ASTÉRISQUE 280

COHOMOLOGY OF SIEGEL VARIETIES

Abdellah Mokrane
Patrick Polo
Jacques Tilouine

Société Mathématique de France 2002
Publié avec le concours du Centre National de la Recherche Scientifique

A. Mokrane
Département de Mathématiques, C.N.R.S. UMR 7539, Institut Galilée, Université de Paris 13, 93430 Villetaneuse.
E-mail : `mokrane@math.univ-paris13.fr`

P. Polo
Département de Mathématiques, C.N.R.S. UMR 7539, Institut Galilée, Université de Paris 13, 93430 Villetaneuse.
E-mail : `polo@math.univ-paris13.fr`

J. Tilouine
Institut Universitaire de France et Département de Mathématiques, C.N.R.S. UMR 7539, Institut Galilée, Université de Paris 13, 93430 Villetaneuse.
E-mail : `tilouine@math.univ-paris13.fr`

2000 *Mathematics Subject Classification.* — 11F46, 11G15, 14F30, 14K22, 17B50, 17B56, 20G30.

Key words and phrases. — Siegel modular forms, p-adic cohomology, Galois representations, BGG complexes, (co)homology of nilpotent groups.

COHOMOLOGY OF SIEGEL VARIETIES

Abdellah Mokrane, Patrick Polo, Jacques Tilouine

Abstract. **—** This volume deals with the question of torsion in the cohomology of Shimura varieties, the coefficient system being $\mathbb{Z}_p$ or, more generally, a certain local system V_λ of flat $\mathbb{Z}_p$-modules. Its goal is to show, for Siegel varieties, that the localization of this cohomology at a non-Eisenstein maximal ideal $\mathfrak{m}$ of the Hecke algebra $\mathbb{T}$ has no p-torsion ($p = \mathrm{char}(\mathbb{T}/\mathfrak{m})$), for p greater than an explicit bound $c(\lambda)$ depending only on the highest weight λ of the coefficient system. This localization, moreover, kills the boundary cohomology.

Two arithmetic applications are presented: one concerns Hida families of Hecke eigensystems and the other is a step towards the existence of certain Taylor-Wiles systems for symplectic groups.

An ingredient in the proof is a version over $\mathbb{Z}_p$ of Bernstein-Gelfand-Gelfand complexes and of Kostant's theorem computing the $\mathfrak{n}$-homology of the Weyl module V_λ, for p greater than the above bound $c(\lambda)$ (which implies that λ belongs to the closure of the fundamental p-alcove).

***Résumé* (Cohomologie des variétés de Siegel). —** Cette monographie traite de la question de la torsion dans la cohomologie des variétés de Shimura, à coefficients dans $\mathbb{Z}_p$ ou, plus généralement, dans un certain système local V_λ de $\mathbb{Z}_p$-modules plats. Son objet est d'établir, pour les variétés de Siegel, que la localisation de cette cohomologie en un idéal maximal de type non-Eisenstein $\mathfrak{m}$ de l'algèbre de Hecke $\mathbb{T}$ n'a pas de p-torsion ($p = \mathrm{char}(\mathbb{T}/\mathfrak{m})$), pour p plus grand qu'une certaine borne explicite $c(\lambda)$ qui ne dépend que du plus haut poids λ du système de coefficients. En outre, cette localisation tue la cohomologie du bord.

On donne deux applications arithmétiques de ce résultat. L'une concerne les familles de Hida de systèmes de valeurs propres de Hecke, l'autre constitue une étape importante dans la construction de certains systèmes de Taylor-Wiles pour les groupes symplectiques.

Un ingrédient de la preuve est une version sur $\mathbb{Z}_p$ de complexes de Bernstein-Gelfand-Gelfand et d'un théorème de Kostant, calculant la $\mathfrak{n}$-homologie du module de Weyl V_λ, pour p plus grand que la borne ci-dessus (ce qui implique que λ appartient à l'adhérence de la p-alcôve fondamentale).

CONTENTS

INTRODUCTION

The first paper of this volume deals with the question of torsion in the cohomology of Siegel varieties S_U with coefficients in a local system V_λ of finite flat $\mathbb{Z}_p$-modules. Its goal is to show that its localization at a non-Eisenstein maximal ideal $\mathfrak{m}$ of the Hecke algebra is torsion-free for p large with respect to the highest weight λ of the coefficient system V_λ. At the same time, as could be expected, besides getting rid of the torsion, the localization has the effect of killing the boundary cohomology (and its torsion), so that we show that

$$H_c^\bullet(S_U, V_\lambda)_\mathfrak{m} = IH^\bullet(S_U, V_\lambda)_\mathfrak{m} = H_!^\bullet(S_U, V_\lambda)_\mathfrak{m} = H^\bullet(S_U, V_\lambda)_\mathfrak{m}$$

and these cohomology modules are concentrated in middle degree d. This question of absence of torsion is important in the construction of p-ordinary families of cuspidal Hecke eigensystems, and in the verification of the first main condition for having a Taylor-Wiles system. These applications are given at the end of the paper. The main assuption is that the p-adic Galois representation associated to a cuspidal cohomological representation does exist (it is known only in genus $\leqslant 2$) and that those corresponding to the maximal ideal $\mathfrak{m}$ have large residual image (it can be verified on examples for $g = 2$). Faltings introduced around 1980 the dual Bernstein-Gelfand-Gelfand complex as a tool for determining the Hodge decomposition of the complex cohomology of locally symmetric varieties. The rational version of this tool appeared in Faltings-Chai book, and they incidentally mention that a p-adic integral version as well exists, but only for so-called p-small weights λ. We developed this idea, and it became our main tool for determining the Fontaine-Laffaille constituents of the modulo p de Rham cohomology of these Siegel varieties. This, allied with Falting's mod. p étale-de Rham comparison theorem together with a Galois-theoretic argument allows us to show the vanishing of the various modulo p cohomologies of S_U localized at $\mathfrak{m}$ in degree $q < d$. For this, we needed a rather detailed study of the Bernstein-Gelfand-Gelfand complex over $\mathbb{Z}_p$ (in p-small weight) and a $\mathbb{Z}_p$-integral version of Kostant theorem decomposing the cohomology of the unipotent radical of a parabolic as a sum of irreducible modules over the Levi quotient. These results are presented in great generality in the second paper, which provides also a useful assortment of results on $\mathbb{Z}_p$-representations of a reductive group in p-small weights.

Astérisque
280, 2002, p. 1–95

COHOMOLOGY OF SIEGEL VARIETIES WITH p-ADIC INTEGRAL COEFFICIENTS AND APPLICATIONS

by

Abdellah Mokrane & Jacques Tilouine

Abstract. — Under the assumption that Galois representations associated to Siegel modular forms exist (it is known only for genus at most 2), we study the cohomology with p-adic integral coefficients of Siegel varieties, when localized at a non-Eisenstein maximal ideal of the Hecke algebra, provided the prime p is large with respect to the weight of the coefficient system. We show that it is torsion-free, concentrated in degree d, and that it coincides with the interior cohomology and with the intersection cohomology. The proof uses p-adic Hodge theory and the dual BGG complex modulo p in order to compute the "Hodge-Tate weights" for the mod. p cohomology. We apply this result to the construction of Hida p-adic families for symplectic groups and to the first step in the construction of a Taylor-Wiles system for these groups.

Résumé **(Cohomologie des variétés de Siegel à coefficients entiers p-adiques et applications)**
Supposant connue l'existence des représentations galoisiennes associées aux formes modulaires de Siegel (elle ne l'est qu'en genre $\leqslant 2$ pour le moment), on étudie la cohomologie des variétés de Siegel à coefficients entiers p-adiques localisée en un idéal maximal non-Eisenstein de l'algèbre de Hecke, lorsque p est grand par rapport au poids du système de coefficients. Plus précisément, on montre qu'elle est sans torsion, concentrée en degré médian, et qu'elle coïncide avec la cohomologie d'intersection et avec la cohomologie intérieure. On utilise pour cela la théorie de Hodge p-adique et le complexe BGG dual modulo p qui calcule « les poids de Hodge-Tate » de la réduction modulo p de cette cohomologie. On applique ce résultat à la construction de familles de Hida p-ordinaires pour les groupes symplectiques et à l'ébauche de la construction d'un système de Taylor-Wiles pour ces groupes.

1. Introduction

1.1. Let G be a connected reductive group over $\mathbb{Q}$. Diamond [**16**] and Fujiwara [**29**] (independently) have axiomatized the Taylor-Wiles method which allows to study some local components $\mathbf{T}_\mathfrak{m}$ of a Hecke algebra $\mathbf{T}$ for G of suitable (minimal) level; when it applies, this method shows at the same time that $\mathbf{T}_\mathfrak{m}$ is complete intersection and that some cohomology module, viewed as a $\mathbf{T}$-module, is locally free at $\mathfrak{m}$. It

2000 ***Mathematics Subject Classification.*** — 11F46, 11G15, 14K22, 14F30.
Key words and phrases. — Siegel modular forms, p-adic cohomology, Galois representations.

has been successfully applied to $\mathrm{GL}(2)_{/\mathbb{Q}}$ [**73**], to some quaternionic Hilbert modular cases [**29**], and to some inner forms of unitary groups [**38**]. If one tries to treat other cases, one can let the Hecke algebra act faithfully on the middle degree Betti cohomology of an associated Shimura variety; then, one of the problems to overcome is the possible presence of torsion in the cohomology modules with p-adic integral coefficients. For $G = \mathrm{GSp}(2g)$ ($g \geqslant 1$), we want to explain in this paper why this torsion is not supported by maximal ideals of $\mathbf{T}$ which are "non-Eisenstein" and ordinary (see below for precise definitions), provided the residual characteristic p is prime to the level and greater than a natural bound. A drawback of our method is that it necessitates to assume that the existence and some local properties of the Galois representations associated to cohomological cuspidal representations on G are established. For the moment, they are proven for $g \leqslant 2$ (see below). In his recent preprint [**43**], Hida explains for the same symplectic groups G how by considering only coherent cohomology, one can let the Hecke algebra act faithfully too on cohomology modules whose torsion-freeness is built-in (without assuming any conjecture). However for some applications (like the relation, for some groups G, between special values of adjoint L-functions, congruence numbers, and cardinality of adjoint Selmer groups), the use of the Betti cohomology seems indispensable.

1.2. Let $G = \mathrm{GSp}(2g)$ be the group of symplectic similitudes given by the matrix $J = \left(\begin{smallmatrix} 0 & s \\ -s & 0 \end{smallmatrix}\right)$, whose entries are $g \times g$-matrices, and s is antidiagonal, with non-zero coefficients equal to 1; the standard Borel B, resp. torus T, in G consists in upper triangular matrices, resp. diagonal matrices in G. For any dominant weight λ for (G, B, T), we write $\widehat{\lambda}$ for its dual (that is, the dominant weight associated to the Weyl representation dual of that of λ). Let ρ be the half-sum of the positive roots. Recall that λ is given by a $(g+1)$-uple $(a_g, \ldots, a_1; c) \in \mathbb{Z}^{g+1}$ with $c \equiv a_1 + \cdots + a_g \bmod 2$, that $\widehat{\lambda} = (a_g, \ldots, a_1; -c)$ and $\rho = (g, \ldots, 1; 0)$ (see section 3.1 below). Throughout this paper, the following integer will be of great importance:

$$\boldsymbol{w} = |\lambda + \rho| = |\lambda| + d = \sum_{i=1}^{g} (a_i + i) = d + \sum_{i=1}^{g} a_i$$

where $d = g(g+1)/2$. It can be viewed as a cohomological weight as follows.

Let $\mathbb{A} = \mathbb{A}_f \times \mathbb{Q}_\infty$ be the ring of rational adèles; let G_f resp. G_∞ be the group of $\mathbb{A}_f$-points resp. $\mathbb{Q}_\infty$-points of G. Let U be a "good" open compact subgroup of $G(\mathbb{A}_f)$ (see Introd. of Sect. 2); let S resp. S_U be the Shimura variety of infinite level, resp. of level U associated to G; then $d = \dim S_U$ is the middle degree of the Betti cohomology of S_U. Let $V_\lambda(\mathbb{C})$ be the coefficient system over S resp. S_U with highest weight λ. See Sect. 2.1 for precise definitions.

Let $\pi = \pi_f \otimes \pi_\infty$ be a cuspidal automorphic representation of $G(\mathbb{A})$ which occurs in $H^d(S_U, V_\lambda(\mathbb{C}))$. This means that

– the π_f-isotypical component $W_\pi = H^d(\pi_f)$ of the G_f-module $H^\bullet(S, V_\lambda(\mathbb{C}))$ is non-zero, and

– $\pi_f^U \neq 0$.

It is known (see Sect. 2.3.1 below) that the first condition is implied by the fact that π_∞ belongs to the L-packet $\Pi_{\widehat{\lambda}+\rho}$ of Harish-Chandra's parameter $\widehat{\lambda}+\rho$ in the discrete series. In fact, it is equivalent to this fact if λ is regular or if $g = 2$.

By a Tate twist, we can restrict ourselves to the case where $c = a_g + \cdots + a_1$. We do this in the sequel. Then, $|\lambda|$ is the Deligne weight of the coefficient system V_λ and $\boldsymbol{w} = |\lambda + \rho|$ is the cohomological weight of W_π, hence the (hypothetical) motivic weight of π.

Let p be a prime. Let us fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let v be the valuation of $\overline{\mathbb{Q}}$ induced by ι_p normalized by $v(p) = 1$; let K be the v-adic completion of a number field containing the Hecke eigenvalues of π. We denote by $\mathcal{O}$ the valuation ring of (K, v); we fix a local parameter $\varpi \in \mathcal{O}$. Let N be the level of U, that is, the smallest positive integer such that the principal congruence subgroup $U(N)$ is contained in U. Let $\mathcal{H}^N$ resp. $\mathcal{H}_U(\mathcal{O})$ be the abstract Hecke algebra outside N generated over $\mathbb{Z}$, resp. over $\mathcal{O}$ by the standard Hecke operators for all primes ℓ prime to N; for any such prime ℓ, let $P_\ell(X) \in \mathcal{H}^N[X]$ be the minimal polynomial of the Hecke-Frobenius element (it is monic, of degree 2^g, see [**13**] page 247). Let $\theta_\pi : \mathcal{H}^N(\mathcal{O}) \to \mathcal{O}$ be the $\mathcal{O}$-algebra homomorphism associated to π_f.

Let $\widehat{G} = \mathrm{GSpin}_{2g+1}$ be the group of spinorial similitudes for the quadratic form

$$\sum_{i=1}^{g} 2x_i x_{2g+1-i} + x_{g+1}^2;$$

it is a split Chevalley group over $\mathbb{Z}[1/2]$ (we won't consider the prime $p = 2$ in the sequel); it can be viewed as the dual reductive group of G (see Sect. 3.2 below); let $\widehat{B}$, $\widehat{N}$, $\widehat{T}$ the standard Borel, its unipotent radical, resp. standard maximal torus therein. The group $\widehat{G}$ acts faithfully irreducibly on a space $V_{/\mathbb{Z}}$ of dimension 2^g, via the spinorial representation. Let B_V be the upper triangular Borel of GL_V. Note that $\widehat{B}$ is mapped into B_V by the spin representation.

1.3. We put $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We assume that

(Gal) there exists a continuous homomorphism

$$\rho_\pi : \Gamma \longrightarrow \mathrm{GL}_V(\mathcal{O})$$

associated to π: that is, unramified outside Np, and such that the characteristic polynomial of the Frobenius element at a prime q not dividing Np is equal to $\theta_\pi(P_q(X))$.

We shall make below an assumption on the reduction of ρ_π modulo the maximal ideal of $\mathcal{O}$ which will imply that ρ_π act absolutely irreducibly on V for each geometric fiber; hence the choice of a stable $\mathcal{O}$-lattice $V_{\mathcal{O}}$ in $V \otimes K$ is unique up to homothety.

Evidences for **(Gal)**. — For $g = 2$, assuming

(Hol) π_∞ is in the holomorphic discrete series,

Weissauer [**87**] (see also [**34**] and [**52**]) has shown the existence of a four-dimensional p-adic Galois representation

$$\rho_\pi : \Gamma \longrightarrow \mathrm{GL}_V(\overline{\mathbb{Q}}_p)$$

Moreover, his construction, relying on trace formulae, shows actually that

$$L(W_\pi, s)^4 = L(\rho_\pi, s)^m \quad \text{for some } m \geqslant 1.$$

From this relation, one sees easily that the irreducibility of $\rho_\pi \otimes \mathrm{Id}_{\overline{\mathbb{Q}}_p}$ implies that the (Galois) semisimplification of $W_{\pi,p}$ is isomorphic to $n.\rho_\pi$ $(m = 4n)$.

Another crucial assumption for us will be that p is prime to N (hence π is unramified at p). Recall that under this assumption, Faltings has shown (Th. 6.2 (iii) of [**13**] and Th. 5.6 of [**22**]) that for any q, the p-adic representation $H^q(S_U \otimes \overline{\mathbb{Q}}_p, V_\lambda(\overline{\mathbb{Q}}_p))$ is crystalline.

Let D_p, resp. I_p be a decomposition subgroup, resp. inertia subgroup of Γ. Via the identification $X^*(T) = X_*(\widehat{T})$, we can view any $\mu \in X^*(T)$ as a cocharacter of $\widehat{T}$, hence as a homomorphism $I_p \to \mathbb{Z}_p^\times \to \widehat{T}(\mathbb{Z}_p) \to \widehat{G}(\mathbb{Z}_p)$ where the first map is the cyclotomic character $\chi : I_p \to \mathbb{Z}_p^\times$. We denote by $\widetilde{\rho}$ the character of T whose semisimple part is that of ρ, but whose central parameter is d. it is the highest weight of an irreducible representation of G given by ρ on the derived group G'. The character $\lambda + \widetilde{\rho}$ has coordinates $(a_g + g, \dots, a_1 + 1; \boldsymbol{w})$. Let us introduce the assumption of Galois ordinarity, denoted in the sequel **(GO)**:

1) The image $\rho_\pi(D_p)$ of the decomposition group is contained in $\widehat{G}$,

2) There exists $\widehat{g} \in \widehat{G}(\mathcal{O})$ such that

$$\rho_\pi(D_p) \subset \widehat{g} \cdot \widehat{B}(\mathcal{O}) \cdot \widehat{g}^{-1},$$

3) the restriction of the conjugate $\rho_\pi^{\widehat{g}}$ to I_p, followed by the quotient by the unipotent radical $\widehat{g} \cdot \widehat{N} \cdot \widehat{g}^{-1}$ of $\widehat{g} \cdot \widehat{B} \cdot \widehat{g}^{-1}$ factors through $-(\lambda + \widetilde{\rho}) : I_p \to \widehat{T}(\mathbb{Z}_p)$.

Comments

1) Let us introduce the condition of automorphic ordinarity:

(AO) For each $r = 1, \dots, g$,

$$v(\theta_\pi(T_{p,r})) = a_{r+1} + \cdots + a_g$$

where $T_{p,r}$ is the classical Hecke operator associated to the double class of

$$\mathrm{diag}(1_r, p \cdot 1_{2g-2r}, p^2 \cdot 1_r).$$

We conjecture that for any g, if ρ_π is residually absolutely irreducible, **(AO)** implies **(GO)**. It is well-known for $g = 1$ ([**89**] Th. 2.2.2, and [**54**]). Moreover, for $g = 2$, it follows from Proposition 7.1 of [**77**] together with a recent result of E. Urban [**80**].

2) The minus sign in front of $(\lambda + \widetilde{\rho})$ comes from the definition of Hodge-Tate weights (for us: the jumps of the Hodge filtration): the weight of the Tate representation $\mathbb{Z}_p(n)$ is $-n$.

Let $\overline{\theta}_\pi = \theta_\pi \mod. \varpi$ and $\mathfrak{m} = \operatorname{Ker} \overline{\theta}_\pi$. Our last assumption concerns "non-Eisenstein-ness" of the maximal ideal $\mathfrak{m}$. It says that the image of the residual representation $\overline{\rho}_\pi$ induced by ρ_π on $V_{\mathcal{O}}/\varpi V_{\mathcal{O}}$ is "large enough". More precisely, let $W_{\widehat{G}}$ be the Weyl group of $\widehat{G}$, viewed as a subgroup of $\widehat{G}$. Recall the standard description $W_{\widehat{G}} \cong S_g \propto \{\pm 1\}^g$. Let $W' \subset \widehat{G}$ corresponding to $\{\pm 1\}^g$. The "residually large image assumption" is as follows:

(RLI) There exists a split (non necessarily connected) reductive Chevalley subgroup H of $\widehat{G}_{/\mathbb{Z}}$ with $W' \propto \widehat{T} \subset H$, and a subfield $k' \subset k$, of order say $|k'| = q' = p^{f'}$ ($f' \geqslant 1$), so that $H(k')_\nu \subset \operatorname{Im} \overline{\rho}_\pi$ and $\overline{\rho}_\pi(I_p) \subset H^0(k')$.

Here, $H(k')_\nu$ denotes the subgroup of $H(k')$ consisting in elements whose ν belongs to $\operatorname{Im} \nu \circ \overline{\rho}_\pi$.

It has the consequence that $\overline{\rho}_\pi$ and ρ_π are absolutely irreducible, hence the uniqueness of the stable lattice $V_{\mathcal{O}}$ up to homothety.

1.4. One defines the sheaf $V_\lambda(\mathcal{O})$ over S_U using the right action of $U_p = G(\mathbb{Z}_p)$ (see [**77**] Sect. 2.1). We put $V_\lambda(A) = V_\lambda(\mathcal{O}) \otimes A$ for any $\mathcal{O}$-module A; these are locally constant sheaves on S_U. Our main result is as follows.

Theorem 1. — *Let π be cuspidal with π_∞ in the discrete series and of good level group U, occuring in*

$$H^d(S_U, V_\lambda(\mathbb{C}));$$

let p be a prime not dividing $N = level(U)$, assume **(Gal)**, **(GO)**, **(RLI)**, *$p > 5$ and that the weight λ is small with respect to p:*

$$p - 1 > |\lambda + \rho|$$

Then, one has:

i) $H^\bullet(S_U, V_\lambda(k))_\mathfrak{m} = H^d(S_U, V_\lambda(k))_\mathfrak{m}$

ii) $H^\bullet(S_U, V_\lambda(\mathcal{O}))_\mathfrak{m} = H^d(S_U, V_\lambda(\mathcal{O}))_\mathfrak{m}$ *and this $\mathcal{O}$-module is free of finite rank. Similarly,*

iii) $H^\bullet(S_U, V_\lambda(K/\mathcal{O}))_\mathfrak{m} = H^d(S_U, V_\lambda(K/\mathcal{O}))_\mathfrak{m}$ *and this $\mathcal{O}$-module is cofree of finite rank.*

The same statements hold for the cohomology with compact supports.

Comments

1) By standard arguments, the whole theorem follows if we show that:

$$H^q_*(S_U, V_\lambda(k))[\mathfrak{m}] = 0 \quad \text{for } q < d$$

where $* = c, \varnothing$, and for any Hecke-module M, $M[\mathfrak{m}]$ stands for its $\mathfrak{m}$-torsion. This is the main result of the text.

2) In several instances in the proof, it is important that the maximal Hodge weights of the cohomology modules involved are distinct for distinct modules, and are smaller than $p-1$; the condition

$$p - 1 > a_1 + \cdots + a_g + d$$

implies this; at the same time, it is also the condition needed to apply a comparison theorem of Faltings (Th. 5.3 of [**22**]). We shall refer to this condition throughout the paper by saying that λ is p-small. This terminology has not the same meaning here than in [**61**], but is in fact stronger than what is called p-smallness there. Hence, under the present assumption, we can make use of Theorem D of [**61**]. In brief, this assumption is unavoidable in our approach. The condition $p > 5$ comes from the theory of modular representations of reductive groups and has been pointed out to us by P. Polo. It is necessary for the validity of Lemma 13 of Section 7.1, as there is a counterexample to this Lemma for $p = 5$ and $G = \mathrm{GSp}(4)$; hence in our approach, the minimal possible p is 7 (for $g = 2$ and $a_1 = a_2 = 0$) but $p = 5$ is also acceptable if $\mathrm{Im}\ \overline{\rho}_\pi$ is "very large" : see the remark following Lemma 12. Observe anyway that our bound on p depends only on λ (not on the level group U). This is crucial for the applications we have in view.

3) The assumption **(RLI)** is used only in Lemma 13 of Section 7.1, but this lemma is crucial for our proof of the Theorem.

4) Note that for λ regular and for $g = 2$, by calculations of [**72**], and results of Schwermer and Franke (see Theorem 3.2(i) of [**77**]), one has $H^q(S_U, V_\lambda(\mathbb{C})) = 0$ for any $q < 3$, while this is not so for the compact support cohomology: the boundary long exact sequence for Borel-Serre compactification relates $H^2_c(S_U, V_\lambda(\mathbb{C}))$ to an H^1 of modular curves, which does not vanish. Our vanishing statement concerns the localization at $\mathfrak{m}$ and means that there is no mixing of Hodge weights between the $\mathfrak{m}$-part of H^2_c and that of H^3_c.

5) For $g = 2$, E. Urban [**79**] has found a completely different proof of the absence of torsion of $H^2(S_U, V_\lambda(\mathcal{O}))_\mathfrak{m}$ under mild assumptions (with $\mathfrak{m}$ non-Eisenstein). His proof is much shorter than ours but relies on the fact that the complement in S_U of the Igusa divisor is affine, which is particular to the Siegel threefold. Whereas our theorem seems to carry over (with the same proof) to various other situations, like the Hilbert (or quaternionic) modular case, or unitary groups $U(2,1)_{/\mathbb{Q}}$.

Evidences

1) If $g = 2$ and π is neither CAP nor endoscopic, one can conjecture that for p sufficiently general, $\mathrm{Im}\,\rho_\pi$ contains the derived group $\widehat{G}(\mathbb{Z}_p)$. Then **(RLI)** is trivially satisfied; if moreover p is also ordinary, the situation is as desired. Such a conjecture is unfortunately presently out of reach.

2) A more tractable situation is the following. See the details in Section 7.3. Let F be a real quadratic field with $\mathrm{Gal}(F/\mathbb{Q}) = \{1, \sigma\}$. Let f be a holomorphic Hilbert cusp form for $\mathrm{GL}(2)_{/F}$, of weight (k_1, k_σ), $k_1, k_\sigma \geqslant 2$, $k_1 = k_\sigma + 2m$ ($m \geqslant 1$). One can show ([**90**] and [**63**]) the existence of a holomorphic theta lift from $\mathrm{GL}(2)_{/F}$ to $G = \mathrm{GSp}(4)_{/\mathbb{Q}}$ for f. Let π be the corresponding automorphic representation of $G(\mathbb{A})$. It is cohomological for a suitable coefficient system. Since f is not a base change from $\mathrm{GL}(2)_{/\mathbb{Q}}$, π is cuspidal, neither CAP nor endoscopic. We allow that f is CM of type $(2,2)$; that is, is a theta series coming from a CM quadratic extension $M = FE$ of F, where E is imaginary quadratic. Moreover, π is stable at ∞ (see [**64**]), ρ_π exists and is motivic, namely: $\rho_\pi = \mathrm{Ind}_F^{\mathbb{Q}} \rho_f$, and it is absolutely irreducible. Moreover, for p sufficiently large (and splitting in E in the $(2,2)$-CM case), the image of the associated Galois representation $\rho_\pi : \Gamma \to \mathrm{GL}_K(V)$ is equal (up to explicit finite index) to the group of points over a finite extension of $\mathbb{Z}_p$ of either the L-group ${}^L(\mathrm{Res}_{\mathbb{Q}}^F \mathrm{GL}(2)_{/F}) = \mathrm{Gal}(F/\mathbb{Q}) \propto (\mathrm{GL}(2) \times \mathrm{GL}(2))^0$ (if f is not CM), or those of ${}^L \mathrm{Res}_{\mathbb{Q}}^M M^\times = \mathrm{Gal}(M/\mathbb{Q}) \propto (\mathbb{G}_m^2 \times \mathbb{G}_m^2)^0$ if f is CM of type $(2,2)$. The subgroup H of $\widehat{G}$ whose image by the spin representation is ${}^L\,\mathrm{GL}(2)_{/F}$ resp. ${}^L M^\times$, does contain $W' \propto \widehat{T}$; that is, the assumption **(RLI)** is satisfied for H. If p is ordinary for f and splits in F, ρ_π satisfies **(GO)**; assume finally that p satisfies $p - 1 > k_1 - 1$; then, our result applies. See Sect. 7.3 for numerical examples.

In Section 8, we obtain a refinement of Theorem 1 as follows:

Theorem 2. — *Under the assumptions of Theorem 1,*

1) *the finite free $\mathcal{O}$-module $H^\bullet(S_U, V_\lambda(\mathcal{O}))_\mathfrak{m}$ coincides with the $\mathfrak{m}$-localizations of*

– *the middle degree interior cohomology $H_!^d(S_U, V_\lambda(\mathcal{O})) = \mathrm{Im}(H_c^d \to H^d)$,*

– *the middle degree intersection cohomology $IH^d(S_U, V_\lambda(\mathcal{O}))$.*

2) *if λ is regular or if $g = 2$, $H_!^d({}_US, V_\lambda(K))_\mathfrak{m}$ contains only cuspidal eigenclasses, whose infinity type are in the discrete series of HC parameter $\widehat{\lambda} + \rho$.*

The main tool for the proof of the first assertion is the solution by Pink of a conjecture of Harder [**59**], together with a repeated use of our Theorem 1 for $\mathrm{GSp}(2(g-r))$ for all integers $r = 1, \ldots, g$. To apply this argument, we need a mod. p version of Kostant's formula, proven in Theorem B of [**61**] under the assumption of p-smallness. This allows to apply Pink's theorem in a fashion similar to [**37**] (who worked in characteristic zero). The second assertion follows by using a result of Wallach [**85**], resp. direct calculations of [**72**].

We state in Section 9 and 10 several consequences of these results:

– Control theorem and existence of p-ordinary cuspidal Hida families for G, improving upon [**77**],

– Verification of a condition of freeness of a cohomology module occuring in the definition of a Taylor-Wiles system.

1.5. Let us briefly discuss the proof of Theorem 1. Let $V_\lambda(\mathbb{F}_p)$ resp. $V_\lambda(k)$ be the etale sheaf over $X \otimes \mathbb{Q}$ associated to the representation $V_{\lambda\,/\mathbb{F}_p}$ of $G_{\mathbb{F}_p} = G \otimes \mathbb{F}_p$, of highest weight λ, resp. its extension of scalars to k. As mentioned in Comment 1) to Theorem 1, it is enough to show that

$$(*) \qquad W^j_* = H^j_*(X \otimes \overline{\mathbb{Q}}, V_\lambda(k))[\mathfrak{m}] = 0$$

where $* = \varnothing$ or c, and for any $j < d$.

Let $X_{/\mathbb{Z}[1/N]}$ be the moduli scheme classifying g-dimensional p.p.a.v. with level U structure over $\mathbb{Z}[1/N]$. Let $\overline{X}$ be a given toroidal compactification over $\mathbb{Z}[1/N]$ (see Th. 6.7 of Chap. IV [**13**], or Fujiwara [**30**]). Let $X_0 = X \otimes \mathbb{F}_p$, $\overline{X}_0 = \overline{X} \otimes \mathbb{F}_p$.

To the representation $V_{\lambda\,/\mathbb{F}_p}$ (with $|\lambda+\rho| < p-1$), one associates also a filtered log-crystal $\overline{\mathcal{V}}_\lambda$ over $\overline{X}_0$ (see Section 5.2 below); the F-filtration on the dual $\overline{\mathcal{V}}^\vee_\lambda$, satisfies $\mathrm{Fil}^0 = \overline{\mathcal{V}}^\vee_\lambda$ and $\mathrm{Fil}^{|\lambda|+1} = 0$. Then, the main tools for proving $(*)$ are

– Faltings's Comparison Theorem ([**22**], Th. 5.3, see Sect. 6.1). It says that, since $p-1 > \boldsymbol{w}$, for any $j \geqslant 0$, the linear dual of $H^j_*(X \otimes \overline{\mathbb{Q}}_p, V_\lambda(\mathbb{F}_p))$ is the image by the usual contravariant Fontaine-Laffaille functor $\mathbf{V}^*$ of the logarithmic de Rham cohomology

$$M = H^j_{\text{log-dR},*}(\overline{X} \otimes \mathbb{F}_p, \overline{\mathcal{V}}^\vee_\lambda) = H^j(\overline{\mathcal{V}}^\vee_\lambda \otimes \Omega^\bullet_{\overline{X}_0}(\log\infty)).$$

– The mod. p generalized Bernstein-Gelfand-Gelfand dual complex (section 5.4)

$$\kappa : \overline{\mathcal{K}}^\bullet_\lambda \hookrightarrow \overline{\mathcal{V}}^\vee_\lambda \otimes \Omega^\bullet_{\overline{X}_0}.$$

This is the mod. p analogue of a construction carried in Chapter VI of [**13**]. The main result is that κ is a filtered quasi-isomorphism: it provides an explicit description of the jumps of the Hodge filtration in terms of group-theoretic data. In particular for $j < d$, $\boldsymbol{w}$ is not a jump.

