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COHOMOLOGIES p-ADIQUES ET APPLICATIONS ARITHMÉTIQUES (I)

édité par

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Mots clefs. — Courbe hyperbolique, champ de modules, uniformisation fuchsienne, uniformisation de Bers, p-adique, théorie de Serre-Tate, relèvement canonique, représentation galoisienne, action extérieure de Galois, groupe de Teichmüller, espace symétrique p-adique, transformée intégrale, résidu, représentation p-adique, groupe p-divisible, cristaux, modules de Cartier, biextension.

COHOMOLOGIES p-ADIQUES ET APPLICATIONS ARITHMÉTIQUES (I)

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Résumé. — Ce volume est le premier d'une série de trois consacrés aux méthodes p-adiques en géométrie arithmétique. Les thèmes abordés dans ce volume touchent à la théorie des groupes formels et de leurs déformations, au programme de Langlands p-adique, et à la géométrie hyperbolique p-adique.

***Abstract* (*p*-adic cohomologies and arithmetic applications (I))**

This volume is the first of three dealing with p-adic methods in arithmetic geometry. The themes appearing in this volume include the theory of formal groups and their deformations, the p-adic Langlands program, and the p-adic hyperbolic geometry.

TABLE DES MATIÈRES

RÉSUMÉS DES ARTICLES

Dans cet article, nous présentons une théorie concernant *l'uniformisation et les espaces de modules des courbes hyperboliques p-adiques.* D'une part, cette théorie étend aux places non archimédiennes les uniformisations de Fuchs et Bers et les espaces de modules des courbes hyperboliques complexes. Pour cette raison, nous désignerons souvent cette théorie sous le nom de *théorie de Teichmüller p-adique.* D'autre part, cette théorie peut être vue comme un analogue hyperbolique de la théorie de Serre-Tate pour les variétés abéliennes ordinaires et leurs espaces de modules.

L'objet au centre de la théorie de Teichmüller p-adique est le *champ des modules des « nilcurves ».* Ce champ est un recouvrement plat du champ des modules de courbes hyperboliques en caractéristique p. Il paramètre les courbes hyperboliques munies de « données auxiliaires d'uniformisation en caractéristique p ». La géométrie de ce champ de modules peut *s'analyser de manière combinatoire* au voisinage de l'infini. D'autre part, une analyse globale de sa géométrie mène à *une démonstration de l'irréductibilité du champ des modules de courbes hyperboliques via des méthodes de caractéristique p.* Diverses parties de ce champ des « nilcurves » admettent des *relèvements canoniques* au-dessus desquels on obtient *des coordonnées canoniques et des représentations galoisiennes canoniques.* Ces coordonnées canoniques sont l'analogue, pour les courbes hyperboliques, des coordonnées canoniques dans la théorie de Serre-Tate et l'analogue p-adique des coordonnées de Bers dans la théorie de Teichmüller. De plus, les représentations galoisiennes qui apparaissent éclairent d'un jour nouveau l'action extérieure du groupe de Galois d'un corps local sur le complété profini du groupe de Teichmüller.

Nous faisons une étude détaillée de certaines représentations continues naturelles de $G = \mathrm{GL}(n, K)$ dans les espaces vectoriels localement convexes sur un corps non archimédien localement compact de caractéristique 0. Nous construisons des applications “transformées intégrales” entre des sous-quotients de la duale d'une représentation “holomorphe” provenant d'un espace symétrique p-adique, et des représentations “de la série principale” construites à partir de fonctions localement analytiques sur G. Nous caractérisons l'image de chacune de nos transformées intégrales comme un espace de fonctions sur G jouissant de certaines propriétés par rapport aux transformations et vérifiant un système d'équations aux dérivées partielles de type hypergéométrique.

Ce travail constitue une généralisation d'un travail de Morita, qui a étudié ce genre de représentations pour le groupe $\mathrm{SL}(2, K)$. Notre travail étend également celui de Schneider-Stuhler sur la cohomologie de de Rham des espaces symétriques p-adiques. Nous le voyons comme faisant partie d'un programme général visant à développer la théorie de ce type de représentations.

Nous proposons une nouvelle théorie de Dieudonné qui associe à un groupe formel p-divisible X sur un anneau p-adique excellent R un objet d'algèbre linéaire appelé « display ». A partir du « display » on peut exhiber des équations structurelles pour le module de Cartier de X et retrouver son cristal de Grothendieck-Messing. Nous donnons des applications à la théorie des déformations des groupes formels p-divisibles.

ABSTRACTS

In this article, we survey a theory, developed by the author, concerning the *uniformization of p-adic hyperbolic curves and their moduli*. On the one hand, this theory generalizes the Fuchsian and Bers uniformizations of complex hyperbolic curves and their moduli to nonarchimedean places. It is for this reason that we shall often refer to this theory as *p-adic Teichmüller theory*, for short. On the other hand, this theory may be regarded as a fairly precise hyperbolic analogue of the Serre-Tate theory of ordinary abelian varieties and their moduli.

The central object of p-adic Teichmüller theory is the *moduli stack of nilcurves*. This moduli stack forms a finite flat covering of the moduli stack of hyperbolic curves in positive characteristic. It parametrizes hyperbolic curves equipped with auxiliary "uniformization data in positive characteristic." The geometry of this moduli stack may be *analyzed combinatorially* locally near infinity. On the other hand, a global analysis of its geometry gives rise to a *proof of the irreducibility of the moduli stack of hyperbolic curves using positive characteristic methods*. Various portions of this stack of nilcurves admit *canonical p-adic liftings*, over which one obtains *canonical coordinates and canonical p-adic Galois representations*. These canonical coordinates form the analogue for hyperbolic curves of the canonical coordinates of Serre-Tate theory and the p-adic analogue of the Bers coordinates of Teichmüller theory. Moreover, the resulting Galois representations shed new light on the outer action of the Galois group of a local field on the profinite completion of the Teichmüller group.

We study in detail certain natural continuous representations of $G = GL_n(K)$ in locally convex vector spaces over a locally compact, non-archimedean field K of characteristic zero. We construct boundary value maps, or integral transforms, between subquotients of the dual of a "holomorphic" representation coming from a p-adic symmetric space, and "principal series" representations constructed from locally analytic functions on G. We characterize the image of each of our integral transforms as a space of functions on G having certain transformation properties and satisfying a system of partial differential equations of hypergeometric type.

This work generalizes earlier work of Morita, who studied this type of representation of the group $SL_2(K)$. It also extends the work of Schneider-Stuhler on the De Rham cohomology of p-adic symmetric spaces. We view this work as part of a general program of developing the theory of such representations.

We give a new Dieudonné theory which associates to a formal p-divisible group X over an excellent p-adic ring R an object of linear algebra called a display. On the display one can read off the structural equations for the Cartier module of X, and find the crystal of Grothendieck-Messing. We give applications to deformations of formal p-divisible groups.

INTRODUCTION

Un semestre spécial, consacré aux cohomologies p-adiques et à leurs applications arithmétiques, a eu lieu, du 17 février au 11 juillet 1997, dans le cadre du centre Émile Borel, situé à Paris dans les locaux de l'institut Henri Poincaré.

Les principaux thèmes abordés ont été :

– les théorèmes de comparaison entre différentes cohomologies p-adiques des variétés algébriques sur les corps locaux, les représentations p-adiques du groupe de Galois absolu d'un tel corps,
– les groupes p-divisibles et la théorie de Dieudonné cristalline, la cohomologie des $\mathcal{D}$-modules arithmétiques, les équations différentielles p-adiques,
– l'uniformisation p-adique, l'étude des espaces symétriques p-adiques, des courbes hyperboliques p-adiques, de la cohomologie des variétés de Shimura,
– la géométrie et la cohomologie logarithmiques,
– les fonctions L p-adiques, leurs relations avec les systèmes d'Euler, en particulier dans le cas des formes modulaires.

Les activités structurées ont consisté en

a) Douze cours :
– P. Berthelot (Rennes) : *$\mathcal{D}$-modules arithmétiques*,
– C. Breuil (CNRS, Orsay) : *Cohomologie log cristalline et cohomologie étale de torsion* (Cours Peccot du Collège de France),
– G. Christol (Paris VI) : *Equations différentielles p-adiques*,
– G. Faltings (MPI, Bonn) : *Almost étale extensions*,
– J.-M. Fontaine (Orsay) : *Arithmétique des représentations galoisiennes p-adiques*,
– L. Illusie (Orsay) et A. Ogus (Berkeley) : *Géométrie logarithmique*,
– K. Kato (Tokyo) : *Euler systems and p-adic L-functions*,
– W. Messing (Minneapolis) : *Topologie et cohomologie syntomiques et log syntomiques*,

– S. Mochizuki (RIMS, Kyoto) : *The Ordinary and Generalized Ordinary Moduli of Hyperbolic Curves*,
– M. Rapoport (Cologne) : *Aspects p-adiques des variétés de Shimura*,
– P. Schneider (Münster) : *Analysis on p-adic symmetric spaces*,
– T. Zink (Bielefeld) : *Cartier theory and its connection to crystalline Dieudonné theory*.

b) Un séminaire avec un ou deux exposés chaque semaine.

c) Deux colloques :
– *Problèmes de coefficients en cohomologie cristalline et en cohomologie rigide*, du 28 au 30 avril,
– *Arithmétique des fonctions L et méthodes p-adiques*, du 30 juin au 4 juillet.

d) Un groupe de travail sur *le théorème de comparaison de Tsuji*, du 20 au 29 mai.

Les organisateurs ont demandé à tous ceux qui avaient fait un cours de le rédiger ou de nous faire parvenir un texte sur un sujet voisin. Nous avons également invité Takeshi Tsuji à écrire un résumé de sa démonstration, maintenant publiée[(1)], de la *conjecture* C_{st}.

Nous tenons à remercier les auteurs non seulement pour leur contribution mais aussi pour leur patience ; nous espérons qu'ils voudront bien nous excuser du retard avec lequel ces volumes paraissent.

Les articles ont été examinés par des rapporteurs que nous remercions pour leur aide aussi désintéressée qu'utile.

Enfin, nous pensons que tous ceux qui ont participé à ce semestre seront d'accord avec nous pour saluer l'atmosphère agréable dans laquelle il s'est déroulé. Nous remercions chaleureusement Joseph Oesterlé, alors directeur du Centre Émile Borel, son équipe et tout le personnel de l'Institut Henri Poincaré pour leur gentillesse, leur compétence, leur efficacité et leur dévouement. Ils se joindront sûrement à nous pour accorder une mention spéciale à Madame Nocton, notre bibliothécaire — tous les mathématiciens qui ont travaillé à Paris la connaissent et savent combien son rôle a été précieux ; et une autre à notre secrétaire — Florence Damay — qui a quitté le Centre Émile Borel juste à la fin de notre semestre ; elle en fut la cheville ouvrière mais aussi le sourire, avec une formidable aptitude à comprendre et résoudre les problèmes extra-mathématiques rencontrés par les très nombreux participants.

Les éditeurs

(1) *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. math. **137** (1999), 233–411

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AN INTRODUCTION TO p-ADIC TEICHMÜLLER THEORY

by

Shinichi Mochizuki

Abstract. — In this article, we survey a theory, developed by the author, concerning the *uniformization of p-adic hyperbolic curves and their moduli.* On the one hand, this theory generalizes the Fuchsian and Bers uniformizations of complex hyperbolic curves and their moduli to nonarchimedean places. It is for this reason that we shall often refer to this theory as *p-adic Teichmüller theory*, for short. On the other hand, this theory may be regarded as a fairly precise hyperbolic analogue of the Serre-Tate theory of ordinary abelian varieties and their moduli.

The central object of p-adic Teichmüller theory is the *moduli stack of nilcurves.* This moduli stack forms a finite flat covering of the moduli stack of hyperbolic curves in positive characteristic. It parametrizes hyperbolic curves equipped with auxiliary "uniformization data in positive characteristic." The geometry of this moduli stack may be *analyzed combinatorially* locally near infinity. On the other hand, a global analysis of its geometry gives rise to a *proof of the irreducibility of the moduli stack of hyperbolic curves using positive characteristic methods.* Various portions of this stack of nilcurves admit *canonical p-adic liftings*, over which one obtains *canonical coordinates and canonical p-adic Galois representations.* These canonical coordinates form the analogue for hyperbolic curves of the canonical coordinates of Serre-Tate theory and the p-adic analogue of the Bers coordinates of Teichmüller theory. Moreover, the resulting Galois representations shed new light on the outer action of the Galois group of a local field on the profinite completion of the Teichmüller group.

1. From the Complex Theory to the "Classical Ordinary" p-adic Theory

In this §, we attempt to bridge the gap for the reader between the classical uniformization of a hyperbolic Riemann surface that one studies in an undergraduate complex analysis course and the point of view espoused in [**21, 22**].

2000 ***Mathematics Subject Classification.*** — 14H10, 14F30.
Key words and phrases. — Hyperbolic curve, moduli stack, uniformization theory, Fuchsian uniformization, Bers uniformization, p-adic, Serre-Tate theory, canonical liftings, Galois representations, outer Galois actions, Teichmüller group .

1.1. The Fuchsian Uniformization. — Let X be a *hyperbolic* algebraic curve over $\mathbb{C}$, the field of complex numbers. By this, we mean that X is obtained by removing r points from a smooth, proper, connected algebraic curve of genus g (over $\mathbb{C}$), where $2g-2+r>0$. We shall refer to (g,r) as the *type* of X. Then it is well-known that to X, one can associate in a natural way a Riemann surface $\mathbf{X}$ whose underlying point set is $X(\mathbb{C})$. We shall refer to Riemann surfaces $\mathbf{X}$ obtained in this way as "hyperbolic of finite type."

Now perhaps the most fundamental *arithmetic* – read "arithmetic at the infinite prime" – fact known about the *algebraic* curve X is that $\mathbf{X}$ admits a uniformization by the upper half plane $\mathbf{H}$:

$$\mathbf{H} \to \mathbf{X}$$

For convenience, we shall refer to this uniformization of $\mathbf{X}$ in the following as the *Fuchsian uniformization of* $\mathbf{X}$. Put another way, the uniformization theorem quoted above asserts that the universal covering space $\widetilde{\mathbf{X}}$ of $\mathbf{X}$ (which itself has the natural structure of a Riemann surface) is holomorphically isomorphic to the upper half plane $\mathbf{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. This fact was "familiar" to many mathematicians as early as the last quarter of the nineteenth century, but was only proven rigorously much later by Koebe.

> *The fundamental thrust of* [**21, 22**] *is to generalize the Fuchsian uniformization to the p-adic context.*

At this point, the reader might be moved to interject: But hasn't this already been achieved decades ago by Mumford in [**25**]? In fact, however, Mumford's construction gives rise to a p-adic analogue *not of the Fuchsian uniformization, but rather of the Schottky uniformization* of a complex hyperbolic curve. Even in the complex case, the Schottky uniformization is an entirely different sort of uniformization – both geometrically and arithmetically – from the Fuchsian uniformization: for instance, its periods are holomorphic, whereas the periods that occur for the Fuchsian uniformization are only real analytic. This phenomenon manifests itself in the nonarchimedean context in the fact that the construction of [**25**] really has nothing to do with a fixed prime number "p," and in fact, takes place entirely in the formal analytic category. In particular, the theory of [**25**] has nothing to do with "Frobenius." By contrast, the theory of [**21, 22**] depends very much on the choice of a prime "p," and makes essential use of the "action of Frobenius." Another difference between the theory of [**25**] and the theory of [**21, 22**] is that [**25**] only addresses the case of curves whose "reduction modulo p" is totally degenerate, whereas the theory of [**21, 22**] applies to curves whose reduction modulo p is only assumed to be "sufficiently generic." Thus, at any rate, the theory of [**21, 22**] is entirely different from and has little directly to do with the theory of [**25**].

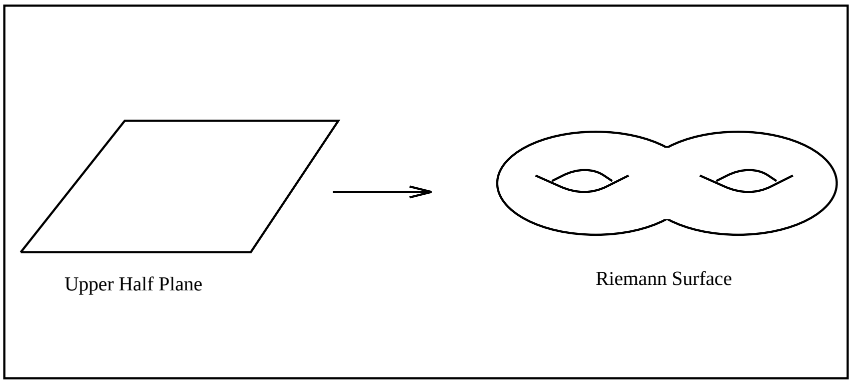

FIGURE 1. The Fuchsian Uniformization

1.2. Reformulation in Terms of Metrics. — Unfortunately, if one sets about trying to generalize the Fuchsian uniformization $\mathbf{H} \to \mathbf{X}$ to the p-adic case in any sort of naive, literal sense, one immediately sees that one runs into a multitude of apparently insurmountable difficulties. Thus, it is natural to attempt to recast the Fuchsian uniformization in a more universal form, a form more amenable to relocation from the archimedean to the nonarchimedean world.

One natural candidate that arises in this context is the notion of a metric – more precisely, the notion of a *real analytic Kähler metric*. For instance, the upper half plane admits a natural such metric, namely, the metric given by

$$\frac{dx^2 + dy^2}{y^2}$$

(where $z = x + iy$ is the standard coordinate on $\mathbf{H}$). Since this metric is invariant with respect to all holomorphic automorphisms of $\mathbf{H}$, it induces a natural metric on $\widetilde{\mathbf{X}} \cong \mathbf{H}$ which is independent of the choice of isomorphism $\widetilde{\mathbf{X}} \cong \mathbf{H}$ and which descends to a metric $\mu_{\mathbf{X}}$ on $\mathbf{X}$.

Having constructed the canonical metric $\mu_{\mathbf{X}}$ on $\mathbf{X}$, we first make the following observation:

> *There is a general theory of canonical coordinates associated to a real analytic Kähler metric on a complex manifold.*

(See, e.g., [**21**], Introduction, § 2, for more technical details.) Moreover, the canonical coordinate associated to the metric $\mu_{\mathbf{X}}$ is precisely the coordinate obtained by pulling back the standard coordinate "z" on the unit disc via any holomorphic isomorphism of $\widetilde{\mathbf{X}} \cong \mathbf{H}$ with the unit disc. Thus, in other words, passing from $\mathbf{H} \to \widetilde{\mathbf{X}}$ to $\mu_{\mathbf{X}}$ is a "faithful operation," i.e., one doesn't really lose any information.

Next, let us make the following observation: Let $\mathcal{M}_{g,r}$ denote the moduli stack of smooth r-pointed algebraic curves of genus g over $\mathbb{C}$. If we order the points that were removed from the compactification of X to form X, then we see that X defines a point $[X] \in \mathcal{M}_{g,r}(\mathbb{C})$. Moreover, it is elementary and well-known that the cotangent space to $\mathcal{M}_{g,r}$ at $[X]$ can be written in terms of square differentials on X. Indeed, if, for simplicity, we restrict ourselves to the case $r = 0$, then this cotangent space is naturally isomorphic to $Q \stackrel{\text{def}}{=} H^0(X, \omega_{X/\mathbb{C}}^{\otimes 2})$ (where $\omega_{X/\mathbb{C}}$ is the algebraic coherent sheaf of differentials on X). Then the observation we would like to make is the following: Reformulating the Fuchsian uniformization in terms of the metric $\mu_{\mathbf{X}}$ allows us to "push-forward" $\mu_{\mathbf{X}}$ to obtain a canonical real analytic Kähler metric $\mu_{\mathbf{M}}$ on the complex analytic stack $\mathbf{M}_{\mathbf{g},\mathbf{r}}$ associated to $\mathcal{M}_{g,r}$ by the following formula: if $\theta, \psi \in Q$, then

$$\langle \theta, \psi \rangle \stackrel{\text{def}}{=} \int_{\mathbf{X}} \frac{\theta \cdot \overline{\psi}}{\mu_{\mathbf{X}}}$$

(Here, $\overline{\psi}$ is the complex conjugate differential to ψ, and the integral is well-defined because the integrand is the quotient of a $(2,2)$-form by a $(1,1)$-form, i.e., the integrand is itself a $(1,1)$-form.)

This metric on $\mathbf{M}_{\mathbf{g},\mathbf{r}}$ is called the *Weil-Petersson metric*. It is known that

> *The canonical coordinates associated to the Weil-Petersson metric coincide with the so-called Bers coordinates on $\widetilde{\mathbf{M}}_{g,r}$ (the universal covering space of $\mathbf{M}_{\mathbf{g},\mathbf{r}}$).*

The Bers coordinates define an anti-holomorphic embedding of $\widetilde{\mathbf{M}}_{g,r}$ into the complex affine space associated to Q. We refer to the Introduction of [**21**] for more details on this circle of ideas.

At any rate, in summary, we see that much that is useful can be obtained from this reformulation in terms of metrics. However, although we shall see later that the reformulation in terms of metrics is not entirely irrelevant to the theory that one ultimately obtains in the p-adic case, nevertheless this reformulation is still not sufficient to allow one to effect the desired translation of the Fuchsian uniformization into an analogous p-adic theory.

1.3. Reformulation in Terms of Indigenous Bundles. — It turns out that the "missing link" necessary to translate the Fuchsian uniformization into an analogous p-adic theory was provided by Gunning ([**13**]) in the form of the notion of an *indigenous bundle*. The basic idea is as follows: First recall that the group $\mathrm{Aut}(\mathbf{H})$ of holomorphic automorphisms of the upper half plane may be identified (by thinking about linear fractional transformations) with $\mathrm{PSL}_2(\mathbb{R})^0$ (where the superscripted "0" denotes the connected component of the identity). Moreover, $\mathrm{PSL}_2(\mathbb{R})^0$ is naturally contained inside $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$. Let $\Pi_{\mathbf{X}}$ denote the (topological) fundamental group of

$\mathbf{X}$ (where we ignore the issue of choosing a base-point since this will be irrelevant for what we do). Then since $\Pi_{\mathbf{X}}$ acts naturally on $\widetilde{\mathbf{X}} \cong \mathbf{H}$, we get a natural representation

$$\rho_{\mathbf{X}} : \Pi_{\mathbf{X}} \to \mathrm{PGL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$$

which is well-defined up to conjugation by an element of $\mathrm{Aut}(\mathbf{H}) \subseteq \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$. We shall henceforth refer to $\rho_{\mathbf{X}}$ as the *canonical representation associated to* $\mathbf{X}$. Thus, $\rho_{\mathbf{X}}$ gives us an action of $\Pi_{\mathbf{X}}$ on $\mathbb{P}^1_{\mathbb{C}}$, hence a diagonal action on $\widetilde{\mathbf{X}} \times \mathbb{P}^1_{\mathbb{C}}$. If we form the quotient of this action of $\Pi_{\mathbf{X}}$ on $\widetilde{\mathbf{X}} \times \mathbb{P}^1_{\mathbb{C}}$, we obtain a $\mathbb{P}^1$-bundle over $\widetilde{\mathbf{X}}/\Pi_{\mathbf{X}} = \mathbf{X}$ which automatically algebraizes to an algebraic $\mathbb{P}^1$-bundle $P \to X$ over X. (For simplicity, think of the case $r = 0$!)

In fact, $P \to X$ comes equipped with more structure. First of all, note that the trivial $\mathbb{P}^1$-bundle $\widetilde{\mathbf{X}} \times \mathbb{P}^1_{\mathbb{C}} \to \widetilde{\mathbf{X}}$ is equipped with the trivial connection. (Note: here we use the "Grothendieck definition" of the notion of a connection on a $\mathbb{P}^1$-bundle: i.e., an isomorphism of the two pull-backs of the $\mathbb{P}^1$-bundle to the first infinitesimal neighborhood of the diagonal in $\widetilde{\mathbf{X}} \times \widetilde{\mathbf{X}}$ which restricts to the identity on the diagonal $\widetilde{\mathbf{X}} \subseteq \widetilde{\mathbf{X}} \times \widetilde{\mathbf{X}}$.) Moreover, this trivial connection is clearly fixed by the action of $\Pi_{\mathbf{X}}$, hence descends and algebraizes to a connection ∇_P on $P \to X$. Finally, let us observe that we also have a section $\sigma : X \to P$ given by descending and algebraizing the section $\widetilde{\mathbf{X}} \to \widetilde{\mathbf{X}} \times \mathbb{P}^1_{\mathbb{C}}$ whose projection to the second factor is given by $\widetilde{\mathbf{X}} \cong \mathbf{H} \subseteq \mathbb{P}^1_{\mathbb{C}}$. This section is referred to as the *Hodge section.* If we differentiate σ by means of ∇_P, we obtain a *Kodaira-Spencer morphism* $\tau_{X/\mathbb{C}} \to \sigma^* \tau_{P/X}$ (where "$\tau_{A/B}$" denotes the relative tangent bundle of A over B). It is easy to see that this Kodaira-Spencer morphism is necessarily an isomorphism.

This triple of data $(P \to X, \nabla_P, \sigma)$ is the prototype of what Gunning refers to as an *indigenous bundle.* We shall refer to this specific $(P \to X, \nabla_P)$ (one doesn't need to specify σ since σ is uniquely determined by the property that its Kodaira-Spencer morphism is an isomorphism) as the *canonical indigenous bundle.* More generally, an *indigenous bundle on* X (at least in the case $r = 0$) is any $\mathbb{P}^1$-bundle $P \to X$ with connection ∇_P such that $P \to X$ admits a section (necessarily unique) whose Kodaira-Spencer morphism is an isomorphism. (In the case $r > 0$, it is natural to introduce log structures in order to make a precise definition.)

Note that the notion of an indigenous bundle has the virtue of being *entirely algebraic* in the sense that at least as an object, the canonical indigenous bundle $(P \to X, \nabla_P)$ exists in the algebraic category. In fact, the space of indigenous bundles forms a torsor over the vector space Q of quadratic differentials on X (at least for $r = 0$). Thus,

The issue of which point in this affine space of indigenous bundles on X corresponds to the canonical indigenous bundle is a deep arithmetic issue, but the affine space itself can be defined entirely algebraically.

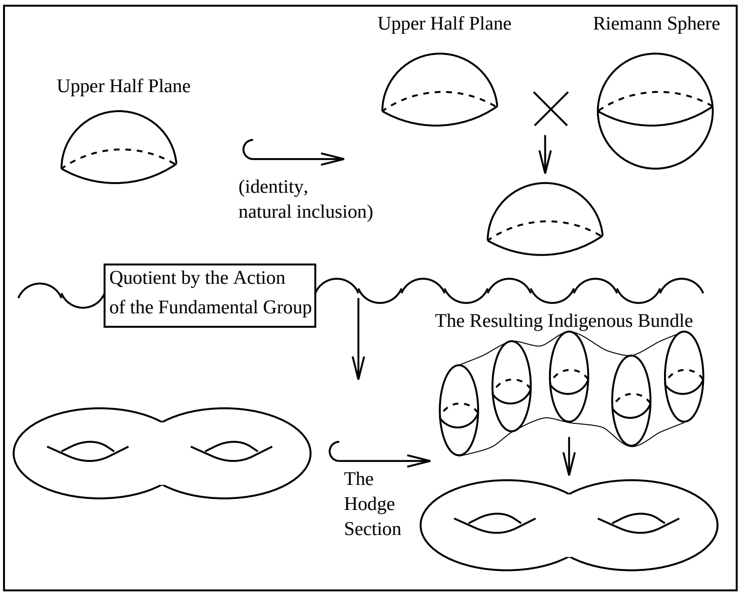

FIGURE 2. The Construction of the Canonical Indigenous Bundle

One aspect of the fact that the notion of an indigenous bundle is entirely algebraic is that indigenous bundles can, in fact, be defined over $\mathbb{Z}[\frac{1}{2}]$, and in particular, over $\mathbb{Z}_p$ (for p odd). In [**21**], Chapter I, a fairly complete theory of indigenous bundles in the p-adic case (analogous to the complex theory of [**13**]) is worked out. To summarize, indigenous bundles are closely related to projective structures and Schwarzian derivatives on X. Moreover, the underlying $\mathbb{P}^1$-bundle $P \to X$ is always the same (for all indigenous bundles on X), i.e., the choice of connection ∇_P determines the isomorphism class of the indigenous bundle. We refer the reader to [**21**], Chapter I, for more details. (Note: Although the detailed theory of [**21**], Chapter I, is philosophically very relevant to the theory of [**22**], most of this theory is technically and logically unnecessary for reading [**22**].)

At any rate, to summarize, the introduction of indigenous bundles allows one to consider the Fuchsian uniformization as being embodied by an object – the canonical indigenous bundle – which exists in the algebraic category, but which, compared to other indigenous bundles, is somehow "special." In the following, we would like to analyze the sense in which the canonical indigenous bundle is special, and to show how this sense can be translated immediately into the p-adic context. Thus, we see that

> *The search for a p-adic theory analogous to the theory of the Fuchsian uniformization can be reinterpreted as the search for a notion*

of "canonical p-adic indigenous bundle" which is special in a sense precisely analogous to the sense in which the canonical indigenous bundle arising from the Fuchsian uniformization is special.

1.4. Frobenius Invariance and Integrality. — In this subsection, we explore in greater detail the issue of what precisely makes the canonical indigenous bundle (in the complex case) so special, and note in particular that a properly phrased characterization of the canonical indigenous bundle (in the complex case) translates very naturally into the p-adic case.

First, let us observe that in global discussions of motives over a number field, it is natural to think of the operation of complex conjugation as a sort of "Frobenius at the infinite prime." In fact, in such discussions, complex conjugation is often denoted by "Fr_∞." Next, let us observe that one special property of the canonical indigenous bundle is that its monodromy representation (i.e., the "canonical representation" $\rho_{\mathbf{X}}$: $\Pi_{\mathbf{X}} \to \mathrm{PGL}_2(\mathbb{C})$) is *real-valued*, i.e., takes its values in $\mathrm{PGL}_2(\mathbb{R})$. Another way to put this is to say that the canonical indigenous bundle is Fr_∞-invariant, i.e.,

The canonical indigenous bundle on a hyperbolic curve is invariant with respect to the Frobenius at the infinite prime.

Unfortunately, as is observed in [**5**], this property of having real monodromy is not sufficient to characterize the canonical indigenous bundle completely. That is to say, the indigenous bundles with real monodromy form a discrete subset of the space of indigenous bundles on the given curve X, but this discrete subset consists (in general) of more than one element.

Let us introduce some notation. Let $\mathcal{M}_{g,r}$ be the stack of r-pointed smooth curves of genus g over $\mathbb{C}$. Let $\mathcal{S}_{g,r}$ be the stack of such curves equipped with an indigenous bundle. Then there is a natural projection morphism $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ (given by forgetting the indigenous bundle) which exhibits $\mathcal{S}_{g,r}$ as an affine torsor on $\mathcal{M}_{g,r}$ over the vector bundle $\Omega_{\mathcal{M}_{g,r}/\mathbb{C}}$ of differentials on $\mathcal{M}_{g,r}$. We shall refer to this torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ as the *Schwarz torsor*.

Let us write $\mathcal{S}_X$ for the restriction of the Schwarz torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ to the point $[X] \in \mathcal{M}_{g,r}(\mathbb{C})$ defined by X. Thus, $\mathcal{S}_X$ is an affine complex space of dimension $3g-3+r$. Let $\mathcal{R}_X \subseteq \mathcal{S}_X$ be the set of indigenous bundles with real monodromy. As observed in [**5**], $\mathcal{R}_X$ is a discrete subset of $\mathcal{S}_X$. Now let $\mathcal{S}'_X \subseteq \mathcal{S}_X$ be the subset of indigenous bundles $(P \to X, \nabla_P)$ with the following property:

(*) The associated monodromy representation $\rho : \Pi_{\mathbf{X}} \to \mathrm{PGL}_2(\mathbb{C})$ is injective and its image Γ is a *quasi-Fuchsian* group. Moreover, if $\Omega \subseteq \mathbb{P}^1(\mathbb{C})$ is the domain of discontinuity of Γ, then Ω/Γ is a disjoint union of two Riemann surfaces of type (g, r).

(Roughly speaking, a "quasi-Fuchsian group" is a discrete subgroup of $\mathrm{PGL}_2(\mathbb{C})$ whose domain of discontinuity Ω (i.e., the set of points of $\mathbb{P}^1(\mathbb{C})$ at which Γ acts discontinuously) is a disjoint union of two topological open discs, separated by a topological circle. We refer to [**10, 27**] for more details on the theory of quasi-Fuchsian groups.)

It is known that $\mathcal{S}'_X$ is a bounded ([**10**], p. 99, Lemma 6), open (cf. the discussion of § 5 of [**27**]) subset of $\mathcal{S}_X$ (in the complex analytic topology). Moreover, since a quasi-Fuchsian group with real monodromy acts discretely on the upper half plane (see, e.g., [**26**], Chapter I, Proposition 1.8), it follows immediately that such a quasi-Fuchsian group is Fuchsian. Put another way, we have that:

> *The intersection $\mathcal{R}_X \bigcap \mathcal{S}'_X \subseteq \mathcal{S}_X$ is the set consisting of the single point corresponding to the canonical indigenous bundle.*

It is this characterization of the canonical indigenous bundle that we will seek to translate into the p-adic case.

To translate the above characterization, let us first recall the point of view of Arakelov theory which states, in effect, that $\mathbb{Z}_p$-integral structures (on say, an affine space over $\mathbb{Q}_p$) correspond to closures of bounded open subsets (of, say, an affine space over $\mathbb{C}$). Thus, from this point of view, one may think of $\mathcal{S}'_X$ as defining a natural integral structure (in the sense of Arakelov theory) on the complex affine space $\mathcal{S}_X$. Thus, from this point of view, one arrives at the following characterization of the canonical indigenous bundle:

> *The canonical indigenous bundle is the unique indigenous bundle which is integral (in the Arakelov sense) and Frobenius invariant (i.e., has monodromy which is invariant with respect to complex conjugation).*

This gives us at last an answer to the question posed earlier: How can one characterize the canonical indigenous bundle in the complex case in such a way that the characterization carries over word for word to the p-adic context? In particular, it gives rise to the following conclusion:

> *The proper p-adic analogue of the theory of the Fuchsian and Bers uniformizations should be a theory of $\mathbb{Z}_p$-integral indigenous bundles that are invariant with respect to some natural action of the Frobenius at the prime p.*

This conclusion constitutes the fundamental philosophical basis underlying the theory of [**22**]. In [**21**], this philosophy was partially realized in the sense that *certain* $\mathbb{Z}_p$-integral Frobenius indigenous bundles were constructed. The theory of [**21**] will be reviewed later (in § 1.6). The goal of [**22**], by contrast, is to lay the foundations for a general theory of *all* $\mathbb{Z}_p$-integral Frobenius indigenous bundles and to say as much as is possible in as much generality as is possible concerning such bundles.

1.5. The Canonical Real Analytic Trivialization of the Schwarz Torsor

In this subsection, we would like to take a closer look at the *Schwarz torsor* $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$. For general g and r, this affine torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ does not admit any algebraic or holomorphic sections. Indeed, this affine torsor defines a class in $H^1(\mathcal{M}_{g,r}, \Omega_{\mathcal{M}_{g,r}/\mathbb{C}})$ which is the Hodge-theoretic first Chern class of a certain ample line bundle $\mathcal{L}$ on $\mathcal{M}_{g,r}$. (See [**21**], Chapter I, § 3, especially Theorem 3.4, for more details on this Hodge-theoretic Chern class and Chapter III, Proposition 2.2, of [**22**] for a proof of ampleness.) Put another way, $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ is the torsor of (algebraic) connections on the line bundle $\mathcal{L}$. However, the map that assigns to X the canonical indigenous bundle on X defines a *real analytic* section

$$s_{\mathbf{H}} : \mathcal{M}_{g,r}(\mathbb{C}) \to \mathcal{S}_{g,r}(\mathbb{C})$$

of this torsor.

The first and most important goal of the present subsection is to remark that

> *The single object $s_{\mathbf{H}}$ essentially embodies the entire uniformization theory of complex hyperbolic curves and their moduli.*

Indeed, $s_{\mathbf{H}}$ by its very definition contains the data of "which indigenous bundle is canonical," hence already may be said to embody the Fuchsian uniformization. Next, we observe that $\overline{\partial} s_{\mathbf{H}}$ is equal to the Weil-Petersson metric on $\mathcal{M}_{g,r}$ (see [**21**], Introduction, Theorem 2.3 for more details). Moreover, (as is remarked in Example 2 following Definition 2.1 in [**21**], Introduction, § 2) since the canonical coordinates associated to a real analytic Kähler metric are obtained by essentially integrating (in the "sense of anti-$\overline{\partial}$-ing") the metric, it follows that (a certain appropriate restriction of) $s_{\mathbf{H}}$ "is" essentially the Bers uniformization of Teichmüller space. Thus, as advertised above, the single object $s_{\mathbf{H}}$ stands at the very center of the uniformization theory of complex hyperbolic curves and their moduli.

In particular, it follows that we can once again reinterpret the fundamental issue of trying to find a p-adic analogue of the Fuchsian uniformization as the issue of *trying to find a p-adic analogue of the section $s_{\mathbf{H}}$*. That is to say, the torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ is, in fact, defined over $\mathbb{Z}[\frac{1}{2}]$, hence over $\mathbb{Z}_p$ (for p odd). Thus, forgetting for the moment that it is not clear precisely what p-adic category of functions corresponds to the real analytic category at the infinite prime, one sees that

> *One way to regard the search for a p-adic Fuchsian uniformization is to regard it as the search for some sort of canonical p-adic analytic section of the torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$.*

In this context, it is thus natural to refer to $s_{\mathbf{H}}$ as the *canonical arithmetic trivialization of the torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ at the infinite prime.*

Finally, let us observe that this situation of a torsor corresponding to the Hodge-theoretic first Chern class of an ample line bundle, equipped with a canonical real analytic section occurs not only over $\mathcal{M}_{g,r}$, but over any individual hyperbolic curve X (say, over $\mathbb{C}$), as well. Indeed, let $(P \to X, \nabla_P)$ be the canonical indigenous bundle on X. Let $\sigma : X \to P$ be its Hodge section. Then by [**21**], Chapter I, Proposition 2.5, it follows that the $T \stackrel{\mathrm{def}}{=} P - \sigma(X)$ has the structure of an $\omega_{X/\mathbb{C}}$-torsor over X. In fact, one can say more: namely, this torsor is the Hodge-theoretic first Chern class corresponding to the ample line bundle $\omega_{X/\mathbb{C}}$. Moreover, if we compose the morphism $\widetilde{\mathbf{X}} \cong \mathbf{H} \subseteq \mathbb{P}^1_{\mathbb{C}}$ used to define σ with the standard complex conjugation morphism on $\mathbb{P}^1_{\mathbb{C}}$, we obtain a new $\Pi_{\mathbf{X}}$-equivariant $\widetilde{\mathbf{X}} \to \mathbb{P}^1_{\mathbb{C}}$ which descends to a real analytic section $s_{\mathbf{X}} : X(\mathbb{C}) \to T(\mathbb{C})$. Just as in the case of $\mathcal{M}_{g,r}$, it is easy to compute (cf. the argument of [**21**], Introduction, Theorem 2.3) that $\overline{\partial} s_{\mathbf{X}}$ is equal to the canonical hyperbolic metric $\mu_{\mathbf{X}}$. Thus, just as in the case of the real analytic section $s_{\mathbf{H}}$ of the Schwarz torsor over $\mathcal{M}_{g,r}$, $s_{\mathbf{X}}$ *essentially "is" the Fuchsian uniformization of* $\mathbf{X}$.

1.6. The Classical Ordinary Theory. — As stated earlier, the purpose of [**22**] is to study all integral Frobenius invariant indigenous bundles. On the other hand, in [**21**], a very important special type of Frobenius invariant indigenous bundle was constructed. This type of bundle will henceforth be referred to as *classical ordinary.* (Such bundles were called "ordinary" in [**21**]. Here we use the term "classical ordinary" to refer to objects called "ordinary" in [**21**] in order to avoid confusion with the more general notions of ordinarity discussed in [**22**].) Before discussing the theory of the [**22**] (which is the goal of § 2), it is thus natural to review the classical ordinary theory. In this subsection, we let p be an odd prime.

If one is to construct p-adic Frobenius invariant indigenous bundles for arbitrary hyperbolic curves, the first order of business is to make precise the notion of Frobenius invariance that one is to use. For this, it is useful to have a prototype. The prototype that gave rise to the classical ordinary theory is the following:

> Let $\mathcal{M} \stackrel{\mathrm{def}}{=} (\mathcal{M}_{1,0})_{\mathbb{Z}_p}$ be the moduli stack of elliptic curves over $\mathbb{Z}_p$. Let $\mathcal{G} \to \mathcal{M}$ be the universal elliptic curve. Let $\mathcal{E}$ be its first de Rham cohomology module. Thus, $\mathcal{E}$ is a rank two vector bundle on $\mathcal{M}$, equipped with a Hodge subbundle $\mathcal{F} \subseteq \mathcal{E}$, and a connection $\nabla_{\mathcal{E}}$ (i.e., the "Gauss-Manin connection"). Taking the projectivization of $\mathcal{E}$ defines a $\mathbb{P}^1$-bundle with connection $(P \to \mathcal{M}, \nabla_P)$, together with a Hodge section $\sigma : \mathcal{M} \to P$. It turns out that (the natural extension over the compactification of $\mathcal{M}$ obtained by using log structures of) the bundle (P, ∇_P) is an *indigenous bundle* on $\mathcal{M}$. In particular, (P, ∇_P) defines a crystal in $\mathbb{P}^1$-bundles on $\mathrm{Crys}(\mathcal{M} \otimes \mathbf{F}_p/\mathbb{Z}_p)$. Thus, one can form the pull-back $\Phi^*(P, \nabla_P)$ via the Frobenius morphism of this crystal. If one then adjusts the integral structure of $\Phi^*(P, \nabla_P)$ (cf. Definition 1.18 of Chapter VI of [**22**]; [**21**], Chapter III, Definition 2.4), one obtains the *renormalized Frobenius pull-back*

$\mathbb{F}^*(P, \nabla_P)$. Then (P, ∇_P) is Frobenius invariant in the sense that $(P, \nabla_P) \cong \mathbb{F}^*(P, \nabla_P)$.

Thus, *the basic idea behind* [**21**] *was to consider to what extent one could construct indigenous bundles on arbitrary hyperbolic curves that are equal to their own renormalized Frobenius pull-backs, i.e., satisfying*

$$\mathbb{F}^*(P, \nabla_P) \cong (P, \nabla_P)$$

In particular, it is natural to try to consider moduli of indigenous bundles satisfying this condition. Since it is not at all obvious how to do this over $\mathbb{Z}_p$, a natural first step was to make the following *key* observation:

> *If (P, ∇_P) is an indigenous bundle over $\mathbb{Z}_p$ preserved by $\mathbb{F}^*$, then the reduction modulo p of (P, ∇_P) has square nilpotent p-curvature.*

(The "p-curvature" of an indigenous bundle in characteristic p is a natural invariant of such a bundle. We refer to [**21**], Chapter II, as well as § 1 of Chapter II of [**22**] for more details.) Thus, if $(\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$ is the *stack of r-pointed stable curves of genus g* (as in [**4, 20**]) in characteristic p, one can define the stack $\overline{\mathcal{N}}_{g,r}$ of such curves equipped with a "nilpotent" indigenous bundle. (Here, "nilpotent" means that its p-curvature is square nilpotent.) In the following, we shall often find it convenient to refer to pointed stable curves equipped with nilpotent indigenous bundles as *nilcurves*, for short. Thus, $\overline{\mathcal{N}}_{g,r}$ is the moduli stack of nilcurves. We would like to emphasize that

> *The above observation – which led to the notion of "nilcurves" – is the key technical breakthrough that led to the development of the "p-adic Teichmüller theory" of* [**21, 22**].

The first major result of [**21**] is the following (cf. [**22**], Chapter II, Proposition 1.7; [**21**], Chapter II, Theorem 2.3):

***Theorem 1.1* (Stack of Nilcurves)**. — *The natural morphism $\overline{\mathcal{N}}_{g,r} \to (\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$ is a finite, flat, local complete intersection morphism of degree p^{3g-3+r}.*

> *In particular, up to "isogeny" (i.e., up to the fact that $p^{3g-3+r} \neq 1$), the stack of nilcurves $\overline{\mathcal{N}}_{g,r} \subseteq \overline{\mathcal{S}}_{g,r}$ defines a canonical section of the Schwarz torsor $\overline{\mathcal{S}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$ in characteristic p.*

Thus, relative to our discussion of complex Teichmüller theory – which we saw could be regarded as the study of a certain canonical real analytic section of the Schwarz torsor – it is natural that "p-adic Teichmüller theory" should revolve around the study of $\overline{\mathcal{N}}_{g,r}$.

Although the structure of $\overline{\mathcal{N}}_{g,r}$ is now been much better understood, at the time of writing of [**21**] (Spring of 1994), it was not so well understood, and so it was natural to do the following: Let $\overline{\mathcal{N}}^{\mathrm{ord}}_{g,r} \subseteq \overline{\mathcal{N}}_{g,r}$ be the open substack where $\overline{\mathcal{N}}_{g,r}$ is *étale* over $(\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$. This open substack will be referred to as the *(classical) ordinary locus of* $\overline{\mathcal{N}}_{g,r}$. If one sets up the theory (as is done in [**21, 22**]) using *stable curves* (as we do here), rather than just smooth curves, and applies the theory of *log structures* (as in [**18**]), then it is easy to show that the ordinary locus of $\overline{\mathcal{N}}_{g,r}$ is *nonempty.*

It is worth pausing here to note the following: The reason for the use of the term "ordinary" is that it is standard general practice to refer to as "ordinary" situations where Frobenius acts on a linear space equipped with a "Hodge subspace" in such a way that it acts with slope zero on a subspace of the same rank as the rank of the Hodge subspace. Thus, we use the term "ordinary" here because the Frobenius action on the cohomology of an ordinary nilcurve satisfies just such a condition. In other words, ordinary nilcurves are ordinary in their capacity as nilcurves. However, it is important to remember that:

> *The issue of whether or not a nilcurve is ordinary is entirely different from the issue of whether or not the Jacobian of the underlying curve is ordinary (in the usual sense). That is to say, there exist examples of ordinary nilcurves whose underlying curves have nonordinary Jacobians as well as examples of nonordinary nilcurves whose underlying curves have ordinary Jacobians.*

Later, we shall comment further on the issue of the incompatibility of the theory of [**21**] with Serre-Tate theory relative to the operation of passing to the Jacobian.

At any rate, since $\overline{\mathcal{N}}^{\mathrm{ord}}_{g,r}$ is étale over $(\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$, it lifts naturally to a p-adic formal stack $\mathcal{N}$ which is étale over $(\overline{\mathcal{M}}_{g,r})_{\mathbb{Z}_p}$. Let $\mathcal{C} \to \mathcal{N}$ denote the tautological stable curve over $\mathcal{N}$. Then the main result (Theorem 0.1 of the Introduction of [**21**]) of the theory of [**21**] is the following:

***Theorem 1.2* (Canonical Frobenius Lifting)**

There exists a unique pair $(\Phi_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}; (P, \nabla_P))$ *satisfying the following:*

(1) *The reduction modulo* p *of the morphism* $\Phi_{\mathcal{N}}$ *is the Frobenius morphism on* $\mathcal{N}$, *i.e.,* $\Phi_{\mathcal{N}}$ *is a* Frobenius lifting.

(2) (P, ∇_P) *is an indigenous bundle on* $\mathcal{C}$ *such that the renormalized Frobenius pull-back of* $\Phi^*_{\mathcal{N}}(P, \nabla_P)$ *is isomorphic to* (P, ∇_P), *i.e.,* (P, ∇_P) *is* Frobenius invariant with respect to $\Phi_{\mathcal{N}}$.

Moreover, this pair also gives rise in a natural way to a Frobenius lifting $\Phi_{\mathcal{C}} : \mathcal{C}^{\mathrm{ord}} \to \mathcal{C}^{\mathrm{ord}}$ *on a certain formal* p*-adic open substack* $\mathcal{C}^{\mathrm{ord}}$ *of* $\mathcal{C}$ *(which will be referred to as the ordinary locus of* $\mathcal{C}$*).*

Thus, this Theorem is a partial realization of the goal of constructing a canonical integral Frobenius invariant bundle on the universal stable curve.

Again, we observe that

> *This canonical Frobenius lifting $\Phi_{\mathcal{N}}$ is by no means compatible (relative to the operation of passing to the Jacobian) with the canonical Frobenius lifting $\Phi_{\mathcal{A}}$ (on the p-adic stack of ordinary principally polarized abelian varieties) arising from Serre-Tate theory (cf., e.g.,* [**22**], *§0.7, for more details).*

At first glance, the reader may find this fact to be extremely disappointing and unnatural. In fact, however, when understood properly, this incompatibility is something which is to be expected. Indeed, relative to the analogy between Frobenius liftings and Kähler metrics implicit in the discussion of § 1.1 $\sim$ 1.5 (cf., e.g., [**22**], § 0.8, for more details) such a compatibility would be the p-adic analogue of a compatibility between the Weil-Petersson metric on $(\mathcal{M}_{g,r})_{\mathbb{C}}$ and the Siegel upper half plane metric on $(\mathcal{A}_g)_{\mathbb{C}}$. On the other hand, it is easy to see in the complex case that these two metrics are far from compatible. (Indeed, if they were compatible, then the Torelli map $(\mathcal{M}_g)_{\mathbb{C}} \to (\mathcal{A}_g)_{\mathbb{C}}$ would be unramified, but one knows that it is ramified at hyperelliptic curves of high genus.)

Another important difference between $\Phi_{\mathcal{N}}$ and $\Phi_{\mathcal{A}}$ is that in the case of $\Phi_{\mathcal{A}}$, by taking the union of $\Phi_{\mathcal{A}}$ and its transpose, one can compactify $\Phi_{\mathcal{A}}$ into an entirely algebraic (i.e., not just p-adic analytic) object, namely a Hecke correspondence on $\mathcal{A}_g$. In the case of $\Phi_{\mathcal{N}}$, however, such a compactification into a correspondence is impossible. We refer to [**23**] for a detailed discussion of this phenomenon.

So far, we have been discussing the differences between $\Phi_{\mathcal{N}}$ and $\Phi_{\mathcal{A}}$. In fact, however, in one very important respect, they are very similar objects. Namely, they are both *(classical) ordinary Frobenius liftings*. A (classical) ordinary Frobenius lifting is defined as follows: Let k be a perfect field of characteristic p. Let $A \stackrel{\text{def}}{=} W(k)$ (the Witt vectors over k). Let S be a formal p-adic scheme which is formally smooth over A. Let $\Phi_S : S \to S$ be a morphism whose reduction modulo p is the Frobenius morphism. Then differentiating Φ_S defines a morphism $\mathrm{d}\Phi_S : \Phi_S^*\Omega_{S/A} \to \Omega_{S/A}$ which is zero in characteristic p. Thus, we may form a morphism

$$\Omega_\Phi : \Phi_S^*\Omega_{S/A} \to \Omega_{S/A}$$

by dividing $\mathrm{d}\Phi_S$ by p. Then Φ_S is called a *(classical) ordinary Frobenius lifting* if Ω_Φ is an isomorphism. Just as there is a general theory of canonical coordinates associated to real analytic Kähler metrics, there is a general theory of canonical coordinates associated to ordinary Frobenius liftings. This theory is discussed in detail in § 1 of Chapter III of [**21**]. The main result is as follows (cf. § 1 of [**21**], Chapter III):

Theorem 1.3 **(Ordinary Frobenius Liftings)**. — *Let $\Phi_S : S \to S$ be a (classical) ordinary Frobenius lifting. Then taking the invariants of $\Omega_{S/A}$ with respect to Ω_Φ gives rise to an étale local system $\Omega_\Phi^{\mathrm{et}}$ on S of free $\mathbb{Z}_p$-modules of rank equal to $\dim_A(S)$.*

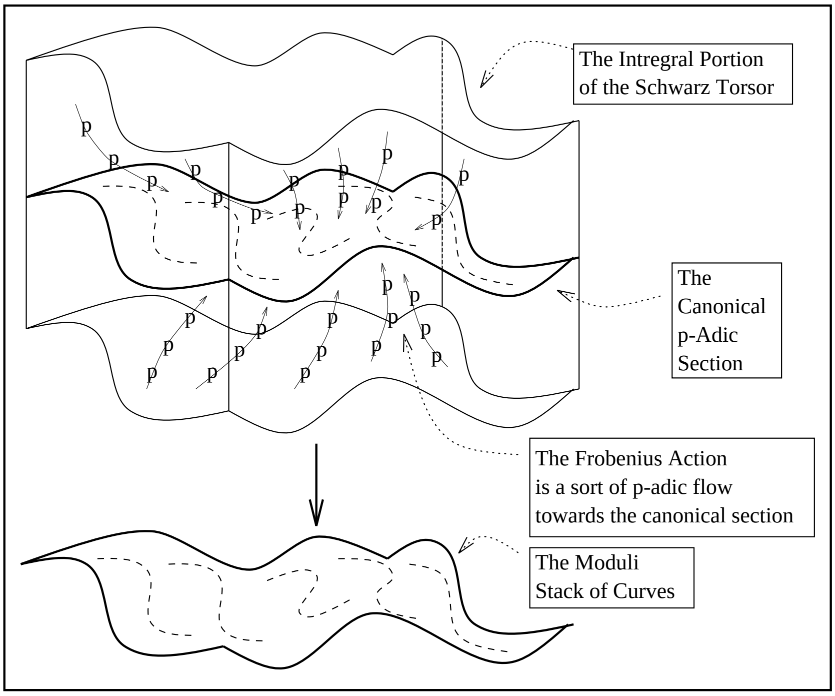

FIGURE 3. The Canonical Frobenius Action Underlying Theorem 1.2

Let $z \in S(\overline{k})$ be a point valued in the algebraic closure of k. Then $\Omega_z \stackrel{\text{def}}{=} \Omega^{\text{et}}_{\Phi}|_z$ may be thought of as a free $\mathbb{Z}_p$-module of rank $\dim_A(S)$; write Θ_z for the $\mathbb{Z}_p$-dual of Ω_z. Let S_z be the completion of S at z. Let $\widehat{\mathbf{G}}_{\text{m}}$ be the completion of the multiplicative group scheme $\mathbf{G}_{\text{m}}$ over $W(\overline{k})$ at 1. Then there is a unique isomorphism

$$\Gamma_z : S_z \cong \widehat{\mathbf{G}}_{\text{m}} \otimes^{\text{gp}}_{\mathbb{Z}_p} \Theta_z$$

such that:

(i) the derivative of Γ_z induces the natural inclusion $\Omega_z \hookrightarrow \Omega_{S/A}|_{S_z}$;

(ii) the action of Φ_S on S_z corresponds to multiplication by p on $\widehat{\mathbf{G}}_{\text{m}} \otimes^{\text{gp}}_{\mathbb{Z}_p} \Theta_z$.

Here, by "$\widehat{\mathbf{G}}_{\text{m}} \otimes^{\text{gp}}_{\mathbb{Z}_p} \Theta_z$," we mean the tensor product in the sense of (formal) group schemes. Thus, $\widehat{\mathbf{G}}_{\text{m}} \otimes^{\text{gp}}_{\mathbb{Z}_p} \Theta_z$ is noncanonically isomorphic to the product of $\dim_A(S) = \text{rank}_{\mathbb{Z}_p}(\Theta_z)$ copies of $\widehat{\mathbf{G}}_{\text{m}}$.

Thus, we obtain canonical multiplicative parameters on $\mathcal{N}$ and $\mathcal{C}^{\text{ord}}$ (from $\Phi_{\mathcal{N}}$ and $\Phi_{\mathcal{C}}$, respectively). If we apply Theorem 1.3 to the canonical lifting $\Phi_{\mathcal{A}}$ of Serre-Tate theory (cf., e.g., [**22**], § 0.7), we obtain the Serre-Tate parameters. Moreover, note that in Theorem 1.3, the identity element "1" of the formal group scheme $\widehat{\mathbf{G}}_{\mathrm{m}} \otimes_{\mathbb{Z}_p} \Omega_z$ corresponds under Γ_z to some point $\alpha_z \in S(W(\overline{k}))$ that lifts z. That is to say,

> *Theorem 1.3 also gives rise to a notion of canonical liftings of points in characteristic p.*

In the case of $\Phi_{\mathcal{A}}$, this notion coincides with the well-known notion of the Serre-Tate canonical lifting of an ordinary abelian variety. In the case of $\Phi_{\mathcal{N}}$, the theory of canonically lifted curves is discussed in detail in Chapter IV of [**21**]. In [**22**], however, the theory of canonical curves in the style of Chapter IV of [**21**] does not play a very important role.

Remark. — Certain special cases of Theorem 1.3 already appear in the work of Ihara ([**14, 15, 16, 17**]). In fact, more generally, the work of Ihara ([**14, 15, 16, 17**]) on the Schwarzian equations of Shimura curves and the possibility of constructing an analogue of Serre-Tate theory for more general hyperbolic curves anticipates, at least at a philosophical level, many aspects of the theory of [**21, 22**].

Thus, in summary, although the classical ordinary theory of [**21**] is not compatible with Serre-Tate theory relative to the Torelli map, it is in many respects deeply structurally analogous to Serre-Tate theory. Moreover, this close structural affinity arises from the fact that in both cases,

> *The ordinary locus with which the theory deals is defined by the condition that some canonical Frobenius action have slope zero.*

Thus, although some readers may feel unhappy about the use of the term "ordinary" to describe the theory of [**21**] (i.e., despite the fact that this theory is incompatible with Serre-Tate theory), we feel that this close structural affinity arising from the common condition of a slope zero Frobenius action justifies and even renders natural the use of this terminology.

Finally, just as in the complex case, where the various indigenous bundles involved gave rise to monodromy representations of the fundamental group of the hyperbolic curve involved, in the p-adic case as well, the canonical indigenous bundle of Theorem 1.2 gives rise to a canonical Galois representation, as follows. We continue with the notation of Theorem 1.2. Let $\mathcal{N}' \to \mathcal{N}$ be the morphism $\Phi_{\mathcal{N}}$, which we think of as a covering of $\mathcal{N}$; let $\mathcal{C}' \stackrel{\mathrm{def}}{=} \mathcal{C} \otimes_{\mathcal{N}} \mathcal{N}'$. Note that $\mathcal{C}$ and $\mathcal{N}$ have natural log structures (obtained by pulling back the natural log structures on $\overline{\mathcal{M}}_{g,r}$ and its tautological curve, respectively). Thus, we obtain $\mathcal{C}^{\log}$, $\mathcal{N}^{\log}$. Let

$$\Pi_{\mathcal{N}} \stackrel{\mathrm{def}}{=} \pi_1(\mathcal{N}^{\log} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p); \quad \Pi_{\mathcal{C}} \stackrel{\mathrm{def}}{=} \pi_1(\mathcal{C}^{\log} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

Similarly, we have $\Pi_{\mathcal{N}'}$; $\Pi_{\mathcal{C}'}$. Then the main result is the following (Theorem 0.4 of [**21**], Introduction):

***Theorem 1.4* (Canonical Galois Representation).** — *There is a natural $\mathbb{Z}_p$-flat, p-adically complete "ring of additive periods" $\mathcal{D}_{\mathcal{N}}^{\mathrm{Gal}}$ on which $\Pi_{\mathcal{N}'}$ (hence also $\Pi_{\mathcal{C}'}$ via the natural projection $\Pi_{\mathcal{C}'} \to \Pi_{\mathcal{N}'}$) acts continuously, together with a twisted homomorphism*

$$\rho : \Pi_{\mathcal{C}'} \to \mathrm{PGL}_2(\mathcal{D}_{\mathcal{N}}^{\mathrm{Gal}})$$

where "twisted" means with respect to the action of $\Pi_{\mathcal{C}'}$ on $\mathcal{D}_{\mathcal{N}}^{\mathrm{Gal}}$. This representation is obtained by taking Frobenius invariants of (P, ∇_P), using a technical tool known as crystalline induction.

Thus, in summary, the theory of [**21**] gives one a fairly good understanding of what happens over the ordinary locus $\overline{\mathcal{N}}_{g,r}^{\mathrm{ord}}$, complete with analogues of various objects (monodromy representations, canonical modular coordinates, etc.) that appeared in the complex case. On the other hand, it begs the following questions:

(1) What does the nonordinary part of $\overline{\mathcal{N}}_{g,r}$ look like? What sorts of nonordinary nilcurves can occur? In particular, what does the p-curvature of such nonordinary nilcurves look like?

(2) Does this "classical ordinary theory" admit any sort of compactification? One sees from [**23**] that it does not admit any sort of compactification via correspondences. Still, since the condition of being ordinary is an "open condition," it is natural to ask what happens to this classical ordinary theory as one goes to the boundary.

The theory of [**22**] answers these two questions to a large extent, not by adding on a few new pieces to [**21**], but by starting afresh and developing from new foundations a general theory of integral Frobenius invariant indigenous bundles. The theory of [**22**] will be discussed in § 2.

2. Beyond the "Classical Ordinary" Theory

2.1. Atoms, Molecules, and Nilcurves. — Let p be an odd prime. Let g and r be nonnegative integers such that $2g - 2 + r \geq 1$. Let $\overline{\mathcal{N}}_{g,r}$ be the stack of nilcurves in characteristic p. We denote by $\mathcal{N}_{g,r} \subseteq \overline{\mathcal{N}}_{g,r}$ the open substack consisting of *smooth nilcurves*, i.e., nilcurves whose underlying curve is smooth. Then the first step in our analysis of $\overline{\mathcal{N}}_{g,r}$ is the introduction of the following notions (cf. Definitions 1.1 and 3.1 of [**22**], Chapter II):

***Definition 2.1*.** — We shall call a nilcurve *dormant* if its p-curvature (i.e., the p-curvature of its underlying indigenous bundle) is identically zero. Let d be a nonnegative integer. Then we shall call a smooth nilcurve *spiked of strength d* if the zero locus of its p-curvature forms a divisor of degree d.

If d is a nonnegative integer (respectively, the symbol ∞), then we shall denote by

$$\mathcal{N}_{g,r}[d] \subseteq \mathcal{N}_{g,r}$$

the locally closed substack of nilcurves that are spiked of strength d (respectively, dormant). It is immediate that there does indeed exist such a locally closed substack, and that if k is an algebraically closed field of characteristic p, then

$$\mathcal{N}_{g,r}(k) = \coprod_{d=0}^{\infty} \mathcal{N}_{g,r}[d]\ (k)$$

Moreover, we have the following result (cf. [**22**], Chapter II, Theorems 1.12, 2.8, and 3.9):

***Theorem 2.2* (Stratification of $\mathcal{N}_{g,r}$).** — *Any two irreducible components of $\overline{\mathcal{N}}_{g,r}$ intersect. Moreover, for $d = 0, 1, \ldots, \infty$, the stack $\mathcal{N}_{g,r}[d]$ is smooth over $\mathbf{F}_p$ of dimension $3g-3+r$ (if it is nonempty). Finally, $\mathcal{N}_{g,r}[\infty]$ is irreducible, and its closure in $\overline{\mathcal{N}}_{g,r}$ is smooth over $\mathbf{F}_p$.*

Thus, in summary, we see that

> *The classification of nilcurves by the size of the zero locus of their p-curvatures induces a natural decomposition of $\mathcal{N}_{g,r}$ into smooth (locally closed) strata.*

Unfortunately, however, Theorem 2.2 still only gives us a very rough idea of the structure of $\mathcal{N}_{g,r}$. For instance, it tells us nothing of the degree of each $\mathcal{N}_{g,r}[d]$ over $\mathcal{M}_{g,r}$.

Remark. — Some people may object to the use of the term "stratification" here for the reason that in certain contexts (e.g., the Ekedahl-Oort stratification of the moduli stack of principally polarized abelian varieties – cf. [**11**], § 2), this term is only used for decompositions into locally closed subschemes whose closures satisfy certain (rather stringent) axioms. Here, we do not mean to imply that we can prove any nontrivial results concerning the closures of the $\mathcal{N}_{g,r}[d]$'s. That is to say, in [**22**], we use the term "stratification" only in the weak sense (i.e., that $\mathcal{N}_{g,r}$ is the union of the $\mathcal{N}_{g,r}[d]$). This usage conforms to the usage of Lecture 8 of [**24**], where "flattening stratifications" are discussed.

In order to understand things more explicitly, it is natural to attempt to do the following:

(1) *Understand the structure – especially, what the p-curvature looks like – of all* molecules *(i.e., nilcurves whose underlying curve is totally degenerate).*

(2) *Understand how each molecule* deforms, *i.e., given a molecule, one can consider its formal neighborhood $\mathcal{N}$ in $\overline{\mathcal{N}}_{g,r}$. Then one wants to know the degree of each $\mathcal{N} \bigcap \mathcal{N}_{g,r}[d]$ (for all d) over the corresponding formal neighborhood $\mathcal{M}$ in $\overline{\mathcal{M}}_{g,r}$.*

Obtaining a complete answer to these two questions is the topic of [**22**], Chapters IV and V.

First, we consider the problem of understanding the structure of *molecules*. Since the underlying curve of a molecule is a totally degenerate curve – i.e., a stable curve obtained by gluing together $\mathbb{P}^1$'s with three nodal/marked points – it is natural to restrict the given nilpotent indigenous bundle on the whole curve to each of these $\mathbb{P}^1$'s with three marked points. Thus, for each irreducible component of the original curve, we obtain a $\mathbb{P}^1$ with three marked points equipped with something very close to a nilpotent indigenous bundle. The only difference between this bundle and an indigenous bundle is that its monodromy at some of the marked points (i.e., those marked points that correspond to nodes on the original curve) might not be nilpotent. In general, a bundle (with connection) satisfying all the conditions that an indigenous bundle satisfies except that its monodromy at the marked points might not be nilpotent is called a *torally indigenous bundle* (cf. [**22**], Chapter I, Definition 4.1). (When there is fear of confusion, indigenous bundles in the strict sense (as in [**21**], Chapter I) will be called *classical indigenous.*) For simplicity, we shall refer to any pointed stable curve (respectively, totally degenerate pointed stable curve) equipped with a nilpotent *torally indigenous* bundle as a *nilcurve* (respectively, *molecule*) (cf. § 0 of [**22**], Chapter V). Thus, when it is necessary to avoid confusion with the toral case, we shall say that "$\overline{\mathcal{N}}_{g,r}$ is the stack of classical nilcurves." Finally, we shall refer to a (possibly toral) nilcurve whose underlying curve is $\mathbb{P}^1$ with three marked points as an *atom.*

At any rate, to summarize, *a molecule may be regarded as being made up of atoms.* It turns out that the monodromy at each marked point of an atom (or, in fact, more generally any nilcurve) has an invariant called the *radius.* The radius is, strictly speaking, an element of $\mathbf{F}_p/\{\pm\}$ (cf. Proposition 1.5 of [**22**], Chapter II) – i.e., the quotient set of $\mathbf{F}_p$ by the action of ± 1 – but, by abuse of notation, we shall often speak of the radius ρ as an element of $\mathbf{F}_p$. Then we have the following answer to (1) above (cf. § 1 of [**22**], Chapter V):

***Theorem 2.3* (The Structure of Atoms and Molecules)**. — *The structure theory of atoms (over any field of characteristic p) may be summarized as follows:*

(1) *The three radii of an atom define a bijection of the set of isomorphism classes of atoms with the set of ordered triples of elements of* $\mathbf{F}_p/\{\pm 1\}$.

(2) *For any triple of elements* $\rho_\alpha, \rho_\beta, \rho_\gamma \in \mathbf{F}_p$, *there exist integers* $a, b, c \in [0, p-1]$ *such that (i)* $a \equiv \pm 2\rho_\alpha$, $b \equiv \pm 2\rho_\beta$, $c \equiv \pm 2\rho_\gamma$; *(ii)* $a+b+c$ *is odd and* $< 2p$. *Moreover, the atom of radii* $\rho_\alpha, \rho_\beta, \rho_\gamma$ *is dormant if and only if the following three inequalities are satisfied simultaneously:* $a + b > c$, $a + c > b$, $b + c > a$.

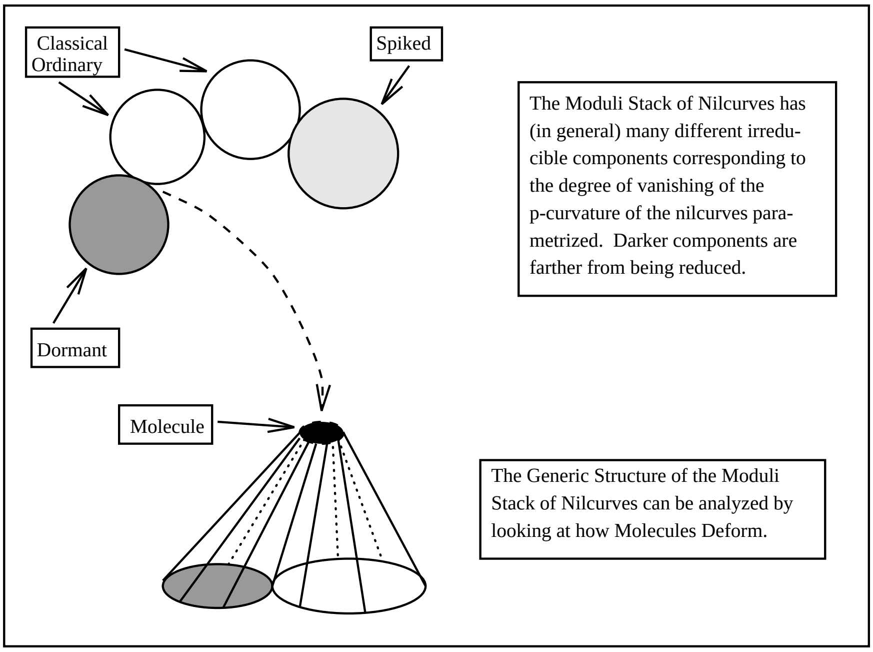

FIGURE 4. The Structure of $\overline{\mathcal{N}}_{g,r}$

(3) *Suppose that the atom of radii $\rho_\alpha, \rho_\beta, \rho_\gamma$ is nondormant. Let $v_\alpha, v_\beta, v_\gamma$ be the degrees of the zero loci of the p-curvature at the three marked points. Then the non-negative integers $v_\alpha, v_\beta, v_\gamma$ are uniquely determined by the following two conditions: (i) $v_\alpha + v_\beta + v_\gamma$ is odd and $< p$; (ii) $v_\alpha \equiv \pm 2\rho_\alpha$, $v_\beta \equiv \pm 2\rho_\beta$, $v_\gamma \equiv \pm 2\rho_\gamma$.*

Molecules are obtained precisely by gluing together atoms at their marked points in such a way that the radii at marked points that are glued together coincide (as elements of $\mathbf{F}_p/\{\pm 1\}$).

In the last sentence of the theorem, we use the phrase "obtained precisely" to mean that all molecules are obtained in that way, and, moreover, any result of gluing together atoms in that fashion forms a molecule. Thus,

> *Theorem 2.3 reduces the structure theory of atoms and molecules to a matter of combinatorics.*

Theorem 2.3 follows from the theory of [**22**], Chapter IV.

Before proceeding, we would like to note the analogy with the theory of "pants" (see [**1**] for an exposition) in the complex case. In the complex case, the term "pants" is used to describe a Riemann surface which is topologically isomorphic to a Riemann sphere minus three points. The *holomorphic* isomorphism class of such a Riemann surface is given precisely by specifying three radii, i.e., the size of its three holes. Moreover, any hyperbolic Riemann surface can be analyzed by decomposing it into a union of pants, glued together at the boundaries. Thus, there is a certain analogy between the theory of pants and the structure theory of atoms and molecules given in Theorem 2.3.

Next, we turn to the issue of understanding how molecules deform. Let M be a nondormant *classical molecule* (i.e., it has nilpotent monodromy at all the marked points). Let us write n_{tor} for the number of "toral nodes" (i.e., nodes at which the monodromy is not nilpotent) of M. Let us write n_{dor} for the number of dormant atoms in M. To describe the deformation theory of M, it is useful to choose a *plot* Π for M. A plot is an ordering (satisfying certain conditions) of a certain subset of the nodes of M (see § 1 of [**22**], Chapter V for more details). This ordering describes the order in which one deforms the nodes of M. (Despite the similarity in notation, plots have nothing to do with the "VF-patterns" discussed below.) Once the plot is fixed, one can contemplate the various *scenarios* that may occur. Roughly speaking, a scenario is an assignment (satisfying certain conditions) of one of the three symbols $\{0, +, -\}$ to each of the branches of each of the nodes of M (see § 1 of [**22**], Chapter V for more details). There are $2^{n_{\mathrm{dor}}}$ possible scenarios. The point of all this terminology is the following:

> *One wants to deform the nodes of M in a such a way that one can always keep track of how the p-curvature deforms as one deforms the nilcurve.*

If one deforms the nodes in the fashion stipulated by the plot and scenario, then each deformation that occurs is one the following four types: *classical ordinary*, *grafted*, *philial*, *aphilial*.

The *classical ordinary* case is the case where the monodromy (at the node in question) is nilpotent. It is also by far the most technically simple and is already discussed implicitly in [**21**]. The *grafted case* is the case where a dormant atom is grafted on to (what after previous deformations is) a nondormant smooth nilcurve. This is the case where the consequent deformation of the p-curvature is the most technically difficult to analyze and is the reason for the introduction of "plots" and "scenarios." In order to understand how the p-curvature deforms in this case, one must introduce a certain technical tool called the *virtual p-curvature*. The theory of virtual p-curvatures is discussed in § 2.2 of [**22**], Chapter V. The *philial case* (respectively, *aphilial case*) is the case where one glues on a nondormant atom to (what after previous deformations is) a nondormant smooth nilcurve, and the parities (i.e., whether the number is even or odd) of the vanishing orders of the p-curvature at the two branches of the node are *opposite* to one another (respectively, the *same*). In the philial case (respectively, aphilial case), deformation gives rise to a spike (respectively, no spike). An illustration of these four fundamental types of deformation is given in Fig. 6. The signs in this

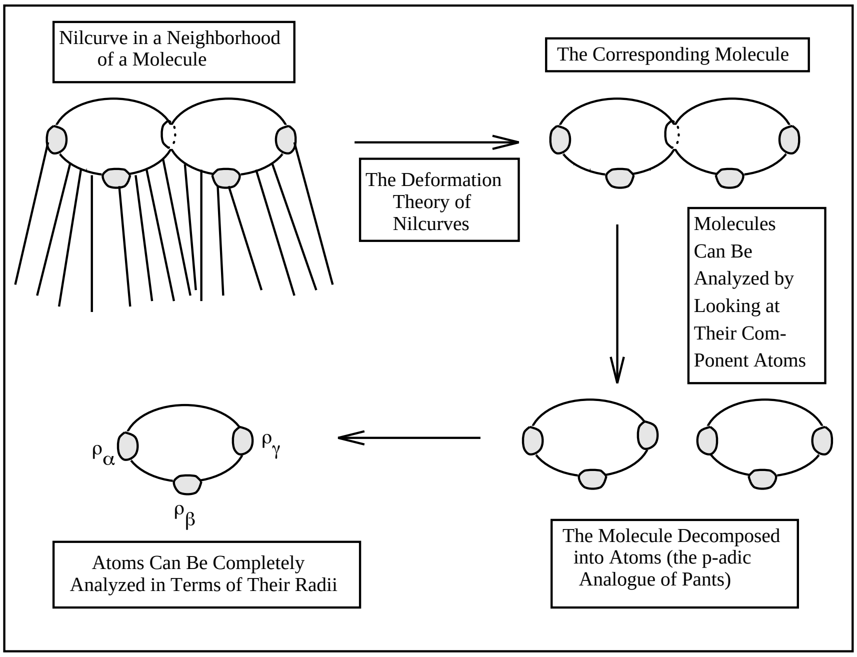

FIGURE 5. The Step Used to Analyze the Structure of $\overline{\mathcal{N}}_{g,r}$

illustration are the signs that are assigned to the branches of the nodes by the "scenario." When the p-curvature is not identically zero (i.e., on the light-colored areas), this sign is the parity (i.e., plus for even, minus for odd) of the vanishing order of the p-curvature. For a given scenario Σ, we denote by $n_{\mathrm{phl}}(\Sigma)$ (respectively, $n_{\mathrm{aph}}(\Sigma)$) the number of philial (respectively, aphilial) nodes that occur when the molecule is deformed according to that scenario.

If $U = \mathrm{Spec}(A)$ is a connected noetherian scheme of dimension 0, then we shall refer to the length of the artinian ring A as the *padding degree of* U. Then the theory just discussed gives rise to the following answer to (2) above (cf. Theorem 1.1 of [**22**], Chapter V):

***Theorem 2.4* (Deformation Theory of Molecules)**. — *Let M be a classical molecule over an algebraically closed field k of characteristic p. Let $\mathcal{N}$ be the completion of $\overline{\mathcal{N}}_{g,r}$ at M. Let $\mathcal{M}$ be the completion of $(\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$ at the point defined by the curve underlying M. Let $\overline{\eta}$ be the strict henselization of the generic point of $\mathcal{M}$. Then the natural morphism $\mathcal{N} \to \mathcal{M}$ is finite and flat of degree $2^{n_{\mathrm{tor}}}$. Moreover:*

(1) *If M is* dormant, *then $\mathcal{N}_{\mathrm{red}} \cong \mathcal{M}$, and $\mathcal{N}_{\overline{\eta}}$ has padding degree 2^{3g-3+r}.*

(2) *If M is* nondormant, *fix a plot Π for M. Then for each of the $2^{n_{\mathrm{dor}}}$ scenarios associated to Π, there exists a natural open substack $\mathcal{N}_\Sigma \subseteq \mathcal{N}_{\overline{\eta}} \stackrel{\mathrm{def}}{=} \mathcal{N} \times_{\mathcal{M}} \overline{\eta}$ such that:*

(i.) $\mathcal{N}_{\overline{\eta}}$ *is the disjoint union of the* $\mathcal{N}_\Sigma$ *(as* Σ *ranges over all the scenarios); (ii.) every residue field of* $\mathcal{N}_\Sigma$ *is separable over (hence equal to)* $k(\overline{\eta})$*; (iii) the degree of* $(\mathcal{N}_\Sigma)_{\mathrm{red}}$ *over* $\overline{\eta}$ *is* $2^{n_{\mathrm{aph}}(\Sigma)}$*; (iv) each connected component of* $\mathcal{N}_\Sigma$ *has padding degree* $2^{n_{\mathrm{phl}}(\Sigma)}$*; (v) the smooth nilcurve represented by any point of* $(\mathcal{N}_\Sigma)_{\mathrm{red}}$ *is spiked of strength* $p \cdot n_{\mathrm{phl}}(\Sigma)$*.*

In particular, this Theorem reduces the computation of the degree of any $\mathcal{N}_{g,r}[d]$ *over* $(\mathcal{M}_{g,r})_{\mathbf{F}_p}$ *to a matter of combinatorics.*

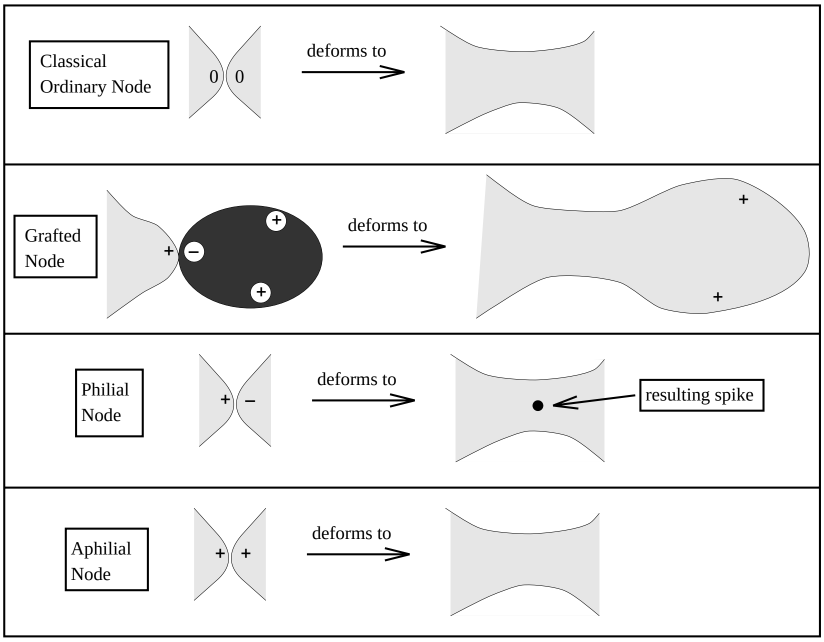

FIGURE 6. The Four Types of Nodal Deformation

For instance, let us denote by $n^{\mathrm{ord}}_{g,r,p}$ the degree of $\mathcal{N}^{\mathrm{ord}}_{g,r}$ (which – as a consequence of Theorem 2.4! (cf. Corollary 1.2 of [**22**], Chapter V) – is open and dense in $\mathcal{N}_{g,r}[0]$) over $(\mathcal{M}_{g,r})_{\mathbf{F}_p}$. Then following the algorithm implicit in Theorem 2.4, $n^{\mathrm{ord}}_{g,r,p}$ is computed explicitly for low g and r in Corollary 1.3 of [**22**], Chapter V (e.g., $n^{\mathrm{ord}}_{1,1,p} = n^{\mathrm{ord}}_{0,4,p} = p$; $n^{\mathrm{ord}}_{0,5,p} = \frac{1}{2}(p^2+1)$; etc.). Moreover, we note the following two interesting phenomena:

(1) *Degrees such as* $n^{\mathrm{ord}}_{g,r,p}$ *tend to be well-behaved – even polynomial, with coefficients equal to various integrals over Euclidean space – as functions of* p*.* Thus, for instance, the limit, as p goes to infinity, of $n^{\mathrm{ord}}_{0,r,p}/p^{r-3}$ exists and is equal to the volume of a certain polyhedron in $(r-3)$-dimensional Euclidean space. See Corollary 1.3 of [**22**], Chapter V for more details.

(2) Theorem 2.4 gives, for every choice of totally degenerate r-pointed stable curve of genus g, an (a priori) distinct algorithm for computing $n^{\mathrm{ord}}_{g,r,p}$. Since $n^{\mathrm{ord}}_{g,r,p}$, of course, does not depend on the choice of underlying totally degenerate curve, we thus obtain equalities between the various sums that occur (to compute $n^{\mathrm{ord}}_{g,r,p}$) in each case. If one writes out these equalities, *one thus obtains various combinatorial identities*. Although the author has yet to achieve a systematic understanding of these combinatorial identities, already in the cases that have been computed (for low g and r), these identities reduce to such nontrivial combinatorial facts as Lemmas 3.5 and 3.6 of [**22**], Chapter V.

Although the author does not have even a conjectural theoretical understanding of these two phenomena, he nonetheless feels that they are very interesting and deserve further study.

2.2. The $\mathcal{MF}^{\nabla}$-Object Point of View. — Before discussing the general theory of canonical liftings of nilpotent indigenous bundles, it is worth stopping to examine the general conceptual context in which this theory will be developed. To do this, let us first recall the theory of $\mathcal{MF}^{\nabla}$*-objects* developed in § 2 of [**6**]. Let p be a prime number, and let S be a smooth $\mathbb{Z}_p$-scheme. Then in *loc. cit.*, a certain category $\mathcal{MF}^{\nabla}(S)$ is defined. Objects of this category $\mathcal{MF}^{\nabla}(S)$ consist of: (1) *a vector bundle $\mathcal{E}$ on S equipped with an integrable connection* $\nabla_{\mathcal{E}}$ (one may equivalently regard the pair $(\mathcal{E}, \nabla_{\mathcal{E}})$ as a crystal on the crystalline site Crys$(S \otimes_{\mathbb{Z}_p} \mathbf{F}_p/\mathbb{Z}_p)$ *valued in the category of vector bundles*); (2) *a filtration $F^{\cdot}(\mathcal{E}) \subseteq \mathcal{E}$ (called the Hodge filtration) of subbundles of* $\mathcal{E}$; (3) *a Frobenius action $\Phi_{\mathcal{E}}$ on the crystal* $(\mathcal{E}, \nabla_{\mathcal{E}})$. Moreover, these objects satisfy certain conditions, which we omit here.

Let Π_S be the fundamental group of $S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (for some choice of base-point). In *loc. cit.*, for each $\mathcal{MF}^{\nabla}(S)$-object $(\mathcal{E}, \nabla_{\mathcal{E}}, F^{\cdot}(\mathcal{E}), \Phi_{\mathcal{E}})$, a certain natural Π_S-module V is constructed by taking invariants of $(\mathcal{E}, \nabla_{\mathcal{E}})$ with respect to its Frobenius action $\Phi_{\mathcal{E}}$. If $\mathcal{E}$ is of rank r, then V is a free $\mathbb{Z}_p$-module of rank r. On typical example of such an $\mathcal{MF}^{\nabla}(S)$-object is the following:

> If $X \to S$ is the tautological abelian variety over the moduli stack of principally polarized abelian varieties, then the relative *first de Rham cohomology module* of $X \to S$ forms an $\mathcal{MF}^{\nabla}(S)$-module whose restriction to the ordinary locus of S is (by Serre-Tate theory) intimately connected to the "p-adic uniformization theory" of S.

In the context of [**22**], we would like to consider the case where $S = (\mathcal{M}_{g,r})_{\mathbb{Z}_p}$. Moreover, just as the first de Rham cohomology module of the universal abelian variety gives rise to a "fundamental uniformizing $\mathcal{MF}^{\nabla}(S)$-module" on the moduli stack of principally polarized abelian varieties, we would like to define and study a corresponding "fundamental uniformizing $\mathcal{MF}^{\nabla}$-object" on $(\mathcal{M}_{g,r})_{\mathbb{Z}_p}$. Unfortunately, as long as one sticks to the conventional definition of $\mathcal{MF}^{\nabla}$-object given in [**6**], it appears that such a natural "fundamental uniformizing $\mathcal{MF}^{\nabla}$-object" simply does not exist on $(\mathcal{M}_{g,r})_{\mathbb{Z}_p}$. This is not so surprising in view of the nonlinear nature of the Teichmüller group (i.e., the fundamental group of $(\mathcal{M}_{g,r})_{\mathbb{C}}$). In order to obtain a natural "fundamental uniformizing $\mathcal{MF}^{\nabla}$-object" on $(\mathcal{M}_{g,r})_{\mathbb{Z}_p}$, one must generalize the "classical" linear notion of [**6**] as follows: Instead of considering crystals (equipped with filtrations and Frobenius actions) valued in the category of *vector bundles*, one must consider crystals (still equipped with filtrations and Frobenius actions in some appropriate sense) valued in the category of *schemes* (or more generally, *algebraic spaces*). Thus,

> *One philosophical point of view from which to view* [**22**] *is that it is devoted to the study of a certain canonical uniformizing $\mathcal{MF}^{\nabla}$-object on $(\mathcal{M}_{g,r})_{\mathbb{Z}_p}$ valued in the category of algebraic spaces.*

Just as in the case of abelian varieties, this canonical uniformizing $\mathcal{MF}^{\nabla}$-object will be obtained by taking some sort of de Rham cohomology of the universal curve over $(\mathcal{M}_{g,r})_{\mathbb{Z}_p}$. The rest of this subsection is devoted to describing this $\mathcal{MF}^{\nabla}$-object in more detail.

Now let S be the spectrum of an algebraically closed field (of characteristic not equal to 2), and let X be a smooth, proper, geometrically curve over S of genus ≥ 2. Let $P \to X$ be a $\mathbb{P}^1$-bundle equipped with a connection ∇_P. If $\sigma : X \to P$ is a section of this $\mathbb{P}^1$-bundle, then we shall refer to the number $\frac{1}{2}\deg(\sigma^*\tau_{P/X})$ (where $\tau_{P/X}$ is the relative tangent bundle of P over X) as the *canonical height of* σ. Moreover, note that by differentiating σ by means of ∇_P, one obtains a morphism $\tau_{X/S} \to \sigma^*\tau_{P/X}$. We shall say that σ is *horizontal* if this morphism is identically zero.

(Roughly speaking) we shall call (P, ∇_P) *crys-stable* if it does not admit any horizontal sections of canonical height ≤ 0 (see Definition 1.2 of [**22**], Chapter I for a precise definition). (Roughly speaking) we shall call (P, ∇_P) *crys-stable of level* 0 (or just *stable*) if it does not admit any sections of canonical height ≤ 0 (see Definition 3.2 of [**22**], Chapter I for a precise definition). Let l be a positive half-integer (i.e., a positive element of $\frac{1}{2}\mathbb{Z}$). We shall call (P, ∇_P) *crys-stable of level l* if it admits a section of canonical height $-l$. If it does admit such a section, then this section is the unique section of $P \to X$ of negative canonical height. This section will be referred to as the *Hodge section* (see Definition 3.2 of [**22**], Chapter I for more details). For instance, if $\mathcal{E}$ is a vector bundle of rank two on X such that $\mathrm{Ad}(\mathcal{E})$ is a stable vector bundle on X (of rank three), and $P \to X$ is the projective bundle associated to $\mathcal{E}$, then (P, ∇_P) will be crys-stable of level 0 (regardless of the choice of ∇_P). On the other hand, an indigenous bundle on X will be crys-stable of level $g - 1$. More generally, these definitions generalize to the case when X is a family of pointed stable curves over an arbitrary base (on which 2 is invertible).

The nonlinear $\mathcal{MF}^{\nabla}$-object on $(\mathcal{M}_{g,r})_{\mathbb{Z}_p}$ (where p is odd) that is the topic of [**22**] is (roughly speaking) the crystal in algebraic spaces given by the considering the *fine moduli space* $\mathcal{Y} \to (\mathcal{M}_{g,r})_{\mathbb{Z}_p}$ *of crys-stable bundles* on the universal curve (cf. Theorem 2.7, Proposition 3.1 of [**22**], Chapter I for more details). Put another way, this crystal is a sort of de Rham-theoretic H^1 with coefficients in PGL_2 of the universal curve over $\mathcal{M}_{g,r}$. The nonlinear analogue of the Hodge filtration on an $\mathcal{MF}^{\nabla}$-object is the collection of subspaces given by the *fine moduli spaces* $\mathcal{Y}^l$ *of crys-stable bundles of level* l (for various l) – cf. [**22**], Chapter I, Proposition 3.3, Lemmas 3.4 and 3.8, and Theorem 3.10 for more details.

Remark. — This collection of subspaces is reminiscent of the stratification (on the moduli stack of smooth nilcurves) of §2.1. This is by no means a mere coincidence. In fact, in some sense, the stratification of $\mathcal{N}_{g,r}$ which was discussed in § 2.1 is the Frobenius conjugate of the Hodge structure mentioned above. That is to say, *the relationship between these two collections of subspaces is the nonlinear analogue of the relationship between the filtration on the de Rham cohomology of a variety in positive characteristic induced by the "conjugate spectral sequence" and the Hodge filtration on the cohomology.* (That is to say, the former filtration is the Frobenius conjugate of the latter filtration.)

Thus, to summarize, relative to the analogy between the nonlinear objects dealt with in this paper and the "classical" $\mathcal{MF}^{\nabla}$-objects of [**6**], the only other piece of data that we need is a *Frobenius action.* It is this issue of defining a natural Frobenius action which occupies the bulk of [**22**].

2.3. The Generalized Notion of a Frobenius Invariant Indigenous Bundle. — In this subsection, we would like to take up the task of describing the Frobenius action on crys-stable bundles. Just as in the case of the linear $\mathcal{MF}^{\nabla}$-objects of [**6**], and as motivated by comparison with the complex case (see the discussion of § 1), we are interested in *Frobenius invariant sections of the* $\mathcal{MF}^{\nabla}$*-object*, i.e., Frobenius invariant bundles. Moreover, since ultimately we are interested in *uniformization theory*, instead of studying general Frobenius invariant crys-stable bundles, we will only consider *Frobenius invariant indigenous bundles.* The reason that we must nonetheless introduce crys-stable bundles is that in order to obtain canonical lifting theories that are valid at generic points of $\mathcal{N}_{g,r}$ parametrizing *dormant* or *spiked* nilcurves, it is necessary to consider indigenous bundles that are fixed not (necessarily) after *one* application of Frobenius, but after *several* applications of Frobenius. As one applies Frobenius over and over again, the bundles that appear at intermediate stages need not be indigenous. They will, however, be crys-stable. This is why we must introduce crys-stable bundles.

In order to keep track of how the bundle transforms after various applications of Frobenius, it is necessary to introduce a certain combinatorial device called a *VF-pattern* (where "VF" stands for "Verschiebung/Frobenius"). VF-patterns may be described as follows. Fix nonnegative integers g, r such that $2g - 2 + r > 0$. Let $\chi \stackrel{\text{def}}{=} \frac{1}{2}(2g - 2 + r)$. Let $\mathcal{L}ev$ be the set of $l \in \frac{1}{2}\mathbb{Z}$ satisfying $0 \leq l \leq \chi$. We shall call $\mathcal{L}ev$ the *set of levels*. (That is, $\mathcal{L}ev$ is the set of possible levels of crys-stable bundles.) Let $\Pi : \mathbb{Z} \to \mathcal{L}ev$ be a map of sets, and let ϖ be a positive integer. Then we make the following definitions:

(i) We shall call (Π, ϖ) a *VF-pattern* if $\Pi(n + \varpi) = \Pi(n)$ for all $n \in \mathbb{Z}$; $\Pi(0) = \chi$; $\Pi(i) - \Pi(j) \in \mathbb{Z}$ for all $i, j \in \mathbb{Z}$ (cf. Definition 1.1 of [**22**], Chapter III).

(ii) A VF-pattern (Π, ϖ) will be called *pre-home* if $\Pi(\mathbb{Z}) = \{\chi\}$. A VF-pattern (Π, ϖ) will be called the *home VF-pattern* if it is pre-home and $\varpi = 1$.

(iii) A VF-pattern (Π, ϖ) will be called *binary* if $\Pi(\mathbb{Z}) \subseteq \{0, \chi\}$. A VF-pattern (Π, ϖ) will be called the *VF-pattern of pure tone* ϖ if $\Pi(n) = 0$ for all $n \in \mathbb{Z}$ not divisible by ϖ.

(iv) Let (Π, ϖ) be a VF-pattern. Then $i \in \mathbb{Z}$ will be called *indigenous (respectively, active; dormant) for this VF-pattern* if $\Pi(i) = \chi$ (respectively, $\Pi(i) \neq 0$; $\Pi(i) = 0$). If $i, j \in \mathbb{Z}$, and $i < j$, then (i, j) will be called *ind-adjacent for this VF-pattern* if $\Pi(i) = \Pi(j) = \chi$ and $\Pi(n) \neq \chi$ for all $n \in \mathbb{Z}$ such that $i < n < j$.

At the present time, all of this terminology may seem rather abstruse, but eventually, we shall see that it corresponds in a natural and evident way to the p-adic geometry defined by indigenous bundles that are Frobenius invariant in the fashion described by the VF-pattern in question. Finally, we remark that often, in order to simplify notation, we shall just write Π for the VF-pattern (even though, strictly speaking, a VF-pattern is a pair (Π, ϖ)).

Now fix an odd prime p. Let (Π, ϖ) be a *VF-pattern.* Let S be a perfect scheme of characteristic p. Let $X \to S$ be a smooth, proper, geometrically connected curve of genus $g \geq 2$. (Naturally, the theory goes through for arbitrary pointed stable curves, but for simplicity, we assume in the present discussion that the curve is smooth and without marked points.) Write $W(S)$ for the (ind-)scheme of Witt vectors with coefficients in S. Let $\mathcal{P}$ be a crystal on $\mathrm{Crys}(X/W(S))$ valued in the category of $\mathbb{P}^1$-bundles. Thus, the restriction $\mathcal{P}|_X$ of $\mathcal{P}$ to $\mathrm{Crys}(X/S)$ may be thought of as a $\mathbb{P}^1$-bundle with connection on the curve $X \to S$. Let us assume that $\mathcal{P}|_X$ defines an *indigenous bundle on* X. Now we consider the following procedure (cf. Fig. 7):

> Using the Hodge section of $\mathcal{P}|_X$, one can form the *renormalized Frobenius pull-back* $\mathcal{P}_1 \stackrel{\text{def}}{=} \mathbb{F}^*(\mathcal{P})$ of $\mathcal{P}$. Thus, $\mathbb{F}^*(\mathcal{P})$ will be a crystal valued in the category of $\mathbb{P}^1$-bundles on $\mathrm{Crys}(X/W(S))$. Assume

> that $\mathcal{P}_1|_X$ is crys-stable of level $\Pi(1)$. Then there are two possibilities: either $\Pi(1)$ is zero or nonzero. If $\Pi(1) = 0$, then let $\mathcal{P}_2$ be the usual (i.e., non-renormalized) Frobenius pull-back $\Phi^*\mathcal{P}_1$ of the crystal $\mathcal{P}_1$. If $\Pi(1) \neq 0$, then $\mathcal{P}_1|_X$ is crys-stable of positive level, hence admits a *Hodge section*; thus, using the Hodge section of $\mathcal{P}_1|_X$, we may form the renormalized Frobenius pull-back $\mathcal{P}_2 \stackrel{\text{def}}{=} \mathbb{F}^*(\mathcal{P}_1)$ of $\mathcal{P}_1$. Continuing inductively in this fashion – i.e., always assuming $\mathcal{P}_i|_X$ to be crys-stable of level $\Pi(i)$, and forming $\mathcal{P}_{i+1}$ by taking the renormalized (respectively, usual) Frobenius pull-back of $\mathcal{P}_i$ if $\Pi(i) \neq 0$ (respectively, $\Pi(i) = 0$), we obtain a sequence $\mathcal{P}_i$ of crystals on $\mathrm{Crys}(X/W(S))$ valued in the category of $\mathbb{P}^1$-bundles.

Then we make the following

Definition 2.5. — We shall refer to $\mathcal{P}$ as Π-*indigenous* (on X) if all the assumptions (on the $\mathcal{P}_i$) necessary to carry out the above procedure are satisfied, and, moreover, $\mathcal{P}_\varpi \cong \mathcal{P}$.

Thus, to say that $\mathcal{P}$ is Π-indigenous (more properly, (Π, ϖ)-indigenous) is to say that it is Frobenius invariant in the fashion specified by the combinatorial data (Π, ϖ).

Now we are ready to define a certain stack that is of central importance in [**22**]. The stack $\mathcal{Q}^\Pi$ – also called the *stack of quasi-analytic self-isogenies of type* (Π, ϖ) – is defined as follows:

> *To a perfect scheme S, $\mathcal{Q}^\Pi(S)$ assigns the category of pairs $(X \to S, \mathcal{P})$, where $X \to S$ is a curve as above and $\mathcal{P}$ is a Π-indigenous bundle on X.*

Thus, $\mathcal{Q}^\Pi$ *is may be regarded as the moduli stack of indigenous bundles that are Frobenius invariant in the fashion specified by the VF-pattern* Π.

We remark that in fact, more generally, one can define $\mathcal{Q}^\Pi$ on the category of *epiperfect schemes S*. (Whereas a perfect scheme is a scheme on which the Frobenius morphism is an isomorphism, an epiperfect scheme is one on which the Frobenius morphism is a closed immersion.) Then instead of using $W(S)$, one works over $B(S)$ – i.e., the "universal PD-thickening of S." For instance, the well-known ring B_{crys} introduced by Fontaine (and generalized to the higher-dimensional case in [**6**]) is a special case of $B(S)$. *The point is that one needs the base spaces that one works with to be $\mathbb{Z}_p$-flat and equipped with a natural Frobenius action.* The advantage of working with arbitrary $B(S)$ (for S epiperfect) is that the theory of crystalline representations (and the fact that B_{crys} is a special case of $B(S)$) suggest that $B(S)$ is likely to be the most general natural type of space having these two properties – i.e., $\mathbb{Z}_p$-flatness and being equipped with a natural Frobenius action. The disadvantage of working with arbitrary $B(S)$ (as opposed to just $W(S)$ for perfect S) is that many properties of

$\mathcal{Q}^{\Pi}$ are technically more difficult or (at the present time impossible) to prove in the epiperfect case. For the sake of simplicity, in this Introduction, we shall only consider the perfect case. For more details, we refer to [**22**], Chapter VI.

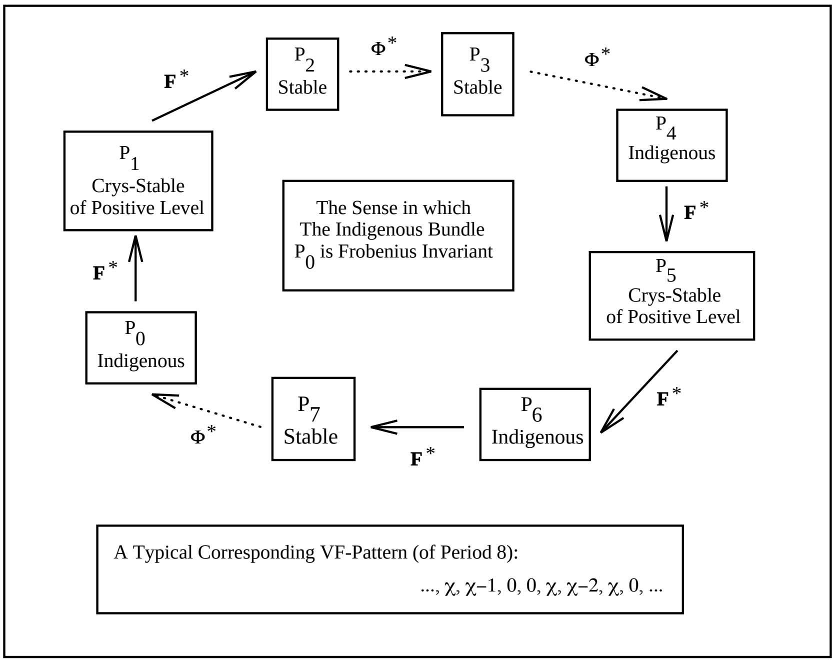

FIGURE 7. The Sense of Frobenius Invariance Specified by a VF-Pattern

Now, we are ready to discuss the main results concerning $\mathcal{Q}^{\Pi}$. The general theory of $\mathcal{Q}^{\Pi}$ is the topic of [**22**], Chapter VI. We begin with the following result (cf. Theorem 2.2 of [**22**], Chapter VI):

***Theorem 2.6* (Representability and Affineness)**. — *The stack $\mathcal{Q}^{\Pi}$ is representable by a perfect algebraic stack whose associated coarse moduli space (as in* [**7**]*, Chapter 1, Theorem 4.10) is quasi-affine. If* Π *is pre-home, then this coarse moduli space is even affine.*

Thus, in the pre-home case, $\mathcal{Q}^{\Pi}$ is *perfect and affine*. In particular, any sort of de Rham/crystalline-type cohomology on $\mathcal{Q}^{\Pi}$ must vanish. It is for this reason that we say (in the pre-home case) that $\mathcal{Q}^{\Pi}$ is *crystalline contractible* (cf. Fig. 8). Moreover, (cf. Theorem 2.12 of [**22**], Chapter III),

***Corollary 2.7* (Irreducibility of Moduli)**. — *(The fact that $\mathcal{Q}^{\Pi}$ is crystalline contractible for the home VF-pattern is intimately connected with the fact that) $\mathcal{M}_{g,r}$ is irreducible.*

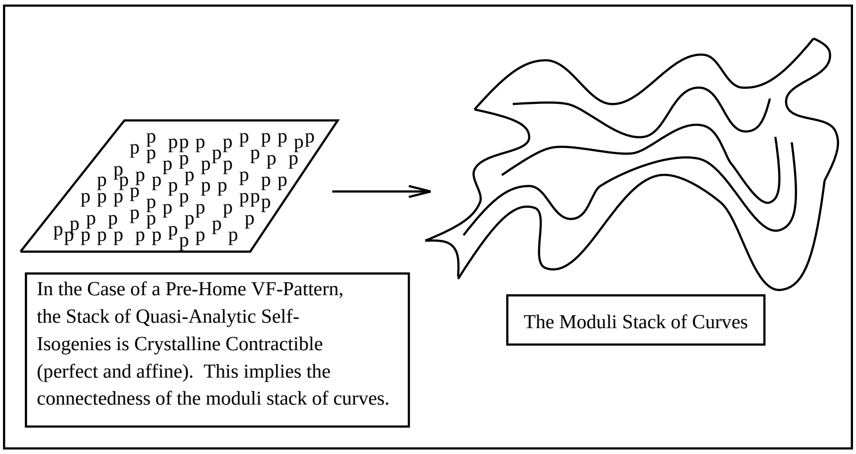

FIGURE 8. Crystalline Contractibility in the Pre-Home Case

The basic idea here is the following: By induction on g, it suffices to prove that $\mathcal{M}_{g,r}$ does not admit any *proper connected components*. But if it did admit such a component J, then one can apply the following analysis to $\mathcal{N}_J \stackrel{\text{def}}{=} \mathcal{N}_{g,r} \times_{\mathcal{M}_{g,r}} J$: First of all, by Theorem 1.1, $\mathcal{N}_J$ is finite and flat of degree p^{3g-3+r} over J. Now let I be an irreducible component of $\mathcal{N}_J$ for which the vanishing locus of the p-curvature of the nilcurve parametrized by the generic point of I is *maximal* (in other words, an irreducible component whose generic point lies in $\mathcal{N}_{g,r}[d]$, for d maximal). It is then a formal consequence of Theorems 1.1 and 2.2 that I is *smooth and proper* over $\mathbf{F}_p$, and that the whole of I (i.e., not just the generic point) lies in some $\mathcal{N}_{g,r}[d]$. Now we apply the fact that $\mathcal{N}_{g,r}[0]$ is *affine* (a fact which belongs to the same circle of ideas as Theorem 2.6). This implies (since I is proper and of positive dimension) that the d such that $I \subseteq \mathcal{N}_{g,r}[d]$ is *nonzero*. Thus, since (by [**21**], Chapter II, Corollary 2.16) $\mathcal{N}_{g,r}$ is *nonreduced* at the generic point of $\mathcal{N}_{g,r}[d]$, it follows that the degree of I over J is $< p^{3g-3+r}$. On the other hand, by using the fact that the Schwarz torsor may also be interpreted as the Hodge-theoretic first Chern class of a certain ample line bundle (cf. [**21**], Chapter I, § 3), it is a formal consequence (of basic facts concerning Chern classes in crystalline cohomology) that $\deg(I/J)$ (which is a positive integer) is divisible by p^{3g-3+r}. This contradiction (i.e., that $\deg(I/J)$ is a positive integer $< p^{3g-3+r}$ which is nevertheless divisible by p^{3g-3+r}) concludes the proof.

