\lambda|^{p'}}. \tag{2.25}$$

Moreover, if $\lambda \in \Delta_t$, we have the expansion

$$(\lambda - B)^{-1} = \left(\lambda - B^{(0)}\right)^{-1} + \left(\lambda - B^{(0)}\right)^{-1} B^{(>0)} \left(\lambda - B^{(0)}\right)^{-1} + \cdots \tag{2.26}$$

and this expansion only contains a finite number of terms. By (2.26), we find that there exists $C > 0, q' \in \mathbf{N}$ such that if $t \geqslant 1, \lambda \in \Delta_t$,

$$\left|(\lambda - B)^{-1}\right| \leqslant C\sqrt{t}^{\,q'}. \tag{2.27}$$

By (2.25), (2.27), there exist $c > 0, C > 0$ such that for $t \geqslant 1$,

$$\left|\frac{1}{2i\pi}\int_{\Delta_t} \frac{k\left(\sqrt{t}\,\lambda\right)}{\lambda - B} d\lambda\right| \leqslant \frac{C}{t^p}. \tag{2.28}$$

Our Proposition follows from (2.28). □

Let $P^{\{0\}}$ be the orthogonal projection operator from E on $\ker B^{(0)} \simeq H^\bullet(E, v)$. Set

$$P^{\{0\}^\perp} = 1 - P^{\{0\}}. \tag{2.29}$$

Then $P^{\{0\}^\perp}$ is the orthogonal projection operator, which projects on the orthogonal bundle $\left(\ker B^{(0)}\right)^\perp$ to $\ker B^{(0)}$ in E. Also $B^{(0)}$ acts as an invertible operator on $\left(\ker B^{(0)}\right)^\perp$. Let $\left(B^{(0)}\right)^{-1}$ denote its inverse. We extend $\left(B^{(0)}\right)^{-1}$ to an operator which acts like 0 on $\ker B^{(0)}$. Now we proceed as in [**B10**, Theorem 9.29]. Let $D_t^{(\geqslant 1)}$ be the component of D_t which has partial degree $\geqslant 1$ in $\Lambda^\bullet(T^*M)$. Recall that H_t was defined in (2.22).

Theorem 2.11. — *Given $t > 0$, the following identity holds,*

$$H_t = \sum_{p=0}^{\dim M} \sum_{\substack{0 \leqslant i_0 \leqslant p+1 \\ j_1, \ldots, j_{p+1-i_0} \geqslant 0 \\ \sum_{m=1}^{p+1-i_0} j_m \leqslant i_0 - 1}} \frac{k^{\left(i_0 - 1 - \sum_{m=0}^{p+1-i_0} j_m\right)}(0)}{\left(i_0 - 1 - \sum_{m=0}^{p+1-i_0} j_m\right)!} (-1)^{p+1-i_0} C_1 D_t^{(\geqslant 1)} C_2 \cdots D_t^{(\geqslant 1)} C_{p+1}. \tag{2.30}$$

In (2.30), i_0 of the C_j are equal to $P^{\{0\}}$, and the other C_j are equal respectively to $\left(\sqrt{t}\, B^{(0)}\right)^{-(1+j_1)}, \ldots, \left(\sqrt{t}\, B^{(0)}\right)^{-(1+j_{p+1-i_0})}$. In particular, H_t is a polynomial in the variable $1/\sqrt{t}$.

As $t \to +\infty$,

$$H_t = P^{\{0\}} k\left(B^{H^\bullet(E,v)}\right) P^{\{0\}} + \mathcal{O}\left(1/\sqrt{t}\right). \tag{2.31}$$

Proof. — Using (2.18), (2.21), we find that for $t \geqslant 1$,

$$\frac{1}{2i\pi} \int_{(d_1/2)\delta} \frac{k\left(\sqrt{t}\,\lambda\right)}{\lambda - B} d\lambda = \frac{1}{2i\pi} \int_{\left(d_1/2\sqrt{t}\right)\delta} \frac{k\left(\sqrt{t}\,\lambda\right)}{\lambda - B} d\lambda = \frac{1}{2i\pi} \int_{(d_1/2)\delta} \frac{k(\lambda)}{\lambda - \sqrt{t}\, B} d\lambda. \tag{2.32}$$

By Proposition 1.20 and by (2.32), for $t \geqslant 1$,

$$H_t = \frac{1}{2i\pi} \int_{(d_1/2)\delta} \frac{k(\lambda)}{\lambda - D_t} d\lambda. \tag{2.33}$$

Now we have the expansion

$$(\lambda - D_t)^{-1} = \left(\lambda - \sqrt{t}\, B^{(0)}\right)^{-1} + \left(\lambda - \sqrt{t}\, B^{(0)}\right)^{-1} D_t^{(\geqslant 1)} \left(\lambda - \sqrt{t}\, B^{(0)}\right)^{-1} + \cdots, \tag{2.34}$$

and the expansion in (2.34) only contains a finite number of terms. By (2.21), 0 is the only element inside the domain bounded by $(d_1/2)\,\delta$ which may lie in the spectrum of $B^{(0)}$. Using (2.33), (2.34) and the theorem of residues, we get (2.30). By (1.38),

$$\omega\left(\nabla^{H^\bullet(E,v)}, g^{H^\bullet(E,v)}\right) = P^{\{0\}} \omega\left(\nabla^E, g^E\right) P^{\{0\}}. \tag{2.35}$$

By (2.30), (2.35), we get (2.31) for $t \geqslant 1$. The general case follows using analyticity. The proof of our Theorem is completed. □

2.3. A convergence result

We make the same assumptions as in Section 2.2. Let $\ell \in [0,1] \to A'_\ell$ be a smooth family of G-invariant flat superconnections on E which have total degree 1. Given $t \in \mathbf{R}_+^*$, we use the notation in Sections 1.6 and 1.7. In particular for $t > 0$, the objects

constructed in (1.42) which are associated to A'_ℓ will be denoted with the subscript ℓ, t. Let $k(x)$ be a holomorphic odd function. Recall that $\psi_t : \Lambda^\bullet(T^*M) \to \Lambda^\bullet(T^*M)$ was defined in (1.50).

Proposition 2.12. — *For $t > 0$, set $k^t(x) = \sqrt{t}k(\sqrt{t}x)$. Then*

$$L_{k,g}\left(A'_\ell, g_t^E\right) = L_{k,g}\left(C'_{\ell,t}, g^E\right), \tag{2.36}$$
$$L_{k^t,g}\left(A'_\ell, g^E\right) = \psi_t L_{k,g}(A'_\ell, g_t^E).$$

Proof. — By (1.46), (1.48), we get the first identity in (2.36). Using Proposition 1.20, we then get the second identity in (2.36). □

Let $h(x)$ be a holomorphic odd function such that $\deg h \geqslant 3$. We still define $k(x)$ as in (2.2). Recall that by Theorem 2.1, if h is real and $g = 1$, the forms $\varphi \mathrm{Tr_s}\left[gk(B_\ell)\frac{\partial}{\partial \ell}A_\ell\right]$ are real.

Theorem 2.13. — *For $t > 0$, the following identity holds,*

$$\frac{\partial}{\partial \ell}\frac{1}{t}h_g^\wedge\left(A'_\ell, g_t^E\right) = \frac{\partial}{\partial t}\varphi \mathrm{Tr_s}\left[gk(B_{\ell,t})\frac{\partial}{\partial \ell}A_{\ell,t}\right] \text{ in } \Omega^\bullet(M)/d\Omega^\bullet(M). \tag{2.37}$$

Proof. — We proceed as in [**BLo1**, Theorem 1.9]. Set $\widetilde{M} = M \times \mathbf{R}_+^*$. Over $M \times \{t\} \subset \widetilde{M}$, we equip E with the metric $g_t^E = t^N g^E$. Let $\widetilde{g}^E$ be the corresponding metric on the pull-back of E to $\widetilde{M}$. The flat superconnection A' lifts to a flat superconnection $\widetilde{A}'$ on $\widetilde{M}$. Its adjoint $\widetilde{A}''_t$ is given by,

$$\widetilde{A}'' = A''_t + \frac{dt}{t}N. \tag{2.38}$$

Therefore,

$$h_g\left(\widetilde{A}', \widetilde{g}^E\right) = h_g\left(A', g_t^E\right) + \frac{dt}{t}h_g^\wedge\left(A', g_t^E\right). \tag{2.39}$$

Now we use Theorem 2.1, and we get (2.37). The proof of our Theorem is completed. □

We make the fundamental assumption that **the rank of $H^\bullet(E, v_\ell)$ does not depend on ℓ**. As in Section 1.6, we identify $H^\bullet(E, v_\ell)$ to a smooth G-invariant subbundle of E. By orthogonal projection on $H^\bullet(E, v_\ell)$, given $x \in M$, we obtain a G-invariant Hermitian connection on the bundle $H^\bullet(E, v_\ell)_x$ over $[0, 1]$. We can then trivialize $H^\bullet(E, v_\ell)$ along $[0, 1]$ by parallel transport. In particular the flat connections $\nabla^{H^\bullet(E,v_\ell)}$ on the vector bundles $H^\bullet(E, v_\ell)$ can now be viewed as a one parameter family of G-invariant flat connections on a fixed Hermitian vector bundle over M. Let $g^{H^\bullet(E,v_\ell)}$ be the metric on $H^\bullet(E, v_\ell)$. We define the connection $\nabla^{H^\bullet(E,v_\ell),u}$ on $H^\bullet(E, v_\ell)$ as in (1.11). In particular $\frac{\partial}{\partial \ell}\nabla^{H^\bullet(E,v_\ell),u}$ is well defined.

Let $k(x)$ be a holomorphic odd function such that (2.20) holds.

Let $\alpha_t, t \in \mathbf{R}_+^* \cup +\infty$ be a family of smooth forms on M. In the sequel, we write that as $t \to +\infty$, $\alpha_t \to \alpha_{+\infty}$ if α_t converges to $\alpha_{+\infty}$ uniformly on the compact subsets of M together with its derivatives.

Theorem 2.14. — *As* $t \to +\infty$,

$$\varphi \mathrm{Tr}_{\mathrm{s}}\left[gk(B_{\ell,t})\frac{\partial}{\partial \ell}A_{\ell,t}\right] \to \varphi \mathrm{Tr}_{\mathrm{s}}\left[gk\left(B_\ell^{H^\bullet(E,v_\ell)}\right)\frac{\partial}{\partial \ell}\nabla^{H^\bullet(E,v_\ell),u}\right]. \tag{2.40}$$

Proof. — By Proposition 1.20,

$$\mathrm{Tr}_{\mathrm{s}}\left[gk(B_{\ell,t})\frac{\partial}{\partial \ell}A_{\ell,t}\right] = \mathrm{Tr}_{\mathrm{s}}\left[gk(D_{\ell,t})\frac{\partial}{\partial \ell}C_{\ell,t}\right]. \tag{2.41}$$

To make our arguments simpler, we will temporarily assume that M is compact. Now we use the results of Section 2.2. Observe that because the rank of $\ker B_\ell^0 \simeq H^\bullet(E, v_\ell)$ is independent of ℓ, we may choose $d_1, d_2 \in \mathbf{R}_+^*$, with $d_1 < d_2/4$, such that (2.21) holds for any $\ell \in [0,1]$, i.e.

$$\left|\mathrm{Sp}\left(B_\ell^{(0)}\right)\right| \subset \{0\} \cup [2d_1, d_2/2]. \tag{2.42}$$

Since our expressions now depend on ℓ, we will add a subscript ℓ to all the expressions we meet in this Section. In particular, by Proposition 2.9,

$$k(D_{\ell,t}) = G_{\ell,t} + H_{\ell,t}. \tag{2.43}$$

Clearly,

$$\frac{\partial}{\partial \ell}C_{\ell,t} = \sqrt{t}\,\frac{\partial}{\partial \ell}A_\ell^{(0)} + \frac{\partial}{\partial \ell}\nabla_\ell^{E,u} + \mathcal{O}\left(1/\sqrt{t}\right). \tag{2.44}$$

By Proposition 2.10 and by (2.44), for $t \geqslant 1$, for any $p \in \mathbf{N}$, there is $C > 0$ such that for $t \geqslant 1$,

$$\left|\mathrm{Tr}_{\mathrm{s}}\left[gG_{\ell,t}\frac{\partial}{\partial \ell}C_{\ell,t}\right]\right| \leqslant \frac{C}{t^p}. \tag{2.45}$$

Let $P_\ell^{\{0\}}$ be the orthogonal projection operator from E on $\ker B_\ell^{(0)} \simeq H^\bullet(E, v_\ell)$. By (2.31) in Theorem 2.11, we find that as $t \to +\infty$,

$$\mathrm{Tr}_{\mathrm{s}}\left[gH_{\ell,t}\frac{\partial}{\partial \ell}\nabla_\ell^{E,u}\right] \to \mathrm{Tr}_{\mathrm{s}}\left[gk\left(B^{H(E,v_\ell)}\right)P_\ell^{\{0\}}\frac{\partial}{\partial \ell}\nabla_\ell^{E,u}P_\ell^{\{0\}}\right]. \tag{2.46}$$

Notice that $\ker B_\ell^{(0)} = \ker A_\ell^{(0)}$. We claim that $\frac{\partial}{\partial \ell}A_\ell^0$ maps $\ker B_\ell^{(0)}$ in its orthogonal. In fact let f be a smooth section of $\ker B_\ell^{(0)}$. Then $A_\ell^{(0)} f = 0$, so that

$$\frac{\partial}{\partial \ell}A_\ell^{(0)} f + A_\ell^{(0)}\frac{\partial}{\partial \ell}f = 0. \tag{2.47}$$

Since $A_\ell^{(0)}$ is self-adjoint, $\mathrm{Im}\, A_\ell^{(0)}$ is orthogonal to $\ker A_\ell^{(0)}$. Our assertion now follows from (2.47). By (2.31), we get

$$\mathrm{Tr}_{\mathrm{s}}\left[gH_{\ell,+\infty}\frac{\partial}{\partial \ell}A_\ell\right] = 0. \tag{2.48}$$

By (2.48), it is clear that

$$\lim_{t\to+\infty} \mathrm{Tr_s}\left[gH_{\ell,t}\frac{\partial}{\partial\ell}A^{(0)}_{\ell,t}\right] = \lim_{t\to+\infty} \mathrm{Tr_s}\left[g\left(H_{\ell,t}-H_{\ell,+\infty}\right)\frac{\partial}{\partial\ell}\sqrt{t}\,A^{(0)}_{\ell}\right]. \tag{2.49}$$

Using (2.30), we find that as $t\to+\infty$,

$$\left(H_t-H_\infty\right)\sqrt{t}\,\frac{\partial}{\partial\ell}A^{(0)}_{\ell} \to \sum_{p=0}^{\dim M}\sum_{\substack{0\leqslant i_0\leqslant p+1\\ j_1,\ldots,j_{p+1-i_0}\geqslant 0\\ \sum_{m=1}^{p+1-i_0} j_m\leqslant i_0-1}} \frac{k^{\left(i_0-1-\sum_{m=0}^{p+1-i_0} j_m\right)}(0)}{\left(i_0-1-\sum_{h=0}^{p+1-i_0} j_m\right)!}(-1)^{p+1-i_0} C_1R_1C_2\cdots R_pC_{p+1}\frac{\partial}{\partial\ell}A^{(0)}_{\ell}, \tag{2.50}$$

where one of the two following possibilities occur:

- All the C_j's are equal to $P^{\{0\}}_\ell$, one of the R_j's is equal to $B^{(2)}$ and the other R_j's are equal to $B^{(1)}$.
- One C_j is equal to $\left(B^{(0)}\right)^{-1}$, the other C_j's are equal to $P^{\{0\}}_\ell$, and the R_j's are all equal to $B^{(1)}$.

Using the same arguments as in (2.48), we find that, in the right-hand side of (2.50), the first sort of term does not contribute to the supertrace. As to the second sort of terms, only those terms where C_1 or C_{p+1} are equal to $\left(B^{(0)}_\ell\right)^{-1}$ contribute to the supertrace.

By (1.29),

$$B^{(1)}_\ell = \frac{1}{2}\omega\left(\nabla^E_\ell, g^E\right). \tag{2.51}$$

Using (1.38) and (2.51), we get

$$P^{\{0\}}_\ell B^{(1)}_\ell P^{\{0\}}_\ell = B^{H^\bullet(E,v_\ell)}_\ell. \tag{2.52}$$

Ultimately, by (2.50)-(2.52), we find that as $t\to+\infty$,

$$\mathrm{Tr_s}\left[g\left(H_{\ell,t}-H_{\ell,+\infty}\right)\frac{\partial}{\partial\ell}\sqrt{t}\,A^{(0)}\right] \to -\mathrm{Tr_s}\left[gk\left(B^{H^\bullet(E,v_\ell)}_\ell\right)P^{\{0\}}_\ell\left(\frac{\partial}{\partial\ell}A^{(0)}_\ell\left(B^{(0)}_\ell\right)^{-1}B^{(1)}_\ell + B^{(1)}_\ell\left(B^{(0)}_\ell\right)^{-1}\frac{\partial}{\partial\ell}A^{(0)}_\ell\right)P^{\{0\}}_\ell\right]. \tag{2.53}$$

By (2.41), (2.44)-(2.46), (2.48), (2.49), (2.53), we find that as $t \to +\infty$,

$$\begin{aligned}(2.54)\quad \mathrm{Tr_s}\left[gk(B_{\ell,t})\frac{\partial}{\partial \ell}A_{\ell,t}\right] \to \\ \mathrm{Tr_s}\Bigg[gk\left(B^{H^\bullet(E,v_\ell)}\right)P_\ell^{\{0\}}\Bigg(\frac{\partial}{\partial \ell}\nabla_\ell^{E,u} - \frac{\partial}{\partial \ell}A_\ell^{(0)}\left(B_\ell^{(0)}\right)^{-1}B_\ell^{(1)} \\ - B_\ell^{(1)}\left(B_\ell^{(0)}\right)^{-1}\frac{\partial}{\partial \ell}A_\ell^{(0)}\Bigg)P_\ell^{\{0\}}\Bigg].\end{aligned}$$

By finite dimensional Hodge theory,

$$(2.55)\qquad \left(\ker B_\ell^{(0)}\right)^\perp = \mathrm{Im}\,(v_\ell) \oplus \mathrm{Im}\,(v_\ell^*)\,.$$

Also on $\left(\ker B_\ell^{(0)}\right)^\perp$, v_ℓ acts as an invertible operator from $\mathrm{Im}\,(v_\ell^*)$ into $\mathrm{Im}\,(v_\ell)$, and v_ℓ^* as an invertible operator from $\mathrm{Im}\,(v_\ell)$ into $\mathrm{Im}\,(v_\ell^*)$. Let $v_\ell^{-1}, (v_\ell^*)^{-1}$ denote the corresponding inverses. As before, we extend these maps by 0 on the corresponding orthogonal bundles. Then, by (2.17), (2.51), we have the identity of operators acting on $\ker B_\ell^{(0)}$,

$$(2.56)\qquad \left(B_\ell^{(0)}\right)^{-1}B_\ell^{(1)} = \left((v_\ell^*)^{-1} - v_\ell^{-1}\right)\omega\left(\nabla^E, g^E\right),$$

and when acting on $\left(\ker B_\ell^{(0)}\right)^\perp$, we have the identity

$$(2.57)\qquad B_\ell^{(1)}\left(B_\ell^{(0)}\right)^{-1} = \omega\left(\nabla^E, g^E\right)\left((v_\ell^*)^{-1} - v_\ell^{-1}\right).$$

Recall that on $M \times [0,1]$, $H^\bullet(E, v_\ell)$ is equipped with a unitary connection, which we denote by $\widetilde{\nabla}^{H^\bullet(E,v_\ell),u}$. Let R be its curvature. By definition, we have the identity of forms on M with values in skew-adjoint elements of $\mathrm{End}\,(E)$,

$$(2.58)\qquad \frac{\partial}{\partial \ell}\nabla^{H^\bullet(E,v_\ell),u} = R\left(\frac{\partial}{\partial \ell}, \cdot\right).$$

Let $\widetilde{\nabla}^{E,u}$ be the obvious connection on the pull-back of E to $M \times [0,1]$, which coincides with $\nabla_\ell^{E,u}$ on $M \times \{\ell\}$, and with $\frac{\partial}{\partial \ell}$ along $[0,1]$. Recall that E splits as

$$(2.59)\qquad E = \ker B_\ell^{(0)} \oplus \left(\ker B_\ell^{(0)}\right)^\perp.$$

Let $\widetilde{\nabla}^{E,u,s}$ be the orthogonal projection of the connection of the connection $\widetilde{\nabla}^{E,u}$ with respect to the splitting (2.59). Then there is a 1-form K on $M \times \mathcal{M}$ with values in skew-adjoint elements of $\mathrm{End}\,(E)$ which interchange $\ker B_\ell^{(0)}$ and $\left(\ker B_\ell^{(0)}\right)^\perp$ such that

$$(2.60)\qquad \widetilde{\nabla}^{E,u} = \widetilde{\nabla}^{E,u,s} + K.$$

By (2.60), we get

$$R = P_\ell^{\{0\}} \left(\widetilde{\nabla}^{E,u,2} - K^2 \right) P_\ell^{\{0\}}. \tag{2.61}$$

Clearly, we have the identity of sections of $\Lambda^\bullet(T^*M) \widehat{\otimes} \operatorname{End}(E)$,

$$\widetilde{\nabla}^{E,u,2} \left(\frac{\partial}{\partial \ell}, . \right) = \frac{\partial}{\partial \ell} \nabla_\ell^{E,u}. \tag{2.62}$$

Let f be a smooth section of $\ker B_\ell^{(0)}$ on $\times]0,1]$. Then

$$v_\ell f = 0, \qquad v_\ell^* f = 0, \tag{2.63}$$

so that taking into account the fact that v_ℓ, v_ℓ^* are odd, we have the identity of forms on M,

$$\begin{aligned} \left(\nabla^{E,u} v_\ell \right) f - v_\ell K f = 0, \\ \left(\nabla^{E,u} v_\ell^* \right) f - v_\ell^* K f = 0 \end{aligned} \tag{2.64}$$

Since v_ℓ is ∇_ℓ^E flat and v_ℓ^* is $\nabla^{E,*}$ flat, by (1.10), (1.11), we get

$$\nabla_\ell^{E,u} v_\ell = \frac{1}{2} \left[\omega \left(\nabla_\ell^E, g^E \right), v_\ell \right], \qquad \nabla_\ell^{E,u} v_\ell^* = -\frac{1}{2} \left[\omega \left(\nabla_\ell^E, g^E \right), v_\ell^* \right]. \tag{2.65}$$

By (2.64), (2.65), we find that the restriction of K to $\Lambda^\bullet(T^*M)$ is such that

$$v_\ell K f = \frac{1}{2} v_\ell \omega \left(\nabla_\ell^E, g^E \right) f, \qquad v_\ell^* K f = -\frac{1}{2} v_\ell^* \omega \left(\nabla_\ell^E, g^E \right) f. \tag{2.66}$$

Similarly, by (2.47),

$$K \left(\frac{\partial}{\partial \ell} \right) f = - \left(A_\ell^{(0)} \right)^{-1} \frac{\partial}{\partial \ell} A_\ell^{(0)} f. \tag{2.67}$$

Observe that since K takes its values in skew-adjoint morphisms, (2.66) and (2.67 entirely characterize K. Using the fact that $A_\ell^{(0)}$ and $\frac{\partial}{\partial \ell} A^{(0)\ell}$ take their values in self-adjoint morphisms, we deduce from (2.66), (2.67) the identity of sections of $\Lambda^\bullet(T^*M) \widehat{\otimes} \operatorname{End} \left(\ker B_\ell^{(0)} \right)$,

$$\begin{aligned} P_\ell^{\{0\}} K \left(\frac{\partial}{\partial \ell} \right) K P_\ell^{\{0\}} &= P_\ell^{\{0\}} \frac{\partial}{\partial \ell} A_\ell^{(0)} \left((v_\ell^*)^{-1} - v_\ell^{-1} \right) \omega \left(\nabla_\ell^E, g^E \right) P_\ell^{\{0\}}, \\ P_\ell^{\{0\}} K K \left(\frac{\partial}{\partial \ell} \right) P_\ell^{\{0\}} &= P_\ell^{\{0\}} \omega \left(\nabla_\ell^E, g^E \right) \left(v_\ell^{-1} - (v_\ell^*)^{-1} \right) \frac{\partial}{\partial \ell} A_\ell^{(0)} P_\ell^{\{0\}}. \end{aligned} \tag{2.68}$$

By (2.56), (2.61), (2.62),(2.68), we get

$$R \left(\frac{\partial}{\partial \ell}, . \right) = P_\ell^{\{0\}} \left(\frac{\partial}{\partial \ell} \nabla_\ell^{E,u} - \frac{\partial}{\partial \ell} A_\ell^{(0)} \left(B_\ell^{(0)} \right)^{-1} B_\ell^{(1)} - B_\ell^{(1)} \left(B_\ell^{(0)} \right)^{-1} \frac{\partial}{\partial \ell} A_\ell^{(0)} \right) P_\ell^{\{0\}}. \tag{2.69}$$

By (2.54), (2.69), we get (2.40) when M is compact. When M is not compact, we obtain our result by restriction to compact subsets of M. The proof of our Theorem is completed. □

2.4. Rigidity properties of the forms $S_{h,g}\left(A', g^E\right)$

We make the same assumptions as in Section 2.3. In particular we still assume the $H^\bullet(E, v_\ell)$ have constant rank.

Let h be a holomorphic odd function with $\deg h \geqslant 3$, which is such that (1.54) holds. We still define $k(x)$ as in (2.2).

Recall that the forms $S_{h,g}\left(A'_\ell, g^E\right)$ were defined in Definition 1.24. We define $L_{k,g}\left(A'_\ell, g^E\right)$ as in Definition 2.4. Also note that the vector bundle $H^\bullet(E, v_\ell)$ has been unitarily trivialized along $[0,1]$, so that $\nabla^{H^\bullet(E,v_\ell)}$ is now considered as a family of flat connections on the fixed vector bundle $H^\bullet(E, v_\ell) \simeq H^\bullet(E, v_0)$, which is equipped with the metric $g^{H^\bullet(E,v_\ell)}$. We define the form $L_{k,g}\left(\nabla^{H^\bullet(E,v_\ell)}, g^{H^\bullet(E,v_\ell)}\right)$ as in Definition 2.4.

Theorem 2.15. — *The following identity holds,*

$$\begin{aligned}(2.70)\quad & S_{h,g}\left(A'_1, g^E\right) - S_{h,g}\left(A'_0, g^E\right) \\ & = L_{k,g}\left(A'_\ell, g^E\right) - L_{k,g}\left(\nabla^{H(E,v_\ell)}, g^{H^\bullet(E,v_\ell)}\right) \text{ in } \Omega^\bullet(M)/d\Omega^\bullet(M).\end{aligned}$$

Proof. — Since h is such that (1.54) holds, k verifies (2.20). Our Theorem follows from (2.15) and from Theorems 2.13 and 2.14. □

Remark 2.16. — Observe that in our Theorem, we only ask that $\deg h \geqslant 3$, while Theorem 2.5 guarantees that the terms in the right-hand side of (2.70) only depend on the homotopy class of $\ell \to A'_\ell$ if $\deg h \geqslant 5$. Theorem 2.15 says that the difference of the classes in the right-hand side of (2.70) only depend on (A'_0, A'_1), and this even in degree 3.

2.5. Rigidity properties of the torsion forms $T_{h,g}\left(A', g^E\right)$

Let now

$$(2.71)\qquad (E, v) : 0 \to E^0 \xrightarrow{v} E^1 \xrightarrow{v} \cdots \xrightarrow{v} E^m \to 0$$

be a flat complex of complex vector bundles on the manifold M, on which G acts fibrewise by flat automorphisms preserving the $\mathbf{Z}$-grading. Let $\nabla^E = \bigoplus_{i=0}^m \nabla^{E^i}$ be the flat connection. Then $A' = v + \nabla^E$ is a flat superconnection of total degree 1. Let $g^E = \bigoplus_{i=0}^m g^{E^i}$ be a G-invariant metric on $E = \bigoplus_{i=0}^m E^i$ such that the E^i's are mutually orthogonal in E.

Let h be a real holomorphic odd function, with $\deg h \geqslant 3$, such that (1.54) holds. We define k as in (2.2). Recall that the torsion forms $T_{h,g}\left(A'_\ell, g^E\right)$ were defined in Definition 1.29.

Let $\ell \in [0,1] \to A'_\ell = v_\ell + \nabla^E_\ell$ be a smooth path of G-invariant flat superconnections on E of the above type. As in Sections 2.3 and 2.4, we make the assumption that

the rank of $H^\bullet(E, v_\ell)$ does not depend on ℓ. We define the forms $L_{k,g}\left(\nabla^E_\ell, g^E\right)$ and $L_{k,g}\left(\nabla^{H(E,v_\ell)}, g^{H^\bullet(E,v_\ell)}\right)$ as in Definition 2.4.

Theorem 2.17. — *The following identity holds,*

$$\begin{aligned}(2.72)\quad & T_{h,g}\left(A'_1, g^E\right) - T_{h,g}\left(A'_0, g^E\right) \\ & = L_{k,g}\left(\nabla^E_\ell, g^E\right) - L_{k,g}\left(\nabla^{H(E,v_\ell)}, g^{H^\bullet(E,v_\ell)}\right) \text{ in } \Omega^\bullet(M)/d\Omega^\bullet(M).\end{aligned}$$

Proof. — Our Theorem follows from (2.15) and from Theorems 2.13 and 2.14. □

For $a \in \mathbf{R}^*$, let $\sqrt{a}$ be any square root of a. Let $R(a)$ be a polynomial. Put

$$(2.73)\qquad h_a(x) = R\left(\frac{\partial}{\partial a}\right)\frac{1}{\sqrt{a}}h(\sqrt{a}x).$$

If $a \in \mathbf{R}^*_+$, $h_a(x)$ verifies the assumptions in (1.54).

Theorem 2.18. — *For $a \in \mathbf{R}^*_+$, the following identity holds,*

$$(2.74)\qquad T_{h_a,g}\left(A', g^E\right) = R\left(\frac{\partial}{\partial a}\right)\psi_a T_{h,g}\left(A', g^E\right).$$

Proof. — Clearly,

$$(2.75)\qquad T_{h_a,g}\left(A', g^E\right) = R\left(\frac{\partial}{\partial a}\right)T_{h(\sqrt{a}.)/\sqrt{a},g}\left(A', g^E\right).$$

By Proposition 1.21 and by (2.75), we get (2.74). □

Remark 2.19. — In this finite dimensional context, equality (2.74) is not a surprise. In fact by [**BLo1**, Theorem A1.2] or by Theorem 1.33, and by Remark 1.34, the forms $T_{h,g}(A', g^E)$ have an axiomatic characterization, which implies equality (2.74). In particular, it is ultimately possible to make sense of $T_{h,g}(A', g^E)$ for any formal power series h. This is in dramatic contrast with the infinite dimensional situation we will consider in Section 3.

Let now h be a holomorphic odd function verifying the conditions in (1.54). Put

$$(2.76)\qquad k(x) = \frac{h'(x) - h'(0)}{2x}.$$

Theorem 2.20. — *The following identity holds,*

$$\begin{aligned}(2.77)\quad & \left[T_{h,g}\left(A'_1, g^E\right) - T_{h,g}\left(A'_0, g^E\right)\right]^{(\geqslant 2)} \\ & = L_{k,g}\left(\nabla^E_\ell, g^E\right) - L_{k,g}\left(\nabla^{H^\bullet(E,v_\ell)}, g^{H^\bullet(E,v_\ell)}\right) \text{ in } \Omega^\bullet(M)/d\Omega^\bullet(M).\end{aligned}$$

Proof. — If $\deg h \geqslant 3$, this is just Theorem 2.17. In general, we cannot apply this result to the function $h(x) - h'(0)x$, since it does not verify the conditions in (1.54). For $a > 0$, set

$$h_a(x) = \frac{1}{\sqrt{a}} h\left(\sqrt{a}x\right). \tag{2.78}$$

Put

$$\overline{h}(x) = \frac{\partial}{\partial a} h_a(x)|_{a=1}. \tag{2.79}$$

Then $\deg \overline{h} \geqslant 3$. By Theorem 2.18,

$$T_{\overline{h},g}\left(A', g^E\right) = \frac{\partial}{\partial a} \psi_a T_{h,g}\left(A', g^E\right)|_{a=1}. \tag{2.80}$$

Set

$$\overline{k}(x) = \frac{\overline{h}'(x)}{2x} = \frac{h''(x)}{4}. \tag{2.81}$$

Theorem 2.17 holds with h replaced by $\overline{h}$. Also, by Proposition 2.12,

$$L_{(h'_a(x)-h'_a(0))/2x,g}\left(\nabla^E_\ell, g^E\right) = \psi_a L_{(h'(x)-h'(0))/2x,g}\left(\nabla^E_\ell, g^E\right). \tag{2.82}$$

By (2.82), we obtain,

$$L_{\overline{k},g}\left(\nabla^E_\ell, g^E\right) = \frac{\partial}{\partial a} \psi_a L_{(h'(x)-h'(0))/2x,g}\left(\nabla^E_\ell, g^E\right)|_{a=1}. \tag{2.83}$$

Of course, when replacing E by $H^\bullet(E, v_\ell)$, an analogue of (2.83) still holds. By Theorem 2.17, (2.80), and by (2.83), we get (2.77). The proof of our Theorem is completed. □

Remark 2.21. — The main interest of the proof of Theorem 2.20 is that, as we shall see in the proof of Theorem 3.45, it can be transferred to infinite dimensions.

2.6. The imaginary part of the odd Chern classes

We make the same assumptions as in Sections 1.1-1.5. Namely, $E = E_+ \oplus E_-$ be a complex $\mathbf{Z}_2$-graded vector bundle, on which a compact Lie group G acts fibrewise by even automorphisms. Also $g^E = g^{E_+} \oplus g^{E_-}$ denotes a G-invariant Hermitian metric on $E = E_+ \oplus E_-$, such that E_+ and E_- are orthogonal in E. Let $\mathcal{M}$ be a smooth manifold of G-invariant flat superconnections A' on E. For $s \in [0,1]$, put

$$A^s = (1-s)A' + sA''. \tag{2.84}$$

Let $\widetilde{A}$ be the corresponding obvious superconnection one obtains from (2.84) on the pull-back of E to $M \times \mathcal{M} \times [0,1]$. Then its curvature $\widetilde{A}^2$ is a smooth section of $\Lambda\left(T^*\left(M \times \mathcal{M} \times [0,1]\right)\right) \widehat{\otimes} \operatorname{End}(E)$.

Proposition 2.22. — *The following identity holds,*

$$\widetilde{A}^2 = -4s\,(1-s)\,B^2 + 2dsB + (1-s)\,d^{\mathcal{M}}A' + sd^{\mathcal{M}}A''. \tag{2.85}$$

Proof. — This is an obvious computation, which is left to the reader. □

In the sequel, we will consider differential forms on $M \times \mathcal{M} \times]0,1]$. The operator φ defined in (1.22) now refers to forms on this manifold. Let f be a holomorphic function.

Definition 2.23. — Let $\alpha_{f,g}(A', g^E)$ be the form on $M \times \mathcal{M} \times [0,1]$,

$$\alpha_{f,g}(A', g^E) = \varphi \mathrm{Tr}_{\mathrm{s}}\left[gf\left(-\widetilde{A}^2\right)\right]. \tag{2.86}$$

By Chern-Weil theory, we know that $\alpha_{f,g}(A', g^E)$ is an even closed form on $M \times \mathcal{M} \times [0,1]$.

If ω is a form on $M \times \mathcal{M} \times [0,1]$, we denote by $\omega|_{s=0}$ the restriction of ω to the submanifold $(s=0)$. Other obvious notation will be used as well. In the sequel, $\int_{[0,1]}$ denotes integration along the fibre $[0,1]$.

Proposition 2.24. — *The following identity of even forms holds on $M \times \mathcal{M}$,*

$$d\int_{[0,1]} \alpha_{f,g}(A', g^E) = \alpha_{f,g}(A', g^E)|_{s=0} - \alpha_{f,g}(A', g^E)|_{s=1}. \tag{2.87}$$

In particular given $\ell_0 \in \mathcal{M}$, the form $\int_{[0,1]} \alpha_{f,g}(A', g^E)|_{\ell=\ell_0}$ is a closed odd form on M, which is purely imaginary if f is real and $g = 1$.

Proof. — Identity (2.87) follows from Stokes formula. Also by (2.85),

$$\widetilde{A}^2|_{s=0} = d^{\mathcal{M}}A', \qquad \widetilde{A}^2_{s=1} = d^{\mathcal{M}}A'', \tag{2.88}$$

and the restriction of both terms to $(\ell = \ell_0)$ vanish. By (2.87), the form

$$\int_{[0,1]} \alpha_{f,g}(A', g^E)|_{\ell=\ell_0}$$

is closed. Also,

$$A^{s,*} = A^{1-s}. \tag{2.89}$$

From (2.89), we deduce easily that if f is real and $g = 1$, the form $\int_{[0,1]} \alpha_{f,g}(A', g^E)$ is purely imaginary. The proof of our Proposition is complete. □

Remark 2.25. — The closed form on M, $\int_{[0,1]} \alpha_{f,g}(A', g^E)$, depends explicitly on the path $s \in [0,1] \to A^s$ defined in (2.84), but its cohomology class only depends on A', A''. In the sequel, the choice of the particular canonical path A^s will play a crucial role.

Assume now that c is an oriented smooth curve in $\mathcal{M}$, starting at ℓ_0 and ending at ℓ_1.

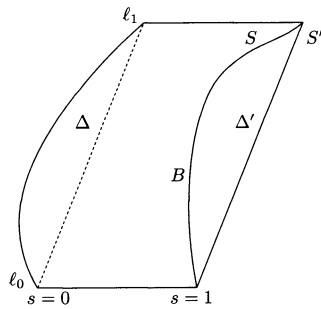

FIGURE 2.2

Proposition 2.26. — *Assume that* $\deg f \geqslant 2$. *Then the following identity of odd forms on* M *holds,*

$$d\int_{c\times[0,1]} \alpha_{f,g}(A', g^E) = \int_{[0,1]} \alpha_{f,g}(A', g^E)|_{\ell=\ell_0} - \int_{[0,1]} \alpha_{f,g}(A', g^E)|_{\ell=\ell_1}. \tag{2.90}$$

Proof. — By Stokes formula,

$$d\int_{c\times[0,1]} \alpha_{f,g}(A', g^E) = \int_{\partial(c\times[0,1])} \alpha_{f,g}(A', g^E). \tag{2.91}$$

If $\deg f \geqslant 2$, using (2.85), the restriction of $\alpha_{f,g}(A', g^E)$ to $c\times\{0\}$ and $c\times\{1\}$ vanishes. From (2.91), we get (2.90). □

Let now S, S' be two oriented surfaces in $\mathcal{M}\times[0,1]$ taken as indicated in Figure 2.2. The surfaces S, S' are of the form $c\times[0,1]$, where the path $c\in\mathcal{M}$ is taken are before. Let B be the region bounded by S, S', Δ, Δ'. The set B is itself of the form $\Delta\times[0,1]$. By Proposition 2.26, if $\deg f \geqslant 2$, we have the identity of odd forms on M,

$$d\int_{S} \alpha_{f,g}(A', g^E) = \int_{[0,1]} \alpha_{f,g}(A', g^E)|_{\ell=\ell_0} - \int_{[0,1]} \alpha_{f,g}(A', g^E)|_{\ell=\ell_1}, \tag{2.92}$$

$$d\int_{S'} \alpha_{f,g}(A', g^E) = \int_{[0,1]} \alpha_{f,g}(A', g^E)|_{\ell=\ell_0} - \int_{[0,1]} \alpha_{f,g}(A', g^E)|_{\ell=\ell_1}.$$

Theorem 2.27. — *If* $\deg f \geqslant 3$, *we have the identity of even forms on* M,

$$d\int_{B} \alpha_{f,g}(A', g^E) = \int_{S'} \alpha_{f,g}(A', g^E) - \int_{S} \alpha_{f,g}(A', g^E). \tag{2.93}$$

Proof. — Using (2.85) and the fact that $\deg f \geqslant 3$, we get

$$\int_{\Delta} \alpha_{f,g}(A', g^E) = 0, \qquad \int_{\Delta'} \alpha_{f,g}(A', g^E) = 0. \tag{2.94}$$

Our Theorem now follows from (2.94) and Stokes formula. □

Proposition 2.26 says that if $\deg f \geqslant 2$, the cohomology class of the closed form on M, $\int_{[0,1]} \alpha_{f,g}(A', g^E)$, is rigid under deformation of A'. More precisely the class of $\int_{c\times[0,1]} \alpha_{f,g}(A', g^E)$ in $\Omega^\bullet(M)/d\Omega^\bullet(M)$ gives a refinement of this rigidity at the level of differential forms. Theorem 2.27 shows that if $\deg f \geqslant 3$, the class of $\int_{c\times[0,1]} \alpha_{f,g}(A', g^E)$ in $\Omega^\bullet(M)/d\Omega^\bullet(M)$ is itself rigid under deformation of c. It still has the properties of a Bott-Chern class.

2.7. Superconnection classes and the Chern character

Definition 2.28. — If $f(x)$ is a holomorphic function, put

$$(Ff)(x) = x\int_0^1 f'\left(4s(1-s)x^2\right) ds. \tag{2.95}$$

Then $(Ff)(x)$ is a holomorphic odd function, which is real if f is real.

Proposition 2.29. — *If* $f(x) = \sum_{p=0}^{+\infty} a_p x^p$, *then*

$$(Ff)(x) = \sum_{p=1}^{+\infty} \frac{p!(p-1)!}{(2p-1)!} a_p 2^{2p-2} x^{2p-1}. \tag{2.96}$$

Proof. — If $f(x) = x^p$, then

$$(Ff)(x) = \int_0^1 s^{p-1}(1-s)^{p-1} ds p 2^{2p-2} x^{2p-1} = \frac{p!(p-1)!}{(2p-1)!} 2^{2p-2} x^{2p-1}, \tag{2.97}$$

which is just (2.96). □

Remark 2.30. — Assume that f is real and that on $\mathbf{R}_+^*$, $f'(-\cdot)$ is of constant sign. Observe that

$$\int_{\mathbf{R}} |(Ff)(ix)|\, dx = \frac{1}{4}\int_{[0,1]} \frac{ds}{s(1-s)} \int_{\mathbf{R}_+} |f'(-y)|\, dy. \tag{2.98}$$

From (2.98), we deduce that if $f \neq 0$, then $Ff(i\cdot) \notin L_1(\mathbf{R})$. This is the case in particular if $f(x) = e^x$. In the case where $f(x) = e^x$, Ff will be denoted $Fe^\bullet$.

It also follows from Proposition 2.29 that

$$(Fe^\bullet)(x) = \sum_{p=1}^{+\infty} \frac{(p-1)!}{(2p-1)!} 2^{2p-2} x^{2p-1}. \tag{2.99}$$

Observe that

$$xe^{x^2} = \sum_{p=1}^{+\infty} \frac{x^{2p-1}}{(p-1)!}. \tag{2.100}$$

The coefficient of x^{2p-1} in $(Fe^\bullet)(x)$ is obtained from the corresponding coefficient in the expansion of xe^{x^2} by multiplication by the factor $2^{2p-2}\frac{[(p-1)!]^2}{(2p-1)!}$.

Now again, the map φ defined in (1.22) refers to forms on M. We define the form $(Ff)_g\left(A', g^E\right)$ as in (1.23).

Proposition 2.31. — *The following identity of forms on M holds,*

$$\frac{1}{2}\int_{[0,1]} \alpha_{f,g}(A', g^E) = \frac{1}{2i\pi}\,(Ff)_g\left(A', g^E\right). \tag{2.101}$$

Proof. — By (2.85), (2.86),

$$\begin{aligned}\frac{1}{2}\int_{[0,1]} \alpha_{f,g}(A', g^E) &= \frac{1}{\sqrt{2i\pi}}\varphi\int_0^1 \frac{1}{2}\mathrm{Tr}_s\left[gf\left(4s(1-s)B^2 - 2dsB\right)\right] \\ &= \frac{1}{\sqrt{2i\pi}}\varphi\int_0^1 \mathrm{Tr}_s\left[gBf'\left(4s(1-s)B^2\right)\right]ds = \frac{1}{2i\pi}\,(Ff)_g\left(A', g^E\right).\end{aligned} \tag{2.102}$$

□

Let f be a holomorphic function. If A is a G-invariant superconnection on E, set

$$f_g(E, A) = \varphi\mathrm{Tr}_s\left[gf\left(-A^2\right)\right]. \tag{2.103}$$

Note here that our notation in (2.103) differs from the notation $h_g\left(A', g^E\right)$ in (1.23). Then by Quillen [**Q1**], $f_g(E, A)$ is a closed form, whose cohomology class does not depend on A.

Let $\ell \in [0,1] \to A_\ell$ be a smooth one parameter path of G-invariant flat superconnections on E. We lift E to a vector bundle on $M \times [0,1]$. If $\widetilde{A} = A_\ell + d\ell\frac{\partial}{\partial \ell}$ is the obvious lift of A_ℓ to $M \times [0,1]$, we define the Chern-Simons form $\mathrm{CS}(f)_g(A_\ell)$ by the formula,

$$\mathrm{CS}(f)_g(A_\ell) = -\int_{[0,1]} f_g\left(E, \widetilde{A}\right). \tag{2.104}$$

Then the Chern-Simons class $\mathrm{CS}(f)_g(A_0, A_1) \in \Omega^\bullet(M)/d\Omega^\bullet(M)$ of $\mathrm{CS}(f)_g(A_\ell)$ depends only on A_0, A_1. It is such that

$$d\mathrm{CS}(f)_g(A_0, A_1) = f_g(E, A_1) - f_g(E, A_0). \tag{2.105}$$

Definition 2.32. — Put

$$f_g^\circ(A_\ell) = -2i\pi\mathrm{CS}(f)_g(E, A_\ell). \tag{2.106}$$

Then

$$\mathrm{Re}\, f_g^\circ(A_\ell) = 2\pi\,\mathrm{Im}\mathrm{CS}(f)_g(E, A_\ell). \tag{2.107}$$

We will denote by $f_g^\circ(A_0, A_1)$ the class of $f_g^\circ(A_\ell)$ in $\Omega^\bullet(M)/d\Omega^\bullet(M)$.

Now we make the same assumptions as in Section 2.6. Recall that $A = \frac{1}{2}(A' + A'')$. Then the path $s \in [0,1] \to A^{s/2}$ defined in (2.84) interpolates between A' and A.

We can as well define the Chern-Simons form $f^\circ\left(A^{s/2}\right)$, which we will denote instead $f^\circ\left(A', g^E\right)$. By by (2.101), (2.104), (2.106), we get

$$f_g^\circ\left(A', g^E\right) = 2i\pi\frac{1}{2}\alpha_{f,g}(A', g^E). \tag{2.108}$$

Then Proposition 2.31 can be written as follows.

Proposition 2.33. — *The following identity of forms on M holds,*

$$f_g^\circ\left(A', g^E\right) = (Ff)_g\left(A', g^E\right). \tag{2.109}$$

If $f(x) = e^x$, we will use the notation $\mathrm{ch}^\circ\left(A', g^E\right)$ instead of $f^\circ\left(A', g^E\right)$, so that

$$\mathrm{ch}_g^\circ\left(A', g^E\right) = \left(Fe^\bullet\right)_g\left(A', g^E\right). \tag{2.110}$$

Then $\mathrm{ch}_g^\circ\left(A', g^E\right)$ is a secondary Chern character.

Remark 2.34. — Using (2.99) and (2.109) for $f = e^x$, we recover a result given in [**BLo1**, Proposition 1.14].

Now we make the same assumptions as in Section 2.1. Let $\ell \in [0,1] \to A'_\ell$ be a smooth one parameter family of G-invariant flat superconnections on E. Let f be a holomorphic function such that $\deg f \geqslant 2$. Then by Proposition 2.29, $\deg Ff \geqslant 3$. Set

$$k(x) = \frac{(Ff)'(x)}{2x}. \tag{2.111}$$

We use the same notation as in Proposition 2.26.

Proposition 2.35. — *The following identity holds,*

$$\frac{1}{2}\int_{c\times[0,1]} \alpha_{f,g}(A', g^E) = \frac{L_{k,g}\left(A'_\ell, g^E\right)}{2i\pi}. \tag{2.112}$$

Proof. — We will assume that $f(x) = x^p$, with $p \geqslant 2$. Using Proposition 2.22, we get

$$\begin{aligned}\int_{c\times[0,1]} &\alpha_{f,g}(A', g^E) \\ &= \frac{2}{2i\pi}\int_{c\times[0,1]} \varphi\mathrm{Tr_s}\left[gf''\left(4s(1-s)B_\ell^2\right)B_\ell\frac{\partial}{\partial\ell}\left((1-s)A'_\ell + sA''_\ell\right)\right]d\ell ds \\ &= 2\frac{p!\,(p-1)!}{(2p-2)!}2^{2p-3}\frac{1}{2i\pi}\int_c \varphi\mathrm{Tr_s}\left[B_\ell^{2p-3}\frac{\partial}{\partial\ell}A_\ell\right]d\ell.\end{aligned} \tag{2.113}$$

By Proposition 2.29,

$$(Ff)(x) = \frac{p!\,(p-1)!}{(2p-1)!}2^{2p-2}x^{2p-1}, \tag{2.114}$$

and so

$$k(x) = \frac{p!\,(p-1)!}{(2p-2)!} 2^{2p-3} x^{2p-3}. \tag{2.115}$$

Using (2.15), (2.113), (2.115), we get (2.112). □

Remark 2.36. — From Propositions 2.26, 2.31 and 2.35, we recover equation (2.16) in Theorem 2.5. Also if $\deg f \geqslant 3$, then $\deg Ff \geqslant 5$, and $\deg k \geqslant 3$. By Theorem 2.27, we recover the remainder of Theorem 2.5.

Definition 2.37. — If $f(x)$ is holomorphic, put

$$Qf(x) = \int_0^1 f\left(4s(1-s)x\right) ds. \tag{2.116}$$

Similarly, we define, $Q : \Lambda\left(T^{*,\mathrm{even}}M\right) \to \Lambda\left(T^{*,\mathrm{even}}M\right)$ by the formula,

$$Q\alpha = \int_0^1 \psi_{4s(1-s)} \alpha ds. \tag{2.117}$$

If $f(x) = \sum_{p=0}^{+\infty} a_p x^p$, then

$$Qf(x) = \sum_{p=0}^{+\infty} \frac{(p!)^2}{(2p+1)!} a_p (4x)^p. \tag{2.118}$$

If $\alpha \in \Lambda^{2p}(T^*M)$, then

$$Q\alpha = \frac{(p!)^2}{(2p+1)!} 4^p \alpha. \tag{2.119}$$

We use the same notation as in Section 1.5. Let $\ell \in [0,1] \to g_\ell^E$ be a smooth family of Hermitian metrics on E taken as before. Let A_ℓ, B_ℓ be the objects defined in (1.17) which are attached to g_ℓ^E.

Definition 2.38. — Put

$$\widetilde{\mathrm{ch}}^{\circ}\left(A', g_\ell^E\right) = \int_0^1 \varphi \mathrm{Tr}_{\mathrm{s}} \left[g \frac{1}{2} \left(g_\ell^E\right)^{-1} \frac{\partial g_\ell^E}{\partial \ell} \left(Fe^{\bullet}\right)'\left(B_\ell\right) \right] d\ell. \tag{2.120}$$

Theorem 2.39. — *The class of the form* $\widetilde{\mathrm{ch}}^{\circ}\left(A', g_\ell^E\right)$ *in* $\Omega^{\bullet}(M)/d\Omega^{\bullet}(M)$ *only depends on* g_0^E, g_1^E. *Moreover,*

$$d\widetilde{\mathrm{ch}}^{\circ}\left(A', g_\ell^E\right) = \mathrm{ch}^{\circ}\left(A', g_1^E\right) - \mathrm{ch}^{\circ}\left(A', g_0^E\right). \tag{2.121}$$

If $g = 1$, *the form* $\widetilde{\mathrm{ch}}^{\circ}\left(A', g_\ell^E\right)$ *is real. Finally, if* $h(x) = xe^{x^2}$,

$$\widetilde{\mathrm{ch}}_g^{\circ}\left(A', g_\ell^E\right) = Q\widetilde{h}_g\left(A', g_\ell^E\right). \tag{2.122}$$

Proof. — Using (2.110), the first part of our Theorem follows from Theorem 1.11. Finally using Remark 2.30 and (2.119), we get (2.122). The proof of our Theorem is completed. □

2.8. The Chern torsion forms

We make the same assumptions as in Section 1.10. As we saw in Remark 2.30, the function $Fe^{\bullet}(i\cdot)$ does not lie in $L_1(\mathbf{R})$. So, a priori, we cannot define the analytic torsion forms $T_{Fe^{\bullet},g}\left(A', g^E\right)$.

In this Section, we set

$$h(x) = xe^{x^2}. \tag{2.123}$$

Definition 2.40. — Put

$$T_{\mathrm{ch},g}\left(A', g^E\right) = QT_{h,g}\left(A', g^E\right). \tag{2.124}$$

The form $T_{\mathrm{ch}}\left(A', g^E\right)$ is called a Chern torsion form.

Proposition 2.41. — *The even form $T_{\mathrm{ch},g}\left(A', g^E\right)$ is such that*

$$dT_{\mathrm{ch},g}\left(A', g^E\right) = \mathrm{ch}_g^{\circ}\left(\nabla^E, g^E\right) - \mathrm{ch}_g^{\circ}\left(\nabla^{H^{\bullet}(E,v)}, g^{H^{\bullet}(E,v)}\right). \tag{2.125}$$

It is real if $g = 1$.

Proof. — This follows from Theorem 1.30, and from Remark 2.30. □

Remark 2.42. — Rigidity formulas similar to Theorem 2.20 obviously hold for the forms $T_{\mathrm{ch},g}\left(A', g^E\right)$. The main point of Definition 2.40 is that it normalizes the torsion forms unambiguously. Note that if (E, v) is acyclic, i.e. if $H^{\bullet}(E, v) = \{0\}$, by [**BLo1**, Theorem A1.2], the class of $T_{\mathrm{ch},g}\left(A', g^E\right)$ in $\Omega^{\bullet}(M)/d\Omega^{\bullet}(M)$ is the unique natural class such that (2.125) holds. The considerations of [**BLo1**] also extend to the non acyclic case.

2.9. Generalized metrics and the forms $U_{h,g}\left(A', g^E\right)$

We make the same assumptions as in Sections 1.6 and 2.4. Let A' be a G-invariant flat superconnection on E which has total degree 1. Recall that the notation $\overline{*}$ was introduced in Section 1.2.

Definition 2.43. — A smooth section $\mathbf{g}^E$ of $\left(\Lambda^{\bullet}(T^*M)\,\widehat{\otimes}\,\mathrm{Hom}\left(E, \overline{E}^*\right)\right)^{\mathrm{even}}$ is said to be a generalized metric if

$$\overline{\mathbf{g}^E}^* = \mathbf{g}^E, \tag{2.126}$$

and if the component $\mathbf{g}^{E,(0)} \in \mathrm{End}\left(E, \overline{E}^*\right)^{\mathrm{even}}$ of $\mathbf{g}^E$ defines a standard Hermitian metric on E. Then E_+ and E_- are orthogonal with respect to $\mathbf{g}^{E,(0)}$.

Since $\mathbf{g}^{E,(0)}$ is invertible, $\mathbf{g}^E$ is also invertible. Also any $\mathbf{g}^E$ can be deformed to the standard metric $\mathbf{g}^{E,(0)}$ by the homotopy

$$\ell \in [0,1] \to \mathbf{g}_{\ell}^E = (1-\ell)\,\mathbf{g}^E + \ell \mathbf{g}^{E,(0)}. \tag{2.127}$$

In the sequel, we assume that $\mathbf{g}^E$ is G-invariant.

Definition 2.44. — The adjoint superconnection A'' with respect to $\mathbf{g}^E$ is defined by the formula,

$$A'' = \left(\mathbf{g}^E\right)^{-1} \overline{A}'^* \mathbf{g}^E. \tag{2.128}$$

Then A'' is a flat superconnection. As in (1.17), we set

$$A = \frac{1}{2}\left(A'' + A'\right), \qquad B = \frac{1}{2}\left(A'' - A'\right). \tag{2.129}$$

Then B is an odd section of $\Lambda^\bullet\left(T^*M\right) \widehat{\otimes} \operatorname{End}(E)$. Also Proposition 1.5 still holds.

Let h be a holomorphic odd function. We define the form $h_g\left(A', \mathbf{g}^E\right)$ by the same formula as in Definition 1.7.

Proposition 2.45. — *The form $h_g\left(A', \mathbf{g}^E\right)$ is odd, closed, and its cohomology class does not depend on $\mathbf{g}^E$, and is equal to $h_g\left(A'\right)$.*

Proof. — The proof of our Proposition is the same as the proof of [**BLo1**, Theorems 1.9 and 1.11] and of Theorem 1.8. □

Let now $t \in [1, +\infty[\to \mathbf{g}_t^E$ be a smooth family of G-invariant generalized metrics on E. Let $g^E = \bigoplus_{i=0}^m g^{E^i}$ be a standard Hermitian metric on $E = \bigoplus_{i=0}^m E^i$. In the sequel we identify E and $\overline{E}^*$ by the metric g^E. We assume that there exist $n \in \mathbf{N}$ and $H \in \left(\Lambda^\bullet\left(T^*M\right) \widehat{\otimes} \operatorname{End}(E)\right)^{\text{even}}$ such that as $t \to +\infty$,

$$\begin{aligned} t^{n/2} t^{-N/2} \mathbf{g}_t^E t^{-N/2} &= 1 + \frac{H}{\sqrt{t}} + \mathcal{O}(1/t), \\ t^{N/2}\left(\mathbf{g}_t^E\right)^{-1} \frac{\partial}{\partial t} \mathbf{g}_t^E t^{-N/2} &= \left(N - \frac{n}{2}\right)\frac{1}{t} + \mathcal{O}\left(1/t^{3/2}\right). \end{aligned} \tag{2.130}$$

Recall that $A'^{(0)} = v, A'^{(1)} = \nabla^E$. Let A_t'' be the adjoint superconnection to A' with respect to $\mathbf{g}_t^E$. Let v^* be the adjoint of v with respect to the standard metric g^E, let $\nabla^{E,*}$ be the adjoint connection to ∇^E with respect to g^E.

Proposition 2.46. — *As $t \to +\infty$,*

$$\begin{aligned} t^{N/2} A' t^{-N/2} &= \sqrt{t}\, v + \nabla^E + \mathcal{O}\left(1/\sqrt{t}\right), \\ t^{N/2} A_t'' t^{-N/2} &= \sqrt{t} v^* + \nabla^{E,*} + [v^*, H] + \mathcal{O}\left(1/\sqrt{t}\right). \end{aligned} \tag{2.131}$$

Proof. — Since A' is of total degree 1, the first identity in (2.131) is trivial. Let A''^0 be the adjoint of A' with respect to the standard metric g^E. Then

$$t^{N/2} A_t'' t^{-N/2} = t^{N/2}\left(\mathbf{g}_t^E\right)^{-1} t^{N/2}\left(t^{-N/2} A''^0 t^{N/2}\right) t^{-N/2} \mathbf{g}_t^E t^{-N/2}. \tag{2.132}$$

By (2.130), (2.132), we get the second equation in (2.131). □

As in (1.42), set

$$A_t = \frac{1}{2}\left(A_t'' + A'\right), \qquad B_t = \frac{1}{2}\left(A_t'' - A'\right). \tag{2.133}$$

Let now $h(x)$ be a holomorphic odd function such that (1.54) holds.

Definition 2.47. — Put

$$b_t = t\varphi \mathrm{Tr_s}\left[g\frac{1}{2}\left(\mathbf{g}_t^E\right)^{-1}\frac{\partial}{\partial t}\mathbf{g}_t^E h'\left(B_t\right)\right]. \tag{2.134}$$

Then b_t is an even form on M. Let $g^{H^\bullet(E,v)}$ be the Hermitian metric induced by g^E on $H^\bullet(E,v)$ as in Section 1.5. Recall that $\chi_g(E), \chi'_g(E)$ were defined in (1.56).

Proposition 2.48. — *The following identity holds,*

$$\frac{\partial}{\partial t}h_g\left(A', \mathbf{g}_t^E\right) = \frac{1}{t}db_t. \tag{2.135}$$

Moreover as $t \to +\infty$,

$$h_g\left(A', \mathbf{g}_t^E\right) = h_g\left(\nabla^{H^\bullet(E,v)}, g^{H^\bullet(E,v)}\right) + \mathcal{O}\left(1/\sqrt{t}\right), \tag{2.136}$$

$$b_t = \left(\frac{1}{2}\chi'_g(E) - \frac{n}{4}\chi_g(E)\right)h'(0) + \mathcal{O}\left(1/\sqrt{t}\right).$$

Proof. — The proof of (2.135) is the same as the proof of [**BLo1**, Theorem 2.9] or of Theorem 1.11, and uses Proposition 2.45. Observe that

$$P^{H^\bullet(E,v)}\left[v^*, H\right]P^{H^\bullet(E,v)} = 0. \tag{2.137}$$

Using (2.130), Proposition 2.46 and (2.137), the proof of (2.136) is the same as the proof of [**BLo1**, Theorem 2.13] and of Theorem 1.22. □

Definition 2.49. — Set

$$U_{h,g}\left(A', \mathbf{g}_t^E\right) = -\int_1^{+\infty}\left(b_t - b_{+\infty}\right)\frac{dt}{t}. \tag{2.138}$$

Theorem 2.50. — *The even form* $U_{h,g}\left(A', \mathbf{g}_t^E\right)$ *is such that*

$$dU_{h,g}\left(A', \mathbf{g}_t^E\right) = h_g\left(A', \mathbf{g}_1^E\right) - h_g\left(\nabla^{H^\bullet(E,v)}, g^{H^\bullet(E,v)}\right). \tag{2.139}$$

Proof. — Our Theorem follows from Proposition 2.48. □

Remark 2.51. — The forms $S_{h,g}\left(A', g^E\right)$ of Section 1.9 are special cases of the forms $U_{h,g}\left(A', \mathbf{g}_t^E\right)$.

If $\ell \in]0,1] \to \mathbf{g}_\ell^E$ is a smooth family of generalized metrics, we define $\widetilde{h}_g\left(A', \mathbf{g}_\ell^E\right)$ as in Definition 1.10. Using Proposition 2.48, the obvious analogue of Theorem 1.11 holds. We denote by $\widetilde{h}_g\left(A', \mathbf{g}_0^E, \mathbf{g}_1^E\right)$ the class of $\widetilde{h}_g\left(A', \mathbf{g}_\ell^E\right)$ in $\Omega^\bullet(M)/d\Omega^\bullet(M)$.

Let now $\ell \in [0,1] \to \left(\mathbf{g}_t^E\right)_\ell$ be a smooth family of generalized metrics on E, such that (2.130) holds uniformly in $\ell \in [0,1]$. Namely we assume that there is a smooth family of standard metrics $\ell \in [0,1] \to g_\ell^E$ and also a smooth family $\ell \in [0,1] \to H_\ell \in \left(\Lambda^\bullet(T^*M) \widehat{\otimes} \mathrm{End}(E)\right)^{\mathrm{even}}$ such that (2.130) holds uniformly in $\ell \in [0,1]$.

Theorem 2.52. — *The following identity holds,*

$$U_{h,g}\left(A',\left(\mathbf{g}_t^E\right)_1\right)-U_{h,g}\left(A',\left(\mathbf{g}_t^E\right)_0\right)=\widetilde{h}_g\left(A',\left(\mathbf{g}^E\right)_0,\left(\mathbf{g}^E\right)_1\right)$$
$$-\widetilde{h}_g\left(\nabla^{H^\bullet(E,v)},\left(g^{H^\bullet(E,v)}\right)_0,\left(g^{H^\bullet(E,v)}\right)_1\right)\text{ in }\Omega^\bullet(M)/d\Omega^\bullet(M). \tag{2.140}$$

Proof. — The proof of our Theorem is the same as the proof of [**BLo1**, Theorem 2.17] or of Theorem 1.26. □

Remark 2.53. — By Theorem 2.52, the class of $U_{h,g}\left(A',\mathbf{g}_t^E\right)$ in $\Omega^\bullet(M)/d\Omega^\bullet(M)$ only depends on $\mathbf{g}_1^E$ and on g^E.

2.10. Generalized metrics and flat complexes

We make the same assumptions as in Sections 1.10 and 2.5, and we use the notation of Section 2.9. Assume that A' is a G-invariant flat superconnection of total degree 1, of the form

$$A'=v+\nabla^E. \tag{2.141}$$

Then ∇^E is a flat connection on E which preserves the grading, and v is a parallel chain map. Let $\mathbf{g}^E$ be a G-invariant generalized metric on E. Let $\nabla^{E,*}$ be the adjoint of the flat connection ∇^E with respect to the generalized metric $\mathbf{g}^E$. Then $\nabla^{E,*}$ is a flat superconnection. Put

$$A_0=\frac{1}{2}\left(\nabla^{E,*}+\nabla^E\right),\qquad B_0=\frac{1}{2}\left(\nabla^{E,*}-\nabla^E\right). \tag{2.142}$$

Proposition 2.54. — *The even form*

$$\eta=\varphi\frac{1}{2}\mathrm{Tr_s}\left[g\frac{1}{2}\left(N+\left(\mathbf{g}^E\right)^{-1}N\mathbf{g}^E\right)h'\left(B_0\right)\right] \tag{2.143}$$

is closed and its cohomology class does not depend on $\mathbf{g}^E$*, and is equal to the constant* $\frac{1}{2}\widetilde{\chi}'_g\left(E\right)h'\left(0\right)$.

Proof. — Since ∇^E preserves the grading of E, we have the obvious

$$\left[A_0,N+\left(\mathbf{g}^E\right)^{-1}N\mathbf{g}^E\right]=\left[B_0,N-\left(\mathbf{g}^E\right)^{-1}N\mathbf{g}^E\right]. \tag{2.144}$$

Moreover, by (1.19),

$$\left[A_0,B_0\right]=0. \tag{2.145}$$

Therefore, using (2.144), (2.145) and the fact that supertraces vanish on supercommutators, we get

$$d\varphi^{-1}\eta=\frac{1}{2}\mathrm{Tr_s}\left[g\left[B_0,\frac{1}{2}\left(N-\left(\mathbf{g}^E\right)^{-1}N\mathbf{g}^E\right)\right]h'\left(B_0\right)\right]=0, \tag{2.146}$$

i.e. the form η is closed. The fact that its cohomology class does not depend on $\mathbf{g}^E$ follows tautologically. By making $\mathbf{g}^E$ to be a standard metric on E, and using the fact that $h'(x)$ is an even function, the last part of our Proposition follows. □

Definition 2.55. — For $u \in \mathbf{R}_+^*$, let $\mathfrak{g}_t^E$ be the generalized metric,

$$\mathfrak{g}_t^E = t^{N/2}\mathbf{g}^E t^{N/2}. \tag{2.147}$$

Let B_t be obtained as in (2.129) with respect to $\mathfrak{g}_t^E$.

Proposition 2.56. — *As* $t \to 0$,

$$\begin{aligned} &h_g\left(A', \mathfrak{g}_t^E\right) = h_g\left(\nabla^E, \mathbf{g}^E\right) + \mathcal{O}\left(\sqrt{t}\right), \\ &\frac{1}{2}\varphi \mathrm{Tr}_{\mathrm{s}}\left[g\left(\mathfrak{g}_t^E\right)^{-1}\frac{\partial}{\partial t}\mathfrak{g}_t^E h'\left(B_t\right)\right] = \frac{\eta}{t} + \mathcal{O}\left(1/\sqrt{t}\right). \end{aligned} \tag{2.148}$$

Proof. — Let $A_t'', \nabla^{E,*}, v^*$ be the adjoints of A', ∇^E, v with respect to $\mathbf{g}^E$. Then

$$\begin{aligned} t^{N/2}A' t^{-N/2} &= \sqrt{t}\, v + \nabla^E, \\ t^{N/2}A_t'' t^{-N/2} &= \sqrt{t}\, v^* + \nabla^{E,*}. \end{aligned} \tag{2.149}$$

From (2.149), we get the first equation in (2.148). Also,

$$t^{N/2}\left(\mathfrak{g}_t^E\right)^{-1}\frac{\partial}{\partial t}\mathfrak{g}_t^E t^{-N/2} = \frac{1}{2t}\left(N + \left(\mathbf{g}^E\right)^{-1} N \mathbf{g}^E\right). \tag{2.150}$$

From (2.149), (2.150), we get the second equation in (2.148). □

Proposition 2.57. — *As* $u \to 0$,

$$\begin{aligned} &\widetilde{h}_g\left(A', \mathbf{g}^E, \mathfrak{g}_u^E\right) - \eta \log(u) \to \\ &\qquad - \int_0^1 \left\{\varphi \mathrm{Tr}_{\mathrm{s}}\left[g\frac{1}{2}\left(\mathfrak{g}_t^E\right)^{-1}\frac{\partial}{\partial t}\mathfrak{g}_t^E h'\left(B_t\right)\right] - \frac{\eta}{t}\right\} dt. \end{aligned} \tag{2.151}$$

Proof. — By the analogue of (1.26),

$$\widetilde{h}_g\left(A', \mathbf{g}^E, \mathfrak{g}_u^E\right) = -\varphi\frac{1}{2}\int_u^1 \mathrm{Tr}_{\mathrm{s}}\left[g\left(\mathfrak{g}_t^E\right)^{-1}\frac{\partial}{\partial t}\mathfrak{g}_t^E h'\left(B_t\right)\right] dt. \tag{2.152}$$

Using Proposition 2.56 and (2.152), we get (2.151). □

Let now $\mathbf{g}^E, \mathbf{g}^{E\prime}$ be two generalized metrics on E. We define the metrics $\mathfrak{g}_u^E, \mathfrak{g}_u^{E\prime}$ as in (2.147).

Proposition 2.58. — *As* $u \to 0$,

$$\widetilde{h}_g\left(A', \mathfrak{g}_u^E, \mathfrak{g}_u^{E\prime}\right) \to \widetilde{h}_g\left(\nabla^E, \mathbf{g}^E, \mathbf{g}^{E\prime}\right). \tag{2.153}$$

Proof. — For $\ell \in [0,1]$, put

$$\mathbf{g}^{E,\ell} = (1-\ell)\,\mathbf{g}^{E} + \ell \mathbf{g}^{E\prime}. \tag{2.154}$$

We also define $\mathfrak{g}_u^{E,\ell}$ as in (2.147). Then

$$\left(\mathfrak{g}_u^{E,\ell}\right)^{-1} \frac{\partial}{\partial \ell} \mathfrak{g}_u^{E,\ell} = u^{-N/2} \left(\mathbf{g}^{E,\ell}\right)^{-1} \left(\mathbf{g}^{E\prime} - \mathbf{g}^{E}\right) u^{N/2}. \tag{2.155}$$

Using the analogue of (1.26), (2.149) and (2.155), we get (2.153). □

CHAPTER 3

ANALYTIC TORSION FORMS: RIGIDITY AND THE CHERN CHARACTER

Let $\pi : M \to S$ be a smooth fibration with compact fibre X. Let F be a complex vector bundle on M, equipped with a flat connection ∇^F. In this Chapter, we extend the construction of analytic torsion forms in de Rham theory to an equivariant context. We show that these forms verify anomaly formulas in degree $\geqslant 2$ under deformation of the flat connection ∇^F, which extend the corresponding results for finite dimensional torsion forms established in Theorem 2.17. Finally we give a Chern normalization of the analytic torsion forms.

This Chapter is organized as follows. In Sections 3.1-3.7, we recall the basic formalism of Bismut-Lott [**BLo1**]. In Section 3.1, we describe the geometric setting. We introduce in particular a horizontal vector bundle $T^H M \subset TM$. In Section 3.2, we interpret the de Rham operator on the total space of M as a flat superconnection A' on the relative de Rham complex $\Omega^\bullet(X, F|_X)$. In Section 3.3, we introduce a metric g^{TX} on the relative tangent bundle TX, and we construct the tensors obtained in [**B3**], which are associated to $(T^H M, g^{TX})$. In Section 3.4, we construct the adjoint superconnection A''. In Section 3.5, we recall elementary facts of Clifford algebras. In Section 3.6, we relate the connection $A = \frac{1}{2}(A'' + A')$ to the Levi-Civita superconnection of [**B3**]. In Section 3.7, given $t > 0$, we consider the metric g^{TX}/t on TX, and the associated metric $g_t^{\Omega^\bullet(X,F|_X)}$ on $\Omega^\bullet(X, F|_X)$, and we construct the objects we just described, which are attached to the rescaled metric.

In Section 3.8, we give the Lichnerowicz formula of [**B3**], [**BLo1**] for the curvature A^2, and we establish another related Lichnerowicz formula. In Section 3.9-3.12, we extend the results of [**BLo1**] to the equivariant setting. In Section 3.10, for $h(x) = xe^{x^2}$, we construct the forms $h_g\left(A', g_t^{\Omega^\bullet(X,F|_X)}\right)$. In Section 3.11, we establish the corresponding transgression formulas. In Section 3.12, we construct the analytic torsion forms $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$.

In Section 3.13, we construct analytic torsion forms associated to more general functions h, and we prove natural compatibility properties of these forms. Sections 3.14-3.16 are devoted to the proof of rigidity of the analytic torsion forms under deformation of ∇^F. These Sections extend Sections 2.2-2.5 to our infinite dimensional setting. Finally in Section 3.17, we construct the Chern analytic torsion forms.

3.1. Equivariant smooth fibrations

Let $\pi : M \to S$ be a submersion of smooth manifolds, with compact fibre X of dimension n. Let $TX \subset TM$ be the tangent bundle to the fibres X.

Let G be a compact Lie group acting on M along the fibres of X, that is if $g \in G$, $\pi g = \pi$. Then G acts on TM and on $TX \subset TM$. Let $T^H M \subset TM$ be a G-invariant horizontal subbundle, so that

$$TM = T^H M \oplus TX. \tag{3.1}$$

Observe that since G is compact, such a $T^H M$ always exists. Let $P^{TX} : TM \to TX$ be the projection associated to the splitting (3.1). Observe that

$$T^H M \simeq \pi^* TS. \tag{3.2}$$

By (3.1), (3.2), we have the identification of bundles of algebras

$$\Lambda^\bullet(T^*M) \simeq \pi^*\Lambda^\bullet(T^*S) \mathbin{\widehat{\otimes}} \Lambda^\bullet(T^*X), \tag{3.3}$$

and this identification is also an identification of G-bundles.

Take $g \in G$. Set

$$M_g = \{x \in M, gx = x\}. \tag{3.4}$$

Since G is compact, M_g is a smooth submanifold of M. Since G acts along the fibres X, it follows that g acts trivially on $T^H M|_{M_g}$. Therefore we have a fibration $\pi : M_g \to S$ with fibre X_g, the fixed point set in the corresponding fibre X, which is either compact or empty. In particular

$$T^H M|_{M_g} \subset TM_g, \tag{3.5}$$

i.e. the restriction of $T^H M$ to M_g defines an horizontal subbundle on M_g. Note that if one fibre X_g is empty, the fibres over the corresponding connected component of S are empty as well.

Let F be a complex flat vector bundle on M, and let ∇^F be the corresponding flat connection. In the sequel, we will consider F as trivially $\mathbf{Z}_2$-graded , i.e. as an even vector bundle. We assume that the action of G on M lifts to F, and that G preserves the connection ∇^F. Let $\left(\Omega^\bullet(M,F), d^M\right)$ be the de Rham complex of smooth differential forms with values in F, equipped with the de Rham operator d. Then G acts on the left on $\Omega^\bullet(M,F)$, so that if $g \in G, s \in \Omega^\bullet(M,F)$, gs is given by

$$(gs)(x) = g_* s\left(g^{-1}x\right). \tag{3.6}$$

Clearly G preserves the $\mathbf{Z}$-grading of $\Omega^\bullet(M,F)$, and commutes with d^M.

Let $\left(\Omega^\bullet\left(X, F|_X\right), d^X\right)$ be the fibrewise de Rham complex of smooth forms along the fibres X with values in $F|_X$, equipped with the fibrewise de Rham operator d^X. Again G acts on $\Omega^\bullet\left(X, F|_X\right)$, preserves the $\mathbf{Z}$-grading and commutes with d^X. Then $\left(\Omega^\bullet\left(X, F|_X\right), d^X\right)$ can be viewed as a family of infinite dimensional complex on S, on which G acts fibrewise. Let $H^\bullet(X, F|_X)$ be the cohomology of the complex $\left(\Omega^\bullet\left(X, F|_X\right), d^X\right)$. Then $H^\bullet(X, F|_X)$ is a finite dimensional $\mathbf{Z}$-graded G-bundle on S.

Let $\Omega^\bullet\left(S,\Omega^\bullet\left(X,F|_X\right)\right)$ be the space of smooth sections of $\Lambda^\bullet(T^*S)\widehat{\otimes}\,\Omega^\bullet\left(X,F|_X\right)$ on S. Using (3.3), we have the identification of $\mathbf{Z}$-graded G-vector spaces,

$$\Omega^\bullet(M,F)\simeq\Omega^\bullet\left(S,\Omega^\bullet\left(X,F|_X\right)\right). \tag{3.7}$$

3.2. A flat superconnection of total degree 1

Here, we follow [**BLo1**, Section 3 (b)]. The operator d^M acting on $\Omega^\bullet(M,F)$ has degree 1 and is such that $d^{M,2}=0$. Also if ω is a smooth section of $\Lambda^\bullet(T^*S)$, and if $s\in\Omega^\bullet(M,F)$, then

$$d^M\left(\pi^*(\omega)\,s\right)=\pi^*\left(d^S\omega\right)s+(-1)^{\deg\omega}\,\pi^*(\omega)\,d^M s. \tag{3.8}$$

Comparing with (1.4), we find that $A'=d^M$ can be considered as a flat superconnection of $\Omega^\bullet\left(X,F|_X\right)$, which has total degree 1.

If $U\in TS$, let $U^H\in T^HM$ be the horizontal lift of U, so that $\pi_*U^H=U$. If U is a smooth section of TS, the Lie derivative operator L_{U^H} acts naturally on $\Omega^\bullet(M,F)$. One verifies easily that if $f\in C^\infty(M,\mathbf{C})$, if $a\in C^\infty\left(M,\Lambda^\bullet(T^*X)\,\widehat{\otimes}\,F\right)$, then

$$L_{(fU)^H}a=(\pi^*f)\,L_{U^H}a. \tag{3.9}$$

Definition 3.1. — Let $\nabla^{\Omega^\bullet\left(X,F|_X\right)}$ be the connection on $\Omega^\bullet\left(X,F|_X\right)$, such that if $U\in TS$ and if s is a smooth section of $\Omega^\bullet\left(X,F|_X\right)$, then

$$\nabla_U^{\Omega^\bullet\left(X,F|_X\right)}s=L_{U^H}s. \tag{3.10}$$

Clearly the connection $\nabla^{\Omega^\bullet\left(X,F|_X\right)}$ preserves the $\mathbf{Z}$-grading of $\Omega^\bullet\left(X,F|_X\right)$.

Definition 3.2. — If U,V are smooth sections of TS, set

$$T^H(U,V)=-P^{TX}\left[U^H,V^H\right]. \tag{3.11}$$

One verifies easily that T^H is a tensor, i.e. it defines a 2-form on S with values in smooth sections of TX. The interior multiplication i_{T^H} acts naturally on $\Lambda^\bullet(T^*M)\,\widehat{\otimes}\,F$. It increases the total degree by 1, while decreasing the vertical degree by 1, and increasing the horizontal degree by 2.

Now we have a classical result stated in [**BLo1**, Proposition 3.4].

Proposition 3.3. — *The following identity of operators acting on* $\Omega^\bullet(M,F)$ *holds,*

$$A'=d^X+\nabla^{\Omega^\bullet\left(X,F|_X\right)}+i_{T^H}. \tag{3.12}$$

Remark 3.4. — Equation (3.4) gives us a decomposition of the superconnection d^M which is a special case of (1.32). Since $A'^2=0$, from (3.12), we get

$$\begin{aligned}&d^{X,2}=0,\qquad\left[\nabla^{\Omega^\bullet\left(X,F|_X\right)},d^X\right]=0,\quad\nabla^{\Omega^\bullet\left(X,F|_X\right),2}+\left[d^X,i_{T^H}\right]=0,\\&\left[\nabla^{\Omega^\bullet\left(X,F|_X\right)},i_{T^H}\right]=0,\quad i_{T^H}^2=0.\end{aligned} \tag{3.13}$$

As in Section 1.6, the flat superconnection A' induces on $H^\bullet(X, F|_X)$ a flat connection $\nabla^{H^\bullet(X,F|_X)}$ which preserves the $\mathbf{Z}$-grading. This is the Gauss-Manin connection $\nabla^{H^\bullet(X,F|_X)}$ on $H^\bullet(X, F|_X)$.

3.3. A metric on TX and the tensors T and S

Let g^{TX} be a G-invariant Euclidean metric on TX. In the sequel, we identify TX and T^*X by the metric g^{TX}.

By [**B3**, Section 1], $\left(T^H M, g^{TX}\right)$ determine an Euclidean connection ∇^{TX} on TX. In fact let g^{TS} be an Euclidean metric on TS. We equip TM with the G-invariant metric $g^{TM} = \pi^* g^{TS} \oplus g^{TX}$. Let $\nabla^{TM,L}$ be the Levi-Civita connection on $\left(TM, g^{TM}\right)$. Let ∇^{TX} be the connection on TX,

$$\nabla^{TX} = P^{TX} \nabla^{TM,L}. \tag{3.14}$$

Let ∇^{TM} be the connection on TM,

$$\nabla^{TM} = \pi^* \nabla^{TS} \oplus \nabla^{TX}. \tag{3.15}$$

Let T be the torsion of ∇^{TM}. Put

$$S = \nabla^{TM,L} - \nabla^{TM}. \tag{3.16}$$

Then S is a 1-form on M with values in antisymmetric elements of $\mathrm{End}\,(TX)$. Classically, if $A, B, C \in TM$,

$$\begin{aligned} &S(A)\,B - S(B)\,A + T(A,B) = 0, \\ &2\,\langle S(A)\,B, C\rangle + \langle T(A,B), C\rangle + \langle T(C,A), B\rangle - \langle T(B,C), A\rangle = 0. \end{aligned} \tag{3.17}$$

By [**B3**, Theorem 1.9], we know that

- The connection ∇^{TX} preserves the metric g^{TX}.
- The connection ∇^{TX} and the tensors T and $\langle S(\cdot)\,\cdot, \cdot\rangle$ do not depend on g^{TS}.
- The tensor T takes its values in TX, and vanishes on $TX \times TX$.
- For any $A \in TM$, $S(A)$ maps TX into $T^H M$.
- For any $A, B \in T^H M$, $S(A)\,B \in TX$.
- If $A \in T^H M$, $S(A)\,A = 0$.

From (3.17), we find that if $A \in T^H M$, $B, C \in TX$,

$$\langle T(A,B), C\rangle = \langle T(A,C), B\rangle = -\langle S(B)\,C, A\rangle. \tag{3.18}$$

By construction, all the above objects are G-invariant. Now, we recall a simple result stated in [**B10**, Theorem 1.1].

Theorem 3.5. — *The connection ∇^{TX} on $\left(TX, g^{TX}\right)$ is characterized by the following two properties:*

- *On each fibre X, it restricts to the Levi-Civita connection.*

– *If* $u \in TS$,

$$\nabla^{TX}_{U^H} = L_{U^H} + \frac{1}{2}\left(g^{TX}\right)^{-1} L_{U^H} g^{TX}. \tag{3.19}$$

If $U, V \in TS$,

$$T\left(U^H, V^H\right) = T^H\left(U, V\right). \tag{3.20}$$

If $U \in TS, A \in TX$,

$$T\left(U^H, A\right) = \frac{1}{2}\left(g^{TX}\right)^{-1} L_{U^H} g^{TX} A. \tag{3.21}$$

Let dv_X be the volume along the fibre X which is associated to the metric g^{TX}. Let $e_1, \ldots, e_n$ be an orthonormal basis of TX. Set

$$e = -\sum_{i=1}^{n} S\left(e_i\right) e_i. \tag{3.22}$$

Then using the properties which were listed after (3.17), $e \in T^H M$.

If $U \in TS$, let $\mathrm{div}_X\left(U^H\right)$ be the smooth function along X such that

$$L_{U^H} dv_X = \mathrm{div}_X\left(U^H\right) dv_X. \tag{3.23}$$

Now we have a result stated in [**BF1**, Proposition 1.4].

Proposition 3.6. — *If* $U \in TS$,

$$\left\langle e, U^H \right\rangle = \mathrm{div}_X\left(U^H\right). \tag{3.24}$$

Take $g \in G$. The metric g^{TX} induces a metric g^{TX_g} on TX_g. Also since ∇^{TX} is G-invariant, $\nabla^{TX}|_{M_g}$ preserves TX_g.

Proposition 3.7. — *The restriction of* ∇^{TX} *to* TX_g *is exactly the Euclidean connection canonically attached to* $\left(T^H M|_{M_g}, g^{TX_g}\right)$. *Moreover the tensors* T *and* $\langle S(\cdot)\cdot, \cdot\rangle$ *restrict to the corresponding tensors associated* $\left(M_g, T^H M|_{M_g}, g^{TX_g}\right)$.

Proof. — Clearly the metric g^{TM} considered in Section 3.3 is G-invariant. Therefore M_g is totally geodesic in M with respect to g^{TM}, and so $\nabla^{TM,L}_{M_g}$ preserves TM_g. The first part of our Proposition follows. The remainder of the Proposition follows from (3.11), (3.17) and from Theorem 3.5. □

3.4. The adjoint superconnection

We make the same assumptions as in Section 3.3. In addition we equip F with a G-invariant Hermitian metric g^F. Let $\nabla^{F,*}$ be the connection adjoint to ∇^F with respect to g^F. As in (1.9), set

$$\omega\left(\nabla^F, g^F\right) = \left(g^F\right)^{-1} \nabla^F g^F. \tag{3.25}$$

Then by (1.10),

$$\nabla^{F,*} = \nabla^F + \omega\big(\nabla^F, g^F\big). \tag{3.26}$$

As in (1.11), we define the connection $\nabla^{F,u}$ by

$$\nabla^{F,u} = \nabla^F + \frac{1}{2}\omega\big(\nabla^F, g^F\big). \tag{3.27}$$

Then $\nabla^{F,u}$ is a unitary connection on F. By (1.30), its curvature is given by

$$\nabla^{F,u,2} = -\frac{1}{4}\omega^2\left(\nabla^F, g^F\right). \tag{3.28}$$

Let $\nabla^{\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}$ be the unitary connection on $\Lambda^\bullet(T^*X)\,\widehat{\otimes}\,F$ induced by ∇^{TX} and $\nabla^{F,u}$.

Let $*^{TX}$ be the Hodge star operator associated to g^{TX}. We equip $\Omega^\bullet\big(X, F|_X\big)$ with the Hermitian product such that if $s, s' \in \Omega^\bullet\big(X, F|_X\big)$,

$$\langle s, s'\rangle = \int_X \langle s \wedge *s'\rangle_F = \int_X \langle s, s'\rangle_{\Lambda^\bullet(T^*X)\widehat{\otimes}F}\, dv_X. \tag{3.29}$$

Let $g^{\Omega^\bullet\left(X,F|_X\right)}$ be the corresponding metric on $\Omega^\bullet\big(X, F|_X\big)$. Then $g^{\Omega^\bullet\left(X,F|_X\right)}$ is G-invariant.

Now we will use the notation in Section 1.3. Let A'' be the adjoint of the superconnection A' with respect to the metric $g^{\Omega^\bullet\left(X,F|_X\right)}$. The adjoint $d^{X,*}$ is just the fibrewise adjoint of d^X. Let $\nabla^{\Omega^\bullet\left(X,F|_X\right),*}$ be the connection on $\Omega^\bullet\big(X, F|_X\big)$ which is adjoint to $\nabla^{\Omega^\bullet\left(X,F|_X\right)}$ with respect to $g^{\Omega^\bullet\left(X,F|_X\right)}$. Recall that T^H, defined in (3.11), is a section of $\Lambda^2\left(T^*S\right)\widehat{\otimes}\,TX$. Since TX and T^*X are identified by g^{TX}, we can consider T^H as a section of $\Lambda^2\left(T^*S\right)\widehat{\otimes}\,T^*X$. Then $T^H\wedge$ acts naturally on $\Lambda^\bullet\left(T^*S\right)\widehat{\otimes}\,\Omega^\bullet\big(X, F|_X\big)$ and increases the total degree by 3.

Then we have the result stated in [**BLo1**, Proposition 3.7].

Proposition 3.8. — *The following identity holds,*

$$A'' = d^{X,*} + \nabla^{\Omega^\bullet\left(X,F|_X\right),*} - T^H\wedge. \tag{3.30}$$

Now we use the formalism of Section 1.5. Namely, set

$$A = \frac{1}{2}\left(A'' + A'\right), \qquad B = \frac{1}{2}\left(A'' - A'\right). \tag{3.31}$$

Then A is a G-invariant superconnection on $\Omega^\bullet\big(X, F|_X\big)$, and B is a smooth G-invariant section of $\left(\Lambda^\bullet\left(T^*S\right)\widehat{\otimes}\,\mathrm{End}\left(\Omega^\bullet\big(X, F|_X\big)\right)\right)^{\mathrm{odd}}$, such that

$$B^* = -B. \tag{3.32}$$

The obvious analogue of Proposition 1.5 still holds.

3.5. Clifford algebras

Recall that TX and T^*X have been identified by the metric g^{TX}. If $A \in TX$, let $c(A), \hat{c}(A)$ be the odd endomorphisms of $\Lambda^\bullet(T^*X)$,

$$c(A) = A \wedge -i_A, \qquad \hat{c}(A) = A \wedge +i_A. \tag{3.33}$$

If A, B in TX,

$$[c(A), c(B)] = -2\langle A, B\rangle, \qquad [\hat{c}(A), \hat{c}(B)] = 2\langle A, B\rangle, \qquad [c(A), \hat{c}(B)] = 0. \tag{3.34}$$

Let $c(TX)$ be the bundle of Clifford algebras on $\left(TX, g^{TX}\right)$. Then $c(TX)$ is the algebra over $\mathbf{R}$ generated by $1, A \in TX$ and the commutation relations for $A, B \in TX$,

$$[A, B] = -2\langle A, B\rangle. \tag{3.35}$$

Then (3.34) says that $A \to c(A)$ and $A \to i\hat{c}(A)$ give two representations of the bundle of Clifford algebras $c(TX)$.

Also $c(TX)$ acts naturally on itself by multiplication on the left and on the right, and these two actions commute. They will be denoted respectively by c^l and c^r. Classically, there is a $\mathbf{Z}$-graded isomorphism of vector spaces $c(TX) \simeq \Lambda^\bullet(T^*X)$. Let τ be the operator on $\Lambda^\bullet(T^*X)$, which is 1 on $\Lambda^{\text{even}}(T^*X)$, -1 on $\Lambda^{\text{odd}}(T^*X)$. Then one verifies easily that under the above isomorphism, if $A \in TX$,

$$c(A) = c^l(A), \qquad \hat{c}(A) = \tau c^r(A). \tag{3.36}$$

In the sequel, we will often use the notation $c(TX)$ and $\hat{c}(TX)$ for the bundle of algebras generated by 1 and the $c(A)$'s and by 1 and the $\hat{c}(A)$'s.

3.6. The Levi-Civita superconnection

By imitating (1.11), put

$$\begin{aligned} D^X &= d^X + d^{X,*}, \\ \nabla^{\Omega^\bullet\left(X, F|_X\right), u} &= \frac{1}{2}\left(\nabla^{\Omega^\bullet\left(X, F|_X\right)} + \nabla^{\Omega^\bullet\left(X, F|_X\right), *}\right), \\ \omega\left(\nabla^{\Omega^\bullet\left(X, F|_X\right)}, g^{\Omega^\bullet\left(X, F|_X\right)}\right) &= \nabla^{\Omega^\bullet\left(X, F|_X\right), *} - \nabla^{\Omega^\bullet\left(X, F|_X\right)}. \end{aligned} \tag{3.37}$$

Then $\nabla^{\Omega^\bullet\left(X, F|_X\right), u}$ is a unitary connection on $\Omega^\bullet\left(X, F|_X\right)$. Moreover D^X is a fibrewise self-adjoint operator acting on $\Omega^\bullet\left(X, F|_X\right)$. By Hodge theory,

$$\ker D^X \simeq H^\bullet(X, F|_X). \tag{3.38}$$

As a finite dimensional subbundle of $\Omega^\bullet\left(X, F|_X\right)$, $\ker D^X$ inherits the L_2 metric of $\Omega^\bullet\left(X, F|_X\right)$. Let $g_{L_2}^{H^\bullet(X, F|_X)}$ be the corresponding Hermitian metric on $H^\bullet(X, F|_X)$. Recall that $e \in T^H M$ was defined in (3.22). The following result was established in [**BLo1**, Section 3 (d)].

Proposition 3.9. — *If s is a smooth section of $\Omega^\bullet\big(X, F|_X\big)$, if $U \in TS$, then*

$$\nabla_U^{\Omega^\bullet\left(X,F|_X\right),u} s = \nabla_{U^H}^{\Lambda^\bullet(T^*X)\widehat{\otimes}F,u} s + \frac{1}{2}\left\langle e, U^H\right\rangle. \tag{3.39}$$

Observe that $c\left(T^H\right)$ is a smooth section of $\left(\Lambda^2\left(T^*S\right)\widehat{\otimes}\operatorname{End}\left(\Lambda^\bullet(T^*X)\right)\right)^{\mathrm{odd}}$. Let $e_1, \ldots, e_n$ be an orthonormal basis of TX. Let $f_1, \ldots, f_m$ be a basis of TS, let $f^1, \ldots, f^m$ be the corresponding dual basis of T^*M. In the next definition, we view S as a form along the fibres X with values in $\operatorname{End}(TM)$.

In the sequel, we adopt Einstein's conventions. By [**BZ1**, Proposition 4.12],

$$D^X = c(e_i)\nabla_{e_i}^{\Lambda^\bullet(T^*X)\widehat{\otimes}F,u} - \frac{1}{2}\widehat{c}(e_i)\omega\big(\nabla^F, g^F\big)\left(e_i\right). \tag{3.40}$$

Let $\nabla^{\Lambda^\bullet(T^*X)}$ be the connection on $\Lambda^\bullet(T^*X)$ induced by ∇^{TX}. Along the fibres X, the vector bundle $\Lambda^\bullet\left(T^*S\right)$ is trivial. Let $\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}$ be the obvious connection on $\Lambda^\bullet\left(T^*S\right)\widehat{\otimes}\,\Lambda^\bullet(T^*X)$ along the fibres X which is induced by $\nabla^{\Lambda^\bullet(T^*X)}$.

Definition 3.10. — Let ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}$ be the connection along the fibres X on $\Lambda^\bullet\left(T^*S\right)\widehat{\otimes}\,\Lambda^\bullet(T^*X)$,

$$\begin{aligned}{}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} = \nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} &+ \frac{1}{2}\left\langle Se_i, f_\alpha^H\right\rangle\sqrt{2}c(e_i)f^\alpha \\ &+ \frac{1}{2}\left\langle Sf_\alpha^H, f_\beta^H\right\rangle f^\alpha f^\beta.\end{aligned} \tag{3.41}$$

The curvature ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X),2}$ of the connection ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}$ is a section of

$$\Lambda^\bullet(T^*X)\,\widehat{\otimes}\,\Lambda^\bullet\left(T^*S\right)\widehat{\otimes}\operatorname{End}\left(\Lambda^\bullet(T^*X)\right).$$

Again, expressions involving $\nabla^{TX,2}$ and S will only be viewed as forms along the fibres X.

Proposition 3.11. — *The following identity holds,*

$$\begin{aligned}{}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X),2} &= \frac{1}{4}\left\langle\nabla^{TX,2}e_i, e_j\right\rangle\left(c(e_i)c(e_j) - \widehat{c}(e_i)\widehat{c}(e_j)\right) \\ &+ \frac{1}{2}\left\langle\left(SP^{TX}S + \nabla^{TX}S\right)f_\alpha^H, f_\beta^H\right\rangle f^\alpha f^\beta + \frac{1}{2}\left\langle\nabla^{TX}Se_i, f_\alpha^H\right\rangle\sqrt{2}c(e_i)f^\alpha.\end{aligned} \tag{3.42}$$

Proof. — This is an easy identity established in [**B10**, Proposition 11.8]. □

The following identity was established in [**B3**, Theorem 4.14], [**B5**, Théorème 2.3].

Proposition 3.12. — *If $A, A' \in TX, B, B' \in TM$, then*

$$\begin{aligned}&\left\langle\nabla^{TX,2}(A, A')P^{TX}B, P^{TX}B'\right\rangle + \left\langle SP^{TX}S(A, A')B, B'\right\rangle \\ &\qquad + \left\langle\left(\nabla_\cdot^{TX}S\right)(A, A')B, B'\right\rangle = \left\langle\nabla^{TX,2}(B, B')A, A'\right\rangle.\end{aligned} \tag{3.43}$$

When we equip F with the connection $\nabla^{F,u}$, the connection ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}$ extends to a connection ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}$ on $\Lambda^\bullet(T^*S)\,\widehat{\otimes}\,\Lambda^\bullet(T^*X)\,\widehat{\otimes}\,F$ along the fibres X.

Definition 3.13. — Put

$$ {}^1\nabla_t^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u} = \psi_t^{-1}\,{}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}\psi_t. \tag{3.44}$$

Now we use the same notation as in (3.41).

Theorem 3.14. — *The following identity holds,*

$$\begin{aligned} A &= \frac{1}{2}D^X + \nabla^{\Omega^\bullet\left(X,F|_X\right),u} - \frac{1}{2}c\left(T^H\right), \\ B &= -\frac{1}{2}\widehat{c}(e_i)\,{}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{1/2,e_i} + \frac{1}{4}c(e_i)\omega\left(\nabla^F,g^F\right)(e_i) \\ &\quad + \frac{1}{2}f^\alpha\omega\left(\nabla^F,g^F\right)\left(f_\alpha^H\right). \end{aligned} \tag{3.45}$$

Proof. — The first identity in (3.45) was established in [**BLo1**, Proposition 3.9]. By the same reference,

$$\begin{aligned} B = -\frac{1}{2}\widehat{c}(e_i)\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{e_i} + \frac{1}{4}c(e_i)\omega\left(\nabla^F,g^F\right)(e_i) \\ + \frac{1}{2}f^\alpha\left(\left\langle S\left(e_i\right)e_j,f_\alpha^H\right\rangle c(e_i)\widehat{c}(e_j) + \omega\left(\nabla^F,g^F\right)\left(f_\alpha^H\right)\right) - \frac{1}{2}\widehat{c}\left(T^H\right). \end{aligned} \tag{3.46}$$

By (3.17), (3.46), we get the second identity in (3.45). □

Remark 3.15. — As observed in [**BLo1**, Remark 3.10], (3.45) shows that A is a special case of a Levi-Civita superconnection in the sense of [**B3**]. The second identity in (3.45) is of special interest. It shows that B is a generalized fibrewise Dirac operator, in which the fibrewise connection $\nabla^{\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}$ is replaced by the connection ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{1/2}$.

3.7. A rescaling of the metric on TX

For $t > 0$, set

$$g_t^{TX} = \frac{g^{TX}}{t}. \tag{3.47}$$

Let $g_t^{\Omega^\bullet\left(X,F|_X\right)}$ be the metric on $\Omega^\bullet\left(X,F|_X\right)$ associated to g_t^{TX}, g^F.

Let N be the number operator of $\Omega^\bullet\left(X,F|_X\right)$, i.e. N acts by multiplication by k on $\Omega^k\left(X,F\right)$. One verifies easily that

$$g_t^{\Omega^\bullet\left(X,F|_X\right)} = t^{N-n/2}g^{\Omega^\bullet\left(X,F|_X\right)}. \tag{3.48}$$

Therefore, up to the constant factor $t^{-n/2}$, the metric $g_t^{\Omega^\bullet(X,F|_X)}$ fits with the conventions used in Section 1.7.

Let A_t'' be the adjoint of A' with respect to $g_t^{\Omega^\bullet(X,F|_X)}$. Clearly $A'' = A_1''$. As in (1.41), we have,

$$A_t'' = t^{-N} A'' t^N. \tag{3.49}$$

As in (1.42), set

$$A_t = \frac{1}{2}(A_t'' + A'), \qquad B_t = \frac{1}{2}(A_t'' - A'). \tag{3.50}$$

Now, we imitate Definition 1.19.

Definition 3.16. — For $t > 0$, set

$$C_t' = t^{N/2} A' t^{-N/2}, \qquad C_t'' = t^{-N/2} A'' t^{N/2}. \tag{3.51}$$

Then C_t' is a flat superconnection on $\Omega^\bullet(X, F|_X)$, and C_t'' is its adjoint with respect to $g^{\Omega^\bullet(X,F|_X)}$. Set

$$C_t = \frac{1}{2}(C_t'' + C_t'), \qquad D_t = \frac{1}{2}(C_t'' - C_t'). \tag{3.52}$$

As in (1.48), we get

$$C_t = t^{N/2} A_t t^{-N/2}, \qquad D_t = t^{N/2} B_t t^{-N/2}. \tag{3.53}$$

Of course, all the objects which we just defined are G-invariant.

Proposition 3.17. — *For $t > 0$, the following identities hold,*

$$C_t = \psi_t^{-1} \sqrt{t}\, A \psi_t, \qquad D_t = \psi_t^{-1} \sqrt{t}\, B \psi_t. \tag{3.54}$$

Proof. — Since A' is of total degree 1, the proof of our Proposition is the same as the proof of Proposition 1.20. □

3.8. A Lichnerowicz formula

Let $R^{F,u}$ be the curvature of the connection $\nabla^{F,u}$ on F. By (1.30),

$$R^{F,u} = -\frac{1}{4}\omega^2(\nabla^F, g^F). \tag{3.55}$$

Set

$$R^{TX} = \nabla^{TX,2}. \tag{3.56}$$

Definition 3.18. — Put

$$\mathcal{R} = \frac{1}{4}\langle e^i, R^{TX} e^j \rangle \hat{c}(e_i)\hat{c}(e_j) + R^{F,u}. \tag{3.57}$$

Then $\mathcal{R}$ is a smooth section of $\Lambda^\bullet(T^*M)\,\widehat{\otimes}\,\hat{c}\,(TX)\,\widehat{\otimes}\,\mathrm{End}\,(F)$. Let $e_1,\ldots,e_n$ be a locally defined smooth orthonormal basis of TX. Let E be a vector bundle on M. Let∇^E be any connection on E along the fibres X. In the sequel we use the notation,

$$\left(\nabla^E_{e_i}\right)^2 = \sum_{i=1}^{n}\left(\nabla^E_{e_i}\right)^2 - \nabla^E_{\sum_{i=1}^n \nabla^{TX}_{e_i} e_i}. \tag{3.58}$$

One verifies easily that (3.58) does not depend on the choice of the basis $e_1,\ldots,e_n$.

Let K be the scalar curvature of the fibre $\left(X, g^{TX}\right)$. Let z be an odd Grassmann variable which anticommutes with all the other odd objects we met before. The following formula was established in [**BLo1**, Theorem 3.11] as a consequence of the Lichnerowicz formula for the curvature of the Levi-Civita superconnection given in [**BLo1**, Theorem 3.6]. Observe that the second equation in (1.30) asserts that the tensor $U,V\in TX\to\nabla^{F,u}\omega\left(\nabla^F,g^F\right)(V)$ is symmetric.