– Lemma 13 in Section 7.1 shows, assuming **(RLI)** and **(GO)**, that if $W^j \neq 0$, its restriction to the inertia group I_p admits $k \otimes \mathbb{Z}/p\mathbb{Z}(-\boldsymbol{w})$ as subquotient.

Thus if $W^j \neq 0$ we obtain a contradiction since the maximal weight $\boldsymbol{w}$ should not occur in W^j.

Theorem 2 is equivalent to the fact that the localization at $\mathfrak{m}$ of the degree d boundary cohomology of $V_\lambda(k)$ vanishes. The argument for this is similar to the previous one, but makes use of the minimal compactification $j : X_{\mathbb{Q}} \hookrightarrow X^*_{\mathbb{Q}}$ of $X_{\mathbb{Q}} = X \otimes \mathbb{Q}$ (instead of the toroidal one). The advantage of this compactification is that Hecke correspondences extend naturally. We use crucially a theorem of R. Pink (Th. 4.2.1 of [**59**]) which describes the Galois action on the cohomology of each stratum with coefficents in the étale sheaves $R^q j_* V_\lambda(k)$; by the spectral sequence of the stratification it is enough to show the vanishing of the localization at $\mathfrak{m}$ of the degree d cohomology of each individual stratum. For this, we follow the same lines as for the proof of Theorem 1: the jumps of the Hodge filtration in the degree d cohomolology with compact support $H^d_c(X_r)$ of the non-open strata X_r cannot contain both $\boldsymbol{w}$ and 0;

on the other hand, if the $\mathfrak{m}$-torsion of $H_c^d(X_r)$ is not 0, Lemma 13 does imply that these weights both occur. Hence, $H_c^d(X_r)_{\mathfrak{m}} = 0$. The last two sections contain two applications which were the original motivations for this work.

Acknowledgements. — To start with, the authors have greatly benefitted of the seminar on toroidal compactifications and cohomology of Siegel varieties held at the University of Paris-Nord in 97-98. Without it, this work wouldn't have existed. They wish to thank the participants thereof, and in particular, A. Abbès. Part of the writing was done by the second author at UCLA, at MPI (Bonn) and at Kyoto University; the excellent working conditions of these institutions were appreciated. A series of lectures on this work at MRI (Allahabad) on the invitation of D. Prasad in the frame of CEFIPRA program 1501B was also quite helpful. The first author would like to express his gratitude to the Mathematic Department of Padova University, and especially B. Chiarellotto for an invitation where a part of this work was exposed. During the preparation of this text, we had useful discussions or correspondence with many persons, in particular, A. Abbès, D. Blasius, M. Dimitrov, A. Genestier, G. Harder, J.C. Jantzen, K. Khuri-Makdisi, K. Kuennemann, A. Nair, B.C. Ngo, R. Pink, P. Polo, B. Roberts, J. Wildeshaus, H. Yoshida. We thank them heartily for their patience and good will. Finally, let the referees be thanked for their impressively careful reading and useful comments.

2. Cohomology of Siegel varieties and automorphic representations

We keep the notations of the introduction. Let us make precise what we mean by a good open compact subgroup of $G(\widehat{\mathbb{Z}})$: U is good if

1) it is neat: the subgroup of $\mathbb{C}^\times$ generated by the eigenvalues of elements in $U \cdot G_\infty \cap G_{\mathbb{Q}}$ does not contain any root of unity other than 1, and

2) $\nu(U) = \widehat{\mathbb{Z}}^\times$.

Let us now recall some properties of the cohomology groups $H_*^\bullet(S_U, V_\lambda(K))$, for K a field of characteristic zero and $* = \varnothing$, c or $!$ (as usual, $H_!^\bullet$ denotes the image of $H_c^\bullet$ in $H^\bullet$). In this section, $\mathfrak{g} = \mathrm{Lie}(G)$ will denote the real Lie algebra.

2.1. Generalities over $\mathbb{C}$. — Let U_∞ be the stabilizer in G_∞ of the map

$$h : \mathbb{C}^\times \longrightarrow G_\infty, \quad z = x + iy \longmapsto \begin{pmatrix} x \cdot 1_g & y \cdot s_g \\ -y \cdot s_g & x \cdot 1_g \end{pmatrix}$$

with s_g the $g \times g$ antidiagonal matrix, with non-zero entries equal to 1. For any good compact open subgroup $U \subset G(\widehat{\mathbb{Z}})$, let

$$S_U = G(\mathbb{Q})\backslash G(\mathbb{A})/UU_\infty \quad \text{and}\, S = G(\mathbb{Q})\backslash G(\mathbb{A})/U_\infty$$

be the Siegel varieties of level U, resp. infinite level. Since U has no torsion, S_U is a smooth quasi-projective algebraic variety of dimension $d = \frac{g(g+1)}{2}$. S is a pro-variety.

For any (rational) irreducible representation V_λ of G of highest weight λ, we define the local system $V_\lambda(\mathbb{C})$ on S_U as the locally constant sheaf of sections of

$$pr_1 : G(\mathbb{Q})\backslash \left(G(\mathbb{A}) \times V_\lambda \otimes \mathbb{C}\right)/UU_\infty \longrightarrow S_U$$

By Prop. 2.7 of [**8**](which does not require cocompactness), one has

$$H^\bullet(S_U, V_\lambda(\mathbb{C})) = H^\bullet(\mathfrak{g}, U_\infty, \mathcal{C}^\infty(G_\mathbb{Q}\backslash G_\mathbb{A}, \mathbb{C}) \otimes V_\lambda(\mathbb{C})).$$

The maps of spaces

$$\mathcal{C}^\infty_{\mathrm{cusp}}(G_\mathbb{Q}\backslash G_\mathbb{A}, \mathbb{C}) \longrightarrow \mathcal{C}^\infty_{c/\mathrm{center}}(G_\mathbb{Q}\backslash G_\mathbb{A}, \mathbb{C}) \subset \mathcal{C}^\infty_{(2)}(G_\mathbb{Q}\backslash G_\mathbb{A}, \mathbb{C}) \subset \mathcal{C}^\infty(G_\mathbb{Q}\backslash G_\mathbb{A}, \mathbb{C})$$

(where the first map denotes a smooth truncation to a large compact mod. center subset, and $\mathcal{C}^\infty_{\mathrm{cusp}} = \mathcal{C}^\infty \cap L^2_0$ and $\mathcal{C}^\infty_{(2)} = \mathcal{C}^\infty \cap L^2$) give rise to maps

$$H^\bullet_{\mathrm{cusp}}(S, V_\lambda(\mathbb{C})) \longrightarrow H^\bullet_c(S, V_\lambda(\mathbb{C})) \longrightarrow H^\bullet_{(2)}(S, V_\lambda(\mathbb{C})) \longrightarrow H^\bullet(S, V_\lambda(\mathbb{C}))$$

and a well-known theorem of Borel [**5**] asserts that their composition is injective:

$$H^\bullet_{\mathrm{cusp}}(S, V_\lambda(\mathbb{C})) \hookrightarrow H^\bullet_!(S, V_\lambda(\mathbb{C})).$$

Moreover, as in the proof of Th. 3.2 (or Th. 5.2) of [**8**], one has a G_f-equivariant decomposition

$$\begin{aligned} H^\bullet_{\mathrm{cusp}}(S, V_\lambda(\mathbb{C})) &= H^\bullet(\mathfrak{g}, U_\infty, \mathcal{C}^\infty_{\mathrm{cusp}}(G_\mathbb{Q}\backslash G_\mathbb{A}, \mathbb{C}) \otimes V_\lambda(\mathbb{C})) \\ &= \bigoplus_\pi \pi_f \otimes H^\bullet(\mathfrak{g}, U_\infty, \pi_\infty^{U_\infty} \otimes V_\lambda(\mathbb{C})) \end{aligned}$$

where $\pi = \pi_f \otimes \pi_\infty$ runs over the set of isomorphism classes of cuspidal representations and $\pi_\infty^{U_\infty}$ is the Harish-Chandra module of π_∞.

Proposition 1. — *If λ is regular dominant or if $g = 2$, the interior, L^2 and cuspidal cohomology groups coincide and are concentrated in middle degree:*

$$H^\bullet_{\mathrm{cusp}}(S, V_\lambda(\mathbb{C})) = H^\bullet_{(2)}(S, V_\lambda(\mathbb{C})) = H^\bullet_!(S, V_\lambda(\mathbb{C})) = H^d_!(S, V_\lambda(\mathbb{C})).$$

Proof. — Recall first that $H^\bullet_{\mathrm{cusp}} = H^\bullet_{(2)}$ implies $H^\bullet_{\mathrm{cusp}} = H^\bullet_{(2)} = H^\bullet_!(S, V_\lambda(\mathbb{C}))$ (see also Cor. to Th. 9 of [**21**]).

By Th. 4 of [**6**] (which applies here since $\mathrm{rk}\, G = \mathrm{rk}\, U_\infty$):

$$\begin{aligned} H^\bullet_{(2)}(S, V_\lambda(\mathbb{C})) &= H^\bullet(\mathfrak{g}, U_\infty, \mathcal{C}^\infty_{(2)}(G_\mathbb{Q}\backslash G_\mathbb{A}, \mathbb{C}) \otimes V_\lambda(\mathbb{C})) \\ &= \bigoplus_\pi \pi_f \otimes H^\bullet(\mathfrak{g}, U_\infty, \pi_\infty^{U_\infty} \otimes V_\lambda(\mathbb{C})) \end{aligned}$$

where π runs over the discrete spectrum of $L^2(Z_\mathbb{A} G_\mathbb{Q}\backslash G_\mathbb{A}, \omega)$ where ω is the central character of $V_\lambda^\vee$.

Let $\pi = \pi_f \otimes \pi_\infty$ be such an automorphic representation; its local components are unitary. Moreover, one must have $H^\bullet(\mathfrak{g}, U_\infty, \pi_\infty^{U_\infty} \otimes V_\lambda(\mathbb{C})) \neq 0$. By [**82**] Th. 5.6, the assumption that λ is regular implies that $\pi_\infty = A_\mathfrak{q}(\lambda)$, is a cohomological induction from a parabolic subalgebra $\mathfrak{q}$ which must be that of the Borel. In that case, this induction provides the discrete series. So, π_∞ is one of the 2^{g-1} unitary representations of G_∞ in the discrete series of HC parameter $\widehat{\lambda}+\rho$. By [**8**] Chap. III, Cor. 5.2 (iii), the tempered unitary π_∞'s contribute only in middle degree; Moreover, since the automorphic representation $\pi=\pi_f\pi_\infty$ occurs in the global discrete spectrum and admits at least one local component which is tempered, it must be cuspidal; indeed, a theorem of Wallach ([**85**], Th. 4.3) asserts that if π_∞ is tempered, the multiplicity of π in L^2_{disc} is equal to that in L^2_0.

If $g=2$, the classification of Vogan-Zuckerman [**82**] as explicited in Section 1 of [**72**] yields the vanishing of H^1 and the temperedness of the π_∞ occuring in H^3. Then one concludes as above.

Remark. — If λ is not regular, there may also be non-tempered representations π_∞ which occur as infinity type of π. However, by Langlands classification ([**8**], Sect. 4.8, Th. 4.11) and Th. 6.1 of [**8**], it implies that $H^q_{(2)}(S,V_\lambda)(\pi_f)\neq 0$ for some $q<d$. Franke's spectral sequence (below) seems to suggest then that $H^q(S,V_\lambda)(\pi_f)\neq 0$ (we leave this as a question).

This proposition will be used in the proof of Theorem 2 (in Section 8 below) to rule out the occurence of non-cuspidal representations in the localization of the middle degree L^2-cohomology $H^\bullet_{(2)}(S_U,V_\lambda)$, at a "non-Eisenstein" maximal ideal of the Hecke algebra (that is, satisfying **(RLI)**).

2.2. Franke's spectral sequence. — This section is not used in the sequel, but it provides a motivation for Section 8. By [**8**] Chap. VII Cor. 2.7, we have

$$H^\bullet(S,V_\lambda(\mathbb{C}))=H^\bullet(\mathfrak{g},U_\infty;\mathcal{C}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))\otimes V_\lambda(\mathbb{C}))$$

By [**7**], one can replace the space of $\mathcal{C}^\infty$-functions by those of uniformly moderate growth. Franke has shown ([**25**], Th. 13, or [**84**] 2.2) that one can even replace this space by the space $\mathcal{A}(G)$ of automorphic forms on G. He has moreover defined a filtration on$\mathcal{A}(G)$, called the Franke filtration (see [**84**] 4.7) whose graded pieces interpret as L^2-cohomology. This yields an hypercohomology spectral sequence associated to a filtered complex; more precisely:

Let Φ^+, resp. Φ^+_L, be the positive root system of G, resp. of a standard Levi L of G, given by (G,B,T), resp. $(L,B\cap L,T)$. The corresponding simple roots are denoted by Δ, resp. Δ_L. For each standard parabolic $P=L\cdot U$, let $\mathfrak{a}_P$ is the Lie algebra of the center of L. Recall then Franke's spectral sequence ([**25**] Th. 19 or [**84**] Corollaire 4.8)

$$E_1^{p,q}=H^{p+q}_{(2)}(S,V_\lambda(\mathbb{C}))\bigoplus_P\bigoplus_{w\in W^P(\lambda,p)}\mathrm{Ind}_{P_f}^{G_f}H^{p+q-\ell(w)}_{(2)}(S(L),V(L;w\cdot(\lambda+\rho)))$$
$$\Longrightarrow H^{p+q}(S,V_\lambda(\mathbb{C}))$$

where

– $P = L \cdot U_P$ runs over the set of proper standard parabolic subgroups,
– P_f, resp. G_f denotes the group of $\mathbb{A}_f$-points of P, resp. G,
– for each p, $W^P(\lambda, p)$ is a certain subset of

$$W^L = \{w \in W \mid w^{-1}(\alpha) > 0, \text{ for all } \alpha \in \Phi_L\},$$

so that $W^L = \coprod_p W^P(\lambda, p)$,

– the locally constant sheaf $V(L; w \cdot (\lambda+\rho))$ on the provariety $S(L)$ is attached to the representation of L of highest weight $w\cdot(\lambda+\rho) = w(\lambda+\rho)-\rho$ (dominant for the order given by $(L, B\cap L, T)$), twisted by $-w(\lambda+\rho)|_L$, that is, by the one-dimensional representation of L attached to the (exponential of the) restriction of $-w\cdot(\lambda+\rho)$ to its (co-)center $\mathfrak{a}_P$.

This spectral sequence is G_f-equivariant. It allows one to represent any G_f-irreducible constituent of $H^{p+q}(S, V_\lambda(\mathbb{C}))$ as $\operatorname{Ind}_{P_f}^{G_f} \pi_f$ where π_f is an irreducible admissible representation of L_f such that $\pi = \pi_f \otimes \pi_\infty$ is automorphic, in the discrete spectrum of $L^2(L_\mathbb{Q} Z_\mathbb{A}\backslash L_\mathbb{A}, \phi)$ with P a rational parabolic in G, L its Levi quotient, and ϕ some unitary Hecke character.

Moreover, by Th. 19(ii) of [**25**], if λ is regular, Franke's spectral sequence degenerates at $E_1^{p,q}$. So, we have a Hecke-equivariant decomposition for each degree $q \in [0, 2d]$:

$$H^q(S, V_\lambda(\mathbb{C})) = IH^q(S_U, V_\lambda(\mathbb{C})) \oplus \bigoplus_P \bigoplus_{p=0}^{q} \bigoplus_{w\in W^P(\lambda,p)} IH^{q-\ell(w)}(S^L, V^L_{w(\lambda+\rho)-\rho}(\mathbb{C})(-w\cdot(\lambda+\rho)_L)).$$

However, unlike the GL_n-case, the question of the rationality of this splitting for the group G is open (with a possibly negative answer). We nevertheless expect that it should yield, after localization at a "non-Eisenstein" maximal prime ideal of the Hecke algebra, an equality of the form

$$IH^q(S_U, V_\lambda(\mathbb{C}))_\mathfrak{m} = H^q(S_U, V_\lambda(\mathbb{C}))_\mathfrak{m}$$

for λ regular. We establish this in Section 8 below for a suitable $\mathfrak{m}$, by a Galois-theoretic argument which in some sense replaces the lacking Jacquet-Shalika theorem.

2.3. Hodge filtration in characteristic zero. — Recall we assumed that U is good, so that its projection to any Levi quotient of G is torsion-free and $\nu(U) = \widehat{\mathbb{Z}}^\times$. By the first condition, S_U is smooth; the second condition implies that S_U admits a geometrically connected canonical model over $\mathbb{Q}$. Let X be this canonical model; it is a geometrically connected smooth quasi-projective scheme over $\mathbb{Q}$. Let $\overline{X}$ a toroidal compactification of X defined by an admissible polyhedral cone decomposition of $\operatorname{Sym}^2 X^*(T)$ ([**1**] Chap. 3 and [**13**] Chap. IV, Th. 5.7). Let $\infty_X = \overline{X} - X$ be the divisor with normal crossings at infinity. Let $f : A \to X$ be the universal principally polarized abelian variety with level U-structure over X (it exists over $\mathbb{Q}$). Let Q be the Siegel parabolic of G, that is, the maximal parabolic associated to the longest simple root for (G, B, T); let M its Levi subgroup. For any B_M-dominant weight μ, let $\mathcal{W}(\mu)$ resp. $\overline{\mathcal{W}}(\mu)$, be the corresponding automorphic vector bundle on X, resp. its canonical Mumford extension to $\overline{X}$ (see Th. 4.2, Chap. VI of [**13**]). These are coherent sheaves. As observed by Harris [**36**], the coherent cohomology $H^\bullet(\overline{X}, \overline{\mathcal{W}}(\mu))$ has a natural action of the Hecke algebra. Let $\lambda = (a_g, \dots, a_1; c)$ as above (recall that for simplicity we assume $c = a_g + \cdots + a_1$). Let $H = \mathrm{diag}(0, \dots, 0, -1, \dots, -1) \in \mathfrak{g}$.

2.3.1. Complex Hodge Filtration. — It results from Deligne's mixed Hodge theory that the complex cohomology $H^m(X, V_\lambda(\mathbb{C}))$ carries a mixed Hodge structure with Hodge weights greater than, or equal to $m + |\lambda|$ and that the interior cohomology (image of $H_c^m \to H^m$) is pure of Hodge weight $m + |\lambda|$. It is studied in greater details in Sect. 6.5 of [**13**]. We won't need any information about its W-filtration, so we concentrate on its F-filtration (Hodge filtration). With the notation of 6.4 of [**13**], de Rham comparison theorem reads:

$$H^m(X(\mathbb{C}), V_\lambda(\mathbb{C})) = H^m_{\mathrm{dR}}(X(\mathbb{C}), \mathcal{V}_\lambda^\vee)$$

where $\mathcal{V}_\lambda$ denotes the coherent sheaf associated to the Q-representation restriction to the Siegel parabolic Q of the G-representation of highest weight λ. The reason for the dual (denoted $^\vee$) is the following. The de Rham comparison theorem sends the local system $R^1 f_* \mathbb{C}$ on $R^1 f_* \Omega^\bullet_{A/X}$; however, as explained on top of page 224 of [**13**], the construction of coherent sheaves from Q-representations associates to the standard representation the dual of $R^1 f_* \Omega^\bullet_{A/X}$, while the locally constant sheaf associated to the standard representation is $R^1 f_* \mathbb{C}$.

Let $\mathfrak{g}$, resp. $\mathfrak{t}$, be the Lie algebra of G, resp. T. Let

$$H = \mathrm{diag}(0, \dots, 0, -1, \dots, -1) \in \mathfrak{t}$$

Let W^M be the set of Kostant representatives of the quotient $W_M \backslash W_G$ of the Weyl groups; for each $w \in W^M$, let $p(w) = -\left(w(\lambda + \rho) - \rho\right)(H)$; it is a non-negative integer. The main result of Sect. 6.5 (Theorem 5.5(i), Chap. VI) of [**13**] gives a Hecke-equivariant description of the graded pieces of the F-filtration in terms of coherent cohomology of automorphic vector bundles extended to a toroidal compactification $\overline{X}$ of X, as follows:

$$(BGG) \qquad \mathrm{gr}^p_F H^\bullet(X, V_\lambda(\mathbb{C})) = \bigoplus_{\substack{w \in W^M \\ p(w) = p}} H^{\bullet - \ell(w)}(\overline{X}, \overline{\mathcal{W}}(w(\lambda + \rho) - \rho)^\vee)$$

Because of our comment on de Rham comparison theorem, we see that contrary to what is mentioned in R. Taylor's paper ([**72**] p. 295, l. 14 from bottom), the statement of Th. 5.5, l. 6 in [**13**] is correct, because the local system denoted V_λ in Faltings-Chai is actually dual to the one denoted V_λ in Taylor's and in the present paper. Our statement, in accordance to Faltings', is that the sum runs over the w such that $w(\lambda + \rho)(H) + p = \rho(H)$. We think therefore that Taylor's statement cited above is incorrect (but correct after a Tate twist, anyway).

For any subset B of $A = \{1, \dots, g\}$, let $(B, \overline{B})$ the corresponding partition of A. We define $w_B \in W_G$ by its action on $(t;\nu) \in T$: for $t = (t_B, t_{\overline{B}})$, one puts $w_B(t;\nu) = (t_B^{-1}, t_{\overline{B}};\nu)$. An easy calculation shows that for any $w \in W_G$, if $w = (\sigma, w_B)$ for some permutation σ of A and B some subset of A, one has:

$$p(w) = -(w(\lambda+\rho)-\rho)(H) = -(w_B(\lambda+\rho)-\rho)(H) = \sum_{i\in B}(a_i+i)$$

We put $j_B = \sum_{i\in B}(a_i+i)$, so $j_A = \boldsymbol{w}$ is the motivic weight defined in the introduction. The j_B's belong to the closed interval $[0, \boldsymbol{w}]$. They are indexed by a set of cardinality 2^g, but need not be mutually distinct, from $g = 3$ on. Note that for any degree m of the cohomology, the jumps of the Hodge filtration occuring in H^m always form a subset of $\{j_B \mid B \subset A\}$.

Let $\pi = \pi_f \otimes \pi_\infty$ be a cuspidal representation of $G(\mathbb{A})$, with π_∞ holomorphic in the discrete series of HC parameter $\widehat{\lambda} + \rho$; let $\theta_\pi : \mathcal{H}^N \to \mathbb{C}$ be the character of the (prime-to-N) Hecke algebra, associated to π and $\mathfrak{p}_\pi = \operatorname{Ker}\theta_\pi$. By [**8**] Chap. III Th. 3.3(ii), the $(\mathfrak{g}, U_\infty)$-cohomology of $\pi_\infty \otimes V_\lambda$ is concentrated in degree d. we put

$$W_\pi = H^d(X, V_\lambda(\mathbb{C}))[\mathfrak{p}_\pi]$$

By cuspidality of π, W_π has a Hodge structure pure of weight $\boldsymbol{w} = d + |\lambda|$:

$$W_\pi = \bigoplus_{p+q=\boldsymbol{w}} W_\pi^{p,q}$$

Let us show that $W_\pi^{\boldsymbol{w},0}$ and $W_\pi^{0,\boldsymbol{w}}$ are both non-zero. More precisely, let $w' \in W^M$ be the Kostant representative of largest length, namely d (it is unique, and if $w'' \in W_M$ is the unique element of largest length, then $w'w''$ is the unique element of largest length in W_G). Then,

Proposition 2. — *There is a $\mathcal{H}^N$-linear embedding*

$$\pi_f^U \subset H^{\boldsymbol{w},0} = H^0(\overline{X}, \overline{\mathcal{W}}_{w'(\lambda+\rho)-\rho}), \quad \pi_f^U \subset H^{0,\boldsymbol{w}} = H^d(\overline{X}, \overline{\mathcal{W}}_\lambda).$$

Proof. — Let $\mathfrak{q}$ be the Lie algebra of the Siegel parabolic. Since π is cuspidal, a calculation of M. Harris, Prop. 3.6 of [**36**] shows that for any q and μ M-dominant, $\pi_f^U \otimes H^q(\mathfrak{q}, U_\infty, \pi_\infty \otimes W_\mu)$ embeds $\mathcal{H}^N$-linearly into $H^q(\overline{X}, \overline{\mathcal{W}}_\mu)$. Moreover by Theorem 3.2.1 of [**9**], $H^q(\mathfrak{q}, U_\infty, \pi_\infty \otimes W_\mu)$ does not vanish in only two cases: $\mu = \lambda$ and $q = d$, or $\mu = w'(\lambda+\rho) - \rho$ and $q = 0$.

Remark. — If π is stable at infinity, that is, if all the possible infinity types π'_∞ in the discrete series of HC parameter $\widehat{\lambda}+\rho$ give rise to automorphic cuspidal representations $\pi' = \pi_f \otimes \pi'_\infty$, then all the possible Hodge weights do occur in W_π:

$$\text{For any } j_B,\ B \subset A,\ A = B \coprod \overline{B} \quad W_\pi^{j_B, j_{\overline{B}}} \neq 0.$$

2.3.2. p-adic Hodge filtration. — The Hodge-to-de Rham spectral sequence

$$(BGG)_{\mathbb{Q}} \quad E_1^{p,q} = \bigoplus_{\substack{w\in W^M \\ p(w)=p}} H^{p+q-\ell(w)}(\overline{X}, \overline{\mathcal{W}}(w(\lambda+\rho)-\rho)) \Longrightarrow H^{p+q}(\overline{X}, \overline{\mathcal{V}}_\lambda \otimes \Omega^\bullet_{\overline{X}/\mathbb{Q}}(\log \infty_X))$$

makes sense over $\mathbb{Q}$ and degenerates in $E_1^{p,q}$ ([**13**] Sect. VI.6, middle of page 238). Here, $\mathcal{V}_\lambda$ denotes the flat vector bundle defined over $\mathbb{Q}$ associated to the rational representation V_λ of G. More explanations on the rational structures involved, as well as integral versions thereof will be given in Sections 5.2 and 5.3.

Actually, let C be the completion of an algebraic closure of $\mathbb{Q}_p$; by Th. 6.2 of [**13**], there is a Hodge-Tate decomposition theorem inducing the splitting of $(BGG)_{\mathbb{C}}$; More precisely:

$$(BGG)_{HT} \quad H^{p+q}(X, V_\lambda(\mathbb{Q}_p))\otimes C \cong \bigoplus_{\substack{w\in W^M \\ p(w)=p}} H^{p+q-\ell(w)}(\overline{X}, \overline{\mathcal{W}}(w(\lambda+\rho)-\rho))\otimes C(p(w)).$$

By a theorem of Harris [**9**], the Hecke algebra $\mathcal{H}^N$ acts naturally on each summand of the LHS of this splitting. Now, the main feature of the above splitting is its naturality for algebraic correspondences on $\overline{X}$. It implies the compatibility of the decomposition $(BGG)_{HT}$ with the action of $\mathcal{H}^N$. Let $K_0 \subset \mathbb{C}$ be a number field containing the image of θ_π. Let $W_{\pi,K_0} = H^d(X, V_\lambda(K_0))[\mathfrak{p}_\pi]$. We fix a p-adic embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let K be the corresponding completion of K_0 ; put $W_{\pi,p} = W_{\pi,K_0} \otimes_{K_0} K$. The restriction of $(BGG)_{HT}$ to the part killed by $\mathfrak{p}_\pi$ is still a $\mathcal{H}^N$-equivariant decomposition of $W_{\pi,p} \otimes_K C$. If we assume **(Hol)**, we see from Prop. 1 above that the Hodge-Tate weights $\boldsymbol{w}$ and 0 do occur; indeed,

$$W_{\pi,p}^{\boldsymbol{w},0} = H^0(\overline{X}, \overline{\mathcal{W}}_{w'(\lambda+\rho)-\rho})[\mathfrak{p}_\pi] \quad \text{and} \quad W_{\pi,p}^{0,\boldsymbol{w}} = H^d(\overline{X}, \overline{\mathcal{W}}_\lambda)[\mathfrak{p}_\pi]$$

by comparing to complex cohomology, we see from Prop. 1 that these two spaces are non-zero.

Let us remark that if π is stable at infinity, the analogue of Prop. 2 for all possible infinity types in the discrete series of HC parameter $\widehat{\lambda} + \rho$ (in number 2^g, but isomorphic two by two) implies that all the possible Hodge-Tate weights j_B $(B \subset A)$ do occur in the Hodge-Tate decomposition of $W_{\pi,p}$.

3. Galois representations

3.1. Relation between ρ_π and $W_{\pi,p}$. — The absolute Galois group Γ acts on $W_{\pi,p}$. Let us first recall, for later use, the following well-known fact.

Lemma 1. — *$W_{\pi,p}$ is pure of weight $\boldsymbol{w}$. That is, for any ℓ prime to Np, all the eigenvalues of the geometric Frobenius at ℓ have archimedean absolute value $\ell^{\boldsymbol{w}/2}$.*

Proof. — Since π is cuspidal, we know by a theorem of Borel (see Sect. 2.1) that $W_{\pi,p}$ is contained in the interior cohomology $H_!^d(X, V_\lambda)$. By Th. 1.1 of Chap. VI of [**13**], there is a toroidal compactification $Y \subset \overline{Y}$ of the $|\lambda|$-times fiber product $Y = A^{|\lambda|}$ of the universal abelian variety A above a toroidal compactification of the Siegel variety $X \subset \overline{X}$, all these schemes being flat over $\mathbb{Z}[1/N]$; over this base, $\overline{Y}$ is smooth and $\overline{Y} - Y$ is a divisor with normal crossings. One can interpret the etale sheaf as cut by algebraic correspondences in $(R^1\pi_*\mathbb{Q}_p)^{\otimes d}$ (see [**13**] p. 235, and 238, or this text, Sect. 5.2), hence $H_*^d(X, V_\lambda) \subset H_*^{\boldsymbol{w}}(Y, \mathbb{Q}_p)$ $(* = \varnothing, c)$. By the classical commutative diagram (coming from the degeneracy of the Leray spectral sequence):

$$\begin{tikzcd} H_c^{\boldsymbol{w}}(Y, \mathbb{Q}_p) \arrow[r] & H^{\boldsymbol{w}}(\overline{Y}, \mathbb{Q}_p) \arrow[r] & H^{\boldsymbol{w}}(Y, \mathbb{Q}_p) \\ H_c^d(X, V_\lambda) \arrow[u, hook] \arrow[rr] & & H^d(X, V_\lambda) \arrow[u, hook] \end{tikzcd}$$

We conclude that $H_!^d(X, V_\lambda)$ is pure of weight $\boldsymbol{w}$; recall that this can be interpreted either in the sense of Deligne (take ℓ unramified and different from p, then the eigenvalues of geometric Fr_ℓ have archimedean absolute values $\ell^{\boldsymbol{w}/2}$) or in a p-adic sense (in the crystalline case, say: that the eigenvalues of the crystalline Frobenius have archimedean absolute values $p^{\boldsymbol{w}/2}$).

Assume now that π admits an associated p-adic Galois representation $\rho_\pi : \Gamma \to \mathrm{GL}_V(\overline{\mathbb{Q}}_p)$; we assume that ρ_π is irreducible. We don't know a priori whether ρ_π is a Galois constituent of $W_{\pi,p}$ although, by [**13**] Chapter VII Th. 6.2, we know that the characteristic polynomial of ρ_π annihilates the global p-adic representation $W_{\pi,p}$. If moreover p does not divide N, we know by Faltings [**22**] Th. 5.2 that $W_{\pi,p}$ is crystalline but we don't know this a priori for ρ_π. However, for $g \leqslant 2$, if ρ_π is absolutely irreducible, we do know that it is a constituent of $W_{\pi,p}$ (by [**72**] and [**53**] or [**87**]). Indeed, for $g = 2$, Laumon [**53**] and also Weissauer (completing works of [**34**], [**72**] and [**52**]) has shown the existence of a four-dimensional p-adic Galois representation

$$\rho_\pi : \Gamma \longrightarrow \mathrm{GL}_V(\overline{\mathbb{Q}}_p)$$

such that

$$L(W_\pi, s)^4 = L(\rho_\pi, s)^m$$

thus, the assumption of irreducibility for ρ_π implies that the Galois semisimplification $W_{\pi,p}^{s.s.}$ of $W_{\pi,p}$ is isomorphic to $n.\rho_\pi$, for $m = 4n$. In turn, it also implies that ρ_π is pure of weight $\boldsymbol{w}$ and is crystalline at p if p is prime to N.