As remarked earlier, this derivation of the irreducibility of the moduli of $\mathcal{M}_{g,r}$ from the basic theorems of p-adic Teichmüller theory is reminiscent of the proof of the irreducibility of $\mathcal{M}_{g,r}$ given by using complex Teichmüller theory to show that Teichmüller space is contractible (cf., e.g., [**2, 4**]). Moreover, it is also interesting in that it suggests that perhaps at some future date the theory (or some extension of the theory) of [**22**] may be used to compute other cohomology groups of $\mathcal{M}_{g,r}$. Other proofs of the irreducibility of $\mathcal{M}_{g,r}$ include those of [**8, 9**], but (at least as far the author knows) the proof given here is *the first that relies on essentially characteristic p methods* (i.e., "Frobenius").

Before proceeding, we must introduce some more notation. If Z is a smooth stack over $\mathbb{Z}_p$, let us write Z_W for the stack on the category of perfect schemes of characteristic p that assigns to a perfect S the category $Z(W(S))$. We shall refer to Z_W as the *infinite Weil restriction of* Z. It is easy to see that Z_W is representable by a perfect stack (Proposition 1.13 of [**22**], Chapter VI). Moreover, this construction generalizes immediately to the logarithmic category. Write $\mathcal{M}_W$ (respectively, $\mathcal{S}_W$) for $((\overline{\mathcal{M}}^{\log}_{g,r})_{\mathbb{Z}_p})_W$ (respectively, $((\overline{\mathcal{S}}^{\log}_{g,r})_{\mathbb{Z}_p})_W$). (Here $\overline{\mathcal{S}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$ is the Schwarz torsor over $\overline{\mathcal{M}}_{g,r}$; we equip it with the log structure obtained by pulling back the log structure of $\overline{\mathcal{M}}^{\log}_{g,r}$.) Now if $\mathcal{P}$ is Π-indigenous on X, it follows immediately from the elementary theory of indigenous bundles that there exists a unique curve $X_W \to W(S)$ whose restriction to $S \subseteq W(S)$ is $X \to S$ and such that the restriction of the crystal $\mathcal{P}$ to X_W defines an indigenous bundle on X_W. The assignment $\mathcal{P} \mapsto (X_W \to W(S), \mathcal{P}|_{X_W})$ (respectively, $\mathcal{P} \mapsto \{X_W \to W(S)\}$) thus defines a natural morphism $\mathcal{Q}^{\Pi} \to \mathcal{S}_W$ (respectively, $\mathcal{Q}^{\Pi} \to \mathcal{M}_W$). Now we have the following results (cf. Propositions 2.3, 2.9; Corollaries 2.6 and 2.13 of [**22**], Chapter VI):

Theorem 2.8 **(Immersions)**. — *The natural morphism $\mathcal{Q}^{\Pi} \to \mathcal{S}_W$ is an immersion in general, and a closed immersion if the VF-pattern is pre-home or of pure tone. The morphism $\mathcal{Q}^{\Pi} \to \mathcal{M}_W$ is a closed immersion if the VF-pattern is the home VF-pattern.*

Theorem 2.9 **(Isolatedness in the Pre-Home Case)**. — *In the pre-home case, $\mathcal{Q}^{\Pi}$ is closed inside $\mathcal{S}_W$ and disjoint from the closure of any non-pre-home $\mathcal{Q}^{\Pi'}$'s.*

We remark that in both of these cases, much more general theorems are proved in [**22**]. Here, for the sake of simplicity, we just selected representative special cases of the main theorems in [**22**] so as to give the reader a general sense of the sorts of results proved in [**22**].

The reason that Theorem 2.9 is interesting (or perhaps a bit surprising) is the following: The reduction modulo p of a Π-indigenous bundle (in the pre-home case) is an *admissible* nilpotent indigenous bundle. (Here, the term "admissible" means that the p-curvature has no zeroes.) Moreover, the admissible locus $\overline{\mathcal{N}}^{\text{adm}}_{g,r}$ of $\overline{\mathcal{N}}_{g,r}$ is by no means closed in $\overline{\mathcal{N}}_{g,r}$, nor is its closure disjoint (in general) from the closure of the dormant or spiked loci of $\overline{\mathcal{N}}_{g,r}$. On the other hand, the reductions modulo p of Π'-indigenous bundles (for non-pre-home Π') may, in general, be dormant or spiked nilpotent indigenous bundles. Thus, *Theorem 2.9 states that considering $\mathbb{Z}_p$-flat Frobenius invariant liftings of indigenous bundles (as opposed to just nilpotent indigenous bundles in characteristic p) has the effect of "blowing up" $\overline{\mathcal{N}}_{g,r}$ in such a way that the genericization/specialization relations that hold in $\overline{\mathcal{N}}_{g,r}$ do not imply such relations among the various $\mathcal{Q}$'s.*

We shall come back to this phenomenon again in the following subsection (cf. Fig. 9).

2.4. The Generalized Ordinary Theory. — In this subsection, we maintain the notations of the preceding subsection. Unfortunately, it is difficult to say much more about the explicit structure of the stacks $\mathcal{Q}^{\Pi}$ without making more assumptions. Thus, just as in the classical ordinary case (reviewed in § 1.6), it is natural to define an open substack – the *ordinary locus of* $\mathcal{Q}^{\Pi}$ – and to see if more explicit things can be said concerning this open substack. This is the topic of [**22**], Chapter VII. We shall see below that in fact much that is interesting can be said concerning this ordinary locus.

We begin with the definition of the ordinary locus. First of all, we observe that there is a natural algebraic stack

$$\overline{\mathcal{N}}^{\Pi,s}_{g,r}$$

(of finite type over $\mathbf{F}_p$) that parametrizes "data modulo p for $\mathcal{Q}^{\Pi}$" (Definition 1.11 of [**22**], Chapter III). That is to say, roughly speaking, $\overline{\mathcal{N}}^{\Pi,s}_{g,r}$ parametrizes the reductions modulo p of the $\mathcal{P}_i$ appearing in the discussion preceding Definition 2.5. We refer to [**22**], Chapter III for a precise definition of this stack. At any rate, by reducing modulo p the data parametrized by $\mathcal{Q}^{\Pi}$, we obtain a natural morphism of stacks

$$\mathcal{Q}^{\Pi} \to \overline{\mathcal{N}}^{\Pi,s}_{g,r}$$

On the other hand, since $\overline{\mathcal{N}}^{\Pi,s}_{g,r}$ parametrizes curves equipped with certain bundles, there is a natural morphism $\overline{\mathcal{N}}^{\Pi,s}_{g,r} \to (\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$. Let $\mathcal{N}^{\mathrm{ord}} \subseteq \overline{\mathcal{N}}^{\Pi,s}_{g,r}$ denote the open substack over which the morphism $\overline{\mathcal{N}}^{\Pi,s}_{g,r} \to (\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$ is *étale*. Let $\mathcal{Q}^{\mathrm{ord}} \subseteq \mathcal{Q}^{\Pi}$ denote the open substack which is the inverse image of $\mathcal{N}^{\mathrm{ord}} \subseteq \overline{\mathcal{N}}^{\Pi,s}_{g,r}$.

Definition 2.10. — We shall refer to $\mathcal{Q}^{\mathrm{ord}}$ as the *(*Π*-)ordinary locus of* $\mathcal{Q}^{\Pi}$.

Just as in the classical ordinary case, there is an equivalent definition of Π-ordinarity given by looking at the action of Frobenius on the first de Rham cohomology modules of the $\mathcal{P}_i$ (cf. Lemma 1.4 of [**22**], Chapter VII). Incidentally, the classical ordinary theory corresponds to the Π-ordinary theory in the case of the home VF-pattern. (In particular, $\mathcal{N}^{\mathrm{ord}}$ is simply the ordinary locus $\overline{\mathcal{N}}^{\mathrm{ord}}_{g,r}$ of $\overline{\mathcal{N}}_{g,r}$.) Thus, in some sense, the theory of [**21**] is a special case of the generalized ordinary theory.

Our first result is the following (cf. Theorem 1.6 of [**22**], Chapter VII):

Theorem 2.11 **(Basic Structure of the Ordinary Locus)**. — *$\mathcal{Q}^{\mathrm{ord}}$ is naturally isomorphic to the perfection of $\mathcal{N}^{\mathrm{ord}}$.*

Thus, already one has a much more explicit understanding of the structure of $\mathcal{Q}^{\mathrm{ord}}$ than of the whole of $\mathcal{Q}^{\Pi}$. That is to say, Theorem 2.11 already tells us that $\mathcal{Q}^{\mathrm{ord}}$ is the perfection of a smooth algebraic stack of finite type over $\mathbf{F}_p$.

Our next result – which is somewhat deeper than Theorem 2.11, and is, in fact, one of the main results of [**22**] – is the following (cf. Theorem 2.11 of [**22**], Chapter VII):

Theorem 2.12 **(ω-Closedness of the Ordinary Locus)**. — *If Π is* binary*, then $\mathcal{Q}^{\mathrm{ord}}$ is ω-*closed *(roughly speaking, "closed as far as the differentials are concerned" – cf.* [**22**]*, Chapter VII, § 0, § 2.3 for more details) in $\mathcal{Q}^{\Pi}$. In particular,*

(1) *If $3g-3+r=1$, then $\mathcal{Q}^{\mathrm{ord}}$ is actually* closed *in $\mathcal{Q}^{\Pi}$.*

(2) *If $\mathcal{R} \subseteq \mathcal{Q}^{\Pi}$ is a subobject containing $\mathcal{Q}^{\mathrm{ord}}$ and which is "pro" (cf.* [**22**]*, Chapter VI, Definition 1.9) of a fine algebraic log stack which is locally of finite type over $\mathbf{F}_p$, then $\mathcal{Q}^{\mathrm{ord}}$ is* closed *in $\mathcal{R}$.*

In other words, at least among perfections of fine algebraic log stacks which are locally of finite type over $\mathbf{F}_p$, $\mathcal{Q}^{\mathrm{ord}}$ is already "complete" inside $\mathcal{Q}^{\Pi}$.

Thus, if Π is pre-home or of pure tone, then $\mathcal{Q}^{\mathrm{ord}}$ is an ω-closed substack of $\mathcal{S}_W$. If the VF-pattern in question is the home pattern, then $\mathcal{Q}^{\mathrm{ord}}$ is an ω-closed substack of $\mathcal{M}_W$.

This is a rather surprising result in that the definition of $\mathcal{Q}^{\mathrm{ord}}$ was such that $\mathcal{Q}^{\mathrm{ord}}$ is naturally an open substack of $\mathcal{Q}^{\Pi}$ which has no a priori reason to be closed (in any sense!) inside $\mathcal{Q}^{\Pi}$. Moreover, $\overline{\mathcal{N}}^{\mathrm{ord}}_{g,r}$ is most definitely *not* closed in $\overline{\mathcal{N}}_{g,r}$. Indeed, one of the original motivations for trying to generalize the theory of [**21**] was to try to compactify it. Thus, Theorem 2.12 states that if, instead of just considering ordinary nilpotent indigenous bundles modulo p, one considers $\mathbb{Z}_p$-flat Frobenius invariant indigenous bundles, the theory of [**21**] is, in some sense, already compact! Put another way, if one thinks in terms of the Witt vectors parametrizing such $\mathbb{Z}_p$-flat Frobenius invariant indigenous bundles, then although the scheme defined by the *first* component of the Witt vector is not "compact," if one considers *all* the components of the Witt vector, the resulting scheme is, in some sense, "compact" (i.e., ω-closed in the space $\mathcal{S}_W$ of all indigenous bundles over the Witt vectors). This phenomenon is similar to the phenomenon observed in Theorem 2.9. In fact, if one combines Theorem 2.9 with Theorem 2.12, one obtains that:

> *In the home (i.e., classical ordinary) case, the stack $\mathcal{Q}^{\mathrm{ord}}$ is ω-closed in $\mathcal{S}_W$ and disjoint from the closures of all $\mathcal{Q}^{\Pi'}$ for all non-pre-home Π'. Moreover, $\mathcal{Q}^{\mathrm{ord}}$ is naturally an ω-closed substack of $\mathcal{Q}^{\Pi'}$ for all pre-home Π'.*

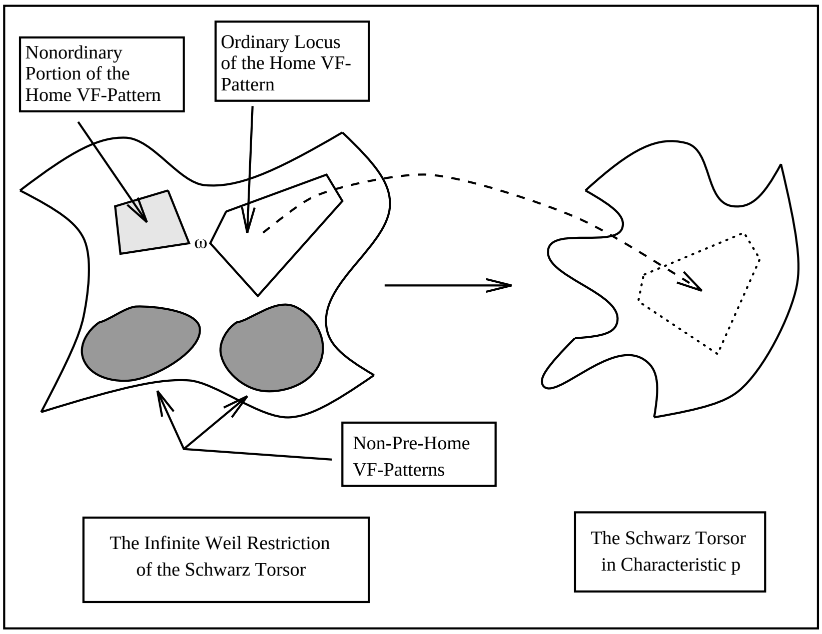

FIGURE 9. The ω-Closedness and Isolatedness of the Classical Ordinary Theory

This fact is rendered in pictorial form in Fig. 9; cf. also the discussion of § 3 below.

The next main result of the generalized ordinary theory is the generalized ordinary version of Theorem 1.2. First, let us observe that since the natural morphism $\mathcal{N}^{\mathrm{ord}} \to (\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$ is *étale*, it admits a unique lifting to an étale morphism

$$\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p} \to (\overline{\mathcal{M}}_{g,r})_{\mathbb{Z}_p}$$

of smooth p-adic formal stacks over $\mathbb{Z}_p$. Unlike in the classical ordinary case, however, where one obtains a single canonical modular Frobenius lifting, in the generalized case, one obtains a whole *system of Frobenius liftings* (cf. Theorem 1.8 of [**22**], Chapter VII) on $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$:

***Theorem 2.13* (Canonical System of Frobenius Liftings)**. — *Over $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$, there is a canonical system of Frobenius liftings and indigenous bundles: i.e., for each indigenous i (i.e., such that $\Pi(i) = \chi$), a lifting*

$$\Phi_i^{log} : \mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p} \to \mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$$

of a certain power of the Frobenius morphism, together with a collection of indigenous bundles $\mathcal{P}_i$ on the tautological curve (pulled back from $(\overline{\mathcal{M}}_{g,r})_{\mathbb{Z}_p}$) over $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$. Moreover, these Frobenius liftings and bundles are compatible*, in a natural sense (Definition 1.7 of* [**22**]*, Chapter VII).*

See Fig. 10 for an illustration of the system of Frobenius liftings obtained for the VF-pattern illustrated in Fig. 7.

At this point, one very important question arises:

To what extent are the stacks $\mathcal{N}^{\mathrm{ord}}$ nonempty?

Needless to say, this is a very important issue, for if the $\mathcal{N}^{\mathrm{ord}}$ are empty most of the time, then the above theory is meaningless. In the classical ordinary case, it was rather trivial to show the nonemptiness of $\overline{\mathcal{N}}^{\mathrm{ord}}_{g,r}$. In the present generalized ordinary setting, however, it is much more difficult to show the nonemptiness of $\mathcal{N}^{\mathrm{ord}}$. In particular, one needs to make use of the extensive theory of [**22**], Chapters II and IV. Fortunately, however, one can show the nonemptiness of $\mathcal{N}^{\mathrm{ord}}$ in a fairly wide variety of cases (Theorems 3.1 and 3.7 of [**22**], Chapter VII):

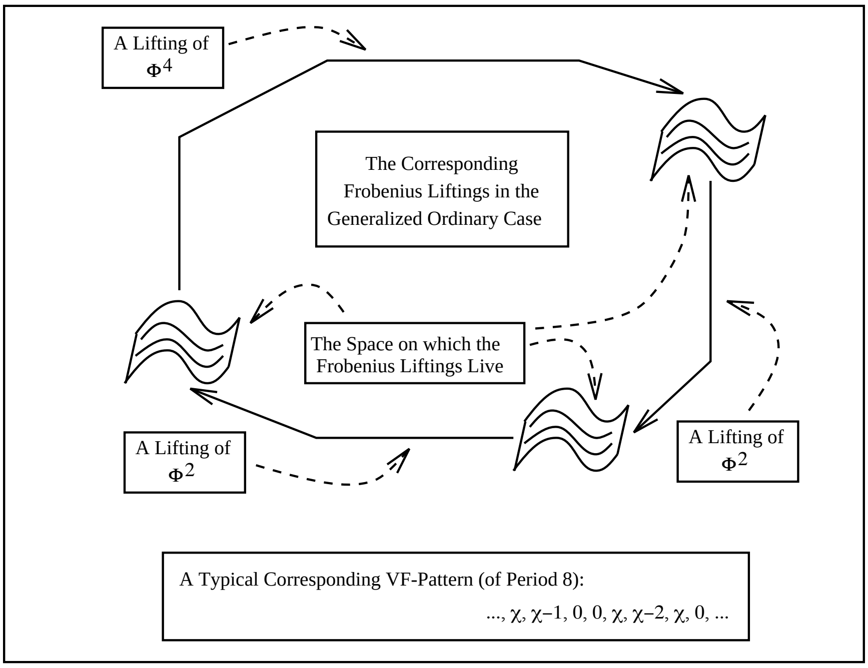

FIGURE 10. The Canonical System of Modular Frobenius Liftings

***Theorem 2.14* (Binary Existence Result).** — *Suppose that $g \geq 2$; $r = 0$; and $p > 4^{3g-3}$. Then for any* binary *VF-pattern (i.e., VF-pattern such that $\Pi(\mathbb{Z}) \subseteq \{0, \chi\}$), the stack $\mathcal{N}^{\mathrm{ord}}$ is nonempty.*

***Theorem 2.15* (Spiked Existence Result).** — *Suppose that $2g-2+r \geq 3$ and $p \geq 5$. Then there exists a "spiked VF-pattern" of period 2 (i.e., $\varpi = 2$ and $0 < \Pi(1) < \chi$) for which $\mathcal{N}^{\mathrm{ord}}$ is nonempty.*

In fact, there is an open substack of $\mathcal{N}^{\mathrm{ord}}$ called the very ordinary locus *(defined by more stringent conditions than ordinarity); moreover, one can choose the spiked VF-pattern so that not only $\mathcal{N}^{\mathrm{ord}}$, but also the "very ordinary locus of $\mathcal{N}^{\mathrm{ord}}$" is nonempty.*

These cases are "fairly representative" in the following sense: In general, in the binary case, the reduction modulo p of a Π-indigenous bundle will be *dormant*. In the spiked case (of Theorem 2.15), the reduction modulo p of a Π-indigenous bundle will be *spiked*. Thus, in other words,

> *Roughly speaking, these two existence results show that for each type (admissible, dormant, spiked) of nilcurve, there exists a theory (in fact, many theories) of canonical liftings involving that type of nilcurve.*

Showing the existence of such a theory of canonical liftings for *each generic point of* $\overline{\mathcal{N}}_{g,r}$ was one of the original motivations for the development of the theory of [**22**].

Next, we observe that just as in Theorem 1.2 (the classical ordinary case),

> *In the cases discussed in Theorems 2.14 and 2.15, one can also construct canonical systems of Frobenius liftings on certain "ordinary loci" of the universal curve over $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$. Moreover, these systems of canonical Frobenius lifting lie over the canonical system of modular Frobenius liftings of Theorem 2.13.*

We refer to Theorem 3.2 of [**22**], Chapter VIII and Theorem 3.4 of [**22**], Chapter IX for more details.

We end this subsection with a certain philosophical observation. In [**22**], Chapter VI,

> *The stack $\mathcal{Q}^{\Pi}$ is referred to as the stack of* quasi-analytic self-isogenies.

That is to say, in some sense it is natural to regard the Frobenius invariant indigenous bundles parametrized by $\mathcal{Q}^{\Pi}$ as *isogenies* of the curve (on which the bundles are defined) onto itself. Indeed, this is suggested by the fact that over the ordinary locus (i.e., relative to the Frobenius invariant indigenous bundle in question) of the curve, the bundle actually does define a literal morphism, i.e., a Frobenius lifting (as discussed in the preceding paragraph). Thus, one may regard a Frobenius invariant indigenous bundle as the appropriate way of compactifying such a self-isogeny to an object defined over the whole curve. This is why we use the adjective "quasi-analytic" in describing the self-isogenies. (Of course, such self-isogenies can never be p-adic analytic over the whole curve, for if they were, they would be algebraic, which, by the Riemann-Hurwitz formula, is absurd.) Note that this point of view is in harmony with the situation in the parabolic case ($g = 1$, $r = 0$), where there is an algebraically defined canonical choice of indigenous bundle, and having a Frobenius invariant indigenous bundle really does correspond to having a lifting of Frobenius (hence a self-isogeny of the curve in question).

Moreover, note that in the case where the VF-pattern has several $\chi = \frac{1}{2}(2g-2+r)$'s in a period, so that there are various indigenous $\mathcal{P}_i$'s in addition to the original Frobenius invariant indigenous bundle $\mathcal{P}$, one may regard the situation as follows. Suppose that $\mathcal{P}$ is indigenous over a curve $X \to W(S)$, whereas $\mathcal{P}_i$ is indigenous over $X_i \to W(S)$. Then one can regard the "quasi-analytic self-isogeny" $\mathcal{P} : X \to X$ as the composite of various quasi-analytic isogenies $\mathcal{P}_i : X_i \to X_j$ (where i and j are "ind-adjacent" integers). Note that this point of view is consistent with what literally occurs over the ordinary locus (cf. Theorem 3.2 of [**22**], Chapter VIII). Finally, we observe that

> *The idea that $\mathcal{Q}^{\Pi}$ is a moduli space of some sort of p-adic self-isogeny which is "quasi-analytic" is also compatible with the analogy between $\mathcal{Q}^{\Pi}$ and Teichmüller space (cf. the discussion of Corollary 2.7) in that Teichmüller space may be regarded as a moduli space of quasiconformal maps (cf., e.g.,* [**2**]*).*

2.5. Geometrization. — In the classical ordinary case, once one knows the existence of the canonical modular Frobenius lifting (Theorem 1.2), one can apply a general result on ordinary Frobenius liftings (Theorem 1.3) to conclude the existence of canonical multiplicative coordinates on $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$. We shall refer to this process of passing (as in Theorem 1.3) from a certain type of Frobenius lifting to a local uniformization/canonical local coordinates associated to the Frobenius lifting as the *geometrization of the Frobenius lifting*. In the generalized ordinary context, Theorem 2.13 shows the existence of a canonical *system of Frobenius liftings* on the $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$ associated to a VF-pattern (Π, ϖ). Thus, the following question naturally arises:

> *Can one geometrize the sort of system of Frobenius liftings that one obtains in Theorem 2.13 in a fashion analogous to the way in which ordinary Frobenius liftings were geometrized in Theorem 1.3?*

Unfortunately, we are not able to answer this question in general. Nevertheless, in the cases discussed in Theorems 2.14 and 2.15, i.e., the binary and very ordinary spiked cases, we succeed (in [**22**], Chapters VIII and IX) in geometrizing the canonical system of modular Frobenius liftings. The result is *uniformizations/geometries based not on $\widehat{\mathbf{G}}_{\mathrm{m}}$ as in the classical ordinary case, but rather on more general types of Lubin-Tate groups, twisted products of Lubin-Tate groups, and fibrations whose bases are Lubin-Tate groups and whose fibers are such twisted products.* In the rest of this subsection, we would like to try to give the reader an idea of what sorts of geometries occur in the two cases studied.

In the following, we let k be a perfect field of characteristic p, A its ring of Witt vectors $W(k)$, and S a smooth p-adic formal scheme over A. Let λ be a positive integer, and let $\mathcal{O}_\lambda \stackrel{\text{def}}{=} W(\mathbf{F}_{p^\lambda})$. For simplicity, we assume that $\mathcal{O}_\lambda \subseteq A$. Let $\mathcal{G}_\lambda$ be the *Lubin-Tate formal group associated to* $\mathcal{O}_\lambda$. (See [**3**] for a discussion of Lubin-Tate formal groups.) Then $\mathcal{G}_\lambda$ is a formal group over $\mathcal{O}_\lambda$, equipped with a natural action by $\mathcal{O}_\lambda$ (i.e., a ring morphism $\mathcal{O}_\lambda \hookrightarrow \mathrm{End}_{\mathcal{O}_\lambda}(G_\lambda)$). Moreover, it is known that the space of invariant differentials on $\mathcal{G}_\lambda$ is canonically isomorphic to $\mathcal{O}_\lambda$. Thus, in the following, we shall identify this space of differentials with $\mathcal{O}_\lambda$.

We begin with the simplest case, namely, that of a *Lubin-Tate Frobenius lifting.* Let $\Phi : S \to S$ be a morphism whose reduction modulo p is the λ^{th} power of the Frobenius morphism. Then differentiating Φ_S defines a morphism $\mathrm{d}\Phi_S : \Phi_S^*\Omega_{S/A} \to \Omega_{S/A}$ which is zero in characteristic p. Thus, we may form a morphism

$$\Omega_\Phi : \Phi_S^*\Omega_{S/A} \to \Omega_{S/A}$$

by dividing $\mathrm{d}\Phi_S$ by p. Then Φ_S is called a *Lubin-Tate Frobenius lifting (of order λ)* if Ω_Φ is an isomorphism. If Φ_S is a Lubin-Tate Frobenius lifting, then it induces a "Lubin-Tate geometry" – i.e., a geometry based on $\mathcal{G}_\lambda$ – on S. That is to say, one has the following analogue of Theorem 1.3 (cf. Theorem 2.17 of [**22**], Chapter VIII):

***Theorem 2.16* (Lubin-Tate Frobenius Liftings).** — *Let $\Phi_S : S \to S$ be a Lubin-Tate Frobenius lifting of order λ. Then taking the invariants of $\Omega_{S/A}$ with respect to Ω_Φ gives rise to an étale local system Ω_Φ^{et} on S of free $\mathcal{O}_\lambda$-modules of rank equal to* $\dim_A(S)$.

Let $z \in S(\overline{k})$ be a point valued in the algebraic closure of k. Then $\Omega_z \stackrel{\text{def}}{=} \Omega_\Phi^{\text{et}}|_z$ may be thought of as a free $\mathcal{O}_\lambda$-module of rank $\dim_A(S)$; write Θ_z for the $\mathcal{O}_\lambda$-dual of Ω_z. Let S_z be the completion of S at z. Then there is a unique isomorphism

$$\Gamma_z : S_z \cong \mathcal{G}_\lambda \otimes^{\text{gp}}_{\mathcal{O}_\lambda} \Theta_z$$

such that:

(i) *the derivative of Γ_z induces the natural inclusion $\Omega_z \hookrightarrow \Omega_{S/A}|_{S_z}$;*

(ii) *the action of Φ_S on S_z corresponds to multiplication by p on $\mathcal{G}_\lambda \otimes^{\text{gp}}_{\mathcal{O}_\lambda} \Theta_z$.*

Here, by "$\mathcal{G}_\lambda \otimes^{\text{gp}}_{\mathcal{O}_\lambda} \Theta_z$," we mean the tensor product over $\mathcal{O}_\lambda$ of (formal) group schemes with $\mathcal{O}_\lambda$-action. Thus, $\mathcal{G}_\lambda \otimes^{\text{gp}}_{\mathcal{O}_\lambda} \Theta_z$ is noncanonically isomorphic to the product of $\dim_A(S) = \mathrm{rank}_{\mathcal{O}_\lambda}(\Theta_z)$ *copies of $\mathcal{G}_\lambda$.*

Of course, this result has nothing to do with the moduli of curves. In terms of VF-patterns, Theorem 2.13 gives rise to a Lubin-Tate Frobenius lifting of order ϖ when the VF-pattern is of pure tone ϖ.

The next simplest case is the case of an *anabelian system of Frobenius liftings*. Let n be a positive integer. Then an *anabelian system of Frobenius liftings of length* n *and order* λ is a collection of n Lubin-Tate Frobenius liftings

$$\Phi_1, \ldots, \Phi_n : S \to S$$

each of order λ. Of course, in general such Frobenius liftings will not commute with one another. In fact, it can be shown that two Lubin-Tate Frobenius liftings of order λ commute with each other if and only if they are equal (Lemma 2.24 of [**22**], Chapter VIII). This is the reason for the term "anabelian." Historically, this term has been used mainly in connection with Grothendieck's Conjecture of Anabelian Geometry ([**12**]). The reason why we thought it appropriate to use the term here (despite the fact that anabelian geometries as discussed here have nothing to do with the Grothendieck Conjecture) is the following: (Just as for the *noncommutative* fundamental groups of Grothendieck's anabelian geometry) *the sort of noncommutativity that occurs among the* Φ_i*'s* (at least in the modular case – cf. Theorem 2.13) *arises precisely as a result of the hyperbolicity of the curves* on whose moduli the Φ_i's act.

Let $\delta_i \stackrel{\text{def}}{=} \frac{1}{p}\mathrm{d}\Phi_i$. Let $\Delta \stackrel{\text{def}}{=} \delta_n \circ \cdots \circ \delta_1$. Then taking invariants of $\Omega_{S/A}$ with respect to Δ gives rise to an étale local system $\Omega^{\text{et}}_{\Phi}$ on S in free $\mathcal{O}_{n\lambda}$-modules of rank $\dim_A(S)$. Next let S_{PD} denote the p-adic completion of the PD-envelope of the diagonal in the product (over A) of n copies of S; let S_{FM} denote the p-adic completion of the completion at the diagonal of the product (over A) of n copies of S. Thus, we have a natural morphism

$$S_{\text{PD}} \to S_{\text{FM}}$$

Moreover, one may think of S_{PD} as a sort of *localization* of S_{FM}. Write $\Phi_{\text{PD}} : S_{\text{PD}} \to S_{\text{PD}}$ for the morphism induced by sending

$$(s_1, \ldots, s_n) \mapsto (\Phi_1(s_2), \Phi_2(s_3), \ldots, \Phi_n(s_1))$$

(where $(s_1, \ldots, s_n)$ represents a point in the product of n copies of S). Then we have the following result (cf. Theorem 2.17 of [**22**], Chapter VIII):

***Theorem 2.17* (Anabelian System of Frobenius Liftings)**. — *Let* $\Phi_1, \ldots, \Phi_n : S \to S$ *be a system of anabelian Frobenius liftings of length* n *and order* λ. *Let* $z \in S(\overline{k})$ *be a point valued in the algebraic closure of* k. *Then* $\Omega_z \stackrel{\text{def}}{=} \Omega^{\text{et}}_{\Phi}|_z$ *may be thought of as a free* $\mathcal{O}_{n\lambda}$*-module of rank* $\dim_A(S)$; *write* Θ_z *for the* $\mathcal{O}_{n\lambda}$*-dual of* Ω_z. *Let* $(S_{\text{PD}})_z$ *be the completion of* S_{PD} *at* z. *Then there is a unique morphism*

$$\Gamma_z : (S_{\text{PD}})_z \to \mathcal{G}_\lambda \otimes^{\text{gp}}_{\mathcal{O}_\lambda} \Theta_z$$

such that:

(i) *the derivative of* Γ_z *induces a certain (see Theorem 2.15 of* [**22**], *Chapter VIII for more details) natural inclusion of* Ω_z *into the restriction to* $(S_{\text{PD}})|_z$ *of the differentials of* $\prod_{i=1}^n S$ *over* A;

(ii) *the action of* Φ_{PD} *on* $(S_{\mathrm{PD}})_z$ *is compatible with multiplication by* p *on* $\mathcal{G}_\lambda \otimes^{\mathrm{gp}}_{\mathcal{O}_\lambda} \Theta_z$.

Here, by "$\mathcal{G}_\lambda \otimes^{\mathrm{gp}}_{\mathcal{O}_\lambda} \Theta_z$*," we mean the tensor product over* $\mathcal{O}_\lambda$ *of (formal) group schemes with* $\mathcal{O}_\lambda$*-action. Thus,* $\mathcal{G}_\lambda \otimes^{\mathrm{gp}}_{\mathcal{O}_\lambda} \Theta_z$ *is noncanonically isomorphic to the product of* $n \cdot \dim_A(S) = \mathrm{rank}_{\mathcal{O}_\lambda}(\Theta_z)$ *copies of* $\mathcal{G}_\lambda$.

Moreover, in general, Γ_z does not descend to $(S_{\mathrm{FM}})_z$ (cf. [**22**], Chapter VIII, § 2.6, 3.1).

One way to envision anabelian geometries is as follows: The various Φ_i's induce various linear Lubin-Tate geometries on the space S that (in general) do not commute with one another. Thus, *the anabelian geometry consists of various linear geometries on* S *all tangled up inside each other*. If one localizes in a sufficiently drastic fashion – i.e., all the way to $(S_{\mathrm{PD}})_z$ – then one can untangle these tangled up linear geometries into a single $\mathcal{O}_{n\lambda}$-linear geometry (via Γ_z). However, the order λ Lubin-Tate geometries are so tangled up that even localization to a relatively localized object such as $(S_{\mathrm{FM}})_z$ is not sufficient to untangle these geometries.

Finally, to make the connection with Theorem 2.13, we remark that the system of Theorem 2.13 gives rise to an anabelian system of length n and order λ in the case of a VF-pattern (Π, ϖ) for which $\varpi = n \cdot \lambda$, and $\Pi(i) = \chi$ (respectively, $\Pi(i) = 0$) if and only if i is divisible (respectively, not divisible) by λ.

In fact, both Lubin-Tate geometries and anabelian geometries are special cases of *binary ordinary geometries* (the sorts of geometries that occur for *binary* VF-patterns, i.e., Π whose image $\subseteq \{0, \chi\}$). A general geometrization result for binary ordinary geometries is given in Theorem 2.17 of [**22**], Chapter VIII. Here, we chose to concentrate on the Lubin-Tate and anabelian cases (in fact, of course, Lubin-Tate geometries are a special case of anabelian geometries) since they are relatively representative and relatively easy to envision.

The other main type of geometry that is studied in [**22**] is the geometry associated to a *very ordinary spiked Frobenius lifting* $\Phi : S \to S$. Such a Frobenius lifting reduces modulo p to the square of the Frobenius morphism and satisfies various other properties which we omit here (see Definition 1.1 of [**22**], Chapter IX for more details). In particular, such a Frobenius lifting comes equipped with an invariant called the *colevel*. The colevel is a nonnegative integer c. Roughly speaking,

> *A very ordinary spiked Frobenius lifting is a Frobenius lifting which is "part Lubin-Tate of order* 2*" and "part anabelian of length* 2 *and order* 1*."*

The colevel c is the number of dimensions of S on which Φ is Lubin-Tate of order 2. The main geometrization theorem (roughly stated) on this sort of Frobenius lifting is as follows (cf. Theorems 1.5 and 2.3 of [**22**], Chapter IX):

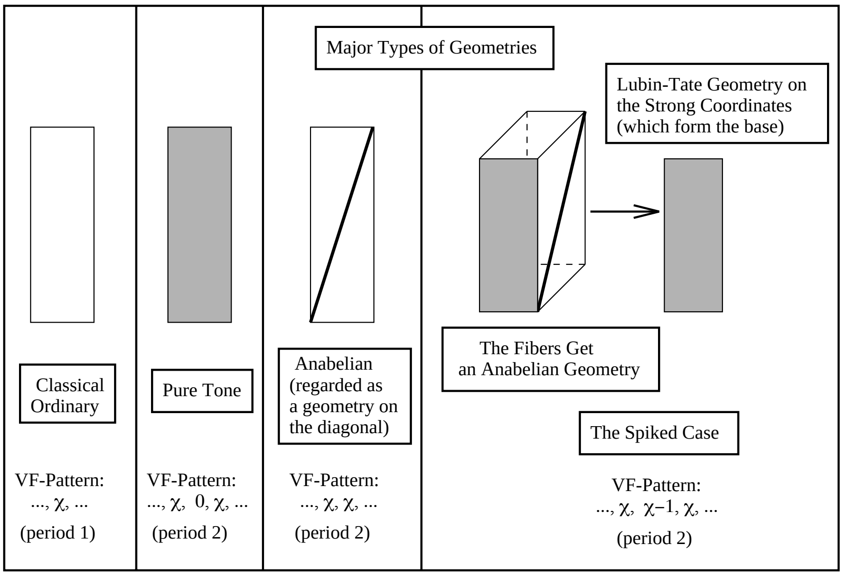

FIGURE 11. Major Types of p-adic Geometries

***Theorem 2.18* (Very Ordinary Spiked Frobenius Liftings)**. — *Let $\Phi : S \to S$ be a very ordinary spiked Frobenius lifting of colevel c. Then Φ defines an étale local system $\Omega_\Phi^{\mathrm{st}}$ on S of free $\mathcal{O}_2$-modules of rank c equipped with a natural inclusion $\Omega_\Phi^{\mathrm{st}} \hookrightarrow \Omega_{S/A}$.*

Let $z \in S(\overline{k})$ be a point valued in the algebraic closure of k. Then $\Omega_z^{\mathrm{st}} \stackrel{\mathrm{def}}{=} \Omega_\Phi^{\mathrm{st}}|_z$ may be thought of as a free $\mathcal{O}_2$-module of rank c; write Θ_z^{st} for the $\mathcal{O}_2$-dual of Ω_z^{st}. Let S_z be the completion of S at z. Then there is a unique morphism

$$\Gamma_z : S_z \to \mathcal{G}_2 \otimes_{\mathcal{O}_2}^{\mathrm{gp}} \Theta_z^{\mathrm{st}}$$

such that:

(i) *the derivative of Γ_z induces the natural inclusion of Ω_z^{st} into $\Omega_{S/A}$;*

(ii) *the action of Φ on S_z is compatible with multiplication by p on $\mathcal{G}_2 \otimes_{\mathcal{O}_2}^{\mathrm{gp}} \Theta_z^{\mathrm{st}}$.*

Here, the variables on S_z obtained by pull-back via Γ_z carry a Lubin-Tate geometry of order 2*, and are called the* strong variables *on S_z. Finally, the fiber of Γ_z over the identity element of the group object $\mathcal{G}_2 \otimes_{\mathcal{O}_2}^{\mathrm{gp}} \Theta_z^{\mathrm{st}}$ admits an anabelian geometry of length* 2 *and order* 1 *determined by Φ (plus a "Hodge subspace" for Φ – cf.* [**22**]*, Chapter IX, § 1.5, for more details). The variables in these fibers are called the* weak variables*.*

Thus, in summary, Φ defines a *virtual fibration* on S to a base space (of dimension c) naturally equipped with a Lubin-Tate geometry of order 2; moreover, (roughly speaking) the fibers of this fibration are naturally equipped with an anabelian geometry of length 2 and order 1. In terms of VF-patterns, this sort of Frobenius lifting occurs in the case $\varpi = 2$, $\Pi(1) \neq 0$ (cf. Theorem 2.15). The colevel is then given by $2(\chi - \Pi(1))$.

Next, we note that as remarked toward the end of § 2.4, in the binary ordinary and very ordinary spiked cases one obtains geometrizable systems of Frobenius liftings not only over $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$ (which is étale over $(\overline{\mathcal{M}}_{g,r})_{\mathbb{Z}_p}$) but also on the ordinary locus of the universal curve over $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$. (More precisely, in the very ordinary spiked case, one must replace $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$ by the formal open substack defined by the very ordinary locus.) Thus, in particular,

> *In the binary ordinary and very ordinary spiked cases, one obtains geometries as discussed in the above theorems not only on the moduli of the curves in question, but also on the ordinary loci of the universal curves themselves.*

See Fig. 11 for a pictorial representation of the major types of geometries discussed.

Finally, we observe that one way to understand these generalized ordinary geometries is the following:

> *The "Lubin-Tate-ness" of the resulting geometry on the moduli stack is a reflection of the extent to which the p-curvature (of the indigenous bundles that the moduli stack parametrizes) vanishes.*

That is to say, the more the p-curvature vanishes, the more Lubin-Tate the resulting geometry becomes. For instance, in the case of a Lubin-Tate geometry, the order of the Lubin-Tate geometry (cf. Theorem 2.16) corresponds precisely to the number of dormant crys-stable bundles in a period (minus one). In the case of a spiked geometry, the number of "Lubin-Tate dimensions" is measured by the colevel. Moreover, this colevel is proportional to the degree of vanishing of the p-curvature of the indigenous bundle in question.

2.6. The Canonical Galois Representation. — Finally, since we have been considering Frobenius invariant indigenous bundles,

> *We would like to construct representations of the fundamental group of the curve in question into* PGL_2 *by looking at the Frobenius invariant sections of these indigenous bundles.*

Such representations will then be the p-adic analogue of the canonical representation in the complex case of the topological fundamental group of a hyperbolic Riemann surface into $\mathrm{PSL}_2(\mathbb{R}) \subseteq \mathrm{PGL}_2(\mathbb{C})$ (cf. the discussion at the beginning of § 1.3). Unfortunately, things are not so easy in the p-adic (generalized ordinary) case because *a priori* the canonical indigenous bundles constructed in Theorem 2.13 only have connections and Frobenius actions with respect to the *relative* coordinates of the tautological curve over $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$. This means, in particular, that we cannot immediately apply the theory of [**6**], § 2, to pass to representations of the fundamental group. To overcome this difficulty, we must employ the technique of *crystalline induction* developed in [**21**]. Unfortunately, in order to carry out crystalline induction, one needs to introduce an object called the *Galois mantle* which can only be constructed when the system of Frobenius liftings on $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$ is geometrizable. Thus, in particular, we succeed (in [**22**], Chapter X) in constructing representations of the sort desired only in the binary ordinary and very ordinary spiked cases.

First, we sketch what we mean by the *Galois mantle*. The Galois mantle can be constructed for any geometrizable system of Frobenius liftings (e.g., any of the types discussed in § 2.5). In particular, the notion of the Galois mantle has nothing to do with curves or their moduli. For simplicity, we describe the Galois mantle in the *classical ordinary case*. Thus, let S and A be as in § 2.5. Let Π_S be the fundamental group of $S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (for some choice of base-point). Let Φ be a *classical ordinary Frobenius lifting* (in other words, Lubin-Tate of order 1) on S. Then by taking Frobenius invariant sections of the tangent bundle, one obtains an étale local system $\Theta^{\mathrm{et}}_{\Phi}$ on S of free $\mathbb{Z}_p$-modules of rank $\dim_A(S)$. Moreover, Φ defines a natural exact sequence of continuous Π_S-modules

$$0 \to \Theta^{\mathrm{et}}_{\Phi}(1) \to E_{\Phi} \to \mathbb{Z}_p \to 0$$

where the "(1)" denotes a Tate twist, and "$\mathbb{Z}_p$" is equipped with the trivial Π_S-action. Roughly speaking, this extension of Π_S-modules is given by taking the p^{th} power roots of the canonical multiplicative coordinates of Theorem 1.3 (cf. § 2.2 of [**22**], Chapter VIII for a detailed discussion of the p-divisible group whose Tate module may be identified with E_{Φ}). Let $\mathcal{B}'$ be the affine space of dimension $\dim_A(S)$ over $\mathbb{Z}_p$ parametrizing splittings of the above exact sequence. Then the action of Π_S on the above exact sequence induces a natural action of Π_S on $\mathcal{B}'$. Roughly speaking, the *Galois mantle* $\mathcal{B}$ *associated to* Φ is the p-adic completion of a certain kind of p-adic localization of $\mathcal{B}$.

> *More generally, to any geometrizable system of Frobenius liftings (as in § 2.5) on S, one can associate a natural p-adic space* $\mathcal{B}$ – the Galois mantle associated to the system of Frobenius liftings – *with a continuous* Π_S*-action. In the binary ordinary case,* $\mathcal{B}$ *will have a natural affine structure over some finite étale extension of* $\mathbb{Z}_p$*. In the very ordinary spiked case,* $\mathcal{B}$ *will be fibred over an affine space over* $\mathcal{O}_2$ *with fibers that are also equipped with an affine structure over* $\mathcal{O}_2$*.*

In fact, to be more precise, $\mathcal{B}$ is only equipped with an action by a certain open subgroup of Π_S, but we shall ignore this issue here since it is rather technical and not so important. We refer to § 2.3 and § 2.5 of [**22**], Chapter IX for more details on the

Galois mantle. So far, for simplicity, we have been ignoring the logarithmic case, but everything is compatible with log structures.

We are now ready to state the main result on the canonical Galois representation in the generalized ordinary case, i.e., the generalized ordinary analogue of Theorem 1.4 (cf. Theorems 1.2 and 2.2 of [**22**], Chapter X). See Fig. 12 for a graphic depiction of this theorem.

***Theorem 2.19* (Canonical Galois Representation)**. — *Let p be an odd prime. Let g and r be nonnegative integers such that $2g-2+r \geq 1$. Fix a VF-pattern (Π, ϖ) which is either binary ordinary or spiked of order 2. Let $S \stackrel{\text{def}}{=} \mathcal{N}^{\text{ord}}_{\mathbb{Z}_p}$ in the binary ordinary case, and let S be the very ordinary locus of $\mathcal{N}^{\text{ord}}_{\mathbb{Z}_p}$ in the spiked case. Let $Z \to S$ be a certain appropriate finite covering which is log étale in characteristic zero (cf. the discussion preceding Theorems 1.2 and 2.2 of* [**22**]*, Chapter X for more details). Let $X^{\log}_Z \to Z^{\log}$ be the tautological log-curve over $Z^{\log}$. Let Π_{X_Z} (respectively, Π_Z) be the fundamental group of $X^{\log}_Z \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (respectively, $Z^{\log} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$) for some choice of base-point. (Of course, despite the similarity in notation, these fundamental groups have no direct relation to the VF-pattern "Π.") Thus, there is a natural morphism $\Pi_{X_Z} \to \Pi_Z$. Let $\mathcal{B}$ be the* Galois mantle *associated to the canonical system of Frobenius liftings of Theorem 2.13. The morphism $\Pi_{X_Z} \to \Pi_Z$ allows us to regard $\mathcal{B}$ as being equipped with a Π_{X_Z}-action.*

Let $\mathcal{P}$ be the tautological Π-indigenous bundle on X. Then by taking Frobenius invariants of $\mathcal{P}$, one obtains a $\mathbb{P}^1$-bundle

$$\mathbb{P}_{\mathcal{B}} \to \mathcal{B}$$

equipped with a natural continuous Π_{X_Z}-action compatible with the above-mentioned action of Π_{X_Z} on the Galois mantle $\mathcal{B}$.

Put another way, one obtains a twisted homomorphism of Π_{X_Z} into PGL_2 of the functions on $\mathcal{B}$. (Here, "twisted" refers to the fact that the multiplication rule obeyed by the homomorphism takes into account the action of Π_{X_Z} on the functions on $\mathcal{B}$.) Finally, note that for any point of $Z \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (at which the log structure is trivial), one also obtains similar representations by restriction. This gives one canonical Galois representations even in the non-universal case.

Finally, in [**22**], Chapter X, § 1.4, 2.3, we show that:

> *The Galois representation of Theorem 2.19 allows one to relate the various p-adic analytic structures constructed throughout* [**22**] *(i.e., canonical Frobenius liftings, canonical Frobenius invariant indigenous bundles, etc.) to the* algebraic/arithmetic *Galois action on the* profinite Teichmüller group (cf. [**22**], Chapter X, Theorems 1.4, 2.3).

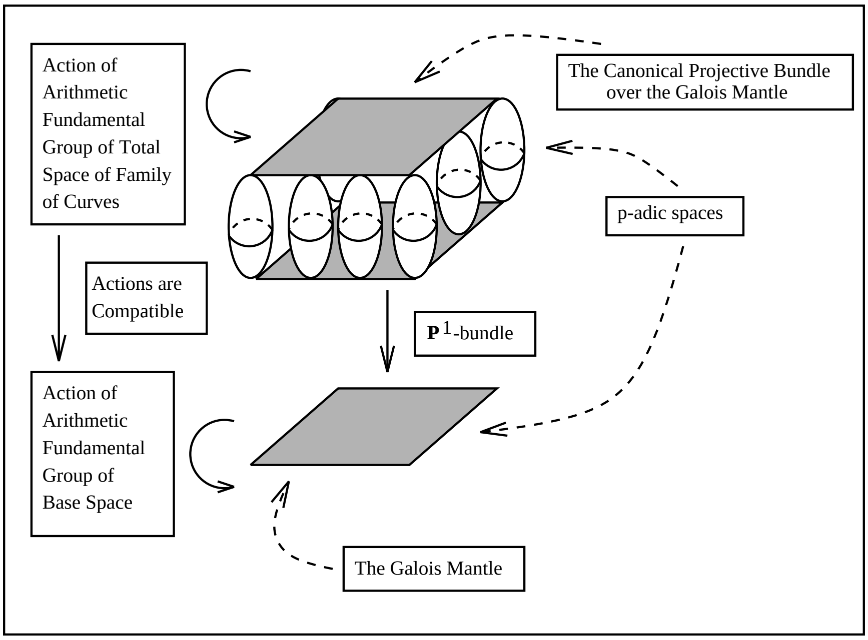

FIGURE 12. The Canonical Galois Representation

More precisely: By iterating the canonical Frobenius liftings on $\mathcal{N} \stackrel{\text{def}}{=} \mathcal{N}^{\text{ord}}_{\mathbb{Z}_p}$, we obtain a certain natural infinite covering

$$\mathcal{N}[\infty] \to \mathcal{N}$$

(i.e., projective limit of finite coverings which are log étale in characteristic zero). On the other hand, if we denote by

$$\mathcal{C}_{\mathbb{Z}_p} \to \mathcal{M}_{\mathbb{Z}_p} \stackrel{\text{def}}{=} (\overline{\mathcal{M}}^{\log}_{g,r})_{\mathbb{Z}_p}$$

the *universal log-curve over the moduli stack*, and by $\mathcal{C}_{\overline{\eta}}$ the geometric generic fiber of this morphism, then the *natural outer action* $\pi_1(\mathcal{M}_{\mathbb{Q}_p})$ (i.e., action on a group defined modulo inner automorphisms of the group) on $\pi_1(\mathcal{C}_{\overline{\eta}})$ defines an action of $\pi_1(\mathcal{M}_{\mathbb{Q}_p})$ on

$$\mathbf{Rep}_{\mathbb{Q}_p} \stackrel{\text{def}}{=} \text{Rep}(\pi_1^{\text{top}}(\mathcal{X}), PGL_2(\mathcal{O}_{\varpi}))$$

(where $\mathcal{O}_{\varpi}$ is defined to be the ring of Witt vectors with coefficients in the finite field of p^{ϖ} elements, and "Rep" denotes the set of isomorphism classes of homomorphisms $\pi_1^{\text{top}}(\mathcal{X}) \to PGL_2(\mathcal{O}_{\varpi})$; two such homomorphisms are regarded as isomorphic if they differ by composition with an inner automorphism of $PGL_2(\mathcal{O}_{\varpi})$). Moreover, this action defines (by the "definition of π_1") an infinite étale covering $\mathcal{R}_{\mathbb{Q}_p} \to \mathcal{M}_{\mathbb{Q}_p}$. We denote the normalization of $\mathcal{M}_{\mathbb{Z}_p}$ in $\mathcal{R}_{\mathbb{Q}_p}$ by $\mathcal{R}_{\mathbb{Z}_p}$. Let $\widehat{\mathcal{M}}$ be the *p-adic completion of* $\mathcal{M}_{\mathbb{Z}_p}$, and $\widehat{\mathcal{R}} \stackrel{\mathrm{def}}{=} \mathcal{R}_{\mathbb{Z}_p} \times_{\mathcal{M}_{\mathbb{Z}_p}} \widehat{\mathcal{M}}_{\mathbb{Z}_p}$. Then the main results on this topic (i.e., [**22**], Chapter X, Theorems 1.4, 2.3) state that the Galois of representation of Theorem 2.19 induces a commutative diagram

$$\begin{array}{ccc} \mathcal{N}[\infty] & \xrightarrow{\widehat{\kappa}} & \widehat{\mathcal{R}} \\ \big\downarrow & & \big\downarrow \\ \mathcal{N} & \longrightarrow & \widehat{\mathcal{M}} \end{array}$$

in which the horizontal morphism (which is denoted $\widehat{\kappa}$ in [**22**], Chapter X) on top is an *open immersion.*

The proof that $\widehat{\kappa}$ is an open immersion divides naturally into *three parts*, corresponding to the three "layers" of the morphism $\mathcal{N}[\infty] \to \widehat{\mathcal{M}}$. The *first layer* is the quasi-finite (but not necessarily finite) étale morphism $\mathcal{N} \to \widehat{\mathcal{M}}$. Because the morphism $\mathcal{N} \to \widehat{\mathcal{M}}$ is étale even in characteristic p, this layer is rather easy to understand. The *second layer* corresponds to the finite covering $Z \to S$ of Theorem 2.19. Together, the first and second layers correspond to the "mod p portion" of the Galois representation of Theorem 2.19 – i.e., the first layer corresponds to the "slope zero portion" of this representation modulo p, while the second layer corresponds to the "positive slope portion" of this representation modulo p. From the point of view of the "$\mathcal{MF}^{\nabla}$-objects" over $B(\mathcal{N})$ (cf. the discussion following Definition 2.5 in § 2.3) corresponding to the representation of Theorem 2.19, this *slope zero portion* (i.e., the first layer) parametrizes the isomorphism class of these $\mathcal{MF}^{\nabla}$-objects over $(B(\mathcal{N})_{\mathbf{F}_p})_{\mathrm{red}}$, while the *positive slope portion* (i.e., the second layer) parametrizes the isomorphism class of the *deformations* of these $\mathcal{MF}^{\nabla}$-objects from bundles on curves over $(B(\mathcal{N})_{\mathbf{F}_p})_{\mathrm{red}}$ to bundles on curves over $B(\mathcal{N})_{\mathbf{F}_p}$.

Finally, the *third layer* of the covering is what remains between $\mathcal{N}[\infty]$ and the "Z" of Theorem 2.19. This portion is the *analytic* portion of the covering (i.e., the portion of the covering equipped with a natural "analytic structure"). Put another way, this portion is the portion of the covering which is dealt with by the technique of *crystalline induction* (which is concerned precisely with equipping this portion of the covering with a natural "crystalline" analytic structure – cf. [**22**], Chapter IX, § 2.3 – especially the Remark following Theorem 2.11 – for more details).

Thus, the fact that the morphism $\widehat{\kappa}$ "does not omit any information" at all three layers is essentially a tautological consequence of the various aspects of the extensive theory developed throughout [**22**]. From another point of view, by analyzing this morphism $\widehat{\kappa}$, we obtain a rather detailed understanding of a certain portion of the *canonical tower of coverings of* $\mathcal{M}_{\mathbb{Q}_p} \stackrel{\mathrm{def}}{=} (\overline{\mathcal{M}}_{g,r}^{\log})_{\mathbb{Q}_p}$ given by

$$\mathcal{R}_{\mathbb{Q}_p} \to \mathcal{M}_{\mathbb{Q}_p}$$

analogous to the analysis given in [**19**] of coverings of the moduli stack of elliptic curves over $\mathbb{Z}_p$ obtained by considering p-power torsion points (cf. the Remark following [**22**], Chapter X, Theorem 1.4, for more details).

Thus, in summary, Theorem 2.19 concludes our discussion of "p-adic Teichmüller theory" as exposed in [**22**] by constructing a p-adic analogue of the canonical representation discussed at the beginning of § 1.3, that is to say, a p-adic analogue of something very close to the Fuchsian uniformization itself – which was where our discussion began (§ 1.1).

3. Conclusion

Finally, we pause to take a look at what we have achieved. Just as in § 1, we would like to *describe the p-adic theory by comparing it to the classical theory at the infinite prime.* Thus, let us write

$$\mathcal{C}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{C}} \stackrel{\text{def}}{=} (\overline{\mathcal{M}}_{g,r}^{\log})_{\mathbb{C}}$$

for the *universal log-curve over the moduli stack* $(\overline{\mathcal{M}}_{g,r}^{\log})_{\mathbb{C}}$ of r-pointed stable log-curves of genus g over the complex numbers. Let us fix a "base-point" (say, in the interior – i.e., the open substack parametrizing smooth curves – of $\mathcal{M}_{\mathbb{C}}$) $[X] \in \mathcal{M}_{\mathbb{C}}(\mathbb{C})$ corresponding to some hyperbolic algebraic curve X over $\mathbb{C}$. Let us write $\mathcal{X} \stackrel{\text{def}}{=} X(\mathbb{C})$ for the corresponding *hyperbolic Riemann surface.* Next, let us consider the space

$$\mathbf{Rep}_{\mathbb{C}} \stackrel{\text{def}}{=} \mathrm{Rep}(\pi_1^{\text{top}}(\mathcal{X}), PGL_2(\mathbb{C}))$$

of isomorphism classes of representations of the *topological fundamental group* $\pi_1^{\text{top}}(\mathcal{X})$ into $PGL_2(\mathbb{C})$. This space has the structure of an *algebraic variety* over $\mathbb{C}$, induced by the algebraic structure of $PGL_2(\mathbb{C})$ by choosing generators of $\pi_1^{\text{top}}(\mathcal{X})$. Note, moreover, that as $[X]$ varies, the resulting spaces $\mathrm{Rep}(\pi_1(\mathcal{X}), PGL_2(\mathbb{C}))$ form a *local system* on $\mathcal{M}_{\mathbb{C}}$ (valued in the category of algebraic varieties over $\mathbb{C}$) which we denote by

$$\mathcal{R}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{C}}$$

One can also think of $\mathcal{R}_{\mathbb{C}}$ as the local system defined by the natural action of $\pi_1^{\text{top}}(\mathcal{M}_{\mathbb{C}}(\mathbb{C}))$ on $\mathbf{Rep}_{\mathbb{C}} \stackrel{\text{def}}{=} \mathrm{Rep}(\pi_1^{\text{top}}(\mathcal{X}), PGL_2(\mathbb{C}))$ which is induced by the *natural outer action* of $\pi_1^{\text{top}}(\mathcal{M}_{\mathbb{C}}(\mathbb{C}))$ on $\pi_1^{\text{top}}(\mathcal{X})$ – cf. the discussion of the p-adic case at the end of § 2.6 above (for more details, see [**22**], Chapter X, § 1.4, 2.3).

Next, let us denote by

$$\mathcal{QF} \subseteq \mathcal{R}_{\mathbb{C}}$$

the subspace whose fiber over a point $[Y] \in \mathcal{M}_{\mathbb{C}}(\mathbb{C})$ is given by the representations $\pi_1^{\text{top}}(\mathcal{Y}) \to PGL_2(\mathbb{C})$ that define *quasi-Fuchsian groups* (cf. § 1.4), i.e., simultaneous uniformizations of pairs of Riemann surfaces (of the same type (g,r)), for which one (say, the "first" one) of the pair of Riemann surfaces uniformized is the Riemann surface $\mathcal{Y}$ corresponding to $[Y]$. Thus, whereas the fibers of $\mathcal{R}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{C}}$ are of *dimension* $2(3g-3+r)$ over $\mathbb{C}$, the fibers of $\mathcal{QF} \to \mathcal{M}_{\mathbb{C}}$ are of *dimension* $3g-3+r$ over $\mathbb{C}$.

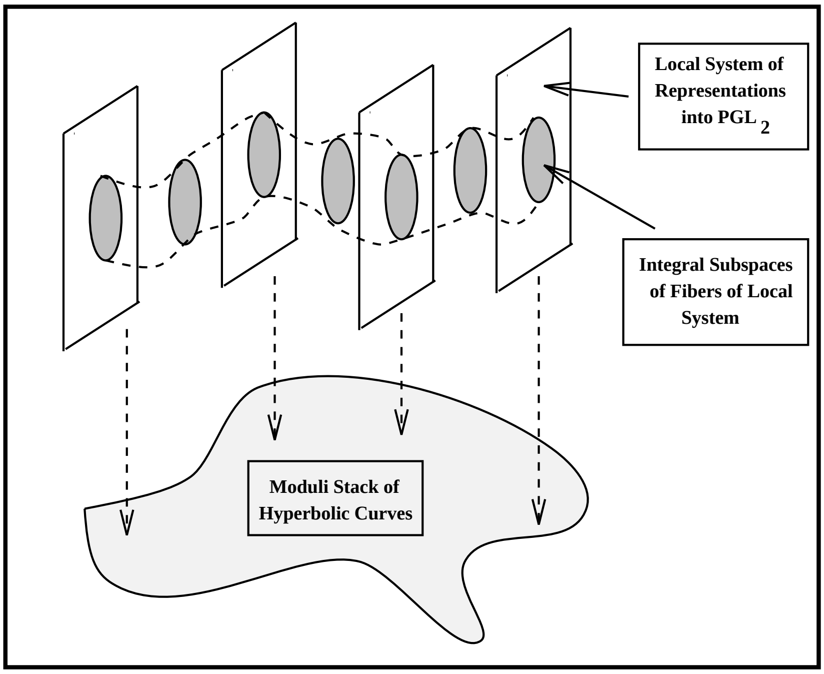

FIGURE 13. Integral Subspaces of the Local System of Representations

Then, relative to the notation of [**22**], Chapter X, § 1.4, 2.3, the *analogy between the complex and p-adic cases* may be summarized by the following diagram:

$$\begin{array}{ccc} \mathcal{QF} & \Longleftrightarrow & \mathcal{N}[\infty] \\ \cap & & \cap \\ (\mathcal{R}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{C}}) & \Longleftrightarrow & (\widehat{\mathcal{R}} \to \widehat{\mathcal{M}}) \end{array}$$

(where the vertical inclusion on the left is the natural one; and the vertical inclusion on the right is the morphism $\widehat{\kappa}$ of [**22**], Chapter X, Theorems 1.4, 2.3). We also give an *illustration* (Fig. 13) of this sort of situation. Relative to this illustration, the "integral (or bounded) subspaces" of the local system are $\mathcal{QF}$ and $\mathcal{N}[\infty]$ (cf. § 1.4 for an explanation of the term "integral"). Note that just as in the complex case, the fibers of the covering $\mathcal{N}[\infty] \to \widehat{\mathcal{M}}$ have, so to speak, "Galois dimension" $3g-3+r$ over $\mathcal{O}_{\varpi}$ (cf. the crystalline induction portion of the proof of [**22**], Chapter X, Theorem 1.4), whereas the fibers of the covering $\widehat{\mathcal{R}} \to \widehat{\mathcal{M}}$ are of "Galois dimension" $2(3g-3+r)$

over $\mathcal{O}_{\varpi}$. In the p-adic case, $\mathcal{N}[\infty]$ denotes the "*crystalline*" or "*Frobenius invariant indigenous bundle*" locus of $\widehat{\mathcal{R}}$ – cf. the discussion of § 1.4.

In the complex case, the "Frobenius" (i.e., complex conjugation) invariant portion of $\mathcal{QF}$ is the space of *Fuchsian groups*, hence defines the *Bers uniformization* of Teichmüller space (cf. § 1.5). On the other hand, in the p-adic case, the covering $\mathcal{N}[\infty] \to \widehat{\mathcal{M}}$ is "made up of" composites of Frobenius liftings, by *forgetting* that these Frobenius liftings are morphisms from a single space to itself, and just thinking of them as *coverings*. If one then invokes the structure of Frobenius liftings as *morphisms from a single space to itself*, one so-to-speak recovers the original Frobenius liftings, which (by the theory of [**22**], Chapters VIII and IX) define *p-adic uniformizations of* $(\overline{\mathcal{M}}^{\log}_{g,r})_{\mathbb{Z}_p}$.

In the complex case, the space of quasi-Fuchsian groups $\mathcal{QF}$ may also be interpreted in terms of *quasi-conformal maps*. Similarly, in the p-adic case, one may interpret integral Frobenius invariant indigenous bundles as *quasi-analytic self-isogenies* of hyperbolic curves (cf. the end of § 2.4).

Finally, in the complex case, although $\mathcal{QF}$ is *not closed* in $\mathcal{R}_{\mathbb{C}}$, the space $\mathcal{QF}$ (when regarded as a space of representations) is *complete* relative to the condition that the representations always define *indigenous bundles* for some conformal structures on the *two* surfaces being uniformized. Note that one may think of these two surfaces as *reflections* of another, i.e., translates of one another by some action of *Frobenius* at the infinite prime (i.e., complex conjugation). Similarly, although $\mathcal{N}[\infty]$ is *not closed* in $\widehat{\mathcal{R}}$, it is *complete* (at least for binary VF-patterns Π) in the sense discussed at the end of [**22**], Chapter X, § 1.4, i.e., relative to the condition that the representation always defines an indigenous bundle on the universal thickening $B^+(-)$ of the base. Note that this thickening $B^+(-)$ is in some sense the *minimal thickening of* (the normalization of the maximal log étale in characteristic zero extension of) "$(-)$" *that admits an action of Frobenius* (cf. the theory of [**22**], Chapter VI, § 1; $B^+(-)$ is the PD-completion of the rings $B(-)$ discussed in [**22**], Chapter VI, § 1; in fact, instead of using $B^+(-)$ here, it would also be quite sufficient to use the rings $B(-)$ of [**22**], Chapter VI, § 1). In other words, just as in the complex case,

> $\mathcal{N}[\infty]$ *is already complete* relative *to the condition that the representations it parametrizes always define indigenous bundles on the given curve* and all of its Frobenius conjugates.

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p-ADIC BOUNDARY VALUES

by

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Abstract. — We study in detail certain natural continuous representations of $G = GL_n(K)$ in locally convex vector spaces over a locally compact, non-archimedean field K of characteristic zero. We construct boundary value maps, or integral transforms, between subquotients of the dual of a "holomorphic" representation coming from a p-adic symmetric space, and "principal series" representations constructed from locally analytic functions on G. We characterize the image of each of our integral transforms as a space of functions on G having certain transformation properties and satisfying a system of partial differential equations of hypergeometric type.

This work generalizes earlier work of Morita, who studied this type of representation of the group $SL_2(K)$. It also extends the work of Schneider-Stuhler on the De Rham cohomology of p-adic symmetric spaces. We view this work as part of a general program of developing the theory of such representations.

Introduction

In this paper, we study in detail certain natural continuous representations of $G = GL_n(K)$ in locally convex vector spaces over a locally compact, non-archimedean field K of characteristic zero. We construct boundary value maps, or integral transforms, between subquotients of the dual of a "holomorphic" representation coming from a p-adic symmetric space, and "principal series" representations constructed from locally analytic functions on G. We characterize the image of each of our integral transforms as a space of functions on G having certain transformation properties and satisfying a system of partial differential equations of hypergeometric type.

This work generalizes earlier work of Morita, who studied this type of representation of the group $SL_2(K)$. It also extends the work of Schneider-Stuhler on the De Rham cohomology of p-adic symmetric spaces. We view this work as part of a general program of developing the theory of such representations.

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A major motivation for studying continuous representations of p-adic groups comes from the observation that, in traditional approaches to the representation theory of p-adic groups, one separates representations into two essentially disjoint classes – the smooth representations (in the sense of Langlands theory) and the finite dimensional rational representations. Such a dichotomy does not exist for real Lie groups, where the finite dimensional representations are "smooth." The category of continuous representations which we study is broad enough to unify both smooth and rational representations, and one of the most interesting features of our results is the interaction between these two types of representations.

The principal tools of this paper are non-archimedean functional analysis, rigid geometry, and the "residue" theory developed in the paper [ST]. Indeed, the boundary value maps we study are derived from the residue map of [ST].

Before summarizing the structure of our paper and discussing our main results, we will review briefly some earlier, related results.

The pioneering work in this area is due to Morita ([Mo1-Mo6]). He intensively studied two types of representations of $SL_2(K)$. The first class of representations comes from the action of $SL_2(K)$ on sections of rigid line bundles on the one-dimensional rigid analytic space $\mathfrak{X}$ obtained by deleting the K-rational points from $\mathbf{P}^1_{/K}$; this space is often called the p-adic upper half plane. The second class of representations is constructed from locally analytic functions on $SL_2(K)$ which transform by a locally analytic character under the right action by a Borel subgroup P of $SL_2(K)$. This latter class make up what Morita called the (p-adic) principal series.

Morita showed that the duals of the "holomorphic" representations coming from the p-adic upper half plane occur as constituents of the principal series. The simplest example of this is Morita's pairing

$$\Omega^1(\mathfrak{X}) \times C^{\mathrm{an}}(\mathbf{P}^1(K), K)/K \longrightarrow K \tag{$*$}$$

between the locally analytic functions on $\mathbf{P}^1(K)$ modulo constants (a "principal series" representation, obtained by induction from the trivial character) and the 1-forms on the one-dimensional symmetric space (a holomorphic representation.)

Morita's results illustrate how continuous representation theory extends the theory of smooth representations. Under the pairing (*), the locally constant functions on $\mathbf{P}^1(K)$ modulo constants (a smooth representation known as the Steinberg representation) are a G-invariant subspace which is orthogonal to the subspace of $\Omega^1(\mathfrak{X})$ consisting of exact forms. In particular, this identifies the first De Rham cohomology group of the p-adic upper half plane over K with the K-linear dual of the Steinberg representation.

The two types of representations considered by Morita (holomorphic discrete series and principal series) have been generalized to GL_n.

The "holomorphic" representations defined in [Sch] use Drinfeld's d-dimensional p-adic symmetric space $\mathfrak{X}$. The space $\mathfrak{X}$ is the complement in $\mathbf{P}^d_{/K}$ of the K-rational hyperplanes. The action of the group $G := GL_{d+1}(K)$ on $\mathbf{P}^d$ preserves the missing hyperplanes, and therefore gives an action of G on $\mathcal{X}$ and a continuous action of G on the infinite dimensional locally convex K-vector space $\mathcal{O}(\mathcal{X})$ of rigid functions on $\mathcal{X}$. The (p-adic) holomorphic discrete series representations are modelled on this example, and come from the action of G on the global sections of homogeneous vector bundles on $\mathbf{P}^d$ restricted to $\mathcal{X}$. There is a close relationship between these holomorphic representations and classical automorphic forms, coming from the theory of p-adic uniformization of Shimura varieties ([RZ], [Var]).

The second type of representation we will study are the "locally analytic" representations. Such representations are developed systematically in a recent thesis of Féaux de Lacroix ([Fea]). He defines a class of representations (which he calls "weakly analytic") in locally convex vector spaces V over K, relying on a general definition of a V-valued locally analytic function. Such a representation is a continuous linear action of G on V with the property that, for each v, the orbit maps $f_v(g) = g \cdot v$ are locally analytic V-valued functions on G. Notice that locally analytic representations include both smooth representations and rational ones.

Féaux de Lacroix's thesis develops some of the foundational properties of this type of representation. In particular, he establishes the basic properties of an induction functor (analytic coinduction). If we apply his induction to a one-dimensional locally analytic representation of a Borel subgroup of G, we obtain the p-adic principal series.

In this paper, we focus on one holomorphic representation and analyze it in terms of locally analytic principal series representations. Specifically, we study the representation of $G = GL_{d+1}(K)$ on the space $\Omega^d(\mathcal{X})$ of d-forms on the d-dimensional symmetric space $\mathcal{X}$. Our results generalize Morita, because we work in arbitrary dimensions, and Schneider-Stuhler, because we analyze all of $\Omega^d(\mathcal{X})$, not just its cohomology. Despite our narrow focus, we uncover new phenomena not apparent in either of the other works, and we believe that our results are representative of the general structure of holomorphic discrete series representations.

Our main results describe a d-step, G-invariant filtration on $\Omega^d(\mathcal{X})$ and a corresponding filtration on its continuous linear dual $\Omega^d(\mathcal{X})'$. We establish topological isomorphisms between the $d+1$ subquotients of the dual filtration and subquotients of members of the principal series. The j-th such isomorphism is given by a "boundary value map" $I^{[j]}$.

The filtration on $\Omega^d(\mathcal{X})$ comes from geometry and reflects the fact that $\mathcal{X}$ is a hyperplane complement. The first proper subspace $\Omega^d(\mathcal{X})^1$ in the filtration on $\Omega^d(\mathcal{X})$ is the space of exact forms, and the first subquotient is the d-th De Rham cohomology group.

The principal series representation which occurs as the j-th subquotient of the dual of $\Omega^d(\mathcal{X})$ is a hybrid object blending rational representations, smooth representations, and differential equations. The construction of these principal series representations is a three step process. For each $j = 0, \dots, d$, we first construct a representation V_j of the maximal parabolic subgroup $P_{\underline{j}}$ of G having a Levi subgroup of shape $GL_j(K) \times GL_{d+1-j}(K)$. The representation V_j (which factors through this Levi subgroup) is the tensor product of a simple rational representation with the Steinberg representation of one of the Levi factors. In the second step, we apply analytic coinduction to V_j to obtain a representation of G.

The third step is probably the most striking new aspect of our work. For each j, we describe a pairing between a generalized Verma module and the representation induced from V_j. We describe a submodule $\mathfrak{d}_{\underline{j}}$ of this Verma module such that $I^{[j]}$ is a topological isomorphism onto the subspace of the induced representation annihilated by $\mathfrak{d}_{\underline{j}}$:

$$I^{[j]} : [\Omega^d(\mathcal{X})^j / \Omega^d(\mathcal{X})^{j+1}]' \xrightarrow{\sim} C^{\mathrm{an}}(G, P_{\underline{j}}; V_j)^{\mathfrak{d}_{\underline{j}}=0}$$

The generators of the submodules $\mathfrak{d}_{\underline{j}}$ make up a system of partial differential equations. Interestingly, these differential equations are hypergeometric equations of the type studied by Gelfand and his collaborators (see [GKZ] for example). Specifically, the equations which arise here come from the adjoint action of the maximal torus of G on the (transpose of) the unipotent radical of $P_{\underline{j}}$.

For the sake of comparison with earlier work, consider the two extreme cases when $j = 0$ and $j = d$. When $j = 0$, the group $P_{\underline{j}}$ is all of G, the representation V_j is the Steinberg representation of G, and the induction is trivial. The submodule $\mathfrak{d}_{\underline{0}}$ is the augmentation ideal of $U(\mathfrak{g})$, which automatically kills V_j because Steinberg is a smooth representation.

When $j = d$, V_d is an one-dimensional rational representation of $P_{\underline{d}}$, and the module $\mathfrak{d}_{\underline{d}}$ is zero, so that there are no differential equations. In this case we obtain an isomorphism between the bottom step in the filtration and the locally analytic sections of an explicit homogeneous line bundle on the projective space $G/P_{\underline{d}}$. When $d = 1$, these two special cases ($j = 0$ and $j = 1$) together for $SL_2(K)$ are equivalent to Morita's theory applied to $\Omega^1(\mathcal{X})$.

We conclude this introduction with an outline of the sections of this paper. In sections one and two, we establish fundamental properties of $\Omega^d(\mathcal{X})$ as a topological vector space and as a G-representation. For example, we show that $\Omega^d(\mathcal{X})$ is a reflexive Fréchet space.

We introduce our first integral transform in section 2. Let ξ be the logarithmic d-form on $\mathbf{P}^d$ with first order poles along the coordinate hyperplanes. We study the map

$$\begin{array}{rcl} I : \Omega^d(\mathcal{X})' & \longrightarrow & C^{\mathrm{an}}(G, K) \\ \lambda & \longmapsto & [g \mapsto \lambda(g_*\xi)]. \end{array}$$

We show that functions in the image of I satisfy both discrete relations and differential equations, although we are unable to precisely characterize the image of the map I.

In section 3, we study the map I in more detail. We make use of the kernel function introduced in [ST], and attempt to clarify the relationship between the transform I and the results of that paper. Properties of the kernel function established in [ST], augmented by some new results, yield a map

$$I_{\mathrm{o}} : \Omega^d(\mathcal{X})' \longrightarrow C(G/P, K)/C_{\mathrm{inv}}(G/P, K)$$

where $C(G/P, K)$ denotes the continuous functions on G/P and $C_{\mathrm{inv}}(G/P, K)$ denotes the subspace generated by those continuous functions invariant by a larger parabolic subgroup. Using the "symmetrization map" of Borel and Serre, we show that the map I_{o} contains the same information as the original transform I. The map I_{o} has the advantage of targeting the possibly simpler space of functions on the compact space G/P. However, as was shown in [ST], the kernel function is locally analytic only on the big cell; it is continuous on all of G/P, but has complicated singularities at infinity. For this reason, the image of the map I_{o} does not lie inside the space of locally analytic functions. Introducing a notion of "analytic vectors" in a continuous representation, we prove that the image of I_{o} lies inside the subspace of analytic vectors, and so we can make sense of what it means for a function in the image of I_{o} to satisfy differential equations. However, as with I, we cannot completely describe the image of this "complete" integral transform, and to obtain precise results we must pass to subquotients of $\Omega^d(\mathcal{X})'$.

In the course of our analysis in section 3, we obtain the important result that the space of logarithmic forms (generated over K by the $g_*\xi$) is dense in $\Omega^d(\mathcal{X})$, and consequently our maps I and I_{o} are injective.

In section 4, we focus our attention on the differential equations satisfied by the functions in the image of the transform I. More precisely, let $\mathfrak{b}$ be the annihilator in $U(\mathfrak{g})$ of the special logarithmic form ξ. Any function in the image of I is killed by $\mathfrak{b}$. The key result in this section is the fact that the left $U(\mathfrak{g})$-module $U(\mathfrak{g})/\mathfrak{b} = U(\mathfrak{g})\xi$ has one-dimensional weight spaces for each weight in the root lattice of G. In some weak sense, the $U(\mathfrak{g})$-module $U(\mathfrak{g})\xi$ plays the role of a Harish-Chandra $(\mathfrak{g}, K)$-module in our p-adic setting.

The filtration on $\Omega^d(\mathcal{X})$ is closely related to a descending filtration of $U(\mathfrak{g})$ by left ideals

$$U(\mathfrak{g}) = \mathfrak{b}_0 \supset \mathfrak{b}_1 \supset \cdots \supset \mathfrak{b}_{d+1} = \mathfrak{b}.$$

By combinatorial arguments using weights, we show that the subquotients of this filtration are finite direct sums of irreducible highest weight $U(\mathfrak{g})$-modules. Each of these modules has a presentation as a quotient of a generalized Verma module by a certain submodule. These submodules are the modules $\mathfrak{d}_{\underline{j}}$ which enter into the statement of the main theorem.

In section 5, we obtain a "local duality" result. Let $\Omega_b^d(U^0)$ be the Banach space of bounded differential forms on the admissible open set U^0 in $\mathcal{X}$ which is the inverse image, under the reduction map, of an open standard chamber in the Bruhat-Tits building of G. Let B be the Iwahori group stabilizing this chamber, and let $\mathcal{O}(B)^{\mathfrak{b}=0}$ be the (globally) analytic functions on B annihilated by the (left invariant) differential operators in $\mathfrak{b}$. We construct a pairing which induces a topological isomorphism between the dual space $(\mathcal{O}(B)^{\mathfrak{b}=0})'$ and $\Omega_b^d(U^0)$.

We go on in section 5 to study the filtration of $\mathcal{O}(B)^{\mathfrak{b}=0}$ whose terms are the subspaces killed by the successively larger ideals $\mathfrak{b}_i$. We compute the subquotients of this local filtration, and interpret them as spaces of functions satisfying systems of partial differential equations. These local computations are used in a crucial way in the proof of the main theorem.

In section 6, we return to global considerations and define our G-invariant filtration on $\Omega^d(\mathfrak{X})$. We define this filtration first on the algebraic differential forms on $\mathfrak{X}$. These are the rational d-forms having poles along an arbitrary arrangement of K-rational hyperplanes. The algebraic forms are dense in the rigid forms, and we define the filtration on the full space of rigid forms by taking closures. A "partial fractions" decomposition due to Gelfand-Varchenko ([GV]) plays a key role in the definition of the filtration and the proof of its main properties.

In section 7, we use rigid analysis to prove that the first step in the global filtration coincides with the space of exact forms; this implies in particular that the exact forms are closed in $\Omega^d(\mathfrak{X})$. The desired results follow from a "convergent partial fractions" decomposition for global rigid forms on $\Omega^d(\mathfrak{X})$. One major application of this characterization of the first stage of the filtration is that it allows us to relate the other stages with subspaces of forms coming by pull-back from lower dimensional p-adic symmetric spaces. Another consequence of the results of this section is an analytic proof of that part of the main theorem of [SS] describing $H^d_{\mathrm{DR}}(\mathfrak{X})$ in terms of the Steinberg representation.

In section 8, we prove the main theorem, identifying the subquotients of the filtration on the dual of $\Omega^d(\mathfrak{X})$ with the subspaces of induced representations killed by the correct differential operators. All of the prior results are brought to bear on the problem. We show that the integral transform is bijective by showing that an element of the induced representation satisfying the differential equations can be written as a finite sum of G-translates of elements of a very special form, and then explicitly exhibiting an inverse image of such a special element. The fact that the map is a topological isomorphism follows from continuity and a careful application of an open-mapping theorem.

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0. Notations and conventions

For the reader's convenience, we will begin by summarizing some of the notation we use in this paper. In general, we have followed the notational conventions of [ST].

Let K denote a fixed, non-archimedean locally compact field of characteristic zero, residue characteristic $p > 0$ and ring of integers o. Let $|\cdot|$ be the absolute value on K, let $\omega : K \to \mathbf{Z}$ be the normalized additive valuation, and let π be a uniformizing parameter. We will use $\mathbf{C}_p$ for the completion of an algebraic closure of K.

Fix an integer $d \geq 1$ and let $\mathbf{P}^d$ be the projective space over K of dimension d. We let $G := GL_{d+1}(K)$, and adopt the convention that G acts on $\mathbf{P}^d$ through the left action $g([q_0 : \cdots : q_d]) = [q_0 : \cdots : q_d]g^{-1}$. We let T be the diagonal torus in G, and $\overline{T}$ the image of T in $PGL_{d+1}(K)$. We use $\epsilon_0, \ldots, \epsilon_d$ for the characters of T, where, if $t = (t_{ii})_{i=0}^d$ is a diagonal matrix, then $\epsilon_i(t) = t_{ii}$.

The character group $X^*(\overline{T})$ is the root lattice of G. It is spanned by the set $\Phi := \{\epsilon_i - \epsilon_j : 0 \leq i \neq j \leq d\}$ of roots of G. Let $\Xi_0, \ldots, \Xi_d$ be homogeneous coordinates for $\mathbf{P}^d$. Suppose that $\mu \in X^*(\overline{T})$, and write $\mu = \sum_{i=0}^d m_i\epsilon_i$. We let

$$\Xi_\mu = \prod_{i=0}^{d} \Xi_i^{m_i}.$$

Since μ belongs to the root lattice, we know that $\sum_{i=0}^d m_i = 0$, and therefore Ξ_μ is a well-defined rational function on $\mathbf{P}^d$.

Certain choices of μ arise frequently and so we give them special names. For $i = 0, \ldots, d-1$ we let $\beta_i = \epsilon_i - \epsilon_d$ and $\beta = \beta_0 + \cdots + \beta_{d-1}$. We also let $\alpha_i = \epsilon_{i+1} - \epsilon_i$, for $i = 0, \ldots, d-1$. The set $\{\alpha_i\}_{i=0}^{d-1}$ is a set of simple roots. We also adopt the convention that $\alpha_d = \epsilon_0 - \epsilon_d$. Any weight μ in $X^*(\overline{T})$ may be written uniquely as a sum $\mu = \sum_{i=0}^d m_i\alpha_i$ with integers $m_i \geq 0$ of which at least one is equal to 0. If μ is written in this way, we let $\ell(\mu) := m_d$.

As mentioned in the introduction, we let $\mathcal{X}$ denote Drinfeld's d-dimensional p-adic symmetric space. The space $\mathcal{X}$ is the complement in $\mathbf{P}^d$ of the K-rational hyperplanes. The G-action on $\mathbf{P}^d$ preserves $\mathcal{X}$. The structure of $\mathcal{X}$ as a rigid analytic space comes from an admissible covering of $\mathcal{X}$ by an increasing family of open K-affinoid subvarieties $\mathcal{X}_n$. To define the subdomains $\mathcal{X}_n$, let $\mathcal{H}$ denote the set of hyperplanes in $\mathbf{P}^d$ which are defined over K. For any $H \in \mathcal{H}$ let ℓ_H be a unimodular linear form in $\Xi_0, \ldots, \Xi_d$ such that H is the zero set of ℓ_H. (Here, and throughout this paper, a linear form ℓ_H is called unimodular if it has coefficients in o and at least one coefficient is a unit.)

The set $\mathcal{X}_n$ consists of the set of points $q \in \mathbf{P}^d$ such that

$$\omega(\ell_H([q_0 : \cdots : q_d])) \leq n$$

for any $H \in \mathcal{H}$ whenever $[q_0 : q_1 : \cdots : q_d]$ is a unimodular representative for the homogeneous coordinates of q. We denote by $\mathcal{O}(\mathcal{X})$ the ring of global rigid analytic functions on $\mathcal{X}$, and by $\Omega^i(\mathcal{X})$ the global i-forms. By $H^*_{\mathrm{DR}}(\mathcal{X})$ we mean the rigid-analytic De Rham cohomology of $\mathcal{X}$.

The space $\mathcal{X}$ has a natural G-equivariant map (the reduction map) $r : \mathcal{X} \to \overline{X}$ to the Bruhat-Tits building $\overline{X}$ of $PGL_{d+1}(K)$. For the definition of this map, see Definition 2 of [ST].

The torus $\overline{T}$ stabilizes a standard apartment $\overline{A}$ in $\overline{X}$. The Iwahori group

$$B := \{g \in GL_{d+1}(o) : g \text{ is lower triangular mod } \pi\}$$

is the pointwise stabilizer of a certain closed chamber $\overline{C}$ in $\overline{A} \subset X$. Following the conventions of [ST], we mean by $(\overline{C}, 0)$ the chamber $\overline{C}$ together with the vertex 0 stabilized by $GL_{d+1}(o)$. We will frequently denote a random closed chamber in X with the letter Δ, while Δ^0 will denote the interior of Δ. The inverse image $U^0 = r^{-1}(\overline{C}^0)$ of the open standard chamber $\overline{C}^0$ under the reduction map is an admissible open subset in $\mathcal{X}$.

In addition to these conventions regarding roots and weights of G, we use the following letters for various objects associated with G:

$$\begin{aligned}
P &:= \text{the lower triangular Borel subgroup of } G\\
U &:= \text{the lower triangular unipotent group of } G\\
N &:= \text{the normalizer of } T \text{ in } G\\
W &:= \text{the Weyl group } N/T \text{ of } G\\
w_{d+1} &:= \text{the longest element in } W\\
P_s &:= P \cup PsP \text{ for any simple reflection } s \in W
\end{aligned}$$

For an element $g \in Uw_{d+1}P$ in the big cell we define $u_g \in U$ by the identity $g = u_g w_{d+1} h$ with $h \in P$.

Corresponding to a root $\alpha = \epsilon_i - \epsilon_j$ we have a homomorphism $\widetilde{\alpha} : K^+ \to G$ sending $u \in K^+$ to the matrix (u_{rs}) with:

$$u_{rs} = \begin{cases} 1 & \text{if } r = s\\ u & \text{if } r = i \text{ and } s = j\\ 0 & \text{otherwise.}\end{cases}$$

The image U_α of $\widetilde{\alpha}$ in G is the root subgroup associated to α. It is filtered by the subgroups $U_{\alpha,r} := \widetilde{\alpha}(\{u \in K : \omega(u) \geq r\})$ for $r \in \mathbf{R}$. For a point $x \in \overline{A}$ we define U_x to be the subgroup of G generated by all $U_{\alpha,-\alpha(x)}$ for $\alpha \in \Phi$.

1. $\Omega^d(\mathcal{X})$ as a locally convex vector space

We begin by establishing two fundamental topological properties of $\Omega^d(\mathcal{X})$. We construct a family of norms on $\Omega^d(\mathcal{X})$, parameterized by chambers of the building $\overline{X}$, which defines the natural Fréchet topology (coming from its structure as a projective limit of Banach spaces) on $\Omega^d(\mathcal{X})$. We further show the fundamental result that $\Omega^d(\mathcal{X})$ is a reflexive Fréchet space.

We first look at the space $\mathcal{O}(\mathcal{X})$. For any open K-affinoid subvariety $\mathcal{Y} \subseteq \mathcal{X}$ its ring $\mathcal{O}(\mathcal{Y})$ of analytic functions is a K-Banach algebra with respect to the spectral norm. We equip $\mathcal{O}(\mathcal{X})$ with the initial topology with respect to the family of restriction maps $\mathcal{O}(\mathcal{X}) \to \mathcal{O}(\mathcal{Y})$. Since the increasing family of open K-affinoid subvarieties $\mathcal{X}_n$ forms an admissible covering of $\mathcal{X}$ ([SS] Sect. 1) we have

$$\mathcal{O}(\mathcal{X}) = \varprojlim_n \mathcal{O}(\mathcal{X}_n)$$

in the sense of locally convex K-vector spaces. It follows in particular that $\mathcal{O}(\mathcal{X})$ is a Fréchet space. Using a basis η_0 of the free $\mathcal{O}(\mathcal{X})$-module $\Omega^d(\mathcal{X})$ of rank 1 we topologize $\Omega^d(\mathcal{X})$ by declaring the linear map

$$\begin{array}{ccc} \mathcal{O}(\mathcal{X}) & \xrightarrow{\sim} & \Omega^d(\mathcal{X}) \\ F & \longmapsto & F\eta_0 \end{array}$$

to be a topological isomorphism; the resulting topology is independent of the choice of η_0. In this way $\Omega^d(\mathcal{X})$ becomes a Fréchet space, too. Similarly each $\Omega^d(\mathcal{X}_n)$ becomes a Banach space. In the following we need a certain G-invariant family of continuous norms on $\Omega^d(\mathcal{X})$. First recall the definition of the weights

$$\beta_i := \varepsilon_i - \varepsilon_d \quad \text{for } 0 \le i \le d-1.$$

We have

$$\Omega^d(\mathcal{X}) = \mathcal{O}(\mathcal{X})\, d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}.$$

The torus $\overline{T}$ acts on the form $d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$ through the weight

$$\beta := \beta_0 + \cdots + \beta_{d-1}.$$

For any point $q \in \mathcal{X}$ such that $z := r(q) \in \overline{A}$ we define a continuous (additive) semi-norm γ_q on $\Omega^d(\mathcal{X})$ by

$$\gamma_q(\eta) := \omega(F(q)) + \beta(z) \text{ if } \eta = F d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}.$$

Lemma 1.1. — *Let $q \in \mathcal{X}$ such that $\overline{x} := r(q) \in \overline{A}$; we then have*

$$\gamma_{gq} = \gamma_q \circ g^{-1} \textit{ for any } g \in N \cup U_x.$$

Proof. — First let $g \in G$ be any element such that $g\overline{x} \in \overline{A}$. Using [ST] Cor. 4 and the characterizing property of the function $\mu(g^{-1}, .)$ ([ST] Def. 28) one easily computes

$$\gamma_{gq} - \gamma_q \circ g^{-1} = \omega\left(\frac{g_*^{-1}\Xi_0}{\Xi_0}(q) \cdots \frac{g_*^{-1}\Xi_d}{\Xi_d}(q)\right) + \omega(\det g).$$

Obviously the right hand side vanishes if g is a diagonal or permutation matrix and hence for any $g \in N$. It remains to consider a $g = \widetilde{\alpha}(u) \in U_{\alpha, -\alpha(x)}$ for some root $\alpha \in \Phi$. Then the right hand side simplifies to $\omega(1 - u\Xi_\alpha(q)) = \omega(\Xi_\alpha(gq)) - \omega(\Xi_\alpha(q))$. According to [ST] Cor. 4 this is equal to $\alpha(r(gq)) - \alpha(r(q)) = \alpha(\overline{x}) - \alpha(\overline{x}) = 0$. $\square$

This allows us to define, for any point $q \in \mathcal{X}$, a continuous semi-norm γ_q on $\Omega^d(\mathcal{X})$ by

$$\gamma_q := \gamma_{gq} \circ g$$

where $g \in G$ is chosen in such a way that $r(gq) \in \overline{A}$. Moreover, for any chamber Δ in $\overline{X}$, we put

$$\gamma_\Delta := \inf_{r(q)\in\Delta^0} \gamma_q.$$

Since $r^{-1}(\Delta)$ is an affinoid ([ST] Prop. 13) this is a continuous semi-norm. To see that it actually is a norm let us look at the case of the standard chamber $\overline{C}$. Let $\eta = F \cdot d\Xi_{\alpha_{d-1}} \wedge \cdots \wedge d\Xi_{\alpha_0} \in \Omega^d(\mathcal{X})$. Since $F|U^0$ is bounded we have the expansion

$$F|U^0 = \sum_{\mu \in X^*(\overline{T})} a(\mu)\Xi_\mu$$

with $a(\mu) \in K$ and $\{\omega(a(\mu)) - l(\mu)\}_\mu$ bounded below. Since the restriction map $\Omega^d(\mathcal{X}) \longrightarrow \Omega^d(U^0)$ is injective we have the norm

$$\omega_C(\eta) := \inf_\mu \{\omega(a(\mu)) - l(\mu)\} = \inf_{q\in U^0} \omega(F(q))$$

on $\Omega^d(\mathcal{X})$.

Lemma 1.2. — $\omega_C \leq \gamma_{\overline{C}} \leq \omega_C + 1$.

Proof. — Let $\eta := F \cdot d\Xi_{\alpha_{d-1}} \wedge \cdots \wedge d\Xi_{\alpha_0}$. The identity

$$d\Xi_{\alpha_{d-1}} \wedge \cdots \wedge d\Xi_{\alpha_0} = \pm\Xi_{-\beta-\alpha_d} d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$$

together with [ST] Cor. 4 implies

$$\gamma_q(\eta) = \omega(F(q)) + \omega(\Xi_{-\beta-\alpha_d}(q)) + \beta(z) = \omega(F(q)) - \alpha_d(z)$$

for $r(q) = z \in \overline{C}^0$. Because of $-1 \leq \alpha_d|\overline{C} \leq 0$ we obtain

$$\omega(F(q)) \leq \gamma_q(\eta) \leq \omega(F(q)) + 1$$

for any $q \in U^0$. It remains to recall that $\omega_C(\eta) = \inf_{q\in U^0} \omega(F(q))$. □

This shows that all the γ_Δ are continuous norms on $\Omega^d(\mathcal{X})$. In fact the family of norms $\{\gamma_\Delta\}_\Delta$ defines the Fréchet topology of $\Omega^d(\mathcal{X})$. In order to see this it suffices to check that the additively written spectral norm ω_Δ for the affinoid $r^{-1}(\Delta)$ satisfies

$$\omega_\Delta(F) = \inf_{r(q)\in\Delta^0} \omega(F(q)) \text{ for } F \in \mathcal{O}(\mathcal{X}).$$

Let $\mathcal{X}_B$ denote Berkovich's version of the rigid analytic variety $\mathcal{X}$. Each point $q \in \mathcal{X}_B$ gives rise to the multiplicative semi-norm $F \mapsto \omega(F(q))$ on $\mathcal{O}(\mathcal{X})$. If one fixes $F \in \mathcal{O}(\mathcal{X})$ then the function $q \mapsto \omega(F(q))$ is continuous on $\mathcal{X}_B$. We need the following facts from [Be2]:

– The reduction map $r : \mathcal{X} \to \overline{X}$ extends naturally to a continuous map $r_B : \mathcal{X}_B \to \overline{X}$.

– The map r_B has a natural continuous section $s_B : \overline{X} \to \mathfrak{X}_B$ such that $F \mapsto \omega(F(s_B(z)))$, for $z \in r(\mathfrak{X})$, is the spectral norm $\omega_{r^{-1}(z)}$ for the affinoid $r^{-1}(z)$.

In particular, for a fixed $F \in \mathcal{O}(\mathfrak{X})$, the map $z \mapsto \omega(F(s_B(z)))$ is continuous on $\overline{X}$. Since $r(\mathfrak{X})$ is dense in $\overline{X}$ it follows that

$$\begin{aligned}\inf_{r(q)\in\Delta^0} \omega(F(q)) &= \inf_{z\in r(\mathfrak{X})\cap\Delta^0} \omega_{r^{-1}(z)}(F) = \inf_{z\in r(\mathfrak{X})\cap\Delta} \omega_{r^{-1}(z)}(F)\\ &= \inf_{r(q)\in\Delta} \omega(F(q)) = \omega_\Delta(F).\end{aligned}$$

Lemma 1.3. — *The G-action $G \times \Omega^d(\mathfrak{X}) \to \Omega^d(\mathfrak{X})$ is continuous.*

Proof. — Clearly each individual element $g \in G$ induces a continuous automorphism of $\Omega^d(\mathfrak{X})$. As a Fréchet space $\Omega^d(\mathfrak{X})$ is barrelled ([Tie] Thm. 3.15). Hence the Banach-Steinhaus theorem ([Tie] Thm. 4.1) holds for $\Omega^d(\mathfrak{X})$ and we only have to check that the maps

$$\begin{array}{ccc} G & \longrightarrow & \Omega^d(\mathfrak{X}) \\ g & \longmapsto & g\eta \end{array} \qquad \text{for } \eta \in \Omega^d(\mathfrak{X})$$

are continuous (compare the reasoning in [War] p. 219). By the universal property of the projective limit topology this is a consequence of the much stronger local analyticity property which we will establish in Prop. 1' of the next section. $\square$

Proposition 1.4. — *$\mathcal{O}(\mathfrak{X})$ is reflexive and its strong dual $\mathcal{O}(\mathfrak{X})'$ is the locally convex inductive limit*

$$\mathcal{O}(\mathfrak{X})' = \varinjlim_n \mathcal{O}(\mathfrak{X}_n)'$$

of the dual Banach spaces $\mathcal{O}(\mathfrak{X}_n)'$.

The proof is based on the following concepts.

Definition. — *A homomorphism $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ between K-Banach spaces is called compact if the image under ψ of the unit ball $\{f \in \mathcal{A} : |f|_{\mathcal{A}} \leq 1\}$ in $\mathcal{A}$ is relatively compact in $\mathcal{B}$.*

We want to give a general criterion for a homomorphism of affinoid K-algebras to be compact. Recall that an affinoid K-algebra $\mathcal{A}$ is a Banach algebra with respect to the residue norm $|\ |_a$ induced by a presentation

$$a\ : K\langle T_1, \ldots, T_m\rangle \twoheadrightarrow \mathcal{A}$$

as a quotient of a Tate algebra. All these norms $|\ |_a$ are equivalent.

***Definition* ([Ber] 2.5.1)**. — *A homomorphism $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ of affinoid K-algebras is called inner if there is a presentation $a : K\langle T_1, \ldots, T_m\rangle \twoheadrightarrow \mathcal{A}$ such that*

$$\inf\{\omega(\psi a(T_i)(y)) : y \in \mathrm{Sp}(\mathcal{B}), 1 \leq i \leq m\} > 0.$$

Lemma 1.5. — *Any inner homomorphism $\psi : \mathcal{A} \to \mathcal{B}$ of affinoid K-algebras is compact.*

Proof. — First of all we note that if the assertion holds for one residue norm on $\mathcal{A}$ then it holds for all of them. If ψ is inner we find, according to [Ber] 2.5.2, a presentation $a : K\langle T_1, \dots, T_m\rangle \twoheadrightarrow \mathcal{A}$ such that

$$\inf\{\omega(\psi a(T_i)(y)) : y \in \mathrm{Sp}(\mathcal{B}), 1 \leq i \leq m\} > 1.$$

This means that we actually have a commutative diagram of affinoid K-algebras

$$\begin{tikzcd} K\langle T_1, \dots, T_m\rangle \arrow[r, "\imath"] \arrow[d, "a"'] & K\langle \pi^{-1}T_1, \dots, \pi^{-1}T_m\rangle \arrow[d] \\ \mathcal{A} \arrow[r, "\psi"] & \mathcal{B} \end{tikzcd}$$

where $\imath$ is the obvious inclusion of Tate algebras. Since the valuation of K is discrete the unit ball in $K\langle T_1, \dots, T_m\rangle$ (with respect to the Gauss norm) is mapped surjectively, by a, onto the unit ball in $\mathcal{A}$ (with respect to $|\ |_a$). Hence it suffices to prove that the inner monomorphism $\imath$ is compact. But this is a straightforward generalization of the argument in the proof of [Mo1] 3.5. □

Proof of Proposition 4. — In the proof of [SS] § 1 Prop. 4 the following two facts are established:

- The restriction maps $\mathcal{O}(\mathfrak{X}_{n+1}) \to \mathcal{O}(\mathfrak{X}_n)$ are inner;
- $\mathfrak{X}_n$ is a Weierstraß domain in $\mathfrak{X}_{n+1}$ for each n.

The second fact implies that the restriction map $\mathcal{O}(\mathfrak{X}_{n+1}) \to \mathcal{O}(\mathfrak{X}_n)$ has a dense image. It then follows from Mittag-Leffler ([B-TG3] II § 3.5 Thm. 1) that the restriction maps $\mathcal{O}(\mathfrak{X}) \to \mathcal{O}(\mathfrak{X}_n)$ have dense images. Using Lemma 5 we see that the assumptions in [Mo1] 3.3(i) and 3.4(i) are satisfied for the sequence of Banach spaces $\mathcal{O}(\mathfrak{X}_n)$. Our assertion results.

Of course then also $\Omega^d(\mathfrak{X})$ is reflexive with $\Omega^d(\mathfrak{X})' = \varinjlim_n \Omega^d(\mathfrak{X}_n)'$. □

2. $\Omega^d(\mathfrak{X})$ as a locally analytic G-representation

In this section, we study the G-action on $\Omega^d(\mathfrak{X})$ and investigate in which sense it is locally analytic. Using this property of the G-action, we construct a continuous map I from $\Omega^d(\mathfrak{X})'$ to the space of locally analytic K-valued functions on G. It follows from the construction of this map that its image consists of functions annihilated by a certain ideal $\mathfrak{a}$ in the algebra of punctual distributions on G. In particular, this means that functions in the image of I satisfy both discrete relations (meaning that their values at certain related points of G cannot be independently specified) and differential equations. We will study these relations in more detail in later sections.

We will use the notion of a locally analytic map from a locally K-analytic manifold into a Hausdorff locally convex K-vector space as it is defined in [B-VAR] 5.3.1. But we add the attribute "locally" in order to make clearer the distinction from rigid analytic objects.

Proposition 2.1. — *For any $\eta \in \Omega^d(\mathfrak{X})$ and any $\lambda \in \Omega^d(\mathfrak{X})'$ the function $g \mapsto \lambda(g_*\eta)$ on G is locally analytic.*

Since, by Prop. 1.4, λ comes from a continuous linear form on some $\Omega^d(\mathfrak{X}_n)$ this is an immediate consequence of the following apparently stronger fact.

Proposition 2.1'. — *Whenever $\Omega^d(\mathfrak{X})$ is equipped with the coarser topology coming from the spectral norm on $\mathfrak{X}_n$ for some fixed but arbitrary $n \in \mathbb{N}$ then the map $g \mapsto g_*\eta$, for any $\eta \in \Omega^d(\mathfrak{X})$, is locally analytic.*

Proof. — For the moment being we fix a natural number $n \in \mathbb{N}$. In the algebraic, and hence rigid analytic, K-group GL_{d+1} we have the open K-affinoid subgroup

$$H_n := \{h \in GL_{d+1}(o_{\mathbf{C}_p}) : h \equiv g \bmod \pi^{n+1} \text{ for some } g \in GL_{d+1}(o)\}$$

which contains the open K-affinoid subgroup

$$D_n := 1 + \pi^{n+1} M_{d+1}(o_{\mathbf{C}_p});$$

here o, resp. $o_{\mathbf{C}_p}$, denotes the ring of integers in K, resp. $\mathbf{C}_p$. As a rigid variety over K the latter group D_n is a polydisk of dimension $r := (d+1)^2$. Since H_n preserves the K-affinoid subdomain $\mathfrak{X}_n$ of $\mathbf{P}^d$ the algebraic action of GL_{d+1} on $\mathbf{P}^d$ restricts to a rigid analytic action $m : H_n \times \mathfrak{X}_n \to \mathfrak{X}_n$ which corresponds to a homomorphism of K-affinoid algebras

$$\begin{array}{ccc} \mathcal{O}(\mathfrak{X}_n) & \longrightarrow & \mathcal{O}(H_n \times \mathfrak{X}_n) = \mathcal{O}(H_n)\widehat{\otimes}_K \mathcal{O}(\mathfrak{X}_n) \\ F & \longmapsto & m^*F. \end{array}$$

For any $h \in H_n$ we clearly have

$$[(\text{evaluation in } h) \otimes \text{id}] \circ m^*F = hF \ .$$

For a fixed $g \in GL_{d+1}(o)$ we consider the rigid analytic "chart"

$$\begin{array}{rcl} \imath_g : D_n & \longrightarrow & H_n \\ h & \longmapsto & gh \ . \end{array}$$

Fixing coordinates $T_1, \dots, T_r$ on the polydisk D_n we have

$$\mathcal{O}(D_n)\widehat{\otimes}_K \mathcal{O}(\mathfrak{X}_n) \cong \mathcal{O}(\mathfrak{X}_n)\langle T_1, \dots, T_r\rangle.$$

The power series

$$\mathcal{F}_g(T_1, \dots, T_r) := (\imath_g^* \otimes \text{id}) m^*F \in \mathcal{O}(\mathfrak{X}_n)\langle T_1, \dots, T_r\rangle$$

has the property that $ghF = \mathcal{F}_g(T_1(h), \dots, T_r(h))$ for any $h \in D_n$. This shows that, for any $F \in \mathcal{O}(\mathfrak{X}_n)$, the map

$$\begin{array}{ccc} GL_{d+1}(o) & \longrightarrow & \mathcal{O}(\mathfrak{X}_n) \\ g & \longmapsto & gF \end{array}$$

is locally analytic.