Theorem 3.19. — *Given $t\in\mathbf{R}_+^*$, the following identity holds,*

$$\begin{aligned}
C_t^2 - zD_t = &-\frac{t}{4}\left({}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{t/2,e_i} - \frac{z}{\sqrt{t}}\widehat{c}(e_i)\right)^2 + t\frac{K}{16}\\
&+\frac{t}{8}c(e_i)c(e_j)\mathcal{R}\left(e_i,e_j\right)+\frac{1}{2}f^\alpha f^\beta\mathcal{R}\left(f^H_\alpha,f^H_\beta\right)+\frac{\sqrt{t}}{2}c(e_i)f^\alpha\mathcal{R}\left(e_i,f^H_\alpha\right)\\
&+\frac{t}{16}\left[\omega\left(\nabla^F,g^F\right)(e_i)\right]^2-\frac{\sqrt{t}}{4}f^\alpha\widehat{c}(e_i)\nabla^{TX\otimes F,u}_{f^H_\alpha}\omega\left(\nabla^F,g^F\right)(e_i)\\
&+\frac{t}{32}\widehat{c}(e_i)\widehat{c}(e_j)\omega^2\left(\nabla^F,g^F\right)(e_i,e_j)-\frac{t}{8}c(e_i)\widehat{c}(e_j)\nabla^{TX\otimes F,u}_{e_i}\omega\left(\nabla^F,g^F\right)(e_j)\\
&-\frac{1}{4}z\sqrt{t}\,c(e_i)\omega\left(\nabla^F,g^F\right)(e_i)-\frac{1}{2}zf^\alpha\omega\left(\nabla^F,g^F\right)\left(f^H_\alpha\right).
\end{aligned}\tag{3.59}$$

Now we will establish another, but essentially equivalent Lichnerowicz formula. Recall that ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}_{t/2}$ is a connection on $\Lambda^\bullet\left(T^*S\right)\widehat{\otimes}\Lambda^\bullet(T^*X)$ along the fibres X. Its curvature lies in $\Lambda^\bullet(T^*X)\,\widehat{\otimes}\,\Lambda^\bullet\left(T^*S\right)\widehat{\otimes}\,\mathrm{End}\left(\Lambda^\bullet(T^*X)\right)$, and was computed in Proposition 3.11.

Theorem 3.20. — *Given $t\in\mathbf{R}_+^*$, the following identity holds,*

$$\begin{aligned}
C_t^2 - zD_t = &-\frac{t}{4}\left({}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{t/2,e_i} - \frac{z}{\sqrt{t}}\widehat{c}(e_i)\right)^2 + t\frac{K}{16}\\
&-\frac{t}{8}\widehat{c}(e_i)\widehat{c}(e_j)\left({}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X),2}_{t/2}\left(e_i,e_j\right)\right.\\
&\left.+\frac{1}{4}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle\widehat{c}\left(e_k\right)\widehat{c}\left(e_l\right)-\frac{1}{4}\omega^2\left(\nabla^F,g^F\right)\left(e_i,e_j\right)\right)+
\end{aligned}\tag{3.60}$$

$$
\begin{aligned}
&+\frac{t}{16}\left[\omega\left(\nabla^F,g^F\right)(e_i)\right]^2-\frac{t}{32}c(e_i)c(e_j)\omega^2\left(\nabla^F,g^F\right)(e_i,e_j)\\
&-\frac{\sqrt{t}}{8}c(e_i)f^\alpha\left[\omega\left(\nabla^F,g^F\right)(e_i),\omega\left(\nabla^F,g^F\right)\left(f_\alpha^H\right)\right]-\frac{1}{8}\omega^2\left(\nabla^F,g^F\right)\left(f_\alpha^H,f_\beta^H\right)\\
&+\frac{t}{8}\widehat{c}(e_i)c(e_j)\,{}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{t/2,e_i}\omega\left(\nabla^F,g^F\right)(e_j)\\
&+\frac{\sqrt{t}}{4}\widehat{c}(e_i)f^\alpha\,{}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{t/2,e_i}\omega\left(\nabla^F,g^F\right)\left(f_\alpha^H\right)\\
&-\frac{1}{4}z\sqrt{t}\,c(e_i)\omega\left(\nabla^F,g^F\right)(e_i)-\frac{1}{2}zf^\alpha\omega\left(\nabla^F,g^F\right)\left(f_\alpha^H\right).
\end{aligned}
$$

Proof. — We use formula (3.45) for B. We will consider here the $i\widehat{c}(e_j)$ as standard Clifford variables and the $c(e_j)$ as auxiliary Clifford variables. The operator

$$
i\widehat{c}(e_i)\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{1/2,e_i}
$$

is then an ordinary fibrewise Dirac operator, to which the classical Lichnerowicz formula in [**BF2**, Proposition 2.1] can be applied. Our Theorem follows. □

Remark 3.21. — The comparison of Theorems 3.19 and 3.20 is interesting. To prove directly that formulas (3.59) and (3.60) are in fact identical, one should use Propositions 3.11 and 3.12.

3.9. A unitary connection on $H^\bullet(X,F|_X)$

We define $\omega\left(\nabla^{H^\bullet(X,F|_X)},g_{L_2}^{H^\bullet(X,F|_X)}\right)$ as in (1.9). Let $\nabla^{H^\bullet(X,F|_X),*}$ be the connection on $H^\bullet(X,F|_X)$ which is the adjoint of $\nabla^{H^\bullet(X,F|_X)}$ with respect to the metric $g_{L_2}^{H^\bullet(X,F|_X)}$. Set

$$
V=\frac{1}{2}\left(d^{X,*}-d^X\right). \tag{3.61}
$$

Then $\ker V=\ker D^X$, and so by (3.38),

$$
\ker V\simeq H^\bullet(X,F|_X). \tag{3.62}
$$

Let $P^{\ker V}$ be the orthogonal projection operator from $\Omega^\bullet\left(X,F|_X\right)$ on $\ker V$. By [**BLo1**, Proposition 3.14], the obvious analogue of (1.38) holds, i.e.

$$
\begin{aligned}
&\nabla^{H^\bullet(X,F|_X)}=P^{\ker V}\nabla^{\Omega^\bullet\left(X,F|_X\right)},\\
&\nabla^{H^\bullet(X,F|_X),*}=P^{\ker V}\nabla^{\Omega^\bullet\left(X,F|_X\right),*},\\
&\omega\left(\nabla^{H^\bullet(X,F|_X)},g_{L_2}^{H^\bullet(X,F|_X)}\right)=P^{\ker V}\omega\left(\nabla^{\Omega^\bullet\left(X,F|_X\right)},g^{\Omega^\bullet\left(X,F|_X\right)}\right)P^{\ker V}.
\end{aligned} \tag{3.63}
$$

3.10. The odd closed forms $h_g\left(A', g_t^{\Omega^\bullet(X,F|X)}\right)$

First we state a result established in [**BLo1**, Theorem 3.15], which is the obvious extension of Proposition 1.6 in this infinite dimensional context. We define the Lefschetz number $\chi_g(F)$ by the formula

$$\chi_g(F) = \sum_{j=0}^{n} (-1)^j \operatorname{Tr}^{H^j(X,F|X)}[g]. \tag{3.64}$$

Then $\chi_g(F)$ is a locally constant function on S.

Proposition 3.22. — *For any $t > 0$,*

$$\operatorname{Tr}_s\left[g \exp\left(-A_t^2\right)\right] = \chi_g(F). \tag{3.65}$$

Proof. — The proof is the same as the corresponding proof in [**BLo1**]. □

Now we follow [**BLo1**, Section 3] and also Chapter 1. In the sequel, we set

$$h(x) = xe^{x^2}. \tag{3.66}$$

Take $g \in G$.

Definition 3.23. — For $t > 0$, set

$$h_g\left(A', g_t^{\Omega^\bullet(X,F|X)}\right) = (2i\pi)^{1/2} \varphi \operatorname{Tr}_s\left[gh\left(B_t\right)\right]. \tag{3.67}$$

Similarly, set

$$h_g\left(C_t', g^{\Omega^\bullet(X,F|X)}\right) = (2i\pi)^{1/2} \varphi \operatorname{Tr}_s\left[gh\left(D_t\right)\right]. \tag{3.68}$$

By (3.53), as in (1.49),

$$h_g\left(A', g_t^{\Omega^\bullet(X,F|X)}\right) = h_g\left(C_t', g^{\Omega^\bullet(X,F|X)}\right). \tag{3.69}$$

Let $e\left(TX_g, \nabla^{TX_g}\right)$ be the closed Euler form in Chern-Weil theory, which represents the Euler class of TX_g associated to the Euclidean connection ∇^{TX_g}. Let R^{TX_g} be the curvature of ∇^{TX_g}. Then

$$\begin{aligned} e\left(TX_g, \nabla^{TX_g}\right) &= \operatorname{Pf}\left[\frac{R^{TX_g}}{2\pi}\right] \text{ if } \dim X_g \text{ is even,} \\ &= 0 \text{ if } \dim X_g \text{ is odd.} \end{aligned} \tag{3.70}$$

Let $e(TX_g) \in H^\bullet(M_g, \mathbf{Q})$ be the cohomology class of $e\left(TX_g, \nabla^{TX_g}\right)$, i.e. the Euler class of TX_g.

Let $G(g) = \overline{\langle g \rangle}$ be the closed Lie subgroup of G generated by g. Then $G(g)$ keeps M_g fixed and acts on F by flat automorphisms. Therefore the forms $h_g\left(\nabla^F, g^F\right)$ on M_g can be defined as in Definition 1.7. They are closed odd forms on M_g.

Now we state an extension of [**BLo1**, Theorem 3.16], where only the case $g = 1$ was considered. Recall that as we saw in Sections 3.2 and 3.6, $H^\bullet(X, F|_X)$ is a $\mathbf{Z}$-graded vector bundle on S, equipped with the flat Gauss-Manin connection $\nabla^{H^\bullet(X,F|_X)}$, and with the metric $g_{L_2}^{H^\bullet(X,F|_X)}$.

Theorem 3.24. — *The forms* $h_g\left(A', g_t^{\Omega^\bullet(X,F|_X)}\right)$ *are odd, closed, and their cohomology class does not depend on* $t > 0$. *They are real if* $g = 1$. *Moreover as* $t \to 0$,

$$h_g\left(A', g_t^{\Omega^\bullet(X,F|_X)}\right) = \int_{X_g} e\left(TX_g, \nabla^{TX_g}\right) h_g\left(\nabla^F, g^F\right) + \mathcal{O}\left(\sqrt{t}\right). \tag{3.71}$$

As $t \to +\infty$,

$$h_g\left(A', g_t^{\Omega^\bullet(X,F|_X)}\right) = h_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2}^{H^\bullet(X,F|_X)}\right) + \mathcal{O}\left(1/\sqrt{t}\right). \tag{3.72}$$

Proof. — The proof of the first part of our Theorem is the same as the proof of [**BLo1**, Theorem 3.16] and of Theorem 1.8. The proof of (3.72) is the same as the proof of the corresponding result in [**BLo1**, Theorem 3.16] with $g = 1$.

Now we concentrate on the proof of (3.71). When $g = 1$, our result was already established in [**BLo1**]. In the case of a general g, we proceed as follows. We view z as the generator of $\mathbf{R}^*$. If $\alpha \in \Lambda\left(T^{*S} \times \mathbf{R}^*\right)$, if

$$\alpha = \beta + z\gamma, \quad \text{with} \quad \beta, \gamma \in \Lambda^\bullet\left(T^*S\right), \tag{3.73}$$

set

$$\alpha^z = \gamma. \tag{3.74}$$

Clearly

$$\mathrm{Tr_s}\left[gh\left(D_t\right)\right] = \mathrm{Tr_s}\left[g \exp\left(-C_t^2 + zD_t\right)\right]^z. \tag{3.75}$$

Let $P_t\left(x, x'\right)$ be the smooth kernel associated to $\exp\left(-C_t^2 + zD_t\right)$ with respect to $dv_X / (2\pi)^{\dim X}$, so that if $s \in \Omega^\bullet\left(X, F|_X\right)$,

$$\exp\left(-C_t^2 + zD_t\right) s\left(x\right) = \int_X P_t\left(x, x'\right) s\left(x'\right) \frac{dv_X\left(x'\right)}{(2\pi)^{\dim X'}}. \tag{3.76}$$

Then

$$\mathrm{Tr_s}\left[g \exp\left(-C_t^2 + zD_t\right)\right] = \int_X \mathrm{Tr_s}\left[gP_t\left(g^{-1}x, x\right)\right] \frac{dv_X\left(x\right)}{(2\pi)^{\dim X}}. \tag{3.77}$$

Standard results on heat kernels show that as $t \to 0$, the integral in (3.77) localizes near the fixed point fibre X_g. Then we combine the techniques of the local families index theorem of [**B3**] with the techniques used in the proof of the Lefschetz fixed point formulas to obtain (3.71). We refer to Section 13.3-13.5 for a detailed account of the techniques which are needed in the proof of (3.71). This involves in particular the use of the rescaling techniques of Berline-Getzler-Vergne [**BeGeV**, Chapter 10].

The proof of our Theorem is completed. □

Now we obtain an equivariant extension of the Riemann-Roch-Grothendieck formula of Bismut-Lott [**BLo1**, Theorem 3.17].

Theorem 3.25. — *The following identity holds,*

$$h_g\left(\nabla^{H^\bullet(X,F|X)}\right) = \int_{X_g} e\left(TX_g\right) h_g\left(\nabla^F\right) \text{ in } H^{\text{odd}}\left(S, \mathbf{C}\right). \tag{3.78}$$

Proof. — Our result follows from Theorem 3.24. □

Remark 3.26. — From (3.78), we get, for any odd $k \in \mathbf{N}$,

$$h_g\left(\nabla^{H^\bullet(X,F|X)}\right)^{(k)} = \int_{X_g} e\left(TX_g\right) h_g\left(\nabla^F\right)^{(k)} \text{ in } H^k\left(S, \mathbf{C}\right). \tag{3.79}$$

Since $\deg\left(e\left(TX_g\right)\right) = \dim X_g$, we deduce from (3.79) that this identity still holds when replacing $h\left(x\right) = xe^{x^2}$ by $h\left(x\right) = x^k$ for any odd $k \in \mathbf{N}$, and more generally, by any arbitrary holomorphic odd function $h\left(x\right)$.

3.11. A transgression formula

As in [**BLo1**, Section 3 (i)], we imitate the constructions of Section 1.7 in an infinite dimensional context.

Definition 3.27. — For $t > 0$, set

$$h_g^\wedge\left(A', g_t^{\Omega^\bullet\left(X,F|X\right)}\right) = \varphi \text{Tr}_{\text{s}}\left[\frac{N}{2} g h'\left(B_t\right)\right]. \tag{3.80}$$

Similarly, put

$$h_g^\wedge\left(C'_t, g^{\Omega^\bullet\left(X,F|X\right)}\right) = \varphi \text{Tr}_{\text{s}}\left[\frac{N}{2} g h'\left(D_t\right)\right]. \tag{3.81}$$

By (3.53), as in (1.49),

$$h_g^\wedge\left(A', g_t^{\Omega^\bullet\left(X,F|X\right)}\right) = h_g^\wedge\left(C'_t, g^{\Omega^\bullet\left(X,F|X\right)}\right). \tag{3.82}$$

Proposition 3.28. — *The following identities hold,*

$$\begin{aligned} h_g\left(A', g_t^{\Omega^\bullet\left(X,F|X\right)}\right) &= (2i\pi)^{1/2} \varphi \psi_t^{-1} \text{Tr}_{\text{s}}\left[g h\left(\sqrt{t}\, B\right)\right], \\ h_g^\wedge\left(A', g_t^{\Omega^\bullet\left(X,F|X\right)}\right) &= \varphi \psi_t^{-1} \text{Tr}_{\text{s}}\left[\frac{N}{2} g h'\left(\sqrt{t}\, B\right)\right]. \end{aligned} \tag{3.83}$$

Proof. — Using Proposition 3.17, the proof of our Proposition is the same as the proof of Proposition 1.21. □

Now we have the obvious extension of [**BLo1**, Theorem 3.20] and of Theorem 1.18.

Theorem 3.29. — *The form* $h_g^{\wedge}\left(A', g_t^{\Omega^\bullet(X,F|X)}\right)$ *is even. It is real if* $g = 1$. *Moreover,*

$$\frac{\partial}{\partial t} h_g\left(A', g_t^{\Omega^\bullet(X,F|X)}\right) = d\,\frac{h_g^{\wedge}\left(A', g_t^{\Omega^\bullet(X,F|X)}\right)}{t}. \tag{3.84}$$

Clearly, the function $\mathrm{Tr}^F[g]$ is locally constant on M_g. By the Lefschetz fixed point formula, we get

$$\chi_g(F) = \int_{X_g} \mathrm{Tr}^F[g]\, e(TX_g). \tag{3.85}$$

Put

$$\chi'_g(F) = \sum_{j=0}^{m} (-1)^j\, j \mathrm{Tr}^{H^j(X,F|X)}[g]. \tag{3.86}$$

Then $\chi_g(F), \chi'_g(f)$ are locally constant functions on S.

Now we state an extension of [**BLo1**, Theorem 3.21].

Theorem 3.30. — *As* $t \to 0$,

$$h_g^{\wedge}\left(A', g_t^{\Omega^\bullet(X,F|X)}\right) = \frac{n}{4}\chi_g(F)\, h'(0) + \mathcal{O}\left(\sqrt{t}\right). \tag{3.87}$$

As $t \to +\infty$,

$$h_g^{\wedge}\left(A', g_t^{\Omega^\bullet(X,F|X)}\right) = \frac{1}{2}\chi'_g(F)\, h'(0) + \mathcal{O}\left(1/\sqrt{t}\right). \tag{3.88}$$

Proof. — Using Theorem 3.24, and by proceeding as in the proof of [**BLo1**, Theorem 3.21], our Theorem follows. □

3.12. The equivariant analytic torsion forms

Now we follow [**BLo1**, Section 3 (j)] and Sections 1.9 and 1.10. Recall that $h(x)$ is still given by (3.66).

Definition 3.31. — Set

$$\begin{aligned}\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right) = -\int_0^{+\infty} \Bigg[& h_g^{\wedge}\left(A', g_t^{\Omega^\bullet(X,F|X)}\right) - \frac{1}{2}\chi'_g(F)\, h'(0) \\ & - \left(\frac{n}{4}\chi_g(F) - \frac{1}{2}\chi'_g(F)\right) h'\left(i\sqrt{t}/2\right)\Bigg]\frac{dt}{t}.\end{aligned} \tag{3.89}$$

From Theorem 3.30, we find that the integral in the right-hand side of (3.89) is well defined.

Now we establish an extension of [**BLo1**, Theorem 3.23].

Theorem 3.32. — *The form* $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ *is even, and real if* $g = 1$. *Moreover,*

$$d\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right) = \int_{X_g} e\left(TX_g, \nabla^{TX_g}\right) h_g\left(\nabla^F, g^F\right) - h_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2}^{H^\bullet(X,F|_X)}\right). \tag{3.90}$$

Proof. — Our Theorem follows from Theorems 3.29 and 3.30. □

Remark 3.33. — The forms $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ are called analytic torsion forms. By (3.90), we find that for any even $k \in \mathbf{N}$,

$$d\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)^{(k)} = \int_{X_g} e\left(TX_g, \nabla^{TX_g}\right) h_g\left(\nabla^F, g^F\right)^{(k+1)} - h_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2}^{H^\bullet(X,F|_X)}\right)^{(k+1)}. \tag{3.91}$$

Let $T'^H M, g'^{TX}, g'^F$ be another triple of data. We will denote with an extra prime the objects canonically attached to this new triple. Let $\widetilde{e}\left(TX_g, \nabla^{TX_g}, \nabla'^{TX_g}\right) \in \Omega^\bullet(M_g)/d\Omega^\bullet(M_g)$ be the corresponding Chern-Simons class, so that

$$d\widetilde{e}\left(TX_g, \nabla^{TX_g}, \nabla'^{TX_g}\right) = e\left(TX_g, \nabla'^{TX_g}\right) - e\left(TX_g, \nabla^{TX_g}\right). \tag{3.92}$$

Now we extend [**BLo1**, Theorem 3.24].

Theorem 3.34. — *The following identity holds,*

$$\mathcal{T}_{h,g}\left(T'^H M, g'^{TX}, \nabla^F, g'^F\right) - \mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right) = \int_{X_g} \widetilde{e}\left(TX_g, \nabla^{TX_g}, \nabla'^{TX_g}\right) h_g\left(\nabla^F, g^F\right) + \int_{X_g} e\left(TX_g, \nabla'^{TX_g}\right) \widetilde{h}_g\left(\nabla^F, g^F, g'^F\right) - \widetilde{h}_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2}^{H^\bullet(X,F|_X)}, g_{L_2}'^{H^\bullet(X,F|_X)}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{3.93}$$

Proof. — The proof of Theorem is an easy consequence of functoriality of the forms $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ and and Theorem 3.32. □

Remark 3.35. — Suppose that the connected components of X_g have odd dimension. This is true if X is orientable, and either X is odd dimensional and g preserves the orientation, or X is even dimensional and g reverses the orientation. If $H^\bullet(X, F|_X) = 0$, by Theorems 3.32 and 3.34, $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ is a closed form on S whose cohomology class does not depend on $\left(T^H M, g^{TX}, g^F\right)$.

Remark 3.36. — Let $\left(D^X\right)^{-1}$ be the inverse of D^X acting on the orthogonal bundle to $\ker D^X$ in $\Omega^\bullet\left(X, F|_X\right)$. For $s \in \mathbf{C}, \mathrm{Re}\,(s) > \dim(X)/2$, set

$$\vartheta_g(s) = -\mathrm{Tr}_s\left[N\left(D^{X,2}\right)^{-s}\right]. \tag{3.94}$$

Then $\vartheta_g(s)$ extends to a meromorphic function of $s \in \mathbf{C}$, which is holomorphic near $s = 0$. By definition, the equivariant Ray-Singer analytic torsion [**RS1**], [**BZ2**] of the de Rham complex $\left(\Omega^\bullet(X, F|_X), d^X\right)$ is given by $\frac{\partial \vartheta_g}{\partial s}(0)$. In the case where $g = 1$, it was shown by Bismut and Lott [**BLo1**, Theorem 3.29] that

$$\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)^{(0)} = \frac{1}{2}\frac{\partial \vartheta_g}{\partial s}(0). \tag{3.95}$$

The arguments in [**BLo1**] extend to the case of a general g. From equation (3.95), we derive the anomaly formulas for equivariant Ray-Singer metrics given in [**BZ2**, Theorem 0.1].

3.13. Analytic torsion forms associated to arbitrary functions

In [**BLo1**] and in Section 3.12, we defined the form $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$ only for $h(x) = xe^{x^2}$. We claim that if $h(x)$ is any holomorphic odd function such that (1.54) holds, we can still define the torsion forms $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$, and that the obvious analogues of the results of Sections 3.10-3.12 still hold. In fact because of the decay condition (1.54), the operators $h(B_t)$ and $h'(B_t)$ are trace class. The arguments on the behaviour of the considered forms as $t \to +\infty$ can be adapted word for word. As to the arguments on the behaviour if the forms as $t \to 0$, they can also be reproduced. In fact, by using finite propagation speed as in Chapter 13, one shows easily that the problem of convergence is local near X_g. As in Chapter 13, we then work with the resolvent equation, which leads to these convergence results.

Then we still have the degree by degree equation (3.91). We can write $h(x)$ in the form,

$$h(x) = \sum_{\substack{k \in \mathbf{N} \\ k \text{ odd}}} b_k x^k, \tag{3.96}$$

for $k \in \mathbf{N}$, k even, the right-hand side of (3.90) depends only on the a_{k+1}. The question then arises of knowing if $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)^{(k)} \in \Omega^\bullet(S)/d\Omega^\bullet(S)$ depends on h only via b_{k+1}. We will provide a partial answer to this question.

From now on, we still take $h(x)$ as in (3.66), i.e.

$$h(x) = xe^{x^2}. \tag{3.97}$$

For $a \in \mathbf{C}^*$, let $\sqrt{a}$ be any square root of a. Let $R(a)$ be a polynomial. As in (2.73), set

$$h_a(x) = R\left(\frac{\partial}{\partial a}\right)\frac{1}{\sqrt{a}}h(\sqrt{a}x). \tag{3.98}$$

Then if $a \in \mathbf{R}_+^*$, $h_a(x)$ verifies the assumptions in (1.54). We can then define the analytic torsion forms $\mathcal{T}_{h_a,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)$. Also by Theorem 3.32, the even

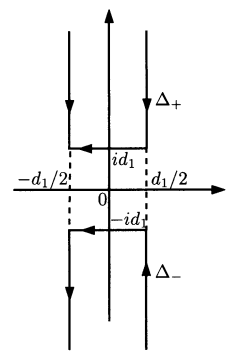

FIGURE 3.1

form $\mathcal{T}_{h_a}\left(T^H M, g^{TX_g}, g^F\right)$ is such that

$$d\mathcal{T}_{h_a,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right) = \int_{X_g} e\left(TX_g, \nabla^{TX_g}\right)(h_a)_g\left(\nabla^F, g^F\right) - (h_a)_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2}^{H^\bullet(X,F|_X)}\right). \tag{3.99}$$

Theorem 3.37. — *For $a \in \mathbf{R}_+^*$, the following identity holds,*

$$\mathcal{T}_{h_a,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right) = R\left(\frac{\partial}{\partial a}\right)\psi_a \mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right). \tag{3.100}$$

Proof. — Clearly,

$$\mathcal{T}_{h_a,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right) = R\left(\frac{\partial}{\partial a}\right)\mathcal{T}_{h(\sqrt{a}.)/\sqrt{a},g}\left(T^H M, g^{TX}, \nabla^F, g^F\right). \tag{3.101}$$

Also by Proposition 3.17,

$$\sqrt{a}C_t = \psi_a C_{at}\psi_a^{-1}. \tag{3.102}$$

From (3.89), (3.101), (3.102), we get (3.100). □

Remark 3.38. — Theorem 3.37 gives a strong indication that for any even $k \in \mathbf{N}$, $\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)^{(k)} \in \Omega^\bullet(S)/d\Omega^\bullet(S)$ depends on h only via b_{k+1}.

3.14. An identity for $k(D_t)$

For simplicity, we assume temporarily that M is compact. With the notation in (3.61),

$$B^{(0)} = V. \tag{3.103}$$

As in (2.18), we get

$$\mathrm{Sp}(B) = \mathrm{Sp}\left(B^{(0)}\right). \tag{3.104}$$

Since $\ker B^{(0)} \simeq H^\bullet(X, F|_X)$ has constant rank, there exists $d_1 \in \mathbf{R}_+^*$ such that

$$\left|\mathrm{Sp}\left(B^{(0)}\right)\right| \subset \{0\} \cup [2d_1, +\infty[. \tag{3.105}$$

Let $Q(x)$ be an odd polynomial. Let $k(x)$ be the holomorphic odd function,

$$k(x) = Q(x)\, e^{x^2}. \tag{3.106}$$

Recall that $\delta \subset \mathbf{C}$ is the circle of centre 0 and radius 1. Let $\Delta = \Delta_+ \cup \Delta_-$ be the contour in $\mathbf{C}$ indicated in Figure 3.1.

Definition 3.39. — For $t > 0$, set

$$\begin{aligned} G_t &= \psi_t^{-1} \frac{1}{2i\pi} \int_\Delta \frac{k\left(\sqrt{t}\,\lambda\right)}{\lambda - B} d\lambda \psi_t, \\ H_t &= \psi_t^{-1} \int_{\frac{d_1}{2}\delta} \frac{k\left(\sqrt{t}\,\lambda\right)}{\lambda - B} d\lambda \psi_t. \end{aligned} \tag{3.107}$$

Proposition 3.40. — *For any $t > 0$,*

$$k(D_t) = G_t + H_t. \tag{3.108}$$

Proof. — The proof is the same as the proof of Proposition 2.9. □

Let I^0 be the vector bundle of L_2 sections of $\Lambda^\bullet(T^*X) \widehat{\otimes} F$ along the fibres X, and let $\| \ \|_0$ be the norm on I^0 associated to the Hermitian product (3.29). If $L \in \mathcal{L}\left(I^0\right)$, for $p \geqslant 1$, put

$$\|L\|_p = \mathrm{Tr}\left[(L^*L)^{p/2}\right]^{1/p}. \tag{3.109}$$

Then (3.109) defines a norm on a vector subspace of $\mathcal{L}\left(I^0\right)$. For $p = 1$, we get the trace class operators. For $p = \infty$, by definition $\| \ \|_\infty = \| \ \|$ is the ordinary operator norm on $\mathcal{L}\left(I^0\right)$.

Now we establish an analogue of Proposition 2.10.

Proposition 3.41. — *There exist $C > 0, c > 0$ such that for $t \geqslant 1$,*

$$\|G_t\|_1 \leqslant Ce^{-ct}. \tag{3.110}$$

Proof. — Take $p \in \mathbf{N}, p > \dim X$. Let $k_p(\lambda)$ be the unique holomorphic function on $\mathbf{C} \setminus \mathbf{R}$ such that

- As $\lambda \to \pm i\infty, k_p(\lambda) \to 0$.
- The following identity holds,

$$\frac{k_p^{(p-1)}(\lambda)}{(p-1)!} = k(\lambda). \tag{3.111}$$

Clearly, if $\lambda \in \Delta$,

$$|\mathrm{Re}(\lambda)| \leqslant \frac{1}{2}|\mathrm{Im}(\lambda)|. \tag{3.112}$$

Using (3.112), we find that there exist $C > 0, C' > 0$ such that if $\lambda \in \Delta$,

$$\left|k_p\left(\sqrt{t}\,\lambda\right)\right| \leqslant C \exp\left(-C' t\,|\lambda|^2\right). \tag{3.113}$$

Clearly,

$$\frac{1}{2i\pi}\int_\Delta \frac{k\left(\sqrt{t}\,\lambda\right)}{\lambda - B} d\lambda = \frac{1}{2i\pi}\int_\Delta \frac{k_p\left(\sqrt{t}\,\lambda\right)}{\sqrt{t}^{\,p-1}(\lambda - B)^p} d\lambda. \tag{3.114}$$

If $\lambda \in \Delta$, we have the expansion,

$$(\lambda - B)^{-1} = \left(\lambda - B^{(0)}\right)^{-1} + \left(\lambda - B^{(0)}\right)^{-1} B^{(\geqslant 1)} \left(\lambda - B^{(0)}\right)^{-1} + \cdots \tag{3.115}$$

and the expansion only contains a finite number of terms. Also by (3.45), $B^{(\geqslant 1)}$ is an operator of order 0. By (3.105), (3.115), we find that there is $C > 0$ such that if $\lambda \in \Delta$,

$$\left\|(\lambda - B)^{-1}\right\|_\infty \leqslant C. \tag{3.116}$$

Fix $\lambda_0 \in \Delta$. Since $p > \dim X$, and B is fibrewise elliptic of order 1,

$$\left\|(\lambda_0 - B)^{-1}\right\|_p < +\infty. \tag{3.117}$$

If $\lambda \in \Delta$,

$$(\lambda - B)^{-1} = (\lambda_0 - B)^{-1} + (\lambda - \lambda_0)(\lambda_0 - B)^{-1}(\lambda - B)^{-1}. \tag{3.118}$$

From (3.116)-(3.118), we find that if $\lambda \in \Delta$,

$$\left\|(\lambda - B)^{-1}\right\|_p \leqslant C(1 + |\lambda|)\left\|(\lambda_0 - B)^{-1}\right\|_p \leqslant C'(1 + |\lambda|). \tag{3.119}$$

Using (3.119), we find that if $\lambda \in \Delta$,

$$\left\|(\lambda - B)^{-p}\right\|_1 \leqslant C(1 + |\lambda|)^p. \tag{3.120}$$

From (3.107), (3.113), (3.114), (3.120), we get (3.110). The proof of our Proposition is completed. □

We still proceed as in Section 2.3. Let $P^{\{0\}}$ be the orthogonal projection operator from $\Omega^\bullet(X, F|_X)$ on $\ker B^{(0)} \simeq H^\bullet(X, F|_X)$. Set

$$P^{\{0\}\perp} = 1 - P^{\{0\}}. \tag{3.121}$$

Then $P^{\{0\}\perp}$ is the orthogonal projection operator, which projects on the orthogonal bundle $\left(\ker B^{(0)}\right)^\perp$ to $\ker B^{(0)}$ in $\Omega^\bullet(X, F|_X)$. Also $B^{(0)}$ acts as an invertible operator on $\left(\ker B^{(0)}\right)^\perp$. Let $\left(B^{(0)}\right)^{-1}$ denote its inverse. We extend $\left(B^{(0)}\right)^{-1}$ to an operator which acts like 0 on $\ker B^{(0)}$.

Theorem 3.42. — *The obvious analogue of Theorem 2.11 holds. In particular, as* $t \to +\infty$,

$$\left\| H_t - P^{\{0\}} k\left(B^{H^\bullet(X,F|_X)}\right) P^{\{0\}} \right\|_1 = \mathcal{O}\left(1/\sqrt{t}\right). \tag{3.122}$$

Proof. — To establish the analogue of (2.30), we proceed as in the proof of Theorem 2.11. To prove (3.122), we also proceed as in the proof of Theorem 2.11. In fact observe that in (2.30), $i_0 \geqslant 1$, so that $P^{\{0\}}$ appears at least once. Now $P^{\{0\}}$ is a projector on a finite dimensional vector bundle, and in particular it is trace class. Since all the operators appearing in the right-hand side of (2.30) are bounded , the proof of (3.122) then proceeds as the proof of (2.31). □

3.15. A convergence result

Let $\ell \in [0,1] \to \nabla^F_\ell$ be a smooth one parameter family of flat connections on the vector bundle F. Let $H^\bullet(X,F)_\ell$ be the cohomology of the fibres X with coefficient in $\left(F, \nabla^F_\ell\right)$. As in Section 2.3, the objects we just considered which are associated to $\left(T^H M, g^{TX}, \nabla^F_\ell, g^F\right)$ will be denoted with the subscript ℓ.

Recall that $h(x) = xe^{x^2}$. Let $k(x)$ be given as in (2.76).

Theorem 3.43. — *For* $t > 0$, *the following identity holds,*

$$\frac{\partial}{\partial \ell}\frac{1}{t} h_g^{\wedge}\left(A'_\ell, g_t^{\Omega^\bullet\left(X,F|_X\right)}\right) = \frac{\partial}{\partial t}\varphi \mathrm{Tr}_{\mathrm{s}}\left[gk(B_{\ell,t})\frac{\partial}{\partial \ell}A_{\ell,t}\right] \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{3.123}$$

Proof. — The proof of our Theorem is the same as the proof of Theorem 2.13. □

Now, we make the assumption that the rank of $H^\bullet(X,F)_\ell$ does not depend on ℓ. Then $H^\bullet(X,F)_\ell$ depends smoothly on ℓ.

As in Section 2.3, we identify $H^\bullet(X,F)_\ell$ to a smooth finite dimensional G-invariant subbundle of $\Omega^\bullet\left(X, F|_X\right)$. By orthogonal projection on $H^\bullet(X,F)_\ell$, given $s \in S$, we obtain a unitary connection on the bundle $H^\bullet(X,F)_{\cdot,s}$ over $[0,1]$. We can then trivialize $H^\bullet(X,F)_{\cdot,s}$ on $[0,1]$ with respect to this connection. In particular the flat connections $\nabla^{H^\bullet(X,F|_X)_\ell}$ can now be viewed as a smooth family of flat connections $\nabla^{H^\bullet(X,F|_X)}_\ell$ on the fixed vector bundle $H^\bullet(X,F|_X)$ over S, which is equipped with a metric $g^{H^\bullet(X,F|_X)}_{L_2}$. We denote by $\ell \in [0,1] \to \nabla^{H^\bullet(X,F|_X)}_\ell$ the corresponding smooth family of flat connections on $H^\bullet(X,F|_X)$.

Let $k(x)$ be a holomorphic odd function given by (3.106).

Theorem 3.44. — *As* $t \to +\infty$,

$$\varphi \mathrm{Tr}_{\mathrm{s}}\left[gk(B_{\ell,t})\frac{\partial}{\partial \ell}A_{\ell,t}\right] \to \varphi \mathrm{Tr}_{\mathrm{s}}\left[gk\left(B^{H^\bullet(E,v_\ell)}_\ell\right)\frac{\partial}{\partial \ell}\nabla^{H^\bullet(E,v_\ell),u}\right]. \tag{3.124}$$

Proof. — We proceed as in the proof of Theorem 2.14. The analogue of (2.41) still holds. First we assume that M is compact. Since the rank of $H^{\bullet}(X,F)_{\ell}$ is constant, we may and we will assume that $d_1 \in \mathbf{R}_+^*$ is such that (3.105) holds for every $\ell \in [0,1]$. By Proposition 3.40,

$$k(D_{\ell,t}) = G_{\ell,t} + H_{\ell,t}. \tag{3.125}$$

Clearly, we have an analogue of (2.44), and the operators which appear in the right-hand side of (2.44) are of order 0. Using Proposition 3.41, we get

$$\left|\mathrm{Tr_s}\left[gG_{\ell,t}\frac{\partial}{\partial \ell}C_{\ell,t}\right]\right| \leqslant Ce^{-ct}. \tag{3.126}$$

Moreover we observe again that in the analogue of (2.30), we have $i_0 \geqslant 1$, i.e. $P^{\{0\}}$ appears at least once. Since $P^{\{0\}}$ has constant rank, it follows that uniform bound estimates can be transformed to trace class estimates. We can then continue the proof of our Theorem in the same way as when we proved Theorem 2.13, and so we get (3.124) when M is compact. By restriction to compact pieces of S, we obtain our Theorem in full generality. □

3.16. Rigidity formulas for the analytic torsion forms

We make the same assumptions as in Section 3.15. Recall that $h(x) = xe^{x^2}$. We still define the function $k(x)$ as in (2.76). Namely

$$k(x) = \frac{h'(x) - h'(0)}{2x}. \tag{3.127}$$

Let $L_{k,g}\left(\nabla_{\ell}^{F}, g^{F}\right), L_{k,g}\left(\nabla_{\ell}^{H^{\bullet}(X,F|X)}, g_{L_2}^{H^{\bullet}(X,F|X)}\right)$ be defined as in Definition 2.4.

Theorem 3.45. — *The following identity holds,*

$$\left[\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla_1^F, g^F\right) - \mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla_0^F, g^F\right)\right]^{(\geqslant 2)} = \tag{3.128}$$

$$\int_{X_g} e\left(TX_g, \nabla^{TX}\right) L_{k,g}\left(\nabla_{\ell}^{F}, g^{F}\right) - L_{k,g}\left(\nabla_{\ell}^{H^{\bullet}(X,F|X)}, g_{L_2}^{H^{\bullet}(X,F|X)}\right) \text{ in } \Omega^{\bullet}(S)/d\Omega^{\bullet}(S).$$

Proof. — We proceed as in the proof of Theorems 2.17 and 2.20. We use the notation in the proof of Theorem 2.20, with $h(x) = xe^{x^2}$. By Theorem 3.37,

$$\mathcal{T}_{\overline{h},g}\left(T^H M, g^{TX}, \nabla^F, g^F\right) = \frac{\partial}{\partial a}\psi_a \mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)|_{a=1}. \tag{3.129}$$

Clearly Theorem 3.43 still holds when replacing h by $\overline{h}$, i.e.

$$\frac{\partial}{\partial \ell}\overline{h}^{\wedge}\left(A'_{\ell,t}, g^{TX}, g^F\right) = \frac{\partial}{\partial t}\mathrm{Tr_s}\left[g\overline{k}\left(B_{\ell,t}\right)\frac{\partial}{\partial \ell}A_{\ell,t}\right] \quad \text{in} \quad \Omega^{\bullet}(S)/d\Omega^{\bullet}(S). \tag{3.130}$$

In fact, since $\deg \overline{h} \geqslant 3$, the obvious analogue of Theorem 2.1 holds. By proceeding as in the proof of Theorem 2.13, we get (3.130).

Now, by proceeding as in [**BLo1**, proof of Theorem 3.16], we obtain,

$$\lim_{t\to 0}\varphi\mathrm{Tr}_{\mathrm{s}}\left[g\overline{k}\left(B_{\ell,t}\right)\frac{\partial}{\partial\ell}A_{\ell,t}\right]=\int_{X_g}e\left(TX_g,\nabla^{TX_g}\right)\varphi\mathrm{Tr}_{\mathrm{s}}\left[g\overline{k}\left(B_\ell^F\right)\frac{\partial}{\partial\ell}\nabla_\ell^{F,u}\right]. \tag{3.131}$$

Also by Theorem 3.44,

$$\lim_{t\to+\infty}\varphi\mathrm{Tr}_{\mathrm{s}}\left[g\overline{k}\left(B_{\ell,t}\right)\frac{\partial}{\partial\ell}A_{\ell,t}\right]=\varphi\mathrm{Tr}_{\mathrm{s}}\left[g\overline{k}\left(B_\ell^{H^\bullet(X,F|_X)}\right)\frac{\partial}{\partial\ell}\nabla_\ell^{H^\bullet(X,F|_X),u}\right]. \tag{3.132}$$

From (3.130)-(3.132), we get

$$\begin{aligned}\left[\mathcal{T}_{\overline{h},g}\left(T^HM,g^{TX_g},\nabla_1^F,g^F\right)-\mathcal{T}_{\overline{h},g}\left(T^HM,g^{TX_g},\nabla_0^F,g^F\right)\right]=\\ \int_{X_g}e\left(TX_g,\nabla^{TX_g}\right)L_{\overline{k},g}\left(\nabla_\ell^F,g^F\right)-L_{\overline{k},g}\left(\nabla_\ell^{H^\bullet(X,F|_X)},g_{L_2,\ell}^{H^\bullet(X,F|_X)}\right)\\ \text{in}\quad\Omega^\bullet(S)/d\Omega^\bullet(S).\end{aligned} \tag{3.133}$$

Using (2.83), (3.129), (3.133) and the fact that the degree of $e\left(TX_g,\nabla^{TX_g}\right)$ is precisely $\dim X_g$, the proof of our Theorem then continues continues as the proof of Theorem 2.20. □

3.17. The Chern analytic torsion forms

Now, we extend Definition 2.40 to our infinite dimensional setting.

Definition 3.46. — Put

$$\mathcal{T}_{\mathrm{ch},g}\left(T^HM,g^{TX},\nabla^F,g^F\right)=Q\mathcal{T}_{h,g}\left(T^HM,g^{TX},\nabla^F,g^F\right). \tag{3.134}$$

The even forms $\mathcal{T}_{\mathrm{ch},g}\left(T^HM,g^{TX},\nabla^F,g^F\right)$ will be called Chern analytic torsion forms.

Theorem 3.47. — *If $g=1$, the forms $\mathcal{T}_{\mathrm{ch},g}\left(T^HM,g^{TX},\nabla^F,g^F\right)$ are real. Also the following identity of forms holds on S,*

$$\begin{aligned}d\mathcal{T}_{\mathrm{ch},g}\left(T^HM,g^{TX},\nabla^F,g^F\right)=\int_{X_g}e\left(TX_g,\nabla^{TX_g}\right)\mathrm{ch}_g^\circ\left(\nabla^F,g^F\right)\\ -\mathrm{ch}_g^\circ\left(\nabla^{H^\bullet(X,F|_X)},g_{L_2}^{H^\bullet(X,F|_X)}\right).\end{aligned} \tag{3.135}$$

Proof. — Our Theorem follows from Theorem 3.32 and from Remark 2.30. □

Remark 3.48. — The main point of Definition 3.46 is that we have now normalized the analytic torsion forms without any ambiguity.

CHAPTER 4

THE ANALYTIC TORSION FORMS OF A $\mathbf{Z}_2$-GRADED VECTOR BUNDLE

The purpose of this Chapter is to evaluate the equivariant analytic torsion forms of an Euclidean $\mathbf{Z}_2$-graded vector bundle equipped with a metric connection in terms of genera $I(\theta, x)$ and $J(\theta, x)$. This evaluation is of critical importance in our proof of the formula relating the higher analytic torsion forms to the combinatorial torsion forms.

The evaluation of these equivariant analytic torsion forms in de Rham cohomology is also related to the evaluation of corresponding holomorphic torsion forms for holomorphic vector bundles in Bismut [**B7, B8**]. These last torsion forms were calculated in terms of the genus $R(x)$ of Gillet and Soulé [**GS1**] in the case where the considered group action is trivial, and of the more general genus $R(\theta, x)$ introduced in [**B8**] in the general case. Let $L(\theta, s) = \sum_{n=1}^{+\infty} e^{in\theta}/n^s$ be the Lerch series. As explained in the introduction, the genus $R(\theta, x)$ is expressed in terms of the real part of $L(\theta, .)$ and its first derivative at odd negative integers, and of its imaginary part and its first derivative at even negative integers. Here, the Chern analytic torsion forms are expressed using the genus $J(\theta, x)$. The remarkable fact about $J(\theta, x)$ is that it is expressed as a power series in which the role of even and odd negative integers are interchanged with respect to the corresponding expression for $R(\theta, x)$ given in [**B7, B8**]. Up to irrelevant normalizing factors, one can even consider that for some mysterious reason, the two series are complementary to each other.

The relation of the present Chapter to [**B7, B8**] is even more obvious at the computational level. In fact we use directly the results of [**B7, B8**] to evaluate our analytic torsion forms.

On the other hand, up to a factor $1/2$, given $r \in \mathbf{N}^*$, the Fourier transform of $J\left(\frac{\cdot}{r}, x\right)$ on $\mathbf{Z}/r\mathbf{Z}$ is directly related to the genus obtained by Bismut-Lott [**BLo1**, Corollary 4.14] in their evaluation of the analytic torsion forms of circle bundles. As we shall see in Chapter 16, this coincidence is not an accident, since we will show that the formulas in [**BLo1**] are in fact consequences of the main result of this paper.

This Chapter is organized as follows. In Section 4.1, we construct the flat superconnection of the considered $\mathbf{Z}_2$-graded vector bundle. In Section 4.2, following [**B8**], we introduce a function $\sigma(u, \eta, x)$, and we evaluate the relevant supertraces in terms of this function. In Section 4.3, we obtain the asymptotics of these supertraces as a parameter T tends to 0 and $+\infty$. In Section 4.4, we construct the higher analytic torsion forms associated to the function $h(x) = xe^{x^2}$ in positive degree, and we evaluate these forms in terms of the additive genus associated to a function $I(\theta, x)$. In Section 4.5, we introduce the corresponding Chern analytic torsion forms, which we express in terms of the genus associated to a function $J(\theta, x)$. In Section 4.6, we compute the function $I(\theta, x)$ in terms of the Lerch series, and in Section 4.8 we make a related computation for the function $J(\theta, x)$. Finally in Section 4.8, we exhibit the striking relation of the genus $J(\theta, x)$ to the genus $R(\theta, x)$ which we have described above.

4.1. The flat superconnection of a $\mathbf{Z}_2$-graded vector bundle

In this Section, we use the notation and the results of Section 3.

Let M be a manifold. Let $\pi : E = E_+ \oplus E_- \to M$ be a $\mathbf{Z}_2$-graded real vector bundle on the manifold M. Set $n_\pm = \dim E_\pm, n = \dim E$, so that $n = n_+ + n_-$. Let $g^E = g^{E_+} \oplus g^{E_-}$ be an Euclidean metric on $E = E_+ \oplus E_-$ such that E_+ and E_- are orthogonal in E. Let $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ be a metric preserving connection on $E = E_+ \oplus E_-$, and let $R^E = R^{E_+} \oplus R^{E_-}$ be its curvature.

Let $g \in \mathrm{End}(E)$ be an isometry of E, which preserves the splitting $E = E_+ \oplus E_-$ and which is parallel with respect to ∇^E. In particular g commutes with R^E. Let $e^{\pm i\theta_j}, 0 < \theta_j < \pi, 1 \leqslant j \leqslant q$ and possibly $1, -1$ be the locally constant eigenvalues of g. Then $E \otimes_{\mathbf{R}} \mathbf{C}$ splits into mutually orthogonal eigenbundles according to the distinct eigenvalues of g. The connection $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ and the curvature $R^E = R^{E_+} \oplus R^{E_-}$ preserve the above splittings. In particular, we have the orthogonal splitting of $\mathbf{Z}_2$-graded complex vector bundles

$$E \otimes_{\mathbf{R}} \mathbf{C} = E^1 \oplus E^{-1} \bigoplus_{1 \leqslant j \leqslant q} \left(E^{e^{i\theta_j}} \oplus E^{e^{-i\theta_j}} \right). \tag{4.1}$$

In (4.1), E^{+1} and E^{-1} are complexifications of real vector bundles (they are possibly zero), and $E^{e^{i\theta_j}} \oplus E^{e^{-i\theta_j}}$ is the complexification of a real vector bundle. The connection ∇^E preserves the splitting (4.1). We will denote by $\nabla^{E^1}, \ldots$ the induced connections on $E^1 \ldots$ and by $R^{E^1}, \ldots$ the corresponding curvatures.

Definition 4.1. — For $x \in M$, let I_x (resp. I_x^0) be the vector space of smooth (resp. square integrable) sections of $\pi^* \Lambda(E^*)$ along the fibre E_x.

We equip I_x^0 with the L_2 scalar product,

$$s, s' \in I_x^0 \mapsto \langle s, s' \rangle = \int_{E_x} \langle s, s' \rangle(x) dv_E(x). \tag{4.2}$$

Let d^E be the de Rham operator acting along the fibres I.

The connection ∇^E induces a horizontal vector bundle $T^H E$ on the total space of E. Then one verifies easily that with the notation in (3.11), if $U, V \in TM, Z \in E$,

$$T^H_Z(U,V) = R^E(U,V)Z. \tag{4.3}$$

Also, with the notation in [**BLo1**, Definition 3.2] and in (3.10), if $U \in TM$, if s is a smooth section of I on M,

$$\nabla^I_U s = \nabla^{\Lambda(E^*)}_U s. \tag{4.4}$$

Let $\mathcal{C}'$ be the canonical flat superconnection on I which is attached to the above situation as in [**BLo1**, Section 3 (b)] and in Section 3.2. By [**BLo1**, Proposition 3.4] or by Proposition 3.3, and by (4.3), (4.4),

$$\mathcal{C}' = d^E + \nabla^I + i_{R^E Z}. \tag{4.5}$$

Let $q : E \mapsto \mathbf{R}$ be the smooth function, such that if $Z = (Z_+, Z_-) \in E = E_+ \oplus E_-$,

$$q(Z) = \frac{1}{2}\left(|Z_+|^2 - |Z_-|^2\right). \tag{4.6}$$

Then q is a fibrewise Morse function, whose only critical point 0 has index $\dim E_-$.

For $T \in \mathbf{R}$, let $g^{\mathbf{R}}_T$ be the metric on $\mathbf{R}$ given by

$$\|1\|^2_{g^{\mathbf{R}}_T} = e^{-2Tq}. \tag{4.7}$$

In the sequel, $\mathbf{R}$ will be considered as a flat twisting bundle on the total space of E. Let $\mathcal{C}''_T$ be the adjoint flat superconnection of $\mathcal{C}'$ with respect to the metrics $g^E, g^{\mathbf{R}}_T$.

A special case of Proposition 3.8 is as follows.

Proposition 4.2. — *The following identity holds,*

$$\mathcal{C}''_T = d^{E*} + 2Ti_{Z_+ - Z_-} + \nabla^I - R^E Z \wedge . \tag{4.8}$$

Proof. — This is an obvious computation, which is left to the reader. □

Put

$$\mathcal{C}_T = \frac{1}{2}(\mathcal{C}''_T + \mathcal{C}'), \qquad \mathcal{D}_T = \frac{1}{2}(\mathcal{C}''_T - \mathcal{C}'). \tag{4.9}$$

A related construction is as follows. Put

$$\overline{\mathcal{C}}'_T = e^{-Tq}\mathcal{C}' e^{Tq}, \qquad \overline{\mathcal{C}}''_T = e^{Tq}\mathcal{C}''_0 e^{-Tq}, \tag{4.10}$$

$$\overline{\mathcal{C}}_T = \frac{1}{2}(\overline{\mathcal{C}}''_T + \overline{\mathcal{C}}'_T), \qquad \overline{\mathcal{D}}_T = \frac{1}{2}(\overline{\mathcal{C}}''_T - \overline{\mathcal{C}}'_T).$$

Then $\overline{\mathcal{C}}'_T$ is a flat superconnection on I, and $\overline{\mathcal{C}}''_T$ is its adjoint with respect to the metric g^E. Also,

$$\overline{\mathcal{C}}_T = e^{-Tq}\mathcal{C}_T e^{Tq}, \qquad \overline{\mathcal{D}}_T = e^{-Tq}\mathcal{D}_T e^{Tq}. \tag{4.11}$$

Now we will use the notation of Section 3.5 on Clifford algebras. Let $e_1, \ldots, e_{n_+}$ be an orthonormal basis of E_+, let $e_{n_+ + 1}, \ldots, e_{n_+ + n_-}$ be an orthonormal basis of

E_-. Let N_+, N_- be the number operators of $\Lambda(E_+^*), \Lambda(E_-^*)$. Then N_+, N_- extend to operators acting on $\Lambda(E)$. One has the trivial identity,

$$\sum_{1\leqslant i\leqslant n_+} c(e_i)\widehat{c}(e_i) = 2N_+ - n_+, \qquad \sum_{n_++1\leqslant i\leqslant n_++n_-} c(e_i)\widehat{c}(e_i) = 2N_- - n_-. \tag{4.12}$$

Proposition 4.3. — *The following identities hold,*

$$\begin{aligned} \overline{\mathcal{C}}'_T &= d^E + T(Z_+ - Z_-)\wedge + \nabla^I + i_{R^EZ}, \\ \overline{\mathcal{C}}''_T &= d^{E*} + Ti_{Z_+-Z_-} + \nabla^I - R^EZ\wedge . \end{aligned} \tag{4.13}$$

Proof. — This follows from (4.5) and from Proposition 4.2. □

Theorem 4.4. — *The following identity holds,*

$$\begin{aligned} \overline{\mathcal{C}}_T^2 = -\frac{1}{4}\left(\nabla_{e_i} + \langle R^EZ, e_i\rangle\right)^2 + \frac{1}{4}\langle e_i, R^Ee_j\rangle\widehat{c}(e_i)\widehat{c}(e_j) \\ + \frac{T^2}{4}|Z|^2 + \frac{T}{4}\left(\sum_{1\leqslant i\leqslant n_+} c(e_i)\widehat{c}(e_i) - \sum_{n_++1\leqslant i\leqslant n_++n_-} c(e_i)\widehat{c}(e_i)\right). \end{aligned} \tag{4.14}$$

Proof. — Our formula follows from [**BLo1**, Theorem 3.11], which was stated in Theorem 3.19, or from a simple direct computation. □

4.2. The superconnection heat kernel and the function σ

Definition 4.5. — For $T \in \mathbf{R}$, let $\mathcal{P}_T(Z, Z')$ be the smooth kernel of $\exp\left(-\overline{\mathcal{C}}_T^2\right)$ with respect to $dv_E(Z')/(2\pi)^{\dim E}$.

For $T \in \mathbf{R}^*$, let Q_T be the obvious square root of $T^2 + R^{E,2}$. Then

$$Q_T = |T|\left(1 + \frac{1}{2}\frac{R^E}{T^2} + \cdots\right). \tag{4.15}$$

Theorem 4.6. — *For $T \in \mathbf{R}^*$, the following identity holds,*

$$\begin{aligned} \mathcal{P}_T(Z, Z') = 2^{n/2}\det\left(\frac{Q_T/2}{\sinh(Q_T/2)}\right)^{1/2} \\ \exp\left(-\left\langle\frac{Q_T/2}{\tanh(Q_T/2)}Z, Z\right\rangle - \left\langle\frac{Q_T/2}{\tanh(Q_T/2)}Z', Z'\right\rangle + 2\left\langle\frac{Q_T/2e^{R^E/2}}{\sinh(Q_T/2)}Z, Z'\right\rangle\right) \\ \exp\left(-\frac{T}{4}\left(\sum_{1\leqslant i\leqslant n_+} c(e_i)\widehat{c}(e_i) - \sum_{n_++1\leqslant i\leqslant n_++n_-} c(e_i)\widehat{c}(e_i)\right)\right. \\ \left. - \frac{1}{4}\langle e_i, R^Ee_j\rangle\,\widehat{c}(e_i)\widehat{c}(e_j)\right). \end{aligned} \tag{4.16}$$

Proof. — Identity (4.16) follows (4.14) and from Mehler's formula as in [**B8**, eq. (4.48)]. □

If $z \in \mathbf{C}$, $\sqrt{z}$ denotes an arbitrary (but fixed) square root of z. Now we follow [**B8**, Definition 4.1].

Definition 4.7. — For $u, \eta, x \in \mathbf{C}$, put

$$\sigma(u,\eta,x) = 4\sinh\left(\frac{x-2\eta+\sqrt{x^2+4u}}{4}\right)\sinh\left(\frac{-x+2\eta+\sqrt{x^2+4u}}{4}\right). \tag{4.17}$$

Clearly $\sigma(u,\eta,x)$ is a holomorphic function of its arguments, which does not depend on the choice of the square root $\sqrt{x^2+4u}$. Also $\sigma(u,i\theta,x)$ is periodic in θ of period 2π. Moreover,

$$\sigma(u,\eta,x) = \sigma(u,-\eta,-x). \tag{4.18}$$

It follows from the above that

$$\sigma\left(u,i\pi,x\right) = \sigma\left(u,-i\pi,x\right). \tag{4.19}$$

Also, one finds easily that as $u \to 0$,

$$\frac{\sigma\left(u,0,x\right)}{u} \to \frac{\sinh\left(x/2\right)}{x/2}. \tag{4.20}$$

If the eigenvalues of g are distinct of -1, there is $B \in \mathrm{End}^{\mathrm{even}}\left(E\right)$, antisymmetric, such that

$$g = e^{B}. \tag{4.21}$$

Suppose now that g is just $-1 \in \mathrm{End}\left(E\right)$. In this case, we write

$$g = e^{i\pi}. \tag{4.22}$$

By extending (4.21) to this case, if

$$B = i\pi, \tag{4.23}$$

then we still have (4.21).

By the above, we can always write g in the form (4.21) , with $B \in \mathrm{End}^{\mathrm{even}}\left(E\right)\otimes_{\mathbf{R}}\mathbf{C}$, with $B = i\pi$ on E^{-1}. We may and we will assume that B is parallel with respect to ∇^{E}, so that B commutes with R^{E}.

Observe that $\sigma\left(T^2/4, B, R^{E}\right)$ is a well defined element of $\mathrm{End}\left(E\right)$, which does not depend on our choice of B.

As in (3.6), g acts on I by the formula,

$$(gs)(Z) = g_{*}s(g^{-1}Z). \tag{4.24}$$

The smooth kernel associated to the operator $g\exp\left(-\overline{\mathcal{C}}_{T}^{2}\right)$ is just $gP_{T}(g^{-1}Z, Z')$. In this last expression, g denotes the obvious action on $\Lambda(E^{*})$.

Proposition 4.8. — *The following identity holds,*

$$
\begin{aligned}
(4.25)\quad g\mathcal{P}_T(g^{-1}Z,Z) = 2^{n/2}\det\left(\frac{Q_T/2}{\sinh(Q_T/2)}\right)^{1/2} \\
\exp\left(-\left\langle\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4,B,R^E\right)Z,Z\right\rangle\right) \\
g\exp\left(-\frac{T}{4}\left(\sum_{1\leqslant i\leqslant n_+}c(e_i)\widehat{c}(e_i)-\sum_{n_++1\leqslant i\leqslant n_++n_-}c(e_i)\widehat{c}(e_i)\right)\right. \\
\left.-\frac{1}{4}\left\langle e_i,R^Ee_j\right\rangle\widehat{c}(e_i)\widehat{c}(e_j)\right).
\end{aligned}
$$

Proof. — This follows from Theorem 4.6. □

In the next Proposition, we evaluate the supertrace of certain operators acting on $\Lambda(E^*)$. Let $N = N_+ + N_-$ be the number operator of $\Lambda(E^*)$.

Proposition 4.9. — *For $T \in \mathbf{R}_+^*$, the following identity holds,*

$$
\begin{aligned}
(4.26)\quad \mathrm{Tr_s}\left[g\exp\left(-\frac{T}{4}\left(\sum_{1\leqslant i\leqslant n_+}c(e_i)\widehat{c}(e_i)-\sum_{n_++1\leqslant i\leqslant n_++n_-}c(e_i)\widehat{c}(e_i)\right)\right.\right. \\
\left.\left.-\frac{1}{4}\left\langle e_i,R^Ee_j\right\rangle\widehat{c}(e_i)\widehat{c}(e_j)\right)\right] \\
= (-1)^{\dim E_-^1}\det\left[\sigma\left(T^2/4,B,R^E\right)\right]^{1/2}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
(4.27)\quad -\frac{1}{2}\mathrm{Tr_s}\left[\left(N-\frac{n}{2}\right)g\exp\left(-\frac{T}{4}\left(\sum_{1\leqslant i\leqslant n_+}c(e_i)\widehat{c}(e_i)-\sum_{n_++1\leqslant i\leqslant n_++n_-}c(e_i)\widehat{c}(e_i)\right)\right.\right. \\
\left.\left.-\frac{1}{4}\left\langle e_i,R^Ee_j\right\rangle\widehat{c}(e_i)\widehat{c}(e_j)\right)\right] \\
= (-1)^{\dim E_-^1}\det\left[\sigma\left(T^2/4,B,R^E\right)\right]^{1/2}\mathrm{Tr_s}\left[\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,B,R^E\right)}{\sigma\left(T^2/4,B,R^E\right)}\right].
\end{aligned}
$$

Proof. — First we assume that E_+ and E_- are even dimensional, and that there is $B \in \mathrm{End}(E)$, preserving the splitting, parallel and antisymmetric, such that $g = e^B$. In particular $\dim E_-^1$ is even. We have the easy formula,

$$
(4.28)\qquad B|_{\Lambda(E^*)} = \frac{1}{4}\left\langle Be_i,e_j\right\rangle\left(c(e_i)c(e_j)-\widehat{c}(e_i)\widehat{c}(e_j)\right).
$$

Also in (4.26) we may as well replace g by $\exp\left(B|_{\Lambda(E^*)}\right)$.

By [**BZ1**, Proposition 4.9], among the monomials in the $c(e_i), \widehat{c}(e_j)$, up to permutation, the only monomial whose supertrace is non zero is $c(e_1)\widehat{c}(e_1)\cdots c(e_n)\widehat{c}(e_n)$, and moreover,

$$\mathrm{Tr_s}\left[c(e_1)\widehat{c}(e_1)\cdots c(e_n)\widehat{c}(e_n)\right] = (-2)^n. \tag{4.29}$$

Assume first that $E_- = 0$, so that $E = E_+$. We will use the results of [**B7**, Theorem 6.4] in the case $B = 0$, and [**B8**, Theorem 4.5] in the general case. In [**B7**] and [**B8**], a similar computation is done when n_+ is even (the underlying vector bundle is complex), with anticommuting Clifford variables $c(e_i), \widehat{c}(e_j)$, such that, as explained in [**B7**, eq. (6.25)], the only monomial up to permutation whose supertrace is non zero is $c(e_1)\cdots c(e_n)\widehat{c}(e_1)\cdots\widehat{c}(e_n)$, and moreover,

$$\mathrm{Tr_s}\left[c(e_1)\cdots c(e_n)\widehat{c}(e_1)\cdots\widehat{c}(e_n)\right] = 2^n. \tag{4.30}$$

From (4.30), since n is even, we get

$$\mathrm{Tr_s}\left[c(e_1)i\widehat{c}(e_1)\cdots c(e_n)i\widehat{c}(e_n)\right] = 2^n. \tag{4.31}$$

Using (3.34), and comparing (4.29) and (4.31), we see we can use the results of [**B7**] and [**B8**] without any change in this case. Therefore we get (4.26).

In the general case, we replace E by $E \oplus E$. Then g acts on $E \oplus E$ as before. Also $(E \oplus E)_+$ and $(E \oplus E)_-$ are trivially even dimensional. We claim that the action of g on $E \oplus E$ verifies the above assumptions. This is clear if no eigenvalue of g is equal to -1. If the action of g on E is equal to -1, then $g|_{E\oplus E}$ is of the form $g = e^{\pi B}$, with $B \in \mathrm{End}\,(E \oplus E)$ the obvious symplectic endomorphism. We can then use formula (4.26) applied to $E \oplus E$. The obtained identity is then just the square of the identity (4.26) we are looking for E. Using analyticity, to verify that (4.26) holds, we only need to verify that the right-hand side of (4.26) has the right sign. However in this case, for $R^E = 0$, the left hand-side of (4.26) is just $(2\cosh(T/4))^{\dim E}$, which fits with (4.26). Therefore, we have established (4.26) in full generality.

Still assuming that $E = E_+$ or $E = E_-$, and using (4.12), we get (4.27) by differentiating (4.26) in T. By summing the corresponding equalities, (4.27) follows. The proof of our Proposition is completed. □

Remark 4.10. — The formula corresponding to (4.26) when $T < 0$ can be obtained from (4.26) by exchanging the roles of E_+ and E_-.

The operator $g\exp\left(-\mathcal{C}_T^2\right)$ is fibrewise trace class. We will now evaluate its supertrace. We use the notation in (4.1). In particular E_-^1 denotes the 1-subbundle of E_- with respect to the action of g.

Proposition 4.11. — *Given $T \in \mathbf{R}_+^*$, the following identities hold,*

$$\begin{aligned}
&\mathrm{Tr_s}\left[g\exp\left(-\mathcal{C}_T^2\right)\right] = (-1)^{\dim E_-^1},\\
&-\frac{1}{2}\mathrm{Tr_s}\left[\left(N-\frac{n}{2}\right)g\exp\left(-\mathcal{C}_T^2\right)\right] = (-1)^{\dim E_-^1}\mathrm{Tr_s}\left[\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,B,R^E\right)}{\sigma\left(T^2/4,B,R^E\right)}\right],\\
&\mathrm{Tr_s}\left[Tqg\exp\left(-\mathcal{C}_T^2\right)\right] = (-1)^{\dim E_-^1}\mathrm{Tr_s}\left[\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,B,R^E\right)}{\sigma\left(T^2/4,B,R^E\right)}\right].
\end{aligned} \tag{4.32}$$

Proof. — Clearly,

$$\mathrm{Tr_s}\left[g\exp\left(-\mathcal{C}_T^2\right)\right] = \int_E \mathrm{Tr_s}\left[g\mathcal{P}_T(g^{-1}Z,Z)\right]\frac{dv_E(Z)}{(2\pi)^{\dim E}}. \tag{4.33}$$

The first equality in (4.32) now follows from Propositions 4.8 and 4.9. The proof of the second equality is similar. Using Propositions 4.8 and 4.9, and the obvious

$$\int_{-\infty}^{+\infty} x^2\exp\left(-x^2\right)\frac{dx}{\sqrt{\pi}} = \frac{1}{2}, \tag{4.34}$$

we get

$$\begin{aligned}
&\mathrm{Tr_s}\left[Tqg\exp\left(-\mathcal{C}_T^2\right)\right]\\
&\quad = (-1)^{\dim E_-^1}\frac{1}{2}\mathrm{Tr_s}\left[\frac{\sinh\left(\frac{Q_T}{2}\right)}{\sinh\left(\frac{R^E-2B+Q_T}{4}\right)\sinh\left(\frac{-R^E+2B+Q_T}{4}\right)}\frac{T}{4Q_T}\right].
\end{aligned} \tag{4.35}$$

Also,

$$\begin{aligned}
&\frac{1}{2}\mathrm{Tr_s}\left[\left(\frac{1}{\tanh\left(\frac{R^E-2B+Q_T}{4}\right)}+\frac{1}{\tanh\left(\frac{-R^E+2B+Q_T}{4}\right)}\right)\frac{T}{4Q_T}\right]\\
&\qquad = \frac{1}{2}\mathrm{Tr_s}\left[\frac{\sinh\left(\frac{Q_T}{2}\right)}{\sinh\left(\frac{R^E-2B+Q_T}{4}\right)\sinh\left(\frac{-R^E+2B+Q_T}{4}\right)}\frac{T}{4Q_T}\right].
\end{aligned} \tag{4.36}$$

From (4.35), (4.36) we get the last equality in (4.32). The proof of our Proposition is completed. □

Remark 4.12. — By Witten [**W**], we know that for $T > 0$, the L_2 cohomology of the complex $(I, d^E + Tdf\wedge)$ is the compactly supported cohomology of E_-. Therefore its equivariant Euler characteristic $\chi_{L_2}(g)$ is given by

$$\chi_{L_2}(g) = (-1)^{\dim E_-}\det\left(g|_{E_-}\right) = (-1)^{\dim E_-^1}. \tag{4.37}$$

Comparing (4.32) and (4.37), we see that the first equality in (4.32) is just a version of [**BLo1**, Theorem 3.15] or of Proposition 3.22 applied to the noncompact fibres E. A direct proof of this equality can be given by arguments similar to the ones in [**BLo1**].

4.3. The asymptotics of the heat equation supertraces

We state a simple result in [**B8**, Proposition 4.2].

Proposition 4.13. — *For $u, \theta, x \in \mathbf{C}$, the following identity holds,*

$$\sigma(u, i\theta, x) = \left(\theta^2 + i\theta x + u\right) \prod_{k \in \mathbf{Z}^*} \left(\frac{\left(\theta + 2k\pi\right)^2 + i\left(\theta + 2k\pi\right) x + u}{4k^2\pi^2} \right). \tag{4.38}$$

Now we will take the logarithmic derivative of identity (4.38).

Proposition 4.14. — *For $\theta \in \mathbf{R}, x \in \mathbf{C}, T \in \mathbf{R}^*$, and $|x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, $|x| < \inf_{k \in \mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$, the following identity holds,*

$$\frac{1}{2} \frac{\frac{\partial}{\partial T} \sigma\left(T^2/4, i\theta, x\right)}{\sigma\left(T^2/4, i\theta, x\right)} = \sum_{k \in \mathbf{Z}} \frac{T}{T^2 + 4i\left(\theta + 2k\pi\right) x + 4\left(\theta + 2k\pi\right)^2}. \tag{4.39}$$

Proof. — This is an obvious consequence of Proposition 4.13. □

Proposition 4.15. — *As $T \to 0$,*

$$\begin{aligned} \frac{1}{2} \frac{\frac{\partial}{\partial T} \sigma\left(T^2/4, i\theta, x\right)}{\sigma\left(T^2/4, i\theta, x\right)} &= \frac{1}{T} + \mathcal{O}(T) \text{ if } \theta \in 2\pi\mathbf{Z}, \\ &= \mathcal{O}(T) \text{ if } \theta \notin 2\pi\mathbf{Z}. \end{aligned} \tag{4.40}$$

As $T \to +\infty$,

$$\frac{1}{2} \frac{\frac{\partial}{\partial T} \sigma\left(T^2/4, i\theta, x\right)}{\sigma\left(T^2/4, i\theta, x\right)} = \frac{1}{4} + \mathcal{O}\left(\frac{1}{T^2}\right). \tag{4.41}$$

Proof. — Using (4.39), we get (4.40). A direct computation leads to (4.41). □

Remark 4.16. — An obvious computation shows that in (4.41), in degree 0, the convergence is in fact $\mathcal{O}(e^{-T/2})$.

Recall that $E^1 = E^1_+ \oplus E^1_-$ is the fixed subbundle of $E = E_+ \oplus E_-$ under g.

Proposition 4.17. — *As $T \to 0$,*

$$-\frac{1}{2} \mathrm{Tr}_{\mathrm{s}} \left[\left(N - \frac{n}{2}\right) g \exp\left(-\mathcal{C}_T^2\right) \right] = (-1)^{\dim E_-} \frac{\dim E^1_+ - \dim E^1_-}{T} + \mathcal{O}(T). \tag{4.42}$$

As $T \to +\infty$,

$$-\frac{1}{2} \mathrm{Tr}_{\mathrm{s}} \left[\left(N - \frac{n}{2}\right) g \exp\left(-\mathcal{C}_T^2\right) \right] = (-1)^{\dim E^1_-} \frac{\dim E_+ - \dim E_-}{4} + \mathcal{O}\left(\frac{1}{T^2}\right). \tag{4.43}$$

Proof. — This follows from Propositions 4.11 and 4.15. □

4.4. The higher analytic torsion forms of a $\mathbf{Z}_2$-graded vector bundle

Now we may develop the rescaling formalism of Section 3.7. Namely, for $t > 0$, we can replace the metric g^E by the metric g^E/t, and consider the associated superconnections. Instead, we will develop directly the equivalent formalism of Definition 3.16. Set

$$\mathcal{C}'_{t,T} = t^{N/2}\mathcal{C}'_T t^{-N/2}, \qquad \mathcal{C}''_{t,T} = t^{-N/2}\mathcal{C}''_T t^{N/2}. \tag{4.44}$$

Then $\mathcal{C}'_{t,T}$ is a flat superconnection, and $\mathcal{C}''_{t,T}$ is its adjoint with respect to the metrics $g^E, g_T^{\mathbf{R}}$. Set

$$\mathcal{C}_{t,T} = \frac{1}{2}\left(\mathcal{C}''_{t,T} + \mathcal{C}'_{t,T}\right), \qquad \mathcal{D}_{t,T} = \frac{1}{2}\left(\mathcal{C}''_{t,T} - \mathcal{C}'_{t,T}\right). \tag{4.45}$$

Recall that ψ_t was defined in (1.50). As in Proposition 3.17,

$$\mathcal{C}_{t,T} = \psi_t^{-1}\sqrt{t}\,\mathcal{C}_T\psi_t, \qquad \mathcal{D}_{t,T} = \psi_t^{-1}\sqrt{t}\,\mathcal{D}_T\psi_t. \tag{4.46}$$

Recall that

$$h(x) = xe^{x^2}, \qquad h'(x) = \left(1 + 2x^2\right)e^{x^2}. \tag{4.47}$$

Proposition 4.18. — *Given $T > 0$, the following identities hold,*

(4.48)

$$-\frac{1}{2}\mathrm{Tr_s}\left[\left(N - \frac{n}{2}\right)gh'\left(\mathcal{D}_{t,T}\right)\right] = \left(1 + 2\frac{\partial}{\partial a}\right)\psi_a\Big(-\frac{1}{2}\mathrm{Tr_s}\Big[\Big(N - \frac{n}{2}\Big)\exp\left(-\mathcal{C}^2_{taT}\right)\Big]\Big)|_{a=1},$$

$$\mathrm{Tr_s}\left[Tqgh'\left(\mathcal{D}_{t,T}\right)\right] = \left(1 + 2\frac{\partial}{\partial a}\right)\psi_a\mathrm{Tr_s}\left[taTqg\exp\left(-\mathcal{C}^2_{taT}\right)\right]|_{a=1}.$$

Proof. — For $v > 0$, let $F_v : I \mapsto I$ be given by

$$F_v s(Z) = s\left(Z/v\right). \tag{4.49}$$

By (4.13), we find that

$$\mathcal{D}_{t,T} = F_{\sqrt{t}}\mathcal{D}_{tT}F_{\sqrt{t}}^{-1}. \tag{4.50}$$

Using (4.46), (4.47), (4.50), we get the first identity in (4.48). The proof of the second identity is similar. □

By (4.50), we get

$$-\frac{1}{2}\mathrm{Tr_s}\left[\left(N - \frac{n}{2}\right)gh'\left(\mathcal{D}_{t,T}\right)\right] = -\frac{1}{2}\mathrm{Tr_s}\left[\left(N - \frac{n}{2}\right)gh'\left(\mathcal{D}_{tT}\right)\right]. \tag{4.51}$$

By Propositions 4.11 and 4.18,

$$-\frac{1}{2}\mathrm{Tr_s}\left[\left(N - \frac{n}{2}\right)gh'\left(\mathcal{D}_{t,T}\right)\right] = \mathrm{Tr_s}\left[Tqgh'\left(\mathcal{D}_{t,T}\right)\right]. \tag{4.52}$$

Using Propositions 4.17 and 4.18, we find that as $T \to 0$,

$$-\frac{1}{2}\mathrm{Tr_s}\left[\left(N - \frac{n}{2}\right)gh'\left(\mathcal{D}_T\right)\right] = -\left(-1\right)^{\dim E^1_-}\frac{\dim E^1_+ - \dim E^1_-}{T} + \mathcal{O}\left(T\right), \tag{4.53}$$

and that as $T \to +\infty$,

$$-\frac{1}{2}\mathrm{Tr_s}\left[\left(N-\frac{n}{2}\right)gh'\left(\mathcal{D}_T\right)\right] = (-1)^{\dim E^1_-}\frac{\dim E_+ - \dim E_-}{4} + \mathcal{O}\left(\frac{1}{T^2}\right). \tag{4.54}$$

If α is a form on M, let $\alpha^{(>0)}$ be the component of α which has positive degree. By Proposition 4.11,

$$-\frac{1}{2}\mathrm{Tr_s}\left[\left(N-\frac{n}{2}\right)gh'\left(\mathcal{D}_{t,T}\right)\right]^{(>0)} = -\frac{1}{2}\mathrm{Tr_s}\left[Ngh'\left(\mathcal{D}_{t,T}\right)\right]^{(>0)}. \tag{4.55}$$

Using (4.51), (4.53)-(4.55), we can construct the higher analytic torsion forms in positive degree of $\left(E, g^E, \nabla^E\right)$ by imitating Definition 3.31.

Definition 4.19. — The analytic torsion forms $\mathcal{T}_{h,g}\left(E, g^E, \nabla^E\right)^{(>0)}$ are defined by the formula,

$$\mathcal{T}_{h,g}\left(E, g^E, \nabla^E\right)^{(>0)} = \varphi\int_0^{+\infty} -\mathrm{Tr_s}\left[\frac{N}{2}gh'\left(\mathcal{D}_T\right)\right]^{(>0)}\frac{dT}{T}. \tag{4.56}$$

By (4.52), (4.55), we know that in (4.56), we may replace $-\mathrm{Tr_s}\left[\frac{N}{2}gh'\left(\mathcal{D}_T\right)\right]^{(>0)}$ by $\mathrm{Tr_s}\left[Tfgh'\left(\mathcal{D}_T\right)\right]^{(>0)}$ and still obtain the same result.

Let $N^{\Lambda^\bullet(T^*M)}$ be the number operator of $\Lambda^\bullet(T^*M)$.

Proposition 4.20. — *The following identity holds,*

(4.57)

$$\mathcal{T}_{h,g}\left(E, g^E, \nabla^E\right)^{(>0)} = \varphi\left(1+N^{\Lambda(T^*M)}\right)\int_0^{+\infty} -\mathrm{Tr_s}\left[\frac{N}{2}g\exp\left(-\mathcal{C}_T^2\right)\right]^{(>0)}\frac{dT}{T}.$$

Proof. — By (4.48), (4.55), (4.56), we get

$$\mathcal{T}_{h,g}\left(E, g^E, \nabla^E\right)^{(>0)} = \varphi\left(1+2\frac{\partial}{\partial a}\right)\psi_a\left(\int_0^{+\infty} -\mathrm{Tr_s}\left[\frac{N}{2}g\exp\left(-\mathcal{C}_T^2\right)\right]^{(>0)}\frac{dT}{T}\right)|_{a=1}, \tag{4.58}$$

which is equivalent to (4.57). □

If $f(x)$ is a holomorphic function of $x \in \mathbf{C}$, we denote by $f^{(>0)}(x)$ the function $f(x) - f(0)$.

Observe that by Proposition 4.15, the function of $s \in \mathbf{C}, 1 < \mathrm{Re}\,(s) < 2$,

$$\frac{1}{2^s\Gamma(s)}\int_0^{+\infty} T^{s-1}\left(\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4, i\theta, -x\right)}{\sigma\left(T^2/4, i\theta, -x\right)} - \frac{1}{4}\right)dT$$

extends to a holomorphic function of s near $s = 0$.

Definition 4.21. — For $x \in \mathbf{C}, \theta \in \mathbf{R}$, $|x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$, put

$$F(\theta,x) = \frac{\partial}{\partial s}\left[\frac{1}{2^s\Gamma(s)}\int_0^{+\infty} T^{s-1}\left(\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4, i\theta, -x\right)}{\sigma\left(T^2/4, i\theta, -x\right)} - \frac{1}{4}\right) dT\right](s)\,|_{s=0}, \tag{4.59}$$

$$I(\theta,x) = \left(1 + 2x\frac{\partial}{\partial x}\right) F(\theta,x).$$

Observe that by (4.40),

$$F(\theta,x)^{(>0)} = \int_0^{+\infty}\left[\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4, i\theta, -x\right)}{\sigma\left(T^2/4, i\theta, -x\right)}\right]^{(>0)}\frac{dT}{T}. \tag{4.60}$$

By (4.18), we find that

$$F(\theta,x) = F(-\theta,-x), \qquad I(\theta,x) = I(-\theta,-x). \tag{4.61}$$

We identify $F(\theta,x), I(\theta,x)$ to the corresponding additive genera. In particular,

$$I\left(\pm\theta_j, \nabla^{E^{e^{\pm i\theta_j}}}\right) = \mathrm{Tr_s}\left[I\left(\pm\theta_j, -\frac{R^{E^{e^{\pm i\theta_j}}}}{2i\pi}\right)\right]. \tag{4.62}$$

By (4.61),

$$I\left(\theta_j, \nabla^{E^{e^{i\theta_j}}}\right) = I\left(-\theta_j, \nabla^{E^{e^{-i\theta_j}}}\right). \tag{4.63}$$

We define $I\left(0, \nabla^{E^1}\right), I\left(\pi, \nabla^{E^{-1}}\right)$ in the same way.

Definition 4.22. — Set

$$I_g(E,\nabla^E) = I\left(0,\nabla^{E^1}\right) + I\left(\pi,\nabla^{E^{-1}}\right) + 2\sum_{1\leqslant j\leqslant q} I\left(\theta_j, \nabla^{E^{e^{i\theta_j}}}\right). \tag{4.64}$$

Equivalently,

$$I_g\left(E,\nabla^E\right) = \mathrm{Tr_s}\left[I\left(-iB, -\frac{R^E}{2i\pi}\right)\right]. \tag{4.65}$$

Theorem 4.23. — *The following identity holds,*

$$\mathcal{T}_{h,g}\left(E,g^E,\nabla^E\right)^{(>0)} = (-1)^{\dim E^1_-} I_g^{(>0)}\left(E,\nabla^E\right). \tag{4.66}$$

In particular the forms $\mathcal{T}_{h,g}\left(E,g^E,\nabla^E\right)^{(>0)}$ are even, closed, and their cohomology class does not depend on g^E or ∇^E.

Proof. — Equation (4.66) follows from Propositions 4.11 and 4.20. Our Theorem follows. □

4.5. The Chern analytic torsion forms of a $\mathbf{Z}_2$-graded vector bundle

Recall that the operator $Q : \Lambda^\bullet(T^*M) \to \Lambda^\bullet(T^*M)$ was defined in (2.117). Now we imitate Definitions 2.40 and 3.46.

Definition 4.24. — Put

$$\mathcal{T}_{\mathrm{ch},g}\left(E, g^E, \nabla^E\right)^{(>0)} = Q\mathcal{T}_{h,g}\left(E, g^E, \nabla^E\right)^{(>0)}. \tag{4.67}$$

In (2.116), if $f : \mathbf{C} \to \mathbf{C}$ is analytic, we also defined $Qf(x)$.

Definition 4.25. — For $x \in \mathbf{C}, \theta \in \mathbf{R}$, $|x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$, put

$$J(\theta, x) = QI(\theta, x). \tag{4.68}$$

We define the form $J_g\left(E, \nabla^E\right)$ as in (4.64), (4.65), using instead the function $J(\theta, x)$. Then

$$J_g\left(E, \nabla^E\right) = \mathrm{Tr_s}\left[J\left(-iB, -\frac{R^E}{2i\pi}\right)\right]. \tag{4.69}$$

By (4.68),

$$J_g\left(E, \nabla^E\right) = QI_g\left(E, \nabla^E\right). \tag{4.70}$$

Theorem 4.26. — *The following identity holds,*

$$\mathcal{T}_{\mathrm{ch},g}\left(E, g^E, \nabla^E\right)^{(>0)} = (-1)^{\dim E^1_-} J_g^{(>0)}\left(E, \nabla^E\right). \tag{4.71}$$

In particular the forms $\mathcal{T}_{\mathrm{ch},g}\left(E, g^E, \nabla^E\right)^{(>0)}$ are even, closed, and their cohomology class does not depend on g^E and ∇^E.

Proof. — This follows from Theorem 4.23. □

4.6. The Lerch series and the function $I(\theta, x)$

For $a \in \mathbf{Z}, x \in \mathbf{R}, y \in \mathbf{R}, s \in \mathbf{C}$, let $S_a(x, y, s)$ be the Kronecker zeta function,

$$S_a(x, y, s) = {\sum_{n\in\mathbf{Z}}}' (x+n)^a |x+n|^{-2s} e^{-2i\pi ny}, \tag{4.72}$$

where in (4.72), $\sum'_{n\in\mathbf{Z}}$ is a sum over $n \in \mathbf{Z}, n \neq -x$. The series in (4.72) converges absolutely for $\mathrm{Re}(s) > \frac{a+1}{2}$, and defines a holomorphic function of s. Also it is periodic of period 1 in both variables x, y. By [**We**, p57],

1. If a is odd, or if a is even and $y \notin \mathbf{Z}$, $s \mapsto S_a(x, y, s)$ has a holomorphic continuation to $\mathbf{C}$.
2. If a is even and if $y \in \mathbf{Z}$, $s \mapsto S_a(x, y, s)$ extends to a meromorphic function on $\mathbf{C}$ with a simple pole at $s = \frac{a+1}{2}$.

Let $\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ be the Riemann zeta function. Then

$$S_0(0,0,s) = 2\zeta(2s), \qquad S_1(0,0,s) = 0. \tag{4.73}$$

Also by [**We**, p57], we have the functional equation for $S_a(x,y,s), a = 0$ or 1,

$$\Gamma(s)S_a(x,y,s) = i^{-a}\pi^{2s-a-1/2}e^{2i\pi xy}\Gamma\left(a-s+\frac{1}{2}\right)S_a\left(y,-x,a-s+\frac{1}{2}\right). \tag{4.74}$$

By (4.74), it follows that for $a = 0,1$, $S_a(x,y,.)$ vanishes at negative integers, and $S_1(x,y,.)$ also vanishes at 0.

Definition 4.27. — For $y \in \mathbf{R}, s \in \mathbf{C}, \mathrm{Re}(s) > 1$, set

$$\zeta(y,s) = \sum_{n=1}^{+\infty} \frac{\cos(ny)}{n^s}, \qquad \eta(y,s) = \sum_{n=1}^{+\infty} \frac{\sin(ny)}{n^s}. \tag{4.75}$$

Then $\zeta(y,s)$ and $\eta(y,s)$ are the real and imaginary parts of the Lerch series $L(y,s) = \sum_{n=1}^{+\infty} \frac{e^{iny}}{n^s}$. Clearly,

$$\zeta(y,s) = \frac{1}{2}S_0\left(0,\frac{y}{2\pi},\frac{s}{2}\right), \qquad \eta(y,s) = \frac{i}{2}S_1\left(0,\frac{y}{2\pi},\frac{s+1}{2}\right). \tag{4.76}$$

If $y \notin 2\pi\mathbf{Z}$, $s \mapsto \zeta(y,s)$ extends to a holomorphic function on $\mathbf{C}$, if $y \in 2\pi\mathbf{Z}$, $s \mapsto \zeta(y,s)$ extends to a meromorphic function on $\mathbf{C}$ with a simple pole at $s = 1$. Also $s \mapsto \eta(y,s)$ extends to a holomorphic function on $\mathbf{C}$. Moreover,

$$\zeta(0,s) = \zeta(s), \qquad \eta(0,s) = 0. \tag{4.77}$$

By the above, $\zeta(y,.)$ vanishes at even negative integers, and $\eta(y,.)$ vanishes at odd negative integers.

Definition 4.28. — For $\theta \in \mathbf{R}^*, x \in \mathbf{C}, |x| < |\theta|$, put

$$F^\theta(x) = \frac{\pi}{4}\frac{1}{|\theta|}\left(1-\frac{ix}{\theta}\right)^{-1/2}, \qquad I^\theta(x) = \frac{\pi}{4}\frac{1}{|\theta|}\left(1-\frac{ix}{\theta}\right)^{-3/2}. \tag{4.78}$$

Observe that

$$I^\theta(x) = \left(1+2x\frac{\partial}{\partial x}\right)F^\theta(x). \tag{4.79}$$

In the sequel, we denote by

$$\sum_{k\in\mathbf{Z}}{}'\left(F^{2k\pi+\theta}(x) - F^{2k\pi}(0)\right)$$

the sum of the corresponding series, where we take as a convention that if $2k\pi+\theta$ or $2k\pi$ vanish, the corresponding term $F^{2k\pi+\theta}(x)$ or $F^{2k\pi}(0)$ is replaced by 0. Note that this series is convergent, while the sum of the series $F^{2k\pi+\theta}(x)$ is not well-defined. The same notation will be used for sums involving other functions than F.

Theorem 4.29. — *For $\theta \in \mathbf{R}, x \in \mathbf{C}$, $|x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$,$|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$, then*

$$F(\theta, x) - F(0,0) = \sum_{k\in\mathbf{Z}}{}' \left(F^{2k\pi+\theta}(x) - F^{2k\pi}(0)\right), \tag{4.80}$$

$$F(\theta, x) = \frac{1}{2}\left[\sum_{\substack{p\in\mathbf{N}\\ p \text{ even}}} \frac{(2p)!}{(p!)^3}\frac{\partial\zeta}{\partial s}(\theta, -p)\left(\frac{x}{4}\right)^p + i\sum_{\substack{p\in\mathbf{N}\\ p \text{ odd}}} \frac{(2p)!}{(p!)^3}\frac{\partial\eta}{\partial s}(\theta, -p)\left(\frac{x}{4}\right)^p\right].$$

Proof. — By (4.39),

$$\frac{1}{T}\left[\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4, i\theta, -x\right)}{\sigma\left(T^2/4, i\theta, -x\right)} - \frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4, i\theta, 0\right)}{\sigma\left(T^2/4, i\theta, 0\right)}\right] = \sum_{k\in\mathbf{Z}} \frac{4i\left(\theta + 2k\pi\right)x}{\left(T^2 - 4i\left(\theta + 2k\pi\right)x + 4\left(\theta + 2k\pi\right)^2\right)\left(T^2 + 4\left(\theta + 2k\pi\right)^2\right)}. \tag{4.81}$$

By (4.81), given θ, x taken as indicated, there is $C > 0$ such that for $0 < T \leqslant 1$,

$$\sum_{k\in\mathbf{Z}} \left|\frac{4i\left(\theta + 2k\pi\right)x}{\left(T^2 - 4i\left(\theta + 2k\pi\right)x + 4\left(\theta + 2k\pi\right)^2\right)\left(T^2 + 4\left(\theta + 2k\pi\right)^2\right)}\right| \leqslant C, \tag{4.82}$$

and that for $T \geqslant 1$,

$$\sum_{k\in\mathbf{Z}} \left|\frac{4i\left(\theta + 2k\pi\right)x}{\left(T^2 - 4i\left(\theta + 2k\pi\right)x + 4\left(\theta + 2k\pi\right)^2\right)\left(T^2 + 4\left(\theta + 2k\pi\right)^2\right)}\right| \leqslant \frac{C}{T^2}. \tag{4.83}$$

From (4.39), (4.59), (4.82), (4.83), we deduce that

$$F(\theta, x)^{(>0)} = \sum_{k\in\mathbf{Z}} \int_0^{+\infty} \left\{\frac{1}{T^2 - 4i\left(\theta + 2k\pi\right)x + 4\left(\theta + 2k\pi\right)^2}\right\}^{(>0)} dT \tag{4.84}$$

We have the trivial,

$$\int_0^{+\infty} \left[\frac{dT}{\left(T^2 - 4i\left(\theta + 2k\pi\right)x + 4\left(\theta + 2k\pi\right)^2\right)}\right]^{(>0)} = \frac{\pi}{4}\left(-i\left(\theta + 2k\pi\right)x + \left(\theta + 2k\pi\right)^2\right)^{-1/2}, \tag{4.85}$$

with the convention that if $\theta + 2k\pi = 0$, the right-hand side of (4.85) vanishes. From (4.84), (4.85) we get the first equality in (4.80) in positive degree.

If $\theta + 2k\pi \neq 0$,

(4.86)
$$\left[\left(-i\left(\theta+2k\pi\right)x+\left(\theta+2k\pi\right)^2\right)^{-1/2}\right]^{(>0)} = \frac{1}{|\theta+2k\pi|}\sum_{p=1}^{+\infty}\frac{(2p)!}{(p!)^2}\left(\frac{ix}{4\left(\theta+2k\pi\right)}\right)^p.$$

By (4.84)-(4.86), and using a dominated convergence argument, we get

$$F(\theta,x)^{(>0)} = \frac{1}{8}\left[\sum_{\substack{p\in\mathbf{N}^*\\ p\text{ even}}}\frac{(2p)!}{(p!)^2}S_0\left(\frac{\theta}{2\pi},0,\frac{p+1}{2}\right)\left(\frac{ix}{8\pi}\right)^p + \sum_{\substack{p\in\mathbf{N}^*\\ p\text{ odd}}}\frac{(2p)!}{(p!)^2}S_1\left(\frac{\theta}{2\pi},0,\frac{p+2}{2}\right)\left(\frac{ix}{8\pi}\right)^p\right]. \tag{4.87}$$

If $n \in \mathbf{N}$, the function $\Gamma(s)$ has a simple pole at $-n$, and the corresponding residue is given by $(-1)^n/n!$. Using the functional equation (4.74), if $n \in \mathbf{N}^*$,

$$\begin{aligned}\frac{\partial}{\partial s}S_0\left(0,\frac{\theta}{2\pi},-n\right) &= (-1)^n\pi^{-2n}\frac{(2n)!}{2^{2n}}S_0\left(\frac{\theta}{2\pi},0,n+\frac{1}{2}\right),\\ \frac{\partial}{\partial s}S_1\left(0,\frac{\theta}{2\pi},-n\right) &= \frac{(-1)^n}{i}\pi^{-2n-1}\frac{(2n+1)!}{2^{2n+1}}S_1\left(\frac{\theta}{2\pi},0,n+\frac{3}{2}\right).\end{aligned} \tag{4.88}$$

From (4.76), (4.87) and (4.88), we get the second equality in (4.80) in positive degree.

For $s \in \mathbf{C}, \mathrm{Re}\,(s) > 1$, put

$$\varphi(s) = \frac{1}{2^s\Gamma(s)}\int_0^{+\infty}T^{s-1}\left(\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,i\theta,0\right)}{\sigma\left(T^2/4,i\theta,0\right)}-\frac{1}{4}\right)dT. \tag{4.89}$$

By Proposition 4.15 and Remark 4.16, $\varphi(s)$ is a holomorphic function of s, which extends to a holomorphic function near $s = 0$. By (4.17),

$$\sigma\left(T^2/4,i\theta,0\right) = 2\left(\cosh\left(T/2\right)-\cos(\theta)\right), \tag{4.90}$$

and so,

$$\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,i\theta,0\right)}{\sigma\left(T^2/4,i\theta,0\right)} = \frac{1}{4}\frac{\sinh\left(T/2\right)}{\cosh\left(T/2\right)-\cos\left(\theta\right)}. \tag{4.91}$$

Now we proceed as in [**BZ2**, proof of Theorem 5.17]. We have the easy equality,

$$\frac{1}{4}\frac{\sinh\left(T/2\right)}{\cosh\left(T/2\right)-\cos\left(\theta\right)} = \frac{1}{4}+\frac{1}{2}\sum_{n\geqslant 1}e^{-nT/2}\cos\left(n\theta\right). \tag{4.92}$$

By (4.89), (4.92), we obtain,

$$\varphi(s) = \frac{1}{2}\zeta(\theta,s). \tag{4.93}$$

From (4.93), we get

$$\varphi'(0) = \frac{1}{2}\frac{\partial \zeta}{\partial s}(\theta, 0). \tag{4.94}$$

From (4.89), (4.94), we get the second equality in (4.80) in degree 0.

For $s \in \mathbf{C}, \mathrm{Re}(s) > \frac{1}{2}$,

$$\sum_{k\in\mathbf{Z}}{}' \frac{1}{|k+\theta/2\pi|^{2s}} - \sum_{k\in\mathbf{Z}}{}' \frac{1}{|k|^{2s}} = S_0(\theta/2\pi, 0, s) - S_0(0, 0, s). \tag{4.95}$$

Also one finds easily that as $s \in]\frac{1}{2}, +\infty[\to \frac{1}{2}$, then

$$\sum_{k\in\mathbf{Z}}{}' \frac{1}{|k+\theta/2\pi|^{2s}} - \sum_{k\in\mathbf{Z}}{}' \frac{1}{|k|^{2s}} \to \sum_{k\in\mathbf{Z}}{}' \left(\frac{1}{|k+\theta/2\pi|} - \frac{1}{|k|}\right). \tag{4.96}$$

By Lerch's formula [**We**, Chapter 7, eqs. (15)-(23)] as used in [**BZ2**, eqs (5.51)-(5.54)], we get, for any $\theta \in \mathbf{R}$,

$$\zeta(\theta, 0) = -\frac{1}{2}. \tag{4.97}$$

By (4.74), (4.76) and (4.97), at $s = 1/2$, $S_0(\theta/2\pi, 0, s)$ has a simple pole with residue 1. Therefore the right-hand side of (4.95) extends to a holomorphic function near $s = 1/2$. More precisely,

$$\begin{aligned}\Big[S_0(\theta/2\pi, 0, s) - S_0(0, 0, s)\Big]|_{s=1/2} \\ = \frac{\partial}{\partial s}\Big[(s-1/2)\Big(S_0(\theta/2\pi, 0, s) - S_0(0, 0, s)\Big)\Big]|_{s=1/2}.\end{aligned} \tag{4.98}$$

By (4.74), (4.76) and (4.97), we obtain,

$$\Big[S_0(\theta/2\pi, 0, s) - S_0(0, 0, s)\Big]|_{s=1/2} = 4\left(\frac{\partial \zeta}{\partial s}(\theta/2\pi, 0) - \frac{\partial \zeta}{\partial s}(0, 0)\right). \tag{4.99}$$

By the second equality in (4.80), and by (4.95)-(4.99), we get the first equality in (4.80) also in degree 0. The proof of our Theorem is completed. □

Theorem 4.30. — *The following identities hold,*

(4.100)

$$I(\theta, x) - I(0, 0) = \sum_{k\in\mathbf{Z}}{}' \left(I^{2k\pi+\theta}(x) - I^{2k\pi}(0)\right),$$

$$I(\theta, x) = \frac{1}{2}\left[\sum_{\substack{p\in\mathbf{N}\\ p \text{ even}}} \frac{(2p+1)!}{(p!)^3}\frac{\partial \zeta}{\partial s}(\theta, -p)\left(\frac{x}{4}\right)^p + i\sum_{\substack{p\in\mathbf{N}\\ p \text{ odd}}} \frac{(2p+1)!}{(p!)^3}\frac{\partial \eta}{\partial s}(\theta, -p)\left(\frac{x}{4}\right)^p\right].$$

Proof. — Our Theorem follows from (4.59), (4.79) and from Theorem 4.29. □

Theorem 4.31. — *If $r \in \mathbf{N}^*$, and if $a \in \mathbf{N}$, then*

$$F\left(2\pi\frac{a}{r},x\right)^{(>0)} = \frac{1}{4}\sum_{0\leqslant m\leqslant r-1}\exp\left(2i\pi\frac{ma}{r}\right)\left[\sum_{\substack{p\in\mathbf{N}^*\\ p\,\text{even}}}\frac{(2p)!}{(p!)^2}\zeta\left(2\pi\frac{m}{r},p+1\right)\left(\frac{irx}{8\pi}\right)^p\right.$$
$$\left. - i\sum_{\substack{p\in\mathbf{N}^*\\ p\,\text{odd}}}\frac{(2p)!}{(p!)^2}\eta\left(2\pi\frac{m}{r},p+1\right)\left(\frac{irx}{8\pi}\right)^p\right],$$

(4.101)

$$I\left(2\pi\frac{a}{r},x\right)^{(>0)} = \frac{1}{4}\sum_{0\leqslant m\leqslant r-1}\exp\left(2i\pi\frac{ma}{r}\right)\left[\sum_{\substack{p\in\mathbf{N}^*\\ p\,\text{even}}}\frac{(2p+1)!}{(p!)^2}\zeta\left(2\pi\frac{m}{r},p+1\right)\left(\frac{irx}{8\pi}\right)^p\right.$$
$$\left. - i\sum_{\substack{p\in\mathbf{N}^*\\ p\,\text{odd}}}\frac{(2p+1)!}{(p!)^2}\eta\left(2\pi\frac{m}{r},p+1\right)\left(\frac{irx}{8\pi}\right)^p\right].$$

Proof. — Clearly, if $a \in \mathbf{N}$,

$$S_0\left(\frac{a}{r},0,s\right) = {\sum_{n\in\mathbf{Z}}}'\left|\frac{a}{r}+n\right|^{-2s}$$
$$= \frac{1}{r}\sum_{0\leqslant m,m'\leqslant r-1}\exp\left(2i\pi m\frac{a-m'}{r}\right){\sum_{n\in\mathbf{Z}}}'\left|n+\frac{m'}{r}\right|^{-2s} \tag{4.102}$$

From (4.102), we get

$$S_0\left(\frac{a}{r},0,s\right) = r^{2s-1}\sum_{0\leqslant m\leqslant r-1}\exp\left(2i\pi\frac{ma}{r}\right)S_0\left(0,\frac{m}{r},s\right). \tag{4.103}$$

By (4.76), (4.103), we obtain,

$$S_0\left(\frac{a}{r},0,s\right) = 2r^{2s-1}\sum_{0\leqslant m\leqslant r-1}\exp\left(2i\pi\frac{ma}{r}\right)\zeta\left(2\pi\frac{m}{r},2s\right). \tag{4.104}$$

A similar argument shows that

$$S_1\left(\frac{a}{r},0,s\right) = -2ir^{2s-2}\sum_{0\leqslant m\leqslant r-1}\exp\left(2i\pi\frac{ma}{r}\right)\eta\left(2\pi\frac{m}{r},2s-1\right). \tag{4.105}$$

By (4.87), (4.104), (4.105), we get the first equality in (4.101). The second equality in (4.101) is then a consequence of (4.59). □

Remark 4.32. — If k is a function $\mathbf{Z}/r\mathbf{Z} \mapsto \mathbf{C}$, its Fourier transform $\widehat{k} : \mathbf{Z}/r\mathbf{Z} \to \mathbf{C}$ is defined by the relation,

$$\widehat{k}(m) = \frac{1}{r}\sum_{a\in\mathbf{Z}/r\mathbf{Z}}\exp\left(-2i\pi\frac{ma}{r}\right)k(a). \tag{4.106}$$

From (4.101), we find that if $\widehat{I}^r(.,x)$ denotes the Fourier transform of $I(2\pi\frac{.}{r},x)$ considered as a function on $\mathbf{Z}/m\mathbf{Z}$, then

$$\widehat{I}^r\left(\frac{m}{r},x\right)^{(0)} = \frac{1}{4}\Bigg[\sum_{\substack{p\in\mathbf{N}^*\\ p\,\text{even}}} \frac{(2p+1)!}{(p!)^2}\zeta\left(2\pi\frac{m}{r},p+1\right)\left(\frac{irx}{8\pi}\right)^p - i\sum_{\substack{p\in\mathbf{N}^*\\ p\,\text{odd}}} \frac{(2p+1)!}{(p!)^2}\eta\left(2\pi\frac{m}{r},p+1\right)\left(\frac{irx}{8\pi}\right)^p\Bigg]. \tag{4.107}$$

Formula (4.107) is of special interest. In fact up to a factor $1/4$, it coincides with the power series obtained by Bismut-Lott [**BLo1**, Corollary 4.14], in their evaluation of the analytic torsion forms of a circle bundle equipped with the flat line bundle whose holonomy along the fibre is $e^{2i\pi m/r}$. As we shall see in Section 16, this is not an accident, since we will show that the result of [**BLo1**] is in fact a consequence of our main result.

4.7. The Lerch series and the function $J(\theta,x)$

Definition 4.33. — For $\theta\in\mathbf{R}^*, x\in\mathbf{C}, |x|<\theta$, put

$$J^\theta(x) = \frac{\pi}{4|\theta|}\left(1-\frac{ix}{\theta}\right)^{-1}. \tag{4.108}$$

Equivalently,

$$J^\theta(x) = \frac{\pi}{4}\frac{\operatorname{sgn}(\theta)}{(\theta-ix)}. \tag{4.109}$$

Observe that

$$J^\theta(0) = F^\theta(0). \tag{4.110}$$

Recall that the operator Q was defined in Definition 2.37.

