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TEIMURAZ PIRASHVILI

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POLYNOMIAL FUNCTORS OVER FINITE FIELDS
[after Franjou, Friedlander, Henn, Lannes, Schwartz, Suslin]

by **Teimuraz PIRASHVILI**

INTRODUCTION

Let K be a field and let **Vect** be the category of vector spaces over K. Let $\mathcal{V}$ be the full subcategory of **Vect** consisting of finite dimensional vector spaces and let $\mathcal{F}$ be the category of all covariant functors $\mathcal{V} \to \mathbf{Vect}$. A functor $F : \mathcal{V} \to \mathcal{V}$ is called a *polynomial functor* if the maps $\mathrm{Hom}(V, W) \to \mathrm{Hom}(FV, FW)$ given by $f \mapsto F(f)$ are polynomial for all V and W. If these maps are homogeneous of degree d, then F is called *homogeneous of degree d.* Let us recall that a map $g : X \to Y$ between finite dimensional vector spaces is called *polynomial* if after choosing bases it is given by m polynomials with n variables, where $n = \dim X$ and $m = \dim Y$. Over finite fields there is an essential difference between polynomials and maps obtained by polynomials. This yields another version of polynomial functors – strict polynomial functors, which was recently proposed by Friedlander and Suslin [14]. There is a functor $\mathcal{P} \to \mathcal{F}$ from the category of strict polynomial functors to the category of functors, which is no longer an embedding. The category $\mathcal{F}$ is closely related to the category of unstable modules over the Steenrod algebra thanks to the remarkable result of Henn, Lannes and Schwartz [16], while the category $\mathcal{P}$ is related to the theory of polynomial representations of the general linear group.

The aim of this talk is to describe some important properties of the categories $\mathcal{F}$ and $\mathcal{P}$. We start with some general remarks on the category $\mathcal{F}$ and then we describe relationship between the category $\mathcal{F}$ and the category of unstable modules over the Steenrod algebra. In the section 2 we deal with the category $\mathcal{P}$ and its relations with the category of polynomial representation of GL_n. In the next section we discuss the theorem of Franjou-Friedlander-Scorichenko-Suslin [13] on the comparison between $\mathrm{Ext}_{\mathcal{P}}$ and $\mathrm{Ext}_{\mathcal{F}}$. In the last section we discuss the Betley-Pirashvili conjecture [2] involving Waldhausen's stable K-theory. Recently this conjecture was proved for finite fields by Betley [1] and Suslin (see Appendix of [13]) independently.

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1. THE CATEGORY $\mathcal{F}$

1.1. General Properties

In what follows $\otimes_K$ and Hom_K are denoted by $\otimes$ and Hom respectively. For any vector space V one denotes the dual vector space by $V^{\diamondsuit}$.

The category $\mathcal{F}$ of all functors from the category $\mathcal{V}$ of finite dimensional vector spaces to the category of all vector spaces is an abelian category. For any $V \in \mathcal{V}$ one defines $P_V := K[\mathrm{Hom}(V,-)]$. Here $K[S]$ denotes the free vector space generated by a set S. For any $F \in \mathcal{F}$ one has a natural isomorphism $\mathrm{Hom}_{\mathcal{F}}(P_V, F) \cong F(V)$ thanks to the Yoneda lemma and therefore P_V is a projective object in $\mathcal{F}$.

For an object $F \in \mathcal{F}(K)$ one defines the *dual* DF of F by $V \mapsto (F(V^{\diamondsuit}))^{\diamondsuit}$. Let $J_W = DP_W$. Then $\mathrm{Hom}_{\mathcal{F}}(F, J_V) \cong (DF)(V)$ for all $F \in \mathcal{F}$. Hence J_W is injective. Moreover P_V and J_V, $V \in \mathcal{V}$, are respectively projective and injective generators of the category $\mathcal{F}$. Clearly $P_V \otimes P_W \cong P_{V \oplus W}$ and $J_V \otimes J_W \cong J_{V\oplus W}$.

Let us recall the definition of the degree of functors due to Eilenberg and Mac Lane [10]. Let $F \in \mathcal{F}$ be a functor with $F(0) = 0$. The *second cross-effect* of F is given by

$$F(X \mid Y) := ker(F(X \oplus Y) \longrightarrow F(X) \oplus F(Y)).$$

Clearly $F(X \oplus Y) \cong F(X) \oplus F(Y) \oplus F(X \mid Y)$. In order to define the third cross-effect of F, one can consider the second cross-effect $F(X \mid Y)$ as a functor on X and then take the second cross-effect of it. In this way one gets all higher cross-effects. One easily shows that the n-th cross-effect $F(X_1 \mid \cdots \mid X_n)$ of F is the same as the kernel of the natural homomorphism:

$$F(X_1 \oplus \cdots \oplus X_n) \to \bigoplus_{i=1}^{n} F(X_1 \oplus \cdots \oplus \hat{X}_i \oplus \cdots \oplus X_n).$$

A functor F is called of *degree* n if the $(n+1)$-st cross-effect of F vanishes and the n-th cross-effect is nonzero. In this case we write $deg(F) = n$. We let $\mathcal{F}_n$ be the category of all functors $F \in \mathcal{F}$ of degree $\leq n$. The examples of functors of degree n are functors T^n, S^n, Λ^n, Γ^n. Here $T^n(V)$ (resp. S^nV, Λ^nV, Γ^nV) is the n-th tensor (resp. symmetric, exterior, divided) power of V. Clearly $DS^n \cong \Gamma^n$, while $D\Lambda^n \cong \Lambda^n$ and $DT^n \cong T^n$.

A functor F is called *analytic* if it is an union of subfunctors of finite degrees. We let $\mathcal{F}_\omega$ be the category of all analytic functors. The functors $T^*(V) = \bigoplus_{d\geq 0} T^d(V)$, $\Lambda^*(V)$, $\Gamma^*(V)$ are examples of analytic functors.

If K is a finite field, then any map between finite dimensional vector spaces is polynomial. A consequence of this phenomenon is the fact that over such fields the functor J_V is analytic for all $V \in \mathcal{V}$. Indeed, since $J_V \cong J_K^{\otimes(\dim V)}$ it is sufficient to consider only J_K. Clearly $J_K(V) \cong Maps(V^{\diamondsuit}, K)$ and therefore it has a natural commutative algebra structure. Moreover if q is the number of elements in K, then the equality $a^q = a$ holds in $J_K(V)$. Hence the natural embedding $V \subset Maps(V^{\diamondsuit}, K)$

yields a homomorphism of algebras $S_q^*(V) \to J_K(V)$. It is not difficult to prove that this homomorphism is an isomorphism. Here $S_q^*(V)$ is the quotient of the symmetric algebra $S^*(V)$ by the ideal generated by $v^q - v, v \in V$. Since the quotient of an analytic functor is analytic, J_V is analytic too. Thus for finite fields the category $\mathcal{F}_\omega$ is an abelian category with sufficiently many injective objects.

1.2. The quotient category $\mathcal{F}_d/\mathcal{F}_{d-1}$

In this section we assume for simplicity that K is a prime field, although almost everything works for general rings (see [26]).

Let $F \in \mathcal{F}_n$. Since $F(X_1 \mid \cdots \mid X_n)$ is additive with respect to each variable we see that

$$F(X_1 \mid \cdots \mid X_n) \cong (cr_n F) \otimes (X_1 \otimes \cdots \otimes X_n),$$

where $cr_n F := F(K \mid \cdots \mid K) \subset F(\bigoplus_{i=1}^n K)$. The action of the symmetric group Σ_n on $\bigoplus_{i=1}^n K$ by permutation of summands yields an action of Σ_n on $cr_n F$. In this way one obtains an exact functor $cr_n : \mathcal{F}_n \to \Sigma_n\text{-}mod$. Clearly $cr_n F = 0$ if and only if $F \in \mathcal{F}_{n-1}$. For any $M \in \Sigma_n\text{-}mod$, one denotes by $M^\flat$ and $M^\sharp$ the functors given by $M^\flat(X) = (M \otimes X^{\otimes n})_{\Sigma_n}$ and $M^\sharp(X) := (M \otimes X^{\otimes n})^{\Sigma_n}$. Here for a Σ_n-module A, we let A_{Σ_n} and A^{Σ_n} be the coinvariants and invariants under the action of Σ_n. These functors are related by the *norm* homomorphism $N : M^\flat \to M^\sharp$, which is induced by the action of $\sum_{\sigma \in \Sigma_n} \sigma$.