This construction varies in an obvious way with the natural number n. In particular if we start with a function $F \in \mathcal{O}(\mathcal{X}) \subseteq \mathcal{O}(\mathcal{X}_n)$ then the coefficients of the power series $\mathcal{F}_g$ also lie in $\mathcal{O}(\mathcal{X})$. It follows that actually, for any $F \in \mathcal{O}(\mathcal{X})$, the map

$$\begin{array}{rcl} GL_{d+1}(o) & \longrightarrow & \mathcal{O}(\mathcal{X}) \\ g & \longmapsto & gF \end{array}$$

is locally analytic provided the right hand side is equipped with the sup-norm on $\mathcal{X}_n$ for a fixed but arbitrary $n \in \mathbb{N}$. Since F was arbitrary and $GL_{d+1}(o)$ is open in G the full map

$$\begin{array}{rcl} G & \longrightarrow & \mathcal{O}(\mathcal{X}) \\ g & \longmapsto & gF \end{array}$$

has to have the same local analyticity property.

This kind of reasoning extends readily to any GL_{d+1}-equivariant algebraic vector bundle $\mathbb{V}$ on $\mathbf{P}^d$. Then the space of rigid analytic sections $\mathbb{V}(\mathcal{X})$ is a Fréchet space as before on which G acts continuously and such that the maps

$$\begin{array}{rcl} G & \longrightarrow & \mathbb{V}(\mathcal{X}) \\ g & \longmapsto & gs \end{array}$$

for any $s \in \mathbb{V}(\mathcal{X})$ have the analogous local analyticity property. The reason is that the algebraic action induces a rigid analytic action

$$H_n \times \mathbb{V}_{/\mathcal{X}_n} \longrightarrow \mathbb{V}_{/\mathcal{X}_n}$$

which is compatible with the action of H_n on $\mathcal{X}_n$ via m. But this amounts to the existence of a vector bundle isomorphism

$$m^*(\mathbb{V}_{/\mathcal{X}_n}) \xrightarrow{\cong} pr_2^*(\mathbb{V}_{/\mathcal{X}_n})$$

satisfying a certain cocycle condition (compare [Mum] 1.3); here $pr_2 : H_n \times \mathcal{X}_n \to \mathcal{X}_n$ is the projection map. Hence similarly as above the H_n-action on the sections $\mathbb{V}(\mathcal{X}_n)$ is given by a homomorphism

$$\mathbb{V}(\mathcal{X}_n) \longrightarrow m^*(\mathbb{V}_{/\mathcal{X}_n})(H_n \times \mathcal{X}_n) \xrightarrow{\cong} pr_2^*(\mathbb{V}_{/\mathcal{X}_n})(H_n \times \mathcal{X}_n) = \mathcal{O}(\mathcal{X}_n)\widehat{\otimes}_K \mathbb{V}(\mathcal{X}_n)\ .$$

The rest of the argument then is the same as above. □

That result has two important consequences for our further investigation. In the first place it allows us to introduce the basic map for our computation of the dual space $\Omega^d(\mathcal{X})'$. Let

$$C^{\mathrm{an}}(G, K) := \text{ space of locally } K\text{-analytic functions on } G.$$

We always consider this space as the locally convex inductive limit

$$C^{\mathrm{an}}(G, K) = \varinjlim_{\mathcal{U}} C^{\mathrm{an}}_{\mathcal{U}}(G, K).$$

Here $\mathcal{U} = \{U_i\}_{i\in I}$ is a disjoint covering of the locally K-analytic manifold G by closed balls (in the sense of charts) and

$$C^{\mathrm{an}}_{\mathcal{U}}(G,K) := \{f \in C^{\mathrm{an}}(G,K)\ :\ f|U_i \text{ is analytic for any } i \in I\}$$

is the direct product of the Banach spaces of analytic functions on each U_i (where the Banach norm is the spectral norm on U_i). The group G acts by left translations on $C^{\mathrm{an}}(G,K)$.

Lemma 2.2. — *The G-action $G \times C^{\mathrm{an}}(G,K) \to C^{\mathrm{an}}(G,K)$ is continuous.*

Proof. — Clearly, each group element $g \in G$ acts continuously on $C^{\mathrm{an}}(G,K)$. Being the locally convex inductive limit of a direct product of Banach spaces, $C^{\mathrm{an}}(G,K)$ is barrelled. Hence it suffices (as in the proof of Lemma 1.3) to check that the maps

$$\begin{array}{rcl} G & \longrightarrow & C^{\mathrm{an}}(G,K) \quad \text{for } f \in C^{\mathrm{an}}(G,K) \\ g & \longmapsto & gf \end{array}$$

are continuous. But those maps actually are differentiable ([Fea] 3.3.4). □

In all that follows, the d-form

$$\xi := \frac{d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}}{\Xi_{\beta_0} \cdots \Xi_{\beta_{d-1}}}$$

on $\mathfrak{X}$ is the basic object. Because of Prop. 1 we have the G-equivariant map

$$\begin{array}{rcl} I : \Omega^d(\mathfrak{X})' & \longrightarrow & C^{\mathrm{an}}(G,K) \\ \lambda & \longmapsto & [g \mapsto \lambda(g_*\xi)]. \end{array}$$

Lemma 2.3. — *The map I is continuous.*

Proof. — Since $\Omega^d(\mathfrak{X})'$ is the locally convex inductive limit of the Banach spaces $\Omega^d(\mathfrak{X}_n)'$ it suffices to establish the corresponding fact for $\Omega^d(\mathfrak{X}_n)$. In the proof of Prop. 1' we have seen that the map

$$\begin{array}{rcl} G & \longrightarrow & \Omega^d(\mathfrak{X}_n) \\ g & \longmapsto & g_*\xi \end{array}$$

is analytic on the right cosets of $G \cap D_n$ in G. We obtain that, for $\lambda \in \Omega^d(\mathfrak{X}_n)'$, the function $g \mapsto \lambda(g_*\xi)$ lies in $C^{\mathrm{an}}_{\mathcal{U}}(G,K)$ with $\mathcal{U} := \{(G \cap D_n)g\}_{g\in G}$ and that on a fixed coset $(G \cap D_n)g$ the spectral norms satisfy the inequality

$$\|\lambda(._*\xi)\| \leq \|\lambda\| \cdot \|._*\xi\|.$$ □

We also have the right translation action of G on $C^{\mathrm{an}}(G,K)$ which we write as

$$\delta_g f(h) = f(hg).$$

In addition we have the action of the Lie algebra $\mathfrak{g}$ of G by left invariant differential operators; for any $\mathfrak{x} \in \mathfrak{g}$ the corresponding operator on $C^{\mathrm{an}}(G,K)$ is given by

$$(\mathfrak{x}f)(g) := \frac{d}{dt} f(g \exp(t\mathfrak{x}))_{|t=0}\,;$$

here $\exp : \mathfrak{g} \dashrightarrow G$ denotes the exponential map which is defined locally around 0. This extends by the universal property to a left action of the universal enveloping algebra $U(\mathfrak{g})$ on $C^{\mathrm{an}}(G,K)$. For any $f \in C^{\mathrm{an}}(G,K)$, any $g \in G$, and any $\mathfrak{x} \in \mathfrak{g}$ sufficiently close to 0 (depending on g) we have Taylor's formula

$$f(g\exp(\mathfrak{x})) = \sum_{n=0}^{\infty} \frac{1}{n!}(\mathfrak{x}^n f)(g)$$

(compare, for example, the proof in [Hel] II.1.4 which goes through word for word for p-adic Lie groups). We actually find for any $h \in G$ a neighbourhood N_0 of h in G and a neighbourhood $\mathfrak{n}$ of 0 in $\mathfrak{g}$ such that the above formula holds for all $(g,\mathfrak{x}) \in N_0 \times \mathfrak{n}$.

The right translation action of G and the $U(\mathfrak{g})$-action on $C^{\mathrm{an}}(G,K)$ combine into an action of the algebra $\mathcal{D}(G)$ of punctual distributions on G ([B-GAL] III § 3.1). Any $D \in \mathcal{D}(G)$ can be written in a unique way as a finite sum $D = \mathfrak{z}_1\delta_{g_1} + \cdots + \mathfrak{z}_r\delta_{g_r}$ with $\mathfrak{z}_i \in U(\mathfrak{g})$ and $g_i \in G$, δ_g denoting the Dirac distribution supported at $g \in G$. Then one has $Df = \sum_i \mathfrak{z}_i(f(.g_i))$ for $f \in C^{\mathrm{an}}(G,K)$; observe that

$$\delta_g(\mathfrak{z}(f)) = (\mathrm{ad}(g)\mathfrak{z})(\delta_g(f)).$$

This $\mathcal{D}(G)$-action commutes with the left translation action of G on $C^{\mathrm{an}}(G,K)$. Moreover $\mathcal{D}(G)$ acts by continuous endomorphisms on $C^{\mathrm{an}}(G,K)$; this is again a simple application of the Banach-Steinhaus theorem (compare [Fea] 3.1.2).

The second consequence of Prop. 1' is that the map $g \mapsto g_*\eta$ from G into $\Omega^d(\mathfrak{X})$ is differentiable ([B-VAR] 1.1.2) for any $\eta \in \Omega^d(\mathfrak{X})$. It follows that $\mathfrak{g}$ and hence $U(\mathfrak{g})$ act on $\Omega^d(\mathfrak{X})$ from the left by

$$\mathfrak{x}\eta := \frac{d}{dt}\exp(t\mathfrak{x})_*\eta_{|t=0}.$$

Obviously the G-action and the $U(\mathfrak{g})$-action again combine into a left $\mathcal{D}(G)$-action by continuous endomorphisms on $\Omega^d(\mathfrak{X})$. Note that $\Omega^d(\mathfrak{X})$ as a Fréchet space is barrelled, too. We define now

$$\mathfrak{a} := \{D \in \mathcal{D}(G) : D\xi = 0\}$$

to be the annihilator ideal of ξ in $\mathcal{D}(G)$; it is a left ideal. On the other hand

$$C^{\mathrm{an}}(G,K)^{\mathfrak{a}=0} := \{f \in C^{\mathrm{an}}(G,K) : \mathfrak{a}f = 0\}$$

then is a G-invariant closed subspace of $C^{\mathrm{an}}(G,K)$. The formula

$$[D(I(\lambda))](g) = \lambda(g_*(D\xi)) \text{ for } D \in \mathcal{D}(G), \lambda \in \Omega^d(\mathfrak{X})' \text{ and } g \in G$$

implies that that subspace contains the image of the map I, i.e., that I induces a G-equivariant continuous linear map

$$\Omega^d(\mathfrak{X})' \longrightarrow C^{\mathrm{an}}(G,K)^{\mathfrak{a}=0}.$$

3. The kernel map

In the previous section we constructed a map I from $\Omega^d(\mathcal{X})'$ to a certain space of locally analytic functions on G. We see this map as a "boundary value" map, but this interpretation needs clarification. In particular, the results of [SS] and [ST] suggest that a more natural "boundary" for the symmetric space $\mathcal{X}$ is the compact space G/P. In this section, we study a different boundary value map I_o, which carries $\Omega^d(\mathcal{X})'$ to (a quotient of) a space of functions on G/P. Our objective is to relate I_o to I. The major complications come from the fact that the image of I_o does not consist of locally analytic functions, a phenomenon essentially due to the fact that the kernel function on G/P studied in [ST] is locally analytic on the big cell with continuous, not locally analytic, extension to G/P. We relate I_o to I using a "symmetrization map," due to Borel and Serre, which carries functions on G/P into functions on G, together with a theory of "analytic vectors" in a continuous G-representation. One crucial consequence of our work in this section is the fact that the integral transform I_o (and I) is injective.

Recall the definition, in [ST] Def. 27, of the integral kernel function $k(g,q)$ on $G/P \times \mathcal{X}$. This function is given by

$$k(g,.) = \begin{cases} (u_g)_* \dfrac{1}{\Xi_{\beta_0}\cdots\Xi_{\beta_{d-1}}} & \text{if } g = u_g w_{d+1} p \text{ is in the big cell,} \\ 0 & \text{otherwise.} \end{cases}$$

Here we rather want to consider the map

$$\begin{aligned} \mathbf{k} : G/P &\longrightarrow \Omega^d(\mathcal{X}) \\ g &\longmapsto k(g,.)\, d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}. \end{aligned}$$

Since the numerator of the form ξ is invariant under lower triangular unipotent matrices (compare the formula after Def. 28 in [ST]) we can rewrite our new map as

$$\mathbf{k}(g) = \begin{cases} (u_g)_*\xi & \text{if } g = u_g w_{d+1} p \text{ is in the big cell,} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.1. — *The map* $\mathbf{k}$ *is continuous and vanishes outside the big cell. Moreover whenever* $\Omega^d(\mathcal{X})$ *is equipped with the coarser topology coming from the spectral norm on* $\mathcal{X}_n$ *for some fixed but arbitrary* $n \in \mathbb{N}$ *then* $\mathbf{k}$ *is locally analytic on the big cell.*

Proof. — The vanishing assertion holds by definition. The assertion about local analyticity of course is a consequence of Prop. 2.1'. But we will give another argument which actually produces explicitly the local series expansions. This will be needed in the subsequent considerations.

Let U denote the unipotent radical of P. According to [ST] Lemma 12 the sets $\mathbf{B}(u,r) = uw_{d+1}t^rBP/P$, for a fixed $u \in U$, t the diagonal matrix with entries

$(\pi^d, \ldots, \pi, 1)$, and varying $r \in \mathbb{N}$, form a fundamental system of open neighbourhoods of the point $uw_{d+1}P/P$ in the big cell. One easily checks that

$$D(u,r) := \{v \in U : vw_{d+1}P/P \in \mathbf{B}(u,r)\}$$

is a polydisk in the affine space U. Hence the maps

$$\begin{array}{ccc} D(u,r) & \xrightarrow{\sim} & \mathbf{B}(u,r) \subseteq \text{ big cell} \\ v & \longmapsto & vw_{d+1}P/P \end{array}$$

constitute an atlas for the big cell as a locally analytic manifold. Fix $n \in \mathbb{N}$. We have to show that given a $u \in U$ we find an $r \in \mathbb{N}$ such that the map

$$\begin{array}{ccc} D(u,r) & \longrightarrow & \Omega^d(\mathfrak{X}) \\ v & \longmapsto & \mathbf{k}(vw_{d+1}) \end{array}$$

is analytic with respect to the coarser topology on the right hand side corresponding to n. Recall that this amounts to the following ([B-VAR]). Let v_{ji} for $0 \leq i < j \leq d$ denote the matrix entries of the matrix $v \in U$. Moreover we use the usual abbreviation

$$(v-u)^{\underline{m}} := \prod_{0 \leq i < j \leq d} (v_{ji} - u_{ji})^{m_{ji}}$$

for any multi-index $\underline{m} = (m_{10}, \ldots, m_{dd-1}) \in \mathbb{N}_0^{d(d+1)/2}$. We have to find an $r \in \mathbb{N}$ such that there is a power series expansion

$$k(vw_{d+1}, q) = \sum_{\underline{m}} (v-u)^{\underline{m}} \cdot F_{\underline{m}}(q)$$

with $F_{\underline{m}} \in \mathcal{O}(\mathfrak{X})$ which is uniformly convergent on $D(u,r) \times \mathfrak{X}_n$. From now on we fix $u \in U$. We choose $r \in \mathbb{N}$ such that

$$\omega(v_{ji} - u_{ji}) > 2n \text{ for all } v \in D(u,r) \text{ and } 0 \leq i < j \leq d.$$

We write

$$k(vw_{d+1}, q) = \prod_{i=0}^{d-1} \frac{1}{f_i(v,q)}$$

where

$$f_i(v,q) := \sum_{j=i}^{d-1} a_{ji}(v) \Xi_{\beta_j}(q) + a_{di}(v)$$

with

$$a_{ji}(v) := \begin{cases} v_{ji} & \text{for } j > i, \\ 1 & \text{for } j = i. \end{cases}$$

We also write

$$f_i(v,q) = f_i(u,q) + \sum_{j=i+1}^{d-1} b_{ji}(v) \Xi_{\beta_j}(q) + b_{di}(v)$$

with

$$b_{ji}(v) := a_{ji}(v) - u_{ji} = v_{ji} - u_{ji}.$$

Observe that

$$\omega(b_{ji}(.)) > 2n \text{ on } D(u,r).$$

As was already discussed in the proof of [ST] Prop. 47 we have

$$\omega(f_i(u,q)) \leq n \text{ for } q \in \mathfrak{X}_n, \text{ and}$$
$$\omega\left(\sum_{j=i+1}^{d-1} b_{ji}(v)\Xi_{\beta_j}(q) + b_{di}(v)\right) > n \text{ for } (v,q) \in D(u,r) \times \mathfrak{X}_n.$$

Consequently

$$\frac{1}{f_i(v,q)} = \frac{1}{f_i(u,q)} \sum_{m\geq 0} (-1)^m \left(\frac{\sum_{j=i+1}^{d-1} b_{ji}(v)\Xi_{\beta_j}(q) + b_{di}(v)}{f_i(u,q)}\right)^m$$

is an expansion into a series uniformly convergent on $D(u,r) \times \mathfrak{X}_n$. We rewrite this as

$$\frac{1}{f_i(v,q)} = \sum_{m_{i+1i},\ldots,m_{di}\geq 0} c_{\underline{m}(i)} \frac{\Xi_{m_{i+1i}\beta_{i+1}+\cdots+m_{d-1i}\beta_{d-1}}(q)}{f_i(u,q)^{1+m_{i+1i}+\cdots+m_{di}}} \cdot \prod_{i<j\leq d} (v_{ji} - u_{ji})^{m_{ji}}$$

where $\underline{m}(i) := (m_{i+1i}, \ldots, m_{di})$ and the $c_{\underline{m}(i)}$ are certain nonzero integer coefficients. By multiplying together we obtain the expansion

$$(*) \qquad k(vw_{d+1},q) = \sum_{\underline{m}} \frac{c_{\underline{m}}\Xi_{\mu(\underline{m})}(q)}{f_0(u,q)^{s_0(\underline{m})}\cdots f_{d-1}(u,q)^{s_{d-1}(\underline{m})}} \cdot (v-u)^{\underline{m}}$$

which is uniformly convergent on $D(u,r) \times \mathfrak{X}_n$; here we have set

$$\mu(\underline{m}) := m_{10}\beta_1 + (m_{20} + m_{21})\beta_2 + \cdots + (m_{d-10} + \cdots + m_{d-1d-2})\beta_{d-1}$$

if $d > 1$, resp. $\mu(\underline{m}) := 0$ if $d = 1$, and

$$s_i(\underline{m}) := 1 + m_{i+1i} + \cdots + m_{di} \text{ for } 0 \leq i \leq d-1;$$

again the $c_{\underline{m}}$ are appropriate nonzero integer coefficients. This establishes the asserted local analyticity on the big cell. It follows immediately that $\mathbf{k}$ is continuous on the big cell (with respect to the original Fréchet topology on $\Omega^d(\mathfrak{X})$). It therefore remains to prove, for all $n \in \mathbb{N}$, the continuity of k viewed as a map from G into $\mathcal{O}(\mathfrak{X}_n)$ in all points outside the big cell. But this is the content of [ST] Lemma 45. □

Corollary 3.2. — *The function $\lambda \circ \mathbf{k} : G/P \to K$, for any continuous linear form λ on $\Omega^d(\mathfrak{X})$, is continuous, vanishes outside the big cell, and is locally analytic on the big cell.*

Proof. — The continuity and the vanishing are immediately clear. The local analyticity follows by using [B-VAR] 4.2.3 and by observing that, according to Prop. 1.4, λ comes from a continuous linear form on some $\Omega^d(\mathfrak{X}_n)$. □

Proposition 3.3. — *The image of $\mathbf{k}$ generates $\Omega^d(\mathfrak{X})$ as a topological K-vector space.*

Proof. — We first consider the map $k : G \to \mathcal{O}(\mathfrak{X})$. Let $\mathcal{K} \subseteq \mathcal{O}(\mathfrak{X})$ be the vector subspace generated by the image of k and let $\overline{\mathcal{K}}$ denote its closure. The formula $(*)$ in the proof of Proposition 1 for the matrix $u = 1$ says that, given the natural number $n \in \mathbb{N}$, we find an $r \in \mathbb{N}$ such that the expansion

$$k(vw_{d+1}, q) = \sum_{\underline{m}} \frac{c_{\underline{m}} \Xi_{\mu(\underline{m})}(q)}{\Xi_{\beta_0}(q)^{s_0(\underline{m})} \dots \Xi_{\beta_{d-1}}(q)^{s_{d-1}(\underline{m})}} \cdot (v-1)^{\underline{m}}$$

holds uniformly for $(v, q) \in D(1, r) \times \mathfrak{X}_n$. The coefficients of this expansion up to a constant are the value at $u = 1$ of iterated partial derivatives of the function $k(.w_{d+1}, .) : D(1, r) \to \mathcal{K}$ (momentarily viewed in $\mathcal{O}(\mathfrak{X}_n)$). Since increasing n just means decreasing r it follows that all the functions Ξ_μ with $\mu = \mu(\underline{m}) - s_0(\underline{m})\beta_0 - \dots - s_{d-1}(\underline{m})\beta_{d-1}$ lie in $\overline{\mathcal{K}}$. This includes, for those $\underline{m}$ for which only the m_{i+1i} may be nonzero, all the functions

$$\frac{\Xi_{\beta_1}^{m_0} \cdots \Xi_{\beta_{d-1}}^{m_{d-2}}}{\Xi_{\beta_0}^{1+m_0} \cdots \Xi_{\beta_{d-1}}^{1+m_{d-1}}} = \Xi_{\alpha_0}^{m_0} \cdots \Xi_{\alpha_{d-1}}^{m_{d-1}} \cdot \frac{1}{\Xi_{\beta_0} \cdots \Xi_{\beta_{d-1}}} \quad \text{with } m_0, \dots, m_{d-1} \geq 0.$$

Passing now to d-forms we therefore know that the closed K-vector subspace Ω of $\Omega^d(\mathfrak{X})$ generated by the image of $\mathbf{k}$ contains all forms $\Xi_\mu \xi$ where $\mu = m_0\alpha_0 + \dots + m_{d-1}\alpha_{d-1}$ with $m_0, \dots, m_{d-1} \geq 0$. As a consequence of [ST] Cor. 40 the subspace Ω is G-invariant. By applying Weyl group elements w and noting that $w_*\xi = \pm\xi$ we obtain $\Xi_\mu \xi$, for any $\mu \in X^*(\overline{T})$, in Ω. Using the G-invariance of Ω again we then have the subset

$$\{u_*(\Xi_\mu \xi) : \mu \in X^*(\overline{T}), u \in U\} = \{(u_*\Xi_\mu) d\Xi_{\beta_0} \wedge \dots \wedge d\Xi_{\beta_{d-1}} : \mu \in X^*(\overline{T}), u \in U\} \subseteq \Omega.$$

According to the partial fraction expansion argument in [GV] Thm. 21 the $u_*\Xi_\mu K$-linearly span all rational functions of $\Xi_{\beta_0}, \dots, \Xi_{\beta_{d-1}}$ whose denominator is a product of polynomials of degree 1. Moreover the proof of § 1 Prop. 4 in [SS] shows that those latter functions are dense in $\mathcal{O}(\mathfrak{X})$. It follows that $\Omega = \Omega^d(\mathfrak{X})$. $\square$

Put

$$C(G/P, K) := \text{ space of continuous } K\text{-valued functions on } G/P;$$

it is a Banach space with respect to the supremum norm on which G acts continuously by left translations. The subspace

$$C_{\text{inv}}(G/P, K) := \sum_s C(G/P_s, K) \subseteq C(G/P, K)$$

is closed; actually one has the topological direct sum decomposition

$$C(G/P, K) = C_{\text{inv}}(G/P, K) \oplus C_{\text{o}}(Pw_{d+1}P/P, K)$$

where the second summand on the right hand side is the space of K-valued continuous functions vanishing at infinity on the big cell ([BS] § 3). We recall that a continuous function on a locally compact space Y is said to vanish at infinity if its extension by zero to the one-point compactification of Y is continuous. We equip the quotient space $C(G/P,K)/C_{\mathrm{inv}}(G/P,K)$ with the quotient topology. By Proposition 1 the map

$$\begin{array}{rcl} I'_{\mathrm{o}} : \Omega^d(\mathcal{X})' & \longrightarrow & C(G/P,K) \\ \lambda & \longmapsto & [g \mapsto \lambda(\mathbf{k}(g))] \end{array}$$

is well defined; by [ST] Cor. 30 it is P-equivariant. Moreover it follows from [ST] Prop. 29.3 and the Bruhat decomposition that the induced map

$$I_{\mathrm{o}} : \Omega^d(\mathcal{X})' \longrightarrow C(G/P,K)/C_{\mathrm{inv}}(G/P,K)$$

is G-equivariant.

Lemma 3.4. — *The maps I'_{o} and I_{o} are continuous.*

Proof. — We only need to discuss the map I'_{o}. Because of Prop. 1.4 we have to check that, for each $n \in \mathbb{N}$, the map

$$\begin{array}{rcl} \mathcal{O}(\mathcal{X}_n)' & \longrightarrow & C(G/P,K) \\ \lambda & \longmapsto & [g \mapsto \lambda(k(g,.))] \end{array}$$

is continuous. The norm of λ is equal to

$$c_1 := \inf\{\omega(\lambda(F)) : F \in \mathcal{O}(\mathcal{X}_n), \inf_{q\in\mathcal{X}_n} \omega(F(q)) \geq 0\}.$$

On the other hand the norm of the image of λ under the above map is equal to

$$c_2 := \inf_{g\in G} \omega(\lambda(k(g,.))) = \inf_{u\in U} \omega(\lambda(k(uw_{d+1},.)))$$

where U denotes, as before, the unipotent radical of P. But we have

$$\inf_{\substack{u\in U\\ q\in\mathcal{X}_n}} \omega(k(uw_{d+1},q)) \geq -dn$$

(compare the proof of [ST] Prop. 47). It follows that $c_2 \geq c_1 - dn$. ☐

Lemma 3.5. — *The maps I'_{o} and I_{o} are injective.*

Proof. — For I'_{o} this is an immediate consequence of Prop. 3. According to Cor. 2 the image of I'_{o} is contained in $C_{\mathrm{o}}(Pw_{d+1}P/P,K)$ which is complementary to $C_{\mathrm{inv}}(G/P,K)$. Hence I_{o} is injective, too. ☐

In order to see the relation between I_{o} and the map I in the previous section we first recall part of the content of [BS] § 3:

Fact 1. — The “symmetrization”

$$(\Sigma\phi)(g) := \sum_{w\in W} (-1)^{\ell(w)} \phi(gww_{d+1})$$

induces a G-equivariant injective map

$$C(G/P,K)/C_{\mathrm{inv}}(G/P,K) \xhookrightarrow{\Sigma} C(G,K).$$

Here and in the following we let $C(Y,K)$, resp. $C_{\mathrm{o}}(Y,K)$, denote, for any locally compact space Y, the K-vector space of K-valued continuous functions, resp. of K-valued continuous functions vanishing at infinity, on Y; the second space is a Banach space with respect to the supremum norm.

We also let

$$\begin{array}{ccc} C(G,K) & \xrightarrow{\mathrm{res}} & C(U,K) \\ \phi & \longmapsto & \phi|U \end{array}$$

and

$$\begin{array}{ccc} C_{\mathrm{o}}(U,K) & \longrightarrow & C(G/P,K)/C_{\mathrm{inv}}(G/P,K) \\ \phi & \longmapsto & \phi^{\#}(g) := \begin{cases} \phi(u) & \text{if } g = uw_{d+1}p \in Uw_{d+1}P, \\ 0 & \text{otherwise .} \end{cases} \end{array}$$

Fact 2. — $^{\#}$ is an isomorphism whose inverse is $\mathrm{res} \circ \Sigma$.

It follows in particular that $^{\#}$ is an isometry.

Consider now the diagram

$$\begin{tikzcd} C^{\mathrm{an}}(G,K) \arrow[r, "\subseteq"] & C(G,K) \\ \Omega^d(\mathfrak{X})' \arrow[u, "I"] \arrow[r, "I_{\mathrm{o}}"] & C(G/P,K)/C_{\mathrm{inv}}(G/P,K) \arrow[u, "\Sigma"'] \end{tikzcd}$$

in which all maps are G-equivariant and injective. We claim that the diagram is commutative; for that it suffices to prove the identity

$$(**) \qquad g_*\xi = \sum_{w\in W} (-1)^{\ell(w)} u_{gww_{d+1}*}\xi.$$

From Prop. 1 we know that each summand on the right hand side is a continuous function in $g \in G$ (where $u_{gw*}\xi := 0$ if gw is not in the big cell). Hence it suffices to check the identity for g in the dense open subset $\bigcap_{w\in W} Pw_{d+1}Pw$. On the other hand it is an identity between logarithmic d-forms which can be checked after having applied the G-equivariant map "dis" into distributions on G/P; according to [ST] Remark on top of p. 423 the left hand side becomes

$$\sum_{w\in W} (-1)^{\ell(w)} \delta_{gw} = \sum_{w\in W} (-1)^{\ell(w)} \delta_{u_{gw}w_{d+1}}$$

whereas the right hand side becomes

$$\sum_{w\in W} (-1)^{\ell(w)} \cdot \sum_{v\in W} (-1)^{\ell(v)} \delta_{u_{gww_{d+1}}v} = \sum_{w\in W} (-1)^{\ell(w)} \cdot \sum_{v\in W} (-1)^{\ell(v)} \delta_{u_{gw}w_{d+1}v}.$$

The image of "dis" actually consists of linear forms on the Steinberg representation (see [ST]) and so any identity in that image can be checked by evaluation on locally constant and compactly supported functions on the big cell. But for those, all terms on the right hand side with $v \neq 1$ obviously vanish.

We view the above diagram as saying that any locally analytic function in the image of I is the symmetrization of a continuous "boundary value function" on G/P. In order to make this more precise we first have to discuss the concept of an "analytic vector". Let V be a K-Banach space on which G acts continuously (by which we always mean that the map $G \times V \to V$ describing the action is continuous). As in the case $V = K$ we have the Hausdorff locally convex vector space $C^{\mathrm{an}}(G, V)$ of all V-valued locally K-analytic functions on G (apart from replacing K by V everywhere the definition is literally the same). It is barrelled, so that the same argument as in the proof of Lemma 2.2 shows that the left translation action of G on $C^{\mathrm{an}}(G, V)$ is continuous.

Definition. — *A vector $v \in V$ is called analytic if the V-valued function $g \mapsto gv$ on G is locally analytic.*

We denote by V_{an} the vector subspace of all analytic vectors in V. It is clearly G-invariant. Moreover the G-equivariant linear map

$$\begin{array}{ccc} V_{\mathrm{an}} & \longrightarrow & C^{\mathrm{an}}(G, V) \\ v & \longmapsto & [g \mapsto g^{-1}v] \end{array}$$

is injective. We always equip V_{an} with the subspace topology with respect to this embedding. (Warning: That topology in general is finer than the topology which the Banach norm of V would induce on V_{an}. Evaluating a function at $1 \in G$ defines a continuous map $C^{\mathrm{an}}(G, V) \to V$.) Of course the G-action on V_{an} is continuous. By functoriality any G-equivariant continuous linear map $L : V \to \widetilde{V}$ between Banach spaces with continuous G-action induces a G-equivariant continuous linear map $L_{\mathrm{an}} : V_{\mathrm{an}} \to \widetilde{V}_{\mathrm{an}}$. A useful technical observation is that the locally convex vector space V_{an} does not change if we pass to an open subgroup $H \subseteq G$. First of all it follows from the continuity of the G-action on V that the function $g \mapsto g^{-1}v$ is locally analytic on G if and only if its restriction to H is locally analytic. Fixing a set of representatives R for the cosets in $H \setminus G$ we have the isomorphism of locally convex vector spaces

$$C^{\mathrm{an}}(G, V) = \prod_{g \in R} C^{\mathrm{an}}(Hg, V)$$

([Fea] 2.2.4). Hence the embedding $V_{\mathrm{an}} \hookrightarrow C^{\mathrm{an}}(G, V)$ coincides with the composite of the embedding $V_{\mathrm{an}} \hookrightarrow C^{\mathrm{an}}(H, V)$ and the "diagonal embedding"

$$\begin{array}{ccc} C^{\mathrm{an}}(H, V) & \longrightarrow & \displaystyle\prod_{g \in R} C^{\mathrm{an}}(Hg, V) \\ f & \longmapsto & (g^{-1}(f(.g^{-1})))_{g \in R}. \end{array}$$

Remark 3.6. — V_{an} is closed in $C^{\mathrm{an}}(G, V)$.

Proof. — Let $(v_i)_{i \in I}$ be a Cauchy net in V_{an} which in $C^{\mathrm{an}}(G, V)$ converges to the function f. By evaluating at $h \in G$ we see that the net $(h^{-1}v_i)_{i \in I}$ converges to $f(h)$ in V. Put $v := f(1)$. Since h is a continuous endomorphism of V it follows on the other hand that $(h^{-1}v_i)_{i\in I}$ converges to $h^{-1}v$. Hence $f(h) = h^{-1}v$ which means that f comes from $v \in V_{\mathrm{an}}$. $\square$

Lemma 3.7. — *If each vector in V is analytic then $V_{\mathrm{an}} = V$ as topological vector spaces.*

Proof. — We have to show that the map

$$\begin{array}{ccc} V & \longrightarrow & C^{\mathrm{an}}(G, V) \\ v & \longmapsto & [g \mapsto g^{-1}v] \end{array}$$

is continuous. According to our earlier discussion we are allowed to replace G by whatever open subgroup H is convenient. By [Fea] 3.1.9 our assumption implies that the G-action on V defines a homomorphism of Lie groups $\rho : G \to \mathbf{GL}(V)$ (with the operator norm topology on the right hand side). On a sufficiently small compact open subgroup $H \subseteq G$ this homomorphism is given by a power series

$$\rho(g) = \sum_{\underline{n}} A_{\underline{n}} \cdot \underline{x}(g)^{\underline{n}} \text{ for } g \in H$$

which is convergent in the operator norm topology on $\mathrm{End}_K(V)$; here $\underline{x}$ is a vector of coordinate functions from H onto some polydisk of radius 1, the $\underline{n}$ are corresponding multi-indices, and the $A_{\underline{n}}$ lie in $\mathrm{End}_K(V)$. In particular the operator norm of the $A_{\underline{n}}$ is bounded above by some constant $c > 0$. If we insert a fixed vector $v \in V$ into this power series then we obtain the expansion

$$gv = \sum_{\underline{n}} A_{\underline{n}}(v) \cdot \underline{x}(g)^{\underline{n}}$$

as a function of $g \in H$ and the spectral norm of the right hand side is bounded above by $c \cdot \|v\|$. $\square$

Proposition 3.8. — *The map I_{o} induces a G-equivariant injective continuous linear map*

$$\Omega^d(\mathcal{X})' \longrightarrow [C(G/P, K)/C_{\mathrm{inv}}(G/P, K)]_{\mathrm{an}}.$$

Proof. — For the purposes of this proof we use the abbreviation $V := C(G/P, K)/C_{\mathrm{inv}}(G/P, K)$. We have to show that the image of I_{o} is contained in V_{an} and that the induced map into V_{an} is continuous. As before it suffices to discuss the corresponding map $\Omega^d(\mathcal{X}_n)' \to V_{\mathrm{an}}$ for a fixed but arbitrary $n \in \mathbb{N}$. Both spaces, V as well as $\Omega^d(\mathcal{X}_n)'$, are Banach spaces with an action of the group $GL_{d+1}(o)$; the map between them induced by I_{o} is equivariant and continuous by Lemma 4. Since $GL_{d+1}(o)$ is open in G it can be used, by the above observation, instead of G to compute the locally convex vector space V_{an}. If we show that $GL_{d+1}(o)$ acts continuously on $\Omega^d(\mathcal{X}_n)'$ then I_{o} certainly induces a continuous map $[\Omega^d(\mathcal{X}_n)']_{\mathrm{an}} \to V_{\mathrm{an}}$. What we therefore have to show in addition is that the identity

$$[\Omega^d(\mathcal{X}_n)']_{\mathrm{an}} = \Omega^d(\mathcal{X}_n)'$$

holds as topological vector spaces.

We know already from the proof of Prop. 2.1' that every vector in $\Omega^d(\mathcal{X}_n)$ is analytic. By [Fea] 3.1.9 this means that the $GL_{d+1}(o)$-action on $\Omega^d(\mathcal{X}_n)$ is given by a homomorphism of Lie groups

$$GL_{d+1}(o) \longrightarrow \mathbf{GL}(\Omega^d(\mathcal{X}_n))$$

(recall that the right hand side carries the operator norm topology). Since passing to the adjoint linear map is a continuous linear map between Banach spaces it follows that also the $GL_{d+1}(o)$-action on $\Omega^d(\mathcal{X}_n)'$ is given by a corresponding homomorphism of Lie groups. This means in particular that the latter action is continuous and that every vector in $\Omega^d(\mathcal{X}_n)'$ is analytic. We therefore may apply the previous lemma. $\square$

Corollary 3.9. — *The G-action $G \times \Omega^d(\mathcal{X})' \to \Omega^d(\mathcal{X})'$ is continuous and, for any $\lambda \in \Omega^d(\mathcal{X})'$, the map $g \mapsto g\lambda$ on G is locally analytic.*

Proof. — Since $\Omega^d(\mathcal{X})'$ is barrelled as a locally convex inductive limit of Banach spaces the first assertion follows from the second by the same argument which we have used already twice. In the proof of the previous proposition we have seen that each function $g \mapsto g\lambda$ is locally analytic on $GL_{d+1}(o)$. But this is sufficient for the full assertion. $\square$

Since G/P is compact we may view the symmetrization map as a map

$$C(G/P, K)/C_{\mathrm{inv}}(G/P, K) \xrightarrow{\Sigma} BC(G, K)$$

into the Banach space $BC(G, K)$ of bounded continuous functions on G. It is then an isometry as can be seen as follows. By its very definition Σ is norm decreasing. On the other hand ϕ can be reconstructed from $\Sigma\phi$ by restriction to U followed by $^{\#}$ which again is norm decreasing. Hence Σ must be norm preserving.

We obtain the induced continuous injective map

$$[C(G/P, K)/C_{\mathrm{inv}}(G/P, K)]_{\mathrm{an}} \overset{\Sigma_{\mathrm{an}}}{\hookrightarrow} BC(G, K)_{\mathrm{an}}.$$

Since any $f \in BC(G, K)_{\mathrm{an}}$ is obtained from the locally analytic map $g \mapsto g^{-1}f$ by composition with the evaluation map at $1 \in G$ and hence is locally analytic we see that $BC(G, K)_{\mathrm{an}}$ in fact is contained in $C^{\mathrm{an}}(G, K)$. We therefore can rewrite the commutative diagram which relates I_{o} and I in the form

$$\begin{tikzcd} \Omega^d(\mathcal{X})' \arrow[r, "I_{\mathrm{o}}"] \arrow[dr, "I"'] & {[C(G/P, K)/C_{\mathrm{inv}}(G/P, K)]_{\mathrm{an}}} \arrow[d, "\Sigma_{\mathrm{an}}"] \\ & C^{\mathrm{an}}(G, K). \end{tikzcd}$$

So far we have explained how to understand on G/P the fact that the functions in the image of I are locally analytic. But the latter also satisfy the differential equations from the ideal $\mathfrak{a}$ in $\mathcal{D}(G)$. How can those be viewed on G/P?

For any Banach space V the right translation action by G on $C^{\mathrm{an}}(G,V)$ induces a corresponding action of the algebra $\mathcal{D}(G)$ by continuous endomorphisms. Any $\mathfrak{x} \in \mathfrak{g}$ acts via the usual formula

$$(\mathfrak{x}f)(g) = \frac{d}{dt} f(g \exp(t\mathfrak{x}))\big|_{t=0}.$$

(Compare [Fea] 3.3.4.) This clearly is functorial in V. If we now look at the case $BC(G,K)$ we have two embeddings

$$\begin{tikzcd} & BC(G,K)_{\mathrm{an}} \arrow[dl] \arrow[dr] & \\ C^{\mathrm{an}}(G, BC(G,K)) \arrow[rr, "\varepsilon"] & & C^{\mathrm{an}}(G,K) \end{tikzcd}$$

which are connected through the map ε which comes by functoriality from the map $\mathrm{ev}_1 : BC(G,K) \to K$ evaluating a function at $1 \in G$. This latter map is $\mathcal{D}(G)$-equivariant. Hence, for any left ideal $\mathfrak{d} \subseteq \mathcal{D}(G)$, we obtain the identity

$$\begin{aligned} &BC(G,K)_{\mathrm{an}} \cap C^{\mathrm{an}}(G,K)^{\mathfrak{d}=0} \\ &\qquad = BC(G,K)_{\mathrm{an}} \cap \{f \in C^{\mathrm{an}}(G, BC(G,K)) : \mathfrak{d}f \subseteq \ker(\varepsilon)\}. \end{aligned}$$

Using the abbreviation $V := C(G/P,K)/C_{\mathrm{inv}}(G/P,K)$ we know from Prop. 8 that $\mathrm{im}(I_{\mathrm{o}}) \subseteq V_{\mathrm{an}}$; on the other hand $\mathrm{im}(I) \subseteq C^{\mathrm{an}}(G,K)^{\mathfrak{a}=0}$. Those images correspond to each other under the map Σ_{an}. It follows that $\mathrm{im}(I_{\mathrm{o}})$ is contained in the subspace

$$V_{\mathrm{an}}^{\sigma\mathfrak{a}=0} := V_{\mathrm{an}} \cap \{f \in C^{\mathrm{an}}(G,V) : \mathfrak{a}f \subseteq \ker(\sigma)\}$$

where $\sigma : C^{\mathrm{an}}(G,V) \to C^{\mathrm{an}}(G,K)$ is the map induced by $\mathrm{ev}_1 \circ \Sigma : V \to K$ which sends $\phi \in C(G/P,K)$ to $\sum_{w \in W} (-1)^{\ell(w)} \phi(ww_{d+1})$. We arrive at the following conclusion.

Theorem 3.10. — *We have the commutative diagram of injective continuous linear maps*

$$\begin{tikzcd} \Omega^d(\mathfrak{X})' \arrow[r, "I_{\mathrm{o}}"] \arrow[dr, "I"'] & {[C(G/P,K)/C_{\mathrm{inv}}(G/P,K)]_{\mathrm{an}}^{\sigma\mathfrak{a}=0}} \arrow[d, "\Sigma_{\mathrm{an}}"] \\ & C^{\mathrm{an}}(G,K)^{\mathfrak{a}=0} \end{tikzcd} .$$

We think of I_{o} in this form as being "the" *boundary value map.* We point out that the ideal $\mathfrak{a}$ contains the following Dirac distributions. For any $g \in G$ put $W(g) := \{w \in W : gww_{d+1} \in Pw_{d+1}P\}$ and consider the element

$$\delta(g) := \delta_g - \sum_{w \in W(g)} (-1)^{\ell(w)} \delta_{u_{gww_{d+1}}}$$

in the algebra $\mathcal{D}(G)$. By interpreting δ_h as the right translation action by h those elements act on $C(G,K)$ and $BC(G,K)$. We claim that any function ϕ in the image of Σ satisfies

$$\phi(g) = \sum_{w\in W(g)} (-1)^{\ell(w)}\phi(u_{gww_{d+1}}) \text{ for any } g \in G.$$

We may write $\phi = \Sigma\psi^{\#}$ and then compute

$$\begin{aligned}\sum_{w\in W(g)} (-1)^{\ell(w)}(\Sigma\psi^{\#})(u_{gww_{d+1}}) &= \sum_{w\in W(g)} (-1)^{\ell(w)} \sum_{v\in W} (-1)^{\ell(v)}\psi^{\#}(u_{gww_{d+1}}vw_{d+1}) \\ &= \sum_{w\in W(g)} (-1)^{\ell(w)}\psi(u_{gww_{d+1}}) = \sum_{w\in W} (-1)^{\ell(w)}\psi^{\#}(gww_{d+1}) = (\Sigma\psi^{\#})(g).\end{aligned}$$

Since Σ is G-equivariant the same identities hold for the functions $\phi(h.)$ for any $h \in G$. In other words any function ϕ in the image of Σ actually satisfies

$$\delta(g)\phi = 0 \text{ for any } g \in G.$$

The relation $(**)$ established after Fact 2 implies that $\mathfrak{a}$ contains the left ideal generated by the $\delta(g)$ for $g \in G$.

4. The ideal $\mathfrak{b}$

As we have learned, the integral transform I carries continuous linear forms on $\Omega^d(\mathfrak{X})$ to locally analytic functions on G. Functions in the image of this map are annihilated by an ideal $\mathfrak{a}$ in the algebra of punctual distributions $\mathcal{D}(G)$. This annihilation condition means that functions in the image of I satisfy a mixture of discrete relations and differential equations.

In this section, we focus our attention on the differential equations satisfied by functions in the image of I. By this, we mean that we will study in detail the structure of the ideal $\mathfrak{b} := \mathfrak{a}\cap U(\mathfrak{g})$. By definition, $\mathfrak{b}$ is the annihilator ideal in $U(\mathfrak{g})$ of the special differential form ξ. We will describe a set of generators for $\mathfrak{b}$ and use this to prove the fundamental result that the weight spaces in $U(\mathfrak{g})/\mathfrak{b}$ (under the adjoint action of the torus $\overline{T}$) are one-dimensional. We will then analyze the left $U(\mathfrak{g})$-module $U(\mathfrak{g})/\mathfrak{b}$, identifying a filtration of this module by submodules and exhibiting the subquotients of this filtration as certain explicit irreducible highest weight $U(\mathfrak{g})$-modules. At the end of the section we prove some additional technical structural results which we will need later.

The results in this section are fundamental preparation for the rest of the paper.

We begin by recalling the decomposition

$$U(\mathfrak{g}) = \oplus_{\mu\in X^*(\overline{T})}U(\mathfrak{g})_\mu$$

of $U(\mathfrak{g})$ into the weight spaces $U(\mathfrak{g})_\mu$ with respect to the adjoint action of the torus $\overline{T}$. For a root $\alpha = \varepsilon_i - \varepsilon_j$ the weight space $\mathfrak{g}_\alpha$ is the 1-dimensional space generated by the element $L_\alpha \in \mathfrak{g}$ which corresponds to the matrix with a 1 in position (i,j) and zeros elsewhere; sometimes we also write $L_{ij} := L_\alpha$. Clearly a monomial $L_{\alpha_1}^{m_1}\cdots L_{\alpha_r}^{m_r} \in$

$U(\mathfrak{g})$ has weight $m_1\alpha_1 + \cdots + m_r\alpha_r$. The Poincaré-Birkhoff-Witt theorem says that once we have fixed a total ordering of the roots α any element in $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{g}_{\mathrm{o}}$ can be written in a unique way as a polynomial in the L_α. We will also need the filtration $U_n(\mathfrak{g})$ of $U(\mathfrak{g})$ by degree; we write $\deg(\mathfrak{z}) := n$ if $\mathfrak{z} \in U_n(\mathfrak{g}) \backslash U_{n-1}(\mathfrak{g})$.

The form ξ is invariant under $\overline{T}$. This implies that the ideal $\mathfrak{b}$ is homogeneous and contains $U(\mathfrak{g})\mathfrak{g}_{\mathrm{o}}$. An elementary calculation shows that L_α acts on $\Omega^d(\mathfrak{X})$ by

$$L_\alpha(F\xi) = \Big(\Xi_i \frac{\partial F}{\partial \Xi_j}\Big)\xi \, - \, \Xi_\alpha F\xi.$$

In particular we obtain $L_\alpha \xi = -\Xi_\alpha \xi$. By iteration that formula implies that the ideal $\mathfrak{b}$ contains the following relations:

cancellation : $L_{ij}L_{jl}$ for any indices $i \neq j \neq l$,
sorting : $L_{ij}L_{k\ell} - L_{i\ell}L_{kj}$ for any distinct indices (i,j,k,l).

Our goal is to show that the weight spaces of $U(\mathfrak{g})/\mathfrak{b}$ are 1-dimensional. For that we need to introduce one more notation. For a weight μ we put

$$d(\mu) := \sum_{m_i > 0} m_i$$

where the m_i are the coefficients of μ in the linear combination

$$\mu = \sum_{i=0}^{d} m_i \varepsilon_i.$$

Lemma 4.1. — *Let $\mathfrak{z} \in U(\mathfrak{g})$ be a monomial in the L_α of weight μ; we then have:*

i. $\deg(\mathfrak{z}) \geq d(\mu)$*;*
ii. *write* $\mathfrak{z} = \prod_{i,j} L_{ij}^{n_{ij}}$ *and put*

$$A(\mathfrak{z}) := \{i : n_{ij} > 0 \text{ for some } j\} \quad \text{and} \quad B(\mathfrak{z}) := \{j : n_{ij} > 0 \text{ for some } i\};$$

then $\deg(\mathfrak{z}) = d(\mu)$ *if and only if* $A(\mathfrak{z})$ *and* $B(\mathfrak{z})$ *are disjoint.*

Proof. — (Recall that we have fixed a total ordering of the roots α.) Since $\mathfrak{z}$ has weight μ we must have

$$\mu = \sum_{i,j} n_{ij}(\varepsilon_i - \varepsilon_j).$$

If on the other hand we write $\mu = \sum_k m_k \varepsilon_k$ we see that

$$(*) \qquad m_k = \sum_j n_{kj} - \sum_i n_{ik}.$$

As a result of this expression it follows that $m_k \leq \sum_j n_{kj}$, so that

$$d(\mu) = \sum_{m_k > 0} m_k \leq \sum_k \sum_j n_{kj} = \deg(\mathfrak{z}).$$

We suppose now that $A(\mathfrak{z})$ and $B(\mathfrak{z})$ are disjoint. Then $(*)$ implies that m_k is positive if and only if $k \in A(\mathfrak{z})$ and for positive m_k we must have

$$m_k = \sum_j n_{kj}.$$

Therefore

$$d(\mu) = \sum_{k \in A(\mathfrak{z})} m_k = \sum_{k \in A(\mathfrak{z})} \sum_j n_{kj} = \sum_{k,j} n_{kj} = \deg(\mathfrak{z}).$$

Conversely, we suppose that $k \in A(\mathfrak{z}) \cap B(\mathfrak{z})$. Then $m_k < \sum_j n_{kj}$. Therefore, if $m_k \geq 0$, we obtain

$$d(\mu) = \sum_{m_k > 0} m_k < \sum_k \sum_j n_{kj} = \deg(\mathfrak{z}).$$

If $m_k \leq 0$ a similar argument, using the fact that $d(\mu)$ may be computed from the m_k with $m_k < 0$, gives the desired result. □

Lemma 4.2. — *Let $\mathfrak{z} \in U(\mathfrak{g})$ be a nonzero polynomial in the L_α of weight μ; then the coset $\mathfrak{z} + \mathfrak{b}$ contains a representative of weight μ which is a linear combination of monomials in the L_α of degree $d(\mu)$.*

Proof. — Among all elements of $\mathfrak{z} + \mathfrak{b}_\mu$ which are polynomials in the L_α let $\mathfrak{x}$ be one of minimal degree. By the preceeding lemma the degree of $\mathfrak{x}$ is greater than or equal to $d(\mu)$. Let $\mathfrak{y}$ be a monomial in the L_α of degree $\deg(\mathfrak{x})$ which occurs with a nonzero coefficient in $\mathfrak{x}$. Assume that the sets $A(\mathfrak{y})$ and $B(\mathfrak{y})$ as defined in the preceeding lemma are not disjoint. Then there exist three indices $i \neq j \neq l$ such that L_{ij} and L_{jl} each occur to nonzero powers in the monomial $\mathfrak{y}$. By the commutation rules in $U(\mathfrak{g})$ we have

$$\mathfrak{y} \in U(\mathfrak{g}) L_{ij} L_{jl} + U_{n-1}(\mathfrak{g}) \text{ with } n := \deg(\mathfrak{x}).$$

Hence the cancellation relations imply that $\mathfrak{y} \in \mathfrak{b} + U_{n-1}(\mathfrak{g})$. This means that modulo $\mathfrak{b}$ we may remove an appropriate scalar multiple of $\mathfrak{y}$ from $\mathfrak{x}$ and pick up only a polynomial of lower degree. But by our minimality assumption on $\deg(\mathfrak{x})$ there has to be at least one such $\mathfrak{y}$ such that $A(\mathfrak{y})$ and $B(\mathfrak{y})$ are disjoint. The previous lemma then implies that $d(\mu) = \deg(\mathfrak{y}) = \deg(\mathfrak{x})$. If we express $\mathfrak{x}$ as a linear combination of monomials in the L_α then each such monomial has weight μ and hence, by the previous lemma again, degree $\geq d(\mu)$. □

A monomial

$$L_{i_0 j_0} L_{i_1 j_1} \cdots L_{i_m j_m}$$

will be called *sorted* if $i_0 \leq \cdots \leq i_m$ and $j_0 \leq \cdots \leq j_m$ and if those two sequences do not overlap (i.e. no i_k is a j_l). For example, the monomial $L_{10} L_{10} L_{32} L_{32}$ is sorted with sequences 1,1,3,3, and 0,0,2,2 whereas the monomial $L_{32} L_{31}$ with sequences 3,3 and 2,1 is not sorted.

Lemma 4.3. — *Among all the monomials in the L_α of weight μ there is exactly one, denoted by $L_{(\mu)}$, which is sorted.*

Proof. — The non-overlapping condition means that any sorted monomial must have degree $d(\mu)$. Write $\mu = \sum m_k \varepsilon_k$. Those k for which m_k is positive must occur as the first index in some L_{ij}, and therefore (by the non-overlapping condition) can only occur as first indices. Similarly, those k for which m_k is negative can only occur as second indices. This determines the lists of first and second indices – for example, the list of first indices consists of precisely those k for which $m_k > 0$, each repeated m_k times, listed in ascending order. Once these two lists are determined the corresponding monomial is determined. $\square$

Proposition 4.4. — *Let $\mathfrak{z} \in U(\mathfrak{g})$ be a polynomial in the L_α of weight μ; we then have $\mathfrak{z} + \mathfrak{b} = aL_{(\mu)} + \mathfrak{b}$ for some $a \in K$.*

Proof. — By Lemma 2 we may assume that $\mathfrak{z}$ is a monomial of degree $d(\mu)$. The sets $A(\mathfrak{z})$ and $B(\mathfrak{z})$, as defined in Lemma 1, then are disjoint, and therefore the individual L_{ij} which occur in $\mathfrak{z}$ commute with one another. Consequently we may rearrange these L_{ij} freely. Using this fact it is easy to see that we may use the sorting relations to transform $\mathfrak{z}$ into $L_{(\mu)}$. $\square$

Corollary 4.5. — *The weight space $(U(\mathfrak{g})/\mathfrak{b})_\mu$ in the left $U(\mathfrak{g})$-module $U(\mathfrak{g})/\mathfrak{b}$, for any $\mu \in X^*(\overline{T})$, has dimension one.*

Proof. — The preceeding proposition says that the weight space in question is generated by the coset $L_{(\mu)} + \mathfrak{b}$. On the other hand an explicit computation shows that $L_{(\mu)}\xi = e2^c\Xi_\mu\xi$ with some integer $c \geq 0$ and some sign $e = \pm 1$ (both depending on μ); hence $L_{(\mu)} \notin \mathfrak{b}$. $\square$

Later on it will be more convenient to use a renormalized $L_{(\mu)}$. We let L_μ denote the unique scalar multiple of $L_{(\mu)}$ which has the property that $L_\mu\xi = -\Xi_\mu\xi$.

Although we now have a completely explicit description of $U(\mathfrak{g})/\mathfrak{b}$ its structure as a $\mathfrak{g}$-module is not yet clear. For a root $\alpha = \varepsilon_i - \varepsilon_j$ and a weight $\mu = \sum_k m_k\varepsilon_k \in X^*(\overline{T})$ our earlier formula implies

$$L_\alpha(\Xi_\mu\xi) = (m_j - 1)\Xi_{\mu+\alpha}\xi$$

and hence

$$(+) \qquad L_\alpha L_\mu \equiv (m_j - 1)L_{\mu+\alpha} \bmod \mathfrak{b}.$$

If we put $J(\mu) := \{0 \leq k \leq d : m_k > 0\}$ then $J(\mu) \subseteq J(\mu + \alpha)$ provided $m_j \neq 1$. It follows that

$$\mathfrak{b}_J := \mathfrak{b} + \sum_{J \subseteq J(\mu)} KL_\mu$$

is, for any subset $J \subseteq \{0, \dots, d\}$, a left ideal in $U(\mathfrak{g})$. We have:

– $\mathfrak{b}_{\{0,\dots,d\}} = \mathfrak{b}$ and $\mathfrak{b}_\varnothing = U(\mathfrak{g})$;
– $\mathfrak{b}_J \subseteq \mathfrak{b}_{J'}$ if and only if $J' \subseteq J$.

For $J \neq \{0,\dots,d\}$ we set

$$\mathfrak{b}_J^> := \sum_{J \subsetneq J'} \mathfrak{b}_{J'}.$$

Moreover we introduce the descending filtration by left ideals

$$U(\mathfrak{g}) = \mathfrak{b}_0 \supseteq \mathfrak{b}_1 \supseteq \cdots \supseteq \mathfrak{b}_{d+1} = \mathfrak{b}$$

defined by

$$\mathfrak{b}_j := \sum_{\# J \geq j} \mathfrak{b}_J.$$

The subquotients of that filtration decompose as $\mathfrak{g}$-modules into

$$\mathfrak{b}_j/\mathfrak{b}_{j+1} = \bigoplus_{\# J = j} (\mathfrak{b}_J + \mathfrak{b}_{j+1})/\mathfrak{b}_{j+1} = \bigoplus_{\# J = j} \mathfrak{b}_J/\mathfrak{b}_J^>.$$

Our aim in the following therefore is to understand the $\mathfrak{g}$-modules $\mathfrak{b}_J/\mathfrak{b}_J^>$. A trivial case is

$$\mathfrak{b}_0/\mathfrak{b}_1 = \mathfrak{b}_\varnothing/\mathfrak{b}_\varnothing^> = K\,.$$

We therefore assume, for the rest of this section, that J is a nonempty proper subset of $\{0,\dots,d\}$. First of all we need the maximal parabolic subalgebra of $\mathfrak{g}$ given by

$$\mathfrak{p}_J := \begin{array}{c}\text{all matrices in } \mathfrak{g} \text{ with a zero entry}\\ \text{in position } (i,j) \text{ for } i \in J \text{ and } j \notin J.\end{array}$$

It follows from the above formula (+) that the subalgebra $\mathfrak{p}_J$ leaves invariant the finite dimensional subspace

$$M_J := \sum_{\mu \in B(J)} K L_\mu$$

of $\mathfrak{b}_J/\mathfrak{b}_J^>$ where

$$B(J) := \begin{array}{c}\text{set of all weights } \mu = \sum_k m_k \varepsilon_k \text{ such that}\\ J(\mu) = J \text{ and } m_k = 1 \text{ for } k \in J.\end{array}$$

Using again the formula (+) the subsequent facts are straightforward. The unipotent radical

$$\mathfrak{n}_J := \begin{array}{c}\text{all matrices with zero entries in}\\ \text{position } (i,j) \text{ with } i \in J \text{ or } j \notin J\end{array}$$

of $\mathfrak{p}_J$ acts trivially on M_J. We have the Levi decomposition $\mathfrak{p}_J = \mathfrak{l}_J + \mathfrak{n}_J$ with $\mathfrak{l}_J = \mathfrak{l}'(J) + \mathfrak{l}(J)$ where

$$\mathfrak{l}'(J) := \begin{array}{c}\text{all matrices with zero entries in}\\ \text{position } (i,j) \text{ with } i \text{ and } j \text{ not both in } J\end{array}$$

and

$$\mathfrak{l}(J) := \begin{array}{c}\text{all matrices with zero entries in}\\ \text{position } (i,j) \text{ with } i \text{ or } j \in J.\end{array}$$

The structure of M_J as a module for the quotient $\mathfrak{p}_J/\mathfrak{n}_J = \mathfrak{l}_J$ is as follows:

- The first factor $\mathfrak{l}'(J) \cong \mathfrak{gl}_{\#J}$ acts on M_J through the trace character;
- as a module for the second factor $\mathfrak{l}(J) \cong \mathfrak{gl}_{d+1-\#J}$ our M_J is isomorphic to the $\#J$-th symmetric power of the contragredient of the standard representation of $\mathfrak{gl}_{d+1-\#J}$ on the $(d+1-\#J)$-dimensional K-vector space.

In particular, M_J is an irreducible $\mathfrak{p}_J/\mathfrak{n}_J$-module. The map

$$\begin{array}{ccc} U(\mathfrak{g}) \underset{U(\mathfrak{p}_J)}{\otimes} M_J & \longrightarrow & \mathfrak{b}_J/\mathfrak{b}_J^{>} \\ (\mathfrak{z}, m) & \longmapsto & \mathfrak{z}m \end{array}$$

is surjective. In fact, $\mathfrak{b}_J/\mathfrak{b}_J^{>}$ is an irreducible highest weight $U(\mathfrak{g})$-module: If we put $\nu := (\sum_{k\in J} \varepsilon_k) - \#J \cdot \varepsilon_\ell$ for some fixed $\ell \notin J$, then one deduces from (+) that $U(\mathfrak{g}) \cdot L_\mu + \mathfrak{b}$, for any μ with $J(\mu) = J$, contains $L_\nu + \mathfrak{b}$. For the subset $J = \{0, \ldots, j-1\}$ the parabolic subalgebra $\mathfrak{p}_J$ is in standard form with respect to our choice of positive roots and the highest weight of $\mathfrak{b}_J/\mathfrak{b}_J^{>}$ is $\varepsilon_0 + \cdots + \varepsilon_{j-1} - j \cdot \varepsilon_j$.

We finish this section by establishing several facts to be used later on about the relation between the left ideals $\mathfrak{b}_J^{>}$ and the subalgebras $U(\mathfrak{n}_J^+)$ for

$$\mathfrak{n}_J^+ := \text{ transpose of } \mathfrak{n}_J.$$

First of all, note that $\mathfrak{n}_J^+$ and hence each $U(\mathfrak{n}_J^+)$ is commutative and $\mathrm{ad}(\mathfrak{l}_J)$-invariant.

Proposition 4.6

i. $U(\mathfrak{n}_J^+) \cap \mathfrak{b}$ *is the ideal in* $U(\mathfrak{n}_J^+)$ *generated by the sorting relations* $L_{ij}L_{k\ell} - L_{i\ell}L_{kj}$ *for* $i, k \in J$ *and* $j, l \notin J$,
ii. *the cosets of the sorted monomials* L_μ *for* $J(\mu) \subseteq J$ *and* $J(-\mu) \cap J = \varnothing$ *form a basis of* $U(\mathfrak{n}_J^+)/U(\mathfrak{n}_J^+) \cap \mathfrak{b}$ *as a* K*-vector space;*
iii. $U(\mathfrak{n}_J^+) \cap \mathfrak{b}_J^{>} = U(\mathfrak{n}_J^+) \cap \mathfrak{b}$*;*
iv. $(U(\mathfrak{n}_J^+) \cap \mathfrak{b}) \cdot \mathfrak{l}(J) \subseteq \mathfrak{b}$*;*
v. $U(\mathfrak{n}_J^+) \cap \mathfrak{b}$ *is* $\mathrm{ad}(\mathfrak{l}(J))$*-invariant.*

Proof. — i. Let $\mathfrak{s} \subseteq U(\mathfrak{n}_J^+)$ denote the ideal generated by those sorting relations. Using the commutativity of $U(\mathfrak{n}_J^+)$ it is easy to see that any monomial in the L_{ij} in $U(\mathfrak{n}_J^+)$ can be transformed into a sorted monomial by relations in $\mathfrak{s}$. In particular any coset in $U(\mathfrak{n}_J^+) \cap \mathfrak{b}/\mathfrak{s}$ has a representative which is a linear combination of sorted monomials. But we know that the sorted monomials are linearly independent modulo $\mathfrak{b}$. We therefore must have $U(\mathfrak{n}_J^+) \cap \mathfrak{b} = \mathfrak{s}$.

ii. The argument just given also shows that the cosets of all sorted monomials contained in $U(\mathfrak{n}_J^+)$ form a basis of the quotient in question. But they are exactly those which we have listed in the assertion.

iii. The sorted monomials listed in the assertion ii. are linearly independent modulo $\mathfrak{b}_J^{>}$.

iv. We have to check that $\mathfrak{z}L_{rs} \in \mathfrak{b}$ for any of the sorting relations $\mathfrak{z} = L_{ij}L_{k\ell} - L_{i\ell}L_{kj}$ from i. and for any L_{rs} such that $r, s \notin J$. If $r \neq j, \ell$ then $\mathfrak{z}L_{rs} = L_{rs}\mathfrak{z} \in \mathfrak{b}$. If $r = \ell$ then we may use the cancellation relations to obtain $\mathfrak{z}L_{rs} = L_{ij}L_{k\ell}L_{\ell s} - L_{kj}L_{i\ell}L_{\ell s} \in \mathfrak{b}$. Similarly if $r = j$ we have $\mathfrak{z}L_{rs} = L_{k\ell}L_{ij}L_{js} - L_{i\ell}L_{kj}L_{js} \in \mathfrak{b}$.

v. Because $\mathrm{ad}(\mathfrak{x})(\mathfrak{z}) = \mathfrak{x}\mathfrak{z} = -\mathfrak{z}\mathfrak{x}$ it follows from iv. that $\mathrm{ad}(\mathfrak{l}(J))(U(\mathfrak{n}_J^+) \cap \mathfrak{b}) \subseteq \mathfrak{b}$. But $U(\mathfrak{n}_J^+)$ is $\mathrm{ad}(\mathfrak{l}(J))$-invariant. Hence $U(\mathfrak{n}_J^+) \cap \mathfrak{b}$ is $\mathrm{ad}(\mathfrak{l}(J))$-invariant, too. □

We have $M_J \subseteq \mathfrak{b}_J/\mathfrak{b}_J^> \subseteq U(\mathfrak{n}_J^+) + \mathfrak{b}_J^>/\mathfrak{b}_J^>$. In fact $L_\mu \in U(\mathfrak{n}_J^+)$ for $\mu \in B(J)$. Let $M_J^\circ \subseteq U(\mathfrak{n}_J^+)$ denote the preimage of M_J under the projection map $U(\mathfrak{n}_J^+) \longrightarrow U(\mathfrak{g})/\mathfrak{b}_J^>$.

Lemma 4.7

i. $M_J^\circ \cdot \mathfrak{l}(J) \subseteq \mathfrak{b}_J^>$*;*
ii. M_J° *and* $M_J^\circ \cap \mathfrak{b}_J^> = M_J^\circ \cap \mathfrak{b}$ *are* $\mathrm{ad}(\mathfrak{l}(J))$*-invariant;*
iii. $\mathrm{ad}(\mathfrak{x})(\mathfrak{z}) = \mathfrak{x}\mathfrak{z} \bmod \mathfrak{b}_J^>$ *for* $\mathfrak{x} \in \mathfrak{l}(J)$ *and* $\mathfrak{z} \in M_J^\circ$.

Proof. — i. Because of Prop. 6 iii. and iv. it suffices to show that $L_\mu L_{k\ell} \in \mathfrak{b}_J^>$ whenever $\mu \in B(J)$ and $k, \ell \notin J$. If $k \notin J(-\mu)$ then $L_\mu L_{k\ell}$ after sorting coincides up to a constant with some L_ν such that $J = J(\mu) \underset{\neq}{\subset} J(\nu)$; hence $L_\mu L_{k\ell} \in \mathfrak{b}_J^>$ in this case. If $k \in J(-\mu)$ then L_μ has a factor L_{ik} and since the factors of the monomial L_μ commute with one another we may use a cancellation relation to conclude that $L_\mu L_{k\ell} \in \mathfrak{b}$.

ii. Using Prop. 6 iii. and v. we are reduced to showing that $\mathrm{ad}(L_{k\ell})(L_\mu) \in M_J^\circ$ whenever $\mu \in B(J)$ and $k, \ell \notin J$. The monomial L_μ is of the form $L_\mu = c \cdot \prod_{i \in J} L_{is_i}$ with $s_i \notin J$ and some nonzero integer c. We have

$$[L_{k\ell}, L_{is_i}] = \begin{cases} -L_{i\ell} & \text{if } k = s_i, \\ 0 & \text{if } k \neq s_i. \end{cases}$$

Since $\mathrm{ad}(L_{k\ell})$ is a derivation it follows that

$$\mathrm{ad}(L_{k\ell})(L_\mu) = -c \cdot \sum_{\substack{i \in J \\ s_i = k}} L_{i\ell} \prod_{\substack{j \in J \\ j \neq i}} L_{js_j}$$

which clearly lies in M_J°.

iii. This is an immediate consequence of the first assertion. □

The last lemma shows that the structure of M_J as an $\mathfrak{l}(J)$-module is induced by the adjoint action of $\mathfrak{l}(J)$ on M_J°. Whenever convenient we will use all the notations introduced above also for the empty set $J = \varnothing$; all the above assertions become trivially true in this case.

5. Local duality

In this section, we study linear forms on the Banach space $\Omega_b^d(U^0)$ of bounded differential forms on the admissible open set $U^0 = r^{-1}(\overline{C}^0)$ which is the inverse image of the open standard chamber in $\overline{X}$ under the reduction map. The restriction map gives a continuous injection from $\Omega^d(\mathfrak{X})$ into this Banach space, and therefore linear forms on $\Omega_b^d(U^0)$ are also elements of $\Omega^d(\mathfrak{X})'$.

Our first principal result of this section identifies $\Omega_b^d(U^0)$ with the dual of the space $\mathcal{O}(B)^{\mathfrak{b}=0}$ of (globally) analytic functions on B which are annihilated by the ideal $\mathfrak{b}$ studied in the preceeding section.The filtration which we introduced on $U(\mathfrak{g})/\mathfrak{b}$ then yields filtrations of $\mathcal{O}(B)^{\mathfrak{b}=0}$ and $\Omega_b^d(U^0)$. Applying our analysis of the subquotients of the filtration on $U(\mathfrak{g})/\mathfrak{b}$ from the preceeding section, we describe each subquotient of the filtration on $\mathcal{O}(B)^{\mathfrak{b}=0}$ as a space of analytic vector-valued functions on the unipotent radical of a specific maximal parabolic subgroup in G satisfying certain explicit differential equations.

Of fundamental importance to this analysis are the linear forms arising from the residue map on the standard chamber.

The space $\Omega_b^d(U^0)$ of bounded d-forms η on U^0 are those which have an expansion

$$\eta = \sum_{\nu \in X^*(\overline{T})} a(\nu)\Xi_\nu d\Xi_{\alpha_{d-1}} \wedge \cdots \wedge d\Xi_{\alpha_0}$$

such that

$$\omega_C(\eta) := \inf_\nu \{\omega(a(\nu)) - \ell(\nu)\} > -\infty.$$

We may and will always view $\Omega_b^d(U^0)$ as a Banach space with respect to the norm ω_C (compare [ST] Remark after Lemma 17). According to Lemma 1.2 the restriction map induces a continuous injective map $\Omega^d(\mathfrak{X}) \to \Omega_b^d(U^0)$. In [ST] Def. 19 we defined the residue of $\eta \in \Omega_b^d(U^0)$ at the pointed chamber $(\overline{C}, 0)$ by

$$\mathrm{Res}_{(\overline{C},0)}\eta := a(\alpha_d).$$

It is then clear that, for any weight $\mu \in X^*(\overline{T})$,

$$\eta \longmapsto \mathrm{Res}_{(\overline{C},0)}\Xi_{-\mu}\eta = a(\alpha_d + \mu)$$

is a continuous linear form on $\Omega_b^d(U^0)$ and a fortiori on $\Omega^d(\mathfrak{X})$. Applying the map I we obtain the locally analytic function

$$f_\mu(g) := \mathrm{Res}_{(\overline{C},0)}(\Xi_{-\mu} \cdot g_*\xi)$$

on G. We collect the basic properties of these functions.

Property 1. — Under the adjoint action of $B \cap T$ the function f_μ has weight $-\mu$, i.e.,

$$f_\mu(t^{-1}gt) = \mu(t^{-1}) \cdot f_\mu(g) \text{ for } g \in G \text{ and } t \in B \cap T.$$

This is straightforward from [ST] Lemma 20.

Property 2. — The restriction $f_\mu|B$ of f_μ to the Iwahori subgroup $B \subseteq G$ is analytic on B.

First of all recall that B is a product of disks and annuli where the matrix entries g_{ij} of $g \in B$ can be used as coordinates (the diagonal entries correspond to the annuli). By construction as well as by the formula

$$d\Xi_{\alpha_{d-1}} \wedge \cdots \wedge d\Xi_{\alpha_0} = (-1)^{d(d+1)/2}\Xi_{-\beta-\alpha_d} d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$$

we have, for a fixed $g \in G$, the expression

$$(a) \qquad (g_*\xi)|U^0 = (-1)^{d(d+1)/2} \sum_{\mu \in X^*(\overline{T})} f_\mu(g)\Xi_\mu \xi$$

in $\Omega_b^d(U^0)$. This is, of course, not a convergent expansion with respect to the norm ω_C. But if we write $g_*\xi|U^0 = F(g)\xi|U^0$ then the series

$$F(g) = (-1)^{d(d+1)/2} \sum_{\mu \in X^*(\overline{T})} f_\mu(g)\Xi_\mu$$

is uniformly convergent on each affinoid subdomain of U^0.

On the other hand a direct calculation shows that

$$g_*\xi = \det(g) \left(\prod_{j=0}^{d} \frac{1}{f_j(g,.)} \right) d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$$

where

$$f_j(g,q) := \sum_{i=0}^{d-1} g_{ij}\Xi_{\beta_i} + g_{dj}.$$

Recall that U^0 is given by the inequalities

$$\omega(\Xi_0(q)) < \cdots < \omega(\Xi_d(q)) < 1 + \omega(\Xi_0(q)).$$

It follows that for $g \in B$ the term $g_{jj}\Xi_{\beta_j}$ in the sum $f_j(g,q)$ is strictly larger in valuation than the other terms (we temporarily put $\beta_d := 0$). We therefore have, for $g \in B$ and $q \in U^0$, the geometric series expansion

$$\frac{1}{f_j(g,q)} = \frac{1}{g_{jj}\Xi_{\beta_j}} \sum_{m \geq 0} \left(-\sum_{\substack{i=0 \\ i \neq j}}^{d} \frac{g_{ij}}{g_{jj}} \Xi_{\varepsilon_i - \varepsilon_j} \right)^m.$$

If we multiply those expansions together and compare the result to (a) we obtain the expansion

$$(b) \qquad f_\mu(g) = \frac{\det(g)}{g_{00} \cdots g_{dd}} \cdot \sum_{\underline{m} \in I(\mu)} c_{\underline{m}} \cdot \prod_{i \neq j} \left(\frac{g_{ij}}{g_{jj}} \right)^{m_{ij}}$$

where the $c_{\underline{m}}$ are certain nonzero integers (given as a sign times a product of polynomial coefficients) and

$$I(\mu) := \begin{array}{c}\text{set of all tuples } \underline{m} = (m_{ij})_{i\neq j} \text{ consisting} \\ \text{of integers } m_{ij} \geq 0 \text{ such that} \\ \mu = \sum_{i\neq j} m_{ij}(\varepsilon_i - \varepsilon_j).\end{array}$$

In order to see that this expansion actually is uniformly convergent in $g \in B$ let

$$\pi(\underline{m}) := \sum_{i<j} m_{ij}.$$

It is clear that if we fix μ and an $n \geq 0$ then the number of $\underline{m} \in I(\mu)$ such that $\pi(\underline{m}) = n$ is finite. But on the other hand, for $g \in B$, the matrix entries g_{ij} for $i < j$ are divisible by π. Hence the valuation of the summand corresponding to the tuple $\underline{m}$ in the expansion (b) is at least $\pi(\underline{m})$.

Property 3. — The restriction of f_μ to B does not vanish identically.

In order to see this we make a choice of simple roots $\alpha'_0, \ldots, \alpha'_{d-1}$ with respect to which μ is positive, i.e., $\mu = n_0\alpha'_0 + \cdots + n_{d-1}\alpha'_{d-1}$ with $n_i \geq 0$. Consider the matrix $g_0 \in B$ which has a 1 on all diagonal positions, a π on the positions $\alpha'_0, \ldots, \alpha'_{d-1}$, and 0 elsewhere. Then

$$f_\mu(g_0) = c\pi^{n_0+\cdots+n_{d-1}} \text{ with some nonzero } c \in \mathbf{Z}.$$

Let ω_B denote the spectral norm on the affinoid algebra $\mathcal{O}(B)$ of K-analytic functions on B. We have to determine the precise value of $\omega_B(f_\mu|B)$.

Lemma 5.1. — *For any $\underline{m} \in I(\mu)$ we have $\pi(\underline{m}) \geq \ell(\mu)$.*

Proof. — Recall ([ST] p. 405) that

$$\ell(\mu) = -\inf_{z\in\overline{C}} \mu(z).$$

It follows that $\ell(\mu+\nu) \leq \ell(\mu) + \ell(\nu)$ holds for any $\mu, \nu \in X^*(\overline{T})$. Hence if $\mu = \sum_{i\neq j} m_{ij}(\varepsilon_i - \varepsilon_j)$ then we have

$$\ell(\mu) \leq \sum_{i\neq j} m_{ij}\ell(\varepsilon_i - \varepsilon_j).$$

It therefore suffices to check that

$$\ell(\varepsilon_i - \varepsilon_j) \leq \begin{cases} 1 & \text{if } i < j, \\ 0 & \text{if } i > j. \end{cases}$$

But that is obvious from the definition of the chamber C. ☐

Property 4. — We have the identity

$$\omega_B(f_\mu|B) = \ell(\mu).$$

The norm ω_B on $\mathcal{O}(B)$ is multiplicative and the first factor $\det(g)\cdot(g_{00}\cdots g_{dd})^{-1}$ in the expansion (b) is a unit in $\mathcal{O}(B)$. It therefore follows from the lemma that $\omega_B(f_\mu|B)\geq \ell(\mu)$ and that it suffices to find an $\underline{m}\in I(\mu)$ such that

$$\omega(c_{\underline{m}})+\sum_{i<j} m_{ij}=\ell(\mu).$$

Let us first consider the special case where $\ell(\mu)=0$. Then $\mu=n_0\alpha_0+\cdots+n_{d-1}\alpha_{d-1}$ with all $n_i\geq 0$. Consider the element $h=(h_{ij})\in B$ where $h_{ii}=1$ for $0\leq i\leq d$ and $h_{i+1,i}=-1$ for $0\leq i\leq d-1$, with all other $h_{ij}=0$. For this matrix, we compute

$$\begin{aligned} h_*\xi &= (1-\tfrac{\Xi_1}{\Xi_0})^{-1}\cdots(1-\tfrac{\Xi_d}{\Xi_{d-1}})^{-1}\xi \\ &= \textstyle\sum_{(i_0,\ldots,i_d)}(\tfrac{\Xi_1}{\Xi_0})^{i_0}\cdots(\tfrac{\Xi_d}{\Xi_{d-1}})^{i_d}\xi \end{aligned}$$

in $\Omega^d_b(U^0)$. Using (a) this shows that $f_\mu(h)=\pm 1$. On the other hand substituting $g=h$ in the series expansion (b) we see that $1=\pm f_\mu(h)=\pm c_{\underline{m}}$ for the particular $\underline{m}\in I(\mu)$ corresponding to the representation $\mu=\sum_{i=0}^{d-1} n_i\alpha_i$ – that is, the $\underline{m}$ with $m_{i+1,i}=n_i$ and other $m_{ij}=0$. This implies that

$$\ell(\mu)=0=\omega(c_{\underline{m}})+\sum_{i<j} m_{ij}$$

in this case. In order to treat the general case we first make the following observations. Let $s_i\in G$ denote the permutation matrix which represents the reflection in the Weyl group corresponding to the simple root α_i. The Coxeter element $s=s_0\cdots s_{d-1}$ permutes the roots $\alpha_0,\ldots,\alpha_d$ cyclically. The same then is true for the element $\rho:=ys$ where y denotes the diagonal matrix in G with entries $\pi,1,\ldots,1$. But ρ normalizes the subgroup B and in particular changes the residue of a d-form only by a sign ([ST] Thm. 24). Let now

$$\nu=n_0\alpha_0+\cdots+n_d\alpha_d \text{ with all } n_i\geq 0 \text{ and } n_a=0 \text{ for some } 0\leq a\leq d$$

be any weight; in particular $\ell(\nu)=n_d$. The weight $\mu:=\rho^{d-a}(\nu)$ then satisfies $\ell(\mu)=0$. Defining $h\in B$ as before we have $f_\mu(h)=\pm 1$. The matrix $(h'_{ij})=h':=\rho^{a-d}h\rho^{d-a}\in B$ is given by

$$h'_{ii}=1, h'_{i+1,i}=-1 \text{ for } i\neq a, h'_{0d}=-\pi, \text{ and all other } h'_{ij}=0.$$

Substituting $g=h'$ in (b) we obtain

$$f_\nu(h')=\pm c_{\underline{n}}\cdot\pi^{n_{0d}}$$

where $\underline{n}\in I(\nu)$ corresponds to the above representation of ν (in particular, $n_{0d}=n_d$). On the other hand we compute

$$\begin{aligned} \omega(f_\nu(h')) &= \omega(\mathrm{Res}_{(\overline{C},0)}\Xi_{-\nu}\cdot(\rho^{a-d}h\rho^{d-a})_*\xi) \\ &= \omega(\mathrm{Res}_{(\overline{C},0)}\Xi_{-\nu}\cdot(\rho^{a-d}h)_*\xi) \\ &= \omega(\mathrm{Res}_{(\overline{C},0)}(\rho^{d-a})_*\Xi_{-\nu}\cdot h_*\xi) \end{aligned}$$

$$
\begin{aligned}
&= \omega\Big(\prod_{i=1}^{d-a} s^i(\nu)(y^{-1}) \cdot \mathrm{Res}_{(\overline{C},0)}\Xi_{-\mu} \cdot h_*\xi\Big) \\
&= -\sum_{i=1}^{d-a} \omega(s^i(\nu)(y)) + \omega(f_\mu(h)) \\
&= -(n_{d-1} - n_d) - (n_{d-2} - n_{d-1}) - \cdots - (n_a - n_{a+1}) \\
&= n_d = \ell(\nu).
\end{aligned}
$$

Property 5. — Since B is open in G the enveloping algebra $U(\mathfrak{g})$ also acts by left invariant differential operators on $\mathcal{O}(B)$. It is an immediate consequence of the definition that

$$f_\mu|B \in \mathcal{O}(B)^{\mathfrak{b}=0}.$$

Proposition 5.2. — *For any $\mu \in X^*(\overline{T})$ the weight space of weight $-\mu$ in $\mathcal{O}(B)^{\mathfrak{b}=0}$ with respect to the adjoint action of $B \cap T$ is the 1-dimensional subspace generated by $f_\mu|B$.*

Proof. — We consider the pairing

$$
\begin{array}{ccc}
U(\mathfrak{g})/\mathfrak{b} \times \mathcal{O}(B)^{\mathfrak{b}=0} & \longrightarrow & K \\
(\mathfrak{z}, f) & \longmapsto & (\mathfrak{z}f)(1).
\end{array}
$$

It is nondegenerate on the right by Taylor's formula. It also is invariant with respect to the adjoint action of $B \cap T$ on both sides. Hence the induced map

$$\mathcal{O}(B)^{\mathfrak{b}=0} \hookrightarrow \mathrm{Hom}_K(U(\mathfrak{g})/\mathfrak{b}, K)$$

is injective and respects weight spaces. It then follows from Corollary 4.5 that the weight spaces on the left hand side are at most 1-dimensional. But we know that $f_\mu|B$ is nonvanishing. $\square$

The meaning of that proposition is that any function $f \in \mathcal{O}(B)^{\mathfrak{b}=0}$ has an expansion of the form

$$f = \sum_{\mu \in X^*(\overline{T})} b(\mu)(f_\mu|B)$$

with $b(\mu) \in K$ such that $\omega(b(\mu)) + \ell(\mu) \to \infty$ with respect to the Fréchet filter of complements of finite subsets in $X^*(\overline{T})$. First expand f into a series in the matrix entries and then collect all terms of a specific weight $-\mu$. We obtain in this way an expansion

$$f = \sum_\mu \widetilde{f}_\mu \text{ with } \omega_B(\widetilde{f}_\mu) \longrightarrow \infty.$$

Since the $U(\mathfrak{g})$-action on $\mathcal{O}(B)$ is by continuous endomorphisms and since the ideal $\mathfrak{b}$ is homogeneous in the weight space decomposition the equation $\mathfrak{b}f = 0$ implies $\mathfrak{b}\widetilde{f}_\mu = 0$ for any μ. It therefore follows from the proposition that $\widetilde{f}_\mu = b(\mu)(f_\mu|B)$ for some $b(\mu) \in K$.

If we now consider a d-form

$$\eta = \sum_{\mu \in X^*(\overline{T})} a(\mu)\Xi_\mu \xi \in \Omega_b^d(U^0)$$

then we see that

$$\langle \eta, f\rangle := \sum_{\mu} a(\mu) b(\mu)$$

converges in K. In this way we obtain a bilinear pairing

$$\langle , \rangle : \Omega_b^d(U^0) \times \mathcal{O}(B)^{\mathfrak{b}=0} \longrightarrow K.$$

Actually the following stronger statement is immediately clear.

Proposition 5.3. — *The pairing $\langle , \rangle$ induces a topological isomorphism*

$$[\mathcal{O}(B)^{\mathfrak{b}=0}]' = \Omega_b^d(U^0).$$

The connection between this local duality and the map I from the second section is provided by the diagram

$$\begin{tikzcd}
\Omega^d(\mathcal{X})' \arrow[r, "I"] & C^{\mathrm{an}}(G,K)^{\mathfrak{a}=0} \arrow[d, "\text{restriction}"] \\
 & C^{\mathrm{an}}(B,K)^{\mathfrak{b}=0} \\
\Omega_b^d(U^0)' \arrow[uu, "\text{restriction}'"] & \mathcal{O}(B)^{\mathfrak{b}=0} \arrow[l, "\langle , \rangle"'] \arrow[u, "\subseteq"']
\end{tikzcd}$$

which, by the very construction of the above pairing, is commutative up to sign.

The ideal filtration $\mathfrak{b} \subseteq \cdots \subseteq \mathfrak{b}_j \subseteq \cdots \subseteq U(\mathfrak{g})$ gives rise to a filtration

$$\mathcal{O}(B)^{\mathfrak{b}=0} \supseteq \cdots \supseteq \mathcal{O}(B)^{\mathfrak{b}_{j+1}=0} \supseteq \mathcal{O}(B)^{\mathfrak{b}_j=0} \supseteq \cdots \supseteq \mathcal{O}(B)^{\mathfrak{b}_0=0} = \{0\}$$

as well as, by duality, to a "local" filtration

$$\Omega_b^d(U^0) = \Omega_b^d(U^0)^0 \supseteq \cdots \supseteq \Omega_b^d(U^0)^j \supseteq \cdots \supseteq \Omega_b^d(U^0)^{d+1} = \{0\}$$

with

$$\Omega_b^d(U^0)^j := [\mathcal{O}(B)^{\mathfrak{b}=0}/\mathcal{O}(B)^{\mathfrak{b}_j=0}]'.$$

We need to understand how the properties of the ideal filtration which we have established in the previous section translate into properties of the other filtrations. Recall that the "bases" $\{f_\mu\}$ of $\mathcal{O}(B)^{\mathfrak{b}=0}$ and $\{L_\mu\}$ of $U(\mathfrak{g})/\mathfrak{b}$, respectively, are "dual" to each other in the sense that

$$L_\nu f_\mu(1) = \begin{cases} \pm 1 & \text{for } \nu = \mu, \\ 0 & \text{for } \nu \neq \mu. \end{cases}$$

If $f \in \mathcal{O}(B)^{\mathfrak{b}=0}$ has the expansion $f = \sum\limits_{\mu} b(\mu)(f_\mu|B)$ we therefore have

$$(L_\nu f)(1) = \sum_\mu b(\mu)(L_\nu f_\mu)(1) = \pm b(\nu).$$

From this one easily deduces that

– $\mathcal{O}(B)^{\mathfrak{b}_J=0} = \{f \in \mathcal{O}(B)^{\mathfrak{b}=0} : f = \sum_{J \not\subseteq J(\mu)} b(\mu)(f_\mu|B)\}$;

– any coset in $\mathcal{O}(B)^{\mathfrak{b}_J^>=0}/\mathcal{O}(B)^{\mathfrak{b}_J=0}$ has a unique representative of the form

$$f = \sum_{J(\mu)=J} b(\mu)(f_\mu|B).$$

This leads to the fact that the map

$$\begin{array}{ccc} \mathcal{O}(B)^{\mathfrak{b}_{j+1}=0}/\mathcal{O}(B)^{\mathfrak{b}_j=0} & \xrightarrow{\cong} & \bigoplus\limits_{\#J=j} \mathcal{O}(B)^{\mathfrak{b}_J^>=0}/\mathcal{O}(B)^{\mathfrak{b}_J=0} \\ f = \sum\limits_{\#J(\mu)=j} b(\mu)(f_\mu|B) & \longmapsto & \Big(\sum\limits_{J(\mu)=J} b(\mu)(f_\mu|B)\Big)_J \end{array}$$

is a continuous linear isomorphism. We will give a reinterpretation of the right hand side which reflects the fact that $\mathfrak{b}_J/\mathfrak{b}_J^>$ is a quotient of the generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_J)} M_J$ via the map which sends $\mathfrak{z} \otimes m$ to $\mathfrak{z}m$. Set

$$\mathfrak{d}_J := \ker(U(\mathfrak{g}) \underset{U(\mathfrak{p}_J)}{\otimes} M_J \longrightarrow \mathfrak{b}_J/\mathfrak{b}_J^>).$$

By the Poincaré-Birkhoff-Witt theorem the inclusion $U(\mathfrak{n}_J^+) \subseteq U(\mathfrak{g})$ induces an isomorphism $U(\mathfrak{n}_J^+) \otimes_K M_J \xrightarrow{\cong} U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_J)} M_J$. In this section we always will view $\mathfrak{d}_J$ as a subspace of $U(\mathfrak{n}_J^+) \otimes_K M_J$.

Let U_J^+ be the unipotent subgroup in G whose Lie algebra is $\mathfrak{n}_J^+$, and let $\mathcal{O}(U_J^+ \cap B)$ denote the K-affinoid algebra of K-analytic functions on the polydisk $U_J^+ \cap B$. Consider the pairing

$$\begin{array}{rcl} \langle\,,\,\rangle : (U(\mathfrak{n}_J^+) \underset{K}{\otimes} M_J) \times (\mathcal{O}(U_J^+ \cap B) \underset{K}{\otimes} M_J') & \longrightarrow & \mathcal{O}(U_J^+ \cap B) \\ (\mathfrak{z} \otimes m, e \otimes E) & \longmapsto & E(m) \cdot \mathfrak{z}e \end{array}$$

and define the Banach space

$$\mathcal{O}(U_J^+ \cap B, M_J')^{\mathfrak{d}_J=0} := \{\varepsilon \in \mathcal{O}(U_J^+ \cap B) \underset{K}{\otimes} M_J' : \langle \mathfrak{d}_J, \varepsilon\rangle = 0\}.$$

Let also $\{L_\mu^*\}_{\mu \in B(J)}$ denote the basis of M_J' dual to the basis $\{L_\mu\}_\mu$ of M_J.

Proposition 5.4. — *The map*

$$\begin{array}{rcl} \nabla_J : \mathcal{O}(B)^{\mathfrak{b}_J^>=0}/\mathcal{O}(B)^{\mathfrak{b}_J=0} & \xrightarrow{\cong} & \mathcal{O}(U_J^+ \cap B, M_J')^{\mathfrak{d}_J=0} \\ f & \longmapsto & \sum\limits_{\mu \in B(J)} [(L_\mu f)|U_J^+ \cap B] \otimes L_\mu^* \end{array}$$

is an isomorphism of Banach spaces.

Proof. — For $\mathfrak{Z} = \sum_\nu \mathfrak{z}_{(\nu)} \otimes L_\nu \in \mathfrak{d}_J \subseteq U(\mathfrak{n}_J^+) \otimes_K M_J$ we have

$$\begin{aligned}\langle \mathfrak{Z}, \sum_\mu [(L_\mu f)|U_J^+ \cap B] \otimes L_\mu^* \rangle &= \sum_{\mu,\nu} L_\mu^*(L_\nu) \cdot (\mathfrak{z}_{(\nu)} L_\mu f)|U_J^+ \cap B \\ &= (\sum_\nu \mathfrak{z}_{(\nu)} L_\nu) f|U_J^+ \cap B = 0\end{aligned}$$

since $\sum_\nu \mathfrak{z}_{(\nu)} L_\nu \in U(\mathfrak{n}_J^+) \cap \mathfrak{b}_J^>$. Morover for $\mu \in B(J)$ we have $L_\mu \in \mathfrak{b}_J$. Hence the map ∇_J is well defined. It clearly is continuous. The Banach space on the left hand side of the assertion has the orthonormal basis $\pi^{-\ell(\nu)} f_\nu|B$ for $J(\nu) = J$. Concerning the right hand side we observe that the above pairing composed with the evaluation in 1 induces an injection

$$\mathcal{O}(U_J^+ \cap B) \underset{K}{\otimes} M_J' \hookrightarrow \mathrm{Hom}_K(U(\mathfrak{n}_J^+) \underset{K}{\otimes} M_J, K)$$

which restricts to an injection

$$\mathcal{O}(U_J^+ \cap B, M_J')^{\mathfrak{d}_J = 0} \hookrightarrow \mathrm{Hom}_K(\mathfrak{b}_J/\mathfrak{b}_J^>, K).$$

Hence the only weights which can occur in the right hand side are those ν with $J(-\nu) = J$ and the corresponding weight spaces are at most 1-dimensional. Moreover the same argument as after Prop. 2 shows that the occurring weight vectors (scaled appropriately) form an orthonormal basis. Since ∇_J visibly preserves weights the assertion follows once we show that

$$\nabla_J(f_\nu|B) \neq 0 \text{ for any } \nu \text{ with } J(\nu) = J.$$

All that remains to be checked therefore is the existence, for a given ν with $J(\nu) = J$, of a $\mu \in B(J)$ such that $L_\mu f_\nu$ does not vanish identically on $U_J^+ \cap B$.

The weight ν is of the form $\nu = \sum_{j=0}^{d} n_j \varepsilon_j$ with $n_j > 0$ for $j \in J$ and $n_j \leq 0$ for $j \notin J$. We have

$$\#J \leq \sum_{j \in J} n_j = -\sum_{j \notin J} n_j.$$

Choose integers $n_j \leq m_j \leq 0$ for $j \notin J$ such that $\#J = -\sum_{j \notin J} m_j$ and define

$$\mu := \sum_{j \in J} \varepsilon_j + \sum_{j \notin J} m_j \varepsilon_j \in B(J).$$

Observe that $J(\nu - \mu) \subseteq J$ and $J(\mu - \nu) \cap J = \varnothing$. This means that $L_{\nu-\mu} \in U(\mathfrak{n}_J^+)$. It suffices to check that $L_{\nu-\mu} L_\mu f_\nu(1) \neq 0$. We compute

$$\begin{aligned} L_{\nu-\mu} L_\mu f_\nu(1) &= \mathrm{Res}_{(\overline{C},0)} \Xi_{-\nu} \cdot L_{\nu-\mu} L_\mu \xi \\ &= -\mathrm{Res}_{(\overline{C},0)} \Xi_{-\nu} \cdot L_{\nu-\mu}(\Xi_\mu \xi).\end{aligned}$$

As a consequence of the formula (+) in section 4 we have $L_{\nu-\mu}(\Xi_\mu \xi) = m \cdot \Xi_\nu \xi$ for some nonzero integer m. Hence we obtain

$$L_{\nu-\mu} L_\mu f_\nu(1) = -m \cdot \mathrm{Res}_{(\overline{C},0)} \xi = \pm m \neq 0. \qquad \square$$

As a consequence of this discussion we in particular have the following map.

Lemma 5.5. — *There is a unique continuous linear map*

$$D_J : \mathcal{O}(U_J^+ \cap B, M_J')^{\mathfrak{d}_J=0} \longrightarrow [\Omega_b^d(U^0)^j / \Omega_b^d(U^0)^{j+1}]',$$

where $j := \#J$*, which sends the weight vector* $\sum_{\mu\in B(J)}[(L_\mu f_\nu)|U_J^+ \cap B] \otimes L_\mu^*$*, for* ν *with* $J(\nu) = J$*, to the linear form* $\lambda_\nu(\eta) := \mathrm{Res}_{(\overline{C},0)} \Xi_{-\nu}\eta$*.*

6. The global filtration

In this section, we find a G-invariant filtration on the full space $\Omega^d(\mathfrak{X})$ that is compatible with the local filtration discussed in the previous section. This "global" filtration is defined first on the subspace of $\Omega^d(\mathfrak{X})$ consisting of algebraic d-forms having poles along a finite set of K-rational hyperplanes; the filtration on the full space is obtained by passing to the closure. We obtain at the same time a filtration on the dual space $\Omega^d(\mathfrak{X})'$. A key tool in our description of this filtration is a "partial fractions decomposition" due to Gelfand and Varchenko.

At the end of the section, we apply general results from the theory of topological vector spaces (in the non-archimedean situation) to show that the subquotients of the global filtration on $\Omega^d(\mathfrak{X})$ are reflexive Fréchet spaces whose duals can be computed by the subquotients of the dual filtration.

Let us first recall some general notions from algebraic geometry. Let $\mathcal{L}$ be an invertible sheaf on $\mathbf{P}^d_{/K}$. With any regular meromorphic section s of $\mathcal{L}$ over $\mathbf{P}^d_{/K}$ we may associate a divisor $\mathrm{div}(s)$ (compare EGA IV.21.1.4). One has $\mathrm{div}(s') = \mathrm{div}(s)$ if and only if $s' = ts$ for some invertible regular (= constant) function t on $\mathbf{P}^d$. Let $\{Y_i\}_{i\in I}$ be the collection of prime divisors on $\mathbf{P}^d_{/K}$ and write

$$\mathrm{div}(s) = \sum_{i\in I} n_i Y_i$$

where almost all of the integers n_i are zero. One has

$$\sum_i n_i \deg(Y_i) = n \text{ if } \mathcal{L} \cong \mathcal{O}(n)$$

([Har] II.6.4). We put

$$\mathrm{div}(s)_\infty := -\sum_{\substack{i\\ n_i<0}} n_i Y_i$$

and

$$\imath_\mathrm{o}(s) := \#\{i \in I : n_i < 0\}\,.$$

By convention let $\mathrm{div}(0)_\infty := 0$ and $\imath_\mathrm{o}(0) = 0$. We want to apply these notions in the case of the canonical invertible sheaf $\mathcal{L} = \Omega^d \cong \mathcal{O}(-d-1)$ on $\mathbf{P}^d_{/K}$. A regular meromorphic global section η in this case is a d-form $\eta = F\xi$ such that F is a nonzero rational function on $\mathbf{P}^d_{/K}$. We will study the subspace

$$\Omega^d_{\mathrm{alg}}(\mathfrak{X}) := \begin{array}{l}\text{all regular meromorphic global sections}\\ \eta \text{ of } \Omega^d \text{ such that } \mathrm{div}(\eta)_\infty \text{ is supported on}\\ \text{a union of } K\text{-rational hyperplanes in } \mathbf{P}^d\\ \text{together with the zero section}\end{array}$$

of "algebraic forms" in $\Omega^d(\mathfrak{X})$. For any $\eta \in \Omega^d_{\mathrm{alg}}(\mathfrak{X})$ we introduce its index as being the nonnegative integer

$$\imath(\eta) := \min \max_k \imath_{\mathrm{o}}(\eta_k)$$

where the minimum is taken over all representations $\eta = \sum\limits_k \eta_k$ of η as a finite sum of other $\eta_k \in \Omega^d_{\mathrm{alg}}(\mathfrak{X})$. By definition we have

$$\imath(\eta + \eta') \le \max(\imath(\eta), \imath(\eta')).$$

Hence $\Omega^d_{\mathrm{alg}}(\mathfrak{X})$ is equipped with the filtration

$$\cdots \supseteq \Omega^d_{\mathrm{alg}}(\mathfrak{X})^0 \supseteq \cdots \supseteq \Omega^d_{\mathrm{alg}}(\mathfrak{X})^d \supseteq \Omega^d_{\mathrm{alg}}(\mathfrak{X})^{d+1} = \{0\}$$

by the subspaces

$$\Omega^d_{\mathrm{alg}}(\mathfrak{X})^j := \{\eta \in \Omega^d_{\mathrm{alg}}(\mathfrak{X}) : \imath(\eta) \le d + 1 - j\}.$$

Lemma 6.1. — *The index $\imath(\eta)$ is G-invariant and takes values between 1 and $d+1$ for all nonzero $\eta \in \Omega^d_{\mathrm{alg}}(\mathfrak{X})$.*

Proof. — The G-invariance is clear since G preserves K-rational hyperplanes. The upper bound for the index follows from the existence of a partial fraction decomposition ([GV] Thm. 21) which says that $\Omega^d_{\mathrm{alg}}(\mathfrak{X})$ as a vector space is spanned by the forms $u_*(\Xi_\mu \xi) = (u_* \Xi_{\mu-\beta}) d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$ with $\mu \in X^*(\overline{T})$ and $u \in P$ unipotent. Each $\Xi_\mu \xi$ has poles along at most the $d+1$ coordinate hyperplanes defined by the equations $\Xi_i = 0$ for $i = 0, \ldots, d$. $\square$

It follows that the subspace $\Omega^d_{\mathrm{alg}}(\mathfrak{X})$ together with its filtration is G-invariant. Moreover the filtration is finite with $\Omega^d_{\mathrm{alg}}(\mathfrak{X}) = \Omega^d_{\mathrm{alg}}(\mathfrak{X})^0$. In order to obtain finer information we need to take a closer look at that partial fraction decomposition. First we introduce, for any subset $J \subseteq \{0, \ldots, d\}$, the subgroup

$$U(J) := \begin{array}{l}\text{all lower triangular unipotent matrices } u = (u_{ij})\\ \text{such that } u_{ij} = 0 \text{ whenever } i > j \text{ and } j \in J\end{array}$$

of U. In particular $U(\{0, \ldots, d\}) = U(\{0, \ldots, d-1\}) = \{1\}$ and $U(\varnothing) = U(\{d\}) = U$.

Proposition 6.2. — *Every differential form $\eta \in \Omega^d_{\mathrm{alg}}(\mathfrak{X})$ may be written as a sum*

$$\eta = \sum_{J \subseteq \{0,\ldots,d\}} \sum_{\substack{\mu \in X^*(\overline{T})\\ J(\mu) = J}} \sum_{u \in U(J)} A(\mu, u) u_*(\Xi_\mu \xi)$$

where the coefficients $A(\mu,u) \in K$ are zero for all but finitely many pairs (μ,u); furthermore, such an expression is unique.

Proof. — We write $\eta = F d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$ and apply that partial fraction decomposition to F obtaining an expression

$$F = \sum_{\mu \in X^*(\overline{T})} \sum_{u \in U} B(\mu,u) u_* \Xi_\mu .$$

The uniqueness part of that Thm. 21 in [GV] says that such an expression even exists and is unique under the following additional requirement: If $\mu = \sum_k m_k \varepsilon_k$ then we sum only over those $u \in U$ whose k-th column consists of zeroes except for the diagonal entry $u_{kk} = 1$ for every k such that $m_k \geq 0$. But

$$\{0 \leq k < d : m_k \geq 0\} = J(\mu + \beta) \backslash \{d\}$$

so that the condition on u becomes exactly that $u \in U(J(\mu+\beta))$. Because of $(u_*\Xi_\mu) d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}} = u_*(\Xi_{\mu+\beta}\xi)$ we obtain the desired unique expression if we put $A(\mu,u) := B(\mu-\beta,u)$. □

Let us temporarily introduce as another invariant of a form $\eta \in \Omega^d_{\mathrm{alg}}(\mathfrak{X})$ the linear subvariety

$$Z(\eta) := \begin{array}{l} \text{the intersection of all hyperplanes} \\ \text{contained in the support of } \mathrm{div}(\eta)_\infty \end{array}$$

in $\mathbf{P}^d_{/K}$. One obviously has:

– codim $Z(\eta) \leq \imath_{\mathrm{o}}(\eta)$;

– codim $Z(g_*(\Xi_\mu \xi)) = \imath_{\mathrm{o}}(g_*(\Xi_\mu\xi))$ for any $g \in G$ and $\mu \in X^*(\overline{T})$.

Write

$$\eta = F_{\mathrm{hom}}(\Xi_0,\ldots,\Xi_d) \cdot \sum_{i=0}^{d} (-1)^i \Xi_i d\Xi_0 \wedge \cdots \wedge \widehat{d\Xi_i} \wedge \cdots \wedge d\Xi_d$$

as a homogeneous form on affine space $\mathbb{A}^{d+1}$ and apply the partial fraction decomposition in [GV] to F_{hom}. Then, at each stage of the construction of the partial fraction decomposition of F_{hom}, the linear forms occurring in the denominator of any term are linear combinations of those in the denominator of F_{hom}. This means that

– $Z(\eta) \subseteq Z(A(\mu,u)u_*(\Xi_\mu\xi))$.