There are other situations, namely when π is a base change of a Hilbert modular eigenform, where one knows that ρ_π is crystalline, although one may not know that it is a constituent of $W_{\pi,p}$; see Sect. 7.3 below. One of the uses of our assumption **(RLI)** will be to relate (residually only) $W_{\pi,p}$ and ρ_π (see Sect. 7.1).

3.2. Spin groups and duality

3.2.1. description. — For the general definitions on spinors, we follow [**32**] Sect. 20.2, and [**18**] VIII.8 and IX.2; however by lack of references for our precise need, we give some details in this section. Although these groups exist over $\mathbb{Z}$, we'll restrict ourselves to $\mathbb{Z}[1/2]$, ($p = 2$ is excluded of our study). Let $\widetilde{V} = \mathbb{A}^{2g+1}_{\mathbb{Z}[1/2]}$ endowed with the quadratic form $q(x) = \sum_{i=1}^{g} 2x_i x_i' + x_0^2$ for $x = \sum_{i=1}^{g} x_i e_i + x_0 e_0 + \sum_{i=1}^{g} x_i' e_i'$; the scalar product is denoted by $\langle x, y\rangle$. The canonical basis is ordered as $(e_g, \ldots, e_1, e_0, e_1', \ldots, e_g')$, so that $\langle e_i, e_j'\rangle = \delta_{i,j}$, e_0 is unitary, $W = \langle e_g, \ldots, e_1\rangle$ and $W' = \langle e_1', \ldots, e_g'\rangle$ are totally isotropic, and the sum $\widetilde{V} = W \oplus W' \oplus \langle e_0\rangle$ is orthogonal. The Clifford algebra $C(\widetilde{V}, q)$ over $\mathbb{Z}[1/2]$ is the quotient of the tensor algebra by the two-sided ideal generated by $x \otimes x - q(x)$, $(x \in \widetilde{V})$; it is $\mathbb{Z}/2\mathbb{Z}$-graded: $C(\widetilde{V}, q) = C^+ \oplus C^-$. The main involutive automorphism Π is defined as Id on C^+ and $-\mathrm{Id}$ on C^-; the main antiinvolution $x \mapsto x^*$ is defined by $v_1 \cdots v_r \mapsto (-1)^r v_r \cdots v_1$. We write $N(x) = x \cdot x^* = x^* \cdot x$ for the spinor norm. The $\mathbb{Z}[1/2]$-group scheme $\mathrm{GSpin}_{\widetilde{V}} = \mathrm{GSpin}_{2g+1}$ (called the regular Clifford group in [**18**] IX.2) is defined as the group of invertible elements g of $C(V, q)$ such that $g \cdot \widetilde{V} \cdot g^{-1} = \widetilde{V}$. The group of orthogonal similitudes $\mathrm{GO}_{\widetilde{V}} = \mathrm{GO}_{2g+1}$ is defined as the group of $h \in \mathrm{GL}_{\widetilde{V}}$ such that $q \circ h = c(h) \cdot q$. Consider the group-scheme morphism

$$\nu : \mathrm{GO}_{2g+1} \longrightarrow \mathbb{G}_m, \quad h \longmapsto \det h \cdot c(h)^{-g}.$$

One has $c(h) = \nu^2(h)$. Moreover, the homomorphism of $\mathbb{Z}[1/2]$-group schemes

$$\psi : \mathrm{GSpin}_{\widetilde{V}} \longrightarrow \mathrm{GO}_{\widetilde{V}}, \quad g \longmapsto (x \mapsto \Pi(g) \cdot x \cdot g^*)$$

is an isogeny of degree two (using [**18**] VIII.8) which satisfies $\nu \circ \psi = N$. The spin representation **spin** is a representation of $\mathrm{GSpin}_{\widetilde{V}}$ on $V = \wedge W$; it can be defined via the universal property of the Clifford algebra, as in [**32**] Lemmata 20.9 and 20.16. We have $\dim V = 2^g$. We write $\widehat{G}$ for $\mathrm{GSpin}_{\widetilde{V}}$. It is a Chevalley group over $\mathbb{Z}[1/2]$; the standard maximal torus $\widehat{T}$, resp. Borel $\widehat{B}$, of $\widehat{G}$ is the inverse image by ψ of the diagonal torus, resp. upper triangular subgroup in GO_{2g+1}.

3.2.2. Dual root data. — We want to recall first the notion of a (reduced) based root datum

$$(M, R, \Delta, M^*, R^\vee, \Delta^\vee),$$

consisting of two free $\mathbb{Z}$-modules M, M^* of rank, say, n with a perfect pairing $M \times M^* \to \mathbb{Z}$ and finite subsets $R \supset \Delta$ in M, resp. $R^\vee \supset \Delta^\vee$ of M^*, together with a bijection $R \to R^\vee$; R is the set of roots, and Δ the simple roots; these data should satisfy two conditions RD I and RD II: *cf.* [**70**] 1.9 or rather, for the degree of generality that we need, Exp. XXI Sect. 1.1 and 2.1.3; here, "reduced" means that in the set of roots R, we allow no multiple of any given root except its opposite.

In order to make some calculations, let us recall briefly the classification given by these data. The main reference is [**17**], whose Exposés are quoted by their roman numbering.

Definition 1. — For any scheme $S \neq \varnothing$, a split reductive group with "épinglage" over S, is a t-uple $(G, B, T, (X_\alpha)_{\alpha\in\Delta})_S$ consisting in a connected reductive group scheme G_S of rank n, together with a Borel B_S and split maximal torus $T_S \subset B_S$: $T \cong \mathbb{G}_m^n$. Let R, resp. $\Delta \subset R$, be the root system, resp. set of simple roots, attached to (G, B, T) (Exp. XIX Sect. 3). The "épinglage" $(X_\alpha)_{\alpha\in\Delta}$ is the datum for each $\alpha \in \Delta$, of a section $X_\alpha \in \Gamma(S, \mathfrak{g}_\alpha)$ which is a basis of $\mathfrak{g}_\alpha$ at each point $s \in S$.

For details on "épinglages", see [**17**] XXII 1.13 and XXIII 1.1. Any such split reductive group defines a reduced based root datum

$$(M, R, \Delta, M^*, R^\vee\, \Delta^\vee).$$

Note that the "épinglage" is not needed in the construction, it comes in only for the fidelity of the functor. The definition runs as follows. Put $M = X^*(T)$, $M^* = X_*(T)$; the duality $\langle\, ,\,\rangle$ between these modules is the composition $(\lambda, \mu) \mapsto \lambda \circ \mu$, R, resp. Δ is the set of roots, resp. simple roots attached to (G, B, T), and $\alpha^\vee$ is defined for each $\alpha \in \Delta$ as follows: let T_α be the connected component of $\operatorname{Ker}\alpha$, let Z_α be its centralizer in G. It is reductive of semisimple rank one, hence its derived group Z'_α is isomorphic to $\mathrm{SL}(2)$ or $\mathrm{PGL}(2)$, and its character group is generated by α; then, $\alpha^\vee : \mathbb{G}_m \to Z'_\alpha \cap T$ is defined as the unique cocharacter of Z'_α such that $\alpha \circ \alpha^\vee = 2$. For details, see Exp. XX, Th. 2.1. As checked in Exp. XXII 1.13, these data satisfy the two conditions (DR I) and (DR II) of Exp. XXI 1.1, hence do form a based root datum (données radicielles épinglées). The system thus obtained is reduced.

Theorem 3. — *There is an equivalence of categories between reduced based root data and split reductive groups with "épinglage".*

This is the main theorem of [**17**], it consists in 4.1 of Exp. XXIII Sect. 4 and Th. 1.1 of Exp. XXV Sect. 1.

Now, given a reduced based root datum, one can form its dual by exchanging (M, R, Δ) and $(M^*, R^\vee, \Delta^\vee)$. This induces a duality of split reductive group schemes with épinglages, over a base S. Let us apply this to our situation. We take $G = \mathrm{GSp}_{2g}$, $(G, B, T)_{/\mathbb{Z}[1/2]}$; $M = X^*(T)$ and $M^* = X_*(T)$, naturally paired by the composition. By using the standard basis of $X^*(T)$, one identifies M to the subgroup of $\mathbb{Z}^g \times \mathbb{Z}$, consisting in $\mu = (\mu_{ss}; \mu_c)$ such that $|\mu| \equiv \mu_c \bmod 2$. This lattice is endowed with the standard scalar product; here $\mathbb{Z}^g$ corresponds to the characters of the semisimple part of T, and the last component to the central variable. In this identification, $R \subset \mathbb{Z}^g \times \{0\}$ and one can write $\alpha^\vee = 2 \cdot \frac{\alpha}{\alpha\cdot\alpha}$ in the space $\mathbb{Q}^g \times \{0\}$. The simple roots of G are $\alpha_g = t_g/t_{g-1}, \ldots, \alpha_1 = t_1^2\nu^{-1}$, for $t = \operatorname{diag}(t_g, \ldots, t_1, t_1\nu^{-1}, \ldots, t_g\nu^{-1}) \in T$; hence their coordinates in $M = \mathbb{Z}^g \times \mathbb{Z}$ are $(1,-1,0,\ldots;0),\ldots,(0,\ldots,2;0)$. The corresponding coroots have therefore coordinates $\alpha_g^\vee = (1,-1,\ldots;0),\ldots,\alpha_1^\vee = (0,\ldots,1;0)$. Then, $X_*(T)$ is identified to $\mathbb{Z}^g \times \mathbb{Z} + \frac{1}{2}\cdot \mathrm{diag}(\mathbb{Z}^{g+1})$.

The resulting dual of $(G,B,T)_{\mathbb{Z}[1/2]}$ is precisely $(\widehat{G},\widehat{B},\widehat{T})_{\mathbb{Z}[1/2]}$ (it is true as well over $\mathbb{Z}$, but we don't need, and don't want to consider characteristic 2 spin groups).

Let $\widehat{\varpi}$ be the minuscule weight of $\widehat{G}$; it belongs to $X^*(\widehat{T}) = X_*(T)$. It satisfies the formulae: $\widehat{\varpi}\cdot\alpha_i^{\vee\vee} = \delta_{1,i}$ for $i = 1,\ldots,g$. Hence, in the basis we have fixed, its coordinates are $(1/2,\ldots,1/2;x)$. The central parameter x must equal $1/2$ as well, because the homomorphism ψ is etale of degree two, and induces the standard representation, whose highest weight is therefore $2\widehat{\varpi}$, but whose central character is $z\mapsto z$. Now, any character $\mu\in X^*(T)$ is identified to a cocharacter of $\widehat{T}$. Then,

Lemma 2. — *In $X^*(\mathbb{G}_m) = \mathbb{Z}$, for any $\mu = (\mu_{ss};\mu_c)\in X^*(T)$, one has:*

$$\widehat{\varpi}\circ\mu = \frac{|\mu_{ss}|}{2} + \frac{\mu_c}{2}. \tag{3.2.2.1}$$

Note that the right-hand side is an integer.

Proof. — Clear.

Let us make simple remarks:

1) Let B_V be the upper triangular Borel of GL_V. Then $\widehat{B}$ is mapped into B_V by the spin representation.

2) In the identification $X_*(T) = X^*(\widehat{T})$, the central cocharacter $\mathbb{G}_m\to T$, $z\mapsto \mathrm{diag}(z,\ldots,z)$ becomes the multiplier $N:\widehat{T}\to\mathbb{G}_m$ of our regular Clifford group $\widehat{G}$; it is clear on the level of tangent maps.

3) If we describe $T_{\mathrm{GO}_{\tilde V}}(\mathbb{C})$ as the torus $\mathbb{G}_m\times T_{O_{\tilde V}}$ of matrices

$$\mathrm{diag}(z\cdot t_g,\ldots,z\cdot t_1,z,z\cdot t_1^{-1},\ldots,z\cdot t_g^{-1})$$

then, $\widehat{T}(\mathbb{C})$ can be described as the set of t-uples $(t_g,\ldots,t_1,[u,\zeta])$ where $u^2 = t_g\cdots t_1$ and $\zeta^2 = z$, the couple (u,ζ) being taken modulo the group generated by $(-1,-1)$. The map $\psi:\widehat{T}(\mathbb{C})\to T_{\mathrm{GO}}(\mathbb{C})$ is then given by $t_i\mapsto t_i$, $[u,\zeta]\mapsto\zeta^2$. All this follows easily from the fact that ψ is dual of the degree two isogeny $T_{ss}\times Z_G\to T$ given by $(t_{ss},z)\mapsto t_{ss}\cdot z$.

Let us apply these considerations to compute the local Langlands correspondence for a representation π_p of $G(\mathbb{Q}_p)$ in the principal series. Let us assume $\pi_p = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\phi$ (unitary induction). If ϕ is unramified, it can be viewed as

$$\phi = (\alpha_g,\ldots,\alpha_1;\gamma)\in\mathbb{C}^g\times\mathbb{C}, \tag{3.2.2.2}$$

the parametrization being given by:

$$\mathrm{diag}(t_g,\ldots,t_1,\nu\cdot t_1^{-1},\ldots,\nu\cdot t_g^{-1})\longmapsto |t_g|_p^{\alpha_g}\cdots|t_1|_p^{\alpha_1}|\nu|_p^{(\gamma-\alpha_g-\cdots\alpha_1)/2}$$

Even if it is ramified, we can make the following identifications

$$\underline{\mathrm{Hom}}(T(\mathbb{Q}_p),\mathbb{C}^\times) = \underline{\mathrm{Hom}}(X_*(T)\otimes\mathbb{Q}_p^\times,\mathbb{C}^\times) = \tag{3.2.2.3}$$

$$\begin{aligned}\mathrm{Hom}(X_*(T), \underline{\mathrm{Hom}}(\mathbb{Q}_p^\times, \mathbb{C}^\times)) &= X^*(T) \otimes \underline{\mathrm{Hom}}(\mathbb{Q}_p^\times, \mathbb{C}^\times)\\ &= \underline{\mathrm{Hom}}(\mathbb{Q}_p^\times, \mathbb{C}^\times \otimes X^*(T)) = \underline{\mathrm{Hom}}(\mathbb{Q}_p^\times, \widehat{T}(\mathbb{C})).\end{aligned}$$

So that we can view ϕ as a cocharacter $\mathbb{Q}_p^\times \to \widehat{T}(\mathbb{C})$. We introduce a twist of this character by d on the central component ($\gamma \mapsto \gamma - d$), in order to get rid of the irrationality inherent to Langlands parameters: $\widetilde{\phi} = \phi \cdot |\nu|_p^{-d}$, it corresponds to the cocharacter $\widetilde{\phi}$ obtained by twisting ϕ by the unramified cocharacter $\mathbb{G}_m \to Z_{\widehat{G}}(\mathbb{C}), t \mapsto |t|_p^{-d}$. In the unramified case, $\widetilde{\phi}$ is given by the formula

$$t \longmapsto (|t|_p^{\alpha_g}, \ldots, |t|_p^{\alpha_1}, [|t|_p^{\frac{\alpha_g+\cdots+\alpha_1}{2}}, |t|_p^{(\gamma-d)/2}]). \tag{3.2.2.4}$$

Consider the canonical map $a : W_{\mathbb{Q}_p} \to \mathbb{Q}_p^\times$ given by class-field theory (sending arithmetic Frobenius to p). The composition $\widetilde{\phi} \circ a$ is denoted $\sigma(\pi_p)$ and is called the image by Langlands local correspondence of π_p.

Let us return now to our Galois representations. Note first that the question whether ρ_π, if absolutely irreducible, factors through the spin representation

$$\widehat{G}(\overline{\mathbb{Q}}_p) \hookrightarrow \mathrm{GL}_V(\overline{\mathbb{Q}}_p)$$

is open.

However, for $g = 2$, if π is stable at ∞ and if π satisfies multiplicity one: $m(\pi) = 1$, then it can be shown that ρ_π takes values in $\widehat{G}$ (see [**72**] p. 295-296). This remark, due to E. Urban (to appear) results from Poincaré duality and the autoduality of π (which is well known, at least, at almost all places).

3.3. Ordinarity. — Let D_p, resp. I_p be a decomposition subgroup, resp. inertia subgroup of Γ. Via the identification $X^*(T) = X_*(\widehat{T})$, we can view any $\mu \in X^*(T)$ as a cocharacter of $\widehat{T}$, hence as a homomorphism $I_p \to \mathbb{Z}_p^\times \to \widehat{T}(\mathbb{Z}_p) \to \mathrm{GL}_{\mathbb{Z}_p}(V)$ where the first map is the cyclotomic character $\chi : I_p \to \mathbb{Z}_p^\times$. Let $\widetilde{\rho} = (g, \ldots, 1; d)$. Thus, $\widetilde{\rho}$ is the sum of the fundamental weights of G; it is the highest weight of an irreducible representation of G contained in $\mathrm{St}^{\otimes d}$. The assumption of Galois ordinarity, denoted **(GO)** in the sequel, is:

– The image $\rho_\pi(D_p)$ of the decomposition group is contained in $\widehat{G}$,

– there exists $\widehat{g} \in \widehat{G}(\mathcal{O})$ such that

$$\rho_\pi(D_p) \subset \widehat{g} \cdot \widehat{B}(\mathcal{O}) \cdot \widehat{g}^{-1},$$

– the restriction of the conjugate $\rho_\pi^{\widehat{g}}$ to I_p, followed by the quotient by the unipotent radical $\widehat{g} \cdot \widehat{N} \cdot \widehat{g}^{-1}$ of $\widehat{g} \cdot \widehat{B} \cdot \widehat{g}^{-1}$ factors through $-(\lambda + \widetilde{\rho}) \circ \chi : I_p \to \widehat{T}(\mathbb{Z}_p)$.

Example. — For $g = 1$, $\lambda = (n; n)$ corresponds to the representation $\mathrm{Sym}^n(\mathrm{St})$ of $\mathrm{GL}(2)$, and $\widetilde{\rho} = (1; 1)$ corresponds to St. Then the weights of the (2-dim.) spin representation of GSpin_3 are $\widehat{\varpi} = (\frac{1}{2}; \frac{1}{2})$ and $\widehat{\varpi}^{w_0} = (-\frac{1}{2}; \frac{1}{2})$; hence the composition of χ, $-(\lambda + \widetilde{\rho})$ and the spin representation (modulo unipotent radical) gives the diagonal matrix $\mathrm{diag}(\chi^{-(n+1)},1)$ (modulo Weyl group), which is the usual formula for an ordinary representation coming from an ordinary cusp form of weight $k=n+2$:

$$\rho_f|_{D_p} \cong \begin{pmatrix} 1 & * \\ 0 & \chi^{-n-1} \end{pmatrix}.$$

Convention. — In the rest of the paper, we make the abuse of notation to write $\widehat{B}$, resp. $\widehat{N}$, $\widehat{T}$, instead of their respective conjugates by $\widehat{g}$: $\widehat{g}\cdot\widehat{B}\cdot\widehat{g}^{-1}$ and so on. With this convention, we have $\overline{\rho}_\pi(I_p)\subset\widehat{B}(k)$.

Relative to the triple $(\widehat{G},\widehat{B},\widehat{T})$, we have the notion of dominant characters $\mu\in X^*(\widehat{T})$ and Weyl classification of highest weight $\mathcal{O}$-representations of $\widehat{G}$, provided $p-1>|\mu+\rho|$ (see Polo-T. [**61**]). Let $\widehat{\varpi}$ be the minuscule weight of $\widehat{G}$. As already calculated, its coordinates are:

$$\Big(\frac{1}{2},\dots,\frac{1}{2};\frac{1}{2}\Big)$$

Lemma 3. — *For any $\sigma\in I_p$,*

$$\widehat{\varpi}(\overline{\rho}_\pi(\sigma)) \ \textit{mod.}\ \widehat{N}(k)) = \omega^{-\boldsymbol{w}}(\sigma) \tag{3.3.1}$$

and similarly, for the lowest weight $\widehat{\varpi}^{w_0}$

$$\widehat{\varpi}^{w_0}(\overline{\rho}_\pi(\sigma)) \ \textit{mod.}\ \widehat{N}(k)) = 1. \tag{3.3.2}$$

Proof. — By **(GO)**, the left-hand side is given by $\widehat{\varpi}\circ[-(\lambda+\widetilde{\rho})]\circ\omega(\sigma)$; therefore, the desired relation follows from Lemma 2, with $\mu=\lambda+\widetilde{\rho}$. Indeed, the coordinates of $\lambda+\widetilde{\rho}$ in $\mathbb{Z}^g\times\mathbb{Z}$ are $(a_g+g,\dots,a_1+1;a_g+\cdots+a_1+d)$, hence the scalar product $\langle\widehat{\varpi},\lambda+\widetilde{\rho}\rangle$ is equal to $\sum_i\frac{a_i+i}{2}+\frac{(\sum_i a_i)+d}{2}$, that is, $\frac{\boldsymbol{w}}{2}+\frac{\boldsymbol{w}}{2}$ *i.e.* $\boldsymbol{w}$. Similarly for (3.3.2).

Comments

1) Let us introduce the condition of automorphic ordinarity:

(AO) For each $r=1,\dots,g$, $v(\theta_\pi(T_{p,r}))=a_{r+1}+\cdots+a_1$,

where $T_{p,r}$ is the classical Hecke operator associated to the double class of

$$\mathrm{diag}(1_r,p\cdot 1_{2g-2r},p^2\cdot 1_r).$$

We conjecture that for any g, if ρ_π is a subquotient of $W_{\pi,p}$, then **(AO)** implies **(GO)**. It is well-known for $g=1$ ([**89**] Th. 2.2.2, [**41**] and [**54**]).

Consider the statement

$\mathbf{KM_g}(\pi_f,p)$. — *If p is prime to N, the slopes of the crystalline Frobenius on the isotypical component $\mathbf{D}_{\mathrm{crys}}(W_{\pi,p})$ are the p-adic valuations of the roots of the polynomial $\theta_\pi(P_p(X))$, reciprocal of the p-Euler factor of the automorphic L-function of π.*

For $g=2$, we have seen in 3.1 that $W^{s.s.}_{\pi,p}$ is ρ_π-isotypical (assuming its absolute irreducibility). We have observed (Proposition 7.1 of [**77**]) that if $\mathbf{KM_2}(\pi,p)$ holds and if π is stable at infinity, the condition **(AO)** for π implies **(GO)**. In a recent preprint, E. Urban [**80**] has proven $\mathbf{KM_2}(\pi, p)$; thus, for $g = 2$, if π is stable at ∞, **(AO)** implies **(GO)**.

2) If π_p is in the principal series (for instance, if π is unramified at p), and if the p-adic representation ρ_π is, say, potentially crystalline at p (for instance, crystalline), one can ask in general the following question.

On one hand, the local component π_p of π at p is unitarily induced from ϕ for a character $\phi : T(\mathbb{Q}_p) \to \mathbb{C}^\times$; we defined in Sect. 3.2.2 the local Galois representation $\sigma(\pi_p)$ of the Weil group $W_{\mathbb{Q}_p}$ given by

$$W_{\mathbb{Q}_p} \longrightarrow \mathbb{Q}_p^\times \longrightarrow \widehat{T}(\mathbb{C}) \subset \widehat{G}(\mathbb{C})$$

where $\mathbb{Q}_p^\times \to \widehat{T}(\mathbb{C})$ is given by the twist $\widetilde{\phi}$ through the identification (3.2.2.2). This representation is rational (the traces belong to some number field).

Let us consider on the other hand the restriction to D_p of ρ_π. By applying the (covariant) Fontaine's functor $D_{p\mathrm{crys}}$ (*cf.* Fontaine, Exposé III, Astérisque 223), we obtain a representation ${}'\rho_{\pi,p}$ of the Weil group $W_{\mathbb{Q}_p}$:

$${}'\rho_{\pi,p} : W_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_V\, .$$

One can conjecture a compatibility at (p, p) between the local and global Langlands correspondences, namely that the F-semisimplification of the two rational representations ${}'\rho_{\pi,p}$ and $\sigma(\pi_p)$ are isomorphic (where $a : W_{\mathbb{Q}_p} \to \mathbb{Q}_p^\times$ is the map induced by class-field theory, sending arithmetic Frobenius to p, and the twist is to pass from Langlands parameters to "Hecke" parameters). This fact is known in the following cases:

– for $g = 1$, by well-known theorems of Scholl and Katz-Messing,

– for $g = 2$, for a representation π on GSp(4) which is the base change from $\mathrm{GL}(2, F)$ (F real quadratic) of a Hilbert modular form which is in the discrete series at some finite place, and which is unramified at places above p (in which case $\rho_{f,p}$, hence $\rho_{\pi,p}$ is crystalline at p by Breuil's theorem [**11**]). This is a particular case of a theorem of T. Saito [**66**].

Note however that this statement does not allow one to recover the representation $\rho_{\pi,p} = \rho_\pi|_{D_p}$ (because it says nothing about the Hodge filtration) unless we assume it is ordinary (in the usual geometric sense, see [**60**]). More precisely, we have two parallel observations:

– Let us assume that $\rho_{\pi,p}$ is crystalline; then the assumption of geometric ordinarity means that the eigenvalues $(\xi_B^{-1})_{B\subset\{1,\ldots,g\}}$ of the crystalline Frobenius are such that the $\mathrm{ord}_p(\xi_B)$ $(B \subset \{1, \ldots, g\})$ coincide (with multiplicities) with the Hodge-Tate weights; these numbers, if π is stable at infinity, should be (as mentioned at the end of Sect. 2.3.2) $j_B = \sum_{i\in B}(a_i + i)$ $(B \subset A = \{1, \ldots, g\})$. These quantities can also be written

$$\langle \widehat{\varpi}^{w_B}, (\lambda + \widetilde{\rho}) \rangle = \widehat{\varpi}^{w_B} \circ (\lambda + \widetilde{\rho})$$

where $w_B \in W_{\widehat{G}}$ is the element of the Weyl group such that for $\widehat{t} = (t_g, \ldots, t_1, [u, \zeta]) \in \widehat{T}$ and $w_B(\widehat{t}) = \widehat{\theta}$, $\theta_i = t_i^{-1}$ if and only if $i \in B$ and all its other components are those of $\widehat{t}$. Therefore, it implies by Fontaine-Laffaille theory that ρ_π is ordinary at p in the precise sense of (**GO**). Thus the conjunction of geometric ordinarity, and of stability of π at ∞ (together with the complete determination of Hodge-Tate weights of ρ_π) implies **(GO)**.

– Let us assume π is unramified at p; let us introduce complex numbers θ_i's and ζ, such that for any $t \in \widehat{T}(\mathbb{C})$ mod. $W_{\widehat{G}}$,

$$|t_i|_p^{\alpha_i} = \theta_i^{-\operatorname{ord}_p(t_i)} \quad \text{and} \quad |z|_p^{\gamma} = \zeta^{-\operatorname{ord}_p(z)},$$

we can rewrite (3.2.2.4) as

$$\widetilde{\phi}(p) = (\theta_g^{-1}, \ldots, \theta_1^{-1}, [(\theta_g \cdots \theta_1)^{-1/2}, p^{d/2} \cdot \zeta^{-1}])$$

The composition with **spin** gives a complex diagonal matrix whose entries are inverse to the 2^g algebraic integers

$$\xi_J = \Big(\prod_{i \in J} \theta_i^{-1} \cdot \prod_{i \notin J} \theta_i\Big)^{1/2} \cdot \zeta.$$

The Automorphic Ordinarity Conjecture for the p-adic embedding ι_p states

$$\operatorname{ord}_p(\iota_p(\xi_J)) = \sum_{i \in J} (a_i + i), \quad \text{for any } J.$$

Therefore, the quantities $x_i = -\operatorname{ord}_p(\iota_p(\theta_i))$ and $y = \operatorname{ord}_p(\iota_p(\zeta))$ satisfy the linear system in $(x_g, \ldots, x_1; y) \in \mathbb{Z}^{g+1}$:

$$-\frac{y + d + \sum_{i \in J} x_i - \sum_{i \notin J} x_i}{2} = \sum_{i \in J} (a_i + i).$$

It contains a Cramer system. Therefore, assumption **(AO)** implies

$$\operatorname{ord}_p \theta_i = -(a_i + i), \quad \operatorname{ord}_p(\zeta) = a_g + \cdots + a_1$$

up to permutation of the coordinates. This can be rewritten as an equality in $\operatorname{Hom}(\mathbb{Q}_p^\times, \widehat{T}(K)/\widehat{T}(\mathcal{O}))$:

$$\iota_p \circ \widetilde{\phi} = -(\lambda + \widetilde{\rho}).$$

We conclude that **(AO)** together with $\mathbf{KM_g}(\pi, p)$ implies (part of) the compatibility conjecture at (p, p): the (p-adic orders of) the eigenvalues counted with multiplicities of $D_{\mathrm{crys}}(\rho_\pi)(\mathrm{Frob}_p)$ coincide with those of $\sigma(\pi_p)(\mathrm{Frob}_p)$.

4. Crystals and connections

4.1. de Rham and crystalline cohomology of open varieties. — Let $f : \overline{X} \to S$ be a smooth proper morphism of schemes; $X \subset \overline{X}$ be an open immersion above S, with complement a relative Cartier divisor $D \to S$ with normal crossings and smooth irreducible components. Let $\overline{\mathcal{V}}$ be a coherent sheaf over $\overline{X}$ endowed with an integrable connection ∇ with logarithmic poles along D; let $\mathcal{V}$ its restriction to X. Let $\mathcal{I}(D)$ be the sheaf of ideals defining D. Then the relative de Rham cohomology sheaves $\mathcal{H}^j_{\mathrm{dR}}(X/S,\mathcal{V})$ are defined as

$$(2.1)_\varnothing \qquad \mathbf{R}^j f_*(\overline{\mathcal{V}}\otimes_{\mathcal{O}_{\overline{X}}}\Omega^\bullet_{\overline{X}/S}(\log D)).$$

Let us now introduce a complex

$$\Omega^\bullet_{\overline{X}/S}(-\log D)=\Omega^\bullet_{\overline{X}/S}(\log D)\otimes_{\mathcal{O}_{\overline{X}}}\mathcal{I}(D)$$

We define the cohomology sheaves with compact support $\mathcal{H}^j_{\mathrm{dR},c}(X/S,\mathcal{V})$ by:

$$(2.1)_c \qquad \mathbf{R}^j f_*(\overline{\mathcal{V}}\otimes_{\mathcal{O}_{\overline{X}}}\Omega^\bullet_{\overline{X}/S}(-\log D)).$$

If $S=\operatorname{Spec} k$ is the spectrum of a field k, we write H^j_{dR} instead of $\mathcal{H}^j_{\mathrm{dR}}$. A priori, these definitions depend on the compactification $\overline{X}$ of X. One can show for $S=\operatorname{Spec} k$ and $\mathcal{V}$ trivial that the resolution of singularities implies the independence of the compactification (Théorème 2.11 of [**57**]).