Proposition 4.34. — *The following identity holds,*

$$QI^\theta(x) = J^\theta(x). \tag{4.111}$$

Proof. — By (4.86),

$$F^\theta(x) = \frac{\pi}{4|\theta|}\sum_{p=0}^{+\infty}\frac{(2p)!}{(p!)^2}\left(\frac{ix}{4\theta}\right)^p. \tag{4.112}$$

Using (2.118) and (4.71), we get

$$QF^\theta(x) = \frac{\pi}{4|\theta|}\sum_{p=0}^{+\infty}\frac{(ix/\theta)^p}{2p+1}. \tag{4.113}$$

By (4.79), (4.113), we obtain,

$$QI^{\theta}(x) = \frac{\pi}{4|\theta|} \sum_{p=1}^{+\infty} \left(\frac{ix}{\theta}\right)^{p}, \tag{4.114}$$

which is just (4.111). □

Theorem 4.35. — *The following identity holds,*

$$J(\theta, x) - J(0,0) = \sum_{k \in \mathbf{Z}}{}' \left(J^{2k\pi+\theta}(x) - J^{2k\pi}(0)\right), \tag{4.115}$$

$$J(\theta, x) = \frac{1}{2}\left[\sum_{\substack{p \in \mathbf{N} \\ p \text{ even}}} \frac{\partial \zeta}{\partial s}(\theta, -p)\frac{x^p}{p!} + i \sum_{\substack{p \in \mathbf{N} \\ p \text{ odd}}} \frac{\partial \eta}{\partial s}(\theta, -p)\frac{x^p}{p!}\right].$$

Proof. — Our identity follows from (2.118), from Theorem 4.30 and from Proposition 4.34. □

Theorem 4.36. — *If $r \in \mathbf{N}^*$, and if $a \in \mathbf{N}$, then*

$$J\left(2\pi\frac{a}{r}, x\right)^{(>0)} = \frac{1}{4} \sum_{0 \leqslant m \leqslant r-1} \exp\left(2i\pi\frac{ma}{r}\right) \left[\sum_{\substack{p \in \mathbf{N}^* \\ p \text{ even}}} \zeta\left(2\pi\frac{m}{r}, p+1\right)\left(\frac{irx}{2\pi}\right)^{p} - i \sum_{\substack{p \in \mathbf{N}^* \\ p \text{ odd}}} \eta\left(2\pi\frac{m}{r}, p+1\right)\left(\frac{irx}{2\pi}\right)^{p}\right]. \tag{4.116}$$

Proof. — This is a trivial consequence of (2.118) and of Theorem 4.31. □

Observe that $\frac{\partial \zeta}{\partial s}(\theta, 0)$ is a smooth function of $\theta \in \mathbf{R} \setminus 2\pi\mathbf{Z}$. Therefore $\frac{\partial \zeta}{\partial s}(\theta - ix, 0)$ makes sense as a formal power series, so that

$$\frac{\partial \zeta}{\partial s}(\theta - ix, 0) = \sum_{p=0}^{+\infty} \frac{\partial^p}{\partial \theta^p}\frac{\partial \zeta}{\partial s}(\theta, 0)\frac{(-ix)^p}{p!}. \tag{4.117}$$

Theorem 4.37. — *If $\theta \in \mathbf{R} \setminus 2\pi\mathbf{Z}$, the following identity holds,*

$$J(\theta, x) = \frac{1}{2}\frac{\partial \zeta}{\partial s}(\theta - ix, 0). \tag{4.118}$$

Proof. — By (4.115), (4.118) holds in degree 0. By Theorem 4.35, if $\theta \in \mathbf{R} \setminus 2\pi\mathbf{Z}$, if $ix \in \mathbf{R}$, for $|x|$ small enough,

$$J(\theta, x) = \left[\frac{1}{8}\sum_{k \in \mathbf{Z}} \frac{1}{\left|k + \frac{\theta - ix}{2\pi}\right|}\right]^{(>0)} = \left[\frac{1}{8} S_0\left(\frac{\theta - ix}{2\pi}, 0, \frac{1}{2}\right)\right]^{(>0)}. \tag{4.119}$$

Using the functional equation (4.74), if $y \notin 2\pi\mathbf{Z}$,

$$\frac{\partial}{\partial s} S_0(0, y, 0) = S_0\left(y, 0, \frac{1}{2}\right). \tag{4.120}$$

Also by (4.76),

$$\frac{\partial}{\partial s} S_0(0, y, 0) = 4\frac{\partial \zeta}{\partial s}(2\pi y, 0). \tag{4.121}$$

By (4.119)-(4.121), we get (4.118) also in positive degree. □

Now, we will establish for the genus $J(\theta, x)$ a formula which is closely related to a corresponding formula proved in [**BGo1**, Theorem 4.2] in the context of holomorphic torsion.

Theorem 4.38. — *If $\theta \in \mathbf{R} \setminus 2\pi\mathbf{Z}$, if $\theta' \in \mathbf{R}, x \in \mathbf{C}$ are such that $|\theta'|, |x|$ are small enough, then*

$$J(\theta + \theta', x) = J(\theta, x + i\theta'). \tag{4.122}$$

Also for $\theta' \in]-2\pi, 2\pi[\setminus\{0\}$, for $x \in \mathbf{C}, |x| < \inf_{k\in\mathbf{Z}} |\theta' + 2k\pi|$, then

$$J(\theta', x) = J(0, x + i\theta') + J^{\theta'}(x). \tag{4.123}$$

Proof. — If $\theta \in]-2\pi, 2\pi[\setminus\{0\}$, for $\theta' \in \mathbf{R}$, and $|\theta'|$ small enough, for any $k \in \mathbf{Z}$,

$$\operatorname{sgn}(\theta + \theta' + 2k\pi) = \operatorname{sgn}(\theta + 2k\pi). \tag{4.124}$$

From (4.109), Theorem 4.35 and from (4.124), we get (4.122). Also if $\theta' \in]-2\pi, 2\pi[\setminus 0$, for $k \in \mathbf{Z}^*$,

$$\operatorname{sgn}(\theta' + 2k\pi) = \operatorname{sgn}(k). \tag{4.125}$$

Using the same arguments as before and (4.125) instead of (4.124), we get (4.123). The proof of our Theorem is completed. □

Remark 4.39. — Of course, there are corresponding statements for $I(\theta, x)$. However the formulas are more complicate. It is interesting to observe that while the Chern normalization of analytic torsion forms is conceptually natural, here and later, their evaluation leads to simpler formulas than with any other normalization.

4.8. Formal relation to the R genus

As explained in the Introduction, in equivariant Arakelov theory, a power series $R(\theta, x)$ appears naturally [**B8, B9**], which is given by the formula,

$$R(\theta,x) = \sum_{\substack{p\geqslant 0\\ p \text{ even}}} i\left\{\sum_{j=1}^{p}\frac{1}{j}\eta(\theta,-p) + 2\frac{\partial\eta}{\partial s}(\theta,-p)\right\}\frac{x^p}{p!} + \sum_{\substack{p\geqslant 0\\ p \text{ odd}}}\left\{\sum_{j=1}^{p}\frac{1}{j}\zeta(\theta,-p) + 2\frac{\partial\zeta}{\partial s}(\theta,-p)\right\}\frac{x^p}{p!}, \tag{4.126}$$

so that $R(x) = R(x,0)$ is the R series obtained by Gillet and Soulé [**GS1**] in the non equivariant context. Recall that the Lerch series is given by $L(y,s) = \sum_{n=1}^{+\infty}\frac{e^{iny}}{n^s}$.

Proposition 4.40. — *The following identity holds,*

$$R(\theta,x) + 4J(\theta,x) = \sum_{p\in\mathbf{N}}\Big(\sum_{j=1}^{p}\frac{1}{j}L(\theta,-p) + 2\frac{\partial L}{\partial s}(\theta,-p)\Big)\frac{x^p}{p!}. \tag{4.127}$$

Proof. — After (4.77), we saw that $\zeta(y,s)$ vanishes when s is an even negative integer, and that $\eta(y,s)$ vanishes when s is an odd negative integer. Our identity then follows from (4.115) and from (4.126). □

Assume now that M is complex manifold, that $E = E_+ \oplus E_-$ is a holomorphic $\mathbf{Z}_2$ graded vector bundle on M, and that $g^E = g^{E_+} \oplus g^{E_-}$ is a Hermitian metric on $E = E_+ \oplus E_-$. We denote by $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ the holomorphic Hermitian connection on $E = E_+ \oplus E$. Let $g \in \mathrm{End}(E)$ be a holomorphic isometry of E, which preserves E_+ and E_-.

By [**B8**, Section 7 c)], the holomorphic analytic torsion forms $\widetilde{\mathrm{ch}}\left(E, g^E\right)$ are well defined. Similarly to the underlying real $\mathbf{Z}_2$-graded vector bundle E, we can associate the Chern de Rham analytic torsion forms $\mathcal{T}_{\mathrm{ch},g}\left(E, g^E\right)$ (here we omit ∇^E, since g^E determines ∇^E.

To the genus $J(\theta,x)$, we can associate the closed form $S_g\left(E, g^E\right)$.

Proposition 4.41. — *The following identity holds,*

$$\widetilde{\mathrm{ch}}\left(E, g^E\right)^{(>0)} + 2\mathcal{T}_{\mathrm{ch}^\circ,g}\left(E, g^E\right)^{(>0)} = (R+4J)_g\left(E, g^E\right)^{(>0)}. \tag{4.128}$$

Proof. — This is a trivial consequence of Theorem 4.26 and of Proposition 4.40. □

Remark 4.42. — The fact that the genera $R(\theta,x)$ and $J(\theta,x)$ fit so well in formulas (4.127), (4.128) is maybe more than coincidental.

CHAPTER 5

A FAMILY OF THOM-SMALE GRADIENT VECTOR FIELDS

The purpose of this Chapter is to establish various results on families of Thom-Smale complexes, associated to the gradient field of a fibrewise Morse function, which is supposed to be Morse-Smale in every fibre.

This Chapter is organized as follows. In Section 5.1, we recall the construction of the Thom-Smale complex associated to a Morse-Smale vector field. In Section 5.2, we state the results of Laudenbach [**La**], which guarantee that, under natural assumptions, there is a de Rham map, which is a quasi-isomorphism from the de Rham complex into the Thom-Smale complex. The key fact is that the closure of the stable and unstable cells are manifolds with conical singularities in C^1 coordinates. In Section 5.3, we introduce a group action on the Morse-Smale complex. In Section 5.4, we consider the case of an equivariant fibration, and we briefly describe the corresponding Leray spectral sequence. In Section 5.5, we suppose that this fibration is equipped with fibrewise Morse-Smale vector fields. We construct the corresponding family of Morse-Smale complexes. Finally in Section 5.6, we establish a families version of Laudenbach's results [**La**]. In particular we show that, under standard assumptions, the integral of a smooth form along the closure of the stable or unstable cells is a smooth form. Also we compare various natural spectral sequences. We show in particular that the fundamental group of the base S of the fibration acts on the cohomology of the fibre X as a finite group.

5.1. The Thom-Smale complex of a gradient vector field

Let X be a compact manifold of dimension n. Let $f : X \to \mathbf{R}$ be a Morse function. Let B be the set of critical points of f,

$$B = \{x \in X, df(x) = 0\}. \tag{5.1}$$

If $x \in B$, recall that the index $\mathrm{ind}\,(x)$ is such that the quadratic form $d^2 f\,(x)$ on $T_x M$ has signature $(n - \mathrm{ind}\,(x)\,, \mathrm{ind}\,(x))$.

Let h^{TX} be a metric on TX, and let ∇f be the gradient vector field of f with respect to h^{TX}. Set

$$Y = -\nabla f. \tag{5.2}$$

Consider the differential equation

$$\frac{dy}{dt} = Y(y). \tag{5.3}$$

Equation (5.3) defines a group of diffeomorphisms $\Psi_t|_{t\in\mathbf{R}}$ of X.

If $x \in B$, put

$$\begin{aligned} W^u(x) &= \{y \in X, \lim_{t\to-\infty} \Psi_t(y) = x\}, \\ W^s(x) &= \{y \in X, \lim_{t\to+\infty} \Psi_t(y) = x\}. \end{aligned} \tag{5.4}$$

The cells $W^u(x)$ and $W^s(x)$ are called the unstable and stable cells. They are embedded submanifolds of X, and moreover,

$$W^u(x) \simeq \mathbf{R}^{\mathrm{ind}(x)}, \qquad W^s(x) \simeq \mathbf{R}^{n-\mathrm{ind}(x)}. \tag{5.5}$$

Also $W^u(x)$ and $W^s(x)$ intersect transversally at x. If $x \in B$, set

$$T_xX^u = T_xW^u(x), \qquad T_xX^s = T_xW^s(x). \tag{5.6}$$

Then

$$T_xX = T_xX^s \oplus T_xX^u. \tag{5.7}$$

Assume that Y verifies the Smale transversality conditions [**Sm1, Sm2**]. Namely we suppose that if $x, y \in B$,, $W^u(x)$ and $W^s(y)$ intersect transversally. In particular if $\mathrm{ind}(y) = \mathrm{ind}(x) - 1$, $W^u(x) \cap W^s(y)$ consists of a finite set $\Gamma(x, y)$ of integral curves γ of the vector field Y, with $\gamma_{-\infty} = x, \gamma_{+\infty} = y$, along which $W^u(x)$ and $W^s(y)$ intersect transversally.

By [**Sm1**, Theorem A], [**Mi2**, Theorems 4.4 and 5.2], given a Morse function f, there exists a metric h^{TX} such that the corresponding vector field Y verifies the Smale transversality conditions.

If $x \in B$, let o^u_x, o^s_x be the orientation lines of T_xX^u, T_xX^s. Then o^u_x, o^s_x are $\mathbf{Z}_2$-lines. Also by (5.5), (5.6), o^u_x, o^s_x can be identified with the orientation lines of $W^u(x), W^s(x)$. In the sequel, we will identify these lines to the corresponding complex lines. Note that the lines o^u_x, o^s_x are canonically identified to their duals.

Let $x, y \in B$, with $\mathrm{ind}(y) = \mathrm{ind}(x) - 1$. Take $\gamma \in \Gamma(x, y)$. The orientation bundle of the orthogonal bundle $T^\perp W^s(y)$ to $TW^s(y)$ in $TX|_{W^s(y)}$ is canonically isomorphic to $o^u(y)$. Let $T'W^s(x)$ be the orthogonal bundle to Y in $TW^u(x)$. Its orientation bundle $o(T'W^s(x))$ is canonically isomorphic to o^u_x, so that $s \in o(T'W^s(x))$ corresponds to $Y \mathbin{\widehat{\otimes}} s \in o^u_x$. Since $T^\perp W^s(y)$ and $T'W^s(x)$ have the same orientation bundle, to $\gamma \in \Gamma(x, y)$, we can associate $n_\gamma(x, y) \in o^u_x \otimes o^u_y$.

Let $\left(F, \nabla^F\right)$ be a complex flat vector bundle on X, and let $\left(F^*, \nabla^{F^*}\right)$ be the corresponding dual flat vector bundle. Set

$$C_\bullet\left(W^u, F^*\right) = \bigoplus_{x \in B} F_x^* \otimes o_x^u, \tag{5.8}$$
$$C_i\left(W^u, F^*\right) = \bigoplus_{\substack{x \in B \\ \mathrm{ind}(x)=i}} F_x^* \otimes o_x^u.$$

By (5.5), on $W^u(x)$, the flat vector bundle F can be canonically trivialized by parallel transport. In particular, if $x, y \in B$ are such that $\mathrm{ind}(y) = \mathrm{ind}(x) - 1$ and if $\gamma \in \Gamma(x,y), e^* \in F_x^*$, let $\tau_\gamma(e^*) \in F_y^*$ be the parallel transport of $f \in F_x^*$ along γ with respect to the flat connection ∇^{F^*}.

If $x \in B, s \in o_x^u, e^* \in F_x^*$, set

$$\partial(s \otimes e^*) = \sum_{\substack{y \in B \\ \mathrm{ind}(y)=\mathrm{ind}(x)-1}} \sum_{\gamma \in \Gamma(x,y)} n_\gamma(x,y) s \otimes \tau_\gamma(e^*). \tag{5.9}$$

Then ∂ maps $C_i(W^u, F^*)$ into $C_{i-1}(W^u, F^*)$.

Now we recall a basic result of Thom [**T**] and Smale [**Sm2**].

Theorem 5.1. — *We have the identity*

$$\partial^2 = 0, \tag{5.10}$$

so that $(C_\bullet(W^u, F^*), \partial)$ *is a chain complex. Moreover there is a canonical isomorphism of* $\mathbf{Z}$*-graded vector spaces,*

$$H_\bullet(C_\bullet(W^u, F^*), \partial) \simeq H_\bullet(X, F^*). \tag{5.11}$$

Let $(C^\bullet(W^u, F), \partial)$ be the complex dual to the complex $(C_\bullet(W^u, F^*), \partial)$. By (5.8), we get

$$C^\bullet\left(W^u, F\right) = \bigoplus_{x \in B} F_x \otimes o_x^u, \tag{5.12}$$
$$C^i\left(W^u, F\right) = \bigoplus_{\substack{x \in B \\ \mathrm{ind}(x)=i}} F_x \otimes o_x^u.$$

By Theorem 5.1, we get the canonical isomorphism,

$$H^\bullet(C^\bullet(W^u, F), \partial) \simeq H^\bullet(X, F). \tag{5.13}$$

The complex $\left(C^\bullet\left(W^u, F\right), \partial\right)$ will be called the Thom-Smale complex attached to Y.

Let $o(TX)$ be the orientation bundle of TX. Then $o(TX)$ is a $\mathbf{Z}_2$-line bundle. We will consider $o(TX)$ as a complex line bundle. If we replace f by $-f$ and Y by $-Y$, the roles of W^u and W^s are interchanged. Comparing (5.8) and (5.12), we obtain easily the isomorphism of complexes,

$$\left(C_\bullet\left(W^u, F\right), \partial\right) \simeq \left(C^{n-\bullet}\left(W^s, F \otimes o(TX)\right), \partial\right). \tag{5.14}$$

On the other hand, by (5.13),

$$H^{\bullet}\left(C^{\bullet}\left(W^{s}, F \otimes o\left(TX\right)\right), \partial\right) \simeq H^{\bullet}\left(X, F \otimes o\left(TX\right)\right). \tag{5.15}$$

One verifies easily that (5.11)-(5.15) is just Poincaré duality.

Let $\left(\Omega^{\bullet}\left(X, F|_X\right), d^X\right)$ be the complex of smooth sections of $\Lambda^{\bullet}(T^*X) \widehat{\otimes} F$ on X, equipped with the de Rham operator d^X, so that we have the canonical isomorphism,

$$H^{\bullet}\left(\Omega^{\bullet}\left(X, F|_X\right), d^X\right) \simeq H^{\bullet}\left(X, F\right). \tag{5.16}$$

Let us recall that a Morse function f is said to be nice if f takes the value i on the set of critical points of index i. By a result of Smale [**Sm1**, Theorem B], [**Mi2**, Theorem 4.8], given a gradient vector field $Y = -\nabla f$ which verifies the above transversality conditions, there is a nice Morse function $\widetilde{f}$ and a metric $\widetilde{h}^{TX}$ on TX such that Y is also the gradient vector field for $\widetilde{f}$ with respect to $\widetilde{h}^{TX}$. So, if necessary, we may as well assume now that f is a nice Morse function.

Now we follow Milnor [**Mi1**, Section 9] and [**BZ1**, Chapter I c)]. Suppose that f is nice. For $p \in \mathbf{N}$, set

$$U_p = f^{-1}\left[p - 1/2, +\infty\right[. \tag{5.17}$$

The decreasing family of closed sets U_p defines a decreasing filtration on the de Rham complex $\left(\Omega^{\bullet}\left(X, F|_X\right), d\right)$. By definition $F^p\Omega^{\bullet}\left(X, F|_X\right)$ is the set of elements of $\Omega^{\bullet}\left(X, F|_X\right)$ whose support is included in U_p. Let us construct the corresponding spectral sequence. By definition,

$$E_0^{p,q} = F^p\Omega^{p+q}\left(X, F\right)/F^{p+1}\Omega^{p+q}\left(X, F\right). \tag{5.18}$$

Then

$$E_1^{p,q} = H^{p+q}\left(U_i, U_{i+1}, F\right). \tag{5.19}$$

The basic result of Morse theory shows that

$$\begin{aligned} E_1^{p,q} &= C^p\left(W^u, F\right) \text{ if } q = 0, \\ &= 0 \text{ if } q \neq 0. \end{aligned} \tag{5.20}$$

By (5.20), one finds easily that

$$\left(E_1, d_1\right) \simeq \left(C^{\bullet}\left(W^u, F\right), \partial\right), \tag{5.21}$$

and that the spectral sequence degenerates at E_2. In particular we have established the existence of the canonical isomorphism (5.13). Also equation (5.21) gives a purely algebraic construction of the complex $\left(C^{\bullet}\left(W^u, F\right), \partial\right)$.

5.2. The de Rham map of Laudenbach

If $x \in X, \varepsilon > 0$, let $B^X(x, \varepsilon)$ be the open ball of centre x and radius ε with respect to h^{TX}. By a simple argument by Helffer-Sjöstrand [**HSj**, Proposition 5.1], if $x \in B$, for $\varepsilon_0 > 0$ small enough, there exists an identification of $B^X(x, 3\varepsilon_0)$ with an open neighbourhood $V_{3\varepsilon_0}$ of x in X, a Morse function $f_{\varepsilon_0} : X \to \mathbf{R}$, and a metric $h^{TX}_{\varepsilon_0}$ on TX, which have the following properties:

- $\left(f_{\varepsilon_0}, h^{TX}_{\varepsilon_0}\right)$ coincide with f, h^{TX} on $X \setminus V_{3\varepsilon_0}$.
- The critical set for the function f_{ε_0} is still equal to $\mathbf{B}$. Also the stable and unstable cells associated to $\left(f_{\varepsilon_0}, h^{TX}_{\varepsilon_0}\right)$ coincide with the corresponding stable and unstable cells for $\left(f, h^{TX}\right)$.
- Under the above coordinate system, on $V_{2\varepsilon_0}$, the metric $h^{TX}_{\varepsilon_0}$ comes a from a standard metric on TX such that T_xX^s and T_xX^u are orthogonal in TX. Moreover if $Z \in T_xX$, if $Z = (Z_+, Z_-)$, with $Z_+ \in T_xX^s, Z_- \in T_xX^u$, and $|Z| \leqslant 2\varepsilon_0$, then

$$f_{\varepsilon_0}(Z) = f(x) + \frac{1}{2}\left(|Z_+|^2 - |Z_-|^2\right). \tag{5.22}$$

Note that the above constructions can easily be done equivariantly. Also observe that if f is nice, the new function f_{ε_0} is also nice.

Observe the the Thom-Smale complex associated to $\left(f_{\varepsilon_0}, h^{TX}_{\varepsilon_0}\right)$ coincides with the given Thom-Smale complex $(C^\bullet(W^u, F), \partial)$. In the sequel we will assume that $\left(f, h^{TX}\right) = \left(f_{\varepsilon_0}, h^{TX}_{\varepsilon_0}\right)$.

In [**La**], Laudenbach proved that under the above conditions, the closed cells $\overline{W^u(x)}$ are submanifolds of X with conical singularities. Also he showed that if $x \in B$, $\overline{W^u(x)} \setminus W^u(x)$ is stratified as a union of $W^u(y)$, with $\operatorname{ind}(y) < \operatorname{ind}(x)$. An important point in [**La**] is that the coordinate charts in which the above description of $\overline{W^u(x)}$ is valid are in general only C^1. Once an orientation of $W^u(x)$ is fixed, $\overline{W^u(x)}$ defines a current on X. Equivalently, smooth forms can be integrated on $\overline{W^u(x)}$. Moreover if $x \in B$, the boundary of $\overline{W^u(x)}$ considered as a current coincides with its geometric boundary, i.e. it is the current of integration on the $\overline{W^u(y)}$, with $\overline{W^u(y)} \subset \overline{W^u(x)} \setminus W^u(x)$ and $\operatorname{ind}(y) = \operatorname{ind}(x) - 1$, the orientations being obtained as in (5.9).

Clearly, we can trivialize the vector bundle F on $W^u(x)$ with respect to the connection ∇^F. In particular, if $\alpha \in \Omega^\cdot(M, F)$, the integral $\int_{\overline{W^u(x)}} \alpha$ lies in $o^u_x \otimes F_x$.

Definition 5.2. — Let P^∞ be the map,

$$\alpha \in \Omega^\cdot(M, F) \to P^\infty \alpha = \sum_{x \in B} \int_{\overline{W^u(x)}} \alpha \in C^\bullet(W^u, F). \tag{5.23}$$

Now we have the key result by Laudenbach [**La**, Propositions 6 and 7], [**BZ1**, Theorem 2.9].

Theorem 5.3. — *The map P^{∞} is a quasi-isomorphism of complexes, which provides the canonical identification of the cohomology groups of both complexes.*

5.3. Equivariant Thom-Smale complexes

We make the same assumptions as in Sections 5.1 and 5.2. Let G be a compact Lie group. We assume that G acts on X, and that this action lifts to F, and preserves the flat connection ∇^F. Then by (3.6), G acts on $\left(\Omega^{\bullet}\left(X, F|_X\right), d\right)$. Therefore G acts on $H^{\bullet}(X, F)$ and preserves its $\mathbf{Z}$-grading.

Recall that if $g \in G$, the Lefschetz number $\chi_g(F)$ was defined in (3.64) by the formula

$$\chi_g(F) = \sum_{j=0}^{n} (-1)^j \operatorname{Tr}^{H^j(X,F|_X)}[g]. \tag{5.24}$$

Take $g \in G$. Set

$$X_g = \{x \in X, gx = x\}. \tag{5.25}$$

Then X_g is a finite union of compact submanifolds of X. Also, if $x \in X_g$,

$$T_x X_g = \{U \in T_x X, g_* U = U\}. \tag{5.26}$$

Also $\operatorname{Tr}^{F|_{X_g}}[g]$ is a locally constant function on X_g.

Let $e(TX_g) \in H^{\bullet}(X_g, \mathbf{Q})$ be the Euler class of TX_g. The Lefschetz fixed point formula asserts that

$$\chi_g(F) = \int_{X_g} \operatorname{Tr}^F[g]\, e(TX_g). \tag{5.27}$$

Now we assume that $f : X \to \mathbf{R}$ is a G-invariant Morse function, that h^{TX} is a G-invariant metric on TX, and that $Y = -\nabla f$ verifies the Thom-Smale transversality conditions. Note that Y is then a G-invariant vector field.

By [**Sm1, Sm2**], [**Mi2**, Theorems 4.4 and 5.2], generically, a gradient vector field for f verifies the Smale transversality conditions. As explained in [**BZ2**, Section 1 d)], if G is a non trivial group, a G-invariant generic gradient vector field is not necessarily Thom-Smale. However, as shown in [**BZ2**, Theorem 1.10] using results of Illman [**Il**], if G is a finite group, there exists a G-invariant Morse function f and a G-invariant metric h^{TX} such that $Y = -\nabla f$ verifies the Thom-Smale transversality conditions.

Set

$$B_g = B \cap X_g. \tag{5.28}$$

Since ∇f is G-invariant, we deduce from (5.26) that $\nabla f|_{X_g} \in TX_g$. Therefore the restriction $f|_{X_g}$ of f to X_g is a Morse function, and the restriction of ∇f to X_g is a gradient field for $f|_{X_g}$. The set B_g is exactly the set of critical points of $f|_{X_g}$. Also if $x \in B_g$, $T_x X^s$ and $T_x X^u$ are g-invariant.

Now we use the notation of Sections 5.1-5.2. Clearly G acts on the finite set B, and interchanges the $W^u(x)$'s, and also the $W^s(x)$'s. It then follows easily that G acts on the complex $(C^\bullet(W^u, F), \partial)$, and so it acts on its cohomology. Then (5.13) is an identification of G-vector spaces.

By (1.56), (1.57) and by the above, it follows that

$$\chi_g(F) = \sum_{j=0}^{n} (-1)^j \operatorname{Tr}^{C^j(W,F)}[g]. \tag{5.29}$$

If $x \in B_g$, g acts on $o^u(x) \otimes F_x$. Then (5.29) can be written in the form,

$$\chi_g(F) = \sum_{x \in B_g} (-1)^{\operatorname{ind}(x)} \operatorname{Tr}^{F_x \otimes o^u_x}[g]. \tag{5.30}$$

If $x \in B_g$, let $\operatorname{ind}_g(x)$ be the index of $f|_{X_g}$ at x. The action of g on $o^u(x)$ is given by

$$g|_{o^u(x)} = \det g|_{T^u_x X}. \tag{5.31}$$

Also, one has the trivial,

$$(-1)^{\operatorname{ind}(x)} \det g|_{T^u_x X} = (-1)^{\operatorname{ind}_g(x)}. \tag{5.32}$$

By (5.31), (5.32) we get

$$(-1)^{\operatorname{ind}(x)} \operatorname{Tr}^{F_x \otimes o^u_x}[g] = (-1)^{\operatorname{ind}_g(x)} \operatorname{Tr}^{F_x}[g]. \tag{5.33}$$

By (5.30)-(5.33), we get

$$\chi_g(F) = \sum_{x \in B_g} (-1)^{\operatorname{ind}_g(x)} \operatorname{Tr}^{F_x}[g]. \tag{5.34}$$

Of course, by Chern-Gauss-Bonnet, we obtain directly the equality of the right-hand sides of (5.27) and (5.34).

Proposition 5.4. — *The vector field $Y|_{X_g}$ is Morse-Smale.*

Proof. — We claim that if $x \in B_g$, $W^u(x) \cap X_g$ is just the unstable cell for $Y|_{X_g}$ at x. In fact, if $y \in W^u(x) \cap X_g$, since Y is G-invariant, the integral curve of Y through y lies in X_g, so that our assertion follows. The same property holds for the corresponding stable cells.

Let now $x, x' \in B_g$, and let $y \in W^u(x) \cap W^s(x') \cap X_g$. Since Y is Morse-Smale, $W^u(x)$ and $W^s(x')$ intersect transversally at y. By considering the corresponding $+1$ eigenspaces under the obvious action of g, our Proposition follows. □

5.4. A smooth fibration

We make the same assumptions and we use the same notation as in Section 3. In particular $\pi : M \to S$ is a submersion of smooth manifolds with compact fibre X of dimension n, and $\left(F, \nabla^F\right)$ is a complex flat vector bundle on M. Also G is a Lie group acting on M along the fibres X, whose action lifts to F and preserves ∇^F.

The de Rham complex $\left(\Omega^\bullet(M,F), d^M\right)$ is a filtered complex. Namely if $m, p \in \mathbf{N}, 0 \leqslant m \leqslant \dim M, 0 \leqslant p \leqslant \dim S$, set

$$F'^p\Omega^m(M,F) = \Big\{ s \in \Omega^\bullet(M,F),\ \text{if } X_1, \ldots, X_{m-p+1} \in TX, \text{ then } i_{X_1} \cdots i_{X_{m-p+1}} s = 0 \Big\}. \tag{5.35}$$

Then the $F'^p\Omega^\bullet(M,F)$ defines a filtration on $\Omega^\bullet(M,F)$. The corresponding spectral sequence, which we will note $\mathbf{E}'^{(\bullet,\bullet)}_\bullet$, is the Leray spectral sequence. In particular,

$$\mathbf{E}'^{(p,q)}_0 = C^\infty\left(M, \pi^*\Lambda^p(T^*S) \mathbin{\widehat{\otimes}} \Lambda^q(T^*X) \mathbin{\widehat{\otimes}} F\right), \tag{5.36}$$

and d_0 is the fibrewise de Rham operator d^X.

Recall that $H^\bullet(X, F|_X)$ is the cohomology of the fibre X with coefficients in F. Then $H^\bullet(X, F|_X)$ is a $\mathbf{Z}$-graded flat vector bundle on S, which is equipped with a flat connection, the Gauss-Manin connection $\nabla^{H^\bullet(X,F|_X)}$. It follows from the above considerations that

$$\mathbf{E}'^{(p,q)}_1 = \Omega^p\left(S, H^q(X, F|_X)\right). \tag{5.37}$$

Also d_1 acting on $\mathbf{E}'^{(\bullet,\bullet)}_1$ is just the de Rham operator d^S. In particular,

$$\mathbf{E}'^{(p,q)}_2 = H^p\left(S, H^q(X, F|_X)\right). \tag{5.38}$$

Of course, the Lie group G acts naturally on the spectral sequence $\mathbf{E}'^{(\bullet,\bullet)}_r$.

5.5. A family of gradient vector fields

We make the same assumptions as in Section 5.4. Let $f : M \to \mathbf{R}$ be a G-invariant smooth function. We assume that f is Morse along every fibre X.

Let h^{TX} be a G-invariant metric on TX. Let $\nabla f \subset TX$ be the gradient field of f along the fibre X with respect to h^{TX}. Then $\nabla f \in TX$. We make the fundamental assumption that $Y = -\nabla f$ is Thom-Smale along every fibre X.

By proceeding as in [**Mi2**, Section 4], one verifies easily that there is a G-invariant smooth function $\widetilde{f} : M \to \mathbf{R}$ such that $\nabla\widetilde{f}$ is a gradient vector field for $\widetilde{f}$, and $\widetilde{f}$ is fibrewise nice. So, if necessary, we may as well assume that f itself is fibrewise nice.

Let $\mathbf{B}$ be the zero set of Y, i.e. the set of fibrewise critical points of f. Let $\mathbf{B}^i$ be the set of critical points of f which have index i along the fibres X. Then $\mathbf{B}, \mathbf{B}^i$ are finite covers of S. We denote by B, B^i the corresponding fibres.

Observe that TX^u, TX^s are now vector bundles on $\mathbf{B}$, and that

$$TX|_{\mathbf{B}} = TX^u \oplus TX^s. \tag{5.39}$$

Let o^u, o^s be the $\mathbf{Z}_2$-lines on $\mathbf{B}$, which are the orientation lines of TX^u, TX^s. In the sequel we will still denote by o^u, o^s the corresponding complexifications.

Now we temporarily assume that f is fibrewise nice. Since f takes the constant value i on $\mathbf{B}^i$, we see that $\mathbf{B}$ is exactly the set of critical points of f on M, and that f is a Morse-Bott function. For $p \in \mathbf{N}$, set

$$U_p = f^{-1}]p - 1/2, +\infty[. \tag{5.40}$$

As in Section 5.1, the U_p's define a decreasing filtration F'' on $\Omega^\bullet(M, F)$. Namely $F''^p\Omega^\bullet(M, F)$ is the set of $s \in \Omega^\bullet(M, F)$ whose support is included in U_p. Let $\mathbf{E}''^{(\bullet,\bullet)}_\bullet$ be the corresponding spectral sequence. Using the analogue of (5.19) and the Thom isomorphism, we get

$$\mathbf{E}''^{(p,q)}_1 = H^q\left(\mathbf{B}^p, F|_{\mathbf{B}^p} \otimes o^u\right). \tag{5.41}$$

Now we no longer suppose f to be fibrewise nice. Put

$$\begin{aligned} C^\bullet(W^u, F) &= \bigoplus_{x \in B} F_x \otimes o^u_x, \\ C^i(W^u, F) &= \bigoplus_{x \in B^i} F_x \otimes o^u_x. \end{aligned} \tag{5.42}$$

Then $C^\bullet(W^u, F)$ is a flat $\mathbf{Z}$-graded vector bundle on S. Let $\nabla^{C^\bullet(W^u,F)}$ be the corresponding flat connection on $C^\bullet(W^u, F)$. The identification (5.21) of the fibrewise complexes $(C^\bullet(W^u, F), \partial)_s$ to an algebraic complex shows that the chain map ∂ depends smoothly on $s \in S$, and is flat with respect to $\nabla^{C^\bullet(W^u,F)}$. Therefore $C^\bullet(W^u, F)$ is an example of a $\mathbf{Z}$-graded flat complex in the sense of Sections 1.10 and 2.5. In particular $A' = \partial + \nabla^{C^\bullet(W^u,F)}$ is a flat superconnection of total degree 1 on $C^\bullet(W^u, F)$. By (5.13),

$$H^\bullet\left(C^\bullet(W^u, F), \partial\right) \simeq H^\bullet(X, F|_X). \tag{5.43}$$

As we saw in Section 1.6, the flat connection $\nabla^{C^\bullet(W^u,F)}$ induces a flat connection on $H^\bullet\left(C^\bullet(W^u, F), \partial\right)$. Using (5.43), one verifies easily that this connection is just the Gauss-Manin connection $\nabla^{H^\bullet(X,F|_X)}$.

The complex $\left(\Omega^\bullet(S, C^\bullet(W^u, F)), A'\right)$ is naturally bigraded. The partial grading in $\Lambda^\bullet(T^*S)$ defines a filtration on $\left(\Omega^\bullet(S, C^\bullet(W^u, F)), \nabla^{C^\bullet(W^u,F)} + \partial\right)$. Let $E'^{(\bullet,\bullet)}_\bullet$ be the corresponding spectral sequence. Then

$$E'^{(p,q)}_0 = \Omega^p\left(S, C^q(W^u, F)\right), \tag{5.44}$$

and d'_0 is just ∂.By (5.13),

$$E'^{(p,q)}_1 = \Omega^p\left(S, H^q(X, F|_X)\right), \tag{5.45}$$

and the differential d'^1 is the Gauss-Manin connection $\nabla^{H^\bullet(X,F|_X)}$. In particular

$$E_2'^{(p,q)} = H^p\left(S, H^q\left(X, F|_X\right)\right). \tag{5.46}$$

Similarly the grading in $C^\bullet(W^u, F)$ defines another filtration on

$$\left(\Omega^\bullet(S, C^\bullet(W^u, F))\nabla^{C^\bullet(W^u,F)} + \partial\right).$$

Let $E_\bullet''^{(\bullet,\bullet)}$ be the corresponding spectral sequence. By construction

$$E_1''^{(p,q)} = H^q\left(\mathbf{B}^p, F|_{\mathbf{B}^p} \otimes o^u|_{\mathbf{B}^p}\right). \tag{5.47}$$

Comparing (5.37), (5.38), (5.41) with (5.45), (5.46), (5.47), we get

$$\begin{aligned} &\mathbf{E}_r'^{(\bullet,\bullet)} \simeq E_r'^{(\bullet,\bullet)} \text{ with } r = 1, 2, \\ &\mathbf{E}_1''^{(\bullet,\bullet)} \simeq E_1''^{(\bullet,\bullet)}. \end{aligned} \tag{5.48}$$

Of course, (5.48) gives identifications of G-vector spaces. As we shall see in Section 5.6, this result extends to the full spectral sequences.

5.6. A families version of the results of Laudenbach

Observe that $TX|_{\mathbf{B}}$ is just the normal bundle $N_{\mathbf{B}/M}$ to $\mathbf{B}$ in M. Let h^{TX} be a G-invariant metric on TX. Given $\varepsilon > 0$, let U_ε be the ε-neighbourhood of $\mathbf{B}$ in $TX|_{\mathbf{B}}$.

By proceeding as in Helffer-Sjöstrand [**HSj**, Proposition 5.1], as in Section 5.2, for $\varepsilon_0 > 0$ small enough, there exists a G-equivariant identification of $U_{3\varepsilon_0}$ with a tubular neighbourhood $V_{3\varepsilon_0}$ of $\mathbf{B}$ in the fibres X, which maps the fibres $TX|_{\mathbf{B}}$ into the corresponding fibres X, a G-invariant fibrewise Morse function $f_{\varepsilon_0} : M \to \mathbf{R}$ and a G-invariant metric $h^{TX}_{\varepsilon_0}$ on TX which have the following properties:

- $\left(f_{\varepsilon_0}, h^{TX}_{\varepsilon_0}\right)$ coincide with f, h^{TX} on $M \setminus V_{3\varepsilon_0}$.
- The fibrewise critical set for the function f_{ε_0} is still equal to $\mathbf{B}$. Also the stable and unstable cells associated to $\left(f_{\varepsilon_0}, h^{TX}_{\varepsilon_0}\right)$ coincide with the corresponding stable and unstable cells for $\left(f, h^{TX}\right)$.
- Under the above coordinate system, on $U_{2\varepsilon_0}$, the metric $h^{TX}_{\varepsilon_0}$ comes a from a standard metric on $TX|_{\mathbf{B}}$ such that $TX^s|_{\mathbf{B}}$ and $TX^u|_{\mathbf{B}}$ are orthogonal in $TX|_{\mathbf{B}}$. Moreover if $x \in \mathbf{B}, Z \in T_xX|_{\mathbf{B}}$, if $Z = (Z_+, Z_-)$, with $Z_+ \in T_xX|_{\mathbf{B}}, Z_- \in T_xX|_{\mathbf{B}}$, if $|Z| \leqslant 2\varepsilon_0$, then

$$f_{\varepsilon_0}(Z) = f(x) + \frac{1}{2}\left(|Z_+|^2 - |Z_-|^2\right). \tag{5.49}$$

The Thom-Smale complex associated to $\left(f_{\varepsilon_0}, h^{TX}_{\varepsilon_0}\right)$ coincides with $\left(C^\bullet(W^u, F), \partial\right)$. In the sequel, we will then assume that $\left(f, h^{TX}\right) = \left(f_{\varepsilon_0}, h^{TX}_{\varepsilon_0}\right)$.

We will consider the unstable cells $W^u(x)$ as subsets of M, which fibre on S. The fibrewise closures $\overline{W^u(x)}$ also patch into closed sets in M, which fibre over S. This is because all the arguments used in [**La**] can be applied to the vector field Y, viewed as a gradient vector field for the Morse-Bott function $f : M \to \mathbf{R}$. If x is a locally trivial

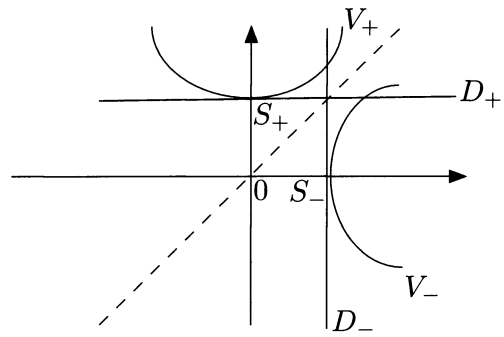

FIGURE 5.1

section of $\mathbf{B}$, we still denote by $W^u(x)$ the associated unstable cell, and $\overline{W^u(x)}$ its closure.

A word for word reproduction of the arguments of Laudenbach [**La**] show that the $\overline{W^u(x)}$ are submanifolds of M with conical singularities, that the $\overline{W^u(x)} \setminus W^u(x)$ are stratified by W^u of strictly lower index. Again, the description of $\overline{W^u}$ as a manifold with conical singularities can only be done in C^1 charts over M. Therefore $\overline{W^u(x)}$ is a well defined current on M. Its boundary as a current coincides with its geometric boundary.

A subtle point is to understand how the global description of the $\overline{W^u(x)}$ fit with the fibrewise description of the $\overline{W^u(x)}$. We will address this point in our next result.

By the above, if $\alpha \in \Omega^\bullet(M, F)$, if x is a locally trivial section of $\mathbf{B}$, the pushforward current $\int_{\overline{W^u(x)}} \alpha$ is well defined. This is a locally defined current on S with values in o_x^u.

Theorem 5.5. — *If $\alpha \in \Omega^\bullet(M, F)$, then $\int_{\overline{W^u(x)}} \alpha$ is C^∞ on S.*

Proof. — We will closely follow the general strategy of Laudenbach [**La**], while almost entirely avoiding the description of the compactified cells $\overline{W}^u$ as manifolds with conical singularities.

We will first consider the case of a single fibre, and give a description of the compactification of the descending cells. This description will then immediately extend to the case of families.

First, we consider the standard Morse model in (5.22). Namely let $E = E_+ \oplus E_-$ be a $\mathbf{Z}_2$-graded finite dimensional real vector space, equipped with an Euclidean metric g^E such that E_+ and E_- are orthogonal in E. If $Z = (Z_+, Z_-) \in E = E_+ \oplus E_-$, set

$$f(Z) = \frac{1}{2}\left(|Z_+|^2 - |Z_-|^2\right). \tag{5.50}$$

In Figure 5.1, we have represented the level sets $V_+ = f^{-1}\{1\}, V_- = f^{-1}\{-1\}$, the ascending and descending unit spheres $S_+ = V_+ \cap E_+, S_- = V_- \cap E_-$, and also $D_+ = S_+ \times E_-, D_- = E_+ \times S_-$.

If $Y = -\nabla f$, then

$$Y = (-Z_+, Z_-)\,. \tag{5.51}$$

The corresponding flow Ψ_t is given by

$$\phi_t\,(Z_+, Z_-) = \left(e^{-t} Z_+, e^t Z_-\right). \tag{5.52}$$

It will be convenient to identify V_+ to D_+, V_- to D_- by the flow Ψ_t. These identifications are given by

$$\begin{aligned}(Z_+, Z_-) \in V_+ &\to \left(\frac{Z_+}{|Z_+|}, |Z_+|\, Z_-\right),\\ (Z_+, Z_-) \in V_- &\to \left(|Z_-|\, Z_+, \frac{Z_-}{|Z_-|}\right).\end{aligned} \tag{5.53}$$

Clearly the flow identifies $V_+ \setminus S_+$ to $V_- \setminus S_-$, and $D_+ \setminus S_+$ to $D_- \setminus S_-$. This last identification is just given by

$$(Z_+, Z_-) \in D_+ \setminus S_+ \to \left(|Z_-|\, Z_+, \frac{Z_-}{|Z_-|}\right) \in D_+ \setminus S_-. \tag{5.54}$$

Now, we consider the real blow up of E along E_+ or E_-. Namely, let $p_+ : \mathbf{R}_+ \times E_+ \times S_- \to E, p_- : \mathbf{R}_+ \times S_+ \times E_- \to E$ be given by

$$\begin{aligned}p_+\,(r, Z_+, Z_-) &= (Z_+, rZ_-)\,,\\ p_-\,(r, Z_+, Z_-) &= (rZ_+, Z_-)\,.\end{aligned} \tag{5.55}$$

Then $\widehat{D}_+ = p_+^{-1}\,(D_+)\,, \widehat{D}_- = p_-^{-1}\,(D_-)$ are obtained by blowing up D_+, D_- along S_+, S_-. They both coincide with $\mathbf{R}_+ \times S_+ \times S_-$. Moreover the identification of $\widehat{D}_+$ to $\widehat{D}_-$ via the flow is just given by the identity map of $\mathbf{R} \times S_+ \times S_-$.

Let now F_+ be a submanifold of D_+ which intersects S_+ transversally. Set $H_+ = F_+ \cap S_+$. Our first goal is to describe a compactification of the image of F_+ by the flow $\Phi_t, t \geqslant 0$. Near H_+, there is a projection $r_+ : E_+ \to H_+$. Using transversality, near H_+, the map $Z = (Z_+, Z_-) \in F_+ \to (r_+ Z_+, Z_-)$ is a diffeomorphism. Therefore there is a smooth locally defined map $h : H_+ \times E_- \to E_+$, such that locally near H_+, F_+ is the image of $H_+ \times E_-$ by the map $(x, Z_-) \in H_+ \times E_- \to (h\,(x, Z_-)\,, Z_-)$, and moreover $h\,(x, 0) = x$. It follows that locally, the image of F_+ by the flow in E can be compactified as the image of $[0,1] \times [0,1] \times H_+ \times S_-$ by the map,

$$(a, b, x, Z_-) \to (ah\,(x, abZ_-)\,, bZ_-)\,. \tag{5.56}$$

From (5.56), we deduce that locally, the image of F_+ by the flow can be compactified into the image of a smooth manifold with boundary by a smooth map. In particular this compactification produces a well-defined current on E. The same argument shows

that the image of F_+ in D_- can be compactified into a set F_-, which is locally the image of $[0,1] \times H_+ \times S_-$ by the map

$$(a, x, Z_-) \to (ah(x, aZ_-), Z_-). \tag{5.57}$$

In particular, it still defines a current on D_-. Also $S_- \subset F_-$.

Now, we show how to 'iterate' the above procedure. Let $E^1 = E^1_+ \oplus E^1_-$ be another $\mathbf{Z}_2$-graded Euclidean vector space of dimension n. The corresponding objects associated to E^1 will be denoted with a superscript ^{1}. Let $\Phi = (\Phi_+, \Phi_-)$ be a smooth diffeomorphism from a neighbourhood of S_- in D_- into D^1_+. Set $F^1_+ = \Phi(S_-)$. We assume that F^1_+ intersects S^1_+ transversally. Put $H^1_+ = F^1_+ \cap S^1_+$. Let r^1_+ be a projection of E^1_+ on H^1_+, which is well defined near H^1_+. Set $K_- = \Phi^{-1}(H^1_+)$. Then K_- is a smooth submanifold of S_-. Using transversality, we find that near $\{0\} \times H_+ \times K_- \subset [0,1] \times H_+ \times S_-$, the map

$$\begin{aligned}(a, x, Z_-) \in [0,1] \times H_+ \times S_- \to \\ \left(a, x, r^1_+ \Phi_+(ah(x, aZ_-), Z_-), \Phi_-(ah(x, aZ_-), Z_-)\right) \\ = \left(a, x, x^1, Z^1_-\right) \in [0,1] \times H_+ \times H^1_+ \times E^1_-\end{aligned} \tag{5.58}$$

is a diffeomorphism.

Then a compactification of the image of $\Phi(F_-)$ by the flow Ψ^1_t associated to the vector field Y^1 can be described locally as the image of $[0,1]^3 \times H_+ \times H^1_+ \times S^1_-$ by the map,

$$\left(a, a^1, b^1, x, x^1, Z^1_-\right) \to \left(a^1 \Phi_+(ah(x, aZ_-), Z_-), b^1 Z^1_-\right), \tag{5.59}$$

where Z_- is itself evaluated at $\left(a, x, x^1, a^1 b^1 Z^1_-\right)$ under the inverse of the diffeomorphism in (5.58). In particular, this compactification defines a current on E^1. Moreover, a compactification F^1_- of the image of $\Phi(F_-)$ in D^1_- by the flow Ψ^1_t can be described locally as the image of $[0,1]^2 \times H_+ \times H^1_+ \times S^1_-$ by the map

$$\left(a, a^1, x, x^1, Z^1_-\right) \to \left(a^1 \Phi_+(ah(x, aZ_-), Z_-), Z^1_-\right), \tag{5.60}$$

where in (5.60), Z_- is evaluated at $\left(a, x, x^1, a^1 Z^1_-\right)$ under the inverse of the diffeomorphism defined in (5.58).

It should now be clear that the above process can be iterated. From the above, we recover the fact established in Laudenbach [**La**] that in the case of a single fibre X, the descending cells can be compactified, and define currents on X. Observe here that we have not established as in [**La**] that the compactifications are C^1 manifolds with conical singularities.

Assume now that the assumptions of our Theorem are verified. Locally over S, we can trivialize $TX|_{\mathbf{B}} = TX^s|_{\mathbf{B}} \oplus TX^u|_{\mathbf{B}}$ into a $\mathbf{Z}_2$-graded vector space. In our description of the compactification of the descending cells under the given transversality assumptions, it follows that all the maps in (5.57)-(5.60) describing the compactification of the descending cells and their intersection with the D_- will depend smoothly

on the extra parameter $s \in S$. By integration, it follows that if $\alpha \in \Omega^\bullet\big(X, F|_X\big)$ has a small support, its integral along the compactification of any of the descending cells is well-defined, and depends smoothly on the parameter s. The proof of our Theorem is completed. □

Remark 5.6. — By a well-known result of Palis and Smale [**PSm**], Morse-Smale dynamical systems are structurally stable. However, this does not lead to a simple proof of Theorem 5.5, because, in general, the conjugation homeomorphisms are not smooth.

In view of Theorem 5.5, we can now set the following definition.

Definition 5.7. — Let $\mathbf{P}^\infty = \Omega^\bullet(M,F) \to \Omega^\bullet\left(S, C^\bullet\left(W^u, F\right)\right)$ be given by

$$\mathbf{P}^\infty \alpha = \sum_{x \in B} \int_{\overline{W^u(x)}} \alpha. \tag{5.61}$$

Theorem 5.8. — *The map $\mathbf{P}^\infty$ is a quasi-isomorphism of $\mathbf{Z}$-graded G-complexes mapping $\left(\Omega^\bullet(M,F), d^M\right)$ into $\left(\Omega^\bullet\left(S, C^\bullet\left(W^u, F\right)\right), A'\right)$, and mapping the filtrations F', F'' on $\Omega^\bullet(M,F)$ into the corresponding filtrations in $\left(\Omega^\bullet(S, C^\bullet\left(W^u, F\right)), A'\right)$. Also $\mathbf{P}^\infty$ maps the spectral sequences on $\Omega^\bullet(M,F)$ into the corresponding spectral sequences on $\Omega^\bullet\left(S, C^\bullet\left(W^u, F\right)\right)$. In particular the map $\mathbf{E}_0'^{(\bullet,\bullet)} \to E_0'^{(\bullet,\bullet)}$ corresponds to the fibrewise quasi-isomorphisms $P^\infty : \left(\Omega^\bullet\big(X, F|_X\big), d^X\right) \to \left(C^\bullet\left(W^u, F\right), \partial\right)$. Finally $\mathbf{P}^\infty$ induces the identification of spectral sequences,*

$$\begin{aligned} \mathbf{E}_r'^{(\bullet,\bullet)} &\simeq E_r'^{(\bullet,\bullet)}, r \geqslant 1, \\ \mathbf{E}_r''^{(\bullet,\bullet)} &\simeq E_r''^{(\bullet,\bullet)}, r \geqslant 1. \end{aligned} \tag{5.62}$$

Proof. — First we show that $\mathbf{P}^\infty$ is a morphism of $\mathbf{Z}$-graded filtered complexes, i.e.

$$\mathbf{P}^\infty d^M = A' \mathbf{P}^\infty. \tag{5.63}$$

Take $\alpha \in \Omega^\bullet(M,F)$. We claim that we have the quality of currents on S,

$$\mathbf{P}^\infty d^M \alpha = A' \mathbf{P}^\infty \alpha. \tag{5.64}$$

In fact, by proceeding as in Laudenbach [**La**], or by using the arguments in the proof of Theorem 5.5, we find easily that the boundary of $\overline{W}^u$ as a current coincides with its geometric boundary. Using Stokes formula, we get the equality of currents in (5.64). Also by Theorem 5.5, these currents are smooth. Therefore, they coincide in $\Omega^\bullet\left(S, C^\bullet\left(W^u, F\right)\right)$. It is clear that $\mathbf{P}^\infty$ is a filtered morphism with respect to the filtrations F' and F''. The fact that the map $\mathbf{E}_0'^{(\bullet,\bullet)} \to E_0'^{(\bullet,\bullet)}$ is just P^∞ is trivial.

Using (5.48), a standard result in homological algebra [**CaE**, Chapter XIII, Theorem 3.2] shows that $\mathbf{P}^\infty$ is a quasi-isomorphism, and that (5.62) holds. The proof of our Theorem is completed. □

Observe that if S is simply connected, the flat vector bundle $H^\bullet(X, F|_X)$ on S is trivial as a flat vector bundle.

Theorem 5.9. — *If S is simply connected,*

$$H^\bullet(M, F) \simeq H^\bullet(M, \mathbf{C}) \widehat{\otimes} H^\bullet(X, F|_X), \tag{5.65}$$

and the spectral sequences $\mathbf{E}_\bullet'^{(\bullet,\bullet)}, \mathbf{E}_\bullet''^{(\bullet,\bullet)}$ and $E_\bullet'^{(\bullet,\bullet)}, E_\bullet''^{(\bullet,\bullet)}$ degenerate at E_2.

Even if S is not simply connected , for $F = \mathbf{C}$, the above spectral sequences degenerate at E_2. Moreover $\pi_1(S)$ acts as a finite group on $H^\bullet(X, \mathbf{Z})$.

Proof. — The flat vector bundle $C^\bullet(W^u, F)$ is trivial as a flat $\mathbf{Z}$-graded vector bundle on S. It follows that the spectral sequence $E_\bullet'^{(\bullet,\bullet)}$ degenerates at E_2, and that

$$H^\bullet(\Omega(S, C^\bullet(W^u, F)), A') \simeq H^\bullet(M, \mathbf{C}) \widehat{\otimes} H^\bullet(X, F|_X). \tag{5.66}$$

Using Theorem 5.8 and (5.66), we obtain the first part of our Theorem.

Let us now assume that $F = \mathbf{C}$. Let $H \subset \pi_1(S)$ be the subgroup of $\pi_1(S)$ which is the kernel of the monodromy representation on the complex $(C^\bullet(W^u, \mathbf{C}), \partial)$. Then H is a normal subgroup of finite index, i.e. $\pi_1(S)/H$ is a finite group. Let S' be the corresponding $\pi_1(S)/H$ cover of S. The lift of $(C^\bullet(W^u, \mathbf{C}), \partial)$ to S' is a trivial flat complex. The arguments in the proof of the first part of our Theorem show that the corresponding spectral sequences degenerate at E_2. Since our spectral sequences are just the $\pi_1(S)/H$-invariant parts of these spectral sequences and $\pi_1(S)/H$ is finite, the same property holds for our spectral sequences. Finally since $\pi_1(S)/H$ is finite, $\pi_1(S)$ acts as a finite group on $H^\bullet(X, \mathbf{Z})$. The proof of our Theorem is completed. □

Remark 5.10. — Let E be a $\mathrm{SL}(n, \mathbf{Z})$ vector bundle on S, let $\Lambda \subset E$ be the corresponding lattice. Let M be the total space of E/Λ. Then M fibres on S with fibre the torus $X = E/\Lambda$. In this case the action of $\pi_1(S)$ on $H^\bullet(X, \mathbf{R})$ is the monodromy representation on $\Lambda^\bullet(E^*)$, which is in general not finite. Therefore, in general, such torus fibrations do not verify our assumptions.

CHAPTER 6

FIBRATIONS, BEREZIN INTEGRALS AND EULER CURRENTS

In this Chapter, we recall the construction by Mathai-Quillen [**MQ**] of the Thom forms and of the transgressed Euler forms for Euclidean vector bundles with connection in the Berezin integral formalism. Under the assumptions of Section 3, we apply these results to the vector bundle TX on M. This point of view was already developed in [**BZ1**, Chapter III], which we follow closely. In fact let us recall that in [**BZ1**], which only involved one single fibre X, two copies of the exterior bundle of algebras $\Lambda^{\bullet}(T^*X)$ appeared naturally. An involution exchanging these two copies was then used in [**BZ1**], in particular in local index theoretic computations. In our relative situation, the problem is less symmetric. Still, a less obvious symmetry is preserved. We explain here what is needed for the local families index theoretic computations which will be done in the following Chapters. The results established in this Chapter will be used in particular in Chapters 7, 9 and 13.

This Chapter is organized as follows. In Section 6.1, we recall the Berezin integral formalism. In Section 6.2, we give describe the Mathai-Quillen Thom forms. In Section 6.3, we recall the results of [**BGS3**] on the convergence of the Thom forms as a parameter T tends to $+\infty$. In Section 6.4, we construct the Mathai-Quillen transgressed Euler form. In Section 6.5, in the context of fibrations, we establish curvature identities. Finally, in Section 6.6, we prove a fibrewise Stokes formula, which is based on the symmetries mentioned above.

6.1. The Berezin integral

Let E and V be real finite dimensional vector spaces of dimension n and m. Let g^E be an Euclidean metric on E. We will often identify E and E^* by the metric g^E. Let $e_1, \cdots, e_n$ be an orthonormal basis of E, and let $e^1, \cdots, e^n$ be the corresponding dual basis of E^*.

Let $\Lambda^{\bullet}(E^*)$ be the exterior algebra of E^*. It will be convenient to introduce another copy $\widehat{\Lambda}^{\bullet}(E^*)$. If $e \in E^*$, we will denote by $\hat{e}$ the corresponding element in $\widehat{\Lambda}^{\bullet}(E^*)$.

Suppose temporarily that E is oriented and that $e_1, \cdots, e_n$ is an oriented basis of E. Let $\int^B$ be the linear map from $\Lambda^\bullet(V^*) \widehat{\otimes} \widehat{\Lambda}^\bullet(E^*)$ into $\Lambda(V^*)$, such that if $\alpha \in \Lambda(V^*), \beta \in \widehat{\Lambda}(E^*)$,

$$\int^B \alpha\beta = 0 \text{ if } \deg\beta < \dim E, \tag{6.1}$$

$$\int^B \alpha \widehat{e^1} \wedge \cdots \wedge \widehat{e^n} = \frac{(-1)^{n(n+1)/2}}{\pi^{n/2}} \alpha.$$

More generally, let $o(E)$ be the orientation line of E. Then $\int^B$ defines a linear map from $\Lambda^\bullet(V^*) \widehat{\otimes} \widehat{\Lambda}^\bullet(E^*)$ into $\Lambda^\bullet(V^*) \widehat{\otimes} o(E)$. The linear map $\int^B$ is called a Berezin integral.

Let A be an antisymmetric endomorphism of E. We identify A with the element of $\Lambda(E^*)$,

$$A = \frac{1}{2} \sum_{1 \leqslant i,j \leqslant n} \langle e_i, Ae_j \rangle \widehat{e^i} \wedge \widehat{e^j}. \tag{6.2}$$

By definition, the Pfaffian $\mathrm{Pf}\left[\frac{A}{2\pi}\right]$ of $\frac{A}{2\pi}$ is given by,

$$\int^B \exp(-A/2) = \mathrm{Pf}\left[\frac{A}{2\pi}\right]. \tag{6.3}$$

Then $\mathrm{Pf}\left[\frac{A}{2\pi}\right]$ lies in $o(E)$. Moreover $\mathrm{Pf}\left[\frac{A}{2\pi}\right]$ vanishes if n is odd.

6.2. The Mathai-Quillen Thom forms

Let M be a manifold of dimension m. Let $\pi : E \to M$ be a real vector bundle of dimension n. Let g^E be an Euclidean metric on E. Let ∇^E be an Euclidean connection on (E, g^E), and let $R^E = \nabla^{E,2}$ be its curvature. Then R^E is a smooth section of $\Lambda^2(T^*M) \widehat{\otimes} \mathrm{End}(E)$. Also $\pi^*\nabla^E$ is an Euclidean connection on $\pi^*(E, g^E)$, and $\pi^* R^E$ is the curvature of $\pi^*\nabla^E$.

Let $e_1, \cdots, e_n$ be an orthonormal basis of E and let $e^1, \cdots, e^n$ be the corresponding dual basis of E^*. Let $f_1, \cdots, f_m$ be a basis of TM, and let $f^1, \cdots, f^m$ be the corresponding dual basis of T^*M. We identify R^E with the section $\dot{R}^E$ of $\Lambda^2(T^*M) \widehat{\otimes} \widehat{\Lambda}^2(E^*)$,

$$\dot{R}^E = \frac{1}{4} \sum_{\substack{1 \leqslant \alpha,\beta \leqslant m \\ 1 \leqslant i,j \leqslant n}} \langle e_i, R^E(f_\alpha, f_\beta) e_j \rangle f^\alpha \wedge f^\beta \wedge \widehat{e^i} \wedge \widehat{e^j}. \tag{6.4}$$

The connection ∇^E defines a horizontal subspace $T^H E$ of TE such that $TE = T^H E \oplus E$. Let P^E be the projection $TE \to E$ and let $P^{E*} : E^* \to T^*E$ be the transpose of P^E. Then P^E is a section of $T^*E \otimes E$. If we identify E with E^* by the

metric g^E, P^E can be considered as a section of $T^*E \otimes E^*$, so that

$$P^E = \sum_1^n \left(P^{E*} e^i\right) \widehat{e^i}. \tag{6.5}$$

Let Y be the generic element of E.

Definition 6.1. — For $T \geqslant 0$, let A_T be the section of $\Lambda(T^*E) \widehat{\otimes} \pi^* \widehat{\Lambda}(E^*)$,

$$A_T = \frac{\pi^* \dot{R}^E}{2} + \sqrt{T} P^E + T|Y|^2. \tag{6.6}$$

The connection $\pi^* \nabla^E$ acts as a differential operator on smooth sections of $\Lambda(T^*E) \widehat{\otimes} \pi^* \widehat{\Lambda}(E^*)$. Also if $e \in E$, the interior multiplication i_e acts naturally on $\widehat{\Lambda}(E^*)$, and so it acts as a derivation of the graded algebra $\Lambda(T^*E) \widehat{\otimes} \pi^* \widehat{\Lambda}(E^*)$. To indicate clearly that i_e only acts on the second factor $\widehat{\Lambda}(E^*)$, we will write $i_{\hat{e}}$ instead of i_e.

The following result was proved in [**MQ**, Section 6] and [**BeGeV**, Lemma 1.85 and Propositions 1.87 and 1.88].

Theorem 6.2. — *The following identities hold,*

$$\left[\pi^* \nabla^E + 2\sqrt{T} i_{\widehat{Y}}, A_T\right] = 0, \qquad \frac{\partial A_T}{\partial T} = \left[\pi^* \nabla^E + 2\sqrt{T} i_{\widehat{Y}}, \frac{\widehat{Y}}{2\sqrt{T}}\right]. \tag{6.7}$$

We will use the formalism of the Berezin integral developed in Section 6.1, with $V = TE$. If ω is a smooth section of $\Lambda(T^*E) \widehat{\otimes} \pi^* \widehat{\Lambda}(E^*)$ over E, $\int^B \omega$ is a smooth section of $\Lambda(T^*E) \widehat{\otimes} \pi^* o(E)$, i.e. a smooth differential form over E with values in $\pi^* o(E)$.

Put

$$e\left(E, \nabla^E\right) = \mathrm{Pf}\left[\frac{R^E}{2\pi}\right]. \tag{6.8}$$

Then $e\left(E, \nabla^E\right)$ is a smooth closed section of $\Lambda^{\dim E}(T^*M) \widehat{\otimes} o(E)$. It is a Chern-Weil representative of the rational Euler class of E. Of course, if $n = \dim E$ is odd, then

$$e\left(E, \nabla^E\right) = 0. \tag{6.9}$$

Definition 6.3. — Let a_T and b_T be the forms on E,

$$a_T = \int^B \exp(-A_T), \qquad b_T = \int^B \frac{\widehat{Y}}{2\sqrt{T}} \exp(-A_T). \tag{6.10}$$

Let π_* denote the integral along the fibre of forms on E taking value in $\pi^* o(E)$. Now we state a result of Mathai-Quillen [**MQ**, Theorem 6.4], also given in [**BZ1**, Theorem 3.4].

Theorem 6.4. — *The forms a_T have degree n, they are closed, and their cohomology class does not depend on T. For $T > 0$, the forms a_T represent the Thom class of E, so that*

$$\pi_* a_T = 1. \tag{6.11}$$

The forms b_T have degree $n-1$. Moreover,

$$\begin{aligned} a_0 &= \pi^* e\left(E, \nabla^E\right) \\ b_T &= -\frac{i_Y a_T}{2T}, \ T > 0, \\ \frac{\partial \alpha_T}{\partial T} &= -d b_T, \ T > 0. \end{aligned} \tag{6.12}$$

6.3. Convergence of the Mathai-Quillen currents

Let $o(TM)$ be the orientation bundle of TM. We identify M to the zero section of E. If $k \in \mathbf{N}$ and if K is a compact subset of E, let $\| \ \|_{C^k_K(E)}$ be a natural norm on the Banach space $C^k_K(E)$ of forms in E with values in $\pi^* o(TM)$, which are continuous with k continuous derivatives, and whose support is included in K. Let δ_M be the current of integration on M. If μ is a smooth compactly supported form on E with values in $\pi^* o(TM)$, then $\int_E \mu \delta_M = \int_M \mu$. Observe that δ_M can also be viewed as a current on E with values in $o(E)$.

Now we state a convergence result for the currents α_T, β_T, which was proved in [**BGS3**, Theorem 3.12] and stated in the present form in [**BZ1**, Theorem 3.5].

Theorem 6.5. — *Let K be a compact subset of E. There is a constant $C > 0$ such that for any smooth form μ on E with values in $\pi^* o(TM)$, whose support is included in K, for $T \geqslant 1$, then*

$$\begin{aligned} \left| \int_E \mu \left(a_T - \delta_M\right) \right| &\leqslant \frac{C}{\sqrt{T}} \|\mu\|_{C^1_K(E)}, \\ \left| \int_E \mu b_T \right| &\leqslant \frac{C}{T^{3/2}} \|\mu\|_{C^1_K(E)}. \end{aligned} \tag{6.13}$$

Remark 6.6. — By proceeding as in [**B6**, Theorem 3.2], one can give a microlocal refinement to Theorem 6.5. Namely one can show that as $T \to +\infty$, a_T converges to δ_M in the space of currents whose wave front set is included in $N^*_{M/E} \simeq E^*$, with similar estimates.

6.4. A transgressed Euler class

We now construct a Mathai-Quillen current[**MQ**, Section 7].

Definition 6.7. — Let $\psi\left(E,\nabla^E\right)$ be the current on E with values in $o(E)$,

$$\psi\left(E,\nabla^E\right)=\int_0^{+\infty} b_T dT. \tag{6.14}$$

By Theorem 6.5, the current $\psi\left(E,\nabla^E\right)$ is well-defined.

Recall that M is identified to the zero section of E. The normal bundle to M in E is exactly E.

Let g'^E be another metric on E, and let ∇'^E be an Euclidean connection on E with respect to g'^E. Let $\widetilde{e}\left(E,\nabla^E,\nabla'^E\right)$ denote the Chern-Simons class of forms of degree $n-1$ over M with values in $o(E)$, which is defined modulo smooth exact forms, such that

$$d\widetilde{e}\left(E,\nabla^E,\nabla'^E\right)=e\left(E,\nabla'^E\right)-e\left(E,\nabla^E\right). \tag{6.15}$$

For the definition and properties of the wave front set of a current, we refer to [**Hö**, Chapter VIII]. The following result was stated in [**BZ1**, Theorem 3.7].

Theorem 6.8. — *The current $\psi\left(E,\nabla^E\right)$ is of degree $n-1$. If λ is a smooth function on E with values in $\mathbf{R}^*$, under the map $e\in E\to\lambda e\in E$, $\psi\left(E,\nabla^E\right)$ is changed into $\operatorname{sign}(\lambda)^n\,\psi\left(E,\nabla^E\right)$. The current $\psi\left(E,\nabla^E\right)$ is locally integrable on E. Its wave front set of is included in E^*. Also,*

$$d\psi\left(E,\nabla^E\right)=\pi^*e\left(E,\nabla^E\right)-\delta_M. \tag{6.16}$$

The restriction of $-\psi\left(E,\nabla^E\right)$ to the fibres of E coincides with the solid angle form of the fibre associated to the metric g^E.

If g'^E is another metric on E, and if ∇'^E is a connection on E which preserves the metric g'^E, then

$$\psi\left(E,\nabla'^E\right)-\psi\left(E,\nabla^E\right)=\pi^*\widetilde{e}\left(E,\nabla^E,\nabla'^E\right)\ \textit{modulo exact currents.} \tag{6.17}$$

Remark 6.9. — Assume that $\dim E\leqslant\dim M$. Let s be a smooth section of E. Set

$$M'=\{x\in M, s(x)=0\}. \tag{6.18}$$

Suppose that over M', ds is of maximal rank $\dim E$. Equivalently, the graph of s intersects the 0 section of E transversally. Then M' is a smooth submanifold of M. Let $N_{M'/M}$ be the normal bundle to M' in M. Then $ds: N_{M'/M}\to E_{|M'}$ is an identification of vector bundles. Since the wave front set of $\psi\left(E,\nabla^E\right)$ is included in E^*, by [**Hö**, Theorem 8.2.4], the pulled-back current $s^*\psi\left(E,\nabla^E\right)$ on M is well-defined, and its wave front set is included in $N^*_{M'/M}$. Moreover,

$$ds^*\psi\left(E,\nabla^E\right)=e\left(E,\nabla^E\right)-\delta_{M'}. \tag{6.19}$$

Here $\delta_{M'}$ should be viewed as a current on M with values in $o(TM)$. By proceeding as in [**BGS2**, Theorem 3.15], we find that the current $s^*\psi\left(E,\nabla^E\right)$ is locally integrable on M.

6.5. Fibrations and curvature identities

Now we make the same assumptions and we use the same notation as in Section 3. In particular $\pi : M \to S$ is a submersion with compact fibre X, and $T^H M \subset TM$ is a horizontal vector bundle on TM, so that $T^H M \simeq \pi^* TS$. Then $TX = T^H M \oplus TX$, so that $\Lambda^\bullet(T^*M) \simeq \pi^*\Lambda^\bullet(T^*S) \mathbin{\widehat{\otimes}} \Lambda^\bullet(T^*X)$. In the sequel, we will often write $\Lambda^\bullet(T^*S)$ instead of $\pi^*\Lambda^\bullet(T^*S)$, to emphasize the fact that $\Lambda^\bullet(T^*S)$ is viewed as a trivial vector bundle along the fibres X.

We will denote by $\widehat{\Lambda}^\bullet(T^*X)$ another copy of the exterior algebra of the fibre X. The bundle of algebras $\widehat{\Lambda}^\bullet(T^*X)$ will be the exterior algebra of the fibres X. If $e \in T^*X$, we denote by $\hat{e}$ the corresponding element in $\widehat{\Lambda}^\bullet(T^*X)$. All the objects which are naturally differential forms along the fibres X will be denoted with a $\widehat{\ }$.

Recall that the fibrewise connection ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}$ on $\Lambda^\bullet(T^*S) \mathbin{\widehat{\otimes}} \Lambda^\bullet(T^*X)$ was defined in Definition 3.10 by the formula

$$
\begin{aligned}
{}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} &= \nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} + \frac{1}{2}\left\langle Se_i, f_\alpha^H \right\rangle \sqrt{2}c(e_i) f^\alpha \\
&\quad + \frac{1}{2}\left\langle Sf_\alpha^H, f_\beta^H \right\rangle f^\alpha f^\beta .
\end{aligned} \tag{6.20}
$$

In particular ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X),2}$ should be considered as a smooth section of $\widehat{\Lambda}^\bullet(T^*X) \mathbin{\widehat{\otimes}} \Lambda^\bullet(T^*S) \mathbin{\widehat{\otimes}} \mathrm{End}\left(\Lambda^\bullet(T^*X)\right)$. Also $\widehat{\nabla}^{TX,2}$ now denotes the restriction of $\nabla^{TX,2}$ to the fibres X as a hatted form. This is a section of $\widehat{\Lambda}^2(T^*X) \mathbin{\widehat{\otimes}} \mathrm{End}\,(TX)$. Of course, we still consider $\nabla^{TX,2}$ as a smooth section of $\Lambda^\bullet(T^*M) \mathbin{\widehat{\otimes}} \mathrm{End}\,(TX)$.

Definition 6.10. — Let $\mathcal{A}$ be the tensor obtained from

$$
-\frac{1}{2}\left({}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X),2} + \frac{1}{4}\left\langle \widehat{\nabla}^{TX,2} e_i, e_j \right\rangle \widehat{c}(e_i)\widehat{c}(e_j)\right), \tag{6.21}
$$

by replacing $c(e_i)$ by $\sqrt{2}e^i\wedge$ for $1 \leqslant i \leqslant n$.

Then $\mathcal{A} \in \widehat{\Lambda}^2(T^*X) \mathbin{\widehat{\otimes}} \Lambda^\bullet(T^*S) \mathbin{\widehat{\otimes}} \Lambda^\bullet(T^*X) \simeq \widehat{\Lambda}^2(T^*X) \mathbin{\widehat{\otimes}} \Lambda(T^*M)$.

Theorem 6.11. — *The following identity holds,*

$$
\mathcal{A} = \frac{1}{4}\left\langle e_i, \nabla^{TX,2} e_j \right\rangle \widehat{e^i}\widehat{e^j}. \tag{6.22}
$$

Proof. — By (3.42), we get

$$
\begin{aligned}
-2\mathcal{A} &= \frac{1}{4}\left\langle \nabla^{TX,2}(e_i, e_j)\, e_k, e_\ell \right\rangle \widehat{e^i}\widehat{e^j} e^k e^\ell \\
&\quad + \frac{1}{4}\left\langle \left(SP^{TX}S + \nabla^{TX}_{\cdot} S\right)(e_i, e_j) f_\alpha^H, f_\beta^H \right\rangle \widehat{e^i}\widehat{e^j} f^\alpha f^\beta \\
&\quad + \frac{1}{2}\left\langle \left(\nabla^{TX}_{\cdot} S\right)(e_i, e_j)\, e_k, f_\alpha^H \right\rangle \widehat{e^i}\widehat{e^j} e^k f^\alpha .
\end{aligned} \tag{6.23}
$$

Recall that as we saw after (3.17), S maps TX into $T^H M$. Using Proposition 3.12 and (6.23), we get (6.22). □

Now we will establish a version of Bianchi's identity for ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}$. Put

$$\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} = \widehat{e^i} \wedge \nabla_{e_i}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}. \tag{6.24}$$

Then $\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}$ acts on the smooth sections of $\widehat{\Lambda}(T^*X) \widehat{\otimes} \Lambda^\bullet(T^*S) \widehat{\otimes} \Lambda^\bullet(T^*X)$. Since ∇^{TX} is fibrewise torsion free, $\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}$ acts on the smooth sections of $\widehat{\Lambda}(T^*X)$ as the fibrewise hatted de Rham operator $\widehat{d}^X$.

In the sequel, we view $T(e_i, f_\alpha)$ as a section of TX. The interior multiplication operator $i_{T(e_i,f_\alpha)}$ acts on $\Lambda^\bullet(T^*X)$. Then the operator $\widehat{e}^i f^\alpha i_{T(e_i,f_\alpha^H)}$ acts on $\widehat{\Lambda}^\bullet(T^*X) \widehat{\otimes} \Lambda^\bullet(T^*S) \widehat{\otimes} \Lambda^\bullet(T^*X)$. It increases the degree in $\widehat{\Lambda}(T^*X)$ by 1, and decreases the degree in $\Lambda(T^*X)$ by 1.

Theorem 6.12. — *The following identity holds,*

$$\left(\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} + \widehat{e}^i f^\alpha i_{T(e_i,f_\alpha^H)}\right) \mathcal{A} = 0. \tag{6.25}$$

Proof. — Bianchi's identity for ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}$ asserts that

$$\left[{}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}, {}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X),2}\right] = 0. \tag{6.26}$$

Clearly there is an identification of $\mathbf{Z}$-graded vector spaces $c(TX) \simeq \Lambda^\bullet(T^*X)$. Given $r \in \mathbf{N}$, let $c_r(TX) \subset c(TX)$ correspond to $\Lambda^r(T^*X)$ by this isomorphism. Let $j : c(TX) \to \Lambda^\bullet(T^*X)$ be the bundle isomorphism which to $\alpha \in c_r(TX)$ associates $2^{r/2}\alpha \in \Lambda^r(T^*X)$. Under this isomorphism, if $Y \in TX$, the map $\alpha \in c(TX) \to [c(Y), \alpha] \in c(TX)$ corresponds to the map $\beta \in \Lambda(T^*X) \to -\sqrt{2} i_Y \beta \in \Lambda^\bullet(T^*X)$.

Using (6.20), under the isomorphism j, the operator which corresponds to the commutator with ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}$ is just the commutator with

$$\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} + \left\langle S(e_i) e_j, f_\alpha^H \right\rangle \widehat{e^i} f^\alpha i_{e_j} + \frac{1}{2} \left\langle S(e_i) \widehat{e^i} f_\alpha^H, f_\beta^H \right\rangle f^\alpha f^\beta. \tag{6.27}$$

From (6.26) and from the Bianchi identity for $\widehat{\nabla}^{TX}$, we conclude that

$$\left[\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} + \left\langle S(e_i) e_j, f_\alpha^H \right\rangle \widehat{e^i} f^\alpha i_{e_j}, \mathcal{A}\right] = 0. \tag{6.28}$$

Also by (3.18),

$$\left\langle S(e_i) e_j, f_\alpha^H \right\rangle = \left\langle T\left(e_i, f_\alpha^H\right), e_j \right\rangle. \tag{6.29}$$

By (6.28), (6.29), we get (6.25). The proof of our Theorem is completed. □

Now we use the formalism of Sections 6.1-6.4 on the manifold M, by taking here $\left(E, g^E, \nabla^E\right) = \left(TX, g^{TX}, \nabla^{TX}\right)$. Recall that for $T \geqslant 0$, A_T, a section of $\Lambda^\bullet(T^*TX) \widehat{\otimes} \widehat{\Lambda}^\bullet(T^*X)$, was defined in Definition 6.1.

Let $f : M \to \mathbf{R}$ be a smooth function. Let $\nabla f \in TX$ be the fibrewise gradient vector field of f with respect to g^{TX}.

Definition 6.13. — Put

$$B_T = (\nabla f)^* A_T. \tag{6.30}$$

Then B_T is a section of $\Lambda^\bullet(T^*M)\,\widehat{\otimes}\,\widehat{\Lambda}^\bullet(T^*X)$. By (6.4)-(6.6), we get

$$B_T = \frac{1}{4}\left\langle e_i, \nabla^{TX,2} e_j\right\rangle \widehat{e^i}\widehat{e^j} + \sqrt{T}\left\langle \nabla^{TX}\nabla f, e_i\right\rangle \widehat{e^i} + T\left|\nabla f\right|^2. \tag{6.31}$$

Take $T \in \mathbf{R}_+$. Observe that ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} + \sqrt{2T}c(\nabla f)$ is a superconnection along the fibres X. Its curvature $\left({}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} + \sqrt{2T}c(\nabla f)\right)^2$ is a section of $\widehat{\Lambda}^\bullet(T^*X)\,\widehat{\otimes}\,\Lambda^\bullet(T^*S)\,\widehat{\otimes}\,\mathrm{End}\left(\Lambda^\bullet(T^*X)\right)$. Let $\mathcal{B}_T$ be obtained from

$$-\frac{1}{2}\left(\left({}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} + \sqrt{2T}c(\nabla f)\right)^2 + \frac{1}{4}\left\langle \nabla^{TX,2} e_i, e_j\right\rangle \widehat{c}(e_i)\widehat{c}(e_j)\right), \tag{6.32}$$

by replacing $c(e_i)$ by $\sqrt{2}e^i\wedge$, for $1 \leqslant i \leqslant n$ as in Definition 6.10.

Observe that $f^\alpha \nabla_{f_\alpha^H} f$ is just the horizontal component of df.

Theorem 6.14. — *The following identity holds,*

$$\mathcal{B}_T = B_T + \sqrt{T}\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}\left(f^\alpha \nabla_{f_\alpha^H} f\right). \tag{6.33}$$

Proof. — Clearly,

$$\left({}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} + \sqrt{2T}c(\nabla f)\right)^2 = \tag{6.34}$$

$${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X),2} + \sqrt{2T}\widehat{e^i}c\left(\nabla^{TX}_{e_i}\nabla f\right) + 2\sqrt{T}\left\langle S(e_i)\nabla f, f_\alpha^H\right\rangle \widehat{e^i} f^\alpha - 2T\left|\nabla f\right|^2.$$

Also, since ∇^{TX} is fibrewise torsion free,

$$\left\langle \nabla^{TX}_{e_i}\nabla f, e_j\right\rangle = \left\langle \nabla^{TX}_{e_j}\nabla f, e_i\right\rangle. \tag{6.35}$$

Moreover, using (6.29), we get

$$\left\langle S(e_i)\nabla f, f_\alpha^H\right\rangle = \nabla_{T(e_i, f_\alpha^H)} f. \tag{6.36}$$

From Theorem 6.11 and from (6.34)-(6.36), we obtain,

$$\mathcal{B}_T = \frac{1}{4}\left\langle e_i, \nabla^{TX,2} e_j\right\rangle \widehat{e^i}\widehat{e^j} + \sqrt{T}e^i\widehat{e^j}\left\langle \nabla^{TX}_{e_i}\nabla f, e_j\right\rangle - \sqrt{T}\widehat{e^i} f^\alpha \nabla_{T(e_i, f_\alpha^H)} f + T\left|\nabla f\right|^2. \tag{6.37}$$

Also,

$$\nabla^{TX}\widehat{\nabla f} = e^i \wedge \widehat{e^j}\left\langle \nabla^{TX}_{e_i}\nabla f, e_j\right\rangle + f^\alpha \wedge \widehat{e^i}\nabla_{f_\alpha^H}\nabla_{e_i} f. \tag{6.38}$$

Moreover,

$$\nabla_{f_\alpha^H}\nabla_{e_i} f = \nabla_{e_i}\nabla_{f_\alpha^H} f + \nabla_{T(e_i, f_\alpha^H)} f. \tag{6.39}$$

From (6.38), (6.39), we conclude that

$$\begin{aligned}\nabla^{TX}\widehat{\nabla f} &= e^i \wedge \widehat{e^j}\left\langle \nabla^{TX}_{e_i}\nabla f, e_j\right\rangle - \widehat{e^i} \wedge \nabla_{e_i}\left(f^\alpha \nabla_{f_\alpha^H} f\right) - \widehat{e^i} \wedge f^\alpha \nabla_{T(e_i, f_\alpha^H)} f \\ &= e^i \wedge \widehat{e^j}\left\langle \nabla^{TX}_{e_i}\nabla f, e_j\right\rangle - \widehat{e^i} \wedge f^\alpha \nabla_{T(e_i, f_\alpha^H)} f - \widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}\left(f^\alpha \nabla_{f_\alpha^H} f\right).\end{aligned} \tag{6.40}$$

Using (6.37)-(6.40), we get (6.33). The proof of our Theorem is completed. □

Theorem 6.15. — *The following identity holds,*

$$\left(\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} + \widehat{e^i} f^\alpha i_{T(e_i, f_\alpha^H)} - 2\sqrt{T} i_{\nabla f}\right) B_T = 0 \tag{6.41}$$

Proof. — By proceeding as in the proof of Theorem 6.12, and using Theorem 6.14, we find that

$$\left(\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} + \widehat{e^i} f^\alpha i_{T(e_i, f_\alpha^H)} - 2\sqrt{T} i_{\nabla f}\right) \cdot \left(B_T + \sqrt{T}\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} \left(f^\alpha \nabla_{f_\alpha^H} f\right)\right) = 0. \tag{6.42}$$

Also, since ∇^{TX} is fibrewise torsion free,

$$\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X),2} \left(f^\alpha \nabla_{f_\alpha^H} f\right) = 0. \tag{6.43}$$

Finally, if $U \in TX$,

$$i_U \widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} \left(f^\alpha \nabla_{f_\alpha^H} f\right) = 0. \tag{6.44}$$

From (6.42)-(6.44), we get (6.41). The proof of our Theorem is completed. □

Remark 6.16. — Needless to say, a more direct proof of (6.41) can be given.

6.6. A Stokes formula

In the sequel, $\int_X$ denotes integration along the fibre of $\pi : M \to X$. Let r be the algebra endomorphism in $\widehat{\Lambda}^\bullet(T^*X) \widehat{\otimes} \Lambda^\bullet(T^*S) \widehat{\otimes} \Lambda^\bullet(T^*X)$ such that

$$r\left(e^i\right) = \widehat{e^i}, \qquad r\left(\widehat{e^i}\right) = e^i, \quad r\left(f^\alpha\right) = f^\alpha. \tag{6.45}$$

Proposition 6.17. — *If β is a smooth section of $\widehat{\Lambda}^\bullet(T^*X) \widehat{\otimes} \Lambda^\bullet(T^*M)$, then*

$$\int_X \int^B \beta = (-1)^{n(n+1)/2} \int_X \int^B r\beta. \tag{6.46}$$

Proof. — Let $\beta^{\max} \in \Lambda^\bullet(T^*S)$ be the component of β such that

$$\beta^{\max} e^1 \wedge \cdots \wedge e^n \wedge \widehat{e^1} \wedge \cdots \wedge \widehat{e^n}$$

is the component of β which is a factor of

$$e^1 \wedge \cdots \wedge e^n \wedge \widehat{e^1} \wedge \cdots \wedge \widehat{e^n}.$$

Then by (6.1),

$$\int_X \int^B \beta = \frac{(-1)^{n(n+1)/2}}{\pi^{\dim X/2}} \int_X \beta^{\max} e^1 \wedge \cdots \wedge e^n. \tag{6.47}$$

Also, one has the trivial,

$$(r\beta)^{\max} = (-1)^{n(n+1)/2} \beta^{\max}. \tag{6.48}$$

From (6.47), (6.48), we get (6.46). □

Theorem 6.18. — *If α is a smooth section of $\Lambda^\bullet(T^*X)\,\widehat{\otimes}\,\Lambda^\bullet(T^*M)$, then*

$$\int_X \int^B \widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}\alpha = 0. \tag{6.49}$$

Proof. — We denote by $\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\widehat{\Lambda}^\bullet(T^*X)}$ the connection along the fibres X on $\Lambda^\bullet(T^*S)\widehat{\otimes}\widehat{\Lambda}^\bullet(T^*X)$. Here $\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\widehat{\Lambda}^\bullet(T^*X)}$ increases the degree in unhatted Grassmann variables in $\Lambda^\bullet(T^*X)$ by 1. Clearly,

$$r\left(\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}\alpha\right) = \nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\widehat{\Lambda}^\bullet(T^*X)} r\alpha. \tag{6.50}$$

Also, if d^X denotes the de Rham operator along the fibre, which acts naturally on smooth sections along the fibre of $\Lambda^\bullet(T^*S)\,\widehat{\otimes}\,\Lambda^\bullet(T^*X)$, we have the identity,

$$\int^B \nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} r\alpha = d^X \int^B r\alpha. \tag{6.51}$$

Using Proposition 6.17 and (6.50), (6.51), we get

$$\int_X \int^B \widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}\alpha = (-1)^{n(n+1)/2} \int_X d^X \int^B r\alpha = 0, \tag{6.52}$$

i.e. we have established (6.49). The proof of our Theorem is completed. □

CHAPTER 7

ANALYTIC TORSION FORMS AND MORSE-SMALE VECTOR FIELDS

The purpose of this Chapter is to state the main result of this paper, and to check its compatibility to known results, like the anomaly formulas for analytic torsion forms and the rigidity results which were established in Sections 2.5 and 3.16. Also we verify that our main result is compatible to Poincaré duality, and also to taking products. Finally, we state a number of consequences of our main result.

This Chapter is organized as follows. In Section 7.1, we describe the geometric setting. In Section 7.2, we state our main result, which relates the analytic torsion forms $\mathcal{T}_{h,g}\left(T^HM, g^{TX}, \nabla^F, g^F\right)$ to the torsion forms $T_{h,g}\left(A^{C^\bullet(W^u,F)\prime}, g^{C^\bullet(W^u,F)}\right)$ of the given family of fibrewise Thom-Smale complexes. In Section 7.3, we give the corresponding formula for the Chern analytic torsion forms $\mathcal{T}_{\mathrm{ch},g}\left(T^HM, g^{TX}, \nabla^F, g^F\right)$.

In Section 7.4, we show that our formula is compatible to the anomaly formulas and the rigidity results established in the previous Sections, in Section 7.5, we prove its compatibility to products, and in Section 7.6, we show it is compatible to Poincaré duality.

In Section 7.7, we derive a relation between the torsion forms associated with two families of Morse-Smale vector fields. Finally, in Sections 7.8-7.10, we derive consequences of our main formula when the X_g are odd dimensional or even dimensional.

7.1. Assumptions and notation

In the whole Section, we use the notation,

$$h(x) = xe^{x^2}. \tag{7.1}$$

We make the same assumptions as in Chapter 3 and in Section 5.5, and we use the corresponding notation. In particular, let T^HM be a G-invariant horizontal subbundle of TM, let g^{TX}, g^F be G-invariant metrics on TX, F. Also we assume that $f : M \to \mathbf{R}$ is G-invariant and fibrewise Morse, that h^{TX} is a G-invariant metric on TX, and that ∇f is the fibrewise gradient vector field of f with respect to h^{TX}. We assume that $Y = -\nabla f$ is Morse-Smale in every fibre X. Finally, we fix $g \in G$.

Let $\mathcal{T}_{h,g}\left(T^HM, g^{TX}, \nabla^F, g^F\right)$ be the analytic torsion forms which are constructed as in Definition 3.31.

Recall that $\nabla^{C^\bullet(W^u,F)}$ denotes the flat connection on $C^\bullet(W^u,F)$. Using (5.42), we equip the complex of flat vector bundles $(C^\bullet(W^u,F),\partial)$ with the metric $g^{C^\bullet(W^u,F)}$ induced by the metric g^F on $F|_{\mathbf{B}}$. Then the $C^i(W,F)$'s are mutually orthogonal in $C^\bullet(W^u,F)$. Let $T_{h,g}\left(A^{C^\bullet(W^u,F)\prime}, g^{C^\bullet(W^u,F)}\right)$ be the associated torsion forms on S, which we construct as in Definition 1.29.

Let $g_{L_2}^{H^\bullet(X,F|_X)}$ be the Hermitian metric on $H^\bullet(X,F|_X)$ which was constructed in Section 3.6. Let $g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}$ be the metric on $H^\bullet(X,F|_X)$ which is obtained from the metric $g^{C^\bullet(W^u,F)}$ as in Section 1.6 via the isomorphism $H^\bullet(C^\bullet(W^u,F),\partial)\simeq H^\bullet(X,F|_X)$ which was stated in (5.43). Let $\widetilde{h}_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}, g_{L_2}^{H^\bullet(X,F|_X)}\right)\in \Omega^\bullet(S)/d\Omega^\bullet(S)$ be the class constructed in Definition 1.10.

Let $\psi\left(TX_g,\nabla^{TX_g}\right)$ be the Euler current on M_g, which is obtained as in Definition 6.7.

By the results of Section 5.3, $f|_{M_g}$ is fibrewise Morse, and $\nabla f|_{M_g}$ is a section of TX_g, which is a fibrewise gradient vector field for $f|_{M_g}$. By Proposition 5.4, if $Y=-\nabla f$, $Y|_{Mg}$ is Morse-Smale along the fibres X_g.

Set

$$\mathbf{B}_g=\mathbf{B}\cap M_g. \tag{7.2}$$

Then $\mathbf{B}_g$ is exactly the zero set for $Y|_{M_g}$. Also $\mathbf{B}_g$ is a finite covering of S, with fibre B_g. By Remark 6.9, the current $(\nabla f)^*\psi\left(TX_g,\nabla^{TX_g}\right)$ is well-defined, and its wave front set is included in $T^*X_g|_{\mathbf{B}_g}$. By (6.19), we have the equation of currents on M_g,

$$d(\nabla f)^*\psi\left(TX_g,\nabla^{TX_g}\right)=e\left(TX_g,\nabla^{TX_g}\right)-\sum_{x\in B_g}(-1)^{\mathrm{ind}_g(x)}\delta_x. \tag{7.3}$$

Also using the properties of the wave front set of $(\nabla f)^*\psi\left(TX_g,\nabla^{TX_g}\right)$ and [**Hö**, Theorem 8.2.13], we find that the integral along the fibre,

$$\int_{X_g} h_g\left(\nabla^F,g^F\right)(\nabla f)^*\psi\left(TX_g,\nabla^{TX_g}\right)$$

is smooth on S. Moreover, by (5.33) and (7.3), we get

$$\begin{aligned} d\int_{X_g} h_g\left(\nabla^F,g^F\right)(\nabla f)^*\psi\left(TX_g,\nabla^{TX_g}\right) &= -\int_{X_g} h_g\left(\nabla^F,g^F\right)e\left(TX_g,\nabla^{TX_g}\right)\\ &\quad + h_g\left(C^\bullet(W^u,F),g^{C^\bullet(W^u,F)}\right). \end{aligned} \tag{7.4}$$

Recall that the function $I(\theta,x)$ was defined in Definition 4.21, and evaluated in Theorems 4.30 and 4.31. By Theorem 4.29,

$$I(\theta,0)=\frac{1}{2}\frac{\partial\zeta}{\partial s}(\theta,0). \tag{7.5}$$

Definition 7.1. — Put

$$^0I(\theta,x)=I(\theta,x)-I(0,0). \tag{7.6}$$

By (7.5), (7.6),

$$ {}^0I(\theta,0) = \frac{1}{2}\left(\frac{\partial\zeta}{\partial s}(\theta,0) - \frac{\partial\zeta}{\partial s}(0,0)\right). \tag{7.7} $$

Also, by Lerch's formula as used in [**We**, Chapter 7, eqs. (15)-(23)] and in [**BZ2**, eq. (5.54)],

$$ \begin{aligned} &\frac{\partial\zeta}{\partial s}(0,0) + \frac{1}{2}\log(2\pi) = 0, \\ &\frac{\partial\zeta}{\partial s}(\theta,0) + \frac{1}{2}\log(2\pi) = \frac{1}{2}\Gamma'(1) - \frac{1}{4}\Big(\frac{\Gamma'}{\Gamma}(\theta/2\pi) + \frac{\Gamma'}{\Gamma}(1-\theta/2\pi)\Big),\ 0<\theta<2\pi. \end{aligned} \tag{7.8} $$

Consider the $\mathbf{Z}_2$-graded vector bundle $TX|_{\mathbf{B}_g} = TX^s|_{\mathbf{B}_g} \oplus TX^u|_{\mathbf{B}_g}$ over the manifold $\mathbf{B}_g$. To avoid any ambiguity, let us just say that $TX^s|_{\mathbf{B}_g}$ is the even part of $TX|_{\mathbf{B}_g}$, and $TX^u|_{\mathbf{B}_g}$ is the corresponding odd part. The vector bundle $TX|_{\mathbf{B}_g}$ is naturally equipped with an action of g. We define the form ${}^0I_g\left(TX|_{\mathbf{B}_g}, \nabla^{TX|_{\mathbf{B}_g}}\right)$ as in (4.64)-(4.65). Let ${}^0I_g\left(TX|_{\mathbf{B}_g}\right)$ be the corresponding cohomology class.

7.2. Statement of the main result

For convenience, we state the main result in this paper. This result was already given in the Introduction as Theorem 0.1.

Theorem 7.2. — *For any $g \in G$, the following identity holds,*

$$ \begin{aligned} &\mathcal{T}_{h,g}\left(T^HM, g^{TX}, \nabla^F, g^F\right) - T_{h,g}\left(A^{C^\bullet(W^u,F)\prime}, g^{C^\bullet(W^u,F)}\right) \\ &\qquad + \widetilde{h}_g\left(\nabla^{H^\bullet(X,F|_X)}, g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F)}, g^{H^\bullet(X,F|_X)}_{L_2}\right) \\ &= -\int_{X_g} h_g\left(\nabla^F, g^F\right)(\nabla f)^*\,\psi\left(TX_g, \nabla^{TX_g}\right) \\ &\quad + \sum_{x\in B_g} (-1)^{\mathrm{ind}(x)}\,\mathrm{Tr}^{F_x\otimes o^u_x}[g]\,{}^0I_g\left(T_xX|_{\mathbf{B}_g}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \end{aligned} \tag{7.9} $$

7.3. The formula for the Chern analytic torsion forms

Recall that the function $J(\theta,x)$ was defined in Definition 4.25, and evaluated in Theorems 4.35 and 4.36. By Theorem 4.35,

$$ J(\theta,0) = \frac{1}{2}\frac{\partial\zeta}{\partial s}(\theta,0). \tag{7.10} $$

Definition 7.3. — Put

$$ {}^0J(\theta,x) = J(\theta,x) - J(0,0). \tag{7.11} $$

By (7.10),

$$ {}^0J(\theta,0) = \frac{1}{2}\left(\frac{\partial\zeta}{\partial s}(\theta,0) - \frac{\partial\zeta}{\partial s}(0,0)\right). \tag{7.12} $$

By (7.7), (7.12),

$$ {}^0J(\theta,0) = {}^0I(\theta,0). \tag{7.13} $$

We define ${}^0J_g\left(TX|_{\mathbf{B}_g}\right)$ as before.

Theorem 7.4. — *For any $g \in G$, the following identity holds,*

$$ \begin{aligned} \mathcal{T}_{\mathrm{ch},g}\left(T^HM, g^{TX}, \nabla^F, g^F\right) - T_{\mathrm{ch},g}\left(A^{C^\bullet(W^u,F)\prime}, g^{C^\bullet(W^u,F)}\right) \\ + \widetilde{\mathrm{ch}}^\circ_g\left(H^\bullet(X,F|_X), g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F)}, g^{H^\bullet(X,F|_X)}_{L_2}\right) \\ = -\int_{X_g} \mathrm{ch}^\circ_g\left(\nabla^F, g^F\right)(\nabla f)^*\,\psi\left(TX_g, \nabla^{TX_g}\right) \\ + \sum_{x\in B_g} (-1)^{\mathrm{ind}(x)}\,\mathrm{Tr}^{F_x\otimes o^u_x}[g]\,{}^0J_g\left(TX|_{\mathbf{B}_g}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \end{aligned} \tag{7.14} $$

Proof. — Using (2.122), (3.134), (4.70) and (7.13), and applying the operator Q to the left-hand side of (7.9), we get (7.14). □

7.4. Compatibility of Theorem 7.2 to deformations and rigidity results on analytic torsion forms

Using Theorems 1.11, 1.30 and 3.32, and also (7.4), we find that when applying the de Rham operator d to the left-hand side of (7.9), we get 0. Therefore (7.9) is a refinement of a known equality on differential forms. This argument shows tautologically that Theorem 7.2 is compatible with the variation formulas for the terms which appear in the left-hand side of (7.9) with respect to the data T^HM, g^{TX}, g^F.

In degree 0, Theorems 7.2 and 7.4 are equivalent. They are in fact equivalent to the result of Bismut-Zhang [**BZ2**, Theorem 0.2], which is an extension of a result of Lott and Rothenberg [**LoRo**] to the case of flat vector bundles which are not necessarily unitarily flat. In particular, in degree 0 and for $g = 1$, both Theorems are equivalent to the extension of the Theorem of Cheeger [**C**] and Müller [**Mü1, Mü2**] to arbitrary flat vector bundles, which was given in [**BZ1**, Theorem 0.2].

We claim that our Theorem is compatible to the rigidity results in Theorems 2.20 and 3.45. Let $k(x)$ be given by (3.127). Let $\ell \in [0,1] \to \nabla^F_\ell$ be a smooth family of flat connections on F. As in Section 3.16, we assume that the rank of $H^\bullet(X,F|_X)$ does not depend on ℓ. Clearly, $C^\bullet(W^u,F)$ is now equipped with a smooth family of flat superconnections $\ell \in [0,1] \to A^{C^\bullet(W^u,F)\prime}_\ell$ of total degree 1.

Here, we use the notation of Sections 3.15 and 3.16. By Theorem 2.5, we get easily,

$$\widetilde{h}_g\left(\nabla_1^{H^\bullet(X,F|_X)}, g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F),1}, g^{H^\bullet(X,F|_X)}_{L_2,1}\right)^{(\geqslant 2)} - \widetilde{h}_g\left(\nabla_0^{H^\bullet(X,F|_X)}, g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F),0}, g^{H^\bullet(X,F|_X)}_{L_2,0}\right)^{(\geqslant 2)} = L_k\left(\nabla_\ell^{H^\bullet(X,F|_X)}, g^{H^\bullet(X,F|_X)}_{L_2}\right) - L_k\left(\nabla_\ell^{H^\bullet(X,F|_X)}, g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F)}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{7.15}$$

Since $\deg \psi\left(TX_g, \nabla^{TX_g}\right) = \dim X - 1$,

$$\left(\int_{X_g} h_g\left(\nabla^F, g^F\right)(\nabla f)^* \psi\left(TX_g, \nabla^{TX_g}\right)\right)^{(\geqslant 2)} = \int_{X_g} h_g\left(\nabla^F, g^F\right)^{\geqslant 3}(\nabla f)^* \psi\left(TX_g, \nabla^{TX_g}\right). \tag{7.16}$$

By Theorem 2.5 and by (6.19), we get

$$\left(\int_{X_g}\left(h_g\left(\nabla_1^F, g^F\right) - h_g\left(\nabla_0^F, g^F\right)\right)(\nabla f)^* \psi\left(TX_g, \nabla^{TX_g}\right)\right)^{(\geqslant 2)} = -\int_{X_g} e\left(TX_g, \nabla^{TX_g}\right) L_{k,g}\left(\nabla_\ell^F, g^F\right) + \sum_{x\in B_g}(-1)^{\mathrm{ind}_g(x)} L_{k,g}\left(\nabla_\ell^F, g^F\right)_x \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{7.17}$$

Using (5.33), one finds easily that

$$L_{k,g}\left(A_\ell^{C^\bullet(W^u,F)'}, g^{C^\bullet(W^u,F)}\right) = L_{k,g}\left(\nabla_\ell^{C^\bullet(W^u,F)}, g^{C^\bullet(W^u,F)}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{7.18}$$

By Theorems 2.20, 3.45 and (7.15)-(7.18), we find that Theorem 7.2 is compatible with deformations of the flat connection ∇^F.