Together these three observations imply that

$$\imath_{\mathrm{o}}(\eta) \geq \imath_{\mathrm{o}}(A(\mu,u)u_*(\Xi_\mu\xi)) .$$

It then follows from the unicity of the partial fraction decomposition that we actually have

$$\imath(\eta) = \max_{\mu,u} \imath_{\mathrm{o}}(A(\mu,u)u_*(\Xi_\mu\xi)) .$$

But $\operatorname{div}(\xi)_\infty = \sum_{i=0}^{d}\{\Xi_i = 0\}$ and therefore $\imath_{\mathrm{o}}(\Xi_\mu\xi) = d+1-\#J(\mu)$. We obtain the following explicit formula

$$\imath(\eta) = \max\{d+1-\#J(\mu) : \mu \text{ such that } A(\mu,u) \neq 0 \text{ for some } u \in U(J(\mu))\}$$

for the index of any d-form $\eta \neq 0$. Another consequence of this discussion that we will need later is the inequality

$$\imath(\eta) \leq \operatorname{codim} Z(\eta)\,.$$

Corollary 6.3. — *$\Omega^d_{\mathrm{alg}}(\mathfrak{X})^j$ as a K-vector space is spanned by the forms $u_*(\Xi_\mu\xi)$ where $(\mu,u) \in X^*(\overline{T}) \times U$ runs over those pairs for which $u \in U(J(\mu))$ and $\#J(\mu) \geq j$; in particular*

$$\Omega^d_{\mathrm{alg}}(\mathfrak{X})^j = \sum_{g\in G} g_*(\mathfrak{b}_j\xi).$$

Our "global" G-equivariant filtration

$$\Omega^d(\mathfrak{X}) = \Omega^d(\mathfrak{X})^0 \supseteq \cdots \supseteq \Omega^d(\mathfrak{X})^d \supseteq \Omega^d(\mathfrak{X})^{d+1} = \{0\}$$

of $\Omega^d(\mathfrak{X})$ now is defined in the following way by taking closures.

Definition. — $\Omega^d(\mathfrak{X})^j :=$ *closure of* $\Omega^d_{\mathrm{alg}}(\mathfrak{X})^j$ *in* $\Omega^d(\mathfrak{X})$. *The dual filtration*

$$\{0\} = \Omega^d(\mathfrak{X})'_0 \subseteq \Omega^d(\mathfrak{X})'_1 \subseteq \cdots \subseteq \Omega^d(\mathfrak{X})'_{d+1} = \Omega^d(\mathfrak{X})'$$

is given by

$$\Omega^d(\mathfrak{X})'_j := [\Omega^d(\mathfrak{X})/\Omega^d(\mathfrak{X})^j]'.$$

The second statement in Corollary 3 immediately implies that the latter filtration corresponds under our map I to the filtration of $C^{\mathrm{an}}(G,K)$ defined through annihilation conditions with respect to the left invariant differential operators in the ideal sequence $\mathfrak{b}_0 \supseteq \cdots \supseteq \mathfrak{b}_{d+1} = \mathfrak{b}$, i.e.,

$$I(\Omega^d(\mathfrak{X})'_j) \subseteq C^{\mathrm{an}}(G,K)^{\mathfrak{b}_j=0} \text{ for } 0 \leq j \leq d+1.$$

The compatibility between the local and the global filtration is established in the subsequent lemma.

Lemma 6.4. — $\Omega^d(\mathfrak{X})^j \subseteq \Omega^d(\mathfrak{X}) \cap \Omega^d_b(U^0)^j$.

Proof. — Consider any d-form $u_*(\Xi_\mu\xi)$ with $u \in U(J(\mu))$ and write

$$u_*(\Xi_\mu\xi)|U^0 = \sum_{\nu\in X^*(\overline{T})} a(\nu)\Xi_\nu\xi.$$

We claim that $a(\nu) \neq 0$ implies that $\#J(\nu) \geq \#J(\mu)$. In order to see this let $\mu = \sum_k m_k\varepsilon_k$. Because of the condition on u we have

$$u_*(\Xi_\mu\xi) = \Big(\prod_{k\notin J(\mu)} \Xi_k\Big)\Big(\prod_{k\in J(\mu)} \Xi_k^{m_k}\Big)\Big(\prod_{k\notin J(\mu)} u_*\Xi_k^{m_k-1}\Big)\xi.$$

The first two products together contain each Ξ_k with a positive exponent. In the third product the exponents are negative. On U^0 the summands of the linear form $u_*\Xi_k = \Xi_k + u_{k+1k}\Xi_{k+1} + \cdots + u_{dk}\Xi_d$ differ pairwise in valuation. Hence after factoring out the largest summand we can develop $(u_*\Xi_k)^{-1}$, on U^0, into a geometric series. The terms of the resulting series have powers of a single $\Xi_{k'}$ in the denominator. It follows that each of the $d+1-\#J(\mu)$ factors in the third product can cancel out at most one of the Ξ_k's in the first two products so that at least $d+1-(d+1-\#J(\mu)) = \#J(\mu)$ others remain. This establishes our claim which was that

$$u_*(\Xi_\mu \xi)|U^0 \in \Omega_b^d(U^0)^{\#J(\mu)} \text{ for } u \in U(J(\mu)).$$

(For this slight reformulation one only has to observe that $\Omega_b^d(U^0)^j$ has a nonvanishing weight space exactly for those ν with $\#J(\nu) \geq j$.) It is then a consequence of Cor. 3 that

$$\Omega^d_{\mathrm{alg}}(\mathfrak{X})^j \subseteq \Omega_b^d(U^0)^j.$$

As a simultaneous kernel of certain of the continuous linear forms $\eta \mapsto \mathrm{Res}_{(\overline{C},0)}\Xi_{-\mu}\eta$ on $\Omega_b^d(U^0)$ the right hand side is closed in $\Omega_b^d(U^0)$. It therefore follows that

$$\Omega^d(\mathfrak{X})^j \subseteq \Omega_b^d(U^0)^j. \qquad \square$$

As a consequence of this fact we may view the map D_J from Lemma 5.5 as a continuous linear map

$$D_J : \mathcal{O}(U_J^+ \cap B, M_J')^{\mathfrak{d}_J = 0} \longrightarrow [\Omega^d(\mathfrak{X})^j/\Omega^d(\mathfrak{X})^{j+1}]'\,,$$

where $j := \#J$, which sends the weight vector $\sum_{\mu\in B(J)}[(L_\mu f_\nu)|U_J^+ \cap B] \otimes L_\mu^*$, for ν with $J(\nu) = J$, to the linear form $\lambda_\nu(\eta) := \mathrm{Res}_{(\overline{C},0)}\Xi_{-\nu}\eta$.

We finish this section by collecting the basic properties which the subquotients of our global filtration have as locally convex vector spaces.

Proposition 6.5. — *Each subquotient $\Omega^d(\mathfrak{X})^i/\Omega^d(\mathfrak{X})^j$ for $0 \leq i \leq j \leq d+1$ is a reflexive Fréchet space; in particular its strong dual is barrelled and complete.*

Proof. — In section 1 we deduced the reflexivity of $\Omega^d(\mathfrak{X})$ from the fact that it is the projective limit of a sequence of Banach spaces with compact transition maps. It is a general fact (the proofs of Theorems 2 and 3 in [Kom] carry over literally to the nonarchimedean situation) that in such a Fréchet space every closed subspace along with its corresponding quotient space are projective limits of this type, too. $\square$

Lemma 6.6. — *Let $A : V \to \widetilde{V}$ be a strict continuous linear map between the K-Fréchet spaces V and $\widetilde{V}$; if $\widetilde{V}$ is reflexive then the dual map $A' : \widetilde{V}' \to V'$ between the strong duals is strict as well.*

Proof. — (Recall that A is strict if on $\mathrm{im}(A)$ the quotient topology from V coincides with the subspace topology from $\widetilde{V}$.) The subspace $\mathrm{im}(A)$ of $\widetilde{V}$ being a quotient of the Fréchet space V is complete by the open mapping theorem and hence is closed.

Let now $\widetilde{\Sigma} \subseteq \widetilde{V'}$ be any open o-submodule. We have to find an open o-submodule $\Sigma \subseteq V'$ such that $A'(\widetilde{\Sigma}) \supseteq \operatorname{im}(A') \cap \Sigma$. We may assume that $\ker(A') \subseteq \widetilde{\Sigma}$. By the definition of the strong dual we also may assume that

$$\widetilde{\Sigma} = \widetilde{\Gamma}^{\mathrm{o}} := \{\lambda \in \widetilde{V}' : |\lambda(\widetilde{v})| \leq 1 \text{ for any } \widetilde{v} \in \widetilde{\Gamma}\}$$

for some closed and bounded o-submodule $\widetilde{\Gamma} \subseteq \widetilde{V}$. Since $\widetilde{V}$ is reflexive $\widetilde{\Gamma}$ is weakly compact ([Tie] Thms 4.20.b, 4.21, and 4.25.2) and hence compact ([DeG] Prop. 3.b). Since $\operatorname{im}(A)$ is closed in $\widetilde{V}$ the Hahn-Banach theorem ([Tie] Thm. 3.5) implies that $\ker(A')^{\mathrm{o}} = \operatorname{im}(A)$. Using [Tie] Thm. 4.14 we deduce form the inclusion $\ker(A') \subseteq \widetilde{\Sigma}$ that

$$\widetilde{\Gamma} = \widetilde{\Gamma}^{\mathrm{oo}} = \widetilde{\Sigma}^{\mathrm{o}} \subseteq \ker(A')^{\mathrm{o}} = \operatorname{im}(A).$$

In fact, $\widetilde{\Gamma}$ is a compact subset of $\operatorname{im}(A)$. According to [B-GT] IX 2.10, Prop. 18 we find a compact subset $\Gamma \subseteq V$ such that $A(\Gamma) = \widetilde{\Gamma}$. Then $\Sigma := \Gamma^{\mathrm{o}}$ is an open o-submodule in V' such that $\operatorname{im}(A') \cap \Sigma = A'(\widetilde{\Sigma})$. □

Proposition 6.7

i. *For $0 \leq j \leq d+1$ the natural map $\Omega^d(\mathfrak{X})'_j \hookrightarrow \Omega^d(\mathfrak{X})'$ is a topological embedding as a closed subspace;*

ii. *for $0 \leq i \leq j \leq d+1$ the natural map $\Omega^d(\mathfrak{X})'_j/\Omega^d(\mathfrak{X})'_i \xrightarrow{\cong} [\Omega^d(\mathfrak{X})^i/\Omega^d(\mathfrak{X})^j]'$ is a topological isomorphism.*

Proof. — i. This follows immediately from Prop. 5 and Lemma 6. ii. The natural exact sequence

$$0 \longrightarrow \Omega^d(\mathfrak{X})^i/\Omega^d(\mathfrak{X})^j \longrightarrow \Omega^d(\mathfrak{X})/\Omega^d(\mathfrak{X})^j \longrightarrow \Omega^d(\mathfrak{X})/\Omega^d(\mathfrak{X})^i \longrightarrow 0$$

consists of strict linear maps between Fréchet spaces which are reflexive by Prop. 5. The dual sequence is exact by Hahn-Banach and consists of strict linear maps by Lemma 6. □

Corollary 6.8. — *If V denotes one of the locally convex vector spaces appearing in the previous Proposition then the G-action $G \times V \to V$ is continuous and the map $g \mapsto g\lambda$ on G, for any $\lambda \in V$, is locally analytic.*

Proof. — Because of Prop. 7 this is a consequence of Cor. 3.9. □

7. The top filtration step

The purpose of this section is to describe the first stage of the global filtration in various different ways. This information (for all the p-adic symmetric spaces of dimension $\leq d$) will be used in an essential way in our computation of all the stages of the global filtration in the last section.

Theorem 7.1. — *The following three subspaces of $\Omega^d(\mathfrak{X})$ are the same:*

1. *The subspace $d(\Omega^{d-1}(\mathfrak{X}))$ of exact forms in $\Omega^d(\mathfrak{X})$;*

2. *The first stage* $\Omega^d(\mathcal{X})^1$ *in the global filtration;*
3. *The subspace of forms* η *such that* $\operatorname{Res}_{(\overline{C},0)} g_*\eta = 0$ *for any* $g \in G$.

In particular all three are closed subspaces.

The proof requires a series of preparatory statements which partly are of interest in their own right. We recall right away that any exact form of course has vanishing residues. The subspace $\Omega^d(\mathcal{X})^1$ is closed by construction. The subspace in 3. is closed as the simultaneous kernel of a family of continuous linear forms.

Lemma 7.2. — *An algebraic differential form* $\eta \in \Omega^d_{\mathrm{alg}}(\mathcal{X})$ *is exact if and only if* η *belongs to* $\Omega^d_{\mathrm{alg}}(\mathcal{X})^1$.

Proof. — Suppose first that η is exact. Expand η in its partial fractions decomposition (Prop. 6.2). From Cor. 6.3 we see that η is congruent to a finite sum of logarithmic forms $u_*\xi$ modulo $\Omega^d_{\mathrm{alg}}(\mathcal{X})^1$, where u is in the subgroup U of lower triangular unipotent matrices. However, by [ST] Thm. 24, Cor. 40, and Cor. 50 the forms $u_*\xi$ are linearly independent modulo exact forms. Since η is exact, therefore, no logarithmic terms can appear in its partial fractions expansion and η belongs to $\Omega^d_{\mathrm{alg}}(\mathcal{X})^1$. Conversely it suffices, by Cor. 6.3 and G-invariance, to consider a form $\Xi_\mu\xi$ with $\mu \neq 0$. Since the Weyl group acts through the sign character on ξ we may use G-equivariance again and assume that ε_0 occurs in μ with a positive coefficient $m_0 > 0$. Then $\Xi_\mu\xi = d\theta$ with $\theta := \frac{1}{m_0}\Xi_{\beta_0}\Xi_{\mu-\beta}d\Xi_{\beta_1} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$. ☐

In the following we let $\Omega^d(\mathcal{X}_n)^j$, for $n \in \mathbb{N}$, denote the closure of $\Omega^d_{\mathrm{alg}}(\mathcal{X}_n)^j$ in the Banach space $\Omega^d(\mathcal{X}_n)$.

Lemma 7.3. — *For a form* $\eta \in \Omega^d(\mathcal{X})$ *we have:*

i. η *is exact if and only if* $\eta|\mathcal{X}_n$ *is exact for any* $n \in \mathbb{N}$*;*
ii. $\eta \in \Omega^d(\mathcal{X})^1$ *if and only if* $\eta|\mathcal{X}_n \in \Omega^d(\mathcal{X}_n)^1$ *for any* $n \in \mathbb{N}$.

Proof. — i. By the formula on the bottom of p. 64 in [SS] we have

$$H^*_{\mathrm{DR}}(\mathcal{X}) \;=\; \varprojlim_n H^*_{\mathrm{DR}}(\mathcal{X}^{\circ}_n)$$

where the $\mathcal{X}^{\circ}_n \subseteq \mathcal{X}$ are certain admissible open subvarieties such that

– $\mathcal{X} = \bigcup_n \mathcal{X}^{\circ}_n$ is an admissible covering, and
– $\mathcal{X}_{n-1} \subseteq \mathcal{X}^{\circ}_n \subseteq \mathcal{X}_n$.

The second property of course implies that

$$\varprojlim_n H^*_{\mathrm{DR}}(\mathcal{X}^{\circ}_n) \;=\; \varprojlim_n H^*_{\mathrm{DR}}(\mathcal{X}_n)\ .$$

ii. This follows by a standard argument about closed subspaces of projective limits of Banach spaces (compare the proof of Thm. 2 in [Kom]). ☐

The main technique for the proof of Theorem 1 will be a "convergent partial fractions" decomposition for rigid d-forms on $\mathcal{X}$. We begin by recalling the explicit description of rigid forms on $\mathcal{X}_n$ given in [SS] p. 53. Fix a set $\mathcal{H} = \{\ell_0, \ldots, \ell_s\}$ of unimodular representatives for the hyperplanes modulo π^{n+1} in such a way that it contains the coordinate hyperplanes $\{\Xi_i = 0\}$ for $0 \leq i \leq d$. A rigid d-form η on the affinoid $\mathcal{X}_n$ is represented by a convergent expansion

$$(*) \qquad \eta \;=\; \sum_{I,J} a_{I,J} \, \frac{\Xi_0^{j_0} \cdots \Xi_d^{j_d}}{\ell_0^{i_0} \cdots \ell_s^{i_s}} \, \Theta$$

in homogeneous coordinates where I and J run over all $(s+1)$-tuples $(i_0, \ldots, i_s)$ and $(d+1)$-tuples $(j_0, \ldots, j_d)$ of non-negative integers respectively with $\sum i_k - \sum j_k = d+1$ and where

$$\Theta := \sum_{i=0}^{d} (-1)^i \Xi_i d\Xi_0 \wedge \cdots \wedge \widehat{d\Xi_i} \wedge \cdots \wedge d\Xi_d \; .$$

The convergence means that the coefficients $a_{I,J}$ satisfy $\omega(a_{I,J}) - n(\sum_{k=0}^{d} j_k) \to \infty$ as $\sum_{k=0}^{d} j_k \to \infty$.

Lemma 7.4. — *In the expansion* $(*)$ *we may assume* $a_{I,J} = 0$ *unless the corresponding set of "denominator forms"* $\{\ell_k : i_k \geq 1\}$ *is linearly independent.*

Proof. — Suppose that $\ell_0, \ldots, \ell_r$ are linearly dependent, and that $i_k \geq 1$ for $0 \leq k \leq r$. Write

$$\sum_{k=0}^{r} b_k \ell_k \;=\; 0$$

with the $b_k \in o$ and at least one $b_k = 1$. Suppose for example that $b_0 = 1$. Then

$$\ell_0 \;=\; -\sum_{k=1}^{r} b_k \ell_k$$

and

$$\frac{\Xi_0^{j_0} \cdots \Xi_d^{j_d}}{\ell_0^{i_0} \cdots \ell_r^{i_r}} \, \Theta \;=\; -\sum_{k=1}^{r} \frac{b_k \Xi_0^{j_0} \cdots \Xi_d^{j_d}}{\ell_0^{i_0+1} \cdots \ell_k^{i_k-1} \cdots \ell_r^{i_r}} \, \Theta \quad .$$

The individual terms on the right side of this sum have the same degree as the term on the left. This, together with the fact that the b_k belong to o, implies that the expression on the right may be substituted into the series expansion for η and the sum re-arranged. Further, this process may be iterated until the denominators occurring on the right side are linearly independent. ☐

Using this Lemma, we see that any $\eta \in \Omega^d(\mathcal{X}_n)$ can be written as a finite sum of forms

$$(**) \qquad \eta_L \;=\; \sum_{I,J} a_{I,J} \, \frac{\Xi_0^{j_0} \cdots \Xi_d^{j_d}}{\ell_0^{i_0} \cdots \ell_r^{i_r}} \, \Theta$$

where $L = \{\ell_0, \ldots, \ell_r\}$ is a fixed linearly independent set chosen from $\mathcal{H}$, I runs through the $(r+1)$-tuples of positive integers, and $\omega(a_{I,J}) - n(\sum_{k=0}^{d} j_k) \to \infty$ as $\sum_{k=0}^{d} j_k \to \infty$.

Lemma 7.5. — *A form η_L as in $(**)$ belongs to $\Omega^d(\mathfrak{X}_n)^{d+1-\#L}$.*

Proof. — This is clear from the inequality $\imath(\eta) \leq \operatorname{codim} Z(\eta)$ in section 6. ☐

Definition. — *A form $\eta \in \Omega^d(\mathfrak{X}_n)$ is called decomposable if it has a convergent expansion of the form*

$$\eta = \sum_{g \in G} \sum_{\mu \in X^*(\overline{T})} c(g,\mu)(g_*(\Xi_\mu \xi)|\mathfrak{X}_n)$$

where

1. *$c(g,\mu) \in K$ and $= 0$ for all but finitely many $g \in G$ which are independent of μ,*
2. *if $c(g,\mu) \neq 0$ for some μ then the columns of the matrix g are unimodular,*
3. *$\omega(c(g,\mu)) - nd(\mu) \to \infty$ as $d(\mu) \to \infty$ ($d(\mu)$ was defined in section 4, just before the statement of Lemma 4.1).*

Lemma 7.6. — *Suppose that η_L is given by a series as in $(**)$ on $\mathfrak{X}_{2n}$. Then the restriction of η_L to $\mathfrak{X}_{n-1}$ is either decomposable or may be written as a (finite) sum of series $\eta_{L'}$ converging on $\mathfrak{X}_{n-1}$ and with $\#L' < \#L$.*

Proof. — The dichotomy in the statement of the Lemma arises out of the following two possibilities:
Case I. There is a unimodular relation

$$\sum_{k=0}^{r} b_k \ell_k \equiv 0 \pmod{\pi^n} .$$

Case II. Whenever there is a relation

$$\sum_{k=0}^{r} b_k \ell_k \equiv 0 \pmod{\pi^n} , \text{ with } b_k \in o ,$$

we must have all b_k divisible by π.

Let us treat Case I first. Suppose that b_0 is a unit in the unimodular relation, and write

$$\ell_0 = -\sum_{k=1}^{r} (b_k/b_0)\ell_k + \pi^n h .$$

To simplify the notation, set

$$\ell := -\sum_{k=1}^{r} (b_k/b_0)\ell_k .$$

The fact that ℓ_0 is unimodular means that ℓ is unimodular as well. We have

$$\frac{1}{\ell_0} = \frac{1}{\ell}(1 + \pi^n h/\ell)^{-1} ,$$

and, since $\pi^n h/\ell$ has sup-norm $\leq |\pi| < 1$ on $\mathcal{X}_{n-1}$, using the geometric series we see that we may rewrite the series expansion for η_L so that it converges on $\mathcal{X}_{n-1}$:

$$\eta_L|\mathcal{X}_{n-1} \;=\; \sum a'_{I,J}\,\frac{\Xi_0^{j_0}\cdots\Xi_d^{j_d}}{\ell^i\ell_1^{i_1}\cdots\ell_r^{i_r}}\,\Theta$$

But since ℓ is a linear combination of ℓ_k for $k \neq 0$, the proof of Lemma 4 shows that $\eta_L|\mathcal{X}_{n-1}$ is a sum of series $\eta_{L'}$ where L' is a linearly independent subset of the dependent set $\{\ell, \ell_1, \ldots, \ell_r\}$; such a set has fewer than $r+1$ elements.

For Case II we take a different approach. Apply elementary divisors to find linear forms $f_0, \ldots, f_d$ which form a basis for the o-lattice spanned by $\Xi_0, \ldots, \Xi_d$ and such that $\pi^{e_0}f_0, \ldots, \pi^{e_r}f_r$ form a basis for the span of $\ell_0, \ldots, \ell_r$. Since any monomial in the Ξ_i is an integral linear combination of monomials in the f_i, we may rewrite η_L using the f_i for coordinates:

$$\eta_L \;=\; \sum\; a''_{I,J}\,\frac{f_0^{j_0}\cdots f_d^{j_d}}{\ell_0^{i_0}\cdots\ell_r^{i_r}}\,\Theta\,.$$

Using our Case II hypothesis, we know that $e_k < n$ for $0 \leq k \leq r$. Therefore $\pi^{n-1}f_k$, for each $0 \leq k \leq r$, is an integral linear combination of $\ell_0, \ldots, \ell_r$. Let $g \in G$ be the matrix such that $g_*\Xi_i = \ell_i$ for $0 \leq i \leq r$ and $g_*\Xi_i = f_i$ for $r+1 \leq i \leq d$. By construction the columns of g are unimodular. Rewriting the series for η_L in terms of the $\pi^{1-n}\ell_0, \ldots, \pi^{1-n}\ell_r, f_{r+1}, \ldots, f_d$ we see that

$$(***)\qquad \eta_L \;=\; \sum a''_{I,J}\det(g)^{-1}\pi^{(1-n)(\sum_{k=0}^r j_k)}\sum_\mu c_{\mu,I,J}\,g_*(\Xi_\mu\xi)$$

where each of the inner sums is finite and the coefficients $c_{\mu,I,J}$ are integral. Since the original sum for η_L converges on $\mathcal{X}_{2n}$, we have

$$\omega(a''_{I,J}) \;=\; H\Big(\sum_{k=0}^d j_k\Big) \;+\; 2n\Big(\sum_{k=0}^d j_k\Big)$$

where $H(m)$ is a function which goes to infinity as m goes to infinity. But then

$$\omega\Big(a''_{I,J}\det(g)^{-1}\pi^{(1-n)(\sum_{k=0}^r j_k)}\Big) \geq H\Big(\sum_{k=0}^d j_k\Big) + 2n\Big(\sum_{k=0}^d j_k\Big) + (1-n)\Big(\sum_{k=0}^d j_k\Big) + C$$

which shows that, after rearrangement according to μ, the series $(***)$ converges on $\mathcal{X}_{n+1}$ (if $c_{\mu,I,J} \neq 0$ then $\sum_{k=0}^d j_k \geq d(\mu)$). Thus in Case II η_L is decomposable on $\mathcal{X}_{n+1}$. □

Proposition 7.7. — *Let η be a rigid d-form on $\mathcal{X}$; then $\eta|\mathcal{X}_n$, for any $n > 0$, is decomposable.*

Proof. — This follows by induction from Lemma 6. Indeed, any rigid form η on $\mathcal{X}_m$ with $m := 2^{d+1}(n+2)$ is decomposable on $\mathcal{X}_n$. □

Definition. — *A form $\eta \in \Omega^d_{\mathrm{alg}}(\mathcal{X})$ is called logarithmic if it lies in the smallest G-invariant vector subspace containing ξ.*

Corollary 7.8. — *Let η be a rigid d-form on $\mathcal{X}$. Then, for any $n > 0$, the restriction of η to $\mathcal{X}_n$ has a decomposition*

$$\eta \;=\; \eta_0 \;+\; \eta_1$$

where η_0 is the restriction of a logarithmic form and η_1 is an exact form in $\Omega^d(\mathcal{X}_n)^1$.

Proof. — Applying the convergent partial fractions decomposition of Prop. 7, write η on $\mathcal{X}_{n+1}$ as

$$\eta|\mathcal{X}_{n+1} \;=\; \sum_g c(g,0)(g_*\xi|\mathcal{X}_{n+1}) \;+\; \sum_g \sum_{\mu\neq 0} c(g,\mu)(g_*(\Xi_\mu\xi)|\mathcal{X}_{n+1})\ .$$

Let η_0 be the first of these sums, and η_1 the second. Clearly η_0 is logarithmic and η_1, by Cor. 6.3, belongs to $\Omega^d(\mathcal{X}_{n+1})^1$. Thus we need only show that η_1 is exact on $\mathcal{X}_n$. However, one sees easily that the series for η_1 may be integrated term-by-term to obtain a rigid $(d-1)$-form θ on $\mathcal{X}_n$ with $d\theta = \eta_1$ (compare the proof of Lemma 2). □

As a last preparation we need the following result on logarithmic forms.

Proposition 7.9. — *For any $n > 0$ we have:*

i. *There is a compact open set $V_n \subset U$ such that*

$$u_*\xi \in \Omega^d(\mathcal{X}_n)^1 \cap d(\Omega^{d-1}(\mathcal{X}_n))$$

for all $u \in U\backslash V_n$;

ii. *there is a finite set $u^{(1)}, \ldots, u^{(k)}$ of elements of U and a disjoint covering of V_n by sets $\{D(u^{(\ell)}, r)\}_{\ell=1}^k$ such that*

$$v_*\xi \equiv u_*^{(\ell)}\xi \pmod{\Omega^d(\mathcal{X}_n)^1 \cap d(\Omega^{d-1}(\mathcal{X}_n))}$$

if $v \in D(u^{(\ell)}, r)$;

iii. *the image of*

$$\Omega^d(\mathcal{X}) \longrightarrow \Omega^d(\mathcal{X}_n)/(\Omega^d(\mathcal{X}_n)^1 \cap d(\Omega^{d-1}(\mathcal{X}_n)))$$

and the space $\Omega^d(\mathcal{X}_n)/\Omega^d(\mathcal{X}_n)^1$ both are finite dimensional; more precisely, the classes of the forms $u_*^{(1)}\xi, \ldots, u_*^{(k)}\xi$ span both spaces.

Proof. — i. In homogeneous coordinates, we write

$$u_*\xi = \frac{\Theta}{\ell_0 \cdots \ell_d}$$

where $\ell_j = \sum_{i=j}^d u_{ij}\Xi_i$ and the u_{ij} are the matrix entries of the lower triangular unipotent matrix u. Let

$$V_n := \{u \in U : \omega(u_{lk}) \geq -(n+1)d \text{ for all } d \geq l \geq k \geq 0\}.$$

We claim V_n has the desired property. Suppose that $u \notin V_n$, so that, for some pair $d \geq l > k \geq 0$ we have $\omega(u_{lk}) < -(n+1)d$. Focus attention for the moment on the linear form ℓ_k. Since $u_{kk} = 1$, we may split the set of row indices $k, \ldots, d$ into two nonempty sets A and B such that

$$\inf_{l \in A} \omega(u_{lk}) > \sup_{l \in B} \omega(u_{lk}) + n + 1.$$

We point out two facts for later use. First, the index k automatically belongs to the set A, and so ℓ_k^B is a linear combination of the Ξ_i with $i > k$. Second, and for the same reason, the set of linear forms $\{\ell_j\}_{j \neq k} \cup \{\ell_k^A\}$ is a triangular basis for the full space of K-linear forms in the Ξ_i. Continuing with the main line of argument, write

$$\ell_k = \ell_k^A + \ell_k^B = \Big(\sum_{l \in A} u_{lk} \Xi_l\Big) + \Big(\sum_{l \in B} u_{lk} \Xi_l\Big).$$

Then

$$\frac{1}{\ell_k} = \frac{1}{\ell_k^B} \left(\frac{1}{1 + (\ell_k^A / \ell_k^B)} \right).$$

The linear forms $\pi^{-\inf_{l \in A} \omega(u_{lk})} \ell_k^A$ and $\pi^{-\inf_{l \in B} \omega(u_{lk})} \ell_k^B$ are unimodular. From this, we obtain the following estimate on $\mathfrak{X}_n$:

$$(1) \qquad \begin{array}{rl} \omega(\ell_k^A / \ell_k^B) & \geq \inf_{l \in A} \omega(u_{lk}) - \inf_{l \in B} \omega(u_{lk}) - n \\ & \geq \inf_{l \in A} \omega(u_{lk}) - \sup_{l \in B} \omega(u_{lk}) - n \\ & > 1 \,. \end{array}$$

At this point, it will be convenient to change from homogeneous to inhomogeneous coordinates. Let

$$\overline{\ell}_j := \ell_j / \Xi_d = \sum_{i=0}^{d-1} u_{ij} \Xi_{\beta_i} + u_{dj},$$

and similarly let $\overline{\ell}_k^A := \ell_k^A / \Xi_d$ and $\overline{\ell}_k^B := \ell_k^B / \Xi_d$. Then we may expand the form $u_* \xi$ as a convergent series on $\mathfrak{X}_n$:

$$(2) \qquad u_* \xi = \sum_{m=0}^{\infty} c_m F_m d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$$

where the coefficients $c_m \in \mathbf{Z}$,

$$F_m = \frac{(\overline{\ell}_k^A)^m}{\overline{\ell}_0 \cdots (\overline{\ell}_k^B)^{m+1} \cdots \overline{\ell}_{d-1}}$$

and $\overline{\ell}_k^B$ has taken the place of $\overline{\ell}_k$ in the denominators of these forms (observe that $\Theta = (-1)^d \Xi_d^{d+1} d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$). Our estimate (1) tells us that there is a constant C so that the functions F_m satisfy $\inf_{q \in \mathfrak{X}_n} \omega(F_m(q)) \geq m - C$ in the sup norm on $\mathfrak{X}_n$.

To finish the proof, we will show that the expansion (2) may be integrated term by term on $\mathfrak{X}_n$. This shows that $u_* \xi$ is exact on $\mathfrak{X}_n$. In addition, since it proves that each algebraic form in the expansion (2) is exact, we see from Lemma 2 that these forms belong to $\Omega^d_{\mathrm{alg}}(\mathcal{X}_n)^1$ and so $u_*\xi$ belongs to $\Omega^d(\mathcal{X}_n)^1$ as well.

As we remarked earlier, the forms $\ell_0, \ldots, \ell_k^A, \ldots, \ell_d$ are a triangular basis for the space of all linear forms. Therefore, we may choose $v \in U$ so that $v_*\Xi_j = \ell_j$ for all $0 \leq j \leq d$ except for $j = k$, and $v_*\Xi_k = \ell_k^A$. Let $f := v_*^{-1}(\ell_k^B/\Xi_d)$. The form f does not involve Ξ_{β_k}. Then we compute

$$F_m = v_*\left(\frac{\Xi_{\beta_k}^m}{\Xi_{\beta_0}\cdots\Xi_{\beta_{k-1}} f^{m+1}\Xi_{\beta_{k+1}}\cdots\Xi_{\beta_{d-1}}}\right).$$

Using this and the estimate for the F_m, we see that

$$\theta = \left(\sum_{m=0}^{\infty}\frac{c_m}{m+1}F_m\right) v_*((-1)^k\Xi_{\beta_k} d\Xi_{\beta_0}\wedge\cdots\wedge\widehat{d\Xi_{\beta_k}}\wedge\cdots\wedge d\Xi_{\beta_{d-1}})$$

is a convergent expansion for a rigid $(d-1)$-form θ on $\mathcal{X}_n$, and that $d\theta = u_*\xi$.

ii. In the notation of Prop. 3.1, let $u^{(1)}, \ldots, u^{(k)}$ be finitely many elements of U so that the open sets $\{D(u^{(\ell)}, r)\}_{\ell=1}^k$ form a disjoint covering of V_n and so that, for each $\ell = 1, \ldots, k$,

$$\omega(v_{ji} - u_{ji}^{(\ell)}) > 2(n+1) \text{ for all } v \in D(u^{(\ell)}, r) \text{ and all } 0 \leq i < j \leq d. \tag{3}$$

Then, for $v \in D(u^{(\ell)}, r)$, we have the uniformly convergent expansion (*) on $\mathcal{X}_n$ from the proof of Prop. 3.1, where, to simplify the notation, we write $u = u^{(\ell)}$:

$$v_*\xi = k(vw_{d+1}, \cdot)d\Xi_{\beta_0}\wedge\cdots\wedge d\Xi_{\beta_{d-1}}$$

with

$$k(vw_{d+1}, q) = \sum_{\underline{m}} c_{\underline{m}} h_{\underline{m}}\cdot(v-u)^{\underline{m}},$$

$$h_{\underline{m}} = \frac{\Xi_{\mu(\underline{m})}(q)}{f_0(u,q)^{s_0(\underline{m})}\cdots f_{d-1}(u,q)^{s_{d-1}(\underline{m})}},$$

and $c_{\underline{m}} \in \mathbf{Z}$. In this expansion, the term with $\underline{m} = (0, \ldots, 0)$ is $u_*\xi = u_*^{(\ell)}\xi$. Also, comparing the estimate in (3) with those used in Prop. 3.1, we see that we have

$$\inf_{q\in\mathcal{X}_n}\omega(h_{\underline{m}}(q)\cdot(v-u)^{\underline{m}}) \geq \Big(\sum_{0\leq i<j\leq d} m_{ji}\Big) - nd.$$

We claim that, except for the term with $\underline{m} = (0, \ldots, 0)$, this series may be integrated term by term to yield a convergent $(d-1)$-form on $\mathcal{X}_n$. This means that $(v_*\xi - u_*\xi)|\mathcal{X}_n$ is an exact form, and further that (just as in the proof of the first assertion) each term in the expansion of $v_*\xi - u_*\xi$ is an exact algebraic form, so that $v_*\xi - u_*\xi$ belongs to $\Omega^d(\mathcal{X}_n)^1$. In other words,

$$v_*\xi \equiv u_*^{(\ell)}\xi \pmod{\Omega^d(\mathcal{X}_n)^1 \cap d(\Omega^{d-1}(\mathcal{X}_n))}.$$

To prove our claim, let S_j be the set of $\underline{m}$ such that $s_i(\underline{m}) = 1$ for $i = 0, \ldots, j-1$ but $s_j(\underline{m}) > 1$. Let

$$F_j := \sum_{\underline{m} \in S_j} h_{\underline{m}} \cdot (v-u)^{\underline{m}}$$

and

$$\eta_j := F_j d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}} .$$

Because

$$v_*\xi = \eta_0 + \cdots + \eta_{d-1} + u_*^{(\ell)}\xi ,$$

it suffices to integrate each η_j term by term. Notice that if $\underline{m} \in S_j$, then $\Xi_{\mu(\underline{m})}$ does not involve any of Ξ_{β_i} for $i = 0, \ldots, j$. We may choose a matrix $g \in U$ so that $g_*\Xi_{\beta_i} = f_i(u, \cdot)$ for $i = 0, \ldots, j$ and $g_*\Xi_{\beta_i} = \Xi_{\beta_i}$ for $i = j+1, \ldots, d-1$. Now set

$$G_j := \sum_{\underline{m} \in S_j} \frac{c_{\underline{m}}}{1 - s_j(\underline{m})} h_{\underline{m}} (v-u)^{\underline{m}} .$$

The estimate on the sup norm for $h_{\underline{m}}$ implies that this is the convergent expansion of a rigid function on $\mathfrak{X}_n$. Therefore

$$\theta_j := (-1)^j G_j g_*(\Xi_{\beta_j} d\Xi_{\beta_0} \wedge \cdots \wedge \widehat{d\Xi_{\beta_j}} \wedge \cdots \wedge d\Xi_{\beta_{d-1}})$$

is a rigid $(d-1)$-form on $\mathfrak{X}_n$. Furthermore, a simple computation shows that $d\theta_j = \eta_j$. Indeed, a typical term in the series for θ_j is

$$\frac{c_{\underline{m}}}{1 - s_j(\underline{m})} h_m (v-u)^{\underline{m}} g_*((-1)^j \Xi_{\beta_j} d\Xi_{\beta_0} \wedge \cdots \wedge \widehat{d\Xi_{\beta_j}} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}) . \tag{4}$$

Let

$$H_{\underline{m}} := \left(\frac{\Xi_{\mu(\underline{m})}}{\Xi_{\beta_0} \cdots \Xi_{\beta_{j-1}} \Xi_{\beta_j}^{s_j(\underline{m})} f_{j+1}(u, \cdot)^{s_{j+1}(\underline{m})} \cdots f_{d-1}(u, \cdot)^{s_{d-1}(\underline{m})}} \right),$$

so that $h_{\underline{m}} = g_* H_{\underline{m}}$. Then the term in (4) is

$$(-1)^j \frac{c_{\underline{m}}}{1 - s_j(\underline{m})} (v-u)^{\underline{m}} g_*(H_{\underline{m}} \Xi_{\beta_j} d\Xi_{\beta_0} \wedge \cdots \wedge \widehat{d\Xi_{\beta_j}} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}) .$$

We leave it as an exercise to verify that applying d to this expression one obtains the term

$$c_{\underline{m}} h_m (v-u)^{\underline{m}} d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}} .$$

iii. By Cor. 8 and [ST] Cor. 40, a form $\eta \in \Omega^d(\mathfrak{X})$, restricted to $\mathfrak{X}_n$, may be written

$$(\eta | \mathfrak{X}_n) = \eta_0 + \eta_1$$

where

$$\eta_1 \in \Omega^d(\mathfrak{X}_n)^1 \cap d(\Omega^{d-1}(\mathfrak{X}_n))$$

and η_0 is (the restriction of) a finite sum of logarithmic forms $u_*\xi$. Thus the image of

$$\Omega^d(\mathfrak{X}) \to \Omega^d(\mathfrak{X}_n)/(\Omega^d(\mathfrak{X}_n)^1 \cap d(\Omega^{d-1}(\mathfrak{X}_n)))$$

is spanned by logarithmic forms $u_*\xi$. Similarly, from Prop. 3.3 we know that the logarithmic forms $u_*\xi$ generate $\Omega^d(\mathfrak{X})$ as a topological vector space. Since the image of $\Omega^d(\mathcal{X})$ in $\Omega^d(\mathcal{X}_n)$ under restriction is dense the same forms $u_*\xi$ generate the quotient $\Omega^d(\mathcal{X}_n)/\Omega^d(\mathcal{X}_n)^1$ as a Banach space. In both cases we hence may conclude that, using the first assertion, the $u_*\xi$ for $u \in V_n$ and then, using the second assertion, even the $u_*^{(1)}\xi, \ldots, u_*^{(k)}\xi$ span the two vector spaces in question. □

Proof of Theorem 1. — We show that each η in the third space also lies in the intersection of the first two spaces. By Lemma 3, it suffices to show that the restriction of η to $\mathcal{X}_n$ belongs to $E_n := \Omega^d(\mathcal{X}_n)^1 \cap d(\Omega^{d-1}(\mathcal{X}_n)))$ for all $n > 0$. We fix an n and choose a finite set $u^{(1)}, \ldots, u^{(k)}$ of elements of V_n as in Prop. 9. We also choose $m \geq n$ so that the image of $\mathcal{X}_m$ in the building contains the chambers $u^{(\ell)}(\overline{C}, 0)$.

Apply Cor. 8 to write $\eta|\mathcal{X}_m = \eta_0 + \eta_1$ on $\mathcal{X}_m$, with η_0 logarithmic and $\eta_1 \in E_m \subset E_n$. Our hypothesis on m implies that the linear form $\mathrm{Res}_{u^{(\ell)}(\overline{C},0)}$ is continuous on $\Omega^d(\mathcal{X}_m)$, and since η_1 is exact on $\mathcal{X}_m$ we must have

$$\mathrm{Res}_{u^{(\ell)}(\overline{C},0)}(\eta_1) = 0 \text{ for } \ell = 0, \ldots, k.$$

Since all residues of η are zero, we conclude that

$$\mathrm{Res}_{u^{(\ell)}(\overline{C},0)}(\eta_0) = 0 \text{ for } \ell = 0, \ldots, k.$$

We now need to show that, under our residue hypothesis, the restriction to $\mathcal{X}_n$ of the logarithmic form η_0 belongs to E_n. Since η_0 is a logarithmic form, we may write it as a sum of forms $u_*\xi$ with $u \in U$ ([ST] Cor. 40), and for our purposes we may (by Prop. 9.i) assume that all $u \in V_n$. Thus, for each ℓ, we have finitely many distinct $v_{\ell j} \in V_n$ and constants $c_{\ell j}$ so that

$$\eta_0 = \sum_{\ell=1}^{k} \sum_{j=0}^{s_\ell} c_{\ell j}((v_{\ell j})_*\xi)$$

where, for $j = 0, \ldots, s_\ell$, we have $v_{\ell j} \in D(u^{(\ell)}, r)$. By the proofs of Facts A and B of [ST], page 430-431, we see that

$$\mathrm{Res}_{u^{(\ell)}(\overline{C},0)}(\eta_0) = \sum_{j=0}^{s_\ell} \mathrm{Res}_{u^{(\ell)}(\overline{C},0)} c_{\ell j}((v_{\ell j})_*\xi) = \sum_{j=0}^{s_\ell} c_{\ell j} = 0.$$

It then follows from Prop. 9.ii that

$$\begin{aligned} \eta_0|\mathcal{X}_n &\equiv \sum_{\ell=1}^{k} \sum_{j=0}^{s_\ell} c_{\ell j} u_*^{(\ell)}\xi \pmod{E_n} \\ &\equiv 0 \pmod{E_n} \end{aligned}$$

as claimed. □

From section 3, in particular Lemma 3.5 and Fact 2, we have the injective G-equivariant map

$$\begin{array}{rcl} I_\circ : \Omega^d(\mathcal{X})' & \hookrightarrow & C_\circ(U, K) \cong C(G/P, K)/C_{\mathrm{inv}}(G/P, K) \\ \lambda & \longmapsto & [u \mapsto \lambda(u_*\xi)] \,. \end{array}$$

Let $C^\infty(G/P,K) \subseteq C(G/P,K)$ denote the subspace of all locally constant functions and put $C^\infty_{\mathrm{inv}}(G/P,K) := C^\infty(G/P,K) \cap C_{\mathrm{inv}}(G/P,K)$ and $C^\infty_{\mathrm{o}}(U,K) := C^\infty(U,K) \cap C_{\mathrm{o}}(U,K)$. The quotient

$$\mathrm{St} := C^\infty(G/P,K)/C^\infty_{\mathrm{inv}}(G/P,K)$$

is an irreducible smooth G-representation known as the Steinberg representation of the group G. The above isomorphism for the target of I_{o} restricts to an isomorphism

$$C^\infty_{\mathrm{o}}(U,K) \;\cong\; \mathrm{St}\ .$$

Proposition 7.10. — *If $\lambda \in \Omega^d(\mathfrak{X})'$ vanishes on exact forms then the function $I_{\mathrm{o}}(\lambda)$ on U is locally constant with compact support.*

Proof. — Such a linear form λ extends continuously to $\Omega^d(\mathfrak{X}_n)$ for some n. Since, by Thm. 1, it vanishes on $\Omega^d(\mathfrak{X})^1$, it vanishes on $\Omega^d(\mathfrak{X}_n)^1$. Then from Prop. 9.i it vanishes on $u_*\xi$ outside of V_n, and therefore the function in question is compactly supported. Prop. 9.ii shows that there is a finite disjoint covering of V_n by sets $D(u^{(\ell)},r)$ such that $\lambda(v_*\xi) = \lambda(u^{(\ell)}_*\xi)$ for $v \in D(u^{(\ell)},r)$. Therefore the function in question is locally constant. ☐

It follows that I_{o} induces an injective G-equivariant map

$$[\Omega^d(\mathfrak{X})/{}^{\text{exact}}_{\text{forms}}]' \;\hookrightarrow\; C^\infty_{\mathrm{o}}(U,K) \cong \mathrm{St}\ .$$

Since $\mathrm{Res}_{(\overline{C},0)}\xi$ is nonzero the left hand side contains a nonzero vector. But the right hand side is algebraically irreducible as a G-representation. Hence we see that this map must be bijective.

From our Theorem 1 and from the nonarchimedean version of [Kom] Thm. 3 we have the identifications of locally convex vector spaces

$$\Omega^d(\mathfrak{X})/{}^{\text{exact}}_{\text{forms}} \;=\; \Omega^d(\mathfrak{X})/\Omega^d(\mathfrak{X})^1 \;=\; \varprojlim_n \Omega^d(\mathfrak{X}_n)/\Omega^d(\mathfrak{X}_n)^1\ .$$

On the other hand, Prop. 9 says that, for any $n > 0$, the space $\Omega^d(\mathfrak{X}_n)/\Omega^d(\mathfrak{X}_n)^1$ is finite dimensional. We conclude that $\Omega^d(\mathfrak{X})/{}^{\text{exact}}_{\text{forms}}$, resp. its dual space, is a projective, resp. injective, limit of finite dimensional Hausdorff spaces. In particular the topology on $[\Omega^d(\mathfrak{X})/{}^{\text{exact}}_{\text{forms}}]'$ is the finest locally convex topology. In this way we have computed the top step of our filtration as a topological vector space.

Theorem 7.11. — *The G-equivariant map*

$$\begin{array}{rcl} [\Omega^d(\mathfrak{X})/{}^{\text{exact}}_{\text{forms}}]' & \xrightarrow{\cong} & \mathrm{St} \\ \lambda & \longmapsto & [u \mapsto \lambda(u_*\xi)] \end{array}$$

is an isomorphism; morover, the topology of the strong dual on the left hand side is the finest locally convex topology.

8. The partial boundary value maps

In this section we will introduce and study, for any $0 \le j \le d$, a "partial boundary value map" $I^{[j]}$ from $[\Omega^d(\mathcal{X})^j/\Omega^d(\mathcal{X})^{j+1}]'$ into a space of functions on G. Recall that we denoted by $\mathfrak{p}_J$, for any subset $J \subseteq \{0,\ldots,d\}$, the parabolic subalgebra in $\mathfrak{g}$ of all matrices which have a zero entry in position (i,j) for $i \in J$ and $j \notin J$; moreover $\mathfrak{n}_J \subseteq \mathfrak{p}_J$ denoted the unipotent radical. Let $P_J \subseteq G$ be the parabolic subgroup whose Lie algebra is $\mathfrak{p}_J$ and let $U_J \subseteq P_J$ be its unipotent radical. We have the Levi decomposition $P_J = U_J L_J$ with $L_J := L'(J) \times L(J)$ and

$$\begin{aligned} L'(J) :=\ & \text{all matrices in } G \text{ with} \\ & \text{– a zero entry in position } (i,j) \\ & \quad \text{for } i \neq j \text{ and not both in } J, \text{ and} \\ & \text{– an entry 1 in position } (i,i) \text{ for } i \notin J \end{aligned}$$

and

$$\begin{aligned} L(J) :=\ & \text{all matrices in } G \text{ with} \\ & \text{– a zero entry in position } (i,j) \\ & \quad \text{for } i \neq j \text{ and } i \text{ or } j \in J, \text{ and} \\ & \text{– an entry 1 in position } (i,i) \text{ for } i \in J. \end{aligned}$$

Clearly, $L'(J) \cong GL_{\#J}(K)$ and $L(J) \cong GL_{d+1-\#J}(K)$. With these new notations, the subgroup $U(J)$ from section 6 is the subgroup $U(J) = U \cap L(J)$ of lower triangular unipotent matrices in $L(J)$, and $\mathfrak{l}_J, \mathfrak{l}'(J)$, and $\mathfrak{l}(J)$ are the Lie algebras of L_J, $L'(J)$, and $L(J)$ respectively. In the following we are mostly interested in the subsets $\underline{j} := \{0,\ldots,j-1\}$ for $0 \le j \le d$. Let

$$\begin{aligned} V_j :=\ & \text{closed subspace of } \Omega^d(\mathcal{X})^j/\Omega^d(\mathcal{X})^{j+1} \text{ spanned by} \\ & \text{the forms } g_*(\Xi_\mu \xi) \text{ for } \mu \in B(\underline{j}) \text{ and } g \in L(\underline{j}) \end{aligned}$$

viewed as a locally convex vector space with respect to the subspace topology.

Lemma 8.1

i. *The subgroup $P_{\underline{j}}$ preserves V_j;*
ii. *$U_{\underline{j}} L'(\underline{j})$ acts through the determinant character on V_j.*

Proof. — Only the second assertion requires a proof. We have $\Xi_\mu \xi = \Xi_{\mu-\beta} d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$. For $\mu \in B(\underline{j})$ the product $\Xi_{\mu-\beta}$ does not contain any Ξ_i for $i \in \underline{j}$. On the other hand the elements $h \in U_{\underline{j}} L'(\underline{j})$ have columns i for $i \notin \underline{j}$ consisting of zeroes except the entry 1 in position (i,i). It follows that $h_* \Xi_{\mu-\beta} = \Xi_{\mu-\beta}$ for those h and μ. And on $d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$ such an h acts through multiplication by $\det(h)$ (see the last formula on p. 416 in [ST]). Since $U(\underline{j})$ normalizes $U_{\underline{j}} L'(\underline{j})$ we more generally obtain $h_*(u_*(\Xi_\mu \xi)) = \det(h) \cdot u_*(\Xi_\mu \xi)$ for $h \in U_{\underline{j}} L'(\underline{j}), u \in U(\underline{j})$, and $\mu \in B(\underline{j})$. □

In order to compute the space V_j we use the rigid analytic morphism

$$\begin{array}{rccc} \mathrm{pr}_j : & \mathcal{X} & \longrightarrow & \mathcal{X}^{d+1-j} \\ & q = [q_0 : \cdots : q_d] & \longmapsto & [q_j : \cdots : q_d]; \end{array}$$

here $\mathcal{X}^{d+1-j}$ denotes the p-adic symmetric space of the group $GL_{d+1-j}(K)$. This morphism is $P_{\underline{j}}$-equivariant if $P_{\underline{j}}$ acts on $\mathcal{X}$, resp. on $\mathcal{X}^{d+1-j}$, through the inclusion $P_{\underline{j}} \subseteq G$, resp. the projection $P_{\underline{j}} \twoheadrightarrow L(\underline{j}) \cong GL_{d+1-j}(K)$. In section 4 we introduced the irreducible $\mathfrak{p}_{\underline{j}}$-submodule $M_{\underline{j}}$ of $\mathfrak{b}_{\underline{j}}/\mathfrak{b}_{\underline{j}}^{>}$. For general reasons, it integrates to a rational representation of $P_{\underline{j}}$. We will work with the following explicit model for this representation. Consider an element $g = (g_{rs}) \in L(\underline{j})$. The adjoint action of g^{-1} on any $L_{i\ell} \in \mathfrak{n}_{\underline{j}}^{+}$, i.e., with $0 \leq i < j \leq \ell \leq d$, is given by

$$\mathrm{ad}(g^{-1})L_{i\ell} = g_{\ell j}L_{ij} + \cdots + g_{\ell d}L_{id}.$$

We may deduce from this that the adjoint action of $L(\underline{j})$ on $U(\mathfrak{g})$ preserves $M_{\underline{j}}^{\circ}$ as well as $M_{\underline{j}}^{\circ} \cap \mathfrak{b}_{\underline{j}}^{>} = U(\mathfrak{n}_{\underline{j}}^{+}) \cap \mathfrak{b}$. Indeed, the sorting relations $L_{ik}L_{i'\ell} - L_{i\ell}L_{i'k}$ generate $U(\mathfrak{n}_{\underline{j}}^{+}) \cap \mathfrak{b}_{\underline{j}}^{>} = U(\mathfrak{n}_{\underline{j}}^{+}) \cap \mathfrak{b}$ according to Prop. 4.6, and the image of such a relation $\mathrm{ad}(g^{-1})(L_{ik}L_{i'\ell} - L_{i\ell}L_{i'k})$ is a linear combination of sorting relations of the same type involving only i and i' as first indices. It follows that

$$g(\mathfrak{z} + \mathfrak{b}_{\underline{j}}^{>}) := \mathrm{ad}(g)(\mathfrak{z}) + \mathfrak{b}_{\underline{j}}^{>} \ \text{ for } \mathfrak{z} \in M_{\underline{j}}^{\circ}$$

is a well defined action of the group $L(\underline{j})$ on the space $M_{\underline{j}}$. We extend this to a rational representation of $P_{\underline{j}}$ by letting $U_{\underline{j}}L'(\underline{j})$ act through the determinant character. The corresponding derived action of the Lie algebra $\mathfrak{p}_{\underline{j}}$ on $M_{\underline{j}}$ is trivial on $\mathfrak{n}_{\underline{j}}$, is through the trace character on $\mathfrak{l}'(\underline{j})$, and on $\mathfrak{l}(\underline{j})$ is induced by the adjoint action. But in Lemma 4.7 we have seen that this latter action coincides with the left multiplication action.

We now consider the continuous linear map

$$\begin{array}{rcl} A_j : \Omega^{d-j}(\mathcal{X}^{d+1-j}) \underset{K}{\otimes} M_{\underline{j}} & \longrightarrow & \Omega^d(\mathcal{X})/\Omega^d(\mathcal{X})^{j+1} \\ \eta \otimes (L_\mu + \mathfrak{b}_{\underline{j}}^{>}) & \longmapsto & L_\mu \left(\frac{d\Xi_{\beta_0}}{\Xi_{\beta_0}} \wedge \cdots \wedge \frac{d\Xi_{\beta_{j-1}}}{\Xi_{\beta_{j-1}}} \wedge \mathrm{pr}_j^*(\eta) \right). \end{array}$$

According to Lemma 1.3 the $P_{\underline{j}}$-action on both sides (diagonally on the left side) is continuous. In the following we will use the abbreviations

$$\xi_{d-j} := \frac{d\Xi_{\beta_j}}{\Xi_{\beta_j}} \wedge \cdots \wedge \frac{d\Xi_{\beta_{d-1}}}{\Xi_{\beta_{d-1}}} \ \text{ as a } (d-j)\text{-form on } \mathcal{X}^{d+1-j}$$

and

$$\xi^{(j)} := \frac{d\Xi_{\beta_0}}{\Xi_{\beta_0}} \wedge \cdots \wedge \frac{d\Xi_{\beta_{j-1}}}{\Xi_{\beta_{j-1}}} \ \text{ as a } j\text{-form on } \mathcal{X}.$$

For $g \in G$ we have

$$g_*\xi = \det(g) \cdot \left(\prod_{i=0}^{d} \frac{\Xi_i}{g_*\Xi_i} \right) \cdot \xi.$$

For $g \in L(\underline{j})$ there is a corresponding formula for $g_*\xi_{d-j}$ and the two together imply

$$g_*\xi = \xi^{(j)} \wedge \mathrm{pr}_j^*(g_*\xi_{d-j}).$$

By Prop. 3.3 the $u_*\xi_{d-j}$ for $u \in U(\underline{j})$ generate a dense subspace in $\Omega^{d-j}(\mathfrak{X}^{d+1-j})$. After we establish various properties of the map A_j, this fact will allow us to assume that $\eta = u_*\xi_{d-j}$ for some $u \in U(\underline{j})$. First of all we note that the definition of A_j is independent of the particular representative L_μ for the coset $L_\mu + \mathfrak{b}_{\underline{j}}^{>}$ as long as this representative is chosen in $M_{\underline{j}}^{\mathrm{o}}$: For $\mathfrak{z} \in M_{\underline{j}}^{\mathrm{o}} \cap \mathfrak{b}_{\underline{j}}^{>} = U(\mathfrak{n}_{\underline{j}}^{+}) \cap \mathfrak{b}$ and $u \in U(\underline{j})$ we have $\mathrm{ad}(u^{-1})(\mathfrak{z}) \in \mathfrak{b}$ and consequently

$$\mathfrak{z}(\xi^{(j)} \wedge \mathrm{pr}_j^*(u_*\xi_{d-j})) = \mathfrak{z}(u_*\xi) = u_*([\mathrm{ad}(u^{-1})(\mathfrak{z})]\xi) = 0.$$

Next we compute

$$\begin{aligned} A_j(g_*h_*\xi_{d-j} \otimes \mathrm{ad}(g)(L_\mu)) &= [\mathrm{ad}(g)(L_\mu)](\xi^{(j)} \wedge \mathrm{pr}_j^*(g_*h_*\xi_{d-j})) \\ &= [\mathrm{ad}(g)(L_\mu)](g_*h_*\xi) \\ &= g_*(L_\mu(h_*\xi)) \\ &= g_*(L_\mu(\xi^{(j)} \wedge \mathrm{pr}_j^*(h_*\xi_{d-j}))) \\ &= g_*(A_j(h_*\xi_{d-j} \otimes L_\mu)) \end{aligned}$$

for $g, h \in L(\underline{j})$. This shows that the map A_j is $L(\underline{j})$-equivariant. As special cases of the above identity we have

$$A_j(g_*\xi_{d-j} \otimes L_\mu) = g_*([\mathrm{ad}(g^{-1})(L_\mu)]\xi)$$

and

$$A_j(g_*\xi_{d-j} \otimes \mathrm{ad}(g)(L_\mu)) = g_*(L_\mu\xi) = -g_*(\Xi_\mu\xi)$$

for $g \in L(\underline{j})$ and $\mu \in B(\underline{j})$. The former, together with the fact that $M_{\underline{j}}^{\mathrm{o}} \cdot \xi \subseteq \sum_{\mu \in B(\underline{j})} K \cdot \Xi_\mu\xi$, shows that the image of A_j is contained in V_j. The latter shows that this image is dense in V_j. By Lemma 1.ii, the group $U_{\underline{j}}L'(\underline{j})$ acts on the domain of A_j, as well as on V_j, through the determinant character. Hence A_j in fact is $P_{\underline{j}}$-equivariant.

By Thm. 7.1 the exact $(d-j)$-forms on $\mathfrak{X}^{d+1-j}$ coincide with the subspace $\Omega^{d-j}(\mathfrak{X}^{d+1-j})^1$. According to Cor. 6.3, this latter space is topologically generated, as an $L(\underline{j})$-representation, by the forms $\Xi_\nu\xi_{d-j}$ for the weights $0 \neq \nu = \sum_{k=j}^{d} n_k\varepsilon_k \in X^*(\overline{T})$. We have $A_j(\Xi_\nu\xi_{d-j} \otimes L_\mu) = L_\mu(\Xi_\nu\xi)$. Let $\mu = \varepsilon_0 + \cdots + \varepsilon_{j-1} - \sum_{k=j}^{d} m_k\varepsilon_k$ with $m_k \geq 0$. By an iteration of the formula (+) in section 4 one has

$$L_\mu(\Xi_\nu\xi) = c(\mu) \cdot \Big(\prod_{k=j}^{d} \prod_{m=1}^{m_k} (n_k - m)\Big) \cdot \Xi_{\mu+\nu}\xi$$

for some constant $c(\mu) \in K^\times$. There are two cases to distinguish. If $n_k \leq m_k$ for all $j \leq k \leq d$ then we choose a $j \leq \ell \leq d$ such that $n_\ell \geq 1$ and see that the product on the right hand side of the above identity contains the factor 0. Hence $L_\mu(\Xi_\nu\xi) = 0$ in this case. Otherwise there is some $j \leq \ell \leq d$ such that $m_\ell < n_\ell$. Then $J(\mu+\nu) \supseteq \{0,\ldots,j-1,\ell\}$ so that, by Cor. 6.3, $\Xi_{\mu+\nu}\xi$ and hence $L_\mu(\Xi_\nu\xi)$ lies in $\Omega^d(\mathfrak{X})^{j+1}$. This shows that $\substack{\text{exact}\\ \text{forms}} \otimes M_{\underline{j}}$ lies in the kernel of A_j.

Remark 8.2. — V_j is topologically generated by the forms $u_*(\Xi_\mu\xi)$ for $u \in U(\underline{j})$ and $\mu \in B(\underline{j})$.

Proof. — We in fact will show that in $\Omega^d(\mathfrak{X})$ any form $g_*(\Xi_\mu\xi)$ with $g \in L(\underline{j})$ and $\mu \in B(\underline{j})$ is a (finite) linear combination of forms $u_*(\Xi_\nu\xi)$ with $u \in U(\underline{j})$ and $\nu \in B(\underline{j})$. First of all we have

$$g_*(\Xi_\mu\xi) = -g_*(L_\mu\xi) = -[\mathrm{ad}(g)(L_\mu)](g_*\xi).$$

From the discussion after Lemma 3.5 we know that $g_*\xi_{d-j}$ is an alternating sum of forms $u_*\xi_{d-j}$ with $u \in U(\underline{j})$. Using the identity $g_*\xi = \xi^{(j)} \wedge \mathrm{pr}_j^*(g_*\xi_{d-j})$ again we see that $g_*\xi$ is an alternating sum of forms $u_*\xi$ with $u \in U(\underline{j})$. Inserting this into the above equation we are reduced to treating a form $[\mathrm{ad}(g)(L_\mu)](u_*\xi) = u_*([\mathrm{ad}(u^{-1}g)(L_\mu)]\xi)$. But $\mathrm{ad}(u^{-1}g)(L_\mu)$ lies in $\sum_{\nu\in B(\underline{j})} K \cdot L_\nu + \mathfrak{b}$. $\square$

Proposition 8.3. — *The linear map A_j induces a $P_{\underline{j}}$-equivariant topological isomorphism*

$$[\Omega^{d-j}(\mathfrak{X}^{d+1-j})/\substack{\text{exact}\\ \text{forms}}] \underset{K}{\otimes} M_{\underline{j}} \xrightarrow{\cong} V_j.$$

Proof. — So far we know that A_j induces a continuous $P_{\underline{j}}$-equivariant map with dense image between the two sides in the assertion. For simplicity we denote this latter map again by A_j. Both sides are Fréchet spaces (the left hand side as a consequence of Thm. 7.1). We claim that it suffices to show that the dual map A_j' is surjective. We only sketch the argument since it is a straightforward nonarchimedean analog of [B-TVS] IV.28, Prop. 3. Let us assume A_j' to be surjective for the moment being. The Hahn-Banach theorem ([Tie] Thm. 3.6) then immediately implies that A_j is injective. Actually $A_j' : V_j' = \mathrm{im}(A_j)' \xrightarrow{\cong} V'$ then is a linear bijection where we abbreviate by V the space on the left hand side of the assertion. This means that A_j' induces a topological isomorphism $\mathrm{im}(A_j)_s' \to V_s'$ between the weak dual spaces. Since the Mackey topology ([Tie] p. 282) is defined in terms of the weak dual it follows that $A_j : V \to \mathrm{im}(A_j)$ is a homeomorphism for the Mackey topologies. But on metrizable spaces the Mackey topology coincides with the initial topology ([Tie] Thm. 4.22). Therefore $A_j : V \xrightarrow{\cong} \mathrm{im}(A_j)$ is a topological isomorphism for the initial topologies. With V also $\mathrm{im}(A_j)$ then is complete. Because of the density we have to have $\mathrm{im}(A_j) = V_j$. $\square$

Before we establish the surjectivity of A_j' we interrupt the present proof in order to discuss the strong dual of the left hand side in our assertion.

Let

$$\mathrm{St}_{d+1-j} := C^\infty(L(\underline{j})/L(\underline{j}) \cap P, K)/C^\infty_{\mathrm{inv}}(L(\underline{j})/L(\underline{j}) \cap P, K)$$

denote the Steinberg representation of the group $L(\underline{j})$ equipped with the finest locally convex topology (in particular, St_1 is the trivial character of the group $K^\times$). Recall that identifying $U(\underline{j})$ with the big cell in $L(\underline{j})/L(\underline{j}) \cap P$ induces an isomorphism $\mathrm{St}_{d+1-j} \cong C^\infty_{\mathrm{o}}(U(\underline{j}), K)$. We know from Thm. 7.11 that

$$\begin{array}{rcl} [\Omega^{d-j}(\mathfrak{X}^{d+1-j})/{\scriptstyle\text{exact}\atop\scriptstyle\text{forms}}]' & \xrightarrow{\cong} & C^\infty_{\mathrm{o}}(U(\underline{j}), K) \cong \mathrm{St}_{d+1-j} \\ \lambda & \longmapsto & [u \mapsto \lambda(u_*\xi_{d-j})] \end{array}$$

is a $L(\underline{j})$-equivariant topological isomorphism. In particular, the strong dual of the left hand side in Prop. 3 carries the finest locally convex topology and may be identified with the space $\mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})$ of all K-linear maps from $M_{\underline{j}}$ into St_{d+1-j}. With this identification, the map A'_j becomes the map

$$\begin{array}{rcl} I^{[j]}_{\mathrm{o}} : V'_j & \longrightarrow & \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j}) \\ \lambda & \longmapsto & \{L_\mu \mapsto [u \mapsto \lambda(L_\mu(u_*\xi))]\} \end{array}$$

and ist surjectivity will be proved in the course of the proof of Prop. 4 below.

Recall that $M_{\underline{j}}$ is isomorphic to the contragredient of the j-th symmetric power $\mathrm{Sym}^j(K^{d+1-j})$ of the standard representation of $L(\underline{j}) \cong GL_{d+1-j}(K)$ on K^{d+1-j}.

Proposition 8.4

i. *V_j is a reflexive Fréchet space;*
ii. *the linear map*

$$\begin{array}{rcl} I^{[j]}_{\mathrm{o}} : V'_j & \xrightarrow{\cong} & \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j}) \\ \lambda & \longmapsto & \{L_\mu \longmapsto [u \mapsto \lambda(L_\mu(u_*\xi))]\} \end{array}$$

is a $P_{\underline{j}}$-equivariant isomorphism;
iii. *the topology of V'_j is the finest locally convex one;*
iv. *$V'_j \cong \mathrm{St}_{d+1-j} \underset{K}{\otimes} \mathrm{Sym}^j(K^{d+1-j})$ (with $U_{\underline{j}}L'(\underline{j})$ acting on the right hand side through the inverse of the determinant character);*
v. *$V_j \cong \mathrm{Hom}_K(\mathrm{St}_{d+1-j}, M_{\underline{j}})$ (with the weak topology on the right hand side).*

Proof. — The first assertion follows by the same argument as for Prop. 6.5. The only other point to establish is the surjectivity of $I^{[j]}_{\mathrm{o}}$. This then settles Prop. 3 which in turn implies the rest of the present assertions by dualizing.

Let $\varphi \in C^\infty_{\mathrm{o}}(U(\underline{j}), K) \cong \mathrm{St}_{d+1-j}$ denote the characteristic function of the compact open subgroup $U(\underline{j}) \cap B$ in $U(\underline{j})$. Since St_{d+1-j} is an irreducible (in the algebraic sense) $L(\underline{j})$-representation it is generated by φ as a $L(\underline{j})$-representation. Hence the finitely many linear maps

$$\begin{aligned} E_\mu : M_{\underline{j}} &\longrightarrow \mathrm{St}_{d+1-j} \\ L_\nu &\longmapsto \begin{cases} \varphi & \text{if } \nu = \mu, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for $\mu \in B(\underline{j})$ generate $\mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})$ as a $L(\underline{j})$-representation. For the surjectivity of $I_{\mathrm{o}}^{[j]}$ it therefore suffices, by $L(\underline{j})$-equivariance, to find a preimage for each E_μ. At the beginning of section 5, we introduced the continuous linear forms

$$\eta \longmapsto \mathrm{Res}_{(\overline{C},0)} \Xi_{-\mu}\eta$$

on $\Omega^d(\mathfrak{X})$ for any $\mu \in X^*(\overline{T})$. In terms of the pairing $\langle , \rangle$ defined before Prop. 5.3 this linear form is given as

$$\eta \longmapsto \langle \eta | U^{\mathrm{o}}, f_\mu | B \rangle.$$

We now fix a $\mu \in B(\underline{j})$. Since $f_\mu | B$ has weight $-\mu$ we have $(L_\nu(f_\mu|B))(1) = 0$ for all $\nu \neq \mu$ (compare the proof of Prop. 5.2); in particular $(\mathfrak{z}(f_\mu|B))(1) = 0$ for any $\mathfrak{z} \in \mathfrak{b}_{j+1}$. Taylor's formula then implies that

$$f_\mu | B \in \mathcal{O}(B)^{\mathfrak{b}_{j+1}=0}.$$

By Lemma 6.4, the above linear form vanishes on $\Omega^d(\mathfrak{X})^{j+1}$ and consequently induces a continuous linear form λ_μ on V_j. We compute

$$I_{\mathrm{o}}^{[j]}(\lambda_\mu)(L_\nu)(u) = \mathrm{Res}_{(\overline{C},0)} \Xi_{-\mu} \cdot L_\nu(u_* \xi) = \mathrm{Res}_{u^{-1}(\overline{C},0)} \theta$$

with

$$\theta := (u^{-1}\Xi_{-\mu}) \cdot (\mathrm{ad}(u^{-1})(L_\nu))(\xi).$$

Since, by Thm. 7.1, forms in $\Omega^d(\mathfrak{X})^1$ have no residues it suffices to determine θ modulo $\Omega^d(\mathfrak{X})^1$. The subspace $M_{\underline{j}}^* := \sum_{\mu \in B(\underline{j})} K \cdot \Xi_{-\mu}$ of $\mathcal{O}(\mathfrak{X})$ is $L(\underline{j})$-invariant. In fact, one easily computes that, for $g = (g_{rs}) \in L(\underline{j})$ and $0 \leq i < j \leq \ell \leq d$, one has

$$g_* \Xi_{-(\varepsilon_i - \varepsilon_\ell)} = g_{j\ell} \Xi_{-(\varepsilon_i - \varepsilon_j)} + \cdots + g_{d\ell} \Xi_{-(\varepsilon_i - \varepsilon_d)}.$$

This formula and our previous formula for $\mathrm{ad}(g^{-1})L_{i\ell}$ together show that the pairing

$$\begin{aligned} M_{\underline{j}} \times M_{\underline{j}}^* &\longrightarrow K \\ (L_\mu + \mathfrak{b}_{\underline{j}}^{>}, \Xi_{-\nu}) &\longmapsto \begin{cases} 1 & \text{if } \mu = \nu, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

is $L(\underline{j})$-equivariant. It therefore exhibits $M_{\underline{j}}^*$ as the $L(\underline{j})$-representation dual to $M_{\underline{j}}$. The point of this pairing is that, by Cor. 6.3, we have $\Xi_{-\mu} \cdot (L_\nu \xi) = -\Xi_{\nu-\mu}\xi \in \Omega^d(\mathfrak{X})^1$ for $\mu \neq \nu$. Applying this together with the equivariance to the above form θ we obtain that

$$\theta \in \begin{cases} -\xi + \Omega^d(\mathfrak{X})^1 & \text{if } \mu = \nu, \\ \Omega^d(\mathfrak{X})^1 & \text{if } \mu \neq \nu \end{cases}$$

and consequently that

$$I_{\mathrm{o}}^{[j]}(\lambda_\mu)(L_\nu)(u) = \begin{cases} -\mathrm{Res}_{u^{-1}(\overline{C},0)}\xi & \text{if } \mu = \nu, \\ 0 & \text{if } \mu \neq \nu. \end{cases}$$

By [ST] Lemma 23 the form ξ has residues only on the standard apartment and those are equal to ± 1. The chamber $u^{-1}\overline{C}$ lies in the standard apartment if and only if u fixes $\overline{C}$. It follows that $I_{\mathrm{o}}^{[j]}(\lambda_\mu)(L_\mu)$ is supported on $U(\underline{j}) \cap B$ where it is a constant function with value ± 1. All in all we see that

$$I_{\mathrm{o}}^{[j]}(\lambda_\mu) = \pm E_\mu$$

(the sign depending on the parity of d). $\square$

The natural $P_{\underline{j}}$-equivariant linear map

$$[\Omega^d(\mathfrak{X})^j/\Omega^d(\mathfrak{X})^{j+1}]' \longrightarrow V_j'$$

is surjective (by Hahn-Banach) and is strict (by the same argument as for Prop. 6.7). Moreover both sides are inductive limits of sequences of Banach spaces (see the proof of Prop. 6.5) and are locally analytic $P_{\underline{j}}$-representations in the sense of Cor. 6.8. Therefore the assumptions of the Frobenius reciprocity theorem 4.2.6 in [Fea] are satisfied and we obtain the G-equivariant continuous linear map

$$\begin{aligned} I^{[j]} : [\Omega^d(\mathfrak{X})^j/\Omega^d(\mathfrak{X})^{j+1}]' &\longrightarrow C^{\mathrm{an}}(G, P_{\underline{j}}; V_j') \\ \lambda &\longmapsto [g \mapsto (g^{-1}\lambda)|V_j]. \end{aligned}$$

Here $C^{\mathrm{an}}(G, P_{\underline{j}}; V_j')$ – the "induced representation in the locally analytic sense" – denotes the vector space of all locally analytic maps $f : G \to V_j'$ such that $f(gh) = h^{-1}(f(g))$ for any $g \in G$ and $h \in P_{\underline{j}}$ on which G acts by left translations. Its natural locally convex topology is constructed in [Fea] 4.1.3 (to avoid confusion we should point out that [Fea] uses a more restrictive notion of a V-valued locally analytic map but which coincides with the notion from Bourbaki provided V is quasi-complete – see loc.cit. 2.1.4 and 2.1.7).

Definition. — *The above map $I^{[j]} : [\Omega^d(\mathfrak{X})^j/\Omega^d(\mathfrak{X})^{j+1}]' \to C^{\mathrm{an}}(G, P_{\underline{j}}; V_j')$ is called the j-th partial boundary value map.*

Lemma 8.5. — *$I^{[j]}$ is injective.*

Proof. — It is an immediate consequence of Cor. 6.3 that $\sum_{g \in G} g(V_j)$ is dense in $\Omega^d(\mathfrak{X})^j/\Omega^d(\mathfrak{X})^{j+1}$. $\square$

In order to describe the image of $I^{[j]}$ we first need to understand in which sense we can impose left invariant differential equations on vectors in an induced representation. For any Hausdorff locally convex K-vector space V the right translation action of G on $C^{\mathrm{an}}(G, V) := C^{\mathrm{an}}(G, \{1\}; V)$ is differentiable and induces an action of $U(\mathfrak{g})$ by left invariant and continuous operators ([Fea] 3.3.4). For $V := V'_j \cong \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})$ we therefore may consider the K-bilinear map

$$\begin{array}{rcl}\langle , \rangle : (U(\mathfrak{g}) \underset{K}{\otimes} M_{\underline{j}}) \times C^{\mathrm{an}}(G, \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})) & \longrightarrow & C^{\mathrm{an}}(G, \mathrm{St}_{d+1-j}) \\ (\mathfrak{z} \otimes m, f) & \longmapsto & [g \mapsto (\mathfrak{z}f)(g)(m)].\end{array}$$

Note that, for a fixed $\mathfrak{Z} \in U(\mathfrak{g}) \underset{K}{\otimes} M_{\underline{j}}$, the "differential operator"

$$\langle \mathfrak{Z}, \rangle : C^{\mathrm{an}}(G, \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})) \longrightarrow C^{\mathrm{an}}(G, \mathrm{St}_{d+1-j})$$

is continuous and G-equivariant (for the left translation actions). The action of $P_{\underline{j}}$ on $\mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j}) = M'_{\underline{j}} \underset{K}{\otimes} \mathrm{St}_{d+1-j}$ is differentiable and the derived action of $\mathfrak{p}_{\underline{j}}$ is given by

$$(\mathfrak{x}E)(m) = -E(\mathfrak{x}m) \tag{1}$$

for $\mathfrak{x} \in \mathfrak{p}_{\underline{j}}, E \in \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})$, and $m \in M_{\underline{j}}$. This is immediate from the fact that any vector in St_{d+1-j} is fixed by an open subgroup of $P_{\underline{j}}$ so that the derived action of $\mathfrak{p}_{\underline{j}}$ on St_{d+1-j} is trivial.

Now recall that the induced representation $C^{\mathrm{an}}(G, P_{\underline{j}}; \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j}))$ is the closed subspace of $C^{\mathrm{an}}(G, \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j}))$ of all those maps f which satisfy $f(gh) = h^{-1}(f(g))$ for $g \in G$ and $h \in P_{\underline{j}}$. For such an f we therefore have

$$\begin{aligned}(\mathfrak{x}f)(g) &= \frac{d}{dt} f(g \exp(t\mathfrak{x}))_{|t=0} \\ &= \frac{d}{dt} \exp(t\mathfrak{x})^{-1}(f(g))_{|t=0} \\ &= -\mathfrak{x}(f(g))\end{aligned}$$

for $\mathfrak{x} \in \mathfrak{p}_{\underline{j}}$ and slightly more generally

$$\begin{aligned}(\mathfrak{z}(\mathfrak{x}f))(g) &= \frac{d}{dt}(\mathfrak{x}f)(g \exp(t\mathfrak{z}))_{|t=0} \\ &= -\frac{d}{dt}\mathfrak{x}(f(g \exp(t\mathfrak{z})))_{|t=0} \\ &= -\mathfrak{x}(\frac{d}{dt} f(g \exp(t\mathfrak{z}))_{|t=0}) \\ &= -\mathfrak{x}((\mathfrak{z}f)(g))\end{aligned} \tag{2}$$

for $\mathfrak{x} \in \mathfrak{p}_{\underline{j}}$ and $\mathfrak{z} \in \mathfrak{g}$; the third equality is a consequence of the continuity of the operator $\mathfrak{x}$. Combining (1) and (2) we obtain

$$(\mathfrak{z}(\mathfrak{x}f))(g)(m) = (-\mathfrak{x}((\mathfrak{z}f)(g)))(m) = ((\mathfrak{z}f)(g))(\mathfrak{x}m)$$

or equivalently

$$\langle \mathfrak{z}\mathfrak{x} \otimes m, f \rangle = \langle \mathfrak{z} \otimes \mathfrak{x}m, f \rangle$$

for $f \in C^{\mathrm{an}}(G, P_{\underline{j}}; \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})), m \in M_{\underline{j}}, \mathfrak{x} \in \mathfrak{p}_{\underline{j}}$, and $\mathfrak{z} \in \mathfrak{g}$. This means that the above pairing restricts to a pairing

$$\langle\, , \,\rangle : (U(\mathfrak{g}) \underset{U(\mathfrak{p}_{\underline{j}})}{\otimes} M_{\underline{j}}) \times C^{\mathrm{an}}(G, P_{\underline{j}}; \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})) \longrightarrow C^{\mathrm{an}}(G, \mathrm{St}_{d+1-j})$$

and enables us to consider, for any subset $\mathfrak{d} \subseteq U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_{\underline{j}})} M_{\underline{j}}$, the G-invariant closed subspace

$$\begin{aligned} &C^{\mathrm{an}}(G, P_{\underline{j}}; \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j}))^{\mathfrak{d}=0} := \\ &\qquad\qquad \{f \in C^{\mathrm{an}}(G, P_{\underline{j}}; \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})) : \langle \mathfrak{Z}, f\rangle = 0 \text{ for any } \mathfrak{Z} \in \mathfrak{d}\}. \end{aligned}$$

The relevant subset for our purposes is the kernel

$$\mathfrak{d}_{\underline{j}} = \ker(U(\mathfrak{g}) \underset{U(\mathfrak{p}_{\underline{j}})}{\otimes} M_{\underline{j}} \longrightarrow \mathfrak{b}_{\underline{j}}/\mathfrak{b}_{\underline{j}}^{>})$$

of the natural surjection sending $\mathfrak{z} \otimes m$ to $\mathfrak{z}m$. By the Poincaré-Birkhoff-Witt theorem the inclusion $U(\mathfrak{n}_{\underline{j}}^{+}) \subseteq U(\mathfrak{g})$ induces an isomorphism $U(\mathfrak{n}_{\underline{j}})^{+} \underset{K}{\otimes} M_{\underline{j}} \xrightarrow{\cong} U(\mathfrak{g}) \underset{U(\mathfrak{p}_{\underline{j}})}{\otimes} M_{\underline{j}}$. We mostly will view $\mathfrak{d}_{\underline{j}}$ as a subspace of $U(\mathfrak{n}_{\underline{j}}^{+}) \underset{K}{\otimes} M_{\underline{j}}$.

Theorem 8.6. — *The map $I^{[j]}$ (together with $I_{\mathrm{o}}^{[j]}$) induces a G-equivariant topological isomorphism*

$$\begin{array}{rcl} I^{[j]} : [\Omega^d(\mathfrak{X})^j/\Omega^d(\mathfrak{X})^{j+1}]' & \xrightarrow{\cong} & C^{\mathrm{an}}(G, P_{\underline{j}}; \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j}))^{\mathfrak{d}_{\underline{j}}=0} \\ \lambda & \longmapsto & [g \mapsto I_{\mathrm{o}}^{[j]}((g^{-1}\lambda)|V_j)]. \end{array}$$

Proof. — We start by showing that the image of $I^{[j]}$ satisfies the relations $\mathfrak{d}_{\underline{j}} = 0$. Let $\mathfrak{Z} = \sum_{\mu \in B(\underline{j})} \mathfrak{z}_{(\mu)} \otimes L_\mu \in \mathfrak{d}_{\underline{j}} \subseteq U(\mathfrak{n}_{\underline{j}}^{+}) \otimes_K M_{\underline{j}}$; then $\mathfrak{z} = \sum_\mu \mathfrak{z}_{(\mu)} L_\mu \in U(\mathfrak{n}_{\underline{j}}^{+}) \cap \mathfrak{b}_{\underline{j}}^{>} = U(\mathfrak{n}_{\underline{j}}^{+}) \cap \mathfrak{b}$ (Prop. 4.6.iii). Note that

$$[I^{[j]}(\lambda)(g)](L_\mu)(u) = (g^{-1}\lambda)(L_\mu(u_*\xi)) = \lambda(g_*(L_\mu(u_*\xi)))$$

for $g \in G, \mu \in B(\underline{j})$, and $u \in U(\underline{j})$. We compute

$$\begin{aligned} \langle \mathfrak{Z}, I^{[j]}(\lambda)\rangle(g)(u) &= \sum_\mu [(\mathfrak{z}_{(\mu)}(I^{[j]}(\lambda)))(g)](L_\mu)(u) \\ &= \sum_\mu \lambda(g_*(\mathfrak{z}_{(\mu)} L_\mu(u_*\xi))) \\ &= \lambda(g_*(\mathfrak{z}(u_*\xi))) \\ &= \lambda(g_* u_*((\mathrm{ad}(u^{-1})(\mathfrak{z}))\xi)) \end{aligned}$$

which is zero because $U(\mathfrak{n}_{\underline{j}}^{+}) \cap \mathfrak{b}$ is $\mathrm{ad}(U(\underline{j}))$-invariant as we have seen earlier in this section.

We know already that $I^{[j]}$ is continuous, G-equivariant, and injective. Next we establish surjectivity. Let f be a map in the right hand side of the assertion. By a series of simplifications we will show that it suffices to consider an f of a very particular form for which we then will exhibit an explicit preimage under $I^{[j]}$.

We show first that we may assume that

– f is supported on $BP_{\underline{j}}$ and

– $f|U^+_{\underline{j}} \cap B$ is analytic (not merely locally analytic).

By the Iwasawa decomposition we have the finite disjoint open covering

$$G/P_{\underline{j}} = \dot{\bigcup_g} gBP_{\underline{j}}/P_{\underline{j}}$$

where g runs through a set of representatives for the cosets in $GL_{d+1}(o)/B$. As before let $U^+_{\underline{j}}$ denote the transpose of $U_{\underline{j}}$. Then $U^{(0)}_{\underline{j}} := U^+_{\underline{j}} \cap B$ is the congruence subgroup of all matrices in $U^+_{\underline{j}}$ whose non-diagonal entries are integral multiples of π. Consider the higher congruence subgroups $U^{(n)}_{\underline{j}}$, for $n \geq 0$, of all matrices in $U^+_{\underline{j}}$ whose non-diagonal entries are integral multiples of π^n. These $U^{(n)}_{\underline{j}}$ are polydisks in an obvious way, and we have $U^{(n)}_{\underline{j}} = y^n(U^+_{\underline{j}} \cap B)y^{-n}$ where $y \in G$ is the diagonal matrix with entries $(\pi, \ldots, \pi, 1, \ldots, 1)$. The Iwahori decomposition for B implies that the map

$$\begin{array}{ccc} gU^{(0)}_{\underline{j}} & \xrightarrow{\sim} & gBP_{\underline{j}}/P_{\underline{j}} \\ gu & \longmapsto & guP_{\underline{j}} \end{array}$$

is a homeomorphism. Our map f restricted to $gU^{(0)}_{\underline{j}}$ still only is locally analytic. But we find a sufficiently big $n \in \mathbb{N}$ such that $f|ghU^{(n)}_{\underline{j}}$ is analytic for all g as above and all h in a system of representatives for the cosets in $U^{(0)}_{\underline{j}}/U^{(n)}_{\underline{j}}$. If we put

$$f_{g,h} := ((gh)^{-1}f)|U^{(n)}_{\underline{j}}P_{\underline{j}} \text{ extended by zero to } G$$

then these maps lie in the right hand side of our assertion and we have

$$f = \sum_{g,h} (gh)f_{g,h}.$$

The reason for this of course is that

$$G = \dot{\bigcup_{g,h}} ghU^{(n)}_{\underline{j}}P_{\underline{j}}$$

is a disjoint finite open covering. By linearity and G-equivariance of $I^{[j]}$ it therefore suffices to find a preimage for each $f_{g,h}$. This means we may assume that our map f is supported on $U^{(n)}_{\underline{j}}P_{\underline{j}}$ and is analytic on $U^{(n)}_{\underline{j}}$. Using G-equivariance again, we may translate f by y^{-n} so that it has the desired properties.

For our next reduction, we will show that we may further assume that

– f is supported on $BP_{\underline{j}}$ with $f|U^+_{\underline{j}} \cap B = \varepsilon \otimes \varphi$ for some $\varepsilon \in \mathcal{O}(U^+_{\underline{j}} \cap B, M'_{\underline{j}})^{\mathfrak{d}_{\underline{j}}=0}$ and $\varphi \in \mathrm{St}_{d+1-j}$.

If we consider an analytic map on $U^+_{\underline{j}} \cap B$ with values in the locally convex vector space $\mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})$ then the coefficients in its power series expansion multiplied by appropriate powers of π form a bounded subset of $\mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})$. The topology of that vector space is the finest locally convex one. Hence any bounded subset and therefore the set of coefficients lies in a finite dimensional subspace. This means that our $f|U^+_{\underline{j}} \cap B$ is an element of $\mathcal{O}(U^+_{\underline{j}} \cap B) \otimes_K \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})$. Moreover, viewing $\mathfrak{d}_{\underline{j}}$ as a subspace of $U(\mathfrak{n}^+_{\underline{j}}) \otimes_K M_{\underline{j}}$ it is clear that with respect to the obvious pairing

$$\begin{aligned}\langle\ ,\ \rangle : (U(\mathfrak{n}^+_{\underline{j}}) \underset{K}{\otimes} M_{\underline{j}}) \times (\mathcal{O}(U^+_{\underline{j}} \cap B) \underset{K}{\otimes} \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})) &\longrightarrow \mathcal{O}(U^+_{\underline{j}} \cap B) \underset{K}{\otimes} \mathrm{St}_{d+1-j} \\ (\mathfrak{z} \otimes m, e \otimes E) &\longmapsto \mathfrak{z}e \otimes E(m)\end{aligned}$$

we have $\langle \mathfrak{d}_{\underline{j}}, f|U^+_{\underline{j}} \cap B\rangle = 0$. We now decompose

$$f|U^+_{\underline{j}} \cap B = \sum_i e_i \otimes E_i$$

into a finite sum with $e_i \in \mathcal{O}(U^+_{\underline{j}} \cap B)$ and $E_i \in \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})$ such that the images $E_i(M_{\underline{j}})$ are linearly independent 1-dimensional subspaces of St_{d+1-j}. Then each $e_i \otimes E_i$ satisfies the relations $\langle \mathfrak{d}_{\underline{j}}, e_i \otimes E_i\rangle = 0$. We define maps f_i on G with values in $\mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})$ by setting

$$f_i(uh) := e_i(u) \cdot h^{-1}(E_i) \text{ for } u \in U^+_{\underline{j}} \cap B \text{ and } h \in P_{\underline{j}}$$

and extending this by zero to G. Since the map $h \mapsto h^{-1}(E_i)$ is locally analytic on $P_{\underline{j}}$ it easily follows that $f_i \in C^{\mathrm{an}}(G, P_{\underline{j}}; \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j}))$. By construction f_i is supported on $BP_{\underline{j}}$ with $f_i|U^+_{\underline{j}} \cap B = e_i \otimes E_i$. Clearly

$$f = \sum_i f_i.$$

We claim that each f_i satisfies the relations $\mathfrak{d}_{\underline{j}} = 0$. This will be a consequence of the following observation. The group $P_{\underline{j}}$ acts diagonally on $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_{\underline{j}})} M_{\underline{j}}$ via $h(\mathfrak{z} \otimes m) := \mathrm{ad}(h)\mathfrak{z} \otimes hm$. The point to observe is that the subspace $\mathfrak{d}_{\underline{j}}$ is $P_{\underline{j}}$-invariant. Note first that because $U(\mathfrak{n}^+_{\underline{j}}) \cap \mathfrak{b}^>_{\underline{j}} \subseteq \mathfrak{b}$ (Prop. 4.6. iii) an element $\sum_\mu \mathfrak{z}_{(\mu)} \otimes L_\mu \in U(\mathfrak{n}^+_{\underline{j}}) \otimes_K M_{\underline{j}}$ lies in $\mathfrak{d}_{\underline{j}}$ if and only if $\sum_\mu \mathfrak{z}_{(\mu)} L_\mu \xi = 0$. Let now $\sum_\mu \mathfrak{z}_{(\mu)} \otimes L_\mu \in \mathfrak{d}_{\underline{j}} \subseteq U(\mathfrak{n}^+_{\underline{j}}) \otimes_K M_{\underline{j}}$ and $h \in P_{\underline{j}}$. We distinguish two cases. If $h \in L(\underline{j})$ then using the $\mathrm{ad}(L(\underline{j}))$-invariance of $U(\mathfrak{n}^+_{\underline{j}}) \cap \mathfrak{b}$ we obtain

$$\Big(\sum_\mu \mathrm{ad}(h)(\mathfrak{z}_{(\mu)}) \cdot hL_\mu\Big)\xi = \Big(\mathrm{ad}(h)\Big(\sum_\mu \mathfrak{z}_{(\mu)} L_\mu\Big)\Big)\xi = 0.$$

If $h \in L'(\underline{j})U_{\underline{j}}$ then using Lemma 1.ii we obtain

$$\begin{aligned}\Big(\sum_\mu \mathrm{ad}(h)(\mathfrak{z}_{(\mu)}) \cdot hL_\mu\Big)\xi &= \det(h) \cdot h_* \Big(\sum_\mu \mathfrak{z}_{(\mu)} h_*^{-1} L_\mu \xi\Big)\\ &= h_*\Big(\sum_\mu \mathfrak{z}_{(\mu)} L_\mu \xi\Big) = 0.\end{aligned}$$

Going back to our maps f_i and letting again $\sum_\mu \mathfrak{z}_{(\mu)} \otimes L_\mu \in \mathfrak{d}_{\underline{j}} \subseteq U(\mathfrak{n}_{\underline{j}}^+) \otimes_K M_{\underline{j}}$ we now compute

$$\begin{aligned}\Big(\Big(\sum_\mu \mathfrak{z}_{(\mu)} \otimes L_\mu\Big) f_i\Big)(uh) &= \sum_\mu (\mathfrak{z}_{(\mu)} f_i)(uh)(L_\mu)\\ &= \sum_\mu ((\mathrm{ad}(h)\mathfrak{z}_{(\mu)})e_i)(u) \cdot h^{-1}(E_i)(L_\mu)\\ &= h^{-1}\Big(\sum_\mu ((\mathrm{ad}(h)\mathfrak{z}_{(\mu)})e_i)(u) \cdot E_i(hL_\mu)\Big)\\ &= h^{-1}\Big(\Big\langle \sum_\mu \mathrm{ad}(h)\mathfrak{z}_{(\mu)} \otimes hL_\mu, e_i \otimes E_i\Big\rangle(u)\Big)\\ &= h^{-1}\Big(\Big\langle h(\sum_\mu \mathfrak{z}_{(\mu)} \otimes L_\mu), e_i \otimes E_i\Big\rangle(u))\\ &= 0.\end{aligned}$$

This establishes our claim.

We want to further normalize the component φ in this last expression. Let $\varphi_{\mathrm{o}} \in C_{\mathrm{o}}^\infty(U(\underline{j}), K) \cong \mathrm{St}_{d+1-j}$ denote the characteristic function of $U(\underline{j}) \cap B$. Then φ can be written as a linear combination of vectors of the form $g^{-1}\varphi_{\mathrm{o}}$ with $g \in L(\underline{j})$. A straightforward argument shows that f can be decomposed accordingly so that we may assume $\varphi = g^{-1}\varphi_{\mathrm{o}}$ for some $g \in L(\underline{j})$. We now find a finite disjoint open covering

$$g(U_{\underline{j}}^+ \cap B)P_{\underline{j}} = \dot{\bigcup_i} u_i y^n (U_{\underline{j}}^+ \cap B) P_{\underline{j}}$$

with appropriate $n \in \mathbb{N}$ and $u_i \in U_{\underline{j}}^+$. The map gf is supported on $gBP_{\underline{j}}$ and its restriction $gf|g(U_{\underline{j}}^+ \cap B)g^{-1}$ is analytic with values in $M'_{\underline{j}} \otimes K\varphi_{\mathrm{o}}$. If we put

$$f_i := ((u_i y^n)^{-1} gf)|BP_{\underline{j}} \text{ extended by zero to } G$$

then these maps lie in the induced representation on the right hand side of our assertion and we have

$$f = \sum_i g^{-1} u_i y^n f_i.$$

The restriction of f_i to $U_{\underline{j}}^+ \cap B$ satisfies

$$f_i(u) = (gf)(u_i y^n u) = \pi^{-jn} \cdot (gf)(u_i y^n u y^{-n}).$$

But $u_i y^n u y^{-n} \in gBP_{\underline{j}} \cap U_{\underline{j}}^+ \subseteq g(U_{\underline{j}}^+ \cap B)g^{-1}$. It follows that $f_i|U_{\underline{j}}^+ \cap B$ is analytic with values in $M'_j \otimes K\varphi_{\mathrm{o}}$. At this point we have arrived at the conclusion that we may assume that

– f is supported on $BP_{\underline{j}}$ with $f|U_{\underline{j}}^+ \cap B = \varepsilon \otimes \varphi_{\mathrm{o}}$ for some $\varepsilon \in \mathcal{O}(U_{\underline{j}}^+ \cap B, M'_{\underline{j}})^{\mathfrak{d}_{\underline{j}}=0}$.

We rephrase the above discussion in the following way. We have the linear map

$$\mathrm{Ext}_{\underline{j}} : \mathcal{O}(U_{\underline{j}}^+ \cap B, M'_{\underline{j}})^{\mathfrak{d}_{\underline{j}}=0} \longrightarrow C^{\mathrm{an}}(G, P_{\underline{j}}; \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j}))^{\mathfrak{d}_{\underline{j}}=0}$$

defined by

$$\mathrm{Ext}_{\underline{j}}(\varepsilon)(g) := \begin{cases} h^{-1}(\varepsilon(u) \otimes \varphi_{\mathrm{o}}) & \text{for } g = uh \text{ with } u \in U_{\underline{j}}^+ \cap B, h \in P_{\underline{j}}, \\ 0 & \text{otherwise .} \end{cases}$$

Its image generates the right hand side (algebraically) as a G-representation. An argument analogous to the proof of [Fea] 4.3.1 shows that $\mathrm{Ext}_{\underline{j}}$ is continuous. On the other hand, in section 6 after Lemma 4 we had constructed a continuous linear map

$$D_{\underline{j}} : \mathcal{O}(U_{\underline{j}}^+ \cap B, M'_{\underline{j}})^{\mathfrak{d}_{\underline{j}}=0} \longrightarrow [\Omega^d(\mathfrak{X})^j / \Omega^d(\mathfrak{X})^{j+1}]'.$$

The surjectivity of $I^{[j]}$ therefore will follow from the identity

$$\mathrm{Ext}_{\underline{j}} = I^{[j]} \circ D_{\underline{j}}.$$

By the continuity of all three maps involved it suffices to check this identity on weight vectors. Fix a weight ν with $J(\nu) = \{0, \ldots, j-1\}$. By construction the map $D_{\underline{j}}$ sends the weight vector $\sum_{\mu \in B(\underline{j})} [(L_\mu f_\nu)|U_{\underline{j}}^+ \cap B] \otimes L_\mu^*$ to the linear form $\lambda_\nu(\eta) = \mathrm{Res}_{(\overline{C},0)} \Xi_{-\nu} \eta$. What we therefore have to check is that $I^{[j]}(\lambda_\nu)$ is supported on $BP_{\underline{j}}$ with

$$I^{[j]}(\lambda_\nu)|U_{\underline{j}}^+ \cap B = \sum_{\mu \in B(\underline{j})} [(L_\mu f_\nu)|U_{\underline{j}}^+ \cap B] \otimes L_\mu^* \otimes \varphi_{\mathrm{o}}.$$

By definition we have

$$\begin{aligned} [I^{[j]}(\lambda_\nu)(g)](L_\mu)(u) &= [(g^{-1}\lambda_\nu)|V_j](L_\mu(u_*\xi)) \\ &= \lambda_\nu(g(L_\mu(u_*\xi))) \\ &= \mathrm{Res}_{(\overline{C},0)} \Xi_{-\nu} \cdot g_*(L_\mu(u_*\xi)) \\ &= \mathrm{Res}_{(\overline{C},0)} \Xi_{-\nu} \cdot g_* u_*((\mathrm{ad}(u^{-1})(L_\mu))\xi) \end{aligned}$$

for $\mu \in B(\underline{j})$ and $u \in U(\underline{j}) \subseteq P_{\underline{j}}$. First we deal with the vanishing of this expression for $g \notin BP_{\underline{j}}$. Observe that

– $g \notin BP_{\underline{j}}$ if and only if $gu \notin BP_{\underline{j}}$, and
– $\mathrm{ad}(u^{-1})(L_\mu)\xi \in \sum_{\mu' \in B(\underline{j})} K \cdot \Xi_{\mu'}\xi$.

Hence it suffices to show that

$$\mathrm{Res}_{(\overline{C},0)} \Xi_{-\nu} \cdot g_*(\Xi_\mu \xi) = 0 \text{ for } g \notin BP_{\underline{j}}.$$

We distinguish two cases. First we assume that $g \notin U^+_{\underline{j}} P_{\underline{j}}$. Then the divisor $\operatorname{div}(\Xi_{-\nu} \cdot g_*(\Xi_\mu \xi))_\infty$ is supported among the hyperplanes $\Xi_0 = 0, \ldots, \Xi_{j-1} = 0$ and $g_*\Xi_j = 0, \ldots, g_*\Xi_d = 0$. Those are linearly dependent if $g \notin U^+_{\underline{j}} P_{\underline{j}}$ and hence have a nonempty intersection, i.e., $Z(\Xi_{-\nu} \cdot g_*(\Xi_\mu \xi)) \neq \varnothing$. According to the discussion after Prop. 6.2 the form $\Xi_{-\nu} \cdot g_*(\Xi_\mu \xi)$ therefore lies in $\Omega^d_{\mathrm{alg}}(\mathfrak{X})^1$, hence is exact by Lemma 7.2, and consequently has zero residue. Second we consider the case $g \in U^+_{\underline{j}} \setminus (U^+_{\underline{j}} \cap B)$. Then g fixes $\Xi_0, \ldots, \Xi_{j-1}$ so that $g^{-1}\Xi_{-\nu}$ is a linear combination of $\Xi_{-\nu'}$ with $J(\nu') \subseteq \{0, \ldots, j-1\}$. It follows that $\Xi_{-\nu} \cdot g_*(\Xi_\mu \xi)$ is a linear combination of forms $\Xi_{\nu''}\xi$ among which the only possible non-exact one is ξ! (compare the proof of Lemma 7.2). We obtain

$$\operatorname{Res}_{(\overline{C},0)} \Xi_{-\nu} \cdot g_*(\Xi_\mu \xi) = \operatorname{Res}_{g^{-1}(\overline{C},0)} (g^{-1}\Xi_{-\nu}) \Xi_\mu \xi = c \cdot \operatorname{Res}_{g^{-1}(\overline{C},0)} \xi$$

with some constant $c \in K$. But ξ has residues only on the standard apartment and $g^{-1}(\overline{C}, 0)$ lies in the standard apartment only if $g \in U^+_{\underline{j}} \cap B$. This establishes the assertion about the support of $I^{[j]}(\lambda_\nu)$.

Fix now a $g \in U^+_{\underline{j}} \cap B$ and let $u \in U(\underline{j})$. Repeating the last argument for gu instead of g we obtain that $\operatorname{Res}_{(\overline{C},0)} \Xi_{-\nu} \cdot g_* u_*(\Xi_\mu \xi) = 0$ unless gu and hence u fixes $(\overline{C}, 0)$. This means that, for $g \in U^+_{\underline{j}} \cap B$, the function $[I^{[j]}(\lambda_\nu)(g)](L_\mu) \in C^\infty_{\mathrm{o}}(U(\underline{j}), K)$ vanishes outside $U(\underline{j}) \cap B$. For $u \in U(\underline{j}) \cap B$ we have

$$\begin{aligned} [I^{[j]}(\lambda_\nu)(g)](L_\mu)(u) &= \operatorname{Res}_{(\overline{C},0)} (u^{-1} g^{-1} \Xi_{-\nu})((\operatorname{ad}(u^{-1})(L_\mu))\xi) \\ &= \sum_{J(\nu') \subseteq \underline{j}} c(\nu') \operatorname{Res}_{(\overline{C},0)} (u^{-1} \Xi_{-\nu'})((\operatorname{ad}(u^{-1})(L_\mu))\xi) \end{aligned}$$

where

$$g^{-1}\Xi_{-\nu} = \sum_{J(\nu') \subseteq \underline{j}} c(\nu') \Xi_{-\nu'}.$$

If $\nu' \in B(\underline{j})$ then we computed the corresponding summand already in the proof of Prop. 8.4 and, in particular, showed that it is independent of $u \in U(\underline{j}) \cap B$. On the other hand the subspace

$$\sum_{\substack{J(\nu') \subseteq \underline{j} \\ \nu' \notin B(\underline{j})}} K \cdot \Xi_{-\nu'}$$

of $\mathcal{O}(\mathfrak{X})$ is preserved by the action of $U(\underline{j})$. This means that, for $\nu' \notin B(\underline{j})$, the form $(u^{-1}\Xi_{-\nu'})((\operatorname{ad}(u^{-1})(L_\mu))\xi)$ cannot contain ξ and therefore must have zero residue. This computation says that, for fixed $g \in U^+_{\underline{j}} \cap B$ and fixed $\mu \in B(\underline{j})$, the function $[I^{[j]}(\lambda_\nu)(g)](L_\mu)(u)$ is constant in $u \in U(\underline{j}) \cap B$. In other words we have

$$I^{[j]}(\lambda_\nu)(g) = \sum_{\mu \in B(\underline{j})} [I^{[j]}(\lambda_\nu)(g)](L_\mu)(1) \otimes L^*_\mu \otimes \varphi_{\mathrm{o}}$$

for $g \in U_{\underline{j}}^{+} \cap B$. But using the various definitions we compute

$$[I^{[j]}(\lambda_\nu)(g)](L_\mu)(1) = \mathrm{Res}_{(\overline{C},0)} \Xi_{-\nu} \cdot g_*(L_\mu \xi) = (L_\mu f_\nu)(g).$$

This establishes the surjectivity and hence bijectivity of the map $I^{[j]}$. Finally, that $I^{[j]}$ is open and hence a topological isomorphism is a consequence of the open mapping theorem in the form given in [GK] Thm. 3.1(A_3) provided we show that both sides of $I^{[j]}$ are (LB)-spaces, i.e., a locally convex inductive limit of a sequence of Banach spaces. For the left hand side this fact is implicitly contained in our earlier arguments: In the proof of Prop. 6.5 we had noted that $\Omega^d(\mathcal{X})^j/\Omega^d(\mathcal{X})^{j+1}$ is the projective limit of a sequence of Banach spaces with compact transition maps. We certainly may assume in addition that these transition maps have dense images. By the same argument as in the proof of Prop. 2.4 it then follows that the strong dual $[\Omega^d(\mathcal{X})^j/\Omega^d(\mathcal{X})^{j+1}]'$ is an (LB)-space. We now turn to the right hand side. Using [GKPS] Thm. 3.1.16 (compare also [Kom] Thm. 7') it suffices to show that $C^{\mathrm{an}}(G, P_{\underline{j}}; \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j}))$ is the locally convex inductive limit of a sequence of Banach spaces with compact transition maps. To see this it is convenient to identify this space, as a locally convex vector space (without the G-action), with the space $C^{\mathrm{an}}(G/P_{\underline{j}}, \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j}))$ of all locally analytic functions on $G/P_{\underline{j}}$ with values in $\mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})$. The recipe how to do this is given in [Fea] 4.3.1. One fixes a section $\imath$ of the projection map $G \twoheadrightarrow G/P_{\underline{j}}$ such that

$$\begin{array}{rcl} G/P_{\underline{j}} \times P_{\underline{j}} & \xrightarrow{\sim} & G \\ (gP_{\underline{j}}, h) & \longmapsto & \imath(gP_{\underline{j}})h \end{array}$$

is an isomorphism of locally analytic manifolds ([Fea] 4.1.1). We then have the continuous injection

$$\begin{array}{rcl} C^{\mathrm{an}}(G, P_{\underline{j}}; V) & \longrightarrow & C^{\mathrm{an}}(G/P_{\underline{j}}, V) \\ f & \longmapsto & [gP_{\underline{j}} \mapsto f(\imath(gP_{\underline{j}}))] \end{array}$$

writing $V := \mathrm{Hom}_K(M_{\underline{j}}, \mathrm{St}_{d+1-j})$ for short. In fact we will need that V is of the form $V = V_{\mathrm{fin}} \otimes_K V_{\mathrm{sm}}$ for two $P_{\underline{j}}$-representations V_{fin} and V_{sm} which are finite dimensional algebraic and smooth, respectively. If V_f runs over the finite dimensional subspaces of V_{sm} then

$$V = \varinjlim_{V_f} V_{\mathrm{fin}} \otimes_K V_f$$

and each $V_{\mathrm{fin}} \otimes_K V_f$ is invariant under some open subgroup of $P_{\underline{j}}$. A possible inverse of the above map has to be given by

$$\phi \longmapsto f_\phi(g) := (g^{-1}\imath(gP_{\underline{j}}))(\phi(gP_{\underline{j}})).$$

Since

$$C^{\mathrm{an}}(G/P_{\underline{j}}, V) = \varinjlim_{V_f} C^{\mathrm{an}}(G/P_{\underline{j}}; V_{\mathrm{fin}} \otimes_K V_f)$$

it suffices to check that

$$\begin{array}{rcl} C^{\mathrm{an}}(G/P_{\underline{j}}; V_{\mathrm{fin}} \underset{K}{\otimes} V_f) & \longrightarrow & C^{\mathrm{an}}(G, P_{\underline{j}}; V) \\ \phi & \longmapsto & f_\phi \end{array}$$

is well defined and continuous. Consider the obvious bilinear map

$$\beta : [V_{\mathrm{fin}} \underset{K}{\otimes} V_f] \times [\mathrm{End}_K(V_{\mathrm{fin}}) \underset{K}{\otimes} \mathrm{Hom}_K(V_f, V_{\mathrm{sm}})] \longrightarrow V$$

between vector spaces equipped with the finest locally convex topology. By [Fea] 2.4.3 (the condition BIL is trivially satisfied) it induces a continuous bilinear map

$$\begin{array}{r} C^{\mathrm{an}}(G/P_{\underline{j}}; V_{\mathrm{fin}} \otimes V_f) \times C^{\mathrm{an}}(P_{\underline{j}}, \mathrm{End}(V_{\mathrm{fin}}) \otimes \mathrm{Hom}(V_f, V_{\mathrm{sm}})) \longrightarrow C^{\mathrm{an}}(G/P_{\underline{j}} \times P_{\underline{j}}, V). \\ (\phi, \Psi) \longmapsto \beta \circ (\phi \times \Psi) \end{array}$$

Using the section $\imath$ we obtain the continuous bilinear map

$$\widehat{\beta} : C^{\mathrm{an}}(G/P_{\underline{j}}; V_{\mathrm{fin}} \otimes V_f) \times C^{\mathrm{an}}(P_{\underline{j}}, \mathrm{End}(V_{\mathrm{fin}}) \otimes \mathrm{Hom}(V_f, V_{\mathrm{sm}})) \longrightarrow C^{\mathrm{an}}(G, V)$$

defined by $\widehat{\beta}(\phi, \Psi)(g) := \beta(\phi(gP_{\underline{j}}), \Psi(\imath(gP_{\underline{j}})^{-1}g))$. It remains to observe that $\Psi_{\mathrm{o}}(h) := h^{-1}.\otimes h^{-1}.$ lies in $C^{\mathrm{an}}(P_{\underline{j}}, \mathrm{End}(V_{\mathrm{fin}}) \otimes \mathrm{Hom}(V_f, V_{\mathrm{sm}}))$ and that $\widehat{\beta}(\phi, \Psi_{\mathrm{o}}) = f_\phi$.