For the crystalline cohomology there is a similar definition. Our reference is [**48**], section 5, 6. We use the language of logarithmic schemes; as noted by Kato in Complement 1 of his paper, his results are compatible with Faltings theory of crystalline cohomology of open varieties [**23**]: in Faltings approach, a logarithmic structure on $\overline{X}$ is a family $(\mathcal{L}_i,x_i)_{1\leqslant i\leqslant r}$ where $\mathcal{L}_i$ is an invertible sheaf and x_i a global section thereof, these data always define a logarithmic scheme in Kato's sense (while the converse is false). Let (S,I,γ) a triple where S is a scheme, $\mathcal{I}$ is a quasi-coherent nilpotent ideal of $\mathcal{O}_S$ and γ is a divided power structure on $\mathcal{I}$ (PD-structure, for short). Let S_0 the closed subscheme defined by $\mathcal{I}$; we consider a smooth morphism $\overline{X_0}\to S_0$ and D_0 a relative Cartier divisor with normal crossings. It defines a logarithmic structure $M=\{g\in\mathcal{O}_{\overline{X_0}}\mid g \text{ invertible outside } D_0\}\subset\mathcal{O}_{\overline{X_0}}$. One defines the logarithmic crystalline site of $(\overline{X}_0/S)^{\log}_{\mathrm{crys}}$ as in Kato [**48**] Sect. 5.2. The objects are 5-uples (U,T,M_T,i,δ) where $U\to\overline{X_0}$ is étale, (T,M_T) is a scheme with fine logarithmic structure over S, $i:(U,M|_U)\to(T,M_T)$ is an exact closed immersion over S and δ is a divided power structure compatible with γ. Recall that a closed immersion of log-schemes $f:(X,M)\to(T,N)$ is called exact if $f^*N\to M$ is an isomorphism. Morphisms are the natural ones. On this site, the structural sheaf $\mathcal{O}_{\overline{X}_0/S}$ is defined by

$$\mathcal{O}_{\overline{X}_0/S}(U,T,M_T,i,\delta)=\Gamma(T,\mathcal{O}_T).$$

Definition 2. — A crystal on $(\overline{X}_0/S)^{\log}_{\mathrm{crys}}$ is a sheaf $\mathcal{V}$ of $\mathcal{O}_{\overline{X}_0/S}$-modules satisfying the following condition: for any morphism $g:T'\to T$ in $(\overline{X}_0/S)^{\log}_{\mathrm{crys}}$, $g^*\mathcal{V}_T\to\mathcal{V}_{T'}$ is an isomorphism. Here $\mathcal{V}_T$ and $\mathcal{V}_{T'}$ denote the sheaves on $T_{\mathrm{\acute{e}t}}$ and $T'_{\mathrm{\acute{e}t}}$ defined by $\mathcal{V}$.

Let $(\overline{X},D)$ be a lifting of $(\overline{X}_0,D_0)$ to S, that is, a smooth S-scheme together with a divisor with normal crossings flat over S such that $(X\times_S S_0, D\times_S S_0)=(X_0,D_0)$. Note that since $\mathcal{I}$ is nilpotent, the étale sites of X and X_0, resp. of S and S_0 are equivalent by $U \mapsto U \times_S S_0$. By Th. 6.2 of [**48**] (see Sect. 4.2 for more details), the data of a crystal on $(\overline{X}_0/S)^{\log}_{\mathrm{crys}}$ is equivalent to that of an $\mathcal{O}_{\overline{X}}$-module $\mathcal{M}$ endowed with a quasi-nilpotent integrable connection with logarithmic singularities

$$\nabla : \mathcal{M} \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_{\overline{X}}} \Omega^1_{\overline{X}/S}(\log D).$$

For any sheaf $\mathcal{V}$ on $(\overline{X}_0/S)^{\log}_{\mathrm{crys}}$, we denote by $f_{\mathrm{crys},*}\mathcal{V}$ its direct image by $f : X_0 \to S$; it is a sheaf on S. We write $f_{\text{ét},*}\mathcal{V}$ for the etale sheaf on $S_{\text{ét}}$ which is the direct image of the etale sheaf $\mathcal{V}$ on X_0. To compute the cohomology sheaves of a crystal, we apply the spectral sequence

$$Rf_{\mathrm{crys},*}\mathcal{V} = Rf_{\text{ét},*}(Ru_*\mathcal{V})$$

where u is the canonical projection from the site $(\overline{X}_0/S)^{\log}_{\mathrm{crys}}$ to $\overline{X}_{0\,\text{ét}}$. It is defined, for a sheaf $\mathcal{V}$ on $(\overline{X}_0/S)^{\log}_{\mathrm{crys}}$, and for any étale morphism $U \to \overline{X}_0$, by

$$(u_*\mathcal{V})(U) = \Gamma(U, \mathcal{V}_U).$$

Moreover, if $\mathcal{V}$ is a crystal, we have

$$Ru_*\mathcal{V} \cong \mathcal{M} \otimes_{\mathcal{O}_{\overline{X}}} \Omega^{\bullet}_{\overline{X}/S}(\log D).$$

Again, by Th. 2.11 of [**57**], one can show, assuming the resolution of singularities that for $S = \mathbb{Z}/p^n\mathbb{Z}$, $S_0 = \mathbb{Z}/p\mathbb{Z}$ this definition does not depend on the compactification.

Remark. — In our case, one even does not need the resolution of singularities. It will be a consequence of the comparison theorem!

These definitions transfer to the compact support case; it is mentioned in [**22**] p. 58. We explain this in Kato's setting. For a log-scheme (T, N), we denote by $\mathcal{I}(N)$ the sheaf of ideals in $\mathcal{O}_T$ generated by N. We define a sheaf of ideals $\mathcal{I}(D_0)$ on $(\overline{X}_0/S)^{\log}_{\mathrm{crys}}$ as:

$$\mathcal{I}(D_0)(U, T, M_T, i, \delta) = \Gamma(T, \mathcal{I}(M_T)).$$

$\mathcal{I}(D_0)$ is a crystal of $\mathcal{O}_{\overline{X}_0/S}$-modules. By definition, the cohomology with compact support of a crystal $\mathcal{V}$ is the cohomology of the crystal

$$\mathcal{V} \otimes_{\mathcal{O}_{\overline{X}_0/S}} \mathcal{I}(D_0).$$

The cohomology sheaves

$$Rf_{\mathrm{crys},*,c}\mathcal{V} = Rf_{\mathrm{crys},*}(\mathcal{V} \otimes_{\mathcal{O}_{\overline{X}_0/S}} \mathcal{I}(D_0))$$

are computed by a similar spectral sequence

$$Rf_{\mathrm{crys},*,c}\mathcal{V} = Rf_{\text{ét},*}(Ru_{*,c}\mathcal{V})$$

where $u_{*,c}$ is defined, for a sheaf $\mathcal{V}$ on $(\overline{X}_0/S)^{\log}_{\mathrm{crys}}$ and an étale morphism $g : U \to \overline{X}_0$, by

$$(u_{*,c}(\mathcal{V})(U) = \Gamma(U, \mathcal{V}_U \otimes_{\mathcal{O}_U} g^*\mathcal{I}(D_0)).$$

One has also:

$$Rf_{\mathrm{crys},*,c}\mathcal{V} = Rf_{\text{ét},*}(\mathcal{M}\otimes_{\mathcal{O}_{\overline{X}}}\Omega^\bullet_{\overline{X}/S}(-\log D)).$$

This result can be proven as in the case without support; it will be explained in the next section.

4.2. L-construction. — In the proof of Theorem 6 below, we will apply the crystalline L-construction in the logarithmic setting (in the classical crystalline setting, *cf.* Chap. 6 of [**4**]); we want to explain the definitions and results here.

Let $(S,\mathcal{I},\gamma)$ a triple where S is a scheme, $\mathcal{I}$ is a quasi-coherent ideal of $\mathcal{O}_S$ and γ is a PD-structure on $\mathcal{I}$. Let S_0 the closed subscheme defined by $\mathcal{I}$; we consider a smooth morphism $\overline{X}_0 \to S_0$ and Y_0 a relative Cartier divisor with normal crossings. Let $(\overline{X},Y)$ be a lifting of $(\overline{X}_0,Y_0)$ to S; we suppose that there exists an integer $m>0$ such that $p^m\mathcal{O}_{\overline{X}}=0$. Let $Z_1,\dots,Z_a$ be the irreducible components of Y. Let Ξ be the blowing-up of $\overline{X}\times_S\overline{X}$ along the subscheme $\sum_i(Z_i\times_S Z_i)$. Let $\overline{X}\widehat{\times}_S\overline{X}$ be the complement in Ξ of the strict transforms of $\overline{X}\times Z_i$ and $Z_i\times\overline{X}$, $1\leqslant i\leqslant r$ and let $\widetilde{Y}$ be the exceptional divisor in $\overline{X}\widehat{\times}_S\overline{X}$; it is a divisor with normal crossings. The couple $(\overline{X}\widehat{\times}_S\overline{X},\widetilde{Y})$ is the categorical fiber product of $(\overline{X},Y)$ by itself over S, in the category of logarithmic schemes (*cf.* [**22**] IV, c). Locally, if $x_1,\dots,x_d$ are local coordinates of $\overline{X}$ over S such that Y is defined by the equation $x_1\cdots x_a=0$, then $\overline{X}\widehat{\times}_S\overline{X}$ is the relative affine scheme given as spectrum of

$$S[x_i\otimes 1, 1\otimes x_i]_{1\leqslant i\leqslant d}[u_j^{\pm1}]_{1\leqslant j\leqslant a}/(x_j\otimes 1\cdot u_j - 1\otimes x_j)_{1\leqslant j\leqslant a}$$

and $\widetilde{Y}$ is defined by the equation $x_1\otimes 1\cdots x_a\otimes 1=0$ (or $1\otimes x_1\cdots 1\otimes x_a=0$).

The product $\overline{X}\widehat{\times}_S\overline{X}$ is the "exactification" of the diagonal embedding of log-schemes $\overline{X}\hookrightarrow\overline{X}\times\overline{X}$ and $\widetilde{Y}$ is the inverse image of $Y\times_S Y$ in this exactification.
Recall that if $f:(X,M)\to(T,N)$ is a closed immersion, there exists locally a unique exact closed immersion $\widetilde{f}:(X,M)\to(\widetilde{T},\widetilde{N})$ which is universal in the following obvious meaning:
For any commutative triangle

$$\begin{tikzcd}
(X,M) \arrow[rr, "g"] \arrow[dr] & & (Z,P) \arrow[dl] \\
& (T,N) &
\end{tikzcd}$$

such that g is an exact closed immersion, there exists a unique morphism $(Z,P)\to(\widetilde{T},\widetilde{N})$ which lifts $(Z,P)\to(T,N)$.
The log-scheme $(\widetilde{T},\widetilde{N})$ is the "exactification" of (T,N).
We endow $\overline{X}\widehat{\times}_S\overline{X}$ with a PD-structure as follows. Let $\mathcal{D}_{\overline{X}}$ be the PD-envelope of the diagonal immersion $\overline{X}\to\overline{X}\widehat{\times}_S\overline{X}$. In the local coordinates above, $\mathcal{D}_{\overline{X}}$ is the PD-polynomial algebra $\mathcal{O}_{\overline{X}}\langle v_1,\dots,v_a,\xi_{a+1},\dots,\xi_d\rangle$ where $v_i=u_i-1$ and $\xi_i=x_i\otimes 1-1\otimes x_i$.

We denote by $\mathcal{D}^n_{\overline{X}}$ the n^{th} order divided power neighborhood: $\mathcal{D}^n_{\overline{X}} = \mathcal{D}_{\overline{X}}/\mathcal{I}_\Delta^{[n+1]}$ where $\mathcal{I}_\Delta$ is the ideal of the diagonal immersion and the exponent with brackets denotes the $(n+1)^{\text{th}}$ PD power of $\mathcal{I}_\Delta$.

Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\overline{X}}$-modules. We recall the interpretation of a connection on $\mathcal{M}$ in terms of an HPD-stratification in our context. For us, the notion of an HPD stratification on $\mathcal{M}$ is defined word for word as in [**4**] Sect. 4.3 (which treats the crystalline situation on $\overline{X}_0$, without the divisor Y_0). It consists namely in the datum of a $\mathcal{D}_{\overline{X}}$-linear isomorphism

$$\epsilon : \mathcal{D}_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{M} \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{D}_{\overline{X}}$$

such that ϵ reduces to identity modulo $\mathcal{I}_\Delta$ and the natural cocycle condition on $\overline{X}\widehat{\times}_S\overline{X}\widehat{\times}_S\overline{X}$ holds ([**4**] def. 2.10). In the case $\mathcal{M} = \mathcal{D}_{\overline{X}}$, we have two canonical HPD stratifications. The first is induced by extending by (left) $\mathcal{D}_{\overline{X}}$-linearity the map $\theta : c\otimes d \mapsto ((1\otimes d)\otimes(1\otimes c)$

for c and d in $\mathcal{O}_{\overline{X}}$; it makes use of the right module structure of $\mathcal{D}_{\overline{X}}$ over $\mathcal{O}_{\overline{X}}$. The second is given similarly by tensoring on the left by $\mathcal{D}_{\overline{X}}$ over $\mathcal{O}_{\overline{X}}$ the left-hand side of $\iota : c\otimes d \mapsto ((c\otimes 1)\otimes(1\otimes d)$;

it uses the structure of left $\mathcal{O}_{\overline{X}}$-module of $\mathcal{D}_{\overline{X}}$.

Also, as in [**4**] 4.4, one recalls the notion of PD-differential operator. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{O}_{\overline{X}}$-modules.

A PD-differential operator $\mathcal{M} \to \mathcal{N}$ of order $\leqslant n$ (resp. HPD-differential operator) is a $\mathcal{O}_{\overline{X}}$-linear map $\mathcal{D}^n_{\overline{X}} \otimes \mathcal{M} \to \mathcal{N}$ (resp. $\mathcal{D}_{\overline{X}} \otimes \mathcal{M} \to \mathcal{N}$). Every PD-differential operator $\delta : \mathcal{D}^n_{\overline{X}} \otimes \mathcal{M} \to \mathcal{N}$ induces a classical differential operator $\delta^b : \mathcal{M} \to \mathcal{N}$ of order n with "cologarithmic zeroes" along Y.

The importance of these notions for us stems from the following theorem whose proof runs exactly as in the "classical" case ([**4**] Theorem 4.12). For that, we introduce the notion of a quasi-nilpotent connection in the sense of [**4**] 4.10 (but in our log setting, again):

Definition 3. — A connection ∇ on $\mathcal{M}$ is quasi-nilpotent if for any local section s of $\mathcal{M}$ with local coordinates $x_1, \ldots, x_d$ on X such that Y is defined by the equation $x_1 \cdots x_a = 0$, there exists a positive integer k such that

$$\prod_{0\leqslant j\leqslant k-1} (\nabla(x_i\partial/\partial x_i) - j)^k(s) = 0$$

for $1 \leqslant i \leqslant a$ and $(\nabla(\partial/\partial x_i))^k(s) = 0$ for $a+1 \leqslant i \leqslant d$).

Theorem 4. — *The data of an HPD stratification on $\mathcal{M}$ is equivalent to the data of a logarithmic integrable connection ∇ on $\mathcal{M}$ wich is quasi-nilpotent.*

Then, Grothendieck's linearization functor L is defined as follows. Let $\mathcal{H}$ be the category of $\mathcal{O}_{\overline{X}}$-modules with morphisms given by HPD-differential operators and $\mathcal{C}$

to the category of crystals over $(\overline{X}_0/S)^{\log}_{\text{crys}}$. For any sheaf $\mathcal{M}$ of $\mathcal{O}_{\overline{X}}$-modules, we endow the $\mathcal{O}_{\overline{X}}$-module $\mathcal{D}_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{M}$ with the HPD-stratification $\epsilon_{L(\mathcal{M})}$

$$\mathcal{D}_{\overline{X}} \otimes \mathcal{D}_{\overline{X}} \otimes \mathcal{M} \xrightarrow{\iota \otimes \mathrm{Id}_{\mathcal{M}}} \mathcal{D}_{\overline{X}} \otimes \mathcal{D}_{\overline{X}} \otimes \mathcal{M} \xrightarrow{\mathrm{Id}_{\mathcal{D}_{\overline{X}}} \otimes f} \mathcal{D}_{\overline{X}} \otimes \mathcal{M} \otimes \mathcal{D}_{\overline{X}}$$

where $f : \mathcal{M} \otimes \mathcal{D}_{\overline{X}} \to \mathcal{D}_{\overline{X}} \otimes \mathcal{M}$ interchanges the factors. In other words, the HPD-stratification is given by:

$$(a \otimes b) \otimes (c \otimes d) \otimes m \longmapsto (ac \otimes b) \otimes m \otimes (1 \otimes d)$$

Definition 4. — The covariant functor $L : \mathcal{H} \to \mathcal{C}$ is defined by:

– For any sheaf $\mathcal{M}$ of $\mathcal{O}_{\overline{X}}$-modules, $L(\mathcal{M})$ is the crystal corresponding to the $\mathcal{O}_{\overline{X}}$-module with HPD-stratification $(\mathcal{D}_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{M}, \epsilon_{L(\mathcal{M})})$.

– For an HPD-differential operator $\varphi : \mathcal{M} \to \mathcal{N}$ (that is, an $\mathcal{O}_{\overline{X}}$-linear map $\varphi : \mathcal{D}_{\overline{X}} \otimes \mathcal{M} \to \mathcal{N}$), $L(\varphi) : L(\mathcal{M}) \to L(\mathcal{N})$ is the morphism of crystals corresponding to the $\mathcal{O}_{\overline{X}}$-linear morphism compatible with HPD-stratifications, given by the composition:

$$\mathcal{D}_{\overline{X}} \otimes \mathcal{M} \xrightarrow{\iota \otimes \mathrm{Id}_{\mathcal{M}}} \mathcal{D}_{\overline{X}} \otimes \mathcal{D}_{\overline{X}} \otimes \mathcal{M} \xrightarrow{\mathrm{Id}_{\mathcal{D}_{\overline{X}}} \otimes \varphi} \mathcal{D}_{\overline{X}} \otimes \mathcal{N}.$$

We refer to [**4**] Sect. 2, Sect. 6 for more details. Note that since $\mathcal{D}_{\overline{X}}$ is locally free, the functor L is exact.

The correspondence between crystals on $(\overline{X}_0/S)^{\log}_{\text{crys}}$ and $\mathcal{O}_{\overline{X}}$-module $\mathcal{M}$ endowed with a quasi-nilpotent integrable connection with logarithmic singularities, is then given by the following rule: Let $pr_1, pr_2 : \mathcal{D}_{\overline{X}} \to \overline{X}$ be the two canonical projections. If $\mathcal{V}$ is a crystal on $(\overline{X}_0/S)^{\log}_{\text{crys}}$, let $\mathcal{M} = \mathcal{V}_{\overline{X}}$ be the evaluation of $\mathcal{V}$ on $\overline{X}$. The defining condition of a crystal produces an isomorphism:

$$\epsilon : pr_2^* \mathcal{M} \simeq pr_1^* \mathcal{M}$$

This induces an integrable quasi-nilpotent logarithmic connection on $\mathcal{M}$ as explained above. Conversely, by theorem 4, every logarithmic integrable connection on $\mathcal{M}$ wich is quasi-nilpotent induces an HPD stratification on $\mathcal{M}$. If (U, T, M_T, i, δ) is an object of the crystalline site, then by smoothness, etale locally on T, the morphism $(\overline{X}_0, D_0) \to (\overline{X}, D)$ extend to a morphism $h : (T, M_T) \to (\overline{X}, D)$. We define $\mathcal{V}_T$ to be $h^*\mathcal{M}$. If we have two such $h_i : (T, M_T) \to (\overline{X}, D)$ $(i = 1, 2)$, then there exists $h' : (T, M_T) \to (\mathcal{D}_{\overline{X}}, M_{\mathcal{D}_{\overline{X}}})$ such that $h_i = h' pr_i$ and ϵ induces an isomorphism $h_1^*\mathcal{M} \simeq h_2^*\mathcal{M}$. Thus $\mathcal{V}$ is well-defined.

It is not hard from the classical case (Theorem 6.12 of [**4**]), to deduce the following crystalline Poincaré lemma.

Lemma 4. — *Let $\mathcal{V}$ be a crystal on $(X_0/S)^{\log}_{\text{crys}}$ and $\mathcal{M}$ the associated $\mathcal{O}_{\overline{X}}$-module with its integrable connection. Then the complex of crystals $L(\mathcal{M} \otimes \Omega^\bullet_{\overline{X}}(\log Y))$ is a resolution of $\mathcal{V}$.*

Example. — For $S = \operatorname{Spec} k$, $X_0 = \operatorname{Spec} k[t]$, $D_0 = \{0\}$, the L-construction applied to the logarithmic de Rham complex gives the following Poincaré resolution:

$$0 \longrightarrow \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_{X_0}\langle v\rangle \longrightarrow \mathcal{O}_{X_0}\langle v\rangle dv \longrightarrow 0$$

where $d : \mathcal{O}_{X_0}\langle v\rangle \to \mathcal{O}_{X_0}\langle v\rangle dv$ is $\mathcal{O}_{X_0}$-linear and maps v to dv. Here, $L(\mathcal{O}_{X_0}) = \mathcal{O}_{X_0}\langle v\rangle$ and $L(\Omega_{X_0/k}(\log D_0)) = \mathcal{O}_{X_0}\langle v\rangle dv$ where v should be thought of as $\log t$.

Finally, the same argument as in the classical theory ([**4**] Sect. 5.27) shows also the following useful lemma:

Lemma 5. — *Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\overline{X}}$-modules and $\mathcal{I}(Y)$ the ideal of definition of Y. Then:*

$$Ru_*L(\mathcal{M}) = \mathcal{M} \text{ and } Ru_{*,c}L(\mathcal{M}) = \mathcal{M}\otimes\mathcal{I}(Y).$$

Combining Lemma 4 and 5 above, we deduce:

$$Ru_*\mathcal{V} \cong \mathcal{M}\otimes_{\mathcal{O}_{\overline{X}}}\Omega^\bullet_{\overline{X}/S}(\log D) \quad \text{and} \quad Ru_{*,c}\mathcal{V} \cong \mathcal{M}\otimes_{\mathcal{O}_{\overline{X}}}\Omega^\bullet_{\overline{X}/S}(-\log D).$$

4.3. The Gauss-Manin connection. — As in section 4.1, $\overline{X}$ is a smooth S-scheme (not necessarily proper), X an S-open scheme of $\overline{X}$ such that $D = \overline{X} - X$ is a divisor with normal crossings over S. Let $f : \overline{\mathcal{X}} \to \overline{X}$ be a proper morphism such that $\overline{\mathcal{X}}$ is smooth over S, f is smooth over X and $\mathcal{D} = \overline{\mathcal{X}} \times_{\overline{X}} D$ is a relative divisor with normal crossings (such f is called semi-stable, see [**44**]). We have a relative de Rham complex with logarithmic poles

$$\Omega^\bullet_{\overline{\mathcal{X}}/\overline{X}}(\log \mathcal{D}/D) = \Omega^\bullet_{\overline{\mathcal{X}}/S}(\log \mathcal{D})/f^*\Omega^\bullet_{\overline{X}/S}(\log D).$$

As explained in [**49**] (see also [**44**]), we have a Gauss-Manin connection with logarithmic poles along D, on the coherent $\mathcal{O}_{\overline{X}}$-module:

$$\mathcal{E}^\alpha = R^\alpha f_*(\Omega^\bullet_{\overline{\mathcal{X}}/\overline{X}}(\log \mathcal{D}/D)).$$

In fact, this sheaf is locally free either if S is over $\mathbb{Q}$ or if S is over a field of characteristic p greater than α. The restriction of $\mathcal{E}^\alpha$ to X is the usual Gauss-Manin sheaf $R^\alpha f_{|\mathcal{X}*}\Omega^\bullet_{\mathcal{X}/X}$ and $\mathcal{E}^\alpha$ is the Deligne's canonical extension to $\overline{X}$. The Gauss-Manin connection on $\mathcal{E}$ is integrable and if $\mathcal{O}_S$ is killed by a power of p, then this connection is quasi-nilpotent ([**49**]).

5. BGG resolutions for crystals

Let $B = T.N$ resp. $Q = M\cdot U$ be the Levi decomposition of the upper triangular subgroup of G, resp. of the Siegel parabolic, viewed as group schemes over $\mathbb{Z}$. We keep the notations of the introduction for the weights of G. Let $\mathbf{V} = \langle e_g, \dots, e_1, e_1^*, \dots, e_g^*\rangle$ be the standard $\mathbb{Z}$-lattice on which G acts; given two vectors $v, w \in \mathbf{V}$, we write $\langle v, w\rangle = {}^t vJw$ for their symplectic product. Q is the stabilizer of the standard lagrangian lattice $\mathbf{W} = \langle e_g, \dots, e_1\rangle$; we have $\mathbf{V} = \mathbf{W}\oplus\mathbf{W}^*$; $M = L_I$ is the stabilizer of the decomposition $(\mathbf{W}, \mathbf{W}^*)$; one has $M \cong \mathrm{GL}(g) \times \mathbf{G}_m$. Let $B_M = B \cap M$ be the standard Borel of M. Let Φ, resp. Φ_M be the set of roots of (G, B), resp. (M, B_M) and let $\Phi^M = \Phi - \Phi_M$. We denote by $\Phi^\pm$, resp. $\Phi^\pm_M$, $\Phi^{M\pm}$, the set of positive/negative roots in Φ, resp. Φ_M, Φ^M.

5.1. Weyl modules over $\mathbb{Z}_p$. — From this section on, the notations $\mathfrak{g}$, $\mathfrak{q}$, (and $\mathfrak{m}$ but there should not be confusion with the maximal ideal of the Hecke algebra) stand for the Lie algebras over $\mathbb{Z}$ of the corresponding group schemes. The Kostant-Chevalley algebra $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ (resp. $\mathcal{U}(\mathfrak{q})$, $\mathcal{U}(\mathfrak{m})$) is the subring of the rational enveloping algebra $U(\mathfrak{g}_\mathbb{Q})$ (resp. $U(\mathfrak{q}_\mathbb{Q})$, resp. $\mathcal{U}(\mathfrak{m}_\mathbb{Q})$) generated over $\mathbb{Z}$ by $X^n/n!$ with $X \in \mathbf{g}_\alpha$, $\alpha \in \Phi$ (resp. $\alpha \in \Phi - \Phi^{M-}$, Φ_M), $n \geqslant 0$ an integer. There is a natural ring epimorphism $\mathcal{U}(\mathfrak{q}) \to \mathcal{U}(\mathfrak{m})$. A $\mathfrak{g}$-stable lattice of a $G_\mathbb{Q}$-representation which is $\mathcal{U}$-stable is called $\mathfrak{g}$-admissible (see [**12**], Sect. VIII.12.7 and 8) same thing for a $\mathfrak{m}$-lattice which is $\mathcal{U}(\mathfrak{q})$-stable.

5.1.1. Admissible lattices. — In this section, we explain how one can construct Weyl modules over $\mathbb{Z}_{(p)}$ by plethysms when the highest weight is p-small: $|\lambda| < p$. This construction is used in Appendix II to give a construction by plethysms of the crystals (resp. filtered vector bundles) over a toroidal compactification of the Siegel variety over $\mathbb{Z}_p$, associated to irreducible representations whose highest weights are p-small.

If λ is a fundamental weight, then the irreducible representation V_λ of G has a canonical admissible lattice $V(\lambda)_\mathbb{Z}$ for the Chevalley order $\mathfrak{g}$ [**12**] p. 206. For another dominant weight $\lambda \in X^+$, several admissible lattices exist over $\mathbb{Z}$. However, given an prime p, we have shown in [**61**], Sect. 1.2, that for $\lambda = (a_g, \ldots, a_1; c)$ such that $a_g + a_{g-1} + g + (g-1) < p$, these lattices all coincide after tensoring by the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at p. Note that our condition $|\lambda + \rho| < p-1$ implies $a_g + a_{g-1} + g + (g-1) < p$.

For such a weight, let us recall the construction by plethysms of this unique admissible $\mathbb{Z}_p$-lattice $V_{\lambda, \mathbb{Z}_p}$. It will be used systematically in the sequel as it fits well in the construction of sheaves over the Siegel modular variety.

Let $s = |\lambda|$; hence $s < p$. For any (i, j) with $1 \leqslant i < j \leqslant n$, let $\phi_{i,j} : \mathbf{V}^{\otimes s} \to \mathbf{V}^{\otimes (s-2)}$ the contraction given by

$$v_1 \otimes \cdots \otimes v_s \longmapsto \langle v_i, v_j \rangle v_1 \otimes \cdots \otimes \widehat{v}_i \otimes \cdots \otimes \widehat{v}_j \otimes \cdots \otimes v_s;$$

Let $\psi \in \mathbf{V}^{\otimes 2}$ be the image of the symplectic form $\langle\, ,\, \rangle \in (\mathbf{V} \otimes \mathbf{V})^*$ via the identifications

$$(\mathbf{V} \otimes \mathbf{V})^* \;\cong\; \mathbf{V}^* \otimes \mathbf{V}^* \;\cong\; \mathbf{V} \otimes \mathbf{V}$$

the last one being given by $\mathbf{V} \cong \mathbf{V}^*$, $v \mapsto \langle v, \bullet \rangle$.

We consider for any $s \geqslant 2$ the maps $\psi_{i,j} : \mathbf{V}^{\otimes s-2} \to \mathbf{V}^{\otimes s}$ obtained by inserting ψ at ith and jth components. Observe that $\psi_{i,j}$ is injective. Let $\theta_{i,j} = \psi_{i,j} \circ \phi_{i,j} \in \mathrm{End}(\mathbf{V}^{\otimes s})$. Let $\mathbf{V}^{\langle s \rangle}$ be the submodule of $\mathbf{V}^{\otimes s}$ defined as intersection of the kernels of the $\theta_{i,j}$'s (note that $\mathrm{Ker}\, \theta_{i,j} = \mathrm{Ker}\, \phi_{i,j}$).

As we shall see below, for $p > 2 \cdot g$, $\mathbf{V}^{\langle s\rangle}_{\mathbb{Z}_{(p)}}$ is the image of $\mathbf{V}^{\otimes s}$ by an idempotent in the $\mathbb{Z}_p$-algebra generated by the $\theta_{i,j}$'s inside $\mathrm{End}_{\mathbb{Z}_{(p)}}(\mathbf{V}^{\otimes s})$. Finally, by applying the Young symmetrizer $c_\lambda = a_\lambda \cdot b_\lambda$ (see [**32**] 15.3 and 17.3), whose coefficients are in $\mathbb{Z}_{(p)}$, to $\mathbf{V}^{\langle s\rangle} \otimes \mathbb{Z}_{(p)}$, one obtains the sought-for lattice $V_{\lambda,\mathbb{Z}_{(p)}}$.

Lemma 6. — *There exists an idempotent e_s in the $\mathbb{Z}[\frac{1}{g}]$-subalgebra of $\mathrm{End}_{\mathbb{Z}[\frac{1}{g}]}(\mathbf{V}^{\otimes s})$ generated by the $\theta_{i,j}$'s $(1 \leqslant i < j \leqslant g)$, such that*

$$\mathbf{V}^{\langle s\rangle} = e_s \cdot \mathbf{V}^{\otimes s}.$$

Proof. — Let

$$\Phi = \bigoplus \phi_{i,j} : \mathbf{V}^{\otimes s} \longrightarrow \bigoplus_{1\leqslant i<j\leqslant s} \mathbf{V}^{\otimes(s-2)}$$

Thus,

$$\mathbf{V}^{\langle s\rangle} = \mathrm{Ker}\,\Phi.$$

Similarly, put

$$\Psi : \sum_{i<j} \psi_{i,j} : \bigoplus_{1\leqslant i<j\leqslant s} \mathbf{V}^{\otimes(s-2)} \longrightarrow \mathbf{V}^{\otimes s}.$$

and

$$\Theta = \Psi \circ \Phi = \sum_{1\leqslant i<j\leqslant s} \theta_{i,j}.$$

Since

$$\Phi \circ \Psi = (\times g),$$

we see that $\frac{1}{g} \cdot \Theta$ is an idempotent. It belongs to the $\mathbb{Z}[\frac{1}{g}]$-algebra generated by the $\theta_{i,j}$'s.

Thus,

$$\mathbf{V}^{\otimes s} = \mathbf{V}^{\langle s\rangle} \oplus \mathrm{Im}\,\Psi, \quad x = \Big(x - \frac{1}{g} \cdot \Theta(x)\Big) + \frac{1}{g} \cdot \Theta(x).$$

This decomposition of $\mathbb{Z}_{(p)}$-modules is G-stable. We put $e_s = \mathrm{Id} - \frac{1}{g} \cdot \Theta$. This is the desired projector to $\mathbf{V}^{\langle s\rangle}$.

To conclude:

Corollary 1. — *For any prime p which does not divide $2 \cdot g$ and such that $p > s = |\lambda|$, the module $V_{\lambda,\mathbb{Z}_{(p)}}$ obtained by Construction 5.1 is the image of $\mathbf{V}^{\otimes s}_{\mathbb{Z}_{(p)}}$ by an idempotent in the $\mathbb{Z}_{(p)}$-subalgebra of $\mathrm{End}_{\mathbb{Z}_{(p)}}(\mathbf{V}^{\otimes s})$ generated by permutations and the $\theta_{i,j}$'s. This algebra commutes to the G-action.*

We apply a similar construction for a B_M-dominant weight μ of M with $|\mu| < p$. We denote by $W_{\mu,\mathbb{Z}_{(p)}}$ the canonical admissible lattice of W_μ over $\mathbb{Z}_{(p)}$ given by the Young symmetrizer. It can be regarded as a $\mathcal{U}(\mathfrak{q})$-module via $\mathcal{U}(\mathfrak{q}) \to \mathcal{U}(\mathfrak{m})$.