Proposition 1.1 ([26]). — *The functors $M \mapsto M^\flat$ and $M \mapsto M^\sharp$ define left and right adjoints to the functor $cr_n : \mathcal{F}_n \to \Sigma_n\text{-}mod$. Moreover $cr_n M^\flat \cong M \cong cr_n M^\sharp$. Thus the kernels and cokernels of the natural maps $(cr_n F)^\flat \to F$, $F \to (cr_n F)^\sharp$ are functors of degree $< n$. For any $F \in \mathcal{F}_n$ the composition $(cr_n F)^\flat \to F \to (cr_n F)^\sharp$ coincides with the norm homomorphism.*

Since the norm homomorphism is an isomorphism over the rationals it follows that in this case for any functor F of degree d one has an isomorphism $F \cong cr_d(F)^\flat \oplus F'$, where $deg(F') \le d-1$. Using this fact it is not difficult to prove that any functor of finite degree has a unique decomposition $\bigoplus_d F_d$, where F_d is of the form $M^\flat$ and M is a representation of Σ_d. Moreover any natural transformation between functors F_d and F_l for $d \neq l$ vanishes. Hence over the rationals one has an equivalence of categories $\mathcal{F}_n \cong \prod_{i=1}^n (\Sigma_i\text{-}mod)$ (see also Appendix A of [24]).

For $char(K) > 0$ we only have an equivalence of categories $\mathcal{F}_n/\mathcal{F}_{n-1} \cong \Sigma_n\text{-}mod$.

If M is a simple representation of Σ_n then the image of the norm homomorphism $M^\flat \to M^\sharp$ is a simple object in $\mathcal{F}$ and any simple object of $\mathcal{F}$ is of this form. This is a formal consequence of Proposition 1.1 (compare with Proposition 4.7 of [21]). Recently Piriou and Schwartz [30] proved that for any simple functor $F \in \mathcal{F}$ one has $\mathrm{Ext}^1_{\mathcal{F}}(F, F) = 0$, unlike to the situation with representations of symmetric groups.

1.3. Unstable modules over the Steenrod algebra and $\mathcal{F}$

In the 90's Henn, Lannes and Schwartz [16] discovered a remarkable link between the category $\mathcal{F}(\mathbf{F}_p)$ and representations of the Steenrod algebra. For simplicity we consider only the case $K = \mathbf{F}_2$.

We let $\mathcal{A}$ denote the mod-2 Steenrod algebra. Let us recall that it is generated by the elements Sq^i of degree i, $i > 0$ and these generators satisfy the Adem relations (see [S1]). A (graded) module $M^* = \bigoplus_{i\geq 0} M^i$ over $\mathcal{A}$ is said to be *unstable* if for every $x \in M$ one has $Sq^i x = 0$ if $i > |x|$. Here $|x|$ denotes the degree of x. Note that for any space X, the mod-2 cohomology $H^*(X)$ is an example of an unstable $\mathcal{A}$-module. Let $\mathcal{U}$ be the category of unstable $\mathcal{A}$-modules. One observes that for any $n \geq 0$ there exists a unique (up to an isomorphism) unstable $\mathcal{A}$-module $F(n)$ such that for any $M^* \in \mathcal{U}$ one has a natural isomorphism

$$\mathrm{Hom}_{\mathcal{U}}(F(n), M^*) \cong M^n.$$

An unstable A-module M^* is called *nilpotent* if for any $x \in M$ there exists $k > 0$ such that $(Sq_0)^k x = 0$. Here $Sq_0 : M^n \to M^{2n}$ is given by $m \mapsto Sq^n m$ and $(Sq_0)^k$ denotes the k-th iteration of Sq_0. Let us notice that $H^*(X)$ is nilpotent as an unstable module if and only if $H^*(X)$ is nilpotent as a commutative algebra.

If V is a finite dimensional vector space, then by H_*V and H^*V we denote the homology and cohomology of V considered as an abelian group. Let us recall that one has the following isomorphisms of functors

$$H^*(V) \cong S^*(V^\diamondsuit[1]),\ H_*(V) \cong \Gamma^*(V[1]).$$

Here $V[1]$ indicates the fact that V is concentrated in degree 1. Since H^*V is an unstable module, any $\theta \in \mathcal{A}$ yields a morphism $S^n \to S^{n+|\theta|}$ and therefore it induces by duality a natural transformation $\Gamma^{n+|\theta|} \to \Gamma^n$. Hence for any $F \in \mathcal{F}$ the graded vector space $\mathbf{m}(F) = \bigoplus_n \mathrm{Hom}_{\mathcal{F}}(\Gamma^n, F)$ has a natural $\mathcal{A}$-module structure and $\mathbf{m}(F) \in \mathcal{U}$. In this way one obtains a functor $\mathbf{m} : \mathcal{F} \to \mathcal{U}$. It follows from the very definition that

$$\mathbf{m}J_V = \bigoplus_n \mathrm{Hom}_{\mathcal{F}}(\Gamma^n, J_V) \cong \bigoplus_n (\Gamma^n V)^\diamondsuit = S^*V.$$

The functor $\mathbf{m}$ has a left adjoint $\mathbf{f} : \mathcal{U} \to \mathcal{F}$. So one has an isomorphism

$$\mathrm{Hom}_{\mathcal{F}}(\mathbf{f}M^*, F) \cong \mathrm{Hom}_{\mathcal{U}}(M^*, \mathbf{m}F).$$

Putting $F = J_V$ one obtains $(\mathbf{f}(M^*)(V))^\diamondsuit \cong \mathrm{Hom}_{\mathcal{U}}(M^*, S^*V)$. One observes that if M^* is a finitely generated $\mathcal{A}$-module then $\mathrm{Hom}_{\mathcal{U}}(M^*, S^*V)$ is a finite dimensional vector space. Hence in this case $\mathbf{f}(M^*)(V)$ is isomorphic to $(\mathrm{Hom}_{\mathcal{U}}(M^*, S^*V))^\diamondsuit$. In general M^* is a union of finitely generated $\mathcal{A}$-modules and therefore $\mathrm{Hom}_{\mathcal{U}}(M^*, S^*V)$ has a structure of a profinite vector space, whose continuous dual is denoted by $\mathrm{Hom}_{\mathcal{U}}(M^*, S^*V)'$. Since $\mathbf{f}$ preserves colimits one has the following isomorphism

$$\mathbf{f}(M^*)(V) \cong \mathrm{Hom}_{\mathcal{U}}(M^*, S^*V)'.$$

For example, $\mathbf{f}(F(n)) = \Gamma^n$.

THEOREM 1.2 ([16]). — *The functor* $\mathbf{f}$ *is exact and preserves tensor products. For any* $V \in \mathcal{V}$ *one has a natural isomorphism* $\mathbf{f}(S^*V) \cong J_V$. *Furthermore the values of* $\mathbf{f}$ *are analytic functors and for any analytic* F *the natural map* $\mathbf{f}(\mathbf{m}(F)) \to F$ *is an isomorphism. An object* $M^* \in \mathcal{U}$ *is nilpotent if and only if* $\mathbf{f}(M^*) = 0$.

Let $\mathcal{N}$ be the category of nilpotent unstable modules. Obviously it is a Serre subcategory in $\mathcal{U}$. Let us note that the theorem implies the equivalence of categories $\mathcal{U}/\mathcal{N} \cong \mathcal{F}_\omega$.

The original proof of this theorem used deep structural results on injective objects in the category of unstable modules due to Carlsson, Miller, Lannes, Zarati and others. Kuhn [20] gave a reasonably simple proof of Theorem 1.2. His proof can be divided in several steps. The first one is the following important result.

THEOREM 1.3 ([20]). — *Let* $F \in \mathcal{F}$ *be a finite functor. Then one can embed* F *in a sum of the form* $\bigoplus_{i=1}^k S^{n_i}$.