We now are reduced to show that $C^{\mathrm{an}}(G/P_{\underline{j}}, V)$ is the locally convex inductive limit of a sequence of Banach spaces with compact transition maps. Since $G/P_{\underline{j}}$ is compact this is a special case of [Fea] 2.3.2. ◻

To finish let us reconsider the bottom filtration step. By definition $\mathrm{St}_1 = K$ is the trivial representation, and $L(\underline{d}) = K^\times$ acts on the one dimensional space $M_{\underline{d}}$ through the character $a \mapsto a^{-d}$. Let therefore K_χ denote the one dimensional $P_{\underline{d}}$-representation given by the locally analytic character

$$\begin{array}{rcl} \chi : P_{\underline{d}} & \longrightarrow & K^\times \\ g & \longmapsto & \dfrac{(g_{dd})^{d+1}}{\det(g)}. \end{array}$$

By comparing weights one easily checks that the natural map $U(\mathfrak{n}_{\underline{d}}^+) \underset{K}{\otimes} M_{\underline{d}} \longrightarrow \mathfrak{b}_{\underline{d}}/\mathfrak{b}$ is bijective which means that $\mathfrak{d}_{\underline{d}} = 0$. Our theorem therefore specializes in this case to the assertion that the map

$$\begin{array}{rcl} I^{[d]} : [\Omega^d(\mathfrak{X})^d]' & \xrightarrow{\cong} & C^{\mathrm{an}}(G, P_{\underline{d}}; K_\chi) \\ \lambda & \longmapsto & [g \mapsto -\lambda(g_*(d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}))] \end{array}$$

is a G-equivariant topological isomorphism.

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THE DISPLAY OF A FORMAL p-DIVISIBLE GROUP

by

Thomas Zink

Abstract. — We give a new Dieudonné theory which associates to a formal p-divisible group X over an excellent p-adic ring R an object of linear algebra called a display. On the display one can read off the structural equations for the Cartier module of X, and find the crystal of Grothendieck-Messing. We give applications to deformations of formal p-divisible groups.

Introduction

We fix throughout a prime number p. Let R be a commutative unitary ring. Let $W(R)$ be the ring of Witt vectors. The ring structure on $W(R)$ is functorial in R and has the property that the Witt polynomials are ring homomorphisms:

$$\begin{aligned}\mathbf{w}_n : \qquad & W(R) \longrightarrow R \\ (x_0, \dots x_i, \dots) & \longmapsto x_0^{p^n} + p x_1^{p^{n-1}} + \dots + p^n x_n\end{aligned}$$

Let us denote the kernel of the homomorphism $\mathbf{w}_0$ by I_R. The Verschiebung is a homomorphism of additive groups:

$$\begin{aligned}{}^V : \qquad & W(R) \longrightarrow W(R) \\ (x_0, \dots x_i, \dots) & \longmapsto (0, x_0, \dots x_i, \dots)\end{aligned}$$

The Frobenius endomorphism ${}^F : W(R) \to W(R)$ is a ring homomorphism. The Verschiebung and the Frobenius are functorial and satisfy the defining relations:

$$\begin{aligned}&\mathbf{w}_n({}^F x) = \mathbf{w}_{n+1}(x), \quad \text{for } n \geq 0 \\ &\mathbf{w}_n({}^V x) = p\mathbf{w}_{n-1}(x), \text{ for } n > 0, \quad \mathbf{w}_0({}^V x) = 0.\end{aligned}$$

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Moreover the following relations are satisfied:

$$^{F}{}^{V} = p, \qquad {}^{V}({}^{F}xy) = x\,{}^{V}y, \quad x, y \in W(R)$$

We note that $I_R = {}^{V}W(R)$.

Let P_1 and P_2 be $W(R)$-modules. An ^{F}-linear homomorphism $\phi : P_1 \to P_2$ is a homomorphism of abelian group which satisfies the relation $\phi(wm) = {}^{F}w\phi(m)$, where $m \in P$, $w \in W(R)$. Let

$$\phi^{\sharp} : W(R) \otimes_{F,W(R)} P_1 \longrightarrow P_2$$

be the linearization of ϕ. We will call ϕ an ^{F}-linear epimorphism respectively an ^{F}-linear isomorphism if $\phi^{\sharp}$ is an epimorphism respectively an isomorphism.

The central notion of these notes is that of a display. The name was suggested by the displayed structural equations for a reduced Cartier module introduced by Norman [N]. In this introduction we will assume that p is nilpotent in R.

Definition 1. — A $3n$-display over R is a quadruple (P, Q, F, V^{-1}), where P is a finitely generated projective $W(R)$-module, $Q \subset P$ is a submodule and F and V^{-1} are ^{F}-linear maps $F : P \to P, \quad V^{-1} : Q \to P$.

The following properties are satisfied:

(i) $I_R P \subset Q \subset P$ and P/Q is a direct summand of the $W(R)$-module $P/I_R P$.

(ii) $V^{-1} : Q \longrightarrow P$ is a ^{F}-linear epimorphism.

(iii) For $x \in P$ and $w \in W(R)$, we have

$$V^{-1}({}^{V}wx) = wFx.$$

If we set $w = 1$ in the relation (iii) we obtain:

$$Fx = V^{-1}({}^{V}1x)$$

One could remove F from the definition of a 3n-display. But one has to require that the ^{F}-linear map defined by the last equation satisfies (iii).

For $y \in Q$ one obtains:

$$Fy = p \cdot V^{-1}y$$

We note that there is no operator V. The reason why we started with V^{-1} is the following example of a 3n-display. Let $R = k$ be a perfect field and let M be a Dieudonné module. It is a finitely generated free $W(k)$-module which is equipped with operators F and V. Since V is injective, there is an inverse operator $V^{-1} : VM \to M$. Hence one obtains a display (M, VM, F, V^{-1}). In fact this defines an equivalence of the category of Dieudonné modules with the category of 3n-displays over k.

Let us return to the general situation. The $W(R)$-module P always admits a direct decomposition

$$P = L \oplus T,$$

such that $Q = L \oplus I_R T$. We call it a normal decomposition. For a normal decomposition the following map is a ^{F}-linear isomorphism:

$$V^{-1} \oplus F : L \oplus T \longrightarrow P$$

Locally on $\operatorname{Spec} R$ the $W(R)$-modules L and T are free. Let us assume that T has a basis $e_1, \ldots, e_d$ and L has a basis $e_{d+1}, \ldots, e_h$. Then there is an invertible matrix (α_{ij}) with coefficients in $W(R)$, such that the following relations hold:

$$\begin{aligned} Fe_j &= \sum_{i=1}^{h} \alpha_{ij} e_i, \quad \text{for } j = 1, \ldots, d \\ V^{-1} e_j &= \sum_{i=1}^{h} \alpha_{ij} e_i \quad \text{for } j = d+1, \ldots, h \end{aligned}$$

Conversely for any invertible matrix (α_{ij}) these relations define a 3n-display.

Let (β_{kl}) the inverse matrix of (α_{ij}). We consider the following matrix of type $(h-d) \times (h-d)$ with coefficients in R/pR:

$$B = (\mathbf{w}_0(\beta_{kl}) \text{ modulo } p)_{k,l=d+1,\ldots,h}$$

Let us denote by $B^{(p)}$ be the matrix obtained from B by raising all coefficients of B to the power p. We say that the 3n-display defined by (α_{ij}) satisfies the V-nilpotence condition if there is a number N such that

$$B^{(p^{N-1})} \cdots B^{(p)} \cdot B = 0.$$

The condition depends only on the display but not on the choice of the matrix.

Definition 2. — A 3n-display which locally on $\operatorname{Spec} R$ satisfies the V-nilpotence condition is called a display.

The 3n-display which corresponds to a Dieudonné module M over a perfect field k is a display, iff V is topologically nilpotent on M for the p-adic topology. In the covariant Dieudonné theory this is also equivalent to the fact that the p-divisible group associated to M has no étale part.

Let S be a ring such that p is nilpotent in S. Let $\mathfrak{a} \subset S$ be an ideal which is equipped with divided powers. Then it makes sense to divide the Witt polynomial $\mathbf{w}_m$ by p^m. These divided Witt polynomials define an isomorphism of additive groups:

$$W(\mathfrak{a}) \longrightarrow \mathfrak{a}^{\mathbb{N}}$$

Let $\mathfrak{a} \subset \mathfrak{a}^{\mathbb{N}}$ be the embedding via the first component. Composing this with the isomorphism above we obtain an embedding $\mathfrak{a} \subset W(\mathfrak{a})$. In fact $\mathfrak{a}$ is a $W(S)$-submodule of $W(\mathfrak{a})$, if $\mathfrak{a}$ is considered as a $W(S)$-module via $\mathbf{w}_0$. Let $R = S/\mathfrak{a}$ be the factor ring. We consider a display $\widetilde{\mathcal{P}} = (\widetilde{P}, \widetilde{Q}, \widetilde{F}, \widetilde{V}^{-1})$ over S. By base change we obtain a display over R:

$$\widetilde{\mathcal{P}}_R = \mathcal{P} = (P, Q, F, V^{-1})$$

By definition one has $P = W(R) \otimes_{W(S)} \widetilde{P}$. Let us denote by $\widehat{Q} = W(\mathfrak{a})\widetilde{P} + \widetilde{Q} \subset \widetilde{P}$ the inverse image of Q. Then we may extend the operator $\widetilde{V}^{-1}$ uniquely to the domain of definition $\widehat{Q}$, such that the condition $\widetilde{V}^{-1}\mathfrak{a}\widetilde{P} = 0$ is fulfilled.

Theorem 3. — *With the notations above let $\widetilde{\mathcal{P}}' = (\widetilde{P}', \widetilde{Q}', \widetilde{F}, \widetilde{V}^{-1})$ be a second display over S, and $\mathcal{P}' = (P', Q', F, V^{-1})$ the display over R obtained by base change. Assume we are given a morphism of displays $u : \mathcal{P} \to \mathcal{P}'$ over R. Then u has a unique lifting $\widetilde{u}$ to a morphism of quadruples:*

$$\widetilde{u} : (\widetilde{P}, \widehat{Q}, \widetilde{F}, \widetilde{V}^{-1}) \longrightarrow (\widetilde{P}', \widehat{Q}', \widetilde{F}, \widetilde{V}^{-1}).$$

This allows us to associate a crystal to a display: Let R be a ring, such that p is nilpotent in R. Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a display over R. Consider a surjection $S \to R$ whose kernel $\mathfrak{a}$ is equipped with a divided power structure. If p is nilpotent in S we call such a surjection a pd-thickening of R. Let $\widetilde{\mathcal{P}} = (\widetilde{P}, \widetilde{Q}, \widetilde{F}, \widetilde{V}^{-1})$ be any lifting of the display $\mathcal{P}$ to S. By the theorem the module $\widetilde{P}$ is determined up to canonical isomorphism by $\mathcal{P}$. Hence we may define:

$$\mathcal{D}_{\mathcal{P}}(S) = S \otimes_{W(S)} \widetilde{P}$$

This gives a crystal on $\operatorname{Spec} R$ if we sheafify the construction.

Next we construct a functor BT from the category of 3n-displays over R to the category of formal groups over R. A nilpotent R-algebra $\mathcal{N}$ is an R-algebra (without unit), such that $\mathcal{N}^N = 0$ for a sufficiently big number N. Let Nil_R denote the category of nilpotent R-algebras. We will consider formal groups as functors from the category Nil_R to the category of abelian groups. Let us denote by $\widehat{W}(\mathcal{N}) \subset W(\mathcal{N})$ the subgroup of all Witt vectors with finitely many nonzero components. This is a $W(R)$-submodule. We consider the functor $\mathbf{G}^0_{\mathcal{P}}(\mathcal{N}) = \widehat{W}(\mathcal{N}) \otimes_{W(R)} P$ on Nil_R with values in the category of abelian groups. Let $\mathbf{G}^{-1}_{\mathcal{P}}$ be the subgroup functor which is generated by all elements in $\widehat{W}(\mathcal{N}) \otimes_{W(R)} P$ of the following form:

$${}^V\xi \otimes x, \quad \xi \otimes y, \qquad \xi \in \widehat{W}(\mathcal{N}),\ y \in Q,\ x \in P.$$

Then we define a map:

$$V^{-1} - \mathrm{id} : \mathbf{G}^{-1}_{\mathcal{P}} \longrightarrow \mathbf{G}^0_{\mathcal{P}} \tag{1}$$

On the generators above the map $V^{-1} - \mathrm{id}$ acts as follows:

$$(V^{-1} - \mathrm{id})({}^V\xi \otimes x) = \xi \otimes Fx - {}^V\xi \otimes x$$
$$(V^{-1} - \mathrm{id})(\xi \otimes y) = {}^F\xi \otimes V^{-1}y - \xi \otimes y$$

Theorem 4. — *Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a 3n-display over R. The cokernel of the map (1) is a formal group $BT_{\mathcal{P}}$. Moreover one has an exact sequence of functors on Nil_R:*

$$0 \longrightarrow \mathbf{G}^{-1}_{\mathcal{P}} \xrightarrow{V^{-1} - \mathrm{id}} \mathbf{G}^0_{\mathcal{P}} \longrightarrow BT_{\mathcal{P}} \longrightarrow 0$$

If $\mathcal{N}$ is equipped with nilpotent divided powers we define an isomorphism:

$$\exp_{\mathcal{P}} : \mathcal{N} \otimes_R P/Q \longrightarrow BT_{\mathcal{P}}(\mathcal{N}),$$

which is called the exponential map. In particular the tangent space of the formal group $BT_{\mathcal{P}}$ is canonically identified with P/Q.

Let $\mathbb{E}_R$ be the local Cartier ring with respect to the prime p. Then $BT_{\mathcal{P}}$ has the following Cartier module:

$$\mathbf{M}(\mathcal{P}) = \mathbb{E}_R \otimes_{W(R)} P/(F \otimes x - 1 \otimes Fx, V \otimes V^{-1}y - 1 \otimes y)_{\mathbb{E}_R},$$

where x runs through all elements of P and y runs through all elements of Q, and $(\ )_{\mathbb{E}_R}$ indicates the submodule generated by all these elements.

Theorem 5. — *Let $\mathcal{P}$ be a display over R. Then $BT_{\mathcal{P}}$ is a formal p-divisible group of height equal to $\operatorname{rank}_{W(R)} P$.*

The restriction of the functor BT to the category of displays is faithful. It is fully faithful, if the ideal of nilpotent elements in R is a nilpotent ideal.

The following main theorem gives the comparison of our theory and the crystalline Dieudonné theory of Grothendieck and Messing.

Theorem 6. — *Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a display over a ring R. Then there is a canonical isomorphism of crystals over R:*

$$\mathcal{D}_{\mathcal{P}} \xrightarrow{\sim} \mathbb{D}_{BT_{\mathcal{P}}}$$

Here the right hand side is the crystal from Messing's book [Me]. If $W(R) \to S$ is a morphism of pd-thickenings of R, we have a canonical isomorphism

$$S \otimes_{W(R)} P \cong \mathbb{D}_{BT_{\mathcal{P}}}(S).$$

In this theorem we work with the crystalline site whose objects are pd-thickenings $S \to R$, such that the kernel is a nilpotent ideal. We remark that the crystal $\mathbb{D}_{BT_{\mathcal{P}}}$ is defined in [Me] only for pd-thickenings with nilpotent divided powers. But if one deals with p-divisible groups without an étale part this restriction is not necessary (see corollary 97 below). In particular this shows, that the formal p-divisible group $BT_{\mathcal{P}}$ lifts to a pd-thickening $S \to R$ with a nilpotent kernel, iff the Hodge filtration of the crystal lifts (compare [Gr] p.106).

The functor BT is compatible with duality in the following sense. Assume we are given 3n-displays $\mathcal{P}_1$ and $\mathcal{P}_2$ over a ring R, where p is nilpotent.

Definition 7. — A bilinear form $(\ ,\ )$ on the pair of 3n-displays $\mathcal{P}_1, \mathcal{P}_2$ is a bilinear form of $W(R)$-modules:

$$P_1 \times P_2 \longrightarrow W(R),$$

which satisfies

$$^{V}(V^{-1}y_1, V^{-1}y_2) = (y_1, y_2) \quad \text{for} \quad y_1 \in Q_1,\ y_2 \in Q_2.$$

Let us denote by $\mathrm{Bil}(\mathcal{P}_1, \mathcal{P}_2)$ the abelian group of these bilinear forms. Then we will define a homomorphism:

$$\mathrm{Bil}(\mathcal{P}_1, \mathcal{P}_2) \longrightarrow \mathrm{Biext}^1(BT_{\mathcal{P}_1} \times BT_{\mathcal{P}_2}, \widehat{\mathbb{G}}_m) \tag{2}$$

Here the right hand side denotes the group of biextensions of formal groups in the sense of Mumford [Mu].

To do this we consider the exact sequences for $i = 1, 2$:

$$0 \longrightarrow \mathbf{G}^{-1}_{\mathcal{P}_i} \xrightarrow{V^{-1} - \mathrm{id}} \mathbf{G}^0_{\mathcal{P}_i} \longrightarrow BT_{\mathcal{P}_i} \longrightarrow 0$$

To define a biextension in $\mathrm{Biext}^1(BT_{\mathcal{P}_1} \times BT_{\mathcal{P}_2}, \widehat{\mathbb{G}}_m)$, it is enough to give a pair of bihomomorphisms (compare [Mu]):

$$\begin{aligned}\alpha_1 &: \mathbf{G}^{-1}_{\mathcal{P}_1}(\mathcal{N}) \times \mathbf{G}^0_{\mathcal{P}_2}(\mathcal{N}) \longrightarrow \widehat{\mathbb{G}}_m(\mathcal{N}),\\ \alpha_2 &: \mathbf{G}^0_{\mathcal{P}_1}(\mathcal{N}) \times \mathbf{G}^{-1}_{\mathcal{P}_2}(\mathcal{N}) \longrightarrow \widehat{\mathbb{G}}_m(\mathcal{N}),\end{aligned}$$

which agree on $\mathbf{G}^{-1}_{\mathcal{P}_1}(\mathcal{N}) \times \mathbf{G}^{-1}_{\mathcal{P}_2}(\mathcal{N})$, if we consider $\mathbf{G}^{-1}_{\mathcal{P}_i}$ as a subgroup of $\mathbf{G}^0_{\mathcal{P}_i}$ via the embedding $V^{-1} - id$, for $i = 1, 2$. To define α_1 and α_2 explicitly we use the Artin-Hasse exponential $\mathrm{hex} : \widehat{W}(\mathcal{N}) \to \widehat{\mathbb{G}}_m(\mathcal{N})$:

$$\begin{aligned}\alpha_1(y_1, x_2) &= \mathrm{hex}(V^{-1}y_1, x_2) \qquad \text{for } y_1 \in \mathbf{G}^{-1}_{\mathcal{P}_1}(\mathcal{N}),\ x_2 \in \mathbf{G}^0_{\mathcal{P}_2}(\mathcal{N})\\ \alpha_2(x_1, y_2) &= -\,\mathrm{hex}(x_1, y_2) \qquad \text{for } x_1 \in \mathbf{G}^0_{\mathcal{P}_1}(\mathcal{N}),\ y_2 \in \mathbf{G}^{-1}_{\mathcal{P}_2}(\mathcal{N})\end{aligned}$$

This completes the definition of the map (2).

Theorem 8. — *Let R be a ring, such that p is nilpotent in R, and such that the ideal of its nilpotent elements is nilpotent. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be displays over R. Assume that the display $\mathcal{P}_2$ is F-nilpotent, i.e. there is a number r such that $F^r P_2 \subset I_R P_2$. Then the map (2) is an isomorphism.*

I would expect that BT induces an equivalence of categories over any noetherian ring. We have the following result:

Theorem 9. — *Let R be an excellent local ring or a ring such that R/pR is an algebra of finite type over a field k. Assume that p is nilpotent in R. Then the functor BT is an equivalence from the category of displays over R to the category of formal p-divisible groups over R.*

We will now define the obstruction to lift a homomorphism of displays. Let $S \to R$ be a pd-thickening. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be displays over S, and let $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$ be their reductions over R. We consider a morphism of displays $\overline{\varphi} : \overline{\mathcal{P}}_1 \to \overline{\mathcal{P}}_2$. Let $\varphi : P_1 \to P_2$ the unique map which exists by theorem 3. It induces a map, which we call the obstruction to lift $\overline{\varphi}$:

$$\mathrm{Obst}\,\overline{\varphi} : Q_1/I_S P_1 \longrightarrow \mathfrak{a} \otimes_S P_2/Q_2$$

This morphism vanishes iff $\overline{\varphi}$ lifts to a homomorphism of displays $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$.

We will now assume that $pS = 0$ and that $\mathfrak{a}^p = 0$. We equip $S \to R$ with the trivial divided powers. Then $p \operatorname{Obst} \overline{\varphi} = 0$. Therefore $p\overline{\varphi}$ lifts to a homomorphism of displays $\psi : \mathcal{P}_1 \to \mathcal{P}_2$. Let us assume moreover that we are given a second surjection $T \to S$ with kernel $\mathfrak{b}$, such that $\mathfrak{b}^p = 0$, and such that $pT = 0$. Let $\widetilde{\mathcal{P}}_1$ and $\widetilde{\mathcal{P}}_2$ be two displays, which lift $\mathcal{P}_1$ and $\mathcal{P}_2$. Then we give an easy formula (proposition 73), which computes $\operatorname{Obst} \psi$ directly in terms of $\operatorname{Obst} \overline{\varphi}$. This formula was suggested by the work of Gross and Keating [GK], who considered one-dimensional formal groups. We demonstrate how some of the results in [G] and [K] may be obtained from our formula.

Finally we indicate how p-divisible groups with an étale part may be treated using displays. Let R be an artinian local ring with perfect residue class field k of characteristic $p > 0$. We assume moreover that $2R = 0$ if $p = 2$. The exact sequence

$$0 \longrightarrow W(\mathfrak{m}) \longrightarrow W(R) \xrightarrow{\pi} W(k) \longrightarrow 0,$$

admits a unique section $\delta : W(k) \to W(R)$, which is a ring homomorphism commuting with ^{F}.

We define as above:

$$\widehat{W}(\mathfrak{m}) = \{(x_0, x_1, \dots) \in W(\mathfrak{m}) \quad | \quad x_i = 0 \quad \text{for almost all } i\}$$

Since $\mathfrak{m}$ is a nilpotent algebra, $\widehat{W}(\mathfrak{m})$ is a subalgebra stable by F and V. Moreover $\widehat{W}(\mathfrak{m})$ is an ideal in $W(R)$.

We define a subring $\widehat{W}(R) \subset W(R)$:

$$\widehat{W}(R) = \{\xi \in W(R) \quad | \quad \xi - \delta\pi(\xi) \in \widehat{W}(\mathfrak{m})\}.$$

Again we have a split exact sequence

$$0 \longrightarrow \widehat{W}(\mathfrak{m}) \longrightarrow \widehat{W}(R) \xrightarrow{\pi} W(k) \longrightarrow 0,$$

with a canonical section δ of π. Under the assumptions made on R the subring $\widehat{W}(R) \subset W(R)$ is stable by ^{F} and ^{V}. Therefore we may replace in the definition of a 3n-display the ring $W(R)$ by $\widehat{W}(R)$. The resulting object will be called a Dieudonné display over R. In a forthcoming publication we shall prove:

Theorem. — *Let R be an artinian local ring with perfect residue field k of characteristic $p > 0$. We assume moreover that $2R = 0$ if $p = 2$. Then the category of Dieudonné displays over R is equivalent to the category of p-divisible groups over R.*

I introduced displays after discussions with M. Rapoport on the work of Gross and Keating [GK]. I thank Rapoport for his questions and comments and also for his constant encouragement, which made this work possible. I also thank J. de Jong, G.Faltings, and B.Messing for helpful remarks, and O.Gabber for his helpful questions, which he asked during lectures. The remarks of the referee helped me to correct an error in the first version of this paper. I forgot that Messing [Me] assumes nilpotent divided powers, which is necessary in the presence of an étale part (see the remarks

above). I am very grateful to him. Finally I thank the organizers of the "P-adic Semester" in Paris 1997 for giving me the possibility to present my results there. At this time a preliminary version of this work entitled "Cartier Theory and Crystalline Dieudonné Theory" was distributed.

Note added in March 2001: A proof of the last theorem above is given in [Z3]. The relation of the theory of Ch. Breuil [Br] to the theory given here is explained in [Z4]. A construction of the display associated to an abelian scheme over R is given in [LZ], by means of a de Rham-Witt complex relative to R.

1. Displays

1.1. Generalities. — Let A and B be commutative rings and $\rho : A \to B$ be a homomorphism. If N is a B-module, we denote by $N_{[\rho]}$ the A-module obtained by restriction of scalars. Let M be a A-module. A ρ-linear map $\alpha : M \to N$ is an A-linear map $\alpha : M \to N_{[\rho]}$. It induces a B-linear map $\alpha^{\#} : B \otimes_{\rho, A} M \to N$. We will say that α is a ρ-linear isomorphism (respectively epimorphism), if $\alpha^{\#}$ is an isomorphism (respectively epimorphism).

Let R be a unitary commutative ring, which is a $\mathbb{Z}_{(p)}$-algebra. Let $W(R)$ be the Witt ring with respect to the prime number p. We apply the definitions above to the case where $A = B = W(R)$, and where ρ is the Frobenius endomorphism ${}^{F} : W(R) \to W(R)$. (For notations concerning the Witt ring we refer to the introduction.) As an example we consider the Verschiebung ${}^{V} : W(R) \to W(R)$. It induces a $W(R)$-linear isomorphism

$$^{V} : W(R)_{[F]} \longrightarrow I_R.$$

Its inverse is a ^{F}-linear map:

$$^{V^{-1}} : I_R \longrightarrow W(R)$$

This map is a ^{F}-linear epimorphism, but it is not a ^{F}-linear isomorphism (!) unless R is a perfect ring.

We define base change for ^{F}-linear maps as follows. Let $S \to R$ be a homomorphism of commutative rings. Assume $\alpha : Q \to P$ is a ^{F}-linear homomorphism of $W(S)$-modules. Then the base change α_R is

$$\begin{array}{rcl} \alpha_R : W(R) \otimes_{W(S)} Q & \longrightarrow & W(R) \otimes_{W(S)} P. \\ w \otimes x & \longmapsto & {}^{F}w \otimes \alpha(x) \end{array}$$

We have

$$(\alpha^{\#})_{W(R)} = (\alpha_R)^{\#},$$

where the index $W(R)$ is base change for linear maps.

We are now ready to define the notion of a display.

Definition 1. — A $3n$-display over R is a quadrupel (P, Q, F, V^{-1}), where P is a finitely generated projective $W(R)$-module, $Q \subset P$ is a submodule and F and V^{-1} are ^{F}-linear maps $F : P \to P$, $V^{-1} : Q \to P$.

The following properties are satisfied:

(i) $I_R P \subset Q \subset P$ and there exists a decomposition of P into a direct sum of $W(R)$-modules $P = L \oplus T$, such that $Q = L \oplus I_R T$.

(ii) $V^{-1} : Q \to P$ is a ^{F}-linear epimorphism.

(iii) For $x \in P$ and $w \in W(R)$, we have

$$V^{-1}({}^V wx) = wFx. \tag{1}$$

We make some formal remarks on this definition. The 3n-displays form an additive category. We are mainly interested in the case, where R is a $\mathbb{Z}_p$-algebra. Then we have $\mathbb{Z}_p \subset W(R)$ and hence the category is $\mathbb{Z}_p$-linear.

The operator F is uniquely determined by V^{-1} because of the relation:

$$V^{-1}({}^V 1x) = Fx, \quad \text{for } x \in P.$$

If we apply this to the case $x = y \in Q$ and apply the ^{F}-linearity of V^{-1}, we obtain the relation:

$$Fy = p \cdot V^{-1} y. \tag{2}$$

A decomposition $P = L \oplus T$ as required in (i), we will call a normal decomposition. We set $\overline{P} = P/I_R P$ and $\overline{Q} = Q/I_R P$. Then we get a filtration of R-modules

$$0 \subset \overline{Q} \subset \overline{P}, \tag{3}$$

whose graded pieces are projective finitely generated R-modules. This is the Hodge filtration associated to a display.

Lemma 2. — *Let R be a p-adically complete and separated ring. Let us replace in the definition* 1 *the condition (i) by the weaker condition that $I_R P \subset Q \subset P$ and that the filtration (3) has finitely generated projective R-modules as graded pieces. Then (P, Q, F, V^{-1}) is a 3n-display.*

Before proving the lemma we need a general fact about the Witt ring.

Proposition 3. — *Let R be a p-adic ring, i.e. complete and separated in the p-adic topology. Then the ring $W(R)$ is p-adic. Moreover it is complete and separated in the I_R-adic topology.*

Proof. — We begin to show that $W(R)$ is separated in the p-adic topology. Since $W(R)$ is the projective limit of the rings $W_n(R/p^m R)$ for varying n and m it is enough to show that that p is nilpotent in each of the rings $W_n(R/p^m R)$. To see this we consider a ring $\mathfrak{a}$ without unit such that $p^m \mathfrak{a} = 0$. An easy induction on m shows that p is nilpotent in $W_n(\mathfrak{a})$.

It is enough to prove our assertion for a ring R which has no p-torsion. Indeed in the general case we may choose a surjection $S \to R$ where S is a torsion free p-adic ring. But then we obtain a surjection $W(S) \to W(R)$ from the p-adic ring $W(S)$ to the p-adically separated ring $W(R)$. This implies that $W(R)$ is a p-adic ring.

To treat the case of a p-adic ring we need a few lemmas:

Lemma 4. — *Let S be a ring without p-torsion. Let $x = (x_0, \ldots, x_m) \in W_{m+1}(S)$ be a Witt vector. Then for any fixed number $s \geq 1$ the following conditions are equivalent:*

(i) $p^s \mid x_i$ *for* $i = 0, \ldots, m$

(ii) $p^{n+s} \mid \mathbf{w}_n(x)$ *for* $n = 0, \ldots, m$.

Proof. — The first condition clearly implies the second. Assume the second condition holds. By induction we may assume $p^s \mid x_i$ for $i = 0, \ldots, n-1$. Then we write

$$\mathbf{w}_n(x) = \mathbf{w}_{n-1}(x_0^p \ldots x_{n-1}^p) + p^n x_n.$$

By the obvious implication and by induction the first term on the right hand side is $\equiv 0$ mod $p^{(n-1)+ps}$. Since $(n-1) + ps \geq n + s$, we conclude $p^n x_n \equiv 0 \mod p^{n+s}S$. $\square$

Lemma 5. — *Let R be a p-torsion free ring. Let $a \in W_m(R)$ be a given Witt vector. Let u be a number. We assume that the equation*

$$p^u x = a \tag{4}$$

has for each s a solution in the ring $W_m(R/p^sR)$. Then the equation (4) has a solution in $W_m(R)$.

Proof. — Let us consider a fixed s. By assumption there is a $z \in W_m(R)$, such that $p^u z = a$ holds in the ring $W_m(R/p^{s+u}R)$. We let x_s be the image of z in the ring $W_m(R/p^sR)$. Then we claim that x_s is independent of the choice of z.

Indeed, let z' be a second choice and set $\xi = z - z'$. The Witt components of $p^u\xi$ are elements of $p^{s+u}R$. Hence the lemma implies

$$p^{n+s+u} \mid \mathbf{w}_n(p^u\xi) \text{ for } n = 0 \ldots m-1.$$

It follows that $p^{n+s} \mid \mathbf{w}_n(\xi)$. But applying the lemma again we obtain the $p^s \mid \xi_i$ for all Witt components of ξ.

This shows the uniqueness of x_s. We set $x = \varprojlim x_s \in W(R)$ and obtain the desired solution of (4). $\square$

Lemma 6. — *Let S be without p-torsion. We will denote by I_r the ideal ${}^{V^r}W(S) \subset W(S)$. Let $\mathcal{T}$ be the linear topology on $W(S)$, such that the following ideals form a fundamental set of open neighbourhoods of zero:*

$$I_r + W(p^sS) \tag{5}$$

Here, r, s runs through all pairs of numbers.

Then $p^uW(S)$ is for each number u closed in the topology $\mathcal{T}$.

Proof. — We have to show

$$\bigcap_{r,s\in\mathbb{N}} p^u W(S) + I_r + W(p^s S) = p^u W(S) \tag{6}$$

Let x be an element from the left hand side.

We denote for a fixed number r by $\overline{x}$ the image of x in $W_r(S)$. Then the equation

$$p^u z = \overline{x}$$

has a solution z in the ring $W_r(S/p^s S)$ for each number s. By the last lemma we have a solution in $W_r(S)$ too. This shows $x \in p^u W(S) + I_r$.

We take the unique solution $z_r \in W_r(S)$ of $p^u z_r = x$ in $W_r(S)$, and we set $z = \varprojlim z_r$. Hence $x = p^u z \in p^u W(S)$. □

Let S be a torsion free p-adic ring. Clearly the Witt ring $W(S)$ is complete and separated in the topology $\mathcal{T}$. The assertion that $W(S)$ is p-adic is a consequence of the last lemma and the following elementary topological fact (see Bourbaki Topologie III §3 Cor 1):

Lemma 7. — *Let G be an abelian group. Let $\mathcal{A}$ resp. $\mathcal{B}$ be linear topologies on G, which are given by the fundamental systems of neighbourhood of zero $\{A_n\}$ resp. $\{B_n\}$, where A_n and B_n are subgroups.*

We make the following assumptions:

a) *Each A_n is open in the $\mathcal{B}$-topology, i.e. the $\mathcal{B}$ topology is finer.*
b) *Each B_n is closed in the $\mathcal{A}$-topology.*
c) *G is complete and separated in the $\mathcal{A}$-topology.*

Then G is complete and separated in the $\mathcal{B}$-topology.

We omit the easy proof.

We note that in the Witt ring $W(R)$ of any ring we have an equality of ideals for any natural number n:

$$I_R^n = p^{n-1} I_R \tag{7}$$

If R is a p-adic ring the additive group I_R is p-adically complete and separated, because it is by the Verschiebung isomorphic to $W(R)$. This shows that $W(R)$ is then also complete in the I_R-adic topology. This completes the proof of proposition 3. □

Corollary 8. — *Assume that p is nilpotent in R. Then the p-adic and the I_R-adic topology on $W(R)$ coincide. This topology is finer than the V-adic topology, which has the ideals $I_n = {}^{V^n}W(R)$ as a fundamental system of neighbourhoods of zero.*

Proof. — This is clear. □

We turn now to the proof of lemma 2. The proposition 3 implies in particular that $W(R)$ is complete and separated in the I_R-adic topology. We set $A_n = W(R)/I_R^n$. We start with a decomposition $\overline{P} = \overline{L} \oplus \overline{T}$ such that $Q/I_R P = \overline{L}$ over $A_1 = R$ and lift it step by step to a decomposition $A_n \otimes_{W(R)} P = L_n \oplus T_n$ over A_n using the surjections with nilpotent kernel $A_n \to A_{n-1}$. Then we obtain the desired decomposition by taking the projective limit.

Lemma 9. — *Let (P, Q, F, V^{-1}) be a 3n-display over a ring R, and $P = L \oplus T$ be a normal decomposition. Then the map*

$$V^{-1} \oplus F : L \oplus T \longrightarrow P \tag{8}$$

is a ^{F}-linear isomorphism.

Proof. — Since source and target of $V^{-1} \oplus F$ are projective modules of the same rank, it is enough to show, that we have a ^{F}-linear epimorphism. Indeed, by the property (ii) of the definition 1 the $W(R)$-module P is generated by $V^{-1}l$, for $l \in L$ and $V^{-1}({}^{V}wt)$ for $t \in T$ and $w \in W$. The lemma follows, since $V^{-1}({}^{V}wt) = wFt$. □

Using this lemma we can define structural equations for a 3n-display, whose Hodge filtration (3) has free graded pieces. Let (P, Q, F, V^{-1}) be a 3n-display over R with this property. Then the modules L and T in a normal decomposition $P = L \oplus T$, are free. We choose a basis $e_1, \ldots, e_d$ of T, and basis $e_{d+1} \ldots e_h$ of L. Then there are elements $\alpha_{ij} \in W(R)$, $\quad i, j = 1, \ldots, h$, such that the following relations hold.

$$\begin{aligned} Fe_j &= \sum_{i=1}^{h} \alpha_{ij} e_i, \quad \text{for } j = 1, \ldots, d \\ V^{-1}e_j &= \sum_{i=1}^{h} \alpha_{ij} e_i \quad \text{for } j = d+1, \ldots, h \end{aligned} \tag{9}$$

By the lemma 9 the matrix (α_{ij}) is invertible.

Conversely assume we are given an invertible $h \times h$-matrix (α_{ij}) over the ring $W(R)$ and a number d, such that $0 \leq d \leq h$. Let T be the free $W(R)$-module with basis $e_1, \ldots e_d$ and L be the free $W(R)$-module with basis $e_{d+1}, \ldots, e_h$. We set $P = L \oplus T$ and $Q = L \oplus I_R T$, and we define the ^{F}-linear operators F and V^{-1} by the equations (9) and the following equations

$$\begin{aligned} Fe_j &= \sum_{i=1}^{h} p\alpha_{ij} e_i, \quad j = d+1, \ldots, h \\ V^{-1}({}^{V}we_j) &= \sum_{i=1}^{h} w\alpha_{ij} e_i, \quad j = 1, \ldots, d \end{aligned}$$

One verifies easily, that this defines a 3n-display over R.

For a 3n-display (P, Q, F, V^{-1}) we do not have an operator V as in Dieudonné or Cartier theory. Instead we have a $W(R)$-linear operator:

$$V^{\sharp} : P \longrightarrow W(R) \otimes_{F, W(R)} P \tag{10}$$

Lemma 10. — *There exists a unique $W(R)$-linear map (10), which satisfies the following equations:*

$$\begin{aligned} V^\sharp(wFx) &= p\cdot w\otimes x, \quad \textit{for } w\in W(R), x\in P \\ V^\sharp(wV^{-1}y) &= w\otimes y, \quad \textit{for } y\in Q \end{aligned} \tag{11}$$

Moreover we have the identities

$$F^\sharp V^\sharp = p\,\mathrm{id}_P, \quad V^\sharp F^\sharp = p\,\mathrm{id}_{W(R)\otimes_{F,W(R)}P}\,. \tag{12}$$

Proof. — Clearly $V^\sharp$ is uniquely determined, if it exists. We define the map $V^\sharp$ by the following commutative diagram, where $W=W(R)$:

$$\begin{tikzcd} W\otimes_{F,W}L\oplus W\otimes_{F,W}T \arrow[r, "(V^{-1}+F)^\sharp"] \arrow[d, "\mathrm{id}+p\,\mathrm{id}"'] & P \arrow[d, "V^\sharp"] \\ W\otimes_{F,W}L\oplus W\otimes_{F,W}T \arrow[r] & W\otimes_{F,W}P \end{tikzcd} \tag{13}$$

Here the lower horizontal map is the identity.

We need to verify (11) with this definition. We write $x=l+t$, for $l\in L$ and $t\in T$.

$$\begin{aligned} V^\sharp(wFx) = V^\sharp(wFl)+V^\sharp(wFt) &= V^\sharp(V^{-1}({}^Vwl))+V^\sharp(wFt) \\ &= 1\otimes {}^Vwl + pw\otimes t = pw\otimes(l+t) = pw\otimes x. \end{aligned}$$

Next take y to be of the form $y=l+{}^Vut$.

$$\begin{aligned} V^\sharp(wV^{-1}y) &= V^\sharp(wV^{-1}l)+V^\sharp(wuFt) \\ &= w\otimes l + pwu\otimes t = w\otimes l + w^{FV}u\otimes t \\ &= w\otimes(l+{}^Vut) = w\otimes y. \end{aligned}$$

The verification of (12) is left to the reader. □

Remark. — The cokernel of $V^\sharp$ is a projective $W(R)/pW(R)$-module of the same rank as the R-module P/Q.

Let us denote by ${}^{F^i}V^\sharp$ the $W(R)$-linear map

$$\mathrm{id}\otimes_{F^i,W(R)}V^\sharp : W\otimes_{F^i,W}P \longrightarrow W\otimes_{F^{i+1},W}P,$$

and by $V^{n\sharp}$ the composite ${}^{F^{n-1}}V^\sharp\circ\cdots\circ{}^F V^\sharp\circ V^\sharp$.

We say that a 3n-display satisfies the nilpotence (or V-nilpotence) condition, if there is a number N, such that the map

$$V^{N\sharp} : P \longrightarrow W(R)\otimes_{F^N,W(R)}P$$

is zero modulo $I_R+pW(R)$. Differently said, the map

$$R/pR\otimes_{\mathbf{w}_0,W(R)}P \longrightarrow R/pR\otimes_{\mathbf{w}_N,W(R)}P \tag{14}$$

induced by $V^{N\sharp}$ is zero.

Definition 11. — Let p be nilpotent in R. A display (P, Q, F, V^{-1}) is a 3n-display, which satisfies the nilpotence condition above.

Let us choose a normal decomposition $P = L \oplus T$. It is obvious from the diagram (13) that the map

$$R/pR \otimes_{\mathbf{w}_\circ, W(R)} P \xrightarrow{V^\sharp} R/pR \otimes_{\mathbf{w}_1, W(R)} P \xrightarrow{pr} R/pR \otimes_{\mathbf{w}_1, W(R)} T$$

is zero. Therefore it is equivalent to require the nilpotence condition for the following map:

$$U^\sharp : L \hookrightarrow L \oplus T = P \xrightarrow{V^\sharp} W \otimes_{F,W} P \xrightarrow{pr} W \otimes_{F,W} L$$

Less invariantly but more elementary the nilpotence condition may be expressed if we choose a basis as in (9). Let $(\beta_{k,l})$ be the inverse matrix to $(\alpha_{i,j})$. Consider the following $(h-d) \times (h-d)$-matrix with coefficients in R/pR:

$$B = (\mathbf{w}_0(\beta_{kl}) \text{ modulo } p)_{k,l=d+1,\ldots,h}$$

Let $B^{(p^i)}$ be the matrix obtained by raising the coefficients to the p^i-th power. Then the nilpotence condition says exactly that for a suitable number N :

$$B^{(p^{N-1})} \cdots B^{(p)} \cdot B = 0 \tag{15}$$

Corollary 12. — *Assume that p is nilpotent in R. Let (P, Q, F, V^{-1}) be a display over R. Then for any given number n there exists a number N, such that the following map induced by $V^{N\sharp}$ is zero:*

$$W_n(R) \otimes_{W(R)} P \longrightarrow W_n(R) \otimes_{F^N, W(R)} P$$

Proof. — Indeed, by the proof of proposition 3 the ideal $I_R + pW_n(R)$ in $W_n(R)$ is nilpotent. □

We will also consider displays over linear topological rings R of the following type. The topology on R is given by a filtration by ideals:

$$R = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_n \ldots, \tag{16}$$

such that $\mathfrak{a}_i \mathfrak{a}_j \subset \mathfrak{a}_{i+j}$. We assume that p is nilpotent in $R/\mathfrak{a}_1$ and hence in any ring $R/\mathfrak{a}_i$. We also assume that R is complete and separated with respect to this filtration. In the context of such rings we will use the word display in the following sense:

Definition 13. — Let R be as above. A 3n-display $\mathcal{P} = (P, Q, F, V^{-1})$ over R is called a display, if the 3n-display obtained by base change over $R/\mathfrak{a}_1$ is a display in sense of definition 11.

Let $\mathcal{P}$ be a display over R. We denote by $\mathcal{P}_i$ the 3n-display over $R/\mathfrak{a}_i$ induced by base change. Then $\mathcal{P}_i$ is a display in the sense of definition 11. There are the obvious transition isomorphisms

$$\phi_i : (\mathcal{P}_{i+1})_{R/\mathfrak{a}_i} \longrightarrow \mathcal{P}_i \tag{17}$$

Conversely assume we are given for each index i a display $\mathcal{P}_i$ over the discrete ring $R/\mathfrak{a}_i$, and transition isomorphisms ϕ_i as above. Then the system $(\mathcal{P}_i, \phi_i)$ is obtained from a display $\mathcal{P}$ over R. In fact this is an equivalence of the category of systems of displays $(\mathcal{P}_i, \phi_i)$ and the category of displays over R.

If R is for example complete local ring with maximal ideal $\mathfrak{m}$, such that $pR = 0$, we can consider the category of displays over R in the sense of definition 11 but we can also consider the category of displays over the topological ring R, with its $\mathfrak{m}$-adic topology. The last category is in general strictly bigger.

1.2. Examples

Example 14. — Let $R = k$ be a perfect field. A Dieudonné module over k is a finitely generated free $W(k)$-module M, which is equipped with a ^{F}-linear map $F : M \to M$, and a ${}^{F^{-1}}$-linear map $V : M \to M$, such that:

$$FV = VF = p$$

We obtain a 3n-display by setting $P = M,\ Q = VM$ with the obvious operators $F : M \to M$ and $V^{-1} : VM \to M$. Moreover (P, Q, F, V^{-1}) is a display if the map $V : M/pM \to M/pM$ is nilpotent. The map $V^{\sharp}$ is given by

$$\begin{aligned} V^{\sharp} : M &\longrightarrow W(k) \otimes_{F,W(k)} M. \\ m &\longmapsto 1 \otimes Vm \end{aligned}$$

In the other direction starting with a display (P, Q, F, V^{-1}) we obtain a Dieudonné module structure on P if we define V as the composite:

$$\begin{array}{rcl} V : P \xrightarrow{V^{\sharp}} W(k) \otimes_{F,W(k)} P & \longrightarrow & P \\ w \otimes x & \longmapsto & {}^{F^{-1}}w \cdot x \end{array} \tag{18}$$

This makes sense because the Frobenius endomorphism ^{F} is an automorphism of $W(k)$. We see that the category of 3n-displays over a perfect field is naturally equivalent to the category of Dieudonné modules.

More generally let k be a perfect ring of characteristic p. Then ^{F} is an automorphism on $W(k)$ and $pW(k) = I_k$. We call a Dieudonné module k a finitely generated projective $W(k)$-module M equipped with two $\mathbb{Z}$-linear operators

$$F : M \longrightarrow M,$$

$$V : M \longrightarrow M,$$

which satisfy the relation $F(wx) = {}^FwFx,\ V({}^Fwx) = wVx,\ FV = VF = p$.

If we are given a homomorphism of $k \to k'$ of perfect rings, we obtain the structure of a Dieudonné module on $M' = W(k') \otimes_{W(k)} M$.

Since p is injective on $W(k)$, there is an exact sequence of k-modules:

$$0 \longrightarrow M/FM \xrightarrow{V} M/pM \longrightarrow M/VM \longrightarrow 0$$

If we tensorize this sequence with k' we obtain the corresponding sequence for M'. In particular this sequence remains exact. We also see from the sequence that M/VM is of finite presentation. Hence we conclude that M/VM is a finitely generated projective k-module. Therefore we obtain a 3n-display (M, VM, F, V^{-1}).

Proposition 15. — *The category of 3n-displays over a perfect ring k is equivalent to the category of Dieudonné modules over k. Moreover the displays correspond exactly to the Dieudonné modules, such that V is topologically nilpotent for the p-adic topology on M.*

The proof is obvious. We remark that a Dieudonné module M, such that V is topologically nilpotent is a reduced Cartier module. The converse is also true by [Z1] Korollar 5.43.

We note that Berthelot [B] associates to any p-divisible group over a perfect ring a Dieudonné module. In the case of a formal p-divisible group his construction gives the Cartier module (compare [Z2] Satz 4.15).

Example 16. — The multiplicative display $\mathcal{G}_m = (P, Q, F, V^{-1})$ over a ring R is defined as follows. We set $P = W(R)$, $Q = I_R$ and define the maps $F : P \to P$, $V^{-1} : Q \to P$ by:

$$\begin{aligned} Fw &= {}^F w \quad \text{for } w \in W(R) \\ V^{-1}({}^V w) &= w \end{aligned}$$

We note that in this case the map $V^\sharp$ is given by:

$$\begin{gathered} V^\sharp : W(R) \longrightarrow W(R) \otimes_{F, W(R)} W(R) \overset{\kappa}{\cong} W(R) \\ V^\sharp w = 1 \otimes {}^V w = pw \otimes 1 \end{gathered}$$

Hence using the canonical isomorphism κ the map $V^\sharp$ is simply multiplication by p. Therefore we have a display, if p is nilpotent in R, or more generally in the situation of definition 13.

Example 17. — To any 3n-display we can associate a dual 3n-display. Assume we are given two 3n-displays $\mathcal{P}_1$ and $\mathcal{P}_2$ over R.

Definition 18. — A bilinear form of 3n-displays

$$(\ ,\ ) : \mathcal{P}_1 \times \mathcal{P}_2 \longrightarrow \mathcal{G}_m$$

is a bilinear form of $W(R)$-modules

$$(\ ,\ ) : P_1 \times P_2 \longrightarrow W(R),$$

which satisfies the following relation:

$$ {}^V(V^{-1}y_1, V^{-1}y_2) = (y_1, y_2), \quad \text{for } y_1 \in Q_1,\ y_2 \in Q_2 \tag{19}$$

We will denote the abelian group of bilinear forms by $\mathrm{Bil}(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{G}_m)$.

The last relation implies the following:

$$
\begin{aligned}
(V^{-1}y_1, Fx_2) &= {}^F(y_1, x_2) \quad \text{for } y_1 \in Q_1,\ x_2 \in P_2 \\
(Fx_1, Fx_2) &= p\,{}^F(x_1, x_2) \text{ for } x_1 \in P_1, \\
(Fx_1, V^{-1}y_2) &= {}^F(x_1, y_2) \quad \text{for } y_2 \in Q_2,
\end{aligned} \tag{20}
$$

Indeed,

$$
{}^V(V^{-1}y_1, Fx_2) = {}^V(V^{-1}y_1, V^{-1}({}^V1x_2)) = (y_1, {}^V1x_2) = {}^V1(y_1, x_2) = {}^{VF}(y_1, x_2)
$$

implies the first relation of (20) because ^{V} is injective. The other relations are verified in the same way. We note that $(Q_1, Q_2) \subset I_R$ by (19). Assume we are given a finitely generated projective $W(R)$-module P. Then we define the dual module:

$$
P^* = \mathrm{Hom}_{W(R)}(P, W(R))
$$

Let us denote the resulting perfect pairing by $(\ ,\ )$:

$$
\begin{array}{ccc}
P \times P^* & \longrightarrow & W(R) \\
x \times z & \longmapsto & (x, z)
\end{array} \tag{21}
$$

There is also an induced pairing

$$
(\ ,\ ) : W(R) \otimes_{F,W(R)} P \times W(R) \otimes_{F,W(R)} P^* \longrightarrow W(R),
$$

which is given by the formula:

$$
(w \otimes x, v \otimes z) = wv\,{}^F(x, z), \quad x \in P,\ z \in P^*,\ w, v \in W(R)
$$

Let us consider a 3n-display $\mathcal{P} = (P, Q, F, V^{-1})$ over R. We set $\widehat{Q} = \{\phi \in P^* \mid \phi(Q) \subset I_R\}$. Then $\widehat{Q}/I_R P^*$ is the orthogonal complement of $Q/I_R P$ by the induced perfect pairing:

$$
P/I_R P \times P^*/I_R P^* \longrightarrow R
$$

Definition 19. — There is a unique 3n-display $\mathcal{P}^t = (P^*, \widehat{Q}, F, V^{-1})$, such that the operators F and V^{-1} satisfy the following relations with respect to the pairing (21):

$$
\begin{aligned}
(V^{-1}x, Fz) &= {}^F(x, z) \quad \text{for } x \in Q,\ z \in P^* \\
(Fx, Fz) &= p\,{}^F(x, z) \text{ for } x \in P,\ z \in P^* \\
(Fx, V^{-1}z) &= {}^F(x, z) \quad \text{for } x \in P,\ \ z \in \widehat{Q} \\
{}^V(V^{-1}x, V^{-1}z) &= (x, z) \quad\ \ \text{for } x \in Q,\ \ z \in \widehat{Q}
\end{aligned} \tag{22}
$$

Hence we have a bilinear form of displays

$$
\mathcal{P} \times \mathcal{P}^t \longrightarrow \mathcal{G}_m
$$

We call $\mathcal{P}^t$ the dual 3n-display.

As for ordinary bilinear forms one has a canonical isomorphism:

$$\mathrm{Bil}(\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{G}_m) \longrightarrow \mathrm{Hom}(\mathcal{P}_2, \mathcal{P}_1^t) \tag{23}$$

From the relations of definition 19 we easily deduce that the $W(R)$-linear maps $F^\sharp$ and $V^\sharp$ for $\mathcal{P}$ respectively $\mathcal{P}^t$ are dual to each other:

$$\begin{aligned} (V^\sharp x, v \otimes z) &= (x, F^\sharp(v \otimes z)) \\ (F^\sharp(w \otimes x), z) &= (w \otimes x, V^\sharp z) \end{aligned} \tag{24}$$

Let us assume that p is nilpotent in R. In terms of the dual 3n-display we may rephrase the nilpotence condition as follows. Iterating the homomorphism $F^\sharp$ for the dual 3n- display we obtain a map:

$$F^{N\,\sharp} : W(R) \otimes_{F^N, W(R)} P^* \longrightarrow P^* \tag{25}$$

Then the 3n-display $\mathcal{P}$ satisfies the V-nilpotence condition, iff for any number n there exists a number N, such that the following map induced by (25) is zero:

$$F^{N\,\sharp} : W_n(R) \otimes_{F^N, W(R)} P^* \longrightarrow W_n(R) \otimes_{W(R)} P^*$$

In this case we will also say that $\mathcal{P}^t$ satisfies the F-nilpotence condition.

Next we define base change for a 3n-display. Suppose we are given a ring homomorphism $\varphi : S \to R$. Let P be a $W(S)$-module. If $\varphi : P \to P'$ is a ^{F}-linear map of $W(S)$-modules, we define the base change $\varphi_{W(R)}$ as follows:

$$\begin{aligned} \varphi_{W(R)} : W(R) \otimes_{W(S)} P &\longrightarrow W(R) \otimes_{W(S)} P' \\ w \otimes x \quad &\longmapsto \quad {}^F w \otimes \varphi(x) \end{aligned}$$

Then we have $(\varphi_{W(S)})^\sharp = \mathrm{id}_{W(R)} \otimes_{W(S)} \varphi^\sharp$ for the linearizations.

Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a 3n-display over S. Let $\varphi : S \to R$ be any ring morphism. We will now define the 3n-display obtained by base change with respect to φ.

Definition 20. — We define $\mathcal{P}_R = (P_R, Q_R, F_R, V_R^{-1})$ to be the following quadruple:

We set $P_R = W(R) \otimes_{W(S)} P$.

We define Q_R to be the kernel of the morphism $W(R) \otimes_{W(S)} P \to R \otimes_S P/Q$.

We set $F_R = {}^F \otimes F$.

Finally we let $V_R^{-1} : Q_R \to P_R$ be the unique $W(R)$-linear homomorphism, which satisfies the following relations:

$$\begin{aligned} V_R^{-1}(w \otimes y) &= {}^F w \otimes V^{-1} y, \ \text{for } w \in W(R), y \in Q \\ V_R^{-1}({}^V w \otimes x) &= w \otimes Fx, \ \text{for } x \in P \end{aligned} \tag{26}$$

Then $\mathcal{P}_R$ is a 3n-display over R, which is called the 3n-display obtained by base change.

To show that this definition makes sense we have only to prove the existence and uniqueness of V_R^{-1},. The uniqueness is clear. For the existence we choose a normal decomposition $P = L \oplus T$. Then we have an isomorphism:

$$Q_R \simeq W(R) \otimes_{W(S)} L \oplus I_R \otimes_{W(S)} T$$

We define V_R^{-1} on the first summand by the first equation of (26) and on the second direct summand by the second equation. We leave the verification that (26) holds with this definition to the reader.

In the case where φ is surjective the image of the morphism $W(R) \otimes_{W(S)} Q \to W(R) \otimes_{W(S)} P = P_R$, is simply Q_R, but in general this image is strictly smaller than Q_R.

By looking for example at (15) it is clear that $\mathcal{P}_R$ is a display if $\mathcal{P}$ was a display. There is also an obvious converse statement.

Lemma 21. — *Let $\phi : S \to R$ be a ring homomorphism, such that any element in the kernel of ϕ is nilpotent. Then $\mathcal{P}$ is a display if $\mathcal{P}_R$ is a display.*

Remark. — Before we turn to the next example, we collect some general facts about the liftings of projective modules. Let $S \to R$ be a surjective ring homomorphism, such that any element in the kernel is nilpotent, or such that S is complete and separated in the adic topology defined by this kernel. Assume we are given a finitely generated projective module P over R. Then P lifts to S, i.e. there is a finitely generated projective S-module $\widetilde{P}$ together with an isomorphism $\phi : R \otimes_S \widetilde{P} \to P$. By the lemma of Nakayama the pair $(\widetilde{P}, \phi)$ is uniquely determined up to isomorphism. The existence follows from the well-known fact that idempotent elements lift with respect the surjection of matrix algebras $\mathrm{End}_S(S^u) \to \mathrm{End}_R(R^u)$, where u is some number (e.g. H.Bass, Algebraic K-Theory, W.A. Benjamin 1968, Chapt. III Prop. 2.10).

Let L be a direct summand of P. A lifting of L to a direct summand of P is obtained as follows. Let $\widetilde{L}$ be any lifting of L to S. Let $\widetilde{L} \to \widetilde{P}$ be any lifting of $L \to P$, whose existence is guaranteed by the universal property of projective modules. In this way $\widetilde{L}$ becomes a direct summand of $\widetilde{P}$. This is easily seen, if one lifts in the same way a complement T of L in P. Indeed the natural map $\widetilde{L} \oplus \widetilde{T} \to \widetilde{P}$ is by Nakayama an isomorphism.

Let us now assume that the kernel of $S \to R$ consists of nilpotent elements. We also assume that p is nilpotent in S. Let now P denote a projective $W(R)$-module. We set $P_R = R \otimes_{\mathbf{w_0}, W(R)} P$. We have seen that P_R may be lifted to a finitely generated projective S-module $\widetilde{P}_S$. Since $W(S)$ is complete and separated in the I_S-adic topology by proposition 3, we can lift $\widetilde{P}_S$ to a projective finitely generated $W(S)$-module $\widetilde{P}$. We find an isomorphism $W(R) \otimes_{W(S)} \widetilde{P} \to P$, because liftings of P_R to $W(R)$ are uniquely determined up to isomorphism. Hence finitely generated projective modules lift with respect to $W(S) \to W(R)$. Since the kernel of the last morphism lies in the radical of $W(S)$, this lifting is again unique up to isomorphism. We also may lift direct summands as described above.

Let $(\widetilde{P}, \widetilde{Q}, F, V^{-1})$ be a 3n-display over S and (P, Q, F, V^{-1}) be the 3n-display obtained by base change over R. Then any normal decomposition $P = L \oplus T$ may be lifted to a normal decomposition $\widetilde{P} = \widetilde{L} \oplus \widetilde{T}$. Indeed choose any finitely generated projective $W(S)$-modules $\widetilde{L}$ and $\widetilde{T}$, which lift L and T. Because $\widetilde{Q} \to Q$ is surjective, we may lift the inclusion $L \to Q$ to a $W(S)$-module homomorphism $\widetilde{L} \to \widetilde{Q}$. Moreover we find a $W(S)$-module homomorphism $\widetilde{T} \to \widetilde{P}$, which lifts $T \to P$. Clearly this gives the desired normal decomposition $\widetilde{P} = \widetilde{L} \oplus \widetilde{T}$.

Example 22. — Let $S \to R$ be a surjection of rings with kernel $\mathfrak{a}$. We assume that p is nilpotent in S, and that each element $a \in \mathfrak{a}$ is nilpotent.

Let $\mathcal{P}_0 = (P_0, Q_0, F, V^{-1})$ be a 3n-display over R. A deformation (or synonymously a lifting) of $\mathcal{P}_0$ to S is a 3n-display $\mathcal{P} = (P, Q, F, V^{-1})$ over S together with an isomorphism:

$$\mathcal{P}_R \cong \mathcal{P}_0.$$

Let us fix a deformation $\mathcal{P}$. To any homomorphism

$$\alpha \in \operatorname{Hom}_{W(S)}(P, W(\mathfrak{a}) \otimes_{W(S)} P),$$

we associate another deformation $\mathcal{P}_\alpha = (P_\alpha, Q_\alpha, F_\alpha, V_\alpha^{-1})$ as follows:

We set $P_\alpha = P$, $Q_\alpha = Q$, and

$$\begin{aligned} F_\alpha x &= Fx - \alpha(Fx), \quad \text{for } x \in P \\ V_\alpha^{-1} y &= V^{-1}y - \alpha(V^{-1}y), \quad \text{for } y \in Q. \end{aligned} \tag{27}$$

The surjectivity of $(V_\alpha^{-1})^\sharp$ follows the kernel of $W(S) \to W(R)$ is in the radical of $W(S)$ and therefore Nakayama's lemma is applicable.

Since F and F_α respectively V^{-1} and V_α^{-1} are congruent modulo $W(\mathfrak{a})$ the 3n-display $\mathcal{P}_{\alpha,R}$ obtained by base change is canonically isomorphic to $\mathcal{P}_0$.

We note that any deformation is isomorphic to $\mathcal{P}_\alpha$ for a suitable homomorphism α. Indeed, let $\mathcal{P}_1 = (P_1, Q_1, F_1, V_1^{-1})$ be any other deformation of $\mathcal{P}_0$. We find an isomorphism of the pairs (P, Q) and (P_1, Q_1), which reduces to the identity on (P_0, Q_0). Indeed, we fix a normal decomposition $P_0 = L_0 \oplus T_0$ and lift it to a normal decomposition of $\mathcal{P}$ respectively of $\mathcal{P}_1$. Then any isomorphism between the lifted normal decompositions is suitable. Hence we may assume that $(P, Q) = (P_1, Q_1)$. Then we define ^{F}-linear homomorphisms

$$\xi : P \longrightarrow W(\mathfrak{a}) \otimes_{W(S)} P, \quad \eta : Q \longrightarrow W(\mathfrak{a}) \otimes_{W(S)} P,$$

by the equations:

$$\begin{aligned} F_1 x &= Fx - \xi(x) \quad \text{for } x \in P \\ V_1^{-1} y &= V^{-1}y - \eta(y) \quad \text{for } y \in Q. \end{aligned} \tag{28}$$

Then ξ and η must satisfy the relation:

$$\eta({}^V wx) = w\xi(x), \quad \text{for } x \in P.$$

It is then easily checked that there is a unique homomorphism α as above, which satisfies the relations:

$$\begin{aligned} \alpha(V^{-1}y) &= \eta(y), \quad \text{for } y \in Q \\ \alpha(Fx) &= \xi(x), \quad \text{for } x \in P \end{aligned}$$

Then the deformations $\mathcal{P}_\alpha$ and $\mathcal{P}_1$ are isomorphic.

Example 23. — Let R be a ring such that $p \cdot R = 0$. Let us denote by $\mathrm{Frob} : R \to R$ the absolute Frobenius endomorphism, i.e. $\mathrm{Frob}(r) = r^p$ for $r \in R$.

Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a 3n-display over R. We denote the 3n-display obtained by base change with respect to Frob by $\mathcal{P}^{(p)} = (P^{(p)}, Q^{(p)}, F, V^{-1})$. More explicitly we have

$$\begin{aligned} P^{(p)} &= W(R) \otimes_{F,W(R)} P \\ Q^{(p)} &= I_R \otimes_{F,W(R)} P + \mathrm{Image}\,(W(R) \otimes_{F,W(R)} Q) \end{aligned}$$

The operators F and V^{-1} are uniquely determined by the relations:

$$\begin{aligned} F(w \otimes x) &= {}^F w \otimes Fx, \quad \text{for } w \in W(R), x \in P \\ V^{-1}({}^V w \otimes x) &= w \otimes Fx, \\ V^{-1}(w \otimes y) &= {}^F w \otimes V^{-1}y, \quad \text{for } y \in Q. \end{aligned}$$

(At the first glance it might appear that this explicit definition does not use $p \cdot R = 0$. But without this condition $Q^{(p)}/I_R P^{(p)}$ would not be a direct summand of $P^{(p)}/I_R P^{(p)}$. The elements $1 \otimes {}^V wx = pw \otimes x$ would cause trouble, if ^{F} and ^{V} do not commute.)

The map $V^\# : P \to W(R) \otimes_{F,W(R)} P$ of lemma 1.5 satisfies $V^\#(P) \subset Q^{(p)}$. Using the fact that P is generated as a $W(R)$-module by the elements $V^{-1}y$ for $y \in Q$ a routine calculation shows that $V^\#$ commutes with F and V^{-1}. Hence $V^\#$ induces a homomorphism of 3n-displays

$$Fr_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P}^{(p)}, \tag{29}$$

which is called the Frobenius homomorphism of $\mathcal{P}$.

Similarly the map $F^\# : W(R) \otimes_{F,W(R)} P \to P$ satisfies $F^\#(Q^{(p)}) \subset I_R P$. One can check that $F^\#$ commutes with the operators F and V^{-1}. Therefore $F^\#$ induces a map of 3n-displays, which is called the Verschiebung.

$$\mathrm{Ver}_{\mathcal{P}} : \mathcal{P}^{(p)} \longrightarrow \mathcal{P}. \tag{30}$$

From the lemma 1.5 we obtain the relations:

$$Fr_{\mathcal{P}} \cdot \mathrm{Ver}_{\mathcal{P}} = p \cdot \mathrm{id}_{\mathcal{P}^{(p)}}, \qquad \mathrm{Ver}_{\mathcal{P}}\, Fr_{\mathcal{P}} = p \cdot \mathrm{id}_{\mathcal{P}}\,. \tag{31}$$

Example 24. — We will define displays, which correspond to the Lubin-Tate groups. Let O_K be a complete discrete valuation ring with finite residue class field k, and field of fractions K of characteristic 0. We fix a prime element $\pi \in O_K$. Let R be a p-adic ring, which is equipped with a structure $\phi : O_K \to R$ of a O_K-algebra. We set $u = \phi(\pi)$.

The displays we are going to construct are displays $\mathcal{P}$ over the topological ring R with its p-adic topology. Moreover they will be equipped with an action $\iota : O_K \to \operatorname{End} \mathcal{P}$ of O_K. This implies an action of the ring $O_K \otimes_{\mathbb{Z}_p} W(R)$ on P. Let us extend the operators ^{F} and ^{V} on the ring $W(R)$ O_K-linearly to the ring $O_K \otimes W(R)$. We need the following easy lemma:

Lemma 25. — *Consider the ring homomorphism:*

$$O_K \otimes_{\mathbb{Z}_p} W(R) \longrightarrow O_K/\pi O_K \otimes R/uR. \tag{32}$$

It is the residue class map on the first factor, and it is the composite of $\mathbf{w}_0$ *with the natural projection* $R \to R/uR$ *on the second factor.*

Then an element in $O_K \otimes W(R)$ *is a unit, iff its image by (32) is a unit.*

Proof. — By proposition 3 the ring $O_K \otimes_{\mathbb{Z}_p} W(R)$ is complete in the I_R -adic topology. Hence an element in this ring is a unit, iff its image in $O_K \otimes_{\mathbb{Z}_p} R$ is a unit. Since this last ring is complete in the p-adic topology, we get easily our result. □

Let us first do the construction of the Lubin-Tate display in a special case:

Proposition 26. — *Let us assume that* $O_K/\pi O_K = \mathbb{F}_p$. *Let* R *be a* p*-torsion free* p*-adic ring, with an* O_K*-algebra structure* $\phi : O_K \to R$. *Then there is a unique display* $\mathcal{P}_R = (P_R, Q_R, F, V^{-1})$ *over the topological ring* R*, with the following properties:*

(i) $P_R = O_K \otimes_{\mathbb{Z}_p} W(R)$.
(ii) Q_R *is the kernel of the map* $\phi \otimes \mathbf{w}_0 : O_K \otimes_{\mathbb{Z}_p} W(R) \to R$.
(iii) *The operators* F *and* V^{-1} *are* O_K*-linear.*
(iv) $V^{-1}(\pi \otimes 1 - 1 \otimes [u]) = 1$.

To prove this proposition we need two lemmas:

Lemma 27. — *With the assumptions of proposition 26 we set* $e = [O_K : \mathbb{Z}_p]$. *Then the element:*

$$\tau = \frac{1}{p}(\pi^e \otimes 1 - 1 \otimes [u^{ep}]) \in K \otimes_{\mathbb{Z}_p} W(R)$$

is a unit in $O_K \otimes_{\mathbb{Z}_p} W(R)$.

Proof. — The statement makes sense because $O_K \otimes_{\mathbb{Z}_p} W(R)$ has no p-torsion. First we prove that the element $\pi^e \otimes 1 - 1 \otimes [u^{ep}]$ is divisible by p. We have $\pi^e = \varepsilon p$ for some unit $\varepsilon \in O_K^*$. Therefore it is enough to show that p divides $1 \otimes [u^{ep}]$. Since $u^{ep} = \phi(\varepsilon)^p p^p$, it is enough to show that p divides $[p^p]$ in $W(R)$. This will follow from the lemma below.

To show that τ is a unit we consider its image by the map $\phi \otimes \mathbf{w}_0 : O_K \otimes_{\mathbb{Z}_p} W(R) \to R$. It is equal to $\frac{1}{p}(u^e - u^{ep})$, which is a unit in R. It follows immediately from lemma 25 that τ must be a unit too. □

Lemma 28. — *The element $[p^p] \in W(\mathbb{Z}_p)$ is divisible by p.*

Proof. — Let $g_m \in \mathbb{Z}_p$ for $m \geq 0$ be p-adic integers. By a well–known lemma [BAC] IX.3 Proposition 2 there exists a Witt vector $x \in W(\mathbb{Z}_p)$ with $\mathbf{w}_m(x) = g_m$, for all $m \geq 0$, if and only if the following congruences are satisfied:

$$g_{m+1} \equiv g_m \bmod p^{m+1}$$

Hence our assertion follows if we verify the congruences:

$$\frac{(p^p)^{p^{m+1}}}{p} \equiv \frac{(p^p)^{p^m}}{p} \bmod p^{m+1} \quad m = 0, 1, \ldots$$

But both sides of these congruences are zero. □

Proof of proposition 26. — Let $L_R \subset P_R$ be the free $W(R)$-submodule of P_R with the following basis

$$\pi^i \otimes 1 - 1 \otimes [u^i], \quad i = 1, \ldots, e-1.$$

Let us denote by $T_R \subset P_R$ the $W(R)$-submodule $W(R)(1 \otimes 1)$. Then $P_R = T_R \oplus L_R$ is a normal decomposition.

To define a display we need to define ^{F}-linear maps

$$V^{-1} : L_R \longrightarrow P_R$$
$$F : T_R \longrightarrow P_R,$$

such that the map $V^{-1} \oplus F$ is an ^{F}-linear epimorphism.

Since we want V^{-1} to be O_K-linear we find by condition (iv) that for $i = 1, \ldots, e-1$:

$$V^{-1}(\pi^i \otimes 1 - 1 \otimes [u^i]) = \frac{\pi^i \otimes 1 - 1 \otimes [u^{ip}]}{\pi \otimes 1 - 1 \otimes [u^p]} = \sum_{k+l=i-1} \pi^k \otimes [u^{lp}]. \tag{33}$$

Here k and l run through nonnegative integers and the fraction in the middle is by definition the last sum. The equation makes sense because by lemma 27 the element $\pi \otimes 1 - 1 \otimes [u^p]$ is not a zero divisor in $O_K \otimes W(R)$.

If we multiply the equation (iv) by p we find

$$F(\pi \otimes 1 - 1 \otimes [u]) = p,$$

and by the required O_K-linearity of F:

$$(\pi \otimes 1 - 1 \otimes [u^p]) \cdot F1 = p.$$

Therefore we are forced to set:

$$F1 = \tau^{-1} \frac{\pi^e \otimes 1 - 1 \otimes [u^{ep}]}{\pi \otimes 1 - 1 \otimes [u^p]} \tag{34}$$

The ^{F}-linear operators $V^{-1} : L_R \to P_R$ and $F : T_R \to P_R$ defined by the equations (33) and (34) may be extended to ^{F}-linear operators

$$V^{-1} : Q_R \longrightarrow P_R, \quad F : P_R \longrightarrow P_R$$

using the equations (1) and (2). Then V^{-1} is the restriction of the operator V^{-1} : $P_R[\frac{1}{p}] \to P_R[\frac{1}{p}]$ defined by $V^{-1}x = \frac{{}^F x}{\pi\otimes 1 - 1\otimes[u^p]}$ and F is the restriction of pV^{-1} : $P_R[\frac{1}{p}] \to P_R[\frac{1}{p}]$. This shows that the operators F and V^{-1} are O_K-linear. Since 1 is in the image of $(V^{-1})^{\#} : W(R) \otimes_{W(R)} Q_R \to P_R$, and since this map is $O_K \otimes W(R)$-linear, we conclude that $(V^{-1})^{\#}$ is an epimorphism. It follows that (P_R, Q_R, F, V^{-1}) is a 3n-display, which satisfies the conditions of the proposition. The uniqueness is clear by what we have said.

It remains to be shown that we obtained a display in the topological sense. By base change it is enough to do this for $R = O_K$. Let us denote by $\overline{\mathcal{P}} = (\overline{P}, \overline{Q}, F, V^{-1})$ the 3n-display over $\mathbb{F}_p$ obtained by base change $O_K \to \mathbb{F}_p$. Then $\overline{P} = O_K$ and F is the O_K-linear map defined by $F\pi = p$. Hence the map V is multiplication by π. Hence $\overline{\mathcal{P}}$ is a display. □

Finally we generalize our construction to the case where the residue class field k of O_K is bigger than $\mathbb{F}_p$. In this case we define for any torsionfree O_K-algebra $\phi : O_K \to R$ a display

$$\mathcal{P}_R = (P_R, Q_R, F, V^{-1}).$$

Again we set

$$P_R = O_K \otimes_{\mathbb{Z}_p} W(R),$$

and we define Q_R to be the kernel of the natural map

$$\phi \otimes \mathbf{w}_0 : O_K \otimes_{\mathbb{Z}_p} W(R) \longrightarrow R. \tag{35}$$

We identify $W(k)$ with a subring of O_K. The restriction of ϕ to $W(k)$ will be denoted by the same letter:

$$\phi : W(k) \longrightarrow R.$$

Applying the functor W to this last map we find a map (compare (89))

$$\rho : W(k) \longrightarrow W(W(k)) \longrightarrow W(R), \tag{36}$$

which commutes with the Frobenius morphism defined on the first and the third ring of (36) (for a detailed discussion see [Gr] Chapt IV Proposition 4.3).

Let us denote the Frobenius endomorphism on $W(k)$ also by σ. We have the following decomposition in a direct product of rings

$$O_K \otimes_{\mathbb{Z}_p} W(R) = \prod_{i\in\mathbb{Z}/f\mathbb{Z}} O_K \otimes_{\sigma^i, W(k)} W(R). \tag{37}$$

Here f denotes the degree $f = [k : F_p]$ and the tensor product is taken with respect to ρ.

The operators ^{F} and ^{V} on $W(R)$ act via the second factor on the left hand side of (37). On the right–hand side they are operators of degree -1 and $+1$ respectively:

$$\begin{aligned}{}^F &: O_K \otimes_{\sigma^i,W(k)} W(R) \longrightarrow O_K \otimes_{\sigma^{i-1},W(k)} W(R)\\ {}^V &: O_K \otimes_{\sigma^i,W(k)} W(R) \longrightarrow O_K \otimes_{\sigma^{i+1},W(k)} W(R).\end{aligned}$$

We obtain from (37) a decomposition of the $O_K \otimes_{\mathbb{Z}_p} W(R)$-module P_R:

$$P_R = \bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} P_i, \quad P_i = O_K \otimes_{\sigma^i,W(k)} W(R)$$

Therefore we obtain also a decomposition

$$Q_R = Q_0 \oplus P_1 \oplus \cdots \oplus P_{f-1}.$$

The map (35) factors through

$$O_K \otimes_{W(k)} W(R) \longrightarrow R, \tag{38}$$

and Q_0 is the kernel of (38). The following elements form a basis of P_0 as $W(R)$-module

$$\begin{aligned}&\omega_i = \pi^i \otimes 1 - 1 \otimes [u^i], \qquad i = 1, \ldots, e-1\\ &e_0 = 1 \otimes 1.\end{aligned}$$

Here u denotes as before the image of π by the map $O_K \to R$, and e is the ramification index $e = [O_K : W(k)]$. Let $T = W(R)e_0 \subset P_0$, and let $L_0 \subset Q_0$ the free $W(R)$ submodule generated by $\omega_1, \ldots, \omega_{e-1}$. We have a normal decomposition

$$P_R = T \oplus L,$$

where $L = L_0 \oplus P_1 \oplus \cdots \oplus P_{f-1}$.

Now we may define the O_K-linear operators F and V^{-1}. We set $e_i = 1 \otimes 1 \in P_i$. Then V^{-1} is uniquely defined by the following properties:

$$\begin{gathered}V^{-1}\omega_1 = e_{f-1},\\ V^{-1}e_i = e_{i-1} \quad \text{for } i \neq 0 \quad i \in \mathbb{Z}/f\mathbb{Z},\\ V^{-1} \text{ is } O_K\text{-linear.}\end{gathered} \tag{39}$$

Multiplying the first of these equations by p we obtain the following equation in the ring $O_K \otimes_{\sigma^{f-1},W(k)} W(R)$:

$${}^F\omega_1 Fe_0 = pe_{f-1}$$

To see that this equation has a unique solution Fe_0 it suffices to show that:

$$\frac{1}{p}(\pi^e \otimes 1 - 1 \otimes [u^{ep}]) \in O_K \otimes_{\sigma^{f-1},W(k)} W(R)$$

is a unit. This is seen exactly as before, using that $\frac{1}{p}(1 \otimes [u^{pe}])$ is mapped to zero by the map $W(R) \to R/u$.

Hence we have defined the desired ^{F}-linear operators $F : P_R \to P_R$ and $V^{-1} : Q_R \to P_R$. Again V^{-1} extends to a ^{F}-linear endomorphism of $K \otimes_{\mathbb{Z}_p} W(R)$, which is given by the formula:

$$V^{-1}x = {}^F\!\left(\frac{x}{\theta}\right),$$

where $\theta \in O_K \otimes_{\mathbb{Z}_p} W(R)$ is the element, which has with respect to the decomposition (37) the component ω_1 for $i = 0$ and the component e_i for $i \neq 0$.

As before this proves the following proposition:

Proposition 29. — *Let K be a finite extension of $\mathbb{Q}_p$ with ramification index e and index of inertia f. Let O_K, π, k have the same meaning as before.*

Let R be torsion free O_K-algebra, such that R is p-adically complete and separated. Denote by u the image of π by the structure morphism $\phi : O_K \to R$. Let $\rho : W(k) \to W(R)$ be the homomorphism induced by the structure morphism. Then we have a decomposition

$$O_K \otimes_{\mathbb{Z}_p} W(R) \xrightarrow{\sim} \prod_{i \in \mathbb{Z}/f\mathbb{Z}} O_K \otimes_{\sigma^i, W(k)} W(R)$$

Let $\theta \in O_K \otimes_{\mathbb{Z}_p} W(R)$ be the element, which has the component 1 for $i \neq 0$ and the component $\pi \otimes 1 - 1 \otimes [u]$ for $i = 0$.

Then there is a uniquely defined display $\mathcal{P}_R = (P_R, Q_R, F, V^{-1})$ over the topological ring R, which satisfies the following conditions:

(i) $P_R = O_K \otimes_{\mathbb{Z}_p} W(R)$.

(ii) *Q_R is the kernel of the map $\phi \otimes \mathbf{w}_0 : O_K \otimes_{\mathbb{Z}_p} W(R) \to R$.*

(iii) *The operators F and V^{-1} are O_K-linear.*

(iv) $V^{-1}\theta = 1$.

1.3. Descent.

— We will now study the faithfully flat descent for displays.

Lemma 30. — *Let M be a flat $W(S)$-module, and let $S \to R$ be a faithfully flat ring extension. Then there is an exact sequence*

$$0 \to M \to W(R) \otimes_{W(S)} M \rightrightarrows W(R \otimes R) \otimes_{W(S)} M \overset{\to}{\underset{\to}{\to}} W(R \otimes R \otimes R) \otimes_{W(S)} M \cdots \tag{40}$$

Here the $\otimes$ without index means $\otimes_S$.

Proof. — The arrows are induced by applying the functor W to the usual exact sequence for descent:

$$0 \to S \to R \rightrightarrows R \otimes_S R \overset{\to}{\underset{\to}{\to}} \cdots$$

Since M is a direct limit of free modules, we are reduced to the case $M = W(S)$. In this case any term of the sequence (40) comes with the filtration by the ideals $I_{R \otimes_S \cdots \otimes_S R, n} \subset W(R \otimes_S \cdots \otimes_S R)$. We obtain by the usual f.p.q.c. descent an exact sequence, if we go to the graded objects. $\square$

Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a display over S. Then the modules P_R and Q_R obtained by base change fit into an exact sequence

$$0 \longrightarrow Q_R \longrightarrow P_R \longrightarrow R \otimes_S P/Q \longrightarrow 0$$

Proposition 31. — *Let $S \to R$ be a faithfully flat ring morphism. Consider a display (P, Q, F, V^{-1}) over S. Then we have a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 \to & P & \to & P_R & \rightrightarrows & P_{R\otimes_S R} & \overset{\to}{\underset{\to}{\to}} & P_{R\otimes_S R\otimes_S R} & \overset{\to}{\underset{\to}{\overset{\to}{\to}}} \cdots \\ & \cup & & \cup & & \cup & & \cup & \\ 0 \to & Q & \to & Q_R & \rightrightarrows & Q_{R\otimes_S R} & \overset{\to}{\underset{\to}{\to}} & Q_{R\otimes_S R\otimes_S R} & \overset{\to}{\underset{\to}{\overset{\to}{\to}}} \cdots \end{array}$$

Proof. — Indeed, the first row is exact by the lemma. The second row is the kernel of the canonical epimorphism from the first row to:

$$0 \to P/Q \to R \otimes_S P/Q \rightrightarrows R \otimes_S R \otimes_S P/Q \overset{\to}{\underset{\to}{\to}} R \otimes_S R \otimes_S R \otimes P/Q \overset{\to}{\underset{\to}{\overset{\to}{\to}}} \cdots$$

This proves the proposition and more:

***Theorem 32* (descent for displays)**. — *Let $S \to R$ be a faithfully flat ring extension. Let $\mathcal{P} = (P, Q, F, V^{-1})$ and $\mathcal{P}' = (P', Q', F, V^{-1})$ be two displays over S. Then we have an exact sequence*

$$0 \to \operatorname{Hom}(\mathcal{P}, \mathcal{P}') \to \operatorname{Hom}(\mathcal{P}_R, \mathcal{P}'_R) \rightrightarrows \operatorname{Hom}(\mathcal{P}_{R\otimes_S R}, \mathcal{P}'_{R\otimes_S R}). \qquad \square$$

Let N be a $W(R)$-module. Then we may define a variant of the usual descent datum relative to $S \to R$.

Let us give names to the morphisms in the exact sequence (40):

$$W(S) \to W(R) \underset{p_2}{\overset{p_1}{\rightrightarrows}} W(R \otimes_S R) \overset{p_{12}}{\underset{p_{13}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{p_{23}}}}}} W(R \otimes_S R \otimes_S R). \tag{41}$$

Here the index of p_{ij} indicates, that the first factor of $R \otimes_S R$ is mapped to the factor i, and the second is mapped to the factor j. The notation p_i is similar. In the context of descent we will often write $\otimes$ instead of $\otimes_S$ We also use the notation

$$p_i^* N = W(R \otimes R) \otimes_{p_i, W(R)} N.$$

We define a W-descent datum on N to be a $W(R \otimes R)$-isomorphism

$$\alpha\colon p_1^* N \longrightarrow p_2^* N,$$

such that the following diagram is commutative (cocycle condition):

$$
\begin{tikzcd}
p_{12}^*p_1^*N \arrow[r, "p_{12}^*\alpha"] \arrow[d, equal] & p_{12}^*p_2^*N \arrow[d, equal] \\
p_{13}^*p_1^*N \arrow[d, "p_{13}^*\alpha"'] & p_{23}^*p_1^*N \arrow[d, "p_{23}^*\alpha"] \\
p_{13}^*p_2^*N \arrow[r, equal] & p_{23}^*p_2^*N
\end{tikzcd}
\tag{42}
$$

To any descent datum we may associate a sequence of morphisms

$$W(R) \otimes_{W(R)} N \overset{\partial^0}{\underset{\partial^1}{\rightrightarrows}} W(R \otimes R) \otimes_{W(R)} N \overset{\partial^0}{\underset{\partial^2}{\overset{\longrightarrow}{\underset{\longrightarrow}{\overset{\partial^1}{\longrightarrow}}}}} W(R \otimes R \otimes R) \otimes_{W(R)} N \cdots,$$

where the tensor product is always taken with respect to the map

$$W(R) \longrightarrow W(R \otimes \cdots \otimes R) \quad \text{induced by} \quad a \in R \longmapsto 1 \otimes \cdots \otimes 1 \otimes a \in R \otimes \cdots \otimes R.$$

The maps $\partial^i \colon W(R^{\otimes n}) \otimes_{W(R)} N \longrightarrow W(R^{\otimes(n+1)}) \otimes_{W(R)} N$, for $i < n$ are simply the tensorproduct with N of the map $W(R^{\otimes n}) \longrightarrow W(R^{\otimes(n+1)})$ induced by

$$a_1 \otimes \cdots \otimes a_n \longmapsto a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots a_n.$$

Finally the map $\partial^n \colon W(R^{\otimes n}) \otimes_{W(R)} N \to W(R^{\otimes n+1}) \otimes_{W(R)} N$ is obtained as follows. The descent datum α induces a map $u(x) = \alpha(1 \otimes x)$:

$$u \colon N \longrightarrow W(R \otimes R) \otimes_{W(R)} N,$$

which satisfies $u(rx) = p_1(r)u(x)$. Consider the commutative diagram

$$
\begin{tikzcd}
R \arrow[r] \arrow[d] & R \otimes R \arrow[d] \\
R^{\otimes n} \arrow[r] & R^{\otimes n+1}
\end{tikzcd}
$$

The upper horizontal map is $r \mapsto r \otimes 1$ and the lower horizontal map is $r_1 \otimes \cdots \otimes r_n \mapsto r_1 \otimes \cdots \otimes r_n \otimes 1$. The left vertical map is $r \mapsto 1 \otimes \cdots \otimes 1 \otimes r$ and finally the right vertical map is $r_1 \otimes r_2 \mapsto 1 \otimes \cdots \otimes 1 \otimes r_1 \otimes r_2$.

If we apply the functor W we obtain:

$$
\begin{tikzcd}
W(R) \arrow[r] \arrow[d, "q"'] & W(R \otimes R) \arrow[d, "q_2"] \\
W(R^{\otimes n}) \arrow[r] & W(R^{\otimes(n+1)})
\end{tikzcd}
$$

Since u is equivariant with respect to the upper horizontal arrow, we may tensorize u by this diagram to obtain

$$W(R^{\otimes n}) \otimes_{q, W(R)} N \longrightarrow W(R^{\otimes n+1}) \otimes_{q_2, W(R \otimes R)} W(R \otimes R) \otimes_{W(R)} N.$$

This is the map we wanted to define.

We set

$$\delta_n = \sum_{i=0}^{n} (-1)^i \delta^i : W(R^{\otimes n}) \otimes N \longrightarrow W(R^{\otimes (n+1)}) \otimes N.$$

The cocycle condition assures that we get a complex:

$$W(R) \otimes_{W(R)} N \xrightarrow{\delta_1} W(R \otimes R) \otimes_{W(R)} N \xrightarrow{\delta_2} W(R \otimes R \otimes R) \otimes_{W(R)} N \cdots \tag{43}$$

Proposition 33. — *Let $S \to R$ be a faithfully flat ring homomorphism. Assume that p is nilpotent in S. Let P be a finitely generated projective $W(R)$-module with a W-descent datum α relative to $R \to S$. Then the complex (43) for $N = P$ is exact. The kernel P_0 of δ_1 is a projective finitely generated $W(S)$-module and the natural map*

$$W(R) \otimes_{W(S)} P_0 \longrightarrow P$$

is an isomorphism.

We prove this a little later.

Corollary 34. — *The functor which associates to a finitely generated projective $W(S)$-module P_0 the $W(R)$-module $P = W(R) \otimes_{W(S)} P_0$ with its canonical descent datum is an equivalence of categories.*

Proposition 35. — *The following conditions for a W(R)-module P are equivalent:*

(i) *P is finitely generated and projective.*

(ii) *P is separated in the topology defined by the filtration I_nP for $n \in \mathbb{N}$ (same notation as in the proof of proposition 3), and for each n the $W_n(R)$-module P/I_nP is projective and finitely generated.*

(iii) *P is separated as above, and there exist elements $f_1, \ldots, f_m \in R$, which generate the unit ideal, and such that for each $i = 1, \ldots, m$ $W(R_{f_i})$-module $W(R_{f_i}) \otimes_{W(R)} P$ is free and finitely generated.*

Proof. — For any number n and any $f \in R$ we have a natural isomorphism $W_n(R_f) \cong W_n(R)_{[f]}$. This fact shows, that (iii) implies (ii). Next we assume (ii) and show that (i) holds. We find elements $u_1, \ldots, u_h$, which generate P/IP as an R-module. They define a map $L = W(R)^h \to P$. Since L is complete in the topology defined by the ideals I_n this map is surjective and P is complete. By the lemma below we find for each number n a section σ_n of $L/I_nL \to P/I_nP$, such that σ_{n+1} reduces to σ_n. The projective limit of these sections is a section of the $W(R)$-module homomorphism $L \to P$. For the proof of the implication (i) implies (iii), we may assume that $R \otimes_{W(R)} P$ is free. But then the same argument as above shows that any basis of $R \otimes_{W(R)} P$ lifts to a basis of P. ☐

Lemma 36. — *Let $S \to R$ be a surjective ring homomorphism. Let $\pi : P_1 \to P_2$ be a surjective S-module homomorphism. Suppose that P_2 is a projective S-module. Let*

$\overline{\pi} : \overline{P_1} \to \overline{P_2}$ be the R-module homomorphism obtained by tensoring π by $R \otimes_S$. Then any section $\overline{\sigma} : \overline{P_2} \to \overline{P_1}$ lifts to a section $\sigma : P_1 \to P_2$.

Proof. — Let us denote by K the kernel of π, and set $\overline{K} = R \otimes_S K$. Let τ be any section of π. Consider the morphism $\overline{\sigma} - \overline{\tau} : \overline{P_2} \to \overline{K}$. This lifts to a S-module homomorphism $\rho : P_2 \to K$, because P_2 is projective. We set $\sigma = \tau + \rho$. □

Proof. — (of proposition 33): We begin to prove the statement on the exactness of (43) under the additional assumption that $p \cdot S = 0$. On each term of the sequence (43) we consider the filtration by $I_{R^{\otimes n},m} \otimes_{W(R)} P$. Since P is projective the associated graded object is

$$I_{R,m}/I_{R,m+1} \otimes_{W(R)} P \longrightarrow (I_{R\otimes R,m}/I_{R\otimes R,m+1}) \otimes_{W(R)} P \longrightarrow \cdots$$

Applying the assumption $p \cdot R = 0$ we may rewrite this as

$$R \otimes_{p^m,R} P/I_R P \rightrightarrows R \otimes_S R \otimes_{p^m,R} P/I_R P \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \cdots$$

The symbol p^m indicates, that the tensor product is taken with respect to the m-th power of the Frobenius endomorphism. The last sequence comes from a usual descent datum on $R \otimes_{p^m,R} P/I_R P$ and is therefore exact, except for the first place. Now we will get rid of the assumption $p \cdot S = 0$. We consider any ideal $\mathfrak{a} \subset S$ such that $p \cdot \mathfrak{a} = 0$. Let us denote by a bar the reduction modulo p (i.e. $\overline{R} = R/pR$ etc.), and by a dash the reduction modulo $\mathfrak{a}$.

We have an exact sequence

$$\begin{aligned} 0 \to \mathfrak{a} \otimes R \otimes R \cdots \otimes R &\longrightarrow R \otimes R \otimes \cdots \otimes R \to R' \otimes_{S'} R' \otimes \cdots_{S'} \otimes R' \to 0 \\ a \otimes r_1 \otimes \cdots \otimes r_n &\longmapsto a r_1 \otimes \cdots \otimes r_n \end{aligned}$$

An obvious modification of the complex (43) yields a complex

$$W(\mathfrak{a} \otimes R) \otimes_{W(R)} P \xrightarrow{\delta_1} W(\mathfrak{a} \otimes R \otimes R) \otimes_{W(R)} P \xrightarrow{\delta_2} \cdots, \tag{44}$$

where the factor $\mathfrak{a}$ is untouched in the definition of δ_i.

We set $\overline{P} = W(\overline{R}) \otimes_{W(R)} P$. Then the complex (44) identifies with the complex

$$W(\mathfrak{a} \otimes_{\overline{S}} \overline{R}) \otimes_{W(\overline{R})} \overline{P} \xrightarrow{\delta_1} W(\mathfrak{a} \otimes_{\overline{S}} \overline{R} \otimes_{\overline{S}} \overline{R}) \otimes_{W(\overline{R})} \overline{P} \xrightarrow{\delta_2} \cdots \tag{45}$$

given by the induced descent datum on $\overline{P}$. Since $p \cdot \mathfrak{a} = p \cdot \overline{S} = p \cdot \overline{R} = 0$ the argument before applies to show that (45) is exact except for the first place. Now an easy induction argument using the exact sequence of complexes

$$0 \to W(\mathfrak{a} \otimes_{\overline{S}} \overline{R}^{\otimes n}) \otimes_{W(\overline{R})} \overline{P} \to W(R^{\otimes n}) \otimes_{W(R)} P \to W(R'^{\otimes n}) \otimes_{W(R')} P' \to 0$$

proves the exactness statement for the complex in the middle.

In fact our method gives slightly more, namely that we have also, for each m, exactness of the complex of the augmentation ideals

$$I_{R^{\otimes n},m} \otimes_{W(R)} P.$$

Now we set $P_0 = \ker \delta_1 \colon (P \to W(R \otimes R) \otimes_{W(R)} P)$ and $P_0^1 = P_0 \cap I_R P$. By the exact cohomology sequence we have a diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & P_0^1 \arrow[r] \arrow[d] & I_R \underset{W(R)}{\otimes} P \arrow[r] \arrow[d] & I_{R\otimes R} \underset{W(R)}{\otimes} P \arrow[r] \arrow[d] & I_{R\otimes R\otimes R} \underset{W(R)}{\otimes} P \arrow[d] \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & W(R) \underset{W(R)}{\otimes} P \arrow[r] \arrow[d] & W(R\otimes R) \underset{W(R)}{\otimes} P \arrow[r] \arrow[d] & W(R\otimes R\otimes R) \underset{W(R)}{\otimes} P \arrow[d] \\
0 \arrow[r] & P_0/P_0^1 \arrow[r] \arrow[d] & P/I_R P \arrow[r] \arrow[d] & R\otimes R \underset{R}{\otimes} P/I_R P \arrow[r] \arrow[d] & R\otimes R\otimes R \underset{R}{\otimes} P/I_R P \arrow[d] \\
 & 0 & 0 & 0 & 0
\end{tikzcd}$$

By the usual descent P_0/P_0^1 is a finitely generated projective S-module. We may lift it to a projective $W(S)$-module F, by lifting it step by step with respect to the surjections $W_{n+1}(S) \to W_n(S)$ and then taking the projective limit. By the projectivity of F we obtain a commutative diagram

$$\begin{tikzcd}
F \arrow[rr] \arrow[dr, two heads] & & P_0 \arrow[dl, two heads] \\
 & P_0/P_0^1 &
\end{tikzcd}$$

From the upper horizontal arrow we obtain a map $W(R) \otimes_{W(S)} F \to P$, which may be inserted into a commutative diagram

$$\begin{tikzcd}
W(R) \otimes_{W(S)} F \arrow[r] \arrow[d] & P \arrow[d] \\
R \otimes_S P_0/P_0^1 \arrow[r, "\sim"] & P/IP
\end{tikzcd}$$

Since the lower horizontal arrow is an isomorphism by usual descent theory we conclude by Nakayama that the upper horizontal arrow is an isomorphism. Comparing the exact sequence (40) for $M = F$ with the exact sequence (43) for $N = P$, we obtain that $F \to P_0$ is an isomorphism. Since also the graded sequence associated to (40) is exact, we obtain moreover that $P_0^1 = IP_0$. Hence the proof of the proposition 33 is complete. $\square$

We may define a descent datum for 3n-displays. Let S be a ring, such that p is nilpotent in S and let $S \to R$ be a faithfully flat morphism of rings. We consider the usual diagram (compare (41)):

$$R \underset{q_2}{\overset{q_1}{\rightrightarrows}} R \otimes_S R \overset{\overset{q_{12}}{\longrightarrow}}{\underset{\underset{\longrightarrow}{q_{23}}}{\overset{q_{13}}{\longrightarrow}}} R \otimes_S R \otimes_S R$$

Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a 3n-display over R. We denote the 3n-displays obtained by base change by $q_1^*\mathcal{P}$ etc.. Then a descent datum on $\mathcal{P}$ relative to $R \to S$ is an isomorphism of 3n-displays

$$\alpha \colon q_1^*\mathcal{P} \longrightarrow q_2^*\mathcal{P},$$

such that the cocycle condition holds, i.e. the diagram (42) is commutative if the letter p is replaced by q and the letter N is replaced by $\mathcal{P}$. Clearly for any 3n-display $\mathcal{P}_0$ over S we have a canonical descent datum $\alpha_{\mathcal{P}_0}$ on the base change $\mathcal{P}_{0,R}$ over R.

Theorem 37. — *The functor $\mathcal{P}_0 \mapsto (\mathcal{P}_{0,R}, \alpha_{\mathcal{P}_0})$ from the category of displays over S to the category of displays over R equipped with a descent datum relative to $S \to R$ is an equivalence of categories. The same assertion holds for the category of 3n-displays.*

Proof. — Let $(\mathcal{P}, \alpha)$ be a display over R with a descent datum relative to $S \to R$. We define a $W(S)$-module P_0 and a S-module K_0, such that the rows in the following diagram are exact

(46)
$$\begin{tikzcd} 0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P \arrow[r, "\delta"] \arrow[d] & W(R \otimes_S R) \otimes_{W(R)} P \arrow[d] \\ 0 \arrow[r] & K_0 \arrow[r] & P/Q \arrow[r, "\delta"] & R \otimes_S R \otimes_R P/Q \end{tikzcd}$$

Here the maps δ are given by the descent datum α as explained above. That we have also a descent datum on P/Q follows just from our assumption that α is an isomorphism of displays and therefore preserves Q. We claim that the map $P_0 \to K_0$ is surjective. Indeed, since $R \to S$ is faithfully flat, it suffices to show that $R \otimes_S P_0/I_S P_0 \to R \otimes_S K_0$ is surjective. But this can be read of from the commutative diagram:

$$\begin{tikzcd} W(R) \otimes_{W(S)} P_0 \arrow[r, two heads] \arrow[d] & R \otimes_S P_0/IP_0 \arrow[r] \arrow[d] & R \otimes_S K_0 \arrow[d] \\ P \arrow[r, two heads] & P/IP \arrow[r, two heads] & P/Q \end{tikzcd}$$

Note that the vertical arrows are isomorphisms by proposition 33 or the usual descent theory.

Let us denote by Q_0 the kernel of the surjection $P_0 \to K_0$. Then we obtain a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & Q_0 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & Q_2 \arrow[d] \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & W(R \otimes_S R) \otimes_{W(R)} P = p_2^* P \arrow[d] \\
0 \arrow[r] & P_0/Q_0 \arrow[r] \arrow[d] & P/Q \arrow[r] \arrow[d] & R \otimes_S R \otimes_R (P/Q) = q_2^*(P/Q) \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

Here Q_2 and $p_2^* P$ are parts of the display $q_2^* \mathcal{P} = (p_2^* P, Q_2, F, V^{-1})$ which is obtained by base change.

To get a display $\mathcal{P}_0 = (P_0, Q_0, F, V^{-1})$ we still have to define the operators F and V^{-1}. First since α commutes with F by assumption we have a commutative diagram

$$\begin{tikzcd}
P \arrow[r, "\delta"] \arrow[d, "F"'] & p_2^* P \arrow[d, "F"] \\
P \arrow[r, "\delta"] & p_2^* P
\end{tikzcd}$$

This shows that F induces a map on the kernel of δ:

$$F \colon P_0 \longrightarrow P_0$$

Secondly α commutes with V^{-1}, i.e. we have a commutative diagram

$$\begin{tikzcd}
Q_1 \arrow[r, "\underset{\sim}{\alpha}"] \arrow[d, "V^{-1}"'] & Q_2 \arrow[d, "V^{-1}"] \\
p_1^* P \arrow[r, "\alpha"] & p_2^* P
\end{tikzcd}$$

Recalling the definition of δ one obtains a commutative diagram

$$\begin{tikzcd}
Q \arrow[rr, "\delta"] \arrow[d, "V^{-1}"'] & & Q_2 \arrow[d, "V^{-1}"] \\
P \arrow[r, "\delta"] & p_2^* P = W(R \otimes_S R) \otimes_{W(R)} P. &
\end{tikzcd}$$

Hence we obtain $V^{-1}\colon Q_0 \longrightarrow P_0$ as desired. Finally we need to check the nilpotence condition. Since the maps V^{-1} and F are compatible with $P_0 \hookrightarrow P$, the same is true for $V^{\#}$ by the characterization of lemma 10. Hence we have a commutative diagram

$$\begin{tikzcd} P_0 \arrow[r] \arrow[d, "V^{\#}"'] & P \arrow[d, "V^{\#}"] \\ W(S) \otimes_{F,W(S)} P_0 \arrow[r] & W(R) \otimes_{F,W(R)} P \end{tikzcd}$$

The nilpotence follows now from the injectivity of the map

$$S/pS \otimes_{\mathbf{w}_n, W(S)} P_0 \longrightarrow R/pR \otimes_{\mathbf{w}_n, W(R)} P$$

and the form (14) of the nilpotence condition. □

1.4. Rigidity. — Our next aim is a rigidity theorem for displays in the sense of rigidity for p-divisible groups. Let S be a ring, such that p is nilpotent in S. Assume we are given an ideal $\mathfrak{a} \subset S$ with a divided power structure γ_n ([BO] 3.1). We set $\alpha_{p^n}(a) = (p^n - 1)!\gamma_{p^n}(a)$. We may "divide" the $n-th$ Witt polynomial $\mathbf{w}_n(X_0, \dots, X_n)$ by p^n:

$$\mathbf{w}'_n(X_0, \dots, X_n) = \alpha_{p^n}(X_0) + \alpha_{p^{n-1}}(X_1) + \dots + X_n. \tag{47}$$

Let us denote by $\mathfrak{a}^{\mathbb{N}}$ the additive group $\prod_{i \in \mathbb{N}} \mathfrak{a}$. We define a $W(S)$-module structure on $\mathfrak{a}^{\mathbb{N}}$:

$$\xi[a_0, a_1 \cdots] = [\mathbf{w}_0(\xi)a_0, \mathbf{w}_1(\xi)a_1, \dots], \quad \text{where } \xi \in W(S),\ [a_0, a_1, \dots] \in \mathfrak{a}^{\mathbb{N}}.$$

The $\mathbf{w}'_n$ define an isomorphism of $W(S)$-modules:

$$\begin{aligned} \log : W(\mathfrak{a}) &\longrightarrow \mathfrak{a}^{\mathbb{N}} \\ \underline{a} = (a_0, a_1, a_2 \cdots) &\longmapsto [\mathbf{w}'_0(\underline{a}), \mathbf{w}'_1(\underline{a}), \dots] \end{aligned} \tag{48}$$

We denote the inverse image $\log^{-1}[\mathfrak{a}, 0, \dots, 0, \dots]$ simply by $\mathfrak{a} \subset W(\mathfrak{a})$. Then $\mathfrak{a}$ is an ideal of $W(S)$.