Lemma 7. — *The subcategory of the category of M-representations, free and of finite rank over $\mathbb{Z}_p$, consisting of representations of highest weight $< p$ is semisimple.*

Proof. — We have to show that there is no nontrivial extensions in this subcategory. Let λ and μ be two M-dominant weights such that $|\lambda| < p$ and $|\mu| < p$. λ and μ are not in the same orbit for the action of the affine Weyl group ([**46**], Part II, 6.1). Let W_λ and W_μ be the corresponding canonical admissible lattices over $\mathbb{Z}_p$, then $\mathrm{Ext}^1(W_\lambda, W_\mu) = 0$ by the linkage principle ([**46**], Part II, 6.17, see also [**61**], Sect. 1.10, Lemma).

5.1.2. The BGG complex. — We are interested in a variant of the "BGG complex" constructed in [**3**] where one replaces the Borel subgroup by the parabolic Q. Over the field $\mathbb{Q}$, it is defined in [**13**] Chapter VI, Prop. 5.3 as the eigenspace for the infinitesimal character $\chi_{\lambda+\rho}$ inside the standard bar resolution of $V_{\lambda,\mathbb{Q}}$:

$$D(\lambda)_{\mathbb{Q}} := \mathcal{U}_{\mathbb{Q}} \otimes_{\mathcal{U}(\mathfrak{q})_{\mathbb{Q}}} (\Lambda^\bullet(\mathfrak{g}/\mathfrak{q}) \otimes V_{\lambda,\mathbb{Q}}).$$

Following [**3**], we show in [**61**] that this BGG complex admits a natural $\mathbb{Z}_{(p)}$-structure in terms of integral Verma modules:

$$C(\lambda)_{\mathbb{Z}_{(p)}} = \bigoplus_{w \in W^M} \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{q})} W_{w(\lambda+\rho)-\rho,\mathbb{Z}_{(p)}}$$

and we prove in Theorem D and Sect. 4 of [**61**] the following result. Let $D(\lambda)_{\mathbb{Z}_{(p)}} := \mathcal{U}_{\mathbb{Z}_{(p)}} \otimes_{\mathcal{U}(\mathfrak{q})_{\mathbb{Z}_{(p)}}} (\Lambda^\bullet(\mathfrak{g}/\mathfrak{q}) \otimes V_{\lambda,\mathbb{Z}_{(p)}})$ be the standard $\mathbb{Z}_{(p)}$-complex, a natural $\mathbb{Z}_{(p)}$-version of the standard bar resolution over $\mathbb{Q}$ of $V_{\lambda,\mathbb{Q}}$.

Theorem 5. — *Let $\lambda \in X^+$ and let $p > |\lambda + \rho|$. Then there is a canonical morphism of complexes $j : C(\lambda)_{\mathbb{Z}_{(p)}} \hookrightarrow D(\lambda)_{\mathbb{Z}_{(p)}}$ such that*

– it is injective and it admits a retraction of $\mathbb{Z}_{(p)}$-complexes (i.e. $\mathrm{Im}\, j$ is direct factor as a $\mathbb{Z}_{(p)}$-subcomplex),

– $\mathrm{Im}(j_{\mathbb{Q}})$ is the BGG complex over $\mathbb{Q}$.

Remarks

1) The BGG complex mentioned here is a variant for the parabolic Q of the one defined in lemma 9.8 of [**3**] in the Borel case. For details concerning the differential maps, see Sect. 2 of [**61**].

2) The bound on λ needed for proving this theorem is actually looser than $(\sum_{i=1}^g a_i) + d < p$: it is enough that $a_g + a_{g-1} + g + (g-1) < p$.

3) We do not claim that these complexes are exact, as they are not. However, as we will see in Sect. 5.4, after applying the functor L to a sheaf construction (Sect. 4.2), we will transform the dual of $C(\lambda)_\bullet$ into a resolution of the sheafification of the dual of $V_{\lambda,\mathbb{Z}_{(p)}}$.

5.1.3. Kostant-Chevalley algebra and universal enveloping algebra. — We fix the same notations as in 5.1. In particular, $\mathcal{U}$ is the Kostant-Chevalley algebra of $\mathfrak{g}$ over $\mathbb{Z}$. $\mathcal{U}$ can be identified with the algebra $\mathrm{Dist}(G)$ of distributions of G ([**46**], Part II, 1.12). Recall that

$$\mathrm{Dist}(G) = \bigcup_{n\geqslant 0} (\mathbb{Z}[G]/\mathcal{M}^{n+1})^*$$

where $\mathcal{M}$ is the maximal ideal of regular functions vanishing at the unit element. Let $\widetilde{\mathcal{U}}$ be the universal enveloping algebra of $\mathfrak{g}$. By the universal property of $\widetilde{\mathcal{U}}$, we have a natural homomorphism $\gamma : \widetilde{\mathcal{U}} \to \mathcal{U} = \mathrm{Dist}(G)$ which is injective. It is surjective over $\mathbb{Z}_p$ when restricted to the $< p$-step of the filtrations of $\widetilde{\mathcal{U}}$ resp. $\mathcal{U} = \mathrm{Dist}(G)$:

$$\gamma : \widetilde{\mathcal{U}}^{<p} \cong \mathcal{U}^{<p}.$$

It will imply the following lemma:

Lemma 8. — *Let $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ be the Kostant-Chevalley algebra and universal enveloping algebra over $\mathbb{Z}_p$ respectively and V_p, W_p be two Q-representations over $\mathbb{Z}_p$ whose semisimplifications have p-small highest weights (a sufficient condition on the highest weights is $|\lambda_i| < p$), then the canonical map*

$$\mathrm{Hom}_{\mathfrak{q}}(V_p, \widetilde{\mathcal{U}} \otimes_{\widetilde{\mathcal{U}}(\mathfrak{q})} W_p) \longrightarrow \mathrm{Hom}_{\mathfrak{q}}(V_p, \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{q})} W_p)$$

induced by γ, is an isomorphism.

Proof. — By Poincaré-Birkhoff-Witt over $\mathbb{Z}_p$, we have

$$\widetilde{\mathcal{U}} \otimes_{\widetilde{\mathcal{U}}(\mathfrak{q})} W_p = \widetilde{\mathcal{U}}\mathfrak{u}^- \otimes_{\mathbb{Z}_p} W_p$$

where $\mathfrak{u}^-$ is the unipotent radical of the parabolic Lie algebra opposite of $\mathfrak{q}$. It is enough to show

$$\mathrm{Hom}_{\mathfrak{q}}(V_p, \widetilde{\mathcal{U}}(\mathfrak{u}^-) \otimes_{\mathbb{Z}_p} W_p) = \mathrm{Hom}_{\mathfrak{q}}(V_p, \widetilde{\mathcal{U}}(\mathfrak{u}^-)^{<p} \otimes_{\mathbb{Z}_p} W_p)$$

Recall that the decomposition of W_p as a direct sum of $\mathfrak{t}$-eigenmodules W_σ is valid over $\mathbb{Z}_p$ by diagonalizability of tori over any base.

For any $H \in \mathfrak{t}$, $\underline{X}^{\underline{n}} \in \widetilde{\mathcal{U}}(\mathfrak{u}^-)$ ($\underline{n} = (n_\alpha)_{\alpha\in\Phi^{M+}}$) and $w \in W_\sigma$, we have

$$H \cdot (\underline{X}^{\underline{n}} \otimes w) = \Big(\sigma - \sum_{\alpha\in\Phi^{M+}} n_\alpha \alpha\Big)(H) \cdot (\underline{X}^{\underline{n}} \otimes w)$$

For any $\mathfrak{q}$-equivariant $\phi : V_p \to \widetilde{\mathcal{U}}(\mathfrak{u}^-) \otimes_{\mathbb{Z}_p} W_p$, the image of a highest weight vector $v \in V_p$ is of the form

$$\phi(v) = \sum_i \underline{X}_i^{\underline{n}_i} \otimes w_i \quad \text{with } w_i \in W_{\sigma_i}$$

Comparing the weights we have relations of the type

$$\lambda = \sigma_i - \sum_{\alpha\in\Phi^{M+}} n_\alpha^{(i)} \alpha$$

by increasing the coordinates of $n^{(i)}$, we can assume that σ_i is the highest weight of W_p, hence is p-small. Solving a linear system of inequations, we see that for any $\alpha \in \Phi^{M+}$, $n_\alpha^{(i)} < p$ as desired.

5.2. p-adic integral automorphic vector bundles. — Let $f : A \to X$ be the universal principally polarized abelian variety over X (with a U-level structure). Recall that $R^1 f_* \Omega^\bullet_{A/X}$ is endowed with the Gauss-Manin connection, which is integrable and quasi-nilpotent (see Section 4.3). Let $\overline{X}$ be a toroidal compactification of X over $\mathbb{Z}_p$. Let $\overline{X}_n = \overline{X} \otimes \mathbb{Z}/p^n\mathbb{Z}$; let $(\overline{X} \otimes \mathbb{F}_p/(\mathbb{Z}/p^n\mathbb{Z}))^{\log}_{\mathrm{crys}}$ be the logarithmic crystalline site associated to the scheme $\overline{X} \otimes \mathbb{F}_p$ and its divisor at infinity. Note that $\overline{X} \otimes \mathbb{F}_p$ is a toroidal compactification of $X \otimes \mathbb{F}_p$. As recalled in Sect. 4.1 above, there is an equivalence of category between crystals on this site and locally free $\mathcal{O}_{\overline{X}_n}$-modules endowed with an integrable and "quasi-nilpotent" logarithmic connection. Let $\mathbf{Rep}_{\mathbb{Z}_p}(G)$, resp. $\mathbf{Rep}_{\mathbb{Z}_p}(Q)$, be the category of algebraic representations of G, resp. Q, on finitely generated free modules. Consider the respective full subcategories $\mathbf{Rep}^{\leqslant p-1}_{\mathbb{Z}_p}(G)$ and $\mathbf{Rep}^{\leqslant p-1}_{\mathbb{Z}_p}(Q)$ consisting in objects whose highest weights are p-small (in fact, whose highest weights μ satisfy $|\mu| \leqslant p-1$).

For each $n \geqslant 1$, let $\mathcal{V}_n^\nabla$, resp. $\overline{\mathcal{V}}_n^\nabla$ be the category of locally free $\mathcal{O}_{X_n}$-modules, resp. $\mathcal{O}_{\overline{X}_n}$-modules, endowed with an integrable and "quasi-nilpotent", resp. integrable, "quasi-nilpotent" logarithmic connection, and $\mathcal{F}_n$, resp. $\overline{\mathcal{F}}_n$ that of locally free $\mathcal{O}_{X_n}$-modules, resp. $\mathcal{O}_{\overline{X}_n}$-modules endowed with a filtration with locally free graded pieces.

The goal of this section is to define for each $n \geqslant 1$ two functors

$$\overline{V}_{\mathbb{Z}/p^n\mathbb{Z}} : \mathbf{Rep}^{\leqslant p-1}_{\mathbb{Z}_p}(G) \longrightarrow \overline{\mathcal{V}}_n^\nabla$$

and another

$$\overline{F}_{\mathbb{Z}/p^n\mathbb{Z}} : \mathbf{Rep}^{\leqslant p-1}_{\mathbb{Z}_p}(Q) \longrightarrow \overline{\mathcal{F}}_n$$

We first define functors on $\mathbf{Rep}_{\mathbb{Z}_p}(G)$, resp. $\mathbf{Rep}_{\mathbb{Z}_p}(Q)$ with values in vector bundles over X_n. Then we proceed to show that these vector bundles extend to $\overline{X}_n$ provided they come from representations in $\mathbf{Rep}^{\leqslant p-1}_{\mathbb{Z}_p}(G)$ resp. $\mathbf{Rep}^{\leqslant p-1}_{\mathbb{Z}_p}(Q)$.

5.2.1. "Flat vector bundles" on X. — Let us define

$$V_{\mathbb{Z}/p^n\mathbb{Z}} : \mathbf{Rep}_{\mathbb{Z}_p}(G) \longrightarrow \mathcal{V}_n^\nabla$$

Let $\mathcal{O}_X^{2g}$ be the trivial vector bundle of rank $2g$ on X endowed with the canonical symplectic pairing (see section 5.1) and its natural action of G on the left. Let us put

$$\mathcal{T} = \underline{\mathrm{Isom}}_X(\mathcal{O}_X^{2g}, (R^1 f_* \Omega^\bullet_{A/X})^\vee)$$

where the isomorphisms are symplectic similitudes. It is an algebraic G-torsor over X for the right action

$$\mathcal{T} \times G \longrightarrow \mathcal{T}, \quad (\phi, g) \longmapsto \phi \circ g.$$

For any $V \in \mathbf{Rep}_{\mathbb{Z}_p}(G)$, we define $\mathcal{V}$ as the contracted product

$$\mathcal{V} = \mathcal{T} \overset{G}{\times} V$$

that is, the quotient of the cartesian product by the relation $(\phi, g \cdot v) \sim (\phi \circ g, v)$. It is a vector bundle on X hence over X_n for any $n \geqslant 1$.

Fact

1) *$\mathcal{V}$ is equipped with a connection of the desired type.*
2) *The image of the standard representation is $(R^1 f_* \Omega^\bullet_{A/X})^\vee$.*
3) *The correspondence $V \mapsto \mathcal{V}$ is functorial.*

Proof

1) Let $\mathcal{A} = (R^1 f_* \Omega^\bullet_{A/X})^\vee$; we consider the (dual) Gauss-Manin connection:

$$\nabla : \mathcal{A} \longrightarrow \mathcal{A} \otimes_{\mathcal{O}_X} \Omega_X$$

It is symplectic in the sense that for two sections f, g of $\mathcal{A}$, we have

$$\langle \nabla f, g \rangle + \langle f, \nabla g \rangle = d \langle f, g \rangle$$

where the symplectic product is extended to

$$\mathcal{A} \otimes \mathcal{A} \otimes \Omega_X \longrightarrow \Omega_X$$

Therefore, given a point ϕ of $\mathcal{T}$ over an X-scheme Y, we can transport ∇ to an element ∇_ϕ of $\mathfrak{g} \otimes \Omega_X \subset \mathrm{End}_{\mathcal{O}_Y}(\mathcal{O}_Y^{2g}) \otimes_{\mathcal{O}_X} \Omega_X$ defined by the diagram

$$\begin{tikzcd} \mathcal{A}_Y \arrow[r, "\nabla"] & \mathcal{A}_Y \otimes \Omega_X \\ \mathcal{O}_Y^{2g} \arrow[u, "\phi"] \arrow[r, "\nabla_\phi"] & \mathcal{O}_Y^{2g} \otimes \Omega_X^1 \arrow[u] \end{tikzcd}$$

Given $(V, \rho_V) \in \mathbf{Rep}_{\mathbb{Z}_p}(G)$, the representation ρ_V viewed on the Lie algebra $\mathfrak{g}$ enables us to define

$$\nabla_{V,\phi} = (\rho_V \otimes \mathrm{Id}_{\mathcal{O}_Y} \otimes_{\mathcal{O}_X}) \, \mathrm{Id}_{\Omega_X} \circ \nabla \in \mathrm{End}(V) \otimes \mathcal{O}_Y \otimes_{\mathcal{O}_X} \Omega_X$$

It is a connection on $V \otimes \mathcal{O}_Y$. For $Y = \mathcal{T}$, and ϕ the canonical point of $\mathcal{T}$, we can descend this connection to the contracted product because

$$\nabla_{\phi \circ h} = h^{-1} \circ \nabla_\phi \circ h$$

The resulting $\nabla_{\mathcal{V}}$ is integrable and quasi-nilpotent because it is so for the Gauss-Manin connection.

2) Consider the morphism of X-schemes

$$\mathcal{T} \times \mathcal{O}_X^{2g} \longrightarrow \mathcal{A}, \quad (\phi, v) \longmapsto \phi(v)$$

It descends to the contracted product since $\phi \circ g(v) = \phi(g \cdot v)$. It defines therefore a morphism of vector bundles over X: $\mathcal{V}_{\mathrm{st}} \to \mathcal{A}$. This morphism is an isomorphism over $\mathcal{T}$ and $\mathcal{T} \to X$ is faithfully flat, therefore it is an isomorphism over X.

3) is obvious.

5.2.2. Comparison with the transcendental definitions. — Let $\widetilde{\mathcal{T}} = G(\mathbb{Q})\backslash G(\mathbb{A}) \times G(\mathbb{C})/UU_\infty$, the left action of $G(\mathbb{Q})$ on $G(\mathbb{A})\times G(\mathbb{C})$ being diagonal, while the right one of UU_∞ being only on the $G(\mathbb{A})$-factor; the first projection $pr_1 : G(\mathbb{A})\times G(\mathbb{C}) \to G(\mathbb{A})$ induces a structure of principal $G(\mathbb{C})$-bundle over the analytic Siegel variety S_U by

$$\overline{pr_1} : \widetilde{\mathcal{T}} \longrightarrow S_U$$

Moreover, let $\check{\mathcal{Z}}$ be the compact dual domain of the Siegel half-space $\mathcal{Z}$. Let $c = \left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right) \in \mathrm{GSp}_{2g}(\mathbb{C})$ be the standard Cayley matrix which defines the Cayley transform $\beta : \mathcal{Z} \hookrightarrow \check{\mathcal{Z}}$. Consider the twisted multiplication

$$\mu : G(\mathbb{A}) \times G(\mathbb{C}) \longrightarrow G(\mathbb{C}), \quad (g, g') \longmapsto g'c \cdot g_\infty \cdot c^{-1}$$

for $g = (g_f, g_\infty) \in G(\mathbb{A})$; it induces a morphism $\overline{\mu} : \widetilde{\mathcal{T}} \to \check{\mathcal{Z}}$.

Recall the transcendental definition of the automorphic vector bundle associated to $V \in \mathbf{Rep}_{\mathbb{C}}(Q)$: one forms the contracted product

$$\check{\mathcal{V}} = G(\mathbb{C}) \overset{Q(\mathbb{C})}{\times} V$$

which is a vector bundle over $\check{\mathcal{Z}}$. Then one forms its pull-back $\beta^*(\check{\mathcal{V}})$ to $\mathcal{Z}$ by the Cayley transform $\beta : \mathcal{Z} \hookrightarrow \check{\mathcal{Z}}$. One takes the product $\beta^*(\check{\mathcal{V}}) \times G_f/U$, and one defines the holomorphic vector bundle $\widetilde{\mathcal{V}} \to S_U$ by

$$\widetilde{\mathcal{V}} = G(\mathbb{Q})\backslash(\beta^*(\check{\mathcal{V}}) \times G_f/U) \longrightarrow G(\mathbb{Q})\backslash(\mathcal{Z} \times G_f/U) = S_U.$$

We refer to $V \mapsto \widetilde{\mathcal{V}}$ as the transcendental construction. It is valid for $V \in \mathbf{Rep}_{\mathbb{C}}(G)$ as well.

Note that we could avoid the use of the Cayley transform, and use instead the more direct (but equivalent) Borel transform, at the expense of replacing the Siegel parabolic Q by its conjugate $c^{-1}Qc$ in the definition of the compact dual of $\mathcal{Z}$.

Lemma 9. — *Over $\mathbb{C}$, the functor $V_{\mathbb{C}}$ is canonically isomorphic to the one defined by the standard transcendental construction.*

Proof. — We prove two statements

1) There is a canonical isomorphism of $G(\mathbb{C})$-principal bundles $\widetilde{\mathcal{T}} \to \mathcal{T}$.

2) The transcendental construction can be described as

$$\widetilde{\mathcal{V}} = \overline{pr_1}_* \circ \overline{\mu}^* \check{\mathcal{V}} = \widetilde{\mathcal{T}} \overset{G(\mathbb{C})}{\times} V.$$

1) Recall that the description of the Siegel variety for a level subgroup $U \subset G(\widehat{\mathbb{Z}})$ can be done integrally: Note that $G = \mathrm{GSp}(2g)$ and $G' = \mathrm{Sp}(2g)$ are defined over $\mathbb{Z}$. It is a simple exercise to see that

$$S_U = G(\mathbb{Q})^+\backslash(G(\mathbb{A}_f)/U \times \mathcal{Z}) = G'(\mathbb{Z})\backslash(G(\widehat{\mathbb{Z}})/U \times \mathcal{Z}).$$

Let $\mathcal{Z}' = G(\widehat{\mathbb{Z}})/U \times \mathcal{Z}$ Let V_{st} be the (complex) standard representation of G. We recall first that the pull-back by $\mathcal{Z}' \to S_U$ of the vector bundle $\mathcal{A}$ endowed with the dual Gauss-Manin connection is isomorphic to the vector bundle of the local system

$\mathcal{Z}' \times V_{\mathrm{st}}$ endowed with its obvious flat connection. By lack of an adequate reference, we recall the proof. The description of the universal abelian variety over the Siegel variety of level a congruence subgroup $U \subset G(\widehat{\mathbb{Z}})$ is as follows. Let $\widetilde{G}(\mathbb{Z}) = \mathbb{Z}^{2g} \lhd G'(\mathbb{Z})$ be the Jacobi group, that is, the semidirect product of the symplectic lattice $(\mathbb{Z}^{2g}, J)$ by $G'(\mathbb{Z})$ for the action $\gamma \cdot v$ denoting the usual product of matrices. It acts on the left on $G(\widehat{\mathbb{Z}})/U \times \mathcal{Z} \times \mathbb{C}^g$ by

$$(0, \gamma) \cdot (g, z, w) = (\gamma g, \gamma(z), {}^t j(\gamma, z)^{-1} w), \quad (v, 1) \cdot (g, z, w) = (g, z, (z, 1) \cdot J \cdot v)$$

it is indeed an action because for any $\gamma \in G'(\mathbb{Z})$, we have ${}^t\gamma \cdot J \cdot \gamma = J$.

Consider the first projection

$$\mathcal{Z}' \times \mathbb{C}^g \longrightarrow \mathcal{Z}'$$

and take the quotient for the left action of $\widetilde{G}(\mathbb{Z})$ resp. $G(\mathbb{Z})$. We obtain the analytic description of the universal abelian variety A over S_U. For $f : A \to S_U$, the locally constant sheaf $(R^1 f_* \mathbb{Z})^\vee$ which identifies to the relative homology inside $\mathrm{Lie}(A/S_U)$ can be viewed as

$$G'(\mathbb{Z}) \backslash \mathbb{Z}^{2g} \cdot (\mathcal{Z}' \times \{0\}) \quad \text{inside } G'(\mathbb{Z}) \backslash (\mathcal{Z}' \times \mathbb{C}^g)$$

Therefore, its sections identify to the sections s of the trivial covering

$$\mathcal{Z}' \times \mathbb{Z}^{2g} \longrightarrow \mathcal{Z}'$$

satisfying $s(\gamma(g, z)) = \gamma \cdot s(g, z)$.

Therefore, the pull-back of $\widetilde{\mathcal{T}}$ is isomorphic to $\underline{\mathrm{Isom}}_{\mathcal{Z}'}(\mathcal{Z}' \times V_{\mathrm{st}}, \mathcal{Z}' \times V_{\mathrm{st}}) = \mathcal{Z}' \times G(\mathbb{C})$, with action of $G(\mathbb{Q})$ diagonally on the left. Hence, by quotienting by $G(\mathbb{Q})$, we obtain a canonical isomorphism $\widetilde{\mathcal{T}} \cong \mathcal{T}$.

2) Let $V \in \mathbf{Rep}_{\mathbb{C}}(G)$. In this situation, only the $\mathcal{C}^\infty$-structure of $\widetilde{\mathcal{V}}$ matters (indeed, only the structure of the underlying locally constant sheaf). On one hand, it is well-known that $\widetilde{\mathcal{V}}$ is the vector bundle, associated to the V-covering $G(\mathbb{Q}) \backslash (\mathcal{Z}' \times V) \to S_U$. On the other hand, the pull-back by $G(\mathbb{C}) \times \mathcal{Z}' \to \widetilde{\mathcal{T}}$ of $\widetilde{\mathcal{T}} \overset{G(\mathbb{C})}{\times} V$ identifies to $\mathcal{Z}' \times V$; it is endowed with a free action of $G(\mathbb{Q})$ (diagonally on the left), and of U on the right. The resulting quotient is again the vector bundle associated to the V-covering $G(\mathbb{Q}) \backslash (\mathcal{Z}' \times V) \to S_U$ as desired.

5.2.3. $\mathbb{Z}_p$-Integral extension to $\overline{X}$ for p-small weights. — Let us finally define the functor

$$\overline{V}_{\mathbb{Z}_p} : \mathbf{Rep}_{\mathbb{Z}_p}^{\leqslant p-1}(G) \longrightarrow \overline{\mathcal{V}}^{\nabla}$$

which induces the functors $\overline{V}_{\mathbb{Z}/p^n\mathbb{Z}}$ mentioned at the beginning of this section.

We have the diagram

$$\begin{tikzcd} X_{\mathbb{Q}_p} \arrow[r, hook] \arrow[d, "j"'] \arrow[dr, "k"] & X_{\mathbb{Z}_p} \arrow[d] \\ \overline{X}_{\mathbb{Q}_p} \arrow[r, hook, "i"] & \overline{X}_{\mathbb{Z}_p} \end{tikzcd} \tag{5.2.1}$$

On one hand, for any Q-representation W, we have constructed a vector bundle $\mathcal{W}$ over $X_{\mathbb{Z}_p}$; on the other hand, M. Harris ([**37**]) has defined a functor from Q-representations defined over $\mathbb{Q}$ to vector bundles over $\overline{X}_{\mathbb{Q}}$ coinciding with ours on $X_{\mathbb{Q}_p}$. We first glue the vector bundles $\overline{\mathcal{W}}_{\mathbb{Q}_p}$ with $\mathcal{W}_{\mathbb{Z}_p}$ into a vector bundle $\widetilde{\mathcal{W}}_{\mathbb{Z}_p}$ over the cofibered product $\widetilde{X}_{\mathbb{Z}_p} = \overline{X}_{\mathbb{Q}_p} \cup_{X_{\mathbb{Q}_p}} X_{\mathbb{Z}_p}$.

Then, we observe that $\widetilde{X}_{\mathbb{Z}_p} = \overline{X}_{\mathbb{Z}_p} - D_{\mathbb{F}_p}$ is an open subset with complement of codimension 2 in $\overline{X}_{\mathbb{Z}_p}$. Therefore, by [**33**] Cor. 5.11.4, the direct image of $\widetilde{\mathcal{W}}_{\mathbb{Z}_p}$ is a coherent sheaf on $\overline{X}_{\mathbb{Z}_p}$. Let us see it is locally free, at least if V has p-small highest weight. By dévissage, it is enough to consider irreducible M-representations with such p-small highest weight. By Appendix II, it is enough to consider the standard representation. In that case, the coherent sheaf on $\overline{X}_{\mathbb{Z}_p}$ is $\mathrm{Lie}(\mathcal{G}/\overline{X})^\vee$, which is locally free. This concludes the proof.

In particular, for any dominant weight λ, we have attached to the representation V_λ of G of highest weight λ a vector bundle $\mathcal{O}_{\overline{X}_n}$-module $\overline{\mathcal{V}}_{\lambda,n}$ on $\overline{X}_n$ together with a connection with logarithmic poles along D_n, hence a logarithmic crystal $\overline{\mathcal{V}}_{\lambda,n}$ on $(\overline{X}/(\mathbb{Z}/p^n\mathbb{Z}))^{\log}_{\mathrm{crys}}$. Moreover, it carries a natural filtration since V_λ is also a Q-representation.

5.2.4. Differential operators over $\mathbb{Z}_{(p)}$. — Let V and W be two rational representations of Q, and $\mathcal{V}_{/\mathbb{Q}}$, $\mathcal{W}_{/\mathbb{Q}}$ the corresponding automorphic vector bundles over $X_{\mathbb{Q}}$ (see previous subsection) and $\overline{\mathcal{V}}_{/\mathbb{Q}}$, $\overline{\mathcal{W}}_{/\mathbb{Q}}$ their canonical extension to the toroidal compactification $\overline{X}$. According to Proposition 5.1 of [**13**] VI.5, we have a functorial homomorphism

$$\Psi : \mathrm{Hom}_{U(\mathfrak{g}_{\mathbb{Q}})}(U(\mathfrak{g}_{\mathbb{Q}}) \otimes_{\mathcal{U}(\mathfrak{q}_{\mathbb{Q}})} V, U(\mathfrak{g}_{\mathbb{Q}}) \otimes_{\mathcal{U}(\mathfrak{q}_{\mathbb{Q}})} W) \longrightarrow \mathrm{Diff.\,Operators}(\overline{\mathcal{W}}^\vee_{/\mathbb{Q}}, \overline{\mathcal{V}}^\vee_{/\mathbb{Q}}).$$

Actually, in Proposition 5.1 of Chap. VI, the construction of Ψ is explained over $\mathbb{C}$. The $\mathbb{Q}$-rationality statement is explained in Remark 5.2 following the proof of Proposition 5.1 of Sect. VI.5. We now prove a variant thereof over $\mathbb{Z}_{(p)}$.

We treat first the case of degree 0 differential operators by referring to 5.2.2:

Lemma 10. — *Let V, W be two Q-representations of p-small highest weights (in fact, $|\lambda_V|$ and $|\lambda_W| < p$ is enough), V_p and W_p their canonical $\mathcal{U}$-stable lattices and $\overline{\mathcal{V}}_n$, $\overline{\mathcal{W}}_n$ the corresponding automorphic vector bundles over $\overline{X}_n$, $n > 0$. There is a functorial injective homomorphism*

$$\mathrm{Hom}_{\mathfrak{q}}(V_p, W_p) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\overline{X}_n}}(\overline{\mathcal{W}}^\vee_n, \overline{\mathcal{V}}^\vee_n)$$

compatible with the transcendental construction.

Then, the case of general differential operators can be treated as follows:

Lemma 11. — *Let V, W be two irreducible Q-representations of p-small highest weights, V_p and W_p their canonical $\mathcal{U}$-stable lattices and $\overline{\mathcal{V}}_n$, $\overline{\mathcal{W}}_n$ the corresponding* *automorphic vector bundles over* $\overline{X}_n$, $n > 0$. *Then* Ψ *induces for each* $n > 0$, *a homomorphism*

$$\mathrm{Hom}_{\mathcal{U}}(\mathcal{U} \otimes_{\mathcal{U}(\mathfrak{q})} V_p, \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{q})} W_p) \longrightarrow \text{P.D. Diff. Operators}(\overline{\mathcal{W}}_n^\vee, \overline{\mathcal{V}}_n^\vee)$$

Remark. — By p-smallness of the highest weights, the only possible degrees of morphisms in $\mathrm{Hom}_{\mathcal{U}}(\mathcal{U} \otimes_{\mathcal{U}(\mathfrak{q})} V_p, \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{q})} W_p)$ are $< p$, hence, the corresponding PD differential operators, are in fact usual differential operators.

Proof. — We start with operators of order one. Note that the de Rham differential $d : \mathcal{O}_{\overline{X}_n} \to \Omega^1_{\overline{X}_n}$ is the image by Ψ of the obvious map

$$\delta : \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{q})} \mathfrak{g}_{\mathbb{Z}_p}/\mathfrak{q}_{\mathbb{Z}_p} \longrightarrow \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{Z}_p, \quad 1 \otimes X \longmapsto X \otimes 1$$

(compare with [**13**] VI, remark 5.2). By Lemma 10, this implies that each homomorphism $\phi : V_p \to \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{q})} W_p$ of degree one is mapped by Ψ to a $\mathbb{Z}_p$-integral differential operator of order one. Indeed any ϕ as above factors as $\phi = \delta \otimes \mathrm{Id}_{W_p} \circ (\mathrm{Id}_{\widetilde{\mathcal{U}}} \otimes \psi)$ for a $\psi \in \mathrm{Hom}_{\mathfrak{q}}(V_p, \mathfrak{g}/\mathfrak{q} \otimes W_p)$.