Here a functor $T \in \mathcal{F}$ is called *finite* if it is of finite degree and has values in $\mathcal{V}$. The proof of Theorem 1.3 consists of several reductions. One observes that 2nd power map and multiplication in the symmetric algebra yield a natural transformation: $S^i \otimes S^j \to S^{i2^r+j}$, which is a monomorphism for $2^r > j$. Hence the class of functors for which Theorem 1.3 is true is closed with respect to tensor products. One observes also that F embeds in a finite sum of J_V, because $J_V, V \in \mathcal{V}$ are injective cogenerators. Since $J_V = J_K^{\otimes(\dim V)}$ and F has finite degree it suffices to consider the case when F is the image of S^m into $J_K, m \geq 1$ under the projection $S^*(V) \to S_2^*(V)$, because any sub-functor of finite degree of J_K is of such type. Using a combinatorial argument Kuhn was able to give an explicit embedding in this case (see [20]).

The second step in the proof of Theorem 1.2 is the following

PROPOSITION 1.4. — *The full subcategory of* $\mathcal{F}$, *whose objects are* Γ^n, $n \geq 0$, *and the full subcategory of* $\mathcal{U}$, *whose objects are* $F(n)$, $n \geq 0$ *are isomorphic categories. The isomorphism takes* Γ^n *to* $F(n)$.

Sketch of the proof. — Let $a \in \mathcal{A}$ be a homogeneous element of degree m. It yields a natural transformation $a^* : S^n \to S^{n+m}$, thanks to the isomorphism $S^*(V[1]) \cong H^*(V^\diamond)$. It is not too difficult to show that in this way we get a homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(F(n+m), F(n)) \to \mathrm{Hom}_{\mathcal{F}}(S^n, S^{n+m})$$

and counting argument shows that this map is an isomorphism. □

The next step is the following variant of the Gabriel-Popescu theorem on abelian categories. Let us recall that a *ringoid* is a small category with abelian group structure on Hom-sets, such that the composition is biadditive. Clearly a ringoid with one object is the same ring considered as a category with one object. Moreover any small subcategory of an additive category is a ringoid. A *right module* over a ringoid $\mathbf{C}$ is a contravariant functor $M : \mathbf{C} \to Ab$ with the property that the maps $\mathrm{Hom}_{\mathbf{C}}(c,d) \to \mathrm{Hom}(M(d), M(c))$ given by $f \mapsto M(f)$ are homomorphisms of abelian groups, $c, d \in \mathbf{C}$. For a ringoid $\mathbf{C}$ we let *mod*-$\mathbf{C}$ be the category of all right $\mathbf{C}$-modules.

THEOREM 1.5 ([20]). — *Let* $\mathbf{A}$ *be an abelian category with exact directed colimits and let* $\mathbf{C} \subset \mathbf{A}$ *be a small subcategory, such that any object in* $\mathbf{A}$ *is a quotient of an object of the form* $\bigoplus c_i$, *where* $c_i \in \mathbf{C}$. *Let* $r : \mathbf{A} \to mod\text{-}\mathbf{C}$ *be the functor given by* $r(a)(c) = Hom_{\mathbf{A}}(c,a)$. *Then*

i) *The functor* r *has a left adjoint* $l : mod\text{-}\mathbf{C} \to \mathbf{A}$.

ii) *The functor* l *is exact,* r *is fully faithful and* $lra \to a$ *is an isomorphism for any* $a \in \mathbf{A}$. *Hence in this case* $mod\text{-}\mathbf{C}/Ker(l) \cong \mathbf{A}$.

iii) $\mathbf{A}$ *has enough injectives and* r *preserves injectives. Moreover if* I *and* J *are injective in* $\mathbf{A}$ *then* r *yields an isomorphism*

$$\mathrm{Hom}_{\mathbf{A}}(I,J) \cong \mathrm{Hom}_{mod-\mathbf{C}}(rI, rJ).$$

If additionally all objects from $\mathbf{C}$ *are small and projective in* $\mathbf{A}$ *then* r *(and hence* l*) is an equivalence of categories.*

Now we are in the position to give a sketch of the proof of Theorem 1.2. First one takes $\mathbf{A} = \mathcal{U}$ and $\mathbf{C} = \{F(n)\}_{n\geq 0}$. Here $\{F(n)\}_{n\geq 0}$ is the full subcategory of $\mathcal{U}$ whose objects are $F(n)$, $n \geq 0$. The last statement of Theorem 1.5 shows that $\mathcal{U} \cong mod\text{-}\{F(n)\}_{n\geq 0}$. Then we put $\mathbf{A} = \mathcal{F}_\omega$ and take $\mathbf{C} = \{\Gamma^n\}_{n\geq 0}$, where $\{\Gamma^n\}_{n\geq 0}$ is the full subcategory of $\mathcal{F}_\omega$ whose objects are $\Gamma^n, n \geq 0$. By Proposition 1.4 one has an equivalence of categories $mod\text{-}\mathbf{C} \cong \mathcal{U}$. It follows from Theorem 1.3 that the condition of Theorem 1.5 holds. Applying Theorem 1.5 we obtain an adjoint pair of functors (l, r) between the category $\mathcal{U}$ and $\mathcal{F}_\omega$. One checks readily that $r = \mathbf{m}$ and we get all statements of Theorem 1.2 except the statement about the tensor product and the characterization of nilpotent modules. We refer to [20] and [34] for the proof of these facts.

Remark 1.6. — By Proposition 1.4 one can reconstruct the category $\mathcal{U}$ and the Steenrod algebra (without the Bokstein operations) from the endomorphism ring $End_{\mathcal{F}}(S^*, S^*)$, which has a meaning in a far more general setting. Based on this observation an appropriate notion of "Steenrod algebra over a finite field K" was developed in [20] in such a way that the whole material in 1.3 works for any finite field.

Remark 1.7. — Among other applications of Theorem 1.2 we would like to mention the important article of Henn-Lannes-Schwartz [17] and also the work of Kuhn [23] and Schwartz [35] on topological realizability of unstable modules over the Steenrod algebra.

1.4. Artinian conjecture

The following conjecture was posed by L. Schwartz.

If K is a finite field then the category $\mathcal{F}$ is locally noetherian.

Here is another formulation:

If K is a finite field and $V \in \mathcal{V}$ then the injective object J_V is artinian.

It is known that the conjecture is true when $\dim V = 1$. As was proved by Powell [31] the conjecture is also true when $K = \mathbf{F}_2$ and $\dim V = 2$.

The conjecture implies the following assertions

Any injective object in $\mathcal{F}_\omega$ is also injective in $\mathcal{F}$ and the functor $\mathbf{f} : \mathcal{U} \to \mathcal{F}$ respects injective objects.

2. THE CATEGORY $\mathcal{P}$

In this section K is a field. We give several equivalent interpretations of the category of *strict polynomial functors*. We start with recalling the definition of Schur algebras.

2.1. Polynomial representations of GL_n and Schur algebras.

Let $GL_n = GL_{n,K}$ be the general linear group over K considered as an algebraic group. We recall that the coordinate ring $K[GL_n]$ is the quotient of the polynomial algebra $K[x_{ij}, y]_{1\leq i,j\leq n}$ by the ideal generated by $(y\, det(x_{ij}) - 1)$. It has a Hopf algebra structure, whose comultiplication is given by $\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}$. Moreover we let M_n be the algebraic monoid of $n \times n$-matrices. The coordinate ring $K[M_n]$ is the polynomial algebra $K[x_{ij}]_{1\leq i,j\leq n}$. Clearly it is a subbialgebra of $K[GL_n]$. Comodules over $K[GL_n]$ are called *rational representations* of Gl_n. A rational representation of GL_n is called *polynomial* if it is also a comodule over $K[M_n]$. One observes that for each $d \geq 0$ the subspace $A(n,d) \subset k[M_n]$ of homogeneous polynomials of degree d is a subcoalgebra of $k[M_n]$. The linear dual $A(n,d)^{\diamondsuit}$ of $A(n,d)$ is known as *Schur algebra* and it is denoted by $S(n,d)$ (see [15]). A polynomial representation of GL_n is said to be *homogeneous of degree d* if it is a comodule over $A(n,d)$. Clearly the category of finite dimensional homogeneous polynomial representations of degree d is equivalent to the category of finite dimensional modules over the Schur algebra $S(n,d)$. We refer the reader to [25] and references there for extensive information on Schur algebras.