By going to a universal situation it is not difficult to compute what multiplication, Frobenius homomorphism, and Verschiebung on the Witt ring induce on the right hand side of (48):

$$\begin{aligned} [a_0, a_1, \dots][b_0, b_1, \dots] &= [a_0 b_0, p a_1 b_1, \dots, p^i a_i b_i, \dots] \\ {}^F[a_0, a_1, \dots] &= [p a_1, p a_2, \dots . p a_i, \dots] \\ {}^V[a_0, a_1, \dots] &= [0, a_0, a_1, \dots, a_i, \dots] \end{aligned}$$

The following fact is basic:

Lemma 38. — *Let (P, Q, F, V^{-1}) be a display over S. Then there is a unique extension of the operator V^{-1}:*

$$V^{-1}\colon W(\mathfrak{a})P + Q \longrightarrow P,$$

such that $V^{-1}\mathfrak{a}P = 0$.

Proof. — Choose a normal decomposition

$$P = L \oplus T.$$

Then $W(\mathfrak{a})P + Q = \mathfrak{a}T \oplus L \oplus I_S T$. We define V^{-1} using this decomposition. To finish the proof we need to verify that $V^{-1}\mathfrak{a}L = 0$. But ${}^F\mathfrak{a} = 0$, since the Frobenius map on the right hand side of (48) is

$${}^F[u_0, u_1, \dots] = [pu_1, pu_2, \dots].$$

$\square$

Lemma 39. — *Let S be a ring, such that p is nilpotent in S. Let $\mathfrak{a} \subset S$ be an ideal with divided powers. We consider two displays $\mathcal{P} = (P, Q, F, V^{-1})$ and $\mathcal{P}' = (P', Q', F, V^{-1})$ over S. Then the natural map*

$$\operatorname{Hom}(\mathcal{P}, \mathcal{P}') \longrightarrow \operatorname{Hom}(\mathcal{P}_{S/\mathfrak{a}}, \mathcal{P}'_{S/\mathfrak{a}}) \tag{49}$$

is injective. Moreover let M be a natural number, such that $a^{p^M} = 0$ for any $a \in \mathfrak{a}$. Then the group $p^M \operatorname{Hom}(\mathcal{P}_{S/\mathfrak{a}}, \mathcal{P}'_{S/\mathfrak{a}})$ lies in the image of (49).

Proof. — As explained above the map $V^{-1} : Q' \to P'$ extends to the map $V^{-1} : W(\mathfrak{a})P' + Q' \to P'$, which maps $W(\mathfrak{a})P'$ to $W(\mathfrak{a})P'$. Let $u : \mathcal{P} \to \mathcal{P}'$ be a map of displays, which is zero modulo $\mathfrak{a}$, i.e. $u(P) \subset W(\mathfrak{a})P'$. We claim that the following diagram is commutative:

$$\begin{tikzcd}
P \arrow[r, "u"] \arrow[d, "V^{\#}"'] & W(\mathfrak{a})P' \\
W(S) \otimes_{F,W(S)} P \arrow[r, "1 \otimes u"] & W(S) \otimes_{F,W(S)} W(\mathfrak{a})P' \arrow[u, "(V^{-1})^{\#}"']
\end{tikzcd} \tag{50}$$

Indeed, since $P = W(S)V^{-1}Q$, it is enough to check the commutativity on elements of the form $wV^{-1}l$, where $l \in Q$. Since $V^{\#}(wV^{-1}l) = w \otimes l$. the commutativity is readily checked. Let us denote by $1 \otimes_{F^N} u : W(R) \otimes_{F^N, W(R)} P \to W(R) \otimes_{F^N, W(R)} W(\mathfrak{a})P'$ the map obtained by tensoring. Iterating the diagram (50) we obtain

$$(V^{-N})^{\#}(1 \otimes_{F^N} u)(V^{N\#}) = u \tag{51}$$

By the nilpotence condition for each number$M$, there exists a number N, such that

$$V^{N\#}(P) \subset I_{S,M} \otimes_{F^N, W(S)} P.$$

But since $I_{S,M} \cdot W(\mathfrak{a}) = 0$ for big M, we obtain that the left hand side of (51) is zero. This proves the injectivity.

The last assertion is even true without the existence of divided powers. Indeed, it follows from the assumption that $p^M W(\mathfrak{a}) = 0$. Let now $\overline{u} : \mathcal{P}_{S/\mathfrak{a}} \to \mathcal{P}'_{S/\mathfrak{a}}$ be a morphism of displays.

For $x \in P$ let us denote by $\overline{x} \in W(S/\mathfrak{a}) \otimes_{W(S)} P$ its reduction modulo $\mathfrak{a}$. Let $y \in P'$ be any lifting of $u(\overline{x})$. Then we define

$$v(x) = p^M \cdot y.$$

Since $p^M W(\mathfrak{a}) = 0$ this is well–defined. One checks that v is a morphism of displays $\mathcal{P} \to \mathcal{P}'$, and that $\overline{v} = p^M \overline{u}$. □

Proposition 40. — *Let S be a ring such that p is nilpotent in S. Let $\mathfrak{a} \subset S$ be a nilpotent ideal, i.e. $\mathfrak{a}^N = 0$ for some integer N. Let $\mathcal{P}$ and $\mathcal{P}'$ be displays over S. The the natural map*

$$\operatorname{Hom}(\mathcal{P}, \mathcal{P}') \longrightarrow \operatorname{Hom}(\mathcal{P}_{S/\mathfrak{a}}, \mathcal{P}'_{S/\mathfrak{a}})$$

is injective, and the cokernel is a p-torsion group.

Proof. — By induction one restricts to the case, where $\mathfrak{a}^p = 0$. Then we have a unique divided power structure on $\mathfrak{a}$, such that $\gamma_p(a) = 0$ for $a \in \mathfrak{a}$. One concludes by the lemma. □

Corollary 41. — *Assume again that p is nilpotent in S and that the ideal generated by nilpotent elements is nilpotent. Then the group* $\operatorname{Hom}(\mathcal{P}, \mathcal{P}')$ *is torsionfree.*

Proof. — By the proposition we may restrict to the case where the ring S is reduced. Then the multiplication by p on $W(S)$ is the injective map:

$$(s_0, s_1, s_2 \dots) \longmapsto (0, s_0^p, s_1^p \dots)$$

Therefore the multiplication by p on P' is also injective, which proves the corollary. □

2. Lifting Displays

In this chapter we will consider a surjective homomorphism of rings $S \to R$. The kernel will be denoted by $\mathfrak{a}$. We assume that the fixed prime number p is nilpotent in S.

To a display over R we will associate the crystal, which controls the deformation theory of this display in a way which is entirely similar to the deformation theory of Grothendieck and Messing for p-divisible groups.

2.1. The main theorem.

— We begin by a lemma which demonstrates what we are doing in a simple situation.

Lemma 42. — *Let $S \to R$ be as above and assume that there is a number N, such that $a^N = 0$ for any $a \in \mathfrak{a}$. Let (P_i, F_i) for $i = 1, 2$ be projective finitely generated $W(S)$-modules P_i, which are equipped with ^{F}-linear isomorphisms $F_i : P_i \to P_i$. We set $\overline{P}_i = W(R) \otimes_{W(S)} P_i$ and define ^{F}-linear isomorphisms $\overline{F}_i : \overline{P}_i \to \overline{P}_i$, by $\overline{F}_i(\xi \otimes x) = {}^F\xi \otimes F_i x$, for $\xi \in W(R), x \in P_i$.*

Then any homomorphism $\overline{\alpha} : (\overline{P}_1, \overline{F}_1) \to (\overline{P}_2, \overline{F}_2)$ *admits a unique lifting* $\alpha : (P_1, F_1) \to (P_2, F_2)$.

Proof. — First we choose a lifting $\alpha_0 : P_1 \to P_2$, which does not necessarily commute with the F_i. We look for a $W(S)$-linear homomorphism $\omega \in \mathrm{Hom}_{W(S)}(P_1, W(\mathfrak{a})P_2)$, such that

$$F_2(\alpha_0 + \omega) = (\alpha_0 + \omega)F_1. \tag{52}$$

Since $\overline{\alpha}$ commutes with $\overline{F}_1$ the ^{F}-linear map $\eta = F_2\alpha_0 - \alpha_0 F_1$ maps P_1 to $W(\mathfrak{a})P_2$. The equation (52) becomes

$$\omega F_1 - F_2\omega = \eta,$$

or equivalently

$$\omega - F_2^{\#}(W(S) \otimes_{F,W(S)} \omega)(F_1^{\#})^{-1} = \eta^{\#}(F_1^{\#})^{-1}. \tag{53}$$

We define now a $\mathbb{Z}_p$-linear endomorphism U of $\mathrm{Hom}_{W(S)}(P_1, W(\mathfrak{a})P_2)$ by

$$U\omega = F_2^{\#}(W(S) \otimes_{F,W(S)} \omega)(F_1^{\#})^{-1}.$$

Then U is nilpotent. Indeed for this it suffices to show that F_2 is nilpotent on $W(\mathfrak{a})P_2$. Clearly we need only to show that the Frobenius ^{F} is nilpotent on $W(\mathfrak{a})$. Since p is nilpotent an easy reduction reduces this statement to the case, where $p \cdot \mathfrak{a} = 0$. It is well–known that in this case the Frobenius on $W(\mathfrak{a})$ takes the form

$${}^F(a_0, a_1, \ldots, a_i, \ldots) = (a_0^p, a_1^p, \ldots, a_i^p, \ldots).$$

Since this is nilpotent by assumption the operator U is nilpotent, too.

Then the operator $1 - U$ is invertible, and therefore the equation (53)

$$(1 - U)\omega = \eta^{\#}(F_1^{\#})^{-1}$$

has a unique solution. □

Corollary 43. — *Assume that we are given an ideal* $\mathfrak{c} \subset W(\mathfrak{a})$, *which satisfies* ${}^F\mathfrak{c} \subset \mathfrak{c}$ *and a* $W(S)$*-module homomorphism* $\alpha_0 : P_1 \to P_2$, *which satisfies the congruence*

$$F_2\alpha_0(x) \equiv \alpha_0(F_1 x) \mod \mathfrak{c}P_2.$$

Then we have $\alpha \equiv \alpha_0 \mod \mathfrak{c}P_2$.

Proof. — One starts the proof of the lemma with α_0 given by the assumption of the corollary and looks for a solution $\omega \in \mathrm{Hom}_{W(S)}(P_1, \mathfrak{c}P_2)$ of the equation (52). □

Theorem 44. — *Let* $S \to R$ *be a surjective homomorphism of rings, such that* p *is nilpotent in* S. *Assume the kernel* $\mathfrak{a}$ *of this homomorphism is equipped with divided powers. Let* $\mathcal{P}$ *be a display over* R *and let* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *be liftings to* S. *Let us denote by* $\widehat{Q}_i$ *the inverse image of* Q *by the map* $P_i \to P$ *for* $i = 1, 2$. *Let* $V^{-1} : \widehat{Q}_i \to P_i$ *be the extension of the operator* $V^{-1} : Q_i \to P_i$ *given by the divided powers. Then there is a unique isomorphism* $\alpha : (P_1, \widehat{Q}_1, F, V^{-1}) \to (P_2, \widehat{Q}_2, F, V^{-1})$, *which lifts the identity of* $\mathcal{P}$.

Proof. — The uniqueness follows from the proof of lemma 39. Indeed one has only to observe that the commutative diagram (50) still makes sense. By assumption we have $p^M \cdot \mathfrak{a} = 0$ for some number M. We make an induction on the number N to show the following assertion:

There exists a $W(S)$-linear lifting $\alpha : P_1 \to P_2$ of the identity such that

$$\begin{aligned} F\alpha(x) &= \alpha(Fx) \mod p^N W(\mathfrak{a}) \quad \text{for } x \in P_1 \\ V^{-1}\alpha(y) &= \alpha(V^{-1}y) \mod p^N W(\mathfrak{a}) \quad \text{for } y \in \widehat{Q}_1. \end{aligned} \tag{54}$$

We note that the divided powers give us an isomorphism $\prod_n \mathbf{w'}_n : W(\mathfrak{a}) \simeq \mathfrak{a}^{\mathbb{N}}$. From this we see that

$$^{F}W(\mathfrak{a}) \subset pW(\mathfrak{a}), \qquad I_S \cdot W(\mathfrak{a}) \subset pW(\mathfrak{a}).$$

In order to have a start for our induction, we consider the equations (54) to be fulfilled in the case $N = 0$ for any $W(S)$-linear lifting α. Hence we may assume that we have already constructed a $W(S)$-linear homomorphism α_N, which lifts the identity and satisfies (54). To prove the theorem we have to construct a $W(S)$-linear lifting α' of the identity, which satisfies (54) with N replaced by $N + 1$. We choose a normal decomposition $P_1 = L_1 \oplus T_1$ and we put $L_2 = \alpha_N(L_1)$ and $T_2 = \alpha_N(T_1)$. Then $P_2 = L_2 \oplus T_2$ will in general not be a normal decomposition for the display $\mathcal{P}_2$. But we can replace the display $\mathcal{P}_2$ by the display $(P_2, L_2 + I_S T_1, F, V^{-1})$, which is defined because $L_2 + I_S T_1 \subset \widehat{Q}_2$. Hence we may assume without loss of generality that $P_2 = L_2 + T_2$ is a normal decomposition.

For $i = 1, 2$ we consider the ^{F}-linear isomorphisms

$$U_i = V_i^{-1} + F_i : L_i \oplus T_i \longrightarrow P_i.$$

Then we define α to be the unique $W(S)$-linear map $P_1 \to P_2$, lifting the identity which satisfies

$$\alpha(U_1 x) = U_2 \alpha(x), \qquad \text{for } x \in P_1. \tag{55}$$

One readily verifies that α_N satisfies this equation modulo $p^N W(\mathfrak{a})$. By the corollary to the lemma 42 we obtain:

$$\alpha \equiv \alpha_N \mod p^N W(\mathfrak{a}) \tag{56}$$

We will verify that α commutes with F modulo $p^{N+1}W(\mathfrak{a})$. We verify this for elements $l_1 \in L_1$ and $t_1 \in T_1$ separately. We write $\alpha(l_1) = l_2 + t_2$, where $l_2 \in L_2$ and $t_2 \in T_2$. Since $\alpha_N(l_1) \in L_2$ we conclude from the congruence (56) that $t_2 \equiv 0 \mod p^N W(\mathfrak{a})$. Therefore we obtain

$$Ft_2 \equiv 0 \mod p^{N+1} W(\mathfrak{a}).$$

Also since $V^{-1}(W(\mathfrak{a})P_2) \subset W(\mathfrak{a})P_2$, we find

$$V^{-1}t_2 \equiv 0 \mod p^N W(\mathfrak{a}).$$

Now we can compute:

$$\begin{aligned}\alpha(V^{-1}l_1) &= \alpha(U_1 l_1) = U_2\alpha(l_1) = V^{-1}l_2 + Ft_2 \\ &\equiv V^{-1}l_2 = V^{-1}\alpha(l_1) - V^{-1}t_2 \mod p^{N+1}W(\mathfrak{a}).\end{aligned} \tag{57}$$

If we multiply the last equation by p, we obtain

$$\alpha(Fl_1) \equiv F\alpha(l_1) \quad \text{modulo } p^{N+1}W(\mathfrak{a}), \quad \text{for } l_1 \in L_1.$$

To treat the elements in T_1 we write $\alpha(t_1) = l_2' + t_2'$. The same argument as before now yields $l_2' \equiv 0 \mod p^N W(\mathfrak{a})$. Since our operator V^{-1} is ^{F}-linear on $\widehat{Q}_2$ and since l_2' is a sum of elements of the form $\xi \cdot y$, where $\xi \in p^N W(\mathfrak{a})$ and $y \in L_2'$, we obtain

$$V^{-1}l_2' \equiv 0 \mod p^{N+1}W(\mathfrak{a}).$$

Now we compute as above:

$$\begin{aligned}\alpha(Ft_1) = \alpha(U_1t_1) = U_2\alpha(t_1) = V^{-1}l_2' + Ft_2' \\ \equiv Ft_2' = F\alpha(t_1) - Fl_2' \equiv F\alpha(t_1) \mod p^{N+1}W(\mathfrak{a}).\end{aligned}$$

Altogether we have proved

$$\alpha(Fx) \equiv F\alpha(x) \mod p^{N+1}W(\mathfrak{a}), \quad \text{for } x \in P_1. \tag{58}$$

From this equation we conclude formally

$$\alpha(V^{-1}y) \equiv V^{-1}\alpha(y) \mod p^{N+1}W(\mathfrak{a}) \quad \text{for } y \in I_S P_1. \tag{59}$$

Indeed, it is enough to check this congruence for y of the form ${}^V\xi{\cdot}x$. Since $V^{-1}({}^V\xi x) = \xi Fx$, we conclude easily by (58). The following equation holds because both sides are zero:

$$\alpha(V^{-1}y) = V^{-1}\alpha(y) \quad \text{for } y \in \mathfrak{a}\cdot P_1. \tag{60}$$

The equation (57) shows that α does not necessarily commute with V^{-1} on L_1 modulo $p^{N+1}W(\mathfrak{a})$. Indeed, the map $L_1 \xrightarrow{\alpha} L_2 \oplus T_2 \xrightarrow{pr} T_2$ factors through $p^N W(\mathfrak{a})T_2$. Let us denote by η the composite:

$$\eta : L_1 \longrightarrow p^N W(\mathfrak{a})T_2 \xrightarrow{V^{-1}} p^N W(\mathfrak{a})P_2$$

Then we may rewrite the formula (57) as

$$\alpha(V^{-1}l_1) \equiv V^{-1}\alpha(l_1) - \eta(l_1) \mod p^{N+1}W(\mathfrak{a}). \tag{61}$$

We look for a solution α' of our problem, which has the form

$$\alpha' = \alpha + \omega,$$

where ω is a $W(S)$-linear map

$$\omega : P_1 \longrightarrow p^N W(\mathfrak{a})P_2. \tag{62}$$

First of all we want to ensure that the equation (58) remains valid for α'. This is equivalent with

$$\omega(Fx) = F\omega(x) \mod p^{N+1}W(\mathfrak{a}) \quad \text{for } x \in P_1.$$

But the right hand side of this equation is zero $\mod p^{N+1}W(\mathfrak{a})$. Hence α' satisfies (58), if

$$\omega(Fx) \equiv 0 \mod p^{N+1}W(\mathfrak{a}).$$

We note that any $W(S)$-linear map (62) satisfies trivially $\omega(FL_1) = \omega(pV^{-1}L_1) = p\omega(V^{-1}L_1) \equiv 0 \mod p^{N+1}W(\mathfrak{a})$. Hence α' commutes with $F \mod p^{N+1}W(\mathfrak{a})$, if $\omega \mod p^{N+1}W(\mathfrak{a})$ belongs to the $W(S)$-module

$$\operatorname{Hom}(P_1/W(S)FT_1,\ p^N W(\mathfrak{a})/p^{N+1}W(\mathfrak{a}) \otimes_{W(S)} P_2). \tag{63}$$

Moreover α' commutes with $V^{-1} \mod p^{N+1}W(\mathfrak{a})$, if ω satisfies the following congruence

$$\omega(V^{-1}l_1) - V^{-1}\omega(l_1) = \eta(l_1) \mod p^{N+1}W(\mathfrak{a}), \quad \text{for } l_1 \in L_1. \tag{64}$$

Indeed, we obtain from (64)

$$\alpha'(V^{-1}y) = V^{-1}\alpha'(y) \mod p^{N+1}W(\mathfrak{a}), \quad \text{for } y \in \widehat{Q}_1,$$

because of (61) for $y \in L_1$ and because of (59) and (60) for $y \in I_S P_1 + \mathfrak{a}P_1$. Hence our theorem is proved if we find a solution ω of the congruence (64) in the $W(S)$-module (63).

The map V^{-1} induces an ^{F}-linear isomorphism

$$V^{-1} : L_1 \longrightarrow P_1/W(S)FT_1.$$

Hence we may identify the $W(S)$-module (63) with

$$\operatorname{Hom}_{F-\text{linear}}(L_1,\ p^N W(\mathfrak{a})/p^{N+1}W(\mathfrak{a}) \otimes_{W(S)} P_2), \tag{65}$$

by the map $\omega \mapsto \omega V^{-1}$.

We rewrite now the congruence (64) in terms of $\widetilde{\omega} = \omega V^{-1}$. The map $V^{-1}\omega$ is in terms of $\widetilde{\omega}$ the composite of the following maps:

$$\begin{array}{ccc} L_1 \xhookrightarrow{\iota} P_1 \xrightarrow{pr} W(S)V^{-1}L_1 & \xrightarrow{\quad V^{\#} \quad} & W(S) \otimes_{F,W(S)} L_1 \\ & & \big\downarrow \widetilde{\omega}^{\#} \\ p^N W(\mathfrak{a})/p^{N+1}W(\mathfrak{a}) \otimes_{W(S)} P_2 & \xleftarrow{V^{-1}} & p^N W(\mathfrak{a})/p^{N+1}W(\mathfrak{a}) \otimes_{W(S)} P_2 \end{array} \tag{66}$$

The map ι in this diagram is the canonical injection. The map pr is the projection with respect to the following direct decomposition

$$P_1 = W(S)V^{-1}L_1 \oplus W(S)FT_1.$$

Finally the lower horizontal ^{F}-linear map V^{-1} is obtained as follows. The divided powers provide an isomorphism (compare (48)):

$$p^N W(\mathfrak{a})/p^{N+1}W(\mathfrak{a}) \xrightarrow{\sim} (p^N\mathfrak{a}/p^{N+1}\mathfrak{a})^{\mathbb{N}}.$$

Using the notation $[a_0, a_1, \dots, a_n, \dots]$ for a vector of $(p^N\mathfrak{a}/p^{N+1}\mathfrak{a})^{\mathbb{N}}$, the map V^{-1} is given by:

$$V^{-1}[a_0, a_1, \dots] \otimes x = [a_1, a_2, \dots] \otimes Fx.$$

Let us denote by $B = V^{\#} \circ pr \circ \iota$ the composite of the upper horizontal maps in the diagram (66). Then we may write

$$V^{-1}\omega = V^{-1}\widetilde{\omega}^{\#}B.$$

We define a $\mathbb{Z}$-linear operator U on the space

$$\mathrm{Hom}_{F-\mathrm{linear}}(L_1, p^N W(\mathfrak{a})/p^{N+1}W(\mathfrak{a})^{\otimes}_{W(S)}P_2), \tag{67}$$

by

$$U\widetilde{\omega} = V^{-1}\widetilde{\omega}^{\#}B. \tag{68}$$

Hence the equation (64) which we have to solve now reads as follows:

$$(1-U)\widetilde{\omega} = \eta \mod p^{N+1}W(\mathfrak{a}).$$

Here 1 denotes the identity operator on the group (67) and $\widetilde{\omega}$ and η are considered as elements of this group. Clearly this equation has a solution $\widetilde{\omega}$ for any given η, if the operator U is nilpotent on (67).

To see the nilpotency we rewrite the space (67). We set $D_i = P_i/I_S P_i + pP_i = S/pS \otimes_{w_0, W(S)} P_i$, and we denote the image of Q_i in this space by D_i^1. Then our group (67) is isomorphic to

$$\mathrm{Hom}_{\mathrm{Frobenius}}(D_1^1, p^N W(\mathfrak{a})/p^{N+1}W(\mathfrak{a}) \otimes_{S/pS} D_2),$$

where Hom denotes the Frobenius linear maps of S/pS-modules. Now the operator U is given by the formula (68) modulo $pW(S) + I_S$. But then locally on $\operatorname{Spec} S/pS$, the operator B, is just given by the matrix B of (15). Hence the nilpotency follows from (15). $\square$

2.2. Triples and crystals. — Let R be a ring such that p is nilpotent in R, and let $\mathcal{P} = (P, Q, F, V^{-1})$ be a display over R. Consider a pd-thickening $S \to R$ with kernel $\mathfrak{a}$, i.e. by definition that p is nilpotent in S and that the ideal $\mathfrak{a}$ is equipped with divided powers. In particular this implies that all elements in $\mathfrak{a}$ are nilpotent. We will now moreover assume that the divided powers are compatible with the canonical divided powers on $p\mathbb{Z}_p \subset \mathbb{Z}_p$.

A $\mathcal{P}$-triple $\mathcal{T} = (\widetilde{P}, F, V^{-1})$ over S consists of a projective finitely generated $W(S)$-module $\widetilde{P}$, which lifts P, i.e. is equipped with an isomorphism $W(R) \otimes_{W(S)} \widetilde{P} \simeq P$. Hence we have a canonical surjection $\widetilde{P} \to P$ with kernel $W(\mathfrak{a})\widetilde{P}$. Let us denote by $\widehat{Q}$ the inverse image of Q. Moreover a triple consists of two ^{F}-linear operators of $W(S)$-modules $F : \widetilde{P} \to \widetilde{P}$ and $V^{-1} : \widehat{Q} \to \widetilde{P}$. The following relations are required:

$$\begin{aligned} V^{-1}({}^V wx) &= wFx, \quad \text{for } w \in W(S), w \in \widetilde{P}. \\ V^{-1}(\mathfrak{a}\widetilde{P}) &= 0 \end{aligned}$$

Here $\mathfrak{a} \subset W(S)$ is the ideal given by the divided powers (48).

There is an obvious notion of a morphism of triples. Let $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ be a morphism of displays. Let $\mathcal{T}_1$ respectively $\mathcal{T}_2$ be a $\mathcal{P}_1$-triple respectively a $\mathcal{P}_2$-triple over S. An α-morphism $\widetilde{\alpha} : \widetilde{P}_1 \to \widetilde{P}_2$ is a homomorphism of $W(S)$-modules which lifts α and which commutes with F and V^{-1}. We note that $\widetilde{\alpha}(\widehat{Q}_1) \subset \widehat{Q}_2$. Therefore the requirement that $\widetilde{\alpha}$ commutes with V^{-1} makes sense. With this definition the $\mathcal{P}$-triples over S form a category, where $\mathcal{P}$ is allowed to vary in the category of displays over R. We call it the category of triples relative to $S \to R$.

Let us now define base change for triples. Let $\varphi : R \to R'$ be a ring homomorphism. Let $S \to R$ respectively $S' \to R'$ be pd-thickenings. Assume that we are given a homomorphism of pd-thickenings:

$$\begin{array}{ccc} S & \xrightarrow{\psi} & S' \\ \downarrow & & \downarrow \\ R & \xrightarrow{\varphi} & R' \end{array} \tag{69}$$

Let $\mathcal{T}$ be a $\mathcal{P}$-triple over S as before. Let $\mathcal{P}_{R'}$ be the display obtained by base change from $\mathcal{P}$. Then we define a $\mathcal{P}_{R'}$-triple $\mathcal{T}_{S'}$ over S' as follows. We set $\mathcal{T}_{S'} = (W(S') \otimes_{W(S)} \widetilde{P}, F, V^{-1})$ with the following definition of F and V^{-1}. The operator F is simply the ^{F}-linear extension of $F : \widetilde{P} \to \widetilde{P}$. The operator V^{-1} on $\widehat{Q}'$ is uniquely determined by the equations:

$$\begin{aligned} V^{-1}(w \otimes y) &= {}^F w \otimes V^{-1} y, \quad \text{for } y \in \widehat{Q}, w \in W(S') \\ V^{-1}({}^V w \otimes x) &= w \otimes Fx, \qquad \text{for } x \in \widetilde{P} \\ V^{-1}(a \otimes x) &= 0, \qquad \text{for } a \in \mathfrak{a}' \subset W(\mathfrak{a}'). \end{aligned}$$

Here $\mathfrak{a}'$ is the kernel of $S' \to R'$ with its pd-structure.

Let $S \to R$ be a pd-thickening and $\mathcal{P}$ be a display over R. Let $\mathcal{T}$ be a $\mathcal{P}$-triple over S. By theorem 44 it is determined up to unique isomorphism. We can construct all liftings of $\mathcal{P}$ to a display over S as follows. We consider the Hodge filtration of $\mathcal{P}$.

$$Q/I_R P \quad \subset \quad P/I_R P \tag{70}$$

Let L be a direct summand of $\widetilde{P}/I_S\widetilde{P}$, such that the filtration of S-modules

$$L \quad \subset \widetilde{P}/I_S\widetilde{P} \tag{71}$$

lifts the filtration (70). We call this a lifting of the Hodge filtration to $\mathcal{T}$. If we denote by $\widetilde{Q}_L \subset \widetilde{P}$ the inverse image of L by the projection $\widetilde{P} \to \widetilde{P}/I_S\widetilde{P}$ we obtain a display $(\widetilde{P}, \widetilde{Q}_L, F, V^{-1})$. By theorem 44 we conclude:

Proposition 45. — *The construction above gives a bijection between the liftings of the display $\mathcal{P}$ to S and the liftings of the Hodge filtration to $\mathcal{T}$.*

We will now formulate an enriched version of theorem 44.

Theorem 46. — *Let $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ be a morphism of displays over R. Let $S \to R$ be a pd-thickening and consider for $i = 1, 2$ a $\mathcal{P}_i$-triple $\mathcal{T}_i$ over S. Then there is a unique α-morphism of triples $\widetilde{\alpha} : \mathcal{T}_1 \to \mathcal{T}_2$.*

Proof. — To prove the uniqueness we may assume $\alpha = 0$. Then we consider the diagram 50 with P respectively P' replaced by $\widetilde{P}_1$ respectively $\widetilde{P}_2$ and u replaced by $\widetilde{\alpha}$. There is a map $V^{\#}$ on $\widetilde{P}$ which is uniquely determined by

$$V^{\#}(wV^{-1}y) = w \otimes y, \quad \text{for } w \in W(S), y \in \widehat{Q}.$$

Its existence follows by choosing a lifting of the Hodge filtration of $\mathcal{P}$ to $\mathcal{T}$. With these remarks the arguments of lemma 39 apply, and show the uniqueness. To show the existence we first consider the case where α is an isomorphism. By choosing liftings $\widetilde{\mathcal{P}}_1$ respectively $\widetilde{\mathcal{P}}_2$ of $\mathcal{P}_1$ respectively $\mathcal{P}_2$ to S this case is easily reduced to theorem 44. The general case is reduced to the first case by considering the isomorphism of displays:

$$\begin{aligned} \mathcal{P}_1 \oplus \mathcal{P}_2 &\longrightarrow \mathcal{P}_1 \oplus \mathcal{P}_2 \quad , \\ (x, y) &\longmapsto (x, \alpha(x) + y) \end{aligned}$$

where $x \in \mathcal{P}_1$ and $y \in \mathcal{P}_2$. □

Remark. — This theorem extends trivially to the case where S is a topological ring as in definition 13. More precisely let R be as in the last theorem, and let $S \to R$ be any surjection, such that the kernel $\mathfrak{a}$ is equipped with divided powers. If p is not nilpotent in S this is not a pd-thickening in our sense (compare section 2.2). Assume that there is a sequence of sub pd-ideals $\ldots \mathfrak{a}_n \supset \mathfrak{a}_{n+1} \ldots$, such that p is nilpotent in $S/\mathfrak{a}_n$ and such that S in complete and separated in the linear topology defined by the ideals $\mathfrak{a}_n$. Then the theorem above is true for the surjection $S \to R$. We note that S is a p-adic ring. We will call $S \to R$ a topological pd-thickening. We are particularly interested in the case where S has no p-torsion.

Let us fix $S \to R$ as before. To any display $\mathcal{P}$ we may choose a $\mathcal{P}$-triple $\mathcal{T}_{\mathcal{P}}(S)$. By the theorem $\mathcal{P} \mapsto \mathcal{T}_{\mathcal{P}}(S)$ is a functor from the category of displays to the category of triples. It commutes with arbitrary base change in the sense of (69). If we fix $\mathcal{P}$ we may view $S \mapsto \mathcal{T}_{\mathcal{P}}(S)$ as a crystal with values in the category of triples. We deduce from it two other crystals.

Let X be a scheme, such that p is locally nilpotent in $\mathcal{O}_X$. Then we will consider the crystalline site, whose objects are triples (U, T, δ), where $U \subset X$ is an open subscheme, $U \to T$ is a closed immersion defined by an ideal $\mathcal{J} \subset \mathcal{O}_T$, and δ is a divided power structure on $\mathcal{J}$. We assume that p is locally nilpotent on T, and that the divided powers δ are compatible with the canonical divided power structure on the ideal $p\mathbb{Z}_p \subset \mathbb{Z}_p$. The reason for this last condition, which was not necessary in theorem 46 will become apparent later. Let $W(\mathcal{O}_X^{\mathrm{crys}})$ be the sheaf on the crystalline site, which associates to a pd-thickening $U \to T$ the ring $W(\Gamma(T, \mathcal{O}_T))$. A crystal in $W(\mathcal{O}_X^{\mathrm{crys}})$-modules will be called a Witt crystal.

Sometimes we will restrict our attention to the crystalline site which consists of pd-thickenings (U, T, δ), such that the divided power structure is locally nilpotent in the sense of [Me] Chapt. III definition 1.1. We call this the nilpotent crystalline site.

Let $\mathcal{P}$ be a display over R. Then we define a Witt crystal $\mathcal{K}_{\mathcal{P}}$ on $\operatorname{Spec} R$ as follows. It is enough to give the value of $\mathcal{K}_{\mathcal{P}}$ on pd-thickenings of the form $\operatorname{Spec} R' \to \operatorname{Spec} S'$, where $\operatorname{Spec} R' \hookrightarrow \operatorname{Spec} R$ is an affine open neighbourhood. The triple over S' associated to $\mathcal{P}'_R$ is of the form

$$\mathcal{T}_{\mathcal{P}_{R'}}(S') = (\widetilde{P}, F, V^{-1}).$$

We define

$$\mathcal{K}_{\mathcal{P}}(\operatorname{Spec} R' \to \operatorname{Spec} S') = \widetilde{P}. \tag{72}$$

For the left hand side we will also write $\mathcal{K}_{\mathcal{P}}(S')$.

Definition 47. — The sheaf $\mathcal{K}_{\mathcal{P}}$ on the crystalline situs of $\operatorname{Spec} R$ is called the Witt crystal associated to $\mathcal{P}$. We also define a crystal of $\mathcal{O}^{\mathrm{crys}}$-modules on $\operatorname{Spec} R$ by

$$\mathcal{D}_{\mathcal{P}}(S') = \mathcal{K}_{\mathcal{P}}(S')/I_{S'}\mathcal{K}(S').$$

$\mathcal{D}_{\mathcal{P}}$ is called the (covariant) Dieudonné crystal.

More generally we may evaluate these crystals for any topological pd-thickening in the sense of the last remark. If $(S, \mathfrak{a}_n)$ is a topological pd-thickening we set:

$$\begin{aligned} \mathcal{K}_{\mathcal{P}}(S) &= \varprojlim_n \mathcal{K}_{\mathcal{P}}(R/\mathfrak{a}_n) \\ \mathcal{D}_{\mathcal{P}}(S) &= \varprojlim_n \mathcal{D}_{\mathcal{P}}(R/\mathfrak{a}_n) \end{aligned} \tag{73}$$

The Witt crystal and the Dieudonné crystal are compatible with base change. This means that for an arbitrary homomorphism of pd-thickenings (69) we have canonical isomorphisms:

$$\begin{aligned} \mathcal{K}_{\mathcal{P}_{\mathcal{R}'}}(S') &\simeq W(S') \otimes_{W(S)} \mathcal{K}_{\mathcal{P}}(S) \\ \mathcal{D}_{\mathcal{P}_{\mathcal{R}'}}(S') &\simeq S' \otimes_S \mathcal{D}_{\mathcal{P}}(S). \end{aligned} \tag{74}$$

This follows from the definition of the $\mathcal{P}_{R'}$-triple $\mathcal{T}_{S'}$. The R-module $\mathcal{D}_{\mathcal{P}}(R)$ is identified with $P/I_R P$ and therefore inherits the Hodge filtration

$$\mathcal{D}^1_{\mathcal{P}}(R) \subset \mathcal{D}_{\mathcal{P}}(R). \tag{75}$$

The proposition 45 may be reformulated in terms of the Dieudonné crystal.

Theorem 48. — *Let $S \to R$ be a pd-thickening. Consider the category $\mathcal{C}$ whose objects are pairs $(\mathcal{P}, E)$, where $\mathcal{P}$ is a display over R, and E is a direct summand of the S-module $\mathcal{D}_{\mathcal{P}}(S)$, which lifts the Hodge filtration (75). A morphism $\phi : (\mathcal{P}, E) \to (\mathcal{P}', E')$ in the category $\mathcal{C}$ is a morphism of displays $\phi : \mathcal{P} \to \mathcal{P}'$, such that the induced morphism of the associated Dieudonné crystals (definition 47) maps E to E'. Then the category $\mathcal{C}$ is canonically equivalent to the category of displays over S.*

The description of liftings of a display $\mathcal{P}$ over R is especially nice in the following case: Let $S \to R$ be surjection with kernel $\mathfrak{a}$, such that $\mathfrak{a}^2 = 0$. Then we consider the abelian group:

$$\operatorname{Hom}(\mathcal{D}^1_{\mathcal{P}}(R), \mathfrak{a} \otimes_R (\mathcal{D}_{\mathcal{P}}(R)/\mathcal{D}^1_{\mathcal{P}}(R))) \tag{76}$$

We define an action of this group on the set of liftings of $\mathcal{P}$ to S as follows. Two liftings correspond by theorem 48 to two liftings E_1 and E_2 of the Hodge filtration. We need to define their difference in the group (76). Consider the natural homomorphism:

$$E_1 \subset \mathcal{D}_{\mathcal{P}}(S) \longrightarrow \mathcal{D}_{\mathcal{P}}(S)/E_2$$

Since E_1 and E_2 lift the same module $\mathcal{D}^1_{\mathcal{P}}(R)$ the last map factors through

$$E_1 \longrightarrow \mathfrak{a}(\mathcal{D}_{\mathcal{P}}(S)/E_2). \tag{77}$$

The right hand side is canonically isomorphic to $\mathfrak{a} \otimes_R (\mathcal{D}_{\mathcal{P}}(R)/\mathcal{D}^1_{\mathcal{P}}(R))$, since $\mathfrak{a}^2 = 0$. Hence the map (77) may be identified with a map:

$$u : \mathcal{D}^1_{\mathcal{P}}(R) \longrightarrow \mathfrak{a} \otimes_R \mathcal{D}_{\mathcal{P}}(R)/\mathcal{D}^1_{\mathcal{P}}(R)$$

We define $E_1 - E_2 = u$. It follows immediately that:

$$E_2 = \{e - \widetilde{u(e)} \mid e \in E_1\}, \tag{78}$$

where $\widetilde{u(e)} \in \mathfrak{a}\mathcal{D}_{\mathcal{P}}(S)$ denotes any lifting of $u(e)$. This proves the following

Corollary 49. — *Let $\mathcal{P}$ be a display over R. Let $S \to R$ be a surjective ring homomorphism with kernel $\mathfrak{a}$, such that $\mathfrak{a}^2 = 0$. The action of the group* (76) *on the set of liftings of $\mathcal{P}$ to a display over S just defined is simply transitive. If $\mathcal{P}_0$ is a lifting of $\mathcal{P}$ and u an element in* (76) *we denote the action by $\mathcal{P}_0 + u$.*

Using example 1.17 it is easy to give a description of $\mathcal{P}_0 + u$ in the situation of the last corollary. Let $\mathfrak{a} \subset W(\mathfrak{a})$ be the subset of all Teichmüller representatives of elements of $\mathfrak{a}$. If we equip $\mathfrak{a}$ with the divided powers $\alpha_p(\mathfrak{a}) = 0$ this agrees with our definition after equation (48). We restrict our attention to homomorphisms $\alpha : P_0 \to \mathfrak{a}P_0 \subset W(\mathfrak{a})P_0$ and consider the display defined by (27):

$$\begin{aligned} F_\alpha x &= Fx - \alpha(Fx), \quad \text{for } x \in P_0 \\ V_\alpha^{-1} y &= V^{-1}y - \alpha(V^{-1}y), \quad \text{for } y \in Q_0. \end{aligned} \tag{79}$$

Then there is an element u in the group (76) such that:

$$\mathcal{P}_\alpha = \mathcal{P}_0 + u \tag{80}$$

It is easily described: There is a natural isomorphism $\mathfrak{a}P_0 \cong \mathfrak{a} \otimes_R P/I_R P$. Hence α factors uniquely through a map:

$$\overline{\alpha} : P/I_R P \longrightarrow \mathfrak{a} \otimes_R P/I_R P.$$

Conversely any R-module homomorphism $\overline{\alpha}$ determines uniquely a map α. Let u be the composite of the following maps:

$$u : Q/I_R P \subset P/I_R P \xrightarrow{\overline{\alpha}} \mathfrak{a} \otimes_R P/I_R P \longrightarrow \mathfrak{a} \otimes_R P/Q. \tag{81}$$

Then the equation (80) holds. To see this consider the isomorphism :

$$\tau : (P_0, \widehat{Q}_0, F_\alpha, V_\alpha^{-1}) \longrightarrow (P_0, \widehat{Q}_0, F, V^{-1}),$$

which exists by theorem 46 . Using the relations:

$$F\mathfrak{a}P_0 = V^{-1}\mathfrak{a}P_0 = 0, \quad \alpha^2 = 0,$$

it is easily verified that $\tau(x) = x + \alpha(x)$ for $x \in P_0$. It follows that $\mathcal{P}_\alpha$ is isomorphic to the display $(P_0, \tau(Q_0), F, V^{-1})$. Since

$$\tau(Q_0) = \{x + \alpha(x) | x \in Q_0\}$$

the equation (80) follows with the u defined above (81).

Next we define the universal deformation of a display. Let $S \to R$ be a surjection of rings, such that the kernel is a nilpotent ideal $\mathfrak{a}$. For a display $\mathcal{P}$ over R, we define the functor of deformations of $\mathcal{P}$:

$$\operatorname{Def}_{\mathcal{P}}(S)$$

as the set of isomorphism classes of pairs $(\widetilde{\mathcal{P}}, \iota)$, where $\widetilde{\mathcal{P}}$ is a display over S and $\iota : \mathcal{P} \to \widetilde{\mathcal{P}}_R$ is an isomorphism with the display obtained by base change.

We will consider the deformation functor on the following categories $\operatorname{Aug}_{\Lambda\to R}$. Let Λ be a topological ring of type (16). The ring R is equipped with the discrete topology. Suppose we are given a continuous surjective homomorphism $\varphi : \Lambda \to R$.

Definition 50. — Let $\operatorname{Aug}_{\Lambda\to R}$ be the category of morphisms of discrete Λ-algebras $\psi_S : S \to R$, such that ψ_S is surjective and has a nilpotent kernel. If $\Lambda = R$, we will denote this category simply by Aug_R.

A nilpotent R-algebra $\mathcal{N}$ is an R-algebra (without unit), such that $\mathcal{N}^N = 0$ for a sufficiently big number N. Let Nil_R denote the category of nilpotent R-algebras. To a nilpotent R-algebra $\mathcal{N}$ we associate an object $R|\mathcal{N}|$ in Aug_R. As an R-module we set $R|\mathcal{N}| = R \oplus \mathcal{N}$. The ring structure on $R|\mathcal{N}|$ is given by the rule:

$$(r_1 \oplus n_1)(r_2 \oplus n_2) = (r_1 r_2 \oplus r_1 n_2 + r_2 n_1 + n_1 n_2) \quad \text{for } n_i \in \mathcal{N},\ r_i \in R.$$

It is clear that this defines an equivalence of the categories Nil_R and Aug_R. An R-module M is considered as an element of Nil_R by the multiplication rule: $M^2 = 0$. The corresponding object in Aug_R is denoted by $R|M|$. We have natural fully faithful embeddings of categories

$$(R-\text{modules}) \subset \mathrm{Aug}_R \subset \mathrm{Aug}_{\Lambda\to R}$$

Let F be a set-valued functor on $\mathrm{Aug}_{\Lambda\to R}$. The restriction of this functor to the category of R-modules is denoted by $\mathbf{t}_F$ and is called the tangent functor. If the functor $\mathbf{t}_F$ is isomorphic to a functor $M \mapsto M \otimes_R t_F$ for some R-module t_F, we call t_F the tangent space of the functor F (compare [Z1] 2.21).

Let T be a topological Λ-algebra of type (16) and $\psi_T : T \to R$ be a surjective homomorphism of topological Λ-algebras. For an object $S \in \mathrm{Aug}_{\Lambda\to R}$, we denote by $\mathrm{Hom}(T, S)$ the set of continuous Λ-algebra homomorphisms, which commute with the augmentations ψ_T and ψ_S. We obtain a set-valued functor on $\mathrm{Aug}_{\Lambda\to R}$:

$$\mathrm{Spf}\, T(S) = \mathrm{Hom}(T, S) \tag{82}$$

If a functor is isomorphic to a functor of the type $\mathrm{Spf}\, T$ it is called prorepresentable.

We will now explain the prorepresentability of the functor $\mathrm{Def}_{\mathcal{P}}$. Let us first compute the tangent functor. Let M be an R-module. We have to study liftings of our fixed display $\mathcal{P}$ over R with respect to the homomorphism $R|M| \to R$. The corollary 49 applies to this situation. We have a canonical choice for $\mathcal{P}_0$:

$$\mathcal{P}_0 = \mathcal{P}_{R|M|}.$$

Let us denote by $\mathrm{Def}_{\mathcal{P}}(R|M|)$ the set of isomorphism classes of liftings of $\mathcal{P}$ to $R|M|$. Then we have an isomorphism :

$$\mathrm{Hom}_R(Q/I_R P, M \otimes_R P/Q) \longrightarrow \mathrm{Def}_{\mathcal{P}}(R|M|), \tag{83}$$

which maps a homomorphism u to the display $\mathcal{P}_0 + u$. Hence the functor $\mathrm{Def}_{\mathcal{P}}$ has a tangent space, which is canonically isomorphic to the finitely generated projective R-module $\mathrm{Hom}_R(Q/I_R P, P/Q)$. Consider the dual R-module $\omega = \mathrm{Hom}_R(P/Q, Q/I_R P)$. Then we may rewrite the isomorphism (83):

$$\mathrm{Hom}_R(\omega, M) \longrightarrow \mathrm{Def}_{\mathcal{P}}(R|M|)$$

Hence the identical endomorphism of ω defines a morphism of functors:

$$\mathrm{Spf}\, R|\omega| \longrightarrow \mathrm{Def}_{\mathcal{P}} \tag{84}$$

We lift ω to a projective finitely generated Λ-module $\widetilde{\omega}$. We consider the symmetric algebra $\mathcal{S}_\Lambda(\widetilde{\omega})$. Its completion A with respect to the augmentation ideal is a topological Λ-algebra of type (16), which has a natural augmentation $A \to \Lambda \to R$. Since the deformation functor is smooth, i.e. takes surjections $S_1 \to S_2$ to surjective maps of sets, the morphism (84) may be lifted to a morphism:

$$\operatorname{Spf} A \longrightarrow \operatorname{Def}_{\mathcal{P}} \tag{85}$$

It is not difficult to show, that this is an isomorphism using the fact that it induces by construction an isomorphism on the tangent spaces (compare [CFG]). It is easy to describe the universal display over $\mathcal{P}^{\text{univ}}$ over A. Let $u : Q/I_R P \to \omega \otimes_R P/Q$ the map induced by the identical endomorphism of ω. Let $\alpha : P \to \omega \otimes_R P/Q$ be any map, which induces u as described by (81). The we obtain a display $\mathcal{P}_\alpha$ over $R|\omega|$. For $\mathcal{P}^{\text{univ}}$ we may take any lifting of $\mathcal{P}_\alpha$ to A.

Let us assume that the display $\mathcal{P}$ is given by the equations (9). In this case the universal deformation is as follows. We choose an arbitrary lifting $(\widetilde{\alpha_{ij}}) \in Gl_h(W(\Lambda))$ of the matrix (α_{ij}). We choose indeterminates (t_{kl}) for $k = 1, \dots d,\ l = d+1, \dots h$. We set $A = \Lambda[\![t_{kl}]\!]$. For any number n we denote by E_n the unit matrix. Consider the following invertible matrix over $Gl_h(A)$:

$$\begin{pmatrix} E_d & [t_{kl}] \\ 0 & E_{h-d} \end{pmatrix} (\alpha_{ij}) \tag{86}$$

As usual $[t_{kl}] \in W(A)$ denotes the Teichmüller representative. This matrix defines by (9) display $\mathcal{P}^{\text{univ}}$ over the topological ring A. The the pair $(A, \mathcal{P}^{\text{univ}})$ prorepresents the functor $\operatorname{Def}_{\mathcal{P}}$ on the category $\operatorname{Aug}_{\Lambda \to R}$.

2.3. Witt and Dieudonné crystals. — Our next aim is to explain how the Witt crystal may be reconstructed from the Dieudonné crystal.

The ideal $I_R \subset W(R)$ will be equipped with the divided powers (see [Gr] Chapt. IV 3.1):

$$\alpha_p({}^V w) = p^{p-2}\,{}^V(w^p), \qquad \text{for } w \in W(R). \tag{87}$$

The morphism $\mathbf{w}_0 : W(R) \to R$ is a topological pd-thickening, in the sense of the remark after theorem 46, because (87) defines a pd-thickenings $\mathbf{w}_0 : W_n(R) \to R$. We note that the last pd-thickenings are nilpotent, if $p \neq 2$.

If we evaluate a crystal on $\operatorname{Spec} R$ in $W(R)$ we have the topological pd-structure above in mind (compare (73)).

More generally we may consider a pd-thickening $S \to R$, where we assume p to be nilpotent in S. Let $\mathfrak{a} \subset S$ be the kernel. The divided powers define an embedding $\mathfrak{a} \subset W(S)$, which is an ideal of $W(S)$ equipped with the same divided powers as $\mathfrak{a} \subset S$. The kernel of the composite $W(S) \overset{\mathbf{w}_0}{\to} S \to R$ is the orthogonal direct sum $I_S \oplus \mathfrak{a}$. Since we have defined divided powers on each direct summand, we obtain a pd-structure on the kernel of:

$$W(S) \longrightarrow R. \tag{88}$$

Again this induces pd-thickenings $W_m(S) \to R$. Therefore me may consider (88) as a topological pd-thickening, and evaluate crystals in $W(S)$.

In the case $p \neq 2$ the divided powers on the kernel of $W_m(S) \to R$ are nilpotent, if the divided powers on the ideal $\mathfrak{a}$ were nilpotent.

Proposition 51. — *Let $S \to R$ be a pd-thickening. There is a canonical isomorphism*

$$\mathcal{K}_{\mathcal{P}}(S) \cong \mathcal{D}_{\mathcal{P}}(W(S)).$$

This will follow from the more precise statement in proposition 53.

For later purposes it is useful to note that this proposition makes perfect sense if we work inside the nilpotent crystalline site.

To define the isomorphism of proposition 51 we need the following ring homomorphism defined by Cartier:

$$\triangle : W(R) \longrightarrow W(W(R)). \tag{89}$$

It is defined for any commutative ring R. In order to be less confusing we use a hat in the notation, if we deal with the ring $W(W(R))$.

The homomorphism $\triangle$ is functorial in R and satisfies

$$\widehat{\mathbf{w}}_n(\triangle(\xi)) = {}^{F^n}\xi, \qquad \xi \in W(R). \tag{90}$$

As usual these properties determine $\triangle$ uniquely. We leave the reader to verify that the equation:

$$W(\mathbf{w}_n)(\triangle(\xi)) = {}^{F^n}\xi, \tag{91}$$

holds too.

Lemma 52. — *The following relations hold:*

$$\triangle({}^F\xi) = {}^{\widehat{F}}(\triangle(\xi)) = W({}^F)(\triangle(\xi)),$$
$$\triangle({}^V\xi) -{}^{\widehat{V}}(\triangle(\xi)) = \left[{}^V\xi, 0, 0, \ldots\right] \in W(I_R)$$

Here on the right hand side we have used logarithmic coordinates with respect to the divided powers on I_R.

Proof. — We use the standard argument. By functoriality we may restrict to the case where R is torsion free (as $\mathbb{Z}$-module). Then $W(R)$ is torsion free too. Hence it is enough to show that for each integer $n \geq 0$ the equations of the lemma hold after applying $\widehat{\mathbf{w}}_n$. This is readily verified. $\square$

Proposition 53. — *Let $S \to R$ be a pd-thickening with kernel $\mathfrak{a}$, and let $\mathcal{P} = (P, Q, F, V^{-1})$ be a display over R. Let $\mathcal{T} = (\widetilde{P}, F, V^{-1})$ be the unique (up to canonical isomorphism) $\mathcal{P}$-triple over S. Consider the pd-thickening $W(S) \to R$ with kernel $I_S \oplus \mathfrak{a}$. Let $\overline{\mathcal{T}}$ denote the unique $\mathcal{P}$-triple related to this pd-thickening. Then $\overline{\mathcal{T}}$ is of the form*

$$\overline{\mathcal{T}} = (W(W(S)) \otimes_{\triangle, W(S)} \widetilde{P}, F, V^{-1}),$$

where the operators F and V^{-1} are uniquely determined by the following properties:

$$F(\widehat{\xi} \otimes x) = {}^{F}\widehat{\xi} \otimes Fx, \qquad \widehat{\xi} \in W(W(S)), x \in \widetilde{P} \tag{92}$$

$$V^{-1}(\widehat{\xi} \otimes y) = {}^{F}\widehat{\xi} \otimes V^{-1}y, \quad y \in \widehat{Q}$$

$$V^{-1}({}^{V}\widehat{\xi} \otimes x) = \widehat{\xi} \otimes Fx.$$

Here as usual $\widehat{Q}$ denotes the inverse image of Q by the morphism $\widetilde{P} \to P$.

The triple $\overline{\mathcal{T}}$ provides the isomorphism of proposition 51:

$$\mathcal{K}_{\mathcal{P}}(S) = \widetilde{P} = W(S) \otimes_{\widehat{\mathbf{w}}_0} (W(W(S)) \otimes_{\triangle, W(S)} \widetilde{P}) = \mathcal{D}_{\mathcal{P}}(W(S)) \tag{93}$$

Proof. — Let $\alpha : W(S) \to R$ be the pd-thickening (88). It follows that from (91) that $W(W(S)) \otimes_{\triangle, W(S)} \widetilde{P} = \overline{P}$ is a lifting of P relative to α. We have homomorphisms

$$\overline{P} \xrightarrow{\pi} \widetilde{P} \longrightarrow P,$$

where the first arrow is induced by $W(\mathbf{w}_0) : W(W(S)) \to W(S)$. Let $\widehat{Q}$ be the inverse image of Q in $\widetilde{P}$.

We choose a normal decomposition $P = L \oplus T$, and we lift it to a decomposition $\widetilde{P} = \widetilde{L} \oplus \widetilde{T}$. Then we have the decomposition

$$\widehat{Q} = \widetilde{L} \oplus I_S \widetilde{T} \oplus \mathfrak{a}\widetilde{T}. \tag{94}$$

The divided power structure on the ideal $I_S \oplus \mathfrak{a} \subset W(S)$ induces an embedding of this ideal in $W(W(S))$. We will denote the images of I_S respectively $\mathfrak{a}$ by $\widehat{I}_S$ respectively $\widehat{\mathfrak{a}}$. The analogue of the decomposition (94) for the pd-thickening $W(S) \to R$ gives for the inverse image of Q:

$$\pi^{-1}(\widehat{Q}) = W(W(S)) \underset{\triangle, W(S)}{\otimes} \widetilde{L} \oplus I_{W(S)} \underset{\triangle, W(S)}{\otimes} \widetilde{T} \oplus \widehat{I}_S \underset{\triangle, W(S)}{\otimes} \widetilde{T} \oplus \widehat{\mathfrak{a}} \underset{\triangle, W(S)}{\otimes} \widetilde{T}. \tag{95}$$

By the definition of $\overline{\mathcal{T}}$ the operator V^{-1} must be defined on $\pi^{-1}(\widehat{Q})$ and it must be a lifting of V^{-1} on $\widetilde{P}$.

Let us assume for a moment that V^{-1} exists as required in the proposition. We claim that this implies that V^{-1} vanishes on the last two direct summands on (95). To see that V^{-1} vanishes on $\widehat{I}_S \otimes_{\triangle, W(S)} \widetilde{T}$, we remark that by lemma 52 any element of $\widehat{I}_S$ may be written in the form $\triangle({}^{V}\xi) -{}^{\widehat{V}} \triangle(\xi)$, for $\xi \in W(S)$. Hence it suffices to show that for $t \in \widetilde{T}$

$$V^{-1}(\triangle({}^{V}\xi) - {}^{\widehat{V}}\triangle(\xi) \otimes t) = 0.$$

But this follows from the equation (92).

Let $a \in \mathfrak{a} \subset W(S)$ be an element. The same element considered as element of $\widehat{\mathfrak{a}} \subset W(W(S))$ will be denoted by $\widehat{a}$. We have the following lemma, which we prove later.

Lemma 54. — *We have* $\triangle(a) = \widehat{a}$.

Hence $V^{-1}(\widehat{a} \otimes t) = V^{-1}(1 \otimes at) = 1 \otimes V^{-1}at = 0$, by the second equation of (92) for $y = at$. Now we see from the decomposition (95) that the operator V^{-1} from the triple $\overline{T}$ is uniquely determined by the requirements (92). Moreover we can check now that V^{-1} (if it exists) is a lift of $V^{-1} : \widehat{Q} \to \widetilde{P}$ relative to $W(\mathbf{w}_0) : W(W(S)) \to W(S)$. In fact our proof of the uniqueness shows that $\pi^{-1}(\widehat{Q})$ is generated by all elements of the form $\widehat{\xi} \otimes y$, for $\widehat{\xi} \in W(W(S))$ and $y \in \widehat{Q}$ and of the form ${}^{V}\widehat{\xi} \otimes x$, for $x \in \widetilde{P}$. Since $W(\mathbf{w}_0)$ commutes with ^{F} and ^{V}, we see from (92) that V^{-1} is indeed a lift. It remains to show the existence of a V^{-1} as asserted in the proposition.

To prove the existence of V^{-1}, we define an ^{F}-linear operator V^{-1} on $\pi^{-1}(\widehat{Q})$. On the first direct summand of (95) it will be defined by the second equation of (92), and on the second direct summand by the third equation of (92). On the last two direct summands of (95) we set V^{-1} equal to zero. We only have to check, that the last two equations of (92) hold with this definition. We will write down here only some parts of this routine calculation. Let us verify for example that the second equation of (92) holds for $y \in I_S\widetilde{T}$. We may assume that y is of the form $y = {}^{V}\eta t$, where $\eta \in W(S)$ and $t \in \widetilde{T}$. Then we have to decompose $\widehat{\xi} \otimes^{V} \eta t$ according to the decomposition (92):

$$\widehat{\xi} \otimes^{V} \eta t = \triangle({}^{V}\eta) \cdot \widehat{\xi} \otimes t = \left(\triangle\left({}^{V}\eta\right) - {}^{\widehat{V}}\triangle(\eta)\right) \cdot \widehat{\xi} \otimes t + {}^{\widehat{V}}\triangle(\eta)\widehat{\xi} \otimes t$$

Here the first summand is in the third direct summand of (95) and the second summand is in the second direct summand of the decomposition (95). The definition of V^{-1} therefore gives:

$$\begin{aligned} V^{-1}\left(\widehat{\widehat{\xi}} \otimes y\right) &= V^{-1}\left({}^{\widehat{V}}\triangle(\eta) \cdot \widehat{\xi} \otimes t\right) \\ &= V^{-1}\left({}^{\widehat{V}}\left(\triangle(\eta)^{\widehat{F}}\,\widehat{\xi}\right) \otimes t\right) = \triangle(\eta)^{\widehat{F}}\,\widehat{\xi} \otimes Ft \\ &= {}^{\widehat{F}}\widehat{\xi} \otimes \eta Ft = {}^{\widehat{F}}\widehat{\xi} \otimes V^{-1}\left({}^{V}\eta t\right) = {}^{\widehat{F}}\widehat{\xi} \otimes V^{-1}y \end{aligned}$$

Hence the second equation of (92) holds with the given definition of V^{-1} for $y \in I_S\widetilde{T}$. For $y \in \widetilde{L}$ this second equation is the definition of V^{-1} and for $y \in \mathfrak{a}\widetilde{T}$ the lemma 54 shows that both sides of the equation

$$V^{-1}\left({}^{\widehat{F}}\widehat{\xi} \otimes y\right) = \widehat{\xi} \otimes Fy$$

are zero. Because we leave the verification of the third equation (92) to the reader we may write modulo the lemma 54: □

Let us now prove the lemma 54. The ideal $W(\mathfrak{a}) \subset W(S)$ is a pd-ideal, since it is contained in the kernel $\mathfrak{a} \oplus I_S$ of (88). One sees that $W(\mathfrak{a})$ inherits a pd-structure from this ideal. One checks that in logarithmic coordinates on $W(\mathfrak{a})$ this pd-structure has the form:

$$\alpha_p[a_0, a_1, \dots] = [\alpha_p(a_0), p^{(p-1)}\alpha_p(a_1), \dots, p^{i(p-1)}\alpha_p(a_i), \dots]$$

where $\alpha_p(a_i)$ for $a_i \in \mathfrak{a}$ denotes the given pd-structure on $\mathfrak{a}$.

On $W(\mathfrak{a})$ the operator F^n becomes divisible by p^n. We define an operator $\frac{1}{p^n}F^n$ on $W(\mathfrak{a})$ as follows:

$$\begin{array}{rrcl} \frac{1}{p^n}F^n : & W(\mathfrak{a}) & \longrightarrow & W(\mathfrak{a}) \\ & [a_0, a_1, a_2 \dots] & \longmapsto & [a_n, a_{n+1}, a_{n+2}, \dots] \end{array}$$

Since $W(\mathfrak{a}) \subset W(S)$ is a pd-ideal, we have the divided Witt polynomials

$$\widehat{\mathbf{w}}'_n : W(W(\mathfrak{a})) \longrightarrow W(\mathfrak{a})$$

If $a \in \mathfrak{a} \subset W(\mathfrak{a})$ the element $\widehat{a} \in \widehat{\mathfrak{a}} \subset W(W(\mathfrak{a}))$ used in the lemma 54 is characterized by the following properties

$$\widehat{\mathbf{w}}'_0(\widehat{a}) = a, \quad \widehat{\mathbf{w}}'_n(\widehat{a}) = 0 \quad \text{for } n > 0.$$

Therefore the lemma 54 follows from the more general fact:

Lemma 55. — *Let S be a $\mathbb{Z}_p$-algebra and $\mathfrak{a} \subset S$ be a pd-ideal. Then the canonical homomorphism*

$$\triangle : W(\mathfrak{a}) \longrightarrow W(W(\mathfrak{a}))$$

satisfies

$$\widehat{\mathbf{w}}'_n (\triangle (\underline{a})) = \frac{1}{p^n}{}^{F^n}\underline{a}, \quad \textit{for } \underline{a} \in W(\mathfrak{a}), n \geq 0.$$

Proof. — One may assume that S is the pd-polynomial algebra in variables $a_0, a_1, \dots$ over $\mathbb{Z}_p$. Since this ring has no p-torsion the formula is clear from (90) □

Corollary 56. — *Under the assumptions of proposition 53 let $\varphi : W(R) \to S$ be a homomorphism of pd-thickenings. Then the triple $\mathcal{T} = (\widetilde{P}, F, V^{-1})$ may be described as follows: Let δ be the composite of the homomorphisms*

$$\delta : W(R) \xrightarrow{\triangle} W(W(R)) \xrightarrow{W(\varphi)} W(S) \tag{96}$$

This is a ring homomorphism, which commutes with ^{F}.

We define $\widetilde{P} = W(S) \otimes_{\delta, W(R)} P$. Then $\widetilde{P}$ is a lifting of P with respect to the morphism $S \to R$. For the operator F on $\widetilde{P}$ we take the ^{F}-linear extension of the operator F on P. Let $\widehat{Q} \subset \widetilde{P}$ be the inverse image of Q. Finally we define $V^{-1} : \widehat{Q} \to \widetilde{P}$ to be the unique ^{F}-linear homomorphism, which satisfies the following relations.

$$\begin{aligned} V^{-1}(w \otimes y) &= {}^F w \otimes V^{-1}y, && w \in W(S),\, y \in Q \\ V^{-1}({}^V w \otimes x) &= w \otimes Fx, && w \in W(S),\, x \in P \\ V^{-1}(a \otimes x) &= 0 && a \in \mathfrak{a} \subset W(S). \end{aligned} \tag{97}$$

In particular we obtain the following isomorphisms:

$$\mathcal{K}_{\mathcal{P}}(S) \cong W(S) \otimes_{W(R)} \mathcal{K}_{\mathcal{P}}(R)$$
$$\mathcal{D}_{\mathcal{P}}(S) \cong S \otimes_{W(R)} \mathcal{K}_{\mathcal{P}}(R).$$

Proof. — We apply proposition 53 to the trivial pd-thickening $R \to R$, to obtain the triple $\overline{\mathcal{T}}$. Then we make base change with respect to $\varphi : W(R) \to S$. □

We will now see that the isomorphism of proposition 51 (compare (93)) is compatible with Frobenius and Verschiebung.

Let R be a ring such that $p \cdot R = 0$. For a display $\mathcal{P}$ over R we have defined Frobenius and Verschiebung.

$$\mathcal{F}r_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P}^{(p)} \quad \mathrm{Ver}_{\mathcal{P}} : \mathcal{P}^{(p)} \longrightarrow \mathcal{P}$$

They induce morphisms of the corresponding Witt and Dieudonné crystals:

$$Fr_{\mathcal{D}_{\mathcal{P}}} : \mathcal{D}_{\mathcal{P}} \longrightarrow \mathcal{D}_{\mathcal{P}^{(p)}}, \quad Fr_{\mathcal{K}_{\mathcal{P}}} : \mathcal{K}_{\mathcal{P}} \longrightarrow \mathcal{K}_{\mathcal{P}^{(p)}} \tag{98}$$

$$\mathrm{Ver}_{\mathcal{D}_{\mathcal{P}}} : \mathcal{D}_{\mathcal{P}^{(p)}} \longrightarrow \mathcal{D}_{\mathcal{P}}, \quad \mathrm{Ver}_{\mathcal{K}_{\mathcal{P}}} : \mathcal{K}_{\mathcal{P}^{(p)}} \longrightarrow \mathcal{K}_{\mathcal{P}} \tag{99}$$

Let us make the morphisms more explicit. We set $\mathcal{P} = (P, Q, F, V^{-1})$. Let $S \to R$ be a pd-thickening, such that p is nilpotent in S. We denote by $\mathcal{T} = (\widetilde{P}, F, V^{-1})$ the unique $\mathcal{P}$-triple over S. The unique $\mathcal{P}^{(p)}$-triple over S is given as follows

$$\mathcal{T}^{(p)} = \left(W(S) \otimes_{F,W(S)} \widetilde{P}, F, V^{-1}\right),$$

where F and V^{-1} will now be defined:

$$F(\xi \otimes x) = {}^{F}\xi \otimes Fx, \quad \text{for } \xi \in W(S), x \in \widetilde{P}.$$

The domain of definition of V^{-1} is the kernel $\widehat{Q}^{(p)}$ of the canonical map

$$W(S) \otimes_{F,W(S)} \widetilde{P} \longrightarrow R \otimes_{\mathrm{Frob},R} P/Q,$$

which is induced by $W(S) \xrightarrow{w_0} S \longrightarrow R$. The operator V^{-1} on $\widehat{Q}^{(p)}$ is uniquely determined by the following formulas

$$\begin{aligned} V^{-1}(\xi \otimes y) &= {}^{F}\xi \otimes V^{-1}y, && \text{for } \xi \in W(S), y \in \widehat{Q} \\ V^{-1}\left({}^{V}\xi \otimes x\right) &= \xi \otimes Fx, && x \in \widetilde{P} \\ V^{-1}\left(\mathfrak{a} \otimes_{F,W(S)} \widetilde{P}\right) &= 0. \end{aligned} \tag{100}$$

Even though it makes the text long, we do not leave the verification of the existence of V^{-1} to the reader: We take a normal decomposition $\widetilde{P} = \widetilde{L} \oplus \widetilde{T}$. Then we obtain the decompositions

$$\widehat{Q} = \widetilde{L} \oplus I_S\widetilde{T} \oplus \mathfrak{a}\widetilde{T}$$
$$\widehat{Q}^{(p)} = W(S) \otimes_{F,W(S)} \widetilde{L} \oplus I_S \otimes_{F,W(S)} \widetilde{T} \oplus \mathfrak{a} \otimes_{F,W(S)} \widetilde{T}$$

We define the operator V^{-1} on $\widehat{Q}^{(p)}$ by taking the first formula of (100) as a formula on the first direct summand, the second formula on the second direct summand and so on. Then we have to verify that V^{-1} defined on this way satisfies (100). To verify the first formula (100) it is enough to check the cases $y \in \widetilde{L}, y \in I_S\widetilde{T}$ and $y \in \mathfrak{a}\widetilde{T}$ separately. For $y \in \widetilde{L}$ the assertion is the definition of V^{-1} and for $y \in \mathfrak{a}\widetilde{T}$ both sides of the equation become zero. Therefore we may assume $y = {}^V\eta x, \quad$ for $\eta \in W(S)$ and $x \in \widetilde{T}$. We have

$$\xi \otimes {}^V\eta x = p\xi\eta \otimes x.$$

Now in the ring $W(\mathbb{Z}_p) = W\left(W\left(\mathbb{F}_p\right)\right)$ we have the equation

$$p - [p, 0, 0 \cdots] = \triangle({}^V1) - [{}^V1, 0 \cdots 0] = {}^V\triangle 1 = {}^V1.$$

Since $\mathbb{Z}_p \to S$ is a pd-morphism the same equation holds in $W(S)$. We obtain

$$p\xi\eta \otimes x = \left([p, 0 \cdots 0] + {}^V1\right) \xi\eta \otimes x.$$

Since $[p, 0 \cdots 0]\xi\eta \otimes x \in \mathfrak{a} \otimes \widetilde{T}$ we obtain by the definition of V^{-1}

$$\begin{aligned} V^{-1}\left(p\xi\eta \otimes x\right) = V^{-1}\left({}^V1 \cdot \xi\eta \otimes x\right) = V^{-1}\left({}^{VF}(\xi\eta) \otimes x\right) = {}^F(\xi\eta) \otimes Fx \\ = {}^F\xi \otimes \eta Fx = {}^F\xi \otimes V^{-1}\left({}^V\eta x\right). \end{aligned}$$

This proves the assertion. The verification of the last two equations of (100) is done in the same way, but much easier.

Hence we have proved the existence of V^{-1}. It follows that $\mathcal{T}^{(p)}$ is a $\mathcal{P}^{(p)}$-triple.

To the triple $\mathcal{T} = (\widetilde{P}, F, V^{-1})$ there is by lemma 1.5 an associated $W(S)$-linear map

$$V^{\#} : \widetilde{P} \longrightarrow W(S) \otimes_{F,W(S)} \widetilde{P}, \tag{101}$$

which satisfies the relations

$$\begin{aligned} V^{\#}(wV^{-1}y) &= w \otimes y, \qquad \text{for } y \in \widehat{Q}, w \in W(S) \\ V^{\#}(wFx) &= p \cdot w \otimes x. \end{aligned}$$

Indeed, to conclude this from lemma 1.5 we complete $\mathcal{T}$ to a display $(\widetilde{P}, \widetilde{Q}, F, V^{-1})$ and note that $\widehat{Q} = \widetilde{Q} + \mathfrak{a}\widetilde{P}$.

Then we claim that (101) induces a map of triples:

$$Fr_{\mathcal{T}} : \mathcal{T} \longrightarrow \mathcal{T}^{(p)} \tag{102}$$

We have to verify that the morphism (101) commutes with F and V^{-1}. Let us do the verification for V^{-1}. The assertion is the commutativity of the following diagram:

$$\begin{array}{ccc} \widehat{Q} & \xrightarrow{V^{\#}} & \widehat{Q}^{(p)} \subset W(S) \otimes_{F,W(S)} \widetilde{P} \\ {\scriptstyle V^{-1}}\downarrow & & \downarrow{\scriptstyle V^{-1}} \\ \widetilde{P} & \xrightarrow{V^{\#}} & W(S) \otimes_{F,W(S)} \widetilde{P} \end{array}$$

We take any $y \in \widehat{Q}$ and we write it in the form

$$y = \sum_{i=1}^{m} \xi_i V^{-1} z_i,$$

for $\xi_i \in W(S)$ and $z_i \in \widehat{Q}$. Then we compute

$$V^{\#}(V^{-1}y) = 1 \otimes y$$
$$V^{-1}(V^{\#}y) = V^{-1}\Big(\sum_{i=1}^{m} \xi_i \otimes z_i\Big) = \sum_{i=1}^{m} {}^{F}\xi_i \otimes V^{-1}z_i = 1 \otimes y$$

We leave to the reader the verification that

$$F^{\#} : W(S) \otimes_{F,W(S)} \widetilde{P} \longrightarrow \widetilde{P}$$

induces a morphism of triples

$$\mathrm{Ver}_{\mathcal{T}} : \mathcal{T}^{(p)} \longrightarrow \mathcal{T}$$

Then $Fr_{\mathcal{T}}$ and $\mathrm{Ver}_{\mathcal{T}}$ are liftings of $Fr_{\mathcal{P}}$ and $\mathrm{Ver}_{\mathcal{P}}$ and may therefore be used to compute the Frobenius and the Verschiebung on the Witt crystal and the Dieudonné crystal:

Proposition 57. — *Let R be a ring, such that $p \cdot R = 0$. Let $\mathcal{P}$ be a display over R. We consider a $\mathcal{P}$-triple $\mathcal{T} = (\widetilde{P}, F, V^{-1})$ relative to a pd-thickening $S \to R$. Then the Frobenius morphism on the Witt crystal $Fr_{\mathcal{K}_{\mathcal{P}}}(S) : \mathcal{K}_{\mathcal{P}} \to \mathcal{K}_{\mathcal{P}^{(p)}}(S)$ is canonically identified with the map $V^{\#} : \widetilde{P} \to W(S) \otimes_{F,W(S)} \widetilde{P}$, and the Verschiebung morphism $\mathrm{Ver}_{\mathcal{K}_{\mathcal{P}}}(S) : \mathcal{K}_{\mathcal{P}^{(p)}}(S) \to \mathcal{K}_{\mathcal{P}}(S)$ is canonically identified with $F^{\#} : W(S) \otimes_{F,W(S)} \widetilde{P} \to \widetilde{P}$. The Frobenius and Verschiebung on the Dieudonné crystal are obtained by taking the tensor product with $S \otimes_{\mathbf{w}_0, W(S)}$.*

This being said we formulate a complement to the proposition 53.

Corollary 58. — *Let us assume that $p \cdot R = 0$. Then for any pd-extension $S \to R$ the isomorphism of the proposition 53:*

$$\mathcal{K}_{\mathcal{P}}(S) \xrightarrow{\sim} \mathcal{D}_{\mathcal{P}}(W(S))$$

is compatible with the Frobenius and the Verschiebung on these crystals.

Proof. — We will check this for the Frobenius. The commutativity of the following diagram is claimed:

$$\begin{tikzcd}
\mathcal{K}_{\mathcal{P}}(S) \arrow[r] \arrow[d, "Fr_{\mathcal{K}_{\mathcal{P}}}"'] & \mathcal{D}_{\mathcal{P}}(W(S)) \arrow[d, "Fr_{\mathcal{D}_{\mathcal{P}}}"] \\
\mathcal{K}_{\mathcal{P}^{(p)}}(S) \arrow[r] & \mathcal{D}_{\mathcal{P}^{(p)}}(W(S)).
\end{tikzcd}$$

Now we take a $\mathcal{P}$-triple $(\widetilde{P}, F, V^{-1})$ over S. Taking the proposition 57 into account, we may rewrite the last diagram as follows:

$$\begin{tikzcd}
\widetilde{P} \arrow[dd, "V^{\#}"'] & & W(S) \underset{\mathbf{w}_0}{\otimes} (W(W(S)) \underset{\triangle, W(S)}{\otimes} \widetilde{P}) \arrow[ll] \arrow[d, "1 \otimes V^{\#}"] \\
& & W(S) \underset{\mathbf{w}_0}{\otimes} (W(W(S)) \underset{\widehat{F}, W(W(S))}{\otimes} W(W(S)) \underset{\triangle, W(S)}{\otimes} \widetilde{P}) \arrow[d, equal] \\
W(S) \underset{F, W(S)}{\otimes} \widetilde{P} & & W(S) \underset{\mathbf{w}_0}{\otimes} (W(W(S)) \underset{\triangle, W(S)}{\otimes} W(S) \underset{F, W(S)}{\otimes} \widetilde{P}) \arrow[ll]
\end{tikzcd}$$

It is enough to check the commutativity of this diagram on elements of the form $1 \otimes V^{-1}(\widehat{\xi} \otimes y), \widehat{\xi} \in W(W(S)) y \in \widehat{Q}$ and $V^{-1}({}^{\widehat{V}}\widehat{\xi} \otimes x), x \in \widetilde{P}$. This is easy. $\square$

We will now study the functor which associates to a display its Dieudonné crystal over a base R of characteristic p. In this case the Dieudonné crystal is equipped with the structure of a filtered F-crystal. We will prove that the resulting functor from displays to filtered F-crystals is almost fully faithful.

Let R be a ring, such that $p \cdot R = 0$, and let $\mathcal{P}$ be a display over R. The inverse image of the Witt crystal $\mathcal{K}_{\mathcal{P}}$ by the Frobenius morphism $\mathrm{Frob} : R \to R$ may be identified with $\mathcal{K}_{\mathcal{P}^{(p)}}$. To see this we look at the commutative diagram:

$$\begin{tikzcd}
W(S) \arrow[r, "F"] \arrow[d] & W(S) \arrow[d] \\
R \arrow[r, "\mathrm{Frob}"] & R
\end{tikzcd}$$

The vertical map is a pd-thickening by (88) and ^{F} is compatible with the pd-structure. This diagram tells us ([BO] Exercise 6.5), that

$$\mathrm{Frob}^* \mathcal{K}_{\mathcal{P}}(W(S)) = W(W(S)) \otimes_{W(F), W(W(S))} \mathcal{K}_{\mathcal{P}}(W(S)).$$

The pd-morphism $w_0 : W(S) \to S$ gives an isomorphism

$$W(S) \otimes_{W(w_0), W(W(S))} \mathrm{Frob}^* \mathcal{K}_{\mathcal{P}}(W(S)) = \mathrm{Frob}^* \mathcal{K}_{\mathcal{P}}(S)$$

Combining the last two equations we get as desired identification:

$$\mathrm{Frob}^* \mathcal{K}_{\mathcal{P}}(S) = W(S) \otimes_{F, W(S)} \mathcal{K}_{\mathcal{P}}(S) = \mathcal{K}_{\mathcal{P}^{(p)}}(S). \tag{103}$$

From this we also deduce:

$$\mathrm{Frob}^* \mathcal{D}_{\mathcal{P}}(S) = \mathcal{D}_{\mathcal{P}^{(p)}}(S)$$

Remark. — This computation of $\mathrm{Frob}^* \mathcal{D}_{\mathcal{P}}$ may be carried out inside the nilpotent crystalline site, if $p \neq 2$. The point is that we need that $W(S) \to R$ is a topological nilpotent pd-thickening, if S is a nilpotent pd-thickening. The result is the same.

Definition 59. — Let X be a scheme, such that $p \cdot \mathcal{O}_X = 0$. Let us denote by $\mathrm{Frob} : X \to X$ the absolute Frobenius morphism. A filtered F-crystal on X is a triple $(\mathcal{D}, G, Fr)$, where $\mathcal{D}$ is a crystal in $\mathcal{O}_X^{\mathrm{crys}}$-modules $G \subset \mathcal{D}_X$ is an $\mathcal{O}_X$-submodule of the $\mathcal{O}_X$-module $\mathcal{D}_X$ associated to $\mathcal{D}$, such that G is locally a direct summand. Fr is a morphism of crystals

$$Fr : \mathcal{D} \longrightarrow \mathrm{Frob}^* \mathcal{D} = \mathcal{D}^{(p)}.$$

We also define a filtered F-Witt crystal as a triple $(\mathcal{K}, Q, Fr)$, where $\mathcal{K}$ is a crystal in $W(\mathcal{O}_X^{\mathrm{crys}})$-modules, $Q \subset \mathcal{K}_X$ is a $W(\mathcal{O}_X)$-submodule, such that $I_X\mathcal{K}_X \subset Q$ and $Q/I_X\mathcal{K}_X \subset \mathcal{O}_X \otimes_{\mathbf{w}_0, W(\mathcal{O}_X)} \mathcal{K}_X$ is locally a direct summand as $\mathcal{O}_X$-module. Fr is a morphism of $W(\mathcal{O}_X^{\mathrm{crys}})$-crystals

$$Fr : \mathcal{K} \longrightarrow \mathcal{K}^{(p)} = \mathrm{Frob}^* \mathcal{K}.$$

With the same definition we may also consider filtered F-crystals (resp. F-Witt crystals), if $p \neq 2$.

The same argument which leads to (103) shows that for any pd-thickening $T \leftarrow U \hookrightarrow X$ there is a a canonical isomorphism:

$$\mathcal{K}^{(p)}(T) = W(\mathcal{O}_T) \otimes_{F, W(\mathcal{O}_T)} \mathcal{K}(T)$$

From a filtered F-Witt crystal we get a filtered F-crystal by taking the tensor product $\mathcal{O}_X^{\mathrm{crys}} \otimes_{W(\mathcal{O}_X^{\mathrm{crys}})}$. Let R be a ring such that $p \cdot R = 0$ and $\mathcal{P} = (P, Q, F, V^{-1})$ be a display over R as above. Then we give the Witt crystal $\mathcal{K}_{\mathcal{P}}$ the structure of a filtered F-Witt crystal, by taking the obvious Q, and by defining $Fr : \mathcal{K}_{\mathcal{P}} \to {\mathcal{K}_{\mathcal{P}}}^{(p)}$ as the map (98). By taking the tensor product $\mathcal{O}_X^{\mathrm{crys}} \otimes_{\mathbf{w}_0, W(\mathcal{O}_X^{\mathrm{crys}})}$ we also equip the Dieudonné crystal $\mathcal{D}_{\mathcal{P}}$ with the structure of a filtered F crystal.

We will say that a pd-thickening (resp. nilpotent pd-thickening) $S \to R$ is liftable, if there is a morphism of topological pd-thickenings (resp. topological nilpotent pd-thickenings) $S' \to S$ of the ring R, such that S' is a torsionfree p-adic ring. We prove that the functors $\mathcal{K}$ and $\mathcal{D}$ are "fully faithful" in the following weak sense:

Proposition 60. — *Let R be a $\mathbb{F}_p$-algebra. Assume that there exists a topological pd-thickening $S \to R$, such that S is a torsionfree p-adic ring.*

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be displays over R. We denote the filtered F-crystal associated to $\mathcal{P}_i$ by $(\mathcal{D}_i, G_i, Fr_i)$ for $i = 1, 2$ and by $(\mathcal{K}_i, Q_i, Fr_i)$ the filtered F-Witt crystal.

Let $\alpha : (\mathcal{D}_1, G_1, Fr_1) \to (\mathcal{D}_2, G_2, Fr_2)$ be a morphism of filtered F-crystals. Then there is a morphism $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ of displays, such that the morphism of filtered F-crystals $\mathcal{D}(\varphi) : (\mathcal{D}_1, G_1, Fr_1) \to (\mathcal{D}_2, G_2, Fr_2)$, which is associated to φ has the following property:

For any liftable pd-thickening $S' \to R$, we have

$$\alpha_{S'} = \mathcal{D}(\varphi)_{S'}. \tag{104}$$

The similar statement for the filtered F-Witt crystals is also true.

Remark. — The result will later be used to show that the functor BT of the introduction is fully faithful under the assumptions of the proposition. In fact we will use the following variant of the proposition: Assume that $p \neq 2$ and that we are given a topological nilpotent pd-thickening, such that S is a torsionfree p-adic ring. Then it is enough to have a morphism α on the nilpotent crystalline site to conclude the existence of φ, such that for any liftable nilpotent pd-thickening $S' \to R$ the equality (104) holds.

Proof. — First we prove the result for the filtered F-Witt crystals. Let $(\widetilde{P}_i, F, V^{-1})$ be the $\mathcal{P}_i$-triple over S for $i = 1, 2$. We may identify $\mathcal{K}_i(S)$ with $\widetilde{P}_i$ and $Fr_i(S)$ with the morphism $V^{\#} : \widetilde{P}_i \to W(S) \otimes_{F,W(S)} \widetilde{P}_i$. Then we may regard α_S as a homomorphism of $W(S)$-modules

$$\alpha_S : \widetilde{P}_1 \longrightarrow \widetilde{P}_2,$$

which commutes with $V^{\#}$:

$$V^{\#}\alpha_S = (1 \otimes \alpha_S)V^{\#}. \tag{105}$$

Since α_R respects the filtrations Q_1 and Q_2, we get

$$\alpha_S(\widehat{Q}_1) \subset \widehat{Q}_2.$$

Because the ring S is torsionfree we conclude from the equations $F^{\#} \cdot V^{\#} = p$ and $V^{\#} \cdot F^{\#} = p$, which hold for any display, that the maps $F^{\#} : W(S) \otimes_{F,W(S)} \widetilde{P}_i \to \widetilde{P}_i$ and $V^{\#} : \widetilde{P}_i \to W(S) \otimes_{F,W(S)} \widetilde{P}_i$ are injective. Hence the equation

$$F^{\#}(1 \otimes \alpha_S) = \alpha_S F^{\#} \tag{106}$$

is verified by multiplying it from the left by $V^{\#}$ and using (105). We conclude that α_S commutes with F. Finally α_S also commutes with V^{-1} because we have $pV^{-1} = F$ on $\widehat{Q}$.

We see from the following commutative diagram

$$\begin{array}{ccc} \widetilde{P}_1 & \xrightarrow{\alpha_S} & \widetilde{P}_2 \\ \downarrow & & \downarrow \\ P_1 & \xrightarrow{\alpha_R} & P_2 \end{array}$$

that α_R induces a homomorphism of displays and that α_S is the unique lifting of α_R to a morphism of triples. This proves the proposition in the case of filtered F-Witt crystals. Finally a morphism $\beta : \mathcal{D}_1 \to \mathcal{D}_2$ of the filtered F-crystals also provides a morphism $\alpha : \mathcal{K}_1 \to \mathcal{K}_2$ of the Witt crystals by the proposition (53), which commutes with Fr by the corollary (58). It is clear that α also respects the filtrations. Hence the assertion of the theorem concerning filtered F-crystals is reduced to the case of filtered F-Witt crystals. □

2.4. Isodisplays. — Let R be a ring and let $\mathfrak{a} \subset R$ be an ideal, such that p is nilpotent in $R/\mathfrak{a}$. We assume that R is complete and separated in the $\mathfrak{a}$-adic topology. In this section we will consider displays over the topological ring R with its $\mathfrak{a}$-adic topology (see definition 13).

We consider the ring $W_{\mathbb{Q}}(R) = W(R) \otimes_{\mathbb{Z}} \mathbb{Q}$. The Frobenius homomorphism ^{F} and the Verschiebung ^{V} extend from $W(R)$ to $W_{\mathbb{Q}}(R)$.

Definition 61. — An isodisplay over R is a pair $(\mathcal{I}, F)$, where $\mathcal{I}$ is a finitely generated projective $W_{\mathbb{Q}}(R)$-module and

$$F : \mathcal{I} \longrightarrow \mathcal{I}$$

is an ^{F}-linear isomorphism.

Let us assume for a moment that R is torsionfree (as an abelian group). Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_R & \longrightarrow & W(R) & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_R \otimes \mathbb{Q} & \longrightarrow & W_{\mathbb{Q}}(R) & \longrightarrow & R \otimes \mathbb{Q} & \longrightarrow & 0, \end{array}$$

where the vertical maps are injective. In particular $W(R) \cap I_R \otimes \mathbb{Q} = I_R$.

Definition 62. — Let R be torsionfree. A filtered isodisplay over R is a triple $(\mathcal{I}, E, F)$, where $(\mathcal{I}, F)$ is an isodisplay over R and $E \subset \mathcal{I}$ is a $W_{\mathbb{Q}}(R)$ submodule, such that

(i) $I_R\mathcal{I} \subset E \subset \mathcal{I}$
(ii) $E/I_R\mathcal{I} \subset \mathcal{I}/I_R\mathcal{I}$ is a direct summand as $R \otimes \mathbb{Q}$-module.

Example 63. — Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a 3n-display over R. Obviously F extends to an ^{F}-linear homomorphism $F : P \otimes \mathbb{Q} \to P \otimes \mathbb{Q}$.

The pair $(P \otimes \mathbb{Q}, F)$ is an isodisplay. Indeed, to see that F is an ^{F}-linear isomorphism we choose a normal decomposition $P = L \oplus T$. We present $F : P \to P$ as a composite of two morphisms

$$L \oplus T \xrightarrow{p\,\mathrm{id}_L \oplus \mathrm{id}_T} L \oplus T \xrightarrow{V^{-1} \oplus F} L \oplus T.$$

The last morphism is already an ^{F}-linear isomorphism and the first morphism becomes an ^{F}-linear isomorphism, if we tensor by $\mathbb{Q}$.

Example 64. — If R is torsionfree, we get a filtered isodisplay $(P \otimes \mathbb{Q}, Q \otimes \mathbb{Q}, F)$.

Example 65. — Let $\mathfrak{a} \subset R$ be an ideal, such that R is complete and separated in the $\mathfrak{a}$-adic topology. We assume that $pR \subset \mathfrak{a} \subset R$.

Let k be a perfect field, such that $k \subset R/\mathfrak{a}$. Then we find by the universality of Witt vectors a commutative diagram

$$\begin{array}{ccc} W(k) & \xrightarrow{\rho} & R \\ \big\downarrow & & \big\downarrow \\ k & \longrightarrow & R/\mathfrak{a} \end{array} \tag{107}$$

The map $\delta : W(k) \xrightarrow{\triangle} W(W(k)) \xrightarrow{W(\rho)} W(R)$ commutes with ^{F}. Hence if we are given an isodisplay (N, F) over k, we obtain an isodisplay $(\mathcal{I}, F)$ over R if we set

$$\mathcal{I} = W_{\mathbb{Q}}(R) \otimes_{\delta, W_{\mathbb{Q}}(k)} N, \quad F(\xi \otimes x) = {}^F\xi \otimes Fx.$$

We will write $(\mathcal{I}, F) = W_{\mathbb{Q}}(R) \otimes_{\delta, W_{\mathbb{Q}}(k)} (N, F)$.

Let $Qisg_R$ be the category of displays over R up to isogeny. The objects of this category are the displays over R and the homomorphisms are $\mathrm{Hom}_{Qisg}(\mathcal{P}, \mathcal{P}') = \mathrm{Hom}(\mathcal{P}, \mathcal{P}') \otimes \mathbb{Q}$. We note that the natural functor $(\text{Displays})_R \to Qisg_R$ is by corollary 41 faithful if the nilradical of R/pR is nilpotent. It is clear that the construction of example 63 provides a functor:

$$Qisg_R \longrightarrow (\text{Isodisplays})_R \tag{108}$$

Proposition 66. — *If p is nilpotent in R, the functor (108) is fully faithful*

Proof. — The faithfulness means that for any morphism of displays $\alpha : \mathcal{P} \to \mathcal{P}'$, such that the induced map $\alpha_{\mathbb{Q}} : P_{\mathbb{Q}} \to P'_{\mathbb{Q}}$ is zero, there is a number N, such that $p^N\alpha = 0$. This is obvious. To prove that the functor is full, we start with a homomorphism of isodisplays $\alpha_0 : (P_{\mathbb{Q}}, F) \to (P'_{\mathbb{Q}}, F)$. Let $\mathrm{Im}\, P'$ be the image of the map $P' \to P'_{\mathbb{Q}}$. Since we are allowed to multiply α_0 with a power of p, we may assume that α_0 maps $\mathrm{Im}\, P$ to $\mathrm{Im}\, P'$. Since P is projective we find a commutative diagram:

$$\begin{array}{ccc} P_{\mathbb{Q}} & \xrightarrow{\alpha_0} & P'_{\mathbb{Q}} \\ \big\uparrow & & \big\uparrow \\ P & \xrightarrow{\alpha} & P' \end{array} \tag{109}$$

Since $F\alpha - \alpha F$ is by assumption in the kernel of $P' \to P'_{\mathbb{Q}}$, we find a number N, such that $p^N(F\alpha - \alpha F) = 0$. Multiplying α and α_0 by p^N, we may assume without loss of generality that α commutes with F. Moreover, since p is nilpotent in P'/I_RP' we may assume that $\alpha(P) \subset I_RP'$ and hence a fortiori that $\alpha(Q) \subset Q'$. Finally since $pV^{-1} = F$ on Q it follows that $p\alpha$ commutes with V^{-1}. Therefore we have obtained a morphism of displays. $\square$

Let us now consider the case of a torsionfree ring R. Then we have an obvious functor

$$Qisg_R \longrightarrow (\text{filtered Isodisplays})_R. \tag{110}$$

Proposition 67. — *Let R be torsionfree. Then the functor (110) is fully faithful.*

Proof. — Again it is obvious that this functor is faithful. We prove that the functor is full.

Let $\mathcal{P}$ and $\mathcal{P}'$ be displays over R. Assume that we are given a morphism of the corresponding filtered isodisplays

$$\alpha_0 : (P_{\mathbb{Q}}, Q_{\mathbb{Q}}, F) \longrightarrow (P'_{\mathbb{Q}}, Q'_{\mathbb{Q}}, F).$$

We have to show that α_0, if we replace it possibly by $p^N\alpha_0$, is induced by a homomorphism

$$\alpha : (P, Q, F, V^{-1}) \longrightarrow (P', Q', F, V^{-1}).$$

The proof of proposition 66 works except for the point where the inclusion $\alpha(Q) \subset Q'$ is proved. But this time we already know that $\alpha(Q) \subset Q'_{\mathbb{Q}}$. We choose finitely many elements $x_1, \ldots, x_M \in Q$, whose images generate the R-module Q/I_RP. Since it suffices to show that $\alpha(x_i) \in Q'$, if we possibly multiply α by p^N we are done. □

Definition 68. — An isodisplay (resp. filtered isodisplay) is called effective, if it is in the image of the functor (108) (resp. (110)).

Proposition 69. — *Let R be torsionfree. Let $\mathfrak{a} \subset R$ be an ideal, such that there exists a number N, such that $\mathfrak{a}^N \subset pR$ and $p^N \in \mathfrak{a}$. Let $(\mathcal{I}_1, F)$ and $(\mathcal{I}_2, F)$ be effective isodisplays over R. Then any homomorphism $\overline{\alpha}_0 : (\mathcal{I}_1, F)_{R/\mathfrak{a}} \to (\mathcal{I}_2, F)_{R/\mathfrak{a}}$ lifts uniquely to a homomorphism $\alpha_0 : (\mathcal{I}_1, F) \to (\mathcal{I}_2, F)$.*

Proof. — We choose displays $\mathcal{P}_1$ and $\mathcal{P}_2$ over R together with isomorphisms of isodisplays $(P_{i,\mathbb{Q}}, F) \simeq (\mathcal{I}_i, F)$ for $i = 1, 2$. By the proposition 66 we may assume that $\overline{\alpha}_0$ is induced by a morphism of displays $\overline{\alpha} : \mathcal{P}_{1,R/\mathfrak{a}} \to \mathcal{P}_{2,R/\mathfrak{a}}$. Indeed, to prove the proposition it is allowed to multiply $\overline{\alpha}_0$ by a power of p.

Next we remark, that for the proof we may assume that $\mathfrak{a} = p \cdot R$. Indeed, let $S \to T$ be a surjection of rings with nilpotent kernel and such that p is nilpotent in S. Then the induced map $W_{\mathbb{Q}}(S) \to W_{\mathbb{Q}}(T)$ is an isomorphism and hence an isodisplay on S is the same as an isodisplay on T. Applying this remark to the diagram

$$R/\mathfrak{a}R \;\longrightarrow\; R/\mathfrak{a} + pR \;\longleftarrow\; R/pR,$$

we reduce our assertion to the case, where $\mathfrak{a} = pR$.

Since $pR \subset R$ is equipped canonically with divided powers the morphism of displays $\overline{\alpha} : \mathcal{P}_{1,R/pR} \to \mathcal{P}_{2,R/pR}$ lifts by theorem 46 uniquely to a morphism of triples $(P_1, F, V^{-1}) \to (P_2, F, V^{-1})$ which gives a morphism of isodisplays $\alpha_0 : (P_{1,\mathbb{Q}}, F) \to (P_{2,\mathbb{Q}}, F)$. This shows the existence of α_0.

To prove the uniqueness we start with any lifting $\alpha_0 : (\mathcal{I}_1, F) \to (\mathcal{I}_2, F)$ of $\overline{\alpha}_0$. Since it is enough to show the uniqueness assertion for $p^N\overline{\alpha}_0$ and some number N, we may assume that $\alpha_0(P_1) \subset P_2$. Since P_1 and P_2 are torsionfree as abelian groups it follows that α_0 commutes with F and with V^{-1}, which is defined on $\widehat{Q}_1 \subset P_1$ resp. $\widehat{Q}_2 \subset P_2$ taken with respect to $R \to R/pR$. Hence α_0 is a morphism of triples $(P_1, F, V^{-1}) \to (P_2, F, V^{-1})$, which is therefore uniquely determined by the morphism of displays $\overline{\alpha} : \mathcal{P}_{1,R/pR} \to \mathcal{P}_{2,R/pR}$. $\square$

We will now explain the period map. Let us fix an effective isodisplay (N, F) over a perfect field k. We consider the diagram (107) and make the additional assumption that $\mathfrak{a}^t \subset pR$ for some number t. We consider the category $\mathcal{M}(R)$ of pairs $(\mathcal{P}, r)$, where $\mathcal{P} \in Qisg_R$ and r is an isomorphism $r : \mathcal{P}_{R/\mathfrak{a},\mathbb{Q}} \to (N, F)_{R/\mathfrak{a}}$ in the category of isodisplays over $R/\mathfrak{a}$. By the proposition 69 any homomorphism between pairs $(\mathcal{P}, r)$ is an isomorphism and there is at most one isomorphism between two pairs.

The period map will be injection from the set of isomorphism classes of pairs $(\mathcal{P}, r)$ to the set $\mathrm{Grass}_{W_{\mathbb{Q}}(k)}N(R\otimes\mathbb{Q})$, where $\mathrm{Grass}_{W_{\mathbb{Q}}(k)}N$ is the Grassmann variety of direct summands of the $W_{\mathbb{Q}}(k)$-module N.

The definition is as follows. The lemma below will show that the isodisplay $W_{\mathbb{Q}}(R) \otimes_{\delta, W_{\mathbb{Q}}(k)} (N, F)$ is effective. Hence by the proposition 69 there is a unique isomorphism of isodisplays, which lifts r

$$\widetilde{r} : (P_{\mathbb{Q}}, F) \longrightarrow W_{\mathbb{Q}}(R) \otimes_{\delta, W_{\mathbb{Q}}(k)} (N, F).$$

The map

$$W_{\mathbb{Q}}(R) \otimes_{\delta, W_{\mathbb{Q}}(k)} N \xrightarrow{\widetilde{r}^{-1}} P_{\mathbb{Q}} \longrightarrow P_{\mathbb{Q}}/Q_{\mathbb{Q}}$$

factors through the map induced by $\mathbf{w}_0$

$$W_{\mathbb{Q}}(R) \otimes_{\delta, W_{\mathbb{Q}}(k)} N \longrightarrow R_{\mathbb{Q}} \otimes_{\delta, W_{\mathbb{Q}}(k)} N.$$

Hence we obtain the desired period:

$$R_{\mathbb{Q}} \otimes_{\delta, W_{\mathbb{Q}}(k)} N \twoheadrightarrow P_{\mathbb{Q}}/Q_{\mathbb{Q}} \tag{111}$$

Hence if $\mathrm{Iso}\,\mathcal{M}(R)$ denotes the set of isomorphism classes in $\mathcal{M}(R)$ we have defined a map

$$\mathrm{Iso}\ \mathcal{M}(R) \longrightarrow \mathrm{Grass}_{W_{\mathbb{Q}}(k)}N(R_{\mathbb{Q}}).$$

This map is injective by the proposition 67.

Now we prove the missing lemma.

Lemma 70. — *Let (N, F) be an effective isodisplay over a perfect field k (i.e. the slopes are in the interval $[0, 1]$). Then in the situation of the diagram (107) the isodisplay $W_{\mathbb{Q}}(R) \otimes_{\delta, W_{\mathbb{Q}}(k)} (N, F)$ is effective.*

Proof. — One can restrict to the case $R = W(k)$ and $\rho = \mathrm{id}$. Indeed, if we know in the general situation that $W_{\mathbb{Q}}(W(k)) \otimes_{\triangle, W_{\mathbb{Q}}(k)} (N, F)$ is the isocrystal of a display $\mathcal{P}_0$, then $\rho_* \mathcal{P}_0$ is a display with isodisplay $W_{\mathbb{Q}}(R) \otimes_{\delta, W_{\mathbb{Q}}(k)} (N, F)$. In the situation $\rho = \mathrm{id}$ let$(M, \overline{Q}, F, V^{-1})$ be a display with the isodisplay (N, F). Then the associated triple with respect to the pd-thickening $W(k) \to k$ is the form $(W(W(k)) \otimes_{\triangle, W(k)} M, F, V^{-1})$, where V^{-1} is given by (92). This triple gives the desired display if we take some lift of the Hodge-filtration of M/pM to M. The isodisplay of this display is $(W_{\mathbb{Q}}(W(k)) \otimes_{\triangle, W_{\mathbb{Q}}(k)} N, F)$. $\square$

Finally we want to give an explicit formula for the map (111). The map $\widetilde{r}^{-1}$ is uniquely determined by the map:

$$\rho : N \longrightarrow P_{\mathbb{Q}}, \tag{112}$$

which is given by $\rho(m) = \widetilde{r}^{-1}(1 \otimes m)$, for $m \in N$. This map ρ may be characterized by the following properties:

(i) ρ is equivariant with respect to the ring homomorphism $\delta : W_{\mathbb{Q}}(k) \to W_{\mathbb{Q}}(R)$.
(ii) $\rho(Fm) = F\rho(m)$, for $m \in N$
(iii) The following diagram is commutative:

$$\begin{tikzcd} P_{\mathbb{Q}} \arrow[r] & W_{\mathbb{Q}}(R/\mathfrak{a}) \otimes_{W_{\mathbb{Q}}(k)} N \\ & N \arrow[u] \arrow[ul, "\rho"] \end{tikzcd} \tag{113}$$

We equip $P_{\mathbb{Q}}$ with the p-adic topology, i.e. with the linear topology, which has as a fundamental system of neighbourhoods of zero the subgroups $p^i P$. Because $W(R)$ is a p-adic ring, P is complete for this linear topology.

Proposition 71. — *Let $\rho_0 : N \to P$ be any δ-equivariant homomorphism, which makes the diagram (113) commutative. Then the map ρ is given by the following p-adic limit:*

$$\rho = \lim_{i \to \infty} F^i \rho_0 F^{-i}.$$

Proof. — We use ρ to identify $P_{\mathbb{Q}}$ with $W_{\mathbb{Q}}(R) \otimes_{\delta, W_{\mathbb{Q}}(k)} N$, i.e. the map ρ becomes $m \mapsto 1 \otimes m$, for $m \in N$. We write $\rho_0 = \rho + \alpha$. Clearly it is enough to show that:

$$\lim_{i \to \infty} F^i \alpha F^{-i}(m) = 0, \quad \text{for } m \in N. \tag{114}$$

Since ρ and ρ_0 make the diagram (113) commutative, we have $\alpha(N) \subset W_{\mathbb{Q}}(\mathfrak{a}) \otimes_{\delta, W_{\mathbb{Q}}(k)} N$. We note that $W_{\mathbb{Q}}(\mathfrak{a}) = W_{\mathbb{Q}}(pR)$.

We choose a $W(k)$-lattice $M \subset N$, which has a $W(k)$-module decomposition $M = \oplus M_j$, and such that there exists nonnegative integers $s, r_j \in \mathbb{Z}$ with $F^s M_j = p^{r_j} M_j$. We take an integer a, such that

$$\alpha(M) \subset p^a W(pR) \otimes_{\delta, W(k)} M.$$

It suffices to prove (114) for elements $m \in M_j$. We compute for any number u:

$$F^{us}\alpha(F^{-us}m) \in p^{-ur_j}F^{us}\alpha(M_j) \subset p^{a-ur_j}F^{us}(W(pR)\otimes_{\delta,W(k)} M). \tag{115}$$

But using the logarithmic coordinates for the pd-ideal pR we find:

$$^{F}W(pR) = W(p^2R) = pW(pR).$$

This shows that the right hand side of (115) is included in

$$p^{a-ur_j+us}W(pR)\otimes_{\delta,W(k)} M.$$

Since N is an effective isodisplay we conclude $s > r_j$ for each j. This proves that $F^{us}\alpha F^{-us}(m)$ converges to zero if u goes to ∞.