7.5. Compatibility of Theorem 7.2 to products

Let $\pi' : M' \to S'$ be another fibration with compact fibre X'. We assume that this fibration is equipped with data similar to the data for $\pi : X \to S$, including a flat Hermitian vector bundle $\left(F', g^{F'}\right)$, a horizontal vector bundle $T^H M'$, a fibrewise Morse function $f' : M' \to \mathbf{R}$ and a fibrewise Morse-Smale gradient field $\nabla' f'$. These data will be denoted with a $'$.

We can then form the product $\pi'' : M'' = M \times M' \to S'' = S \times S'$ with fibre $X'' = X \times X'$. Let $p : M'' \to M, p' : M'' \to M'$ be the obvious projections. Let $T^H M''$ be the horizontal bundle on M'' which is induced by $T^H M, T^H M'$. Let $\left(F'', g^{F''}\right)$ be the tensor product of $p^*\left(F, g^F\right)$ and $p'^*\left(F', g^{F'}\right)$. More generally, we denote with

a $''$ the objects which are attached to this new fibration. In particular the function $f'' = p^*f + p'^*f'$ is a fibrewise Morse function on M'' and $\nabla'' f'' = p^*\nabla f + p'^*\nabla' f'$ is a fibrewise Morse-Smale vector field.

By an obvious extension of [**BLo1**, Proposition 3.28] to the equivariant setting, we get

$$\mathcal{T}_{h,g}\left(T^H M'', g^{TX''}, \nabla^{F''}, g^{F''}\right) = \chi_g(F)\,\mathcal{T}_{h,g}\left(T^H M', g^{TX'}, \nabla^{F'}, g^{F'}\right) + \chi_g(F')\,\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{7.19}$$

Similarly one can establish the easy formula,

$$T_{h,g}\left(A^{C^\bullet(W''^u,F'')\prime}, g^{C^\bullet(W''^u,F'')}\right) = \chi_g(F)\,T_{h,g}\left(A^{C^\bullet(W'^u,F')\prime}, g^{C^\bullet(W'^u,F')}\right) + \chi_g(F')\,T_{h,g}\left(A^{C^\bullet(W^u,F)\prime}, g^{C^\bullet(W^u,F)}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{7.20}$$

An identity similar to (7.19) and (7.20) holds for

$$\widetilde{h}_g\left(\nabla^{H^\bullet(X'',F''|_{X''})}, g^{H^\bullet(X'',F''|_{X''})}_{C^\bullet(W''^u,F'')}, g^{H^\bullet(X'',F''|_{X''})}_{L_2}\right).$$

By using transitivity properties of the currents $\psi\left(E, g^E\right)$ similar to [**BGS3**, Theorem 3.20], we get the identity of currents on M'',

$$(\nabla'' f'')^*\,\psi\left(TX''_g, \nabla^{TX''_g}\right) = (\nabla f)^*\,\psi\left(TX_g, \nabla^{TX_g}\right) e\left(TX'_g, \nabla^{TX'_g}\right) + \sum_{x\in B_g} (-1)^{\mathrm{ind}_g(x)}\,\delta_x\,(\nabla' f')^*\,\psi'\left(TX'_g, \nabla^{TX'_g}\right), \tag{7.21}$$

modulo exact currents whose wave front set does not intersect $\pi''^*T^*(S\times S')$. By (7.21), we find that

$$\int_{X''_g} h_g\left(\nabla^{F''}, g^{F''}\right)(\nabla'' f'')^*\,\psi\left(TX''_g, \nabla^{TX''_g}\right)$$

verifies an identity similar to (7.19) and (7.20).

Finally using (5.30), we get

$$\sum_{x\in B''_g} (-1)^{\mathrm{ind}(x)}\,\mathrm{Tr}^{F''_x\otimes o''^u_x}[g]\,{}^0I_g\left(TX''|_{\mathbf{B}''_g}\right) = \chi_g(F)\sum_{x\in\mathbf{B}'_g}(-1)^{\mathrm{ind}(x)}\,\mathrm{Tr}^{F'_x\otimes o'^u_x}[g]\,{}^0I_g\left(T_xX'|_{\mathbf{B}'_g}\right) + \chi_g(F')\sum_{x\in\mathbf{B}_g}(-1)^{\mathrm{ind}(x)}\,\mathrm{Tr}^{F_x\otimes o^u_x}[g]\,{}^0I_g\left(T_xX|_{\mathbf{B}_g}\right), \tag{7.22}$$

i.e. (7.22) also verifies an identity similar to (7.19) and (7.20).

We have thus proved that Theorem 7.2 is compatible to taking products.

7.6. Compatibility of Theorem 7.2 to Poincaré duality

Let $o(TX)$ be the orientation bundle of TX. Then $o(TX)$ is a $\mathbf{Z}_2$-bundle, on which g acts naturally. In particular $o(TX)$ can be considered as a flat complex G-line bundle, equipped with a flat G-invariant metric. Let $g^{\overline{F}^* \otimes o(TX)}$ be the corresponding metric on $\overline{F}^* \otimes o(TX)$.

Theorem 7.2 will be shown to be compatible with the transformation $f \to -f, F \to \overline{F}^* \otimes o(TX)$.

By using fibrewise Poincaré duality, we have a canonical isomorphism of flat Hermitian G-bundles over S,

$$\overline{H^\bullet(X, F|_X)}^* \simeq H^{n-\bullet}\left(X, \overline{F}^* \otimes o(TX)\right). \tag{7.23}$$

Also, one verifies easily that

$$\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^{\overline{F}^* \otimes o(TX)}, g^{\overline{F}^* \otimes o(TX)}\right) = (-1)^{n+1}\,\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right). \tag{7.24}$$

Observe that when replacing f by $-f$ and F by $\overline{F}^* \otimes o(TX)$, the complex $C^\bullet(W^u, F)$ is replaced by the complex $C^\bullet\left(W^s, \overline{F}^* \otimes o(TX)\right)$. Moreover, by (5.14), we have the canonical isomorphism of flat Hermitian G-complexes over S,

$$\left(\overline{C^\bullet(W^u, F)}^*, \partial\right) \simeq \left(C^{n-\bullet}\left(W^s, \overline{F}^* \otimes o(TX)\right), \partial\right). \tag{7.25}$$

From (7.25), we obtain easily

$$T_{h,g}\left(A^{C^\bullet(W^s, \overline{F}^* \otimes o(TX))'}, g^{C^\bullet(W^s, \overline{F}^* \otimes o(TX))}\right) = (-1)^{n+1}\,T_{h,g}\left(A^{C^\bullet(W^u, F)'}, g^{C^\bullet(W^u, F)}\right). \tag{7.26}$$

On M_g, the action of g on $o(TX)$ is given by

$$g|_{o(TX)|_{M_g}} = \det\left[g|_{TX}\right]|_{M_g}, \tag{7.27}$$

and (7.27) is equal to ± 1. More precisely,

$$g|_{o(TX)|_{M_g}} = (-1)^{\dim X - \dim X_g}. \tag{7.28}$$

Using in particular Theorem 6.8, one has the obvious equalities,

$$\begin{aligned}
&\widetilde{h}_g\left(\nabla^{H^\bullet(X, \overline{F}^* \otimes o(TX))}, g^{H^\bullet(X, \overline{F}^* \otimes o(TX))}_{C^\bullet(W^s, \overline{F}^* \otimes o(TX))}, g^{H^\bullet(X, \overline{F}^* \otimes o(TX))}_{L_2}\right) \\
&= (-1)^{n+1}\,\widetilde{h}_g\left(\nabla^{H^\bullet(X, F|_X)}, g^{H^\bullet(X, F|_X)}_{C^\bullet(W^u, F)}, g^{H^\bullet(X, F|_X)}_{L_2}\right), \\
&h_g\left(\nabla^{\overline{F}^* \otimes o(TX)}, g^{\overline{F}^* \otimes o(TX)}\right) = -g|_{o(TX)|_{M_g}}\, h_g\left(\nabla^F, g^F\right), \\
&(-\nabla f)^*\,\psi\left(TX_g, \nabla^{TX_g}\right) = (-1)^{\dim X_g}\,(\nabla f)^*\,\psi\left(TX_g, \nabla^{TX_g}\right).
\end{aligned} \tag{7.29}$$

We saw in (7.24) and (7.26) that, when replacing f by $-f$ and F by $\overline{F}^*$, the first two terms in the left-hand side of (7.9) are multiplied by $(-1)^{n+1}$. By (7.28), (7.29),

the third term in the left-hand side of (7.9) and the first two terms in the right-hand side of (7.9) are also multiplied by $(-1)^{n+1}$.

When replacing f by $-f$, and F by $\overline{F}^* \otimes o(TX)$, $(-1)^{\mathrm{ind}(x)}$ is changed into $(-1)^{n-\mathrm{ind}(x)}$, $\mathrm{Tr}^{F_x \otimes o_x^u}[g]$ is unchanged, and ${}^0I_g\left(TX|_{\mathbf{B}_g}\right)$ is now $-{}^0I_g\left(TX|_{\mathbf{B}_g}\right)$. So we find that under the above transformation, the last term in the right-hand side of (7.9) is also multiplied by $(-1)^{n+1}$.

We have thus proved that Theorem 7.2 is compatible with the transformation $f \to -f, F \to \overline{F}^* \otimes o(TX)$.

7.7. Changing the Morse gradient field

Let now f_1 be another G-invariant fibrewise Morse function, and let $\nabla_1 f_1$ be another associated G-invariant fibrewise Morse-Smale vector field. More generally, we denote with the subscript $_1$ the objects which are associated to $\nabla_1 f_1$.

Theorem 7.5. — *For any $g \in G$, the following identity holds,*

$$
\begin{aligned}
(7.30)\quad & T_{h,g}\left(A^{C^\bullet(W_1^u,F)\prime}, g^{C^\bullet(W_1^u,F)}\right) - T_{h,g}\left(A^{C^\bullet(W^u,F)\prime}, g^{C^\bullet(W^u,F)}\right) \\
& \qquad + \widetilde{h}_g\left(\nabla^{H^\bullet(X,F|_X)}, g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F)}, g^{H^\bullet(X,F|_X)}_{C^\bullet(W_1^u,F)}\right) = \\
& \int_{X_g} h_g\left(\nabla^F, g^F\right)\left(\nabla_1 f_1\right)^* \psi\left(TX_g, \nabla^{TX_g}\right) - \int_{X_g} h_g\left(\nabla^F, g^F\right)(\nabla f)^* \psi\left(TX_g, \nabla^{TX_g}\right) \\
& \qquad - \sum_{x \in B_{1,g}} (-1)^{\mathrm{ind}_1(x)} \mathrm{Tr}^{F_x \otimes o_{1,x}^u}[g]\, {}^0I_g\left(T_xX|_{\mathbf{B}_{1,g}}\right) \\
& \qquad + \sum_{x \in B_g} (-1)^{\mathrm{ind}(x)} \mathrm{Tr}^{F_x \otimes o_x^u}[g]\, {}^0I_g\left(T_xX|_{\mathbf{B}_g}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S).
\end{aligned}
$$

Proof. — Our Theorem is a trivial consequence of Theorem 7.2. □

Remark 7.6. — In degree 0, in the case where $g = 1$, Bismut and Zhang [**BZ1**, Theorem 16.1] gave a direct proof of Theorem 7.5, by arguments of Laudenbach [**La**]. In [**La**], if X is a compact manifold, Laudenbach describes the bifurcation of the Thom-Smale complex along a smooth family $t \in [0,1] \to Y_t$ of gradient fields associated to a given Morse function f, which verify the Thom-Smale transversality conditions, except at a finite family of values of t, where generic singularities may occur. If f, f_1 are two Morse functions on a compact manifold X, Laudenbach also considers a Cerf path [**Ce**] $t \in [0,1] \to f_t \in C^\infty(X, \mathbf{R})$ connecting f to f_1, the functions f_t are then Morse except at a finite family of values of t where birth or death of critical points may occur. Laudenbach also describes the bifurcation of the Thom-Smale complex along such paths. By using these two kinds arguments, Bismut and Zhang could give a direct proof of Theorem 7.5 in degree 0 for $g = 1$.

In arbitrary degree, for $g = 1$, we do not know how to give a direct proof of Theorem 7.5. Needless to say, it is possible to establish Theorem 7.5 along a smooth path $t \in [0,1] \to Y_t$ of fibrewise gradient fields which are fibrewise Morse-Smale except at a finite family of values of t where the singularities are at most of the type considered in [**La**] and[**BZ1**].

7.8. The case where $g^{F|_{\mathbf{B}}}$ is flat and $\dim X_g$ is odd

In this Section, we will assume that $F|_{\mathbf{B}}$ carries a G-invariant flat metric. Then we may and we will assume that the given metric $g^{F|_{\mathbf{B}}}$ is flat and G-invariant.

In this Section, we will also assume that $\dim X_g$ is odd. By (7.28), this is the case in particular if the fibres X are odd dimensional, orientable, and if g preserves the orientation.

Since $\dim X_g$ is odd, $e\left(TX_g, \nabla^{TX_g}\right) = 0$. Also since $g^{F|_{\mathbf{B}}}$ is flat,

$$h_g\left(C^\bullet\left(W^u, F\right), g^{C^\bullet(W^u,F)}\right) = 0. \tag{7.31}$$

Then it follows from (7.4) that

$$d\Big(\int_{X_g} h_g\left(\nabla^F, g^F\right)\left(\nabla f\right)^* \psi\left(TX_g, \nabla^{TX_g}\right)\Big) = 0. \tag{7.32}$$

Theorem 7.7. — *The cohomology class,*

$$\sum_{x \in B_g} (-1)^{\mathrm{ind}(x)} \mathrm{Tr}^{F_x \otimes o_x^u}\left[g\right]{}^0 I_g\left(TX|_{\mathbf{B}_g}\right)^{(\geqslant 2)} \in H^{\mathrm{even}}\left(S, \mathbf{C}\right) \tag{7.33}$$

does not depend on ∇f, as long as $g^{F|_{\mathbf{B}}}$ is flat. Moreover the following identities hold,

(7.34)

$$\begin{aligned}
&\mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right)^{(\geqslant 2)} + \widetilde{h}_g\Big(\nabla^{H^\bullet(X,F|_X)}, g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F)}, g^{H^\bullet(X,F|_X)}_{L_2}\Big)^{(\geqslant 2)} \\
&\qquad = \sum_{x \in B_g} (-1)^{\mathrm{ind}(x)} \mathrm{Tr}^{F_x \otimes o_x^u}\left[g\right]{}^0 I_g\left(TX|_{\mathbf{B}_g}\right)^{(\geqslant 2)} \text{ in } \Omega^\bullet\left(S\right)/d\Omega^\bullet\left(S\right), \\
&\Big(\int_{X_g} h_g\left(\nabla^F, g^F\right)\left(\nabla f\right)^* \psi\left(TX_g, \nabla^{TX_g}\right)\Big)^{(\geqslant 2)} = 0 \text{ in } H^{\mathrm{even}}\left(S, \mathbf{C}\right).
\end{aligned}$$

Proof. — Since the metric $g^{F|_{\mathbf{B}}}$ is flat, the metric $g^{C^\bullet(W^u,F)}$ is flat. In particular $C^\bullet\left(W^u, F\right)$ is a flat Hermitian complex of vector bundles. So the chain map ∂ and its adjoint ∂^* are flat. Using the identification

$$\ker\left(\partial + \partial^*\right) \simeq H^\bullet(X, F|_X),$$

we find that the metric $g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F)}$ on $H^\bullet(X, F|_X)$ is also flat.

It follows from (1.70) and from the above considerations that

$$T_{h,g}\left(A^{C^\bullet(W^u,F)\prime}, g^{C^\bullet(W^u,F)}\right)^{(\geqslant 2)} = 0. \tag{7.35}$$

Now we will use Theorem 7.5 applied to f and to $f_1 = -f$. Since the metrics $g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F)}$ and $g^{H^\bullet(X,F|_X)}_{C(W^s,F)}$ are flat,

$$\widetilde{h}_g\Big(\nabla^{H^\bullet(X,F|_X)}, g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F)}, g^{H^\bullet(X,F|_X)}_{C(W^s,F)}\Big)^{(\geqslant 2)} = 0 \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S) \tag{7.36}$$

Therefore, in degree $\geqslant 2$, the left-hand side of (7.30) vanishes in $\Omega^\bullet(S)/d\Omega^\bullet(S)$. Using Theorem 6.8, (7.30) and the fact that $\dim X_g$ is odd, we get the second equality in (7.34). The first equality in (7.34) then follows from Theorem 7.2, from the second equality in (7.34) and from (7.35).

Now, we use Theorem 7.5 again. It follows from the above that the left-hand side of (7.30) vanishes in $\Omega^\bullet(S)/d\Omega^\bullet(S)$. Also, by the above the first two terms in the right-hand side of (7.30) vanish in $\Omega^\bullet(S)/d\Omega^\bullet(S)$. Therefore the sum of the last two terms also vanishes in $\Omega^\bullet(S)/d\Omega^\bullet(S)$. So we have established the first part of our Theorem.

The proof of our Theorem is completed. □

Remark 7.8. — Since the metric $g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F)}$ is flat, one finds easily that

$$\widetilde{h}_g\Big(\nabla^{H^\bullet(X,F|_X)}, g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F)}, g^{H^\bullet(X,F|_X)}_{L_2}\Big)^{(\geqslant 2)} \in \Omega^\bullet(S)/d\Omega^\bullet(S)$$

does not depend on the choice of ∇f. From the first identity in (7.34), we reobtain a proof of the first part of our Theorem.

Put

$$I(x) = I(0,x). \tag{7.37}$$

Recall that by Theorem 4.31,

$$I(x)^{(>0)} = \frac{1}{4}\sum_{\substack{p\in\mathbf{N}^*\\ p \text{ even}}} \frac{(2p+1)!}{(p!)^2}\zeta(p+1)\left(\frac{ix}{8\pi}\right)^p. \tag{7.38}$$

Observe that for $p \geqslant 1$, $\zeta(p+1) \neq 0$. We identify $I(x)$ to the corresponding additive genus.

Recall that $KR_0(S)$ is the stable real K-theory of S. Also $TX|_{\mathbf{B}}$ is a $\mathbf{Z}_2$-graded vector field. Then

$$\sum_{x\in B} (-1)^{\mathrm{ind}(x)} T_x X \in KR_0(S). \tag{7.39}$$

Equation (7.22) with $g = 1$ says that the vector bundle in (7.39) behaves like a derived Euler characteristic with values in $KR_0(S)$. Namely, let $\chi(X), \chi(X')$ be the Euler characteristics of X, X'. Then

$$\sum_{x\in B''} (-1)^{\mathrm{ind}(x)} T_x X'' = \chi(X)\sum_{x\in B'} (-1)^{\mathrm{ind}(x)} T_x X' + \chi(X')\sum_{x\in B} (-1)^{\mathrm{ind}(x)} T_x X. \tag{7.40}$$

Also we will write $\mathcal{T}_h\left(T^H M, g^{TX}\right)$ instead of $\mathcal{T}_{h,1}\left(T^H M, g^{TX}, \nabla^{\mathbf{R}}, g^{\mathbf{R}}\right)$.

Theorem 7.9. — *Assume that* $\dim X$ *is odd. Then the vector bundle,*

$$\sum_{x\in B}(-1)^{\mathrm{ind}(x)}\,T_xX\in KR_0(S)\otimes_{\mathbf{R}}\mathbf{Q} \tag{7.41}$$

does not depend on ∇f. *Moreover, the following identity holds,*

$$\begin{aligned}\mathcal{T}_h\left(T^HM,g^{TX}\right)^{(\geqslant 2)}+\widetilde{h}_g\left(\nabla^{H^\bullet(X,\mathbf{R})},g^{H^\bullet(X,\mathbf{R})}_{C(W^u,\mathbf{R})},g^{H^\bullet(X,\mathbf{R})}_{L_2}\right)^{(\geqslant 2)}\\ =I\Big(\sum_{x\in B}(-1)^{\mathrm{ind}(x)}\,T_xX\Big)^{(\geqslant 2)}\ \text{in}\ \Omega^\bullet(S)/d\Omega^\bullet(S).\end{aligned} \tag{7.42}$$

Proof. — Let $E\in KR_0(S)\otimes_{\mathbf{R}}\mathbf{Q}$ be the virtual vector bundle in (7.41). By Theorem 7.7, $I(E)\in H^{\mathrm{even}}(S,\mathbf{R})$ does not depend on ∇f. Since the coefficients of $I^{(>0)}(x)$ in (7.38) are non zero, $I(E)^{(\geqslant 2)}$ is a sum with non zero coefficients of the Newton classes of E. Therefore the Pontryagin classes of E in positive degree do not depend on ∇f, so that E itself does not depend on ∇f. From (7.34), we get (7.42). The proof of our Theorem is completed. □

7.9. The case where $g^{F|\mathbf{B}}$ is flat and $\dim X_g$ is even

We still assume that $g^{F|\mathbf{B}}$ is flat. Also we assume that $\dim X_g$ is even. By (7.28), this is the case if the fibres X are even dimensional, orientable, and g preserves the orientation.

Theorem 7.10. — *The following identities hold,*

(7.43)

$$\begin{aligned}&\mathcal{T}_{h,g}\left(T^HM,g^{TX},\nabla^F,g^F\right)^{(\geqslant 2)}+\widetilde{h}_g\left(\nabla^{H^\bullet(X,F|_X)},g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F)},g^{H^\bullet(X,F|_X)}_{L_2}\right)^{(\geqslant 2)}\\ &\qquad=-\Big(\int_{X_g}h_g\left(\nabla^F,g^F\right)(\nabla f)^*\,\psi\left(TX_g,\nabla^{TX_g}\right)\Big)^{(\geqslant 2)}\ \text{in}\ \Omega^\bullet(S)/d\Omega^\bullet(S),\\ &\sum_{x\in B_g}(-1)^{\mathrm{ind}(x)}\,\mathrm{Tr}^{F_x\otimes o^u_x}[g]\,{}^0I_g\left(T_xX|_{\mathbf{B}_g}\right)^{(\geqslant 2)}=0\ \text{in}\ H^{\mathrm{even}}(S,\mathbf{C}).\end{aligned}$$

Proof. — We use the same arguments as in the proof of Theorem 7.7. Using Theorem 7.5 leads here to the second identity in (7.43). From Theorem 7.2, we then get the first identity. The proof of our Theorem is completed. □

Theorem 7.11. — *If* $\dim X$ *is even, then*

$$\sum_{x\in B}(-1)^{\mathrm{ind}(x)}\,T_xX=0\ \text{in}\ KR_0(S)\otimes_{\mathbf{Z}}\mathbf{Q}. \tag{7.44}$$

Moreover,

$$\mathcal{T}_h\left(T^HM, g^{TX}\right)^{(\geqslant 2)} + \widetilde{h}\left(\nabla^{H^\bullet(X,\mathbf{R})}, g^{H^\bullet(X,\mathbf{R})}_{C(W^u,\mathbf{R})}, g^{H^\bullet(X,\mathbf{R})}_{L_2}\right)^{(\geqslant 2)} = 0 \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{7.45}$$

Proof. — We use Theorem 7.10 applied to $F = \mathbf{R}$. We get

$$I\left(\sum_{x\in B}(-1)^{\mathrm{ind}(x)}\, T_xX\right)^{(\geqslant 2)} = 0 \text{ in } H^{\mathrm{even}}(S,\mathbf{R}). \tag{7.46}$$

The same argument as in the proof of Theorem 7.9 then shows that (7.44) holds. Equation (7.45) also follows Theorem 7.10. □

7.10. The case where $g^{F|_B}$ is flat, $\dim X$ is odd and F is acyclic

In this Section, we assume that G is trivial, that X is odd dimensional, and that F is fibrewise acyclic.

Theorem 7.12. — *The following identity holds,*

$$\sum_{x\in B}(-1)^{\mathrm{ind}(x)}\, T_xX = 0 \text{ in } KR_0(S)\otimes_{\mathbf{Z}}\mathbf{Q}. \tag{7.47}$$

Moreover if the metric $g^{F|_B}$ is flat,

$$\mathcal{T}_h\left(T^HM, g^{TX}, \nabla^F, g^F\right)^{(\geqslant 2)} = 0 \text{ in } H^{\mathrm{even}}(X,\mathbf{R}). \tag{7.48}$$

Proof. — As in the proof of Theorem 5.66, we replace S by a finite normal covering S', which is such that $\pi_1(S')$ acts trivially on $(C^\bullet(W^u,\mathbf{C}),\partial)$. Take $x\in B, y\in B$, with $\mathrm{ind}(y) = \mathrm{ind}(x)-1$. Recall that the finite set of gradient lines $\Gamma(x,y)$ connecting x and y was defined in Section 5.1. Assume that $\Gamma(x,y)$ is non empty. Then we can take a $\gamma\in\Gamma(x,y)$, which is defined globally on S'. The construction of Section 5.1 shows that γ provides an identification of the $\mathbf{Z}_2$ graded vector bundles T_xX to T_yY in $KR_0(S')$. More precisely,

$$T^s_xX\oplus\mathbf{R}\simeq T^s_yX, \qquad T^u_xX\simeq T^u_yX\oplus\mathbf{R}. \tag{7.49}$$

Since F is acyclic, the Euler characteristic of $(C^\bullet(W^u,F),\partial)$ vanishes. More precisely, by splitting the complex $(C^\bullet(W^u,F),\partial)$ into 'connected' pieces (i.e. into direct sums of indexed by $x\in B$ which can connected by a sequence of gradient lines), the Euler characteristic of each connected piece of $(C^\bullet(W^u,F),\partial)$ vanishes. The same vanishing property holds for the corresponding 'connected' parts of $(C^\bullet(W^u,\mathbf{C}),\partial)$.

It then follows from (7.49) and from the above considerations that

$$\sum_{x\in B}(-1)^{\mathrm{ind}(x)}\, T_xX = 0 \text{ in } KR_0(S'). \tag{7.50}$$

From (7.50), we get (7.47).

Then by (7.34) in Theorem 7.7, and by (7.47), we get (7.48). The proof of our Theorem is completed. □

Remark 7.13. — Equation (7.47), in the case where $\dim X$ is odd, should be compared with equation (7.44) for the case where $\dim X$ is even.

CHAPTER 8

A CONTOUR INTEGRAL

In this Chapter, under the assumptions of Section 7, we construct a closed form γ on $\mathbf{R}_+^* \times \mathbf{R}$ with values in $\Omega^\bullet(S)/d\Omega^\bullet(S)$. The proof of our main result will be obtained by integrating γ on a large rectangular contour in $\mathbf{R}_+^* \times \mathbf{R}$. This Chapter is the obvious extension of [**BZ1**, Chapter V] and of [**BZ2**, Section 3].

In Section 8.1, we construct γ, and in Section 8.2, we obtain the contour integral. As before, we write

$$h(x) = xe^{x^2}. \tag{8.1}$$

Also the assumptions of Section 7 will be in force.

8.1. A closed form

For $T \in \mathbf{R}$, let g_T^F be the metric on F,

$$g_T^F = e^{-2Tf} g^F. \tag{8.2}$$

Let $C_{t,T}$ be the superconnection on $\Omega^\bullet(X, F|_X)$ attached to $(T^H M, g^{TX}, g_T^F)$, and let $D_{t,T}$ be the corresponding morphism, which are obtained as in (3.52).

Put

$$\widetilde{M} = M \times \mathbf{R}_+^* \times \mathbf{R}, \qquad \widetilde{S} = S \times \mathbf{R}_+^* \times \mathbf{R}. \tag{8.3}$$

Let $\widetilde{\pi} : \widetilde{M} \mapsto \widetilde{S}$ be the obvious projection with fibre X. Put

$$T^H \widetilde{M} = T^H M \oplus \mathbf{R}^2. \tag{8.4}$$

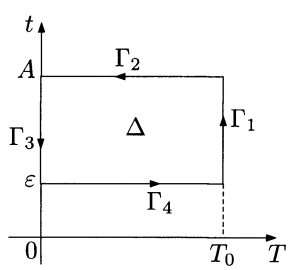

FIGURE 8.1

Over $(t,T) \in \mathbf{R}_+^* \times \mathbf{R}$, we equip TX with the metric $\frac{g^{TX}}{t}$, and the vector bundle F with the metric g_T^F. Let $\widetilde{g}^{TX}, \widetilde{g}^F$ be the corresponding metrics on TX, F.

Let $\widetilde{A}$ be the superconnection on $\Omega^\bullet\left(X, F|_X\right)$ associated to $\left(T^H\widetilde{M}, \widetilde{g}^{TX}, \widetilde{g}^F\right)$, and let $\widetilde{B}$ be the corresponding morphism, which are defined as in (3.31). By (3.53), one verifies easily that

$$t^{N/2}\widetilde{B}t^{-N/2} = D_{t,T} + \frac{dt}{2t}\left(N - \frac{n}{2}\right) - dTf. \tag{8.5}$$

Definition 8.1. — Let γ be the form on $\widetilde{S}$,

$$\gamma = \mathrm{Tr_s}\left[gh\left(\widetilde{B}\right)\right] + \frac{n}{2}\frac{dt}{2t}\chi_g(F). \tag{8.6}$$

Proposition 8.2. — *There is a smooth form odd form $\delta_{t,T}$ on S, depending smoothly on $(t,T) \in \mathbf{R}_+^* \times \mathbf{R}$, such that*

$$\gamma = \mathrm{Tr_s}\left[gh\left(D_{t,T}\right)\right] + \frac{dt}{2t}\mathrm{Tr_s}\left[Ngh'\left(D_{t,T}\right)\right] - dT\mathrm{Tr_s}\left[fgh'\left(D_{t,T}\right)\right] + \frac{dtdT}{2t}\delta_{t,T}. \tag{8.7}$$

Proof. — Our Proposition follows from (8.5) and (8.6). □

Proposition 8.3. — *The form γ on $\widetilde{S}$ is odd and closed.*

Proof. — This is a trivial consequence of Theorem 3.24. □

8.2. A contour integral

We fix constants ε, A, T_0 such that $0 < \varepsilon < 1 < +\infty, 0 \leqslant T_0 < +\infty$. Let $\Gamma = \Gamma_{\varepsilon,A,T_0}$ be the contour shown in Figure 8.1. The contour Γ is made of four oriented pieces $\Gamma_1, \ldots, \Gamma_4$. Let Δ be the oriented interior of Γ.

Theorem 8.4. — *The following identity of even forms on S holds,*

$$\int_\Gamma \gamma = -d\int_\Delta \gamma. \tag{8.8}$$

Proof. — This is an obvious consequence of Proposition 8.3. □

For $1 \leqslant k \leqslant 4$, put

$$I_k^0 = \int_{\Gamma_k} \gamma. \tag{8.9}$$

Then Theorem 8.4 says that

$$\sum_{k=1}^4 I_k^0 = -d\int_\Delta \gamma. \tag{8.10}$$

From (8.10), we get

$$\sum_{k=1}^4 I_k^0 = 0 \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{8.11}$$

Our proof of Theorem 7.2 will consist in making $A \to +\infty, T_0 \to +\infty, \varepsilon \to 0$ in this order in (8.11).

CHAPTER 9

A PROOF OF THE MAIN RESULT

In this Chapter , we establish the main result of this paper, which was stated in the Introduction and also in Theorem 7.2. Our starting point is equation (8.11). We then state seven intermediate results, which are needed in our proof of Theorem 7.2. The proofs of six of these results are delayed to the next six Sections.

The general organization of this Chapter is closely related to [**BZ1**, Chapter VII] and to [**BZ2**, Section 5]. In Section 9.1, we make simplifying assumptions on the metrics g^{TX}, g^F and on $T^H M$. In Section 9.2, we establish convergence results on certain differential forms. In Section 9.3, we state our seven intermediate results. In Section 9.4, we compute the asymptotics of the I_k^0's as $A \to +\infty, T_0 \to +\infty, \varepsilon \to 0$. In Section 9.5, we show that the divergences which appear in this limit process are compatible with equation (8.11), and we obtain an identity of forms in $\Omega^\bullet(S)/d\Omega^\bullet(S)$. In Section 9.6, we show that this identity is just Theorem 7.2.

In this Chapter, we use the notation of Chapter 3, Section 5.5, and of Chapters 7 and 8. Also, to complement equation (1.23) in Definition 1.7, we will use the notation,

$$h_g^*\left(A', g^E\right) = \mathrm{Tr}_s\left[gh\left(B\right)\right], \tag{9.1}$$

the * indicating that the factors $2i\pi$ are omitted.

In the rest of the paper, we still use the notation,

$$h\left(x\right) = xe^{x^2}. \tag{9.2}$$

9.1. Some simplifying assumptions on the metrics g^{TX}, g^F

As we saw in Chapter 7, equation (7.9) in Theorem 7.2 is compatible with many natural operations on the data, including the deformation of the metrics g^{TX}, g^F. In particular, we may and we will assume in the sequel that $g^{TX} = h^{TX}$, so that ∇f is the fibrewise gradient field of f with respect to the metric g^{TX}.

Also, using the arguments we gave after (5.49), we may and we will assume the simplifying assumptions of Section 5.6 to be in force. Namely we assume that $\varepsilon_0 > 0$ is small enough so that we have a fibrewise G-equivariant identification of the $3\varepsilon_0$-neighbourhood $U_{3\varepsilon_0}$ of $\mathbf{B}$ in $TX|_{\mathbf{B}}$ with a corresponding tubular neighbourhood $V_{3\varepsilon_0}$ of $\mathbf{B}$ in M, which maps the fibres $TX|_{\mathbf{B}}$ into the corresponding fibres X, so that

- $TX^s|_{\mathbf{B}}$ and $TX^u|_{\mathbf{B}}$ are mutually orthogonal in $TX|_{\mathbf{B}}$, and moreover, under the above identification, on $V_{2\varepsilon_0}$, the metric g^{TX} is just the obvious flat metric of $TX|_{\mathbf{B}}$.
- If $x \in \mathbf{B}$, if $Z = (Z_+, Z_-) \in T_xX = T_xX^s \oplus T_xX^u$, if $|Z| \leqslant 2\varepsilon_0$, then equation (5.49) holds, i.e.

$$f(Z) = f(x) + \frac{1}{2}\left(|Z_+|^2 - |Z_-|^2\right). \tag{9.3}$$

As in Section 5.6, we can modify the metric g^{TX} into another G-invariant metric, and choose $\varepsilon_0 > 0$ small enough so that if $x \in \mathbf{B}$, T_xX^s and T_xX^u are mutually orthogonal in T_xX, and that on $U_{2\varepsilon_0} \simeq V_{2\varepsilon_0}$, g^{TX} is just the obvious flat metric along the fibres of $TX|_{\mathbf{B}}$.

Also, $F|_{\mathbf{B}}$ is a flat vector bundle on $\mathbf{B}$. On $U_{2\varepsilon_0} \simeq V_{2\varepsilon_0}$, we identify F to $F|_{\mathbf{B}}$ by parallel transport with respect to the flat connection ∇^F along radial lines in $TX|_{\mathbf{B}}$. Since F is flat on M, the above identification also identifies the flat connections. Also it is G-equivariant. The metric $g^{F|_{\mathbf{B}}}$ on $F|_{\mathbf{B}}$ is G-invariant. We may and we will assume that ε_0 is small enough so that on $U_{2\varepsilon_0} \simeq V_{2\varepsilon_0}$, the metric g^F is just the metric $g^{F|_{\mathbf{B}}}$. In particular, on $U_{2\varepsilon_0} \simeq V_{2\varepsilon_0}$, the metric g^F will be fibrewise unitarily flat.

Let $\nabla^{TX|_{\mathbf{B}}} = \nabla^{TX^s|_{\mathbf{B}}} \oplus \nabla^{TX^u|_{\mathbf{B}}}$ be a G-invariant Euclidean connection on $TX|_{\mathbf{B}}$ which preserves the splitting $TX|_{\mathbf{B}} = TX^s|_{\mathbf{B}} \oplus TX^u|_{\mathbf{B}}$. As in Section 4.1, the connection $\nabla^{TX|_{\mathbf{B}}}$ induces a horizontal subbundle $T^H(TX|_{\mathbf{B}}) \subset T(TX|_{\mathbf{B}})$. We may and we will assume that ε_0 is small enough so that under the identification $U_{2\varepsilon_0} \simeq V_{2\varepsilon_0}$, then $T^HM|_{U_{\varepsilon_0}} \simeq T^H(TX|_{\mathbf{B}})|_{V_{\varepsilon_0}}$.

9.2. Convergence results on integrals of differential forms

Let $X_{g,k}, 1 \leqslant k \leqslant p$ be the connected components of X_g. For $1 \leqslant k \leqslant p$, let $\chi(X_{g,k})$ be the Euler characteristic of $X_{g,k}$. The Lefschetz fixed point formula in (3.85) asserts that

$$\chi_g(F) = \sum_{k=1}^{p} \mathrm{Tr}^{F|_{X_{g,k}}}[g]\,\chi(X_{g,k}). \tag{9.4}$$

Also by (5.34),

$$\chi_g(F) = \sum_{x \in B_g} (-1)^{\mathrm{ind}_g(x)} \mathrm{Tr}^{F_x}[g]. \tag{9.5}$$

Set

$$\chi'_g(F) = \sum_{i=0}^{\dim X} (-1)^i \, i\mathrm{Tr}^{H^i(X,F|X)}[g],$$

$$\widetilde{\chi}'^{+/-}_g(F) = \sum_{x\in B_g} (-1)^{\mathrm{ind}_g(x)} \dim\left(T_x X^{s/u}|_{\mathbf{B}_g}\right) \mathrm{Tr}^{F_x}[g], \tag{9.6}$$

$$\mathrm{Tr}_{\mathrm{s}}^{B_g}[f] = \sum_{x\in B_g} (-1)^{\mathrm{ind}_g(x)} \mathrm{Tr}^{F_x}[g] f(x).$$

Recall that $n = \dim X$. Then

$$\chi'^{+}_g(F) + \chi'^{-}_g(F) = n\chi_g(F). \tag{9.7}$$

In the sequel we use the notation of Section 6 with respect to the Euclidean vector bundle (TX_g, g^{TX_g}) on M_g, equipped with the Euclidean connection ∇^{TX_g}. In particular, for $T \in \mathbf{R}$ or $T \in \mathbf{R}\setminus 0$, the forms a_T, b_T are smooth forms on the total space of TX_g. They are defined as in Definition 6.3. Recall that $\nabla f|_{M_g}$ is a section of TX_g.

Definition 9.1. — Let α_T, β_T be the forms on M_g,

$$\alpha_T = (\nabla f)^* a_T, \qquad \beta_T = (\nabla f)^* b_T. \tag{9.8}$$

As explained in Remark 6.6, as $T \to +\infty$, the convergence results of Theorem 6.4 also hold in the sense of microlocal convergence. Also on the zero set $\mathbf{B}_g$ of $\nabla f|_{M_g}$, $d^2 f_{M_g}$ has maximal rank. By [**Hö**, Theorem 8.2.4], the convergence results of Theorem 6.5 can be pulled back into corresponding convergence results for the currents α_T, β_T. A more direct easy argument is given in [**BZ1**, Theorem 3.18]. In particular, from Theorem 6.5, we deduce that

$$\int_{X_g} \mathrm{Tr}^F[g] f \alpha_{+\infty} = \mathrm{Tr}_{\mathrm{s}}^{B_g}[f]. \tag{9.9}$$

Let $N_{X_g/X}$ be the normal bundle to X_g in X. We identify $N_{X_g/X}$ to the orthogonal bundle to TX_g in $TX|_{X_g}$. Then g acts on $N_{X_g/X}$. Let $e^{\pm i\theta_j}, 0 < \theta_j \leqslant \pi$ be the distinct locally constant eigenvalues of g on $N_{X_g/X}$. Let $N^{\theta_j}_{X_g/X}$ be the corresponding real eigenbundles. With the notation in (1.16),

$$N^{\theta_j}_{X_g/X} \otimes_{\mathbf{R}} \mathbf{C} = N^{e^{i\theta_j}}_{X_g/X} \oplus N^{e^{-i\theta_j}}_{X_g/X} \text{ if } 0 < \theta_j < \pi,$$
$$N^{\pi}_{X_g/X} = N^{-1}_{X_g/X}. \tag{9.10}$$

If $x \in \mathbf{B}_g$, $T_x X_g$ and the $N^{\theta_j}_{X_g/X}$'s are mutually orthogonal with respect to the fibrewise Hessian $d^2 f(x)$. Let $(n_+(0)(x), n_-(0)(x))$ be the signature of the restriction of $d^2 f(x)$ to $T_x X_g$. Similarly let $(n_+(\theta_j)(x), n_-(\theta_j)(x))$ be the signature of

$d^2 f(x)$ on $N^{\theta_j}_{X_g/X}$. Set

$$n_+(x) = \dim T_x X^s, \qquad n_-(x) = \dim T_x X^u. \tag{9.11}$$

Let $\overline{h}(x)$ be the holomorphic odd function such that

$$\overline{h}'(x) = e^{x^2}. \tag{9.12}$$

Definition 9.2. — For $T \in \mathbf{R}_+^*$, put

$$m_T = \frac{\partial}{\partial T}\left[T\int_{X_g} \mathrm{Tr}^F[g]\, f\alpha_{T^2/4}\right], \tag{9.13}$$

$$n_T = -\frac{1}{2}\int_{X_g} h_g^*\left(\nabla^F, g^F\right) T\beta_{T^2/4} - \frac{1}{2}\frac{\partial}{\partial T}\int_{X_g} \overline{h}_g^*\left(\nabla^F, g^F\right) T^2\beta_{T^2/4}.$$

Proposition 9.3. — *As* $T \to +\infty$,

$$T\int_{X_g} f(\alpha_T - \alpha_{+\infty}) \to \sum_{x\in B_g} \frac{(-1)^{\mathrm{ind}_g(x)}}{4}\left(n_+(0)(x) - n_-(0)(x)\right), \tag{9.14}$$

$$T^2\int_{X_g} df\beta_T \to -\sum_{x\in B_g} \frac{(-1)^{\mathrm{ind}_g(x)}}{4}\left(n_+(0)(x) - n_-(0)(x)\right).$$

Proof. — Observe that the integrals in (9.14) are $\mathbf{R}$-valued integrals. Then we use [**BZ1**, Theorem 3.20], with $g = 1$ and $\mu = df$. □

Proposition 9.4. — *As* $T \to +\infty$,

$$T^2\left(m_T - \mathrm{Tr_s}^{B_g}[f]\right) \to -\sum_{x\in B_g} (-1)^{\mathrm{ind}_g(x)}\left(n_+(0)(x) - n_-(0)(x)\right)\mathrm{Tr}^{F_x}[g],$$

$$Tn_T \to 0, \tag{9.15}$$

$$\int_{X_g} \overline{h}_g^*\left(\nabla^F, g^F\right) T^2\beta_{T^2/4} \to 0.$$

Proof. — By (6.12) and by (9.13), we get

$$m_T = \int_{X_g} \mathrm{Tr}^F[g] f\alpha_{T^2/4} + \frac{T^2}{2}\int_{X_g} \mathrm{Tr}^F[g] df\beta_{T^2/4}. \tag{9.16}$$

By Proposition 9.3 and by (9.16), we get the first identity in (9.15).

Recall that on $U_{2\varepsilon_0}$, the metric g^F is the pull-back of the metric $g^{F|_B}$. Therefore, on $U_{2\varepsilon_0}$, the forms $h_g^*\left(F, g^F\right)$ and $\overline{h}_g^*\left(F, g^F\right)$ are pull-backs of forms on S. Also $\deg\left(\beta_{T^2/4}\right) = \dim(X_g) - 1$. It follows that for $\eta \in [0,1]$ small enough, the integrals along the fibre X_g of $h_g^*\left(F, g^F\right)\beta_{T^2/4}$ and of $\overline{h}_g^*\left(F, g^F\right)\beta_{T^2/4}$ on $U_{2\varepsilon_0}$ vanish identically. Also on the complement of this neighbourhood, as $T \to +\infty$, $\beta_{T^2/4}$ and $\frac{\partial}{\partial T}\beta_{T^2/4}$ converge to 0 like e^{-cT^2}, with $c > 0$. Therefore, we have established the last two convergence results in (9.15). □

Remark 9.5. — Note that the last convergence result in (9.15) also follows from Theorem 6.5.

9.3. Seven intermediate results

To make our arguments simpler, we will assume from now on that S is compact, or, equivalently, that M is compact. If this is not the case, the constants C which will appear in the results which follow will depend on the choice of a compact subset $K \subset S$.

On $\mathbf{B}_g$, consider the $\mathbf{Z}_2$-vector bundle $TX|_{\mathbf{B}_g} = TX^s|_{\mathbf{B}_g} \oplus TX^u_{\mathbf{B}_g}$. This vector bundle is equipped with a metric such that the splitting $TX|_{\mathbf{B}_g} = TX^s|_{\mathbf{B}_g} \oplus TX^u_{\mathbf{B}_g}$ is orthogonal, with a connection $\nabla^{TX|_{\mathbf{B}_g}}$ preserving the splitting and also the metric, and also with an isometric and parallel action of g which preserves the splitting. In the sequel, we use the notation of Chapter4 for this vector bundle. In particular if $x \in \mathbf{B}_g$, the function $q : TX|_{\mathbf{B}_g} \to \mathbf{R}$ is defined as in (4.6), and $\mathcal{D}^x_T$ denotes the operator defined in (4.9).

Let $d^{X,*}_T$ be the adjoint of d^X with respect to the metrics g^{TX}, g^F_T on (TX, F). Set

$$D^X_T = d^X + d^{X,*}_T. \tag{9.17}$$

As in (3.38)

$$\ker D^X_T \simeq H^\bullet(X, F|_X). \tag{9.18}$$

Let $P^{\{0\}}_T$ be the orthogonal projection operator from $\Omega^\bullet(X, F|_X)$ on $\ker D^X_T$ with respect to the Hermitian metric on $\Omega^\bullet(X, F|_X)$ defined in (3.29), which is attached to the metrics g^{TX}, g^F_T. Let $g^{H^\bullet(X,F|_X)}_{L_2,T}$ be the metric induced by this Hermitian metric on $H^\bullet(X, F|_X)$ via the identification (9.18), so that $g^{H^\bullet(X,F|_X)}_{L_2} = g^{H^\bullet(X,F|_X)}_{L_2,T}$. Now consider the $\mathbf{Z}$-graded flat Hermitian vector bundle $\left(H^\bullet(X, F|_X), g^{H^\bullet(X,F|_X)}_{L_2,T}\right)$. We can then use the notation in Remark 1.12. In particular, by following the notation in (1.29), set

$$B^{H^\bullet(X,F|_X)}_T = \frac{1}{2}\omega\left(H^\bullet(X, F|_X), g^{H^\bullet(X,F|_X)}_{L_2,T}\right). \tag{9.19}$$

Similarly to the flat Hermitian complex $C^\bullet(W^u, F)$ equipped with the flat superconnection $A^{C^\bullet(W^u,F)\prime}$, for $t > 0$, we can construct $B^{C^\bullet(W^u,F)}_t$ as in (1.41). Let $N^{C^\bullet(W^u,F)}$ be the number operator of $C^\bullet(W^u, F)$.

Theorem 9.6. — *Given $M > 0$, there exists $C > 0$ such that for $t \geqslant 1, 0 \leqslant T \leqslant M$,*

$$\left|\mathrm{Tr_s}\left[fgh'\left(D_{t,T}\right)\right] - \mathrm{Tr_s}\left[P^{\{0\}}_T f P^{\{0\}}_T g h'\left(B^{H^\bullet(X,F|_X)}_T\right)\right]\right| \leqslant \frac{C}{\sqrt{t}}. \tag{9.20}$$

Proof. — For a given T, the existence of $C > 0$ follows from the methods of [**BLo1**, Theorem 3.16]. Uniformity in $T \in [0, M]$ is trivial. □

Theorem 9.7. — *There exists $\delta \in]0,1/2]$ such that if ε, A are such that $0 < \varepsilon < A < +\infty$, there exists $C > 0$ such that if $t \in [\varepsilon, A], T \geqslant 1$, then*

$$\left|\mathrm{Tr_s}\left[Ngh'\left(D_{t,T}\right)\right] - \widetilde{\chi}'^{-}_{g}\left(F\right)\right| \leqslant \frac{C}{T^{\delta}}. \tag{9.21}$$

Theorem 9.8. — *The following identity holds,*

$$\begin{aligned}
\lim_{T\to+\infty}\Bigg\{&\int_1^{+\infty}\left(\mathrm{Tr_s}\left[Ngh'\left(D_{t,T}\right)\right] - \chi'_g\left(F\right)\right)\frac{dt}{2t} \\
&- \widetilde{h}^{*}_{g}\left(\nabla^{H^\bullet(X,F|X)}, g^{H^\bullet(X,F|X)}_{L_2,0}, g^{H^\bullet(X,F|X)}_{L_2,T}\right) - \mathrm{Tr_s}^{B_g}\left[f\right]T \\
&- \frac{1}{4}\left(\widetilde{\chi}'^{+}_{g}\left(F\right) - \widetilde{\chi}'^{-}_{g}\left(F\right)\right)\log\left(T\right)\Bigg\} \\
&= \int_0^1\left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}gh'\left(B_t^{C^\bullet(W^u,F)}\right)\right] - \widetilde{\chi}'^{-}_{g}\left(F\right)\right)\frac{dt}{2t} \\
&+ \int_1^{+\infty}\left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}h'\left(B_t^{C^\bullet(W^u,F)}\right)\right] - \chi'_g\left(F\right)\right)\frac{dt}{2t} \\
&- \widetilde{h}^{*}_{g}\left(\nabla^{H^\bullet(X,F|X)}, g^{H^\bullet(X,F|X)}_{L_2,0}, g^{H^\bullet(X,F|X)}_{C^\bullet(W^u,F)}\right) + \frac{1}{4}\left(\widetilde{\chi}'^{-}_{g}\left(F\right) - \widetilde{\chi}'^{+}_{g}\left(F\right)\right)\log\left(\pi\right) \\
&\qquad\qquad \text{in } \Omega^\bullet\left(S\right)/d\Omega^\bullet\left(S\right).
\end{aligned} \tag{9.22}$$

Theorem 9.9. — *There exists $t_0 \in]0,1]$ such that if $t \in]0,t_0]$, as $T \to +\infty$,*

$$\mathrm{Tr_s}\left[fgh'\left(D_{t,T}\right)\right] = \mathrm{Tr_s}^{B_g}[f] + \frac{1}{4}\left(\widetilde{\chi}'^{+}_{g}(F) - \widetilde{\chi}'^{-}_{g}(F)\right)\frac{1}{T} + \mathcal{O}\left(\frac{1}{T^3}\right). \tag{9.23}$$

Theorem 9.10. — *There exists $C > 0$ such that for $0 < t \leqslant 1, 0 \leqslant T \leqslant \frac{1}{\sqrt{t}}$, then*

$$\left|\frac{1}{t}\left(\mathrm{Tr_s}\left[fgh'\left(D_{t,T/\sqrt{t}}\right)\right] - m_T - \sqrt{t}n_T\right)\right| \leqslant C. \tag{9.24}$$

Theorem 9.11. — *For any $T > 0$, the following identity holds,*

$$\lim_{t\to 0}\frac{1}{t}\left(\mathrm{Tr_s}\left[fgh'\left(D_{t,T/t}\right)\right] - \mathrm{Tr_s}^{B_g}[f]\right) = \sum_{x\in B_g}\mathrm{Tr_s}\left[qgh'\left(\mathcal{D}^x_T\right)\right]\mathrm{Tr}^{F_x}\left[g\right]. \tag{9.25}$$

Theorem 9.12. — *There exist $t_0 \in]0,1], C > 0$ such that for $t \in]0,t_0], T \geqslant 1$, then*

$$\left|\frac{1}{t}\left(\mathrm{Tr_s}\left[fgh'\left(D_{t,T/t}\right)\right] - \mathrm{Tr_s}^{B_g}[f] - \frac{t}{4T}\left(\widetilde{\chi}'^{+}_{g}(F) - \widetilde{\chi}'^{-}_{g}(F)\right)\right)\right| \leqslant \frac{C}{T^3}. \tag{9.26}$$

Remark 9.13. — We will give two different proofs of Theorem 9.8 in Chapters 10 and 11. Theorems 9.7 and 9.9 will be proved in Chapter 12, Theorem 9.10 in Chapter 13, Theorem 9.11 in Chapter 14, and Theorem 9.12 in Chapter 15.

9.4. The asymptotics of the I_k^0

We start from the identity (8.11),

$$\sum_{k=1}^{4} I_k^0 = 0 \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{9.27}$$

Note that if $\alpha_n \in d\Omega^\bullet(S)$ is a family of smooth exact forms on S which converge uniformly to a smooth form α, then $\alpha \in d\Omega^\bullet(S)$.

Now we will study individually each $I_k^0, 1 \leqslant k \leqslant 4$, by making in succession $A \to +\infty$, $T_0 \to +\infty, \varepsilon \to 0$.

1) The term $\mathbf{I_1^0}$

Clearly,

$$I_1^0 = \int_\varepsilon^A \mathrm{Tr_s}\left[Ngh'\left(D_{t,T}\right)\right] \frac{dt}{2t}. \tag{9.28}$$

α) $\underline{A \to +\infty}$

By Theorem 3.30, as $A \to +\infty$,

$$I_1^0 - \frac{1}{2}\chi_g'(F)\log(A) \to I_1^1 = \int_\varepsilon^1 \mathrm{Tr_s}\left[Ngh'\left(D_{t,T_0}\right)\right] \frac{dt}{2t} + \int_1^{+\infty} \left(\mathrm{Tr_s}\left[Ngh'\left(D_{t,T_0}\right)\right] - \chi_g'(F)\right) \frac{dt}{2t}. \tag{9.29}$$

β) $\underline{T_0 \to +\infty}$

By Theorem 9.7, we see that as $T_0 \to +\infty$,

$$\int_\varepsilon^1 \mathrm{Tr_s}\left[Ngh'\left(D_{t,T_0}\right)\right] \frac{dt}{2t} \to -\frac{1}{2}\widetilde{\chi}_g'^{-}(F)\log(\varepsilon). \tag{9.30}$$

By (9.29), (9.30), we find that as $T_0 \to +\infty$,

$$I_1^1 - \int_1^{+\infty} \left(\mathrm{Tr_s}\left[Ngh'\left(D_{t,T_0}\right)\right] - \chi_g'(F)\right) \frac{dt}{2t} \to I_1^2 = -\frac{1}{2}\widetilde{\chi}_g'^{-}(F)\log(\varepsilon). \tag{9.31}$$

γ) $\underline{\varepsilon \to 0}$

We get

$$I_1^2 + \frac{1}{2}\widetilde{\chi}_g'^{-}(F)\log(\varepsilon) = 0. \tag{9.32}$$

2) The term $\mathbf{I_2^0}$

Clearly,

$$I_2^0 = \int_0^{T_0} \mathrm{Tr_s}\left[fgh'\left(D_{A,T}\right)\right] dT. \tag{9.33}$$

$\alpha)$ $\underline{A \to +\infty}$

By Theorem 9.6, as $A \to +\infty$,

$$I_2^0 \to I_2^1 = \int_0^{T_0} \mathrm{Tr_s}\left[P_T^{\{0\}} f P_T^{\{0\}} g h'\left(B_T^{H^\bullet(X,F|_X)}\right)\right] dT. \tag{9.34}$$

Proposition 9.14. — *The following identity holds,*

$$I_2^1 = -\widetilde{h}_g^*\left(H^\bullet(X,F|_X), g_{L_2,0}^{H^\bullet(X,F|_X)}, g_{L_2,T}^{H^\bullet(X,F|_X)}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{9.35}$$

Proof. — Using Definition 1.10 and proceeding as in [**BL**, Theorem 6.12] and in [**BZ1**, Proposition 7.16] , we get (9.35). □

$\beta)$ $\underline{T_0 \to +\infty}$

Tautologically,

$$I_2^1 + \widetilde{h}_g^*\left(H^\bullet(X,F|_X), g_{L_2,0}^{H^\bullet(X,F|_X)}, g_{L_2,T}^{H^\bullet(X,F|_X)}\right) = 0. \tag{9.36}$$

3) The term I_3^0

We have the identity,

$$I_3^0 = -\int_\varepsilon^A \mathrm{Tr_s}\left[N g h'(D_t)\right] \frac{dt}{2t}. \tag{9.37}$$

$\alpha)$ $\underline{A \to +\infty}$

By Theorem 3.30, as $A \to +\infty$, then

$$\begin{aligned} I_3^0 + \frac{1}{2}\chi_g'(F)\log(A) \to I_3^1 = &-\int_\varepsilon^1 \mathrm{Tr_s}\left[N g h'(D_t)\right] \frac{dt}{2t} \\ &-\int_1^{+\infty} \left(\mathrm{Tr_s}\left[N g h'(D_t)\right] - \chi_g'(F) h'(0)\right) \frac{dt}{2t}. \end{aligned} \tag{9.38}$$

$\beta)$ $\underline{T_0 \to +\infty}$

As $T_0 \to +\infty$, I_3^1 remains constant and equal to I_3^2.

$\gamma)$ $\underline{\varepsilon \to +0}$

By Theorem 3.30, we see that as $t \to 0$,

$$\begin{aligned} I_3^2 - \frac{n}{4}\chi_g(F)\log(\varepsilon) \\ \to I_3^3 = &-\int_0^1 \left(\mathrm{Tr_s}\left[N g h'(D_t)\right] - \frac{1}{2} n \chi_g(F) h'(0)\right) \frac{dt}{2t} \\ &-\int_1^{+\infty} \left(\mathrm{Tr_s}\left[N g h'(D_t)\right] - \chi_g'(F) h'(0)\right) \frac{dt}{2t}. \end{aligned} \tag{9.39}$$

δ) Evaluation of I_3^3

Theorem 9.15. — *The following identity holds,*

$$(9.40)\quad \varphi I_3^3 = \mathcal{T}_{h,g}\left(T^H M, g^{TX}, \nabla^F, g^F\right) - \left(\frac{n}{4}\chi_g(F) - \frac{1}{2}\chi_g'(F)\right)\left[\int_0^1 \left(h'\left(i\sqrt{t}/2\right) - h'(0)\right)\frac{dt}{t} + \int_1^{+\infty} h'\left(i\sqrt{t}/2\right)\frac{dt}{t}\right].$$

Proof. — Our identity follows from (3.89) and from (9.39). □

4) The term I_4^0

Clearly,

$$(9.41)\qquad I_4^0 = -\int_0^{T_0} \mathrm{Tr_s}\left[fgh'\left(D_{\varepsilon,T}\right)\right] dT.$$

α) $A \to +\infty$

The term I_4^0 remains constant and is equal to I_4^1.

β) $T_0 \to +\infty$

By Theorem 9.9, we find that if $\varepsilon \in]0, t_0]$, as $T_0 \to +\infty$,

$$(9.42)\quad I_4^1 + \mathrm{Tr_s}^{B_g}[f]T_0 + \frac{1}{4}\left(\widetilde{\chi}_g'^{+}(F) - \widetilde{\chi}_g'^{-}(F)\right)\log(T_0) \to I_4^2 = -\int_0^1 \left(\mathrm{Tr_s}\left[fgh'\left(D_{\varepsilon,T}\right)\right] - \mathrm{Tr_s}^{B_g}[f]\right) dT - \int_1^{+\infty}\left\{\mathrm{Tr_s}\left[fgh'\left(D_{\varepsilon,T}\right)\right] - \mathrm{Tr_s}^{B_g}[f] - \frac{1}{4}\left(\widetilde{\chi}_g'^{+}(F) - \widetilde{\chi}_g'^{-}(F)\right)\frac{1}{T}\right\} dT.$$

γ) $\varepsilon \to 0$

Put

$$(9.43)\quad \begin{aligned} J_1^0 &= -\int_0^1 \frac{1}{\sqrt{\varepsilon}}\left(\mathrm{Tr_s}\left[fgh'\left(D_{\varepsilon,T/\sqrt{\varepsilon}}\right)\right] - \mathrm{Tr_s}^{B_g}[f]\right) dT,\\ J_2^0 &= -\int_1^{1/\sqrt{\varepsilon}} \frac{1}{\sqrt{\varepsilon}}\left(\mathrm{Tr_s}\left[fgh'\left(D_{\varepsilon,T/\sqrt{\varepsilon}}\right)\right] - \mathrm{Tr_s}^{B_g}[f]\right) dT,\\ J_3^0 &= -\int_1^{+\infty} \frac{1}{\varepsilon}\left\{\mathrm{Tr_s}\left[fgh'\left(D_{\varepsilon,T/\varepsilon}\right)\right] - \mathrm{Tr_s}^{B_g}[f] - \frac{\varepsilon}{4}\left(\widetilde{\chi}_g'^{+}(F) - \frac{1}{2}\widetilde{\chi}_g'^{-}(F)\right)\frac{1}{T}\right\} dT. \end{aligned}$$

Then

$$(9.44)\qquad I_4^2 = J_1^0 + J_2^0 + J_3^0 - \frac{1}{4}\left(\widetilde{\chi}_g'^{+}(F) - \widetilde{\chi}_g'^{-}(F)\right)\log(\varepsilon).$$

By (9.9) and by Theorem 9.10, we find that as $\varepsilon \to 0$,

$$(9.45)\quad J_1^0 + \int_{X_g} \mathrm{Tr}^F[g] f(\alpha_{1/4} - \alpha_{+\infty}) \frac{1}{\sqrt{\varepsilon}} \to J_1^1 = \frac{1}{2}\int_{X_g} \overline{h}_g^*(\nabla^F, g^F)\beta_{1/4} + \frac{1}{2}\int_0^1 \int_{X_g} h_g^*(\nabla^F, g^F) T\beta_{T^2/4} dT.$$

Also, using (9.13), we get

$$(9.46)\quad J_2^0 = -\int_{\sqrt{\varepsilon}}^1 \left(\mathrm{Tr}_s\left[fgh'(D_{\varepsilon,T/\varepsilon})\right] - m_{T/\sqrt{\varepsilon}} - \sqrt{\varepsilon} n_{T/\sqrt{\varepsilon}}\right)\frac{dT}{\varepsilon} - \left(\int_{X_g} \mathrm{Tr}^F[g] f(\alpha_{1/4\varepsilon} - \alpha_{+\infty})\frac{1}{\varepsilon} - \int_{X_g} \mathrm{Tr}^F[g] f(\alpha_{1/4} - \alpha_{+\infty})\frac{1}{\sqrt{\varepsilon}}\right) + \frac{1}{2\varepsilon}\int_{X_g} \overline{h}_g^*(F, g^F)\beta_{1/4\varepsilon} - \frac{1}{2}\int_{X_g} \overline{h}_g^*(\nabla^F, g^F)\beta_{1/4} + \frac{1}{2}\int_1^{1/\sqrt{\varepsilon}} \left[\int_{X_g} h_g^*(\nabla^F, g^F) T\beta_{T^2/4}\right] dT.$$

By Propositions 9.3, 9.4 and by Theorems 9.10 and 9.11, we find that as $\varepsilon \to 0$,

$$(9.47)\quad J_2^0 - \int_{X_g} \mathrm{Tr}^F[g] f(\alpha_{1/4} - \alpha_{+\infty})\frac{1}{\sqrt{\varepsilon}} \to J_2^1 = -\sum_{x\in B_g} \mathrm{Tr}^{F_x}[g]\left[\int_0^1 \left(\mathrm{Tr}_s[qgh'(\mathcal{D}_T^x)] + (-1)^{\mathrm{ind}_g(x)}(n_+(0)(x) - n_-(0)(x))\frac{1}{T^2}\right) dT + (-1)^{\mathrm{ind}_g(x)}(n_+(0)(x) - n_-(0)(x))\right] - \frac{1}{2}\int_{X_g} \overline{h}_g^*(\nabla^F, g^F)\beta_{1/4} + \frac{1}{2}\int_1^{+\infty} h_g^*(\nabla^F, g^F) T\beta_{T^2/4} dT.$$

By Theorems 9.11 and 9.12, using the notation in (9.11), we see that as $\varepsilon \to 0$,

$$(9.48)\quad J_3^0 \to J_3^1 = -\sum_{x\in B_g} \mathrm{Tr}^{F_x}[g]\int_1^{+\infty} \left(\mathrm{Tr}_s\left[qgh'\left(\mathcal{D}_T^x\right)\right] - (-1)^{\mathrm{ind}_g(x)}(n_+(x) - n_-(x))\frac{1}{4T}\right) dT.$$

From (9.44)-(9.48), we find that as $\varepsilon \to 0$,

$$(9.49)\quad I_4^2 + \frac{1}{4}\left(\widetilde{\chi}_g'^{+}(F) - \widetilde{\chi}_g'^{-}(F)\right)\log(\varepsilon) \to I_4^3 = \int_0^{+\infty} h_g^*\left(F, g^F\right)\beta_u du$$
$$- \sum_{x\in B_g} \mathrm{Tr}^{F_x}[g]\left[\int_0^1 \left(\mathrm{Tr}_s\left[qgh'\left(\mathcal{D}_T^x\right)\right] + (-1)^{\mathrm{ind}_g(x)}\left(n_+(0)(x) - n_-(0)(x)\right)\frac{1}{T^2}\right)dT\right.$$
$$+ \int_1^{+\infty} \left(\mathrm{Tr}_s\left[qgh'\left(\mathcal{D}_T^x\right)\right] - (-1)^{\mathrm{ind}_g(x)}\left(n_+(x) - n_-(x)\right)\frac{1}{4T}\right)dT$$
$$\left. + (-1)^{\mathrm{ind}_g(x)}\left(n_+(0)(x) - n_-(0)(x)\right)\right].$$

δ) Evaluation of I_4^3

Theorem 9.16. — *The following identity holds,*

$$(9.50)\quad \varphi I_4^3 = \int_{X_g} h_g\left(F, g^F\right)(\nabla f)^*\,\psi\left(TX, \nabla^{TX}\right)$$
$$- \sum_{x\in B_g} (-1)^{\mathrm{ind}_g(x)}\,\mathrm{Tr}^{F_x}[g]\left[\sum_{0\leqslant\theta\leqslant\pi} \frac{1}{2}\left(n_+(\theta)(x) - n_-(\theta)(x)\right)\left(\frac{\partial\zeta}{\partial s}(\theta, 0)\right.\right.$$
$$\left.\left. + \log(2)\zeta(\theta, 0) + 1 - \frac{\Gamma'(1)}{2}\right) + I_g^{(>0)}\left(TX|_{\mathbf{B}_g}\right)\right].$$

Proof. — Clearly,

$$(9.51)\qquad \varphi\int_{X_g} h_g^*(F, g^F)\beta_u = \int_{X_g} \varphi h_g^*(F, g^F)\,(2i\pi)^{\dim X_g/2}\,\varphi\beta_u.$$

Since $\deg(\beta_u) = \dim X_g - 1$,

$$(9.52)\qquad (2i\pi)^{\dim X_g}\,\varphi\beta_u = (2i\pi)^{1/2}\,\beta_u.$$

By (1.23), (9.51), (9.52), we get

$$(9.53)\qquad \varphi\int_{X_g} h_g^*(F, g^F)\beta_u = \int_{X_g} h_g(F, g^F)\beta_u.$$

Using (6.14) and (9.53), we obtain,

$$(9.54)\qquad \varphi\int_0^{+\infty}\left[\int_{X_g} h_g^*\left(F, g^F\right)\beta_u\right]du = \int_{X_g} h_g\left(F, g^F\right)(\nabla f)^*\,\psi\left(TX, g^{TX}\right).$$

Also by (4.52), (4.55), (4.56) and by Theorem 4.23,

$$(9.55)\qquad \varphi\int_0^{+\infty} \mathrm{Tr}_s\left[qgh'\left(\mathcal{D}_T\right)\right]^{(>0)}dT = (-1)^{\mathrm{ind}_g(x)}\,I_g^{(>0)}\left(TX|_{\mathbf{B}_g}\right).$$

Using (9.49), (9.54), (9.55), we get (9.50) in positive degree.

Now we use the notation of Section 4.2 with respect to the action of g on $TX|_{\mathbf{B}_g}$. By Propositions 4.11 and 4.18, if $x \in B_g$,

(9.56)
$$\mathrm{Tr_s}\left[Tqgh'\left(\mathcal{D}_T\right)\right]^{(0)} = \left(1+2\frac{\partial}{\partial a}\right)(-1)^{\mathrm{ind}_g(x)}\,\mathrm{Tr_s}\left[\frac{1}{2}\frac{\frac{\partial}{\partial T'}\sigma\left(T'^2/4,B,0\right)}{\sigma\left(T'^2/4,B,0\right)}\right]|_{T'=aT,a=1}.$$

By Proposition 4.15 and by Remark 4.16, we find that as $T \to 0$,

$$\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,i\theta,0\right)}{\sigma\left(T^2/4,i\theta,0\right)} = \frac{1_{\theta\in 2\pi\mathbf{Z}}}{T} + \mathcal{O}\left(T\right), \tag{9.57}$$

and that as $T \to +\infty$,

$$\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,i\theta,0\right)}{\sigma\left(T^2/4,i\theta,0\right)} = \frac{1}{4} + \mathcal{O}\left(e^{-T/2}\right). \tag{9.58}$$

For $s \in \mathbf{C}, \mathrm{Re}\,(s) > 1$, put

$$\psi(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} T^{s-1}\left(\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,i\theta,0\right)}{\sigma\left(T^2/4,i\theta,0\right)} - \frac{1}{4}\right)dT. \tag{9.59}$$

By (9.57), (9.58), $\psi(s)$ is a holomorphic function of s, which extends to a holomorphic function near $s = 0$, and moreover, by (9.57) and (9.58),

$$\begin{aligned}\psi'(0) = &\int_0^1 \left(\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,i\theta,0\right)}{\sigma\left(T^2/4,i\theta,0\right)} - \frac{1_{\theta\in 2\pi\mathbf{Z}}}{T}\right)\frac{dT}{T} \\ &+ \int_1^{+\infty}\left(\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,i\theta,0\right)}{\sigma\left(T^2/4,i\theta,0\right)} - \frac{1}{4}\right)\frac{dT}{T} - 1_{\theta\in 2\pi\mathbf{Z}} + \frac{\Gamma'(1)}{4}.\end{aligned} \tag{9.60}$$

By (4.89), (4.93), (9.59), we obtain,

$$\psi(s) = 2^{s-1}\zeta(\theta,s). \tag{9.61}$$

So by (9.60), (9.61), we get

(9.62)
$$\begin{aligned}&\int_0^1 \left(\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,i\theta,0\right)}{\sigma\left(T^2/4,i\theta,0\right)} - \frac{1_{\theta\in 2\pi\mathbf{Z}}}{T}\right)\frac{dT}{T} + \int_1^{+\infty}\left(\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,i\theta,0\right)}{\sigma\left(T^2/4,i\theta,0\right)} - \frac{1}{4}\right)\frac{dT}{T} \\ &\qquad = 1_{\theta\in 2\pi\mathbf{Z}} - \frac{\Gamma'(1)}{4} + \frac{1}{2}\left(\frac{\partial\zeta}{\partial s}(\theta,0) + \log(2)\zeta(\theta,0)\right).\end{aligned}$$

Now, we have the obvious,

$$
\begin{aligned}
(9.63)\quad & \left(1+2\frac{\partial}{\partial a}\right)\left[\int_0^1 \left(\frac{1}{2}\frac{\frac{\partial}{\partial T'}\sigma\left(T'^2/4,i\theta,0\right)}{\sigma\left(T'^2/4,i\theta,0\right)}|_{T'=aT}-\frac{1_{\theta\in 2\pi\mathbf{Z}}}{aT}\right)\frac{dT}{T}\right. \\
& \left.+\int_1^{+\infty}\left(\frac{1}{2}\frac{\frac{\partial}{\partial T'}\sigma\left(T'^2/4,i\theta,0\right)}{\sigma\left(T'^2/4,i\theta,0\right)}|_{T'=aT}-\frac{1}{4}\right)\frac{dT}{T}\right]|_{a=1} = \\
& \int_0^1\left(\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,i\theta,0\right)}{\sigma\left(T^2/4,i\theta,0\right)}-\frac{1_{\theta\in 2\pi\mathbf{Z}}}{T}\right)\frac{dT}{T}+\int_1^{+\infty}\left(\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4,i\theta,0\right)}{\sigma\left(T^2/4,i\theta,0\right)}-\frac{1}{4}\right)\frac{dT}{T} \\
& -2\times 1_{\theta\in 2\pi\mathbf{Z}}+\frac{1}{2}.
\end{aligned}
$$

By (9.49), (9.56) and (9.63), we get (9.50). The proof of our Theorem is completed. □

9.5. Matching the divergences

Theorem 9.17. — *The following identity holds,*

$$
\begin{aligned}
(9.64)\quad & I_3^3+I_4^3+\int_0^1\left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}gh'\left(B_t^{C^\bullet(W^u,F)}\right)\right]-\widetilde{\chi}_g'^{-}(F)\right)\frac{dt}{2t} \\
& +\int_1^{+\infty}\left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}gh'\left(B_t^{C^\bullet(W^u,F)}\right)\right]-\chi_g'(F)\right)\frac{dt}{2t} \\
& -\widetilde{h}_g^*\left(H^\bullet(X,F|_X),g_{L_2,0}^{H^\bullet(X,F|_X)},g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}\right) \\
& -\frac{1}{4}\left(\widetilde{\chi}_g'^{+}(F)-\widetilde{\chi}_g'^{-}(F)\right)\log(\pi)=0 \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S).
\end{aligned}
$$

Proof. — Recall that by (9.27),

$$
(9.65)\qquad \sum_{k=1}^4 I_k^0=0 \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S).
$$

As $A\to+\infty$, the following diverging terms appear in (9.29), (9.38),

$$
(9.66)\qquad -\frac{1}{2}\chi_g'(F)\log(A)+\frac{1}{2}\chi_g'(F)\log(A)=0.
$$

By (9.65), (9.66), we get

$$
(9.67)\qquad \sum_{k=1}^4 I_k^1=0 \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S).
$$

Using Theorem 9.8 and (9.31), (9.36), (9.42), we find that

$$
\begin{aligned}
(9.68)\quad & I_1^2+I_3^2+I_4^2+\int_0^1\left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}gh'\left(B_t^{H^\bullet(X,F|_X)}\right)\right]-\widetilde{\chi}_g'^{-}(F)\right)\frac{dt}{2t}\\
&+\int_1^{+\infty}\left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}gh'\left(B_t^{H^\bullet(X,F|_X)}\right)\right]-\chi_g'(F)\right)\frac{dt}{2t}\\
&-\widetilde{h}_g^*\left(H^\bullet(X,F|_X),g_{L_2,0}^{H^\bullet(X,F|_X)},g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}\right)\\
&+\frac{1}{4}\left(\widetilde{\chi}_g'^{-}(F)-\chi_g'^{+}(F)\right)\log(\pi)=0 \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S).
\end{aligned}
$$

By (9.32), (9.39), (9.49), as $\varepsilon\to 0$, we get the diverging terms,

$$
(9.69)\qquad \frac{1}{4}\left(\widetilde{\chi}_g'^{-}(F)-\widetilde{\chi}_g'^{+}(F)+\widetilde{\chi}_g'^{+}(F)-\widetilde{\chi}_g'^{-}(F)\right)\log(\varepsilon)=0.
$$

From (9.68), (9.69), we obtain (9.64). The proof of our Theorem is completed. □

9.6. A proof of Theorem 7.2

By (1.70) and by Theorems 9.15, 9.16 and 9.17, we get

$$
\begin{aligned}
(9.70)\quad & \mathcal{T}_{h,g}\left(T^HM,g^{TX},\nabla^Fg^F\right)-T_{h,g}\left(C^\bullet(W^u,F),\nabla^{C^\bullet(W^u,F)},g^{C^\bullet(W^u,F)}\right)-\\
&\widetilde{h}_g\left(H^\bullet(X,F|_X),g_{L_2,0}^{H^\bullet(X,F|_X)},g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}\right)+\int_{X_g}h_g\left(F,g^F\right)(\nabla f)^*\psi\left(TX,\nabla^{TX}\right)\\
&-\sum_{x\in B_g}(-1)^{\mathrm{ind}_g(x)}\mathrm{Tr}^{F_x}[g]\left[\sum_{0\leqslant\theta\leqslant\pi}\frac{1}{2}\Big(n_+(\theta)(x)-n_-(\theta)(x)\Big)\left(\frac{\partial\zeta}{\partial s}(\theta,0)\right.\right.\\
&\left.\left.+\log(2)\zeta(\theta,0)+1+\frac{\log(\pi)-\Gamma'(1)}{2}\right)+I_g^{(>0)}\left(TX|_{\mathbf{B}_g}\right)\right]-\frac{1}{4}\left(\chi_g'^{+}(F)-\chi_g'^{-}(F)\right)\\
&\left[\int_0^1\left(h'\left(i\sqrt{t}/2\right)-h'(0)\right)\frac{dt}{t}+\int_1^{+\infty}h'\left(i\sqrt{t}/2\right)\frac{dt}{t}\right]=0 \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S).
\end{aligned}
$$

Also, one has the trivial,

$$
\begin{aligned}
(9.71)\quad & \int_0^1\left(h'\left(i\sqrt{t}/2\right)-h'(0)\right)\frac{dt}{t}+\int_1^{+\infty}h'\left(i\sqrt{t}/2\right)\frac{dt}{t}\\
&=\int_0^1\left(e^{-t/4}-1\right)\frac{dt}{t}+\int_1^{+\infty}e^{-t/4}\frac{dt}{t}-\frac{1}{2}\int_0^{+\infty}e^{-t/4}dt\\
&=\Gamma'(1)+2\big(\log(2)-1\big).
\end{aligned}
$$

Moreover by (4.97), for any $\theta\in\mathbf{R}$,

$$
(9.72)\qquad \zeta(\theta,0)=-\frac{1}{2}.
$$

From (5.33), (7.7), (7.8), (9.70)-(9.72), we get (7.9). The proof of Theorem 7.2 is completed.

CHAPTER 10

GENERALIZED METRICS: A FIRST PROOF OF THEOREM 9.8

The purpose of this Chapter is to give a proof of Theorem 9.8 using the theory of generalized metrics. It is close in spirit to the proof of the corresponding result in degree 0 which was established in [**BZ1**, Theorem 5.5]. The idea is to show that for $T \geqslant 0$ large enough, we can construct a filtered finite dimensional subbundle $\mathbf{F}_T^{[0,1]} \subset \Lambda^\bullet(T^*S) \widehat{\otimes} \Omega^\bullet\big(X, F|_X\big)$, which we identify via $\mathbf{P}^\infty$ to $\Lambda^\bullet(T^*S) \widehat{\otimes} C^\bullet(W^u, F)$. The Gr of the vector bundle $\mathbf{F}_T^{[0,1]}$ is just the bundle of sum of the eigenspaces of the fibrewise Laplacian $A_T^{2,(0)}$ associated to small eigenvalues. The difficulty is then to control the behaviour of torsion forms for $C^\bullet(W^u, F)$ associated to generalized metrics $\mathbf{g}_T^{C^\bullet(W^u,F)}$ as $T \to +\infty$.

In Chapter 11, when f is supposed to be fibrewise nice, and also parallel with respect to $T^H M$ (a choice of such a f is always possible), another proof of Theorem 9.8 will be given, which relies on the precise estimates on the eigenvalues of $A_T^{2,(0)}$ obtained in Helffer-Sjöstrand [**HSj**], which were given a simpler direct proof in [**BZ2**, Theorem 6.12].

This Chapter is organized as follows. In Section 10.1, using the simplifying assumptions of Section 9.1, we identify the geometric setting near $\mathbf{B}$ using the results of Chapter 4. In Section 10.2, we introduce the eigenbundles associated to small eigenvalues. In Section 10.3, we give various algebraic properties of the curvature A_T^2 , associated to the metrics g^{TX}, g_T^F. In Section 10.4, we introduce a projector $\mathbf{P}_T^{[0,1]}$ on $\mathbf{F}_T^{[0,1]}$, and in Section 10.5 their obvious analogues $\mathbf{P}_{t,T}^{[0,1]}$.

In Section 10.6, by restriction of $\mathbf{P}^\infty$, we construct the maps $\mathbf{P}_T^\infty : \mathbf{F}_T^{[0,1]} \to \Lambda^\bullet(T^*S) \widehat{\otimes} C^\bullet(W^u, F)$. In Section 10.7, we construct the generalized metrics $\mathbf{g}_T^{C^\bullet(W^u,F)}$. In Section 10.8, we obtain the obvious extensions $\mathbf{P}_{t,T}^\infty$ of $\mathbf{P}_T^\infty$, and we study their asymptotics as $t \to +\infty$, and also the asymptotics of the corresponding generalized metrics $\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}$. In Section 10.9, we include $t > 0$ as a base parameter, so that S is replaced by $S \times \mathbf{R}_+^*$. In Section 10.10, we construct superconnection

forms for $\mathbf{F}_T^{[0,1]}$ similar to the forms of Chapters 1 and 2. In Section 10.11, we relate these forms to superconnection forms associated to the generalized metrics $\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}$ on $C^\bullet(W^u,F)$.

In Section 10.12, we state a simple identity on generalized torsion forms. In Section 10.13, we state a result on generalized metrics, which implies Theorem 9.8. The next Sections are devoted to the proof of this result. In Section 10.14, we introduce the projectors $\overline{\mathbb{P}}_T$, which are analogues of the projectors $\mathbf{P}_T^\infty$ with respect to the harmonic oscillators of Chapter 4. In Section 10.15, we extend the instanton results of [**HSj, BZ2**] to the present geometric setting. Our projectors $\mathbf{P}_T^\infty$ being perturbation of ordinary self adjoint projectors $P_T^{[0,1]}$ by nilpotent operators, the methods of [**BZ2**] have to be adequately modified. Finally, in Sections 10.16 and 10.17, we complete the proof of the result stated in Section 10.13.

We assume the assumptions of Chapter 9 to be in force, and in particular the simplifying assumptions of Section 9.1. Also we use the notation of the previous Chapters. As in Chapter 9, we suppose that S is compact, so that M is also compact. As before, we use the notation,

$$h(x) = xe^{x^2}. \tag{10.1}$$

10.1. The harmonic oscillator near B

Recall that $\mathbf{B}$ is the set of fibrewise critical points of f. Now we use the notation of Chapter 4, with $M = \mathbf{B}$. Also the $\mathbf{Z}_2$-graded vector bundle $E = E_+ \oplus E_-$ will be here $TX|_{\mathbf{B}} = TX^s|_{\mathbf{B}} \oplus TX^u|_{\mathbf{B}}$. The metric g^{TX} induces a metric $g^{TX|_{\mathbf{B}}} = g^{TX^s|_{\mathbf{B}}} \oplus g^{TX^u|_{\mathbf{B}}}$. If $x \in \mathbf{B}$, the vector spaces I_x, I_x^0 were defined in Definition 4.1.

Clearly, $F|_{\mathbf{B}}$ is equipped with a flat connection $\nabla^{F|_{\mathbf{B}}}$. We denote by $\nabla^{I\widehat{\otimes}F|_{\mathbf{B}}}$ the connection on $I\widehat{\otimes}F|_{\mathbf{B}}$ induced by ∇^I and $\nabla^{F|_{\mathbf{B}}}$. The metric g^F induces a metric $g^{F|_{\mathbf{B}}}$ on $F|_{\mathbf{B}}$. Let $\mathcal{F}$ be the restriction of f to $\mathbf{B}$. Then by (9.3), using the notation in (4.6), if $x \in B, Z \in (TX|_{\mathbf{B}})_x, |Z| \leqslant \varepsilon$, set

$$f(Z) = \mathcal{F}(x) + q(Z). \tag{10.2}$$

For $T \in \mathbf{R}$, the metric g_T^F induces the metric $g_T^{F|_{\mathbf{B}}}$ on $F|_{\mathbf{B}}$, which is given by

$$g_T^{F|_{\mathbf{B}}} = e^{-2T\mathcal{F}} g^{F|_{\mathbf{B}}}. \tag{10.3}$$

Let $\mathcal{C}^{I\widehat{\otimes}F|'_{\mathbf{B}}}$ be the canonical flat superconnection on $I\widehat{\otimes}F|_{\mathbf{B}}$ which is attached to the above situation. As in (4.5), we have the identity,

$$\mathcal{C}^{I\widehat{\otimes}F|'_{\mathbf{B}}} = d^{TX|_{\mathbf{B}}} + \nabla^{I\widehat{\otimes}F|_{\mathbf{B}}} + i_{R^{TX|_{\mathbf{B}}}Z}. \tag{10.4}$$

Given $T \in \mathbf{R}$, let $\mathcal{C}_T^{I\widehat{\otimes}F|''_{\mathbf{B}}}$ be the adjoint flat superconnection with respect to the metrics $g^{TX|_{\mathbf{B}}}, g_T^{F|_{\mathbf{B}}}$. By an obvious extension of (4.8), we get

$$\mathcal{C}_T^{I\widehat{\otimes}F|''_{\mathbf{B}}} = d^{TX|_{\mathbf{B}}*} + 2Ti_{Z_+ - Z_-} + \nabla^{I\widehat{\otimes}F|_{\mathbf{B}}} + \omega\left(F|_{\mathbf{B}}, g^{F|_{\mathbf{B}}}\right) - 2Td\mathcal{F} - R^{TX|_{\mathbf{B}}}Z\wedge. \tag{10.5}$$

As in (4.9), we set

$$\mathcal{C}_T^{I\widehat{\otimes}F|_{\mathbf{B}}} = \frac{1}{2}\left(\mathcal{C}_T^{I\widehat{\otimes}F|''_{\mathbf{B}}} + \mathcal{C}^{I\widehat{\otimes}F|'_{\mathbf{B}}}\right), \qquad \mathcal{D}_T^{I\widehat{\otimes}F|_{\mathbf{B}}} = \frac{1}{2}\left(\mathcal{C}_T^{I\widehat{\otimes}F|''_{\mathbf{B}}} - \mathcal{C}^{I\widehat{\otimes}F|'_{\mathbf{B}}}\right). \tag{10.6}$$

A related construction, which extends (4.10) is as follows. First, we extend temporarily f given by (10.2) to $TX|_{\mathbf{B}}$. Set

$$\begin{aligned} &\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|'_{\mathbf{B}}} = e^{-Tf}\mathcal{C}^{I\widehat{\otimes}F|'_{\mathbf{B}}}e^{Tf}, && \overline{\mathcal{C}}^{I\widehat{\otimes}F|''_{\mathbf{B}}} = e^{Tf}\mathcal{C}_0^{I\widehat{\otimes}F|''_{\mathbf{B}}}e^{-Tf}, \\ &\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}}} = \frac{1}{2}\left(\overline{\mathcal{C}}^{I\widehat{\otimes}F|''_{\mathbf{B}}} + \overline{\mathcal{C}}_T^{I\widehat{\otimes}F|'_{\mathbf{B}}}\right), && \overline{\mathcal{D}}_T^{I\widehat{\otimes}F|_{\mathbf{B}}} = \frac{1}{2}\left(\overline{\mathcal{C}}^{I\widehat{\otimes}F|''_{\mathbf{B}}} - \overline{\mathcal{C}}_T^{I\widehat{\otimes}F|'_{\mathbf{B}}}\right). \end{aligned} \tag{10.7}$$

As in (4.11),

$$\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}}} = e^{-Tf}\mathcal{C}_T e^{Tf}, \qquad \overline{\mathcal{D}}_T^{I\widehat{\otimes}F|_{\mathbf{B}}} = e^{-Tf}\mathcal{D}_T e^{Tf}. \tag{10.8}$$

Then, by (4.13),

$$\begin{aligned} \overline{\mathcal{C}}_T^{I\widehat{\otimes}F|'_{\mathbf{B}}} &= d^{TX|_{\mathbf{B}}} + T(Z_+ - Z_-)\wedge + \nabla^{I\widehat{\otimes}F|_{\mathbf{B}}} + Td\mathcal{F} + i_{R^{TX|_{\mathbf{B}}}Z}, \\ \overline{\mathcal{C}}^{I\widehat{\otimes}F|''_{\mathbf{B}}} &= d^{TX|_{\mathbf{B}}*} + Ti_{Z_+-Z_-} + \nabla^{I\widehat{\otimes}F|_{\mathbf{B}}} + \omega\left(F|_{\mathbf{B}}, g^{F|_{\mathbf{B}}}\right) - Td\mathcal{F} - R^{TX|_{\mathbf{B}}}Z.\wedge. \end{aligned} \tag{10.9}$$

By (10.7), we get

$$\begin{aligned} \overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}}} &= \frac{1}{2}\left(d^{TX|_{\mathbf{B}},*} + d^{TX|_{\mathbf{B}}}\right) + \frac{T}{2}\left(i_{Z_+-Z_-} + (Z_+ - Z_-)\wedge\right) + \nabla^{I\widehat{\otimes}F|_{\mathbf{B}}} \\ &\quad + \frac{1}{2}\omega\left(F|_{\mathbf{B}}, g^{F|_{\mathbf{B}}}\right) + \frac{1}{2}\left(i_{R^{TX|_{\mathbf{B}}}Z} - R^{TX|_{\mathbf{B}}}Z\wedge\right), \\ \overline{\mathcal{D}}_T^{I\widehat{\otimes}F|_{\mathbf{B}}} &= \frac{1}{2}\left(d^{TX|_{\mathbf{B}},*} - d^{TX|_{\mathbf{B}}}\right) + \frac{T}{2}\left(i_{Z_+-Z_-} - (Z_+ - Z_-)\wedge\right) + \frac{1}{2}\omega\left(F|_{\mathbf{B}}, g^{F|_{\mathbf{B}}}\right) \\ &\quad - Td\mathcal{F} - \frac{1}{2}\left(i_{R^{TX|_{\mathbf{B}}}Z} + R^{TX|_{\mathbf{B}}}Z\wedge\right). \end{aligned} \tag{10.10}$$

Let $e_1, \ldots, e_{n_+}$ be an orthonormal basis of $TX^s|_{\mathbf{B}}$, let $e_{n_+1}, \ldots, e_n$ be an orthonormal basis of $TX^u|_{\mathbf{B}}$. Then by (4.14),

$$\begin{aligned} \overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2} &= -\frac{1}{4}\left(\nabla_{e_i} + \langle R^{TX|_{\mathbf{B}}}Z, e_i\rangle\right)^2 + \frac{1}{4}\langle e_i, R^{TX|_{\mathbf{B}}}e_j\rangle\widehat{c}(e_i)\widehat{c}(e_j) \\ &\quad - \frac{1}{4}\omega^2\left(F|_{\mathbf{B}}, g^{F|_{\mathbf{B}}}\right) + \frac{T^2}{4}|Z|^2 + \frac{T}{4}\left(\sum_{1\leqslant i\leqslant n_+} c(e_i)\widehat{c}(e_i) - \sum_{n_++1\leqslant i\leqslant n_++n_-} c(e_i)\widehat{c}(e_i)\right). \end{aligned} \tag{10.11}$$

In particular, by (10.11), we get

$$\begin{aligned} \overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2,(0)} &= -\frac{1}{4}\Delta^{TX|_{\mathbf{B}}} + \frac{T^2}{4}|Z|^2 \\ &\quad + \frac{T}{4}\left(\sum_{1\leqslant i\leqslant n_+} c(e_i)\widehat{c}(e_j) - \sum_{n_++1\leqslant i\leqslant n_++n_-} c(e_i)\widehat{c}(e_i)\right). \end{aligned} \tag{10.12}$$

Clearly,

$$\mathrm{Sp}\left(\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2}\right) = \mathrm{Sp}\left(\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2,(0)}\right). \tag{10.13}$$

The operator $\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2,(0)}$ is a harmonic oscillator. Take ρ non zero in $\Lambda^{\max}(T^*X^u|_{\mathbf{B}})$. Let $\overline{\mathfrak{f}}_T$ be the one dimensional vector space spanned by $\exp\left(-T|Z|^2/2\right)\rho$. Then by [**W**], [**BZ1**, Proposition 8.3],

$$\ker \overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2,(0)} = \overline{\mathfrak{f}}_T \otimes F|_{\mathbf{B}}. \tag{10.14}$$

Let $\overline{\mathfrak{p}}_T$ be the orthogonal projection operator from I on $\overline{\mathfrak{f}}_T$. Then $\overline{\mathfrak{p}}_T$ extends to an endomorphism of $I \widehat{\otimes} F|_{\mathbf{B}}$. Note that when acting on $I \widehat{\otimes} F|_{\mathbf{B}}$, $\overline{\mathfrak{p}}_T$ is of the form $\overline{\mathfrak{p}}_T \otimes 1$.

For $T \in \mathbf{R}_+^*$, using (4.12), we get

$$\mathrm{Sp}\left(\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2,(0)}\right) = \frac{T}{2}\mathbf{N}. \tag{10.15}$$

10.2. The eigenbundles associated to small eigenvalues

For $T \geqslant 0$, recall that $d_T^{X,*}$ is the adjoint of d^X with respect to the metrics g^{TX}, g_T^F. Let $\nabla_T^{\Omega^\bullet\left(X,F|_X\right),*}$ be the corresponding adjoint connection to $\nabla^{\Omega^\bullet\left(X,F|_X\right)}$, and let A_T'' be the adjoint superconnection on $\Omega^\bullet\left(X,F|_X\right)$ to A' with respect g^{TX}, g_T^F. By (3.12) and (3.30),

$$\begin{aligned} A' &= d^X + \nabla^{\Omega^\bullet\left(X,F|_X\right)} + i_{T^H}, \\ A_T'' &= d_T^{X,*} + \nabla_T^{\Omega^\bullet\left(X,F|_X\right),*} - T^H \wedge . \end{aligned} \tag{10.16}$$

As in (3.31), set

$$A_T = \frac{1}{2}\left(A_T'' + A'\right), \qquad B_T = \frac{1}{2}\left(A_T'' - A'\right). \tag{10.17}$$

Clearly,

$$A_T^2 = \frac{1}{4}\left[A_T'', A'\right]. \tag{10.18}$$

In the sequel, we will also use the notation,

$$\overline{A}_T = e^{-Tf} A_T e^{Tf}, \qquad \overline{B}_T = e^{-Tf} B_T e^{Tf}. \tag{10.19}$$

Also Sp is our notation for the spectrum. If $H \in \Lambda^\bullet(T^*S) \widehat{\otimes} \mathrm{End}\left(\Omega^\bullet\left(X,F|_X\right)\right)$, let $H^{(0)}$ be the component of H in $\mathrm{End}\left(\Omega^\bullet\left(X,F|_X\right)\right)$. Clearly,

$$\mathrm{Sp}\left(A_T^2\right) = \mathrm{Sp}\left(\overline{A_T}^2\right) = \mathrm{Sp}\left(A_T^{2,(0)}\right) = \mathrm{Sp}\left(\overline{A_T}^{2,(0)}\right). \tag{10.20}$$

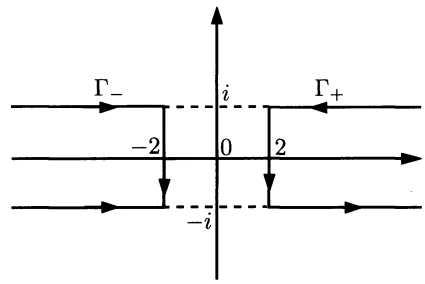

FIGURE 10.1

Definition 10.1. — If $s \in S$, let $F_{T,s}^{[0,1]}$ (resp. $\overline{F}_{T,s}^{[0,1]}$) be the direct sum of the eigenspaces of $A_T^{2,(0)}$ (resp. $\overline{A}_T^{2,(0)}$) associated to eigenvalues $\lambda \in [0,1]$, let $P_{T,s}^{[0,1]}$ (resp. $\overline{P}_{T,s}^{[0,1]}$) be the orthogonal projection operator from $\left(\Omega^\bullet\left(X_s, F|_{X_s}\right), g_T^{\Omega^\bullet(X_s,F|_{X_s})}\right)$ on $F_{T,s}^{[0,1]}$ (resp. from $\left(\Omega^\bullet\left(X_s, F|_{X_s}\right), g^{\Omega^\bullet(X_s,F|_{X_s})}\right)$ on $\overline{F}_{T,s}^{[0,1]}$).

Clearly, we have the obvious orthogonal splittings,

$$F_{T,s}^{[0,1]} = \bigoplus_{i=0}^{\dim X} F_{T,s}^{[0,1],i}, \qquad \overline{F}_{T,s}^{[0,1]} = \bigoplus_{i=0}^{\dim X} \overline{F}_{T,s}^{[0,1],i}. \tag{10.21}$$

Also,

$$\overline{P}_{T,s}^{[0,1]} = e^{-Tf} P_{T,s}^{[0,1]} e^{Tf}. \tag{10.22}$$

Let M^i be the number of elements in B^i. Equivalently, M^i is the number of critical points of f in a given fibre X whose index is equal to i.

Theorem 10.2. — *There exists $T_0 \geqslant 0$ such that for $T \geqslant T_0$,*

$$\begin{aligned} &\mathrm{Sp}\left(A_T^{2,(0)}\right) \subset [0, \frac{1}{4}] \cup [4, \infty[, \\ &\mathrm{rk}\left(F_{T,s}^i\right) = M^i,\ 1 \leqslant i \leqslant \dim X. \end{aligned} \tag{10.23}$$

Proof. — For a given $s \in S$, this result was established in [**BZ1**, Theorems 7.8 and 7.9]. Since S is compact, a trivial uniformity argument shows that we can find $T_0 \in \mathbf{R}_+$ such that (10.23) holds for any $s \in S, T \geqslant T_0$. □

By the above, it follows that the $F_{T,s}^{[0,1]}, \overline{F}_{T,s}^{[0,1]}$'s are the fibres of smooth $\mathbf{Z}$-graded vector bundles $F_T^{[0,1]}, \widetilde{F}_T^{[0,1]}$ on S, which are subbundles of $\Omega^\bullet\left(X, F|_X\right)$. Clearly,

$$h'(x) = \left(1 + 2x^2\right) e^{x^2}. \tag{10.24}$$

Put

$$r(x) = (1 - 2x)\, e^{-x}. \tag{10.25}$$

Let δ be the unit circle in $\mathbf{C}$. Let $\Gamma = \Gamma_+ \cup \Gamma_-$ be the contour indicated in Figure 10.1.

Definition 10.3. — For $t \in \mathbf{R}_+^*, T \geqslant T_0$, put

$$K_{t,T} = \psi_t^{-1} \frac{1}{2i\pi} \int_\Gamma \frac{r(t\lambda)}{\lambda - A_T^2} d\lambda \psi_t, \tag{10.26}$$

$$L_{t,T} = \frac{1}{2i\pi} \int_{\sqrt{t}\,\delta} \frac{h'(\lambda)}{\lambda - D_{t,T}} d\lambda.$$

Proposition 10.4. — *The following identity holds,*

$$h'(D_{t,T}) = K_{t,T} + L_{t,T}. \tag{10.27}$$

Proof. — By (10.24), (10.25),

$$h'(D_{t,T}) = r\left(C_{t,T}^2\right). \tag{10.28}$$

Using Proposition 3.17 and (10.20), (10.23), we get (10.27). □

By (10.27),

$$\mathrm{Tr_s}\left[Ngh'(D_{t,T})\right] = \mathrm{Tr_s}\left[NgK_{t,T}\right] + \mathrm{Tr_s}\left[NgL_{t,T}\right]. \tag{10.29}$$

Theorem 10.5. — *There exists $C > 0, c > 0, \delta \in]0,1]$ such that for $t \geqslant 1, T \geqslant T_0$,*

$$\left|\mathrm{Tr_s}\left[NgK_{t,T}\right]\right| \leqslant \frac{Ce^{-ct}}{T^\delta}. \tag{10.30}$$

Proof. — Set

$$M_{t,T,a} = \psi_t^{-1} \frac{1}{2i\pi} \int_\Gamma \frac{\exp(-ta\lambda)}{\lambda - A_T^2} d\lambda \psi_t. \tag{10.31}$$

Then by(10.25),

$$K_{t,T} = \left(1 + 2\frac{\partial}{\partial a}\right) M_{t,T,a}|_{a=1}. \tag{10.32}$$

Put

$$\overline{M}_{t,T,a} = e^{-Tf} M_{t,T,a} e^{Tf}. \tag{10.33}$$

By (10.19), (10.31),

$$\overline{M}_{t,T,a} = \psi_t^{-1} \frac{1}{2i\pi} \int_\Gamma \frac{\exp(-ta\lambda)}{\lambda - \overline{A}_T^2} d\lambda \psi_t. \tag{10.34}$$

By (10.33),

$$\mathrm{Tr_s}\left[NgM_{t,T,a}\right] = \mathrm{Tr_s}\left[Ng\overline{M}_{t,T,a}\right]. \tag{10.35}$$

Clearly if $d^{X,*}$ is the adjoint of d^X with respect to $g^{\Omega^\bullet\left(X,F|_X\right)}$, then

$$d_T^{X,*} = d^{X,*} + 2Ti_{\nabla f}. \tag{10.36}$$

Recall that $\nabla^{\Omega^\bullet(X,F|_X),*} = \nabla_0^{\Omega^\bullet(X,F|_X),*}$ is the connection adjoint to $\nabla^{\Omega^\bullet(X,F|_X)}$ with respect to $g^{\Omega^\bullet(X,F|_X)}$. Let $(df)^H$ be the horizontal component of df. Then

$$\nabla_T^{\Omega^\bullet(X,F|_X),*} = \nabla^{\Omega^\bullet(X,F|_X),*} - 2T\,(df)^H\,. \tag{10.37}$$

From (10.16), (10.17), (10.36), (10.37), we get

$$A_T = \frac{1}{2}\left(D^X + 2Ti_{\nabla f}\right) + \nabla^{\Omega^\bullet(X,F|_X),u} - T\,(df)^H - \frac{1}{2}c\left(T^H\right). \tag{10.38}$$

If $\nabla^{\Omega^\bullet(X,F|_X),u} = \nabla_0^{\Omega^\bullet(X,F|_X),u}$, by (10.19), 10.38), we obtain,

$$\overline{A}_T = \frac{1}{2}\left(D^X + T\widehat{c}(\nabla f)\right) + \nabla^{\Omega^\bullet(X,F|_X),u} - \frac{1}{2}c\left(T^H\right). \tag{10.39}$$

The essential point in (10.39) is that the term $T\,(df)^H$ has disappeared.