2.2. Polynomial laws and the category $\mathcal{P}$

For a vector space V we let $\underline{V}$ be the functor from the category $-\ K/alg$ of commutative K-algebras to the category of sets given by $R \mapsto R \otimes V$. A *polynomial law* from a vector space V to a vector space W is by definition a natural transformation of functors $\underline{V} \to \underline{W}$ (see [32]). Let Pol be the category of vector spaces and polynomial laws. A polynomial law $f \in \mathrm{Hom}_{Pol}(V, W)$ is called *homogeneous of degree* d if for any algebra R, $r \in R$ and $x \in R \otimes V$ one has $f_R(rx) = r^d f_R(x)$.

Following Friedlander and Suslin [14] *a strict polynomial functor* $F : \mathcal{V} \to \mathcal{V}$ is a rule which associates to each finite dimensional vector space V a vector space $F(V)$ and to each pair (V, W) of finite dimensional vector spaces a polynomial law

$$F(V, W) \in \mathrm{Hom}_{Pol}(\mathrm{Hom}(V, W), \mathrm{Hom}(F(V), F(W))$$

in such a way that for any $V \in \mathcal{V}$ one has $F(V, V)(1_V) = 1_{F(V,V)}$ and for a any $U, V, W \in \mathcal{V}$ the following diagram of polynomial laws commutes:

$$\begin{array}{ccc}
\mathrm{Hom}(V, W) \times \mathrm{Hom}(U, V) & \to & \mathrm{Hom}(U, W) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(FV, FW) \times \mathrm{Hom}(FU, FV) & \to & \mathrm{Hom}(FU, FW),
\end{array}$$

where the vertical arrows are induced by F, while the horizontal arrows come from the composition.

One says that a strict polynomial functor F is of finite degree if the degrees of the polynomial laws $F(X, Y)$ are bounded above. A strict polyno mial functor F is called homogeneous of degree n if for any $V, W \in \mathcal{V}$ the polynomial $F(V, W)$ is homogeneous of degree n. We let $\mathcal{P}$ (resp. $\mathcal{P}_d$) be the category of strict polynomial functors of finite degree (resp. strict homogeneous polynomial functors of degree d). Since any polynomial law is a sum of homogeneous ones, it follows that $\mathcal{P} \cong \bigoplus_d \mathcal{P}_d$.

2.3. The category $\Gamma^d\mathcal{V}$

The vector space S^dV is the module of coinvariants of $V^{\otimes d}$ under the action of the symmetric group Σ_d on d symbols. Similarly Γ^dV is the module of invariants $\Gamma^d(V) = (V^{\otimes d})^{\Sigma_d}$. For any vector space V one has a map $\gamma^d : V \to \Gamma^d(V)$, which is given by $x \mapsto \gamma^d(x) := x^{\otimes n} \in (V^{\otimes n})^{\Sigma_n}$. Moreover for any vector spaces V and W there exists a unique linear map

$$\mu : \Gamma^dV \otimes \Gamma^dW \to \Gamma^d(V \otimes W)$$

with the property $\mu(\gamma^d(x) \otimes \gamma^d(y)) = \gamma^d(x \otimes y)$. One can use the transformation μ to define the composition in the category $\Gamma^d\mathcal{V}$, whose objects are those of $\mathcal{V}$ while morphisms are $\mathrm{Hom}_{\Gamma^d\mathcal{V}}(V, W) = \Gamma^d(\mathrm{Hom}(V, W))$. The identity arrows in $\Gamma^d\mathcal{V}$ are $\gamma^d(1_V)$. Clearly $\Gamma^d\mathcal{V}$ is a K-linear category, that is a category whose set of morphisms between two objects has a vector space structure and the composition is bilinear.

PROPOSITION 2.1. — *The following categories are equivalent:*

- *the category of K-linear functors $\Gamma^d\mathcal{V} \to \mathcal{V}$,*
- *the category of homogeneous strict polynomial functors of degree d,*
- *the category of homogeneous polynomial representations of degree d of GL_n, provided $n \geq d$,*
- *the category of finite dimensional modules over $S(n, d)$, provided $n \geq d$.*

Proof. — Clearly $\underline{V}$ is isomorphic to the functor $\mathrm{Hom}_{K/alg}(S^*(V^\diamondsuit), -)$ as soon as $V \in \mathcal{V}$. Hence by the Yoneda lemma one has $\mathrm{Hom}_{Pol}(V, W) \cong S^*(V^\diamondsuit) \otimes W \cong \bigoplus_d \mathrm{Hom}(\Gamma^d V, W)$. Elements of $\mathrm{Hom}(\Gamma^d V, W)$ correspond to homogeneous polynomial laws of degree d. This shows that a homogeneous strict polynomial functor of degree d is the same as a collection of $F(V) \in \mathcal{V}, V \in \mathcal{V}$ together with linear maps $\Gamma^d \mathrm{Hom}(V, W) \otimes F(V) \to F(W)$ satisfying associativity and unity conditions. Therefore the first two categories are equivalent. The last two categories are also equivalent by the discussion in 2.1. It remains to show the equivalence between the first and the third category. For each $m \geq 0$ we let $\Gamma^{d,m} \in \mathcal{P}_d$ be the functor given by $V \mapsto \Gamma^d(\mathrm{Hom}(K^m, V))$. By the Yoneda lemma one has an isomorphism $\mathrm{Hom}_{\mathcal{P}}(\Gamma^{d,m}, F) \cong F(K^m)$ for any $F \in \mathcal{P}_d$. So $\Gamma^{d,m}$, $m \geq 0$ are small projective generators in the abelian category $\mathcal{P}_d$. By looking at cross-effects of $\Gamma^{d,m}$ it is clear that each $\Gamma^{d,m}$ is a direct summand of $\Gamma^{d,n}$ provided $m \leq n$ and each $\Gamma^{d,m}$ is a direct summand of an object of the form $\bigoplus_{k=1}^d \Gamma^{d,n}$ provided $n \geq d$. Therefore $\Gamma^{d,n}$ is a projective generator. Thus the category of K-linear functors $\Gamma^d\mathcal{V} \to \mathcal{V}$ is equivalent to the category of finite dimensional modules over $End_{\mathcal{P}}(\Gamma^{d,n}) = \Gamma^d(End(K^n))$. But the last algebra is isomorphic to the Schur algebra $S(n, d)$ and hence the result. □

2.4. Elementary properties of $\mathcal{P}$

Clearly $f \mapsto \gamma^d(f)$ defines the (nonlinear) functor $\gamma^d : \mathcal{V} \to \Gamma^d\mathcal{V}$. The precomposition with γ^d yields the functor $\mathcal{P}_d \to \mathcal{F}_d$, which is a full embedding provided the field K contains at least d elements. By abuse of notation we denote the image of F under this functor by the same letter F.

The natural transformation $\Gamma^{d+l} \to \Gamma^d \otimes \Gamma^l$ coming from the Hopf algebra structure on the divided power algebra can be used to define the tensor product of strict homogeneous polynomial functors. It yields the functor $\mathcal{P}_d \times \mathcal{P}_l \to \mathcal{P}_{k+l}$, which corresponds to the obvious tensor functor in $\mathcal{F}$.

The dual of the natural transformation $S^d(S^l V) \to S^{dl}(V)$ is a transformation $\Gamma^{dl} \to \Gamma^d \circ \Gamma^l$, which can be used to define the composition of strict homogeneous polynomial functors. One observes also that the dual of a strict homogeneous polynomial functor is a strict homogeneous polynomial functor. Since Γ^d carries obvious structure of a strict homogeneous polynomial functor of degree d, we see that $T^d, S^d, \Lambda^d \in \mathcal{P}_d$. As was observed in the proof of Proposition 2.1 the functors $\Gamma^{d,m}$ are projective generators of $\mathcal{P}_d$. Therefore $S^{d,m} := D\Gamma^{d,m}$ are injective cogenerators of $\mathcal{P}_d, m \geq 0$. Another system of projective generators is $\Gamma^{d_1} \otimes \cdots \otimes \Gamma^{d_k}$, $d_1 + \cdots + d_k = d$.