More generally we can consider the limit (114), where i runs through a sequence $i = us + q$ for some fixed number q. By the same argument we obtain that this limit is zero too. □

2.5. Lifting homomorphisms. — Consider a pd-thickening $S \to R$ with kernel $\mathfrak{a}$. We assume that p is nilpotent in S.

We consider two displays $\mathcal{P}_i = (P_i, Q_i, F, V^{-1})$ for $i = 1, 2$ over S. The base change to R will be denoted by $\overline{\mathcal{P}}_i = \overline{\mathcal{P}}_{i,R} = (\overline{P}_i, \overline{Q}_i, F, V^{-1})$. Let $\overline{\varphi} : \overline{\mathcal{P}}_1 \to \overline{\mathcal{P}}_2$ be a morphism of displays. It lifts to a morphism of triples:

$$\varphi : (P_1, F, V^{-1}) \longrightarrow (P_2, F, V^{-1}) \tag{116}$$

We consider the induced homomorphism:

$$\operatorname{Obst}\overline{\varphi} : Q_1/I_SP_1 \hookrightarrow P_1/I_SP_1 \xrightarrow{\varphi} P_2/I_SP_2 \longrightarrow P_2/Q_2$$

This map is zero modulo $\mathfrak{a}$, because $\overline{\varphi}(\overline{Q}_1) \subset \overline{Q}_2$. Hence we obtain a map:

$$\operatorname{Obst}\overline{\varphi} : Q_1/I_SP_1 \longrightarrow \mathfrak{a}\otimes_S P_2/Q_2 \tag{117}$$

Clearly this map is zero, iff $\overline{\varphi}$ lifts to a morphism of displays $\mathcal{P}_1 \to \mathcal{P}_2$

Definition 72. — The map $\operatorname{Obst}\overline{\varphi}$ above (117) is called the obstruction to lift $\overline{\varphi}$ to S.

This depends on the divided powers on $\mathfrak{a}$ by the definition of φ.

The obstruction has the following functorial property: Assume we are given a morphism $\alpha : \mathcal{P}_2 \to \mathcal{P}_3$ of displays over S. Let $\overline{\alpha} : \overline{\mathcal{P}}_2 \to \overline{\mathcal{P}}_3$ be its reduction over R. Then $\operatorname{Obst}\overline{\alpha}\overline{\phi}$ is the composite of the following maps:

$$Q_1/I_SP_1 \xrightarrow{\operatorname{Obst}\overline{\varphi}} \mathfrak{a}\otimes_S P_2/Q_2 \xrightarrow{1\otimes\alpha} \mathfrak{a}\otimes_S P_3/Q_3$$

We will denote this fact by:

$$\operatorname{Obst}\overline{\alpha\varphi} = \alpha\operatorname{Obst}\overline{\varphi} \tag{118}$$

In the case $\mathfrak{a}^2 = 0$ we have an isomorphism $\mathfrak{a} \otimes_S P_2/Q_2 \cong \mathfrak{a} \otimes_R \overline{P}_2/\overline{Q}_2$. Hence the obstruction may be considered as a map:

$$\text{(119)} \qquad \operatorname{Obst} \overline{\varphi} : \overline{Q}_1/I_R\overline{P}_1 \longrightarrow \mathfrak{a} \otimes_R \overline{P}_2/\overline{Q}_2$$

In this case the equation (118) simplifies:

$$\text{(120)} \qquad \operatorname{Obst} \overline{\alpha\varphi} = \overline{\alpha} \operatorname{Obst} \overline{\varphi}$$

Let S be a ring, such that $p \cdot S = 0$ for our fixed prime number p. Let $S \to R$ be a surjective ring homomorphism with kernel $\mathfrak{a}$. We assume that $\mathfrak{a}^p = 0$. In this section we will use the trivial divided powers on $\mathfrak{a}$, i.e. $\alpha_p(a) = 0$ for $a \in \mathfrak{a}$.

Let us consider a third ring $\widetilde{S}$, such that $p \cdot \widetilde{S} = 0$. Let $\widetilde{S} \to S$ be a surjection with kernel $\mathfrak{b}$, such that $\mathfrak{b}^p = 0$. Again we equip $\mathfrak{b}$ with the trivial divided powers.

Assume we are given liftings $\widetilde{\mathcal{P}}_i$ over $\widetilde{S}$ of the displays $\mathcal{P}_i$ over S for $i = 1, 2$. The morphism $p\overline{\varphi} : \overline{\mathcal{P}}_1 \to \overline{\mathcal{P}}_2$ lifts to the morphism $p\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ of displays. Hence we obtain an obstruction to lift $p\varphi$ to a homomorphism of displays $\widetilde{\mathcal{P}}_1 \to \widetilde{\mathcal{P}}_2$:

$$\operatorname{Obst}(p\varphi) : \widetilde{Q}_1/I_S\widetilde{P}_1 \longrightarrow \widetilde{P}_2/\widetilde{Q}_2.$$

We will compute this obstruction in terms of $\operatorname{Obst} \overline{\varphi}$. For this we need to define two further maps: The operator V^{-1} on $\widetilde{P}_1$ induces a surjection

$$\text{(121)} \qquad (V^{-1})^{\#} : \widetilde{S} \otimes_{\mathrm{Frob}, \widetilde{S}} \widetilde{Q}_1/I_{\widetilde{S}}\widetilde{P}_1 \longrightarrow \widetilde{P}_1/I_{\widetilde{S}}\widetilde{P}_1 + W(\widetilde{S})F\widetilde{P}_1.$$

Here we denote by Frob the Frobenius endomorphism of $\widetilde{S}$. The map (121) is an isomorphism. To see this it is enough to verify that we have on the right hand side a projective $\widetilde{S}$-module of the same rank as on the left hand side. Let $\widetilde{P} = \widetilde{L} \oplus \widetilde{T}$ be a normal decomposition. Because $p\widetilde{S} = 0$, we have $W(\widetilde{S})F\widetilde{L} \subset pW(\widetilde{S})\widetilde{P} \subset I_{\widetilde{S}}\widetilde{P}$. Since we have a decomposition $\widetilde{P} = W(\widetilde{S})V^{-1}\widetilde{L} \oplus W(\widetilde{S})F\widetilde{T}$, one sees that the right hand side of (121) is isomorphic to $W(\widetilde{S})V^{-1}\widetilde{L}/I_{\widetilde{S}}V^{-1}\widetilde{L}$. This is indeed a projective $\widetilde{S}$-module of the right rank.

The ideal $\mathfrak{b}$ is in the kernel of Frob. Therefore the left hand side of (121) may be written as $\widetilde{S} \otimes_{\mathrm{Frob}, S} Q_1/I_S P_1$. We consider the inverse of the map (121)

$$V^{\#} : \widetilde{P}_1/I_{\widetilde{S}}\widetilde{P}_1 + W(\widetilde{S})F\widetilde{P}_1 \longrightarrow \widetilde{S} \otimes_{\mathrm{Frob}, S} Q_1/I_S P_1,$$

which we will also consider as a homomorphism of $W(\widetilde{S})$-modules

$$\text{(122)} \qquad V^{\#} : \widetilde{P}_1 \longrightarrow \widetilde{S} \otimes_{\mathrm{Frob}, S} Q_1/I_S P_1.$$

Now we define the second homomorphism. Since $\mathfrak{b}^p = 0$, the operator F on $\widetilde{P}_2/I_{\widetilde{S}}\widetilde{P}_2$ factors as follows:

$$\begin{tikzcd}
\widetilde{P}_2/I_{\widetilde{S}}\widetilde{P}_2 \arrow[rr, "F"] \arrow[dr] & & \widetilde{P}_2/I_{\widetilde{S}}\widetilde{P}_2 \\
& P_2/I_S P_2 \arrow[ur, "F^b"'] &
\end{tikzcd}$$

The module $\widetilde{Q}_2/I_{\widetilde{S}}\widetilde{P}_2$ is in the kernel of F. Hence we obtain a Frobenius linear map

$$F^b : P_2/Q_2 \longrightarrow \widetilde{P}_2/I_{\widetilde{S}}\widetilde{P}_2,$$

whose restriction to $\mathfrak{a}(P_2/Q_2)$ induces

$$F^b : \mathfrak{a}(P_2/Q_2) \longrightarrow \mathfrak{b}(\widetilde{P}_2/I_{\widetilde{S}}\widetilde{P}_2).$$

If we use our embedding $\mathfrak{b} \subset W(\mathfrak{b})$, we may identify the target of F^b with $\mathfrak{b} \cdot \widetilde{P}_2 \subset W(\mathfrak{b})\widetilde{P}_2$. Let us denote the linearization of F^b simply by

$$F^{\#} : \quad \widetilde{S} \otimes_{\mathrm{Frob},S} \mathfrak{a}(P_2/Q_2) \longrightarrow \mathfrak{b}\widetilde{P}_2 \tag{123}$$

Proposition 73. — *The obstruction to lift $p\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ to a homomorphism of displays $\widetilde{\mathcal{P}}_1 \to \widetilde{\mathcal{P}}_2$ is given by the composition of the following maps:*

$$\begin{tikzcd} \widetilde{Q}_1/I_S\widetilde{P}_1 \arrow[r, "V^{\#}"] & \widetilde{S} \otimes_{\mathrm{Frob},S} Q_1/I_S P_1 \arrow[r, "\widetilde{S} \otimes \mathrm{Obst}\,\overline{\varphi}"] & \widetilde{S} \otimes_{\mathrm{Frob},S} \mathfrak{a}(P_2/Q_2) \arrow[d, "F^{\#}"] \\ & & \mathfrak{b}(\widetilde{P}_2/\widetilde{Q}_2) \end{tikzcd}$$

Here the horizontal map is induced by the restriction of the map (122) to $\widetilde{Q}_1/I_{\widetilde{S}}\widetilde{P}_1$, and the map $F^{\#}$ is the map (123) followed by the factor map $\mathfrak{b}\widetilde{P}_2 \to \mathfrak{b}(\widetilde{P}_2/\widetilde{Q}_2)$.

Before giving the proof, we state a more precise result, which implies the proposition.

Corollary 74. — *The morphism of displays $p\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ lifts by theorem 46 to a morphism of triples $\widetilde{\psi} : (\widetilde{P}_1, F, V^{-1}) \to (\widetilde{P}_2, F, V^{-1})$. This morphism may be explicitly obtained as follows. We define $\omega : \widetilde{P}_1 \to \mathfrak{b}\widetilde{P}_2 \subset W(\mathfrak{b})\widetilde{P}_2$ to be the composite of the following maps*

$$\widetilde{P}_1 \xrightarrow{V^{\#}} \widetilde{S} \otimes_{\mathrm{Frob},S} Q_1/I_S P_1 \xrightarrow{\widetilde{S} \otimes \mathrm{Obst}\,\overline{\varphi}} \widetilde{S} \otimes_{\mathrm{Frob},S} \mathfrak{a}(P_2/Q_2) \xrightarrow{F^{\#}} \mathfrak{b}\widetilde{P}_2.$$

Then we have the equation

$$\widetilde{\psi} = p\widetilde{\varphi} + \omega,$$

where $\widetilde{\varphi} : \widetilde{P}_1 \to \widetilde{P}_2$ is any $W(\widetilde{S})$-linear map, which lifts $\varphi : P_1 \to P_2$.

We remark that $p\widetilde{\varphi}$ depends only on φ and not on the particular lifting $\widetilde{\varphi}$.

Proof. — It is clear that the proposition follows from the corollary. Let us begin with the case, where $\overline{\varphi}$ is an isomorphism. We apply the method of the proof of theorem 44 to $p\widetilde{\varphi}$.

We find that $p\widetilde{\varphi}$ commutes with F.

$$F(p\widetilde{\varphi}) = (p\widetilde{\varphi})F \tag{124}$$

Indeed, since φ commutes with F, we obtain

$$F\widetilde{\varphi}(x) - \widetilde{\varphi}(Fx) \in W(\mathfrak{b})\widetilde{P}_2.$$

Since $p \cdot W(\mathfrak{b}) = 0$, we obtain (124). We have also that $p\widetilde{\varphi}(Q_1) \subset Q_2$.

We need to understand how much the commutation of $p\widetilde{\varphi}$ and V^{-1} fails. For this purpose we choose normal decompositions as follows. Let $\overline{P}_1 = \overline{L}_1 \oplus \overline{T}_2$ be any normal decomposition. We set $\overline{L}_2 = \overline{\varphi}(\overline{L})$ and $\overline{T}_2 = \overline{\varphi}(\overline{T}_1)$. Since $\overline{\varphi}$ is an isomorphism we have the normal decomposition $\overline{P}_2 = \overline{L}_2 \oplus \overline{T}_2$. We take liftings of these decompositions to normal decompositions

$$P_1 = L_1 \oplus T_1 \qquad \text{and} \qquad P_2 = L_2 \oplus T_2.$$

Finally we lift the last decomposition further to normal decompositions

$$\widetilde{P}_1 = \widetilde{L}_1 \oplus \widetilde{T}_1 \qquad \text{and} \qquad \widetilde{P}_2 = \widetilde{L}_2 \oplus \widetilde{T}_2.$$

We write the restriction of φ to L_1 as follows:

$$\varphi(l_1) = \lambda(l_1) + \mu(l_1), \quad \lambda(l_1) \in L_2,\ \mu(l_1) \in W(\mathfrak{a})T_2$$

Since $\mathfrak{a}^p = 0$, we have $I_S \cdot W(\mathfrak{a}) = 0$ and the Witt addition on $W(\mathfrak{a})$ is the usual addition of vectors. Let us denote by $\mathfrak{a}_n$ the S-module obtained from $\mathfrak{a}$ via restriction of scalars by $\mathrm{Frob}^n : S \to S$. Then we have a canonical isomorphism of S-modules

$$W(\mathfrak{a})T_2 \simeq \prod_{n \geq 0} \mathfrak{a}_n \otimes_S T_2/I_2T_2$$

Hence μ is a map

$$\mu : L_1/I_SL_1 \longrightarrow \prod_{n \geq 0} \mathfrak{a}_n \otimes_S T_2/I_ST_2.$$

We denote by μ_n its $n-th$ component. Then

$$\mu_0 : L_1/I_SL_1 \longrightarrow \mathfrak{a} \otimes_S T_2/I_ST_2$$

may be identified with the obstruction $\eta = \mathrm{Obst}\,\overline{\varphi}$.

Since φ commutes with V^{-1} we have

$$\varphi(V^{-1}l_1) = V^{-1}\lambda(l_1) + V^{-1}\mu(l_1). \tag{125}$$

Let us denote by $\mathfrak{c}$ the kernel of the map $\widetilde{S} \to R$. We choose any lifting $\widetilde{\tau} : \widetilde{L}_1 \to W(\mathfrak{c})\widetilde{P}_2$ of the Frobenius linear map:

$$V^{-1}\mu : L_1 \longrightarrow W(\mathfrak{a})T_2 \xrightarrow{V^{-1}} W(\mathfrak{a})P_2.$$

We write the restriction of $\widetilde{\varphi}$ to $\widetilde{L}_1$ in the form

$$\widetilde{\varphi} = \widetilde{\lambda} + \widetilde{\mu},$$

where $\widetilde{\lambda} : \widetilde{L}_1 \to \widetilde{L}_2$ and $\widetilde{\mu} : \widetilde{L}_1 \to W(\mathfrak{a})\widetilde{T}_2$. Then we obtain from the equation (125) that

$$\widetilde{\varphi}(V^{-1}\widetilde{l}_1) - (V^{-1}\widetilde{\lambda}(\widetilde{l}_1) + \widetilde{\tau}(\widetilde{l}_1)) \in W(\mathfrak{b})\widetilde{P}_2, \quad \text{for } \widetilde{l}_1 \in \widetilde{L}_1.$$

Since $pW(\mathfrak{b}) = 0$, we deduce the equation

$$p\widetilde{\varphi}(V^{-1}\widetilde{l}_1) = pV^{-1}\widetilde{\lambda}(\widetilde{l}_1) + p\widetilde{\tau}(\widetilde{l}_1). \tag{126}$$

On the other hand we have obviously

$$V^{-1}p\widetilde{\varphi}(\widetilde{l}_1) = pV^{-1}\widetilde{\lambda}(\widetilde{l}_1) + F\widetilde{\mu}(\widetilde{l}_1).$$

If we subtract this form (126), we get an information on the commutation of $p\widetilde{\varphi}$ and V^{-1}:

$$p\widetilde{\varphi}(V^{-1}\widetilde{l}_1) - V^{-1}p\widetilde{\varphi}(\widetilde{l}_1) = (p\widetilde{\tau} - F\widetilde{\mu})(\widetilde{l}_1). \tag{127}$$

We set $\mu' = \mu - \mu_0$, with the map μ_0 defined above and consider it as a map $\mu' : L_1 \to {}^V W(\mathfrak{a})T_2$. We choose any lifting of μ' to a $W(\widetilde{S})$-linear map

$$\widetilde{\mu}' : \widetilde{L}_1 \longrightarrow {}^V W(\mathfrak{c})\widetilde{T}_2.$$

Then $V^{-1}\widetilde{\mu}'$ is defined and is a lifting of $V^{-1}\mu$, since by definition $V^{-1}\mu_0 = 0$. Therefore we may take $\tau = V^{-1}\widetilde{\mu}'$. Hence we may rewrite the right hand side of (127):

$$p\tau - F\widetilde{\mu} = F(\widetilde{\mu}' - \widetilde{\mu}). \tag{128}$$

Then $\widetilde{\mu} - \widetilde{\mu}'$ is a lifting of the map

$$\mu_0 : L_1 \longrightarrow \mathfrak{a} \otimes_S (T_2/I_S T_2) \subset W(\mathfrak{a})T_2,$$

to a map

$$\widetilde{\mu}_0 : \widetilde{L}_1 \longrightarrow W(\mathfrak{c})\widetilde{T}_2.$$

In fact the expression $F\widetilde{\mu}_0$ is independent of the particular lifting $\widetilde{\mu}_0$ of μ_0. Therefore we may rewrite the formula (127)

$$V^{-1}p\widetilde{\varphi}(\widetilde{l}_1) - p\widetilde{\varphi}(V^{-1}\widetilde{l}_1) = F\widetilde{\mu}_0(\widetilde{l}_1). \tag{129}$$

Let $\mathfrak{u} \subset W(\mathfrak{c})$ be the kernel of the following composite map:

$$W(\mathfrak{c}) \longrightarrow W(\mathfrak{a}) = \prod_{n\geq 0} \mathfrak{a}_n \xrightarrow{pr} \prod_{n\geq 1} \mathfrak{a}_n.$$

$\mathfrak{u}$ is the ideal consisting of vectors in $W(\mathfrak{c})$, whose components at places bigger than zero are in $\mathfrak{b}$. We see that ${}^F\mathfrak{u} \subset \mathfrak{b} = \mathfrak{b}_0 \subset W(\mathfrak{b})$. We find:

$$F\widetilde{\mu}_0(\widetilde{l}_1) \in \mathfrak{b}(\widetilde{P}_2/I_{\widetilde{S}}\widetilde{P}_2) \subset W(\mathfrak{b})\widetilde{P}_2.$$

More invariantly we may express $F\widetilde{\mu}_0$ as follows.

We have a factorization:

$$\begin{tikzcd} F : \widetilde{P}_2/I_{\widetilde{S}}\widetilde{P}_2 \arrow[rr] \arrow[dr] & & \widetilde{P}_2/I_{\widetilde{S}}\widetilde{P}_2 \\ & P_2/I_S P_2 \arrow[ur, "F^b"'] & \end{tikzcd}$$

Then F^b induces by restriction a map

$$F^b : \mathfrak{a}(P_2/I_S P_2) \longrightarrow \mathfrak{b}(\widetilde{P}_2/I_{\widetilde{S}}\widetilde{P}_2).$$

The map $F\widetilde{\mu}_0$ is the following composite map.

$$\widetilde{L}_1 \longrightarrow L_1 \xrightarrow{\mu_0} \mathfrak{a}(T_2/I_S T_2) \xrightarrow{F^b} \mathfrak{b}(\widetilde{P}_2/I_{\widetilde{S}}\widetilde{P}_2).$$

By a slight abuse of notation we may write

$$F\widetilde{\mu}_0 = F^b\mu_0.$$

We obtain the final form of the commutation rule

$$V^{-1}p\widetilde{\varphi}(\widetilde{l}_1) - p\widetilde{\varphi}(V^{-1}\widetilde{l}_1) = F^b\mu_0(\widetilde{l}_1). \tag{130}$$

We want to know the map of triples

$$\widetilde{\psi} : (\widetilde{P}_1, F, V^{-1}) \longrightarrow (\widetilde{P}_2, F, V^{-1}),$$

which lifts $p\varphi$.

As in the proof of 2.2 we write $\widetilde{\psi} = p\widetilde{\varphi} + \omega$, where $\omega : \widetilde{P}_1 \to W(\mathfrak{b})\widetilde{P}_2$ is a $W(\widetilde{S})$-linear map. The condition that $\widetilde{\psi}$ should commute with F is equivalent to $\omega(W(\widetilde{S})F\widetilde{T}_1) = 0$. We consider only these ω. To ensure that V^{-1} and $\widetilde{\psi}$ commute is enough to ensure

$$V^{-1}\widetilde{\psi}(\widetilde{l}_1) = \widetilde{\psi}(V^{-1}\widetilde{l}_1) \quad \text{for } \widetilde{l}_1 \in \widetilde{L}_1. \tag{131}$$

On $I_{\widetilde{S}}\widetilde{T}_1$ the commutation follows, because $\widetilde{\psi}$ already commutes with F. Using (130) we see that the equality (131) is equivalent with:

$$\omega(V^{-1}\widetilde{l}_1) - V^{-1}\omega(\widetilde{l}_1) = F^b\mu_0(\widetilde{l}_1) \tag{132}$$

We look for a solution of this equation in the space of $W(\widetilde{S})$- linear maps

$$\omega : \widetilde{P}_1/W(\widetilde{S})F\widetilde{T}_1 \longrightarrow \mathfrak{b}_0 \otimes_{\widetilde{S}} \widetilde{P}_2/I_{\widetilde{S}}\widetilde{P}_2 \subset W(\mathfrak{b})\widetilde{P}_2$$

Then we have $V^{-1}\omega(\widetilde{l}_1) = 0$, by definition of the extended V^{-1}. Hence we need to find ω, such that

$$\omega(V^{-1}\widetilde{l}_1) = F^b\mu_0(\widetilde{l}_1). \tag{133}$$

We linearize this last equation as follows. The operator V^{-1} induces an isomorphism

$$(V^{-1})^{\#} : W(\widetilde{S}) \otimes_{F,W(\widetilde{S})} \widetilde{L}_1 \longrightarrow \widetilde{P}_1/W(\widetilde{S})F\widetilde{T}_1,$$

whose inverse will be denoted by $V^{\#}$.

We will also need the tensor product μ_0' of μ_0 with the map $w_0 : W(\widetilde{S}) \to \widetilde{S}$:

$$\mu_0' : W(\widetilde{S}) \otimes_{F,W(S)} L_1 \longrightarrow \widetilde{S} \otimes_{\mathrm{Frob},S} \mathfrak{a}(T_2/I_S T_2).$$

Finally we denote the linearization of F^b simply by $F^{\#}$:

$$F^{\#} : \widetilde{S} \otimes_{\mathrm{Frob},S} \mathfrak{a}(P_2/I_S P_2) \longrightarrow \mathfrak{b}(\widetilde{P}_2/I_{\widetilde{S}}\widetilde{P}_2).$$

Noting that we have a natural isomorphism $W(\widetilde{S}) \otimes_{F,W(\widetilde{S})} \widetilde{L}_1 \cong W(\widetilde{S}) \otimes_{W(S)} L_1$, we obtain the following equivalent linear form of the equation (133):

$$\omega(V^{-1})^{\#} = F^{\#}\mu_0'.$$

It follows that the unique lifting of $p\varphi$ to a homomorphism of triples is

$$\widetilde{\psi} = p\widetilde{\varphi} + F^{\#}\mu_0' V^{\#}.$$

In this equation $V^{\#}$ denotes the composite map

$$\widetilde{P}_1 \longrightarrow \widetilde{P}_1 / W(\widetilde{S}) F \widetilde{T}_1 \longrightarrow W(\widetilde{S}) \otimes_{F, W(S)} \widetilde{L}_1.$$

This map $\widetilde{\varphi}$ induces the obstruction to lift $p\varphi$:

$$\tau : \widetilde{Q}_1 / I_{\widetilde{S}} \widetilde{P}_1 \hookrightarrow \widetilde{P}_1 / I_{\widetilde{S}} \widetilde{P}_1 \xrightarrow{\widetilde{\psi}} \widetilde{P}_2 / I_{\widetilde{S}} \widetilde{P}_2 \to \widetilde{P}_2 / \widetilde{Q}_2.$$

Since $p\widetilde{\varphi}$ maps $\widetilde{Q}_1$ to $\widetilde{Q}_2$, we may replace $\widetilde{\psi}$ in the definition of the obstruction τ by $F^{\#}\mu_0' V^{\#}$. This proves the assertion of the corollary in the case where $\overline{\varphi}$ is an isomorphism.

If $\overline{\varphi}$ is not an isomorphism we reduce to the case of an isomorphism by the standard construction: Consider in general a homomorphism $\psi : \mathcal{P}_1 \to \mathcal{P}_2$ of displays over S. Then we associate to it the isomorphism

$$\begin{aligned} \psi_1 : \mathcal{P}_1 \oplus \mathcal{P}_2 &\longrightarrow \mathcal{P}_1 \oplus \mathcal{P}_2 \\ x \oplus y &\longmapsto x \oplus y + \psi(x) \end{aligned}$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are liftings to $\widetilde{S}$ as in the lemma, we denote by $\widetilde{\psi} : (\widetilde{P}_1, F, V^{-1}) \to (\widetilde{P}_2, F, V^{-1})$ the unique lifting to a homomorphism of triples. Then

$$\widetilde{\psi}_1(\widetilde{x} \oplus \widetilde{y}) = \widetilde{x} \oplus (\widetilde{y} + \widetilde{\psi}(\widetilde{x})), \quad \widetilde{x} \in \widetilde{P}_1, \quad \widetilde{y} \in \widetilde{P}_2.$$

It follows that $\operatorname{Obst} \psi_1$ is the map

$$0 \oplus \operatorname{Obst} \widetilde{\psi} : \widetilde{Q}_1 / I_{\widetilde{S}} \widetilde{P}_1 \oplus \widetilde{Q}_2 / I_{\widetilde{S}} \widetilde{P}_2 \longrightarrow \widetilde{P}_1 / \widetilde{Q}_1 \oplus \widetilde{P}_2 / \widetilde{Q}_2.$$

Applying these remarks the reduction to the case of an isomorphism follows readily. □

We will now apply the last proposition to obtain the following result of Keating:

Proposition 75. — *Let k be an algebraically closed field of characteristic $p > 2$. Let $\mathcal{P}_0$ be the display over k of dimension 1 and height 2. The endomorphism ring O_D of $\mathcal{P}_0$ is the ring of integers in a quaternion division algebra D with center $\mathbb{Q}_p$. Let $\alpha \mapsto \alpha^*$ for $\alpha \in O_D$ be the main involution. We fix $\alpha \in O_D$, such that $\alpha \notin \mathbb{Z}_p$ and we set $i = \operatorname{ord}_{O_D}(\alpha - \alpha^*)$. We define $c(\alpha) \in \mathbb{N}$:*

$$c(\alpha) = \begin{cases} p^{i/2} + 2p^{(i/2-1)} + 2p^{(i/2-2)} + \cdots + 2 & \text{for } i \text{ even} \\ 2p^{\frac{i-1}{2}} + 2p^{\left(\frac{i-1}{2}-1\right)} + \cdots + 2 & \text{for } i \text{ odd} \end{cases}$$

Let $\mathcal{P}$ over $k[\![t]\!]$ be the universal deformation of $\mathcal{P}_0$ in equal characteristic. Then α lifts to an endomorphism of $\mathcal{P}$ over $k[\![t]\!]/t^{c(\alpha)}$ but does not lift to an endomorphism of $\mathcal{P}$ over $k[\![t]\!]/t^{c(\alpha)+1}$.

Proof. — The display $\mathcal{P}_0 = (P_0, Q_0, F, V^{-1})$ is given by the structural equations

$$Fe_1 = e_2$$
$$V^{-1}e_2 = e_1.$$

For any $a \in W(\mathbb{F}_{p^2})$ we have an endomorphism φ_a of $\mathcal{P}_0$, which is given by

$$\varphi_a(e_1) = ae_1 \qquad \varphi_a(e_2) = \sigma(a)e_2 \tag{134}$$

Here σ denotes the Frobenius endomorphism $W(\mathbb{F}_{p^2})$, and a is considered as an element of $W(k)$ with respect to a fixed embedding $\mathbb{F}_{p^2} \subset k$.

We denote by Π the endomorphism of $\mathcal{P}_0$ defined by

$$\Pi e_1 = e_2 \;\; \Pi e_2 = pe_1. \tag{135}$$

The algebra O_D is generated by Π and the φ_a. The following relations hold:

$$\Pi^2 = p\ , \qquad \Pi\varphi_a = \varphi_{\sigma(a)}\Pi.$$

The display $\mathcal{P}^u = (P^u, Q^u, F, V^{-1})$ of X over $k[\![t]\!]$ is given by the structural equations

$$Fe_1 = [t]e_1 + e_2 \qquad , \quad V^{-1}e_2 = e_1.$$

To prove our assertion on the liftability of α it is enough to consider the following cases:

$$\begin{aligned} \alpha &= \varphi_a p^s\ , a \not\equiv \sigma(a) \mod p\ ,\ s \in \mathbb{Z}, s \geq 0 \\ \alpha &= \varphi_a p^s \Pi;\ , \quad a \in W(\mathbb{F}_{p^2})^*\ , \quad s \in \mathbb{Z}\ , s \geq 0 \end{aligned} \tag{136}$$

Let us begin by considering the two endomorphisms α for $s = 0$. The universal deformation $\mathcal{P}^u$ induces by base change $k[\![t]\!] \to k[\![t]\!]/t^p$ a display $\mathcal{P} = (P, Q, F, V^{-1})$. Then α induces an obstruction to the liftability to $S = k[\![t]\!]/t^p$:

$$\begin{aligned} \operatorname{Obst}\alpha : Q/I_S P &\longrightarrow t(P/Q), \\ e_2 &\longmapsto o(\alpha)\cdot e_1 \end{aligned} \tag{137}$$

where $o(\alpha) \in tk[\![t]\!]/t^p$. To compute the obstruction, we need to find the extension of α to a morphism of triples

$$\widetilde{\alpha} : (P, F, V^{-1}) \longrightarrow (P, F, V^{-1}).$$

Let $\widetilde{e}_1, \widetilde{e}_2 \in P$ be defined, by

$$\widetilde{e}_1 = e_1 \qquad \text{and} \qquad \widetilde{e}_2 = [t]e_1 + e_2.$$

This is a basis of P and the extended operator V^{-1} is defined on $\widetilde{e}_2$. We find the equations

$$F\widetilde{e}_1 = \widetilde{e}_2, \qquad V^{-1}\widetilde{e}_2 = \widetilde{e}_1.$$

Then obviously $\widetilde{\alpha}$ is given by the same equations as α:

$$\widetilde{\alpha}(\widetilde{e}_1) = a\widetilde{e}_1, \quad \widetilde{\alpha}(\widetilde{e}_2) = \sigma(a)\widetilde{e}_2, \tag{138}$$

respectively

$$\widetilde{\alpha}(\widetilde{e}_1) = \sigma(a)\widetilde{e}_2, \quad \widetilde{\alpha}(\widetilde{e}_2) = ap\widetilde{e}_1. \tag{139}$$

For the first endomorphism α of (136) we find

$$\begin{aligned}\widetilde{\alpha}(e_2) &= \widetilde{\alpha}(\widetilde{e}_2 - [t]\widetilde{e}_1) = \sigma(a)\widetilde{e}_2 + [t]a\widetilde{e}_1 \\ &= \sigma(a)e_2 + [t](\sigma(a) - a)e_1\end{aligned}$$

Hence the obstruction to lift α to $k[\![t]\!]/t^p$ is $o(\varphi_a) = o(\alpha) = (\sigma(a) - a)t \in tk[\![t]\!]/t^p$. For the second endomorphism α of (136) we find

$$\begin{aligned}\widetilde{\alpha}(e_2) &= \widetilde{\alpha}(\widetilde{e}_2 - [t]\widetilde{e}_1) = ap\widetilde{e}_1 - [t]\sigma(a)\widetilde{e}_2 \\ &= ape_1 - [t]\sigma(a)\left([t]e_1 + e_2\right).\end{aligned}$$

Hence we obtain the obstruction

$$o\left(\varphi_a\Pi\right) = o(\alpha) = -t^2\sigma(a) \in tk[\![t]\!]/t^p.$$

Now we consider the first endomorphism of (136) for $s = 1$. It lifts to an endomorphism over $k[\![t]\!]/t^p$. We compute the obstruction to lift it to $k[\![t]\!]/t^{p^2}$. We can apply the lemma to the situation

$$\begin{tikzcd} k & k[\![t]\!]/t^p \arrow[l] & k[\![t]\!]/t^{p^2} \arrow[l] \\ R \arrow[u, equal] & S \arrow[u, equal] & \widetilde{S} \arrow[u, equal] \end{tikzcd}$$

We set $\overline{\varphi} = \varphi_a$ and $\widetilde{\mathcal{P}} = \mathcal{P}^u_{\widetilde{S}}$. Then we have the following commutative diagram of obstructions

$$\begin{tikzcd} \widetilde{Q}/I_{\widetilde{S}}\widetilde{P} \arrow[r] \arrow[drr, "\mathrm{Obst}(p\varphi_a)"'] & \widetilde{S} \otimes_{\mathrm{Frob},S} Q/I_S P \arrow[r, "\widetilde{S} \otimes \mathrm{Obst}(\varphi_a)"] & \widetilde{S} \otimes_{\mathrm{Frob},S} t(P/Q) \arrow[d, "F^{\#}"] \\ & & t^p(\widetilde{P}/\widetilde{Q}) \end{tikzcd} \tag{140}$$

The first horizontal map here is computed as follows:

$$\begin{tikzcd} \widetilde{Q}/I_{\widetilde{S}}\widetilde{P} \arrow[r] & \widetilde{P}/I_{\widetilde{S}}\widetilde{P} + W(\widetilde{S})F\widetilde{P} & \widetilde{S} \otimes_{\mathrm{Frob},S} Q/I_S P \arrow[l, "(V^{-1})^{\#}"'] \\ e_2 \arrow[r, mapsto] & e_2 = -te_1 & \\ & -te_1 & -t \otimes e_2 \arrow[l, mapsto] \end{tikzcd}$$

We obtain that the maps in the diagram (140) are as follows

$$e_2 \longmapsto -t \otimes e_2 \longmapsto -t \otimes t\left(\sigma(a) - a\right)e_1 \longmapsto -t \cdot t^p\left(-\sigma(a) + a\right)Fe_1$$

Therefore we obtain for $\operatorname{Obst}(p\varphi_a)$:

$$\operatorname{Obst} p\varphi_a = t^{p+1}\,(\sigma(a)-a)\,Fe_1 = t^{p+2}(\sigma a - a)e_1 \in t^p\left(\widetilde{P}/\widetilde{Q}\right).$$

With the same convention as in (137) we write $o(p\varphi_a) = (\sigma(a)-a)t^{p+2}$. Then we prove by induction that $p^s\varphi_a$ lifts to $k[\![t]\!]/t^{p^2} + 2(p^{s-1}+\cdots+1)$ and that the obstruction to lift it to $k[\![t]\!]/t^{p^{s+1}}$ is $(\sigma(a)-a)\cdot t^{p^s+2(p^{s-1}+\cdot+1)}$. For the induction step we apply our lemma to the situation

$$\begin{array}{ccccc}
k[\![t]\!]/t^{p^s+2(p^{s-1}+\cdots+1)} & \longleftarrow & k[\![t]\!]/t^{p^{s+1}} & \longleftarrow & k[\![t]\!]/t^{p^{s+2}} \\
\Vert & & \Vert & & \Vert \\
R & & S & & \widetilde{S}
\end{array}$$

We set $\overline{\varphi} = p^s\varphi_a$ over R and $\widetilde{\mathcal{P}} = \mathcal{P}^u_{\widetilde{S}}$. Then the maps in the diagram (140) are as follows

$$\begin{array}{ccccc}
e_2 & \longmapsto & -t\otimes e_2 & \longmapsto & -t\otimes(\sigma(a)-a)t^{p^s+2(p^{s-1}+\cdots+1)}e_1 \\
 & & & & \big\downarrow \\
 & & & & -t\,(a-\sigma(a))\,t^{p\left(p^s+2\left(p^{s-1}+\cdots+1\right)\right)}Fe_1
\end{array}$$

This gives the asserted obstruction for $p^{s+1}\varphi_a$:

$$\operatorname{Obst}\left(p^{s+1}\varphi_a\right) = \sigma(a)-a)t^{p^{s+1}+2(p^s+\cdots+p)+1}\cdot te_1.$$

Next we consider the case of the endomorphisms $p^s\varphi_a\Pi$. In the case $s=1$ we apply the lemma to the situation

$$\begin{array}{ccccc}
k & \longleftarrow & k[\![t]\!]/t^p & \longleftarrow & k[\![t]\!]/t^{p^2}, \\
\Vert & & \Vert & & \Vert \\
R & \longleftarrow & S & \longleftarrow & \widetilde{S}
\end{array}$$

and the endomorphism $\overline{\varphi} = \varphi_a\Pi$. Then the maps in the diagram (140) are as follows:

$$\begin{array}{ccccc}
e_2 & \longmapsto & -t\otimes e_2 & \longmapsto & -t\otimes -t^2\sigma(a)e_1 \\
 & & & & \big\downarrow \\
 & & & & tt^{2p}aFe_1
\end{array}$$

This gives $\operatorname{Obst}(p\varphi_a\Pi) = t^{2p+2}a$. Now one makes the induction assumption that for even s the obstruction to lift $p^s\varphi_a\Pi$ from $k[\![t]\!]/t^{2(p^s+\cdots+1)}$ to $k[\![t]\!]/t^{p^{s+1}}$ is $-t^{2(p^s+\cdots+1)}\cdot$ $\sigma(a)$ and for odd s is $t^{2(p^s+\cdots+1)}\cdot a$. We get the induction step immediately from the lemma applied to the situation

$$k[\![t]\!]/t^{2(p^s+\cdots+1)} \longleftarrow k[\![t]\!]/p^{s+1} \longleftarrow k[\![t]\!]/p^{s+2}. \qquad \square$$

We finish this section with a result of B. Gross on the endomorphism ring of the Lubin-Tate groups. Let A be a $\mathbb{Z}_p$-algebra. Let S be an A-algebra.

Definition 76. — An A-display over S is a pair $(\widetilde{\mathcal{P}}, \iota)$, where $\widetilde{\mathcal{P}}$ is a display over S, and $\iota : A \to \operatorname{End} \widetilde{\mathcal{P}}$ is a ring homomorphism, such that the action of A on $\widetilde{P}/\widetilde{Q}$ deduced from ι coincides with the action coming from the natural S-module structure on $\widetilde{P}/\widetilde{Q}$ and the homomorphism $A \to S$ giving the A-algebra structure.

Let $a \in A$ be a fixed element. We set $R = S/a$ and $R_i = S/a^{i+1}$. Then we have a sequence of surjections

$$S \longrightarrow \cdots \longrightarrow R_i \longrightarrow R_{i-1} \longrightarrow \cdots \longrightarrow R = R_0$$

Let $\widetilde{\mathcal{P}}_1$ and $\widetilde{\mathcal{P}}_2$ be displays over S. They define by base change displays $\mathcal{P}_1^{(i)}$ and $\mathcal{P}_2^{(i)}$ over R_i. We set $\mathcal{P}_1 = \mathcal{P}_1^{(0)}$ and $\mathcal{P}_2 = \mathcal{P}_2^{(0)}$.

Assume we are given a morphism $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$, which lifts to a morphism $\varphi^{(i-1)} : \mathcal{P}_1^{(i-1)} \to \mathcal{P}_2^{(i-1)}$. The obstruction to lift $\varphi^{(i-1)}$ to a morphism $\mathcal{P}_1^{(i)} \to \mathcal{P}_2^{(i)}$ is a homomorphism:

$$\operatorname{Obst} \varphi^{(i-1)} : Q_1^{(i)}/I_{R_i} P_1^{(i)} \longrightarrow (a^i)/(a^{i+1}) \otimes_{R_i} P_2^{(i)}/Q_2^{(i)}.$$

Clearly $\operatorname{Obst} \varphi^{(i-1)}$ factors through a homomorphism:

$$\operatorname{Obst}_i \varphi : Q_1/I_R P_1 \longrightarrow (a^i)/(a^{i+1}) \otimes_R P_2/Q_2.$$

Proposition 77. — *Assume that $(\widetilde{\mathcal{P}}_2, \iota)$ is an A-display over S. Let $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ be a morphism of displays, which lifts to a morphism $\mathcal{P}_1^{(i-1)} \to \mathcal{P}_2^{(i-1)}$. Then $\iota(a)\varphi$ lifts to a homomorphism $\mathcal{P}_1^{(i)} \to \mathcal{P}_2^{(i)}$ and moreover we have a commutative diagram if $i \geq 2$ or $p > 2$:*

$$\begin{tikzcd} Q_1/I_R P_1 \arrow[r, "\operatorname{Obst}_i \varphi"] \arrow[dr, "\operatorname{Obst}_{i+1}(\iota(a)\varphi)"'] & (a^i)/(a^{i+1}) \otimes_R P_2/Q_2 \arrow[d, "a \otimes \mathrm{id}"] \\ & (a^{i+1})/(a^{i+1}) \otimes_R P_2/Q_2 \end{tikzcd} \tag{141}$$

Loosely said we have $\operatorname{Obst}_{i+1}(\iota(a)\varphi) = a \operatorname{Obst}_i(\varphi)$.

Proof. — We consider the surjection $R_{i+1} \to R_{i-1}$. The kernel $a^i R_{i+1}$ has divided powers if $i \geq 2$ or $p > 2$. Hence the obstruction to lift $\varphi^{(i-1)}$ to $R^{(i+1)}$ is defined:

$$\operatorname{Obst} \varphi^{(i-1)} : Q_1^{(i+1)}/I_{R_{i+1}} P_1^{(i+1)} \longrightarrow (a^i)/(a^{i+2}) \otimes_{R_{i+1}} P_2^{(i+1)}/Q_2^{(i+1)}$$

is defined. Since $\iota(a)$ induces on the tangent space $P^{(i+1)}/Q^{(i+1)}$ the multiplication by a we obtain

$$\operatorname{Obst} \iota(a)\varphi^{(i-1)} = a \operatorname{Obst} \varphi^{(i-1)}$$

This proves the proposition. □

We will now apply this proposition to the case of a Lubin–Tate display. Let $K/\mathbb{Q}_p$ be a totally ramified extension of degree $e \geq 2$. We consider the ring of integers $A = O_K$. The rôle of the element a in the proposition will be played by a prime element $\pi \in O_K$. For S we take the ring $S = O_K \otimes_{\mathbb{Z}_p} W(\overline{\mathbb{F}}_p)$. Now we take a notational difference between π and its image in S, which we denote by a.

Let $\widetilde{\mathcal{P}} = (\widetilde{P}, \widetilde{Q}, F, V^{-1})$ be the Lubin–Tate display over S. We recall that $\widetilde{P} = O_K \otimes_{\mathbb{Z}_p} W(S)$, $\widetilde{Q} = \operatorname{kernel}(O_K \otimes_{\mathbb{Z}_p} W(S) \to S)$, and $V^{-1}(\pi \otimes 1 - 1 \otimes [a]) = 1$.

Let $\mathcal{P}$ be the display obtained by base change over $R = S/aS = \overline{\mathbb{F}}_p$. The operator V^{-1} of $\mathcal{P}$ satisfies

$$V^{-1}\pi^i = \pi^{i-1},$$

where $\pi = \pi \otimes 1 \in O_K \otimes_{\mathbb{Z}_p} W(R)$. (One should not be confused by the fact that this ring happens to be S). We note that $Q = \pi P$.

We consider an endomorphism $\varphi : \mathcal{P} \to \mathcal{P}$, and compute the obstruction to lift φ to $R_1 = S/a^2S$:

$$\operatorname{Obst}_1(\varphi) : Q/I_R P \longrightarrow (a)/(a^2) \otimes_R P/Q.$$

The endomorphism φ induces an endomorphism on P/Q, which is the multiplication by some element $\operatorname{Lie}\varphi \in \overline{\mathbb{F}}_p$. Let us denote by σ the Frobenius endomorphism of $\overline{\mathbb{F}}_p$.

Lemma 78. — $\operatorname{Obst}_1(\varphi)$ *is the composition of the following maps:*

$$\begin{tikzcd} Q/I_R P = Q/pP \arrow[r, "1/\pi"] & P/\pi P = P/Q \arrow[r, "\sigma^{-1}(\operatorname{Lie}\varphi) - \operatorname{Lie}\varphi"] & P/Q \arrow[d, "a"] \\ & & (a)/(a^2) \otimes_R P/Q \end{tikzcd}$$

Proof. — We write

$$\varphi(1) = \xi_0 + \xi_1\pi + \cdots + \xi_{e-1}\pi^{e-1}, \quad \xi_i \in W(\overline{\mathbb{F}}_p).$$

Applying the operator V we obtain:

$$\varphi(\pi^i) = {}^{F^{-i}}\xi_0^{\pi^i} + {}^{F^{-i}}\xi_1\pi^{i+1} + \ldots, \quad \text{for } i = 0, 1 \ldots \tag{142}$$

By theorem 46 this φ admits a unique extension to an endomorphism of the triple $(P^{(1)}, F, V^{-1})$, where $P^{(1)} = O_K \otimes_{\mathbb{Z}_p} W(R_1)$. For the definition of the extension $\widetilde{\varphi}$ we use here the obvious divided powers on the ideal $aR_1 \subset R_1 = S/a^2S$ given by $\alpha_p(a) = 0$. Then we have $V^{-1}[a]P^{(1)} = 0$, for the extended V^{-1}. Hence we find for the triple $(P^{(1)}, F, V^{-1})$ the equations:

$$V^{-1}\pi^i = \pi^{i-1}, \quad \text{for } i \geq 1, \qquad F1 = \frac{p}{\pi}.$$

The last equation follows because the unit τ of lemma 27 specializes in R_1 to π^e/p. Hence we can define $\widetilde{\varphi}$ on $P^{(1)}$ by the same formulas (142) as φ. In other words:

$$\widetilde{\varphi} = \varphi \otimes_{W(\overline{\mathbb{F}}_P)} W(R_1). \tag{143}$$

This formula may also be deduced from the fact that $\widetilde{\varphi}$ is an endomorphism of the display $\mathcal{P}_{R_1}$ obtained by base change via the natural inclusion $R \to R_1$.

The map $\widetilde{\varphi}$ induces an $O_K \otimes_{\mathbb{Z}_p} R_1$-module homomorphism

$$Q^{(1)}/I_{R_1}P^{(1)} \longrightarrow P^{(1)}/Q^{(1)}. \tag{144}$$

By definition the module on the left hand side has the following basis as an R_1-module:

$$\pi - a, \quad \pi^2 - a^2, \ldots, \pi^{e-1} - a^{e-1},$$

where we wrote π for $\pi \otimes 1 \in O_K \otimes_{\mathbb{Z}_p} R_1$ and a for $1 \otimes a$. We note that $\pi^i \in Q^{(1)}$ for $i \geq 2$, because $a^2 = 0$ in R_1 and because $Q^{(1)}$ is an O_K-module. By (143) and (142) we find

$$\begin{aligned} \widetilde{\varphi}(\pi - a) &= {}^{F^{-1}}\xi_0\pi + {}^{F^{-1}}\xi_1\pi^2 + \cdots - a(\xi_0 + \xi_1\pi + \cdots) \\ &\equiv \left({}^{F^{-1}}\xi_0 - \xi_0\right) a \qquad \bmod Q^{(1)} \end{aligned}$$

Since $\widetilde{\varphi}$ is an $O_K \otimes_{\mathbb{Z}_p} W(R_1)$-module homomorphism we have $\widetilde{\varphi}(\pi^i) = 0 \mod Q^{(1)}$. This gives the result for $\operatorname{Obst}_1 \varphi$ because $\xi_0 \mod p = \operatorname{Lie}\varphi$. □

We can obtain a result of B. Gross [G] in our setting:

Proposition 79. — *Let us assume that $p > 2$. Assume that K is a totally ramified extension of $\mathbb{Q}_p$, which has degree $e = [K : \mathbb{Q}_p]$. We fix a prime element $\pi \in O_K$. Let $\widetilde{\mathcal{P}}$ be the corresponding Lubin–Tate display over O_K. Let $\mathcal{P} = \widetilde{\mathcal{P}}_{\overline{\mathbb{F}}_p}$ the display obtained by base change via $O_K \to \mathbb{F}_p \subset \overline{\mathbb{F}}_p$. Let $O_D = \operatorname{End}\mathcal{P}$ be the endomorphism ring. Let $\breve{K}$ be the completion of the maximal unramified extension of K with residue class field $\overline{\mathbb{F}}_p$. Then we have*

$$\operatorname{End}\widetilde{\mathcal{P}}_{O_{\breve{K}}/(\pi^{m+1})} = O_K + \pi^m O_D \quad m \geq 0.$$

Proof. — We use the notation of proposition 77, and set $R_i = O_{\breve{K}}/(\pi^{+1})$. Let $\varphi \in O_D$ be an endomorphism of $\mathcal{P}$. It follows from the formula (2.61) that $\pi^m\varphi$ lifts to an endomorphism of $\widetilde{\mathcal{P}}$ over $O_{\breve{K}}/\pi^{m+1}$. From (77) we obtain by induction:

$$\operatorname{Obst}_{m+1} \pi^m\varphi = \pi^m \operatorname{Obst}_1 \varphi,$$

where π^m on the right hand side denotes the map

$$\pi^m : (\pi)/(\pi^2) \otimes_R P/Q \longrightarrow (\pi^{m+1})/(\pi^{m+2}) \otimes_R P/Q.$$

We recall that $R = R_0 = \overline{\mathbb{F}}_p$ by definition.

Now assume we are given an endomorphism

$$\psi \in (O_K + \pi^m O_D) - (O_K + \pi^{m+1}O_D).$$

Since π is a prime element of O_D we have the expansion

$$\psi = [a_0] + \lfloor a_1 \rceil \pi + \cdots + [a_m]\pi^m + \cdots, \quad \text{where} \quad a_i \in F_{p^e}.$$

We have $a_i \in \mathbb{F}_p$ for $i < m$ and $a_m \notin \mathbb{F}_p$ since $\psi \notin O_K + \pi^{m+1}O_D$. Then we find

$$\operatorname{Obst}^{m+1} \psi = \operatorname{Obst}_{m+1}([a_m]\pi^m + \cdots) = \pi^m \operatorname{Obst}_1([a_m]] + \pi[a_{m+1}] + \cdots)$$

Since $\sigma(a_m) \neq a_m$ the obstruction $\mathrm{Obst}_1\,(\lfloor a_m\rceil + \pi\lfloor a_{m+1}\rceil \cdots)$ does not vanish. Hence $\mathrm{Obst}_{m+1}\,\psi$ does not vanish. $\square$

3. The p-divisible group of a display

3.1. The functor BT. — Let R be a unitary commutative ring, such that p is nilpotent in R. Consider the category Nil_R introduced after definition 50. We will consider functors $F : \mathrm{Nil}_R \to \mathrm{Sets}$, such that $F(0)$ consists of a singe point denoted by 0 and such that F commutes with finite products. Let us denote this category by $\mathcal{F}$. If $\mathcal{N}^2 = 0$, we have homomorphisms in Nil_R:

$$\mathcal{N}\times\mathcal{N} \xrightarrow{\text{addition}} \mathcal{N}, \quad \mathcal{N} \xrightarrow{\ \tau\ } \mathcal{N}, \quad \text{where} \quad \tau \in R.$$

The last arrow is multiplied by τ. Applying F we obtain a R-module structure on $F(\mathcal{N})$. A R-module M will be considered as an object of Nil_R by setting $M^2 = 0$. We write $t_F(M)$ for the R-module $F(M)$.

We view a formal group as a functor on Nil_R (compare [Z1]).

Definition 80. — A (finite dimensional) formal group is a functor $F : \mathrm{Nil}_R \to$ (abelian groups), which satisfies the following conditions.

(i) F(0) =0.

(ii) For any sequence in Nil_R

$$0 \longrightarrow \mathcal{N}_1 \longrightarrow \mathcal{N}_2 \longrightarrow \mathcal{N}_3 \longrightarrow 0,$$

which is exact as a sequence of R-modules the corresponding sequence of abelian groups

$$0 \longrightarrow F(\mathcal{N}_1) \longrightarrow F(\mathcal{N}_2) \longrightarrow F(\mathcal{N}_3) \longrightarrow 0$$

is exact.

(iii) The functor t_F commutes with infinite direct sums.

(iv) $t_F(R)$ is a finitely generated projective R-module.

Our aim is to associate a formal group to a 3n-display.

Let us denote by $\widehat{W}(\mathcal{N}) \subset W(\mathcal{N})$ the subset of Witt vectors with finitely many non–zero components. This is a $W(R)$-subalgebra.

Let us fix $\mathcal{N}$ and set $S = R|\mathcal{N}| = R \oplus \mathcal{N}$. Then we introduce the following $W(R)$-modules

$$\begin{aligned}
P_{\mathcal{N}} &= W(\mathcal{N}) \otimes_{W(R)} P \subset P_S\\
Q_{\mathcal{N}} &= (W(\mathcal{N}) \otimes_{W(R)} P) \cap Q_S\\
\widehat{P}_{\mathcal{N}} &= \widehat{W}(\mathcal{N}) \otimes_{W(R)} P \subset P_S\\
\widehat{Q}_{\mathcal{N}} &= \widehat{P}_{\mathcal{N}} \cap Q_S
\end{aligned}$$

We will denote by $I_{\mathcal{N}} \subset W(\mathcal{N})$ resp. $\widehat{I}_{\mathcal{N}} \subset \widehat{W}(\mathcal{N})$ the $W(R)$-submodules ${}^V W(\mathcal{N})$ and ${}^V \widehat{W}(\mathcal{N})$. We note that ^{F} and ^{V} act also on $\widehat{W}(\mathcal{N})$. Hence the restriction of the operators $F : P_S \to P_S$ and $V^{-1} : Q_S \to P_S$ define operators

$$\begin{array}{ll} F : P_{\mathcal{N}} \longrightarrow P_{\mathcal{N}} & V^{-1} : Q_{\mathcal{N}} \longrightarrow P_{\mathcal{N}} \\ F : \widehat{P}_{\mathcal{N}} \longrightarrow \widehat{P}_{\mathcal{N}} & V^{-1} : \widehat{Q}_{\mathcal{N}} \longrightarrow \widehat{P}_{\mathcal{N}}. \end{array}$$

If we choose a normal decomposition

$$P = L \oplus T,$$

we obtain:

$$\begin{aligned} Q_{\mathcal{N}} &= W(\mathcal{N}) \otimes_{W(R)} L \oplus I_{\mathcal{N}} \otimes_{W(R)} T \\ \widehat{Q}_{\mathcal{N}} &= \widehat{W}(\mathcal{N}) \otimes_{W(R)} L \oplus \widehat{I}_{\mathcal{N}} \otimes_{W(R)} T \end{aligned} \tag{145}$$

Theorem 81. — *Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a 3n-display over R. Then the functor from Nil_R to the category of abelian groups, which associates to an object $\mathcal{N} \in \mathrm{Nil}_R$ the cokernel of the homomorphism of additive groups:*

$$V^{-1} - \mathrm{id} : \widehat{Q}_{\mathcal{N}} \longrightarrow \widehat{P}_{\mathcal{N}},$$

is a finite dimensional formal group. Here id *is the natural inclusion $\widehat{Q}_{\mathcal{N}} \subset \widehat{P}_{\mathcal{N}}$. We denote this functor be $BT_{\mathcal{P}}$. One has an exact sequence:*

$$0 \longrightarrow \widehat{Q}_{\mathcal{N}} \xrightarrow{V^{-1} - \mathrm{id}} \widehat{P}_{\mathcal{N}} \longrightarrow BT_{\mathcal{P}}(\mathcal{N}) \longrightarrow 0. \tag{146}$$

We will give the proof of this theorem and of the following corollary later in this section.

Corollary 82. — *Let $\mathcal{P}$ be a 3n-display, such that there is a number N with the property $F^N P \subset I_R P$. Then we have an exact sequence compatible with (146):*

$$0 \longrightarrow Q_{\mathcal{N}} \xrightarrow{V^{-1} - \mathrm{id}} P_{\mathcal{N}} \longrightarrow BT_{\mathcal{P}}(\mathcal{N}) \longrightarrow 0$$

Remark. — The F-nilpotence condition $F^N P \subset I_R P$ is equivalent to the condition that $F : P \to P$ induces a nilpotent (Frobenius linear) map $R/pR \otimes_{\mathbf{w}_0} P \to R/pR \otimes_{\mathbf{w}_0} P$ of R/pR-modules.

Assume that $\mathcal{N}$ is equipped with divided powers, i.e. the augmentation ideal of the augmented R-algebra $R|\mathcal{N}|$ is equipped with divided powers. Then the divided Witt polynomials define an isomorphism:

$$\prod \mathbf{w}'_n : W(\mathcal{N}) \longrightarrow \prod_{i \geq 0} \mathcal{N} \tag{147}$$

This induces a homomorphism:

$$\begin{aligned} \widehat{W}(\mathcal{N}) &\longrightarrow \bigoplus_{i \geq 0} \mathcal{N} \\ (n_0, n_1, n_2, \dots) &\longmapsto [\mathbf{w}'_0(n_0), \mathbf{w}'_1(n_0, n_1), \dots]. \end{aligned} \tag{148}$$

To see that the homomorphism (147) takes $\widehat{W}(\mathcal{N})$ to the direct sum, it is enough to check, that for a fixed element $n \in \mathcal{N}$ the expression $\alpha_{p^k}(n) = $ "$\frac{n^{p^k}}{p^k}$" becomes zero, if k is big enough. But in terms of the divided powers γ_m on $\mathcal{N}$ this expression is $\frac{(p^k)!}{p^k}\gamma_{pk}(n)$. Since the exponential valuation $\operatorname{ord}_p(\frac{p^k!}{p^k})$ tends with k to infinity, we conclude that (148) is defined.

If we assume moreover that the divided powers on $\mathcal{N}$ are nilpotent in the sense that $\gamma_{p^k}(n)$ is zero for big k, for a fixed $n \in \mathcal{N}$, the homomorphism (148) is an isomorphism. Indeed, for the surjectivity of (148) it is enough to verify that elements of the form $[x, 0, \ldots, 0, \ldots]$ lie in the image, because the morphism (147) is compatible with Verschiebung. To prove the surjectivity of (148) we may moreover restrict to the case where $p \cdot \mathcal{N} = 0$. Indeed $p\mathcal{N} \subset \mathcal{N}$ is a pd-subalgebra, which is an ideal in $\mathcal{N}$. Hence $\mathcal{N}/p\mathcal{N}$ is equipped with nilpotent divided powers. Therefore an induction with the order of nilpotence of p yields the result. If $p \cdot \mathcal{N} = 0$, we see that any expression "$\frac{n^{p^k}}{p^k}$" is zero for $k \geq 2$ because $\frac{(p^k)!}{p^k}$ is divisible by p. But then the assertion, that $[x, 0, 0, \ldots 0]$ is in the image of (148) means that there is $(n_0, n_1, \ldots) \in \widehat{W}(\mathcal{N})$ satisfies the equations

$$x = n_0, \quad \alpha_p(n_0) + n_1 = 0, \quad \alpha_p(n_1) + n_2 = 0, \quad \alpha_p(n_2) + n_3 = 0 \cdots .$$

We have to show that the solutions of these equations:

$$n_k = (-1)^{1+p+\cdots+p^{k-1}} \alpha_p(\cdots \alpha_p(x)) \cdots) \quad k \geq 1,$$

where α_p is iterated k-times, become zero if k is big. It is easy to see from the definition of divided powers that $\alpha_p(\cdots(\alpha_p(x))\cdots)$ and $\gamma_{p^k}(x)$ differ by a unit in $\mathbb{Z}_{(p)}$. Hence we find a solution in $\widehat{W}(\mathcal{N})$, if $\gamma_{p^k}(x)$ is zero for big k. Hence (148) is an isomorphism in the case of nilpotent divided powers. Assume we are given divided powers on $\mathcal{N}$. They define the embedding

$$\begin{aligned} \mathcal{N} &\longrightarrow \quad W(\mathcal{N}), \\ n &\longmapsto [n, 0 \cdots 0 \cdots] \end{aligned} \tag{149}$$

where we have used logarithmic coordinates on the right hand side. If we have nilpotent divided powers the image of the map (149) lies in $\widehat{W}(\mathcal{N})$. Then we obtain the direct decomposition $\widehat{W}(\mathcal{N}) = \mathcal{N} \oplus {}^V\widehat{W}(\mathcal{N})$.

By lemma 38 the operator $V^{-1} : Q_S \to P_S$ extends to the inverse image of Q, if $\mathcal{N}$ has divided powers. This gives a map

$$V^{-1} : W(\mathcal{N}) \otimes_{W(R)} P \longrightarrow W(\mathcal{N}) \otimes_{W(R)} P. \tag{150}$$

If the divided powers on $\mathcal{N}$ are nilpotent, we obtain a map

$$V^{-1} : \widehat{W}(\mathcal{N}) \otimes_{W(R)} P \longrightarrow \widehat{W}(\mathcal{N}) \otimes_{W(R)} P. \tag{151}$$

In fact the nilpotent divided powers are only needed for the existence of this map.

Lemma 83. — *If $\mathcal{N}$ has nilpotent divided powers the map (151) is nilpotent. If $\mathcal{N}$ has only divided powers but if we assume moreover that $F^N P \subset I_R P$ for some number N, the map (150) is nilpotent.*

Proof. — From the isomorphism (145) we get an isomorphism

$$W(\mathcal{N}) \otimes_{W(R)} P \cong \prod_{i \geq 0} \mathcal{N} \otimes_{\mathbf{w}_i, W(R)} P \tag{152}$$

We describe the action of the operator V^{-1} on the right hand side. Let us denote by F_i the following map

$$\begin{aligned} F_i : \mathcal{N} \otimes_{\mathbf{w}_i, W(R)} P &\longrightarrow \mathcal{N} \otimes_{\mathbf{w}_{i-1}, W(R)} P, \quad i \geq 1. \\ a \otimes x &\longmapsto a \otimes Fx \end{aligned}$$

If we write an element from the right hand side of (152) in the form $[u_0, u_1, u_2, \ldots], u_i \in \mathcal{N} \otimes_{\mathbf{w}_i, W(R)} P$, the operator V^{-1} looks as follows:

$$V^{-1}[u_0, u_1, \ldots] = [F_1 u_1, F_2 u_2, \ldots, F_i u_i \cdots]. \tag{153}$$

In the case where the divided powers on $\mathcal{N}$ are nilpotent, we have an isomorphism

$$\widehat{W}(\mathcal{N}) \otimes_{W(R)} P \longrightarrow \bigoplus_{i \geq 0} \mathcal{N} \otimes_{\mathbf{w}_i, W(R)} P. \tag{154}$$

Since V^{-1} on the right hand side is given by the formula (153), the nilpotency of V^{-1} is obvious in this case.

To show the nilpotency of V^{-1} on (152), we choose a number r, such that $p^r \cdot R = 0$. Then we find $\mathbf{w}_i(I_r) \cdot \mathcal{N} \subset p^r \mathcal{N} = 0$, for any $i \in \mathbb{N}$. By our assumption we find a number M, such that $F^M P \subset I_r P$. This implies $F_{i+1} \cdot \ldots \cdot F_{i+M} = 0$ and hence the nilpotency of V^{-1}. □

Corollary 84. — *Let $\mathcal{P}$ be a 3n-display over R. For any nilpotent algebra $\mathcal{N} \in \mathrm{Nil}_R$ the following map is injective*

$$V^{-1} - \mathrm{id} : \widehat{Q}_{\mathcal{N}} \longrightarrow \widehat{P}_{\mathcal{N}}.$$

Proof. — We remark that the functors $\mathcal{N} \mapsto \widehat{P}_{\mathcal{N}}$ and $\mathcal{N} \mapsto \widehat{Q}_{\mathcal{N}}$ are exact in the sense of definition (80) (ii). For $\widehat{Q}_{\mathcal{N}}$ this follows from the decomposition (145).

Since any nilpotent $\mathcal{N}$ admits a filtration

$$0 = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \cdots \subset \mathcal{N}_\tau = \mathcal{N},$$

such that $\mathcal{N}_i^2 \subset \mathcal{N}_{i-1}$, we may by induction reduce to the case $\mathcal{N}^2 = 0$. Since in this case $\mathcal{N}$ may be equipped with nilpotent divided powers, we get the injectivity because by the lemma (83) the map $V^{-1} - \mathrm{id} : \widehat{W}(\mathcal{N}) \otimes P \to \widehat{W}(\mathcal{N}) \otimes P$ is an isomorphism. □

Corollary 85. — *Let $\mathcal{P}$ be a 3n-display over R, such that $F^N P \subset I_R P$ for some number N, then the map*

$$V^{-1} - \mathrm{id} : Q_{\mathcal{N}} \longrightarrow P_{\mathcal{N}}$$

is injective.

The proof is the same starting from lemma (83).

Proof of theorem (81) and its corollary. — For any 3n-display $\mathcal{P}$ we define a functor $\widehat{G}$ on Nil_R by the exact sequence:

$$0 \longrightarrow \widehat{Q}_{\mathcal{N}} \xrightarrow{V^{-1} - \mathrm{id}} \widehat{P}_{\mathcal{N}} \longrightarrow \widehat{G}(\mathcal{N}) \longrightarrow 0.$$

If $\mathcal{P}$ satisfies the assumption of corollary (85) we define a functor G by the exact sequence:

$$0 \longrightarrow Q_{\mathcal{N}} \xrightarrow{V^{-1} - \mathrm{id}} P_{\mathcal{N}} \longrightarrow G(\mathcal{N}) \longrightarrow 0.$$

We verify that the functors G and $\widehat{G}$ satisfy the conditions (i) – (iv) of the definition (80). It is obvious that the conditions (i) and (ii) are fulfilled, since we already remarked that the functors $\mathcal{N} \mapsto Q_{\mathcal{N}}$ (resp. $\widehat{Q}_{\mathcal{N}}$) and $\mathcal{N} \mapsto P_{\mathcal{N}}$ (resp. $\widehat{P}_{\mathcal{N}}$) are exact.

All what remains to be done is a computation of the functors t_G and $t_{\widehat{G}}$. We do something more general.

Let us assume that $\mathcal{N}$ is equipped with nilpotent divided powers. Then we define an isomorphism, which is called the exponential map

$$\exp_{\mathcal{P}} : \mathcal{N} \otimes_R P/Q \longrightarrow \widehat{G}(\mathcal{N}). \tag{155}$$

It is given by the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \widehat{Q}_{\mathcal{N}} \arrow[r] \arrow[d, equal] & \widehat{P}_{\mathcal{N}} \arrow[r] \arrow[d, "V^{-1} - \mathrm{id}"] & \mathcal{N} \otimes_R P/Q \arrow[r] \arrow[d, "\exp"] & 0 \\
0 \arrow[r] & \widehat{Q}_{\mathcal{N}} \arrow[r, "V^{-1} - \mathrm{id}"] & \widehat{P}_{\mathcal{N}} \arrow[r] & \widehat{G}(\mathcal{N}) \arrow[r] & 0.
\end{tikzcd} \tag{156}$$

If $\mathcal{N}^2 = 0$, we can take the divided powers $\gamma_k = 0$ for $i \geq 2$. Then the exponential map provides an isomorphism of the functor $t_{\widehat{G}}$ with the functor $M \mapsto M \otimes_R P/Q$ on the category of R-modules. Hence the conditions (iii) and (iv) of definition 80 are fulfilled. If the display $\mathcal{P}$ satisfies the condition $F^N \cdot P \subset I_R P$ for some number $\mathcal{N}$, we may delete the hat in diagram (156), because the middle vertical arrow remains an isomorphism by lemma (83). In fact in this case we need only to assume that $\mathcal{N}$ has divided powers. We get an isomorphism

$$\exp : \mathcal{N} \otimes P/Q \longrightarrow G(\mathcal{N}). \tag{157}$$

It follows again that $G(\mathcal{N})$ is a finite dimensional formal group. The obvious morphism $\widehat{G}(\mathcal{N}) \rightarrow G(\mathcal{N})$ is a homomorphism of formal groups, which is by (155) and (157) an isomorphism on the tangent functors $t_{\widehat{G}} \to t_G$. Hence we have an isomorphism $\widehat{G} \cong G$, which proves the theorem 81 completely. $\square$

Corollary 86. — *The functor $\mathcal{P} \mapsto BT_{\mathcal{P}}$ commutes with base change. More precisely if $\alpha : R \to S$ is a ring homomorphism base change provides us with a display $\alpha_*\mathcal{P}$ and a formal group $\alpha_* BT_{\mathcal{P}}$ over S. Then we assert that there is a canonical isomorphism:*

$$\alpha_* BT_{\mathcal{P}} \cong BT_{\alpha_*\mathcal{P}}$$

Proof. — In fact for $\mathcal{M} \in \mathrm{Nil}_S$ we have the obvious isomorphism:

$$\widehat{W}(\mathcal{M}) \otimes_{W(R)} P \cong \widehat{W}(\mathcal{M}) \otimes_{W(S)} W(S) \otimes_{W(R)} P = \widehat{W}(\mathcal{M}) \otimes_{W(S)} \alpha_* P$$

This provides the isomorphism of the corollary. $\square$

Proposition 87. — *Let R be a ring, such that $pR = 0$, and let $\mathcal{P}$ be a display over R. Then we have defined a Frobenius endomorphism (29):*

$$Fr_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{P}^{(p)}. \tag{158}$$

Let $G = BT_{\mathcal{P}}$ be the formal group we have associated to $\mathcal{P}$. Because the functor BT commutes with base change we obtain from (158) a homomorphism of formal groups:

$$BT(Fr_{\mathcal{P}}) : G \longrightarrow G^{(p)}. \tag{159}$$

Then the last map (159) is the Frobenius homomorphism Fr_G of the formal group G.

Proof. — Let $\mathcal{N} \in \mathrm{Nil}_R$ be a nilpotent R-algebra. Let $\mathcal{N}_{[p]} \in \mathrm{Nil}_R$ be the nilpotent R-algebra obtained by base change via the absolute Frobenius $\mathrm{Frob} : R \to R$. Taking the p-th power gives an R-algebra homomorphism

$$Fr_{\mathcal{N}} : \mathcal{N} \longrightarrow \mathcal{N}_{[p]}. \tag{160}$$

The Frobenius of any functor is obtained by applying it to (160). In particular the Frobenius for the functor $\widehat{W}$ is just the usual operator ^{F}:

$${}^F : \widehat{W}(\mathcal{N}) \longrightarrow \widehat{W}(\mathcal{N}_{[p]}) = \widehat{W}(\mathcal{N}).$$

From this remark we obtain a commutative diagram:

$$\begin{array}{ccc} \widehat{W}(\mathcal{N}) \otimes_{W(R)} P & \longrightarrow & G(\mathcal{N}) \\ {}^F\otimes id_P \Big\downarrow & & \Big\downarrow Fr_G \\ \widehat{W}(\mathcal{N}_{[p]}) \otimes_{W(R)} P & \longrightarrow & G(\mathcal{N}_{[p]}) \end{array} \tag{161}$$

The left lower corner in this diagram may be identified with $\widehat{W}(\mathcal{N}) \otimes_{F,W(R)} P \cong \widehat{W} \otimes_{W(R)} P^{(p)}$. All we need to verify is that for $\xi \in \widehat{W}(\mathcal{N})$ and $x \in P$ the elements ${}^F\xi \otimes x \in \widehat{W}(\mathcal{N}) \otimes_{F,W(R)} P$ and $\xi \otimes V^{\#}x \in \widehat{W}(\mathcal{N}) \otimes_{W(R)} P^{(p)}$ have the same image by the lower horizontal map of (161). Since P is generated as an abelian group by elements of the form $uV^{-1}y$, where $y \in Q$ and $u \in W(R)$, it is enough to verify the equality of the images for $x = uV^{-1}y$. But in $\widehat{W}(\mathcal{N}) \otimes_{F,W(R)} P$ we have the equalities:

$$^F\xi \otimes uV^{-1}y = {}^F(\xi u) \otimes V^{-1}y = V^{-1}(\xi u \otimes y)$$

The last element has the same image in $G(\mathcal{N}_{[p]})$ as $\xi u \otimes y$, by the exact sequence (146). Hence our proposition follows from the equality:

$$\xi \otimes V^{\#}(uV^{-1}y) = \xi u \otimes y$$

We note that here the left hand side is considered as an element of $\widehat{W} \otimes_{W(R)} P^{(p)}$, while the right hand side is considered as an element of $\widehat{W} \otimes_{F,W(R)} P$. $\square$

Proposition 88. — *Let R be a ring, such that $pR = 0$. Let $\mathcal{P}$ be a display over R. Then there is a number N and a morphism of displays $\gamma : \mathcal{P} \to \mathcal{P}^{(p^N)}$ such that the following diagram becomes commutative:*

$$\begin{tikzcd} \mathcal{P} \arrow[r, "p"] \arrow[d, "Fr^N_{\mathcal{P}}"'] & \mathcal{P} \arrow[dl, "\gamma"] \\ \mathcal{P}^{(p^N)} & \end{tikzcd}$$

Proof. — By (29) $Fr_{\mathcal{P}}$ is induced by the homomorphism $V^{\#} : P \to W(R) \otimes_{F,W(R)} P$. First we show that a power of this map factors through multiplication by p. By the definition of a display there is a number M, such that $V^{M\#}$ factors through:

$$V^{M\#} : P \to I_R \otimes_{F^M,W(R)} P \tag{162}$$

Hence the homomorphism $V^{(M+1)\#}$ is given by the composite of the following maps:
(163)

$$\begin{tikzcd} P \arrow[r, "V^{\#}"] & W(R) \otimes_{F,W(R)} P \arrow[r, "W(R) \otimes V^{M\#}"] & W(R) \otimes_{F,W(R)} I_R \otimes_{F^M,W(R)} P \arrow[d] \\ & & W(R) \otimes_{F^{M+1},W(R)} P \end{tikzcd}$$

Here the vertical arrow is induced by the map $W(R) \otimes_{F,W(R)} I_R \to W(R)$ such that $\xi \otimes \zeta \mapsto \xi {}^F\zeta$. We note that this map is divisible by p., because there is also the map $\kappa : W(R) \otimes_{F,W(R)} I_R \to W(R)$ given by $\xi \otimes {}^V\eta \mapsto \xi\eta$. Composing the horizontal maps in the diagram (163) with κ we obtain a map $\gamma_0 : P \to W(R) \otimes_{F^{M+1},W(R)} P$, such that $\gamma_0 p = V^{(M+1)\#}$. For any number m we set $\gamma_m = V^{m\#}\gamma_0$. Then we have $\gamma_m p = V^{(M+m+1)\#}$.

Secondly we claim that for a big number m the homomorphism γ_m induces a homomorphism of displays. It follows from the factorization (162) that γ_M respects the Hodge filtration. We have to show that for $m \geq M$ big enough the following ^{F} -linear maps are zero:

$$F\gamma_m - \gamma_m F, \quad V^{-1}\gamma_m - \gamma_m V^{-1} \tag{164}$$

These maps become 0, if we multiply them by p. But the kernel of multiplication by p on $W(R) \otimes_{F^m, W(R)} P$ is $W(\mathfrak{a}) \otimes_{F^m, W(R)} P$, where $\mathfrak{a}$ is the kernel of the absolute Frobenius homomorphism $\mathrm{Frob} : R \to R$. Because $W(\mathfrak{a}) I_R = 0$, we conclude that the composite of the following maps induced by (162) is zero:

$$W(\mathfrak{a}) \otimes_{F^m, W(R)} P \to W(\mathfrak{a}) \otimes_{F^m, W(R)} I_R \otimes_{F^M, W(R)} P \to W(R) \otimes_{F^{M+m}, W(R)} P$$

Hence γ_{2M} commutes with F and V^{-1} and is therefore a morphism of displays. This is the morphism γ we were looking for. □

Applying the functor BT to the diagram in the proposition we get immediately that $BT_{\mathcal{P}}$ is a p-divisible group. If p is nilpotent in R a formal group over R is p-divisible, iff its reduction mod p is p-divisible. Hence we obtain:

Corollary 89. — *Let p be nilpotent in R, and let $\mathcal{P}$ be a display over R. Then $BT_{\mathcal{P}}$ is a p-divisible group.*

We will now compute the Cartier module of the formal group $BT_{\mathcal{P}}$. By definition the Cartier ring $\mathbb{E}_R$ is the ring opposite to the ring $\mathrm{Hom}(\widehat{W}, \widehat{W})$. Any element $e \in \mathbb{E}_R$ has a unique representation:

$$e = \sum_{n,m \geq 0} V^n [a_{n,m}] F^m,$$

where $a_{n,m} \in R$ and for any fixed n the coefficients $a_{n,m} = 0$ for almost all m. We write the action $e : \widehat{W}(\mathcal{N}) \to \widehat{W}(\mathcal{N})$ as right multiplication. It is defined by the equation:

$$ue = \sum_{m,n \geq 0} {}^{V^m}([a_{n,m}]({}^{F^n}u)) \tag{165}$$

One can show by reducing to the case of a $\mathbb{Q}$-algebra that ${}^{F^n}u = 0$ for big n. Hence this sum is in fact finite.

Let G be a functor from Nil_R to the category of abelian groups, such that $G(0) = 0$. The Cartier module of G is the abelian group:

$$\mathbf{M}(G) = \mathrm{Hom}(\widehat{W}, G), \tag{166}$$

with the left $\mathbb{E}_R$-module structure given by:

$$(e\phi)(u) = \phi(ue), \quad \phi \in \mathbf{M}(G),\ u \in \widehat{W}(\mathcal{N}),\ e \in \mathbb{E}_R$$

Let P be a projective finitely generated $W(R)$-module. Let us denote by $\mathbf{G}_P$ the functor $\mathcal{N} \mapsto \widehat{W}(\mathcal{N}) \otimes_{W(R)} P$. Then we have a canonical isomorphism :

$$\mathbb{E}_R \otimes_{W(R)} P \to \mathrm{Hom}(\widehat{W}, \mathbf{G}_P) = \mathbf{M}(\mathbf{G}_P) \tag{167}$$

An element $e \otimes x$ from the left hand side is mapped to the homomorphism $u \mapsto ue \otimes x \in \widehat{W}(\mathcal{N}) \otimes_{W(R)} P$.

Proposition 90. — *Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a 3n-display over R. By definition (146) we have a natural surjection of functors $\mathbf{G}_P \to BT_{\mathcal{P}}$. It induces a surjection of Cartier modules:*

$$\mathbb{E}_R \otimes_{W(R)} P \longrightarrow \mathbf{M}(BT_{\mathcal{P}}) \tag{168}$$

The kernel of this map is the $\mathbb{E}_R$-submodule generated by the elements $F \otimes x - 1 \otimes Fx$, for $x \in P$, and $V \otimes V^{-1}y - 1 \otimes y$, for $y \in Q$.

Proof. — We set $\mathbf{G}^0_{\mathcal{P}} = \mathbf{G}_P$ and we denote by $\mathbf{G}^{-1}_{\mathcal{P}}$ the subfunctor $\mathcal{N} \mapsto \widehat{Q}_{\mathcal{N}}$. Let us denote the corresponding Cartier modules by $\mathbf{M}^0_{\mathcal{P}}$ respectively $\mathbf{M}^{-1}_{\mathcal{P}}$. By the first main theorem of Cartier theory, we obtain from (146) an exact sequence of Cartier modules:

$$0 \longrightarrow \mathbf{M}^{-1}_{\mathcal{P}} \xrightarrow{\rho_{\mathcal{P}}} \mathbf{M}^0_{\mathcal{P}} \longrightarrow \mathbf{M}(BT_{\mathcal{P}}) \longrightarrow 0 \tag{169}$$

We have to compute $\rho_{\mathcal{P}}$ explicitly. Using a normal decomposition $P = L \oplus T$ we may write:

$$\mathbf{G}^{-1}_{\mathcal{P}}(\mathcal{N}) = \widehat{W}(\mathcal{N}) \otimes_{W(R)} L \oplus \widehat{I}_{\mathcal{N}} \otimes_{W(R)} T$$

The Cartier module of the last direct summand may be written as follows:

$$\begin{aligned} \mathbb{E}_R F \otimes_{W(R)} T &\longrightarrow \operatorname{Hom}(\widehat{W}, \widehat{I} \otimes_{W(R)} T) \\ eF \otimes t &\longmapsto (u \mapsto ueF \otimes t) \end{aligned} \tag{170}$$

From this we easily see that $\mathbf{M}^{-1}_{\mathcal{P}} \subset \mathbf{M}^0_{\mathcal{P}}$ is the subgroup generated by all elements $eF \otimes x$, where $e \in \mathbb{E}_R$ and by all elements $e \otimes y$, where $e \in \mathbb{E}_R$ and $y \in Q$.

The map $V^{-1} : \mathbf{G}^{-1}_{\mathcal{P}} \to \mathbf{G}^0_{\mathcal{P}}$ is defined by the equations:

$$\begin{aligned} V^{-1}(u \otimes y) &= uV \otimes V^{-1}y, \quad u \in \widehat{W},\ (\mathcal{N})\ y \in Q \\ V^{-1}(uF \otimes x) &= u \otimes Fx, \qquad\qquad\qquad\quad x \in P \end{aligned} \tag{171}$$

Hence on the Cartier modules $V^{-1} - id$ induces a map $\rho_{\mathcal{P}} : \mathbf{M}^{-1}_{\mathcal{P}} \to \mathbf{M}^0_{\mathcal{P}}$, which satisfies the equations:

$$\begin{aligned} \rho_{\mathcal{P}}(eF \otimes x) &= e \otimes Fx - eF \otimes x, \qquad x \in P \\ \rho_{\mathcal{P}}(e \otimes y) &= eV \otimes V^{-1}y - e \otimes y, \quad y \in Q \end{aligned} \tag{172}$$

This proves the proposition. □

3.2. The universal extension. — Grothendieck and Messing have associated to a p-divisible group G over R a crystal $\mathbb{D}_G$, which we will now compare with the crystal $\mathcal{D}_{\mathcal{P}}$, if $\mathcal{P}$ is a display with associated formal p-divisible group $G = BT(\mathcal{P})$.

Let us first recall the theory of the universal extension [Me] in terms of Cartier theory [Z2].

Let S be a $\mathbb{Z}_p$-algebra and L an S-module. We denote by $C(L) = \prod_{i=0}^{\infty} V^i L$, the abelian group of all formal power series in the indeterminate V with coefficients in L.

We define on $C(L)$ the structure of an $\mathbb{E}_S$-module by the following equations

$$\begin{aligned}
\xi\Big(\sum_{i=0}^{\infty} V^i l_i\Big) &= \sum_{i=0}^{\infty} V^i w_n(\xi) l_i, \quad \text{for } \xi \in W(S), l_i \in L \\
V\Big(\sum_{i=0}^{\infty} V^i l_i\Big) &= \sum_{i=0}^{\infty} V^{i+1} l_i \\
F\Big(\sum_{i=0}^{\infty} V^i l_i\Big) &= \sum_{i=1}^{\infty} V^{i-1} p l_i
\end{aligned}$$

The module $C(L)$ may be interpreted as the Cartier module of the additive group of L:

Let $\widehat{L}^+$ be the functor on the category Nil_S of nilpotent S-algebras to the category of abelian groups, which is defined by

$$\widehat{L}^+(\mathcal{N}) = (\mathcal{N} \otimes_S L)^+.$$

Then one has a functor isomorphism:

$$\begin{array}{ccc}
\mathcal{N} \otimes_S L & \cong & \widehat{W}(\mathcal{N}) \otimes_{\mathbb{E}_S} C(L) \\
n \otimes l & \longmapsto & [n] \otimes V^0 l
\end{array}$$

Consider a *pd*-thickening $S \to R$ with kernel $\mathfrak{a}$. Let G be a p-divisible formal group over R and $M = M_G = \mathbf{M}(G)$ be its Cartier module (166), which we will regard as an $\mathbb{E}_S$-module.

Definition 91. — An extension (L, N) of M by the S-module L is an exact sequence of $\mathbb{E}_S$-modules

$$0 \longrightarrow C(L) \longrightarrow N \longrightarrow M \longrightarrow 0, \tag{173}$$

such that N is a reduced $\mathbb{E}_S$-module, and $\mathfrak{a}N \subset V^0 L$, where $\mathfrak{a} \subset W(S) \subset \mathbb{E}_S$ is the ideal in $W(S)$ defined after (48).

Remark. — We will denote $V^0 L$ simply by L and call it the submodule of exponentials of $C(L)$ respectively N. A morphism of extensions $(L, N) \to (L', N')$ consists of a morphism of S-modules $\varphi : L \to L'$ and a homomorphism of $\mathbb{E}_S$-modules $u : N \to N'$ such that the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & C(L) \arrow[r] \arrow[d, "C(\varphi)"'] & N \arrow[r] \arrow[d, "u"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C(L') \arrow[r] & N' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

More geometrically an extension as in definition 91 is obtained as follows. Let $\widetilde{G}$ be a lifting of the p-divisible formal group G to a p-divisible formal group over S, which may be obtained by lifting the display $\mathcal{P}$ to S. Let $\mathbb{W}$ be the vector group associated to a locally free finite S-module W. Consider an extension of f.p.p.f. sheaves over $\operatorname{Spec} S$:

$$0 \longrightarrow \mathbb{W} \longrightarrow E \longrightarrow \widetilde{G} \longrightarrow 0 \tag{174}$$

The formal completion of (174) is an exact sequence of formal groups (i.e. a sequence of formal groups, such that the corresponding sequence of Lie algebras is exact). Hence we have an exact sequence of Cartier modules.

$$0 \longrightarrow C(W) \longrightarrow M_{\widehat{E}} \longrightarrow M_{\widetilde{G}} \longrightarrow 0,$$

$\widehat{E}$ being the formal completion of E.

We have $\mathfrak{a}M_{\widehat{E}} \simeq \mathfrak{a} \otimes_S \operatorname{Lie} E$. We let $L = W + \mathfrak{a}\operatorname{Lie} E$ as submodule of $\operatorname{Lie} E$ or equivalently of $M_{\widehat{E}}$. Since L is killed by F we obtain an exact sequence

$$0 \longrightarrow C(L) \longrightarrow M_{\widehat{E}} \longrightarrow M_G \longrightarrow 0,$$

which is an extension in the sense of definition 91. Conversely we can start with a sequence (173). We choose a lifting of M/VM to a locally free S-module P. Consider any map ρ making the following diagram commutative.

$$\begin{tikzcd} N/VN \arrow[r] \arrow[dr, "\rho"'] & M/VM \\ & P \arrow[u] \end{tikzcd} \tag{175}$$

Let $W = \ker \rho$. Then $L = W + \mathfrak{a}(N/VN)$ as a submodule of $\operatorname{Lie} N$. The quotient of N by $C(W)$ is a reduced $\mathbb{E}_S$-module and hence the Cartier module of a formal group $\widetilde{G}$ over S, which lifts G. We obtain an extension of reduced $\mathbb{E}_S$-modules

$$0 \longrightarrow C(W) \longrightarrow N \longrightarrow M_{\widetilde{G}} \longrightarrow 0,$$

and a corresponding extension of formal groups over S

$$0 \longrightarrow \widehat{W}^+ \longrightarrow \widehat{E} \longrightarrow \widetilde{G} \longrightarrow 0.$$

Then the push–out by the natural morphism $\widehat{W}^+ \to \mathbb{W}$ is an extension of f.p.p.f. sheaves (174).

These both constructions are inverse to each other. Assume we are given two extensions $(\mathbb{W}, E, \widetilde{G})$ and $(\mathbb{W}_1, \mathbb{E}_1, \widetilde{G}_1)$ of the form (174). Then a morphism between the corresponding extensions of Cartier modules in the sense of definition 91 may be geometrically described as follows. The morphism consists of a pair (u, v_R), where $u : E \to E_1$ is a morphism of f.p.p.f. sheaves and $v_R : \mathbb{W}_R \to \mathbb{W}_{1,R}$ a homomorphism of vector groups over R. The following conditions are satisfied.

1) We have a commutative diagram for the reductions over R:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{W}_R & \longrightarrow & E_R & \longrightarrow & G & \longrightarrow & 0 \\ & & \Big\downarrow v_R & & \Big\downarrow u & & \Big\| & & \\ 0 & \longrightarrow & \mathbb{W}_{1,R} & \longrightarrow & E_{1,R} & \longrightarrow & G_1 & \longrightarrow & 0 \end{array}$$

2) For any lifting $\widetilde{v} : \mathbb{W} \to \mathbb{W}_1$ of v_R to a homomorphism of vector groups the map:

$$\widetilde{v} - u_{|\widehat{\mathbb{W}}} : \widehat{\mathbb{W}}. \longrightarrow \widehat{E}_1$$

factors through a linear map $W \to \mathfrak{a} \otimes \mathrm{Lie} E_1$:

$$\widehat{\mathbb{W}} \longrightarrow (\mathfrak{a} \otimes \mathrm{Lie} E_1)^{\wedge} \xrightarrow{\exp} \widehat{E}_1.$$

Here the second map is given by the natural inclusion of Cartier modules $C(\mathfrak{a} M_{\widehat{E}_1}) \subset M_{\widehat{E}_1}$ or equivalently by the procedure in Messing's book [Me] (see [Z2]). This dictionary between extensions used by Messing and extensions of Cartier modules in the sense of definition 91, allows us to use a result of Messing in a new formulation:

Theorem 92. — *Let $S \to R$ be a pd-thickening with nilpotent divided powers. Let G be a formal p-divisible group over R. Then there exists a universal extension $(L^{\mathrm{univ}}, N^{\mathrm{univ}})$ of G by a S-module L^{univ}.*

Then any other extension (L, N) in sense of definition 91 is obtained by a unique S-module homomorphism $L^{\mathrm{univ}} \to L$.

Proof. — This is [Me] Chapt. 4 theorem 2.2. □

Remark. — The definition of the universal extension over S is based on the exponential map

$$\exp : (\mathfrak{a} \otimes \mathrm{Lie} E)^{\wedge} \longrightarrow E^{\wedge},$$

which we simply defined using Cartier theory and the inclusion $\mathfrak{a} \subset W(S)$ given by the divided powers on $\mathfrak{a}$. In the case of a formal p-divisible group it makes therefore sense to ask whether Messing's theorem 92 makes sense for any pd-thickening and not just nilpotent ones. We will return to this question in proposition 96

Since we consider p-divisible groups without an étale part, this theorem should be true without the assumption that the divided powers are nilpotent. This would simplify our arguments below. But we don't know a reference for this.

The crystal of Grothendieck and Messing deduced from this theorem is defined by

$$\mathbb{D}_G(S) = \mathrm{Lie} N^{\mathrm{univ}}.$$

Lemma 93. — *Let $S \to R$ be a pd-thickening with nilpotent divided powers. Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a display over R. By proposition 44 there exist up to canonical isomorphism a unique triple $(\widetilde{P}, F, V^{-1})$, which lifts (P, F, V^{-1}), such that V^{-1} is defined on the inverse image $\widehat{Q} \subset \widetilde{P}$ of Q.*

Then the universal extension of $BT(\mathcal{P})$ is given by the following exact sequence of $\mathbb{E}_S$-modules

$$0 \longrightarrow C(\widehat{Q}/I_S\widetilde{P}) \longrightarrow \mathbb{E}_S \otimes_{W(S)} \widetilde{P}/(F \otimes x - 1 \otimes Fx)_{x \in \widetilde{P}} \longrightarrow M(\mathcal{P}) \longrightarrow 0, \tag{176}$$

where the second arrow maps $y \in \widehat{Q}$ to $V \otimes V^{-1}y - 1 \otimes y$, and the third arrow is given by the canonical map $\widetilde{P} \longrightarrow P$.

Proof. — By [Z1] the $\mathbb{E}_S$-module N in the middle of the sequence (176) is a reduced Cartier module, and the canonical map $\widetilde{P} \to \mathbb{E}_S \otimes_{W(S)} \widetilde{P}$, $x \mapsto 1 \otimes x$ provides an isomorphism $\widetilde{P}/I_S\widetilde{P} \simeq N/VN$.

Let us verify that the arrow $C(\widehat{Q}/I_S\widetilde{P}) \to N$ in the sequence (176) is well-defined. Clearly $y \mapsto V \otimes V^{-1}y - 1 \otimes y$ is a homomorphism of abelian groups $\widehat{Q} \to N$. The subgroup $I_S\widetilde{P}$ is in the kernel:

$$\begin{aligned} V \otimes V^{-1}\,{}^{V}wx - 1 \otimes {}^{V}wx &= V \otimes wFx - 1 \otimes {}^{V}wx \\ &= VwF \otimes x - 1 \otimes {}^{V}wx = {}^{V}wx - 1 \otimes {}^{V}wx = 0, \end{aligned}$$

for $w \in W(S), x \in \widetilde{P}$.

Moreover one verifies readily that $F(V \otimes V^{-1}y - 1 \otimes y) = 0$ in N. Then the image of $\widehat{Q} \longrightarrow N$ is in a natural way an S-module, $\widehat{Q}/I_S\widetilde{P} \to N$ is an S-module homomorphism, and we have a unique extension of the last map to a $\mathbb{E}_S$-module homomorphism

$$C(\widehat{Q}/I_S\widetilde{P}) \longrightarrow N.$$

We see that (176) is a complex of V-reduced $\mathbb{E}_S$-modules. Therefore it is enough to check the exactness of the sequence (176) on the tangent spaces, which is obvious.

We need to check that (176) is an extension in the sense of definition 91, i.e. $\mathfrak{a} \cdot N \subset \widehat{Q}/I_S\widetilde{P}$, where $\widehat{Q}/I_S\widetilde{P}$ is regarded as a subgroup of N by the second map of (176) and $\mathfrak{a} \subset W(S)$ as an ideal.

Indeed, let $a \in \mathfrak{a}$, $x \in \widetilde{P}$ and $\xi = \sum V^i[\xi_{ij}]F^j \in \mathbb{E}_S$. Then $a\xi \otimes x = a\sum_j [\xi_{0j}]F^j \otimes x = 1 \otimes a \sum [\xi_{0j}]F^j x$. Hence it is enough to verify that an element of the form $1 \otimes ax$ is in the image of $\widehat{Q} \to N$. But we have

$$V \otimes V^{-1}ax - 1 \otimes ax = -1 \otimes ax.$$

It remains to be shown that the extension (176) is universal. Let

$$0 \longrightarrow C(L^{\mathrm{univ}}) \longrightarrow N^{\mathrm{univ}} \longrightarrow M(\mathcal{P}) \longrightarrow 0$$

be the universal extension. For any lifting of $M(\mathcal{P})$ to a reduced Cartier module $\widetilde{M}$ over S, there is a unique morphism $N^{\mathrm{univ}} \longrightarrow \widetilde{M}$, which maps L^{univ} to $\mathfrak{a} \cdot \widetilde{M}$. Let $\widetilde{L}$ be the kernel of $L^{\mathrm{univ}} \longrightarrow \mathfrak{a}\widetilde{M}$. Then it is easy to check that

$$0 \longrightarrow C(\widetilde{L}) \longrightarrow N^{\mathrm{univ}} \longrightarrow \widetilde{M} \longrightarrow 0 \tag{177}$$

is the universal extension of $\widetilde{M}$. Hence conversely starting with a universal extension (177) of $\widetilde{M}$, we obtain the universal extension of M over S as

$$0 \longrightarrow C(\widetilde{L} + \mathfrak{a}N^{\mathrm{univ}}) \longrightarrow N^{\mathrm{univ}} \longrightarrow M \longrightarrow 0,$$

where the sum $\widetilde{L} + \mathfrak{a}N^{\mathrm{univ}}$ is taken in $\mathrm{Lie} N^{\mathrm{univ}}$.

Now let $\widetilde{Q} \subset \widehat{Q}$ be an arbitrary $W(S)$-submodule, such that $\widetilde{\mathcal{P}} = (\widetilde{P}, \widetilde{Q}, F, V^{-1})$ is a display. By the consideration above it suffices to show that

$$0 \longrightarrow C(\widetilde{Q}/I_S\widetilde{P}) \longrightarrow N \longrightarrow M(\widetilde{\mathcal{P}}) \longrightarrow 0 \tag{178}$$

is the universal extension of $M(\widetilde{\mathcal{P}})$ over S. In other words, we may assume $R = S$.

Starting from the universal extension (177) for $\widetilde{M} = M(\widetilde{\mathcal{P}})$, we get a morphism of finitely generated projective modules $\widetilde{L} \to \widetilde{Q}/I_S\widetilde{P}$. To verify that this is an isomorphism it suffices by the lemma of Nakayama to treat the case, where $S = R$ is a perfect field. In this case we may identify $M(\widetilde{\mathcal{P}})$ with $\widetilde{P}$. The map $\widetilde{P} \to \mathbb{E}_S \otimes_{W(S)} \widetilde{P}$, $x \mapsto 1 \otimes x$ induces the unique $W(S)[F]$-linear section σ of

$$\begin{tikzcd} 0 \arrow[r] & C(\widetilde{Q}/I_S\widetilde{P}) \arrow[r] & N \arrow[r] & \widetilde{P} \arrow[l, bend right, "\sigma"'] \arrow[r] & 0, \end{tikzcd}$$

such that $V\sigma(x) - \sigma(Vx) \in \widetilde{Q}/I_S\widetilde{P}$ (compare [Z1], 2, 2.5 or [Ra-Zi] 5.26). The extension is classified up to isomorphism by the induced map $\sigma : \widetilde{P} \to N/VN$. Since this last map is $\widetilde{P} \to \widetilde{P}/I_S\widetilde{P}$ the extension is clearly universal. $\square$

Our construction of the universal extension (176) makes use of the existence of the triple $(\widetilde{P}, F, V^{-1})$. If we have a pd-morphism $\varphi : W(R) \to S$, we know how to write down this triple explicitly (corollary 56). Hence we obtain in this case a complete description of the universal extension over S only in terms of (P, Q, F, V^{-1}). Indeed, let $\overline{Q}_\varphi$ be the inverse image of Q/IP be the map

$$S \otimes_{W(R)} P \longrightarrow R \otimes_{W(R)} P.$$

Then the universal extension is given by the sequence

$$0 \longrightarrow C(\overline{Q}_\varphi) \longrightarrow \mathbb{E}_S \otimes_{W(R)} P/(F \otimes x - 1 \otimes Fx)_{x\in P} \longrightarrow M(\mathcal{P}) \longrightarrow 0, \tag{179}$$

where the tensor product with $\mathbb{E}_S$ is given by $\delta_\varphi : W(R) \to W(S)$ (compare (96)). The second arrow is defined as follows. For an element $\overline{y} \in \overline{Q}_\varphi$ we choose a lifting $y \in Q_\varphi \subset W(S) \otimes_{W(R)} P$. Then we write:

$$1 \otimes y \in \mathbb{E}_S \otimes_{W(S)} (W(S) \otimes_{W(R)} P) = \mathbb{E}_S \otimes_{W(R)} P$$

With this notation the image of $\overline{y}$ by the second arrow of (179) is $V \otimes V_\varphi^{-1} y - 1 \otimes y$.

One may specialize this to the case of the *pd*-thickening $S = W_m(R) \to R$, and then go to the projective limit $W(R) = \varprojlim_m W_m(R)$. Then the universal extension over $W(R)$ takes the remarkable simple form:

$$\begin{array}{ccc} 0 \longrightarrow C(Q) \longrightarrow \mathbb{E}_{W(R)} \otimes_{W(R)} P/(F \otimes x - 1 \otimes Fx)_{x \in P} \longrightarrow M(\mathcal{P}) \longrightarrow 0 \\ y \longmapsto \quad V \otimes V^{-1}y - 1 \otimes y \end{array} \tag{180}$$

3.3. Classification of p-divisible formal groups. — The following main theorem gives the comparison between Cartier theory and the crystalline Dieudonné theory of Grothendieck and Messing.

Theorem 94. — *Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a display over a ring R, such that p is nilpotent in R. Let $G = BT(\mathcal{P})$ be the associated formal p-divisible group. Then there is a canonical isomorphism of crystals on the crystalline site of nilpotent pd-thickenings over* Spec R*:*

$$\mathcal{D}_{\mathcal{P}} \xrightarrow{\sim} \mathbb{D}_G$$

It respects the Hodge filtration on $\mathcal{D}_{\mathcal{P}}(R)$ respectively $\mathbb{D}_G(R)$.

Let $S \to R$ be a pd-thickening with nilpotent divided powers. Assume we are given a morphism $W(R) \to S$ of topological pd-thickenings of R. Then there is a canonical isomorphism:

$$S \otimes_{W(R)} P \cong \mathbb{D}_G(S).$$

Remark. — We will remove the restriction to the nilpotent crystalline site below (corollary 97).

Proof. — In the notation of lemma 93 we find $\mathcal{D}_{\mathcal{P}}(S) = \widetilde{P}/I_S\widetilde{P}$ and this is also the Lie algebra of the universal extension of G over S, which is by definition the value of the crystal $\mathbb{D}_G$ at S. □

Corollary 95. — *Let $S \to R$ be a surjective ring homomorphism with nilpotent kernel. Let $\mathcal{P}$ be a display over R and let G be the associated formal p-divisible group. Let $\widetilde{G}$ be a formal p-divisible group over S, which lifts G. Then there is a lifting of $\mathcal{P}$ to a display $\widetilde{\mathcal{P}}$ over S, and an isomorphism $BT(\widetilde{\mathcal{P}}) \to \widetilde{G}$, which lifts the identity $BT(\mathcal{P}) \to G$.*

Moreover let $\mathcal{P}'$ be a second display over R, and let $\alpha : \mathcal{P} \to \mathcal{P}'$ be a morphism. Assume we are given a lifting $\widetilde{\mathcal{P}'}$ over S of $\mathcal{P}'$. We denote the associated formal p-divisible groups by $\widetilde{G}'$ respectively G'. Then the morphism α lifts to a morphism of displays $\widetilde{\mathcal{P}} \to \widetilde{\mathcal{P}'}$, iff $BT(\alpha) : G \to G'$ lifts to a homomorphism of formal p- divisible groups $\widetilde{G} \to \widetilde{G}'$.

Proof. — Since $S \to R$ may be represented as a composition of nilpotent pd-thickenings, we may assume that $S \to R$ itself is a nilpotent pd-thickening. Then the left hand side of the isomorphism of theorem 94 classifies liftings of the display $\mathcal{P}$ by theorem 48 and the right hand side classifies liftings of the formal p-divisible group G by Messing [Me] Chapt V theorem (1.6). Since the constructions are functorial in $\mathcal{P}$ and G the corollary follows. □

Proposition 96. — *Let $\mathcal{P}$ be a display over R. Let $S \to R$ be a pd-thickening with nilpotent kernel $\mathfrak{a}$. Then the extension of lemma 93 is universal (i.e. in the sense of the remark after Messing's theorem 92).*

Proof. — We denote by G the formal p-divisible group associated to $\mathcal{P}$. Any lifting $\widetilde{G}$ of G to S gives rise to an extension of M_G in the sense of definition 91:

$$0 \longrightarrow C(\mathfrak{a}M_{\widetilde{G}}) \longrightarrow M_{\widetilde{G}} \longrightarrow M_G \longrightarrow 0$$

With the notation of the proof of lemma 93 we claim that there is a unique morphism of extensions $N \to M_{\widetilde{G}}$. Indeed, the last corollary shows that $\widetilde{G}$ is the p-divisible group associated to a display $\widetilde{\mathcal{P}}(\widetilde{G})$ which lifts the display $\mathcal{P}$. Hence $\widetilde{\mathcal{P}}(\widetilde{G})$ is of the form $(\widetilde{P}, \widetilde{Q}, F, V^{-1})$, where $(\widetilde{P}, F, V^{-1})$ is the triple in the formulation of lemma 93. But then the description of the Cartier module $M_{\widetilde{G}}$ in terms of the display gives immediately a canonical morphism of Cartier modules $N \to M_{\widetilde{G}}$. Its kernel is $C(L)$, where L is the kernel of the map $\widetilde{P}/I_S\widetilde{P} \to \operatorname{Lie}\widetilde{G}$, i.e. the Hodge filtration determined by $\widetilde{G}$. This shows the uniqueness of $N \to M_{\widetilde{G}}$.

Now let us consider any extension:

$$0 \longrightarrow C(L_1) \longrightarrow N_1 \longrightarrow M(\mathcal{P}) \longrightarrow 0$$

Using the argument (175), we see that there is a lifting $\widetilde{G}$ of G, such that the extension above is obtained from

$$0 \longrightarrow C(U_1) \longrightarrow N_1 \longrightarrow M_{\widetilde{G}} \longrightarrow 0.$$

Let $\widetilde{Q} \subset \widetilde{P}$ be the display which corresponds to $\widetilde{G}$ by the last corollary. Then by lemma 93 the universal extension of $M_{\widetilde{G}}$ is :

$$0 \longrightarrow C(\widetilde{Q}/I_S\widetilde{P}) \longrightarrow N \longrightarrow M_{\widetilde{G}} \longrightarrow 0$$

This gives the desired morphism $N \to N_1$. It remains to show the uniqueness. But this follows because for any morphism of extensions $N \to N_1$ the following diagram is commutative:

$$\begin{array}{ccc} N & \longrightarrow & N_1 \\ \big\downarrow & & \big\downarrow \\ M_{\widetilde{G}} & \longrightarrow & M_{\widetilde{G}} \end{array}$$

Indeed we have shown, that the morphism of extensions $N \to M_{\widetilde{G}}$ is unique. □

Remark. — Let $\mathcal{P}$ be the display of a p-divisible formal group G. Then we may extend the definition of the crystal $\mathbb{D}_G$ to all pd-thickenings $S \to R$ (not necessarily nilpotent) whose kernel is a nilpotent ideal, by setting:

$$\mathbb{D}_G(S) = \operatorname{Lie} E_S,$$

where E_S is the universal extension of G over S, which exist by the proposition above.