Recall that $\widetilde{\mathcal{U}}$ denotes the universal enveloping algebra of $\mathfrak{g}$. We have seen in Lemma 8 that by p-smallness of the highest weights, the natural algebra homomorphism $\gamma : \widetilde{\mathcal{U}} \to \mathcal{U}$ induces a bijection between $\mathrm{Hom}_{\mathfrak{g}}(V_p, \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{q})} W_p)$ and $\mathrm{Hom}_{\mathfrak{g}}(V_p, \widetilde{\mathcal{U}} \otimes_{\widetilde{\mathcal{U}}(\mathfrak{q})} W_p)$. Now, as a corollary of PBW over $\mathbb{Z}_p$ for $\widetilde{\mathcal{U}}$, we see that every element $\phi \in \mathrm{Hom}_{\mathfrak{g}}(V_p, \widetilde{\mathcal{U}} \otimes_{\widetilde{\mathcal{U}}(\mathfrak{q})} W_p)$ of degree $m > 1$ factors as $\phi = (\delta \otimes \mathrm{Id}_{W_p}) \circ \psi$ where ψ has degree $m-1$: fix a basis $(X_\alpha)_{\alpha \in \Phi^{M-}}$ of $\mathfrak{u}^-$; for $v \in V_p$ and $\phi(v) = \sum_i \underline{X}^{\underline{n}^{(i)}} \otimes w_i$, put $\psi(v) = \sum_i \sum_{\alpha \in \Phi^{M-}} \underline{X}^{\underline{n}^{(i)} - 1_\alpha} \otimes X_\alpha \otimes w_i$ where 1_α is the family $(\delta_{\alpha,\beta})_{\beta \in \Phi^{M-}}$. The conclusion follows by induction on m.

5.3. The Hodge filtration on automorphic sheaves

5.3.1. The geometric aspect. — This paragraph is a recollection of well-known facts about the Hodge filtration in the automorphic setting (see [**15**] Sect. 5).

Let $\underline{S} = R_{\mathbb{C}/\mathbb{R}}\mathbf{G}_m$ and $h_0 : \underline{S}(\mathbb{R}) \to G(\mathbb{R})$ the homomorphism defined by

$$z = x + iy \in \mathbb{C}^\times \longmapsto \begin{pmatrix} xI_g & yI_g \\ -yI_g & xI_g \end{pmatrix} = xI_{2g} + yJ_{2g} \in G(\mathbb{R})$$

The $G(\mathbb{R})$-orbit $\mathcal{Z}$ of h_0 is analytically isomorphic to a double copy of the Siegel upper half-plane of genus g. The pair $(G, \mathcal{Z})$ defines a family of Shimura varieties "à la Deligne", isomorphic to our Shimura varieties S_U for various level structures U. If V is a real representation of G and $h \in X$, then the composition $h : \underline{S}(\mathbb{R}) \to G(\mathbb{R}) \to \mathrm{GL}(V)$ defines a real Hodge structure h_V on V ([**15**]). Let F_h be the filtration on $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$ deduced from h_V. For $V = \mathfrak{g}$ the adjoint representation, $F^0_h(\mathfrak{g}_{\mathbb{C}})$ is a Lie algebra of a parabolic subgroup $P(h)$ of $G_{\mathbb{C}}$. The mapping $h \to P(h)$ identifie $\mathcal{Z}$ as an open subset of its compact dual $\check{\mathcal{Z}} = G(\mathbb{C})/Q(\mathbb{C})$. Now, for general V, the mapping $h \to F_h$ define a $G(\mathbb{R})$-equivariant filtration (the Hodge filtration) on the constant fibre bundle $\mathcal{Z} \times V_{\mathbb{C}}$. Dividing by $G(\mathbb{Q})$ and U, we get a filtration on the coherent sheaf $\mathcal{V}$ over S_U, associated to the representation V. Moreover, if $\overline{\mathcal{V}}$ is the canonical extension of $\mathcal{V}$ to some toroidal compactification of S_U, then this filtration has a canonical extension to $\overline{\mathcal{V}}$. This results from Harris' functoriality [**37**] of the canonical extension (Sect. 5.2.3). In the case where V is the standard representation of G, then, by definition of the functor $V_{\mathbb{C}}$ (see Sect. 5.2.1), we have $\mathcal{V}^\vee = R^1 f_* \Omega^\bullet_{A/X}$; by Deligne's unicity of the canonical extension, we have $\overline{\mathcal{V}}^\vee = R^1 \overline{f}_* \Omega^\bullet_{\overline{A}/\overline{X}}(\log \infty_{\overline{A}/\overline{X}})$ and the Hodge filtration on the dual is the classical one given by

$$F^2(\overline{\mathcal{V}}^\vee) = 0 \subset F^1(\overline{\mathcal{V}}^\vee) = \overline{f}_* \Omega^1_{\overline{A}/\overline{X}}(\log \infty_{\overline{A}/\overline{X}}) \subset F^0(\overline{\mathcal{V}}^\vee) = \overline{\mathcal{V}}^\vee. \tag{5.3.1}$$

Then, for a represention V_λ associated to a dominant weight λ of G, we can use Weyl's invariant theory as in Appendix II, to describe the Hodge filtration on $\overline{\mathcal{V}}^\vee_\lambda$. Actually, Appendix II allows to describe this filtration explicitely over $\mathbb{Z}_p$ as well, for λ p-small. Indeed, we show there that, for λ p-small, each $\overline{\mathcal{V}}^\vee_\lambda$ on $X_{/\mathbb{Z}_p}$ is a direct summand of some higher direct image of the logarithmic de Rham complex over a toroidal compactification of the s-fold product of the universal abelian variety (see [**13**] p. 234).

Recall that for a complex $K^\bullet$, the notation $K^{\bullet \geqslant i}$ denotes the subcomplex of $K^\bullet$ equal to $K^\bullet$ in degre $\geqslant i$ and zero elsewhere.

If $\overline{f}_s : \overline{Y} \to \overline{X}$ is such a toroidal compactification over $\mathbb{Z}_p$, then the coherent sheaf

$$\mathcal{F} = R^{\boldsymbol{w}} \overline{f}_{s,*} \Omega^\bullet_{\overline{Y}/\overline{X}}(\log \infty)$$

is locally free if $\boldsymbol{w}$ is an integer $< p$ (see Illusie, [**44**] Cor. 2.4). It is endowed with the Hodge filtration

$$\mathrm{Fil}^i \mathcal{F} = \mathrm{Im}\left(R^{\boldsymbol{w}} \overline{f}_{s\,*} \Omega^{\bullet, \geqslant i}_{\overline{Y}/\overline{X}}(\log \infty) \longrightarrow R^{\boldsymbol{w}} \overline{f}_{s,*} \Omega^\bullet_{\overline{Y}/\overline{X}}(\log \infty) \right).$$

For a dominant weight λ such that $|\lambda| = s$, we take $\boldsymbol{w} = d+s$; recall that $\boldsymbol{w} < p-1$. We endow the sheaf $\overline{\mathcal{V}}^\vee_\lambda$ with the filtration:

$$\mathrm{Fil}^i \overline{\mathcal{V}}^\vee_\lambda = \overline{\mathcal{V}}^\vee_\lambda \cap \mathrm{Fil}^i \mathcal{F}.$$

Let $\overline{\mathcal{V}}^\vee_{\lambda,n}$ be the $\mathcal{O}_{\overline{X}_n}$-module obtained by reduction mod. p^n of the module $\overline{\mathcal{V}}^\vee_\lambda$.

Definition 5. — The Hodge filtration on the de Rham complex

$$\overline{\mathcal{V}}^\vee_{\lambda,n} \otimes_{\mathcal{O}_{\overline{X}_n}} \Omega^\bullet_{\overline{X}_n/\mathbb{Z}/p^n}(\log \infty)$$

is defined by:

$$F^i(\overline{\mathcal{V}}^\vee_{\lambda,n} \otimes_{\mathcal{O}_{\overline{X}_n}} \Omega^\bullet_{\overline{X}_n/\mathbb{Z}/p^n}(\log \infty)) = \sum_j F^j(\overline{\mathcal{V}}^\vee_{\lambda,n}) \otimes_{\mathcal{O}_{\overline{X}_n}} \Omega^\bullet_{\overline{X}_n/\mathbb{Z}/p^n}(\log \infty)^{\geqslant i-j}.$$

5.3.2. The group-theoretic aspect. — Let $H = \mathrm{diag}(0, \dots, 0, -1, \dots, -1) \in \mathrm{Lie}\, T \subset \mathfrak{g}$ (with g 0's and g -1's). H is a generator of the center of $\mathfrak{q} = \mathrm{Lie}\, Q$ (modulo the center of $\mathrm{Lie}\, G$). For any rational Q-representation V, for any $i \in \mathbb{Z}$, let V^i be the sum of the generalized H-eigenspaces with eigenvalues $\geqslant i$. This defines a decreasing filtration $\{V^i\}$ on V. We shall call this filtration the H-filtration. Note that this filtration is Q-stable.

Two cases are of particular interest for us:

– V is an irreducible M-representation with highest weight μ; the filtration is given by $V^{\mu(H)+1} = 0 \subset V^{\mu(H)} = V$. For instance, the standard representation V_0 of M is filtered by $0 = V_0^1 \subset V_0^0 = V_0$ while its twisted contragredient $V_1 = V_0^\vee \otimes \nu$ is filtered by $0 = V_1^0 \subset V_1^{-1} = V_1$.

– $V = V_\lambda$ is an irreducible representation of G associated to the dominant weight λ. Then the filtration given by H can also be defined by plethysms from the 2-step filtration of the standard representation V_{st}: $F^{-1} = V_{\mathrm{st}}$, $F^0 = V_0$ is its unique simple Q-submodule (in fact, an M-module), and $F^1 = 0$.

We can still define the H-filtration as above for a Q-representation V defined over $\mathbb{Z}_p$ instead of $\mathbb{C}$. If V is p-small, the eigen values of H are invertible and so the V^i's are $\mathbb{Z}_p$-summands in V.

In particular, we endow the standard bar resolution of $V_{\lambda,\mathbb{Z}_p}$ (say, for $|\lambda+\rho| < p-1$)

$$D(\lambda) := (\mathcal{U}_{\mathbb{Z}_p} \otimes_{\mathcal{U}(\mathfrak{q})_{\mathbb{Z}_p}} (\Lambda^\bullet(\mathfrak{g}/\mathfrak{q}) \otimes V(\lambda)_{\mathbb{Z}_p}))$$

with the H-filtration.

Let

$$C(\lambda)_{\mathbb{Z}_p} = \bigoplus_{w \in W^M} \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{q})} W_{w(\lambda+\rho)-\rho,\, \mathbb{Z}_p}$$

be the BGG complex introduced in Sect. 5.1.2 attached to $V_{\lambda,\mathbb{Z}_p}$. The H-filtration is given by

$$F^i C(\lambda)_{\mathbb{Z}_p} = \bigoplus_{\substack{w \in W^M \\ w(\lambda+\rho)(H)-\rho(H) \geqslant i}} \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{q})} W_{w(\lambda+\rho)-\rho, \mathbb{Z}_p}.$$

Then the injection $j : C(\lambda)_{\mathbb{Z}_p} \hookrightarrow D(\lambda)_{\mathbb{Z}_p}$ is a filtered direct factor of $D(\lambda)_{\mathbb{Z}_p}$ by [**61**].

5.3.3. Filtered vector bundles on X. — As in section 5.2.1, we define a second functor

$$F_{\mathbb{Z}/p^n\mathbb{Z}} : \mathbf{Rep}_{\mathbb{Z}_p}(Q) \longrightarrow \mathcal{F}_n$$

wich gives the Hodge filtration, as follows. We endow $\mathcal{O}_X^{2g} = \mathcal{O}_X \otimes V_{\mathrm{st}}$ with the standard symplectic pairing and the H- filtration $(0 \subset F^0 \subset F^{-1})$ and we put:

$$\mathcal{T}_H = \underline{\mathrm{Isom}}_{H,X}(\mathcal{O}_X^{2g}, (R^1 f_* \Omega^\bullet_{A/X})^\vee)$$

where the isomorphisms are symplectic similitudes respecting the Hodge filtrations. $\mathcal{T}_H$ is an algebraic Q-torsor over X. For any $W \in \mathbf{Rep}_{\mathbb{Z}_p}(Q)$, let

$$\mathcal{W} = \mathcal{T}_H \overset{Q}{\times} W$$

It is a vector bundle on X hence over X_n for any $n \geqslant 1$. This construction is functorial. As W is filtered by submodules which are Q-stable (by the H-filtration), the vector bundle $\mathcal{W}$ comes equipped with a filtration. If the representation W is p-small, we show by 5.3.2, that its successive quotients are locally free. Moreover, every morphism $W \to W'$ of Q-representations induces a strict morphism of filtered vector bundles. Following the lines of Lemma 9, one shows that the image of the standard representation is $(R^1 f_* \Omega^\bullet_{A/X})^\vee$ with its standard filtration. The proof of these assertions is similar to the one in the previous section.

Remarks

1) In fact, by the same construction, one can define functors $V_{\mathbb{Z}[1/N]}$ and $F_{\mathbb{Z}[1/N]}$ such that $V_{\mathbb{Z}/p^n\mathbb{Z}} = V_{\mathbb{Z}[1/N]} \otimes \mathbb{Z}/p^n\mathbb{Z}$ and similarly for F.

2) Every M-representation gives rise to a Q-representation by letting the unipotent radical act trivially on W.

Similar tho the complex analytic $G(\mathbb{C})$-torsor $\widetilde{\mathcal{T}} = G(\mathbb{Q})\backslash G(\mathbb{A}) \times G(\mathbb{C})/UU_\infty$ (see Sect. 5.2.2), one can construct a complex analytic $Q(\mathbb{C})$-torsor $\widetilde{\mathcal{T}}_H$ as follows. We start from the $Q(\mathbb{C})$-bundle $\mathcal{Q} : G(\mathbb{C}) \to \check{\mathcal{Z}}$. We form its pull-back $\beta^*(\mathcal{Q}) \to \mathcal{Z}$ by β. It still carries an equivariant action of $G(\mathbb{Q})$ on the left. Then, our $Q(\mathbb{C})$-torsor over S_U is given by

$$\widetilde{\mathcal{T}}_H = G(\mathbb{Q})\backslash \beta^*(\mathcal{Q}) \times G_f/U.$$

Let us compare the functor $F_{\mathbb{C}}$ with the transcendental construction: From the definition of $\widetilde{\mathcal{T}}_H$, it is clear that for any $V \in \mathbf{Rep}_{\mathbb{C}}(Q)$,

$$\widetilde{\mathcal{V}} = \widetilde{\mathcal{T}}_H \overset{Q(\mathbb{C})}{\times} V.$$

Moreover, there is a canonical isomorphism $\widetilde{\mathcal{T}}_H \cong \mathcal{T}_H$ of holomorphic $Q(\mathbb{C})$-bundles. Indeed, the pull-back by $\mathcal{Z}' \to S_U$ of $\mathcal{T}_H$

$$\underline{\mathrm{Isom}}_{\mathcal{Z}'}(\beta^*\mathcal{V}_{\mathrm{st}}, \beta^*\mathcal{V}_{\mathrm{st}}) = \beta^*\mathcal{Q} \times G_f$$

hence, by quotienting, the desired isomorphism.

Fact. — *In the construction $V \mapsto \mathcal{V}$ of the coherent sheaf attached to a Q-representation, the H-filtration defined above gives rise to a decreasing filtration on $\mathcal{V}$. When V is a G-representation, it coincides with the Hodge filtration given by F_{h_0}.*

Proof. — Consider the dual filtration

$$\mathrm{Fil}^i\, \mathcal{V}^\vee = \{\varphi : \mathcal{V} \longrightarrow \mathcal{O}_X \mid \varphi(\mathrm{Fil}^j \mathcal{V}) \subset \mathrm{Fil}^{i+j}\mathcal{O}_X\} \tag{5.3.2.1}$$

where the unit object $\mathcal{O}_X$ is endowed with the trivial filtration: $\mathrm{Fil}^0\mathcal{O}_X = \mathcal{O}_X$ and $\mathrm{Fil}^j\mathcal{O}_X = 0$ for any $j > 0$; When V is the complex standard representation $V_{\mathrm{st}} \otimes \mathbb{C}$

of $G_{\mathbb{C}}$, the dual of the H-filtration coincides with the Hodge filtration (given by F_{h_0}) on $\mathcal{V}^\vee$, indeed, the dual of the H-filtration reads:

$$\begin{aligned}&\mathrm{Fil}^0\mathcal{V}^\vee = \{\varphi \mid \varphi(\mathrm{Fil}^1\mathcal{V}) = 0\} = \mathcal{V}^\vee,\\ &\mathrm{Fil}^1\mathcal{V}^\vee = \{\varphi \mid \varphi(\mathrm{Fil}^0\mathcal{V}) = 0\} = \mathcal{V}_1^\vee, \quad \text{and}\\ &\mathrm{Fil}^2\mathcal{V}^\vee = 0.\end{aligned} \tag{5.3.2.2}$$

This is the Hodge filtration (5.3.1).

Finally, we note that this filtration is compatible with tensor product, duality, etc.

5.3.4. Filtered dual BGG complex. — Let us define the dual BGG complexes $\overline{\mathcal{K}}^\bullet_{\lambda,n}$ and $\overline{\mathcal{K}}^{\bullet,\mathrm{sub}}_{\lambda,n}$. Their graded pieces are the coherent sheaves over $\overline{X}_n$:

$$\overline{\mathcal{K}}^i_{\lambda,n} = \bigoplus_{\substack{w\in W^M\\ l(w)=i}} \overline{\mathcal{W}}^\vee_{w(\lambda+\rho)-\rho,n} \quad \text{resp.} \quad \overline{\mathcal{K}}^{i,\mathrm{sub}}_{\lambda,n} = \bigoplus_{\substack{w\in W^M\\ l(w)=i}} \overline{\mathcal{W}}^{\mathrm{sub},\vee}_{w(\lambda+\rho)-\rho,n}$$

with $\overline{w}^{\mathrm{sub}} = \overline{w} \otimes \mathcal{I}(\infty)$ where $\mathcal{I}(\infty) \subset \mathcal{O}_{\overline{X}}$ denotes the ideal of definition of the divisor at infinity of $\overline{X}$, and the differentials are deduced by lemma 11 (Sect. 5.2.5) from the BGG complex of Sect. 5.1.2. By dualizing the H-filtration, we obtain a natural decreasing filtration on $\overline{\mathcal{K}}^\bullet_{\lambda,n}$, stable by the differentials, given by

$$F^i\overline{\mathcal{K}}^\bullet_{\lambda,n} = \bigoplus_{\substack{w\in W^M\\ w(\lambda+\rho)(H)+i\leqslant\rho(H)}} \overline{\mathcal{W}}^\vee_{w(\lambda+\rho)-\rho,n}$$

Recall that by the Theorem of [**61**], the map j has a retraction of filtered complexes, hence the dual $j^\vee$ has a natural section; its sheafification defines an injection of complexes of coherent $\mathcal{O}_{\overline{X}_n}$-modules:

$$\kappa : \overline{\mathcal{K}}^\bullet_{\lambda,n} = \bigoplus_{w\in W^M} \overline{\mathcal{W}}^\vee_{w(\lambda+\rho)-\rho,n} \hookrightarrow \overline{\mathcal{V}}^\vee_{\lambda,n} \otimes_{\mathcal{O}_{\overline{X}_n}} \Omega^\bullet_{\overline{X}_n/\mathbb{Z}/p^n}(\log\infty)$$

$$\kappa : \overline{\mathcal{K}}^{\bullet,\mathrm{sub}}_{\lambda,n} = \bigoplus_{w\in W^M} \overline{\mathcal{W}}^{\vee,\mathrm{sub}}_{w(\lambda+\rho)-\rho,n} \hookrightarrow \overline{\mathcal{V}}^\vee_{\lambda,n} \otimes_{\mathcal{O}_{\overline{X}_n}} \Omega^\bullet_{\overline{X}_n/\mathbb{Z}/p^n}(-\log\infty)$$

We summarize the considerations of this section in the proposition

Proposition 3. — *The morphism κ of complexes of vector bundles over $\overline{X}_n$ ($n \geqslant 1$) is filtered.*

5.4. BGG resolution. — We denote by $\mathcal{D}_n$ the logarithmic divided power envelope of the diagonal immersion $\overline{X}_n \to \overline{X}_n \widehat{\times}_{\mathbb{Z}/p^n} \overline{X}_n$ where $\overline{X}_n \widehat{\times}_{\mathbb{Z}/p^n} \overline{X}_n$ is the fiber product in the category of logarithmic schemes. Let p_1 and p_2 be the two canonical projections $\mathcal{D}_n \to \overline{X}_n$. Finally, for any B_M-dominant weight μ of M, such that $|\mu| < p$, let $L(\overline{\mathcal{W}}_{\mu,n})$ be the logarithmic crystal on $(\overline{X}/\mathbb{Z}/p^n)^{\log}_{\mathrm{crys}}$ corresponding to $p_1^*\overline{\mathcal{W}}_{\mu,n}$ (Sect. 4.2 for L and 5.2 for $\overline{\mathcal{W}}_{\mu,n}$). For simplicity, in the sequel, we drop the index n in the notations of the sheaves, thus we write $\mathcal{W}_\mu$ for $\mathcal{W}_{\mu,n}$. Note that we cannot consider the situation over $\mathbb{Z}_p$ because we need a nilpotent base for our crystalline arguments.

Proposition 4. — *Let λ be a B-dominant weight of G, such that $|\lambda+\rho|<p$;*

(i) *There is a resolution in the category of logarithmic crystals on $(\overline{X}_0/(\mathbb{Z}/p^n\mathbb{Z}))^{\log}_{\rm crys}$:*

$$0 \longrightarrow \overline{\mathcal{V}}_\lambda^\vee \longrightarrow L(\overline{\mathcal{K}}^0_\lambda) \longrightarrow L(\overline{\mathcal{K}}^1_\lambda) \longrightarrow \cdots$$

where

$$\overline{\mathcal{K}}^i_\lambda = \bigoplus_{\substack{w\in W^M\\ l(w)=i}} \overline{\mathcal{W}}^\vee_{w(\lambda+\rho)-\rho}.$$

(ii) *There is a canonical filtered quasi-isomorphism of complexes of logarithmic crystals*

$$L(\overline{\mathcal{K}}^\bullet_\lambda) \longrightarrow L(\overline{\mathcal{V}}_\lambda^\vee \otimes_{\mathcal{O}_{\overline{X}_n}} \Omega^\bullet_{\overline{X}_n/\mathbb{Z}/p^n}(\log\infty)).$$

Proof. — We transpose the proof given in [**13**], VI, Sect. 5 for the complex case in a $\mathbb{Z}_p$-setting. By Lemma 11, each $\mathfrak{g}_{\mathbb{Z}_{(p)}}$-morphism of order 1:

$$\mathcal{U}\otimes_{\mathcal{U}(\mathfrak{q})} W_1 \longrightarrow \mathcal{U}\otimes_{\mathcal{U}(\mathfrak{q})} W_2$$

induces a logarithmic differential operator of order 1, $\overline{\mathcal{W}}_2^\vee \to \overline{\mathcal{W}}_1^\vee$ for the corresponding locally free $\mathcal{O}_{\overline{X}_n}$-module; therefore, it induces a morphism of crystals $L(\overline{\mathcal{W}}_2^\vee) \to L(\overline{\mathcal{W}}_1^\vee)$. We deduce from theorem 5 (section 5.1.2), that there is a complex of crystals

$$0 \longrightarrow \overline{\mathcal{V}}_\lambda^\vee \longrightarrow L(\overline{\mathcal{K}}^0_\lambda) \longrightarrow L(\overline{\mathcal{K}}^1_\lambda) \longrightarrow \cdots .$$

On the other hand, we know that

$$0 \longrightarrow \overline{\mathcal{V}}_\lambda^\vee \longrightarrow L(\overline{\mathcal{V}}_\lambda^\vee \otimes_{\mathcal{O}_{\overline{X}_n}} \Omega^\bullet_{\overline{X}_n/\mathbb{Z}/p^n}(\log\infty))$$

is a resolution of $\overline{\mathcal{V}}_\lambda^\vee$. Indeed, the exactness of the complex is the crystalline Poincaré's lemma (actually, its logarithmic version: bottom of p. 221 of [**48**], see our section 4.2, lemma 4 above).

By Theorem D of [**61**] (Theorem 5 of section 5.1.2 here), $L(\overline{\mathcal{K}}^\bullet_\lambda)$ is a direct summand, as subcomplex, of $L(\overline{\mathcal{V}}_\lambda^\vee \otimes_{\mathcal{O}_{\overline{X}_n}} \Omega^\bullet_{\overline{X}_n/\mathbb{Z}/p^n}(\log\infty))$.

Therefore, $L(\overline{\mathcal{K}}^\bullet_\lambda)$ is a resolution of $\overline{\mathcal{V}}_\lambda^\vee$. This proves statement (i) of the theorem. The second assertion follows from the fact that H commutes with $Z\mathfrak{g}$. As explained in Section 5.1.2 above.

Theorem 6. — *The natural morphisms*

$$\overline{\mathcal{K}}^\bullet_\lambda \longrightarrow \overline{\mathcal{V}}_\lambda^\vee \otimes_{\mathcal{O}_{\overline{X}_n}} \Omega^\bullet_{\overline{X}_n/\mathbb{Z}/p^n}(\log\infty)$$

and

$$\overline{\mathcal{K}}^{\bullet,\rm sub}_\lambda \longrightarrow \overline{\mathcal{V}}_\lambda^\vee \otimes_{\mathcal{O}_{\overline{X}_n}} \Omega^\bullet_{\overline{X}_n/\mathbb{Z}/p^n}(-\log\infty)$$

are filtered quasi-isomorphisms of complexes of coherent sheaves on $\overline{X}_n$.

Proof. — One applies Ru_* resp. $Ru_{*,c}$ to both members of the quasi-isomorphism (ii) of Prop. 4; then one makes use of the fact that $Ru_*L(\mathcal{V}) \cong \mathcal{V}$ for any $\mathcal{O}_{\overline{X}_n}$-module $\mathcal{V}$ and the properties of the L-construction recalled in Section 4.2.

6. Modulo p crystalline representations

6.1. Etale sheaves associated to crystals. — Let k be a perfect field of char. $p > 0$, $W = W(k)$ the ring of Witt vectors with coefficients in k and K the fraction field of W. K^{ac} is a fixed algebraic closure of K and $G_K = \mathrm{Gal}(K^{\mathrm{ac}}/K)$ is the associated Galois group. Let $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ be the category of G_K-modules of finite type over $\mathbb{Z}_p$ and $MF_W^{[0,p-2]}$ that of finitely generated W-modules M endowed with a filtration $(\mathrm{Fil}^r M)_r$ such that $\mathrm{Fil}^r M$ is a direct factor, $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{p-1} = 0$ together with semi-linear maps $\varphi^r : \mathrm{Fil}^r M \to M$ such that the restriction of φ^r to $\mathrm{Fil}^{r+1} M$ is equal to $p\varphi^{r+1}$ and satisfying the strong divisibility condition: $M = \sum_{i\in\mathbb{Z}} \varphi^r(\mathrm{Fil}^r M)$. Recall that by the theory of Fontaine-Laffaille [**24**], we have a fully faithful covariant functor

$$V_{\mathrm{crys}} : MF_W^{[0,p-2]} \longrightarrow \mathrm{Rep}_{\mathbb{Z}_p}(G_K)$$

This functor has the property that it sends the filtered Tate object of unique Hodge-Tate weight $-i$ (meaning the jumps of the Hodge-filtration) to the Tate module $\mathbb{Z}_p(i)$ and for any abelian variety defined over $\mathbb{Q}_p$,

$$H^1_{\text{ét}}(A \times \overline{\mathbb{Q}_p}, \mathbb{Z}_p)$$

has weights 0 and 1.

The contravariant functor V^*_{crys} obtained by composing V_{crys} with duality is the nice inverse of a not so nice contravariant Dieudonné functor $\mathbf{D}^*$: see [**83**] p. 219-223.

A p-adic representation is called of Fontaine-Laffaille type (or crystalline, by abuse of language) if it is in the essential image of V^*_{crys}.

In our setting, we are interested in the subcategory $MF_k^{[0,p-2]}$ of filtered modules M such that $pM = 0$. $MF_k^{[0,p-2]}$ is an abelian category and the objects are in particular k-vector spaces. The restriction of the functor V^*_{crys} to $MF_k^{[0,p-2]}$ can be describe as follows: Let $S = \mathcal{O}_{K^{\mathrm{ac}}}/p\mathcal{O}_{K^{\mathrm{ac}}}$, choose $\beta \in K^{\mathrm{ac}}$ such that $\beta^p = -p$ and for $i < p$, define a filtration $\mathrm{Fil}^i S = \beta^i S$ and Frobenius $\varphi^i(\beta^i x) = x^p$, then as explained in [**83**], Prop. 2.3.1.2', we have an isomorphism

$$V^*_{\mathrm{crys}}(M) \simeq \mathrm{Hom}_{MF_k^{[0,p-2]}}(M, S)$$

Moreover, $V^*_{\mathrm{crys}}(M)$ is a finite dimension $\mathbb{F}_p$-vector space and $\dim_{\mathbb{F}_p} V^*_{\mathrm{crys}}(M) = \dim_k M$.

Let $\overline{X}$ be a smooth and proper scheme over W of relative dimension d and D a relative divisor with normal crossings of $\overline{X}$, we put $X = \overline{X} - D$. Faltings introduced in [**22**] relative versions of the categories mentioned above: the category $\mathcal{R}ep_{\mathbb{Z}_p}(X \otimes K)$ of étales $\mathbb{Z}_p$-sheaves over the generic fiber $X \otimes K$ and the category $\mathcal{M}F^{\nabla}(\overline{X})$ of filtered transversal logarithmic crystals over $\overline{X}$. Moreover, we have a notion of "associated" between objects of $\mathcal{R}ep_{\mathbb{Z}_p}(X \otimes K)$ and those of $\mathcal{M}F^{\nabla}(\overline{X})$. To get a good theory over $\mathbb{Z}_p$, we need to consider only the full subcategory $\mathcal{M}F^{\nabla,[0,p-2]}(\overline{X})$ of $\mathcal{M}F^{\nabla}(\overline{X})$ of filtered crystals $\mathcal{F}$ such that $\mathrm{Fil}^0\mathcal{F} = \mathcal{F}$ and $\mathrm{Fil}^{p-1}\mathcal{F} = 0$ and we have to add some other technical hypothesis (*cf.* Sect. 6.2). Faltings [**22**] (see also [**78**]) has defined a relative contravariant Fontaine functor

$$\mathbf{V}^* : \mathcal{M}F^{\nabla,[0,p-2]}(\overline{X}) \longrightarrow \mathcal{R}ep_{\mathbb{Z}_p}(X \otimes K)$$

In section 6.2 below, we will recall its definition on the objects of p-torsion.

Definition 6. — For any $\mathcal{F} \in \mathcal{M}F^{\nabla,[0,p-2]}(\overline{X})$, we say that $\mathcal{F}$ and $\mathbf{V}^*(\mathcal{F})$ are associated.

We have the following theorem of Faltings ([**22**] Th. 5.3):

Theorem 7. — *Let $\mathcal{F} \in \mathcal{M}F^{\nabla,[0,p-2]}(\overline{X})$. Let $a \in [0, p-2]$ such that* $\mathrm{Fil}^{a+1}\mathcal{F} = 0$. *Then, for any $i \geqslant 0$, such that $i + a \leqslant p - 2$, there is a natural and functorial isomorphism of G_K-modules:*

$$\big(H^i_{\mathrm{et}}(X \otimes K^{\mathrm{ac}}, \mathbf{V}^*(\mathcal{F}))\big)^* \cong V^*_{\mathrm{crys}}(H^i_{\text{log-crys}}(\overline{X}, \mathcal{F}))$$

6.2. The mod. p case. — As we use only the mod. p version of the previous comparison theorem, we only recall the notion of associated sheaves and the comparison theorem in their mod. p version, following [**22**] and [**78**].

6.2.1. Filtered modules. — Let k be a perfect field of char. $p > 0$, $W = W(k)$ the ring of Witt vectors with coefficients in k and K the fraction field of W. K^{ac} is a fixed algebraic closure of K and $G_K = \mathrm{Gal}(K^{\mathrm{ac}}/K)$ is the associated Galois group.

Let $\overline{X}$ be a smooth and proper scheme over W of relative dimension d and D a relative divisor with normal crossings of $\overline{X}$, we put $X = \overline{X} - D$. Let $\overline{X}_0 = \overline{X} \otimes_W k$ be the special fiber of $\overline{X}$ and D_0 the induced divisor. If $F_{X_0} : \mathcal{O}_{\overline{X}_0} \to \mathcal{O}_{\overline{X}_0}$ is the absolute Frobenius, we denote by

$$\varphi_{\overline{X}_0} : F^{-1}_{\overline{X}_0}(\mathcal{O}_{\overline{X}_0}) \longrightarrow \mathcal{O}_{\overline{X}_0}$$

the $\mathcal{O}_{\overline{X}_0}$-linear homomorphism induced by $F_{\overline{X}_0}$.