2.5. Frobenius twist

In representation theory of algebraic groups over fields of characteristic $p > 0$ the Frobenius twist plays an important rôle (see [18]). By Proposition 2.1 to the Frobenius twist on polynomial representations corresponds a similar operation on strict polynomial functors, which can be described as follows. Let K be a field of characteristic $p > 0$. For a vector space V we let $V^{(1)}$ be the vector space obtained by extending scalars via the Frobenius homomorphism $f : K \to K$ given by $f(\lambda) = \lambda^p$. In other words $V^{(1)}$ is a vector space generated by $v^{(1)}$, $v \in V$ modulo the following relations

$$(v+w)^{(1)} = v^{(1)} + w^{(1)}, \quad (\lambda v)^{(1)} = \lambda^p v^{(1)}, \ \lambda \in K.$$

One defines $V^{(r)}, r \geq 1$ by induction: $V^{(r+1)} = (V^{(r)})^{(1)}$.

The map $V^{(1)} \to S^p V$ given by $v^{(1)} \mapsto v^p$ is linear. By duality one obtains the natural transformation $\Gamma^p \to I^{(1)}$, where $I^{(1)}(V) = V^{(1)}$. Using this transformation one readily checks that $I^{(1)} \in \mathcal{P}_p$. For any $F \in \mathcal{P}_d$ we put $F^{(1)} = F \circ I^{(1)}$ and then by induction one defines $F^{(r+1)} = (F^{(r)})^{(1)}, r \geq 0$. Clearly $F^{(r)} \in \mathcal{P}_{dp^r}$ if $F \in \mathcal{P}_d$.

Let us note that for finite prime fields the functors $I^{(1)}$ and I are isomorphic in $\mathcal{F}$. Here $I(V) = V$. However in $\mathcal{P}$ these functors are different, they even have different degrees: $I \in \mathcal{P}_1$, while $I^{(1)} \in \mathcal{P}_p$. For finite fields the Frobenius twist induces an equivalence of categories $(-)^{(1)} : \mathcal{F}_d \to \mathcal{F}_d$.

2.6. $\mathrm{Ext}_{\mathcal{P}}$ and cohomology of algebraic groups

The following theorem was proved by Donkin (see Section 3 of [14]).

THEOREM 2.2 ([6, 7]). — *Let A and B be finite dimensional polynomial representations. Then*

$$\mathrm{Ext}^*_{K[M_n]}(A, B) \to \mathrm{Ext}^*_{K[GL_n]}(A, B)$$

is an isomorphism. Here Ext *is taken in the category of comodules.*

By Proposition 2.1 we also have

COROLLARY 2.3. — *Let $F, T \in \mathcal{P}_d$. Then*

$$\mathrm{Ext}^*_{\mathcal{P}}(F, T) \to \mathrm{Ext}^*_{K[GL_n]}(F(K^n), T(K^n))$$

is an isomorphism provided $n \geq d$.

3. FUNCTOR COHOMOLOGY

In this section we deal with Ext-groups in the categories $\mathcal{F}$ and $\mathcal{P}$. Calculations in $\mathcal{P}$ are easier, because each piece $\mathcal{P}_d$ of $\mathcal{P}$ is of finite global dimension thanks to a result of Donkin [6], who proved this result in terms of Schur algebras (see also Totaro [36] for an explicit formula for the global dimension of $\mathcal{P}_d$). Calculation in $\mathcal{P}$ is easier also because the functors $S^{n_1} \otimes \cdots \otimes S^{n_k}$ (resp. $\Gamma^{n_1} \otimes \cdots \otimes \Gamma^{n_k}$) are injective (resp. projective) in $\mathcal{P}$. Of course S^n is no longer injective in $\mathcal{F}$, but the following result shows that if one inverts the Frobenius $S^n \to S^{np}$ then one gets an injective object in $\mathcal{F}_\omega$.

LEMMA 3.1 ([22]). — *Assume $F \in \mathcal{F}$ has a projective resolution of finite type. Then, for any $k > 0$, one has*

$$colim\ \operatorname{Ext}^k_{\mathcal{F}}(T, S^{p^n}) = 0,$$

where S^n denotes the n-th symmetric power and the limit is considered with respect to the Frobenius maps $\Phi : S^{p^n} \to S^{p^{n+1}}$.

Proof. — (see [12]). Let P_* be a projective resolution of F which is of finite type. Then DP_* is an injective resolution of DF. Hence

$$\operatorname{Ext}^k_{\mathcal{F}}(F, S^{2^n}) = H^k(Hom_{\mathcal{F}}(P_*, S^{2^n})) \cong H^k((\mathbf{m}(DP_*))^{2^n}) \cong (H^k(\mathbf{m}(DP_*)))^{2^n}$$

and it is enough to show that $H^k(\mathbf{m}(DP_*)) \in \mathcal{N}$. Since

$$\mathbf{f}(H^k(\mathbf{m}(DP_*))) = H^k(\mathbf{f}(\mathbf{m}(DP_*))) = H^k(DP_*) = 0,\ k > 0$$

Theorem 1.2 implies the expected inclusion. □

Lemma 3.1 together with Lemma 3.3 plays a crucial rôle in this section.

Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $l : \mathbf{C} \to \mathbf{D}$ and $r : \mathbf{D} \to \mathbf{C}$ be functors. If l is a left adjoint to r we will say that (l, r) is an adjoint pair from $\mathbf{C}$ to $\mathbf{D}$. Moreover for a small category $\mathbf{C}$ we let $\mathbf{C}$-*mod* be the category of all functors $\mathbf{C} \to \mathbf{Vect}$.

LEMMA 3.2. — *Let (l, r) be an adjoint pair from $\mathbf{C}$ to $\mathbf{D}$. Assume $\mathbf{C}$ and $\mathbf{D}$ are small categories. Then for any $F : \mathbf{C} \to \mathbf{Vect}$ and $G : \mathbf{D} \to \mathbf{Vect}$ one has an isomorphism*

$$\operatorname{Ext}^*_{\mathbf{D}-mod}(F \circ r, G) \cong \operatorname{Ext}^*_{\mathbf{C}-mod}(F, G \circ l).$$

LEMMA 3.3 ([27]). — *Let $A \in \mathcal{F}$ be an additive functor and let $B : \mathcal{V} \times \mathcal{V} \to \mathbf{Vect}$ be a bifunctor with the property $B(X, 0) = 0 = B(0, X)$ for all $X \in \mathcal{V}$. Then $\operatorname{Ext}^*_{\mathcal{F}}(A, B^d) = 0 = \operatorname{Ext}^*_{\mathcal{F}}(B^d, A)$, where $B^d(V) = B(V, V)$.*

Proof. — Apply the previous lemma to the adjoint pairs $(i_k, p_k) : \mathcal{V} \to \mathcal{V} \times \mathcal{V}$, $k = 1, 2, 3$, where $i_1(V) = (V, 0), i_2(V) = (0, V), i_3(V) = (V, V), p_k(V_1, V_2) = V_k, k = 1, 2, p_3(V_1, V_2) = V_1 \bigoplus V_2$ and use the fact that $A \circ p_3 = A \circ p_1 \bigoplus A \circ p_2$. □

A strict polynomial functor T is called additive if T is additive as an object in $\mathcal{F}$. The previous argument shows that Lemma 3.3 is still true in the framework of $\mathcal{P}$. Actually the proof of Lemma 3.3 gives a bit more:

$$\operatorname{Ext}^*_{\mathcal{E}}(K, T_1 \otimes \cdots \otimes T_n) = 0 = \operatorname{Ext}^*_{\mathcal{E}}(T_1 \otimes \cdots \otimes T_n, K),$$

where $degK < n$, $T_1(0) = \cdots = T_n(0) = 0$ and $\mathcal{E}$ is $\mathcal{P}$ or $\mathcal{F}$.

Let us see how to use these facts for explicit calculations. First we consider calculations in the category $\mathcal{P}$. Since $I \in \mathcal{P}$ is projective we have $\operatorname{Ext}^i_{\mathcal{P}}(I, -) = 0$ if $i > 0$. More interesting is $\operatorname{Ext}^*_{\mathcal{P}}(I^{(r)}, -)$ for $r \geq 1$.

For simplicity we restrict ourselves to the case $p = 2$ and refer the reader to [12], [14] and [13] for the case $p > 2$. We follow ideas of [12].