This construction is functorial in the following sense. Let $\mathcal{P}'$ be another display over R and denote the associated formal p-divisible group by G'. Then any homomorphism $a : G \to G'$ induces by the universality of the universal extension a morphism of crystals:

$$\mathbb{D}(a) : \mathbb{D}_G \longrightarrow \mathbb{D}_{G'}.$$

Corollary 97. — *If we extend $\mathbb{D}_G$ to the whole crystalline site as above, the theorem 94 continues to hold, i.e. we obtain a canonical isomorphism of crystals:*

$$\mathcal{D}_{\mathcal{P}} \longrightarrow \mathbb{D}_G \tag{181}$$

Proof. — This is clear. □

Proposition 98. — *The functor BT from the category of displays over R to the category of formal p-divisible groups over R is faithful, i.e. if $\mathcal{P}$ and $\mathcal{P}'$ are displays over R, the map*

$$\operatorname{Hom}(\mathcal{P}, \mathcal{P}') \longrightarrow \operatorname{Hom}(BT(\mathcal{P}), BT(\mathcal{P}'))$$

is injective.

Proof. — Let $\mathcal{P} = (P, Q, F, V^{-1})$ and $\mathcal{P}' = (P', Q', F, V^{-1})$ be the displays and G and G' the associated p-divisible groups. Assume $\alpha : \mathcal{P} \to \mathcal{P}'$ is a morphism of displays. It induces a morphism $a : G \to G'$.

But the last corollary gives α back if we apply to a the functor $\mathbb{D}$:

$$\mathbb{D}_G(W(R)) \longrightarrow \mathbb{D}_{G'}(W(R)).$$ □

Proposition 99. — *Let p be nilpotent in R and assume that the set of nilpotent elements in R form a nilpotent ideal. Then the functor BT of proposition 98 is fully faithful.*

We need a preparation before we can prove this.

Lemma 100. — *Let $\mathcal{P}$ and $\mathcal{P}'$ be displays over R. Let $a : G \to G'$ be a morphism of the associated p-divisible groups over R. Assume that there is an injection $R \to S$ of rings, such that $a_S : G_S \to G'_S$ is induced by a morphism of displays $\beta : \mathcal{P}_S \to \mathcal{P}'_S$. Then a is induced by a morphism of displays $\alpha : \mathcal{P} \to \mathcal{P}'$.*

Proof. — The morphism $W(R) \to R$ is a pd-thickening. By the corollary 97 a induces a map $\alpha : P \to P'$, namely the map induced on the Lie algebras of the universal extensions (180). Therefore α maps Q to Q'. By assumption the map $\beta = W(S) \otimes_{W(R)} \alpha : W(S) \otimes_{W(R)} P \to W(S) \otimes_{W(R)} P'$ commutes with F and V^{-1}. Then the same is true for α because of the inclusions $P \subset W(S) \otimes_{W(R)} P$, $P' \subset W(S) \otimes_{W(R)} P'$. Hence α is a morphism of displays. By proposition 98 $BT(\alpha)$ is a. □

Proof of the proposition. — If $R = K$ is a perfect field, the proposition is true by classical Dieudonné theory. For any field we consider the perfect hull $K \subset K^{\mathrm{perf}}$ and apply the last lemma. Next assume that $R = \prod_{i \in I} K_i$ is a product of fields. We denote the base change $R \to K_i$ by an index i. A morphism of p-divisible groups

$G \to G'$, is the same thing as a family of morphisms of p-divisible groups $G_i \to G'_i$ over each K_i. Indeed, one can think of G in terms of systems of finite locally free group schemes. Then one needs only to observe that any finitely generated projective module L over R is of the form $\prod L_i$, since it is a direct summand of R^n. Next one observes that the same statements are true for morphisms of displays $\mathcal{P} \to \mathcal{P}'$, because $W(R) = \prod W(K_i)$ etc. Hence the case where R is a product of fields is established. Since a reduced ring may be embedded in a product of fields we may apply the lemma to this case. The general case follows from corollary 95 if we divide out the nilpotent ideal of nilpotent elements. □

We now give another criterion for the fully faithfulness of the functor BT, which holds under slightly different assumptions.

Proposition 101. — *Let R be an $\mathbb{F}_p$-algebra. We assume that there exists a topological pd-thickening $(S, \mathfrak{a}_n)$ of R, such that the kernels of $S/\mathfrak{a}_n \to R$ are nilpotent, and such that S is a p-adic torsion free ring.*

Then the functor BT from the category of displays over R to the category of p-divisible formal groups is fully faithful.

Proof. — Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be displays over R, and let G_1 and G_2 be the p-divisible formal groups associated by the functor BT. We show that a given homomorphism of p-divisible groups $a : G_1 \to G_2$ is induced by a homomorphism of displays $\mathcal{P}_1 \to \mathcal{P}_2$.

The homomorphism a induces a morphism of filtered F-crystals $a_{\mathcal{D}} : \mathbb{D}_{G_1} \to \mathbb{D}_{G_2}$ on the crystalline site. Since we have identified (corollary 97) the crystals $\mathbb{D}$ and $\mathcal{D}$ on this site, we may apply proposition 60 to obtain a homomorphism $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ of displays. We consider the triples $(\widetilde{P_1}, F, V^{-1})$ and $(\widetilde{P_2}, F, V^{-1})$, which are associated to $\mathcal{P}_1$ and $\mathcal{P}_2$, and the unique lifting of ϕ to a homomorphism $\widetilde{\phi}$ of these triples. Then $\mathbb{D}_{G_i}(S)$ is identified with $\widetilde{P}_i/I_S\widetilde{P}_i$ for $i = 1, 2$. Let $\mathbb{E}_{1,S}$ and $\mathbb{E}_{2,S}$ be the universal extensions of G_1 and G_2 over S. By the proposition loc.cit. the homomorphism $a_{\mathcal{D}}(S) : \operatorname{Lie} \mathbb{E}_{1,S} \to \operatorname{Lie} \mathbb{E}_{2,S}$ coincides with the identifications made, with the homomorphism induced by $\widetilde{\phi}$:

$$\widetilde{\phi} : \widetilde{P}_1/I_S\widetilde{P}_1 \longrightarrow \widetilde{P}_2/I_S\widetilde{P}_2$$

Let us denote by $b : G_1 \to G_2$ the homomorphism $BT(\phi)$. Then by theorem 94 b induces on the crystals the same morphism as ϕ.

The two maps $\mathbb{E}_{1,S} \to \mathbb{E}_{2,S}$ induced by a and b coincide therefore on the Lie algebras. But then these maps coincide because the ring S is torsionfree. Hence we conclude that a and b induce the same map $\mathbb{E}_{1,R} \to \mathbb{E}_{2,R}$, and finally that $a = b$. □

Proposition 102. — *Let k be a field. Then the functor BT from the category of displays over k to the category of formal p-divisible groups over k is an equivalence of categories.*

Proof. — By proposition 99 we know that the functor BT is fully faithful. Hence we have to show that any p-divisible formal group X over k is isomorphic to $BT(\mathcal{P})$ for some display $\mathcal{P}$ over k. Let ℓ be the perfect closure of k. Let $\overline{X} = X_\ell$ be the formal p-divisible group obtained by base change. By Cartier theory we know that $\overline{X} = BT(\overline{\mathcal{P}})$ for some display $\overline{\mathcal{P}}$ over ℓ.

Now we apply descent with respect to the inclusion $q : k \to \ell$. Let q_1 and q_2 be the two natural maps $\ell \to \ell \otimes_k \ell$. Let $\overline{X}_i$ respectively $\overline{\mathcal{P}}_i$ be the objects obtained by base change with respect to q_i for $i = 1, 2$. Our result would follow if we knew that the functor BT is fully faithful over $\ell \otimes_k \ell$. Indeed in this case the descent datum $\overline{X}_1 \cong \overline{X}_2$ defined by X would provide an isomorphism $\overline{\mathcal{P}}_1 \cong \overline{\mathcal{P}}_2$. This isomorphism would be a descent datum (i.e. satisfy the cocycle condition) because by proposition 98 the functor BT is faithful. Hence by theorem 37 it would give the desired display $\mathcal{P}$ over k.

By proposition 101 it is enough to find a topological pd-thickening $S \to \ell \otimes_k \ell$, such that S is a torsion free p-adic ring. We choose a Cohen ring C of k and embedding $C \to W(\ell)$ [AC] IX, §2, 3. Then we consider the natural surjection:

$$W(\ell) \otimes_C W(\ell) \longrightarrow \ell \otimes_k \ell \tag{182}$$

The ring $A = W(\ell) \otimes_C W(\ell)$ is torsionfree because $W(\ell)$ is flat over C. The kernel of (182) is pA. We define S as the p-adic completion:

$$S = \varprojlim_n A/p^n A.$$

Then S is a torsionfree p-adic ring, such that $S/pS \cong \ell \otimes_k \ell$. This follows by going to the projective limit in the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A/p^n A \arrow[r, "p"] \arrow[d] & A/p^{n+1} A \arrow[r] \arrow[d] & A/pA \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & A/p^{n-1} \arrow[r, "p"] & A/p^n A \arrow[r] & A/pA \arrow[r] & 0
\end{tikzcd}$$

But with the canonical divided powers on pS the topological pd-thickening $S \to \ell \otimes_k \ell$ is the desired object. □

Theorem 103. — *Let R be an excellent local ring or a ring such that R/pR is of finite type over a field k. Then the functor BT is an equivalence from the category of displays over R to the category of formal p-divisible groups over R.*

Proof. — We begin to prove this for an Artinian ring R. Since BT is a fully faithful functor, we need to show that any p-divisible group G over R comes from a display $\mathcal{P}$. Let $S \to R$ be a pd-thickening. Since we have proved the theorem for a field, we may assume by induction that the theorem is true for R. Let G be a p-divisible group over R with $BT(\mathcal{P}) = G$. The liftings of G respectively of $\mathcal{P}$ correspond functorially to the liftings of the Hodge filtration to

$$\mathcal{D}_{\mathcal{P}}(S) = \mathbb{D}_G(S).$$

Hence the theorem is true for S.

More generally if $S \to R$ is surjective with nilpotent kernel the same reasoning shows that the theorem is true for S, if it is true for R.

Next let R be a complete noetherian local ring. We may assume that R is reduced. Let $\mathfrak{m}$ be the maximal ideal of R. We denote by G_n the p-divisible group $G_{R/\mathfrak{m}^n}$ obtained by base change from G. Let $\mathcal{P}_n$ be the display over $R/\mathfrak{m}^n$, which correspond to G_n. Then $\mathcal{P} = \varprojlim \mathcal{P}_n$ is a 3n-display over R. Consider the formal group H over R which belongs to the reduced Cartier module $M(\mathcal{P})$. Since $\mathcal{P}_n$ is obtained by base change from $\mathcal{P}$ and consequently $M(\mathcal{P}_n)$ from $M(\mathcal{P})$ too, we have canonical isomorphisms $H_n \cong G_n$. Hence we may identify H and G. Clearly we are done, if we show the following assertion. Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a 3n-display over R, such that $M(\mathcal{P})$ is the Cartier module of a p-divisible formal group of height equal to the rank of P. Then $\mathcal{P}$ is nilpotent.

Indeed, it is enough to check the nilpotence of $\mathcal{P}_S$ over an arbitrary extension $S \supset R$, such that $p \cdot S = 0$ (compare (15)). Since R admits an injection into a finite product of algebraically closed fields, we are reduced to show the assertion above in the case, where R is an algebraically closed field. In this case we have the standard decomposition

$$\mathcal{P} = \mathcal{P}^{\mathrm{nil}} \oplus \mathcal{P}^{\mathrm{et}}$$

where $\mathcal{P}^{\mathrm{nil}}$ is a display and $\mathcal{P}^{\mathrm{et}}$ is a 3n-display with the structural equations

$$V^{-1}e_i = e_i, \qquad \text{for } i = 1, \ldots, h.$$

Then

$$M(\mathcal{P}^{\mathrm{et}}) = \bigoplus_{i=1}^{h} \mathbb{E}_R e_i/(Ve_i - e_i),$$

is zero, because $V - 1$ is a unit in $\mathbb{E}_R$. We obtain $M(\mathcal{P}) = M(\mathcal{P}^{\mathrm{nil}}) = P^{\mathrm{nil}}$. Hence the height of the p-divisible group G is $\mathrm{rank}_R P^{\mathrm{nil}}$. Our assumption $\mathrm{height} G = \mathrm{rank}_R P$ implies $P = P^{\mathrm{nil}}$. This finishes the case, where R is a complete local ring.

Next we consider the case, where the ring R is an excellent local ring. As above we may assume R is reduced. Then the completion $\widehat{R}$ is reduced. Since the geometric fibres of $\operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$, are regular, for any R-algebra L, which is a field, the ring $\widehat{R} \otimes_R L$ is reduced. Hence if R is reduced, so is $\widehat{R} \otimes_R \widehat{R}$. Consider the diagram:

$$R \xrightarrow{p} \widehat{R} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \widehat{R} \otimes_R \widehat{R}$$

Let G be a p-divisible formal group over R. It gives a descent datum on $p^*G = G_{\widehat{R}}$:

$$a : p_1^* G_{\widehat{R}} \longrightarrow p_2^* G_{\widehat{R}}.$$

We find a display $\widehat{\mathcal{P}}$ over $\widehat{R}$, such that $BT(\widehat{\mathcal{P}}) = G_{\widehat{R}}$. Since the functor BT is fully faithful over $\widehat{R}\otimes_R\widehat{R}$ by proposition 99 the isomorphism a is induced by an isomorphism

$$\alpha : p_1^*\widehat{\mathcal{P}} \longrightarrow p_2^*\widehat{\mathcal{P}}$$

From the corollary 98 it follows that α satisfies the cocycle condition. By theorem 37 there is a display $\mathcal{P}$ over R, which induces $(\widehat{\mathcal{P}}, \alpha)$. Since the application of the functor BT gives us the descent datum for G, it follows by the usual descent theory for p-divisible groups, that $BT(\mathcal{P}) = G$.

Finally we consider the case of a finitely generated $W(k)$-algebra R. We form the faithfully flat R-algebra $S = \prod R_{\mathfrak{m}}$, where $\mathfrak{m}$ runs through all maximal ideals of R. Then we will apply the same reasoning as above to the sequence

$$R \longrightarrow S \quad \overset{p_1}{\underset{p_2}{\rightrightarrows}} \quad S \otimes_R S.$$

We have seen, that it is enough to treat the case, where R is reduced. Assume further that $\operatorname{Spec} R$ is connected, so that G has constant height.

We see as in the proof of proposition 99, that to give a p-divisible group of height h over $\prod R_{\mathfrak{m}}$ is the same thing as to give over each $R_{\mathfrak{m}}$ a p-divisible group of height h. The same thing is true for displays. (One must show that the order N of nilpotence in (15) is independent of $\mathfrak{m}$. But the usual argument in linear algebra shows also in p-linear algebra that $N = h - d$ is enough.) Since each ring $R_{\mathfrak{m}}$ is excellent with perfect residue field, we conclude that $G_S = BT(\widetilde{\mathcal{P}})$ for some display $\widetilde{\mathcal{P}}$ over S. We may apply descent if we prove that the ring $S \otimes_R S$ is reduced. This will finish the proof. Let us denote by $Q(R)$ the full ring of quotients. Then we have an injection

$$\Big(\prod R_{\mathfrak{m}}\Big) \otimes_R \Big(\prod R_{\mathfrak{m}}\Big) \hookrightarrow \Big(\prod Q\,(R_{\mathfrak{m}})\Big) \otimes_{Q(R)} \Big(\prod Q\,(R_{\mathfrak{m}})\Big)$$

The idempotent elements in $Q(R)$ allows to write the last tensor product as

$$\bigoplus_{\substack{\mathfrak{p}\in\operatorname{Spec} R\\ \mathfrak{p}\text{ minimal}}} \Big(\left(\prod\nolimits_{\mathfrak{m}} Q\,(R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}})\right) \otimes_{Q(R/\mathfrak{p}R)} \left(\prod\nolimits_{\mathfrak{m}} Q\,(R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}})\right) \Big)$$

We set $K = Q(R/\mathfrak{p}R)$. Then we have to prove that for any index set I they are no nilpotent elements in the tensor product

$$\Big(\prod_{i\in I} K\Big) \otimes_K \Big(\prod_{i\in I} K\Big).$$

But any product of separable (= geometrically reduced) K-algebras is separable, because $\prod$ commutes with the tensor product by a finite extension E of K. $\square$

4. Duality

4.1. Biextensions. — Biextensions of formal group were introduced by Mumford [Mu]. They may be viewed as a formalization of the concept of the Poincaré bundle in the theory of abelian varieties. Let us begin by recalling the basic definitions (loc.cit.).

Let A, B, C be abelian groups. An element in $\mathrm{Ext}^1(B \otimes^{\mathbb{L}} C, A)$ has an interpretation, which is similar to the usual interpretation of $\mathrm{Ext}^1(B, A)$ by Yoneda.

Definition 104. — A biextension of the pair B, C by the abelian group A consists of the following data:

1) A set G and a surjective map
$$\pi : G \longrightarrow B \times C$$
2) An action of A on G, such that G becomes a principal homogenous space with group A and base $B \times C$.
3) Two maps
$$+_B : G \times_B G \longrightarrow G \quad +_C : G \times_C G \longrightarrow G,$$
where the map $G \to B$ used to define the fibre product, is the composite of π with the projection $B \times C \to B$, and where $G \to C$ is defined in the same way.

One requires that the following conditions are verified:

(i) The maps of 3) are equivariant with respect to the morphism $A \times A \to A$ given by the group law.

(ii_B) The map $+_B$ is an abelian group law of G over B, such that the following sequence is an extension of abelian groups over B:
$$\begin{array}{ccccc} 0 \to B \times A & \longrightarrow & G & \xrightarrow{\pi} & B \times C \to 0 \\ b \times a & \longmapsto & a + 0_B(b) & & \end{array}$$
Here $0_B : B \to G$ denotes the zero section of the group law $+_B$ and $a + 0_B(b)$ is the given action of A on G.

(ii_C) The same condition as (ii_B) but for C.

(iii) The group laws $+_B$ and $+_C$ are compatible in the obvious sense:

Let $x_{i,j} \in G, 1 \leq i, j \leq 2$ be four elements, such that $pr_B(x_{i,1}) = pr_B(x_{i,2})$ and $pr_C(x_{1,i}) = pr_C(x_{2,i})$ for $i = 1, 2$. Then
$$(x_{11} +_B x_{12}) +_C (x_{21} +_B x_{22}) = (x_{11} +_C x_{21}) +_B (x_{12} +_C x_{22}).$$

Remark. — The reader should prove the following consequence of these axioms:
$$0_B(b_1) +_C 0_B(b_2) = 0_B(b_1 + b_2)$$
The biextension of the pair B, C by A form a category which will be denoted by $\mathrm{BIEXT}^1(B \times C, A)$. If $A \to A'$ respectively $B' \to B$ and $C' \to C$ are homomorphism of abelian groups, one obtains an obvious functor
$$\mathrm{BIEXT}^1(B \times C, A) \longrightarrow \mathrm{BIEXT}^1(B' \times C', A').$$
Any homomorphism in the category $\mathrm{BIEXT}^1(B \times C, A)$ is an isomorphism. The automorphism group of an object G is canonically isomorphic of the set of bilinear maps

(183) $$\mathrm{Bihom}(B \times C, A).$$

Indeed if α is a bilinear map in (183), the corresponding automorphism of G is given by $g \mapsto g + \alpha(\pi(g))$.

If $b \in B$, we denote by G_b or $G_{b\times C}$ the inverse image of $b \times C$ by π. Then $+_B$ induces on G_b the structure of an abelian group, such that

$$0 \longrightarrow A \longrightarrow G_b \longrightarrow C \longrightarrow 0$$

is a group extension. Similarly one defines G_c for $c \in C$.

A trivialization of a biextension G is a "bilinear" section $s : B \times C \to G$, i.e. $\pi \circ s = \mathrm{id}_{B\times C}$, and $s(b, -)$ for each $b \in B$ is a homomorphism $C \to G_b$, and $s(-, c)$ for each $c \in C$ is a homomorphism $B \to G_C$. A section s defines an isomorphism of G with the trivial biextension $A \times B \times C$.

We denote by $\mathrm{Biext}^1(B \times C, A)$ the set of isomorphism classes in the category $\mathrm{BIEXT}^1(B\times C, A)$. It can be given the structure of an abelian group (using cocycles or Baer sum). The zero element is the trivial biextension.

An exact sequence $0 \to B_1 \to B \to B_2 \to 0$ induces an exact sequence of abelian groups

$$\begin{aligned} 0 \longrightarrow \mathrm{Bihom}(B_2 \times C, A) \longrightarrow \mathrm{Bihom}(B \times C, A) \longrightarrow \mathrm{Bihom}(B_1 \times C, A) \xrightarrow{\ \delta\ } \\ \mathrm{Biext}^1(B_2 \times C, A) \longrightarrow \mathrm{Biext}^1(B \times C, A) \longrightarrow \mathrm{Biext}^1(B_1 \times C, A) \end{aligned}$$

The connecting homomorphism δ is obtained by taking the push–out of the exact sequence

$$0 \longrightarrow B_1 \times C \longrightarrow B \times C \longrightarrow B_2 \times C \longrightarrow 0,$$

by a bilinear map $\alpha : B_1 \times C \to A$. More explicitly this push-out is the set $A\times B\times C$ modulo the equivalence relation:

$$(a, b_1 + b, c) \equiv (a + \alpha(b_1, c), b, c), \quad a \in A,\ b \in B\ c \in C, b_1 \in B_1$$

If $0 \to A_1 \to A \to A_2 \to 0$ is an exact sequence of abelian groups, one obtains an exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathrm{Bihom}(B \times C, A_1) \longrightarrow \mathrm{Bihom}(B \times C, A) \longrightarrow \mathrm{Bihom}(B \times C, A_2) \xrightarrow{\ \delta\ } \\ \mathrm{Biext}^1(B \times C, A_1) \longrightarrow \mathrm{Biext}^1(B \times C, A) \longrightarrow \mathrm{Biext}^1(B \times C, A_2) \end{aligned}$$

We omit the proof of the following elementary lemma:

Lemma 105. — *If B and C are free abelian groups, one has*

$$\mathrm{Biext}^1(B \times C, A) = 0.$$

This lemma gives us the possibility to compute Biext^1 by resolutions:

Proposition 106. — *(Mumford) Assume we are given exact sequences* $0 \to K_1 \to K_0 \to B \to 0$ *and* $0 \to L_1 \to L_0 \to C \to 0$. *Then one has an exact sequence of abelian groups*

$$\begin{aligned} \mathrm{Bihom}(K_0 \times L_0, A) \longrightarrow \mathrm{Bihom}(K_0 \times L_1, A) \times_{\mathrm{Bihom}(K_1 \times L_1, A)} \mathrm{Bihom}(K_1 \times L_0, A) \\ \longrightarrow \mathrm{Biext}^1(B \times C, A) \longrightarrow \mathrm{Biext}^1(K_0 \times L_0, A) \end{aligned}$$

Proof. — One proves more precisely that to give a biextension G of $B \times C$ together with a trivialization over $K_0 \times L_0$:

$$\begin{tikzcd} G \arrow[r] & B \times C \\ & K_0 \times L_0 \arrow[ul, "s"] \arrow[u] \end{tikzcd}$$

is the same thing as to give bilinear maps $\xi : K_0 \times L_1 \to A$ and $\mu : K_1 \times L_0 \to A$, which have the same restriction on $K_1 \times L_1$. We denote this common restriction by $\varphi : K_1 \times L_1 \to A$.

Using the splitting 0_B of the group extension

$$0 \longrightarrow A \longrightarrow G_{B \times 0} \longrightarrow B \longrightarrow 0,$$

we may write

$$s(k_0, l_1) = 0_B(b_0) + \xi(k_0, l_1), \quad \text{for } k_0 \in K_0, l_1 \in L_1, \tag{184}$$

where b_0 is the image of k_0 in B and $\xi(k_0, l_1) \in A$. This defines the bilinear map ξ. Similarly we define μ:

$$s(k_1, l_0) = 0_C(c_0) + \mu(k_1, l_0), \tag{185}$$

for $k_1 \in K_1$ and $l_0 \in L_0$, where $c_0 \in C$ is the image of l_0. Clearly these maps are bilinear, since s is bilinear. Since $0_B(0) = 0_C(0)$ their restrictions to $K_1 \times L_1$ agree.

Conversely if ξ and μ are given, one considers in the trivial biextension $A \times K_0 \times L_0$ the equivalence relation

$$(a, k_0 + k_1, l_0 + l_1) \equiv (a + \xi(k_0, l_1) + \mu(k_1, l_0) + \xi(k_1, l_1), k_0, l_0).$$

Dividing out we get a biextension G of $B \times C$ by A with an obvious trivialization. □

The following remark may be helpful. Let $l_0 \in L_0$ be an element with image $c \in C$. We embed $K_1 \to A \times K_0$ by $k_1 \mapsto (-\mu(k_1, l_0), k_1)$. Then the quotient $(A \times K_0)/K_1$ defines the group extension $0 \to A \to G_c \to B \to 0$.

Corollary 107. — *There is a canonical isomorphism:*

$$\mathrm{Ext}^1(B \otimes^{\mathbb{L}} C, A) \longrightarrow \mathrm{Biext}^1(B \times C, A).$$

Proof. — If B and C are free abelian groups one can show that any biextension is trivial (see (105)). One considers complexes $K_\bullet = \cdots 0 \to K_1 \to K_0 \to 0 \cdots$ and $L_\bullet = \cdots 0 \to L_1 \to L_0 \to 0 \cdots$ as in the proposition, where K_0 and L_0 are free abelian groups. In this case the proposition provides an isomorphism

$$H^1(\mathrm{Hom}(K_\bullet \otimes L_\bullet, A)) = \mathrm{Biext}^1(K \times L, A). \tag{186}$$

Let $T_\bullet = \cdots 0 \to T_2 \to T_1 \to T_0 \to 0 \cdots$ be the complex $K_\bullet \otimes L_\bullet$. Then the group (186) above is simply the cokernel of the map

$$\mathrm{Hom}(T_0 A) \longrightarrow \mathrm{Hom}(T_1/\operatorname{Im} T_2, A). \tag{187}$$

Let $\cdots P_i \to \cdots \to P_1 \twoheadrightarrow K_1 \to 0$ be any free resolution. We set $P_0 = K_0$ and consider the complex $P_\bullet = \cdots \to P_i \to \cdots P_1 \to P_0 \to 0$. The same process applied to the $L's$ yields $Q_\bullet = \cdots \to Q_1 \to \cdots \to Q_1 \to Q_0 \to 0$. Let $\widetilde{T} = P_\bullet \otimes Q_\bullet$. Then the complex

$$\cdots 0 \longrightarrow \widetilde{T}_1/\operatorname{Im} \widetilde{T}_2 \longrightarrow \widetilde{T}_0 \longrightarrow \cdots$$

is identical with the complex

$$\cdots 0 \longrightarrow T_1/\operatorname{Im} T_2 \longrightarrow T_0 \longrightarrow \cdots$$

Therefore the remark (187) yields an isomorphism

$$H^1(\mathrm{Hom}(K_\bullet \otimes L_\bullet, A)) \simeq H^1(\mathrm{Hom}(P_\bullet \otimes Q_\bullet, A)) = \mathrm{Ext}^1(B \otimes^{\mathbb{L}} C, A). \qquad \square$$

The notion of a biextension has an obvious generalization to any topos. This theory is developed in *SGA*7. We will consider the category Nil_R with the flat topology. To describe the topology it is convenient to consider the isomorphic category Aug_R (see definition 50). Let $(B, \epsilon) \in \mathrm{Aug}_R$ be an object, i.e. a morphism $\epsilon : B \to R$ of R-algebras. We write $B^+ = \operatorname{Ker} \epsilon$ for the augmentation ideal. We will often omit the augmentation from the notation, and write B instead of (B, ϵ).

If we are given two morphisms $(B, \varepsilon) \to (A_i, \varepsilon_i)$ for $i = 1, 2$, we may form the tensorproduct:

$$(A_1, \varepsilon_1) \otimes_{(B,\varepsilon)} (A_2, \varepsilon_2) = (A_1 \otimes_{B_1} A_2, \varepsilon_1 \otimes \varepsilon_2).$$

This gives a fibre product in the opposite category $\mathrm{Aug}_R^{\mathrm{opp}}$:

$$\operatorname{Spf} A_1 \times_{\operatorname{Spf} B} \operatorname{Spf} A_2 = \operatorname{Spf}(A_1 \otimes_B A_2).$$

Via the Yoneda embedding we will also consider $\operatorname{Spf} B$ as a functor on Nil_R:

$$\operatorname{Spf} B(\mathcal{N}) = \mathrm{Hom}_{\mathrm{Nil}\, R}(B^+, \mathcal{N}).$$

We equip $\mathrm{Aug}_R^{\mathrm{opp}}$ with a Grothendieck topology. A covering is simply a morphism $\operatorname{Spf} A \to \operatorname{Spf} B$, such that the corresponding ring homomorphism $B \to A$ is flat. We note that in our context flat morphisms are automatically faithfully flat. We may define a sheaf on $\mathrm{Aug}_R^{\mathrm{opp}}$ as follows.

Definition 108. — A functor $F : \mathrm{Aug}_R \to \mathrm{Sets}$ is called a sheaf, if for any flat homomorphism $B \to A$ in Aug_R the following sequence is exact.

$$F(B) \to F(A) \rightrightarrows F(A \otimes_B A).$$

Recall that a left exact functor $G : \mathrm{Nil}_R \to (\mathrm{Sets})$ is a functor, such that $G(0)$ consists of a single point, and such that each exact sequence in Nil_R

$$0 \longrightarrow \mathcal{N}_1 \longrightarrow \mathcal{N}_2 \longrightarrow \mathcal{N}_3 \longrightarrow 0$$

induces an exact sequence of pointed sets

$$0 \longrightarrow G(\mathcal{N}_1) \longrightarrow G(\mathcal{N}_2) \longrightarrow G(\mathcal{N}_3),$$

i.e. $G(\mathcal{N}_1)$ is the fibre over the point $G(0) \subset G(\mathcal{N}_3)$. It can be shown that such a functor respects fibre products in Nil_R. We remark that any left exact functor on Nil_R is a sheaf.

A basic fact is that an exact abelian functor on Nil_R has trivial Čech cohomology.

Proposition 109. — *Let $F : \mathrm{Nil}_R \to (Ab)$ be a functor to the category of abelian groups, which is exact. Then for any flat morphism $B \to A$ in Aug_R the following complex of abelian groups is exact*

$$F(B) \to F(A) \rightrightarrows F(A \otimes_B A) \substack{\to\\\to\\\to} F(A \otimes_B A \otimes_B A) \substack{\to\\\to\\\to\\\to} \cdots$$

Proof. — Let $\mathcal{N}$ be a nilpotent B-algebra and $B \to C$ be a homomorphism in Aug_R. then we define simplicial complexes:

$$\begin{aligned} &(C^n(\mathcal{N}, B \to A), \theta_i^n) \\ &(C^n(C, B \to A), \theta_i^n) \end{aligned} \tag{188}$$

for $n \geq 0$.

We set

$$C^n(\mathcal{N}, B \to A) = \mathcal{N} \otimes_B A \otimes_B \cdots \otimes_B A$$

$$C^n(C, B \to A) = C \otimes_B A \otimes_B \cdots \otimes_B A,$$

where in both equations we have $n+1$ factors on the right hand side. The operators $\theta_i^n : C^{n-1} \to C^n$ for $i = 0, \ldots, n$ are defined by the formulas:

$$\theta_i^n(x \otimes a_0 \otimes \cdots \otimes a_{n-1}) = (x \otimes a_0 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_i \cdots \otimes a_{n-1}),$$

where $x \in \mathcal{N}$ or $x \in C$.

One knows that the associated chain complexes with differential $\delta^n = \sum (-1)^i \theta_i^n$ are resolutions of $\mathcal{N}$ respectively C, if either $B \to A$ is faithfully flat or $B \to A$ has a section $s : A \to B$. In the latter case one defines

$$s^n : C^n \longrightarrow C^{n-1}, s^n(x \otimes a_0 \otimes \cdots \otimes a_n) = xS(a_0) \otimes a_1 \otimes \cdots \otimes a_n.$$

If one sets $C^{-1} = \mathcal{N}$ respectively $C^{-1} = C$ and $\theta_0^0 : C^{-1} \to C^0, \theta_0^0(x) = x \otimes 1$, one has the formulas:

$$s^n \theta_i^n = \begin{cases} \mathrm{id}_{C^{n-1}}, & \text{for } i = 0 \\ \theta_{i-1}^{n-1} s^{n-1}, & \text{for } i > 0 \quad \text{and} \quad n \geq 1. \end{cases} \tag{189}$$

Let us extend the chain complex (C^n, δ^n) by adding zeros on the left:

$$0 \to \cdots \to 0 \to C^{-1} \xrightarrow{\theta_0^0 = \delta^0} C^0 \xrightarrow{\delta^1} C^1 \xrightarrow{\delta^2} C^2 \to \cdots \tag{190}$$

Since by (189) we have $s^n \delta^n + \delta^{n-1} s^{n-1} = \mathrm{id}_{C^{n-1}}$, we have shown that this complex is homotopic to zero.

If $F : \mathrm{Nil}_R \to (Ab)$ is a functor we can apply F to the simplicial complexes (188), because θ_i^n are R-algebra homomorphisms. The result are simplicial complexes, whose associated simple complexes will be denoted by

$$C^n(\mathcal{N}, B \to A, F) \quad \text{respectively} \quad C^n(C, B \to A, F). \tag{191}$$

Let us assume that $B \to A$ has a section. Then the extended complexes $C^n(F), n \in \mathbb{Z}$ are homotopic to zero by the homotopy $F(s^n)$, since we can apply F to the relations (189).

Let now F be an exact functor and assume that $B \to A$ is faithfully flat. If $\mathcal{N}^2 = 0$, each algebra $C^n(\mathcal{N}, B \to A)$ has square zero. In this case the δ^n in (190) are algebra homomorphisms. Therefore we have the right to apply F to (190). This sequence is an exact sequence in Nil_R, which remains exact, if we apply F. Hence the extended complex $C^n(\mathcal{N}, B \to A, F), n \in \mathbb{Z}$ is acyclic if $\mathcal{N}^2 = 0$.

Any exact sequence $0 \to \mathcal{K} \to \mathcal{M} \to \mathcal{N} \to 0$ is Nil_B, gives an exact sequence of complexes.

$$0 \longrightarrow C^n(\mathcal{K}, B \to A, F) \longrightarrow C^n(\mathcal{M}, B \to A, F) \longrightarrow C^n(\mathcal{N}, B \to A, F) \longrightarrow 0.$$

Hence $C^n(\mathcal{N}, B \to A, F)$ is acyclic for any $\mathcal{N} \in \mathrm{Nil}_B$. Finally let $\mathfrak{a} \subset B$ be the kernel of the augmentation $B \to R$. Then one has an exact sequence of complexes:

$$0 \to C^n(\mathfrak{a}, B \to A, F) \to C^n(B, B \to A, F) \to C^n(B/\mathfrak{a}, B/\mathfrak{a} \to A/\mathfrak{a}, F) \to 0 \tag{192}$$

The augmentation of A induces a section of $B/\mathfrak{a} = R \to A/\mathfrak{a}$. Hence the last complex in the sequence (192) is acyclic. Since we have shown $C^n(\mathfrak{a}, B \to A, F)$ to be acyclic, we get that $C^n(B, B \to A, F)$ is acyclic. This was our assertion. □

We reformulate the result in the language of sheaf theory.

Corollary 110. — *An exact functor $F : \mathrm{Nil}_R \to (Ab)$ is a sheaf on the Grothendieck topology $\mathcal{T} = \mathrm{Aug}_R^{\mathrm{opp}}$. For each covering $T_1 \to T_2$ in $\mathcal{T}$ the Čech cohomology groups $\check{H}^i(T_1/T_2, F)$ are zero for $i \geq 1$. In particular an F-torsor over an object of $\mathcal{T}$ is trivial.*

By $SGA7$ one has the notion of a biextension in the category of sheaves. If F, K, L are abelian sheaves a biextension in $\mathrm{BIEXT}^1(K \times L, F)$ is given by an F-torsor G over $K \times L$ and two maps $t_K : G \times_K G \to G$ and $t_L : G \times_L G \to G$, which satisfy some conditions, which should now be obvious. If F is moreover an exact functor, then any F torsor is trivial. Hence in this case we get for any $\mathcal{N} \in \mathrm{Nil}_R$, that $G(\mathcal{N})$ is a biextension of $K(\mathcal{N})$ and $L(\mathcal{N})$ is the category of abelian groups. This is the definition Mumford [Mu] uses.

4.2. Two propositions of Mumford. — We will now update some proofs and results in Mumford's article. We start with some general remarks. Let F be an exact functor. Let $G \xrightarrow{\pi} H$ be any F-torsor is the category of sheaves on $\mathcal{T}$. If $H = \mathrm{Spf}\, A$ is representable we know that π is trivial and hence smooth because F is smooth. (The word smooth is used in the formal sense [Z1] 2.28.) If H is not representable, π is still smooth since the base change of G by any $\mathrm{Spf}\, A \to H$ becomes smooth.

More generally any F-torsor over H is trivial if H is prorepresentable in the following sense:

There is a sequence of surjections in $\mathrm{Aug}\, R$:

$$\longrightarrow \cdots \longrightarrow A_{n+1} \longrightarrow A_n \cdots \longrightarrow A_1,$$

such that

$$H = \varinjlim \mathrm{Spf}\, A_i. \tag{193}$$

Then π has a section because it has a section over any $\mathrm{Spf}\, A_i$ and therefore over H as is seen by the formula:

$$\mathrm{Hom}(H, G) = \varinjlim \mathrm{Hom}(\mathrm{Spf}\, A_i, G)$$

Hence we have shown:

Lemma 111. — *Let $F : \mathrm{Nil}_R \to (Ab)$ be an exact functor. Then any F-torsor over a prorepresentable object H is trivial.*

For some purposes it is useful to state the first main theorem of Cartier theory in a relative form. From now on R will be a $\mathbb{Z}_{(p)}$-algebra.

Let B be an augmented nilpotent R-algebra. In order to avoid confusion we will write $\mathrm{Spf}_R\, B$ instead of $\mathrm{Spf}\, B$ in the following. Let $G : \mathrm{Nil}_R \to (\mathrm{Sets})$ be a left exact functor. There is an obvious functor $\mathrm{Nil}_B \to \mathrm{Nil}_R$. The composite of this functor with G is the base change G_B.

Assume we are given a morphism $\pi : G \to \mathrm{Spf}_R\, B$, which has a section $\sigma : \mathrm{Spf}_R\, B \to G$. Then we associate to the triple (G, π, σ) a left exact functor on Nil_B:

Let $\mathcal{L} \in \mathrm{Nil}_B$ and let $B|\mathcal{L}| = B \oplus \mathcal{L}$ be the augmented B-algebra associated to it. Then $B|\mathcal{L}|$ is also an augmented R-algebra, with augmentation ideal $B^+ \oplus \mathcal{L}$. Then we define the restriction $\mathrm{Res}_B\, G(\mathcal{L})$ of G to be the fibre over σ of the following map

$$\mathrm{Hom}_{\mathrm{Spf}_R B}(\mathrm{Spf}_R\, B|\mathcal{L}|, G) \longrightarrow \mathrm{Hom}_{\mathrm{Spf}_R B}(\mathrm{Spf}_R\, B, G).$$

The functor $G \mapsto \operatorname{Res}_B G$ defines an equivalence of the category of triples (G, π, σ) with the category of left exact functors on Nil_B. We will call the triple (G, π, σ) a pointed left exact functor over Spf_B. It is useful to explain this formalism a little more.

Let us start with a left exact functor F on Nil_R. Then $F \times \operatorname{Spf} B \xrightarrow{\mathrm{pr}} \operatorname{Spf} B$ is naturally a pointed functor over $\operatorname{Spf} B$. The restriction of this pointed functor is F_B:

$$\operatorname{Res}_B(F \times \operatorname{Spf} B) = F_B.$$

Suppose that the B-algebra structure on $\mathcal{L}$ is given by a morphism $\varphi : B^+ \to \mathcal{L}$. Then we have also a map of augmented R-algebras $B|\mathcal{L}| \to R|\mathcal{L}|$, which is on the augmentation ideals $\varphi + \mathrm{id}_{\mathcal{L}} : B^+ \oplus \mathcal{L} \to \mathcal{L}$.

Lemma 112. — *Let $\varphi : B^+ \to \mathcal{L}$ be a morphism in Nil_R. Via φ we may consider $\mathcal{L}$ as an element of Nil_B. Then $\operatorname{Res}_B G(\mathcal{L})$ may be identified with the subset of elements of $G(\mathcal{L})$, which are mapped to φ by the morphism*

$$\pi_{\mathcal{L}} : G(\mathcal{L}) \longrightarrow \operatorname{Hom}(B^+, \mathcal{L}).$$

Proof. — Consider the two embeddings of nilpotent algebras $\iota_{\mathcal{L}} : \mathcal{L} \to B^+ \oplus \mathcal{L} = B|\mathcal{L}|^+, \iota_{\mathcal{L}}(l) = 0 \oplus l$ and $\iota_{B^+} : B^+ \to B^+ \oplus \mathcal{L} = B|\mathcal{L}|^+, \iota_{B^+}(b) = b \oplus 0$. Let us denote by $G_\sigma(B^+ \oplus \mathcal{L}) \subset G(B^+ \oplus \mathcal{L}) = \operatorname{Hom}(\operatorname{Spf}_R B|\mathcal{L}|, G)$ the fibre at σ of the map

$$\operatorname{Hom}(\operatorname{Spf}_R B|\mathcal{L}|, G) \to \operatorname{Hom}(\operatorname{Spf}_R B, G) \tag{194}$$

We have an isomorphism in Nil_R:

$$\begin{aligned} B^+ \bigoplus \mathcal{L} &\xrightarrow{\sim} B^+ \times \mathcal{L} \\ b \oplus l &\longmapsto b \times (\varphi(b) + l) \end{aligned} \tag{195}$$

Let $G(B^+ \oplus \mathcal{L}) \to G(\mathcal{L})$ be the map induced by $B^+ \oplus \mathcal{L} \to \mathcal{L}, b \oplus l \to \varphi(b) + l$. It follows from the isomorphism (195) and the left exactness of G, that this map induces a bijection $G_\sigma(B^+ \oplus \mathcal{L}) \xrightarrow{\sim} G(\mathcal{L})$. Hence we have identified $G(\mathcal{L})$ with the fibre of (194) at σ. It remains to determine, which subset of $G(\mathcal{L})$ corresponds to $\operatorname{Hom}_{\operatorname{Spf} B}(\operatorname{Spf} B|\mathcal{L}|, G)$. But looking at the following commutative diagram

$$\begin{tikzcd} G(B^+ \oplus \mathcal{L}) \arrow[r] \arrow[d, "\pi_{B^+ \oplus \mathcal{L}}"'] & G(\mathcal{L}) \arrow[d, "\pi_{\mathcal{L}}"] \\ \operatorname{Hom}(B^+, B^+ \oplus \mathcal{L}) \arrow[r] & \operatorname{Hom}(B^+, \mathcal{L}) \\ \iota_{B^+} \arrow[r, mapsto] & \varphi \end{tikzcd}$$

we see that this subset is exactly the fibre of $\pi_{\mathcal{L}}$ at φ. □

Conversely given a functor $H : \mathrm{Nil}_B \to (\mathrm{Sets})$, such that $H(0) = \{\text{point}\}$. Then we obtain a functor $G : \mathrm{Nil}_R \to (\mathrm{Sets})$ by:

$$G(\mathcal{N}) = \bigsqcup_{\varphi : B^+ \to \mathcal{N}} H(\mathcal{N}_\varphi), \quad \mathcal{N} \in \mathrm{Nil}_R,$$

where $\mathcal{N}_\varphi$ is $\mathcal{N}$ considered as a B-algebra via φ. Then we have a natural projection $\pi : G(\mathcal{N}) \to \mathrm{Hom}(B^+, \mathcal{N})$, which maps $H(\mathcal{N}_\varphi)$ to φ. The distinguished point in each $H(\mathcal{N}_\varphi)$ defines a section σ of π.

In particular our remark shows that a group object in the category of arrows $G \to \mathrm{Spf}\, B$, such that G is a left exact functor on Nil_R is the same thing as a left exact functor $H : \mathrm{Nil}_B \to (Ab)$.

In Cartier theory one considers the following functors on Nil_R:

$$D(\mathcal{N}) = \mathcal{N}, \qquad \widehat{\Lambda}(\mathcal{N}) = (1 + t\mathcal{N}[t])^\times, \quad \text{for } \mathcal{N} \in \mathrm{Nil}_R\,.$$

Here t is an indeterminate. The functor D is considered as a set valued functor, while $\widehat{\Lambda}$ takes values in the category (Ab) of abelian groups. We embed D into $\widehat{\Lambda}$ by the map $n \mapsto (1 - nt)$ for $n \in \mathcal{N}$.

***Theorem 113* (Cartier).** — *Let $G \xrightarrow{\pi} H$ be a morphism of functors on Nil_R. Assume that G is left exact and has the structure of an abelian group object over H. The embedding $D \subset \widehat{\Lambda}$ induces a bijection.*

$$\mathrm{Hom}_{groups/H}(\widehat{\Lambda} \times H, G) \longrightarrow \mathrm{Hom}_{\mathrm{pointedfunctors}/H}\,(D \times H, G)\,.$$

Proof. — If H is the functor $H(\mathcal{N}) = \{\mathrm{point}\}$, $\mathcal{N} \in \mathrm{Nil}_R$ this is the usual formulation of Cartier's theorem [Z1]. To prove the more general formulation above, one first reduces to the case $H = \mathrm{Spf}\, B$. Indeed to give a group homomorphism $\widehat{\Lambda} \times H \to G$ over H is the same thing as to give for any morphism $\mathrm{Spf}\, B \to H$ a morphism $\widehat{\Lambda} \times \mathrm{Spf}\, B \to \mathrm{Spf}\, B \times_H G$ of groups over $\mathrm{Spf}\, B$.

Secondly the case $H = \mathrm{Spf}\, B$ is reduced to the usual theorem using the equivalence of pointed left exact functors over $\mathrm{Spf}\, B$ and left exact functors on Nil_B. □

The following map is a homomorphism of abelian functors:

$$\begin{aligned} \widehat{\Lambda}(\mathcal{N}) &\longrightarrow \widehat{W}(\mathcal{N}), \\ \prod(1 - x_i t^i) &\longmapsto (x_{p^0}, x_{p^1}, \ldots, x_{p^k} \ldots) \end{aligned} \tag{196}$$

If we compose this with $D \subset \widehat{\Lambda}$, we obtain an inclusion $D \subset \widehat{W}$.

Let R be a $\mathbb{Q}$-algebra. Then the usual power series for the natural logarithm provides an isomorphism of abelian groups:

$$\log : \widehat{\Lambda}(\mathcal{N}) = (1 + t\mathcal{N}[t])^+ \longrightarrow t\mathcal{N}[t]$$

The formula $\varepsilon_1\Big(\sum_{i\geq 1} n_i t^i\Big) = \sum n_{p^k} t^{p^k}$. defines a projector $\varepsilon_1 : t\mathcal{N}[t] \to t\mathcal{N}[t]$. Then Cartier has shown that ε_1 induces an endomorphism of $\widehat{\Lambda}$ over any $\mathbb{Z}_{(p)}$-algebra. Moreover the homomorphism (196) induces an isomorphism:

$$\epsilon_1 \widehat{\Lambda} \cong \widehat{W}.$$

We use this to embed $\widehat{W}$ into $\widehat{\Lambda}$.

Mumford remarked that Cartier's theorem provides a section κ of the natural inclusion

$$\operatorname{Hom}_{\text{groups}/H}(\widehat{W} \times H, G) \longrightarrow \operatorname{Hom}_{\text{pointed functors /H}}(\widehat{W} \times H, G). \tag{197}$$

Indeed, let $\alpha : \widehat{W} \times H \to G$ be a map of pointed set-valued functors. We define $\widetilde{\kappa}(\alpha) : \widehat{\Lambda} \times H \to G$ to be the unique group homomorphism, which coincides with α on $D \times H$ (use theorem 113). We get $\kappa(\alpha)$ as the composition of $\widetilde{\kappa}(\alpha)$ with the inclusion $\widehat{W} \times H \subset \widehat{\Lambda} \times H$.

Proposition 114. — *Let $F : \mathrm{Nil}_R \to (Ab)$ be an exact functor. Then*

$$\operatorname{Ext}^1(\widehat{W}, F) = 0,$$

where the Ext-*group is taken in the category of abelian sheaves on $\mathcal{T}$.*

Proof. — By the remark (193) a short exact sequence $0 \to F \to G \to \widehat{W} \to 0$ has a set–theoretical section $s : \widehat{W} \to G$. Then $\kappa(s)$ splits the sequence. $\square$

Remark. — It is clear that this proposition also has a relative version. Namely in the category of abelian sheaves over any prorepresentable sheaf H in $\mathcal{T}$. we have:

$$\operatorname{Ext}^1_{\text{groups}/H}(\widehat{W} \times H, F \times H) = 0,$$

if H is prorepresentable. Indeed consider an extension

$$0 \longrightarrow F \times H \longrightarrow G \xrightarrow{\ \pi\ } \widehat{W} \times H \longrightarrow 0. \tag{198}$$

Then G is an F torsor over $\widehat{W} \times H$ and hence trivial. Let σ be any section of π. Let us denote by $\iota : H \to \widehat{W} \times H$ and $s_G : H \to G$ the zero sections of the group laws relative to H. We obtain a morphism $s_G - \sigma\iota : H \to F$. Let $\mathrm{pr}_2 : \widehat{W} \times H \to H$ be the projection. Then we define a new section of π by

$$\sigma_{\text{new}} = \sigma + (s_G - \sigma\iota)\,\mathrm{pr}_2\,. \tag{199}$$

Then σ_{new} is a morphism of pointed functors over H, i.e. it respects the sections s_G and ι. Hence we may apply the section κ of (197) to σ_{new}. This gives the desired section of (198).

If $G : \mathrm{Nil}_R \to (Ab)$ is any functor, we set

$$G^+(\mathcal{N}) = \operatorname{Ker}(G(\mathcal{N}) \longrightarrow G(0)). \tag{200}$$

Because of the map $0 \to \mathcal{N}$ we obtain a functorial decomposition

$$G(\mathcal{N}) = G^+(\mathcal{N}) \oplus G(0),$$

which is then respected by morphisms of functors. If G is in the category of abelian sheaves we find:

$$\operatorname{Ext}^1_{Ab}(\widehat{W}, G) = \operatorname{Ext}^1_{Ab}(\widehat{W}, G^+),$$

which vanishes if G^+ is exact.

Cartier's theorem applies to an abelian functor G, such that G^+ is left exact:

$$\mathrm{Hom}(\widehat{\Lambda}, G) \simeq \mathrm{Hom}(\widehat{\Lambda}, G^+) \simeq G^+(XR[\![X]\!]),$$

where the Hom are taken in the category of abelian functors on Nil_R. If F, G are abelian sheaves on $\mathcal{T}$, the sheaf of local homomorphisms is defined as follows:

$$\begin{aligned} \underline{\mathrm{Hom}}(F,G)(A^+) &= \mathrm{Hom}(F_A, G_A), \quad A \in \mathrm{Aug}\, R. \\ \underline{\mathrm{Hom}}(F,G)^+(A^+) &= \mathrm{Ker}(\mathrm{Hom}(F_A, G_A) \longrightarrow \mathrm{Hom}(F_R, G_R)). \end{aligned} \tag{201}$$

Cartier's theorem tells us that for a left exact functor G:

$$\begin{aligned} \underline{\mathrm{Hom}}(\widehat{\Lambda}, G)(A^+) &= G(XA[\![X]\!]) \\ \underline{\mathrm{Hom}}^+(\widehat{\Lambda}, G)(A^+) &= G(XA^+[\![X]\!]) \end{aligned} \tag{202}$$

In particular the last functor $\underline{\mathrm{Hom}}^+(\widehat{\Lambda}, G)$ is exact if G is exact. Using the projector ϵ_1 we see that $\underline{\mathrm{Hom}}^+(\widehat{W}, G)$ is also exact.

***Proposition 115* (Mumford).** — *Let F be an exact functor. Then*

$$\mathrm{Biext}^1(\widehat{W} \times \widehat{W}, F) = 0.$$

Proof. — We strongly recommend to read Mumford's proof, but here is his argument formulated by the machinery of homological algebra. We have an exact sequence (SGA7):

$$0 \to \mathrm{Ext}^1(\widehat{W}, \underline{\mathrm{Hom}}(\widehat{W}, F)) \longrightarrow \mathrm{Biext}^1(\widehat{W} \times \widehat{W}, F) \longrightarrow \mathrm{Hom}(\widehat{W}, \underline{\mathrm{Ext}}^1(\widehat{W}, F)).$$

The outer terms vanish, by proposition (114) and because the functor $\underline{\mathrm{Hom}}^+(\widehat{W}, F)$ is exact. $\square$

Our next aim is the computation of $\mathrm{Bihom}(\widehat{W} \times \widehat{W}, \widehat{\mathbb{G}}_m)$. Let us start with some remarks about endomorphisms of the functors W and $\widehat{W}$.

Let R be any unitary ring. By definition the local Cartier ring $\mathbb{E}_R$ relative to p acts from the right on $\widehat{W}(\mathcal{N})$. Explicitly this action is given as follows. The action of $W(R)$:

$$\widehat{W}(\mathcal{N}) \times W(R) \longrightarrow \widehat{W}(\mathcal{N}), \tag{203}$$

is induced by the multiplication in the Witt ring $W(R|\mathcal{N}|)$. The action of the operators $F, V \in \mathbb{E}_R$ is as follows

$$\underline{n}F = {}^V\underline{n}, \quad \underline{n}V = {}^F\underline{n}, \tag{204}$$

where on the right hand side we have the usual Verschiebung and Frobenius on the Witt ring. An arbitrary element of $\mathbb{E}_R$ has the form $\sum_{i=0}^{\infty} V^i \xi_i + \sum_{j=1}^{\infty} \mu_j F^j$, $\xi_i, \mu_j \in W(R)$, where $\lim \mu_j = 0$ in the V-adic topology on $W(R)$ (see corollary 8). We may write such an element (not uniquely) in the form: $\sum V^n \alpha_n$, where $\alpha_n \in W(R)[F]$.

By the following lemma we may extend the actions (203) and (203) to a right action of $\mathbb{E}_R$ on $\widehat{W}(\mathcal{N})$.

Lemma 116. — *For any $\underline{n} \in \widehat{W}(\mathcal{N})$ there exists a number r such that ${}^{F^r}\underline{n} = 0$.*

Proof. — Since $\underline{n}$ is a finite sum of elements of the form ${}^{V^s}[n]$, $n \in \mathcal{N}$ it suffices to show the lemma for $\underline{n} = [n]$. This is trivial. □

We note that in the case, where p is nilpotent in R there is a number r, such that ${}^{F^r}W(\mathcal{N}) = 0$. Hence in this case the Cartier ring acts from the right on $W(\mathcal{N})$.

We write the opposite ring to $\mathbb{E}_R$ in the following form:

$$ {}^t\mathbb{E}_R = \left\{ \sum_{i=1}^{\infty} V^i \xi_i + \sum_{j=0}^{\infty} \mu_j F^j \mid \xi_i, \mu_j \in W(R), \lim \xi_i = 0 \right\} \tag{205} $$

The limit is taken in the V-adic topology. The addition and multiplication is defined in the same way as in the Cartier ring, i.e. we have the relations:

$$ FV = p, \quad V\xi F = {}^V\xi, \quad F\xi = {}^F\xi F, \quad \xi V = V\,{}^F\xi. \tag{206} $$

Then we have the antiisomorphism

$$ t : \mathbb{E}_R \longrightarrow {}^t\mathbb{E}_R, $$

which is defined by $t(F) = V$, $t(V) = F$ and $t(\xi) = \xi$ for $\xi \in W(R)$. The ring ${}^t\mathbb{E}_R$ acts from the left on $\widehat{W}(\mathcal{N})$:

$$ F\underline{n} = {}^F\underline{n}, \quad V\underline{n} = {}^V\underline{n}. $$

It is the endomorphism ring of $\widehat{W}$ by Cartier theory.

We define $\overline{\mathbb{E}}_R$ to be the abelian group of formal linear combinations of the form:

$$ \overline{\mathbb{E}}_R = \left\{ \sum_{i=1}^{\infty} V^i \xi_i + \sum_{j=0}^{\infty} \mu_j F^j. \right\} \tag{207} $$

There is in general no ring structure on $\overline{\mathbb{E}}_R$, which satisfies the relations (206). The abelian group ${}^t\mathbb{E}_R$ is a subgroup of $\overline{\mathbb{E}}_R$ by regarding an element from the right hand side of (205) as an element from the right hand side of (207). Obviously the left action of ${}^t\mathbb{E}_R$ on $\widehat{W}(\mathcal{N})$ extends to a homomorphism of abelian groups

$$ \overline{\mathbb{E}}_R \longrightarrow \operatorname{Hom}(\widehat{W}, W). \tag{208} $$

We will write this homomorphism as

$$ \underline{n} \longmapsto u\underline{n} $$

since it extends the left action of ${}^t\mathbb{E}_R$. We could also extend the right action of $\mathbb{E}_R$:

$$ \underline{n} \longmapsto \underline{n}u. $$

Of course we get the formula

$$ \underline{n}u = {}^t u\underline{n}. $$

The first theorem of Cartier theory tells us again that (208) is an isomorphism. By the remark after lemma (116), it is clear that in the case where p is nilpotent in R the homomorphism (208) extends to a homomorphism:

$$\overline{\mathbb{E}}_R \longrightarrow \operatorname{End}(W) \tag{209}$$

The reader can verify that there exists a ring structure on $\overline{\mathbb{E}}_R$ that satisfies (206), if p is nilpotent in R. In this case the map $t : \mathbb{E}_R \to {}^t\mathbb{E}_R$ extends to an antiinvolution of the ring $\overline{\mathbb{E}}_R$. Then (209) becomes a homomorphism of rings.

By Cartier theory we have an exact sequence:

$$0 \longrightarrow \widehat{W}(\mathcal{N}) \xrightarrow{\cdot(F-1)} \widehat{W}(\mathcal{N}) \xrightarrow{\text{hex}} \widehat{\mathbb{G}}_m(\mathcal{N}) \longrightarrow 0 \tag{210}$$

The second arrow is the right multiplication by $(F-1) \in \mathbb{E}_R$, and hex is the so called Artin-Hasse exponential. For the following it is enough to take (210) as a definition of $\widehat{\mathbb{G}}_m$. But we include the definition of hex for completeness. It is the composition of the following maps (compare (196)):

$$\widehat{W}(\mathcal{N}) \xrightarrow{\sim} \varepsilon_1\widehat{\Lambda}(\mathcal{N}) \subset \widehat{\Lambda}(\mathcal{N}) = (1 + t\mathcal{N}[t])^\times \xrightarrow{t=1} (1+\mathcal{N})^\times. \tag{211}$$

It is easy to produce a formula for hex but still easier if one does not know it. The verification of the exactness of (210) is done by reduction to the case of a $\mathbb{Q}$-algebra $\mathcal{N}$. We will skip this.

Proposition 117. — *The Artin-Hasse exponential defines an isomorphism of abelian groups:*

$$\varkappa : W(R) \longrightarrow \operatorname{Hom}(\widehat{W}, \widehat{\mathbb{G}}_m) \tag{212}$$

An element $\xi \in W(R)$ corresponds to the following homomorphism $\varkappa_\xi : \widehat{W} \to \widehat{\mathbb{G}}_m$. If $u \in \widehat{W}(\mathcal{N})$, we have:

$$\varkappa_\xi(u) = \operatorname{hex}(\xi \cdot u).$$

Proof. — This is a well–known application of the first main theorem of Cartier theory of p-typical curves. Let $[X] = (X, 0 \ldots 0 \ldots)$ be the standard p-typical curve in $\widehat{W}(XK[\![X]\!])$. We have to show that $\operatorname{hex}(\xi \cdot [X])$ gives exactly all p-typical curves of $\widehat{\mathbb{G}}_m$ if ξ runs through $W(R)$. We set $\gamma_m = \operatorname{hex}([X])$. This is the standard p-typical curve in $\widehat{\mathbb{G}}_m$. It satisfies $F\gamma_m = \gamma_m$ by (210). By definition of the action of the Cartier ring on the p-typical curves of $\widehat{\mathbb{G}}_m$ we have:

$$\operatorname{hex}(\xi\lfloor X\rceil) = \xi\gamma_m.$$

If $\xi = \sum V^i[\xi_i]F^i$ as elements of $\mathbb{E}_R$ we obtain:

$$\xi\gamma_m = \sum_{i=0}^{\infty} V^i[\xi_i]\gamma_m.$$

These are exactly the p-typical curves of $\widehat{\mathbb{G}}_m$. $\square$

From (200) we deduce the following sheafified version of the proposition:

Corollary 118. — *The homomorphism (212) gives rise to an isomorphism of functors on* Nil_R*:*

$$\varkappa : W(\mathcal{N}) \longrightarrow \underline{\mathrm{Hom}}(\widehat{W}, \widehat{\mathbb{G}}_m)^+(\mathcal{N}).$$

We are now ready to classify the bilinear forms $\mathrm{Bihom}(\widehat{W} \times \widehat{W}, \widehat{\mathbb{G}}_m)$. To each $u \in \overline{\mathbb{E}}_R$ we associate the bilinear form β_u:

$$\begin{array}{ccccccc} \widehat{W}(\mathcal{N}) \times \widehat{W}(\mathcal{N}) & \longrightarrow & W(\mathcal{N}) \times \widehat{W}(\mathcal{N}) & \xrightarrow{\text{mult.}} & \widehat{W}(\mathcal{N}) & \xrightarrow{\text{hex}} & \widehat{\mathbb{G}}_m(\mathcal{N}) \\ \xi \times \eta & \longmapsto & \xi u \times \eta & \longmapsto & (\xi u)\eta & & \end{array}$$

Proposition 119. — *We have the relations:*

$$\beta_u(\xi, \eta) = \beta_{{}^t u}(\eta, \xi)$$
$$\mathrm{hex}(\xi u)\eta = \mathrm{hex}\, \xi(u\eta).$$

Proof. — Clearly the second relation implies the first one. For $u \in W(R)$ we have $(\xi u)\eta = \xi(u\eta)$. Hence the assertion is trivial.

First we do the case $u = F$:

$$\mathrm{hex}(\xi F)\eta = \mathrm{hex}\, {}^V\xi\eta = \mathrm{hex}\, {}^V(\xi^F\eta) = \mathrm{hex}(\xi^F\eta)F = \mathrm{hex}\, \xi^F\eta = \mathrm{hex}\, \xi(F\eta).$$

The fourth equation holds because:

$$\mathrm{hex}(\widehat{W}(\mathcal{N})(F-1)) = 0$$

Secondly let $u = V$:

$$\mathrm{hex}(\xi V)\eta = \mathrm{hex}\, {}^F\xi\eta = \mathrm{hex}\, {}^V({}^F\xi\eta) = \mathrm{hex}\, \xi^V\eta = \mathrm{hex}\, \xi(V\eta).$$

Finally we have to treat the general case $u = \sum_{i=1}^{\infty} V^i w_i + \sum_{i=0}^{\infty} w_{-i}F^i$. For a finite sum there is no problem. The general case follows from the following statement:

For given $\xi, \eta \in \widehat{W}(\mathcal{N})$ there is an integer m_0, such that for any $w \in W(R)$:

$$\mathrm{hex}(\xi w F^m)\eta = 0, \qquad \mathrm{hex}(\xi V^m w)\eta = 0.$$

Indeed, this is an immediate consequence of lemma 116. □

***Proposition 120* (Mumford)**. — *The map:*

$$\begin{aligned} \overline{\mathbb{E}}_R &\longrightarrow \mathrm{Bihom}(\widehat{W} \times \widehat{W}, \widehat{\mathbb{G}}_m), \\ u &\longmapsto \beta_u(\xi, \eta) = \mathrm{hex}(\xi u)\eta \end{aligned} \tag{213}$$

is an isomorphism of abelian groups.

Proof. — One starts with the natural isomorphism.

$$\mathrm{Bihom}(\widehat{W} \times \widehat{W}, \widehat{\mathbb{G}}_m) \simeq \mathrm{Hom}(\widehat{W}, \underline{\mathrm{Hom}}^+(\widehat{W}, \mathbb{G}_m)).$$

The sheaf $\underline{\mathrm{Hom}}^{+}(\widehat{W},\widehat{\mathbb{G}}_m)$ is easily computed by the first main theorem of Cartier theory: Let $A = R \oplus \mathcal{N}$ be an augmented nilpotent R-algebra. Then one defines a homomorphism:

$$W(\mathcal{N}) \longrightarrow \underline{\mathrm{Hom}}^{+}(\widehat{W},\widehat{\mathbb{G}}_m)(\mathcal{N}) \subset \mathrm{Hom}(\widehat{W}_A,\widehat{\mathbb{G}}_{m_A}), \tag{214}$$

as follows. For any nilpotent A-algebra $\mathcal{M}$ the multiplication $\mathcal{N} \times \mathcal{M} \to \mathcal{M}$ induces on the Witt vectors the multiplication:

$$W(\mathcal{N}) \times \widehat{W}(\mathcal{M}) \longrightarrow \widehat{W}(\mathcal{M}).$$

Hence any $\omega \in W(\mathcal{N})$ induces a morphism $\widehat{W}(\mathcal{M}) \to \widehat{\mathbb{G}}_m(\mathcal{M}), \xi \mapsto \mathrm{hex}\,\omega\xi$. Since by the first main theorem of Cartier theory:

$$W(A) \longrightarrow \mathrm{Hom}(\widehat{W}_A,\widehat{\mathbb{G}}_{m_A}),$$

is an isomorphism. One deduces easily that (214) is an isomorphism. If we reinterpret the map (213) in terms of the isomorphism (214) just described, we obtain:

$$\begin{aligned} \overline{\mathbb{E}}_R &\longrightarrow \mathrm{Hom}(\widehat{W}, W) \\ u &\longmapsto (\xi \mapsto \xi u) \end{aligned} \tag{215}$$

But this is the isomorphism (208). □

4.3. The biextension of a bilinear form of displays. — After this update of Mumford's theory we come to the main point of the whole duality theory: Let $\mathcal{P}$ and $\mathcal{P}'$ be 3n-displays over R. We are going to define a natural homomorphism:

$$\mathrm{Bil}(\mathcal{P} \times \mathcal{P}', \mathcal{G}_m) \longrightarrow \mathrm{Biext}^1(BT_{\mathcal{P}} \times BT_{\mathcal{P}'}.\widehat{\mathbb{G}}_m) \tag{216}$$

Let $(\ ,\ ) : P \times P' \to W(R)$ be a bilinear form of 3n-displays (18). For $\mathcal{N} \in \mathrm{Nil}_R$ this induces a pairing

$$(\ ,\ ) : \widehat{P}_{\mathcal{N}} \times \widehat{P}'_{\mathcal{N}} \longrightarrow \widehat{W}(\mathcal{N}), \tag{217}$$

(Compare chapter 3 for the notation). More precisely, if $x = \xi \otimes u \in \widehat{P}_{\mathcal{N}} = \widehat{W}(\mathcal{N}) \otimes_{W(R)} P$ and $x' = \xi' \otimes u' \in \widehat{P}'_{\mathcal{N}} = \widehat{W}(\mathcal{N}) \otimes_{W(R)} P'$, we set $(x', x) = \xi\xi'(u, u') \in \widehat{W}(\mathcal{N})$, where the product on the right hand side is taken in $W(R|\mathcal{N}|)$.

To define the biextension associated to (217), we apply a sheafified version proposition 106 to the exact sequences of functors on Nil_R:

$$0 \longrightarrow \widehat{Q}_{\mathcal{N}} \xrightarrow{V^{-1} - \mathrm{id}} \widehat{P}_{\mathcal{N}} \longrightarrow BT_{\mathcal{P}}(\mathcal{N}) \longrightarrow 0$$
$$0 \longrightarrow \widehat{Q}'_{\mathcal{N}} \xrightarrow{V^{-1} - \mathrm{id}} \widehat{P}'_{\mathcal{N}} \longrightarrow BT_{\mathcal{P}'}(\mathcal{N}) \longrightarrow 0.$$

The proposition 106 combined with proposition 115, tells us that any element in $\mathrm{Biext}^1(BT_{\mathcal{P}} \times BT_{\mathcal{P}'}, \widehat{\mathbb{G}}_m)$ is given by a pair of bihomomorphisms

$$\begin{aligned} \alpha_1 : \widehat{Q}_{\mathcal{N}} \times \widehat{P}'_{\mathcal{N}} &\longrightarrow \widehat{\mathbb{G}}_m(\mathcal{N}) \\ \alpha_2 : \widehat{P}_{\mathcal{N}} \times \widehat{Q}'_{\mathcal{N}} &\longrightarrow \widehat{\mathbb{G}}_m(\mathcal{N}), \end{aligned}$$

which agree on $\widehat{Q}_{\mathcal{N}} \times \widehat{Q}'_{\mathcal{N}}$.

In the following formulas an element $y \in \widehat{Q}_{\mathcal{N}}$ is considered as an element of $\widehat{P}_{\mathcal{N}}$ by the natural inclusion id. We set

$$(218)\qquad \begin{aligned} \alpha_1(y,x') &= \operatorname{hex}(V^{-1}y,x'), \quad \text{for } y \in \widehat{Q}_{\mathcal{N}},\ x' \in \widehat{P}'_{\mathcal{N}}.\\ \alpha_2(x,y') &= -\operatorname{hex}(x,y'), \quad \text{for } x \in \widehat{P}_{\mathcal{N}},\ y \in \widehat{Q}'_{\mathcal{N}}.\end{aligned}$$

We have to verify that α_1 and α_2 agree on $\widehat{Q}_{\mathcal{N}} \times \widehat{Q}'_{\mathcal{N}}$, i.e. that the following equation holds:

$$\alpha_1(y, V^{-1}y' - y') = \alpha_2(V^{-1}y - y, y').$$

This means that:

$$\operatorname{hex}(V^{-1}y, V^{-1}y' - y') = -\operatorname{hex}(V^{-1}y - y, y'),$$

which is an immediate consequence of (1.14):

$$\operatorname{hex}(V^{-1}y, V^{-1}y') = \operatorname{hex}^V(V^{-1}y, V^{-1}y') = \operatorname{hex}(y,y').$$

We define the homomorphism (216) to be the map which associates to the bilinear form $(\ ,\ ) \in \operatorname{Bil}(\mathcal{P} \times \mathcal{P}', \mathcal{G})$ the biextension given by the pair α_1, α_2.

Remark. — Consider the biextension defined by the pair of maps $\beta_1 : \widehat{Q}_{\mathcal{N}} \times \widehat{P}'_{\mathcal{N}} \to \widehat{W}(\mathcal{N})$ and $\beta_2 : \widehat{P}_{\mathcal{N}} \times \widehat{Q}'_{\mathcal{N}} \to \widehat{W}_{\mathcal{N}}$ defined as follows:

$$(219)\qquad \begin{aligned} \beta_1(y,x') &= \operatorname{hex}(y,x'), \qquad y \in \widehat{Q}_{\mathcal{N}},\ x' \in \widehat{P}'_{\mathcal{N}}\\ \beta_2(x,y') &= -\operatorname{hex}(x,V^{-1}y'), \quad x \in \widehat{P}_{\mathcal{N}},\ y \in \widehat{Q}'_{\mathcal{N}}.\end{aligned}$$

We claim that the biextension defined by (219) is isomorphic to the biextension defined by (218). Indeed by the proposition 106 we may add to the pair (β_1, β_2) the bihomomorphism

$$\operatorname{hex}(\ ,\ ) : \widehat{P}_{\mathcal{N}} \times \widehat{P}'_{\mathcal{N}} \longrightarrow \widehat{\mathbb{G}}_m(\mathcal{N})$$

obtained from (217). One verifies readily:

$$\begin{aligned} \beta_1(y,x') + \operatorname{hex}(V^{-1}y - y, x') &= \alpha_1(y,x')\\ \beta_2(x,y') + \operatorname{hex}(x, V^{-1}y' - y') &= \alpha_2(y,y').\end{aligned}$$

Remark. — Let $G \xrightarrow{\pi} B \times C$ be a biextension by an abelian group A, with the relative group laws $+_B$ and $+_C$. Let $s : B \times C \to C \times B,\ (b,c) \mapsto (c,b)$ be the switch of factors, and set $\pi^s = s \circ \pi$. Then $(G, \pi^s, +_C, +_B)$ is an object in $\operatorname{BIEXT}(C \times B, A)$. We will denote this biextension simply by G^s. Let us suppose that $B = C$. Then we call a biextension G symmetric if G and G^s are isomorphic.

Let us start with the bilinear form

$$(\ ,\ ) : \mathcal{P} \times \mathcal{P}' \longrightarrow \mathcal{G}_m.$$

We denote by G the biextension, which corresponds to the pair (218) of bihomomorphisms α_1 and α_2. Clearly the biextension G^s corresponds to the pair of bihomomorphisms $\alpha_1^s : \widehat{Q}'_{\mathcal{N}} \times \widehat{P}_{\mathcal{N}} \to \widehat{W}(\mathcal{N})$ and $\alpha_2^s : \widehat{P}'_{\mathcal{N}} \times \widehat{Q}_{\mathcal{N}} \to \widehat{W}(\mathcal{N})$, which are defined by the equations:

$$\begin{aligned} \alpha_1^s(y',x) &= \alpha_2(x,y') = -\operatorname{hex}(x,y') \\ \alpha_2^s(x',y) &= \alpha_1(y,x') = \operatorname{hex}(Vy,x') \end{aligned} \tag{220}$$

If we define a bilinear form:

$$(\ ,\ )_s : \mathcal{P}' \times \mathcal{P} \longrightarrow \mathcal{G}_m,$$

by $(x',x)_s = (x,x')$, we see by the previous remark that the biextension defined by (220) corresponds to the bilinear form $-(x',x)_s$. We may express this by the commutative diagram:

$$\begin{array}{ccc} \operatorname{Bil}(\mathcal{P} \times \mathcal{P}', \mathcal{G}) & \longrightarrow & \operatorname{Biext}^1(BT_{\mathcal{P}} \times BT_{\mathcal{P}'}, \widehat{\mathbb{G}}_m) \\ {\scriptstyle -s}\downarrow & & \downarrow{\scriptstyle s} \\ \operatorname{Bil}(\mathcal{P}' \times \mathcal{P}, \mathcal{G}) & \longrightarrow & \operatorname{Biext}^1(BT_{\mathcal{P}'} \times BT_{\mathcal{P}}, \widehat{\mathbb{G}}_m) \end{array}$$

Let $\mathcal{P} = \mathcal{P}'$ and assume that the bilinear form $(\ ,\ )$ is alternating, i.e. the corresponding bilinear form of $W(R)$-modules $P \times P \to W(R)$ is alternating. Then it follows that the corresponding biextension G in $\operatorname{Biext}^1(BT_{\mathcal{P}} \times BT_{\mathcal{P}}, \widehat{\mathbb{G}}_m)$ is symmetric.

4.4. The duality isomorphism. — Assume we are given a bilinear form $(\ ,\ ) : \mathcal{P} \times \mathcal{P}' \to \mathcal{G}_m$ as in definition 18. Let $G = BT_{\mathcal{P}}$ and $G' = BT_{\mathcal{P}'}$ be the formal groups associated by theorem 81. The Cartan isomorphism $\operatorname{Biext}^1(G \times G', \widehat{\mathbb{G}}_m) = \operatorname{Ext}^1(G \otimes^{\mathbb{L}} G', \widehat{\mathbb{G}}_m) \xrightarrow{\sim} \operatorname{Ext}^1(G, R\underline{\operatorname{Hom}}(G', \widehat{\mathbb{G}}_m))$ provides a canonical homomorphism

$$\operatorname{Biext}^1(G \times G', \widehat{\mathbb{G}}_m) \longrightarrow \operatorname{Hom}(G, \underline{\operatorname{Ext}}^1(G', \widehat{\mathbb{G}}_m)). \tag{221}$$

Let us describe the element on the right hand side, which corresponds to the biextension defined by the pair of bihomomorphisms α_1 and α_2 given by (218). For this purpose we denote the functor $\mathcal{N} \mapsto \widehat{P}_{\mathcal{N}}$ simply by $\widehat{P}$, and in the same way we define functors $\widehat{Q}, \widehat{P}', \widehat{Q}'$. We obtain a diagram of sheaves:

$$\begin{array}{ccccccccc} & & \underline{\operatorname{Hom}}(\widehat{P}', \widehat{\mathbb{G}}_m) & \xrightarrow{(V^{-1}-\mathrm{id})^*} & \underline{\operatorname{Hom}}(\widehat{Q}', \widehat{\mathbb{G}}_m) & \longrightarrow & \underline{\operatorname{Ext}}^1(G', \widehat{\mathbb{G}}_m) & \longrightarrow & 0 \\ & & {\scriptstyle \alpha_1}\uparrow & & \uparrow{\scriptstyle \alpha_2} & & & & \\ 0 & \longrightarrow & \widehat{Q} & \xrightarrow{(V^{-1}-\mathrm{id})} & \widehat{P} & \longrightarrow & G & \longrightarrow & 0 \end{array} \tag{222}$$

Hence $(V^{-1} - \mathrm{id})^*$ is the homomorphism obtained from $V^{-1} - \mathrm{id} : \widehat{Q}' \to \widehat{P}'$ by applying the functor $\underline{\operatorname{Hom}}(-, \widehat{\mathbb{G}}_m)$. The horizontal rows are exact. The square is commutative because the restriction of α_1 to $\widehat{Q} \times \widehat{Q}'$ agrees with the restriction of α_2 in the sense of the inclusions defined by $V^{-1} - \mathrm{id}$. Hence (222) gives the desired $G \to \underline{\operatorname{Ext}}^1(G', \widehat{\mathbb{G}}_m)$.

The functors in the first row of (221) may be replaced by their $^+$-parts (see (200)). Then we obtain a diagram with exact rows:

$$
\begin{tikzcd}
 & \underline{\mathrm{Hom}}(\widehat{P}', \widehat{\mathbb{G}}_m)^+ \arrow[r] & \underline{\mathrm{Hom}}(\widehat{Q}', \widehat{\mathbb{G}}_m)^+ \arrow[r] & \underline{\mathrm{Ext}}^1(G', \widehat{\mathbb{G}}_m)^+ \arrow[r] & 0 \\
0 \arrow[r] & \widehat{Q} \arrow[r] \arrow[u] & \widehat{P} \arrow[r] \arrow[u] & G \arrow[r] \arrow[u] & 0
\end{tikzcd}
\tag{223}
$$

The first horizontal arrow in this diagram is injective, if $\mathcal{P}'$ is a display. Indeed, the group G' is p-divisible and by the rigidity for homomorphisms of p-divisible groups:

$$\underline{\mathrm{Hom}}(G', \widehat{\mathbb{G}}_m)^+ = 0. \tag{224}$$

Remark. — Let $\mathcal{P}'$ be a display. The following proposition 121 will show that the functor $\underline{\mathrm{Ext}}^1(G', \widehat{\mathbb{G}}_m)^+$ is a formal group. We will call it the dual formal group. The isomorphism (226) relates it to the dual display.

By the corollary 118 one obviously obtains an isomorphism

$$W(\mathcal{N}) \otimes_{W(R)} P^t \longrightarrow \underline{\mathrm{Hom}}(\widehat{P}, \widehat{\mathbb{G}}_m)^+(\mathcal{N}). \tag{225}$$

Here $P^t = \mathrm{Hom}_{W(R)}(P, W(R))$ is the dual $W(R)$-module. Therefore the functor $\underline{\mathrm{Hom}}(\widehat{P}', \widehat{\mathbb{G}}_m)^+$ is exact, and the first row of (223) is by proposition 109 exact in the sense of presheaves, if $\mathcal{P}'$ is a display.

Proposition 121. — *Let $\mathcal{P}$ be a display and $\mathcal{P}^t$ be the dual 3n-display. By definition 19 we have a natural pairing*

$$\langle\, , \rangle : \mathcal{P}^t \times \mathcal{P} \longrightarrow \mathcal{G},$$

which defines by (216) a biextension in $\mathrm{Biext}^1(BT_{\mathcal{P}^t} \times BT_{\mathcal{P}}, \widehat{\mathbb{G}}_m)$. *By (221) this biextension defines a homomorphism of sheaves*

$$BT_{\mathcal{P}^t} \longrightarrow \underline{\mathrm{Ext}}^1(BT_{\mathcal{P}}, \widehat{\mathbb{G}}_m)^+. \tag{226}$$

The homomorphism (226) is an isomorphism.

Proof. — In our situation (223) gives a commutative diagram with exact rows in the sense of presheaves:

$$
\begin{tikzcd}
0 \arrow[r] & \underline{\mathrm{Hom}}(\widehat{P}, \widehat{\mathbb{G}}_m)^+ \arrow[r] & \underline{\mathrm{Hom}}(\widehat{Q}, \widehat{\mathbb{G}}_m)^+ \arrow[r] & \underline{\mathrm{Ext}}^1(G, \widehat{\mathbb{G}}_m)^+ \arrow[r] & 0 \\
0 \arrow[r] & \widehat{Q}^t \arrow[r] \arrow[u] & \widehat{P}^t \arrow[r] \arrow[u] & G^t \arrow[r] \arrow[u] & 0.
\end{tikzcd}
\tag{227}
$$

Here we use the notation $G = BT_{\mathcal{P}}$, $G^t = BT_{\mathcal{P}^t}$. Let us make the first commutative square in (227) more explicit.

The bilinear pairing

$$
\begin{array}{lcccl}
(228) & W(\mathcal{N}) \otimes_{W(R)} P^t \times & \widehat{W}(\mathcal{N}) \otimes_{W(R)} P & \longrightarrow & \widehat{\mathbb{G}}_m(\mathcal{N}), \\
& \xi \otimes x^t \quad \times & u \otimes x & \longmapsto & \operatorname{hex}(\xi u \langle x^t, x\rangle)
\end{array}
$$

provides by the corollary 118 an isomorphism of functors

$$
W(\mathcal{N}) \otimes_{W(R)} P^t \longrightarrow \underline{\mathrm{Hom}}(\widehat{P}, \widehat{\mathbb{G}}_m)^+(\mathcal{N}). \tag{229}
$$

In order to express $\underline{\mathrm{Hom}}(\widehat{Q}, \widehat{\mathbb{G}}_m)^+$ in a similar way, we choose a normal decomposition $P = L \oplus T$. Let us denote by $L^* = \mathrm{Hom}_{W(R)}(L, W(R))$ and $T^* = \mathrm{Hom}_{W(R)}(T, W(R))$ the dual modules. In terms of the chosen normal decomposition the dual 3n-display $\mathcal{P}^t = (P^t, Q^t, F, V^{-1})$ may be described as follows.

We set $P^t = P^*, Q^t = T^* \oplus I_R L^*$. Then we have a normal decomposition

$$
P^t = L^t \oplus T^t,
$$

where $L^t = T^*$ and $T^t = L^*$. To define F and V^{-1} for $\mathcal{P}^t$ it is enough to define ^{F}-linear maps:

$$
V^{-1} : L^t \longrightarrow P^t \quad F : T^t \longrightarrow P^t.
$$

We do this using the direct decomposition

$$
P = W(R)V^{-1}L \oplus W(R)FT.
$$

For $x^t \in L^t = T^*$ we set:

$$
\begin{array}{rl}
\langle V^{-1}x^t, wFy\rangle = {}^{w^F}\langle x^t, y\rangle, & w \in W(R),\ y \in T \\
\langle V^{-1}x^t, wV^{-1}x\rangle = 0, & x \in L.
\end{array}
$$

For $y^t \in T^t = L^*$ we set:

$$
\begin{array}{rl}
\langle Fy^t, wFy\rangle = 0, & y \in T \\
\langle Fy^t, wV^{-1}x\rangle = {}^{w^F}\langle {}^t y, x\rangle, & x \in L.
\end{array}
$$

The bilinear pairing:

$$
\begin{array}{lcl}
W(\mathcal{N}) \otimes_{F,W(R)} T^* \times \widehat{W}(\mathcal{N}) \otimes_{F,W(R)} T & \longrightarrow & \widehat{\mathbb{G}}_m(\mathcal{N}) \\
\xi \otimes x^t \times u \otimes y & \longmapsto & \operatorname{hex}(\xi u^F \langle x^t, y\rangle)
\end{array}
$$

defines a morphism

$$
W(\mathcal{N}) \otimes_{F,W(R)} T^* \longrightarrow \underline{\mathrm{Hom}}(\widehat{W} \otimes_{F,W(R)} T, \widehat{\mathbb{G}}_m)^+(\mathcal{N}), \tag{230}
$$

where $\widehat{W} \otimes_{F,W(R)} T$ denotes the obvious functors on Nil_R. The right hand side of (230) may be rewritten by the isomorphism:

$$
\begin{array}{lcl}
(231) & \widehat{I}_{\mathcal{N}} \otimes_{W(R)} T \longrightarrow \widehat{W}(\mathcal{N}) \otimes_{F,W(R)} T \\
& {}^V u \otimes y \longmapsto u \otimes y
\end{array}
$$

The pairing (228) induces an isomorphism:

$$
W(\mathcal{N}) \otimes_{W(R)} L^* \longrightarrow \underline{\mathrm{Hom}}(\widehat{W} \otimes_{W(R)} L, \widehat{\mathbb{G}}_m)^+(\mathcal{N}) \tag{232}
$$

Taking the isomorphisms (230), (231) and (232) together, we obtain an isomorphism of functors

$$\begin{aligned}\underline{\mathrm{Hom}}(\widehat{Q},\widehat{\mathbb{G}}_m)^+(\mathcal{N}) &\cong W(\mathcal{N})\otimes_{F,W(R)} T^* \oplus W(\mathcal{N})\otimes_{W(R)} L^* \\ &= W(\mathcal{N})\otimes_{F,W(R)} L^t \oplus W(\mathcal{N})\otimes_{W(R)} T^t.\end{aligned}\tag{233}$$

We use the decomposition $P^t = W(R)V^{-1}L^t \oplus W(R)FT^t$ to rewrite the isomorphism (229)

$$\begin{aligned}\underline{\mathrm{Hom}}(\widehat{P},\widehat{\mathbb{G}}_m)^+(\mathcal{N}) &\cong W(\mathcal{N})\otimes_{W(R)} W(R)V^{-1}L^t \oplus W(\mathcal{N})\otimes_{W(R)} W(R)FT^t \\ &\simeq W(\mathcal{N})\otimes_{F,W(R)} L^t \oplus W(\mathcal{N})\otimes_{F,W(R)} T^t.\end{aligned}\tag{234}$$

Here an element $\xi\otimes x^t\oplus\eta\otimes y^t$ from the last module of (236) is mapped to $\xi V^{-1}x^t\oplus \eta F y^t$ from the module in the middle.

We rewrite the first square in (227) using the isomorphism (233) and (234):

$$\begin{array}{ccc} W(\mathcal{N})\otimes_{F,} L^t\oplus W(\mathcal{N})\otimes_{F,} T^t & \xrightarrow{(V^{-1}-\mathrm{id})^*} & W(\mathcal{N})\otimes_{F,} L^t\oplus W(\mathcal{N})\otimes T^t \\ \widetilde{\alpha}_1\big\uparrow & & \big\uparrow\widetilde{\alpha}_2 \\ \widehat{W}(\mathcal{N})\otimes L^t\oplus \widehat{W}(\mathcal{N})\otimes_{F,} T^t & \xrightarrow{V^{-1}-\mathrm{id}} & \widehat{W}(\mathcal{N})\otimes L^t\oplus \widehat{W}(\mathcal{N})\otimes T^t\end{array}\tag{235}$$

In this diagram all tensor products are taken over $W(R)$. We have to figure out what are the arrows in this diagram explicitly. We will first say what the maps are and then indicate how to verify this.

$$\begin{aligned}\widetilde{\alpha}_2 &= -\left({}^F\otimes \mathrm{id}_{L^t}\oplus \mathrm{id}_{W(\mathcal{N})\otimes_{W(R)}T^t}\right)\\ \widetilde{\alpha}_1 &= {}^F\otimes \mathrm{id}_{L^t}\oplus \mathrm{id}_{W(\mathcal{N})\otimes_{F,W(R)}T^t}\end{aligned}\tag{236}$$

The upper horizontal map in (235) is the map $(V^{-1}-\mathrm{id})^* = \underline{\mathrm{Hom}}(V^{-1}-\mathrm{id},\widehat{\mathbb{G}}_m)$: $\underline{\mathrm{Hom}}(\widehat{P},\widehat{\mathbb{G}}_m)\to\underline{\mathrm{Hom}}(\widehat{Q},\widehat{\mathbb{G}}_m)$. We describe the maps $(V^{-1})^*=\underline{\mathrm{Hom}}(V^{-1},\widehat{\mathbb{G}}_m)$ and $\mathrm{id}^*=\underline{\mathrm{Hom}}(\mathrm{id},\widehat{\mathbb{G}}_m)$. Let $\xi\otimes x^t\oplus\eta\otimes y^t\in W(\mathcal{N})\otimes_{F,W} L^t\oplus W(\mathcal{N})\otimes_{F,W} T^t$ be an element. Then we have:

$$(V^{-1})^*(\xi\otimes x^t\oplus\eta\otimes y^t) = \xi\otimes x^t\oplus {}^V\eta\otimes y^t.\tag{237}$$

Finally the map id^* is the composite of the map $(V^{-1})^\#\oplus F^\#: W(\mathcal{N})\otimes_{F,W(R)} L^t\oplus W(\mathcal{N})\otimes_{F,W(R)} T^t\to W(\mathcal{N})\otimes_{W(R)} P^t$ with the extension of $-\widetilde{\alpha}_2$ to the bigger domain $W(\mathcal{N})\otimes_{W(R)} P^t = W(\mathcal{N})\otimes_{W(R)} L^t\oplus W(\mathcal{N})\otimes_{W(R)} T^t$. We simply write:

$$\mathrm{id}^* = -\widetilde{\alpha}_2\left((V^{-1})^\#\oplus F^\#\right).\tag{238}$$

If one likes to be a little imprecise, one could say $(V^{-1})^*=\mathrm{id}$ and $(\mathrm{id})^*=V^{-1}$.

Let us now verify these formulas for the maps in (237). $\widetilde{\alpha}_1$ is by definition (218) the composition of $V^{-1}:\widehat{Q}^t_{\mathcal{N}}\to\widehat{P}^t_{\mathcal{N}}$ with the inclusion $\widehat{P}^t_{\mathcal{N}}\subset W(\mathcal{N})\otimes_{W(R)} P^t =$

$\underline{\mathrm{Hom}}(\widehat{P}, \mathbb{G}_m)^+(\mathcal{N})$. Hence by the isomorphism (234) which was used to define the diagram (235) the map $\widetilde{\alpha}_1$ is:

$$\widetilde{\alpha}_1 : \widehat{Q}^t_{\mathcal{N}} \xrightarrow{V^{-1}} W(\mathcal{N}) \underset{W(R)}{\otimes} P^t \xleftarrow[\sim]{(V^{-1})^{\#} \oplus F^{\#}} W(\mathcal{N}) \underset{F,W(R)}{\otimes} L^t \oplus W(\mathcal{N}) \underset{F,W(R)}{\otimes} T^t$$

Clearly this is the map given by (236).

Consider an element $u \otimes x^t \in \widehat{W}(\mathcal{N}) \otimes_{W(R)} L^t$. This is mapped by $\widetilde{\alpha}_2$ to an element in $\underline{\mathrm{Hom}}(\widehat{Q}, \widehat{\mathbb{G}}_m)^+(\mathcal{N}) = \underline{\mathrm{Hom}}(\widehat{I} \otimes_{W(R)} T, \widehat{\mathbb{G}}_m)^+(\mathcal{N}) \oplus \underline{\mathrm{Hom}}(\widehat{W} \otimes_{W(R)} L, \widehat{\mathbb{G}}_m)^+(\mathcal{N})$, whose component in the second direct summand is zero and whose component in the first direct summand is given by the following bilinear form $\overline{\alpha}_2$:

$$\overline{\alpha}_2(u \otimes x^t, {}^V u' \otimes y) = -\operatorname{hex} {}^V u' u \langle x^t, y\rangle = -\operatorname{hex} u' {}^F u^F \langle {}^t x, y\rangle.$$

Hence the image in the first direct summand is equal to the image of ${}^F u \otimes x^t$ by the homomorphism (230).

Next we compute the map:

$$(V^{-1})^* : W(\mathcal{N}) \otimes_{W(R)} P^t \simeq \underline{\mathrm{Hom}}(\widehat{P}, \widehat{\mathbb{G}}_m)^+(\mathcal{N}) \longrightarrow \underline{\mathrm{Hom}}(\widehat{Q}, \widehat{\mathbb{G}}_m)^+(\mathcal{N}).$$

Let use denote by $(\ ,\ )_D$ the bilinear forms induced by the homomorphism (230) respectively (232). Let $\theta \otimes z^t \in W(\mathcal{N}) \otimes_{W(R)} P^t$ be an element, and let $\theta \otimes x \oplus v \otimes y \in \widehat{W}(\mathcal{N}) \otimes_{W(R)} L \oplus \widehat{W}(\mathcal{N}) \otimes_{F,W(R)} T \simeq \widehat{W}(\mathcal{N}) \otimes_{W(R)} L \oplus \widehat{I}_{\mathcal{N}} \otimes_{W(R)} T = \widehat{Q}_{\mathcal{N}}$. Then we have by definition of $(V^{-1})^*$:

$$\big((V^{-1})^*(\theta \otimes z^t), u \otimes x + v \otimes y\big)_D = \operatorname{hex} \theta^F u \langle z^t, V^{-1}x\rangle + \operatorname{hex} \theta v \langle z^t, Fy\rangle. \tag{239}$$

Since we use the isomorphism (234) we have to write $\theta \otimes z^t$ in the form $\xi \otimes V^{-1}x^t + \eta \otimes Fy^t$, where $\xi, \eta \in W(\mathcal{N}), x^t \in L^t, y^t \in T^t$. Then we find for the right hand side of (239):

$$\begin{aligned} \operatorname{hex} \xi^F u \langle V^{-1}x^t, V^{-1}x\rangle &+ \operatorname{hex} \xi v \langle V^{-1}x^t, Fy\rangle \\ &+ \operatorname{hex} \eta^F u \langle Fy^t, V^{-1}x\rangle + \operatorname{hex} \eta v \langle Fy^t, Fy\rangle \end{aligned} \tag{240}$$

By definition of the dual 3n-display the first and the last summand of (240) vanish. Using (20) we obtain for (240):

$$\operatorname{hex} \xi v^F \langle x^t y\rangle + \operatorname{hex} \eta^F u^F \langle y^t, x\rangle = \operatorname{hex} \xi v^F \langle x^t, y\rangle + \operatorname{hex} {}^V \eta u \langle y^t, x\rangle.$$

Since this is equal to the left hand side of (240), we see that $(V^{-1})^*(\xi \otimes V^{-1}x^t + \eta \otimes Fy^t)$ is the element in $\underline{\mathrm{Hom}}(\widehat{Q}, \widehat{\mathbb{G}}_M)^+(\mathcal{N})$ induced by:

$$\xi \otimes x^t + {}^V \eta \otimes y^t \in \widehat{W}(\mathcal{N}) \otimes_{F,W(R)} L^t \oplus \widehat{W}(\mathcal{N}) \otimes_{W(R)} T^t$$

This is the assertion (237).

Finally we compute id^*. By the isomorphisms (229) and (233) the map id^* identifies with a map

$$\mathrm{id}^* : W(\mathcal{N}) \otimes_{W(R)} P^t \longrightarrow W(\mathcal{N}) \otimes_{F,W(R)} L^t \oplus W(\mathcal{N}) \otimes_{W(R)} T^t \tag{241}$$

The assertion of (238) is that this map is the extension of $-\widetilde{\alpha}_2$, if we identify the left hand side of (241) with $W(\mathcal{N}) \otimes_{W(R)} L^t \oplus W(\mathcal{N}) \otimes_{W(R)} T^t$ using our normal decomposition.

Let $\xi \otimes x^t \oplus \eta \otimes y^t \in W(\mathcal{N}) \otimes_{W(R)} L^t \oplus W(\mathcal{N}) \otimes_{W(R)} T^t$ and $u \otimes x \oplus {}^V v \otimes y \in \widehat{W}(\mathcal{N}) \otimes_{W(R)} L \oplus \widehat{I}_{\mathcal{M}} \otimes_{W(R)} T = \widehat{Q}_{\mathcal{M}}$ for some $\mathcal{N}$-algebra $\mathcal{M}$. We obtain:

$$\begin{aligned} \mathrm{id}^*(\xi \otimes x^t \oplus \eta \otimes y^t)(u \otimes x \oplus {}^V v \otimes y) &= \operatorname{hex}(\xi^V v\langle x^t, y\rangle + \operatorname{hex} \eta u\langle y^t, x\rangle \\ &= \operatorname{hex} v^F \xi^F \langle x^t, y\rangle + \operatorname{hex} \eta u\langle y^t, u\rangle, \end{aligned}$$

which proves that

$$\mathrm{id}^*(\xi \otimes x^t + \eta \otimes y^t) = {}^F\xi \otimes x^t + \eta \otimes y^t.$$

Altogether we have verified that the diagram (235) with the maps described coincides with the first square in (227). We may now write the first row of the diagram (227) as follows:

$$\begin{array}{l} 0 \to W \underset{F,W(R)}{\otimes} L^t \oplus I \underset{W(R)}{\otimes} T^t \xrightarrow{(V^{-1})^* - \mathrm{id}^*} W \underset{F,W(R)}{\otimes} L^t \oplus W \underset{W(R)}{\otimes} T^t \\ \hphantom{0 \to W \underset{F,W(R)}{\otimes} L^t \oplus I \underset{W(R)}{\otimes} T^t \xrightarrow{(V^{-1})^* - \mathrm{id}^*} W \underset{F,W(R)}{\otimes} L^t \oplus W} \downarrow \\ \hphantom{0 \to W \underset{F,W(R)}{\otimes} L^t \oplus I \underset{W(R)}{\otimes} T^t \xrightarrow{(V^{-1})^* - \mathrm{id}^*} W \underset{F,W(R)}{\otimes} L^t \oplus} \underline{\mathrm{Ext}}^1(G, \widehat{\mathbb{G}}_m)^+ \\ \hphantom{0 \to W \underset{F,W(R)}{\otimes} L^t \oplus I \underset{W(R)}{\otimes} T^t \xrightarrow{(V^{-1})^* - \mathrm{id}^*} W \underset{F,W(R)}{\otimes} L^t \oplus W} \downarrow \\ \hphantom{0 \to W \underset{F,W(R)}{\otimes} L^t \oplus I \underset{W(R)}{\otimes} T^t \xrightarrow{(V^{-1})^* - \mathrm{id}^*} W \underset{F,W(R)}{\otimes} L^t \oplus W} 0 \end{array} \tag{242}$$

Here we wrote W and I for the functors $\mathcal{N} \mapsto W(\mathcal{N})$ and $\mathcal{N} \mapsto I_{\mathcal{N}}$. We also used the isomorphism (231) to replace $W \otimes_{F,W(R)} T^t$ by $I \otimes_{W(R)} T^t$. The map $(V^{-1})^*$ is just the natural inclusion.

We know from (227), that $\underline{\mathrm{Ext}}^1(G, \widehat{\mathbb{G}}_m)^+$ is an exact functor on Nil_R. We will now compute the tangent space of this functor.

Let us assume that $\mathcal{N}$ is equipped with a pd-structure. Then the logarithmic coordinates (48) define an isomorphism of $W(R)$-modules

$$\mathcal{N} \oplus I_{\mathcal{N}} \simeq W(\mathcal{N}).$$

Hence we have an isomorphism of abelian groups:

$$\mathcal{N} \otimes_{W(R)} T^t \oplus I_{\mathcal{N}} \otimes_{W(R)} T^t \xrightarrow{\sim} W(\mathcal{N}) \otimes_{W(R)} T^t.$$

We extend id^* to an endomorphism of $W(\mathcal{N}) \otimes_{F,W(R)} L^t \oplus W(\mathcal{N}) \otimes_{W(R)} T^t$ by setting:

$$\mathrm{id}^*(\mathcal{N} \otimes_{W(R)} T^t) = 0.$$

We claim that id^* is then a nilpotent endomorphism. First we verify this in the case, where $p \cdot \mathcal{N} = 0$. Then we have ${}^F W(\mathcal{N}) = 0$ and therefore the map $\widetilde{\alpha}_2$ is zero on the first component. It follows from (238) that the image of id^* lies in

$0 \oplus W(\mathcal{N}) \otimes_{W(R)} T^t \subset W(\mathcal{N}) \otimes_{F,W(R)} L^t \oplus W(\mathcal{N}) \otimes_{W(R)} T^t$. Via the natural inclusion and projection id^* induces an endomorphism

$$\mathrm{id}^*_{22} : W(\mathcal{N}) \otimes_{W(R)} T^t \longrightarrow W(\mathcal{N}) \otimes_{W(R)} T^t.$$

By what we have said it is enough to show that id^*_{22} is nilpotent. The endomorphism

$$F : P^t = L^t \oplus T^t \longrightarrow P^t = L^t \oplus T^t,$$

induces via inclusion and projection an endomorphism

$$\varphi : T^t \longrightarrow T^t.$$

By the formula (238) we find for id^*_{22}:

$$\mathrm{id}^*_{22}\left((n + {}^V\xi) \otimes y^t\right) = \xi \otimes \varphi(y^t),$$

where $n \in \mathcal{N}, \xi \in W(\mathcal{N})$, and $y^t \in T^t$. But since $\mathcal{P}$ is a display the 3n-display $\mathcal{P}^t$ is F-nilpotent, i.e. there is an integer r, such that $\varphi^r(T^t) \subset I_R T^t$. Since $W(\mathcal{N}) \cdot I_R = 0$ it follows that $(\mathrm{id}^*_{22})^r = 0$. In the case where $p\mathcal{N}$ is not necessarily zero, we consider the filtration by pd-ideals

$$0 = p^r\mathcal{N} \subset p^{r-1}\mathcal{N} \subset \cdots \subset \mathcal{N}.$$

Since the functors of (242) are exact on Nil_R an easy induction on r yields the nilpotency of id^* in the general case. This proves our claim that id^* is nilpotent if $p \cdot \mathcal{N} = 0$. Since $(V^{-1})^*$ is the restriction of the identity of

$$W(\mathcal{N}) \otimes_{F,W(R)} L^t \oplus W(\mathcal{N}) \otimes_{W(R)} T^t$$

it follows that $(V^{-1})^* - \mathrm{id}^*$ induces an automorphism of the last group. One sees easily (compare (156)) that the automorphism $(V^{-1})^* - \mathrm{id}^*$ provides an isomorphism of the cokernel of $(V^{-1})^*$ with the cokernel of $(V^{-1})^* - \mathrm{id}^*$. Therefore we obtain for a pd-algebra $\mathcal{N}$ that the composition of the following maps:

$$\mathcal{N} \otimes_{W(R)} T^t \hookrightarrow W(\mathcal{N}) \otimes_{W(R)} T^t \longrightarrow \underline{\mathrm{Ext}}^1(G, \widehat{\mathbb{G}}_m)^+(\mathcal{N})$$

is an isomorphism. This shows that the $\underline{\mathrm{Ext}}^1(G, \widehat{\mathbb{G}}_m)^+$ is a formal group with tangent space $T^t/I_R T^t$ by definition 80. Moreover

$$G^t \longrightarrow \underline{\mathrm{Ext}}^1(G, \widehat{\mathbb{G}}_m)^+$$

is an isomorphism of formal groups because it induces an isomorphism of the tangent spaces. This proves the proposition. $\square$

Let $\mathcal{P}$ be a 3n-display and let $\mathcal{P}'$ be a display. We set $G = BT_{\mathcal{P}}$, $G' = BT_{\mathcal{P}'}$, and $(G')^t = BT_{(\mathcal{P}')^t}$. If we apply the proposition 121 to (221) we obtain a homomorphism:

$$\mathrm{Biext}^1(G \times G', \widehat{\mathbb{G}}_m) \longrightarrow \mathrm{Hom}(G, (G')^t) \tag{243}$$

We note that this map is always injective, because the kernel of (221) is by the usual spectral sequence $\mathrm{Ext}^1(G, \underline{\mathrm{Hom}}(G', \widehat{\mathbb{G}}_m))$. But this group is zero, because $\underline{\mathrm{Hom}}(G', \widehat{\mathbb{G}}_m)^+ = 0$ (compare (224)). A bilinear form $\mathcal{P} \times \mathcal{P}' \to \mathcal{G}$ is clearly the same thing as a homomorphism $\mathcal{P} \to (\mathcal{P}')^t$. It follows easily from the diagram (223) that the injection (243) inserts into a commutative diagram:

$$\begin{array}{ccc} \mathrm{Bil}(\mathcal{P} \times \mathcal{P}', \mathcal{G}) & \xrightarrow{\sim} & \mathrm{Hom}(\mathcal{P}, (\mathcal{P}')^t) \\ \Big\downarrow & & \Big\downarrow BT \\ \mathrm{Biext}^1(G \times G', \widehat{\mathbb{G}}_m) & \longrightarrow & \mathrm{Hom}(G, (G')^t) \end{array} \tag{244}$$

Theorem 122. — *Let R be a ring, such that p is nilpotent in R, and such that the set of nilpotent elements in R are a nilpotent ideal. Let $\mathcal{P}$ and $\mathcal{P}'$ be displays over R. We assume that $\mathcal{P}'$ is F-nilpotent, i.e. the dual 3n-display $(\mathcal{P}')^t$ is a display. Then the homomorphism (216) is an isomorphism:*

$$\mathrm{Bil}(\mathcal{P} \times \mathcal{P}', \mathcal{G}) \longrightarrow \mathrm{Biext}^1(BT_{\mathcal{P}} \times BT_{\mathcal{P}'}, \widehat{\mathbb{G}}_m).$$

Proof. — By proposition 99 the right vertical arrow of the diagram (244) becomes an isomorphism under the assumptions of the theorem. Since we already know that the lower horizontal map is injective every arrow is this diagram must be an isomorphism. □

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