Now we claim that the proof of our Theorem is formally the same as the proof of [**B10**, Theorem 9.5], where a similar problem was considered in the holomorphic category. In fact observe that fibrewise, $\mathbf{B}$ is the zero set of ∇f. Also $\widehat{c}(\nabla f)$ anticommutes with the principal symbol $c(i\xi)$ of D^X. By the simplifying assumptions we made in Section 9.1, near $\mathbf{B}$,

$$\overline{A}_T = \overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}}}. \tag{10.40}$$

By (10.14), the kernel of $\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},(0)}$ can be identified with $F|_{\mathbf{B}} \otimes o^u|_{\mathbf{B}}$. Using (10.10) and the fact, we get the easy formula,

$$\overline{\mathfrak{p}}_T \overline{A}_T \overline{\mathfrak{p}}_T = \nabla^{F|_{\mathbf{B}},u}. \tag{10.41}$$

Observe that

$$\mathrm{Sp}\left(\nabla^{F|_{\mathbf{B}},u,2}\right) = \{0\}. \tag{10.42}$$

Therefore,

$$\frac{1}{2i\pi}\int_\Gamma \frac{\exp(-ta\lambda)}{\lambda - \nabla^{F|_{\mathbf{B}},u,2}} d\lambda = 0. \tag{10.43}$$

By (10.32), (10.35), (10.39), (10.40)-(10.43), we find that we can proceed as in [**B10**, Theorem 9.5] and get (10.30). The proof of our Theorem is completed. □

Remark 10.6. — By Theorem 10.5, we find that as $T \to +\infty$,

$$\int_1^{+\infty} \mathrm{Tr_s}\left[NgK_{t,T}\right]\frac{dt}{2t} \to 0. \tag{10.44}$$

Therefore, to establish Theorem 9.8, we only need to study the limit as $T \to +\infty$ of

$$\int_1^{+\infty} \left(\mathrm{Tr_s}\left[NgL_{t,T}\right] - \chi_g'(F)\right)\frac{dt}{2t}.$$

10.3. The grading of A_T^2

If $k \in \Lambda^\bullet(T^*S)\widehat{\otimes}\mathrm{End}\left(\Omega^\bullet\left(X, F|_X\right)\right)$, we can write k in the form,

$$k = \sum_{j=0}^{\dim S} k^{(j)},\ k^{(j)} \in \Lambda^j(T^*S)\,\widehat{\otimes}\,\mathrm{End}\left(\Omega^\bullet\left(X, F|_X\right)\right). \tag{10.45}$$

Observe that $\Lambda^\bullet(T^*S)\widehat{\otimes}\mathrm{End}\left(\Omega^\bullet\left(X, F|_X\right)\right)$ is a $\mathbf{Z}$-graded bundle of algebras. Namely, if k is taken as in (10.45), $\deg k = p$ if for any j,

$$k^{(j)} \in \Lambda^j(T^*S)\,\widehat{\otimes}\,\mathrm{Hom}\Big(\Omega^\bullet\left(X, F|_X\right), \Omega^{\bullet+p-j}\left(X, F|_X\right)\Big).$$

Moreover $\Lambda^\bullet(T^*S)\widehat{\otimes}\mathrm{End}\left(\Omega^\bullet\left(X, F|_X\right)\right)$ inherits a filtration F from the filtration of $\Lambda^\bullet(T^*S)$. We will say that $\deg(k) \geqslant 0$ if it is the sum of elements of non negative degree. Also we will write that

$$\deg(k) \leqslant 2F(k)$$

if for any $j, \deg\left(k^j\right) \leqslant 2j$.

Proposition 10.7. — *For $T \geqslant 0$, $A_T^2 \in \Lambda^\bullet(T^*S)\widehat{\otimes}\mathrm{End}\left(\Omega^\bullet\left(X, F|_X\right)\right)$ is such that*

$$0 \leqslant \deg\left(A_T^2\right) \leqslant 2F\left(A_T^2\right). \tag{10.46}$$

The only term in A_T^2 which makes the first (resp. the second) inequality in (10.46) be an equality is $\frac{1}{4}\left[A', d_T^{X,*}\right]$ (resp. $\frac{1}{4}\left[d^X, A_T''\right]$).

Proof. — Clearly $\deg(A') = 1$. Also in A_T'', $d_T^{X,*}$ is of degree -1, and $\nabla_T^{\Omega^\bullet\left(X,F|_X\right),*}$ and $T^H\wedge$ have positive degree. Therefore, $\deg\left(A_T^2\right) \geqslant 0$. Moreover the only term in A_T^2 with degree 0 is $\frac{1}{4}\left[A', d_T^{X,*}\right]$.

Clearly,

$$[A', A_T''] = \left[d^X, A_T''\right] + \left[\nabla^{\Omega^\bullet\left(X,F|_X\right)}, A_T''\right] + \left[i_{T^H}, A_T''\right]. \tag{10.47}$$

Now d^X is of total degree 1. The degree of $d_T^{X,*}$ is -1 and its F is 0, the degree of $\nabla_T^{\Omega^\bullet\left(X,F|_X\right),*}$ is 1 and its F is 1, the degree of $T^H\wedge$ is 3 and its F is 2. Therefore equality holds in (10.46) for $\left[d^X, A_T''\right]$. Also $\left[\nabla^{\Omega^\bullet\left(X,F|_X\right)}, d_T^{X,*}\right]$ has degree 0, and its F is 1, $\left[\nabla^{\Omega^\bullet\left(X,F|_X\right)}, \nabla_T^{\Omega^\bullet\left(X,F|_X\right),*}\right]$ has degree 2 and its F is 2, $\left[\nabla^{\Omega^\bullet\left(X,F|_X\right)}, T^H\wedge\right]$ has degree 4 and its F is 3. Finally $\left[i_T, \nabla_T^{\Omega^\bullet\left(X,F|_X\right),*}\right]$ has degree 2 and its F is 3, $\left[i_{T^H}, d^{X,*}\right]$ has degree 0 and its F is 2, and $\left[i_{T^H}, T^H\wedge\right]$ has degree 4 and its F is 4. This completes the proof of our Proposition. □

10.4. The projectors $\mathbf{P}_T^{[0,1]}$

Observe that $\Lambda^\bullet(T^*S)$ acts on $\Lambda^\bullet(T^*S)\widehat{\otimes}\Omega^\bullet\big(X, F|_X\big)$. Let $\delta \subset \mathbf{C}$ be the circle of centre 0 and radius 1/4. Recall that by (10.20) and by Theorem 10.2, for $T \geqslant T_0$,

$$\mathrm{Sp}\left(A_T^2\right) \cap \delta = \varnothing. \tag{10.48}$$

Definition 10.8. — For $T \geqslant T_0$, put

$$\mathbf{P}_T^{[0,1]} = \frac{1}{2i\pi}\int_\delta \frac{d\lambda}{\lambda - A_T^2}. \tag{10.49}$$

Clearly, $\mathbf{P}_T^{[0,1]} \in \Lambda^\bullet(T^*S)\widehat{\otimes}\mathrm{End}\left(\Omega^\bullet\big(X, F|_X\big)\right)$. Then we write,

$$\mathbf{P}_T^{[0,1]} = \sum_{j=1}^{\dim S} \mathbf{P}_T^{[0,1],(j)}, \tag{10.50}$$

with $\mathbf{P}_T^{[0,1],(j)} \in \Lambda^j(T^*S)\widehat{\otimes}\mathrm{End}\left(\Omega^\bullet\big(X, F|_X\big)\right)$.

In the sequel, the operator $*$ acts on $\Lambda^\bullet(T^*S)\widehat{\otimes}\mathrm{End}\left(\Omega^\bullet\big(X, F|_X\big)\right)$ as in (1.8), with respect to the metric $g_T^{\Omega^\bullet\left(X, F|_X\right)}$. We will often say that if k is such that $k^* = k$, then it is self-adjoint.

Theorem 10.9. — *For $T \geqslant T_0$, $\mathbf{P}_T^{[0,1]}$ is an even projection operator acting on $\Lambda^\bullet(T^*S)\widehat{\otimes}\Omega^\bullet\big(X, F|_X\big)$, with finite dimensional range, which commutes with the action of $\Lambda^\bullet(T^*S)$ and with A', A_T'', and is such that*

$$\mathbf{P}_T^{[0,1],(0)} = P_T^{[0,1]}. \tag{10.51}$$

Also,

$$\mathbf{P}_T^{[0,1],*} = \mathbf{P}_T^{[0,1]}, \qquad 0 \leqslant \deg\left(\mathbf{P}_T^{[0,1]}\right) \leqslant 2F\left(\mathbf{P}_T^{[0,1]}\right). \tag{10.52}$$

If

$$\mathbf{F}_T^{[0,1]} = \mathrm{Im}\left(\mathbf{P}_T^{[0,1]}\right), \tag{10.53}$$

then $\mathbf{F}_T^{[0,1]}$ is a $\mathbf{Z}_2$-graded filtered vector subbundle of $\Lambda^\bullet(T^*S)\widehat{\otimes}\Omega^\bullet\big(X, F|_X\big)$, on which $\Lambda^\bullet(T^*S), A', A_T''$ act. The linear map $\alpha \in \Lambda^\bullet(T^*S)\widehat{\otimes}F_T^{[0,1]} \to \mathbf{P}_T^{[0,1]}\alpha \in \mathbf{F}_T^{[0,1]}$ is an isomorphism of $\mathbf{Z}_2$-graded filtered vector bundles.

For any $k' \in \mathbf{N}^*$, we have the identity,

$$\mathbf{P}_T^{[0,1]} = \frac{1}{2i\pi}\int_\delta \frac{d\lambda}{\lambda - B_T^{k'}}. \tag{10.54}$$

Proof. — For $\lambda \in \delta$, $\left(\lambda - A_T^2\right)^{-1}$ is a compact even operator. Therefore $\mathbf{P}_T^{[0,1]}$ is a compact even operator. Also by [**ReSi**, Theorem XII.5], $\mathbf{P}_T^{[0,1]}$ is a projection operator. Therefore the range $\mathbf{F}_T^{[0,1]}$ of $\mathbf{P}_T^{[0,1]}$ is finite dimensional. Moreover $\mathbf{P}_T^{[0,1]}$ commutes

with A_T^2. Since A', A_T'' and the elements of $\Lambda^\bullet(T^*S)$ commute with A_T^2, they also commute with $\mathbf{P}_T^{[0,1]}$. We write A_T^2 in the form,

$$A_T^2 = A_T^{2,(0)} + A_T^{2,(>0)}. \tag{10.55}$$

Then if $\lambda \in \delta$,

$$\left(\lambda - A_T^2\right)^{-1} = \sum_{i=0}^{\dim X} \left(\lambda - A_T^{2,(0)}\right)^{-1} A_T^{2,(>0)} \cdots A_T^{2,(>0)} \left(\lambda - A_T^{2,(0)}\right)^{-1}, \tag{10.56}$$

so that $A_T^{2,(>0)}$ appears i times in the right-hand side of (10.56). The term corresponding to $i = 0$ is obviously equal to the projection operator $P_T^{[0,1]}$, i.e. (10.51) holds. Also since $A_T^{2,*} = A_T^2$, the first identity in (10.52) also holds. Also $A_T^{2,(0)}$ is of degree 0. Using Proposition 10.7, (10.49) and (10.56), we get the second identity in (10.52).

Since the projector $\mathbf{P}_T^{[0,1]}$ depends smoothly on the parameter $s \in S$, its range $\mathbf{F}_T^{[0,1]}$ is a finite dimensional vector subbundle of $\Lambda^\bullet(T^*S) \widehat{\otimes} \Omega^\bullet\left(X, F|_X\right)$. Since $\mathbf{P}_T^{[0,1]}$ is even, $\mathbf{F}_T^{[0,1]}$ is naturally $\mathbf{Z}_2$-graded. Also $\Lambda^\bullet(T^*S), A', A_T''$ act naturally on $\mathbf{F}_T^{[0,1]}$. Moreover $\mathbf{F}_T^{[0,1]}$ inherits a filtration from the filtration of $\Lambda^\bullet(T^*S) \widehat{\otimes} \Omega^\bullet\left(X, F|_X\right)$. If $\beta \in \mathbf{F}_T^{[0,1],\geqslant j}$, then

$$\beta = \beta^{(j)} + \beta^{(j+1)} + \cdots \tag{10.57}$$

Since $\mathbf{P}_T^{[0,1]}\beta = \beta$, using (10.51), we get

$$P_T^{[0,1]}\beta^{(j)} = \beta^{(j)}, \tag{10.58}$$

so that $\beta^{(j)} \in \Lambda^j(T^*S) \widehat{\otimes} F_T^{[0,1]}$. This way, we have defined an injective linear map $\mathrm{Gr}^j\left(\mathbf{F}_T^{[0,1]}\right) \to \Lambda^j(T^*S) \widehat{\otimes} F_T^{[0,1]}$. An obvious inverse for this map is just $\alpha \in \Lambda^j(T^*S) \widehat{\otimes} F_T^{[0,1]} \to \left[\mathbf{P}_T^{[0,1]}\alpha\right] \in \mathrm{Gr}^j\left(\mathbf{F}_T^{[0,1]}\right)$. Therefore, we have established that $\alpha \in \Lambda^\bullet(T^*S) \widehat{\otimes} F_T^{[0,1]} \to \mathbf{P}_T^{[0,1]}\alpha \in \mathbf{F}_T^{[0,1]}$ is an isomorphism of $\mathbf{Z}_2$-graded filtered vector bundles.

Recall that $A_T^2 = -B_T^2$. Using the holomorphic functional calculus [**ReSi**], one shows easily that (10.54) holds. The proof of our Theorem is completed. □

10.5. The projectors $\mathbf{P}_{t,T}^{[0,1]}$

Since $TM = T^H M \oplus TX$, we have a smooth identification $TM \simeq \pi^* TS \oplus TX$. Therefore we have the identification,

$$\Lambda(T^*M) \simeq \pi^*\Lambda^\bullet(T^*S) \widehat{\otimes} \Lambda(T^*X). \tag{10.59}$$

Let $N^{\Lambda^\bullet(T^*M)}$ be the operator defining the $\mathbf{Z}$-grading of $\Lambda(T^*M)$. Using (10.59), we find that $N^{\Lambda^\bullet(T^*M)}$ acts naturally on $\Lambda^\bullet(T^*S) \widehat{\otimes} \Omega^\bullet\left(X, F|_X\right)$.

For $t > 0, T \geqslant 0$, let $A''_{t,T}$ be the adjoint superconnection to A' with respect to the metrics $g^{TX}/t, g^F$.

Proposition 10.10. — *The following identities hold,*

(10.60)

$$t^{-N^{\Lambda^\bullet(T^*M)}/2} A' t^{N^{\Lambda^\bullet(T^*M)}/2} = \frac{1}{\sqrt{t}} A', \quad t^{-N^{\Lambda^\bullet(T^*M)}/2} A''_T t^{N^{\Lambda^\bullet(T^*M)}/2} = \frac{1}{\sqrt{t}} A''_{t,T}.$$

Proof. — Since A' increases the degree in $\Lambda(T^*M)$ by 1, the first identity in (10.60) is trivial. Also by (10.16),

$$t^{-N^{\Lambda^\bullet(T^*M)}/2} A''_T t^{N^{\Lambda^\bullet(T^*M)}/2} = \sqrt{t}\, d_T^{X,*} + \frac{1}{\sqrt{t}} \nabla_T^{\Omega^\bullet\left(X, F|_X\right),*} - \frac{1}{t^{3/2}} T^H \wedge . \tag{10.61}$$

Comparing with (3.49), we get the second identity in (10.60). □

By (10.23), we get

$$\mathrm{Sp}\left(A_{t,T}^2\right) \subset [0, \frac{t}{4}] \cap [2t, +\infty[. \tag{10.62}$$

Definition 10.11. — For $T \geqslant T_0, t > 0$, put

$$\mathbf{P}_{t,T}^{[0,1]} = \frac{1}{2i\pi} \int_{t\delta} \frac{d\lambda}{\lambda - A_{t,T}^2}. \tag{10.63}$$

An analogue of Theorem 10.9 holds for $\mathbf{P}_{t,T}^{[0,1]}$, so that $\mathbf{P}_{t,T}^{[0,1]}$ is an even projector on a finite dimensional $\mathbf{Z}_2$-graded filtered subbundle $\mathbf{F}_{t,T}^{[0,1]}$ of $\Lambda^\bullet(T^*S) \widehat{\otimes} \Omega^\bullet\left(X, F|_X\right)$.

Proposition 10.12. — *The following identity holds,*

$$\begin{aligned} \mathbf{P}_{t,T}^{[0,1]} &= t^{-N^{\Lambda^\bullet(T^*M)}/2} \mathbf{P}_T^{[0,1]} t^{N^{\Lambda^\bullet(T^*M)}/2}, \\ \mathbf{F}_{t,T}^{[0,1]} &= t^{-N^{\Lambda^\bullet(T^*M)}/2} \mathbf{F}_T^{[0,1]}. \end{aligned} \tag{10.64}$$

Proof. — This is a consequence of Proposition 10.10. □

Definition 10.13. — For $T \geqslant T_0$, put

$$\mathbf{P}_{\infty,T}^{[0,1]} = \frac{1}{2i\pi} \int_\delta \frac{d\lambda}{\lambda - \frac{1}{4}\left[A', d_T^{X,*}\right]}. \tag{10.65}$$

Still, the obvious analogue of Theorem 10.9 holds for $\mathbf{P}_{\infty,T}^{[0,1]}$. In particular $\mathbf{P}_{\infty,T}^{[0,1]}$ is a projector on a finite dimensional vector bundle $\mathbf{F}_{\infty,T}^{[0,1]}$. Also A' and $d_T^{X,*}$ act on $\mathbf{F}_{\infty,T}^{[0,1]}$.

Proposition 10.14. — *Given $\alpha \in \Lambda^\bullet(T^*S) \widehat{\otimes} \Omega^\bullet\left(X, F|_X\right)$, as $t \to +\infty$,*

$$\mathbf{P}_{t,T}^{[0,1]} \alpha = \mathbf{P}_{\infty,T}^{[0,1]} \alpha + \mathcal{O}(1/t)\, \alpha. \tag{10.66}$$

Proof. — Set

$$R_T = \frac{1}{4}\left[A', \nabla_T^{\Omega^\bullet\left(X, F|_X\right),*} - T^H \wedge\right]. \tag{10.67}$$

Then if $T \geqslant T_0, \lambda \in \delta$,

$$\begin{aligned}\left(\lambda - A_T^2\right)^{-1} = &\left(\lambda - \frac{1}{4}\left[A', d_T^{X,*}\right]\right)^{-1} \\ &+ \left(\lambda - \frac{1}{4}\left[A', d_T^{X,*}\right]\right)^{-1} R_T \left(\lambda - \frac{1}{4}\left[A', d_T^{X,*}\right]\right)^{-1} \ldots\end{aligned} \tag{10.68}$$

and the expansion (10.68) only contains a finite number of terms. Also $\frac{1}{4}\left[A', d_T^{X,*}\right]$ is of degree 0, while R_T is of positive degree. Therefore as $t \to +\infty$,

$$t^{-N^{\Lambda^\bullet(T^*M)}/2}\left(\lambda - A_T^2\right)^{-1} t^{N^{\Lambda^\bullet(T^*M)}/2} = \left(\lambda - \frac{1}{4}\left[A', d_T^{X,*}\right]\right)^{-1} + \mathcal{O}\left(1/t\right). \tag{10.69}$$

Using (10.49), (10.64) and (10.69), we get (10.66). □

For $t > 0, T \geqslant 0$, let $C_{t,T}$ be defined as in Section 8.1, so that by (3.53),

$$C_{t,T} = t^{N/2} A_{t,T} t^{-N/2}. \tag{10.70}$$

Definition 10.15. — Put

$$\widehat{\mathbf{P}}_{t,T}^{[0,1]} = \frac{1}{2i\pi}\int_{t\delta} \frac{d\lambda}{\lambda - C_{t,T}^2}. \tag{10.71}$$

Again, $\widehat{\mathbf{P}}_{t,T}^{[0,1]}$ is a projection operator on a finite dimensional vector bundle $\widehat{\mathbf{F}}_{t,T}^{[0,1]}$.

Proposition 10.16. — *The following identities hold,*

$$\begin{aligned}\widehat{\mathbf{P}}_{t,T}^{[0,1]} &= \psi_t^{-1}\mathbf{P}_T^{[0,1]}\psi_t, \\ \widehat{\mathbf{P}}_{t,T}^{[0,1]} &= t^{N/2}\mathbf{P}_{t,T}^{[0,1]}t^{-N/2}, \\ \widehat{\mathbf{F}}_{t,T}^{[0,1]} &= \psi_t^{-1}\mathbf{F}_T^{[0,1]} = t^{N/2}\mathbf{F}_{t,T}^{[0,1]}.\end{aligned} \tag{10.72}$$

Also, as $t \to +\infty$,

$$\widehat{\mathbf{P}}_{t,T}^{[0,1]} = P_T^{[0,1]} s + \mathcal{O}\left(1/\sqrt{t}\right). \tag{10.73}$$

Proof. — By (3.54),

$$C_{t,T} = \psi_t^{-1}\sqrt{t}\, A_T \psi_t. \tag{10.74}$$

From (10.49), (10.64) and (10.74), we get (10.72). By (10.51) and (10.72), we get (10.73). □

10.6. The maps $\mathbf{P}_T^\infty$

Recall that $P^\infty : \Omega^\bullet(X, F|_X) \to C^\bullet(W^u, F)$ was defined in Definition 5.2, and that $\mathbf{P}^\infty : \Omega^\bullet(M, F) \to \Lambda^\bullet(T^*S) \widehat{\otimes} C^\bullet(W^u, F)$ was defined in Definition 5.7. Then P^∞ and $\mathbf{P}^\infty$ are chain maps which preserve the $\mathbf{Z}$-grading. Also $\mathbf{P}^\infty$ preserves the filtrations associated to $\Lambda^\bullet(T^*S)$.

Definition 10.17. — For $T \geqslant T_0$, let $\mathbf{P}_T^\infty : \mathbf{F}_T^{[0,1]} \to \Lambda^\bullet(T^*S) \widehat{\otimes} C^\bullet(W^u, F), P_T^\infty : F_T^{[0,1]} \to C^\bullet(W^u, F)$ be the restrictions of $\mathbf{P}^\infty, P^\infty$ to $\mathbf{F}_T^{[0,1]}, F_T^{[0,1]}$.

Observe that, by (10.51), $\mathbf{P}_T^{[0,1]} - P_T^{[0,1]}$ contains only terms of positive degree in the Grassmann variables in $\Lambda^\bullet(T^*S)$.

Theorem 10.18. — *There exists $T_0' \geqslant T_0$ such that for $T \geqslant T_0'$, $\mathbf{P}_T^\infty$ is an isomorphism of $\mathbf{Z}_2$-graded filtered vector bundles, which commutes with the action of $\Lambda^\bullet(T^*S)$. Moreover,*

$$\mathbf{P}_T^\infty A' = A^{C^\bullet(W^u,F)'} \mathbf{P}_T^\infty. \tag{10.75}$$

Also,

$$0 \leqslant \deg\left(\mathbf{P}_T^\infty \mathbf{P}_T^{[0,1]}\right) \leqslant 2F\left(\mathbf{P}_T^\infty \mathbf{P}_T^{[0,1]}\right). \tag{10.76}$$

The map $\mathbf{P}^\infty \mathbf{P}_T^{[0,1]} (P_T^\infty)^{-1} : \Lambda^\bullet(T^*S) \widehat{\otimes} C^\bullet(W^u, F) \to \Lambda^\bullet(T^*S) \widehat{\otimes} C^\bullet(W^u, F)$ is one to one, increases the total degree, and moreover,

$$\left(\mathbf{P}^\infty \mathbf{P}_T^{[0,1]} (P_T^\infty)^{-1}\right)^{(0)} = 1. \tag{10.77}$$

Also,

$$\begin{aligned} (\mathbf{P}_T^\infty)^{-1} &= \mathbf{P}_T^{[0,1]} (P_T^\infty)^{-1} \left(\mathbf{P}^\infty \mathbf{P}_T^{[0,1]} (P_T^\infty)^{-1}\right)^{-1}, \\ (\mathbf{P}_T^\infty)^{-1,(0)} &= (P_T^\infty)^{-1}. \end{aligned} \tag{10.78}$$

Finally

$$0 \leqslant \deg(\mathbf{P}_T^\infty)^{-1} \leqslant 2F(\mathbf{P}_T^\infty)^{-1}. \tag{10.79}$$

Proof. — Clearly $\mathbf{P}_T^\infty$ is a homomorphism of $\mathbf{Z}_2$-graded filtered vector bundles, which commutes with the action of $\Lambda^\bullet(T^*S)$. So it maps $\mathrm{Gr}^i\left(\mathbf{F}_T^{[0,1]}\right)$ into

$$\mathrm{Gr}^i\left(\Lambda^\bullet(T^*S) \widehat{\otimes} C^\bullet(W^u, F)\right) \simeq \Lambda^i(T^*S) \widehat{\otimes} C^\bullet(W^u, F).$$

To prove that $\mathbf{P}_T^\infty$ is an isomorphism, we only need to show that it induces an isomorphism of the corresponding Gr-bundles.

By Theorem 10.9, the map $\alpha \in \Lambda(T^*S) \widehat{\otimes} F_T^{[0,1]} \to \mathbf{P}_T^{[0,1]} \alpha \in \mathbf{F}_T^{[0,1]}$ induces the isomorphism $\Lambda^\bullet(T^*S) \widehat{\otimes} F_T^{[0,1]} \simeq \mathrm{Gr}^\bullet\left(\mathbf{F}_T^{[0,1]}\right)$. If $\alpha \in \Lambda^j(T^*S) \widehat{\otimes} F_T^{[0,1]}$,

$$\mathbf{P}_T^{[0,1]} \alpha = \alpha + \beta, \ \beta \in \mathbf{F}_T^{[0,1],\geqslant j+1}. \tag{10.80}$$

From (10.80), we get

$$\mathbf{P}_T^{\infty}\mathbf{P}_T^{[0,1]}\alpha = P^{\infty}\alpha + P^{\infty}\beta,\ P^{\infty}\beta \in \left(\Lambda^{\bullet}(T^*S)\widehat{\otimes}C^{\bullet}(W^u,F)\right)^{\geqslant j+1}. \tag{10.81}$$

Therefore the canonical identification

$$\mathrm{Gr}^j\left(\mathbf{F}_T^{[0,1]}\right) \simeq \Lambda^j(T^*S)\widehat{\otimes}F_T^{[0,1]} \to \mathrm{Gr}^j\left(\Lambda^{\bullet}(T^*S)\widehat{\otimes}C^{\bullet}(W^u,F)\right) \simeq \Lambda^j(T^*S)\widehat{\otimes}C^{\bullet}(W^u,F)$$

is given by,

$$\alpha \in \Lambda^j(T^*S)\widehat{\otimes}F_T^{[0,1]} \to P^{\infty}\alpha \in \Lambda^j(T^*S)\widehat{\otimes}C^{\bullet}(W^u,F). \tag{10.82}$$

Now, by [**BZ2**, Theorems 6.9 and 6.10], for $T \geqslant T_0$ large enough, the map $\alpha \in F_T^{[0,1]} \to P_T^{\infty}\alpha \in C^{\bullet}(W^u,F)$ is an isomorphism of vector bundles. So we have established that for $T \geqslant T_0$ large enough, $\mathbf{P}_T^{\infty}$ is an isomorphism.

By Theorem 5.8, we get (10.75). Also,

$$\mathbf{P}_T^{\infty}\mathbf{P}_T^{[0,1]} = \mathbf{P}^{\infty}\mathbf{P}_T^{[0,1]}. \tag{10.83}$$

Since P^{∞} is a map of filtered complexes which preserves the degree, from Theorem 10.9, we also get (10.76).

Using (10.51), we get (10.77). By (10.77), $\mathbf{P}^{\infty}\mathbf{P}_T^{[0,1]}(P_T^{\infty})^{-1}$ is one to one. Recall that $(P_T^{\infty})^{-1}$ and $\mathbf{P}^{\infty}$ preserve the total degree. By Theorem 10.9, $\mathbf{P}_T^{\infty}$ increases the total degree. Therefore $\mathbf{P}^{\infty}\mathbf{P}_T^{[0,1]}(P_T^{\infty})^{-1}$ increases the total degree. Using the invertibility of $\mathbf{P}^{\infty}\mathbf{P}_T^{[0,1]}(P_T^{\infty})^{-1}$, the first equation in (10.78) follows. From (10.51) and (10.77), we get the second equation in (10.78).

Using (10.52), we get

$$0 \leqslant \deg\left(\mathbf{P}^{\infty}\mathbf{P}_T^{[0,1]}(P_T^{\infty})^{-1}\right) \leqslant 2F\left(\mathbf{P}^{\infty}\mathbf{P}_T^{[0,1]}(P_T^{\infty})^{-1}\right). \tag{10.84}$$

Using (10.52), (10.77) and (10.84), we get (10.79). The proof of our Theorem is completed. □

Remark 10.19. — Equation (10.75) can be rewritten as,

$$\mathbf{P}_T^{\infty}A'(\mathbf{P}_T^{\infty})^{-1} = A^{C^{\bullet}(W^u,F)'}. \tag{10.85}$$

By (10.85), we have identified the flat $\mathbf{Z}_2$-graded filtered vector bundles $\mathbf{F}_T^{[0,1]}$ and $C^{\bullet}(W^u,F)$ and the corresponding flat superconnections A' and $A^{C^{\bullet}(W^u,F)'}$. In the next section, we will consider the case of A''_T.

Also by proceeding as in the proof of Theorem 10.18, we see that, if $\mathbf{P}_{\infty,T}^{\infty} : \mathbf{F}_{\infty,T}^{[0,1]} \to \Lambda^{\bullet}(T^*S)\widehat{\otimes}C^{\bullet}(W^u,F)$ is the restriction of $\mathbf{P}^{\infty}$ to $\mathbf{F}_{\infty,T}^{[0,1]}$, then $\mathbf{P}_{\infty,T}^{\infty}$ is an isomorphism, and the obvious analogue of the first equation in (10.78) holds, i.e.

$$\left[\mathbf{P}_{\infty,T}^{\infty}\right]^{-1} = \mathbf{P}_{\infty,T}^{[0,1]}(P_T^{\infty})^{-1}\left(\mathbf{P}^{\infty}\mathbf{P}_{\infty,T}^{[0,1]}(P_T^{\infty})^{-1}\right)^{-1}. \tag{10.86}$$

10.7. The generalized metric $\mathbf{g}_T^{C^\bullet(W^u,F)}$

As we saw in Theorem 10.18, the map $(\mathbf{P}_T^\infty)^{-1}$ identifies $\Lambda^\bullet(T^*S)\widehat{\otimes}C^\bullet(W^u,F)$ and $\mathbf{F}_T^{[0,1]} \subset \Lambda^\bullet(T^*S)\widehat{\otimes}\Omega^\bullet\big(X,F|_X\big)$. In the sequel, we will consider $(\mathbf{P}_T^\infty)^{-1}$ as a map from $\Lambda^\bullet(T^*S)\widehat{\otimes}C^\bullet(W^u,F)$ into $\Lambda^\bullet(T^*S)\widehat{\otimes}\Omega^\bullet\big(X,F|_X\big)$.

Let $(\mathbf{P}_T^\infty)^{-1,*}$ be the adjoint of $(\mathbf{P}_T^\infty)^{-1}$ with respect to the metrics $g^{C^\bullet(W^u,F)}$, $g_T^{\Omega^\bullet\left(X,F|_X\right)}$. Then $(\mathbf{P}_T^\infty)^{-1,*}$ maps $\Lambda^\bullet(T^*S)\widehat{\otimes}\Omega^\bullet\big(X,F|_X\big)$ into $\Lambda^\bullet(T^*S)\widehat{\otimes}C^\bullet(W^u,F)$.

Definition 10.20. — For $T \geqslant T_0'$, put

$$\mathbf{g}_T^{C^\bullet(W^u,F)} = (\mathbf{P}_T^\infty)^{-1,*}(\mathbf{P}_T^\infty)^{-1}, \qquad g_T^{C^\bullet(W^u,F)} = (P_T^\infty)^{-1,*}(P_T^\infty)^{-1}. \tag{10.87}$$

Observe that $\mathbf{g}_T^{C^\bullet(W^u,F)}$ is a generalized metric on $C^\bullet(W^u,F)$ in the sense of Section 2.9. Also $g_T^{C^\bullet(W^u,F)}$ is a standard metric on $C^\bullet(W^u,F)$, which is such that the $C^i(W^u,F)$'s are orthogonal in $C^\bullet(W^u,F)$ with respect to $g_T^{C^\bullet(W^u,F)}$.

Theorem 10.21. — *For $T \geqslant T_0'$,*

$$\mathbf{g}_T^{C^\bullet(W^u,F),(0)} = g_T^{C^\bullet(W^u,F)}. \tag{10.88}$$

Moreover,

$$\begin{aligned} &\mathbf{P}_T^\infty A'(\mathbf{P}_T^\infty)^{-1} = A^{C^\bullet(W^u,F)\prime}, \\ &\mathbf{P}_T^\infty A_T''(\mathbf{P}_T^\infty)^{-1} = \left(\mathbf{g}_T^{C^\bullet(W^u,F)}\right)^{-1} A^{C^\bullet(W^u,F)\prime\prime}\mathbf{g}_T^{C^\bullet(W^u,F)}. \end{aligned} \tag{10.89}$$

Proof. — Equation (10.88) follows from (10.78). The first identity in (10.89) was already established in (10.75). Clearly,

$$\mathbf{P}_T^{[0,1]}A'\mathbf{P}_T^{[0,1]} = \mathbf{P}_T^{[0,1]}(\mathbf{P}_T^\infty)^{-1}A^{C^\bullet(W^u,F)\prime}\mathbf{P}^\infty\mathbf{P}_T^{[0,1]}. \tag{10.90}$$

Let $\mathbf{P}^{\infty,*}$ be the adjoint of $\mathbf{P}^\infty$. Using (10.52) and taking adjoints in (10.90), we get

$$\mathbf{P}_T^{[0,1]}A_T''\mathbf{P}_T^{[0,1]} = \mathbf{P}_T^{[0,1]}\mathbf{P}^{\infty,*}A^{C^\bullet(W^u,F)\prime\prime}(\mathbf{P}_T^\infty)^{-1,*}\mathbf{P}_T^{[0,1]}. \tag{10.91}$$

Also,

$$\mathbf{P}^\infty A_T''(\mathbf{P}_T^\infty)^{-1} = \mathbf{P}_T^\infty\mathbf{P}_T^{[0,1]}A_T''\mathbf{P}_T^{[0,1]}(\mathbf{P}_T^\infty)^{-1}. \tag{10.92}$$

From (10.91), (10.92), we obtain,

$$\mathbf{P}_T^\infty A_T''(\mathbf{P}_T^\infty)^{-1} = \mathbf{P}_T^\infty\mathbf{P}_T^{[0,1]}\mathbf{P}^{\infty,*}A^{C^\bullet(W^u,F)\prime\prime}(\mathbf{P}_T^\infty)^{-1,*}\mathbf{P}_T^{[0,1]}(\mathbf{P}_T^\infty)^{-1}. \tag{10.93}$$

Since

$$\mathbf{P}_T^{[0,1]}(\mathbf{P}_T^\infty)^{-1} = (\mathbf{P}_T^\infty)^{-1}, \qquad \mathbf{P}^\infty(\mathbf{P}_T^\infty)^{-1} = 1, \tag{10.94}$$

we get

$$(\mathbf{P}_T^\infty)^{-1,*}\mathbf{P}_T^{[0,1]} = (\mathbf{P}_T^\infty)^{-1,*}, \qquad (\mathbf{P}_T^\infty)^{-1,*}\mathbf{P}^{\infty,*} = 1. \tag{10.95}$$

Therefore,

$$(\mathbf{P}_T^{\infty})^{-1,*}(\mathbf{P}_T^{\infty})^{-1}\mathbf{P}_T^{\infty}\mathbf{P}_T^{[0,1]}\mathbf{P}^{\infty,*} = (\mathbf{P}_T^{\infty})^{-1,*}\mathbf{P}_T^{[0,1]}\mathbf{P}^{\infty,*} = (\mathbf{P}_T^{\infty})^{-1,*}\mathbf{P}^{\infty,*} = 1. \tag{10.96}$$

By (10.96), we deduce that

$$\mathbf{P}_T^{\infty}\mathbf{P}_T^{[0,1]}\mathbf{P}^{\infty,*} = \left[(\mathbf{P}_T^{\infty})^{-1,*}(\mathbf{P}_T^{\infty})^{-1}\right]^{-1}. \tag{10.97}$$

From (10.93), (10.94), (10.97), we get the second identity in (10.89). The proof of our Theorem is completed. □

Remark 10.22. — Equation (10.89) shows that $\mathbf{P}_T^{\infty}A_T''(\mathbf{P}_T^{\infty})^{-1}$ is exactly the adjoint superconnection to A' with respect to the generalized metric $\mathbf{g}_T^{C^\bullet(W^u,F)}$, in the sense of Definition 10.1.

10.8. The maps $\mathbf{P}_{t,T}^{\infty}$ and the generalized metrics $\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}$

Let $N^{\Lambda^\bullet(T^*S)\widehat{\otimes}C^\bullet(W^u,F)}$ be the total number operator of $\Lambda^\bullet(T^*S)\,\widehat{\otimes}\,C^\bullet(W^u,F)$.

Definition 10.23. — Given $T \geqslant T_0', t > 0$, let $\mathbf{P}_{t,T}^{\infty} : \mathbf{F}_{t,T}^{[0,1]} \to \Lambda^\bullet(T^*S)\,\widehat{\otimes}\,C^\bullet(W^u,F)$ be the restriction of $\mathbf{P}^{\infty}$ to $\mathbf{F}_{t,T}^{[0,1]}$.

Recall that by Remark 10.19, $\mathbf{P}_{\infty,T}^{\infty} : \mathbf{F}_{\infty,T}^{[0,1]} \to \Lambda^\bullet(T^*S)\,\widehat{\otimes}\,C^\bullet(W^u,F)$ is an isomorphism.

Proposition 10.24. — *The map $\mathbf{P}_{t,T}^{\infty}$ is invertible. Moreover,*

$$\left(\mathbf{P}_{t,T}^{\infty}\right)^{-1} = t^{-N^{\Lambda^\bullet(T^*M)}/2}\left(\mathbf{P}_T^{\infty}\right)^{-1}t^{N^{\Lambda^\bullet(T^*S)\widehat{\otimes}C^\bullet(W^u,F)}/2}. \tag{10.98}$$

As $t \to +\infty$,

$$\left(\mathbf{P}_{t,T}^{\infty}\right)^{-1} = \left(\mathbf{P}_{\infty,T}^{\infty}\right)^{-1} + \mathcal{O}(1/t). \tag{10.99}$$

Proof. — If $f \in C^\bullet(W^u,F)$, by Proposition 10.12, there is $f' \in \mathbf{F}_T^{[0,1]}$ such that

$$\left(\mathbf{P}_{t,T}^{\infty}\right)^{-1} f = t^{-N^{\Lambda^\bullet(T^*M)}/2} f'. \tag{10.100}$$

Then

$$\mathbf{P}^{\infty}t^{-N^{\Lambda^\bullet(T^*M)}/2}f' = f, \tag{10.101}$$

so that since $\mathbf{P}^{\infty}$ preserves the total degree,

$$\mathbf{P}_T^{\infty}f' = t^{N^{C^\bullet(W^u,F)}/2}f. \tag{10.102}$$

From (10.102), we get

$$f' = (\mathbf{P}_T^{\infty})^{-1}t^{N^{C^\bullet(W^u,F)}/2}f, \tag{10.103}$$

so that (10.98) holds.

Also, by the obvious analogue of the first equation in (10.78),

$$(\mathbf{P}^{\infty}_{t,T})^{-1} = \mathbf{P}^{[0,1]}_{t,T}\,(P^{\infty}_{T})^{-1}\left(\mathbf{P}^{\infty}\mathbf{P}^{[0,1]}_{t,T}\,(P^{\infty}_{T})^{-1}\right)^{-1}. \tag{10.104}$$

Using (10.66), the analogue of (10.77) and (10.104), we find that as $t \to +\infty$,

$$(\mathbf{P}^{\infty}_{t,T})^{-1} = \mathbf{P}^{[0,1]}_{\infty,T}\,(P^{\infty}_{T})^{-1}\left(\mathbf{P}^{\infty}\mathbf{P}^{[0,1]}_{\infty,T}\,(P^{\infty}_{T})^{-1}\right)^{-1} + \mathcal{O}(1/t), \tag{10.105}$$

which, by (10.86), is just (10.99). The proof of our Proposition is completed. □

Definition 10.25. — Put

$$\mathbf{Q}^{[0,1]}_{\infty,T} = \frac{1}{2i\pi}\int_{\delta}\frac{d\lambda}{\lambda - \frac{1}{4}[d^X, A''_T]}. \tag{10.106}$$

Still, $\mathbf{Q}^{[0,1]}_{\infty,T}$ is a projector with finite dimensional range $\mathbf{G}^{[0,1]}_{\infty,T}$. Let $\mathbf{Q}^{\infty}_{\infty,T}$ be the restriction of P^{∞} to $\mathbf{G}^{[0,1]}_{\infty,T}$. Note that here, P^{∞} is used and not $\mathbf{P}^{\infty}$. Then $\mathbf{Q}^{\infty}_{\infty,T}$: $\mathbf{G}^{[0,1]}_{\infty,T} \to \Lambda^{\bullet}(T^*S)\,\widehat{\otimes}\,C^{\bullet}(W^u,F)$ is one to one. Also the obvious analogue of formula (10.78) holds, i.e.

$$(\mathbf{Q}^{\infty}_{\infty,T})^{-1} = \mathbf{Q}^{[0,1]}_{\infty,T}\,(P^{\infty}_{T})^{-1}\left(P^{\infty}\mathbf{Q}^{[0,1]}_{\infty,T}\,(P^{\infty}_{T})^{-1}\right). \tag{10.107}$$

Let $g^{\Omega^{\bullet}(X,F|_X)}_{t,T}$ be the metric on $\Omega^{\bullet}(X,F|_X)$ which is associated to the metrics $g^{TX}/t, g^F_T$ on TX, F. Let $(\mathbf{P}^{\infty}_{t,T})^{-1,*}$ be the adjoint of $(\mathbf{P}^{\infty}_{t,T})^{-1}$ with respect to the metrics $g^{\Omega^{\bullet}(X,F|_X)}_{t,T}, g^{C^{\bullet}(W^u,F)}$. Let $(\mathbf{Q}^{\infty}_{\infty,T})^{-1,*}$ be the adjoint of $(\mathbf{Q}^{\infty}_{\infty,T})^{-1}$ with respect to $g^{\Omega^{\bullet}(X,F|_X)}_{T}, g^{C^{\bullet}(W^u,F)}$.

The next result is not needed in our proof of Theorem 9.8.

Proposition 10.26. — *As* $t \to +\infty$,

$$t^{\left(-N^{C^{\bullet}(W^u,F)}+n/2\right)}(\mathbf{P}^{\infty}_{t,T})^{-1,*} = (\mathbf{Q}^{\infty}_{\infty,T})^{-1,*} + \mathcal{O}(1/t). \tag{10.108}$$

Proof. — Let $(\mathbf{P}^{\infty}_{t,T})^{-1,*}_0$ be the adjoint of $(\mathbf{P}^{\infty}_{t,T})^{-1}$ with respect to to the metrics $g^{\Omega^{\bullet}(X,F|_X)}_{T}, g^{C^{\bullet}(W^u,F)}$. Then

$$(\mathbf{P}^{\infty}_{t,T})^{-1,*} = (\mathbf{P}^{\infty}_{t,T})^{-1,*}_0\, t^{N-n/2}. \tag{10.109}$$

Recall that by (10.79), $\deg\left((\mathbf{P}^{\infty}_{T})^{-1}\right) \geqslant 0$. Then

$$(\mathbf{P}^{\infty}_{T})^{-1} = A_0 + A_2 \cdots + A_{2j} \ldots, \tag{10.110}$$

with $\deg(A_{2j}) = 2j$. Using (10.98), we get

$$(\mathbf{P}^{\infty}_{t,T})^{-1} = A_0 + \frac{A_2}{t} + \cdots + \frac{A_{2j}}{t^j} + \cdots \tag{10.111}$$

Also by (10.79),

$$j \leqslant F(A_{2j}). \tag{10.112}$$

Therefore A_{2j} increases the vertical degree in $\Lambda^\bullet(T^*X)$ by at most j.

Let A_{2j}^* be the adjoint of A_{2j} with respect to the metrics $g^{C^\bullet(W^u,F)}, g_T^{\Omega^\bullet\left(X,F|_X\right)}$. From (10.111), we get

$$\left(\mathbf{P}_{t,T}^{\infty}\right)_0^{-1,*} = A_0^* + \frac{A_2^*}{t} + \cdots + \frac{A_{2j}^*}{t^j} + \cdots, \tag{10.113}$$

and A_{2j}^* decreases the vertical degree by at most j. By (10.109), (10.113), we obtain,

$$t^{\left(-N^{C^\bullet(W^u,F)}+n/2\right)}\left(\mathbf{P}_{t,T}^{\infty}\right)^{-1,*} = t^{-N^{C^\bullet(W^u,F)}}\left(A_0^* + \frac{A_2^*}{t} + \cdots + \frac{A_{2j}^*}{t^j} + \cdots\right) t^N. \tag{10.114}$$

Let $A_{2j}^{(2j)}$ be the component of A_{2j} which increases the vertical degree by j. By (10.114), we find that as $t \to +\infty$,

$$t^{\left(-N^{C^\bullet(W^u,F)}+n/2\right)}\left(\mathbf{P}_{t,T}^{\infty}\right)^{-1,*} = A_0^{(0),*} + \cdots A_{2j}^{(2j),*} + \cdots + \mathcal{O}(1/t). \tag{10.115}$$

By (10.110), (10.115), we find that the leading term in the right-hand side of (10.115) is the adjoint with respect to the metric $g^{\Omega^\bullet\left(X,F|_X\right)}$ of the component R of $(\mathbf{P}^\infty)^{-1}$ which makes the second inequality in (10.79) to be an equality.

Clearly,

$$\mathbf{P}^\infty \left(\mathbf{P}_T^\infty\right)^{-1} = 1. \tag{10.116}$$

Since $\deg\left(\mathbf{P}^\infty\right) = 0$, from (10.116), we get

$$P^\infty R = 1. \tag{10.117}$$

Also,

$$\mathbf{P}_T^{[0,1]}\left(\mathbf{P}_T^\infty\right)^{-1} = \left(\mathbf{P}_T^\infty\right)^{-1}. \tag{10.118}$$

Using Proposition 10.7, (10.106) and (10.118), we obtain,

$$\mathbf{Q}_{\infty,T}^{[0,1]} R = R. \tag{10.119}$$

From (10.117), (10.119), we get

$$R = \left(\mathbf{Q}_{\infty,T}^{\infty}\right)^{-1}. \tag{10.120}$$

Our Proposition follows from the statement we gave after (10.115) and from (10.120). □

The following result is not needed either in our proof of Theorem 9.8.

Proposition 10.27. — *The following identities hold,*

(10.121)

$$\mathbf{P}^{\infty}_{\infty,T} A' \left(\mathbf{P}^{\infty}_{\infty,T}\right)^{-1} = A^{C^\bullet(W^u,F)\prime},$$

$$\mathbf{P}^{\infty}_{\infty,T} d^{X,*} \left(\mathbf{P}^{\infty}_{\infty,T}\right)^{-1} = \left[\left(\mathbf{Q}^{\infty}_{\infty,T}\right)^{-1,*} \left(\mathbf{P}^{\infty}_{\infty,T}\right)^{-1}\right]^{-1} \partial^* \left(\left(\mathbf{Q}^{\infty}_{\infty,T}\right)^{-1,*} \left(\mathbf{P}^{\infty}_{\infty,T}\right)^{-1}\right).$$

Proof. — The first identity in (10.121) follows from Theorem 5.8. Using (10.16), (10.99), we see that as $t \to +\infty$,

$$\mathbf{P}^{\infty}_{t,T} A''_{t,T} \left(\mathbf{P}^{\infty}_{t,T}\right)^{-1} = t\mathbf{P}^{\infty}_{\infty,T} d^{X,*} \left(\mathbf{P}^{\infty}_{\infty,T}\right)^{-1} + \mathcal{O}(1). \tag{10.122}$$

Also by (10.89), we get

$$\begin{aligned}\mathbf{P}^{\infty}_{t,T} A''_{t,T} \left(\mathbf{P}^{\infty}_{t,T}\right)^{-1} = {} & \left[t^{\left(-N^{C^\bullet(W^u,F)}+n/2\right)} \left(\mathbf{P}^{\infty}_{t,T}\right)^{-1,*} \left(\mathbf{P}^{\infty}_{t,T}\right)^{-1}\right]^{-1} \\ & \left(t^{-N^{C^\bullet(W^u,F)}} A^{C^\bullet(W^u,F)\prime\prime} t^{N^{C^\bullet(W^u,F)}}\right) t^{-N^{C^\bullet(W^u,F)}+n/2} \left(\mathbf{P}^{\infty}_{t,T}\right)^{-1,*} \left(\mathbf{P}^{\infty}_{t,T}\right)^{-1}.\end{aligned} \tag{10.123}$$

Clearly,

$$t^{-N^{C^\bullet(W^u,F)}} A^{C^\bullet(W^u,F)\prime\prime} t^{N^{C^\bullet(W^u,F)}} = t\partial^* + \nabla^{C^\bullet(W^u,F)*}. \tag{10.124}$$

From (10.99),(10.108), (10.122)-(10.124), we get the second identity in (10.121). □

Let $(P^{\infty}_T)^{-1,*}$ be the adjoint of $(P^{\infty}_T)^{-1}$ with respect to the metrics $g_T^{\Omega^\bullet\left(X,F|_X\right)}$, $g^{C^\bullet(W^u,F)}$.

Proposition 10.28. — *The following identity holds,*

$$t^{N/2} \left(\mathbf{P}^{\infty}_{t,T}\right)^{-1} t^{-N^{C^\bullet(W^u,F)}/2} = \psi_t^{-1} \left(\mathbf{P}^{\infty}_{T}\right)^{-1} \psi_t. \tag{10.125}$$

Proof. — This follows from Proposition 10.24. □

Proposition 10.29. — *There exists a smooth section J of*

$$\left(\Lambda^\bullet(T^*S) \mathbin{\widehat{\otimes}} \operatorname{Hom}\left(C^\bullet(W^u,F), \Omega^\bullet\left(X,F|_X\right)\right)\right)^{\text{even}},$$

such that as $t \to +\infty$,

$$\begin{aligned} t^{N/2} \left(\mathbf{P}^{\infty}_{t,T}\right)^{-1} t^{-N^{C^\bullet(W^u,F)}/2} &= (P^{\infty}_T)^{-1} + \frac{J}{\sqrt{t}} + \mathcal{O}(1/t), \\ t^{-N^{C^\bullet(W^u,F)}/2} \left(\mathbf{P}^{\infty}_{t,T}\right)_0^{-1,*} t^{N/2} &= (P^{\infty}_T)^{-1,*} + \frac{J_0^*}{t} + \mathcal{O}(1/t). \end{aligned} \tag{10.126}$$

Proof. — Using (10.51), (10.78), and (10.125), we get the first identity in (10.126). By taking adjoints, we obtain the second identity. □

Definition 10.30. — Put

$$\mathbf{g}^{C^\bullet(W^u,F)}_{t,T} = \left(\mathbf{P}^{\infty}_{t,T}\right)^{-1,*} \left(\mathbf{P}^{\infty}_{t,T}\right)^{-1}. \tag{10.127}$$

Then $\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}$ is a generalized metric on $C^\bullet(W^u,F)$. Recall that the metric $g_T^{C^\bullet(W^u,F)}$ on $C^\bullet(W^u,F)$, which was defined in (10.87), is such that the $C^i(W^u,F)$'s are mutually orthogonal.

Now we will show that the generalized metrics $\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}$ verify the assumptions in (2.130).

Theorem 10.31. — *There is a smooth section H of*

$$\left(\Lambda^\bullet(T^*S)\widehat{\otimes}\operatorname{End}\left(C^\bullet(W^u,F)\right)\right)^{\text{even}},$$

such that as $t\to+\infty$,

$$t^{-N^{C^\bullet(W^u,F)}/2+n/2}\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}t^{-N^{C^\bullet(W^u,F)}/2}=g_T^{C^\bullet(W^u,F)}+\frac{H}{\sqrt{t}}+\mathcal{O}\left(1/t\right),$$

$$\begin{aligned}(10.128)\quad & t^{N^{C^\bullet(W^u,F)}/2}\left[\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right]^{-1}\frac{\partial}{\partial t}\left[\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right]t^{-N^{C^\bullet(W^u,F)}/2}\\ &=\left(N^{C^\bullet(W^u,F)}-\frac{n}{2}\right)\frac{1}{t}+\mathcal{O}\left(1/t^{3/2}\right).\end{aligned}$$

Proof. — By (10.109), we get

$$\begin{aligned}(10.129)\quad & t^{-N^{C^\bullet(W^u,F)}/2}\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}t^{-N^{C^\bullet(W^u,F)}/2}\\ &=\left(t^{-N^{C^\bullet(W^u,F)}/2-n/2}\left(\mathbf{P}_{t,T}^{\infty}\right)_0^{-1,*}t^{N/2}\right)\left(t^{N/2}\left(\mathbf{P}_{t,T}^{\infty}\right)^{-1}t^{-N^{C^\bullet(W^u,F)}/2}\right).\end{aligned}$$

By Proposition 10.29 and by (10.129), we get the first identity in (10.128).

Also,

$$\begin{aligned}(10.130)\quad & \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}=\\ & t^{\left(N^{C^\bullet(W^u,F)}-n\right)/2}\left[t^{n/2}t^{-N^{C^\bullet(W^u,F)}/2}\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}t^{-N^{C^\bullet(W^u,F)}/2}\right]t^{N^{C^\bullet(W^u,F)}/2}.\end{aligned}$$

Therefore,

$$\begin{aligned}(10.131)\quad & \frac{\partial}{\partial t}\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}=\frac{N^{C^\bullet(W^u,F)}-n}{2t}\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}+\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\frac{N^{C^\bullet(W^u,F)}}{2t}\\ &+t^{\left(N^{C^\bullet(W^u,F)}-n\right)/2}\frac{\partial}{\partial t}\left(t^{n/2}t^{-N^{C^\bullet(W^u,F)}/2}\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}t^{-N^{C^\bullet(W^u,F)}/2}\right)t^{N^{C^\bullet(W^u,F)}/2}.\end{aligned}$$

By (10.98), (10.126), (10.129), one verifies easily that

$$t^{n/2}t^{-N^{C^\bullet(W^u,F)}/2}\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}t^{-N^{C^\bullet(W^u,F)}/2}$$

is a polynomial in $1/\sqrt{t}$. Therefore, as $t\to+\infty$,

$$(10.132)\qquad \frac{\partial}{\partial t}\left(t^{n/2}t^{-N^{C^\bullet(W^u,F)}/2}\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}t^{-N^{C^\bullet(W^u,F)}/2}\right)=\mathcal{O}\left(1/t^{3/2}\right).$$

From the first identity in (10.128), (10.131) and (10.132), we get the second identity in (10.128). The proof of our Theorem is completed. □

Proposition 10.32. — *The following identity holds,*

$$t^{N^{C^\bullet(W^u,F)}/2}\mathbf{P}^\infty_{t,T}A'\left(\mathbf{P}^\infty_{t,T}\right)^{-1}t^{-N^{C^\bullet(W^u,F)}/2} = \sqrt{t}\partial + \nabla^{C^\bullet(W^u,F)}. \tag{10.133}$$

Also as $t \to +\infty$,

$$t^{N^{C^\bullet(W^u,F)}/2}\mathbf{P}^\infty_{t,T}A''_{t,T}\left(\mathbf{P}^\infty_{t,T}\right)^{-1}t^{-N^{C^\bullet(W^u,F)}/2} = P^\infty_T\sqrt{t}\,d^{X,*}\left(P^\infty_T\right)^{-1} + \mathcal{O}(1). \tag{10.134}$$

Proof. — Identity (10.133) follows from (10.89). By (10.123), we get

$$\begin{aligned} t^{N^{C^\bullet(W^u,F)}/2}\mathbf{P}^\infty_{t,T}A''_{t,T}\left(\mathbf{P}^\infty_{t,T}\right)^{-1}t^{-N^{C^\bullet(W^u,F)}/2} &= t^{N^{C^\bullet(W^u,F)}/2}\left[\mathbf{g}^{C^\bullet(W^u,F)}_{t,T}\right]^{-1} \\ t^{N^{C^\bullet(W^u,F)}/2}&\left(t^{-N^{C^\bullet(W^u,F)}/2}A^{C^\bullet(W^u,F)\prime\prime}t^{N^{C^\bullet(W^u,F)}/2}\right)t^{-N^{C^\bullet(W^u,F)}/2} \\ &\mathbf{g}^{C^\bullet(W^u,F)}_{t,T}t^{-N^{C^\bullet(W^u,F)}/2}. \end{aligned} \tag{10.135}$$

Using Theorem 10.31 and (10.135), we find that as $t \to +\infty$,

$$\begin{aligned} t^{N^{C^\bullet(W^u,F)}/2}\mathbf{P}^\infty_{t,T}\,A''_{t,T}\left(\mathbf{P}^\infty_{t,T}\right)^{-1}t^{-N^{C^\bullet(W^u,F)}/2} &= \left(g^{C^\bullet(W^u,F)}_T\right)^{-1}\sqrt{t}\partial^* g^{C^\bullet(W^u,F)}_T \\ &+ \mathcal{O}(1), \end{aligned} \tag{10.136}$$

which is equivalent to (10.134). The proof of our Proposition is completed. □

10.9. Replacing M by $M \times \mathbf{R}_+^*$

Now we replace M by $M \times \mathbf{R}_+^*$, S by $S \times \mathbf{R}_+^*$, $\pi : M \to S$ by $\widetilde{\pi} : \widetilde{M} \to \widetilde{S}$. We denote by $\widetilde{\mathbf{P}}^\infty$ the map of integration along the fibre from $\widetilde{M}$ to $\widetilde{S}$. Over $M \times \{t\}$, we equip TX with the metric g^{TX}/t, and F with the metric g^F_T. Let $\widetilde{A}'$ be the corresponding flat superconnection, and let $\widetilde{A}''_T$ be its adjoint. Then

$$\begin{aligned} \widetilde{A}' &= A' + dt\frac{\partial}{\partial t}, \\ \widetilde{A}''_T &= A''_{t,T} + dt\left(\frac{\partial}{\partial t} + \left(N - \frac{n}{2}\right)\right). \end{aligned} \tag{10.137}$$

Similarly, let $\widetilde{A}^{C^\bullet(W^u,F)\prime}$ be the canonical flat superconnection on $C^\bullet(W^u,F)$ over $S \times \mathbf{R}_+^*$, and let $\widetilde{A}^{C^\bullet(W^u,F)\prime\prime}$ be its adjoint with respect to the metric $g^{C^\bullet(W^u,F)}$. Then

$$\widetilde{A}^{C^\bullet(W^u,F)\prime} = A^{C^\bullet(W^u,F)\prime} + dt\frac{\partial}{\partial t}, \quad \widetilde{A}^{C^\bullet(W^u,F)\prime\prime} = A^{C^\bullet(W^u,F)\prime\prime} + dt\frac{\partial}{\partial t}. \tag{10.138}$$

By (10.23), for $t \in \mathbf{R}_+^*, T \geqslant T_0$,

$$\mathrm{Sp}\left(\widetilde{A}_T^{\,2}\right)|_{S\times\{t\}} \subset [0, \frac{t}{4}] \cap [4t, +\infty[. \tag{10.139}$$

Definition 10.33. — For $t' > 0$, $\frac{t'}{8} < t < 2t'$, put

$$\widetilde{\mathbf{P}}_T^{[0,1]}|_{S\times\{t\}} = \frac{1}{2i\pi}\int_{t'\delta} \frac{d\lambda}{\lambda - \widetilde{A}_T^2|_{S\times\{t\}}}. \tag{10.140}$$

Using (10.139) and the holomorphic functional calculus, one finds that $\widetilde{\mathbf{P}}_T^{[0,1]}|_{S\times\{t\}}$ does not depend on the choice of t'. This way we obtain a well defined $\widetilde{\mathbf{P}}_T^{[0,1]}$, a section of $\Lambda^\bullet\left(T^*\left(S\times\mathbf{R}_+^*\right)\right)\widehat{\otimes}\,\Omega^\bullet\left(X, F|_X\right)$ over $S\times\mathbf{R}_+^*$. Using Theorem 10.9, we find that $\widetilde{\mathbf{P}}_T^{[0,1]}$ is an even self-adjoint projector with finite dimensional range, which commutes with $\widetilde{A}'$ and $\widetilde{A}''_T$.

Comparing with (10.63), we find that there is an odd section $\mathbf{Q}_{t,T}^{[0,1]}$ of $\Lambda^\bullet(T^*S)$ $\widehat{\otimes}\Omega^\bullet\left(X, F|_X\right)$ over $M\times\mathbf{R}_+^*$ such that

$$\widetilde{\mathbf{P}}_T^{[0,1]} = \mathbf{P}_{t,T}^{[0,1]} + dt\mathbf{Q}_{t,T}^{[0,1]}. \tag{10.141}$$

Since $\widetilde{\mathbf{P}}_T^{[0,1]}$ is self-adjoint, in the sense that $\widetilde{\mathbf{P}}_T^{[0,1],*} = \widetilde{\mathbf{P}}_T^{[0,1]}$, $\mathbf{Q}_{t,T}^{[0,1]}$ is also self-adjoint. Since the range of $\widetilde{\mathbf{P}}_T^{[0,1]}$ is finite dimensional, the same is true for $\mathbf{Q}_{t,T}^{[0,1]}$. Since $\widetilde{\mathbf{P}}_T^{[0,1]}$ is a projector, we deduce from (10.141) that

$$\mathbf{Q}_{t,T}^{[0,1]} = \mathbf{P}_{t,T}^{[0,1]}\mathbf{Q}_{t,T}^{[0,1]} + \mathbf{Q}_{t,T}^{[0,1]}\mathbf{P}_{t,T}^{[0,1]}, \tag{10.142}$$

so that $\mathbf{Q}_{t,T}^{[0,1]}$ interchanges $\mathrm{Im}\left(\mathbf{P}_{t,T}^{[0,1]}\right)$ and $\mathrm{Im}\left(1-\mathbf{P}_{t,T}^{[0,1]}\right)$. From the commutation relations,

$$\left[\widetilde{A}', \widetilde{\mathbf{P}}_T^{[0,1]}\right] = 0, \qquad \left[\widetilde{A}''_T, \widetilde{\mathbf{P}}_T^{[0,1]}\right] = 0, \tag{10.143}$$

we get

$$\frac{\partial}{\partial t}\mathbf{P}_{t,T}^{[0,1]} = \left[A', \mathbf{Q}_{t,T}^{[0,1]}\right], \qquad \frac{\partial}{\partial t}\mathbf{P}_{t,T}^{[0,1]} + \left[N, \mathbf{P}_{t,T}^{[0,1]}\right] = \left[A''_T, \mathbf{Q}_{t,T}^{[0,1]}\right]. \tag{10.144}$$

Set

$$\widetilde{\mathbf{F}}_T^{[0,1]} = \mathrm{Im}\left(\widetilde{\mathbf{P}}_T^{[0,1]}\right). \tag{10.145}$$

Let $\widetilde{\mathbf{P}}_T^\infty$ be the restriction of $\widetilde{\mathbf{P}}^\infty$ to $\widetilde{\mathbf{F}}_T^{[0,1]}$. Then by Proposition 10.24, for $T \geqslant T_0'$, $\widetilde{\mathbf{P}}_T^\infty$ is an invertible morphism from $\widetilde{\mathbf{F}}_T^{[0,1]}$ into $\Lambda^\bullet\left(T^*\left(S\times\mathbf{R}_+^*\right)\right)\widehat{\otimes}\,C^\bullet\left(W^u, F\right)$. Then we can write $\left(\widetilde{\mathbf{P}}_T^\infty\right)^{-1}$ in the form,

$$\left(\widetilde{\mathbf{P}}_T^\infty\right)^{-1} = \left(\mathbf{P}_{t,T}^\infty\right)^{-1} + dt\mathbf{R}_{t,T}^\infty. \tag{10.146}$$

From (10.146), we get

$$\left(\widetilde{\mathbf{P}}_T^\infty\right)^{-1,*} = \left(\mathbf{P}_{t,T}^\infty\right)^{-1,*} + dt\mathbf{R}_{t,T}^{\infty,*}. \tag{10.147}$$

Definition 10.34. — Put

$$\widetilde{\mathbf{g}}_T^{C^\bullet(W^u,F)} = \left(\widetilde{\mathbf{P}}_T^\infty\right)^{-1,*}\left(\widetilde{\mathbf{P}}_T^\infty\right)^{-1}. \tag{10.148}$$

Clearly, we can write $\widetilde{\mathbf{g}}_T^{C^\bullet(W^u,F)}$ in the form

$$\widetilde{\mathbf{g}}_T^{C^\bullet(W^u,F)} = \mathbf{g}_{t,T}^{C^\bullet(W^u,F)} + dt\mathbf{S}_{t,T}. \tag{10.149}$$

Then the obvious analogue of Theorem 10.21 holds. In particular $\widetilde{\mathbf{g}}_T^{C^\bullet(W^u,F)}$ is a generalized metric on $C^\bullet(W^u,F)$ over $S \times \mathbf{R}_+^*$.

Theorem 10.35. — *The following identities hold,*

$$\begin{aligned} &\widetilde{\mathbf{P}}_T^\infty \left(\widetilde{A}'\right) \left(\widetilde{\mathbf{P}}_T^\infty\right)^{-1} = \widetilde{A}^{C^\bullet(W^u,F)\prime}, \\ &\widetilde{\mathbf{P}}_T^\infty \left(\widetilde{A}_T''\right) \left(\widetilde{\mathbf{P}}_T^\infty\right)^{-1} = \left(\widetilde{\mathbf{g}}_T^{C^\bullet(W^u,F)}\right)^{-1} \widetilde{A}^{C^\bullet(W^u,F)\prime\prime} \widetilde{\mathbf{g}}_T^{C^\bullet(W^u,F)}. \end{aligned} \tag{10.150}$$

Proof. — Our Theorem follows from Theorem 10.21. □

10.10. The superconnection forms for $\mathbf{F}_T^{[0,1]}$

Observe that as in (10.54),

$$\mathbf{P}_{t,T}^{[0,1]} = \frac{1}{2i\pi} \int_{\sqrt{t}\delta} \frac{d\lambda}{\lambda - B_{t,T}}. \tag{10.151}$$

Using the holomorphic functional calculus, we find that

$$\begin{aligned} h(B_{t,T})\, \mathbf{P}_{t,T}^{[0,1]} &= \frac{1}{2i\pi} \int_{\sqrt{t}\,\delta} \frac{h(\lambda)}{\lambda - B_{t,T}} d\lambda, \\ h'(B_{t,T})\, \mathbf{P}_{t,T}^{[0,1]} &= \frac{1}{2i\pi} \int_{\sqrt{t}\,\delta} \frac{h'(\lambda)}{\lambda - B_{t,T}} d\lambda. \end{aligned} \tag{10.152}$$

Similarly, with the notation in (10.72),

$$\begin{aligned} \widehat{\mathbf{P}}_{t,T}^{[0,1]} &= \frac{1}{2i\pi} \int_{\sqrt{t}\delta} \frac{d\lambda}{\lambda - D_{t,T}}, \\ h(D_{t,T})\, \widehat{\mathbf{P}}_{t,T}^{[0,1]} &= \frac{1}{2i\pi} \int_{\sqrt{t}\,\delta} \frac{h(\lambda)}{\lambda - D_{t,T}} d\lambda, \\ h'(D_{t,T})\, \widehat{\mathbf{P}}_{t,T}^{[0,1]} &= \frac{1}{2i\pi} \int_{\sqrt{t}\,\delta} \frac{h'(\lambda)}{\lambda - D_{t,T}} d\lambda. \end{aligned} \tag{10.153}$$

By (10.26), (10.153), we get

$$L_{t,T} = h'(D_{t,T})\, \widehat{\mathbf{P}}_{t,T}^{[0,1]}. \tag{10.154}$$

Definition 10.36. — For $t \in \mathbf{R}_+^*, T \geqslant T_0'$, put

$$\begin{aligned} a_{t,T} &= \sqrt{2i\pi}\varphi \mathrm{Tr_s}\left[g h(B_{t,T})\, \mathbf{P}_{t,T}^{[0,1]}\right], \\ b_{t,T} &= \frac{1}{2}\varphi \mathrm{Tr_s}\left[\left(N - \frac{n}{2}\right) g h'(B_{t,T})\, \mathbf{P}_{t,T}^{[0,1]}\right]. \end{aligned} \tag{10.155}$$

In (10.155), we may replace $B_{t,T}$ by $D_{t,T}$ and $\mathbf{P}_{t,T}^{[0,1]}$ by $\widehat{\mathbf{P}}_{t,T}^{[0,1]}$. Then $a_{t,T}, b_{t,T}$ are forms on S.

Theorem 10.37. — *For $T \geqslant T_0'$, the form $a_{t,T}$ is odd and closed, and the form $b_{t,T}$ is even. Moreover,*

$$\frac{\partial}{\partial t} a_{t,T} = d\frac{b_{t,T}}{t}. \tag{10.156}$$

Also as $t \to +\infty$,

$$\begin{aligned} a_{t,T} &= h_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2,T}^{H^\bullet(X,F|_X)}\right) + \mathcal{O}\left(1/\sqrt{t}\right), \\ b_{t,T} &= \frac{1}{2}\chi_g'(F) - \frac{n}{4}\chi_g(F) + \mathcal{O}\left(1/\sqrt{t}\right). \end{aligned} \tag{10.157}$$

Proof. — The proof of the first part of our Theorem is essentially the same as the proof of [**BLo1**, Theorems 1.8 and 2.9], or of Theorems 1.8 and 1.18. Still, we have to be more careful, because of the dependence of $\mathbf{P}_{t,T}^{[0,1]}$ on t. Since

$$[A_{t,T}, B_{t,T}] = 0, \qquad \left[A_{t,T}, \mathbf{P}_{t,T}^{[0,1]}\right] = 0, \tag{10.158}$$

we find easily that $a_{t,T}$ is closed. Using the argument after (10.140), given $t' > 0$, if t is close enough to t', instead of (10.152), we can write,

$$h(B_{t,T})\,\mathbf{P}_{t,T}^{[0,1]} = \frac{1}{2i\pi}\int_{\sqrt{t'}\delta} \frac{h(\lambda)}{\lambda - B_{t,T}} d\lambda, \tag{10.159}$$

the key point being that the contour of integration in (10.159) does not depend on t. We can then proceed as in [**BLo1**] and prove (10.156).

Now, we use the notation of Section 3.14. By proceeding as in Section 3.14, we find that there exists $C > 0$ such that for $t \geqslant 1$,

$$\left\| h(D_{t,T}) - P_T^{[0,1]} h\left(B_T^{H^\bullet(X,F|_X)}\right)\right\|_1 \leqslant \frac{C}{\sqrt{t}}. \tag{10.160}$$

Using (10.73) and (10.160), we get (10.157). The proof of our Theorem is completed. □

Remark 10.38. — An equivalent reformulation of equation (10.156) is that if c is the form on $M \times \mathbf{R}_+^*$,

$$c = a_{t,T} + \frac{dt}{t} b_{t,T}, \tag{10.161}$$

then c is closed.

Definition 10.39. — For $T \geqslant T_0'$, put

$$S_{h,g}^{[0,1]}(T) = -\int_1^{+\infty} (b_{t,T} - b_{\infty,T})\frac{dt}{t}. \tag{10.162}$$

Proposition 10.40. — *The even form $S_{h,g}^{[0,1]}(T)$ on S is such that*

$$dS_{h,g}^{[0,1]}(T) = h_g\left(A^{C^\bullet(W^u,F)'}, \mathbf{g}_T^{C^\bullet(W^u,F)}\right) - h_g\left(H^\bullet(X,F|_X), g_{L_2,T}^{H^\bullet(X,F|_X)}\right). \tag{10.163}$$

Proof. — Observe that by Theorem 10.21,

$$a_{1,T} = h_g\left(A^{C^\bullet(W^u,F)\prime}, \mathbf{g}_T^{C^\bullet(W^u,F)}\right). \tag{10.164}$$

Our Proposition follows from Theorem 10.37 and from (10.164). □

Proposition 10.41. — *The following identity holds,*

$$S_{h,g}^{[0,1]}(T) = -\int_1^{+\infty} \left(\varphi \mathrm{Tr_s}\left[NgL_{t,T}\right] - \chi_g'(F)\right)\frac{dt}{2t}. \tag{10.165}$$

Proof. — By (3.53) and by Proposition 10.16, we get

$$b_{t,T} = \frac{1}{2}\varphi \mathrm{Tr_s}\left[\left(N - \frac{n}{2}\right) gL_{t,T}\right]. \tag{10.166}$$

Also by proceeding as in the proof of Proposition 1.6, and using the fact that $\mathbf{P}_{t,T}^{[0,1]}$ is trace class, we get

$$\mathrm{Tr_s}\left[h'(B_{t,T})\,\mathbf{P}_{t,T}^{[0,1]}\right] = \chi_g'(F). \tag{10.167}$$

By (10.162), (10.166), (10.167), we get (10.165). □

10.11. The form c and the complex $C^\bullet(W^u, F)$

Recall that $\widetilde{\mathbf{g}}_T^{C^\bullet(W^u,F)}$ is a generalized metric on the complex $C^\bullet(W^u, F)$ over $S \times \mathbf{R}_+^*$. Also the form c on $S \times \mathbf{R}_+^*$ was defined in (10.161).

Proposition 10.42. — *The following identity of forms on $M \times \mathbf{R}_+^*$ holds,*

$$c = h_g\left(C^\bullet(W^u, F), \widetilde{\mathbf{g}}_T^{C^\bullet(W^u,F)}\right). \tag{10.168}$$

Proof. — This is an easy consequence of Theorem 10.35. □

By (10.149), the generalized metric $\widetilde{\mathbf{g}}_T^{C^\bullet(W^u,F)}$ contains a term with dt as a factor. Instead we will consider $\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}$ as a generalized metric on $C^\bullet(W^u, F)$ over $S \times \mathbf{R}_+^*$, i.e. we eliminate the term with dt in $\widetilde{\mathbf{g}}_T^{C^\bullet(W^u,F)}$. Then $h_g\left(\widetilde{A}^{C^\bullet(W^u,F)\prime}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right)$ is a closed form on $S \times \mathbf{R}_+^*$. We can write $h_g\left(\widetilde{A}^{C^\bullet(W^u,F)\prime}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right)$ in the form

$$\begin{aligned} h_g\left(\widetilde{A}^{C^\bullet(W^u,F)\prime}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right) &= \left[h_g\left(\widetilde{A}^{C^\bullet(W^u,F)\prime}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right)\right]^S \\ &\quad + dt\left[h_g\left(\widetilde{A}^{C^\bullet(W^u,F)\prime}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right)\right]^{dt}, \end{aligned} \tag{10.169}$$

and $\left[h_g\left(\widetilde{A}^{C^\bullet(W^u,F)\prime}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right)\right]^S$, $\left[h_g\left(\widetilde{A}^{C^\bullet(W^u,F)\prime}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right)\right]^{dt}$ are smooth forms on S.

Proposition 10.43. — *For any $t \in \mathbf{R}_+^*$, the following identity of forms holds on S,*

$$a_{t,T} = \left[h_g\left(\widetilde{A}^{C^\bullet(W^u,F)\prime}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right)\right]^S, \tag{10.170}$$

$$b_{t,T} = \left[h_g\left(\widetilde{A}^{C^\bullet(W^u,F)\prime}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right)\right]^{dt} \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S).$$

Proof. — Using (10.150), we get the first identity in (10.170). Recall that $\mathbf{S}_{t,T}$ was introduced in (10.149). For $\ell \in [0,1]$, put

$$\widetilde{\mathbf{g}}_{T,\ell}^{C^\bullet(W^u,F)} = \mathbf{g}_{t,T}^{C^\bullet(W^u,F)} + \ell dt \mathbf{S}_{t,T}. \tag{10.171}$$

Now we enlarge the total space M to be $M \times \mathbf{R}_+^* \times [0,1]$. In particular $A^{C^\bullet(W^u,F)\prime}$ is now replaced by

$$\overline{A}' = A^{C^\bullet(W^u,F)\prime} + dt\frac{\partial}{\partial t} + d\ell\frac{\partial}{\partial \ell}. \tag{10.172}$$

Let $\overline{A}''_T$ be the adjoint of $\overline{A}'$ with respect to $\widetilde{\mathbf{g}}_{T,\ell}^{C^\bullet(W^u,F)}$. Set

$$\delta = h_g\left(\overline{A}', \widetilde{\mathbf{g}}_{T,\ell}^{C^\bullet(W^u,F)}\right). \tag{10.173}$$

Then δ is a closed form on $M \times \mathbf{R}_+^* \times [0,1]$. Now because in (10.171), ℓ is a factor of dt, we can write δ in the form

$$\delta = \alpha + dt\beta + d\ell dt\gamma, \tag{10.174}$$

i.e. the form δ does not contain a term with just $d\ell$. Since δ is closed, we get

$$\frac{\partial \beta}{\partial \ell} + d\gamma = 0. \tag{10.175}$$

By (10.175), we find that

$$\beta_{\ell=0} = \beta_{\ell=1} \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{10.176}$$

Our Proposition follows. □

Recall that by Theorem 10.31, the generalized metric $\mathbf{g}_{t,T}^{C^\bullet(W^u,F)}$ on $C^\bullet(W^u,F)$ verifies the assumptions in (2.130) with respect to the standard Hermitian metric $g_T^{C^\bullet(W^u,F)}$. Therefore we can define the form $U_{h,g}\left(A^{C^\bullet(W^u,F)\prime}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right)$ as in Definition 2.49.

Theorem 10.44. — *For $T \geqslant T_0'$, the following identity holds,*

$$S_{h,g}^{[0,1]}(T) = U_{h,g}\left(A^{C^\bullet(W^u,F)\prime}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{10.177}$$

Proof. — This follows from (2.138), (10.162), (10.170). □

10.12. An identity on the forms $U_{h,g}\left(A^{C^\bullet(W^u,F)\prime}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right)$

Definition 10.45. — For $u \in \mathbf{R}_+^*$, let $\mathfrak{g}_{T,u}^{C^\bullet(W^u,F)}$ be the generalized metric on $C^\bullet(W^u,F)$,

$$\mathfrak{g}_{T,u}^{C^\bullet(W^u,F)} = u^{N^{C^\bullet(W^u,F)}/2}\mathbf{g}_T^{C^\bullet(W^u,F)}u^{N^{C^\bullet(W^u,F)}/2}. \tag{10.178}$$

Observe that our notation is compatible with (2.147). Let $A_{T,u}^{C^\bullet(W^u,F)\prime\prime}$ be the adjoint of $A^{C^\bullet(W^u,F)\prime}$ with respect to $\mathfrak{g}_{T,u}^{C^\bullet(W^u,F)}$. Put

$$B_{T,u}^{C^\bullet(W^u,F)} = \frac{1}{2}\left(A_{T,u}^{C^\bullet(W^u,F)\prime\prime} - A^{C^\bullet(W^u,F)\prime}\right). \tag{10.179}$$

Let $N^{H^\bullet(X,F|_X)}$ be the number operator of $H^\bullet(X,F|_X)$. We define the form $S_{h,g}\left(A', u^{N^{C^\bullet(W^u,F)}}g^{C^\bullet(W^u,F)}\right)$ as in Definition 1.24.

Proposition 10.46. — *For $T \geqslant T_0', u \in]0,1]$, the following identity holds,*

(10.180)

$$\begin{aligned}
&-U_{h,g}\left(A^{C^\bullet(W^u,F)\prime}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right) - \widetilde{h}_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2,0}^{H^\bullet(X,F|_X)}, g_{L_2,T}^{H^\bullet(X,F|_X)}\right)\\
&= -S_{h,g}\left(A^{C^\bullet(W^u,F)\prime}, u^{N^{C^\bullet(W^u,F)}}g^{C^\bullet(W^u,F)}\right)\\
&\qquad - \widetilde{h}_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2,0}^{H^\bullet(X,F|_X)}, u^{N^{H^\bullet(X,F|_X)}}g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}\right)\\
&\qquad + \widetilde{h}_g\left(A^{C^\bullet(W^u,F)\prime}, \mathbf{g}_T^{C^\bullet(W^u,F)}, \mathfrak{g}_{T,u}^{C^\bullet(W^u,F)}\right)\\
&\qquad + \widetilde{h}_g\left(A^{C^\bullet(W^u,F)\prime}, \mathfrak{g}_{T,u}^{C^\bullet(W^u,F)}, u^{N^{C^\bullet(W^u,F)}}g^{C^\bullet(W^u,F)}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S).
\end{aligned}$$

Proof. — Our identity identity follows from Remark 2.51 and Theorem 2.52. □

Let η_T be the even form associated to $A^{C^\bullet(W^u,F)\prime}, \mathbf{g}_T^{C^\bullet(W^u,F)}$, which was introduced in (2.143). By Proposition 2.54,

$$\eta_T = \frac{1}{2}\widetilde{\chi}_g'^{-}(F) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{10.181}$$

Proposition 10.47. — *As* $u \to 0$,

$$
\begin{aligned}
&- S_{h,g}\left(A^{C^\bullet(W^u,F)\prime}, u^{N^{C^\bullet(W^u,F)}} g^{C^\bullet(W^u,F)}\right) - \widetilde{h}_g\Big(\nabla^{H^\bullet(X,F|_X)}, g_{L_2,0}^{H^\bullet(X,F|_X)},\\
&u^{N^{H^\bullet(X,F|_X)}} g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}\Big) + \frac{1}{2}\widetilde{\chi}_g'^{-}(F)\log(u) \to\\
&\varphi\Big\{\int_0^1 \left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)} g h'\left(B_t^{C^\bullet(W^u,F)}\right)\right] - \widetilde{\chi}_g'^{-}(F)\right)\frac{dt}{2t}\\
&+ \int_1^{+\infty}\left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)} g h'\left(B_t^{C^\bullet(W^u,F)}\right)\right] - \chi_g'(F)\right)\frac{dt}{2t}\Big\}\\
&- \widetilde{h}_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2,0}^{H^\bullet(X,F|_X)}, g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S),\\
&\widetilde{h}_g\left(A^{C^\bullet(W^u,F)\prime}, \mathbf{g}_T^{C^\bullet(W^u,F)}, \mathfrak{g}_{T,u}^{C^\bullet(W^u,F)}\right) - \eta_T\log(u) \to\\
&-\int_0^1\left(\varphi\mathrm{Tr_s}\left[\frac{1}{2}\left(\mathfrak{g}_{T,t}^{C^\bullet(W^u,F)}\right)^{-1}\frac{\partial}{\partial t}\mathfrak{g}_{T,t}^{C^\bullet(W^u,F)} g h'\left(B_{T,t}^{C^\bullet(W^u,F)}\right)\right] - \frac{\eta_T}{t}\right)dt\\
&\qquad\qquad \text{in } \Omega^\bullet(S)/d\Omega^\bullet(S).
\end{aligned}
\tag{10.182}
$$

Proof. — By Proposition 1.6 and Definition 1.10,

$$
\begin{aligned}
&\widetilde{h}_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2,0}^{H^\bullet(X,F|_X)}, u^{N^{H^\bullet(X,F|_X)}} g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}\right) =\\
&\widetilde{h}_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2,0}^{H^\bullet(X,F|_X)}, g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}\right) + \frac{1}{2}\chi_g'(F)\log(u) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S).
\end{aligned}
\tag{10.183}
$$

Our identities follow from Propositions 1.27 and 2.57 and from (10.183). □

Proposition 10.48. — *As* $u \to 0$,

$$
\begin{aligned}
&\widetilde{h}_g\left(A^{C^\bullet(W^u,F)\prime}, \mathfrak{g}_{T,u}^{C^\bullet(W^u,F)}, u^{N^{C^\bullet(W^u,F)}} g^{C^\bullet(W^u,F)}\right) \to\\
&\qquad\qquad \widetilde{h}_g\left(\nabla^{C^\bullet(W^u,F)}, \mathbf{g}_T^{C^\bullet(W^u,F)}, g^{C^\bullet(W^u,F)}\right).
\end{aligned}
\tag{10.184}
$$

Proof. — Our Proposition is a special case of Proposition 2.58. □

Theorem 10.49. — *The following identity holds,*

(10.185)

$$
\begin{aligned}
&- U_{h,g}\left(A^{C^\bullet(W^u,F)'}, \mathbf{g}_{t,T}^{C^\bullet(W^u,F)}\right) - \widetilde{h}_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2,0}^{H^\bullet(X,F|_X)}, g_{L_2,T}^{H^\bullet(X,F|_X)}\right) \\
&\qquad = \varphi\left\{\int_0^1 \left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)} g h'\left(B_t^{C^\bullet(W^u,F)}\right)\right] - \widetilde{\chi}_g'^{-}(F)\right)\frac{dt}{2t}\right. \\
&\qquad\quad \left. + \int_1^{+\infty}\left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)} g h'\left(B_t^{C^\bullet(W^u,F)}\right)\right] - \chi_g'(F)\right)\frac{dt}{2t}\right\} \\
&\quad - \int_0^1 \left(\varphi \mathrm{Tr_s}\left[\frac{1}{2}\left(\mathfrak{g}_{T,t}^{C^\bullet(W^u,F)}\right)^{-1}\frac{\partial}{\partial t}\mathfrak{g}_{T,t}^{C^\bullet(W^u,F)} g h'\left(B_{T,t}^{C^\bullet(W^u,F)}\right)\right] - \frac{\eta_T}{t}\right)dt \\
&\qquad\quad - \widetilde{h}_g\left(\nabla^{H^\bullet(X,F|_X)}, g_{L_2,0}^{H^\bullet(X,F|_X)}, g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}\right) \\
&\qquad\quad + \widetilde{h}_g\left(\nabla^{C^\bullet(W^u,F)}, \mathbf{g}_T^{C^\bullet(W^u,F)}, g^{C^\bullet(W^u,F)}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S).
\end{aligned}
$$

Proof. — Our identity follows from Propositions 10.46-10.48. □

10.13. A fundamental result

The fundamental result of this Chapter is as follows.

Theorem 10.50. — *As* $T \to +\infty$,

$$
\int_0^1 \left(\varphi \mathrm{Tr_s}\left[\frac{1}{2}\left(\mathfrak{g}_{T,t}^{C^\bullet(W^u,F)}\right)^{-1}\frac{\partial}{\partial t}\mathfrak{g}_{T,t}^{C^\bullet(W^u,F)} g h'\left(B_{T,t}^{C^\bullet(W^u,F)}\right)\right] - \frac{\eta_T}{t}\right)dt \to 0,
$$

(10.186)

$$
\begin{aligned}
&\widetilde{h}_g\left(\nabla^{C^\bullet(W^u,F)}, \mathbf{g}_T^{C^\bullet(W^u,F)}, g^{C^\bullet(W^u,F)}\right) - \mathrm{Tr_s}^{B_g}[f]T \\
&- \frac{1}{4}\left(\widetilde{\chi}_g'^{+}(F) - \chi_g'^{-}(F)\right)\log(T) \to \frac{1}{4}\left(\widetilde{\chi}_g'^{-}(F) - \chi_g'^{+}(F)\right)\log(\pi).
\end{aligned}
$$

Proof. — The remainder of the Chapter is devoted to the proof of our Theorem. □

Remark 10.51. — Using Theorems 10.5, Remark 10.6, Proposition 10.41 and Theorems 10.44, 10.49 and 10.50, Theorem 9.8 follows.

10.14. The projectors $\overline{\mathbb{P}}_T$

Recall that the operator $\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2}$ was constructed in Section 10.1.

Given $n' \in \mathbf{N}$, let $Q^{n'}(TX^u|_{\mathbf{B}})$ be the algebra of invariant polynomials on degree n' on $TX^u|_{\mathbf{B}}$.

Definition 10.52. — For $T \geqslant 1$, put

$$
\overline{\mathbb{P}}_T = \frac{1}{2i\pi}\int_\delta \frac{d\lambda}{\lambda - \overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2}}. \tag{10.187}
$$

Theorem 10.53. — *For $T \geqslant 1$, $\overline{\mathbb{P}}_T$ is a projection acting on $\Lambda^\bullet(T^*S) \widehat{\otimes} I \widehat{\otimes} F|_{\mathbf{B}}$, with finite dimensional range $\overline{\mathbb{F}}_T$, which commutes with the action of $\Lambda^\bullet(T^*S)$ and with $\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|'_{\mathbf{B}}}, \overline{\mathcal{C}}^{I\widehat{\otimes}F|''_{\mathbf{B}}}$, and is such that*

$$\overline{\mathbb{P}}_T^{(0)} = \overline{\mathfrak{p}}_T. \tag{10.188}$$

Also $\overline{\mathbb{P}}_T$ lies in $(\Lambda^\bullet(T^*S))^{\mathrm{even}} \otimes \mathrm{End}\left(I^0\right)^{\mathrm{even}}$. For any $k' \in \mathbf{N}^*$,

$$\overline{\mathbb{P}}_T = \frac{1}{2i\pi} \int_\delta \frac{d\lambda}{\lambda - \mathcal{D}_T^{I\widehat{\otimes}F|_{\mathbf{B}},k'}}. \tag{10.189}$$

More generally, the obvious analogue of Theorem 10.9 still holds. In particular the linear map $\alpha \in \Lambda^\bullet(T^*S) \widehat{\otimes} \overline{\mathfrak{f}}_T \to \overline{\mathbb{P}}_T\alpha \in \overline{\mathbb{F}}_T$ is an isomorphism of $\mathbf{Z}_2$-graded vector bundles.

When acting on $I \widehat{\otimes} F|_{\mathbf{B}}$, $\overline{\mathbb{P}}_T$ is of the form $\overline{\mathbb{P}}_T \otimes 1$, and the first factor does not depend on $\left(F|_{\mathbf{B}}, \nabla^{F|_{\mathbf{B}}}, g^{F|_{\mathbf{B}}}\right)$.

For $k \in \mathbf{N}$ large enough,

$$\overline{\mathbb{F}}_T = \ker \overline{\mathcal{D}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},k}. \tag{10.190}$$

In particular, for n' large enough,

$$\overline{\mathbb{F}}_T \subset \exp\left(-T|Z|^2/2\right) Q^{n'}\left(TX^u|_{\mathbf{B}}\right) \widehat{\otimes} \Lambda^\bullet(T^*S) \widehat{\otimes} \Lambda^\bullet(T^*X_B) \widehat{\otimes} F|_{\mathbf{B}}. \tag{10.191}$$

Proof. — The proof of the first part of our Theorem is the same as the proof of Theorem 10.9. The key point is that the resolvent of the harmonic oscillator $\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2,(0)}$ is compact.

Recall that the operators $\overline{\mathcal{C}}_T, \overline{\mathcal{D}}_T$ were introduced in (4.11). We claim that in (10.187), we can replace $\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2}$ by $\overline{\mathcal{C}}_T^2$. In fact by (10.11),

$$\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2} - \overline{\mathcal{C}}_T^2 = -\frac{1}{4}\omega^2\left(F|_{\mathbf{B}}, g^{F|_{\mathbf{B}}}\right). \tag{10.192}$$

so that $\overline{\mathcal{C}}_T^2$ commutes with $\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2} - \overline{\mathcal{C}}_T^2$. Then if $\lambda \in \delta$,

$$\left(\lambda - \overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2}\right)^{-1} = \left(\lambda - \overline{\mathcal{C}}_T^2\right)^{-1} + \left(\lambda - \overline{\mathcal{C}}_T^2\right)^{-1}\left(\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2} - \overline{\mathcal{C}}_T^2\right)\left(\lambda - \overline{\mathcal{C}}_T^2\right)^{-1} + \cdots \tag{10.193}$$

and the expansion terminates after a finite number of terms. By (10.193) using the above commutation properties, we find that, when integrating over δ, only the first term in the expansion contributes to the integral. The above shows that in (10.189), we can as well replace $\overline{\mathcal{D}}_T^{I\widehat{\otimes}F|_{\mathbf{B}}}$ by $\overline{\mathcal{D}}_T$. In particular, when acting on $I^0 \widehat{\otimes} F|_{\mathbf{B}}$, the operator $\overline{\mathbb{P}}_T$ is of the form $\overline{\mathbb{P}}_T \widehat{\otimes} 1$.

Equation (10.190) follows from [**ReSi**, Theorem XII.5]. Clearly,

$$I_x^0 = L_2\left(TX|_{\mathbf{B}}\right) \widehat{\otimes} \Lambda^\bullet(T^*S) \widehat{\otimes} \Lambda\left(T^*X|_{\mathbf{B}}\right) \widehat{\otimes} F|_{\mathbf{B}}. \tag{10.194}$$

Using (10.10), we see that $\overline{\mathcal{D}}_T^{I\widehat{\otimes}F|_{\mathbf{B}}}$ can be expressed as linear combination of creation and annihilation operators acting on $L_2\left(TX|_{\mathbf{B}}\right)$ and on $\Lambda^\bullet(T^*S) \widehat{\otimes} \Lambda\left(T^*X_B\right) \widehat{\otimes} F|_{\mathbf{B}}$.

A simple recursion procedure shows that for any $k \in \mathbf{N}^*$, there is $n' \in \mathbf{N}$ such that (10.195)

$$\ker \overline{\mathcal{D}}_T^{I \widehat{\otimes} F|_{\mathbf{B}},k} \subset \exp\left(-T|Z|^2/2\right) Q^{n'}\left(TX^u|_{\mathbf{B}}\right) \widehat{\otimes} \Lambda^\bullet\left(T^*S\right) \widehat{\otimes} \Lambda^\bullet\left(T^*X_B\right) \widehat{\otimes} F|_{\mathbf{B}}. \tag{10.195}$$

By (10.190) and (10.195), we get (10.191). The proof of our Theorem is completed. □

10.15. The maps J_T and $\overline{\mathbf{e}}_T$

Put

$$\overline{\mathbf{P}}_T^{[0,1]} = \frac{1}{2i\pi} \int_\delta \frac{d\lambda}{\lambda - \overline{A}_T^2}. \tag{10.196}$$

Then, by (10.19),

$$\overline{\mathbf{P}}_T^{[0,1]} = e^{-Tf} \mathbf{P}_T^{[0,1]} e^{Tf}. \tag{10.197}$$

So $\overline{\mathbf{P}}_T^{[0,1]}$ is a projector acting on $\Lambda^\bullet\left(T^*S\right) \widehat{\otimes} \Omega^\bullet\left(X, F|_X\right)$, whose finite dimensional image $\overline{\mathbf{F}}_T^{[0,1]}$ is a finite dimensional subbundle of $\Lambda^\bullet\left(T^*S\right) \widehat{\otimes} \Omega^\bullet\left(X, F|_X\right)$.

Let $\gamma : \mathbf{R} \to [0,1]$ be a smooth function such that

$$\begin{aligned} \gamma(a) &= 1 \text{ for } a < \frac{1}{2}, \\ &= 0 \text{ for } a > 1. \end{aligned} \tag{10.198}$$

If $Z \in \mathbf{R}^n$, set

$$\mu(Z) = \gamma\left(Z/|\varepsilon_0|\right). \tag{10.199}$$

Then

$$\begin{aligned} \mu\left(Z\right) &= 1 \text{ if } |Z| \leqslant \varepsilon_0/2, \\ &= 0 \text{ if } |Z| \geqslant \varepsilon_0. \end{aligned} \tag{10.200}$$

If $T > 0$, set

$$\alpha_T = \int_{\mathbf{R}^n} \mu^2\left(Z\right) \exp\left(-T|Z|^2\right) dZ. \tag{10.201}$$

Then there is $c > 0$ such that as $T \to +\infty$,

$$\alpha_T = \left(\frac{\pi}{T}\right)^{n/2} + \mathcal{O}\left(e^{-cT}\right). \tag{10.202}$$

Take $x \in \mathbf{B}$. Let $\rho_x \in \Lambda^{\max}\left(T^{\mathrm{u},*}X|_{\mathbf{B}}\right)_x$ be of norm 1. Then ρ_x is determined up to sign. It defines a section of $o(x) \otimes \Lambda^{\max}\left(T^{\mathrm{u},*}X|_{\mathbf{B}}\right)_x$.

Definition 10.54. — Let $\mathbf{J}_T : \Lambda^\bullet\left(T^*S\right) \widehat{\otimes} C^\bullet\left(W^u, F\right) \to \Lambda^\bullet\left(T^*S\right) \widehat{\otimes} \Omega^\bullet\left(X, F|_X\right)$ be such that if $h \in o(x) \otimes F_x$, then

$$\mathbf{J}_T h = \frac{\mu\left(Z\right)}{\alpha_T^{1/2}} \overline{\mathbb{P}}_T \left[\exp\left(-T|Z|^2/2\right) \rho_x\right] h. \tag{10.203}$$

Using (10.188), we find that the induced map $J_T : C^\bullet(W^u, F) \to \Omega^\bullet\big(X, F|_X\big)$ is given by

$$J_T h = \frac{\mu(Z)}{\alpha_T^{1/2}} \exp\left(-T|Z|^2/2\right) \rho_x h. \tag{10.204}$$

Definition 10.55. — Let $\overline{\mathbf{e}}_T : \Lambda^\bullet(T^*S) \mathbin{\widehat{\otimes}} C^\bullet(W^u, F) \to \overline{\mathbf{F}}_T^{[0,1]}$ be given by

$$\overline{\mathbf{e}}_T = \overline{\mathbf{P}}_T^{[0,1]} \mathbf{J}_T. \tag{10.205}$$

Clearly $\overline{\mathbf{e}}_T$ commutes with the action of $\Lambda^\bullet(T^*S)$. By (10.22), (10.51), (10.197), (10.204 and (10.205), the induced map $\overline{e}_T : C^\bullet(W^u, F) \to \Omega^\bullet\big(X, F|_X\big)$ is given by

$$\overline{e}_T = \overline{P}_T^{[0,1]} J_T. \tag{10.206}$$

In the sequel, we write that as $T \to +\infty$, a family of smooth sections on M is $\mathcal{O}\left(e^{-cT}\right)$ if the sup norm of the derivatives is $\mathcal{O}\left(e^{-cT}\right)$. We now have an extension of [**BZ2**, Theorem 6.7].

Theorem 10.56. — *There is $c > 0$ such that as $T \to +\infty$, for any $s \in C^\bullet(W^u, F)$,*

$$\left(\overline{\mathbf{e}}_T - \mathbf{J}_T\right) s = \mathcal{O}\left(e^{-cT}\right) \text{ uniformly on } M. \tag{10.207}$$

Proof. — The proof proceeds very much as the proof in [**BZ1**, Theorem 8.8] and in [**BZ2**, Theorem 6.7]. The essential difference is that the considered operators are not 'self-adjoint' in the classical sense. By (10.54) and (10.196), we know that for any $k \in \mathbf{N}^*$,

$$\overline{\mathbf{P}}_T^{[0,1]} = \frac{1}{2i\pi} \int_\delta \frac{d\lambda}{\lambda - \overline{A}_T^{2k}}. \tag{10.208}$$

In the sequel, we choose $k \in \mathbf{N}^*$ large enough so that (10.190) holds.

Take $x \in B, h \in o(x) \otimes F_x$. If $\lambda \in \delta$,

$$\left(\lambda - \overline{A}_T^{2k}\right) \frac{\mathbf{J}_T h}{\lambda} - \mathbf{J}_T h = -\overline{A}_T^{2k} \frac{\mathbf{J}_T h}{\lambda}, \tag{10.209}$$

and so,

$$\frac{\mathbf{J}_T h}{\lambda} - \left(\lambda - \overline{A}_T^{2k}\right)^{-1} \mathbf{J}_T h = -\left(\lambda - \overline{A}_T^{2k}\right)^{-1} \frac{\overline{A}_T^{2k} \mathbf{J}_T h}{\lambda}. \tag{10.210}$$

Now, by our fundamental assumptions in Section 9.1, with the required identifications, on $\left\{x' \in X, d^X(x, x') \leqslant \varepsilon\right\}$, the operator $\overline{A}_T^2$ coincides with $\overline{\mathcal{C}}_T^{I \widehat{\otimes} F|_{\mathbf{B}}, 2}$. Since $\mu(Z) = 1$ for $|Z| \leqslant \varepsilon/2$, using (10.190), we get

$$\overline{A}_T^{2k} \mathbf{J}_T h(Z) = 0 \text{ for } |Z| \leqslant \varepsilon/2. \tag{10.211}$$

By (10.191), we deduce from (10.211) that there exists $c > 0$ such that as $T \to +\infty$,

$$\left|\overline{A}_T^{2k} \mathbf{J}_T h\right| = \mathcal{O}\left(e^{-cT}\right) |h|. \tag{10.212}$$

Using (3.45), we see that

$$\overline{A}_T^{2k} = \left(\frac{D^X}{2}\right)^{2k} + K_T, \tag{10.213}$$

where K_T is a differential operator of order $2k-1$, whose coefficients depend polynomially on T, the polynomial being of degree $2k$.

If $q \in \mathbf{R}$, let $\| \ \|_q$ be a smooth family of norms on the fibrewise q^{th} Sobolev space of sections of $\Lambda^\bullet(T^*X) \widehat{\otimes} \Lambda^\bullet(T^*S)$.

Since $D^{X,2k}$ is elliptic of degree $2k$, given $q \in \mathbf{N}$, there exists $C > 0$ such that for $s \in \Lambda^\bullet(T^*S) \widehat{\otimes} \Omega^\bullet\big(X, F|_X\big)$,

$$\|s\|_{q+2k} \leqslant C\left(\left\|D^{X,2k}s\right\|_q + \|s\|_0\right). \tag{10.214}$$

By the considerations which follow (10.213) and by (10.214), we see that given $q \in \mathbf{N}$, there exists $C > 0$ such that for $\lambda \in \delta, T \geqslant 1$,

$$\|s\|_{q+2k} \leqslant C\left(\left\|\left(\lambda - \overline{A}_T^{2k}\right)s\right\|_q + T^{2k}\|s\|_{q+2k-1}\right). \tag{10.215}$$

Also given $q \in \mathbf{N}$, there exists $C > 0$ such that for $A > 0, s \in \Lambda^\bullet(T^*S) \widehat{\otimes} \Omega^\bullet\big(X, F|_X\big)$,

$$\|s\|_{q+2k-1} \leqslant C\left(\frac{\|s\|_{q+2k}}{A} + A^{q+2k-1}\|s\|_0\right). \tag{10.216}$$

From (10.215), (10.216), we deduce that there exists $C > 0, k' \in \mathbf{N}$ such that for $\lambda \in \delta, s \in \Lambda^\bullet(T^*S) \widehat{\otimes} \Omega^\bullet\big(X, F|_X\big)$,

$$\|s\|_{q+2k} \leqslant C\left(\|\left(\lambda - \overline{A}_T^{2k}\right)s\|_q + T^{2k'}\|s\|_0\right). \tag{10.217}$$

Also by (10.23), for $T \geqslant T_0$, we know that $\mathrm{Sp}\left(\overline{A}_T^{2k,(0)}\right) \cap \delta = \varnothing$. More precisely, since $\overline{A}_T^{2k,(0)}$ is self-adjoint, by (10.23), there exists $C' > 0$ such that for $T \geqslant T_0, \lambda \in \delta, s \in \Omega^\bullet\big(X, F|_X\big)$,

$$\left\|\left(\lambda - \overline{A}_T^{2k,(0)}\right)^{-1} s\right\|_0 \leqslant C'\|s\|_0. \tag{10.218}$$

By (10.217), (10.218), we get

$$\left\|\left(\lambda - \overline{A}_T^{2k,(0)}\right)^{-1} s\right\|_{q+2k} \leqslant C'T^{2k'}\|s\|_q. \tag{10.219}$$

Moreover,

$$\left(\lambda - \overline{A}_T^{2k}\right)^{-1} = \left(\lambda - \overline{A}_T^{2k,(0)}\right)^{-1} + \left(\lambda - \overline{A}_T^{2k,(0)}\right)^{-1} \overline{A}_T^{2k,(>0)} \left(\lambda - \overline{A}_T^{2k,(0)}\right)^{-1} + \cdots \tag{10.220}$$

and the expansion in (10.220) contains a finite number of terms. Also by Theorem 3.19 or by (10.39), $\overline{A}_T^{2k,(>0)}$ is a differential operator of order $2k-1$, which depends polynomially on T.