It is not too hard to show that one has an exact sequence

$$(3.1) \qquad 0 \to S^{2^h(1)} \to S^{2^h} \to \cdots \to S^{2^h-i} \otimes S^i \to \cdots \to S^{2^h} \to 0.$$

Here the first nontrivial map is given by the Frobenius, and the rest are the compositions $S^k \otimes S^l \to S^{k-1} \otimes S^1 \otimes S^l \to S^{k-1} \otimes S^{l+1}$, where first map (resp. second) is induced by the comultiplication (resp. multiplication) in the Hopf algebra $S^*(V)$. By the $\mathcal{P}$-version of Lemma 3.3 $\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, S^k \otimes S^l) = 0$ if $k, l \geq 1$ and a fortiori $\mathrm{Hom}^*_{\mathcal{P}}(I^{(r)}, S^k \otimes S^l) = 0$. Let us note that $\mathrm{Hom}_{\mathcal{P}}(I^{(h)}, S^{2^h}) = \mathrm{Hom}_{\mathcal{F}}(I^{(h)}, S^{2^h})$ is one dimensional spanned by the iterated Frobenius map. Hence it follows from the injective resolution (3.1) that $\mathrm{Ext}^i_{\mathcal{P}}(I^{(h+1)}, S^{2^h(1)}) = K$, if $i = 0$ and $i = 2^{h+1}$ and $\mathrm{Ext}^i_{\mathcal{P}}(I^{(h+1)}, S^{2^h(1)}) = 0$ otherwise. Let us apply the degreewise action by the Frobenious twist on (3.1). Then one obtains a non-injective resolution of $S^{2^h(2)}$, which gives rise to a hypercohomology spectral sequence

$$E^1_{st} = \mathrm{Ext}^t_{\mathcal{P}}(I^{(h+2)}, S^{2^{h+1}-s(1)} \otimes S^{s(1)}) \Longrightarrow \mathrm{Ext}^{s+t}_{\mathcal{P}}(I^{(h+2)}, S^{2^h(2)}).$$

By the $\mathcal{P}$-version of Lemma 3.3 the spectral sequence has only two nontrivial columns corresponding to $s = 0$ and $s = 2^{h+1}$. Moreover the previous calculation shows that in both columns there are only two nontrivial terms corresponding to $t = 0$ and $t = 2^{h+2}$, so there is no space for differentials and we get $\mathrm{Ext}^i_{\mathcal{P}}(I^{(h+2)}, S^{2^h(2)}) = K$, if $i = 0, i = 2^{h+1}, 2^{h+2}, 2^{h+2} + 2^{h+1}$ and $\mathrm{Ext}^i_{\mathcal{P}}(I^{(h+2)}, S^{2^h(2)}) = 0$ otherwise. By iteration one obtains:

PROPOSITION 3.4 ([14]). — *For any* $0 \leq j \leq r$ *one has* $\mathrm{Ext}^q_{\mathcal{P}}(I^{(r)}, S^{2^{r-j}(j)}) = K$ *if* $q \equiv 0 \bmod 2^{r-j+1}$ *and* $q < 2^{r+1}$ *and* $\mathrm{Ext}^q_{\mathcal{P}}(I^{(r)}, S^{2^{r-j}(j)}) = 0$ *otherwise.*

Such type of results was used in [14] to prove the following theorem:

The rational cohomology of any finite group scheme is a finitely generated algebra.

Now we do similar calculations in $\mathcal{F}$.

THEOREM 3.5 ([12]). — $\mathrm{Ext}^i_{\mathcal{F}}(I^{(h)}, S^{2^h}) = K$ *if* $i \equiv$ *mod* 2^{h+1} *and* $= 0$ *otherwise.*

Proof. — The exact sequence (3.1) together with Lemma 3.3 yields the exact sequence

$$\cdots \to \mathrm{Ext}^{k-2^h}_{\mathcal{F}}(I^{(h)}, S^{2^h}) \to$$
$$\to \mathrm{Ext}^k_{\mathcal{F}}(I^{(h)}, S^{2^{h-1}}) \to \mathrm{Ext}^k_{\mathcal{F}}(I^{(h)}, S^{2^h}) \to \mathrm{Ext}^{k-2^h+1}_{\mathcal{F}}(I^{(h)}, S^{2^h}) \to \cdots$$

From this sequence it follows that $\mathrm{Ext}^k_{\mathcal{F}}(I^{(h)}, S^{2^{h-1}}) \to \mathrm{Ext}^k_{\mathcal{F}}(I^{(h)}, S^{2^h})$ is an isomorphism for $k < 2^h$. It is known that any finite functor admits a projective resolution of finite type (see [12]). Thus by Lemma 3.1 one has $\mathrm{Ext}^k_{\mathcal{F}}(I^{(h)}, S^{2^h}) = 0$ as soon as $0 < k < 2^{h+1}$. Now one can use this information in the same exact sequence to get $\mathrm{Ext}^k_{\mathcal{F}}(I^{(h)}, S^{2^h}) = 0$ for $2^{h+1} < k < 2^{h+2}$. By iterating this process one gets the result. $\square$

It is worth to mention that Franjou, Friedlander, Scorichenko and Suslin give in [13] explicit calculations of $\mathrm{Ext}^*(\Gamma^{d(h)}, S^n)$, $\mathrm{Ext}^*(\Gamma^{d(h)}, \Lambda^n)$, $\mathrm{Ext}^*(\Gamma^{d(h)}, \Gamma^n)$ and $\mathrm{Ext}^*(\Lambda^{d(h)}, \Lambda^n)$ in the $\mathcal{F}$ and $\mathcal{P}$ framework. Let us also note that the groups $\mathrm{Ext}^*_{\mathcal{F}}(I, -)$ are known as Mac Lane cohomology (see [19]) and they are dual to the topological Hochschild homology (see [29]). The previous theorem for $h = 0$ was first obtained by Breen [4] in the framework of "Extensions du groupe additif" and Bökstedt [3] in the framework of topological Hochschild homology.

Returning to Proposition 3.4 one sees that the Frobenius twist induces monomorphisms

$$\mathrm{Ext}^i_{\mathcal{P}}(I^{(s+j)}, S^{2^s(j)}) \to \mathrm{Ext}^i_{\mathcal{P}}(I^{(s+j+1)}, S^{2^s(j+1)})$$

which stabilize for each i. This phenomenon is a particular case of the well known fact on cohomology of algebraic groups (see [18], page 347), which by virtue of Corollary 2.3 yields the following

THEOREM 3.6. — *Let $F, T \in \mathcal{P}_d$. Then the Frobenius twist gives a monomorphism $\mathrm{Ext}^i_{\mathcal{P}}(F, T) \to \mathrm{Ext}^i_{\mathcal{P}}(F^{(1)}, T^{(1)})$. Moreover, for all sufficiently large m the map*

$$\mathrm{Ext}^i_{\mathcal{P}}(F^{(m)}, T^{(m)}) \to \mathrm{Ext}^i_{\mathcal{P}}(F^{(m+1)}, T^{(m+1)})$$

is an isomorphism.

By [13] the last assertion holds as soon as $i \leq 2^{m+1} - 1$.

We let $\mathrm{Ext}^i_{Gen}(F, T)$ be the common value of $\mathrm{Ext}^i_{\mathcal{P}}(F^{(m)}, T^{(m)})$, $m \gg 0$. Since the Frobenius twist in $\mathcal{F}_d$ is an equivalence of categories we see that the canonical map $\mathrm{Ext}^*_{\mathcal{P}}(F, T) \to \mathrm{Ext}^*_{\mathcal{F}}(F, T)$ gives rise to the homomorphism $\mathrm{Ext}^*_{Gen}(F, T) \to \mathrm{Ext}^*_{\mathcal{F}}(F, T)$. Now comparing the Proposition 3.4 and Theorem 3.5 one sees that the homomorphism $\mathrm{Ext}^*_{Gen}(I^{(h)}, S^{2^h}) \to \mathrm{Ext}^*_{\mathcal{F}}(I^{(h)}, S^{2^h})$ is an isomorphism. However in general $\mathrm{Ext}^*_{Gen}(F, T) \to \mathrm{Ext}^*_{\mathcal{F}}(F, T)$ is not an isomorphism even when $* = 0$. The following rather surprising result of Franjou, Friedlander, Scorichenko and Suslin gives a condition when it is an isomorphism.