We fix a global lifting $\widetilde{\varphi}_{\overline{X}_0}$ of $\varphi_{\overline{X}_0}$ on $\overline{X} \times_W W_2$. The differential

$$d\widetilde{\varphi}_{\overline{X}_0} : \mathcal{O}_{\overline{X}_0} \longrightarrow \Omega^1_{\overline{X}_0}(\log D_0)$$

is divisible by p. We denote by $d\varphi_{\overline{X}_0}/p$ the reduction mod. p of $d\widetilde{\varphi}_{\overline{X}_0}/p$.

Definition 7. — We define the category $\mathcal{M}F_k^{\nabla,[0,p-2]}(\overline{X}_0)$ of strongly divisible filtered logarithmic modules over $\overline{X}_0$ with Hodge-Tate weights between 0 and $p-2$ as follows: an object is a quadruple $(\mathcal{F}, \mathcal{F}^i, \varphi^i_{\mathcal{F}}, \nabla_{\mathcal{F}})$ where

– $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\overline{X}_0}$-module.

– $\mathcal{F}^i$, $i = 0, \ldots, p-1$, is a decreasing filtration of $\mathcal{F}$ by quasi-coherent $\mathcal{O}_{\overline{X}_0}$-modules such that $\mathcal{F}^0 = \mathcal{F}$ and $\mathcal{F}^{p-1} = 0$.

– $\varphi^i_{\mathcal{F}} : \mathcal{F}^i \to \mathcal{F}$ is a $\varphi_{\overline{X}_0}$-linear homomorphism such that the restriction of $\varphi^i_{\mathcal{F}}$ to $\mathcal{F}^{i+1}$ is zero and such that the induced map

$$\oplus_i \varphi^i_{\mathcal{F}} : \oplus \mathcal{F}^i/\mathcal{F}^{i+1} \longrightarrow \mathcal{F}$$

is an isomorphism (condition of strong divisibility).

– $\nabla_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{\overline{X}_0}} \Omega^1_{\overline{X}_0}(\log D_0)$ is a quasi-nilpotent integrable connection satisfying

1) Griffiths transversality: $\nabla_{\mathcal{F}}(\mathcal{F}^i) \subset \mathcal{F}^{i-1} \otimes_{\mathcal{O}_{\overline{X}_0}} \Omega^1_{\overline{X}_0}(\log D_0)$ for $i = 0, \ldots, p-1$.

2) Compatibility with Frobenius: $\nabla_{\mathcal{F}} \circ \varphi^i_{\mathcal{F}} = \varphi^{i-1}_{\mathcal{F}} \otimes \dfrac{d\varphi_{\overline{X}_0}}{p} \circ \nabla_{\mathcal{F}} | \mathcal{F}^i$.

– $\mathcal{F}$ is uniform: there is an étale covering $(\overline{U}_\alpha)$ of $\overline{X}_0$ together with a log-immersion $\overline{U}_\alpha \to \overline{Z}_\alpha$ with $\overline{Z}_\alpha$ log-smooth and such that the evaluation of the filtered crystal associated to $(\mathcal{F}, \mathcal{F}^i)$ on the thickenings $\overline{U}_\alpha \hookrightarrow \overline{Z}^{DP}_\alpha$ is isomorphic to

$$\bigoplus_{\lambda \in \Lambda} (\mathcal{O}_{\overline{Z}^{DP}_\alpha}, J^{[i-e_\lambda]}_{\overline{Z}^{DP}_\alpha}) \quad \text{with } e_\lambda \geqslant 0,\ |\Lambda| < \infty$$

where $\overline{Z}^{DP}_\alpha$ is the log-divided power envelope of the immersion $\overline{U}_\alpha \to \overline{Z}_\alpha$ and $J_{\overline{Z}^{DP}_\alpha}$ is the corresponding PD-ideal.

Remark. — The uniformity condition is introduced in Sect. 4.f of [**23**]. It is needed to check that the category is abelian.

A morphism of $\mathcal{M}F^{\nabla,[0,p-2]}_k(\overline{X}_0)$ is an $\mathcal{O}_{\overline{X}_0}$-linear homomorphism compatible with filtrations and commuting with Frobenius and connections.

By [**22**], Th. 2.1, each $\mathcal{F}^i$ is locally free and locally (for the Zariski topology) a direct factor of $\mathcal{F}$. Moreover, any morphism of $\mathcal{M}F^{\nabla,[0,p-2]}_k(\overline{X}_0)$ is strict with respect the filtrations. We deduce from this that $\mathcal{M}F^{\nabla,[0,p-2]}_k(\overline{X}_0)$ is an abelian category.

6.2.2. The functor $\mathbf{V}^*$. — To a filtered module $\mathcal{F}$ as above, we associate an étale sheaf $\mathbf{V}(\mathcal{F})$ over $X \otimes K$ as follows:
Let $\overline{U} = \operatorname{Spec}(R)$ be an affine open irreducible subset of $\overline{X}$, $U = \overline{U} \times_{\overline{X}} X$, $\overline{U}_0 = \overline{U} \otimes_W k$. Recall that R is flat, of finite type over W (since $\overline{X}$ is smooth over W); assume that $R/pR \neq 0$. Let $\widehat{R}$ be the p-adic completion of R and $\widehat{R}'$ be the union of all normalizations of $\widehat{R}$ in finite sub-Galois extensions of an algebraic closure $\operatorname{Fr}(\widehat{R})^{\mathrm{ac}}$ of the field of fractions $\operatorname{Fr}(\widehat{R})$ of $\widehat{R}$ such that the normalization of $\widehat{R}[1/p]$ in such finite extension is unramified outside D (*cf.* [**22**], II, i)). On $\overline{U}'_0 = \operatorname{Spec}(\widehat{R}'/p\widehat{R}')$, we have a canonical log-structure defined as follows. Let S be the normalization of $\widehat{R}$ in a finite Galois extension of $\operatorname{Fr}(\widehat{R})$ in $\operatorname{Fr}(\widehat{R})^{\mathrm{ac}}$. The inverse image of the divisor D_0 defines a log-structure on $\operatorname{Spec} S/pS$. By passing to the inverse limit, we obtain a log-structure on $\overline{U}'_0$.

Let $(\mathcal{F}, \mathcal{F}^i, \varphi^i_{\mathcal{F}}, \nabla_{\mathcal{F}})$ be an object of $\mathcal{MF}_k^{\nabla,[0,p-2]}(\overline{X}_0)$. As a crystal, we can evaluate $\mathcal{F}$ on the trivial thickening $\overline{U}'_0 \hookrightarrow \overline{U}'_0$. We obtain an $\mathcal{O}_{\overline{U}'_0}$-module $\mathcal{F}_{\overline{U}'_0}$ endowed with a decreasing filtration $\mathcal{F}^i_{\overline{U}'_0}$.

For $i < p$, we define the $\mathrm{Gal}(\widehat{R}'/\widehat{R})$-module $\mathbf{V}_U(\mathcal{F}, i)$ as the kernel of

$$1 - \varphi^i : H^0(\overline{U}'_0, \mathcal{F}^i_{\overline{U}'_0}) \longrightarrow H^0(\overline{U}'_0, \mathcal{F}_{\overline{U}'_0})$$

Let $E = \widehat{R}'/p\widehat{R}'$; choose $\beta \in K^{\mathrm{ac}}$ such that $\beta^p = -p$ and for $i < p$, define a filtration $\mathrm{Fil}^i E = \beta^i E$ and Frobenius $\varphi^i(\beta^i x) = x^p$, then as explained in [**78**] proof of prop. 4.3.4 or [**22**], II, f), we have an isomorphism

$$\mathbf{V}_U(\mathcal{F}, i)^* \simeq \mathrm{Hom}_{\mathrm{fil},\varphi}(\mathcal{F}[i], E),$$

where:

– $\mathrm{Hom}_{\mathrm{fil},\varphi}$ denotes the group of homomorphisms preserving the filtrations and commuting to Frobenius,

– $\mathcal{F}[i]$ is the twisted module defined by $\mathcal{F}[i]^j = \mathcal{F}^{i+j}$ and $\varphi^j_{\mathcal{F}[i]} = \varphi^{i+j}_{\mathcal{F}}$.

Using this description, we deduce that $\mathbf{V}_U(\mathcal{F}, i)$ is finite of order p^h ([**22**], Th. 2.4) where $h = |\Lambda|$ and Λ is the index set in the definition of a uniform filtered module.

By [**22**], II, g) or [**78**](4.4), if we regard $\mathbf{V}_U(\mathcal{F}, i)$ as a finite locally constant sheaf on $(U \otimes_W K)_{\text{ét}}$, we can glue the local data $\mathbf{V}_U(\mathcal{F}, i)$, for various "small" U (*cf.* [**78**] 3.3.2). There is a unique finite locally constant sheaf $\mathbf{V}_X(\mathcal{F}, i)$ on $X \otimes_W K$ such that the restriction to "small" U is $\mathbf{V}_U(\mathcal{F}, i)$. Finally, we define the covariant comparison functor $\mathbf{V}$ by $\mathbf{V}(\mathcal{F}) = \mathbf{V}_X(\mathcal{F}, p-2)(2-p)$, and its contravariant version $\mathbf{V}^*$ by $\mathbf{V}^*(\mathcal{F}) = \mathbf{V}(\mathcal{F})^*$.

6.3. Association modulo p for Siegel varieties. — Let us come back to the case of Siegel varieties. Let $X_{/\mathbb{Z}[1/N]}$ be the moduli scheme classifying p.p.a.v. with level U-structure over $\mathbb{Z}[1/N]$. Its toroidal compactification over $\mathbb{Z}[1/N]$ is denoted by $\overline{X}$ (for some choice of a smooth $\mathrm{GL}(\mathbb{Z}^g)$-admissible polyhedral cone decomposition of the convex cone of all positive semi-definite symetric bilinear forms on $\mathbb{R}^g$). We have $S_U = X \otimes_{\mathbb{Z}[1/N]} \mathbb{C}$. Recall that, to the representation $V_{\lambda/\mathbb{F}_p}$ of $G_{\mathbb{F}_p} = G \otimes \mathbb{F}_p$ of highest weight λ, one can associate an etale sheaf $V_\lambda(\mathbb{F}_p)$ resp. $V_\lambda(k)$ over $X \otimes \mathbb{Q}$ resp. its extension of scalars to k. One possible construction of this etale sheaf is by the theory of the fundamental group: any representation of the arithmetic fundamental group $\pi_1(X \otimes \mathbb{Q}, \overline{x})$ on a finite abelian group V gives rise to an etale sheaf whose fiber at $\overline{x}$ is V. Let us consider the structural map $f : A \to X \otimes \mathbb{Q}$ given by the universal principally polarized abelian surface with level structure of type U (we assume here U sufficiently deep). The sheaf $R^1 f_* \mathbb{Z}/p\mathbb{Z}$ is étale. It corresponds to an antirepresentation of the fundamental group taking values in $G(\mathbb{Z}/p\mathbb{Z})$. Then, composing with the representation $G_{\mathbb{F}_p} \to \mathrm{GL}(V_{\lambda/\mathbb{F}_p})$, we obtain an étale sheaf denoted by $V_\lambda(\mathbb{F}_p)$. Similarly for $V_\lambda(k)$, by considering the extension of scalars from $\mathbb{F}_p$ to k: $G_k \to \mathrm{GL}_k(V_\lambda(k))$.

For any dominant weight λ of G, we have thus obtained a $V_\lambda(\mathbb{F}_p)$ of $\mathcal{R}ep_{\mathbb{F}_p}(X \otimes K)$. On the other hand, if moreover $|\lambda+\rho| < p-1$, the crystal $\overline{\mathcal{V}}_\lambda^\vee$ constructed in Section 5.2 satisfies the conditions of Definition 7 which turn it into an object of $\mathcal{M}F^{\nabla,[0,p-2]}(\overline{X}_0)$. To verify this, one starts with the standard representation. Consider

$$\overline{\mathcal{V}}_1^\vee = R^1\overline{f}_*\Omega^\bullet_{\overline{A}/\overline{X}}(\log \infty_{\overline{A}/\overline{X}}),$$

On $\overline{\mathcal{V}}_1^\vee \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_{\overline{X}_0}$, the Gauss-Manin connection satisfies Griffiths transversality for the Hodge filtration, compatibility to Frobenius and uniformity. A delicate point is to verify the strong divisibility condition (section 6.2, definition 7). It follows from the degeneracy of the Hodge spectral sequence which is proven in [**22**], Th. 6.2. As for the uniformity condition, it amounts to saying that $R^1\overline{f}_*\Omega^\bullet_{\overline{A}_0/\overline{X}_0}(\log \infty_{\overline{A}_0/\overline{X}_0})$ is indeed a vector bundle over $\overline{X}_0$.

For general λ, we use that $\overline{\mathcal{V}}_\lambda^\vee$ is a sub-object (and quotient) of a first direct image for some Kuga-Sato variety and the fact that $\mathcal{M}F^{\nabla,[0,p-2]}(\overline{X}_0)$ is an abelian category. Note that the objects $\overline{\mathcal{V}}_\lambda \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_{\overline{X}_0}$ (without dualizing) do not belong to this category, as their weights don't fit the bound.

***Theorem 8* ([13] Th. 6.2(iii))**. — $\mathbf{V}^*(\overline{\mathcal{V}}_\lambda^\vee \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_{\overline{X}_0}) = V_\lambda(\mathbb{F}_p)$, *that is,* $V_\lambda(\mathbb{F}_p)$ *and* $\overline{\mathcal{V}}_\lambda^\vee \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_{\overline{X}_0}$ *are associated*

The proof is given in [**13**] Th. 6.2(iii). In fact, there, the result is proven only in the $\mathbb{Q}_p$-coefficients case, but for $|\lambda+\rho| < p-1$ the proof is valid word for word in the integral context. The key argument is the existence of the minimal compactification whose boundary has relative codimension $\geqslant 2$. The next section gives more details about this.

6.4. The Comparison Theorem. — We will explain the relative comparison theorem Th. 6.2 of Faltings [**22**] in our particular setting. In fact we merely extend the arguments sketched in [**13**], p. 241. Before going into our situation, we recall the method of [**22**] (we hope that more details will be given by the experts in the future).

6.4.1. General setting. — Let $\widehat{R}$ be a p-adically complete smooth domain over $\mathbb{Z}_p$. Let $R_0 = \widehat{R} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p\mathbb{Z}$ its reduction mod. p; let F be the field of fractions of $\widehat{R}$; choose an algebraic closure $\overline{F}$ of F and form $\overline{\widehat{R}}$, union of all the normalizations of $\widehat{R}$ in finite sub-Galois extensions of $\overline{F}$. Put $S = \overline{\widehat{R}}/p\overline{\widehat{R}}$.

Let $f : Y \to \mathrm{Spec}(\widehat{R})$ be a smooth and proper morphism of schemes of relative dimension $d < p-1$, $Y_0 = Y \otimes_{\mathbb{Z}_p} \mathbb{Z}/p\mathbb{Z}$ the special fiber, $\overline{Y} = Y \otimes_R \overline{\widehat{R}}$, $\overline{Y}_\eta = Y \otimes_R \overline{F}$, $\overline{Y}_0 = Y_0 \otimes_{R_0} S$ and $f_0 : Y_0 \to \mathrm{Spec}(R_0)$, $\overline{f} : \overline{Y} \to \mathrm{Spec}(\overline{\widehat{R}})$, $\overline{f}_\eta : \overline{Y}_\eta \to \mathrm{Spec}(\overline{F})$, $\overline{f}_0 : \overline{Y}_0 \to \mathrm{Spec}(S)$ the corresponding morphisms. We have the following standard diagram:

$$\begin{tikzcd}
\overline{Y}_\eta \arrow[r, hook, "\overline{j}"] \arrow[d, "\overline{f}_\eta"'] & \overline{Y} \arrow[d, "\overline{f}"] & \overline{Y}_0 \arrow[l, hook', "\overline{i}"'] \arrow[d, "\overline{f}_0"] \\
\mathrm{Spec}(\overline{F}) \arrow[r, hook] & \mathrm{Spec}(\overline{\widehat{R}}) & \mathrm{Spec}(S) \arrow[l, hook']
\end{tikzcd}$$

Let $R\Psi(S(1)) = \overline{i}^* R\overline{j}_*(S(1))$ be the "relative complex of p-adic vanishing cycles" for the constant sheaf $S(1) = \mathbb{Z}/p\mathbb{Z}(1)\otimes S$. This object is not explicitely introduced in [**22**], but as explained in [**45**], we can rewrite the complex computing étale cohomology as a complex of vanishing cycles. Then we have a "Kummer" map:

$$R\Psi(S(1)) \longrightarrow \Omega^\bullet_{\overline{Y}_0/\operatorname{Spec}(S)}.$$

Taking direct images, we obtain natural maps:

$$R^* f_{0*}(\Omega^\bullet_{Y_0/\operatorname{Spec}(R_0)}) \otimes_R S \longrightarrow R^*\overline{f}_{0*}(\Omega^\bullet_{\overline{Y}_0/\operatorname{Spec}(S)}) \longleftarrow R^*\overline{f}_{0*,\text{ét}}(R\Psi) \simeq R^*\overline{f}_{\eta*,\text{ét}}(S)$$

$$R^*\overline{f}_{0*,\text{ét}}(R\Psi) \simeq R^*\overline{f}_{\eta*,\text{ét}}(S) \longleftarrow R^*\overline{f}_{\eta*,\text{ét}}(\mathbb{Z}/p\mathbb{Z}(1)) \otimes_R S.$$

Faltings ([**22**], page 72, see also recent corrections of the corresponding proof in informal notes by the author) shows that the second arrow is an "almost-isomorphism"; wich implies that the modules $R^* f_{0*}(\Omega^\bullet_{Y_0/\operatorname{Spec}(R_0)})$ and $R^*\overline{f}_{\eta*,\text{ét}}(\mathbb{Z}/p\mathbb{Z}(1))$ are associated.

6.4.2. Setting for Siegel varieties. — The notations are those of section 6.3. Let $U = \operatorname{Spec}(R) \subset X$ be an affine open subset and $f : Y_U \to U$ be the restriction of $f_s : Y = A \times_X \cdots \times_X A \to X$, where A is the universal abelian variety, we assume $s < p-1$. Let $\widehat{X}$ be the formal completion of X along the special fiber. Let $\widehat{f} : \widehat{Y}_U \to \widehat{U}$ be the base change of f to the affine formal scheme $\widehat{U} = \operatorname{Spf}(\widehat{R})$. Over $\operatorname{Spec}(\widehat{R}\otimes\mathbb{Q}_p)$, we have two étales sheaves $R^s\widehat{f}_*\mathbb{Z}/p\mathbb{Z}(1)$ and $\mathbf{V}^*(R^s\widehat{f}_*(\Omega^\bullet_{Y_U\otimes\mathbb{F}_p/U\otimes\mathbb{F}_p}))$. As explained in the general setting subsection, there is a functorial isomorphism of étales sheaves:

$$R^s\widehat{f}_*\mathbb{Z}/p\mathbb{Z}(1) \simeq \mathbf{V}^*(R^s\widehat{f}_*(\Omega^\bullet_{Y_U\otimes\mathbb{F}_p/U\otimes\mathbb{F}_p}))$$

over $\widehat{U}$. By functoriality, these local isomorphisms glue to a global one over $\widehat{X}$.

Let X^* be the minimal compactification of X over $\mathbb{Z}_p$. It is defined in [**13**] Th. 2.5 Chapter V. It is projective, normal of finite type; its boundary admits a natural stratification whose strata have codimension at least 2 (since we assume $g \geqslant 2$). We apply Grothendieck's GAGA theorem to deduce that the isomorphism over $\widehat{X}$ between the sheaves $R^s\widehat{f}_*\mathbb{Z}/p\mathbb{Z}(1)$ and $\mathbf{V}^*(R^s\widehat{f}_*(\Omega^\bullet_{Y_U\otimes\mathbb{F}_p/U\otimes\mathbb{F}_p}))$ is algebraic. More precisely, every étale covering of the formal scheme $\widehat{X}$ is defined by an étale finite $\mathcal{O}_{\widehat{X}}$-algebra $\mathcal{A}$. Since the minimal compactification is normal and has boundary of codimension $\geqslant 2$, this algebra extends to $\widehat{X^*}$ ([**33**], Cor 5.11.4) and so defines an algebraic étale covering of X whose base change to $\widehat{X}$ is $\mathcal{A}$, we deduce an equivalence of sites $X_{\text{ét}} \simeq \widehat{X}_{\text{ét}}$. As the morphism f is proper and smooth, the sheaf $R^s\widehat{f}_*\mathbb{Z}/p\mathbb{Z}(1)$ on $\widehat{X}$ is locally constant and so descends to X and gives the sheaf $R^s f_*\mathbb{Z}/p\mathbb{Z}(1)$. By construction, the sheaf $\mathbf{V}^*(R^s\widehat{f}_*(\Omega^\bullet_{Y_U\otimes\mathbb{F}_p/U\otimes\mathbb{F}_p}))$ is also locally constant and also descend to X and gives the sheaf $\mathbf{V}^*(R^s f_*(\Omega^\bullet_{Y_U\otimes\mathbb{F}_p/U\otimes\mathbb{F}_p}))$.

Moreover, as $X_{\text{ét}} \simeq \widehat{X}_{\text{ét}}$, every formal morphism between $R^s\widehat{f}_*\mathbb{Z}/p\mathbb{Z}(1)$ and $\mathbf{V}^*(R^s\widehat{f}_*(\Omega^\bullet_{Y_U\otimes\mathbb{F}_p/U\otimes\mathbb{F}_p}))$ is algebraic. This shows that $R^s f_*\mathbb{Z}/p\mathbb{Z}(1)$ is associated to $R^s f_*(\Omega^\bullet_{Y\otimes\mathbb{F}_p/X\otimes\mathbb{F}_p})$ for the asociation without divisor at infinity and $R^s f_*\mathbb{Z}/p\mathbb{Z}(1)$ is associated to $R^s\overline{f}_*(\Omega^\bullet_{\overline{Y}\otimes\mathbb{F}_p/\overline{X}\otimes\mathbb{F}_p}(\log\infty))$ for the association with divisor at infinity.

7. Proof of Theorem 1

7.1. A lemma on modular representations. — Our reference for results used in this Section are [**12**] VIII.13.2 and [**46**], II.3. Let $\widehat{T}$ be the standard maximal torus in $\widehat{G}$. One has

$$\widehat{T} = \{(t_1,\ldots,t_g,u;x) \mid u^2 = t_1\cdots t_g\}$$

The degree 2 covering $\widehat{G}\to \mathrm{GO}_{2g+1}$ induces on $\widehat{T}$ the projection

$$(t_1,\ldots,t_g,u;x)\longmapsto \mathrm{diag}(t_1,\ldots,t_g,xt_g^{-1},\ldots,xt_1^{-1},x)$$

We view the Weyl group $W_{\widehat{G}}$ as a subgroup of $\widehat{G}_{/\mathbb{Z}}$ by using permutation matrices in a standard way. Let W' be the subgroup of $W_{\widehat{G}}$ consisting in the permutations w_B ($B\subset[1,g]$) acting by $t^{w_B}=t'$ where $t=(t_1,\ldots,t_g,u;x)$ and $t'=(t'_1,\ldots,t'_g,u';x)$ with $t'_i=t_i^{-1}$ if $i\in B$, $t'_i=t_i$ if $i\notin B$, and $u'=u\cdot t_B^{-1}$ where $t_B=\prod_{i\in B}t_i$.

Let $\widehat{B}=\widehat{T}.\widehat{N}$ be the Levi decomposition of the standard Borel subgroup $\widehat{B}$. Recall we assumed $\mathbf{GO}(\omega)$ for $\overline{\rho}_\pi$. We can assume that $\overline{\rho}_\pi(D_p)\subset\widehat{B}(k)$. Throughout this section, we assume that

(RLI) there exists a split (non necessarily connected) reductive Chevalley subgroup H of $\widehat{G}_{/\mathbb{Z}}$ with $W'\propto\widehat{T}\subset H$, and a subfield $k'\subset k$, of order say $|k'|=q'=p^{f'}$ ($f'\geqslant 1$), so that $H(k')_\nu\subset\operatorname{Im}\overline{\rho}_\pi$ and $\overline{\rho}_\pi(I_p)\subset H^0(k')$. Where $H(k')_\nu$ is the subgroup of $H(k')$ consisting in elements whose ν belongs to $\operatorname{Im}\nu\circ\overline{\rho}_\pi$.

Comment. — It has been pointed to us by R. Pink that if H is connected and $W'\propto\widehat{T}\subset H$, then H should contain the derived group of $\widehat{G}$; then, **(RLI)** becomes in some sense an assumption of genericity for π and p, but it cannot be verified in a single example for $g\geqslant 2$, hence our insistance on the possible disconnectedness of H: it allows us to show the existence of concrete examples for the theorem.

Let H^0 be the neutral component of H over $\mathbb{Z}$. Its semisimple rank is g. Recall that in the condition of Galois ordinarity **(GO)**, we introduced an element $\widehat{g}\in\widehat{G}$ so that

$$\rho_\pi(D_p)\subset\widehat{g}\cdot\widehat{B}(\mathcal{O})\cdot\widehat{g}^{-1}$$

Recall the convention (valid since Sect. 3.3) that we omit the conjugation by $\widehat{g}$, thus writing $\widehat{B}$, $\widehat{N}$, $\widehat{T}$ instead of $\widehat{g}\cdot\widehat{B}\cdot\widehat{g}^{-1}$ and so on.

The subdata $(H^0, \widehat{T}, \widehat{B} \cap H^0)$ in $(\widehat{G}, \widehat{T}, \widehat{B})$ induce an inclusion of the set of roots of H^0 into that of $\widehat{G}$: $\Phi^{\pm}_{H^0} \subset \Phi^{\pm}$. Let $\Phi' = \Phi \cap Vect_{\mathbb{Q}}(\Phi_{H^0})$ and Δ' a system of basis made of positive simple roots for Φ'. By [**12**] VI, n° 1.7, Prop. 24, it can be completed into a basis Δ of Φ contained in Φ^+. Note that Φ_{H^0} is a subsystem of maximal rank in Φ'. Let Δ_{H^0} be the basis of Φ_{H^0} contained in $\Phi^+_{H^0}$. A priori, it could be different from Δ' (not in the examples we have in view though). Let

$$\Phi^{\perp}_{H^0} = \{\lambda \in X \mid \langle \lambda, \beta^{\vee} \rangle = 0 \text{ for } \beta \in \Phi_{H^0}\}$$

where $\alpha^{\vee}$ denotes the coroot corresponding to a root β.

Observe that $\Phi^{\perp}_{H^0}$ contains $\mathbb{Z} \cdot \nu$ as a direct summand:

$$\Phi^{\perp}_{H^0} = \Phi^{\perp,1}_{H^0} \oplus \mathbb{Z} \cdot \nu.$$

Let X' be the $\mathbb{Z}$-module generated by Δ'. One has

$$X = X' \oplus \Phi^{\perp}_{H^0}.$$

The irreducible representations of H^0 over k' (or over any perfect extension of $\mathbb{F}_p$) are classified by $X'^{+} \times \Phi^{\perp}_{H^0}$. We shall consider certain (absolutely) irreducible representations over k' of the abstract group $H^0(k')$.

Note that by the formula $\nu \circ \rho_{\pi} = \chi^{-\boldsymbol{w}} \cdot \omega_{\pi}$, the image of $\nu \circ \overline{\rho}_{\pi}$ contains $k'^{\times \boldsymbol{w}}$. Let $e = (k'^{\times} : \mathrm{Im}(\nu \circ \overline{\rho}_{\pi}))$. Note that e is a multiple of $\frac{q'-1}{(\boldsymbol{w}, q'-1)} = (k'^{\times} : k'^{\times \boldsymbol{w}})$.

Let

$$\widetilde{\Phi}^{\perp}_{H^0} = (q'-1) \cdot \Phi^{\perp,1}_{H^0} \oplus e \cdot \mathbb{Z} \cdot \nu$$

It is a finite index lattice in $\Phi^{\perp}_{H^0}$ and the kernel of the homomorphism

$$X \longrightarrow \mathrm{Hom}(\widehat{T}(k')_{\nu}, k'^{\times}), \quad \lambda \longmapsto \overline{\lambda}$$

coincides with

$$(q'-1) \cdot X' \oplus \widetilde{\Phi}^{\perp}_{H^0}$$

It results easily from Steinberg's theorem (see Chapter II, Prop. 3.15 and Cor. 3.17 of [**46**]) that the irreducible representations of the abstract group $H^0(k')_{\nu}$ are classified by

$$X_{H,q'} = \{(v, a) \in X'^{+} \times \Phi^{\perp}_{H^0} / \widetilde{\Phi}^{\perp}_{H^0} \mid 0 \leqslant \langle v, \beta^{\vee} \rangle \leqslant q' - 1 \text{ for all } \beta \in \Delta_{H^0}\}$$

For brevity, we call such weights q'-reduced, although the terminology is not conformal to that of Jantzen's book Chapter II, Section 3. For $\mu \in X_{H,q'}$, we write $W(\mu)$ for the corresponding H^0-representation and $\Pi_{H^0}(\mu) \subset X$ for its set of weights, resp. $\overline{\Pi}_{H^0}(\mu) \subset \mathrm{Hom}(\widehat{T}(k'), k'^{\times})$ the set of their restrictions to $\widehat{T}(k')_{\nu}$.

Let $\widehat{\varpi}_i$ be the fundamental weights in X of $\widehat{G}$. We write $\widehat{\varpi} = \widehat{\varpi}_g$ for the minuscule weight of $\widehat{G}$; it is the highest weight of the spin representation $V_{/\mathbb{F}_p}$ of $\widehat{G}$. Let $\Pi_{\widehat{G}}(\widehat{\varpi})$ resp. $\overline{\Pi}_{\widehat{G}}(\widehat{\varpi})$ the set of weights (resp. of the functions on $\widehat{T}(k')$ that they induce) associated to the spin representation $\mathbf{V}_{/k'}$ of $\widehat{G}$.

Recall that $\Pi_{\widehat{G}}(\widehat{\varpi}) = \{\widehat{\varpi}^{w'} \mid w' \in W'\}$ and that we assumed $W' \propto \widehat{T} \subset H$.

Lemma 12. — *For $p > 5$, if $W(\mu)$ is a simple $H^0_{k'}$-module with highest weight $\mu \in X_{H,q'}$ with $\overline{\widehat{\varpi}} = \overline{\mu}$ and $\overline{\Pi}_{H^0}(\mu) \subset \overline{\Pi}_{\widehat{G}}(\widehat{\varpi})$, then $\mu = \widehat{\varpi}$.*

Remark. — For $p = 5$, $\widehat{G} = Spin(5)$ and $H \subset \widehat{G}$, isomorphic to $\mathrm{SL}(2) \times \mathrm{SL}(2)$ via $\widehat{G} \cong \mathrm{Sp}(4)$, $\mu = 3\widehat{\varpi}_2$, the lemma is false, hence the necessity of the assumption $p > 5$.

Proof. — Since $\overline{\mu} = \overline{\widehat{\varpi}}$, one has $\mu - \widehat{\varpi} \in (q' - 1)X$.

1) Let us first check that $\mu - \widehat{\varpi} \in N \cap \Phi^{\perp}_{H^0} = \widetilde{\Phi}^{\perp}_{H^0}$.

Let $\alpha \in \Delta_{H^0}$. We want $\langle \mu - \widehat{\varpi}, \alpha^{\vee} \rangle = 0$. We start with a preliminary observation:

For any simple root $\alpha \in \Delta_{H^0}$, $\langle \widehat{\varpi}, \alpha^{\vee} \rangle \in \{-1, 0, 1\}$. Indeed, this is true for any fundamental weight $\widehat{\varpi}$ of $\widehat{G}$. In particular for our minuscule weight $\widehat{\varpi}$.

Then, we distinguish three cases

– If $\langle \widehat{\varpi}, \alpha^{\vee} \rangle = 1$, we have $\langle \mu, \alpha^{\vee} \rangle = 1$ because μ is q'-reduced.

– If $\langle \widehat{\varpi}, \alpha^{\vee} \rangle = 0$; let us exclude the possibility $\langle \mu, \alpha^{\vee} \rangle = q' - 1$. Since $q' - 1 \geqslant 1$ we would have $\mu - \alpha \in \Pi_{H^0}(\mu)$ as the α-string of μ has length $q' - 1$. Hence by the assumption, we could write $\mu - \alpha = \widehat{\varpi}^y + (q' - 1)\lambda$ for some $y \in W'$ and $\lambda \in X$.