By (10.219), (10.220), we find that there exists $C'' > 0, k'' \in \mathbf{N}$ such that for $\lambda \in \delta, T \geqslant T_0, s \in \Omega^\bullet\big(X, F|_X\big)$,

$$\| \left(\lambda - \overline{A}_T^{2k}\right)^{-1} s\|_{q+2k} \leqslant C'' T^{2k''} \|s\|_q. \tag{10.221}$$

From (10.212), (10.221), we deduce that there exists $c > 0$ such that to $T \geqslant T_0$,

$$\| \left(\lambda - \overline{A}_T^{2k}\right)^{-1} \overline{A}_T^{2k} \mathbf{J}_T h\|_{q+2k} = \mathcal{O}\left(e^{-cT}\right) \|h\|. \tag{10.222}$$

Using (10.222) and Sobolev's inequalities, we see that there exists $c > 0$ such that for $\lambda \in \delta, T \geqslant T_0, x \in B, h \in F_x$,

$$\left| \left(\lambda - \overline{A}_T^{2k}\right)^{-1} \overline{A}_T^{2k} \mathbf{J}_T h \right| = \mathcal{O}\left(e^{-cT}\right) \|h\|. \tag{10.223}$$

From (10.208), (10.210), (10.223), we get (10.207). The proof of our Theorem is completed. □

Definition 10.57. — For $T \geqslant T_0$, let $\mathbf{e}_T : \Lambda^\bullet(T^*S) \widehat{\otimes} C^\bullet(W^u, F) \to \mathbf{F}_T^{[0,1]}$ be the linear map,

$$\mathbf{e}_T = e^{Tf} \overline{\mathbf{e}}_T. \tag{10.224}$$

By (10.197), (10.205), we get

$$\mathbf{e}_T = \mathbf{P}_T^{[0,1]} e^{Tf} \mathbf{J}_T. \tag{10.225}$$

Then $\mathbf{e}_T$ commutes with $\Lambda^\bullet(T^*S)$. The induced map $e_T : C^\bullet(W^u, F) \to \Omega^\bullet\big(X, F|_X\big)$ is given by,

$$e_T = P_T^{[0,1]} e^{Tf} J_T. \tag{10.226}$$

In the sequel, we consider $\mathbf{e}_T$ as a linear map from $\Lambda^\bullet(T^*S) \widehat{\otimes} C^\bullet(W^u, F)$ into $\Lambda^\bullet(T^*S) \widehat{\otimes} \Omega^\bullet\big(X, F|_X\big)$. Let $\mathbf{e}_T^* : \Lambda^\bullet(T^*S) \widehat{\otimes} \Omega^\bullet\big(X, F|_X\big) \to \Lambda^\bullet(T^*S) \widehat{\otimes} C^\bullet(W^u, F)$ be the adjoint of $\mathbf{e}_T$ with respect to the metrics $g^{C^\bullet(W^u,F)}, g_T^{\Omega^\bullet\left(X,F|_X\right)}$.

Recall that $C^\bullet(W^u, \mathbf{R}) = \bigoplus_{x \in B} o(x)$. In the sequel, we will denote by $\mathcal{O}_D(1/T)$ an element of $\Lambda^\bullet(T^*S) \widehat{\otimes} \mathrm{End}\left(C^\bullet(W^u, \mathbf{R})\right)$ which commutes with $\Lambda^\bullet(T^*S)$, which is of positive degree in $\Lambda^\bullet(T^*S)$, which preserves the $\Lambda^\bullet(T^*S) \widehat{\otimes} o(x)$, which is $\mathcal{O}(1/T)$ as $T \to +\infty$. Of course $\mathcal{O}_D(1/T)$ then acts on $C^\bullet(W^u, F)$, and preserves the $o(x) \otimes F_x, x \in B$. Also the various $\mathcal{O}_D(1/T)$ commute with each other.

Proposition 10.58. — *As* $T \to +\infty$,

$$\mathbf{e}_T^* \mathbf{e}_T = 1 + \mathcal{O}_D(1/T) + \mathcal{O}\left(e^{-cT}\right). \tag{10.227}$$

Proof. — If $\overline{\mathbf{e}}_T^*$ be the adjoint of $\overline{\mathbf{e}}_T$ with respect to the metrics $g^{C^\bullet(W^u,F)}, g^{\Omega^\bullet\left(X,F|_X\right)}$, then

$$\mathbf{e}_T^* \mathbf{e}_T = \overline{\mathbf{e}}_T^* \overline{\mathbf{e}}_T. \tag{10.228}$$

By Theorem 10.56, if $s \in C^\bullet(W^u, F)$,

$$\overline{\mathbf{e}}_T s - \mathbf{J}_T s = \mathcal{O}\left(e^{-cT}\right). \tag{10.229}$$

By Theorem 10.53, $\overline{\mathbb{P}}_T$ is of the form $\overline{\mathbb{P}}_T \otimes 1$. Using (10.203), we get

$$\mathbf{J}_T h = \frac{\mu(Z)}{\alpha_T^{1/2}} \overline{\mathbb{P}}_T \left[\exp\left(-T|Z|^2/2\right) \rho_x\right] h. \tag{10.230}$$

Recall that by Theorem 10.53, when acting on $I \widehat{\otimes} F|_{\mathbf{B}}$, $\overline{\mathbb{P}}_T$ is of the form $\overline{\mathbb{P}}_T \otimes 1$, i.e. we may temporarily assume that $F = \mathbf{R}$. If $v \in \mathbf{R}_+^*$, we define f_v as in (4.49). By (10.11),

$$\overline{\mathcal{C}}_T^2 = \psi_T^{-1} F_{\sqrt{T}}^{-1} T \overline{\mathcal{C}}_1^2 F_{\sqrt{T}} \psi_T. \tag{10.231}$$

Using (10.189) and (10.231), we get

$$\overline{\mathbb{P}}_T = \psi_{\sqrt{T}}^{-1} F_{\sqrt{T}}^{-1} \overline{\mathbb{P}}_1 F_{\sqrt{T}} \psi_{\sqrt{T}}. \tag{10.232}$$

Put

$$\overline{\mathbb{P}}_1 \left(\exp\left(-|Z|^2/2\right) \rho_x\right) = k. \tag{10.233}$$

Using (10.188), we find that $k^{(0)}$ is given by

$$k^{(0)} = \exp\left(-|Z|^2/2\right) \rho_x. \tag{10.234}$$

From (10.231), (10.233), we deduce that

$$\overline{\mathbb{P}}_T \left(\exp\left(-T|Z|^2/2\right) \rho_x\right) = \psi_{\sqrt{T}}^{-1} F_{\sqrt{T}}^{-1} k. \tag{10.235}$$

By (10.191), (10.202), (10.228), (10.229), (10.230), ((10.234), (10.235), we get (10.227). The proof of our Proposition is completed. □

Clearly, $\mathbf{P}_T^\infty \mathbf{e}_T \in \Lambda^\bullet(T^*S) \widehat{\otimes} \operatorname{End}(C^\bullet(W^u, F))$. Now we establish an extension of [**BZ2**, Theorem 6.11].

Theorem 10.59. — *There exists $c > 0$ such that as $T \to +\infty$,*

$$\mathbf{P}_T^\infty \mathbf{e}_T = e^{T\mathcal{F}} \left(\frac{\pi}{T}\right)^{N^{C^\bullet(W^u, F)}/2 - n/4} \left(1 + \mathcal{O}_D(1/T) + \mathcal{O}\left(e^{-cT}\right)\right). \tag{10.236}$$

In particular for $T \geqslant T_0'$ large enough, $\mathbf{P}_T^\infty \mathbf{e}_T \in \left(\Lambda^\bullet(T^*S) \widehat{\otimes} \operatorname{End}(C^\bullet(W^u, F))\right)$ is invertible.

Proof. — Take $h \in o(x) \otimes F_x$. Then by (5.61), (10.224),

$$\mathbf{P}_T^\infty \mathbf{e}_T h = \sum_{y \in B} e^{Tf(y)} \int_{\overline{W^u(y)}} e^{T(f - f(y))} \overline{\mathbf{e}}_T h. \tag{10.237}$$

Observe that on $\overline{W}^u(y)$,

$$f - f(y) \leqslant 0. \tag{10.238}$$

By Theorem 10.56 and by (10.238), given $y \in B$,

$$\int_{\overline{W^u(y)}} e^{T(f-f(y))}\overline{\mathbf{e}}_T h = \int_{\overline{W^u(y)}} e^{T(f-f(y))}\mathbf{J}_T h + \mathcal{O}\left(e^{-cT}\right). \tag{10.239}$$

By (10.2), (10.230), (10.235), we get

$$\begin{aligned}\int_{\overline{W^u(x)}} e^{T(f-f(x))}\mathbf{J}_T h &= \frac{1}{\alpha_T^{1/2}}\psi_{1/\sqrt{T}}\int_{T_x^u X}\mu(Z)\exp\left(-T|Z|^2/2\right)F_{1/\sqrt{T}}k(Z)h \\ &= \frac{1}{\alpha_T^{1/2}T^{\mathrm{ind}(x)/2}}\psi_{1/\sqrt{T}}\int_{T_x^u X}\mu\left(Z/\sqrt{T}\right)\exp\left(-|Z|^2/2\right)k(Z)\,h.\end{aligned} \tag{10.240}$$

Using (10.191), (10.202), (10.234), (10.235), (10.239), (10.240), we find that

$$\int_{\overline{W^u(x)}} e^{T(f-f(x))}\overline{\mathbf{e}}_T h = \left(\frac{\pi}{T}\right)^{\mathrm{ind}(x)/2-n/4}(1+\mathcal{O}(1/T))\,h. \tag{10.241}$$

Since the support of $\mathbf{J}_T h$ is included in a small ball centred at x, if $y \in B$, the integral $\int_{\overline{W^u(y)}} e^{T(f-f(y))}\mathbf{J}_T h$ is non zero if and only if $x \in \overline{W^u(y)}$. As we saw in Section 5.1, if $y \in B, y \neq x$, if $x \in \overline{W^u(y)}$, then $f(x) < f(y)$. More precisely, there exists $c > 0$ such that on the support of μ,

$$f - f(y) \leqslant -c. \tag{10.242}$$

From (10.191), (10.230), (10.235), (10.242), we deduce that if $y \in B, y \neq x$,

$$\int_{\overline{W^u(y)}} e^{T(f-f(y))}\mathbf{J}_T h = \mathcal{O}\left(e^{-cT}\right). \tag{10.243}$$

From (10.237), (10.241), (10.243), we get (10.236). The proof of our Theorem is completed. □

10.16. A proof of the first part of Theorem 10.50

By Theorem 10.59, for $T \geqslant T_0'$ large enough, the map $\mathbf{P}_T^\infty e_T$ is invertible. Therefore, for T large enough,

$$\mathbf{g}_T^{C^\bullet(W^u,F)} = \left(\mathbf{P}_T^\infty \mathbf{e}_T\right)^{*,-1}\left(\mathbf{e}_T^* \mathbf{e}_T\right)\left(\mathbf{P}_T^\infty \mathbf{e}_T\right)^{-1}. \tag{10.244}$$

By (10.178),

$$\begin{aligned}&e^{-T\mathcal{F}}t^{N^{C^\bullet(W^u,F)}/2}A^{C^\bullet(W^u,F)\prime}t^{-N^{C^\bullet(W^u,F)}/2}e^{T\mathcal{F}} \\ &\qquad = \left(\sqrt{t}\,e^{-T\mathcal{F}}\partial e^{T\mathcal{F}} + \nabla^{C^\bullet(W^u,F)} + Td\mathcal{F}\right), \\ &e^{-T\mathcal{F}}t^{N^{C^\bullet(W^u,F)}/2}A_{T,t}^{C^\bullet(W^u,F)\prime\prime}t^{-N^{C^\bullet(W^u,F)}/2}e^{T\mathcal{F}} \\ &\qquad = \left(\mathbf{g}_T^{C^\bullet(W^u,F)}e^{T\mathcal{F}}\right)^{-1}\left(\sqrt{t}\,\partial^* + \nabla^{C^\bullet(W^u,F),*}\right)\mathbf{g}_T^{C^\bullet(W^u,F)}e^{T\mathcal{F}}.\end{aligned} \tag{10.245}$$

Set

$$\mathbf{k}_T = e^{-T\mathcal{F}}\mathbf{P}_T^{\infty}\mathbf{e}_T. \tag{10.246}$$

Clearly, by (10.244), (10.246), we get

$$e^{T\mathcal{F}}\mathbf{g}_T^{C^\bullet(W^u,F)}e^{T\mathcal{F}} = (\mathbf{k}_T)^{-1,*}\left(\mathbf{e}_T^*\mathbf{e}_T\right)(\mathbf{k}_T)^{-1}. \tag{10.247}$$

From (10.244)-(10.247), we obtain,

$$\begin{aligned} e^{-T\mathcal{F}}t^{N^{C^\bullet(W^u,F)}/2}A_{T,t}^{C^\bullet(W^u,F)\prime\prime}t^{-N^{C^\bullet(W^u,F)}/2}e^{T\mathcal{F}} &= \mathbf{k}_T\left(\mathbf{e}_T^*\mathbf{e}_T\right)^{-1}\mathbf{k}_T^* \\ &\left(\sqrt{t}\,e^{T\mathcal{F}}\partial^* e^{-T\mathcal{F}} + \nabla^{C^\bullet(W^u,F),*} - Td\mathcal{F}\right)\left(\mathbf{k}_T\left(\mathbf{e}_T^*\mathbf{e}_T\right)^{-1}\mathbf{k}_T^*\right)^{-1}. \end{aligned} \tag{10.248}$$

Also by Proposition 10.58 and Theorem 10.59, as $T \to +\infty$,

$$\mathbf{k}_T\left(\mathbf{e}_T^*\mathbf{e}_T\right)^{-1}\mathbf{k}_T^* = \left(\frac{\pi}{T}\right)^{N^{C^\bullet(W^u,F)}-n/2}\left(1+\mathcal{O}_D\left(1/T\right)\right) + \mathcal{O}\left(e^{-cT}\right). \tag{10.249}$$

Needless to say, both sides in (10.249) are even, including $\mathcal{O}_D\left(1/T\right)$. In particular $\mathcal{O}_D\left(1/T\right)$ commutes with $Td\mathcal{F}$. By (10.249), we deduce that

$$\mathbf{k}_T\left(\mathbf{e}_T^*\mathbf{e}_T\right)^{-1}\mathbf{k}_T^*\left(-Td\mathcal{F}\right)\left(\mathbf{k}_T\left(\mathbf{e}_T^*\mathbf{e}_T\right)^{-1}\mathbf{k}_T^*\right)^{-1} = -Td\mathcal{F} + \mathcal{O}_D\left(1/T\right) + \mathcal{O}\left(e^{-cT}\right). \tag{10.250}$$

Now by the results of Section 5.1, if $x \in B$, the chain map ∂ maps $o(x)\otimes F_x$ a direct sum of $C^\bullet\left(W^u,F\right)_y$, which are such that $f(y) > f(x)$. So there exists $c > 0$ such that as $T \to +\infty$,

$$e^{-T\mathcal{F}}\partial e^{T\mathcal{F}} = \mathcal{O}\left(e^{-cT}\right), \qquad e^{T\mathcal{F}}\partial^* e^{-T\mathcal{F}} = \mathcal{O}\left(e^{-cT}\right). \tag{10.251}$$

From (10.245)-(10.251), we deduce that given $t \in]0,1]$, as $T \to +\infty$,

$$\begin{aligned} e^{-T\mathcal{F}}t^{N^{C^\bullet(W^u,F)}/2}A_{T,t}^{C^\bullet(W^u,F)}t^{-N^{C^\bullet(W^u,F)}/2}e^{T\mathcal{F}} &= \nabla^{C^\bullet(W^u,F),u} \\ &+ \mathcal{O}_D\left(1/T\right) + \left(1+\sqrt{t}\right)\mathcal{O}\left(e^{-cT}\right). \end{aligned} \tag{10.252}$$

By (2.143), (2.150),

$$\begin{aligned} &\int_0^1\left(\varphi\mathrm{Tr}_{\mathrm{s}}\left[\frac{1}{2}\left(\mathfrak{g}_{T,t}^{C^\bullet(W^u,F)}\right)^{-1}\frac{\partial}{\partial t}\mathfrak{g}_{T,t}^{C^\bullet(W^u,F)}gh'\left(B_{T,t}^{C^\bullet(W^u,F)}\right)\right] - \frac{\eta_T}{t}\right)dt \\ &= \int_0^1\varphi\mathrm{Tr}_{\mathrm{s}}\left[\left(N^{C^\bullet(W^u,F)} + \left(\mathbf{g}_T^{C^\bullet(W^u,F)}\right)^{-1}N^{C^\bullet(W^u,F)}\mathbf{g}_T^{C^\bullet(W^u,F)}\right)\right. \\ &\qquad \left.g\left(h'\left(B_{T,t}^{C^\bullet(W^u,F)}\right) - h'\left(B_{T,0}^{C^\bullet(W^u,F)}\right)\right)\right]\frac{dt}{2t}. \end{aligned} \tag{10.253}$$

Also, by (10.247),

$$\begin{aligned} &e^{-T\mathcal{F}}\left(N^{C^\bullet(W^u,F)} + \left(\mathbf{g}_T^{C^\bullet(W^u,F)}\right)^{-1}N^{C^\bullet(W^u,F)}\mathbf{g}_T^{C^\bullet(W^u,F)}\right)e^{T\mathcal{F}} \\ &= N^{C^\bullet(W^u,F)} + \mathbf{k}_T\left(\mathbf{e}_T^*\mathbf{e}_T\right)^{-1}\mathbf{k}_T^*N^{C^\bullet(W^u,F)}\left(\mathbf{k}_T\left(\mathbf{e}_T^*\mathbf{e}_T\right)^{-1}\mathbf{k}_T^*\right)^{-1}. \end{aligned} \tag{10.254}$$

By (10.249), (10.254), it is clear that

$$\mathbf{k}_T \left(\mathbf{e}_T^* \mathbf{e}_T\right)^{-1} \mathbf{k}_T^* N^{C^\bullet(W^u,F)} \left(\mathbf{k}_T \left(\mathbf{e}_T^* \mathbf{e}_T\right)^{-1} \mathbf{k}_T^*\right)^{-1}$$

remains uniformly bounded as $T \to +\infty$.

From (10.252), and from the above boundedness result, we see that as $T \to +\infty$, the integrand in the right-hand side of (10.253) tends to 0. Moreover, by (10.252), we can use dominated convergence in this integral, which then tends to 0 as $T \to +\infty$. We have thus established the first convergence result in Theorem 10.50.

10.17. A proof of the second part of Theorem 10.50

Clearly,

$$\begin{aligned}(10.255)\quad &\widetilde{h}_g\left(\nabla^{C^\bullet(W^u,F)}, \mathbf{g}_T^{C^\bullet(W^u,F)}, g^{C^\bullet(W^u,F)}\right)\\ &= \widetilde{h}_g\left(\nabla^{C^\bullet(W^u,F)}, \mathbf{g}_T^{C^\bullet(W^u,F)}, e^{-2T\mathcal{F}} g^{C^\bullet(W^u,F)}\right) +\\ &\qquad \widetilde{h}_g\left(\nabla^{C^\bullet(W^u,F)}, e^{-2T\mathcal{F}} g^{C^\bullet(W^u,F)}, g^{C^\bullet(W^u,F)}\right).\end{aligned}$$

By (1.26),

$$(10.256)\qquad \widetilde{h}_g\left(\nabla^{C^\bullet(W^u,F)}, e^{-2T\mathcal{F}} g^{C^\bullet(W^u,F)}, g^{C^\bullet(W^u,F)}\right) = T\mathrm{Tr}_{\mathrm{s}}^{B_g}[f].$$

Also,

$$\begin{aligned}(10.257)\quad &\widetilde{h}_g\left(\nabla^{C^\bullet(W^u,F)}, \mathbf{g}_T^{C^\bullet(W^u,F)}, e^{-2T\mathcal{F}} g^{C^\bullet(W^u,F)}\right)\\ &\qquad = \widetilde{h}_g\left(\nabla^{C^\bullet(W^u,F)} + Td\mathcal{F}, e^{T\mathcal{F}} \mathbf{g}_T^{C^\bullet(W^u,F)} e^{T\mathcal{F}}, g^{C^\bullet(W^u,F)}\right).\end{aligned}$$

Using (10.247), (10.249), (10.250), (10.257), we find that as $T \to +\infty$,

$$\begin{aligned}(10.258)\quad &\widetilde{h}_g\left(\nabla^{C^\bullet(W^u,F)}, \mathbf{g}_T^{C^\bullet(W^u,F)}, e^{-2T\mathcal{F}} g^{C^\bullet(W^u,F)}\right) + \frac{1}{4}\left(\widetilde{\chi}_g'^{-}(F) - \widetilde{\chi}_g'^{+}(F)\right)\log(T)\\ &\qquad \longrightarrow \frac{1}{4}\left(\widetilde{\chi}_g'^{-}(F) - \widetilde{\chi}_g'^{+}(F)\right)\log(\pi).\end{aligned}$$

From (10.255)-(10.258), we get the second equation in (10.186). The proof of Theorem 10.50 is completed.

CHAPTER 11

FIBREWISE NICE FUNCTIONS: A SECOND PROOF OF THEOREM 9.8

In this Chapter, we give a second proof of Theorem 9.8, under the extra assumption that f is fibrewise nice, and parallel with respect to the connection on M associated to $T^H M$. Instead of using generalized metrics as in Chapter 10, we use the precise estimates on the small eigenvalues of D_T^X which were obtained in Helffer-Sjöstrand [**HSj**] and in [**BZ2**], and also a technique due to Ma [**Ma1, Ma2**] to estimate analytic torsion forms in a different context.

This Chapter is organized as follows. In Section 11.1, we state precisely our extra assumption on f. In Section 11.2, we state two fundamental results, of which Theorem 9.8 is a consequence. The next Sections are devoted to the proof of these two results.

In Section 11.3, we recall various results from [**BZ2**], which were obtained in a more general form in Chapter 10. In Section 11.4, we give estimates on the spectrum of $B_T^{(0)}$. In Section 11.5, we give simple results on superconnection supertraces on $C^\bullet(W^u, F)$. In Section 11.6, we split the superconnection associated to B_T into three pieces, corresponding to large eigenvalues of $B_T^{(0)}$, to the very small non zero eigenvalues, and to the zero eigenvalue. Section 11.7 is devoted to the proof of two intermediate results. Sections 11.8-11.10 are devoted to the asymptotics of these three pieces as $T \to +\infty$. In Section 11.11, we establish a compatibility result for the asymptotics of certain supertraces as $t \to 0$. Finally, in Sections 11.12 and 11.13, we establish our two fundamental results.

In this Chapter, we use the notation and results of Chapters 1-9, and of Sections 10.1 and 10.2.

11.1. An extra simplifying assumption

In this Chapter, besides the simplifying assumptions of Section 9.1, we also assume that f is fibrewise nice, i.e. on $\mathbf{B}^i$, f takes the value i. As we saw in Section 5.5, this is not a restrictive assumption.

By the assumptions we made in Section 9.1, we find that since f is fibrewise nice,

$$\nabla^{\Omega^\bullet(X,F|X)} f = 0 \text{ on } U_{2\varepsilon_0} \simeq V_{2\varepsilon_0}. \tag{11.1}$$

By possibly taking a smaller ε_0, we may and we will assume that $T^H M$ is chosen so that

$$\nabla^{\Omega^\bullet(X,F|X)} f = 0. \tag{11.2}$$

Equivalently, $T^H M$ will be assumed to be included in the tangent bundle to the level sets of f.

11.2. Two fundamental results

Theorem 11.1. — *As* $T \to +\infty$,

(11.3)

$$\begin{aligned}\int_1^{+\infty} \left(\mathrm{Tr}_s\left[Ngh'\left(D_{t,T}\right)\right] - \chi_g'^{-}(F)\right)\frac{dt}{2t} - \frac{1}{2}\left(\widetilde{\chi}_g'^{-}(F) - \chi_g'(F)\right)(2T - \log(T)) \to \\ \int_0^1 \left(\mathrm{Tr}_s\left[N^{C^\bullet(W^u,F)} gh'\left(B_t^{C^\bullet(W^u,F)}\right)\right] - \widetilde{\chi}_g'^{-}(F)\right)\frac{dt}{2t} \\ + \int_1^{+\infty} \left(\mathrm{Tr}_s\left[N^{C^\bullet(W^u,F)} gh'\left(B_t^{C^\bullet(W^u,F)}\right)\right] - \chi_g'^{-}(F)\right)\frac{dt}{2t} \\ + \frac{1}{2}\left(\widetilde{\chi}_g'^{-}(F) - \chi_g'(F)\right)\log(\pi).\end{aligned}$$

Theorem 11.2. — *As* $T \to +\infty$,

(11.4)

$$\begin{aligned}\widetilde{h}_g\left(\nabla^{H^\bullet(X,F|X)}, g_{L_2,0}^{H^\bullet(X,F|X)}, g_{L_2,T}^{H^\bullet(X,F|X)}\right) + \chi_g'(F)\,T - \left(\frac{1}{2}\chi_g'(F) - \frac{n}{4}\chi_g(F)\right) \\ \log(T) \to \widetilde{h}_g\left(\nabla^{H^\bullet(X,F|X)}, g^{H^\bullet(X,F|X)} 0, g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|X)}\right) \\ - \left(\frac{1}{2}\chi_g'(F) - \frac{n}{4}\chi_g(F)\right)\log(\pi) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S).\end{aligned}$$

Proof. — The remainder of the Chapter is devoted to the proof of Theorems 11.1 and 11.2. □

Remark 11.3. — Clearly, Theorems 11.3 and 11.4 imply Theorem 9.8.

11.3. Preliminary results

Recall that α_T was defined in (10.201) by the formula

$$\alpha_T = \int_{\mathbf{R}^n} \mu^2(Z)\exp\left(-T|Z|^2\right)dZ. \tag{11.5}$$

Recall that $J_T : C^\bullet(W^u, F) \to \Omega^\bullet\big(X, F|_X\big)$ was defined in (10.204) by the formula

$$J_T h = \frac{\mu(Z)}{\alpha_T^{1/2}} \exp\left(-T|Z|^2/2\right) \rho_x h, \tag{11.6}$$

that $\overline{e}_T : C^\bullet(W^u, F) \to \overline{F}_T^{[0,1]}$ in (10.206) is given by

$$\overline{e}_T = \overline{P}_T^{[0,1]} J_T, \tag{11.7}$$

that $e_T : C^\bullet(W^u, F) \to F_T^{[0,1]}$ was defined in (10.226) by the formula

$$e_T = P_T^{[0,1]} e^{Tf} j_T, \tag{11.8}$$

so that

$$e_T = e^{Tf} \overline{e}_T. \tag{11.9}$$

Also $\overline{e}_T^*$ is the adjoint of $\overline{e}_T$ with respect to the metrics $g^{\Omega^\bullet\left(X,F|_X\right)}, g^{C^\bullet(W^u,F)}$, and e_T^* is the adjoint of e_T with respect to the metrics $g_T^{\Omega^\bullet\left(X,F|_X\right)}, g^{C^\bullet(W^u,F)}$.

The following results were established in [**BZ2**, Theorems 6.7, 6.9, 6.11 and 6.12].

Theorem 11.4. — *There exists $c > 0$ such that as $T \to +\infty$,*

$$\begin{aligned}
&\overline{e}_T - J_T = \mathcal{O}\left(e^{-cT}\right), \\
&e_T^* e_T = 1 + \mathcal{O}\left(e^{-cT}\right), \\
&P_T^\infty e_T = e^{T\mathcal{F}} \left(\frac{\pi}{T}\right)^{N^{C^\bullet(W^u,F)}/2 - n/4} \left(1 + \mathcal{O}\left(e^{-cT}\right)\right), \\
&e_T^{-1} d^X e_T = \left(\frac{T}{\pi}\right)^{1/2} e^{-T} \left(\partial + \mathcal{O}\left(e^{-cT}\right)\right),
\end{aligned} \tag{11.10}$$

Remark 11.5. — The first identity follows from Theorem 10.56. Proposition 10.58 shows that the second identity holds in (11.10). The third identity follows from Theorem 10.59. The fourth identity follows from the second and third identities, and also from the identity

$$P_T^\infty d^X \left(P_T^\infty\right)^{-1} = \partial, \tag{11.11}$$

which itself follows from Theorem 5.3.

Remark 11.6. — For $T \geqslant 0$ large enough, put

$$e'_T = e_T \left[e_T^* e_T\right]^{-1/2}. \tag{11.12}$$

Then

$$e_T'^* e'_T = 1. \tag{11.13}$$

By (11.10),

$$e'_T = e_T \left(1 + \mathcal{O}\left(e^{-cT}\right)\right). \tag{11.14}$$

It follows that in (11.10), we may as well replace e_T by e'_T.

11.4. The spectrum of $B_T^{(0)}$

Clearly,

$$\ker B^{C^\bullet(W^u,F),(0)} \simeq H^\bullet(X, F|_X). \tag{11.15}$$

Let $P_T^{C^\bullet(W^u,F),\{0\}} \to \ker B^{C^\bullet(W^u,F)}$ be the orthogonal projection operator on the vector bundle $\ker B^{C^\bullet(W^u,F)}$ with respect to the metric $g^{C^\bullet(W^u,F)}$. The spectrum of $B^{C^\bullet(W^u,F),(0)}$ is purely imaginary. There exists $d_1, d_2 \in \mathbf{R}_+^*$, with $d_1 < d_2/4$, such that

$$\left|\mathrm{Sp}\left(B^{C^\bullet(W^u,F),(0)}\right)\right| \subset \{0\} \cup [2d_1, d_2/2]. \tag{11.16}$$

Recall that $P_T^{\{0\}} : \Omega^\bullet(X, F|_X) \to \ker B_T^{(0)} \simeq H^\bullet(X, F|_X)$ is the orthogonal projection operator on $\ker B_T^{(0)}$ with respect to the metric $g_T^{\Omega^\bullet(X,F|_X)}$.

Theorem 11.7. — *For $T \geqslant 0$ large enough,*

$$\left|\mathrm{Sp}\left(B_T^{(0)}\right)\right| \subset \{0\} \cup \left(\frac{T}{\pi}\right)^{1/2} e^{-T}\left[\frac{3}{2}d_1, \frac{2}{3}d_2\right] \cup [1, +\infty[. \tag{11.17}$$

Moreover there exists $c > 0$ such that as $T \to +\infty$,

$$\left(e_T'\right)^{-1} P_T^{\{0\}} e_T' = P_T^{C^\bullet(W^u,F),\{0\}} + \mathcal{O}\left(e^{-cT}\right). \tag{11.18}$$

Proof. — By Hodge theory,

$$\ker B_T^{(0)} \simeq H^\bullet(X, F|_X). \tag{11.19}$$

In particular, by (11.15), (11.19, we get

$$\mathrm{rk}\left(\ker B_T^{(0)}\right) = \mathrm{rk}\left(\ker B^{C^\bullet(W^u,F),(0)}\right). \tag{11.20}$$

As observed in Remark 11.6, from (11.10), we get

$$\left(e_T'\right)^{-1} d^X e_T' = (T/\pi)^{1/2} e^{-T}\left(\partial + \mathcal{O}\left(e^{-cT}\right)\right). \tag{11.21}$$

Since $e_T' : C^\bullet(W^u, F) \to F_T^{[0,1]}$ is unitary, from (11.21), we get

$$\left(e_T'\right)^{-1} d_T^{X,*} e_T' = (T/\pi)^{1/2} e^{-T}\left(\partial^* + \mathcal{O}\left(e^{-cT}\right)\right). \tag{11.22}$$

From (11.21), (11.22), we get

$$(T/\pi)^{-1/2} e^T \left(e_T'\right)^{-1} B_T^{(0)} e_T' = B^{C^\bullet(W^u,F),(0)} + \mathcal{O}\left(e^{-cT}\right). \tag{11.23}$$

From (11.16), (11.20), (11.23), we get (11.17).

By (11.17), for $T \geqslant 0$ large enough,

$$P_T^{\{0\}} = \frac{1}{2i\pi}\int_{d_1\delta} \frac{d\lambda}{\lambda - (T/\pi)^{-1/2} e^T B_T^{(0)}}. \tag{11.24}$$

Using (11.23), (11.24), we get

$$\left(e_T'\right)^{-1} P_T^{\{0\}} e_T' = \frac{1}{2i\pi}\int_{d_1\delta} \frac{d\lambda}{\lambda - B^{C^\bullet(W^u,F),(0)}} + \mathcal{O}\left(e^{-cT}\right). \tag{11.25}$$

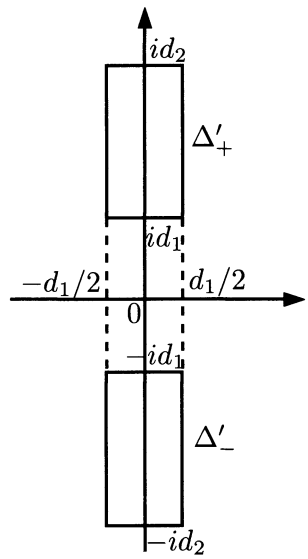

FIGURE 11.1

Also, by (11.16), we obtain,

$$P_T^{C^\bullet(W^u,F),\{0\}} = \frac{1}{2i\pi}\int_{d_1\delta} \frac{d\lambda}{\lambda - B^{C^\bullet(W^u,F),(0)}}. \tag{11.26}$$

From (11.25), (11.26), we get (11.18). The proof of our Theorem is completed. □

11.5. The superconnection supertraces associated to $C^\bullet(W^u, F)$

Recall that δ is the unit circle in $\mathbf{C}$. Let $\Delta' = \Delta'_+ \cup \Delta'_-$ be the contour in $\mathbf{C}$ indicated in Figure 11.1.

Definition 11.8. — Put

$$G_t^{C^\bullet(W^u,F)} = \psi_t^{-1}\frac{1}{2i\pi}\int_{\Delta'} \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - B^{C^\bullet(W^u,F)}} d\lambda \psi_t, \tag{11.27}$$

$$H_t^{C^\bullet(W^u,F)} = \psi_t^{-1}\int_{\frac{d_1}{2}\delta} \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - B^{C^\bullet(W^u,F)}} d\lambda \psi_t.$$

Proposition 11.9. — *For any $t > 0$,*

$$h'\left(D_t^{C^\bullet(W^u,F)}\right) = G_t^{C^\bullet(W^u,F)} + H_t^{C^\bullet(W^u,F)}. \tag{11.28}$$

Proof. — By (11.16), the spectrum of $B^{C^\bullet(W^u,F)}$ is included in the domain bounded by $\Delta' \cup (d_1/2)\,\delta$. Using Proposition 1.20, we get (11.28). □

Put

$$q = [\dim S/2] + 1. \tag{11.29}$$

Proposition 11.10. — *There exist smooth even forms* $a_0^{C^\bullet(W^u,F)}, \ldots, a_q^{C^\bullet(W^u,F)}$ *on* S *such that as* $t \to 0$,

$$\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)} g G_t^{C^\bullet(W^u,F)}\right] = \sum_{k=0}^{q} a_k^{C^\bullet(W^u,F)} t^{-k} + \mathcal{O}(t). \tag{11.30}$$

Proof. — Clearly, there is a holomorphic function $m(x)$ such that $h'(x) = m(x^2)$. By (11.27), we get

$$\psi_t G_t^{C^\bullet(W^u,F)} \psi_t^{-1} = \sum_{k=0}^{q} \frac{m^{(k)}(0)}{k!} \frac{1}{2i\pi} \int_{\Delta'} \frac{\lambda^{2k}}{\lambda - B^{C^\bullet(W^u,F)}} d\lambda t^k + o(t^q). \tag{11.31}$$

Let $\Delta'' \subset \mathbf{C}$ be the image of Δ' by the map $\lambda \to \lambda^2$. Then

$$\frac{1}{2i\pi} \int_{\Delta'} \frac{\lambda^{2k}}{\lambda - B^{C^\bullet(W^u,F)}} d\lambda = \frac{1}{2i\pi} \int_{\Delta''} \frac{\lambda^{k}}{\lambda - B^{C^\bullet(W^u,F),2}} d\lambda. \tag{11.32}$$

Using (11.32), we find that the form

$$\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)} g \frac{1}{2i\pi} \int_{\Delta'} \frac{\lambda^{2k}}{\lambda - B^{C^\bullet(W^u,F)}} d\lambda\right]$$

is even. Equation (11.30) now follows from (11.31). □

Now we proceed as in [**B10**, Theorem 9.29].

Theorem 11.11. — *Given* $t > 0$, *the following identity holds,*

$$H_t^{C^\bullet(W^u,F)} = \sum_{p=0}^{\dim S} \sum_{\substack{0 \leqslant i_0 \leqslant p+1 \\ j_1, \ldots, j_{p+1-i_0} \geqslant 0 \\ \sum_{k=1}^{p+1-i_0} j_k \leqslant i_0 - 1}} \frac{h^{\left(i_0 - 1 - \sum_{k=0}^{p+1-i_0} j_k\right)}(0)}{\left(i_0 - 1 - \sum_{k=0}^{p+1-i_0} j_k\right)!} (-1)^{p+1-i_0} C_1 B^{C^\bullet(W^u,F),(1)} C_2 \cdots B^{C^\bullet(W^u,F),(1)} C_{p+1}. \tag{11.33}$$

In (11.33), i_0 *of the* C_j*'s are equal to* $P^{C^\bullet(W^u,F),\{0\}}$, *the others are equal respectively to* $\left(\sqrt{t}\, B^{C^\bullet(W^u,F),(0)}\right)^{-(1+j_1)}, \ldots, \left(\sqrt{t}\, B^{C^\bullet(W^u,F),(0)}\right)^{-(1+j_{p+1-i_0})}$. *In particular,* $H_t^{C^\bullet(W^u,F)}$ *is a polynomial in the variable* $1/\sqrt{t}$.

As $t \to +\infty$,

$$H_t^{C^\bullet(W^u,F)} = P^{C^\bullet(W^u,F),\{0\}} h'\left(\frac{\omega\left(H^\bullet(X,F|_X), g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}\right)}{2}\right) P^{C^\bullet(W^u,F),\{0\}} + \mathcal{O}\left(1/\sqrt{t}\right). \tag{11.34}$$

Proof. — Using (11.16), we find that for $t \geqslant 1$,

$$\frac{1}{2i\pi}\int_{(d_1/2)\delta}\frac{h'(\sqrt{t}\,\lambda)}{\lambda - B^{C^\bullet(W^u,F)}}d\lambda = \frac{1}{2i\pi}\int_{(d_1/2\sqrt{t})\delta}\frac{h'(\sqrt{t}\,\lambda)}{\lambda - B^{C^\bullet(W^u,F)}}d\lambda = \frac{1}{2i\pi}\int_{(d_1/2)\delta}\frac{h'(\lambda)}{\lambda - \sqrt{t}\,B^{C^\bullet(W^u,F)}}d\lambda. \tag{11.35}$$

By Proposition 1.20 and by (11.35), for $t \geqslant 1$, we get

$$H_t^{C^\bullet(W^u,F)} = \frac{1}{2i\pi}\int_{(d_1/2)\delta}\frac{h'(\lambda)}{\lambda - D_t^{C^\bullet(W^u,F)}}d\lambda. \tag{11.36}$$

Clearly,

$$D_t^{C^\bullet(W^u,F)} = \sqrt{t}\,B^{C^\bullet(W^u,F),(0)} + B^{C^\bullet(W^u,F),(1)}. \tag{11.37}$$

By (11.37), we get

$$\left(\lambda - D_t^{C^\bullet(W^u,F)}\right)^{-1} = \left(\lambda - \sqrt{t}\,B^{C^\bullet(W^u,F),(0)}\right)^{-1} + \left(\lambda - \sqrt{t}\,B^{C^\bullet(W^u,F),(0)}\right)^{-1} B^{C^\bullet(W^u,F),(1)}\left(\lambda - \sqrt{t}\,B^{C^\bullet(W^u,F),(0)}\right)^{-1} + \cdots, \tag{11.38}$$

and the expansion in (11.38) only contains a finite number of terms. By (11.16), 0 is the only element inside the domain bounded by $(d_1/2)\,\delta$ which may lie in the spectrum of $B^{C^\bullet(W^u,F),(0)}$. Using (11.36), (11.38) and the theorem of residues, we get (11.33). Also, by (1.38),

$$P^{C^\bullet(W^u,F),\{0\}}B^{C^\bullet(W^u,F),(1)}P^{C^\bullet(W^u,F),\{0\}} = P^{C^\bullet(W^u,F),\{0\}}\frac{1}{2}\omega\left(\nabla^{H^\bullet(X,F|_X)}, g^{H^\bullet(X,F|_X)}_{C^\bullet(W^u,F)}\right)P^{C^\bullet(W^u,F),\{0\}}. \tag{11.39}$$

From (11.33), (11.39), (11.34) follows when $t \geqslant 1$. By analyticity, we get (11.34) for all $t > 0$. The proof of our Theorem is completed. □

Proposition 11.12. — *There are smooth even forms $b_0^{C^\bullet(W^u,F)}, \ldots, b_q^{C^\bullet(W^u,F)}$ on S such that*

$$\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}gH_t^{C^\bullet(W^u,F)}\right] = \sum_{k=0}^{q} b_k^{C^\bullet(W^u,F)}t^{-k}. \tag{11.40}$$

Also,

$$b_0^{C^\bullet(W^u,F)} = \chi'_g(F). \tag{11.41}$$

Proof. — Observe that in each of the terms in the right-hand side of (11.33), $1/\sqrt{t}$ appears with the power

$$p + 1 - i_0 + \sum_{k=1}^{p+1-i_0} j_k \leqslant p \leqslant q.$$

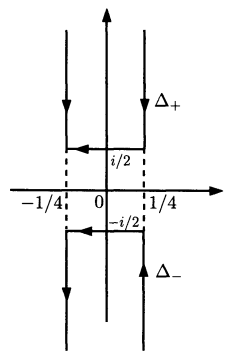

FIGURE 11.2

Also since $B^{C^\bullet(W^u,F),(0)}$ is an odd operator, and $B^{C^\bullet(W^u,F),(1)}$ is even as an operator, when multiplied by $N^{C^\bullet(W^u,F)}g$, the supertrace of the corresponding term is non zero only if the power of $1/\sqrt{t}$ is even. Our Proposition follows. □

Proposition 11.13. — *The following identities hold,*

$$a_k^{C^\bullet(W^u,F)} + b_k^{C^\bullet(W^u,F)} = 0 \text{ if } k > 0, \tag{11.42}$$
$$a_0^{C^\bullet(W^u,F)} = \widetilde{\chi}_g'^{-}(F) - \chi_g'(F).$$

Proof. — By Proposition 11.9,

$$\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)} g h'\left(D_t^{C^\bullet(W^u,F)}\right)\right] = \mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)} g G_t^{C^\bullet(W^u,F)}\right] + \mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)} g H_t^{C^\bullet(W^u,F)}\right]. \tag{11.43}$$

Also as $t \to 0$,

$$\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)} g h'\left(D_t^{C^\bullet(W^u,F)}\right)\right] = \widetilde{\chi}_g'^{-}(F) + \mathcal{O}(t). \tag{11.44}$$

Our Proposition now follows from Propositions 11.10, 11.12 and from (11.44). □

11.6. The superconnection supertraces associated to $F_T^{[0,1]}$

Now we will use a technique due to Ma [**Ma1, Ma2**] to estimate $\mathrm{Tr_s}\left[N g h'\left(D_{t,T}\right)\right]$. In [**Ma1, Ma2**], Ma studied the adiabatic limit of holomorphic torsion forms for a multifibration. In [**Ma1, Ma2**], as $T \to +\infty$, the small eigenvalues of the corresponding Dirac operators tend to 0 like $1/T^k, k = 1, 2, \ldots$, and the question is then to study the contribution to the analytic torsion forms of the various groups of small eigenvalues. Here, the situation will be formally the same, except that the small eigenvalues behave like $e^{-cT}, c > 0$.

Recall that δ is the unit circle in $\mathbf{C}$. Also the contour Δ' was introduced in Figure 11.1. Let $\Delta = \Delta_+ \cup \Delta_-$ be the contour in $\mathbf{C}$ indicated in Figure 11.2.

Definition 11.14. — Put

$$
\begin{aligned}
F_{t,T} &= \psi_t^{-1} \frac{1}{2i\pi} \int_{\Delta} \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - B_T} d\lambda \psi_t, \\
G_{t,T} &= \psi_t^{-1} \frac{1}{2i\pi} \int_{\Delta'} \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - D_{(T/\pi)^{-1} e^{2T}, T}} d\lambda \psi_t, \\
H_{t,T} &= \psi_t^{-1} \frac{1}{2i\pi} \int_{\delta} \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - D_{e^{4T}, T}} d\lambda \psi_t.
\end{aligned} \tag{11.45}
$$

Proposition 11.15. — *The following identity holds,*

$$
h'\left(D_{t,T}\right) = F_{t,T} + G_{\frac{T}{\pi} e^{-2T} t, T} + H_{e^{-4T} t, T}. \tag{11.46}
$$

Proof. — Clearly,

$$
\mathrm{Sp}\left(D_{t,T}\right) = \sqrt{t}\,\mathrm{Sp}\left(B_T^{(0)}\right). \tag{11.47}
$$

By Proposition 3.17, Theorem 11.7 and by (11.47), we find that for $T \geqslant 0$ large enough,

$$
\begin{aligned}
h'\left(D_{t,T}\right) = \psi_t^{-1} \Bigg[& \frac{1}{2i\pi} \int_{\Delta} \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - B_T} d\lambda + \frac{1}{2i\pi} \int_{\left(\frac{T}{\pi}\right)^{1/2} e^{-T} \Delta'} \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - B_T} d\lambda \\
& + \frac{1}{2i\pi} \int_{e^{-2T}\delta} \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - B_T} d\lambda \Bigg] \psi_t.
\end{aligned} \tag{11.48}
$$

Now,

$$
\begin{aligned}
\frac{1}{2i\pi} \int_{\left(\frac{T}{\pi}\right)^{1/2} e^{-T} \Delta'} \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - B_T} d\lambda &= \frac{1}{2i\pi} \int_{\Delta'} \frac{h'\left(\sqrt{\frac{T}{\pi} e^{-2T} t}\,\lambda\right)}{\lambda - \left(\frac{T}{\pi}\right)^{-1/2} e^{T} B_T} d\lambda, \\
\frac{1}{2i\pi} \int_{e^{-2T}\delta} \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - B_T} d\lambda &= \frac{1}{2i\pi} \int_{\delta} \frac{h'\left(\sqrt{e^{-4T} t}\,\lambda\right)}{\lambda - e^{2T} B_T} d\lambda.
\end{aligned} \tag{11.49}
$$

Using Proposition 3.17, (11.48) and (11.49), we get (11.46). □

11.7. Two intermediate results

As in (10.19), we use the notation

$$
\overline{A}_T = e^{-Tf} A_T e^{Tf}, \qquad \overline{B}_T = e^{-Tf} B_T e^{Tf}. \tag{11.50}
$$

Take $c_1 \in]0, 1]$. Put

$$
U_T = \left\{ \lambda \in \mathbf{C}, \frac{1}{4}|\lambda| \leqslant c_1 \sqrt{T}, |\lambda| \geqslant \frac{1}{8} \right\}, \tag{11.51}
$$

Theorem 11.16. — *For $c_1 \in]0,1]$ small enough, for any integer $p \geqslant \dim X + 2$, there exists $C > 0$ such that for $T \geqslant 1$ and $\lambda \in U_T$,*

$$\left|\mathrm{Tr_s}\left[Ng\left(\lambda - B_T\right)^{-p}\right] - \mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)} g\left(\lambda - B_0^{C^\bullet(W^u,F)}\right)^{-p}\right]\right| \leqslant \frac{C}{\sqrt{T}}\left(1+|\lambda|\right)^{p+1}. \tag{11.52}$$

Proof. — By (3.45), since the horizontal component $(df)^H$ vanishes,

$$B_T = -\frac{1}{2}\widehat{c}(e_i)^1\left(\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{1/2,e_i} - T\left\langle \nabla f, e_i\right\rangle\right) - \frac{T}{2}c\left(\nabla f\right) + \frac{1}{4}c(e_i)\omega\left(\nabla^F, g^F\right)(e_i) + \frac{1}{2}f^\alpha\omega\left(\nabla^F, g^F\right)\left(f_\alpha^H\right). \tag{11.53}$$

From (11.53), we get

$$\overline{B}_T = -\frac{1}{2}\widehat{c}(e_i)^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{1/2,e_i} - \frac{T}{2}c\left(\nabla f\right) + \frac{1}{2}f^\alpha\omega\left(\nabla^F, g^F\right)\left(f_\alpha^H\right). \tag{11.54}$$

By (11.54), we find that the $\overline{B}_T^{(>0)}$ does not depend on T. We will write $\overline{B}^{(>0)}$ instead of $\overline{B}_T^{(>0)}$.

Clearly, in (11.52), we can replace B_T by $\overline{B}_T$. By the simplifying assumptions we made in Section 9.1, on the support of μ,

$$\overline{B}_T = \overline{\mathcal{D}}_T^{I\widehat{\otimes}F|_{\mathbf{B}}}. \tag{11.55}$$

As in (10.14),

$$\ker \overline{\mathcal{D}}_T^{I\widehat{\otimes}F|_{\mathbf{B}}} = \bar{\mathfrak{f}}_T \otimes F|_{\mathbf{B}}. \tag{11.56}$$

Using (11.54)-(11.56) , in degree 0, the proof of our Theorem is the same as the proof of [**BL**, Theorem 9.24].

Moreover,

$$\left(\lambda - \overline{B}_T\right)^{-1} = \left(\lambda - \overline{B}_T^{(0)}\right)^{-1} + \left(\lambda - \overline{B}_T^{(0)}\right)^{-1}\left(\overline{B}^{>0}\right)\left(\lambda - \overline{B}_T^{(0)}\right)^{-1} + \cdots \tag{11.57}$$

and the expansion in (11.57) only contains a finite number of terms. Set

$$\overline{P}_T^{]1,+\infty[} = 1 - \overline{P}_T^{[0,1]}. \tag{11.58}$$

If $\lambda \in U_T$, put

(11.59)

$$L_{T,1} = \overline{P}_T^{[0,1]}\left(\lambda - \overline{B}_T^{(0)}\right)^{-1}\overline{P}_T^{[0,1]}, \qquad L_{T,2} = \overline{P}_T^{[0,1]}\left(\lambda - \overline{B}_T^{(0)}\right)^{-1}\overline{P}^{T,]1,+\infty[},$$

$$L_{T,3} = \overline{P}^{T,]1,+\infty[}\left(\lambda - \overline{B}_T^{(0)}\right)^{-1}\overline{P}_T^{[0,1]}, \quad L_{T,4} = \overline{P}^{T,]1,+\infty[}\left(\lambda - \overline{B}_T^{(0)}\right)^{-1}\overline{P}^{T,]1,+\infty[}.$$

We still use the notation in (3.109) to define the norms $\| \ \|_p$. By proceeding as in [**BL**, Theorem 9.21], we can define $m_T(\lambda) \in \mathrm{End}(C^\bullet(W^u, F))$ such that for $T \geqslant 0$ large enough,

$$L_{T,1} = (m_T(\lambda)\,\lambda)^{-1}, \tag{11.60}$$

and moreover by [**BL**, eq. (9.13)], if $c_1 > 0$ is small enough, if $\lambda \in U_T$,

$$\left\| m_T^{-1}(\lambda) - 1 \right\|_\infty \leqslant \frac{C}{\sqrt{T}} (1 + |\lambda|). \tag{11.61}$$

By using (11.55), (11.56) and by proceeding as in the proof of [**BL**, Theorem 9.23], we find that for $2 \leqslant j \leqslant 4$,

$$\|L_{T,j}\|_{p-1} \leqslant C, \qquad \|L_{T,j}\|_\infty \leqslant \frac{C}{\sqrt{T}}. \tag{11.62}$$

From (11.57)-(11.62) we find that to establish (11.52), in (11.57), we may as well replace $\left(\lambda - \overline{B}_T^{(0)}\right)^{-1}$ by $\overline{P}_T^{[0,1]}/\lambda$.

Let $\overline{p}_T$ be the orthogonal projection operator from $\Omega^\bullet\left(X, F|_X\right)$ on $\mathrm{Im}(J_T) \subset \Omega^\bullet\left(X, F|_X\right)$. We claim that

$$\left\| \left(\overline{P}_T^{[0,1]} - 1\right) \overline{p}_T \right\|_\infty = \mathcal{O}\left(e^{-cT}\right). \tag{11.63}$$

In fact, since $J_T : C^\bullet(W^u, F) \to \Omega^\bullet\left(X, F|_X\right)$ is an isometric embedding, (11.63) follows from the first equation in (11.10).

Recall that $\| \ \|_2$ is the Hilbert-Schmidt norm. Then

$$\left\| \overline{P}_T^{[0,1]} - \overline{p}_T \right\|_2^2 = \left\| \overline{P}_T^{[0,1]} \right\|_2^2 + \|\overline{p}_T\|_2^2 - 2\mathrm{ReTr}\left[\overline{P}_T^{[0,1]} \overline{p}_T\right]. \tag{11.64}$$

Since the ranks $\overline{P}_T^{[0,1]}$ and of $\overline{p}_T$ are both equal to $\dim\left(F|_{\mathbf{B}}\right)$, using (11.63), (11.64), we get

$$\left\| \overline{P}_T^{[0,1]} - \overline{p}_T \right\|_2^2 = \mathcal{O}\left(e^{-cT}\right). \tag{11.65}$$

Let Q_T be the orthogonal projection operator from $\Omega^\bullet\left(X, F|_X\right)$ on $F_T^{[0,1]} + \mathrm{Im}(J_T)$. Then

$$\begin{aligned} \left\| \overline{P}_T^{[0,1]} - \overline{p}_T \right\|_1 &= \left\| \left(\overline{P}_T^{[0,1]} - \overline{p}_T\right) Q_T \right\|_1 \\ &\leqslant \left\| \overline{P}_T^{[0,1]} - \overline{p}_T \right\|_2 \|Q_T\|_1 \leqslant 2\dim\left(F|_{\mathbf{B}}\right) \left\| \overline{P}_T^{[0,1]} - \overline{p}_T \right\|_2. \end{aligned} \tag{11.66}$$

From (11.65), (11.66), we get

$$\left\| \overline{P}_T^{[0,1]} - \overline{p}_T \right\|_1 = \mathcal{O}\left(e^{-cT}\right). \tag{11.67}$$

By the above it follows that to establish (11.52), we may as well replace in (11.57) $\left(\lambda - \overline{B}_T^{(0)}\right)^{-1}$ by $\overline{p}_T/\lambda$.

Let $r : \Lambda(T^*X|_{\mathbf{B}}) \to \Lambda^{\max}(T^*X^u|_{\mathbf{B}})$ the obvious orthogonal projection operator. Then using (10.14), (11.6), we find easily that if $s \in \Omega^\bullet(X, F|_X)$,

(11.68)
$$\overline{p}_T s(Z) = \frac{\mu(Z)}{\alpha_T} \exp\left(-T|Z|^2/2\right) r \int_{TX|_{\mathbf{B}}} \mu(Z') \exp\left(-T|Z'|^2/2\right) s(Z')\, dv_{TX}(Z').$$

Using (11.55), (11.68), we get

$$\overline{p}_T \overline{B}_T \overline{p}_T = \overline{p}_T \overline{\mathcal{D}}_T^{I\widehat{\otimes}F|_{\mathbf{B}}} \overline{p}_T. \tag{11.69}$$

By (10.204), since here $d\mathcal{F} = 0$, we get

$$\overline{\mathcal{D}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},(>0)} = \frac{1}{2}\left(\omega\left(F|_{\mathbf{B}}, g^{F|_{\mathbf{B}}}\right) - \widehat{c}\left(R^{TX|_{\mathbf{B}}} Z\right)\right). \tag{11.70}$$

Since $c\left(R^{TX|_{\mathbf{B}}} Z\right)$ is an odd operator, we get

$$rc\left(R^{TX|_{\mathbf{B}}} Z\right) r = 0. \tag{11.71}$$

By (11.68), (11.69), (11.71), we obtain,

$$\overline{p}_T \overline{B}^{(>0)} \overline{p}_T = \frac{1}{2}\omega\left(F|_{\mathbf{B}}, g^{F|_{\mathbf{B}}}\right). \tag{11.72}$$

Using the above, we get (11.52). The proof of our Theorem is completed. □

Theorem 11.17. — *For $c_1 \in]0,1]$ small enough, given any integer $p \geqslant \dim(X) + 1$, there exists $C > 0$ such that for $T \geqslant 1$ large enough, and $\lambda \in U_T$,*

$$\| (\lambda - B_T)^{-1} \|_p \leqslant C(1 + |\lambda|)^p. \tag{11.73}$$

Proof. — Again, we can replace in (11.73) B_T by $\overline{B}_T$. First we claim that (11.73) holds for $\overline{B}_T^{(0)}$. Using (11.55), the proof is the same as the proof of [**BL**, Theorems 9.21 and 9.23]. To get (11.73), we use (11.57) and the fact that $\overline{B}^{(>0)}$ is of order 0. The proof of our Theorem is completed. □

11.8. The term containing $F_{t,T}$

Theorem 11.18. — *There exist $c > 0, C > 0$ such that for $T \geqslant 0$ large enough, and $t \geqslant 1$,*

$$|\mathrm{Tr}_s[NgF_{t,T}]| \leqslant \frac{C}{\sqrt{T}} e^{-ct}. \tag{11.74}$$

Proof. — By (11.45),

$$\mathrm{Tr}_s[NgF_{t,T}] = \psi_t^{-1} \mathrm{Tr}_s\left[Ng \frac{1}{2i\pi} \int_\Delta \frac{h'(\sqrt{t}\,\lambda)}{\lambda - \overline{B}_T} d\lambda\right]. \tag{11.75}$$

Take $p \in \mathbf{N}$. Let $k_p(\lambda)$ be the unique holomorphic function on $\mathbf{C} \setminus \mathbf{R}$ such that

– As $\lambda \to \pm i\infty$, $k_p(\lambda) \to 0$.

– The following identity holds,

$$\frac{k_p^{(p-1)}(\lambda)}{(p-1)!} = h'(\lambda). \tag{11.76}$$

Clearly, if $\lambda \in \Delta$,

$$|\mathrm{Re}(\lambda)| \leqslant \frac{1}{2}|\mathrm{Im}(\lambda)|. \tag{11.77}$$

Using (11.77), we find that there exist $C > 0, C' > 0$ such that if $\lambda \in \Delta$,

$$\left|k_p\left(\sqrt{t}\,\lambda\right)\right| \leqslant C\exp\left(-C't\,|\lambda|^2\right). \tag{11.78}$$

By (11.76),

$$\begin{aligned}\frac{1}{2i\pi}\int_\Delta \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - \overline{B}_T}d\lambda &= \frac{1}{2i\pi}\int_\Delta \frac{k_p\left(\sqrt{t}\,\lambda\right)}{\sqrt{t}^{\,p-1}\left(\lambda - \overline{B}_T\right)^p}d\lambda,\\ \frac{1}{2i\pi}\int_\Delta \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - B_0^{C^\bullet(W^u,F)}}d\lambda &= \frac{1}{2i\pi}\int_\Delta \frac{k_p\left(\sqrt{t}\,\lambda\right)}{\sqrt{t}^{\,p-1}\left(\lambda - B_0^{C^\bullet(W^u,F)}\right)^p}d\lambda.\end{aligned} \tag{11.79}$$

By Theorem 11.16 and by (11.78), we get

$$\begin{aligned}&\left|\mathrm{Tr_s}\left[Ng\frac{1}{2i\pi}\int_{\Delta\cap U_T}\frac{k_p\left(\sqrt{t}\,\lambda\right)}{\sqrt{t}^{\,p-1}\left(\lambda - \overline{B}_T\right)^p}d\lambda\right] - \right.\\ &\left.\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}g\frac{1}{2i\pi}\int_{\Delta\cap U_T}\frac{k_p\left(\sqrt{t}\,\lambda\right)}{\sqrt{t}^{\,p-1}\left(\lambda - B_0^{C^\bullet(W^u,F)}\right)^p}d\lambda\right]\right|\\ &\qquad\leqslant \frac{C}{\sqrt{T}}\int_{\Delta\cap U_T}\frac{\left|k_p\left(\sqrt{t}\,\lambda\right)\right|}{\sqrt{t}^{\,p-1}}(1+|\lambda|)^{p+1}\,d\lambda \leqslant \frac{C'}{\sqrt{T}}e^{-ct}.\end{aligned} \tag{11.80}$$

Take $\lambda \in \Delta$. Then by Theorem 11.7, there exists $C > 0$ such that for $T \geqslant 0$ large enough,

$$\left\|\left(\lambda - \overline{B}_T^{(0)}\right)^{-1}\right\| \leqslant C. \tag{11.81}$$

Since $\overline{B}^{(>0)}$ is an operator of order 0, from (11.57), (11.81), we deduce that if $\lambda \in \Delta$,

$$\|\left(\lambda - \overline{B}_T\right)^{-1}\|_\infty \leqslant C. \tag{11.82}$$

Take $\lambda_0 \in \Delta$. Then

$$\left(\lambda - \overline{B}_T\right)^{-1} = \left(\lambda_0 - \overline{B}_T\right)^{-1} + \left(\lambda_0 - \lambda\right)\left(\lambda_0 - \overline{B}_T\right)^{-1}\left(\lambda - \overline{B}_T\right)^{-1}. \tag{11.83}$$

From Theorem 11.17 and from (11.82), (11.83), if $\lambda \in \Delta$,

$$\left\|\left(\lambda - \overline{B}_T\right)^{-1}\right\|_p \leqslant C(1+|\lambda|). \tag{11.84}$$

Using (11.78), (11.84), we get

(11.85)
$$\left|\mathrm{Tr_s}\left[Ng\int_{\Delta\setminus U_T}\frac{k_p\left(\sqrt{t}\,\lambda\right)}{\sqrt{t}^{\,p-1}}\frac{1}{\left(\lambda-\overline{B}_T\right)^p}\right]d\lambda\right|\leqslant C\exp\left(-ctT\right),$$
$$\left|\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}g\int_{\Delta\setminus U_T}\frac{k_p\left(\sqrt{t}\,\lambda\right)}{\sqrt{t}^{\,p-1}}\frac{1}{\left(\lambda-B_0^{C^\bullet(W^u,F)}\right)^p}d\lambda\right]\right|\leqslant C\exp\left(-ctT\right).$$

From (11.75), (11.79), (11.80), (11.85), we get

(11.86)
$$\left|\mathrm{Tr_s}\left[Ng\frac{1}{2i\pi}\int_\Delta\frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda-B_T}d\lambda\right]-\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}g\frac{1}{2i\pi}\int_\Delta\frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda-B_0^{C^\bullet(W^u,F)}}d\lambda\right]\right|\leqslant\frac{C}{\sqrt{T}}e^{-ct}.$$

Also, since $\mathrm{Sp}\left(B_0^{C^\bullet(W^u,F)}\right)=\{0\}$,

(11.87)
$$\frac{1}{2i\pi}\int_\Delta\frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda-B_0^{C^\bullet(W^u,F)}}d\lambda=0.$$

By (11.86), (11.87), we get (11.74). The proof of our Theorem is completed. □

11.9. The term containing $G_{t,T}$

Theorem 11.19. — *For any $t>0$, as $T\to+\infty$,*

(11.88)
$$\mathrm{Tr_s}\left[NgG_{t,T}\right]\to\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}gG_t^{C^\bullet(W^u,F)}\right].$$

Moreover, there exist $c>0, C>0$ such that for T large enough and $t\geqslant 1$,

(11.89)
$$\left|\mathrm{Tr_s}\left[NgG_{t,T}\right]\right|\leqslant Ce^{-ct},$$
$$\left|\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}gG_t^{C^\bullet(W^u,F)}\right]\right|\leqslant Ce^{-ct}.$$

Proof. — Put

(11.90)
$$\overline{D}_{t,T}=e^{-Tf}D_{t,T}e^{Tf}.$$

To establish the first inequality in (11.89), we may as well replace in formula (11.45) for $G_{t,T}$ the operator $D_{(T/\pi)^{-1}e^{2T},T}$ by $\overline{D}_{(T/\pi)^{-1}e^{2T},T}$.

We use the notation in the proof of Theorem 11.16. By (3.54) and (11.54), we get

(11.91)
$$\overline{D}_{(T/\pi)^{-1}e^{2T},T}=(T/\pi)^{-1/2}\,e^T\overline{B}_T^{(0)}+B^{(1)}+\frac{B^{(2)}}{(T/\pi)^{-1/2}\,e^T}.$$

Then for $T \geqslant 0$ large enough, and $\lambda \in \Delta'$,

$$\left(\lambda - \overline{D}_{(T/\pi)^{-1}e^{2T},T}\right)^{-1} = \left(\lambda - (T/\pi)^{-1/2}\, e^T \overline{B}_T^{(0)}\right)^{-1} + \left(\lambda - (T/\pi)^{-1/2}\, e^T \overline{B}_T^{(0)}\right)^{-1} \left(\overline{B}^{(1)} + \frac{\overline{B}^{(2)}}{(T/\pi)^{-1/2}\, e^T}\right) \left(\lambda - (T/\pi)^{-1/2}\, e^T \overline{B}_T^{(0)}\right)^{-1} \ldots \tag{11.92}$$

and the expansion in (11.92) only contains a finite number of terms. By (11.17), for $T \geqslant 0$ large enough,

$$\left|\mathrm{Sp}\left((T/\pi)^{-1/2}\, e^T \overline{B}_T^{(0)}\right)\right| \subset \{0\} \cup [3/2d_1, 2/3d_2] \cup [(T/\pi)^{-1/2}\, e^T, +\infty[. \tag{11.93}$$

Also

$$\left(\lambda - (T/\pi)^{-1/2}\, e^T \overline{B}_T^{(0)}\right)^{-1} = \left(\lambda - (T/\pi)^{-1/2}\, e^T \overline{B}_T^{(0)}\right)^{-1} \overline{P}_T^{[0,1]} + \left(\lambda - (T/\pi)^{-1/2}\, e^T \overline{B}_T^{(0)}\right)^{-1} \overline{P}_T^{]1,+\infty[}. \tag{11.94}$$

Then we split the sum in (11.92) using (11.94). By (11.93) and by the theorem of residues, we find that for $T \geqslant 0$ large enough, the integral over Δ' of a term where only $\overline{P}_T^{]1,+\infty[}$ appears vanishes identically.

Moreover $\overline{F}_T$ is finite dimensional, and for T large enough. its dimension does not depend on T. It follows that in the terms where $\overline{P}_T^{[0,1]}$ appears, the trace class norm of the corresponding operator can be estimated in terms of the $\| \ \|_\infty$ norm. By (11.93), for $T \geqslant 0$ large enough, and $\lambda \in \Delta'$,

$$\left\| \left(\lambda - (T/\pi)^{-1/2}\, e^T \overline{B}_T^{(0)}\right)^{-1} \overline{P}_T^{]1,+\infty[} \right\|_\infty \leqslant \frac{C}{(T/\pi)^{-1/2}\, e^T}, \qquad \left\| \left(\lambda - (T/\pi)^{-1/2}\, e^T \overline{B}_T^{(0)}\right)^{-1} \overline{P}_T^{[0,1]} \right\|_\infty \leqslant C. \tag{11.95}$$

By (11.92), by the above considerations and by (11.95), we see that

$$\left| \mathrm{Tr_s}\left[N g G_{t,T}\right] - \mathrm{Tr_s}\left[N g \overline{P}_T^{[0,1]} \frac{1}{2i\pi} \int_{\Delta'} \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - P_T^{[0,1]} \left(\overline{B}^{(0)}_{\left(\frac{T}{\pi}\right)^{-1}e^{2T},T} + \overline{B}^{(1)}\right) \overline{P}_T^{[0,1]}} d\lambda \right] \right| \leqslant \frac{C}{(T/\pi)^{-1/2}\, e^T}. \tag{11.96}$$

By (11.53), $B_T^{(>0)}$ does not depend on T. We will write $B^{(>0)}$ instead of $B_T^{(>0)}$. Then for $T \geqslant 0$ large enough,

$$\text{(11.97)}\quad \mathrm{Tr_s}\left[Ng\overline{P}_T^{[0,1]}\frac{1}{2i\pi}\int_{\Delta'}\frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda-\overline{P}_T^{[0,1]}\left(\overline{D}^{(0)}_{\left(\frac{T}{\pi}\right)^{-1}e^{2T},T}+\overline{B}^{(1)}\right)P_T^{[0,1]}}\right]d\lambda$$

$$=\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}g\frac{1}{2i\pi}\int_{\Delta'}\frac{h'\left(\sqrt{t}\lambda\right)}{\lambda-e_T^{-1}P_T^{[0,1]}\left((T/\pi)^{-1/2}e^TB_T^{(0)}+B^{(1)}\right)P_T^{[0,1]}e_T}d\lambda\right].$$

By (11.14), (11.23),

$$\text{(11.98)}\qquad e_T^{-1}\left(T/\pi\right)^{-1/2}e^TB_T^{(0)}e_T=B^{C^\bullet(W^u,F),(0)}+\mathcal{O}\left(e^{-cT}\right).$$

Recall that $J_T : C^\bullet\left(W^u,F\right)\to\Omega^\bullet\left(X,F|_X\right)$ was defined in (10.204), (11.6). By (11.53), (11.55),

$$\text{(11.99)}\qquad B^{(1)}J_T=J_T\omega\left(\nabla^{F|_{\mathbf{B}}},g^{F|_{\mathbf{B}}}\right).$$

By (11.8), (11.99), we obtain,

$$\text{(11.100)}\qquad P_T^{[0,1]}B^{(1)}e^{Tf}J_T=e_T\omega\left(\nabla^{F|_{\mathbf{B}}},g^{F|_{\mathbf{B}}}\right).$$

From (11.100), for $T\geqslant 0$ large enough,

$$\text{(11.101)}\qquad e_T^{-1}P_T^{[0,1]}B^{(1)}e^{Tf}J_T=\omega\left(\nabla^{F|_{\mathbf{B}}},g^{F|_{\mathbf{B}}}\right).$$

Moreover, by (11.10), we get

$$\text{(11.102)}\qquad J_T-\overline{e}_T=\mathcal{O}\left(e^{-cT}\right),$$

where in the right-hand side of (11.102), $\mathcal{O}\left(e^{-cT}\right)$ holds in the ordinary L_2 norm of $\Omega^\bullet\left(X,F|_X\right)$. If we equip $\Omega^\bullet\left(X,F|_X\right)$ with the metric $g_T^{\Omega^\bullet\left(X,F|_X\right)}$, then (11.102) is equivalent to,

$$\text{(11.103)}\qquad e^{Tf}J_T-e_T=\mathcal{O}\left(e^{-cT}\right).$$

Using (11.101) and (11.103), we get, for $T\geqslant 0$ large enough,

$$\text{(11.104)}\qquad e_T^{-1}P_T^{[0,1]}B^{(1)}P_T^{[0,1]}e_T=\frac{1}{2}\omega\left(F|_{\mathbf{B}},g^{F|_{\mathbf{B}}}\right)+\mathcal{O}\left(e^{-cT}\right).$$

Also if $\lambda\in\Delta'$,

$$\text{(11.105)}\qquad \left|\mathrm{Re}\left(\lambda\right)\right|\leqslant\frac{1}{2}\left|\mathrm{Im}\left(\lambda\right)\right|.$$

Therefore, there exist $c>0, C>0$ such that for $t\geqslant 0, \lambda\in\Delta'$,

$$\text{(11.106)}\qquad \left|h'\left(\sqrt{t}\,\lambda\right)\right|\leqslant Ce^{-ct}.$$

By (11.98), (11.104), (11.106), we find that given $t > 0$, as $T \to +\infty$,

$$\text{Tr}_s\left[Ng P_T^{[0,1]} \frac{1}{2i\pi} \int_{\Delta'} \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - P_T^{[0,1]}\left(B^{(0)}_{\left(\frac{T}{\pi}\right)^{-1}e^{2T},T} + B^{(1)}\right) P_T^{[0,1]}} d\lambda\right] \tag{11.107}$$
$$= \text{Tr}_s\left[N^{C^\bullet(W^u,F)} g G_t^{C^\bullet(W^u,F)}\right] + \mathcal{O}\left(e^{-cT}\right).$$

By (11.96), (11.97), (11.107), we get (11.88). By (11.96)-(11.107), we also get (11.89). The proof of our Theorem is completed. □

Theorem 11.20. — *For $T \geqslant 0$ large enough, there are smooth even forms $a_0(T), \ldots, a_q(T)$ such that as $t \to 0$,*

$$\text{Tr}_s\left[Ng G_{t,T}\right] = \sum_{k=0}^{q} a_k(T)\, t^{-k} + \mathcal{O}_T(t). \tag{11.108}$$

There exists $C > 0$ such that for T large enough and $t \in [0,1]$,

$$|\mathcal{O}_T(t)| \leqslant Ct. \tag{11.109}$$

Finally, there exists $c > 0$ such that as $T \to +\infty$,

$$a_k(T) = a_k^{C^\bullet(W^u,F)} + \mathcal{O}\left(e^{-cT}\right) \text{ for } 0 \leqslant k \leqslant q. \tag{11.110}$$

Proof. — By (11.45),

$$\psi_t G_{t,T} \psi_t^{-1} = \frac{1}{2i\pi} \int_{\Delta'} \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - D_{(T/\pi)^{-1}e^{2T},T}} d\lambda. \tag{11.111}$$

The proof of (11.108) continues as the proof of Proposition 11.10.

Also, by the same argument as in the proof of Theorem 11.19,

$$\text{Tr}_s\left[Ng \frac{1}{2i\pi} \int_{\Delta'} \frac{\lambda^k}{\lambda - D_{(T/\pi)^{-1}e^{2T},T}} d\lambda\right] \tag{11.112}$$
$$= \text{Tr}_s\left[N^{C^\bullet(W^u,F)} g \frac{1}{2i\pi} \int_{\Delta'} \frac{\lambda^k}{\lambda - B^{C^\bullet(W^u,F)}} d\lambda\right] + \mathcal{O}\left(e^{-cT}\right).$$

Using (11.30), (11.31), (11.111), (11.112), we get (11.109) and (11.110). The proof of our Theorem is completed. □

11.10. The term containing $H_{t,T}$

Theorem 11.21. — *For $T \geqslant 1$ large enough, and $t > 0$,*

$$H_{t,T} = \sum_{p=0}^{\dim S} \sum_{\substack{0 \leqslant i_0 \leqslant p+1 \\ j_1,\ldots,j_{p+1-i_0} \geqslant 0 \\ \sum_{k=1}^{p+1-i_0} j_k \leqslant i_0 - 1}} \frac{h^{'\left(i_0 - 1 - \sum_{k=0}^{p+1-i_0} j_k\right)}(0)}{\left(i_0 - 1 - \sum_{k=0}^{p+1-i_0} j_k\right)!} (-1)^{p+1-i_0} \\ C_1 \left(B^{(1)} + \frac{B^{(2)}}{\sqrt{e^{4T}t}}\right) C_2 \cdots \left(B^{(1)} + \frac{B^{(2)}}{\sqrt{e^{4T}t}}\right) C_{p+1}. \tag{11.113}$$

In (11.113), *i_0 of the C_j's are equal to $P_T^{\{0\}}$, the others are equal respectively to $\left(\sqrt{e^{4T}t}B_T^{(0)}\right)^{-(1+j_1)}, \ldots, \left(\sqrt{e^{4T}t}B_T^{(0)}\right)^{-\left(1+j_{p+1-i_0}\right)}$. In particular, for $T \geqslant 0$ large enough, $H_{t,T}$ is a polynomial in the variable $1/\sqrt{t}$.*

Finally, for $T \geqslant 0$ large enough, as $t \to +\infty$,

$$H_{t,T} = P_T^{\{0\}} h' \left(B_T^{H^\bullet(X,F|X)}\right) P_T^{\{0\}} + \mathcal{O}\left(1/\sqrt{t}\right). \tag{11.114}$$

Proof. — By (11.17), it is clear that for $T \geqslant 0$ large enough, inside the unit circle, 0 is the only possible element in the spectrum of $D_{e^{4T},T}$. We can then proceed as in the proof of Theorem 11.11 to establish our Theorem. □

Theorem 11.22. — *For $T \geqslant 0$ large enough, there are smooth even forms $b_0(T), \ldots b_q(T)$ such that for $t > 0$,*

$$\mathrm{Tr_s}\left[NgH_{t,T}\right] = \sum_{k=0}^{q} b_k(T)\, t^{-k}. \tag{11.115}$$

Moreover,

$$b_0(T) = \chi_g'(F). \tag{11.116}$$

There exists $c > 0$ such that as $T \to +\infty$,

$$\mathrm{Tr_s}\left[NgH_{t,T}\right] = \chi_g'(F) + \mathcal{O}\left(e^{-cT}\right). \tag{11.117}$$

Also, there exists $c > 0$ such that as $T \to +\infty$,

$$\begin{aligned} b_k(T) &= \mathcal{O}\left(e^{-cT}\right) \text{ for } k > 0, \\ &= \chi_g'(F) \text{ for } k = 0. \end{aligned} \tag{11.118}$$

Proof. — Observe that as operators, $B^{(1)}$ is even, and $B^{(2)}$ is odd. Using Theorem 11.21, the proof of the first part of our Theorem is the same as the proof of Proposition 11.12. Also by (11.17), for $T \geqslant 0$ large enough,

$$\mathrm{Sp}\left(e^{2T}B_T\right) \subset \{0\} \cup \left[\frac{3}{2}(T/\pi)^{1/2} e^T d_1, +\infty\right[. \tag{11.119}$$

Using (11.119) and proceeding as in the proof of Theorem 11.19, we get (11.117). Using (11.113), we see that as $T \to +\infty$, the terms which contain a positive power of $1/\sqrt{t}$ tend to 0 like e^{-cT}. By (11.115)-(11.117), we get (11.118). The proof of our Theorem is completed. □

11.11. A compatibility result

Definition 11.23. — For $T \geqslant 0$ large enough, and $t > 0$, put

$$K_{t,T} = \psi_t^{-1} \frac{1}{2i\pi} \int_{\frac{1}{4}\delta} \frac{h'(\sqrt{t}\lambda)}{\lambda - B_T} d\lambda \psi_t. \tag{11.120}$$

Proposition 11.24. — *For $T \geqslant 0$ large enough, and $t > 0$,*

$$G_{(T/\pi)e^{-2T}t,T} + H_{e^{-4T}t,T} = K_{t,T}. \tag{11.121}$$

Proof. — The proof of our Proposition is the same as the proof of Proposition 11.15. □

Theorem 11.25. — *As $T \to +\infty$, given $t > 0$,*

$$\mathrm{Tr_s}\left[NgK_{t,T}\right] = \widetilde{\chi}_g'^{-}(F) + \mathcal{O}\left(1/\sqrt{T}\right). \tag{11.122}$$

For $T \geqslant 0$ large enough, there exist smooth even forms $c_0(T), \ldots, c_q(T)$ such that as $t \to 0$,

$$\mathrm{Tr_s}\left[NgK_{t,T}\right] = \sum_{k=0}^{q} c_k(T)\, t^{-k} + \mathcal{O}_T(t). \tag{11.123}$$

As $T \to +\infty$,

$$\begin{aligned} c_k(T) &= \mathcal{O}\left(1/\sqrt{T}\right), \ k > 0, \\ c_0(T) &= \widetilde{\chi}_g'^{-}(F) + \mathcal{O}\left(1/\sqrt{T}\right). \end{aligned} \tag{11.124}$$

Finally, for $T \geqslant 0$ large enough,

$$a_k(T)\left(\frac{T}{\pi}\right)^{-k} e^{2kT} + b_k(T)\, e^{4kT} = c_k(T), \ 0 \leqslant k \leqslant q. \tag{11.125}$$

Proof. — By proceeding as in the proofs of Theorems 11.18 and 11.20, we get (11.122) and (11.123). Also by proceeding as in the proof of Theorem 11.20, we find that the expansion in (11.123) is uniform in T. Therefore (11.124) holds. By (11.108), (11.115), (11.121), (11.123), we get (11.125). The proof of our Theorem is completed. □

11.12. A Proof of Theorem 11.1

By Proposition 11.15, Theorems 11.18, 11.19 and by Theorem 11.22, for $T \geqslant 0$ large enough,

$$\int_1^{+\infty} \left(\mathrm{Tr_s}\left[Ngh'\left(D_{t,T}\right)\right] - \chi_g'\left(F\right)\right)\frac{dt}{2t} = \int_1^{+\infty} \mathrm{Tr_s}\left[NgF_{t,T}\right]\frac{dt}{2t} + \int_{\frac{T}{\pi}e^{-2T}}^{+\infty} \mathrm{Tr_s}\left[NgG_{t,T}\right]\frac{dt}{2t} + \int_{e^{-4T}}^{+\infty} \left(\mathrm{Tr_s}\left[NgH_{t,T}\right] - \chi_g'\left(F\right)\right)\frac{dt}{2t}. \tag{11.126}$$

By Theorem 11.18, as $T \to +\infty$,

$$\int_1^{+\infty} \mathrm{Tr_s}\left[NgF_{t,T}\right]\frac{dt}{2t} \to 0. \tag{11.127}$$

Also, using in particular Theorem 11.22, we get

$$\begin{aligned}\int_{\left(\frac{T}{\pi}\right)e^{-2T}}^{+\infty} \mathrm{Tr_s}\left[NgG_{t,T}\right]\frac{dt}{2t} + \int_{e^{-4T}}^{+\infty} \left(\mathrm{Tr_s}\left[NgH_{t,T}\right] - \chi_g'\left(F\right)\right)\frac{dt}{2t} \\ = \int_{\left(\frac{T}{\pi}\right)e^{-2T}}^{1} \left(\mathrm{Tr_s}\left[NgG_{t,T}\right] - \sum_{k=0}^{q} a_k\left(T\right)t^{-k}\right)\frac{dt}{2t} + \int_1^{+\infty} \mathrm{Tr_s}\left[NgG_{t,T}\right]\frac{dt}{2t} \\ + \frac{1}{2}\sum_{k=1}^{q}\frac{a_k\left(T\right)}{k}\left(\left(\frac{T}{\pi}\right)^{-k}e^{2kT} - 1\right) + \frac{1}{2}a_0\left(T\right)\log\left(\left(\frac{T}{\pi}\right)^{-1}e^{2T}\right) \\ + \frac{1}{2}\sum_{k=1}^{q}\frac{b_k\left(T\right)}{k}e^{4kT}.\end{aligned} \tag{11.128}$$

By Theorems 11.19 and 11.20, as $T \to +\infty$,

$$\int_{\frac{T}{\pi}e^{-2T}}^{1} \left(\mathrm{Tr_s}\left[NgG_{t,T}\right] - \sum_{k=0}^{q} a_k\left(T\right)t^{-k}\right)\frac{dt}{2t} \to \int_0^1 \left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}gG_t^{C^\bullet(W^u,F)}\right] - \sum_{k=0}^{q} a_k^{C^\bullet(W^u,F)}t^{-k}\right)\frac{dt}{2t}. \tag{11.129}$$

By Propositions 11.12 and 11.13,

$$-\sum_{k=0}^{q} a_k^{C^\bullet(W^u,F)}t^{-k} = \mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}gH_t^{C^\bullet(W^u,F)}\right] - \widetilde{\chi}_g'^{-}\left(F\right). \tag{11.130}$$

By (11.28), (11.129), (11.130), we see that as $T \to +\infty$,

$$\int_{\frac{T}{\pi}e^{-2T}}^{1} \left(\mathrm{Tr_s}\left[NgG_{t,T}\right] - \sum_{k=0}^{q} a_k\left(T\right)t^{-k}\right)\frac{dt}{2t} \to \int_0^1 \left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}gh'\left(B_t^{C^\bullet(W^u,F)}\right)\right] - \widetilde{\chi}_g'^{-}\left(F\right)\right)\frac{dt}{2t}. \tag{11.131}$$

By (11.88), (11.89) in Theorem 11.19, as $T \to +\infty$,

$$\int_1^{+\infty} \mathrm{Tr_s}\left[NgG_{t,T}\right]\frac{dt}{2t} \to \int_1^{+\infty} \mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}gG_t^{C^\bullet(W^u,F)}\right]\frac{dt}{2t}. \tag{11.132}$$

Using (11.124) and (11.125) in Theorem 11.25, we see that if $k > 0$, as $T \to +\infty$,

$$a_k(T)\left(\frac{T}{\pi}\right)^{-k} e^{2kT} + b_k(T)\, e^{4kT} \to 0. \tag{11.133}$$

Also by (11.110) in Theorem 11.20, as $T \to +\infty$,

$$\frac{1}{2}\sum_{k=1}^{q}\frac{a_k(T)}{k} \to \frac{1}{2}\sum_{k=1}^{q}\frac{a_k^{C^\bullet(W^u,F)}}{k}. \tag{11.134}$$

Using (11.40) in Proposition 11.12, and (11.42) in Proposition 11.13,

$$\int_1^{+\infty}\left(\mathrm{Tr_s}\left[NgH_t^{C^\bullet(W^u,F)}\right] - \chi_g'(F)\right)\frac{dt}{2t} = -\frac{1}{2}\sum_{k=1}^{q}\frac{a_k^{C^\bullet(W^u,F)}}{k}. \tag{11.135}$$

Therefore, using Proposition 11.9 and (11.135), we get

$$\begin{aligned}\int_1^{+\infty} \mathrm{Tr_s}\left[NgG_t^{C^\bullet(W^u,F)}\right]\frac{dt}{2t} - \frac{1}{2}\sum_{k=1}^{q}\frac{a_k^{C^\bullet(W^u,F)}}{k} \\ = \int_1^{+\infty}\left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u,F)}gh'\left(B_t^{C^\bullet(W^u,F)}\right)\right] - \chi_g'(F)\right)\frac{dt}{2t}.\end{aligned} \tag{11.136}$$

Finally, by (11.110), we find that as $T \to +\infty$,

$$\begin{aligned}\frac{1}{2}a_0(T)\log\left(\left(\frac{T}{\pi}\right)^{-1}e^{2T}\right) + \frac{1}{2}a_0^{C^\bullet(W^u,F)}\left(-2T + \log(T)\right) \\ \to \frac{1}{2}a_0^{C^\bullet(W^u,F)}\log(\pi).\end{aligned} \tag{11.137}$$

By (11.42), (11.126)-(11.128), (11.131)-(11.134), (11.136), (11.137), we get (11.3). The proof of Theorem 11.1 is completed.

11.13. A proof of Theorem 11.2

Let $N^{H^\bullet(X,F|_X)}$ be the number operator of $H^\bullet(X, F|_X)$.

Theorem 11.26. — *There exists $c > 0$ such that as $T \to +\infty$,*

$$\frac{g_T^{H^\bullet(X,F|_X)}}{g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}} = \left(\left(\frac{\pi}{T}\right)^{N^{H^\bullet(X,F|_X)}-n/2} e^{2TN^{H^\bullet(X,F|_X)}}\right)^{-1}\left(1 + \mathcal{O}\left(e^{-cT}\right)\right). \tag{11.138}$$

Proof. — Take $f \in \ker B^{C^\bullet(W^u,F),(0)}$. Then

$$\partial f = 0, \tag{11.139}$$

Clearly f represents and element of $H^\bullet(X, F|_X)$. For $T \geqslant 0$ large enough, $(P_T^\infty)^{-1} f \in F_T^{[0,1]}$. Also using Theorem 5.3, from (11.139), we get

$$d^X (P_T^\infty)^{-1} f = 0. \tag{11.140}$$

Therefore $(P_T^\infty)^{-1} f$ also represents an element of $H^\bullet(X, F|_X)$. Clearly,

$$P^\infty (P_T^\infty)^{-1} f = f. \tag{11.141}$$

By Theorem 5.3 and by (11.141), $(P_T^\infty)^{-1} f \in F_T^{[0,1]}$ and $f \in C^\bullet(W^u, F)$ represent the same element in $H^\bullet(X, F|_X)$. Also $P_T^{\{0\}} (P_T^\infty)^{-1} f$ lies in $\ker\left(B_T^{(0)}\right)$, and also represents the same element in $H^\bullet(X, F|_X)$. It follows that $f \in \ker\left(B^{C^\bullet(W^u,F),(0)}\right) \to P_T^{\{0\}} (P_T^\infty)^{-1} f \in \ker B_T^{(0)}$ is the canonical map which identifies the two vector bundles.

By the above, we find that for $T \geqslant 0$ large enough,

$$\frac{g_{L_2,T}^{H^\bullet(X,F|_X)}}{g_{C^\bullet(W^u,F)}^{H^\bullet(X,F|_X)}} = \left[P_T^{\{0\}} (P_T^\infty)^{-1}\right]^* P_T^{\{0\}} (P_T^\infty)^{-1}. \tag{11.142}$$

Clearly,

$$e_T'^{-1} P_T^{\{0\}} (P_T^\infty)^{-1} = e_T'^{-1} P_T^{\{0\}} e_T' (P_T^\infty e_T')^{-1}. \tag{11.143}$$

By Theorems 11.4, by (11.14), by Theorem 11.7 and by (11.143), and using the fact that $\mathcal{F} = N^{C^\bullet(W^u,F)}$, there exists $c > 0$ such that as $T \to +\infty$,

$$e_T'^{-1} P_T^{\{0\}} (P_T^\infty)^{-1} = \left(P^{C^\bullet(W^u,F),(0)} + \mathcal{O}\left(e^{-cT}\right)\right) \left(\left(\frac{\pi}{T}\right)^{N^{C^\bullet(W^u,F)}/2 - n/4} e^{TN^{C^\bullet(W^u,F)}}\right)^{-1}. \tag{11.144}$$

Since e_T' is a unitary map, (11.138) follows from (11.142)-(11.144). □

Remark 11.27. — Theorem 11.2 is now a trivial consequence of (1.26) and of Theorem 11.26.

CHAPTER 12

AN ASYMPTOTIC EXPANSION FOR $\mathrm{Tr_s}\left[fgh'\left(D_{t,T}\right)\right]$ AS $T \to +\infty$

The purpose of this Chapter is to prove Theorem 9.9, i.e. to study the asymptotics of $\mathrm{Tr_s}\left[fgh'\left(D_{t,T}\right)\right]$ as $T \to +\infty$. Also, we prove the much easier Theorem 9.7. This Chapter is the obvious extension of [**BZ1**, Chapter XII] and of [**BZ2**, Section 8] to the relative setting.

This Chapter is organized as follows. In Section 12.1, we give a Lichnerowicz formula. In Section 12.2, we state an intermediate result, from which Theorem 9.9 follows. The remainder of the Section is devoted to the proof of this result. In Section 12.3, we establish an estimate on a heat kernel. In Section 12.4, we prove our intermediate result. Finally, in Section 12.5, we establish Theorem 9.7.

We use the notation of the previous Chapters, and in particular of Sections 3.6 and 3.8. Also, we still assume that S is compact.

12.1. A Lichnerowicz formula

Set

$$\overline{C}_{t,T} = e^{-Tf} C_{t,T} e^{Tf}, \qquad \overline{D}_{t,T} = e^{-Tf} D_{t,T} e^{Tf}. \tag{12.1}$$

Theorem 12.1. — *Given $t \in \mathbf{R}_+^*, T \in \mathbf{R}$, the following identity holds,*

$$\begin{aligned}
\overline{C}_{t,T}^2 = {} & -\frac{t}{4}{}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u,2}_{t/2,e_i} + t\frac{K}{16} + \frac{t}{8}c(e_i)c(e_j)\mathcal{R}\left(e_i,e_j\right) \\
& + \frac{1}{2}f^\alpha f^\beta \mathcal{R}\left(f_\alpha^H, f_\beta^H\right) + \frac{\sqrt{t}}{2}c(e_i)f^\alpha \mathcal{R}\left(e_i, f_\alpha^H\right) \\
& + \frac{t}{4}\left(\frac{1}{4}\left[\omega\left(\nabla^F, g^F\right)\left(e_i\right)\right]^2 + T^2\left|\nabla f\right|^2 - T\omega\left(\nabla^F, g^F\right)\left(\nabla f\right)\right) \\
& - \frac{\sqrt{t}}{4}f^\alpha \widehat{c}(e_i)\left(\nabla^{TX\otimes F,u}_{f_\alpha^H}\omega\left(\nabla^F, g^F\right)\left(e_i\right) - 2T\left\langle \nabla^{TX}_{f_\alpha^H}\nabla f, e_i\right\rangle\right) \\
& + \frac{t}{32}\widehat{c}(e_i)\widehat{c}(e_j)\omega^2\left(\nabla^F, g^F\right)\left(e_i,e_j\right) - \frac{t}{8}c(e_i)\widehat{c}(e_j)\Big(\nabla^{TX\otimes F,u}_{e_i}\omega\left(\nabla^F, g^F\right)\left(e_j\right) \\
& \qquad - 2T\left\langle \nabla^{TX}_{e_i}\nabla f, e_j\right\rangle\Big).
\end{aligned} \tag{12.2}$$

Proof. — By [**BLo1**, Theorem 3.11] or by Theorem 3.19, we have a formula for $C_{t,T}^2$, which coincides with the right-hand side of (12.2), except for the first term, in

which ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{t/2}$ is replaced by ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{t/2} - T\nabla f$, which is the connection ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{t/2}$ on $\Lambda^\bullet(T^*S)\,\widehat{\otimes}\,\Lambda^\bullet(T^*X)\,\widehat{\otimes}\,F$ associated to T^HM, g^{TX}, g^F_T. When replacing $C^2_{t,T}$ by $\overline{C}^2_{t,T}$, we get (12.2). □

12.2. A proof of Theorem 9.9

The main result of this Section is as follows.

Theorem 12.2. — *There exists $t_0 \in]0,1]$ such that if $t \in]0,t_0]$, as $T \to +\infty$,*

$$\mathrm{Tr}_s\left[fg\exp\left(-\overline{C}^2_{t,T}\right)\right] = \mathrm{Tr}_s{}^{B_g}[f] + \frac{1}{4}\left(\widetilde{\chi}'^{+}_g(F) - \widetilde{\chi}'^{-}_g(F)\right)\frac{1}{T} + \mathcal{O}\left(\frac{1}{T^3}\right). \tag{12.3}$$

Proof. — The remainder of the Chapter is devoted to the proof of our Theorem. □

Remark 12.3. — By (4.47),

$$\mathrm{Tr}_s[fgh'(D_{t,T})] = \left(1 + 2\frac{\partial}{\partial a}\right)\mathrm{Tr}_s\left[fg\exp\left(-a\overline{C}^2_{t,T}\right)\right]|_{a=1}. \tag{12.4}$$

Also by (3.102),

$$\mathrm{Tr}_s\left[fg\exp\left(-a\overline{C}^2_{t,T}\right)\right] = \psi_a\mathrm{Tr}_s\left[fg\exp\left(-\overline{C}^2_{at,T}\right)\right]. \tag{12.5}$$

From (12.3)-(12.5), we get (9.23), i.e. we have established Theorem 9.9.

12.3. An estimate on the kernel of $\exp\left(-\overline{C}^2_{t,T}\right)$

Let $d^X(\cdot,\cdot)$ denote the Riemannian distance along the fibres X with respect to the metric g^{TX}. Let dv_X be the volume form along the fibres X associated to the metric g^{TX}. Let $\exp\left(-\overline{C}^2_{t,T}\right)(x,x')$ be the smooth kernel associated to the operator $\exp\left(-\overline{C}^2_{t,T}\right)$ with respect to $dv_X(x')/(2\pi)^{n/2}$, so that if $s \in \Omega^\bullet(X, F|_X)$,

$$\exp\left(-\overline{C}^2_{t,T}\right)s(x) = \int_X \exp\left(-\overline{C}^2_{t,T}\right)(x,x')\,s(x')\,\frac{dv_X(x')}{(2\pi)^{n/2}}. \tag{12.6}$$

Clearly,

$$\mathrm{Tr}_s\left[fg\exp\left(-\overline{C}^2_{t,T}\right)\right] = \int_X f(x)\,\mathrm{Tr}_s\left[g\exp\left(-\overline{C}^2_{t,T}\right)(g^{-1}x,x)\right]\frac{dv_X(x)}{(2\pi)^{\dim X/2}}. \tag{12.7}$$

Theorem 12.4. — *There exists $t_0 \in]0,1]$ such that for any $t \in]0,t_0]$, there exist $C > 0, C' > 0$ such that for any $x \in X$, $d^X(x, B_g) \geqslant \varepsilon_0/4$, then*

$$\left|\exp\left(-\overline{C}^2_{t,T}\right)(g^{-1}x,x)\right| \leqslant C\exp(-C'T). \tag{12.8}$$

Proof. — We will give a simple probabilistic proof of our Theorem. Let Δ^X be the Laplace-Beltrami operator along the fibres X, which is associated to the metric g^{TX}. Recall that by Proposition 3.17,

$$\exp\left(-\overline{C}_{t,T}^2\right) = \psi_t^{-1} \exp\left(-t\overline{C}_{1,T}^2\right) \psi_t. \tag{12.9}$$

Let $p_t(x, x')$ be the smooth kernel associated to the operator $\exp\left(t\Delta^X/4\right)$. Let $H_{t,T}(x)$ be defined by,

$$\exp\left(-t\overline{C}_{1,T}^2\right)\left(g^{-1}x, x\right) = p_t\left(g^{-1}x, x\right) H_{t,T}(x). \tag{12.10}$$

Using Theorem 12.1, we will give a probabilistic representation of $H_{t,T}(x)$ by the Itô stochastic calculus. Our formula will be an extension of a Feynman-Kac formula.

Let $\mathfrak{P}_t^x$ be the probability law on $\mathcal{C}([0,1], X)$ of the Brownian bridge $s \in [0,1] \to x_s \in X$ with $x_0 = g^{-1}x, x_1 = x$, which is associated to the metric $2g^{TX}/t$. Recall that by [**D**], [**It**], [**B1**, Chapter II], even though $\mathfrak{P}_t^x$ a.e., the paths $s \in [0,1] \to x_s \in X$ are nowhere differentiable, we can still define the parallel transport operator τ_0^s from $\left(\Lambda^\bullet(T^*S) \widehat{\otimes} \Lambda^\bullet(T^*X) \widehat{\otimes} F\right)_{x_s}$ into $\left(\Lambda^\bullet(T^*S) \widehat{\otimes} \Lambda^\bullet(T^*X) \widehat{\otimes} F\right)_x$ with respect to the connection ${}^1\nabla_{1/2}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}$ along the path $x_\cdot$, and τ_0^s also depends continuously on s. Similarly, the inverse τ_s^0 of τ_0^s is well defined and depends continuously on s. Set

$$E_T = \overline{C}_{1,T}^2 + \frac{1}{4} {}^1\nabla_{1/2,e_i}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u,2} - \frac{K}{16} - \frac{T^2}{4}|\nabla f|^2. \tag{12.11}$$

Then by (12.2), E_T is an operator of order 0. Let V_s be the solution of the differential equation,

$$\begin{aligned} &\frac{d}{ds}V_s = -V_s \tau_0^s t E_T^{x_s} \tau_s^0, \\ &V_0 = 1. \end{aligned} \tag{12.12}$$

A simple application of Itô's stochastic calculus as in [**B2**, Theorem 2.5] shows that

$$H_{t,T}^x = E^{\mathfrak{P}_t^x}\left[\exp\left(-t\int_0^1 \left(\frac{K}{16} + \frac{T^2}{4}|\nabla f|^2\right)(x_s)\, ds\right) V_1 \tau_0^1\right]. \tag{12.13}$$

To estimate the right-hand side of (12.13), we proceed as in [**BZ1**, proof of Theorem 12.6 and equ. (12.29)]. The main difference with respect to [**BZ1**] is that the parallel transport operator τ_0^s is not unitary.

Let $U \in T_xX$, let $s \in [0,1] \to x_s \in X$ be a smooth curve. Let $\tau_s^0 U \in T_{x_s}X$ be the parallel transport of U with respect to the connection ∇^{TX} along the curve $s \to x_s$. Let ${}^1\frac{D}{Ds}$ be the covariant derivative along $s \to x_s$ with respect to the connection

${}^1\nabla_{1/2}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}$. By (3.41), (3.44), we get

$$
\begin{aligned}
(12.14) \qquad & {}^1\frac{D}{Ds}c\left(\tau_s^0 U\right) = 2\left\langle S_{x_s}\left(\dot{x}_s\right)\tau_s^0 U, f_\alpha^H\right\rangle f^\alpha, \\
& {}^1\frac{D}{Ds}\widehat{c}\left(\tau_s^0 U\right) = 0.
\end{aligned}
$$

By (12.14), we obtain,

$$
\begin{aligned}
(12.15) \qquad & \tau_0^s c\left(\tau_s^0 U\right)\tau_s^0 = c\left(U\right) + \int_0^s 2\left\langle S_{x_u}\left(\dot{x}_u\right)\tau_u^0 U, f_\alpha^H\right\rangle f^\alpha du, \\
& \tau_0^s \widehat{c}\left(\tau_s^0 U\right)\tau_s^0 = \widehat{c}\left(U\right).
\end{aligned}
$$

By (12.15), we get immediately an expression for $\tau_0^s t E_T^{x_s}\tau_s^0$ in equation (12.12). The integral in the right-hand side of (12.15) should then be interpreted as a stochastic integral in the sense of Stratonovitch along the path $u \to x_u$.

Clearly, we can expand V_s in the form,

$$
\begin{aligned}
& V_s = V_s^{(0)} + V_s^{(>0)}, \\
(12.16) \qquad & V_s^{(0)} \in \mathrm{End}\left(\Lambda^\bullet(T^*X)\,\widehat{\otimes}\,F\right)_x, \\
& V_s^{(>0)} \in \Lambda^{(>0)}\left(T^*S\right)\widehat{\otimes}\,\mathrm{End}\left(\Lambda^\bullet(T^*X)\,\widehat{\otimes}\,F\right)_x.
\end{aligned}
$$

Similarly,

$$
(12.17) \qquad \tau_0^s E_T^{x_s}\tau_s^0 = E_s^{(0)} + E_s^{(>0)}.
$$

Then equation (12.12) can be written in the form,

$$
\begin{aligned}
(12.18) \qquad & \frac{d}{ds}V_s^{(0)} = -V_s^{(0)}E_s^{(0)}, \quad V^{(0)} = 1, \\
& \frac{d}{ds}V_s^{(>0)} = -V_s^{(>0)}E_s - V_s^{(0)}E_s^{(>0)}, \quad V_0^{(>0)} = 0.
\end{aligned}
$$

By proceeding as in [**BZ1**, proof of Theorem 12.6], there exists $\gamma > 0$ such that for $t > 0, T > 0$,

$$
\begin{aligned}
(12.19) \qquad & \left|V_s^{(0)}\right| \leqslant \exp\left(\gamma t\left(1+T\right)\right), \\
& \left|\left(V_s^{(0)}\right)^{-1}\right| \leqslant \exp\left(\gamma t\left(1+T\right)\right).
\end{aligned}
$$

Put

$$
(12.20) \qquad W_s^{(>0)} = V_s^{(>0)}\left(V_s^{(0)}\right)^{-1}.
$$

Then by (12.18),

$$
\begin{aligned}
(12.21) \qquad & \frac{d}{ds}W_s^{(>0)} = -W_s^{(>0)}V_s^{(0)}E_s^{(>0)}\left(V_s^{(0)}\right)^{-1} - V_s^{(0)}E_s^{(>0)}\left(V_s^{(0)}\right)^{-1}, \\
& W_0^{(>0)} = 0.
\end{aligned}
$$

Clearly, equation (12.21) can be solved by recursion on the degree in $\Lambda^\bullet\left(T^*S\right)$.

If $d^X(x, B_g) \geqslant \varepsilon_0/4$, then either $d^X(x, B) \geqslant \varepsilon_0/4$, or there exists $x_0 \in B \setminus B_g$ such that $d^X(x, x_0) \leqslant \varepsilon_0$. Then $g^{-1}x_0 \in B \setminus B_g$, and $d^X\left(g^{-1}x, g^{-1}x_0\right) \leqslant \varepsilon_0$.