THEOREM 3.7 ([13]). — *Let $T, F \in \mathcal{P}_d$ be strict polynomial functors, then*

$$\mathrm{Ext}^*_{\mathcal{P}}(F, T) \to \mathrm{Ext}^*_{\mathcal{F}}(F, T)$$

is a monomorphism. Moreover if the field K contains at least d elements then

$$\mathrm{Ext}^*_{Gen}(F, T) \to \mathrm{Ext}^*_{\mathcal{F}}(F, T)$$

is an isomorphism.

The proof of the theorem is quite long and can be divided essentially into two parts. In the first part they prove a weaker version (Proposition 3.8) and then they use a base change argument in a very clever way to finish the proof. Here we prove only Proposition 3.8 and refer the interested reader to the original paper for the second part of the proof of Theorem 3.7.

PROPOSITION 3.8. — *Let $F, T \in \mathcal{P}_d$ and let $m \geq 0$. Assume K has at least $2^m d$ elements. Then*

$$\mathrm{Ext}^i_{\mathcal{P}}(F^{(m)}, T^{(m)}) \to \mathrm{Ext}^i_{\mathcal{F}}(F, T)$$

is an isomorphism provided $i \leq 2^{m-d+1} - 2$.

The rest of this section is devoted to the proof of Proposition 3.8.

DEFINITION 3.9 ([11]). — *A graded object $E^* = \bigoplus_n E^n \in \mathcal{E}$, where $\mathcal{E} = \mathcal{F}$ or $\mathcal{P}$ is called exponential if there exist natural isomorphisms $E^0(V) = K, E^*(V \bigoplus W) \cong E^*(V) \otimes E^*(W)$ and the values of E^n lie in $\mathcal{V}$.*

If E^* is an exponential functor, then $E^n(0) = 0$, for $n > 0$ and $deg(E^n) \leq n$. Typical examples of exponential functors are given by the symmetric algebra, the exterior algebra and the divided power algebra. If E^* and $\bar{E}^*$ are exponential functors then $E^* \otimes \bar{E}^*$ is also an exponential functor.

The same argument as in the proof of Lemma 3.3 gives also the following

LEMMA 3.10 ([11, 13]). — *Let E^* be an exponential functor in $\mathcal{E}$. Then for any functors $B, C \in \mathcal{E}$ and any $i \geq 0$ one has isomorphisms of graded vector spaces*

$$\mathrm{Ext}^*_{\mathcal{E}}(E^n, B \otimes C) \cong \bigoplus_{i+j=n} \mathrm{Ext}^*_{\mathcal{E}}(E^i, B) \otimes \mathrm{Ext}^*_{\mathcal{E}}(E^j, C)$$

$$\mathrm{Ext}^*_{\mathcal{E}}(B \otimes C, E^n) \cong \bigoplus_{i+j=n} \mathrm{Ext}^*_{\mathcal{E}}(B, E^i) \otimes \mathrm{Ext}^*_{\mathcal{E}}(C, E^j).$$

LEMMA 3.11 ([11, 13]). — *For any exponential functor E^* and any injective $J \in \mathcal{P}_n$, $n \equiv 0 \bmod 2^h$ one has $\mathrm{Ext}^i_{\mathcal{F}}(E^d, J) = 0$ provided $0 < i \leq 2^{n-d+2} - 2$.*

Proof. — First we exploit the fact that $S^n \in \mathcal{F}$ behaves like an injective with respect to functors of degree $\ll n$. Assume $n = \epsilon_0 + 2\epsilon_1 + \cdots + 2^k \epsilon_k$, $\epsilon_i = 0, 1$ is the 2-adic expansion of a natural number n and let $\alpha(n)$ be the number of nonzero elements of the set $\{\epsilon_0, \cdots, \epsilon_k, \}$. Then S^n is a retract of $(S^1)^{\otimes \epsilon_0} \otimes \cdots \otimes (S^{2^k})^{\otimes \epsilon_k}$. Hence a discussion after Lemma 3.3 shows that $\mathrm{Ext}^*_{\mathcal{F}}(K, S^n \otimes S^m) = 0$ as soon as $degK < \alpha(n) + \alpha(m)$. Based on this fact and using the exact sequence (3.1) it is not too difficult to prove that the Frobenius transformation yields an isomorphism $\mathrm{Ext}^i_{\mathcal{F}}(K, S^{2^n}) \to \mathrm{Ext}^i_{\mathcal{F}}(K, S^{2^{n+1}})$ provided $degK \leq d$ and $i \leq 2^{n-d+2} - 2$. Since any finite functor has a projective resolution of finite type (see [12]), it follows from Lemma 3.1 that for such K one has $\mathrm{Ext}^i_{\mathcal{F}}(K, S^{2^n}) = 0$ provided $degK \leq d$ and $0 < i \leq 2^{n-d+2} - 2$. Lemma 3.10 together with the fact that the functors $S^{n_1} \otimes \cdots \otimes S^{n_k}$, $n_1 + \cdots + n_k = n$ are injective generators in $\mathcal{P}_n$ can be used to finish the proof. $\square$

Proof of Proposition 3.8. — It consists of several reductions. Using a hypercohomology spectral sequence one can prove that it suffices to restrict ourselves to the case when F is a projective object. So we assume that $F = \Gamma^{d_1}\otimes\cdots\otimes\Gamma^{d_k}$, $d_1+\cdots+d_k = d$. Let $T^{(m)} \to J^*$ be an injective resolution in $\mathcal{P}_{d2^m}$. Then one has a hypercohomology spectral sequence

$$E_1^{ij} = \mathrm{Ext}^i_{\mathcal{F}}(F^{(m)}, J^j) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathcal{F}}(F^{(m)}, T^{(m)}).$$

Since $F^{(m)}$ is a direct summand of the exponential functor $(\Gamma^* \otimes \cdots \otimes \Gamma^*)^{(m)}$ one can use Lemma 3.11 to show that $E_1^{ij} = 0$ for all $0 < i \leq 2^{m-d+1} - 2$. So for all $i \leq 2^{m-d+1} - 2$ one has

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{F}}(F^{(m)}, T^{(m)}) &= H^i(\mathrm{Hom}_{\mathcal{F}}(F^{(m)}, J^*)) =\\ &= H^i(\mathrm{Hom}_{\mathcal{P}}(F^{(m)}, J^*)) = \mathrm{Ext}^i_{\mathcal{P}}(F^{(m)}, T^{(m)})\end{aligned}$$

and hence the proposition. □

4. STABLE K-THEORY FOR FINITE FIELDS

4.1. Stable K-theory

Stable K-theory gives the possibility of reducing calculation of homology of the general linear groups with twisted coefficients to the homology of the general linear group with constant coefficients. This trick goes back to Waldhausen. Let R be a ring and let $\mathcal{V}_R$ be the category of finitely generated free R-modules. Moreover we let $\mathcal{B}$ be the category of all functors $\mathcal{V}_R^{op} \times \mathcal{V}_R \to Ab$. For any such functor D the abelian group $D(R^n, R^n)$ has a natural GL_nR-action, so one can consider the corresponding homology groups $H_*(GL_n(R), D(R^n, R^n))$. These groups form a direct system and we let $H_*(GL(R), D)$ be the corresponding limit when $n \to \infty$. It is a classical theorem of Dwyer that this system always stabilizes as soon as R satisfies some finiteness condition and D is of finite degree with respect to both variables (see [9]). Let us denote $H_0(GL(R), D)$ by $K_0^{st}(R, D)$. The functor

$$K_0^{st}(R, -) : \mathcal{B} \to Ab$$

is a right exact functor, whose left derived functors are denoted by $K_*^{st}(R, -)$ and called the *stable K-theory* of R. Similarly one can introduce the *cohomological version of stable K-theory* $K^*_{st}(R, -) : \mathcal{B} \to Ab$ as a left derived functor of the functor $K^0_{st}(R, -)$. Here $K^0_{st}(R, D) = H^0(GL(R), D)$ for any $D \in \mathcal{B}$. It was proved in [2] that in this way one recovers the original definition of Waldhausen for some class of rings including all fields and for such rings one has the following direct sum decomposition (see [2])

$$H_n(GL(R), D) \cong \bigoplus_{i+j=n} H_i(GL(R), K_j^{st}(R, D)),$$

where **the action of $GL(R)$ on K^{st} is trivial**. It was conjectured in [2] that the groups $K_*^{st}(R,D)$ are isomorphic to the homology $H_*(\mathcal{V}_R, D)$ of the category $\mathcal{V}_R$ with the coefficients in the bifunctor D provided D is of finite degree with respect to both variables. This conjecture is much stronger compared to a previous conjecture from [28] (see also [19], page 293), which was proved by Dundas and McCarthy [8]. We refer the reader to [29] for an explicit definition of $H_*(\mathcal{V}_R, D)$. For purposes of these notes it suffices to note that $H_*(\mathcal{V}_R, D) \cong (\mathrm{Ext}^*_{\mathcal{F}}(T,F))^{\diamondsuit}$ if $D(X,Y) = \mathrm{Hom}_R(FX,TY)$. Here $T, F \in \mathcal{F}(R) := (R-mod)^{\mathcal{V}_R}$ and the values of F lie in $\mathcal{V}_R$. By [2] the conjecture is true for a given ring R if and only if it is true for such bifunctors. Quillen proved in [33] that for a finite field K the (co)homology of $GL(K)$ with coefficients in K is trivial. Thus for a finite field K one has

$$K_*^{st}(K,D) \cong H_*(GL(K),D), \quad K^*_{st}(K,D) \cong H^*(GL(K),D)$$

and hence the following theorem proved by Betley and Suslin independently solves our conjecture for finite fields.

THEOREM 4.1. — *Let K be a finite field and let F and T be strict homogeneous functors of degree d. Then for each $i \geq 0$ the natural map*

$$\mathrm{Ext}^i_{\mathcal{F}}(F,T) \to \mathrm{Ext}^i_{GL_n(K)}(F(K^n), T(K^n))$$

is an isomorphism provided n is big enough.

We give here a sketch of the proof following Suslin (Appendix of [13]), see also [1] for a different argument.

First let us consider the case when $F = T = I$. In this case the theorem follows from the result of Dundas and McCarthy [8]. Following Friedlander and Suslin it can be proved also using the famous result of [5]. In our situation it says that for a fixed $m \geq 0$ there exist numbers $t(m)$ and $d(m)$ such that if $r \geq t(m)$ and $[K : \mathbf{F}_p] \geq d(m)$ then

$$\mathrm{Ext}^i_{\mathcal{P}}(I^{(r)}, I^{(r)}) \to H^i(GL_n(K), M_n(K))$$

is an isomorphism for all $i \leq m$. Here $M_n(K)$ is the adjoint representation of $GL_n(K)$. Now comparing this result to Theorem 3.7 we see that for a fixed m Theorem 4.1 is true for $F = T = I$, $i \leq m$ if the field is big enough. In order to handle small fields one uses the following trick. Since the $GL_n(K)$-module $M_n(K)$ is selfdual, one can pass to homology because $H_i(GL_n(K), M_n(K)) \cong H^i(GL_n(K), M_n(K))^{\diamondsuit}$. Now in homology we have a possibility to change a field as follows.

Let $K \to L$ be an extension of finite fields of degree e. Then it yields a canonical homomorphism

$$can : H_*(GL(K), M(K)) \to H_*(GL(L), M(L)),$$

where $M(K) = colim M_n(K)$. Choosing a basis for L over K one gets an embedding $L \subset M_e(L)$ and hence a homomorphism $H_*(GL_n(L), M_n(L)) \to H_*(GL_{en}(K), M_{en}(K))$ and therefore

$$Tr : H_*(GL(L), M(L)) \to H_*(GL(K), M(K)).$$

PROPOSITION 4.2. — *The composition*

$$Tr \circ can : H_*(GL(K), M(K)) \to H_*(GL(K), M(K))$$

coincides with the multiplication by $e = [L : K]$, while the composition

$$can \circ Tr : H_*(GL(L), M(L)) \to H_*(GL(L), M(L))$$

coincides with $\sum_{\sigma \in Gal(L/K)} \sigma_*$.

In order to finish the case $F = T = I$ we choose d sufficiently big with respect to m and prime to p and consider an extension $K \to L$ of degree d. Then for L the theorem is true in dimensions $i \leq m$. By Theorem 3.5 (for $h = 0$) we have $H_i(GL_n(L), M_n(L)) = 0$ for odd $i \leq m$ and $H_i(GL_n(L), M_n(L)) = L$ for even $i \leq m$. Based on Proposition 4.2 it is not too difficult to see that the same relations hold for $H_i(GL_n(K), M_n(K))$. Hence Theorem 4.1 is true for $F = T = I$.

Following Suslin, for a functor $F \in \mathcal{F}$ we let $aF \in \mathcal{F}$ be the functor which is given on objects by

$$(aF)(V) = \bigoplus_{W \subset V} F(V/W).$$

If $f : V \to V'$ is a linear map, then the W'-component of the induced map

$$(aF)(f) : \bigoplus_{W \subset V} F(V/W) \to \bigoplus_{W' \subset V'} F(V'/W')$$

is zero on all W except $W = f^{-1}(W')$. In this case the corresponding component of $(aF)(f)$ is given by $f_* : F(V/f^{-1}(W')) \to F(V'/W')$.

PROPOSITION 4.3. — *Assume $F \in \mathcal{F}$ has a projective resolution of finite type and let $T \in \mathcal{F}$ be a finite functor. Then one has a natural isomorphism*

$$\mathrm{Ext}^*_{GL(K)}(F, T) \to \mathrm{Ext}^*_{\mathcal{F}}(F, aT).$$

Proof. — By a hypercohomology spectral sequence argument it suffices to consider only the case when $F = P_V$ is a standard projective and $T = J_W$ is a standard injective. One can prove the claim in dimension zero by direct considerations, while in dimensions > 0 it follows from the fact that the corresponding bifunctor $D(X, Y) = \mathrm{Hom}(FX, TY)$ is injective in $\mathcal{B}$ and hence $\mathrm{Ext}^i_{GL(K)}(F, T) \cong K^i_{st}(K, D) = 0$, $i > 0$. □

The canonical homomorphism $T \to aT$ yields a homomorphism $\sigma(F, T)$: $\mathrm{Ext}^*_{\mathcal{F}}(F, T) \to \mathrm{Ext}^*_{\mathcal{F}}(F, aT)$. Thanks to Proposition 4.3 this map is an isomorphism if and only if Theorem 4.1 is true for the pair (F, T). We will also need the following

LEMMA 4.4. — *Let E^* be an exponential functor and let $B_1, \cdots, B_n \in \mathcal{F}$ be functors with the property $B_i(0) = 0$, $i = 1, \cdots, n$. Assume further that the natural homomorphisms $\sigma(E^s, B_i)$ are isomorphisms for all $s < m$. Then $\sigma(A^m, B_1 \otimes \cdots \otimes B_n)$ is an isomorphism as well.*

Proof of Theorem 4.1. — By the first part of the proof we know that $\sigma(I, I)$ is an isomorphism and we have to show that $\sigma(F, T)$ is an isomorphism provided F and T satisfy the conditions of the Theorem. This can be done by induction on the degree of functors based on Lemma 4.4 as follows. First one uses the exact sequence (3.1) (here we start to assume that $p = 2$, for odd p one needs to use De Rham and Koszul complexes). This sequence remains exact after applying the exact functor $a : \mathcal{F} \to \mathcal{F}$. By induction one readily proves that $\sigma(I, S^n)$ is an isomorphism for all n. Indeed it suffices to note that the induction assumption and Lemma 4.4 imply $\mathrm{Ext}^*_{\mathcal{F}}(I, a(S^i \otimes S^j)) = 0$, for $i + j < n$ and one can use the same method as in Section 3. By duality one obtains that $\sigma(\Gamma^n, I)$ is an isomorphism as well. Having these facts in mind and repeating the previous argument one sees that $\sigma(\Gamma^n, S^m)$ is an isomorphism for all n, m too. Now Lemma 4.4 yields that $\sigma(F, T)$ is an isomorphism when F is projective in $\mathcal{P}$ and T is an injective, and the hypercohomology spectral sequence gives the result. □

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Teimuraz PIRASHVILI

A.M. Razmadze Math. Institute
Aleksidze str. 1
Tbilisi, 380093, Georgia
E-mail : pira@rmi.acnet.ge