First we consider the case where $d^X(x, B) \geqslant \varepsilon_0/4$. Clearly there exists $\beta > 0$ such that if $y \in X, d^X(y, B) \geqslant \varepsilon_0/8$, then

$$|\nabla f|^2(y) \geqslant \beta. \tag{12.22}$$

Let S be the stopping time,

$$S = \inf\left\{s \in [0,1], d^X(x, x_s) \geqslant \varepsilon_0/8\right\}. \tag{12.23}$$

By (12.13), there exists $C > 0$ such that if $t \in]0,1], T \geqslant 0, a \in]0,1/2]$,

$$\left|H_{t,T}^x\right| \leqslant C\left(\exp\left(-tT^2a\beta/4\right) + \mathfrak{P}_t^x[S \leqslant a]^{1/2}\right)\left\{E^{\mathfrak{P}_t^x}\left[\left|V_1^{t,T}\tau_0^1\right|^2\right]\right\}^{1/2}. \tag{12.24}$$

By using the estimates of Varadhan [**V**, proof of Theorem 5.1] and by proceeding as in [**BZ1**, equ. (12.23)], we find that there exists $c' > 0$ such that under the above conditions,

$$\mathfrak{P}_t^x[S \leqslant a] \leqslant c' \exp\left(-\varepsilon_0^2/72at\right). \tag{12.25}$$

Moreover by (12.19)-(12.21), and using standard estimates on stochastic integrals, we find that there exists $C > 0, \gamma > 0$ such that under the above conditions,

$$\left\{E^{\mathfrak{P}_t^x}\left[\left|V_1\tau_0^1\right|^2\right]\right\}^{1/2} \leqslant C\exp\left(\gamma t\left(1+T\right)\right). \tag{12.26}$$

By (12.24), (12.25), we get

$$\left|H_{t,T}^x\right| \leqslant C\exp\left(\gamma t\left(1+T\right)\right)\left(\exp\left(-tT^2a\beta/4\right) + \exp\left(-r^2/144at\right)\right). \tag{12.27}$$

Take,

$$a = \frac{\varepsilon_0}{6tT\sqrt{\beta}}. \tag{12.28}$$

Given $t \in]0,1]$, for $T \geqslant 0$ large enough, $a \in]0,1/2]$. Then

$$tT^2a\beta/4 = \varepsilon_0^2/144at = \frac{\varepsilon_0 T\sqrt{\beta}}{24}. \tag{12.29}$$

Put

$$t_0 = \frac{\varepsilon_0\sqrt{\beta}}{30\gamma}. \tag{12.30}$$

Then if $t \in]0, t_0]$,

$$\frac{\varepsilon_0\sqrt{\beta}}{24} - \gamma t > 0. \tag{12.31}$$

By (12.27), (12.29), (12.31), we get (12.8) in the case where $d^X(x, B) \geqslant \varepsilon_0/4$.

When there exists $x_0 \in B \setminus B_g$ such that $d^X(x, x_0) \leqslant \varepsilon_0$, the proof of the estimate (12.8) is essentially the same. We proceed exactly as in the proof of [**BZ1**, proof of Theorem 12.6]. The main point is that a continuous path path $s \in [0,1] \to x_s \in X$

such that $x_0 = g^{-1}x, x_1 = x$ necessarily goes through a region where $|\nabla f|^2$ is large enough, and spends some time there. Details are left to the reader. □

Now, we use the notation in Chapter 4 and in Section 10.1. We define $\overline{\mathcal{C}}_{t,T}^{I\widehat{\otimes}F|_{\mathbf{B}}}$ as in (4.44)-(4.46). Equivalently, set

$$\overline{\mathcal{C}}_{t,T}^{I\widehat{\otimes}F|_{\mathbf{B}}} = \psi_t^{-1}\sqrt{t}\,\overline{\mathcal{C}}_T^{I\widehat{\otimes}F|_{\mathbf{B}}}\psi_t. \tag{12.32}$$

Theorem 12.5. — *There exists $t_0 \in]0,1]$ such that for $t \in]0,t_0]$, there exists $C > 0$, $C' > 0$ such that if $x_0 \in B$, if $x, x' \in X$ are such that $d^X(x,x_0) \leqslant r/4$, $d^X(x',x_0) \leqslant r/4$, then*

$$\left|\left(\exp\left(-\overline{\mathcal{C}}_{t,T}^2\right) - \exp\left(-\overline{\mathcal{C}}_{t,T}^{I\widehat{\otimes}F|_{\mathbf{B}},2}\right)\right)(x,x')\right| \leqslant C\exp\left(-C'T\right). \tag{12.33}$$

Proof. — By proceeding as in [**BZ1**, proof of Theorem 12.6] and in Theorem 12.4, we get (12.33). □

12.4. A proof of Theorem 12.2

By Theorems 12.4 and 12.5, and by (12.7), we find that there exists $t_0 \in]0,1]$ such that if $t \in]0,t_0]$, there exist $C > 0, C' > 0$ such that for $T \geqslant 0$,

$$\left|\mathrm{Tr_s}\left[fg\exp\left(-\overline{\mathcal{C}}_{t,T}^2\right)\right] - \int_{\substack{x\in X\\ d^X(x,B_g)\leqslant\varepsilon_0/4}} f(x)\,\mathrm{Tr_s}\left[g\exp\left(-\overline{\mathcal{C}}_{t,T}^{I\widehat{\otimes}F|_{\mathbf{B}},2}\right)(g^{-1}x,x)\right] \frac{dv_X(x)}{(2\pi)^{\dim X/2}}\right| \leqslant C\exp\left(-C'T\right). \tag{12.34}$$

Now we also use the notation in Chapter 4. By (4.50), (10.11), if $x_0 \in B_g$,

$$\begin{aligned}\int_{\substack{Z\in T_{x_0}X\\ |Z|\leqslant\varepsilon_0/4}} \mathrm{Tr_s}\left[fg\exp\left(-\overline{\mathcal{C}}_{t,T}^{I\widehat{\otimes}F|_{\mathbf{B}},2}\right)(g^{-1}Z,Z)\right]\frac{dv_{TX}(Z)}{(2\pi)^{\dim X/2}} \\ = \int_{\substack{Z\in T_{x_0}X\\ |Z|\leqslant\varepsilon_0/4}} f(Z)\,\mathrm{Tr_s}\left[g\mathcal{P}_{tT}\left(g^{-1}Z/\sqrt{t},Z/\sqrt{t}\right)\right]\frac{dv_{TX}(Z)}{(2\pi t)^{\dim X/2}} \\ \mathrm{Tr}^{F_{x_0}}\left[g\exp\left(-\omega^2\left(\nabla^{F|_{\mathbf{B}}},g^{F|_{\mathbf{B}}}\right)/4\right)\right].\end{aligned} \tag{12.35}$$

By Proposition 1.6,

$$\mathrm{Tr}^{F_{x_0}}\left[g\exp\left(-\omega^2\left(\nabla^{F|_{\mathbf{B}}},g^{F|_{\mathbf{B}}}\right)/4\right)\right] = \mathrm{Tr}^{F_{x_0}}[g]. \tag{12.36}$$

Recall that the function $Z \in TX|_{\mathbf{B}} \to q(Z) \in \mathbf{R}$ was defined in (4.6). By (9.3), for $Z \in T_{x_0}X, |Z| \leqslant \varepsilon_0/4$,

$$f(Z) = f(x_0) + q(Z). \tag{12.37}$$

Also, by (4.25), (4.26),

$$\text{(12.38)}\quad \mathrm{Tr_s}\left[g\mathcal{P}_{tT}(g^{-1}Z,Z)\right]=2^{n/2}\det\left[\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2}$$
$$(-1)^{\mathrm{ind}_g(x_0)}\exp\left(-\left\langle\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)Z,Z\right\rangle\right).$$

By (12.37), (12.38), we get

$$\text{(12.39)}\quad \int_{\substack{Z\in T_{x_0}X\\ |Z|\leqslant\varepsilon_0/4}} f(Z)\,\mathrm{Tr_s}\left[g\mathcal{P}_{tT}\left(g^{-1}Z/\sqrt{t},Z/\sqrt{t}\right)\right]\frac{dv_{TX}(Z)}{(2\pi t)^{\dim X/2}}=(-1)^{\mathrm{ind}_g(x_0)}$$
$$\int_{\substack{Z\in T_{x_0}X\\ |Z|\leqslant\varepsilon_0\sqrt{T}/4}}\left(f(x_0)+\frac{q}{T}(Z)\right)\det\left[\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2}$$
$$\exp\left(-\left\langle\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)Z,Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}}.$$

Clearly,

$$\text{(12.40)}\quad \int_{\substack{Z\in T_{x_0}X\\ |Z|\leqslant\varepsilon_0\sqrt{T}/4}}\det\left[\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2}$$
$$\exp\left(-\left\langle\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)Z,Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}}$$
$$=1-\int_{\substack{Z\in T_{x_0}X\\ |Z|\geqslant\varepsilon_0\sqrt{T}/4}}\det\left[\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2}$$
$$\exp\left(-\left\langle\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)Z,Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}}.$$

Moreover, as $T\to+\infty$, one verifies easily that

$$\text{(12.41)}\qquad \frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\to 1.$$

From (12.40), (12.41), we find that given $t\in\,]0,1]$, there exists $c>0$ such that as $T\to+\infty$,

$$\text{(12.42)}\quad \int_{\substack{Z\in T_{x_0}X\\ |Z|\leqslant\varepsilon_0\sqrt{T}/4}} f(x_0)\det\left[\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2}$$
$$\exp\left(-\left\langle\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)Z,Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}}$$
$$=f(x_0)+\mathcal{O}\left(e^{-cT}\right).$$

By the same argument as before, as $T \to +\infty$,

$$\begin{aligned}(12.43)\quad &\int_{\substack{Z\in T_{x_0}X\\ |Z|\leqslant \varepsilon_0\sqrt{T}/4}} \frac{q}{T}(Z)\det\left[\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2}\\ &\exp\left(-\left\langle\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)Z,Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}}\\ &=\int_{Z\in T_{x_0}X}\frac{q}{T}(Z)\det\left[\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2}\\ &\exp\left(-\left\langle\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)Z,Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}}+\mathcal{O}\left(e^{-cT}\right).\end{aligned}$$

By Propositions 4.8, 4.9 and 4.11,

$$\begin{aligned}(12.44)\quad &\int_{Z\in T_{x_0}X}\frac{q}{T}(Z)\det\left[\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2}\\ &\exp\left(-\left\langle\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)Z,Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}}\\ &=\frac{1}{T}\mathrm{Tr_s}\left[\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)}{\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)}\right].\end{aligned}$$

By (9.11), Proposition 4.15 and by (12.43), (12.44), we find that as $T \to +\infty$,

$$\begin{aligned}(12.45)\quad &\int_{\substack{Z\in T_{x_0}X\\ |Z|\leqslant \varepsilon_0\sqrt{T}/4}} \frac{q}{T}(Z)\det\left[\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2}\\ &\exp\left(-\left\langle\frac{1}{tT}\frac{Q_{tT}/2}{\sinh\left(Q_{tT}/2\right)}\sigma\left(t^2T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)Z,Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}}\\ &=\frac{1}{4T}\left(n_+(x_0)-n_-(x_0)\right)+\mathcal{O}\left(\frac{1}{T^3}\right).\end{aligned}$$

By (12.35), (12.36), (12.39), (12.42), (12.45), as $T \to +\infty$,

$$\begin{aligned}(12.46)\quad &\int_{\substack{Z\in T_{x_0}X\\ |Z|\leqslant \varepsilon_0/4}}\mathrm{Tr_s}\left[fg\exp\left(-\overline{\mathcal{C}}_{t,T}^{I\widehat{\otimes}F|_{\mathbf{B}},2}\right)\left(g^{-1}Z,Z\right)\right]\frac{dv_{TX}(Z)}{(2\pi)^{\dim X/2}}\\ &=(-1)^{\mathrm{ind}_g(x_0)}\left[f(x_0)+\frac{1}{4T}\left(n_+(x_0)-n_-(x_0)\right)\right]\mathrm{Tr}^{F_{x_0}}[g].\end{aligned}$$

By (12.7), (12.34) and (12.46), we get (12.3). The proof of Theorem 12.2 is completed.

12.5. A proof of Theorem 9.7

Now we will prove the much easier Theorem 9.7. First, we assume that the simplifying assumptions of Section 11.1 are in force. By Theorem 11.7, for $T \geqslant 0$ large

enough, for $t > 0$,

$$h'(D_{t,T}) = yt^{-1}\frac{1}{2i\pi}\int_{\Delta}\frac{h'(\sqrt{t}\,\lambda)}{\lambda - B_T}d\lambda\psi_t + \psi_t^{-1}\frac{1}{2i\pi}\int_{\delta/4}\frac{h'(\sqrt{t}\,\lambda)}{\lambda - B_T}d\lambda\psi_t. \tag{12.47}$$

Using Theorems 11.16 and 11.17, (11.78), (12.47), we find that given ε, A with $0 \leqslant \varepsilon \leqslant A \leqslant +\infty$, there exists $C > 0$ such that if $t \in [\varepsilon, A], T \geqslant 1$,

$$\left|\mathrm{Tr}_{\mathrm{s}}\left[Ngh'(D_{t,T})\right] - \widetilde{\chi}_g'^{-}(F)\right| \leqslant \frac{C}{\sqrt{T}}, \tag{12.48}$$

i.e. we have established Theorem 9.7 when the assumptions of Section 11.1 are verified.

In the general case, by proceeding as in Section 12.4, we find that if $t \in [\varepsilon, A], t \leqslant t_0$, Theorems 12.4 and 12.5 show that (9.21) holds, i.e. we have established Theorem 9.7. When $t \in [\varepsilon, A]$ is arbitrary, we may as well use Theorem 12.1, and observe that by the simplifying assumptions in Section 9.1, instead of (11.55), we have the slightly weaker,

$$\overline{B}_T^2 = \overline{\mathcal{D}}_T^{I\widehat{\otimes}F|_{\mathbf{B}},2}. \tag{12.49}$$

Using (12.49), and proceeding as in [**B10**, proof of Theorem 9.5], we get (9.21) for $t \in [\varepsilon, A]$. The proof of Theorem 9.7 is completed.

CHAPTER 13

THE ASYMPTOTICS OF $\mathrm{Tr}_s\left[fgh'\left(D_{t,T/\sqrt{t}}\right)\right]$ AS $t \to 0$

The purpose of this Chapter is to establish Theorem 9.10, i.e. to obtain the first two terms in the asymptotics of $\mathrm{Tr}_s\left[fgh'\left(D_{t,T/\sqrt{t}}\right)\right]$ as $t \to 0$, and also a uniform estimate in the range $0 \leqslant T \leqslant 1/\sqrt{t}$. To establish the required estimates, we use techniques from local families index techniques of Berline-Getzler -Vergne [**BeGeV**], and also functional analytic methods inspired from [**BL**]. First, we obtain the required estimates for bounded T, so as to compute the two leading terms in the asymptotic expansion as quickly as possible. Later, we explain how to obtain uniformity in the range $0 \leqslant T \leqslant 1/\sqrt{t}$.

This Chapter is organized as follows. In Section 13.1, we state a corresponding asymptotic expansion for $\mathrm{Tr}_s\left[fg\exp\left(-\overline{C}^2_{t,T/\sqrt{t}}\right)\right]$, and we prove that Theorem 9.10 can be derived from such a result. The remainder of the Chapter is devoted to the proof of this main result. In Section 13.2, we show that the proof of our estimate can be made local near X_g. As in [**BL, B9, BGo1**], finite propagation speed of solutions of hyperbolic equations plays an essential role. In Section 13.3, we show how to replace the manifold X by T_xX , for $x \in X_g$. In Section 13.4, we introduce a Getzler rescaling on certain Clifford variables. In Section 13.5, we obtain the first term in the asymptotic expansion of $\mathrm{Tr}_s\left[fg\exp\left(-\overline{C}^2_{t,T/\sqrt{t}}\right)\right]$ as $t \to 0$, by using a method due to [**BeGeV**], i.e. by computing the leading term in the asymptotic expansion of the considered rescaled operator as $t \to 0$. In Section 13.6, we obtain the second term in the asymptotic expansion of this operator.

In Section 13.7, we give an estimate on the certain smooth kernels depending on t, T for bounded T, from which our main result follows for bounded T. The proof of this result is deferred to Section 13.9. From this intermediate result, in Section 13.8, we obtain the second term in the asymptotic expansion of $\mathrm{Tr}_s\left[fg\exp\left(-\overline{C}^2_{t,T/\sqrt{t}}\right)\right]$ as $t \to 0$. The results of Chapter 6 are used in the explicit computations.

As mentioned before, in Section 13.9, we establish estimates for certain kernels depending on t, T for bounded T. The proof of these estimates is closely related to [**BL, B9, BGo1**], and involves commutator techniques.

In Section 13.10, we establish our main result in the full range of variation of T. Away from $\mathbf{B}_g$, we refine on the estimates of Section 13.9 by exploiting the fact that $|\nabla f|$ has a positive lower bound. Near $\mathbf{B}_g$, since there is an explicit geometric model already considered in Chapter 4, we use the explicit computation of Chapter 4 to obtain the required estimates in that case too.

In this Chapter, we make the same assumptions as in Chapters 9-10 and 12, and we use the corresponding notation. As before, we also assume that S is compact.

13.1. A convergence result and a proof of Theorem 9.10

Recall that the function $\overline{h}(x)$ was defined in (9.12). We get

$$\overline{h}(x) = \int_0^x e^{t^2} dt. \tag{13.1}$$

Then $\overline{h}(x)$ is a real holomorphic odd function.

The fundamental result of this Chapter is as follows.

Theorem 13.1. — *There exists $C > 0$ such that for $t \in]0,1], 0 \leqslant T \leqslant 2/\sqrt{t}$,*

$$\frac{1}{t}\left|\mathrm{Tr_s}\left[fg\exp\left(-\overline{C}^2_{t,T/\sqrt{t}}\right)\right] - \int_{X_g} \mathrm{Tr}^F[g]\, f\alpha_{T^2/4} + \frac{\sqrt{t}}{2}\int_{X_g} \overline{h}^*_g\left(\nabla^F, g^F\right) T\beta_{T^2/4}\right| \leqslant C. \tag{13.2}$$

Proof. — The next subsections are devoted to the proof of our Theorem. □

Remark 13.2. — We now show how to derive Theorem 9.10 from Theorem 13.1. By (12.4),

$$\mathrm{Tr_s}\left[fgh'\left(D_{t,T}\right)\right] = \left(1 + 2\frac{\partial}{\partial a}\right)\mathrm{Tr_s}\left[fg\exp\left(-a\overline{C}^2_{t,T/\sqrt{t}}\right)\right]|_{a=1}. \tag{13.3}$$

Also by (12.5),

$$\mathrm{Tr_s}\left[fg\exp\left(-a\overline{C}^2_{t,T/\sqrt{t}}\right)\right] = \psi_a \mathrm{Tr_s}\left[fg\exp\left(-\overline{C}^2_{at,T/\sqrt{t}}\right)\right]. \tag{13.4}$$

By Theorem 13.1, if $t \in]0, \frac{1}{2}], a \in]\frac{1}{2}, 2], T \leqslant 1/\sqrt{t}$,

$$\frac{1}{t}\left|\mathrm{Tr_s}\left[fg\exp\left(-\overline{C}^2_{at,T/\sqrt{t}}\right)\right] - \int_{X_g} \mathrm{Tr}^F[g]\, f\alpha_{aT^2/4} + \frac{\sqrt{at}}{2}\int_{X_g} \overline{h}^*_g\left(\nabla^F, g^F\right)\sqrt{a}T\beta_{aT^2/4}\right| \leqslant C. \tag{13.5}$$

Recall that $\alpha_{aT^2/4}$ is of degree $\dim X_g$, so that the form on S,

$$\int_{X_g} \mathrm{Tr}^F[g] f \alpha_{aT^2/4},$$

is of degree 0. From (13.4), (13.5), we get

$$\frac{1}{t}\left|\mathrm{Tr}_s\left[fg\exp\left(-a\overline{C}^2_{t,T/\sqrt{t}}\right)\right] - \int_{X_g}\mathrm{Tr}^F[g] f\alpha_{aT^2/4} + \frac{a\sqrt{t}}{2}\psi_a\int_{X_g}\overline{h}^*_g\left(\nabla^F, g^F\right)T\beta_{aT^2/4}\right| \leqslant C. \tag{13.6}$$

Also,

$$\left(1+2\frac{\partial}{\partial a}\right)\int_{X_g}\mathrm{Tr}^F[g] f\alpha_{aT^2/4}|_{a=1} = \frac{\partial}{\partial T}\left[T\int_{X_g}\mathrm{Tr}^F[g] f\alpha_{T^2/4}\right]. \tag{13.7}$$

Moreover, since $\deg\beta_{aT^2/4} = \dim X_g - 1$,

$$\frac{1}{2}a\psi_a\int_{X_g}\overline{h}^*_g\left(\nabla^F, g^F\right)T\beta_{aT^2/4} = \frac{1}{2}\int_{X_g}\mathrm{Tr}\left[g\overline{h}\left(\sqrt{a}\frac{\omega\left(\nabla^F, g^F\right)}{2}\right)\right]\sqrt{a}T\beta_{aT^2/4}. \tag{13.8}$$

Therefore, by (13.1),

$$\left(1+2\frac{\partial}{\partial a}\right)\frac{1}{2}a\psi_a\int_{X_g}\overline{h}^*_g\left(\nabla^F, g^F\right)T\beta_{aT^2/4}|_{a=1} = \frac{1}{2}\int_{X_g}h^*_g\left(\nabla^F, g^F\right)T\beta_{T^2/4} + \frac{1}{2}\frac{\partial}{\partial T}\left[\int_{X_g}\overline{h}^*_g\left(\nabla^F, g^F\right)T^2\beta_{T^2/4}\right]. \tag{13.9}$$

From (13.3), (13.6), (13.7), (13.9), we get (9.24). The proof of Theorem 9.10 is completed.

13.2. Localization of the problem

We take $\varepsilon_0 > 0$ as in Section 9.1. Let a_X be an upper bound for the injectivity radius of the fibres X. Given $\eta > 0$, let $\mathcal{U}_\eta$ be the ε-neighbourhood of X_g in $N_{X_g/X}$. We identify $N_{X_g/X}$ to the orthogonal bundle to TX_g in $TX|_{X_g}$. Then, there exists $\eta_0 \in]0, \inf\{a_X/32, \varepsilon_0/32\}]$ such that if $\eta \in]0, 2\eta_0]$, the map $(x, Z) \in N_{X_g/X} \to \exp^X_x(Z) \in X$ is a diffeomorphism from $\mathcal{U}_\eta$ on a tubular neighbourhood $\mathcal{V}_\eta$ of X_g in X. In the sequel, we identify $\mathcal{U}_\eta$ and $\mathcal{V}_\eta$. This identification is g-equivariant. Let $\alpha \in]0, \eta_0]$ be small enough so that if $x \in X$, if $d^X\left(g^{-1}x, x\right) \leqslant \alpha$, then $x \in \mathcal{V}_{\eta_0}$.

Clearly, if $a \in \mathbf{C}$,

$$\exp\left(-a^2\right) = \int_{-\infty}^{+\infty}\exp(2isa)\exp\left(-s^2\right)\frac{ds}{\sqrt{\pi}}. \tag{13.10}$$

Let $f : \mathbf{R} \to +\infty$ be a smooth even function such that

$$f(s) = 1 \text{ if } |s| \leqslant \alpha/2, \quad = 0 \text{ if } |s| \geqslant \alpha. \tag{13.11}$$

Put

$$g(s) = 1 - f(s). \tag{13.12}$$

Definition 13.3. — For $t > 0, a \in \mathbf{C}$, put

$$F_t(a) = \int_{-\infty}^{+\infty} \exp(2isa) f\left(\sqrt{t}\, s\right) \exp\left(-s^2\right) \frac{ds}{\sqrt{\pi}}, \tag{13.13}$$

$$G_t(a) = \int_{-\infty}^{+\infty} \exp(2isa) g\left(\sqrt{t}\, s\right) \exp\left(-s^2\right) \frac{ds}{\sqrt{\pi}}.$$

Then

$$\exp\left(-a^2\right) = F_t(a) + G_t(a). \tag{13.14}$$

Moreover F_t, G_t are even holomorphic functions, whose restriction to $\mathbf{R}$ lies in $S(\mathbf{R})$. By (13.13), we find that given $m, m' \in \mathbf{N}, c > 0$, there exist $C > 0, C' > 0$ such that if $t \in]0,1], \alpha \in \mathbf{C}, |\mathrm{Im}(\alpha)| \leqslant c$,

$$|a|^m \left|G_t^{(m')}(a)\right| \leqslant C \exp(-C'/t). \tag{13.15}$$

Put

$$I_t(a) = \int_{-\infty}^{+\infty} \exp(2isa/t) g(s) \exp\left(-s^2/t\right) \frac{ds}{\sqrt{\pi t}}. \tag{13.16}$$

Then

$$I_t(a) = G_t\left(a/\sqrt{t}\right). \tag{13.17}$$

By (13.11), (13.16), we find that given $m, m' \in \mathbf{N}$, there exist $C > 0, C' > 0$ such that if $t \in]0,1], a \in \mathbf{C}, |\mathrm{Im}(a)| \leqslant \frac{\alpha}{8}$,

$$|a|^m \left|I_t^{(m')}(a)\right| \leqslant C \exp(-C'/t). \tag{13.18}$$

Clearly, there exist uniquely defined holomorphic functions $\widetilde{F}_t(a), \widetilde{G}_t(a), \widetilde{I}_t(a)$ such that

$$F_t(a) = \widetilde{F}_t\left(a^2\right), \qquad G_t(a) = \widetilde{G}_t\left(a^2\right), \qquad I_t(a) = \widetilde{I}_t\left(a^2\right). \tag{13.19}$$

By (13.14), (13.17),

$$\exp(-a) = \widetilde{F}_t(a) + \widetilde{G}_t(a), \qquad \widetilde{I}_t(a) = \widetilde{G}_t(a/t). \tag{13.20}$$

Definition 13.4. — Given $c > 0$, set

$$
\begin{aligned}
U_c &= \left\{\lambda \in \mathbf{C}, \mathrm{Re}\,(\lambda) \leqslant \frac{\mathrm{Im}^2\,(\lambda)}{4c^2} - c^2\right\},\\
V_c &= \left\{\lambda \in \mathbf{C}, \mathrm{Re}\,(\lambda) \geqslant \frac{\mathrm{Im}^2\,(\lambda)}{4c^2} - c^2\right\},\\
\Gamma_c &= \left\{\lambda \in \mathbf{C}, \mathrm{Re}\,(\lambda) = \frac{\mathrm{Im}^2\,(\lambda)}{4c^2} - c^2\right\}.
\end{aligned} \tag{13.21}
$$

The set V_c is the image of $\{\lambda \in \mathbf{C}, |\mathrm{Im}\,(\lambda)| \leqslant c\}$ by the map $\lambda \to \lambda^2$. By (13.15), given $m, m' \in \mathbf{N}, c > 0$, there exist $C > 0, C' > 0$ such that if $t \in]0,1], \lambda \in V_c$,

$$
|\lambda|^m \left|\widetilde{G}_t^{(m')}\right| \leqslant C \exp\left(-C'/t\right). \tag{13.22}
$$

By (13.18), given $m, m' \in \mathbf{N}$, there exist $C > 0, C' > 0$ such that if $t \in]0,1], \lambda \in V_{\alpha/8}$,

$$
|\lambda|^m \left|\widetilde{I}_t^{(m')}\,(\lambda)\right| \leqslant C \exp\left(-C'/t\right). \tag{13.23}
$$

By (13.20),

$$
\exp\left(-\overline{C}_{t,T}^2\right) = \widetilde{F}_t\left(\overline{C}_{t,T}^2\right) + \widetilde{I}_t\left(t\overline{C}_{t,T}^2\right). \tag{13.24}
$$

In particular, we deduce from (13.24) that

$$
\mathrm{Tr_s}\left[fg\exp\left(-\overline{C}_{t,T}^2\right)\right] = \mathrm{Tr_s}\left[fg\widetilde{F}_t\left(\overline{C}_{t,T}^2\right)\right] + \mathrm{Tr_s}\left[fg\widetilde{I}_t\left(t\overline{C}_{t,T}^2\right)\right]. \tag{13.25}
$$

Proposition 13.5. — *The following identity holds,*

$$
\mathrm{Tr_s}\left[fg\widetilde{I}_t\left(t\overline{C}_{t,T}^2\right)\right] = \psi_t^{-1}\mathrm{Tr_s}\left[fg\widetilde{I}_t\left(t^2\overline{C}_{1,T}^2\right)\right]. \tag{13.26}
$$

Proof. — This follows from (3.55). □

Now we use the notation in (3.109). Also if $x \in X, r \in \mathbf{R}_+^*$, let $B^X\,(x,r)$ be the open ball of centre x and radius r in the corresponding fibre X, with respect to the Riemannian distance d^X.

Theorem 13.6. — *For any $M \geqslant 0$, there exist $C > 0, C' > 0$ such that for $t \in]0,1], 0 \leqslant T \leqslant M/t$,*

$$
\left\|\widetilde{I}_t\left(t^2\overline{C}_{1,T}^2\right)\right\|_1 \leqslant C \exp\left(-C'/t\right). \tag{13.27}
$$

Proof. — We use formula (12.2) for $\overline{C}_{1,T}^2$. Observe that for $0 \leqslant T \leqslant M/t$, when multiplied by t^2, all the zero order terms which appear in the right-hand side of (12.2) remain uniformly bounded. We can then use (13.23) and proceed as in [**BGo1**, proof of Theorem 7.15] to establish (13.27). □

From (13.26), (13.27), we deduce that there exist $C > 0, C' > 0$ such that for $t \in]0,1], 0 \leqslant T \leqslant 2/t$,

$$\left|\mathrm{Tr}_s\left[fg\widetilde{I}_t\left(\overline{C}^2_{t,T}\right)\right]\right| \leqslant C\exp\left(-C'/t\right). \tag{13.28}$$

Using (13.25), (13.28), we see that to establish Theorem 13.1, we only need to establish the corresponding estimate for $\mathrm{Tr}_s\left[fg\widetilde{F}_t\left(\overline{C}^2_{t,T}\right)\right]$.

Let $\widetilde{F}_t\left(\overline{C}^2_{t,T}\right)(x,x')$ be the smooth kernel of $\widetilde{F}_t\left(\overline{C}^2_{t,T}\right)$ with respect to the volume $dv_X(x')/(2\pi)^{\dim X/2}$. Then

$$\mathrm{Tr}_s\left[fg\widetilde{F}_t\left(\overline{C}^2_{t,T}\right)\right] = \int_X \mathrm{Tr}_s\left[f(x)g\widetilde{F}_t\left(\overline{C}^2_{t,T}\right)\left(g^{-1}x,x\right)\right]\frac{dv_X(x)}{(2\pi)^{\dim X/2}}. \tag{13.29}$$

Using finite propagation speed for solutions of hyperbolic equations [**ChP**, Section7.8], [**T**, Section 4.4] and (13.13), given $x \in X$, we find that the support of $\widetilde{F}_t\left(\overline{C}^2_{t,T}\right)(x,.)$ is included in the ball $B^X(x,\alpha)$. Moreover the kernel $\widetilde{F}_t\left(\overline{C}^2_{t,T}\right)(x,.)$ depends only on the restriction of $\overline{C}^2_{t,T}$ to $B^X(x,\alpha)$. By the choice we made of α, the support of $\widetilde{F}_t\left(\overline{C}^2_{t,T}\right)\left(g^{-1}x,x\right)$ is included in $\mathcal{V}_\eta$.

By the above, we find that the proof of Theorem 9.10 is now a local problem on X, and this only near X_g. In the sequel, we will denote by dv_{X_g} the volume form on the fibre X_g with respect to the metric g^{TX_g}. Similarly, we denote by dv_{TX} the volume form on the fibres TX. Other volume forms will be denoted in the same way.

Let $k(x,Z)$ be the smooth function defined on $\mathcal{V}_\varepsilon$ so that

$$dv_X(x,Z) = k(x,Z)\,dv_{X_g}(x)\,dv_{N_{X_g/X}}(Z). \tag{13.30}$$

Since X_g is totally geodesic in X, as in [**BL**, Proposition 8.9], one finds easily that

$$k(x,Z) = 1 + \mathcal{O}\left(|Z|^2\right). \tag{13.31}$$

By the above, we get

$$\begin{aligned}\mathrm{Tr}_s\left[fg\widetilde{F}_t\left(\overline{C}^2_{t,T/\sqrt{t}}\right)\right] = \int_{X_g}\Bigg\{\int_{\left\{Z\in N_{X_g/X},|Z|\leqslant\varepsilon_0\right\}}\mathrm{Tr}_s\Big[f(x,Z)\,g\widetilde{F}_t\left(\overline{C}^2_{t,T/\sqrt{t}}\right)\\ \left(g^{-1}(x,Z),(x,Z)\right)\Big]k(x,Z)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}/2}}\Bigg\}\frac{dv_{X_g}(x)}{(2\pi)^{\dim X_g/2}}.\end{aligned} \tag{13.32}$$

13.3. Replacing X by TX

Let $\gamma(s): \mathbf{R} \to [0,1]$ be a smooth even function such that

$$\begin{aligned}\gamma(s) &= 1 \text{ if } |s| \leqslant 1/2,\\ &= 0 \text{ if } |s| \geqslant 1.\end{aligned} \tag{13.33}$$

If $Z \in TX$, put

$$\rho(Z) = \gamma(|Z|/4\eta_0). \tag{13.34}$$

Then

$$\begin{aligned} \rho(Z) &= 1 \text{ if } |Z| \leqslant 2\eta_0, \\ &= 0 \text{ if } |Z| \geqslant 4\eta_0. \end{aligned} \tag{13.35}$$

If $x \in X_g, Z \in T_xX$, let $t \in \mathbf{R} \to x_t = \exp_x^X(tZ)$ be the geodesic in the fibre X such that $x_0 = x, \frac{dx}{dt}|_{t=0} = Z$. If $0 < \eta \leqslant 2\eta_0$, the map $Z \in B^{T_xX}(0,\eta) \to \exp_x^X(Z) \in B^X(x,\eta)$ is a diffeomorphism. Let $k'_x(Z)$ be the function defined on $B^{T_xX}(0,2\varepsilon_0)$ such that

$$dv_X(Z) = k'_x(Z)\, dv_{TX}(Z). \tag{13.36}$$

Then

$$k'_x(Z) = 1 + \mathcal{O}\left(|Z|^2\right). \tag{13.37}$$

Recall that the connection ${}^1\nabla_t^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}$ on $\Lambda^\bullet(T^*S) \mathbin{\widehat{\otimes}} \Lambda^\bullet(T^*X) \mathbin{\widehat{\otimes}} F$ was defined in Definition 3.13. Take $x \in X_g$. In the sequel, we trivialize the vector bundle $\Lambda^\bullet(T^*S) \mathbin{\widehat{\otimes}} \Lambda^\bullet(T^*X) \mathbin{\widehat{\otimes}} F$ along geodesics in X centred at x with respect to the connection ${}^1\nabla_{t/2}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}$.

If $x \in X_g$, let $\mathbf{H}_x$ be the vector space of smooth sections of $\left(\Lambda^\bullet(T^*S) \mathbin{\widehat{\otimes}} \Lambda^\bullet(T^*X) \mathbin{\widehat{\otimes}} F\right)_x$ over T_xX. The operator $\rho^2(Z)\, \overline{C}^2_{t,T}$ now acts on $\mathbf{H}_x$. Let Δ^{TX} be the Laplacian on T_xX.

Definition 13.7. — Let $L^{1,x}_{t,T}$ be the operator acting on $\mathbf{H}_x$,

$$L^{1,x}_{t,T} = -\left(1 - \rho^2(Z)\right)\frac{t}{4}\Delta^{TX} + \rho^2(Z)\, \overline{C}^2_{t,T/\sqrt{t}}. \tag{13.38}$$

Let $\widetilde{F}_t\left(\overline{C}^2_{t,T/\sqrt{t}}\right)(Z,Z'), Z, Z' \in T_xX$ be the smooth kernel associated to the operator $\widetilde{F}_t\left(\overline{C}^2_{t,T/\sqrt{t}}\right)$ with respect to $dv_{TX}(Z')/(2\pi)^{\dim X/2}$. If $Z \in N_{X_g/X,x}, |Z| \leqslant \eta_0$, if $x' \in X$ is such that $d^X(Z,x') \leqslant \alpha$, since $\alpha \leqslant \eta_0$, then

$$d^X(x,x') \leqslant 2\eta_0. \tag{13.39}$$

In particular, x' is represented by $Z' \in T_xX$ such that $|Z'| \leqslant 2\eta_0$, so that $\rho(Z') = 1$. Using finite propagation speed as in Section 13.2, we find that if $x \in X_g, Z \in N_{X_g/X,x}, |Z| \leqslant \eta_0$,

$$\widetilde{F}_t\left(L^{1,x}_{t,T}\right)(g^{-1}Z,Z) = \widetilde{F}_t\left(\overline{C}^2_{t,T/\sqrt{t}}\right)(g^{-1}Z,Z)\, k'_x(Z). \tag{13.40}$$

13.4. The Getzler rescaling

Let $H_t : \mathbf{H}_x \to \mathbf{H}_x$ be the linear map,

$$H_t h(Z) = h\left(Z/\sqrt{t}\right). \tag{13.41}$$

Put

$$L_{t,T}^{2,x} = H_t^{-1} L_{t,T}^{1,x} H_t. \tag{13.42}$$

Let Op_x be the space of scalar differential operators on $T_x X$. Clearly,

$$L_{t,T}^{2,x} \in \mathrm{End}\left(\Lambda^\bullet(T^*X)_x\right) \widehat{\otimes} \mathrm{End}\left(F_x\right) \otimes \mathrm{Op}_x. \tag{13.43}$$

Put

$$\ell = \dim X_g. \tag{13.44}$$

Let $e_1, \ldots, e_\ell$ be an orthonormal basis of $T_x X_g$, let $e_{\ell+1}, \ldots, e_n$ be an orthonormal basis of $N_{X_g/X,x}$. Then $e_1, \ldots, e_n$ is an orthonormal basis of $T_x X$.

Recall that as an algebra, $\mathrm{End}\left(\Lambda\left(T^*X_g\right)\right)$ is generated by the $c(e_i), \widehat{c}(e_j), 1 \leqslant i, j \leqslant \ell$, so that $\mathrm{End}\left(\Lambda\left(T^*X_g\right)\right) = c\left(TX_g\right) \widehat{\otimes}\, \widehat{c}\left(TX_g\right)$.

Now we introduce a Getzler rescaling [**Ge**], [**BeGeV**, Chapter 10]. For $t > 0, 1 \leqslant i \leqslant n$, put

$$c_t(e_i) = \sqrt{4/t} e^i \wedge - \sqrt{t/4} i_{e_i}. \tag{13.45}$$

Definition 13.8. — If $x \in X_g$, let $L_{t,T}^{3,x}$ be the operator deduced from $L_{t,T}^{2,x}$ by replacing $c(e_i)$ by $c_t(e_i)$ for $1 \leqslant i \leqslant \ell$, while leaving unchanged the $c(e_i)$'s for $\ell + 1 \leqslant i \leqslant n$ and the $\widehat{c}(e_i)$'s for $1 \leqslant i \leqslant n$.

Clearly, we can write $\widetilde{F}_t\left(L_{t,T}^{3,x}\right)\left(g^{-1}Z, Z\right)$ in the form

$$\widetilde{F}_t\left(L_{t,T}^{3,x}\right)\left(g^{-1}Z, Z\right) = \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots j_q}} e^{i_1} \wedge \ldots e^{i_p} i_{e_{j_1}} \ldots i_{e_{j_q}} Q_{i_1 \ldots i_p}^{j_1 \ldots j_q}, \tag{13.46}$$

with

$$Q_{i_1 \ldots i_p}^{j_1 \ldots j_q} \in \left(\widehat{c}\left(TX_g\right) \widehat{\otimes} \mathrm{End}\left(\Lambda\left(N_{X_g/X}^*\right) \widehat{\otimes} F_x\right)\right)_x. \tag{13.47}$$

Put

$$\widetilde{F}_t\left(L_{t,T}^{3,x}\right)\left(g^{-1}Z, Z\right)^{\max} = Q_{1 \ldots \ell}. \tag{13.48}$$

In (13.48), $Q_{1 \ldots \ell}$ is the coefficient of $e^1 \wedge \cdots \wedge e^\ell$.

Definition 13.9. — Let $\widehat{\mathrm{Tr_s}} : \widehat{c}\left(TX_g\right) \to \mathbf{C}$ be the linear map such that

$$\begin{aligned} \widehat{\mathrm{Tr_s}}\left[\widehat{c}\left(e_{i_1}\right) \ldots \widehat{c}\left(e_{i_p}\right)\right] &= 0 \text{ if } p < \ell, \\ \widehat{\mathrm{Tr_s}}\left[\widehat{c}\left(e_1\right) \ldots \widehat{c}\left(e_\ell\right)\right] &= (-1)^{\ell(\ell+1)/2}. \end{aligned} \tag{13.49}$$

We extend $\widehat{\mathrm{Tr}_s}$ to a linear map from $\Lambda^\bullet(T^*S)\widehat{\otimes}\widehat{c}(TX_g)\widehat{\otimes}\mathrm{End}\left(\Lambda\left(N^*_{X_g/X}\right)\widehat{\otimes}F\right)$ into $\mathbf{C}$, so that it is an ordinary supertrace on the last factor.

Proposition 13.10. — *If $x\in X_g, Z\in N_{X_g/X}, |Z|\leqslant\varepsilon_0$, the following identity holds,*

$$t^{\dim N_{X_g/X}/2}\mathrm{Tr}_s\left[fg\widetilde{F}_t\left(\overline{C}^2_{t,T/\sqrt{t}}\right)\left(g^{-1}\sqrt{t}\,Z,\sqrt{t}\,Z\right)\right]k'_x\left(\sqrt{t}\,Z\right) = \widehat{\mathrm{Tr}_s}\left[fg\widetilde{F}_t(L^{3,x}_{t,T})\left(g^{-1}Z,Z\right)\right]^{\max}. \tag{13.50}$$

Proof. — As in [**BZ1**, Proposition 4.9], one verifies easily, that among the monomials in the $c(e_i),\widehat{c}(e_j), 1\leqslant i,j\leqslant\ell$ acting on $\Lambda(T^*X_g)$, up to permutation,

$$c(e_1)\widehat{c}(e_1)\ldots c(e_\ell)\widehat{c}(e_\ell)$$

is the only monomial with a non zero supertrace, and moreover

$$\mathrm{Tr}_s\left[c(e_1)\widehat{c}(e_1)\ldots c(e_\ell)\widehat{c}(e_\ell)\right] = (-2)^\ell. \tag{13.51}$$

From (13.51), we get

$$\mathrm{Tr}_s\left[c(e_1)\ldots c(e_\ell)\widehat{c}(e_1)\ldots\widehat{c}(e_\ell)\right] = 2^\ell\widehat{\mathrm{Tr}_s}\left[\widehat{c}(e_1)\ldots c(e_\ell)\right]. \tag{13.52}$$

From (13.51), (13.52), we get (13.50). □

Let $i: M_g\to M$ be the obvious embedding. The map i induces the fibrewise embedding $i: X_g\to X$. Recall that since $\nabla f\in TX$ is G-invariant, $\nabla f|_{X_g}\in TX_g$. As we saw in Section 3.3, the connection ∇^{TX} induces a connection ∇^{TX_g} on TX_g. In particular $\nabla^{TX_g}_{\cdot}\nabla f$ is a 1-form on M_g with values in TX_g.

If $(h_1,\ldots,h_m)$ is a basis of TM_g, and if $\left(h^1,\ldots,h^m\right)$ is the corresponding dual basis, set

$$\widehat{c}\left(\nabla^{TX_g}_{\cdot}\nabla f\right) = -h^j\widehat{c}\left(\nabla^{TX_g}_{h_j}\nabla f\right). \tag{13.53}$$

Equivalently,

$$\widehat{c}\left(\nabla^{TX_g}_{\cdot}\nabla f\right) = -\sum_{i=1}^{\ell}e^i\widehat{c}\left(\nabla^{TX_g}_{e_i}\nabla f\right) - f^\alpha\widehat{c}\left(\nabla^{TX_g}_{f^H_\alpha}\nabla f\right). \tag{13.54}$$

Recall that $R^{F,u}$ is given by (3.55).

Definition 13.11. — Put

$$L^{3,x}_{0,T} = -\frac{1}{4}\left(\nabla_{e_i}+\left\langle i^*R^{TX}Z,e_i\right\rangle\right)^2+\frac{1}{4}\left\langle e_i,i^*R^{TX}e_j\right\rangle\widehat{c}(e_i)\widehat{c}(e_j) - \frac{T}{2}\widehat{c}\left(\nabla^{TX_g}_{\cdot}\nabla f\right)+\frac{T^2}{4}|\nabla f|^2+i^*R^{F,u}. \tag{13.55}$$

In the sequel, we will write that a sequence of differential operators on T_xX with smooth coefficients converges if the coefficients converge uniformly over compact sets together with their derivatives of any order.

Now we state a convergence result. A more precise version of this result will be given in Theorem 13.18.

Proposition 13.12. — *As* $t \to 0$,

$$L^{3,x}_{t,T} \to L^{3,x}_{0,T}. \tag{13.56}$$

Proof. — Using Theorem 12.1, and proceeding as in [**BeGeV**, Proposition 10.28], we get (13.56). □

13.5. The first term in the asymptotics of $\mathrm{Tr_s}\left[fg\exp\left(-C^2_{t,T/\sqrt{t}}\right)\right]$

Theorem 13.13. — *If* $x \in X_g, Z \in N_{X_g/X,x}$, *as* $t \to 0$,

$$\widehat{\mathrm{Tr_s}}\left[g\widetilde{F}_t\left(L^{3,x}_{t,T}\right)\left(g^{-1}Z,Z\right)\right]^{\max} \to \widehat{\mathrm{Tr_s}}\left[g\exp\left(-L^{3,x}_{0,T}\right)\left(g^{-1}Z,Z\right)\right]^{\max}. \tag{13.57}$$

Moreover there exist $C > 0, C' > 0$ *such that if* $Z \in N_{X_g/X,x}, |Z| \leqslant \varepsilon_0/\sqrt{t}$,

$$\left|\mathrm{Tr_s}\left[g\widetilde{F}_t\left(L^{3,x}_{t,T}\right)\left(g^{-1}Z,Z\right)\right]\right| \leqslant C\exp\left(-C'|Z|^2\right). \tag{13.58}$$

Proof. — The proof of our Theorem is the same as the proof of [**BGo1**, Theorem 7.43]. □

In the sequel, we will write $R^{TX}, R^{F,u}$ instead of $i^*R^{TX}, i^*R^{F,u}$, since we only deal with forms over M_g.

Proposition 13.14. — *The following identity holds,*

$$\begin{aligned} g\exp\left(-L^{3,x}_{0,T}\right)\left(g^{-1}Z,Z\right) &= 2^{n/2}\det\left(\frac{R^{TX}/2}{\sinh\left(R^{TX}/2\right)}\right)^{1/2} \\ &\exp\left(-\left\langle\frac{R^{TX}/2}{\sinh\left(R^{TX}/2\right)}\sigma\left(0,B,R^{TX}\right)Z,Z\right\rangle\right) \\ g\exp\left(-\frac{1}{4}\left\langle e_i,R^{TX}e_j\right\rangle\widehat{c}(e_i)\widehat{c}(e_j)+\frac{T}{2}\widehat{c}\left(\nabla^{TX_g}_{\cdot}\nabla f\right)-\frac{T^2}{4}|\nabla f|^2\right)&\exp\left(-R^{F,u}\right). \end{aligned} \tag{13.59}$$

Proof. — Our Proposition follows from Proposition 4.8 and from (13.55). □

Theorem 13.15. — *The following identity holds,*

$$\begin{aligned} \int_{N_{X_g/X,x}}\widehat{\mathrm{Tr_s}}\left[g\exp\left(-L^{3,x}_{0,T}\right)\left(g^{-1}Z,Z\right)\right]\frac{dv_{TX}(Z)}{(2\pi)^{\dim N_{X_g/X}/2}} \\ = (2\pi)^{\dim X_g/2}\,\mathrm{Tr}^{F_x}[g]\,\alpha_{T^2/4}. \end{aligned} \tag{13.60}$$

Proof. — Using Proposition 13.14, we get

$$\begin{aligned}(13.61)\quad &\int_{N_{X_g/X,x}} \widehat{\mathrm{Tr}_s}\left[g\exp\left(-L_{0,T}^{3,x}\right)(g^{-1}Z,Z)\right]\frac{dv_{TX}(Z)}{(2\pi)^{\dim N_{X_g/X}/2}}\\ &= 2^{\ell/2}\det\left(\frac{R^{TX_g}/2}{\sinh\left(R^{TX_g}/2\right)}\right)^{1/2}\widehat{\mathrm{Tr}_s}\Big[\exp\Big(-\frac{1}{4}\left\langle e_i,R^{TX_g}e_j\right\rangle\widehat{c}(e_i)\widehat{c}(e_j)\\ &\quad+\frac{T}{2}\widehat{c}\left(\nabla^{TX_g}_{\cdot}\nabla f\right)-\frac{T^2}{4}|\nabla f|^2\Big)\Big]\det\left(\sigma\left(0,B,R^{N_{X_g/X}}\right)\right)^{-1/2}\\ &\mathrm{Tr}_s^{\Lambda\left(N^*_{X_g/X}\right)}\left[g\exp\left(-\frac{1}{4}\left\langle e_i,R^{N_{X_g/X}}e_j\right\rangle\widehat{c}(e_i)\widehat{c}(e_j)\right)\right]\mathrm{Tr}^{F_x}\left[g\exp\left(-R^{F,u}\right)\right].\end{aligned}$$

Now using (6.1), (6.10), (6.30), (6.31),(9.8), (13.49), and proceeding as in [**MQ**, Lemma 2.12], we get

$$\begin{aligned}(13.62)\quad &\det\left(\frac{R^{TX_g}/2}{\sinh\left(R^{TX_g}/2\right)}\right)^{1/2}\widehat{\mathrm{Tr}_s}\Big[\exp\Big(-\frac{1}{4}\left\langle e_i,R^{TX_g}e_j\right\rangle\widehat{c}(e_i)\widehat{c}(e_j)\\ &\qquad+\frac{T}{2}\widehat{c}\left(\nabla^{TX_g}_{\cdot}\nabla f\right)-\frac{T^2}{4}|\nabla f|^2\Big)\Big]=\pi^{\ell/2}\alpha_{T^2/4}.\end{aligned}$$

Also by using (4.26) in Proposition 4.9, we get

$$\begin{aligned}(13.63)\quad &\det\left(\sigma\left(0,B,R^{N_{X_g/X}}\right)\right)^{-1/2}\\ &\qquad\mathrm{Tr}_s^{\Lambda\left(N^*_{X_g/X}\right)}\left[g\exp\left(-\frac{1}{4}\left\langle e_i,R^{N_{X_g/X}}e_j\right\rangle\widehat{c}(e_i)\widehat{c}(e_j)\right)\right]=1.\end{aligned}$$

Finally, by Proposition 1.6,

$$(13.64)\qquad \mathrm{Tr}^{F_x}\left[g\exp\left(-R^{F,u}\right)\right]=\mathrm{Tr}^{F_x}[g].$$

From (13.61)-(13.64), we get (13.60). The proof of our Theorem is completed. □

Theorem 13.16. — *For any $T\geqslant 0$, as $t\to 0$,*

$$(13.65)\qquad \mathrm{Tr}_s\left[fg\exp\left(-C^2_{t,T/\sqrt{t}}\right)\right]\to\int_{X_g}\mathrm{Tr}^F[g]\,f\alpha_{T^2/4}.$$

Proof. — Our Theorem follows from (13.25), (13.28), (13.32), and from Theorems 13.13 and 13.15. □

13.6. The asymptotic expansion of the operator $L_{t,T}^{3,x}$

Definition 13.17. — If $x\in X_g$, put

$$(13.66)\qquad M_T^x=\frac{1}{4}\widehat{c}(e_i)\nabla^{F,u}_{\cdot}\omega\left(\nabla^F,g^F\right)(e_i)-\frac{T}{4}\omega\left(\nabla^F,g^F\right)(\nabla f).$$

In the sequel, we will write some operators in the form $O(Z,Z),P(Z),Q(Z),R(Z)$. These operators are smooth tensors, such that $O(Z,Z)$ depends quadratically on Z, and $P(Z),Q(Z),R(Z)$ depend linearly on Z. The dependence in the variable T will be explicitly written.

Theorem 13.18. — *Given $T \in \mathbf{R}_+$, there are bounded operators $N^{i,x}_{j,k} \in T^*_x X \otimes \mathrm{End}(F_x), 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant \ell, \ell+1 \leqslant k \leqslant n$,and smooth tensors $O^i(Z,Z), 1 \leqslant i \leqslant n, P(Z), Q(Z), R(Z)$, such that if*

(13.67)
$$A^x_T = M^x_T + \sum_{1 \leqslant i \leqslant n} \left[\nabla_{e_i} + \left\langle R^{TX} Z, e_i \right\rangle, \sum_{\substack{1 \leqslant j \leqslant \ell \\ \ell+1 \leqslant k \leqslant n}} N^{i,x}_{jk}(Z)\, e^j \wedge c(e_k) + O^i(Z,Z)\right]_+ + P(Z) + TQ(Z) + T^2 R(Z),$$

as $t \to 0$,

(13.68) $$L^{3,x}_{t,T} = L^{3,x}_{0,T} + \sqrt{t}\, A^x_T + \mathcal{O}(t)\left(1 + T\mathcal{O}(Z) + T\mathcal{O}\left(|Z|^2\right) + T^2\mathcal{O}\left(|Z|^2\right) + \mathcal{O}\left(|Z^4|\right)\right).$$

Proof. — We will still use Theorem 12.1. If $Z \in T_x X, |Z| \leqslant \varepsilon_0, U \in T_x X$, let $\tau U(Z)$ be the parallel transport of U with respect to ∇^{TX} along $t \in [0,1] \to tZ$. In our geodesic coordinates , we can still view $\tau U(Z)$ as lying in $T_x X$. Then classically,

(13.69) $$\tau U(Z) = U + \mathcal{O}\left(|Z|^2\right).$$

In formula (12.2), we then choose,

(13.70) $$e_i(Z) = \tau e_i(Z), \ 1 \leqslant i \leqslant n.$$

In fact $e_1(Z), \ldots, e_n(Z)$ will then be an orthonormal basis of $T_Z X$ with respect to the metric g^{TX}_Z.

By (3.41),

(13.71) $$\begin{aligned} {}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}_Z c(\tau U(Z)) &= \sqrt{2}\left\langle S_Z(Z)U, f^H_\alpha \right\rangle f^\alpha, \\ {}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}_Z \widehat{c}(\tau U(Z)) &= 0. \end{aligned}$$

By (13.71), we find that parallel transport with respect to ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}$ preserves the elements of degree $\leqslant 1$ in $\Lambda^\bullet(T^*S) \widehat{\otimes} c(TX)$.

For $p \in \mathbf{N}, q \in \mathbf{N}$, $\mathcal{O}_p\left(|Z|^q\right)$ denotes an expression in

$$\Lambda^\bullet(T^*S) \widehat{\otimes} c(T_x X_g) \widehat{\otimes} \widehat{c}(TX_g) \widehat{\otimes} \mathrm{End}\left(\Lambda\left(N^*_{X_g/X}\right) \widehat{\otimes} F\right)_x$$

which has the following two properties:

- For $k \in \mathbf{N}, k \leqslant q$, its derivatives of order k are $\mathcal{O}\left(|Z|^{q-k}\right)$.
- It is of length $\leqslant p$ with respect to the $\mathbf{Z}$-grading induced by $\Lambda^\bullet(T^*S) \widehat{\otimes} c(T_x X_g)$.

From (13.71), we deduce that, in the trivialization with respect to the connection ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}$,

(13.72) $$c(\tau U(Z)) = c(U) + \sqrt{2}\left\langle S_x(Z)U, f^H_\alpha \right\rangle f^\alpha + \mathcal{O}_1\left(|Z|^2\right).$$

In the sequel for $1 \leqslant i \leqslant n$, $\left[c\left(\tau e_i\left(\sqrt{t}\,Z\right)\right)\right]_t^3$ denotes the operator $c(\tau)$ written in the considered trivialization with respect to the connection ${}^1\nabla_{t/2}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}$. From (13.72), we deduce that for $1 \leqslant i \leqslant \ell$,

$$\sqrt{t}\left[c\left(\tau e_i\left(\sqrt{t}\,Z\right)\right)\right]_t^3 = 2e^i \wedge -\frac{t}{2} i_{e_i} + 2\sqrt{t}\left\langle S(Z)\, e_i, f_\alpha^H\right\rangle f^\alpha + t\mathcal{O}\left(|Z|^2\right). \tag{13.73}$$

and that for $\ell+1 \leqslant i \leqslant n$,

$$\sqrt{t}\left[c\left(\tau e_i\left(\sqrt{t}\,Z\right)\right)\right]_t^3 = \sqrt{t}\left(c(e_i) + 2\left\langle S(Z)\, e_i, f_\alpha^H\right\rangle f^\alpha\right) + t\mathcal{O}\left(|Z|^2\right). \tag{13.74}$$

We trivialize temporarily $\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F$ by parallel transport with respect to the connection ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}$ along the geodesic $t \in [0,1] \to tZ$. Let Γ be the connection form for ${}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}$ in the considered trivialization. Using (13.71) and the arguments which follow, we find that Γ is a one form with values in elements of $\Lambda^\bullet(T^*S)\widehat{\otimes}c(TX)\widehat{\otimes}\widehat{c}(TX)$ which have length $\leqslant 2$. By [**ABoP**, Proposition 3.7],

$$\Gamma(Z) = \frac{1}{2}{}^1\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,2}(Z,) + \mathcal{O}_2\left(|Z|^2\right). \tag{13.75}$$

Using (3.42), (13.73) and (13.75), we get

$$\left[\sqrt{t}\,\Gamma\left(\sqrt{t}\,Z\right)\right]_t^3 = \left\langle i^* R^{TX} Z, .\right\rangle + \sqrt{t} \sum_{\substack{1\leqslant j\leqslant \ell \\ \ell+1\leqslant k\leqslant n}} e^i \wedge c(e_k)\,\mathcal{O}(Z) + \mathcal{O}\left(\sqrt{t}\right)\mathcal{O}\left(|Z|^2\right) + \mathcal{O}(t)\,\mathcal{O}(|Z|) + \mathcal{O}(t)\,\mathcal{O}\left(|Z|^3\right). \tag{13.76}$$

Similarly one finds easily the asymptotics (up to the order $\sqrt{t}$) of the other terms in the first three lines of the right-hand side of (12.2). The proof of our Theorem is completed. □

13.7. A technical result

Now, we state a result, which will imply Theorem 13.1 for bounded T.

Theorem 13.19. — *For any $T \in \mathbf{R}, x \in X_g, Z \in N_{X_g/X,x}$,*

$$\lim_{t\to 0} \frac{1}{\sqrt{t}}\left[\widehat{\mathrm{Tr}_{\mathrm{s}}}\left[g\widetilde{F}_t\left(L_{t,T}^{3,x}\right)\left(g^{-1}Z, Z\right)\right] - \widehat{\mathrm{Tr}_{\mathrm{s}}}\left[g\exp\left(-L_{0,T}^{3,x}\right)\left(g^{-1}Z, Z\right)\right]\right] = \frac{\partial}{\partial s}\widehat{\mathrm{Tr}_{\mathrm{s}}}\left[g\exp\left(-\left(L_{0,T}^{3,x} + sA_T^x\right)\right)\left(g^{-1}Z, Z\right)\right]|_{s=0}. \tag{13.77}$$

Moreover, given $M \in \mathbf{R}_+$, there exist $C > 0, C' > 0$ such that for $t \in [0,1], 0 \leqslant T \leqslant M, x \in X_g, Z \in N_{X_g/X,x}$,

$$\left|\frac{1}{t}\left[\widehat{\mathrm{Tr}_{\mathrm{s}}}\left[g\widetilde{F}_t\left(L_{t,T}^{3,x}\right)\left(g^{-1}Z, Z\right)\right] - \widehat{\mathrm{Tr}_{\mathrm{s}}}\left[g\exp\left(-L_{0,T}^{3,x}\right)\left(g^{-1}Z, Z\right)\right]\right] - \sqrt{t}\frac{\partial}{\partial s}\widehat{\mathrm{Tr}_{\mathrm{s}}}\left[g\exp\left(-\left(L_{0,T}^{3,x} + sM_T^x\right)\right)\left(g^{-1}Z, Z\right)\right]|_{s=0}\right| \leqslant C\exp\left(-C'|Z|^2\right). \tag{13.78}$$

Proof. — The proof of our Theorem is delayed to Section 13.9. □

13.8. A proof of Theorem 13.1 for bounded T

By (13.25), (13.28), (13.31), (13.32), (13.37), (13.40), (13.59), (13.78), we get, for $t \in]0,1], 0 \leqslant T \leqslant M$,

$$
\begin{aligned}
(13.79)\quad & \frac{1}{t}\left|\mathrm{Tr_s}\left[fg\exp\left(-\overline{C}^2_{t,T/\sqrt{t}}\right)\right]\right. \\
& - \int_{X_g} f\left[\widehat{\mathrm{Tr_s}}\left[g\exp\left(-L^{3,x}_{0,T}\right)\left(g^{-1}Z,Z\right)\right]\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}/2}}\right]\frac{dv_{X_g}(x)}{(2\pi)^{\dim X_g/2}} \\
& - \sqrt{t}\left(\int_{X_g} f\left[\int_{N_{X_g/X}} \frac{\partial}{\partial s}\widehat{\mathrm{Tr_s}}\left[g\exp\left(-\left(L^{3,x}_{0,T}+sA^x_T\right)\right)\left(g^{-1}Z,Z\right)\right]|_{s=0}\right.\right. \\
& \qquad\qquad \left.\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}/2}}\right]\frac{dv_{X_g}(x)}{(2\pi)^{\dim X_g/2}} \\
& \quad + \int_{X_g}\left[\int_{N_{X_g/X}} \langle f'(x),Z\rangle\,\widehat{\mathrm{Tr_s}}\left[g\exp\left(-L^{3,x}_{0,T}\right)\left(g^{-1}Z,Z\right)\right]\right. \\
& \qquad\qquad \left.\left.\left.\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}/2}}\right]\frac{dv_{X_g}(x)}{(2\pi)^{\dim X_g/2}}\right)\right| \leqslant C.
\end{aligned}
$$

Set

$$
(13.80)\qquad \widehat{\omega}\left(\nabla^F,g^F\right) = \sum_{i=1}^{\ell} \widehat{e^i}\wedge\omega\left(\nabla^F,g^F\right)(e_i)\,.
$$

Equivalently, $\widehat{\omega}\left(\nabla^F,g^F\right)$ is the hatted version of the restriction of $\omega\left(\nabla^F,g^F\right)$ to the fibre X_g.

Theorem 13.20. — *The following identities hold,*

$$
\begin{aligned}
(13.81)\quad & \int_{N_{X_g/X}} \langle f'(x),Z\rangle\,\widehat{\mathrm{Tr_s}}\left[g\exp\left(-L^{3,x}_{0,T}\right)\left(g^{-1}Z,Z\right)\right]\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}/2}} = 0, \\
& \int_{N_{X_g/X}} \frac{\partial}{\partial s}\widehat{\mathrm{Tr_s}}\left[g\exp\left(-\left(L^{3,x}_{0,T}+sA^x_T\right)\right)\left(g^{-1}Z,Z\right)\right]|_{s=0}\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}/2}} \\
& = (2\pi)^{\dim X_g/2}\frac{1}{4}\int^B \mathrm{Tr}^{F_x}\left[g\left(\nabla^{TX\otimes F,u}+Ti_{\widehat{\nabla f}}\right)\widehat{\omega}\left(\nabla^F,g^F\right)\exp\left(-B_{T^2/4}-R^{F,u}\right)\right].
\end{aligned}
$$

Proof. — Clearly the operator $L^{3,x}_{0,T}$ is invariant by the map $Z \to -Z$. Therefore we get the first identity in (13.81). We claim that in the left-hand side of the second identity, we can replace A^x_T by M^x_T. In fact by the obvious analogue of (13.51), when acting on $\Lambda\left(N^*_{X_g/X}\right)$, only globally even monomials in the $c(e_i),\widehat{c}(e_i),\ell+1\leqslant i\leqslant n$ have a non zero supertrace. This fact makes the first sort of anticommutator in the

right-hand side of (13.67) does not contribute to the second integral in (13.81). The remaining terms in (13.67) are odd in the variable Z, and so do not contribute to the integral for the same reasons as before.

By Proposition 4.8 and by (13.55), as in Proposition 13.14, we get

$$
\begin{aligned}
(13.82)\quad & g\exp\left(-L_{0,T}^{3,x}-sM_T^x\right)\left(g^{-1}Z,Z\right)=2^{n/2}\det\left(\frac{R^{TX}/2}{\sinh\left(R^{TX}/2\right)}\right)^{1/2}\\
&\exp\left(-\left\langle\frac{R^{TX}/2}{\sinh\left(R^{TX}/2\right)}\sigma\left(0,B,R^{TX}\right)Z,Z\right\rangle\right)\\
&g\exp\left(-\frac{1}{4}\left\langle e_i,R^{TX}e_j\right\rangle\widehat{c}(e_i)\widehat{c}(e_j)-\frac{T^2}{4}\left|\nabla f\right|^2+\frac{T}{2}\widehat{c}\left(\nabla_{\cdot}^{TX_g}\nabla f\right)-R^{F,u}-sM_T^x\right).
\end{aligned}
$$

From (13.82), we obtain,

$$
\begin{aligned}
(13.83)\quad & \int_{N_{X_g/X}}\widehat{\mathrm{Tr}_{\mathrm{s}}}\left[g\exp\left(-L_{0,T}^{3,x}-sM_T^x\right)\left(g^{-1}Z,Z\right)\right]\frac{dv_{N_{X_g/X}}}{(2\pi)^{\dim N_{X_g/X}/2}}\\
&=2^{\ell/2}\det\left(\frac{R^{TX_g/2}}{\sinh\left(R^{TX_g/2}\right)}\right)^{1/2}\det\left(\sigma\left(0,B,R^{N_{X_g/X}}\right)\right)^{-1/2}\\
&\widehat{\mathrm{Tr}_{\mathrm{s}}}\left[g\exp\left(-\frac{1}{4}\left\langle e_i,R^{TX}e_j\right\rangle\widehat{c}(e_i)\widehat{c}(e_j)-\frac{T^2}{4}\left|\nabla f\right|^2+\frac{T}{2}\widehat{c}\left(\nabla_{\cdot}^{TX_g}\nabla f\right)-R^{F,u}-sM_T^x\right)\right].
\end{aligned}
$$

By (13.83), we get

$$
\begin{aligned}
(13.84)\quad & \int_{N_{X_g/X}}\frac{\partial}{\partial s}\widehat{\mathrm{Tr}_{\mathrm{s}}}\left[g\exp\left(-\left(L_{0,T}^{3,x}+sM_T^x\right)\right)\left(g^{-1}Z,Z\right)\right]|_{s=0}\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}/2}}\\
&=-2^{\ell/2}\det\left(\frac{R^{TX_g/2}}{\sinh\left(R^{TX_g/2}\right)}\right)^{1/2}\det\left(\sigma\left(0,B,R^{N_{X_g/X}}\right)\right)^{-1/2}\\
&\widehat{\mathrm{Tr}_{\mathrm{s}}}\left[gM_T^x\exp\left(-\frac{1}{4}\left\langle e_i,R^{TX}e_j\right\rangle\widehat{c}(e_i)\widehat{c}(e_j)-\frac{T^2}{4}\left|\nabla f\right|^2+\frac{T}{2}\widehat{c}\left(\nabla_{\cdot}^{TX_g}\nabla f\right)-R^{F,u}\right)\right].
\end{aligned}
$$

As before, in M_T^x, the term

$$
\sum_{\ell+1\leqslant i\leqslant n}\widehat{c}(e_i)\nabla_{\cdot}^{F,u}\omega\left(\nabla^F,g^F\right)(e_i)
$$

does not contribute to the supertrace in (13.84).

Then, by proceeding as in the proof of Theorem 13.15, by using in particular (13.63) (13.66), (13.84), we get (13.81). The proof of our Theorem is completed. □

Recall that the function $\overline{h}(x)$ was defined in (9.12) or in (13.1).

Theorem 13.21. — *The following identity holds,*

$$\frac{1}{4}\int_{X_g} f \int^B \mathrm{Tr}^{F_x}\left[g\left(\nabla^{TX\otimes F,u} + Ti_{\widehat{\nabla f}}\right)\widehat{\omega}\left(\nabla^F, g^F\right)\exp\left(-B_{T^2/4} - R^{F,u}\right)\right] = -\frac{1}{2}\int_{X_g} \overline{h}_g^*\left(\nabla^F, g^F\right) T\beta_{T^2/4}. \tag{13.85}$$

Proof. — By (1.12),

$$\nabla^{TX\otimes F,u}\widehat{\omega}(F, g^F) = \nabla^{TX\otimes F}\widehat{\omega}(F, g^F) + \frac{1}{2}\left[\omega\left(\nabla^F, g^F\right), \widehat{\omega}(F, g^F)\right]. \tag{13.86}$$

By (1.19),

$$\left[\nabla^F, \omega\left(\nabla^F, g^F\right)\right] = -\omega^2\left(\nabla^F, g^F\right). \tag{13.87}$$

Equivalently, if U, V are smooth sections of TM,

$$\nabla^F_U\omega\left(\nabla^F, g^F\right)(V) - \nabla^F_V\omega\left(\nabla^F, g^F\right)(U) - \omega\left(\nabla^F, g^F\right)([U,V]) = -\left[\omega\left(\nabla^F, g^F\right)(U), \omega\left(\nabla^F, g^F\right)(V)\right]. \tag{13.88}$$

Set

$$\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u} = \sum_{i=1}^{\ell} \widehat{e}^i \nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}_{e_i}. \tag{13.89}$$

Equivalently, $\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}$ is the hatted version of the restriction to X_g of the connection $\nabla^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u}$. By (13.86)-(13.88), we obtain,

$$\nabla^{TX\otimes F,u}\widehat{\omega}(F, g^F) = -\left(\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u} + \widehat{e}^i f^\alpha i_{T(e_i, f^H_\alpha)}\right)\omega\left(\nabla^F, g^F\right). \tag{13.90}$$

Using (13.90), we get

$$\int_{X_g} f\int^B \mathrm{Tr}^{F_x}\left[g\left(\nabla^{TX\otimes F,u} + Ti_{\widehat{\nabla f}}\right)\widehat{\omega}\left(\nabla^F, g^F\right)\exp\left(-B_{T^2/4} - R^{F,u}\right)\right] = -\int_{X_g} f\int^B \mathrm{Tr}^{F_x}\left[\left(\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u} + \widehat{e}^i f^\alpha i_{T(e_i, f^H_\alpha)} - Ti_{\nabla f}\right)\left[\omega\left(\nabla^F, g^F\right)\right] \exp\left(-B_{T^2/4} + \omega^2\left(F, g^F\right)/4\right)\right]. \tag{13.91}$$

By Theorem 6.15, (9.1) and (9.12), we get

$$\left(\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)} + \widehat{e}^i f^\alpha i_{T(e_i, f^H_\alpha)} - Ti_{\nabla f}\right)\left(2\overline{h}_g^*\left(\nabla^F, g^F\right)\exp\left(-B_{T^2/4}\right)\right) = \mathrm{Tr}^{F_x}\left[g\left(\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)\widehat{\otimes}F,u} + \widehat{e}^i f^\alpha i_{T(e_i, f^H_\alpha)} - Ti_{\nabla f}\right)\left[\omega\left(\nabla^F, g^F\right)\right] \exp\left(-B_{T^2/4} + \omega^2\left(F, g^F\right)/4\right)\right]. \tag{13.92}$$

By (13.91), (13.92), we obtain,

$$\int_{X_g} f \int^B \mathrm{Tr}^{F_x}\left[g\left(\nabla^{TX\otimes F,u}+Ti_{\widehat{\nabla f}}\right)\widehat{\omega}\left(\nabla^F,g^F\right)\exp\left(-B_{T^2/4}-R^{F,u}\right)\right] \tag{13.93}$$
$$= -2\int_{X_g}\int^B f\widehat{\nabla}^{\Lambda^\bullet(T^*S)\widehat{\otimes}\Lambda^\bullet(T^*X)}\left(\overline{h}_g^*\left(\nabla^F,g^F\right)\exp\left(-B_{T^2/4}\right)\right).$$

By Theorem 6.18 and by (13.93), we get

$$\int_{X_g} f \int^B \mathrm{Tr}^{F_x}\left[g\left(\nabla^{TX\otimes F,u}+Ti_{\widehat{\nabla f}}\right)\widehat{\omega}\left(\nabla^F,g^F\right)\exp\left(-B_{T^2/4}-R^{F,u}\right)\right] \tag{13.94}$$
$$= -2\int_{X_g}\overline{h}_g^*\left(\nabla^F,g^F\right)\int^B \widehat{\nabla f}\exp\left(-B_{T^2/4}\right),$$

which, in view of (6.10), (9.8) is just (13.85). The proof of our Theorem is completed. □

Remark 13.22. — By (13.79), by Theorems 13.15, 13.20 and 13.21, we get (13.2) for bounded T. The proof of Theorem 13.1 for bounded T is completed.

13.9. A proof of Theorem 13.19

Recall that $\ell = \dim X_g$. Set

$$m = \dim S. \tag{13.95}$$

Let $M \in \mathbf{R}_+^*$.

Definition 13.23. — If $x \in X_g, 0 \leqslant p \leqslant s, 0 \leqslant q \leqslant \ell$, let $\mathbf{K}_{(p,q),x}$ (resp. $\mathbf{K}^0_{(p,q),x}$) be the vector space of smooth (resp. square integrable) sections of

$$\left(\Lambda^p(T^*S)\widehat{\otimes}\Lambda^q(T^*X_g)\widehat{\otimes}\widehat{c}(TX_g)\widehat{\otimes}\Lambda^\bullet\left(N_{X_g/X}\right)\widehat{\otimes}F\right)_x$$

over T_xX.

Let $\mathbf{K}_x, \mathbf{K}^0_x$ be the direct sum of the $\mathbf{K}_{(p,q),x}, \mathbf{K}^0_{(p,q),x}$. More generally, of $a \in \mathbf{R}$, we denote by $\mathbf{K}^a_x$ the a^{th} Sobolev space of sections of the above vector space.

Definition 13.24. — For $t \in]0,1], 0 \leqslant T \leqslant M, x \in X_g$, if $s \in \mathbf{K}_{(p,q),x}$ has compact support, put

$$|s|^2_{t,x,0} = \int_{T_XX}|s|^2\left(1+|Z|\,\rho\left(uZ/2\right)\right)^{2(n+m-p-q)}dv_{TX}(Z), \tag{13.96}$$
$$|s|^2_{t,x,1} = |s|^2_{t,x,0}+\sum_{i=1}^n|\nabla_{e_i}s|^2_{t,x,0}.$$

Then (13.96) define Hilbert norms on the Sobolev spaces $\mathbf{K}_x^0, \mathbf{K}_x^1$. Let $\langle\ \rangle_{t,x,0}$ be the associated Hermitian product on $\mathbf{K}_x^0$. We identify $\mathbf{K}_x^0$ to its antidual by the Hermitian product associated to (13.96). Then $\mathbf{K}_x^1$ embeds as a dense subspace of $\mathbf{K}_x^0$, and the norm of the embedding is less than 1. Let $\mathbf{K}_x^{-1}$ be the antidual of $\mathbf{K}_x^1$, en let $|\ |_{t,x,-1}$ be the corresponding norm. Then $\mathbf{K}_x^0$ embeds a a dense subspace of $\mathbf{K}_x^{-1}$, and the norm of the embedding is less than 1.

Theorem 13.25. — *There exist constants $C_1 > 0, \ldots, C_4 > 0$ such that if $t \in]0,1], 0 \leqslant T \leqslant M, x \in X_g$, if $s, s' \in \mathbf{K}_x$ have compact support, then*

$$\begin{aligned} &\mathrm{Re}\left\langle L_{t,T}^{3,x} s, s\right\rangle_{t,x,0} \geqslant C_1 |s|_{t,x,1}^2 - C_2 |s|_{t,x,0}^2, \\ &\left|\mathrm{Im}\left\langle L_{t,T}^{3,x} s, s\right\rangle_{t,x,0}\right| \leqslant C_3 |s|_{t,x,1} |s|_{t,x,0}, \\ &\left|\left\langle L_{t,T}^{3,x} s, s'\right\rangle_{t,x,0}\right| \leqslant C_4 |s|_{t,x,1} |s'|_{t,x,1}. \end{aligned} \tag{13.97}$$

Proof. — Using (12.2), the proof of our Theorem is the same as the proof of [**BL**, Theorem 11.26]. □

Recall that U_c was defined in Definition 13.4.

Theorem 13.26. — *There exist $c > 0, C > 0$ such that if $t \in]0,1], 0 \leqslant T \leqslant M, \lambda \in U_c$, the resolvent $\left(\lambda - L_{t,T}^{3,x}\right)^{-1}$ exists, extends to a continuous linear operator from $\mathbf{K}_x^{-1}$ to $\mathbf{K}_x^1$, and moreover,*

$$\begin{aligned} &\left\|\left(\lambda - L_{t,T}^{3,x}\right)^{-1}\right\|^{0,0} \leqslant C, \\ &\left\|\left(\lambda - L_{t,T}^{3,x}\right)^{-1}\right\|^{-1,1} \leqslant C\left(1 + |\lambda|\right)^2. \end{aligned} \tag{13.98}$$

Proof. — Using Theorem 13.25, the proof of our Theorem is the same as the proof of [**BL**, Theorem 11.27]. □

From now on, $c > 0$ is fixed as in Theorem 13.26.

Definition 13.27. — Let $\mathcal{Q}_x$ be the family of operators,

$$\mathcal{Q}_x = \left\{\nabla_{e_i}, 1 \leqslant i \leqslant n\right\}. \tag{13.99}$$

For $j \in \mathbf{N}$, let $\mathcal{Q}_x^j$ be the set of operators $Q_1 \ldots Q_j$, with $Q_i \in \mathcal{Q}_x, 1 \leqslant i \leqslant j$.

Proposition 13.28. — *Take $k \in \mathbf{N}$. There exists $C_k > 0$ such that if $t \in]0,1]$, $0 \leqslant T \leqslant M$, $x \in X_g, Q_1, \ldots, Q_k \in \mathcal{Q}_x$, if $s, s' \in \mathbf{K}_x$ have compact support, then*

$$\left|\left\langle [Q_1, [\ldots Q_k, L_{t,T}^{3,x}] \ldots] s, s'\right\rangle_{t,x,0}\right| \leqslant C_k |s|_{t,x,1} |s'|_{t,x,1}. \tag{13.100}$$

Proof. — Using (12.2), the proof of our Proposition is the same as the proof of [**BL**, Proposition 11.29]. □

If $k \in \mathbf{N}$, if $s \in \mathbf{K}_x^k$, put

$$\|s\|_{t,x,k}^2 = \sum_{j=0}^{k} \sum_{Q \in \mathcal{Q}_x^j} |Qs|_{t,x,0}^2 . \tag{13.101}$$

If $A \in \mathcal{L}\left(\mathbf{K}_x^k, \mathbf{K}_x^{k+1}\right)$, let $\|A\|_{t,x}^{k,k+1}$ be the norm of A with respect to the above norms.

Theorem 13.29. — *For any $k \in \mathbf{N}$, there exist $m_k \in \mathbf{N}, C_k > 0$ such that if $t \in]0,1], 0 \leqslant T \leqslant M, x \in X_g, \lambda \in U_c$, the resolvent $\left(\lambda - L_{t,T}^{3,x}\right)^{-1}$ maps $\mathbf{K}_x^k$ into $\mathbf{K}_x^{k+1}$, and moreover*

$$\left\| \left(\lambda - L_{t,T}^{3,x}\right)^{-1} \right\|_{t,x}^{k,k+1} \leqslant C_k \left(1 + |\lambda|\right)^{m_k} . \tag{13.102}$$

Proof. — Using Theorem 13.26 and Proposition 13.28, the proof of our Proposition is the same as the proof of [**BL**, Theorem 11.30]. □

Theorem 13.30. — *There exist $C' > 0$ such that for any $m \in \mathbf{N}$, there is $C > 0$, $r \in \mathbf{N}$ such that for $t \in]0,1], x \in X_g, Z, Z' \in T_x X$,*

$$\sup_{|\alpha|,|\alpha'| \leqslant m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \widetilde{F}_t\left(L_{t,T}^{3,x}\right)(Z,Z') \right| \leqslant C\left(1 + |Z| + |Z'|\right)^r \exp\left(-C'|Z - Z'|^2\right) . \tag{13.103}$$

Proof. — Using Theorem 13.29, the proof of our Theorem is the same as the proof of [**BL**, Theorem 11.31], of [**B9**, Theorem 11.14], or of [**BGo1**, Theorem 7.42]. Note in particular that the exponential factor in the right-hand side of (13.103) can be obtained as in [**B9, BGo1**] by using finite propagation speed for solutions of hyperbolic equations. □

Observe that if $s \in \mathbf{K}_x$ has compact support, $|s|_{t,x,0}, |s|_{t,x,1}$ can also be defined for $t = 0$. Also,

$$|s|_{t,x,0} \leqslant |s|_{0,x,0} , \qquad |s|_{t,x,1} \leqslant |s|_{0,x,1} . \tag{13.104}$$

We denote by $\mathbf{K}'^k, k = 0, 1, -1$ the corresponding Hilbert spaces. If $(\alpha = \alpha_1 \ldots \alpha_n)$ is a multiindex, set $Z^\alpha = Z_1^{\alpha_1} \ldots Z_n^{\alpha_n}$.

Definition 13.31. — If $k = \{-1\} \cup \mathbf{N}, k' \in \mathbf{N}$, let $\mathbf{K}_x'^{k,k'}$ be the set of the $s \in \mathbf{K}'^k$ such that if $|\alpha| \leqslant k'$, then $Z^\alpha s \in \mathbf{K}_x'^k$. If $s \in \mathbf{K}_x'^{k,k'}$, set

$$\|s\|_{0,x,k,(k')}^2 = \sum_{|\alpha| \leqslant k'} \|Z^\alpha s\|_{0,x,k}^2 . \tag{13.105}$$

Proposition 13.32. — *For $k \in \mathbf{N}$, there exist $C > 0, m_k \in \mathbf{N}$ such that for $0 \leqslant T \leqslant M, x \in X_g, \lambda \in U_c$, if $s \in \mathbf{K}_x$ has compact support,*

$$\left\| \left(\lambda - L_{0,T}^{3,x}\right)^{-1} s \right\|_{0,x,1,(k)} \leqslant C\left(1 + |\lambda|\right)^{m_k} \|s\|_{0,x,0,(k)}. \tag{13.106}$$

Proof. — The proof of our Proposition is the same as the proof of [**BL**, Proposition 11.34]. □

Proposition 13.33. — *There exists $C > 0$ such that for $t \in]0,1], 0 \leqslant T \leqslant M, x \in X_g, \lambda \in U_c$, if $s \in \mathbf{K}_x$ has compact support,*

$$\left| \left(L_{t,T}^{3,x} - L_{0,T}^{3,x}\right) s \right|_{x,t,-1} \leqslant C\sqrt{t}\, \|s\|_{0,x,1,(3)}. \tag{13.107}$$

Proof. — Using Theorem 13.18, the proof of our Theorem is the same as [**BL**, Theorem 11.35]. □

Proposition 13.34. — *There exist $C > 0, k \in \mathbf{N}$ such that for $t \in]0,1], 0 \leqslant T \leqslant M, x \in X_g, \lambda \in U_c$, if $s \in \mathbf{K}_x$ has compact support,*

$$\left| \left(\left(\lambda - L_{t,T}^{3,x}\right)^{-1} - \left(\lambda - L_{0,T}^{3,x}\right)^{-1} \right) s \right|_{t,x,0} \leqslant C\sqrt{t}\, \left(1 + |\lambda|\right)^{k} \|s\|_{0,x,0,(3)}. \tag{13.108}$$

Proof. — Using Proposition 13.33, the proof of our Proposition is the same as the proof of [**BL**, Theorem 11.36]. □

Recall that Γ_c was defined in Definition 13.4. By Theorem 13.26,

$$\widetilde{F}_t\left(L_{t,T}^{3,x}\right) = \frac{1}{2i\pi} \int_{\Gamma_c} \frac{\widetilde{F}_t\left(\lambda\right)}{\left(\lambda - L_{t,T}^{3,x}\right)} d\lambda, \quad \widetilde{F}_t\left(L_{0,T}^{3,x}\right) = \frac{1}{2i\pi} \int_{\Gamma_c} \frac{\widetilde{F}_t\left(\lambda\right)}{\lambda - L_{0,T}^{3,x}} d\lambda. \tag{13.109}$$

Using (13.22), the above results, (13.109) and proceeding as in [**BL**, Section 11 p)] and in [**B9**, Section 11] leads to a proof of Theorem 13.13, which is a first step in the proof of Theorem 13.19.

Now we will go one step further to establish (13.77) in Theorem 13.19.

Proposition 13.35. — *If $k \in \mathbf{N}$, there exists $C > 0$ such that for $t \in]0,1], 0 \leqslant T \leqslant M, x \in X_g$, if $s \in \mathbf{K}_x$ has compact support, then*

$$\frac{1}{\sqrt{t}} \left\| (L_{t,T}^{3,x} - L_{0,T}^{3,x}) s \right\|_{t,x,k} \leqslant C\, \|s\|_{0,x,k+2,(3)}\,. \tag{13.110}$$

Proof. — Our Proposition is an easy consequence of Theorem 13.18. □

Proposition 13.36. — *Given $k, k' \in \mathbf{N}$, there exist $C > 0, m_{k,k'} \in \mathbf{N}$ such that for $0 \leqslant T \leqslant M, x \in X_g, \lambda \in U_c$, if $s \in \mathbf{K}_x$ has compact support,*

$$\left\| \left(\lambda - L_{0,T}^{3,x}\right)^{-1} s \right\|_{0,x,k+1,(k')} \leqslant C\left(1 + |\lambda|\right)^{m_{k,k'}} \|s\|_{0,x,k,(k')}\,. \tag{13.111}$$

Proof. — The proof is the same as the proof of [**BL**, Theorem 11.30 and Proposition 11.34], or of Theorem 13.29 and Proposition 13.32. □

Theorem 13.37. — *Given $k \in \mathbf{N}$, there exist $C > 0, m_k \in \mathbf{N}$ such that if $t \in]0,1], 0 \leqslant T \leqslant M, x \in X_g, \lambda \in U_c$, if $s \in \mathbf{K}_x$ has compact support,*

$$\frac{1}{\sqrt{t}} \left\| \left(\left(\lambda - L_{t,T}^{3,x}\right)^{-1} - \left(\lambda - L_{0,T}^{3,x}\right)^{-1} \right) s \right\|_{t,x,k} \leqslant C \left(1 + |\lambda|\right)^{m_k} \|s\|_{0,x,k,(3)} . \tag{13.112}$$

Proof. — Clearly,

$$\frac{1}{\sqrt{t}} \left(\left(\lambda - L_{t,T}^{3,x}\right)^{-1} - \left(\lambda - L_{0,T}^{3,x}\right)^{-1} \right) = \left(\lambda - L_{t,T}^{3,x}\right)^{-1} \frac{L_{t,T}^{3,x} - L_{0,T}^{3,x}}{\sqrt{t}} \left(\lambda - L_{0,T}^{3,x}\right)^{-1} . \tag{13.113}$$

Using Theorems 13.29, Propositions 13.35 and 13.36, and (13.113), we get

$$\begin{aligned} \frac{1}{\sqrt{t}} \left\| \left(\left(\lambda - L_{t,T}^{3,x}\right)^{-1} - \left(\lambda - L_{0,T}^{3,x}\right)^{-1} \right) s \right\|_{t,x,k} &\leqslant C \left(1 + |\lambda|\right)^{m_k} \\ \left\| \frac{L_{t,T}^{3,x} - L_{0,T}^{3,x}}{\sqrt{t}} \left(\lambda - L_{0,T}^{3,x}\right)^{-1} s \right\|_{s,t,k-1} &\leqslant C \left(1 + |\lambda|\right)^{m_k} \left\| \left(\lambda - L_{0,T}^{3,x}\right)^{-1} s \right\|_{0,x,k+1,(3)} \\ &\leqslant C \left(1 + |\lambda|\right)^{m_k + m_{k,3}} \|s\|_{0,x,k,(3)} , \end{aligned} \tag{13.114}$$

which is just (13.112). □

Theorem 13.38. — *Given $k \in \mathbf{N}, q \in \mathbf{N}$, there exist $C > 0, n_{k,q} \in \mathbf{N}$ such that if $t \in]0,1], 0 \leqslant T \leqslant M, x \in X_g, \lambda \in U_c$, if $s \in \mathbf{K}_x$ has compact support,*

$$\frac{1}{\sqrt{t}} \left\| \left(\left(\lambda - L_{0,T}^{3,x}\right)^{-q} - (\lambda - L_{0,T}^{3,x})^{-q} \right) s \right\|_{t,x,k+q-1} \leqslant C \left(1 + |\lambda|\right)^{n_{k,q}} \|s\|_{0,x,k,(3))} . \tag{13.115}$$

Proof. — We have the formula,

$$\begin{aligned} &\frac{1}{\sqrt{t}} \left(\left(\lambda - L_{0,T}^{3,x}\right)^{-q} - (\lambda - L_{0,T}^{3,x})^{-q} \right) \\ &\qquad = \left(\lambda - L_{t,T}^{3,x}\right)^{-q+1} \frac{\left(\lambda - L_{t,T}^{3,x}\right)^{-1} - \left(\lambda - L_{0,T}^{3,x}\right)^{-1}}{\sqrt{t}} \\ &\quad + \left(\lambda - L_{t,T}^{3,x}\right)^{-q+2} \frac{\left(\lambda - L_{t,T}^{3,x}\right)^{-1} - \left(\lambda - L_{0,T}^{3,x}\right)^{-1}}{\sqrt{t}} \left(\lambda - L_{0,T}^{3,x}\right)^{-1} + \cdots \end{aligned} \tag{13.116}$$

Using Theorem 13.29, Propositions 13.32 and 13.36 and Theorem 13.37, we get (13.115). □

Theorem 13.39. — *There exist $C' > 0$ such that for any $m \in \mathbf{N}$, there is $C > 0$, $r \in \mathbf{N}$ such that for $t \in]0,1]$, $x \in X_g$, $Z, Z' \in T_x X$,*

$$\sup_{|\alpha|,|\alpha'|\leqslant m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \frac{1}{\sqrt{t}} \left(\widetilde{F}_t\left(L^{3,x}_{t,T}\right) - \widetilde{F}_t\left(L^{3,x}_{0,T}\right) \right)(Z,Z') \right| \leqslant C\left(1+|Z|+|Z'|\right)^r \exp\left(-C'|Z-Z'|^2\right). \tag{13.117}$$

Proof. — Using Theorems 13.37, 13.38, and proceeding as in [**BL**, Theorem 11.31], we get (13.117). □

Proposition 13.40. — *There exist $C > 0$ such that if $t \in]0,1]$, $0 \leqslant T \leqslant M$, $x \in X_g$, if $s \in \mathbf{K}_x$ has compact support,*

$$\left| \left(\frac{L^{3,x}_{t,T} - L^{3,x}_{0,T}}{\sqrt{t}} - A^x_t \right) s \right|_{0,x,-1} \leqslant C\sqrt{t}\, \|s\|_{0,x,1,(3)}. \tag{13.118}$$

Proof. — Our Proposition follows from Theorem 13.18. □

Theorem 13.41. — *There exist $C > 0, k \in \mathbf{N}$ such that if $t \in]0,1]$, $0 \leqslant T \leqslant M$, $x \in X_g$, $\lambda \in U_c$,*

$$\left| \frac{\left(\lambda - L^{3,x}_{t,T}\right)^{-1} - \left(\lambda - L^{3,x}_{0,T}\right)^{-1}}{\sqrt{t}} - \left(\lambda - L^{3,x}_{0,T}\right)^{-1} A^x_T \left(\lambda - L^{3,x}_{0,T}\right)^{-1} \right|_{x,t,0} \leqslant C\sqrt{t}\,(1+|\lambda|)^k \|s\|_{0,x,1,(6)}. \tag{13.119}$$

Proof. — Using (13.113), we get,

$$\begin{aligned} &\frac{\left(\lambda - L^{3,x}_{t,T}\right)^{-1} - \left(\lambda - L^{3,x}_{0,T}\right)^{-1}}{\sqrt{t}} - \left(\lambda - L^{3,x}_{0,T}\right)^{-1} A^x_T \left(\lambda - L^{3,x}_{0,T}\right)^{-1} \\ &= \left(\lambda - L^{3,x}_{t,T}\right)^{-1} \left(\frac{L^{3,x}_{t,T} - L^{3,x}_{0,T}}{\sqrt{t}} - A^x_T \right) \left(\lambda - L^{3,x}_{0,T}\right)^{-1} \\ &\quad + \left(\left(\lambda - L^{3,x}_{t,T}\right)^{-1} - \left(\lambda - L^{3,x}_{0,T}\right)^{-1} \right) A^x_T \left(\lambda - L^{3,x}_{0,T}\right)^{-1}. \end{aligned} \tag{13.120}$$

By Theorem 13.26, by Proposition 13.32 and Proposition 13.40, there exists $k \in \mathbf{N}$ such that

$$\left| \left(\lambda - L^{3,x}_{t,T}\right)^{-1} \left(\frac{L^{3,x}_{t,T} - L^{3,x}_{0,T}}{\sqrt{t}} - A^x_T \right) \left(\lambda - L^{3,x}_{0,T}\right)^{-1} s \right|_{t,x,0} \leqslant C\,(1+|\lambda|)^k\, |s|_{x,0,(3)}. \tag{13.121}$$

Also by Proposition 13.34,

$$\left|\frac{\left(\lambda - L_{t,T}^{3,x}\right)^{-1} - \left(\lambda - L_{0,T}^{3,x}\right)^{-1}}{\sqrt{t}} s\right|_{t,x,0} \leqslant C\left(1 + |\lambda|\right)^{k} \|s\|_{0,x,0,(3)}. \tag{13.122}$$

So by Theorem 13.18, by Proposition 13.36 and by (13.122), we obtain,

$$\begin{aligned}\left|\left(\left(\lambda - L_{t,T}^{3,x}\right)^{-1} - \left(\lambda - L_{0,T}^{3,x}\right)^{-1}\right) A_T^x \left(\lambda - L_{0,T}^{3,x}\right)^{-1} s\right|_{x,t,0} \\ \leqslant C\sqrt{t}\left(1 + |\lambda|\right)^{k} \left\|A_T^x \left(\lambda - L_{0,T}^{3,x}\right)^{-1} s\right\|_{0,x,0,(3)} \\ \leqslant C\sqrt{t}\left(1 + |\lambda|\right)^{k} \left\|\left(\lambda - L_{0,T}^{3,x}\right)^{-1} s\right\|_{x,0,2,(6)} \\ \leqslant C\sqrt{t}\left(1 + |\lambda|\right)^{k+k'} \|s\|_{0,x,1,(6)}.\end{aligned} \tag{13.123}$$

From (13.120), (13.121), (13.123), we get (13.119) □

Using now (13.15), (13.109), and Theorems 13.39 and 13.41, we find that as $t \to 0$,

$$\begin{aligned}\frac{1}{\sqrt{t}}\left(\widetilde{F}_t(L_{t,T}^{3,x}) - \exp\left(-L_{0,T}^{3,x}\right)\right)(Z, Z') \to \\ \frac{\partial}{\partial s} \exp\left(-L_{0,T}^{3,x} - sA_T^x\right)(Z, Z')|_{s=0},\end{aligned} \tag{13.124}$$

which is just (13.77) in Theorem 13.19.

Equations (13.117) and (13.124) do not still give us the estimate (13.78) for $t \in]0, 1], 0 \leqslant T \leqslant M$. To obtain this estimate, one needs to take extra terms in the asymptotic expansion of $L_{t,T}^{3,x}$ in equation (13.68), so as to obtain a uniform bound. Details are left to the reader.

13.10. A proof of Theorem 13.1

Now we will explain how to establish the estimate (13.2) in the full range $t \in]0, 1], 0 \leqslant T \leqslant 2/\sqrt{t}$.

First we consider the case where $x \in X_g$ is such that $d^X(x, B) \geqslant \varepsilon_0/4$. Then if $Z \in T_x X$ is such that $|Z| \leqslant 4\eta_0$, since $\eta_0 \leqslant \varepsilon_0/32$, if follows that

$$d^X(Z, B) \geqslant \varepsilon_0/8. \tag{13.125}$$

In particular $|\nabla f(Z)|^2$ has a positive lower bound.

Definition 13.42. — For $t \in]0, 1], T \in \mathbf{R}_+^*$, if $s \in \mathbf{K}_x$ has compact support, put

$$|s|_{t,T,x,1}^2 = |s|_{t,x,1}^2 + T^2 \left|\rho\left(\sqrt{t}Z\right)|\nabla f|\, s\right|_{t,x,0}^2. \tag{13.126}$$

Using (12.2) and the above considerations, we find easily that if $t \in]0,1], 0 \leqslant T \leqslant 2/\sqrt{t}$, the estimates in Theorem 13.25 hold, with$|s|_{t,x,1}$ replaced by $|s|_{t,T,x,1}$.

In the asymptotic expansion (13.68) of the operator $L^{3,x}_{t,T}$ as $t \to 0$, we now need to keep track of the dependence of A^x_T on T, since T is no longer uniformly bounded. Using (13.68) and the same arguments as in Section 13.9, we see that in the right-hand side of (13.117), instead of C, we will have $C\left(1+T^2\right)$.

Now we will take advantage of the fact that the estimates in Theorem 13.25 hold, with $|s|_{t,x,1}$ replaced by $|s|_{t,T,x,1}$. Put

$$\mathcal{Q}_{T,x} = \left\{\nabla_{e_i}, 1 \leqslant i \leqslant n; |T|\,\rho\left(\sqrt{t}Z\right)\right\}. \tag{13.127}$$

We define $\mathcal{Q}^j_{T,x}$ as in Definition 13.27, by replacing $\mathcal{Q}_x$ by $\mathcal{Q}_{T,x}$. Also if $s \in \mathbf{K}^k_x$, we define $\|s\|_{t,T,x,k}$ as in (13.101), by replacing $\mathcal{Q}_x$ by $\mathcal{Q}_{T,x}$.

Then all the results of Section 13.9 remain formally true, when doing the obvious changes.

Theorem 13.43. — *There exist $C' > 0$ such that for any $k \in \mathbf{N}, m \in \mathbf{N}$, there is $C > 0$, $r \in \mathbf{N}$ such that for $t \in]0,1], 0 \leqslant T \leqslant 2/\sqrt{t}, x \in X_g$ such that $d^X(x,B) \geqslant \varepsilon_0/4$, $Z, Z' \in T_xX$,*

$$\sup_{|\alpha|,|\alpha'|\leqslant m} \left|\rho\left(\sqrt{t}Z\right)T\right|^k \left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \widetilde{F}_t\left(L^{3,x}_{t,T}\right)(Z,Z')\right| \leqslant C\left(1+T^2\right)\left(1+|Z|+|Z'|\right)^r \exp\left(-C'|Z-Z'|^2\right). \tag{13.128}$$

Proof. — Taking into account the above considerations, the proof of our Theorem is the same as proof of Theorem 13.30. □

Theorem 13.44. — *There exist $C' > 0$ such that for any $k \in \mathbf{N}, m \in \mathbf{N}$, there is $C > 0$, $r \in \mathbf{N}$ such that fort $\in]0,1]$, $0 \leqslant T \leqslant 2/\sqrt{t}$, $x \in X_g$ such that $d^X(x,B) \geqslant \varepsilon_0/4$, $Z, Z' \in T_xX$,*

$$\sup_{|\alpha|,|\alpha'|\leqslant m} \left|\left|\rho\left(\sqrt{t}Z\right)T\right|^k \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \frac{1}{\sqrt{t}}\left(\widetilde{F}_t\left(L^{3,x}_{t,T}\right) - \widetilde{F}_t\left(L^{3,x}_{0,T}\right)\right)(Z,Z')\right| \leqslant C\left(1+T^2\right)\left(1+|Z|+|Z'|\right)^r \exp\left(-C'|Z-Z'|^2\right). \tag{13.129}$$

Proof. — By proceeding as in the proof of Theorem 13.39, we get (13.129). □

Theorem 13.45. — *There exists $C' > 0$ such that for any $k \in \mathbf{N}$, there is $C > 0$ such that for $t \in]0,1]$, $0 \leqslant T \leqslant 2/\sqrt{t}$, $x \in X_g$ with $d^X(x,B) \geqslant \varepsilon_0/4$, $Z \in N_{X_g/X,x}, |Z| \leqslant \eta_0/\sqrt{t}$,*

$$\left|\frac{1}{\sqrt{t}}\left(\widetilde{F}_t\left(L^{3,x}_{t,T}\right) - \widetilde{F}_t\left(L^{3,x}_{0,T}\right)\right)\left(g^{-1}Z,Z\right)\right| \leqslant \frac{C}{1+T^k}\exp\left(-C'|Z|^2\right). \tag{13.130}$$

Proof. — By (13.35), if $|Z| \leqslant \eta_0/\sqrt{t}$, then $\rho\left(\sqrt{t}\,Z\right) = 1$. Our Theorem is now a consequence of Theorem 13.44. □

Theorem 13.46. — *There exists $C' > 0$ such that for any $k \in \mathbf{N}$, there is $C > 0$ such that for $t \in]0,1]$, $0 \leqslant T \leqslant 2/\sqrt{t}$, $x \in X_g$ with $d^X(x,B) \geqslant \varepsilon_0/4$, $Z \in N_{X_g/X,x}$, $|Z| \leqslant \eta_0/\sqrt{t}$,*

$$\left|\frac{1}{t}\left[\widehat{\mathrm{Tr}_{\mathrm{s}}}\left[g\widetilde{F}_t\left(L^{3,x}_{t,T}\right)\left(g^{-1}Z,Z\right)\right] - \widehat{\mathrm{Tr}_{\mathrm{s}}}\left[g\exp\left(-L^{3,x}_{0,T}\right)\left(g^{-1}Z,Z\right)\right]\right.\right.$$
$$\left.\left. - \sqrt{t}\,\frac{\partial}{\partial s}\widehat{\mathrm{Tr}_{\mathrm{s}}}\left[g\exp\left(-\left(L^{3,x}_{0,T} + sM^x_T\right)\right)\left(g^{-1}Z,Z\right)\right]|_{s=0}\right| \leqslant \frac{C}{1+T^k}\exp\left(-C'|Z|^2\right). \tag{13.131}$$

Proof. — The proof of our Theorem proceeds as the proof of Theorems 13.19 and 13.45. Details are left to the reader. □

Theorem 13.47. — *There exist $C > 0$ such that for $t \in]0,1]$, $0 \leqslant T \leqslant 2/\sqrt{t}$,*

$$\frac{1}{t}\left|\int_{\{x\in X_g, d^X(x,B)\leqslant\varepsilon_0/4\}}\left[\int_{\substack{Z\in N_{X_g/X}\\ |Z|\leqslant\eta_0}}\mathrm{Tr}_{\mathrm{s}}\left[fg\widetilde{F}_t\left(\overline{C}^2_{t,T/\sqrt{t}}\right)\left(g^{-1}Z,Z\right)\right]k(x,Z)\right.\right.$$
$$\left.\left.\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}/2}}\right]\frac{dv_{X_g}(x)}{(2\pi)^{\dim X_g/2}} - \int_{\{x\in X_g, d^X(x,B)\leqslant\varepsilon_0/4\}}\mathrm{Tr}^F[g]\,f\alpha_{T^2/4}\right| \leqslant C. \tag{13.132}$$

Proof. — In degree 0, our Theorem was already established in [**BZ1**, equ. (13.62)] and [**BZ2**, equ. (9.18)]. So here, we concentrate on the proof of (13.132) in positive degree.

Take $x \in X_g, d^X(x,B_g) \leqslant \varepsilon_0/4$. Let x_0 be the unique element of B_g such that $d^X(x_0,x) \leqslant \varepsilon_0/4$. Since $\eta_0 \leqslant \varepsilon_0/32$, then $B^X(x,\eta_0) \subset B^X(x_0,\varepsilon_0)$. In particular, by (13.35), on $B^X(x,\eta_0)$, the operator $\overline{C}^2_{t,T/\sqrt{t}}$ coincides with the operator $\overline{C}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/\sqrt{t}}$ defined as in (10.11). Therefore, using finite propagation speed as in Section 13.2, we find that in (13.132), we can as well replace $\widetilde{F}_t\left(\overline{C}^2_{t,T/\sqrt{t}}\right)$ by $\widetilde{F}_t\left(\overline{C}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/\sqrt{t}}\right)$. By proceeding as in Theorem 13.6 and in (13.28), we find that we can as well replace $\widetilde{F}_t\left(\overline{C}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/\sqrt{t}}\right)$ by $\exp\left(-\overline{C}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/\sqrt{t}}\right)$. Namely, we now have to estimate the integral

$$\int_{\{Z\in T_{x_0}X, |Z|\leqslant\varepsilon_0/4\}}\mathrm{Tr}_{\mathrm{s}}\left[fg\exp\left(-\overline{C}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/\sqrt{t}}\right)\left(g^{-1}Z,Z\right)\right]\frac{dv_{TX}(Z)}{(2\pi)^{\dim X/2}}. \tag{13.133}$$

Now we use the notation in Chapter 4. In particular the kernel $\mathcal{P}_T(Z,Z')$ was defined in Definition 4.5 and computed in Theorem 4.6. If $Z \in T_{x_0}X$, we write Z in the form,

$$Z = Z_0 + Z_1 \text{ with } Z_0 \in T_{x_0}X_g, Z_1 \in N_{X_g/X,x_0}. \tag{13.134}$$

By (12.37), if $|Z| \leqslant \varepsilon_0/4$,

$$f(Z) = f(x_0) + q(Z_0) + q(Z_1). \tag{13.135}$$

Now we redefine the norm $|Z|$, so that if Z is given by (13.134), then

$$|Z| = \sup\left\{|Z_0|, |Z_1|\right\}. \tag{13.136}$$

By (12.35), (12.36),

$$\int_{Z\in T_{x_0}X,|Z|\leqslant\varepsilon_0/4} \mathrm{Tr}_s\left[fg\exp\left(\overline{\mathcal{C}}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/\sqrt{t}}\right)(g^{-1}Z,Z)\right]\frac{dv_{TX}(Z)}{(2\pi)^{\dim X/2}} = \tag{13.137}$$
$$\int_{\substack{|Z_0|\leqslant\varepsilon_0/4\\ |Z_1|\leqslant\varepsilon_0/4\sqrt{t}}} f\left(Z_0+\sqrt{t}\,Z_1\right)\mathrm{Tr}_s\left[g\mathcal{P}_{\sqrt{t}T}\left(g^{-1}\left(Z_0/\sqrt{t}+Z_1\right),Z_0/\sqrt{t}+Z_1\right)\right]$$
$$\frac{dv_{TX}(Z)}{t^{\dim X_g/2}(2\pi)^{\dim X/2}}\mathrm{Tr}^{F_{x_0}}[g].$$

By (4.25), (4.26), we get

$$\mathrm{Tr}_s\left[g\mathcal{P}_{\sqrt{t}T}(g^{-1}Z,Z)\right] = 2^{n/2}\det\left(\frac{Q_{\sqrt{t}T}/2}{\sinh\left(Q_{\sqrt{t}T}/2\right)}\right)^{1/2} \tag{13.138}$$
$$(-1)^{\mathrm{ind}_g(x_0)}\det\left[\sigma\left(tT^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2}$$
$$\exp\left(-\left\langle\frac{Q_{\sqrt{t}T}/2}{\sinh\left(Q_{\sqrt{t}T}/2\right)}\sigma\left(tT^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)Z,Z\right\rangle\right).$$

We claim that in (13.137), we can replace the condition $|Z_1|\leqslant\varepsilon_0/4\sqrt{t}$ by $Z_1\in N_{X_g/X,x_0}$. This is because when acting on N_{X_g,x_0}, when $T\leqslant 2/\sqrt{t}$, $\left(\frac{Q_{\sqrt{t}T}/2}{\sinh\left(Q_{\sqrt{t}T}/2\right)}\right)^{(0)}$ and $\sigma\left(tT^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)^{(0)}$ have a positive lower bound. Also,

$$\int_{\substack{|Z_0|\leqslant\varepsilon_0/4\\ Z_1\in N_{X_g/X,x_0}}} f\left(Z_0+\sqrt{t}\,Z_1\right) \tag{13.139}$$
$$\cdot\,\mathrm{Tr}_s\left[g\mathcal{P}_{\sqrt{t}T}\left(g^{-1}\left(Z_0/\sqrt{t}+Z_1\right),\left(Z_0/\sqrt{t}+Z_1\right)\right)\right]\frac{dv_{TX}(Z)}{t^{\dim X_g/2}(2\pi)^{\dim X/2}}$$
$$= t\int_{T_{x_0}X} f(Z)\,\mathrm{Tr}_s\left[g\mathcal{P}_{\sqrt{t}T}\left(g^{-1}Z,Z\right)\right]\frac{dv_{TX}(Z)}{(2\pi)^{\dim X/2}}$$
$$-\int_{\substack{|Z_0|\geqslant\varepsilon_0/4\\ Z_1\in N_{X_g/X,x_0}}} f\left(Z_0+\sqrt{t}\,Z_1\right)$$
$$\cdot\,\mathrm{Tr}_s\left[g\mathcal{P}_{\sqrt{t}T}\left(g^{-1}\left(Z_0/\sqrt{t}+Z_1\right),\left(Z_0/\sqrt{t}+Z_1\right)\right)\right]\frac{dv_{TX}(Z)}{t^{\dim X_g/2}(2\pi)^{\dim X/2}}$$

By (13.135) and by Proposition 4.11, we have the identity,

$$(13.140)\quad t\int_{T_{x_0}X} f(Z)\,\mathrm{Tr}_s\left[g\mathcal{P}_{\sqrt{t}T}\left(g^{-1}Z,Z\right)\right]\frac{dv_{TX}(Z)}{(2\pi)^{\dim X/2}}$$
$$=(-1)^{\mathrm{ind}_g(x)}+(-1)^{\mathrm{ind}_g(x)}\,\frac{\sqrt{t}}{T}\,\frac{1}{2}\,\frac{\frac{\partial}{\partial T'}\sigma\left(T'^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)}{\sigma\left(T'^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)}|_{T'=\sqrt{t}T}.$$

Now by Proposition 4.15, for $t\in\,]0,1],0\leqslant T\leqslant 2/\sqrt{t}$,

$$(13.141)\qquad \frac{\sqrt{t}}{T}\left(\frac{\frac{\partial}{\partial T'}\sigma\left(T'^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)}{\sigma\left(T'^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)}|_{T'=\sqrt{t}T}\right)^{(>0)}=\mathcal{O}(t).$$

Let $Q_T^{TX_g}|_{\mathbf{B}_g}$ be the tensor Q_T associated to $TX_g|_{\mathbf{B}_g}$. By (13.138),

$$(13.142)\quad \int_{\substack{|Z_0|\geqslant\varepsilon_0/4\\ Z_1\in N_{X_g/X,x_0}}}(f(x_0)+q(Z_0))\,\mathrm{Tr}_s\Big[g\mathcal{P}_{\sqrt{t}T}\Big(g^{-1}\left(Z_0/\sqrt{t}+Z_1\right),$$
$$\left(Z_0/\sqrt{t}+Z_1\right)\Big)\Big]\frac{dv_{TX}(Z)}{t^{\dim X_g/2}(2\pi)^{\dim X/2}}$$
$$=(-1)^{\mathrm{ind}_g(x_0)}T^{\dim X_g}\int_{\substack{Z_0\in T_{x_0}X_g\\ |Z_0|\geqslant\varepsilon_0/4}}(f(x_0)+q(Z_0))$$
$$\det\left[\frac{Q_{\sqrt{t}T}^{TX_g|_{\mathbf{B}_g}}/2}{\sinh\left(Q_{\sqrt{t}T}^{TX_g|_{\mathbf{B}_g}}/2\right)}\,\frac{\sigma\left(tT^2/4,0,R^{TX_g|_{\mathbf{B}_g}}\right)}{tT^2}\right]^{1/2}$$
$$\exp\left(-T^2\left\langle\frac{Q_{\sqrt{t}T}^{TX_g|_{\mathbf{B}_g}}/2}{\sinh\left(Q_{\sqrt{t}T}^{TX_g|_{\mathbf{B}_g}}/2\right)}\,\frac{\sigma\left(tT^2/4,0,R^{TX_g|_{\mathbf{B}_g}}\right)}{tT^2}Z_0,Z_0\right\rangle\right)\frac{dv_{TX_g}(Z_0)}{\pi^{\dim X_g/2}}.$$

By (4.20), if $t\in\,]0,1],0\leqslant T\leqslant 2/\sqrt{t}$,

$$(13.143)\qquad \frac{Q_{\sqrt{t}T}^{TX_g|_{\mathbf{B}_g}}/2}{\sinh\left(Q_{\sqrt{t}T}^{TX_g|_{\mathbf{B}_g}}/2\right)}\,\frac{\sigma\left(tT^2/4,0,R^{TX_g|_{\mathbf{B}_g}}\right)}{tT^2}=1+\mathcal{O}\left(tT^2\right),$$

so that

$$\det\left[\frac{Q_{\sqrt{t}T}^{TX_g|_{\mathbf{B}_g}}/2}{\sinh\left(Q_{\sqrt{t}T}^{TX_g|_{\mathbf{B}_g}}/2\right)}\,\frac{\sigma\left(tT^2/4,0,R^{TX_g|_{\mathbf{B}_g}}\right)}{tT^2}\right]$$

remains uniformly bounded, and also the operator

$$\left[\frac{Q_{\sqrt{t}T}^{TX_g|_{\mathbf{B}_g}}/2}{\sinh\left(Q_{\sqrt{t}T}^{TX_g|_{\mathbf{B}_g}}/2\right)}\,\frac{\sigma\left(tT^2/4,0,R^{TX_g|_{B_g}}\right)}{tT^2}\right]^{(0)}$$

has a positive lower bound. From (13.142), (13.143), we find that there exist $C > 0$, $C' > 0$ such that if $t \in]0,1]$, $0 \leqslant T \leqslant 2/\sqrt{t}$,

$$\begin{aligned}(13.144)\quad &\left|\left\{\int_{\substack{|Z_0|\geqslant\varepsilon_0/4\\ Z_1\in N_{X_g/X,x_0}}} (f(x_0)+q(Z_0))\,\mathrm{Tr_s}\left[g\mathcal{P}_{\sqrt{t}T}\left(g^{-1}\left(Z_0/\sqrt{t}+Z_1\right),\right.\right.\right.\\ &\left.\left.\left.\left(Z_0/\sqrt{t}+Z_1\right)\right)\right]\frac{dv_{TX}(Z)}{t^{\dim X_g/2}(2\pi)^{\dim X/2}}\right\}^{>0}\Bigg|\\ &\leqslant C\mathcal{O}\left(tT^2\right)T^{\dim X_g}\exp\left(-C'T^2\right).\end{aligned}$$

By (13.144), there exists $C > 0$ such that for $t \in]0,1]$, $0 \leqslant T \leqslant 2/\sqrt{t}$,

$$\begin{aligned}(13.145)\quad &\frac{1}{t}\left|\left\{\int_{\substack{|Z_0|\geqslant\varepsilon_0/4\\ Z_1\in N_{X_g/X,x_0}}} (f(x_0)+q(Z_0))\,\mathrm{Tr_s}\left[g\mathcal{P}_{\sqrt{t}T}\left(g^{-1}\left(Z_0/\sqrt{t}+Z_1\right),\right.\right.\right.\right.\\ &\left.\left.\left.\left.\left(Z_0/\sqrt{t}+Z_1\right)\right)\right]\frac{dv_{TX}(Z)}{t^{\dim X_g/2}(2\pi)^{\dim X/2}}\right\}^{(>0)}\right| \leqslant C.\end{aligned}$$

By (4.32) and (13.138),

$$\begin{aligned}(13.146)\quad &\int_{\substack{|Z_0|\geqslant\varepsilon_0/4\\ Z_1\in N_{X_g/X,x_0}}} tq(Z_1)\,\mathrm{Tr_s}\left[g\mathcal{P}_{\sqrt{t}T}\left(g^{-1}\left(Z_0/\sqrt{t}+Z_1\right),\left(Z_0/\sqrt{t}+Z_1\right)\right)\right]\\ &\qquad\qquad\frac{dv_{TX}(Z)}{t^{\dim X_g/2}(2\pi)^{\dim X/2}}\\ &= T^{\dim X_g}\int_{\substack{Z_0\in T_{x_0}X_g\\ |Z_0|\geqslant\varepsilon_0/4}} \det\left[\frac{Q^{TX_g|_{\mathbf{B}_g}}_{\sqrt{t}T}/2}{\sinh\left(Q^{TX_g|_{\mathbf{B}_g}}_{\sqrt{t}T}/2\right)}\frac{\sigma\left(tT^2/4,0,R^{TX_g|_{\mathbf{B}_g}}\right)}{tT^2}\right]^{1/2}\\ &\exp\left(-T^2\left\langle\frac{Q^{TX_g|_{\mathbf{B}_g}}_{\sqrt{t}T}/2}{\sinh\left(Q^{TX_g|_{\mathbf{B}_g}}_{\sqrt{t}T}/2\right)}\frac{\sigma\left(tT^2/4,0,R^{TX_g|_{\mathbf{B}_g}}\right)}{tT^2}Z_0,Z_0\right\rangle\right)\frac{dv_{TX_g}(Z_0)}{\pi^{\dim X_g}}\\ &\qquad(-1)^{\mathrm{ind}_g(x_0)}\frac{\sqrt{t}}{T}\mathrm{Tr_s}\left[\frac{1}{2}\frac{\frac{\partial}{\partial T'}\sigma\left(T'^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)}{\sigma\left(T'^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)}|_{T'=\sqrt{t}T}\right].\end{aligned}$$

By Proposition 4.15, for $t \in]0,1]$, $0 \leqslant T \leqslant 2/\sqrt{t}$,

$$\frac{1}{\sqrt{t}T}\mathrm{Tr_s}\left[\frac{1}{2}\frac{\frac{\partial}{\partial T'}\sigma\left(T'^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)}{\sigma\left(T'^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)}|_{T'=\sqrt{t}T}\right]$$

remains uniformly bounded. Using the above arguments, we see that there exists $C > 0$ such that for $t \in]0,1], 0 \leqslant T \leqslant 2/\sqrt{t}$,

$$\frac{1}{t}\left|\int_{\substack{|Z_0|\geqslant\varepsilon_0/4\\ Z_1\in N_{X_g/X,x_0}}} tq(Z_1)\,\mathrm{Tr}_s\left[g\mathcal{P}_{\sqrt{t}T}\left(g^{-1}\left(Z_0/\sqrt{t}+Z_1\right),\left(Z_0/\sqrt{t}+Z_1\right)\right)\right]\frac{dv_{TX}(Z)}{t^{\dim X_g/2}(2\pi)^{\dim X/2}}\right| \leqslant C. \tag{13.147}$$

By (13.139)-(13.141), (13.145), (13.146), (13.147), we get (13.132). The proof of our Theorem is completed. □

Remark 13.48. — As we saw in the proof of Proposition 9.4,

$$\int_{\substack{x\in X_g\\ d^X(x,B)\leqslant\varepsilon_0/4}} \overline{h}_g^*\left(\nabla^F, g^F\right) T\beta_{T^2/4} = 0. \tag{13.148}$$

Using (13.25), Theorem 13.6, (13.29), (13.32), Proposition 13.10, Theorems 13.15, 13.20, 13.21, 13.46 and 13.47, we get (13.2). The proof of Theorem 13.1 is completed.

CHAPTER 14

THE ASYMPTOTICS OF $\mathrm{Tr}_s\left[fgh'\left(D_{t,T/t}\right)\right]$ AS $t \to 0$

The purpose of this Chapter is to establish Theorem 9.11, i.e. to obtain the asymptotics of $\mathrm{Tr}_s\left[fgh'\left(D_{t,T/t}\right)\right]$ as $t \to 0$. As in Chapter 13, finite propagation speed of solutions of hyperbolic equations plays an important role in the proofs.

This Chapter is organized as follows. In Section 14.1, we state the main result of this Chapter, from which Theorem 9.11 follows immediately. The remainder of the Chapter is devoted to its proof. In Section 14.2, we show that the required estimates can be made local near X_g. In Section 14.3, we show that these estimates can be localized near B_g. Finally, in Section 14.4, we complete the proof of our main result.

In this Chapter, we make the same assumptions as in Chapters 9-10 and 12-13, and we use the corresponding notation. In particular $h(x) = xe^{x^2}$, and S is assumed to be compact.

14.1. A convergence result and a proof of Theorem 9.11

The main result of this Chapter is as follows.

Theorem 14.1. — *For any $T > 0$, the following identity holds,*

(14.1)
$$\lim_{t\to 0} \frac{1}{t}\left(\mathrm{Tr}_s\left[fg\exp\left(-\overline{C}^2_{t,T/t}\right)\right] - \mathrm{Tr}_s{}^{B_g}\left[f\right]\right) = \sum_{x\in B_g} \mathrm{Tr}_s\left[qg\exp\left(-C^{x,2}_T\right)\right]\mathrm{Tr}^{F_x}\left[g\right].$$

Proof. — The remainder of the Chapter is devoted to the proof of our Theorem. □

Remark 14.2. — By (12.4),

$$\mathrm{Tr}_s\left[fgh'\left(D_{t,T}\right)\right] = \left(1 + 2\frac{\partial}{\partial a}\right)\mathrm{Tr}_s\left[fg\exp\left(-a\overline{C}^2_{t,T/t}\right)\right]|_{a=1}. \tag{14.2}$$

Moreover, by(12.5),

$$\mathrm{Tr}_s\left[fg\exp\left(-a\overline{C}^2_{t,T/t}\right)\right] = \psi_a \mathrm{Tr}_s\left[fg\exp\left(-\overline{C}^2_{at,T/t}\right)\right]. \tag{14.3}$$

By (14.1), we find that given $a > 0$,

(14.4)
$$\lim_{t\to 0} \frac{1}{t}\left(\mathrm{Tr_s}\left[fg\exp\left(-\overline{C}^2_{at,T/t}\right)\right] - \mathrm{Tr_s}^{B_g}\left[f\right]\right) = \sum_{x\in B_g} a\mathrm{Tr_s}\left[qg\exp\left(-\mathcal{C}^{x,2}_{aT}\right)\right]\mathrm{Tr}^{F_x}\left[g\right].$$

Finally, by Proposition 4.18,

$$\mathrm{Tr_s}\left[qgh'\left(\mathcal{D}_T\right)\right] = \left(1 + 2\frac{\partial}{\partial a}\right)\psi_a a\mathrm{Tr_s}\left[qg\exp\left(-\mathcal{C}^2_{aT}\right)\right]|_{a=1}. \tag{14.5}$$

From (14.2)-(14.5), we get (9.25), i.e. we have established Theorem 9.11.

14.2. Localization of the problem near X_g

We use the same notation as in Chapter 13. In particular the functions $\widetilde{F}_t, \widetilde{G}_t, \widetilde{I}_t$ were defined in Section 13.2. By (13.25),

$$\mathrm{Tr_s}\left[fg\exp\left(-\overline{C}^2_{t,T/t}\right)\right] = \mathrm{Tr_s}\left[fg\widetilde{F}_t\left(\overline{C}^2_{t,T/t}\right)\right] + \mathrm{Tr_s}\left[fg\widetilde{I}_t\left(t\overline{C}^2_{t,T/t}\right)\right]. \tag{14.6}$$

By Proposition 13.5,

$$\mathrm{Tr_s}\left[fg\widetilde{I}_t\left(t\overline{C}^2_{t,T/t}\right)\right] = \psi_t^{-1}\mathrm{Tr_s}\left[fg\widetilde{I}_t\left(t^2\overline{C}^2_{1,T/t}\right)\right]. \tag{14.7}$$

By Theorem 13.6, given $T \geqslant 0$, there exist $C > 0, C' > 0$ such that for $t \in]0,1]$,

$$\left\|\widetilde{I}_t\left(t^2\overline{C}_{1,T/t}\right)\right\|_1 \leqslant C\exp\left(-C'/t\right). \tag{14.8}$$

By (14.7), (14.8), we find that given $T \geqslant 0$, there exist $C > 0, C' > 0$ such that when $t \in]0,1]$,

$$\left|\mathrm{Tr_s}\left[fg\widetilde{I}_t\left(t\overline{C}^2_{t,T/t}\right)\right]\right| \leqslant C\exp\left(-C'/t\right). \tag{14.9}$$

From (14.6), (14.9), we see that to establish Theorem 14.1, we may as well replace in (14.1) $\mathrm{Tr_s}\left[fg\exp\left(-\overline{C}^2_{t,T/t}\right)\right]$ by $\mathrm{Tr_s}\left[fg\widetilde{F}_t\left(\overline{C}^2_{t,T/t}\right)\right]$.

As in (13.29),

$$\mathrm{Tr_s}\left[fg\widetilde{F}_t\left(\overline{C}^2_{t,T/t}\right)\right] = \int_X \mathrm{Tr_s}\left[f(x)g\widetilde{F}_t\left(\overline{C}^2_{t,T/t}\right)\left(g^{-1}x,x\right)\right]\frac{dv_X(x)}{(2\pi)^{\dim X/2}}. \tag{14.10}$$

By the argument we gave after (13.29), we know that the support of $\widetilde{F}_t\left(\overline{C}^2_{t,T/t}\right)(x,.)$ is included in $B^X(x,\alpha)$, that $\widetilde{F}_t\left(\overline{C}^2_{t,T/t}\right)(x,.)$ depends only on the restriction of $\overline{C}^2_{t,T/t}$ to $B^X(x,\alpha)$, and moreover that the support of $\widetilde{F}_t\left(\overline{C}^2_{t,T/t}\right)\left(g^{-1}x,x\right)$ is contained in $\mathcal{V}_{\eta_0}$.

It follows from the above that our proof of Theorem 14.1 is now local on X, and this only near X_g.

14.3. An estimate away from B_g

Proposition 14.3. — *For any $T > 0, k \in \mathbf{N}$, there exists $C > 0$ such that for $t \in]0,1], x \in \mathcal{V}_{\varepsilon_0}, d^X(x,B) \geqslant \varepsilon_0/4$, then*

$$\left|\widetilde{F}_t\left(\overline{C}^2_{t,T/t}\right)(g^{-1}x,x)\right| \leqslant Ct^k. \tag{14.11}$$

Proof. — By (3.54),

$$\widetilde{F}_t\left(\overline{C}^2_{t,T/t}\right) = \psi_t^{-1}\widetilde{F}_t\left(t\overline{C}^2_{1,T/t}\right)\psi_t. \tag{14.12}$$

Recall that as in (12.22), there exists $\beta > 0$ such that if $y \in X, d^X(y,B) \geqslant \varepsilon_0/8$, then

$$|\nabla f|^2(y) \geqslant \beta. \tag{14.13}$$

Also, as we saw above, the support of $\widetilde{F}_t\left(t\overline{C}^2_{1,T/t}\right)(x,.)$ is included in $B^X(x,\alpha)$. Since $\alpha \leqslant \varepsilon_0/32$, if $d^X(x,B) \geqslant \varepsilon_0/4, y \in B^X(x,\alpha)$, then $d^X(y,B) \geqslant \varepsilon_0/8$, so that (14.13) holds.

Let $\varphi : M \to [0,1]$ be a smooth function such that

$$\begin{aligned}\varphi(y) &= 1 \text{ if } d^X(y,B) \geqslant \varepsilon_0/8,\\ &= 0 \text{ if } d^X(y,B) \leqslant \varepsilon_0/16.\end{aligned} \tag{14.14}$$

Put

$$L_{t,T} = \varphi^2 t\overline{C}^2_{1,T/t} + \left(1-\varphi^2\right)\left(-\frac{t}{4}\Delta^{TX} + \frac{T^2}{t}\right). \tag{14.15}$$

Using finite propagation speed as in Section 13.2, we find that if $x \in X, d^X(x,B) \geqslant \varepsilon_0/4$, then

$$\widetilde{F}_t\left(t\overline{C}^2_{1,T}\right)(x,.) = \widetilde{F}_t\left(L_{t,T}\right)(x,.). \tag{14.16}$$

Let $|\ |_0$ denote the standard L_2 norm on $\Omega^\bullet\left(X, F|_X\right)$, and let $|\ |_1$ be a norm on the corresponding Sobolev space of order 1. If $t \in]0,1], T > 0, s \in \Omega^\bullet\left(X, F|_X\right)$, put

$$|s|^2_{t,T,1} = t\,|s|^2_1 + \frac{T^2}{t}\,|s|^2_0. \tag{14.17}$$

Using Theorem 12.1, an analogue of Theorem 13.25 holds for the operator $L_{t,T}$. Namely there exist $C_1, \ldots, C_4 > 0$, such that if $s \in \Omega^\bullet\left(X, F|_X\right)$, then

$$\begin{aligned}&\mathrm{Re}\langle L_{t,T}s, s\rangle_0 \geqslant C_1\,|s|^2_{t,T,1} - C_2\,|s|^2_0,\\ &|\mathrm{Im}\langle L_{t,T}s, s\rangle_0| \leqslant C_3\,|s|_{t,T,1}\,|s|_0,\\ &|\langle L^{3,x}_{t,T}s, s'\rangle_0| \leqslant C_4\,|s|_{t,T,1}\,|s'|_{t,T,1}.\end{aligned} \tag{14.18}$$

Let $U_1, \ldots, U_{n'} \in TX$ be a finite family of smooth sections of TX which span TX at every x. Also we define the family of operators

$$\mathcal{Q} = \left\{\nabla_{U_i}, 1 \leqslant i \leqslant n', \frac{T^2}{t}\right\}. \tag{14.19}$$

It is then elementary to establish estimates similar to the estimates in (13.100). Then by proceeding as in [**BL**, Section 11] and in Sections 13.9 and 13.10, we find that for any $T > 0, k \in \mathbf{N}$, there exists $C > 0$ such that for $t \in]0,1], x \in \mathcal{V}_{\eta_0}, d^X(x,B) \geqslant \varepsilon_0/4$, then

$$\left|\widetilde{F}_t\left(L_{t,T}\right)\left(g^{-1}x,x\right)\right| \leqslant Ct^k. \tag{14.20}$$

By (14.12), (14.16), (14.20), we get (14.11). The proof of our Proposition is completed. □

14.4. A proof of Theorem 14.1

By an argument given in the proof of Theorem 13.47, if $x_0 \in B_g, d^X(x,x_0) \leqslant \varepsilon_0/4$,

$$\widetilde{F}_t\left(\overline{C}^2_{t,T/t}\right)\left(g^{-1}x,x\right) = \widetilde{F}_t\left(\overline{\mathcal{C}}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/t}\right)\left(g^{-1}x,x\right). \tag{14.21}$$

By (14.6), (14.9), by Proposition 14.3 and by (14.21), for any $k \in \mathbf{N}$, there exists $C > 0$ such that for $t \in]0,1]$,

(14.22)
$$\left|\mathrm{Tr_s}\left[fg\exp\left(-\overline{C}^2_{t,T/t}\right)\right] - \int_{\substack{x\in X\\ d^X(x,B_g)\leqslant\varepsilon_0/4}} f(x)\,\mathrm{Tr_s}\left[g\widetilde{F}_t\left(\overline{\mathcal{C}}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/t}\right)\left(g^{-1}x,x\right)\right]\frac{dv_X(x)}{(2\pi)^{\dim X/2}}\right| \leqslant Ct^k.$$

Now, using (13.20), (13.23), in (14.22), we may as well replace $\widetilde{F}_t\left(\overline{\mathcal{C}}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/t}\right)\left(g^{-1}x,x\right)$ by $\exp\left(-\overline{\mathcal{C}}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/t}\right)\left(g^{-1}x,x\right)$. By (12.35), (12.36), if $x_0 \in B_g$,

$$\int_{\substack{x\in X\\ d^X(x,x_0)\leqslant\varepsilon_0/4}} f(x)\,\mathrm{Tr_s}\left[g\exp\left(-\overline{\mathcal{C}}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/t}\right)\left(g^{-1}x,x\right)\right]\frac{dv_X(x)}{(2\pi)^{\dim X/2}} = \int_{\substack{Z\in T_{x_0}X\\ |Z|\leqslant\varepsilon_0/4}} f(Z)\,\mathrm{Tr_s}\left[g\mathcal{P}_T\left(g^{-1}Z/\sqrt{t},Z/\sqrt{t}\right)\right]\frac{dv_{TX}(Z)}{(2\pi t)^{\dim X/2}}\mathrm{Tr}^{F_{x_0}}[g]. \tag{14.23}$$

By (12.37), (12.38), we get

$$\int_{\substack{Z\in T_{x_0}X\\ |Z|\leqslant\varepsilon_0/4}} f(Z)\,\mathrm{Tr_s}\left[g\mathcal{P}_T\left(g^{-1}Z/\sqrt{t},Z/\sqrt{t}\right)\right]\frac{dv_{TX}(Z)}{(2\pi t)^{\dim X/2}} = \tag{14.24}$$

$$(-1)^{\mathrm{ind}_g(x_0)}\int_{\substack{Z\in T_{x_0}X\\ |Z|\leqslant\varepsilon_0/4\sqrt{t}}} \left(f(x_0)+tq(Z)\right)\det\left[\frac{Q_T/2}{\sinh\left(Q_T/2\right)}\sigma\left(T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2}$$
$$\exp\left(-\left\langle\frac{Q_T/2}{\sinh\left(Q_T/2\right)}\sigma\left(T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)Z,Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}}.$$

For a given $T > 0$ the operator

$$\left[\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)\right]^{(0)}$$

has a positive lower bound. So by (14.24), there exists $c > 0$ such that as $t \to 0$,

$$\int_{\substack{Z\in T_{x_0}X \\ |Z|\leqslant \varepsilon_0/4\sqrt{t}}} \det\left[\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2} \tag{14.25}$$
$$\exp\left(-\left\langle\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)Z, Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}} = 1 + \mathcal{O}\left(e^{-c/t}\right).$$

By Propositions 4.8, 4.9 and 4.11,

$$\int_{Z\in T_{x_0}X} q(Z)\det\left[\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2} \tag{14.26}$$
$$\exp\left(-\left\langle\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)Z, Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}} =$$
$$\frac{1}{T}\mathrm{Tr}_{\mathrm{s}}\left[\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)}{\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)}\right].$$

By the same argument as before and by (14.26), we see that as $t \to 0$,

$$\int_{\substack{Z\in T_{x_0}X \\ |Z|\leqslant \varepsilon_0/4\sqrt{t}}} q(Z)\det\left[\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2} \tag{14.27}$$
$$\exp\left(-\left\langle\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)Z, Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}} \to$$
$$\frac{1}{T}\mathrm{Tr}_{\mathrm{s}}\left[\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)}{\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)}\right].$$

By Proposition 4.11, by (14.6), (14.9) and by (14.22)-(14.27), we get (14.1). The proof of Theorem 14.1 is completed.

CHAPTER 15

THE ASYMPTOTICS OF $\mathrm{Tr_s}\left[fgh'\left(D_{t,T/t}\right)\right]$ AS $T \to +\infty$

The purpose of this Chapter is to establish Theorem 9.12, i.e. to study the asymptotics of $\mathrm{Tr_s}\left[fgh'\left(D_{t,T/t}\right)\right]$ as $T \to +\infty$.

This Chapter is organized as follows. In Section 15.1, we state an asymptotic formula which implies Theorem 9.12. The remainder of the Chapter is devoted to the proof of this formula. In Section 15.2, we show that our estimates can be localized near B_g. In Section 15.3, we use the explicit geometric model near B_g which was already considered in Section 4 to complete the proof.

In this Chapter, we make the same assumptions as in Sections 9-10 and 12-14 and we use the corresponding notation. In particular we still assume that S is compact.

15.1. A convergence result and a proof of Theorem 9.12

The main result of this Section is as follows.

Theorem 15.1. — *There exist* $t_0 \in]0,1], C > 0$ *such that for* $t \in]0,t_0], T \geqslant 1$,

$$\left|\frac{1}{t}\left(\mathrm{Tr_s}\left[fg\exp\left(-\overline{C}^2_{t,T/t}\right)\right] - \mathrm{Tr_s}^{B_g}[f] - \frac{t}{4T}\left(\widetilde{\chi}'^{+}_g(F) - \widetilde{\chi}'^{-}_g(F)\right)\right)\right| \leqslant \frac{C}{T^3}. \tag{15.1}$$

Proof. — The remainder of the Chapter is devoted to the proof of our Theorem. □

Remark 15.2. — By proceeding as in Remark 14.2, (9.26) follows from (15.2), i.e. we have proved Theorem 9.12.

15.2. An estimate on the kernel of $\exp\left(-\overline{C}^2_{t,T/t}\right)$

Theorem 15.3. — *There exist* $t_0 \in]0,1], C > 0, C' > 0$ *such that for any* $t \in]0,t_0], T \geqslant 1, x \in X, d^X(x,B_g) \geqslant \varepsilon_0/4$,

$$\left|\exp\left(-\overline{C}^2_{t,T/t}\right)(g^{-1}x,x)\right| \leqslant C\exp\left(-C'T/t\right). \tag{15.2}$$

Proof. — We proceed as in [**BZ1**, Proposition 15.1], [**BZ2**, Proposition 11.1] and in our proof of Theorem 12.4, from which our notation is taken. We still use (12.9),

(12.10), (12.13), with T replaced by T/t. Classically, there exists $C > 0$ such that for $t \in]0,1], x \in X$,

$$p_t\left(g^{-1}x, x\right) \leqslant \frac{C}{t^{n/2}}. \tag{15.3}$$

Instead of (12.19), we now say that there exists $\gamma > 0$ such that for $t \in]0,1], T \geqslant 0$,

$$\begin{aligned} \left|V_s^0\right| &\leqslant \exp\left(\gamma\left(1+T\right)\right), \\ \left|\left(V_s^0\right)^{-1}\right| &\leqslant \exp\left(\gamma\left(1+T\right)\right). \end{aligned} \tag{15.4}$$

As in the proof of Theorem 12.4, we first consider the case where $d^X\left(x, B\right) \geqslant \varepsilon_0/4$. Take $a \in]0, 1/2]$. By proceeding as in the proof of Theorem 12.4, instead of (12.27), we now have,

$$\left|H^x_{t,T/t}\right| \leqslant C \exp\left(\gamma\left(1+T\right)\right)\left(\exp\left(-T^2 a\beta/4t\right) + \exp\left(-\varepsilon_0^2/144at\right)\right). \tag{15.5}$$

Take,

$$a = \frac{\varepsilon_0}{6T\sqrt{\beta}}. \tag{15.6}$$

For $T \geqslant 1$ large enough, $a \in]0, 1/2]$. Also,

$$T^2 a\beta/4t = \varepsilon_0^2/144at = \frac{\varepsilon_0 T\sqrt{\beta}}{24t}. \tag{15.7}$$

Put

$$t_0 = \frac{\varepsilon_0\sqrt{\beta}}{48\gamma}. \tag{15.8}$$

Then, if $t \in]0, t_0]$,

$$\frac{\varepsilon_0\sqrt{\beta}}{24t} - \gamma \geqslant \frac{\varepsilon_0 T\sqrt{\beta}}{48t}. \tag{15.9}$$

By (15.3), (15.5), (15.7), (15.9), we get (15.2) in the case where $d^X\left(x, B\right) \geqslant \varepsilon_0/4$.

As in the proof of Theorem 12.4, we also have to consider the case where there exists $x_0 \in B \setminus B_g$ such that $d^X\left(x, x_0\right) \leqslant \varepsilon_0$. The arguments of the proof of Theorem 12.4 and the above considerations then lead to a proof of (15.2). The proof of our Theorem is completed. □

Theorem 15.4. — *There exist $t_0 \in]0,1], C > 0, C' > 0$ such that if $t \in]0, t_0], T \geqslant 1$, if $x_0 \in B_g$, if $x, x' \in X$ are such that $d^X\left(x_0, x\right) \leqslant \varepsilon_0/4, d^X\left(x_0, x'\right) \leqslant \varepsilon_0/4$, then*

$$\left|\left(\exp\left(-\overline{C}^2_{t,T/t}\right) - \exp\left(-\overline{\mathcal{C}}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/t}\right)\right)\left(x, x'\right)\right| \leqslant C\exp\left(-C'T/t\right). \tag{15.10}$$

Proof. — By proceeding as in the proof of [**BZ1**, Theorem 15.2] and in our proof of Theorem 15.3, we obtain our Theorem. □

15.3. A proof of Theorem 15.1

By (12.7), by Theorems 15.3 and 15.4, there exist $t_0 \in]0,1], C > 0, C' > 0$ such that if $t \in]0,t_0], T \geqslant 1$,

$$\text{(15.11)}\quad \left| \mathrm{Tr}_s\left[f g \exp\left(-\overline{C}^2_{t,T/t}\right)\right] - \int_{\substack{x\in X \\ d^X(x,B_g)\leqslant r/4}} f(x)\,\mathrm{Tr}_s\left[g\exp\left(-\overline{C}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/t}\right)(g^{-1}x,x)\right]\frac{dv_X(x)}{(2\pi)^{\dim X/2}}\right| \leqslant C\exp\left(-C'T/t\right).$$

By (14.23) and (14.24), if $x_0 \in B_g$,

$$\text{(15.12)}\quad \int_{\substack{x\in X \\ d^X(x,x_0)\leqslant r/4}} f(x)\,\mathrm{Tr}_s\left[g\exp\left(\overline{C}^{I\widehat{\otimes}F|_{\mathbf{B}},2}_{t,T/t}\right)(g^{-1}x,x)\right]\frac{dv_X(x)}{(2\pi)^{\dim X/2}} = (-1)^{\mathrm{ind}_g(x_0)} \int_{\substack{Z\in T_{x_0}X \\ |Z|\leqslant r\sqrt{T}/4\sqrt{t}}} \left(f(x_0)+\frac{t}{T}q(Z)\right)\det\left[\frac{1}{T}\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2} \exp\left(-\left\langle\frac{1}{T}\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)Z,Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}}\mathrm{Tr}^{F_{x_0}}[g].$$

By using (12.41), we see that for $T \geqslant 1$, the operators

$$\left[\frac{1}{T}\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{(0)}$$

have a positive lower bound. Therefore there exists $C' > 0$ such that for $t \in]0,1]$, $T \geqslant 1$,

$$\text{(15.13)}\quad \int_{\substack{Z\in T_{x_0}X \\ |Z|\leqslant r\sqrt{T}/4\sqrt{t}}} \det\left[\frac{1}{T}\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2} \exp\left(-\left\langle\frac{1}{T}\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4,B,R^{TX|_{\mathbf{B}_g}}\right)Z,Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}} = 1+\mathcal{O}\left(\exp\left(-C'T/t\right)\right).$$

The same argument shows that for $t \in]0,1], T \geqslant 1$,

$$\begin{aligned}(15.14)\quad &\int_{\substack{Z \in T_{x_0}X \\ |Z| \leqslant r\sqrt{T}/4\sqrt{t}}} q(Z) \det\left[\frac{1}{T}\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2} \\ &\exp\left(-\left\langle \frac{1}{T}\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right) Z, Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}} = \\ &\int_{T_{x_0}X} q(Z) \det\left[\frac{1}{T}\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2} \\ &\exp\left(-\left\langle \frac{1}{T}\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right) Z, Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}} \\ &+ \mathcal{O}\left(\exp\left(-C'T/t\right)\right).\end{aligned}$$

By Propositions 4.8, 4.9 and 4.11,

$$\begin{aligned}(15.15)\quad &\int_{T_{x_0}X} q(Z) \det\left[\frac{1}{T}\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)\right]^{1/2} \\ &\exp\left(-\left\langle \frac{1}{T}\frac{Q_T/2}{\sinh(Q_T/2)}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right) Z, Z\right\rangle\right)\frac{dv_{TX}(Z)}{\pi^{\dim X/2}} = \\ &\frac{1}{T}\mathrm{Tr_s}\left[\frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)}{\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)}\right].\end{aligned}$$

By Proposition 4.15, as $T \to +\infty$,

$$(15.16)\qquad \frac{1}{2}\frac{\frac{\partial}{\partial T}\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)}{\sigma\left(T^2/4, B, R^{TX|_{\mathbf{B}_g}}\right)} = \frac{1}{4}\left(n_+(x_0) - -n_-(x_0)\right) + \mathcal{O}\left(\frac{1}{T^2}\right).$$

By (9.6), (15.11)-(15.16), we get (15.1). The proof of Theorem 15.1 is completed.

CHAPTER 16

THE ANALYTIC TORSION FORMS OF UNIT SPHERE BUNDLES

The purpose of this Chapter is to evaluate the analytic torsion forms of the sphere bundle S^E of a vector bundle E in terms of the additive genus ${}^0I(\theta, x)$. The evaluation of these torsion forms has already been done by Bunke [**Bu1**] in the case where $g = 1$.

Our proof uses an extension of the main formula of this paper, to the case where the function f is now fibrewise Morse-Bott. In fact, we consider instead the suspended sphere bundle $S^{E\oplus\mathbf{R}}$, which is equipped with a fibrewise Morse function m to which Theorem 7.2 can be applied. The function $f = m^2$ is fibrewise Morse-Bott, and $S^E \subset S^{E\oplus\mathbf{R}}$ is one of its fibrewise critical manifolds. We show that an extension of Theorem 7.2 still holds in this very special case. By combining these two computations, we are then able to evaluate the torsion forms of S^E. Let us point out that a proof of this extension of Theorem 7.2 has been established in [**BGo5**, Theorem 5.11] in the context of infinitesimal equivariant torsion, as a consequence of the comparison formula for equivariant torsions given in [**BGo4, BGo5**], in relation with another work of Bunke [**Bu2**].

When the fibres S^E are odd dimensional, we give a second proof which is based on a different principle. The proof uses the evaluation of the torsion forms for S^1 bundles associated to a complex line bundle λ, and also the functoriality properties of analytic torsion forms in de Rham theory, which were established by Ma in [**Ma4**] using the adiabatic limit techniques developed in Berthomieu-Bismut [**BerB**], and also in Ma's previous work on holomorphic torsion forms [**Ma1, Ma2**]. The evaluation of the torsion forms for the above S^1 bundles can of course be obtained using the techniques of the first proof. Another method is to use the evaluation of the equivariant torsion forms for these S^1 bundles at roots of unity by Bismut-Lott [**BLo1**, Corollary 4.14], and also Theorem 4.31 and Remark 4.32 to extend these results to arbitrary elements in S^1.

This Chapter is organized as follows. In Section 16.1, we state our formula for the torsion forms of S^E. In Section 16.2, we construct the functions m and f, and we describe the corresponding stable and unstable cells. In particular, we show that even if f is only fibrewise Morse-Bott, there is analogue of the de Rham map of Definition 5.2. In Section 16.3, we evaluate the torsion forms of $S^{E\oplus\mathbf{R}}$. In Section 16.4, we give an embedding formula which relates the torsion forms for S^E to the torsion forms of $S^{E\oplus\mathbf{R}}$. In Section 16.5, we prove this embedding formula, by extending the arguments in Chapter 9 to the case of the fibrewise Morse-Bott function f. Finally in Section 16.6, when S^E is odd dimensional, we give another proof of our main formula, which is based on adiabatic limit techniques.

16.1. A formula for the analytic torsion forms of a unit sphere bundle

Let S be a manifold. Let E be a real vector bundle on S of dimension $n+1 \geqslant 2$, and let $o(E)$ be its orientation line. Let g^E be an Euclidean metric on E. Let ∇^E be a metric preserving connection on E. Let $g \in \mathrm{Aut}(E)$ be a parallel isometry of E.

Let S^E be the sphere bundle in E, i.e.

$$S^E = \{x \in E, \|x\| = 1\}. \tag{16.1}$$

Let $\mathcal{E}$ be the total space of S^E. Let g^{TS^E} be the metric on TS^E which is induced by the metric g^E. Clearly the only non zero cohomology groups in $H^\bullet\left(S^E, \mathbf{Z}\right)$ are $H^0\left(S^E, \mathbf{Z}\right) = \mathbf{Z}, H^n\left(S^E, \mathbf{Z}\right) = o(E)$.

The connection ∇^E defines a horizontal subbundle $T^H\mathcal{E}$ on $\mathcal{E}$. It then follows from (3.90) that $\mathcal{T}_{h,g}\left(T^H\mathcal{E}, g^{S^{E\oplus\mathbf{R}}}\right)$ is a closed form, i.e. it defines an element of $H^{\mathrm{even}}(S)$, whose cohomology class $\mathcal{T}_{h,g}(\mathcal{E})$ does not depend on the choice of g^E or ∇^E.

If $g \in G$, let $\det(g) = \pm 1$ be the determinant of g acting on the fibres of E.

Theorem 16.1. — *For any $g \in G$, the following identity holds,*

$$\mathcal{T}_{h,g}(\mathcal{E}) = \left(1 - (-1)^n \det(g)\right)\left({}^0I_g(E) - \frac{1}{2}\log\left(\frac{2\pi^{(n+1)/2}}{\Gamma\left((n+1)/2\right)}\right)\right) \tag{16.2}$$
$$\text{in } H^{\mathrm{even}}(S, \mathbf{C}).$$

Proof. — The remainder of the Chapter is devoted to the proof of our Theorem. Our Theorem will in fact follow from Theorems 16.9 and 16.11. □

Remark 16.2. — By equation (7.24), if n is odd and g does not preserve orientation, or if n is even and g preserves orientation,

$$T_{h,g}\left(T^H\mathcal{E}, g^{TS^E}\right) = 0. \tag{16.3}$$

Of course, Theorem 16.1 fits with (16.3).

Remark 16.3. — Theorem 16.1 has already been obtained by Bunke [**Bu1**] in the case where $g = 1$. From Theorem 4.31, Remark 4.32 and Theorem 16.1, we recover the evaluation by Bismut-Lott [**BLo1**, Corollary 4.14] of the analytic torsion forms for circle bundles.

16.2. The suspension of a sphere and a Morse-Bott function

Let $n \geqslant 1$. Let E be a finite dimensional real vector space of dimension $n+1 \geqslant 2$, let g^E be an Euclidean metric on E. Then $\mathrm{O}(n)$ acts on E. Let S^E be the sphere

$$S^E = \left\{x \in E, \|x\| = 1\right\}. \tag{16.4}$$

Let g^{TS^E} be the metric on TS^E which is induced by the metric g^E. Let $\mathbf{R}$ be the standard real line, equipped with the standard metric such that $\|1\| = 1$. Then we

equip $E \oplus \mathbf{R}$ with the obvious scalar product, so that E and $\mathbf{R}$ are orthogonal in $E \oplus \mathbf{R}$. The action of $\mathrm{O}(n)$ on E lifts to $E \oplus \mathbf{R}$. Then $\mathrm{O}(n)$ acts on $S^{E\oplus\mathbf{R}}$.

We denote by (x,t) the standard element in $E \oplus \mathbf{R}$. Then $i : x \in S^E \to (x,0) \in S^{E\oplus\mathbf{R}}$ is an embedding.

If $(x,t) \in S^{E\oplus\mathbf{R}}$, put

$$m(x,t) = t, \qquad f(x,t) = t^2. \tag{16.5}$$

For technical reasons, we do not equip $S^{E\oplus\mathbf{R}}$ with its standard round metric. Let $g^{TS^{E\oplus\mathbf{R}}}$ be a $\mathrm{O}(n)$-invariant metric on $TS^{E\oplus\mathbf{R}}$ which has the following two properties.

- If $x_1 = (0,1), x_2 = (0,-1)$, the map $x \in E \to \left(x, \pm\sqrt{1 - \|x\|^2}\right)$ is a $\mathrm{O}(n)$-equivariant diffeomorphism from an open ball centred at 0 in E into neighbourhoods of x_1 or x_2 in $S^{E\oplus\mathbf{R}}$. We assume that near x_1 or x_2, the metric $g^{TS^{E\oplus\mathbf{R}}}$ is just the flat Euclidean metric on E.
- The set $\mathcal{U} = \{(x,t) \in S^{E\oplus\mathbf{R}}, -1/2 < t < 1/2\}$ is a collar neighbourhood of S^E in $S^{E\oplus\mathbf{R}}$. We identify $\mathcal{U}$ to $S^E \times]-1/2, 1/2[$, and this $\mathrm{O}(n)$ equivariantly. We assume that on $\mathcal{U}$,

$$g^{TS^{E\oplus\mathbf{R}}} = g^{TS^E} + |dm|^2. \tag{16.6}$$

We denote by $\nabla m, \nabla f$ the gradient fields to f, m with respect to the metric $g^{TS^{E\oplus\mathbf{R}}}$.

Proposition 16.4. — *The function m is a Morse function on $S^{E\oplus\mathbf{R}}$, which is $\mathrm{O}(n)$-invariant. It has two critical points, $x_1 = (0,1)$ with $\mathrm{ind}(x_1) = n+1$, and $x_1 = (0,-1)$, with $\mathrm{ind}(x_2) = 0$.*

Proof. — The proof of our Proposition is left to the reader. □

Let $W_m^u(x_1), W_m^u(x_2)$ be the unstable cells at x_1, x_2. Then

$$W_m^u(x_1) = S^{E\oplus\mathbf{R}} \setminus \{x_2\}, \qquad W_m^u(x_2) = \{x_2\}. \tag{16.7}$$

Here $W_m^u(x_1)$ is considered as an orientable $n+1$ cell, so that it can be viewed as a section of $o(E)$, while $W_m^u(x_2)$ is unambiguously defined. The cells in (16.7) are $\mathrm{O}(n)$-invariant.

Consider the complex $(C^\bullet(W_m^u), \partial)$. This complex is spanned over $\mathbf{R}$ by $W_m^u(x_1)$, $W_m^u(x_2)$, with

$$\deg W_m^u(x_1) = n+1, \qquad \deg W_m^u(x_2) = 0. \tag{16.8}$$

Since $n \geqslant 1$,

$$\partial = 0. \tag{16.9}$$

Recall that

$$f = m^2. \tag{16.10}$$

Proposition 16.5. — *The function f is a $\mathrm{O}(n)$-invariant Morse-Bott function on $S^{E\oplus\mathbf{R}}$, whose critical set consists of x_1, x_2, S^E, with*

$$\text{(16.11)} \qquad \operatorname{ind}(x_1) = n+1, \qquad \operatorname{ind}(x_2) = n+1, \quad \operatorname{ind}\left(S^E\right) = 0.$$

Proof. — The proof of our Proposition is left to the reader. □

Let $W_f^u(x_1), W_f^u(x_2)$ be the unstable cells at x_1, x_2. Then

$$\text{(16.12)} \qquad \begin{aligned} W_f^u(x_1) &= \left\{(x,t) \in S^{E\oplus\mathbf{R}}, t > 0\right\}, \\ W_f^u(x_2) &= \left\{(x,t) \in S^{E\oplus\mathbf{R}}, t < 0\right\}. \end{aligned}$$

Now we describe a complex $\left(C_\bullet\left(W_f^u\right), \partial\right)$, whose homology coincides canonically with the homology of $S^{E\oplus\mathbf{R}}$. In fact as explained in [**BZ1**, Chapter I c)], to the Morse-Bott function f, one can associate a spectral sequence whose E^1 just computes the relative homologies of sets of the form $\{c \leqslant f < d\}$, where c, d are non critical values. Here we should take the sets $\{f \geqslant 1/4\}$ and $\{f < 1/4\}$. The relative homology of $\{f \geqslant 1/4\}$ has two generators in degree $n+1$, $W_f^u(x_1)$ and $W_f^u(x_2)$, which can be viewed as sections of $o(E)$. The relative homology of $\{f < 1/4\}$ (which here coincides with the standard homology) has one generator x in degree 0, which generates $H_0\left(S^E\right)$ and one generator S^E in degree n, which generates $H_n\left(S^E\right)$. Note that S^E should also be viewed as a section of $o(E)$.

The complex $\left(C_\bullet\left(W_f^u\right), \partial\right)$ is then generated by $W_f^u(x_1) \in o(E), W_f^u(x_2) \in o(E)$ in degree $n+1$, $S^E \in o(E)$ in degree n and x in degree 0. The chain map ∂ is given by

$$\text{(16.13)} \qquad \begin{aligned} \partial W_f^u(x_1) &= S^E, & \partial W_f^u(x_2) &= -S^E, \\ \partial S^E &= 0, & \partial x &= 0. \end{aligned}$$

Here the above spectral sequence degenerates, i.e. the homology of the complex $\left(C_\bullet\left(W_f^u\right), \partial\right)$ coincides canonically with the homology of $S^{E\oplus\mathbf{R}}$, whose representatives are $S^{E\oplus\mathbf{R}}$ in degree $n+1$ and $\mathbf{1}$ in degree 0.

Let dv_{S^E} be the standard Lebesgue measure on S^E which has total mass 1. Observe that $\overline{W}_f^u(x_1), \overline{W}_f^u(x_2), S^E$ are explicit $\mathrm{O}(n)$-invariant cycles, while x has no canonical $\mathrm{O}(n)$-invariant representative in S^E. Instead we will represent x by the current dv_{S^E}, which should be thought of as the $\mathrm{O}(n)$-average of points in S^E. It is then feasible to replace x by dv_{S^E} in (16.13).

Let $W_f^s(x_1), W_f^s(x_2)$ be the stable cells at x_1, x_2. Then

$$\text{(16.14)} \qquad W_f^s(x_1) = \{x_1\}, \qquad W_f^s(x_2) = \{x_2\}.$$

The stable cell $W_f^s\left(S^E\right)$ originating from S^E is given by

$$\text{(16.15)} \qquad W_f^s\left(S^E\right) = S^{E\oplus\mathbf{R}} \setminus \{x_1, x_2\}.$$

The stable cell originating from $x \in S^E$ is one dimensional and given by the meridian connecting x_1 and x_2 through x. Similarly one can define the stable current flowing from dv_E. It is just the average with respect to dv_E of the above meridians indexed by $x \in S^E$.

Now we describe the complex $\left(C^\bullet\left(W_f^u\right), \partial\right)$ dual to $\left(C_\bullet\left(W_f^u\right), \partial\right)$. It is generated by $\left(o(E)_{x_1}, o(E)_{x_2}\right)$ in degree $n+1$, by $\omega \in o(E)$, the canonical volume form ω on S^E with volume equal to 1 in degree n, and by $\mathbf{1}$ in degree 0. The chain map ∂ vanishes in degree 0 and degree $n+1$. Moreover if $\{x_1\} \in o(E)_{x_1}, \{x_2\} \in o(E)_{x_2}$ are associated to the orientation of S^E defined by ω, then

$$\partial\omega = \{x_1\} - \{x_2\}. \tag{16.16}$$

The above complex can also be viewed as a complex of currents on $S^{E\oplus\mathbf{R}}$. The Dirac masses $\delta_{x_1}, \delta_{x_2}$ can be viewed as $n+1$ currents on $S^{E\oplus\mathbf{R}}$ with values in $o(E)$. Consider the map $j : (x,t) \in S^{E\oplus\mathbf{R}} \setminus \{x_1, x_2\} \to \frac{x}{\|x\|} \in S^E$. The form ω on S^E pulls back to a well defined current $j^*\omega$ on $S^{E\oplus\mathbf{R}}$. Let $\varepsilon(\omega) \in o(E)$ be the obvious image of ω. We identify $\mathbf{1}$ to the constant function equal to 1 on $S^{E\oplus\mathbf{R}}$. Then one verifies easily that we have the equality of currents on $S^{E\oplus\mathbf{R}}$,

$$\begin{aligned} &d\delta_{x_1} = 0, && d\delta_{x_2} = 0, \\ &d\mathbf{1} = 0, && dj^*\omega = \varepsilon(\omega)\left(\delta_{x_1} - \delta_{x_2}\right). \end{aligned} \tag{16.17}$$

Finally, note that $\mathrm{O}(n)$ acts on $\left(C^\bullet\left(W_f^u\right), \partial\right)$ just on the factor $o(E)$. In particular,

$$g\omega = \det(g)\,\omega. \tag{16.18}$$

Remark 16.6. — The spectral sequence associated to the function f degenerates, essentially because the Morse-Bott function f only has two critical values. For arbitrary Morse-Bott functions, the spectral sequence does not degenerate.

Even though f is not Morse, but only Morse-Bott, we will still define an associated de Rham map. Let $\left(\Omega^\bullet\left(S^E\right), d\right)$ be the de Rham complex of S^E.

Definition 16.7. — Let $P^\infty : \Omega^\bullet(M) \to C^\bullet\left(W_f^u\right)$ be the map,

$$\alpha \in \Omega^\bullet(M) \to P^\infty\alpha = \sum_1^2 \int_{\overline{W_f^u(x_i)}} \alpha + \int_{S^E} \alpha + \int_{S^E} \alpha^0 dv_{S^E}. \tag{16.19}$$

Proposition 16.8. — *The map P^∞ in (16.19) is a quasi-isomorphism of $\mathbf{Z}$-graded G-complexes, which provides the canonical identification of the cohomology groups of both complexes.*

Proof. — Using Stokes formula, our Proposition follows easily. □

16.3. The torsion forms of $S^{E\oplus\mathbf{R}}$

We make the same assumptions as in Section 16.1 and we use the corresponding notation. Let M be the total space of $S^{E\oplus\mathbf{R}}$. Then M is equipped with a horizontal bundle T^HM. We equip $TS^{E\oplus\mathbf{R}}$ with the metric $g^{TS^{E\oplus\mathbf{R}}}$ constructed in (16.6).

We first prove a special case of Theorem 16.1.

Theorem 16.9. — *The following identity holds,*

$$\mathcal{T}_{h,g}\left(T^HM, g^{TS^{E\oplus\mathbf{R}}}\right) = \left(1-(-1)^{n+1}\det(g)\right)\left({}^0I_g(E) - \frac{1}{2}\log\left(\mathrm{Vol}\left(S^{E\oplus\mathbf{R}}\right)\right)\right) \text{ in } H^{\mathrm{even}}(S,\mathbf{C}). \tag{16.20}$$

Proof. — Let $m : S^{E\oplus\mathbf{R}} \to \mathbf{R}$ be the smooth function constructed in Section 16.2. Then m is fibrewise Morse, and the fibrewise gradient field $-\nabla m$ is fibrewise Morse-Smale. The critical points of m along the fibres are the sections x_1, x_2 of $S^{E\oplus\mathbf{R}}$. Both critical points are G-invariant, i.e. they lie in B_g. Now we will use Theorem 7.2. As in (7.35),

$$T_{h,g}\left(A^{C^\bullet(W_m^u)}, g^{C^\bullet(W_m^u)}\right)^{(\geqslant 2)} = 0. \tag{16.21}$$

As explained in (7.36), since $H^0\left(S^{E\oplus\mathbf{R}},\mathbf{R}\right), H^{n+1}\left(S^{E\oplus\mathbf{R}},\mathbf{R}\right)$ are flat line bundles,

$$\widetilde{h}_g\left(\nabla^{H^\bullet\left(S^{E\oplus\mathbf{R}},\mathbf{R}\right)}, g^{H^\bullet\left(S^{E\oplus\mathbf{R}},\mathbf{R}\right)}_{C^\bullet(W_m^u)}, g^{H^\bullet\left(S^{E\oplus\mathbf{R}},\mathbf{R}\right)}_{L_2}\right)^{(\geqslant 2)} = 0 \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S). \tag{16.22}$$

Also,

$$\mathrm{ind}_g(x_1) = \dim E^1, \qquad \mathrm{ind}_g(x_2) = 0. \tag{16.23}$$

Moreover,

$$TX^u_{x_1} = E, \qquad TX^s_{x_2} = E. \tag{16.24}$$

One has the trivial,

$$\det g = (-1)^{\dim E - \dim E^1}. \tag{16.25}$$

Using Theorem 7.2 and (16.21)-(16.25), we get (16.20). The proof of our Theorem is completed. □

Remark 16.10. — By applying Theorem 16.9 in degree 0, and by noting that

$$\mathrm{Vol}\left(S^E\right) = \frac{2\pi^{(n+1)2}}{\Gamma\left((n+1)2\right)}, \tag{16.26}$$

we get Theorem 16.1 in degree 0, even in the case $n = 1$.

16.4. The embedding formula

The main technical result of this Chapter is as follows.

Theorem 16.11. — *The following identity holds,*

$$\mathcal{T}_{h,g}\left(T^H M, g^{TS^{E\oplus\mathbf{R}}}\right)^{(\geqslant 2)} = \mathcal{T}_{h,g}\left(T^H\mathcal{E}, g^{TS^E}\right)^{(\geqslant 2)} + 2\,(-1)^n \det(g)\,{}^0I_g(E)^{(\geqslant 2)} \text{ in } H^{\text{even}}(S,\mathbf{C}). \tag{16.27}$$

Proof. — The next Section is devoted to the proof of Theorem 16.11. □

Remark 16.12. — By Theorems 16.9 and 16.11, we get Theorem 16.1.

16.5. A proof of Theorem 16.11

We will obtain Theorem 16.11 by establishing a version of Theorem 7.2, adapted to the case where f is the fibrewise Morse-Bott function introduced in Section 16.2, and ∇f is the associated fibrewise gradient field.

We will develop a machinery, which, is in principle, valid for more general Morse-Bott gradient fields, but we will only check the details in the present situation.

In this Section, we use the notation of Chapters 5-15. In particular M still denotes the total space of $S^{E\oplus\mathbf{R}}$, with fibre $X = S^{E\oplus\mathbf{R}}$. Also, we will consider only the case of the trivial vector bundle $F = \mathbf{R}$.

16.5.1. The geometry of critical submanifolds. — Still, $\mathbf{B}$ denotes the manifold of zeros of ∇f. Then $\mathbf{B}$ fibres on S, with fibre B. Then B is the union of connected critical manifolds $\{x\}$ of f in a given fibre $X = S^{E\oplus\mathbf{R}}$. If $x \in B$, f takes a constant value $f(x)$ on x. Note that here, x is x_1, x_2 or S^E.

Observe that here $T^H M|_{\mathbf{B}} \subset T^{\mathbf{B}}$, i.e. $T^H M|_{\mathbf{B}}$ is a horizontal bundle on $\mathbf{B}$. Note that on $\mathcal{E} \subset M$, $T^H M|_{\mathbf{B}} = T^H\mathcal{E}$. Also, in the present situation, f is parallel, so that the horizontal component $(df)^H$ vanishes.

If $x \in B$, the normal bundle $N_{x/X}$ splits as

$$N_{x/X} = N^s_{x/X} \oplus N^u_{x/X}. \tag{16.28}$$

Let $o\left(N^u_{x/X}\right)$ be the orientation bundle of $N^u_{x/X}$. Put

$$n_+(x) = \dim N^s_{x/X}, \qquad n_-(x) = \dim N^u(x). \tag{16.29}$$

Similarly, set

$$\operatorname{ind}(x) = \dim N^u(x). \tag{16.30}$$

Set

$$\mathbf{B}_g = \mathbf{B} \cap M_g. \tag{16.31}$$

Then $\mathbf{B}_g$ fibres on S, with fibre B_g, which itself is the union of the $x_g, x \in B$. Clearly, $\nabla f|_{X_g} \in TX_g$. The restriction of f to B_g is still a Morse-Bott function. If y is a connected component of $\mathbf{B}_g$, if $x \in B$ is such that $y \subset x$, then $Tx|_y$ and $N_{x/X}|_y$ are both stable by g. We define the $n_+(\theta_j)(y), n_-(\theta_j)(y)$ as in Section 9.2, by replacing TX by $N_{x/X}|_y$.

16.5.2. An extended form of Theorem 7.2. — Let $o\left(N^u_{x/X}\right)$ be the orientation line of $N^u_{x/X}$. If $x \in B$, let $\mathbf{B}_x$ be the local fibration on S with fibre x. We denote by $\mathcal{T}_{h,g}\left(T^H\mathbf{B}_x, g^{Tx}, \nabla^{o\left(N^u_{x/X}\right)}\right)$ the analytic torsion forms on S associated with $\mathbf{B}_x$, where the fibres x are equipped with the metric g^{Tx} induced by g^{TX}, and $\nabla^{o\left(N^u_{x/X}\right)}$ is the obvious flat connection. Note that the line bundle $o\left(N^u_{x/X}\right)$ is canonically equipped with a metric, which we do not write explicitly.

We can embed the $C^\bullet\left(W^u_f\right)$ into $\Omega^\bullet(B)$. This is here especially relevant for the form ω on $S^{E\oplus\mathbf{R}}$, which is harmonic. Let $g^{C^\bullet\left(W^u_f\right)}$ be the corresponding metric on $C^\bullet\left(W^u_f\right)$.

Theorem 16.13. — *The following identity holds,*

$$\begin{aligned}(16.32)\quad &\mathcal{T}_{h,g}\left(T^HM, g^{TS^{E\oplus\mathbf{R}}}\right) - T_{h,g}\left(A^{C^\bullet(W^u,F)'}, g^{C^\bullet\left(W^u_f\right)}\right)\\ &\qquad + \widetilde{h}_g\left(\nabla^{H^\bullet(X)}, g^{H^\bullet(X)}_{C^\bullet\left(W^u_f\right)}, g^{H^\bullet(X)}_{L_2}\right)\\ &= \sum_{x\in B}(-1)^{\mathrm{ind}(x)}\,\mathcal{T}_{h,g}\left(T^H\mathbf{B}_x, g^{Tx}, \nabla^{o\left(N^u_{x/X}\right)}\right)\\ &\qquad + \sum_{x\in B_g}(-1)^{\mathrm{ind}_g(x)}\int_x e(Tx)\,{}^0I_g\left(N_{x/X}|_{\mathbf{B}_g}\right) \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S).\end{aligned}$$

Proof. — The remainder of the Chapter is devoted to the proof of our Theorem. □

Remark 16.14. — We briefly show how to derive Theorem 16.11 from Theorem 16.13. Clearly, $N_{S^E/S^{E\oplus\mathbf{R}}} \simeq \mathbf{R}$, and g acts trivially on $N_{S^E/S^{E\oplus\mathbf{R}}}$. Therefore using Theorem 16.13 and proceeding as in the proof of Theorem 16.9, we get (16.27). The proof of Theorem 16.11 is completed.

Let us also point out that an extension of Theorem 16.13 to compact manifolds equipped with an action of a compact Lie group has been established in [**BGo5**, Theorem 5.11], in the context of infinitesimal equivariant torsion, as a consequence of the comparison formula of [**BGo4, BGo5**] which relates two natural versions of equivariant torsion. The result established in [**BGo5**] is valid even if the spectral sequence considered above does not degenerate. Also note that Theorem 16.13 for unit sphere bundles can be recovered from [**BGo5**, Theorem 5.11]. Finally observe that Theorem 16.13 and the corresponding version established in [**BGo5**] are themselves

related to earlier results by Bunke [**Bu2**], who gives a formula for the infinitesimal equivariant torsion of an odd dimensional compact oriented manifold in terms of contributions of the cells of an associated G-CW complex.

16.5.3. Convergence of currents. — Note that in our geometric setting, the obvious analogue of the simplifying assumptions of 9.1 still hold. In particular, $T^H M|_{\mathbf{B}} \subset T\mathbf{B}$.

Since B is the union of the $x \in B$, we will often write $N^u_{B/X}$ instead of $N^u_{x/X}$. Let $\chi_g\left(x, o\left(N^u_{x/X}\right)\right)$ be the Lefschetz number of x equipped with the flat line bundle $o\left(N^u_{x/X}\right)$. Using the above spectral sequence argument, one finds easily that instead of (9.5), we now have

$$\chi_g(\mathbf{R}) = \sum_{x\in B} (-1)^{\mathrm{ind}(x)} \chi_g\left(x, o\left(N^u_{x/X}\right)\right). \tag{16.33}$$

In the sequel, we will write $H^\bullet(X)$ instead of $H^\bullet(X, \mathbf{R})$. Instead of (9.6), we now set

$$\begin{aligned}
\chi'_g(\mathbf{R}) &= \sum_{i=0}^{\dim X} (-1)^i i \mathrm{Tr}^{H^i(X)}[g], \\
\chi'^c_g\left(o\left(N^u_{B/X}\right)\right) &= \sum_{i=0} (-1)^i i \mathrm{Tr}^{H^i\left(B, o\left(N^u_{B/X}\right)\right)}[g], \\
\widetilde{\chi}'^{+/-}_g(\mathbf{R}) &= \sum_{x\in B} (-1)^{\mathrm{ind}(x)} \dim\left(N^{s/u}_{x/X}\right) \chi_g\left(x, N^{s/u}_{x/X}\right), \\
\mathrm{Tr}_{\mathrm{s}}{}^{B_g}[f] &= \sum_{x\in B} (-1)^{\mathrm{ind}(x)} \chi_g\left(x, o\left(N^u_{x/X}\right)\right) f(x).
\end{aligned} \tag{16.34}$$

One verifies easily that the obvious analogue of (9.9) still holds.

We still define m_T as in Definition 9.2. Note that the since $F = \mathbf{R}$, the obvious analogue of n_T in (9.13) vanishes. Instead of Proposition 9.3, we find that as $T \to +\infty$, if $\chi(x_g)$ denotes the Euler characteristic of x_g,

$$\begin{aligned}
T\int_{X_g} f(\alpha_T - \alpha_{+\infty}) &\to \sum_{x\in B_g} \frac{(-1)^{\mathrm{ind}g(x)}}{4} (n_+(0)(x) - n_-(0)(x)) \chi(x_g), \\
T^2 \int_{X_g} df \beta_T &\to -\sum_{x\in B_g} \frac{(-1)^{\mathrm{ind}_g(x)}}{4} (n_+(0)(x) - n_-(0)(x)) \chi(x_g).
\end{aligned} \tag{16.35}$$

Similarly, in Proposition 9.4, the first equation in (9.15) is now replaced by

$$T^2\left(m_T - \mathrm{Tr}_{\mathrm{s}}{}^{B_g}(f)\right) \to -\sum_{x\in B_g} (n_+(0)(x) - n_-(0)(x)) \int_{x_g} \mathrm{Tr}^F[g]\, e(Tx_g). \tag{16.36}$$

16.5.4. Four intermediate results. — In this Section, to make our statements simpler, we will assume S to be compact. If S is not compact, the constants which appear in our estimates will depend explicitly on a compact subset $K \subset S$.

If $x \in B_g$, consider the $\mathbf{Z}_2$-vector bundle $N_{x/X}|_{\mathbf{B}_g} = N^s_{x/X}|_{\mathbf{B}_g} \oplus N^u_{x/X}|_{\mathbf{B}_g}$. This vector bundle is equipped with a metric such that the splitting $N_{x/X}|_{\mathbf{B}_g} = N^s_{x/X}|_{\mathbf{B}_g} \oplus N^u_{x/X}|_{\mathbf{B}_g}$ is orthogonal, with a connection $\nabla^{N_{x/X}|_{\mathbf{B}_g}}$ preserving the splitting and also the metric, and also with an isometric and parallel action of g which preserves the splitting. In the sequel we use the notation of Chapter 4 for this vector bundle. In particular if $x \in B_g$, $\mathcal{D}^x_T$ denotes the operator defined in (4.9).

Let N^c be the number operator of $\Omega^\bullet(x)$. Recall that $T^H M|_{\mathbf{B}} \subset T\mathbf{B}$, so that $T^H M|_{\mathbf{B}}$ is a horizontal subbundle on $\mathbf{B}$. Let D^c_t be the analogue of D_t defined in (3.52) for $\mathbf{B}$, where of course, $\mathbf{B}$ is still equipped with the flat line bundle $o\left(N^u_{x/X}\right)$.

Note that Theorem 9.6 still holds. We now have an analogue of Theorem 9.7.

Theorem 16.15. — *Given ε, A with $0 < \varepsilon < A < +\infty$, there exists $C > 0$ such that if $t \in [\varepsilon, A], T \geqslant 1$, then*

$$\left|\mathrm{Tr_s}\left[Ngh'\left(D_{t,T}\right)\right] - \sum_{x\in B}(-1)^{\mathrm{ind}(x)}\,\mathrm{Tr_s}\left[N^c g h'\left(D^c_t\right)\right] - \widetilde{\chi}'^{-}_g(\mathbf{R})\right| \leqslant \frac{C}{\sqrt{T}}. \tag{16.37}$$

Theorem 16.16. — *The following identity holds,*

$$\begin{aligned}
&\lim_{T\to+\infty}\left\{\int_1^{+\infty}\left(\mathrm{Tr_s}\left[Ngh'\left(D_{t,T}\right)\right] - \chi'_g(F)\right)\frac{dt}{2t}\right. \\
&\qquad - \widetilde{h}^*_g\left(\nabla^{H^\bullet(X,\mathbf{R})}, g^{H^\bullet(X,\mathbf{R})}_{L_2,0}, g^{H^\bullet(X,\mathbf{R})}_{L_2,T}\right) - \mathrm{Tr_s}^{B_g}[f]\,T \\
&\qquad\qquad \left. + \frac{1}{4}\left(\widetilde{\chi}'^{-}_g(F) - \chi'^{+}_g(F)\right)\log(T)\right\} \\
&= \int_0^1\left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u_f)} g h'\left(B_t^{C^\bullet(W^u_f)}\right)\right] - \widetilde{\chi}'^{-}_g(\mathbf{R})\right)\frac{dt}{2t} \\
&\qquad + \int_1^{+\infty}\left(\mathrm{Tr_s}\left[N^{C^\bullet(W^u_f)} g h'\left(B_t^{C^\bullet(W^u_f)}\right)\right] - \chi'_g(\mathbf{R})\right)\frac{dt}{2t} \\
&\quad - \widetilde{h}^*_g\left(\nabla^{H^\bullet(X,\mathbf{R})}, g^{H^\bullet(X,\mathbf{R})}_{L_2,0}, g^{H^\bullet(X,\mathbf{R})}_{C^\bullet(W^u_f)}\right) + \frac{1}{4}\left(\widetilde{\chi}'^{-}_g(\mathbf{R}) - \chi'^{+}_g(\mathbf{R})\right)\log(\pi) \\
&+ \sum_{x\in B}(-1)^{\mathrm{ind}(x)}\int_1^{+\infty}\left(\mathrm{Tr_s}\left[N^c g h'\left(D^c_t\right)\right] - \chi'^{c}_g\left(o\left(N^u_{B/X}\right)\right)\right)\frac{dt}{2t} \text{ in } \Omega^\bullet(S)/d\Omega^\bullet(S).
\end{aligned} \tag{16.38}$$

Theorem 16.17. — *The obvious analogues of Theorem 9.9, 9.10 and 9.12 hold.*

An analogue of Theorem 9.12 is now.

Theorem 16.18. — *For any $T > 0$, the following identity holds,*

(16.39)
$$\lim_{t\to 0} \frac{1}{t} \left(\mathrm{Tr_s}\left[fgh'\left(D_{t,T/t}\right)\right] - \mathrm{Tr_s}^{B_g}[f] \right) = \sum_{x\in B_g} \int_x e\left(Tx\right) \mathrm{Tr_s}\left[qgh'\left(\mathcal{D}_T^x\right)\right] \mathrm{Tr}^{F_x}\left[g\right].$$

Proof. — The proof of Theorems 16.15-16.18 will be sketched in Sections 16.5.4- □

Remark 16.19. — Using Theorem 9.6, Theorems 16.15-16.18, we derive Theorem 16.13 in the same way as Theorem 7.2 from Theorems 9.8-9.11 in Chapter 9.

16.5.5. A proof of Theorem 16.15. — By proceeding as in [**BL**, Sections 8 and 9], we get Theorem 16.15 easily. The situation is made much easier than in those references, because the situation is product near S^E. We still need to define a map $J_T : C^\bullet\left(W_f^u\right) \to \Omega^\bullet\left(X\right)$ extending [**BZ2**, Definition 6.5] and (10.204), (11.6). At x_1, x_2, which are isolated critical points of f, we define J_T as in (10.204). Let α_T be defined as in (11.5) for $n = 1$, i.e.

$$\alpha_T = \int_{\mathbf{R}} \mu^2\left(Z\right) \exp\left(-T|Z|^2\right) dZ. \tag{16.40}$$

Observe that $\mathbf{1}, \omega$ are harmonic forms on S^E. If $Z = t$ denotes the normal coordinate to S^E in $S^{E\oplus\mathbf{R}}$, for $\eta = \mathbf{1}$ or ω, set

$$J_T\eta = \frac{\mu\left(Z\right)}{\alpha_T^{1/2}} \exp\left(-T\left|Z\right|^2/2\right)\eta. \tag{16.41}$$

The proof of Theorem 16.15 then proceeds as in [**BL**, Sections 8 and 9].

16.5.6. A proof of Theorem 16.16. — We define the maps $\overline{e}_T, e_T$ as in (11.7), (11.8).

We claim that the obvious analogue of Theorem 11.4 still holds. In fact, consider the first equation in (11.10). Near x_1 or x_2, this is just the first equation in Theorem 11.4, which was established in [**BZ2**, Theorem 6.7]. Also, near S^E, $S^{E\oplus\mathbf{R}} \simeq S^E \times \left]-\varepsilon, \varepsilon\right[$, we are in a product situation, so that the analogue of (10.211) still holds. Recall that C^c, D^c are the analogues of C, D for the projection of $\mathbf{B}$ on S with fibre B. Let $\kappa \in \left]0, 1\right]$ be small enough so that

$$\mathrm{Sp}\left(C^{c,2}\right) \subset \{0\} \cup \left[2\kappa, +\infty\right[. \tag{16.42}$$

Then, by proceeding as in [**BL**, Sections 8 and 9], for $T \geqslant 0$ large enough,

$$\mathrm{Sp}\left(\overline{C}_T^2\right) \subset \left[0, \frac{\kappa}{2}\right[\cup \left[\frac{3}{2}\kappa, +\infty\right[. \tag{16.43}$$

Let now δ be the circle of centre 0 and radius $\kappa/2$ in $\mathbf{C}$. Then by the arguments in [**BL**, Sections 8 and 9], there exists $C > 0$ such that for $T \geqslant 0$ large enough, if $\lambda \in \delta$,

$$\left\| \left(\lambda - \overline{C}_T^2\right)^{-1} \right\| \leqslant C. \tag{16.44}$$

The proof of the analogue of [**BZ2**, Theorem 6.7] or of the first equation in (11.10) then continues as in [**BZ2**]. From this result we derive the analogue of [**BZ2**, Theorem 6.9] or of the second equation in (11.10).

Also we claim that the analogue of [**BZ2**, Theorem 6.11] or of the third equation in (11.10) holds. The proof is in fact exactly the same. The proof of the analogue of [**BZ2**, Theorem 6.12] or of the fourth equation in (11.10) is also the same. Therefore we have established the obvious extension of Theorem 11.4.

Now we establish a version of Theorem 11.16. Recall that given $c_1 > 0$, the set $U_T \subset \mathbf{C}$ was defined in (11.51).

Theorem 16.20. — *For $c_1 \in]0,1]$ small enough, for any integer $p \geqslant \dim X + 2$, there exists $C > 0$ such that for $T \geqslant 1$ and $\lambda \in U_T$,*

$$\left| \mathrm{Tr_s}\left[N^X g\,(\lambda - B_T)^{-p}\right] - \sum_{x \in B} (-1)^{\mathrm{ind}(x)}\, \mathrm{Tr_s}\left[N^c g\,(\lambda - B^c)^{-p}\right] \right| \leqslant \frac{C}{\sqrt{T}}\,(1 + |\lambda|)^{p+1}\,. \tag{16.45}$$

Proof. — Using (11.54) (which is valid here since $(df)^H = 0$, and the fact that near S^E, we are in a product situation), the proof of our Theorem is the same as the proof of [**BL**, Theorem 9.24] and of Theorem 11.16. □

Let now $\nu \in]0,1]$ be such that

$$|\mathrm{Sp}\,(B^c)| \subset \{0\} \cup [2\nu, +\infty[\,. \tag{16.46}$$

Let $B^{C^\bullet(W_f^u)}$ be the obvious analogue of $B^{C^\bullet(W^u,F)}$. There are $d_1, d_2 \in \mathbf{R}_+^*$, with $d_1 < d_2/4$, such that

$$\left|\mathrm{Sp}\left(B^{C^\bullet(W_f^u)}\right)\right| \subset \{0\} \cup [2d_1, d_2/2]\,. \tag{16.47}$$

Then instead of Theorem 11.7, we now have.

Theorem 16.21. — *For $T \geqslant 0$ large enough,*

$$\left|\mathrm{Sp}\left(B_T^{(0)}\right)\right| \subset \{0\} \cup \left(\frac{T}{\pi}\right)^{1/2} e^{-T}\left[\frac{3}{2}d_1, \frac{2}{3}d_2\right] \cup \left[\frac{3}{2}\nu, +\infty\right[\,. \tag{16.48}$$

Proof. — Using Theorem 16.20, the proof of our Theorem proceeds as the proof of Theorem 11.7. □

We redefine the contour Δ in Figure 11.2, where $\pm i/2$ is replaced by $\pm\nu$, and $\pm\frac{1}{4}$ by $\pm\nu/2$. Put

$$F_t = \psi_t^{-1} \frac{1}{2i\pi} \int_\Delta \frac{h'\left(\sqrt{t}\,\lambda\right)}{\lambda - D^c} d\lambda \psi_t. \tag{16.49}$$

Observe that over $x_i, 1 \leqslant i \leqslant 2$, $D^c = 0$, so that

$$F_t = 0 \text{ at } x_1 \text{ or } x_2. \tag{16.50}$$

A version of Theorem 11.18 is as follows.

Theorem 16.22. — *There exist $c > 0, C > 0$ such that for $T \geqslant 0$ large enough, and $t \geqslant 1$,*

$$\left| \mathrm{Tr_s}\left[N^X g F_{t,T}\right] - \sum_{x \in B} (-1)^{\mathrm{ind}(x)} \mathrm{Tr_s}\left[N^c g F_t\right] \right| \leqslant \frac{C}{\sqrt{T}} e^{-ct}. \tag{16.51}$$

Proof. — We proceed as in the proof of Theorem 11.18, by using Theorem 16.20 instead of Theorem 11.18. □

We claim that the obvious analogue of Theorem 11.19 still holds. In fact by (16.48),

$$\left| \mathrm{Sp}\left(\left(\frac{T}{\pi}\right)^{-1/2} e^T B_T^{(0)} \right) \right| \subset \{0\} \cap [3/2d_1, 2/3d_2] \cup \left[\frac{3}{2} \left(\frac{T}{\pi}\right)^{-1/2} e^T \nu, +\infty \right[. \tag{16.52}$$

Let Q_T be the orthogonal projection from $\Omega^\bullet\big(X, F|_X\big)$ on the direct sum of the eigenspaces of $B_T^{(0)}$ associated to eigenvalues in $[0, 3/4\nu]$, with respect to to the metric $g_T^{\Omega^\bullet\left(X, F|_X\right)}$. By (16.48), Q_T is a projection operator on a finite dimensional subbundle of $\Omega^\bullet\big(X, F|_X\big)$ which has finite constant dimension for $T \geqslant 0$ large enough. Set $R_T = 1 - Q_T$. In the proof of Theorem 11.19, we replace $P_T^{[0,1]}, P^{]1,+\infty[}$ by Q_T, R_T. Then the estimates in the proof of Theorem 11.19 remain true, so that its conclusion still holds.

The remaining results in Chapter 11 can then be proved as before. In view of the above results, this completes the proof of Theorem 16.16.

16.5.7. A proof of Theorem 16.17. — This consists in checking that analogues of Theorems 9.9, 9.10 and 9.12 still hold.

The proof of the analogue of Theorem 9.9 proceeds as in Chapter 12. In fact, near x_1 or x_2, the analysis is the same as in Chapter 12. Near S^E, we take advantage of the fact that the situation is product. In fact near S^E, we may as well use the arguments of Section 12 in the coordinate t, while the contribution of S^E remains constant. Details are left to the reader.

The proof of the analogue of Theorem 9.10 is even simpler. In fact we proceed as in Chapter 13. Observe first that here $F = \mathbf{R}$, so that $\omega\big(\nabla^F, g^F\big) = 0, n_T = 0, R^{F,u} = 0$. The arguments of Section 13.2 show that the proof of our result is local on X and can be localized near X_g. Near x_1 and x_2, the proof of our Theorem is the same as in Chapter 13. Near S_g^E, we take advantage of the product structure of the problem.

The proof of the analogue of Theorem 9.12 proceeds along the same lines as in Chapter 15.

16.5.8. A proof of Theorem 16.18. — To establish Theorem 16.18, we proceed as in the proof of Theorem 9.11 which was given in Chapter 14. We only need to establish an obvious analogue of Theorem 14.1, the arguments in Remark 14.2 then leading to Theorem 16.18. The arguments in the proofs given in Chapter 14 remain valid near x_1 and x_2. If S^E was an arbitrary critical submanifold, to establish the analogue of Theorem 14.1, we would need the techniques used by Bismut-Lebeau in [**BL**, Section 1]. However, here, the situation being product near S^E, much simpler arguments should be used. One can, for example, use a Getzler rescaling on S^E, while using the scaling arguments of Section 14 in the normal coordinate t. Details are left to the reader.

16.6. Adiabatic limits: the case where n is odd

Now we assume that n is odd. We will give a different proof of Theorem 16.1 using adiabatic limit techniques.

Then $\dim S^E = n$ is odd. By (16.3), we may as well assume that g preserves the orientation.

Let $\mathfrak{g}$ be the Lie algebra of G. Let $Z(g) \subset G$ be the centralizer of G and let $\mathfrak{z}(g) \subset \mathfrak{g}$ be its Lie algebra. Observe that R^E is a 2-form on S with values in $\mathfrak{z}(g)$. By arguments which were given in detail in [**BGo1**, Section 2.6] in the holomorphic context, there is an analytic function $K \in \mathfrak{z}(g) \to \mathcal{T}_{h,g}(K) \in \mathbf{C}$, defined on a neighbourhood of 0, such that

$$\mathcal{T}_{h,g}\left(T^H\mathcal{E}, g^{TS^E}\right) = \mathcal{T}_{h,g}\left(-R^E/2i\pi\right). \tag{16.53}$$

The function $K \to \mathcal{T}_{h,g}(K) \in \mathbf{C}$ is called an equivariant infinitesimal analytic torsion form.

From now on, S^E will simply denote the sphere of centre 0 and radius 1 in the vector space E.

Take $K \in \mathfrak{z}(g)$, with $|K|$ small enough. Under the above assumptions, it is clear that there is a complex structure J on E which commutes with both g and K. Let H be the underlying complex vector space, which has complex dimension $(n+1)/2$.

Then S^1 acts freely on S^E via the semigroup $t \in S^1 \to e^{tJ}$. The quotient of S^E by the action of S^1 is the complex projective space $\mathbf{P}_{(n-1)/2}$. Observe that g acts as a holomorphic map on $\mathbf{P}_{(n-1)/2}$. In particular g preserves the orientation of $\mathbf{P}_{(n-1)/2}$.

Consider the projection $q : S^E \to \mathbf{P}_{(n-1)/2}$, with fibre S^1. The cohomology of the fibre S^1 is concentrated in degree 0 and 1, and the cohomology groups form trivial $\mathbf{Z}$-line bundles on $\mathbf{P}_{(n-1)/2}$, on which g and K act trivially. Since $\mathbf{P}_{(n-1)/2}$ is even-dimensional, and g preserves the orientation of $\mathbf{P}_{(n-1)/2}$, by the same argument as in (7.24), the corresponding equivariant infinitesimal analytic torsion forms of $\mathbf{P}_{(n-1)/2}$ with coefficients in these two line bundles vanish identically.

Consider now the spectral sequence associated to the projection q. Still g and K act trivially on the terms of this spectral sequence.

Since S^E is odd-dimensional, and g is oriented, S^E_g is also odd-dimensional. In particular, its equivariant Euler form of TS^E with respect to the action of K on S^E vanishes identically. The same is true for the associated Chern-Simons forms.

Now we will use a formula due to Berthomieu-Bismut [**BerB**] in the context of holomorphic analytic torsion forms, due to Ma [**Ma1, Ma3**] for holomorphic equivariant torsion, and still to Ma [**Ma4**, Theorem 0.1] for non equivariant analytic torsion forms in de Rham theory. This formula expresses the functoriality of the torsion forms under composition of two proper submersions. We claim that this formula can be adequately adapted to our equivariant infinitesimal analytic torsion forms. Verifying that the arguments of [**BerB**] and [**Ma1, Ma3, Ma4**] can be adapted is not difficult, and uses in particular the local index theoretic techniques of [**BLo1**], and of the previous Sections of this paper, and also the adiabatic limit techniques of Mazzeo-Melrose [**MazMe**] and Dai [**Da**].

Let $\mathbf{P}_{(n-1)/2,g}$ be the submanifold of $\mathbf{P}_{(n-1)/2}$ fixed by g. Then $\mathbf{P}_{(n-1)/2,g}$ is a union of projective spaces in $\mathbf{P}_{(n-1)/2}$.

Let $\mathcal{T}_{h,g}\left(T^H S^E to, g^{TS^1}\right)(K)$ be the infinitesimal analytic torsion forms associated to the fibration $q_g : S^E_g \to \mathbf{P}_{(n-1)/2,g}$ with fibre S^1, where $T^H S^E$ is the orthogonal bundle to TS^1. Note that since the fibres S^1 are one-dimensional, the forms $\mathcal{T}_{h,g}\left(T^H S^E to, g^{TS^1}\right)(K)$ on $\mathbf{P}_{(n-1)/2,g}$ are closed with respect to the equivariant operator $d - 2i\pi i_{K^{\mathbf{P}_{(n-1)/2,g}}}$, and the cohomology class of $\mathcal{T}_{h,g}\left(T^H S^E to, g^{TS^1}\right)(K)$ does not depend on the metric data.

By eliminating the constant term in the expansion of $\mathcal{T}_{h,g}(K)$, to which the spectral sequence contributes, an obvious analogue of [**BerB**, Theorem 3.1] and of [**Ma3**, Theorem 3.1] shows that we have the equality,

$$\mathcal{T}_{h,g}(K)^{(>0)} = \left[\int_{\mathbf{P}_{(n-1)/2,g}} e_K\left(T\mathbf{P}_{(n-1)/2,g}\right)\mathcal{T}_{h,g}\left(T^H S^E, g^{TS^1}\right)(K)\right]^{(>0)}. \tag{16.54}$$

We may and we will assume that K is generic, so that its eigenvalues $x_1, \ldots, x_n$ are distinct. Let $K^{\mathbf{P}_{(n-1)/2,g}}$ be the vector field on $\mathbf{P}_{(n-1)/2,g}$ induced by K. If K is generic, the zero set of $K^{\mathbf{P}_{(n-1)/2,g}}$ consists of a finite family of distinct points $\left(y_1, \ldots, y_{(n+1)/2}\right)$, represented in $H \setminus \{0\}/\mathbf{C}^*$ by the common non zero eigenvectors of g and K. From (16.54) and the Bott localization formulas [**Bo**], [**BeV**], we get

$$\mathcal{T}_{h,g}(K)^{(>0)} = \sum_{i=1}^{p} \mathcal{T}_{h,g}\left(T^H S^E, g^{TS^1}\right)(K)^{(>0)}_{y_i}. \tag{16.55}$$

Now $\mathcal{T}_{h,g}(K)^{(>0)}_{y_i}$ is exactly the equivariant infinitesimal torsion of S^1 evaluated at (g, K). Using [**BLo1**, Corollary 4.14], Theorem 4.31 and Remark 4.32, or by using

instead Theorem 16.1 in the case $n = 1$, we find that if g, K act on y_j by multiplication by $e^{i\theta_j}, x_j$,

$$\mathcal{T}_{h,g}(K)_{y_j}^{(>0)} = 2^0 I(\theta_i, x_j). \tag{16.56}$$

From (16.55), (16.56), we get

$$\mathcal{T}_{h,g}(K)^{(>0)} = 2 \sum_{j=0}^{(n+1)/2} {}^0 I(\theta_j, x_j). \tag{16.57}$$

From (16.57), we get (16.2) in positive degree when n is odd.

BIBLIOGRAPHY

[ABoP] Atiyah, M.F., Bott, R., Patodi, V.K. – On the heat equation and the index theorem. *Invent. Math.* **19**, 279-330 (1973).

[BeGeV] Berline N., Getzler E., Vergne M. – Heat kernels and Dirac operators. Grundl. Math. Wiss. Band 298. Berlin-Heidelberg-New-York: Springer 1992.

[BeV] Berline N., Vergne M. – Zéros d'un champ de vecteurs et classes caractéristiques équivariantes. *Duke Math. J.*, **50**, 539-549 (1983).

[BerB] Berthomieu, A., Bismut, J.-M. – Quillen metrics and higher analytic torsion forms. *J. Reine Angew. Math.*, **457**, 85-184 (1994).

[B1] Bismut, J.-M. – *Large deviations and the Malliavin calculus.* Progress in Mathematics nř 45. Basel-Boston-Stuttgart: Birkhäuser 1984.

[B2] Bismut, J.-M. – The Atiyah-Singer theorems: a probabilistic approach. I. *J. Funct. Anal.*, **57**, 56-99 (1984). II. *J. Funct. Anal.*, **57**, 329-348 (1984).

[B3] Bismut, J.-M. – The index Theorem for families of Dirac operators : two heat equation proofs. *Invent. Math.* **83**, 91-151 (1986).

[B4] Bismut, J.-M. – Localization formulas, superconnections and the index theorem for families. *Comm. Math. Phys.*, *103*, 127-166 (1986).

[B5] Bismut, J.-M. – Formules de Lichnerowicz et théorème de l'indice. *Proceedings of the Conference in honour of A. Lichnerowicz*, D. Bernard, Y. Choquet-Bruhat eds.. Vol. Géométrie Différentielle, p. 11-31. Travaux en cours. Paris: Hermann 1988.

[B6] Bismut, J.-M. – Superconnection currents and complex immersions. *Invent. Math.*, **99**, 59-113 (1990).

[B7] Bismut, J.-M. – Koszul complexes, harmonic oscillators and the Todd class. *J.A.M.S.* **3**, 159-256 (1990).

[B8] Bismut, J.-M. – Equivariant short exact sequences of vector bundles and their analytic torsion forms. *Comp. Math.*, **93**, 291-354 (1994).

[B9] Bismut, J.-M. – Equivariant immersions and Quillen metrics. *J. Diff. Geom.*, **41**, 53-157 (1995).

[B10] Bismut, J.-M. – Holomorphic families of immersions and higher analytic torsion forms. *Astérisque* **244**. Paris SMF 1997.

[BF1] Bismut, J.-M., Freed D. – The analysis of elliptic families. I. Metrics and connections on determinant bundles. *Comm. Math. Phys.*, **106**, 159-176 (1986).

[BF2] Bismut, J.-M., Freed D. – The analysis of elliptic families. II. Eta invariants and the holonomy theorem. *Comm. Math. Phys.*, **107**, 103-163 (1986).

[BGS1] Bismut, J.-M., Gillet, H., Soulé, C. – Analytic torsion and holomorphic determinant bundles, I. *Comm. Math. Phys.*, **115**, 49-78 (1988). II. *Comm. Math. Phys.*, **115**, 79-126 (1988). III. *Comm. Math. Phys.*, **115**, 301-351 (1988).

[BGS2] Bismut, J.-M., Gillet, H., Soulé, C. – Bott-Chern currents and complex immersions. *Duke Math. Journal*, **60**, 255-284 (1990).

[BGS3] Bismut, J.-M., Gillet, H., Soulé, C. – Complex immersions and Arakelov geometry. *The Grothendieck Festschrift*, P. Cartier et al. ed. Vol. I, pp. 249-331. Progress in Math. n° 86. Boston: Birkhaüser 1990.

[BGo1] Bismut, J.-M., Goette, S. – Holomorphic equivariant analytic torsions. *G.A.F.A.*, **10**, 1289-1422 (2000).

[BGo2] Bismut, J.-M., Goette, S. – Rigidité des formes de torsion analytique en théorie de de Rham. *C.R. Acad. Sci. Paris*, Série I, **330**, 471-477 (2000).

[BGo3] Bismut, J.-M., Goette, S. – Formes de torsion analytique en théorie de de Rham et fonctions de Morse. *C.R. Acad. Sci. Paris*, Série I, **330**, 479-484 (2000).

[BGo4] Bismut, J.-M., Goette, S. – Torsions analytiques équivariantes en théorie de de Rham. *C.R. Acad. Sci. Paris*, Série I, **332**, 33-39 (2001).

[BGo5] Bismut, J.-M., Goette, S. – Equivariant de Rham torsions. Preprint Mathématique Orsay 2001-08 (2001).

[BK] Bismut, J.-M., Köhler, K. – Higher analytic torsion forms for direct images and anomaly formulas. *J. Alg. Geom.*, **1**, 647-684 (1992).

[BL] Bismut, J.-M., Lebeau, G. – Complex immersions and Quillen metrics. *Publ. Math. IHES*, **74**, 1-297 (1991).

[BLo1] Bismut, J.-M., Lott, J. – Flat vector bundles, direct images and higher analytic torsion forms. *J.Am. Math. Soc.*, **8**, 291-363 (1995).

[BLo2] Bismut, J.-M., Lott, J. – Torus bundles and the group cohomology of $GL(N,\mathbb{Z})$. *J. Diff. Geom.*, **47**, 196-236 (1997).

[BZ1] Bismut, J.-M., Zhang, W. – An extension of a Theorem of Cheeger and Müller. *Astérisque*, **205**, 1992.

[BZ2] Bismut, J.-M., Zhang, W. – Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle. *G.A.F.A.*,**4**, 137-212 (1994).

[Bo] Bott, R. – A residue formula for holomorphic vector fields. *J. Differential Geometry*, **1**, 311–330 (1967).

[BoCh] Bott, R., Chern, S.S. – Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections. *Acta Math.*, **114**, 71-112 (1965).

[Bu1] Bunke, U – Higher analytic torsion of sphere bundles and continuous cohomology of $\mathrm{Diff}(S^{2n-1})$. Preprint 1998.

[Bu2] Bunke, U – Equivariant higher analytic torsion and equivariant Euler characteristic. *Amer. J. Math.*, **122**, 377-401 (2000).

[CaE] Cartan, H., Eilenberg, S. – *Homological Algebra.* Princeton: Princeton University Press 1956.

[Ce] Cerf, J. – La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie. *Publ. Math. IHES*, **39**, 5-173 (1970).

[ChP] Chazarain, J., Piriou, A. – *Introduction à la théorie des équations aux dérivées partielles linéaires.* Paris: Gauthiers-Villars 1981.

[C] Cheeger, J. – Analytic torsion and the heat equation. *Ann. of Math.*, **109**, 259-322 (1979).

[CSi] Cheeger, J., Simons, J. – Differential characters and geometric invariants. *Geometry and Topology* (J. Alexander and J. Harer eds), Lecture Notes in Mathematics, vol. 1167, 50-80. Berlin-Heidelberg-New-York: Springer 1985.

[Da] Dai, X. – Adiabatic limits, non multiplicativity of signature and Leray spectral sequence. *J.A.M.S.*, **4**, 265-321 (1991).

[D] Dynkin, E.B. – Diffusion of tensors. *Soviet Math. Doklady*, **9**, 532-535 (1968).

[Ge] Getzler, E. – A short proof of the Atiyah-Singer Index Theorem. *Topology*, **25**, 111-117 (1986).

[GS1] Gillet, H., Soulé, C. – Analytic torsion and the arithmetic Todd genus. *Topology*, **30**, 21-54 (1991).

[GS2] Gillet, H. Soulé, C. – An arithmetic Riemann-Roch theorem. *Invent. Math.*, **110**, 473-543 (1992).

[HSj] Helffer, B., Sjöstrand, J. – Puits multiples en mécanique semi-classique, IV. Etude du complexe de Witten. *Comm. in PDE*, **10**, 245-340 (1985).

[Hö] Hörmander, L. – *The analysis of partial differential operators* I. Grundl. Math. Wiss., Band 256. Berlin-Heidelberg-New-York: Springer 1983.

[I] Igusa, K. – Parametrized Morse theory and its applications. *Proc. Internat. Cong. Math. Kyoto 1990*, 643-651. Math. Soc. Japan: Tokyo 1991.

[Il] Illman, S. – Smooth equivariant triangulations of G-manifolds for G a finite group. *Math. Annal.*, **233**, 199-220 (1978).

[It] Itô, K. – Stochastic parallel displacement. In *Probabilistic methods in differential equations.* Lecture Notes in Mathematics nř 45, 1-7. Berlin-Heidelberg-New-York: Springer 1975.

[K] Klein, J. – Higher Franz-Reidemeister torsion: low-dimensional applications. *Mapping class groups and moduli spaces of Riemann surfaces* (Göttingen, 1991/Seattle, WA, 1991), 195–204, Contemp. Math., 150, Providence: Amer. Math. Soc., 1993.

[KöRoe] Köhler, K., Roessler, D. – A fixed point formula of Lefschetz type in Arakelov geometry. I. Statement and proofs. Preprint IHES/M/98/76, November 1998. To appear.

[La] Laudenbach, F. – On the Thom-Smale complex. Appendix to 'An extension of a theorem by Cheeger and Müller', *Astérisque*, **205**, 219-233 (1992).

[Le] Lerch, M. – Note sur la fonction $\mathfrak{R}(w,x,s) = \sum_0^\infty \frac{e^{2i\pi kx}}{(w+k)^s}$. *Acta Math.*, **11**, 19-24 (1887-1888).

[LoRo] Lott, J., Rothenberg, M. – Analytic torsion for group actions. *J. Diff. Geom.*, **34**, 431-481 (1991).

[Ma1] Ma, X. – Formes de torsion analytique et familles de submersions. Thèse Université Paris-Sud: Orsay 1998.

[Ma2] Ma, X. – Formes de torsion analytique et familles de submersion II. To appear in *Asian J. Math.*.

[Ma3] Ma, X. – Submersions and equivariant Quillen metrics. To appear.

[Ma4] Ma, X. – Functoriality of real analytic torsion forms. To appear.

[MQ] Mathai, V. Quillen, D. – Superconnections, Thom classes, and equivariant Differential forms. *Topology*, **25**, 85-110 (1986).

[MazMe] Mazzeo, R., Melrose, R. – The adiabatic limit, Hodge cohomology and Leray's spectral sequence of a fibration. *J. Diff. Geom.*, **31**, 185-213 (1990).

[MKeS] McKean, H., Singer, I.M. – Curvature and the eigenvalues of the Laplacian. *J. Diff. Geom.*, **1**, 43-69 (1967).

[Mi1] Milnor, J. – Whitehead torsion. *Bull. Am. Math. Soc.*, **72**, 358-426 (1966).

[Mi2] Milnor, J. – *Lectures on the h-Cobordism Theorem.* Princeton: Princeton University Press 1965.

[Mü1] Müller, W. – Analytic torsion and R-torsion of Riemannian manifolds. *Adv. in Math.*, **28**, 233-305 (1978).

[Mü2] Müller, W. – Analytic torsion and R-torsion for unimodular representations. *J. Amer. Math. Soc.*, **6**, 721–753 (1993).

[PSm] Palis, J., Smale, S. – Structural stability theorems. *Global Analysis.* Proc. Symp. Pure Math., Vol. XIV, 223-231. Providence: A.M.S. 1970.

[Q1] Quillen, D. – Superconnections and the Chern character. *Topology* , **24**, 89-95 (1985).

[Q2] Quillen, D. – Determinants of Cauchy-Riemann operators over a Riemannian surface. *Funct. Anal. Appl.*, **19**, 31-34 (1985).

[RS1] Ray, D.B., Singer, I.M. – R-torsion and the Laplacian for Riemannian manifolds. *Adv. in Math.*, **7**, 145-210 (1971).

[RS2] Ray, D.B., Singer, I.M. – Analytic torsion for complex manifolds. *Ann. of Math.*, **98**, 154-177 (1973).

[ReSi] Reed, M., Simon, B. – *Methods of modern Mathematical Physics*, Vol. IV: Analysis of operators. Boston: Academic Press 1979.

[Rei] Reidemeister, K. – Homotopieringe und Linsenraüm. *Hamburger Abhandl.*, **11**, 102-109 (1935).

[Ro] Rothenberg, M. – Torsion invariants and finite transformation groups. In *Algebraic and Geometric Topology*, Part I, 267-311. Proceedings Symp. Pure Math. **32**. Providence: AMS 1978.

[Sm1] Smale, S. – On gradient dynamical systems. *Ann. of Math.*, **74**, 199-206 (1961).

[Sm2] Smale, S. – Differentiable dynamical systems. *Bull. Am. Math. Soc.*, **73**, 747-817 (1967).

[T] Taylor, M. – *Pseudodifferential operators.* Princeton: Princeton University Press 1981.

[Th] Thom, R. – Sur une partition en cellules associée à une fonction sur une variété. *C.R. Acad. Sci. Paris*, **228**, 973-975 (1949).

[V] Varadhan, S.R.S. – Diffusion processes in a small time interval. *Comm. Pure Appl. Math.*, **20**, 659-685 (1967).

[We] Weil, A. – *Elliptic functions according to Eisenstein and Kronecker.* Erg. Math. Grenzg. 88. Berlin-Heidelberg-New-York: Springer 1976.

[W] Witten, E. – Supersymmetry and Morse theory. *J. Diff. Geom.*, **17**, 661-692 (1982).

INDEX