But $\langle \widehat{\varpi}^y, \alpha^{\vee} \rangle \in \{-1, 0, 1\}$, and $\langle \mu - \alpha, \alpha^{\vee} \rangle = q' - 3$ hence $q' - 1$ should divide $1, 2$ or 3 impossible since $q' - 1 > 3$.

– Similarly, if $\langle \widehat{\varpi}, \alpha^{\vee} \rangle = -1$, we must exclude $\langle \mu, \alpha^{\vee} \rangle = q' - 2$. Again $\mu - \alpha \in \Pi_{H^0}(\mu)$, hence $\mu - \alpha \equiv \widehat{\varpi}^y \bmod. (q' - 1)X$. But $\langle \widehat{\varpi}^y, \alpha^{\vee} \rangle \in \{-1, 0, 1\}$ and $\langle \mu - \alpha, \alpha^{\vee} \rangle \equiv -3 \bmod (q' - 1)$, hence $(q' - 1)$ should divide $2, 3$ or 4; impossible since $q' - 1 > 4$.

2) Thus, $\mu - \widehat{\varpi} \in \Phi^{\perp}_{H^0} \cap N$ (actually, it shows that $\langle \widehat{\varpi}, \alpha^{\vee} \rangle \geqslant 0$ for any $\alpha \in \Delta_{H^0}$). Since the components of $\widehat{\varpi}$ and μ along $\Phi^{\perp,1}_{H^0}$ resp. $\mathbb{Z}\nu$ are reduced (mod. $q' - 1$) resp. mod. e, and that $\mu - \widehat{\varpi} \in \widetilde{\Phi}^{\perp}_{H^0}$, we conclude $\mu = \widehat{\varpi}$. The lemma is proven.

It is the main ingredient in the proof of the following result.

Lemma 13. — *Let $\sigma : \Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_k(W)$ be a continuous Galois representation such that for any $g \in \Gamma$, the characteristic polynomial of $\overline{\rho}_\pi(g)$ annihilates $\sigma(g)$. Assume that $p - 1 > \max(4, \boldsymbol{w})$, that $\overline{\rho}_\pi$ satisfies* $\mathbf{GO}(\omega)$ *and* **(RLI)**,

then, either $W = 0$, or the two characters 1 and $\omega^{-\boldsymbol{w}}$ restricted to I_p occur as subquotients of W viewed as an I_p-module.

Comment. — One could naturally ask whether the simpler assumptions that $\overline{\rho}_\pi$ is absolutely irreducible and for any $g \in \Gamma$ the characteristic polynomial of $\overline{\rho}_\pi(g)$ annihilates $\sigma(g)$ are sufficient to conclude that all constitutents of σ are copies of $\overline{\rho}_\pi$. This statement is true for $g = 1$, but, it is false for $g = 2$. A counterexample has been found by J.-P. Serre. He lets Γ act on $\mathbb{F}_p^4$ through the so-called cuspidal representation of the non-split central extension ${}_2A_5$ of the icosaedral group A_5. It is four-dimensional, symplectic and absolutely irreducible. Then, (W, σ) is one of the two irreducible 2-dimensional of this group. This is why we introduced **(RLI)**. This assumption is not satisfied in the example there. Also, thanks to the ordinarity assumption **(GO)**, we focused our attention on the highest weight of $\overline{\rho}_\pi$ (which is a local information at p) rather than the global representation $\overline{\rho}_\pi$ itself.

Proof. — Assume $W \neq 0$; let Γ' be the inverse image by $\overline{\rho}_\pi$ of $H(k')$ in Γ and Γ'' the kernel of $\overline{\rho}_\pi$ restricted to Γ'. Then $\sigma(\Gamma'')$ is a nilpotent p-group. Thus, replacing W by its submodule fixed by $\sigma(\Gamma'')$, still denoted by W, one can assume that W is a non-zero module on which Γ' acts through $H(k')_\nu$:

$$\begin{tikzcd}
\Gamma' \arrow[r] \arrow[d, "\overline{\rho}_\pi"'] & \mathrm{GL}_k(W) \\
H(k') \arrow[ur] &
\end{tikzcd}$$

We first treat the case of $\omega^{-\boldsymbol{w}}$. Let H^0 be the neutral component of H. Let $\widetilde{W} = \mathrm{Ind}_{H^0(k')_\nu}^{H^0(k')} W$. It is an $H^0(k')$-module, and for any $t \in \widehat{T}(k')_\nu$, the action of t on $\widetilde{W}$ is annihilated by $\prod_{w \in W'}(X - \widehat{\varpi}^w(t))$. By Steinberg theorem ([**46**] Sect II.3.15), the space W viewed as $H^0(k')$-module has a subquotient $W(\mu)$ which comes from an algebraic simple $H^0_{k'}$-module corresponding to a q'-reduced highest weight μ. We associate to this representation the sets Π_μ resp. $\overline{\Pi}_\mu$ as above. By the assumption $W' \subset H$, one can assume that $\overline{\Pi}_{H^0}(\mu) \subset \overline{\Pi}_{\widehat{G}}(\widehat{\varpi})$ and $\overline{\widehat{\varpi}} = \overline{\mu}$ (if $\overline{\mu} = \overline{\widehat{\varpi}}^{w'}$ for some $w' \in W'$, simply replace $W(\mu)$ by $W(\mu^{w'^{-1}})$ which also occurs as $H^0_{k'}$-subquotient of W). By the previous lemma, for $p > 5$, we have $\widehat{\varpi} = \mu$. Let x be a highest weight vector in $W(\mu)$ for $H^0_{\mathbb{F}_p}$. It is fixed by $H \cap \widehat{N}(k)$. Since $I_p \subset \overline{\rho}_\pi^{-1}(H^0(k))$, the action of I_p on x is through its image by $\widehat{\varpi}_g \circ (\overline{\rho}_\pi \bmod. \widehat{N})$. By the assumption **(GO)**, and Lemma 3, this character is equal to $\omega^{-\boldsymbol{w}}$ on I_p which therefore occurs as a subquotient of $W|_{I_p}$. To treat the case of the trivial character, we consider instead of the highest weight μ by the lowest weight μ' of $W(\mu)$; we can assume that $\overline{\mu}' = \overline{\widehat{\varpi}}^{w_0}$ where w_0 is the longest element of $W_{\widehat{G}}$. Let N_{H^0} be the unipotent radical of a Borel of H^0 adapted to (GO). On the lowest weight quotient $W(\mu)_{N_{H^0}}$ (the vector space of N_{H^0}-coinvariants), $\overline{\rho}_\pi$ acts by $\widehat{\varpi}^{w_0} \circ (\overline{\rho}_\pi \bmod. \widehat{N})$, which is trivial by (3.3.2). QED

7.2. Deducing Theorem 1 from Theorem 6. — Recall we have fixed $\lambda = (a_g, \ldots, a_1; c)$ with $c = a_g + \cdots + a_1$ and $|\lambda + \rho| < p - 1$. We have the following reduction steps:

1) By Poincaré duality, and self-duality of the Hecke operators for ℓ prime to N, Statement (i) of Theorem 1 is equivalent to the vanishing of

$$H^j_*(S_U, V_\lambda(k))_{\mathfrak{m}} = 0 \quad \text{for } q < d$$

where $\star = c, \varnothing$. These modules are artinian over $\mathcal{H}_{\mathfrak{m}}$, so by Nakayama's lemma, it is enough to show that their $\mathfrak{m}$-torsion vanishes:

(7.2.1) $$H^j_*(S_U, V_\lambda(k))[\mathfrak{m}] = 0 \quad \text{for } * = \varnothing \text{ or } c \text{ and } q < d$$

which we will prove below.

2) Then, Statements ii) and iii) are easy consequences of i) as can be seen by induction on $q < d$ using the long exact sequences

$$0 \longrightarrow V_\lambda(\mathcal{O}) \longrightarrow V_\lambda(\mathcal{O}) \longrightarrow V_\lambda(\mathcal{O}/\varpi\mathcal{O}) \longrightarrow 0$$

and

$$0 \longrightarrow V_\lambda(\varpi^{-1}\mathcal{O}/\mathcal{O}) \longrightarrow V_\lambda(K/\mathcal{O}) \longrightarrow V_\lambda(K/\mathcal{O}) \longrightarrow 0.$$

For instance, from the latter, one obtains, with obvious notations: if $H^{q-1}_*(K/\mathcal{O})_\mathfrak{m} = 0$, then $H^q_*(\varpi^{-1}\mathcal{O}/\mathcal{O})_\mathfrak{m} \to H^q(K/\mathcal{O})_\mathfrak{m}[\varpi]$ is an isomorphism; hence by Nakayama's lemma, assertion one implies that $H^q_*(K/\mathcal{O})_\mathfrak{m}$ vanishes for $q < d$.

Note that since $p > j_A > a_g \cdots \geqslant a_1 \geqslant 0$, one knows that $V_{\lambda\,\mathbb{F}_p}$ is absolutely irreducible (see for instance Proposition II.3.15, p. 222, of [**46**]).

3) As in section 6.3, $X_{/\mathbb{Z}[1/N]}$ is the moduli scheme classifying p.p.a.v. with level N structure over $\mathbb{Z}[1/N]$. Its toroidal compactification over $\mathbb{Z}[1/N]$ is denoted by $\overline{X}$. Let $V_\lambda(\mathbb{F}_p)$ resp. $V_\lambda(k)$ be the étale sheaf over $X\otimes\mathbb{Q}$ in $\mathbb{F}_p$- resp. k-vector space corresponding to $V_{\lambda\,\mathbb{F}_p}$. Using the etale-Betti comparison isomorphism (and its equivariance for algebraic correspondences), Theorem 1 will be proven if we show the vanishing of the etale cohomology groups corresponding to (7.2.1).

This interpretation as étale cohomology allows us to view $H^j_*(S_U, V_\lambda(\mathbb{F}_p))$ as a $\mathbb{F}_p[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})] \times \mathcal{H}_K$-module:

$$H^j_*(X, V_\lambda(\mathbb{F}_p)) \cong H^j_{\text{ét},*}(X\otimes\overline{\mathbb{Q}}, V_\lambda(\mathbb{F}_p)).$$

Remark. — The $\mathbb{F}_p$-coefficients are useful to apply Fontaine-Laffaille and Faltings theory, while the k-coefficients will come in when we localize at the maximal ideal $\mathfrak{m}$ of $\mathcal{H}_K(\mathcal{O})$.

Let $\overline{\mathcal{V}}^\vee_\lambda$ be the object of $\mathcal{M}F^{\nabla,[0,p-2]}(X_0)$ associated to λ as in Section 6.3. Recall that $\overline{\mathcal{V}}^\vee_\lambda$ has a filtration of length $|\lambda|$; since $d+|\lambda| < p-1$ and since $\overline{\mathcal{V}}^\vee_\lambda$ and $V_\lambda(\mathbb{F}_p)^\vee$ are associated (Theorem 8 above, section 6.3), we can apply Th. 5.3 of [**22**] (see Theorem 7, Section 6.1), so that for any $j \geqslant 0$:

$$H^j_{\text{ét},*}(X\otimes\overline{\mathbb{Q}}_p, V_\lambda(\mathbb{F}_p))^\vee$$

is the image by the Fontaine functor $\mathbf{V}^*$ of

$$H^j_{\text{log-crys},*}(X\otimes\mathbb{F}_p, \mathcal{V}^\vee_\lambda).$$

Note that since we work mod. p instead of mod. p^n, we have

$$H^j_{\text{log-crys},*}(X\otimes\mathbb{F}_p, \mathcal{V}^\vee_\lambda) = H^j_{\text{log-dR},*}(X\otimes\mathbb{F}_p, \mathcal{V}^\vee_\lambda)$$

We have constructed in Section 5.3.4 a filtered complex of coherent sheaves $\overline{\mathcal{K}}^\bullet_\lambda$ on $\overline{X}\otimes\mathbb{F}_p$ by functoriality from the BGG resolution of the $G_{\mathbb{F}_p}$-module $V_{\lambda_{\mathbb{F}_p}}$. It follows from Theorem 6 that there are isomorphisms of filtered $\mathbb{F}_p$-vector spaces:

$$H^j_{\text{log-dR}}(X\otimes\mathbb{F}_p, \mathcal{V}^\vee_\lambda) \cong H^j(\overline{X}\otimes\mathbb{F}_p, \overline{\mathcal{K}}^\bullet_\lambda)$$

and

$$H^j_{\text{log-dR},c}(X\otimes\mathbb{F}_p,\mathcal{V}_\lambda)\cong H^j(\overline{X}\otimes\mathbb{F}_p,\overline{\mathcal{K}}^{\bullet\,\text{sub}}_\lambda)$$

where $\overline{\mathcal{K}}^\bullet_\lambda$ resp. $\overline{\mathcal{K}}^{\bullet\,\text{sub}}_\lambda$ denotes the canonical, resp. subcanonical Mumford extension of the filtered complex of sheaves $\mathcal{K}^\bullet_\lambda$. The resulting filtration on the right-hand side is called the F-filtration; it corresponds via these isomorphisms to the Hodge filtration on the left-hand side. The weights of this filtration can be computed as in [**72**] (who treats the case $g=2$): Let us consider the map

$$W_G\longrightarrow\mathbb{Z},\quad w\longmapsto p(w)=-(w(\lambda+\rho)(H)-\rho(H))$$

where $H=\operatorname{diag}(0,\dots,0,-1,\dots,-1)$. Let W_M be the Weyl group of the Levi subgroup M of the Siegel parabolic. Observe that this map factors through the quotient $W_M\backslash W_G$; this quotient is in bijection with the set W^M (*cf.* p. 229 of [**13**]). By Theorem 6, Sect. 5.4, we have

$$\operatorname{gr}^p H^j_{\text{log-dR},*}=\bigoplus_{\substack{w\in W^M\\ p(w)=p\\ \ell(w)\leqslant j-p}} H^{j-\ell(w)}(\overline{X}\otimes\mathbb{F}_p,\overline{\mathcal{W}}^\vee_{w(\lambda+\rho)-\rho})$$

Note that, unfortunately, p is not a good notation for the degre of our Hodge filtration. The image $p(W_G)$ of p is therefore the set of possible weights occuring in $H^j_{\text{crys},*}$ for $j\leqslant d$. Moreover, p is injective on $W_M\backslash W_G$, and its values are exactly the j_B ($B\subset A$). The set of possible lengthes $\ell(w)$, $w\in W^M$ is $[0,d]$. For each $j<d$, let us consider the set $W^M(j)=\{w\in W^M\mid \ell(w)\leqslant j\}$; the key observation is that for $j<d$, $W^M(j)$ does not contain the unique element $w\in W^M$ such that $\ell(w)=d$, namely the one acting by $(a_g,\dots,a_1;c)\mapsto(-a_g,\dots,-a_1;c)$. But this element is the unique one for which $p(w)$ takes on its maximal value: j_A. Hence, this maximal weight does not occur in $H^j_{\text{log-dR},*}(X\otimes\mathbb{F}_p,\mathcal{V}^\vee_\lambda)$ for $j<d$.

On the other hand, under assumptions **(Gal)** and **(GO)**, $\overline{\rho}_\pi$ is ordinary with weights given by j_B for all subsets $B\subset A$; in particular j_A and 0 indeed occur with multiplicity one; actually, even if we replaced **(GO)** by geometric ordinarity, it would result from lemma 3, Sect. 3.3, that 0 and j_A do occur in ρ_π). Now, consider the global Galois representation σ^j on $W_j=H^j_*(X\otimes\overline{\mathbb{Q}},V_\lambda(k))[\mathfrak{m}]$, the kernel of $\mathfrak{m}$ in the module $H^j_*(X\otimes\overline{\mathbb{Q}},V_\lambda(k))$. The Eichler-Shimura relations imply for any $g\in\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the characteristic polynomial of $\overline{\rho}_\pi(g)$ annihilates $\sigma^j(g)$. Our lemma 13 Sect. 7.1, shows, assuming **(RLI)**, that this implies that W_j admits $\overline{\rho}_\pi$ as subquotient. This is a contradiction since the maximal weight j_A occurs in $\overline{\rho}_\pi|_{I_p}$ but not in $W_j|_{I_p}$.

7.3. Examples. — Let F be a real quadratic field with Galois group $\{1,\sigma\}$. Let $\Gamma_F=\operatorname{Gal}(\overline{\mathbb{Q}}/F)$. Let f be a holomorphic Hilbert cusp form for $\mathrm{GL}(2)_{/F}$ of weight (k_1,k_σ), $k_1,k_\sigma\geqslant 2$, $k_1=k_\sigma+2m$ for an integer $m\geqslant 1$. Assume it is a new form of conductor $\mathfrak{n}$ which is eigen for Hecke operators T_v (v prime to $\mathfrak{n}$); denote by a_v the corresponding eigenvalues. Since the weight of f is not parallel, f does not come from $\mathbb{Q}$. Let f_σ be the inner conjugate of f by σ. Let ϵ be the finite order part of its central character. We assume that ϵ factors through the norm map. Starting from [**90**], a series of works have established that f admits a holomorphic theta lift π to $G(\mathbb{A})$ where $G = \mathrm{GSp}(4)$ (see [**63**] and [**64**]). Since f does not come from $\mathbb{Q}$, π is cuspidal; moreover, in [**64**], B. Roberts explained to us that in particular such a theta lift π is stable at ∞. The published reference for this fact is [**65**]. It occurs in the H^3 of a Siegel variety of some level, say N, with coefficient system of highest weight $\lambda = (a, b; c)$ where $a = k_\sigma + m - 3$, $b = m - 1$, and $c = a + b$. At the moment, the level N of π can only be said to be multiple of $N(\mathfrak{n})D_F$ where D_F is the discriminant of F; $N(\mathfrak{n})D_F$ should be the conductor of π, but this can not yet be established in general.

Let $\mathbb{Q}(f) = \mathbb{Q}[a_v]_v$ be the number field generated by the eigenvalues of f; one can take $\mathbb{Q}(f)$ as field of definition of π (although this may not be the smallest possible one, as pointed out by Prof. Yoshida). For any prime $\mathfrak{p}_f$ of $\mathbb{Q}(f)$ prime to $\mathfrak{n}$, the $\mathfrak{p}_f$-adic Galois representation associated to π exists; it is given by

$$\rho_\pi = \mathrm{Ind}_{\mathbb{Q}}^{F} \rho_f \tag{7.3.1}$$

it is absolutely irreducible. The conductor of ρ_π is $\mathrm{Norm}(\mathfrak{n}) \cdot D_F$; this results from the fact that $\mathfrak{n}$ is also the (prime-to-p part of the) conductor of ρ_f by Carayol's theorem.

Indeed, π is motivic: by Theorem 2.5.1 of [**10**], for any imaginary quadratic field F', there exists a motive $M_{f,F'}$ defined over $F \cdot F'$, of rank 2 over some extension $\mathbb{Q}(f, F')$ of $F' \cdot \mathbb{Q}(f)$; the motives $M_{f,F'}$ are "associated to f": they give rise to a compatible system of λ-adic representations of Γ_F, which is associated to f. Its Hodge-Tate weights are 0 and $k_1 - 1$ above $\mathrm{Id}_{F'}$, and m and $m + k_\sigma - 1$ above $\sigma \otimes \mathrm{Id}_{F'}$.

Remark. — In fact there should exist M_f defined over $\mathbb{Q}$, of rank 2 over $\mathbb{Q}(f)$, associated to f in the above sense.

Then we consider for each imaginary quadratic F'

$$M_{\pi,F'} = \mathrm{Res}_{F'}^{F \cdot F'} M_{f,F'} \tag{7.3.2}$$

$M_{\pi,F'}$ is defined over F', of rank 4 over $\mathbb{Q}(f, F')$; it is pure of weight $\boldsymbol{w} = k_1 - 1$ and the four Hodge-Tate weights $0 < m < m + k_\sigma - 1 < k_1 - 1$ do occur. These motives define a compatible system of degree 4 λ-adic representations of Γ, associated to π.

Remark. — Similarly, there should exist M_π defined over $\mathbb{Q}$, of rank 4 over $\mathbb{Q}(f)$ with those Hodge-Tate weights, associated to π.

In the CM case, we restrict our attention to the situation where f is a theta series coming from a biquadratic extension $M = EF/F$, E imaginary quadratic. Let $\mathrm{Gal}(E/\mathbb{Q}) = \{1, \tau\}$, $\mathrm{Gal}(F/\mathbb{Q}) = \{1, \sigma\}$ and $\mathrm{Gal}(M/\mathbb{Q}) = \{1, \sigma, \tau, \sigma\tau\}$. We write $f = \theta(\phi)$ where ϕ is a Hecke character of infinity type $n_1 + n_\sigma \sigma + n_{\sigma\tau} \sigma\tau + n_\tau \tau \in \mathbb{N}[\mathrm{Gal}(EF/\mathbb{Q})]$, such that

$$n_1 + n_\tau = n_\sigma + n_{\sigma\tau} \quad \text{and} \quad n_1 > n_\sigma > n_{\sigma\tau} > n_\tau \tag{$*$}$$

and of conductor $\mathfrak{f}$ prime to p in M. In that case, one has $a = n_\sigma - n_\tau - 2$, $b = n_1 - n_\sigma - 1$ and $c = n_1 + n_\tau - 3$; indeed, since $n_\tau = (c - a - b)/2$, we see that the condition $n_\tau = 0$ is equivalent to $c = a + b$, in which case one has $n_1 = \boldsymbol{w}$, $n_\sigma = k_\sigma - 1 + m$, $n_{\sigma\tau} = m$ (and $n_\tau = 0$). We assume in fact in the sequel a condition slightly stronger than $(*)$, namely:

$$(**) \qquad \phi^{(1+\tau)\cdot(1-\sigma)} = 1 \quad \text{and} \quad n_1 > n_\sigma > n_{\sigma\tau} > n_\tau$$

Under these assumptions, we say that f is of $(2,2)$-CM type.

Remark. — If $(*)$ is satisfied for a character ϕ, then $(**)$ is satisfied for $\phi^{h_\mathfrak{f}}$ where $h_\mathfrak{f}$ denotes the ray-class number of EF of conductor $\mathfrak{f}$.

Let $\mathcal{O}_f$ be the ring of integers of $\mathbb{Q}(f)$. For a suitable finite set of primes S of $\mathcal{O}_f$ disjoint of the prime divisors of $\mathfrak{n}$, the localization $S^{-1}\mathcal{O}_f$ is principal. In this principal ring, we choose for each prime v prime to $\mathfrak{n}$ a generator $\{v\}$. Let $I = I_f$ be the ring generated by the normalized eigenvalues $a_v^0 = \{v\}^{-m\cdot\sigma} \cdot a_v$ (v prime to $\mathfrak{n}$) of f in $\mathbb{Q}(f)$. The a_v^0's are eigenvalues for the divided Hecke operators $T_0(v) = \{v\}^{-m\cdot\sigma} \cdot T_v$ as introduced by Hida in the beginning of Sect. 3 of [**40**]. By Th. 4.11 of [**40**], these eigenvalues are still integral.

Let p be a rational prime. We assume hereafter that p splits in F, say, $p \cdot \mathcal{O}_F = \mathfrak{q} \cdot \mathfrak{q}^\sigma$, $\mathfrak{q} \neq \mathfrak{q}^\sigma$, and that $\{\mathfrak{q}, \mathfrak{q}^c\} \cap S = \varnothing$. We fix $\iota_p : \mathbb{Q}(f) \hookrightarrow K \subset \overline{\mathbb{Q}}_p$, a p-adic embedding, and $\mathfrak{p}_f$ the prime of I associated to ι_p.

Recall that by a Theorem of Wiles (Th. 2.2.2 of [**88**]) and a Proposition of Hida (Prop. 2.3 of [**41**]), if

$$\operatorname{ord}_p(\iota_p(a_\mathfrak{q}^0)) = 0 \quad \text{resp. } \operatorname{ord}_p(\iota_p(a_{\mathfrak{q}^\sigma}^0)) = 0$$

(that is, $\operatorname{ord}_p(\iota_p(a_\mathfrak{q})) = 0$ resp. $\operatorname{ord}_p(\iota_p(a_{\mathfrak{q}^\sigma})) = m$), then, the decomposition group $D_\mathfrak{q} \subset \Gamma_F$ at $\mathfrak{q}$ preserving ι_p is sent by $\rho_{f,\mathfrak{p}_f}$, resp. $\rho_{f_\sigma,\mathfrak{p}_f}$ to a Borel subgroup of $\mathrm{GL}(2)$; moreover, $\rho_{f,\mathfrak{p}_f}$ resp. $\rho_{f_\sigma,\mathfrak{p}_f}$ restricted to the inertia subgroup $I_\mathfrak{q}$ has a 1-dimensional unramified quotient.

We put $k' = I/\mathfrak{p}_f$. Let J be the subring generated by the (a_v, a_{v^σ}) in $\mathbb{Q}(f) \times \mathbb{Q}(f)$. For p prime to the index of I in its normalization, and of J in its normalisation, we can view $\rho_{\pi,\mathfrak{p}_f}|_F = (\rho_f, \rho_{f_\sigma})$ as taking values in $\mathrm{GL}(2, I_{\mathfrak{p}_f}) \times \mathrm{GL}(2, I_{\mathfrak{p}_f})$. Let $X \subset k'^\times$ be the subgroup generated by the reduction of $Nv^{k_1-1} \cdot \epsilon(v)$ for all finite places v prime to $\mathfrak{n}p$. Let

$$\overline{\mathcal{H}}^0 = \{(g, g') \in \mathrm{GL}_2(k') \times \mathrm{GL}_2(k') \mid \det g = \det g' \in X\}$$

the two factors being exchanged by σ, and

$$\overline{\mathcal{H}} = \{1, \sigma\} \propto \overline{\mathcal{H}}^0.$$

Similarly, let $\overline{\mathcal{H}}_{CM}$ be the image by the spin representation of

$$\{g \in \widehat{T}(k') \propto W' \mid \nu(g) \in X\}.$$

Proposition 5. — *For f as above and $k_1 > k_\sigma > 2$, with Nebentypus of order at most 2, there exists a (non-effective) finite set S of finite places in $\mathbb{Q}(f)$ such that, for any $p \notin S$, splitting in F, for which a $\mathfrak{p}_f|p$ is ordinary for f and f_σ, the image of $\overline{\rho}_{\pi,\mathfrak{p}_f} : \Gamma \to \mathrm{GL}_{k'}(\overline{V})$ is equal to:*

– *$\overline{\mathcal{H}}$, if f is not CM,*

– *contains a subgroup of $\overline{\mathcal{H}}_{CM}$ of index at most gcd$(p-1, n_1 \cdot n_\sigma)$ if f is of $(2,2)$-CM type.*

Comment. — Let H the subgroup of $\widehat{G}$ whose image by the spin representation is ${}^L(\mathrm{Res}_{\mathbb{Q}}^F \mathrm{GL}(2))$ (in the non-CM case) resp. ${}^L(\mathrm{Res}_{\mathbb{Q}}^M M^\times)$ in the $(2,2)$-CM case. Then, in both cases, the image of W' is the group of type $(2,2)$ generated by

$$\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

Thus, by the previous proposition, the assumption **(RLI)** of sect 7.1, is satisfied for H.

Proof. — Assume first that f has no CM. We follow the method of proof of Ribet's thesis [**62**]. More precisely, we apply Th. 3.1 of [**62**]. We change its statement by replacing $\mathbb{F}_p^{k-1}$ there by our subgroup X; since $X \subset \mathbb{F}_p^\times$, the proof of Th. 3.1 runs identically. Let $\overline{G} = \mathrm{Im}\,\overline{\rho}_{\pi,\mathfrak{p}_f}|_F$. In order to apply Th. 3.1 as in Th. 5.1 and 6.1 of [**62**], we have to check

(a) For almost all p splitting in F and ordinary as above, $\overline{\rho}_{f,\mathfrak{p}_f}$ and $\overline{\rho}_{f_\sigma,\mathfrak{p}_f}$ act irreducibly on k'^2 and their images have order divisible by p,

(b) For almost all p as above, there exists $\gamma \in \overline{G}$ such that $(\mathrm{Tr}\,\gamma)^2$ generates $k' \times k'$.

(a) If $\overline{\rho}_f$ is reducible, we have

$$\overline{\rho}_f \equiv \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix} \text{ mod. } \mathfrak{p}.$$

Let us define a global character ψ of conductor dividing $\mathfrak{n} \cdot p$ by

$$\psi_{\mathrm{gal}} \cdot \omega^{1-k_1} = \overline{\chi}_1 / \overline{\chi}_2.$$

Let $\psi_{\mathfrak{q}}$, resp. $\psi_{\mathfrak{q}^\sigma}$ be the restriction of ψ to $I_{\mathfrak{q}}$ resp. to $I_{\mathfrak{q}^\sigma}$. By the ordinarity of ρ_f at $\mathfrak{q}$ and q^σ, we see that $\psi_{\mathfrak{q}} = 1$ or $\omega^{2(k_1-1)}$ and $\psi_{\mathfrak{q}^\sigma} = \omega^{2m}$ or $\omega^{2(k_1-1)-2m}$. Let ϵ be a fundamental unit of F. Consider the numbers

$$\epsilon^{2m\cdot\sigma} - 1, \epsilon^{[2(k_1-1)-2m]\cdot\sigma} - 1, \epsilon^{2(k_1-1)+2m\cdot\sigma} - 1, \epsilon^{2(k_1-1)+[2(k_1-1)-2m]\cdot\sigma} - 1;$$

If $\mathfrak{q}$ is prime to these numbers, we see by global class-field theory that the global character ψ cannot exist.

Remark. — This reflects the fact that no congruence between f and an Eisenstein series can occur, as there are no non-zero Eisenstein series with non-parallel weight.

To assure that p divides the order of $\operatorname{Im}\overline{\rho}_f$, one proceeds as in Lemma 5.3 of [**62**] to exclude all entries of the list of prime-to-p order subgroups in $\mathrm{GL}_2(k')$. The cases to exclude are

– Case (i) is when the image in PGL(2) is abelian,

– Case (ii) is when it is dihedral,

– Case (iii) means the projective image is A_4, S_4 or A_5.

We have to modify the proof in case (ii) as follows. Since $\overline{\rho}_f$ is totally odd, we would obtain a totally imaginary quadratic extension M/F, of relative Galois group say, $\{1,\tau\}$, and a ray-class group character $\overline{\lambda}:\mathrm{Cl}_{M,\mathfrak{f}\cdot p}\to\overline{\mathbb{F}}_p^\times$ (for some ideal $\mathfrak{f}$ of M) such that $\overline{\rho}_f=\mathrm{Ind}_F^M\,\overline{\lambda}^{\mathrm{gal}}$, with $\mathrm{Norm}_{M/F}(\mathfrak{f})D_{M/F}|\mathfrak{n}\cdot p$. One can lift $\overline{\lambda}$ into a Hecke character λ of M of type adapted to k, so that the theta series $\theta(\lambda)$ belongs to $M_k(\Gamma_0(\mathfrak{n}\cdot.p,\epsilon)$ and

$$(C)\qquad f\equiv\theta(\lambda)\ \mathrm{mod.}\ \mathfrak{p}$$

here again, we use the ordinarity of f at p:

– first, if $D_{M/F}$ is divisible by $\mathfrak{q}$ or $\mathfrak{q}^\sigma$, $\theta(\lambda)$ cannot be ordinary at $\mathfrak{q}$ (because k_1 and k_σ are greater than 1); therefore the field M can only ramify above $\mathfrak{n}$: this leaves a finite set of possibilities for M.

– Moreover, by Hida's p-stabilization lemma (Lemma 7.1 of Bull. SMF 1995), since k_1 and k_σ are greater than 2 (that is, the cohomological weight $(k_1-2,k_\sigma-2)$ is regular), the congruence (C) can only occur if λ has conductor prime to p.

In conclusion, consider the finite set Θ of rational primes p dividing one of the congruence numbers $C(\theta(\lambda'))$ for some Hecke character λ' of a CM field M, such that λ' has the right infinity type, and the conductor $\mathfrak{f}$ of λ' and the discriminant $D_{M/F}$ satisfy

$$\mathrm{Norm}(\mathfrak{f})D_{M/F}\text{ divides }\mathfrak{n}.$$

Then for $p\notin\Theta$, case (ii) does not occur.

Remark. — Note that these congruence numbers should be given as the algebraic part of the special value of the Hecke L-function $L_M(\lambda'\lambda'^{[\tau]},k)$. This is the hypothetical converse of a general divisibility result of Hida-T. (Ann. Sci. ENS 1993). It is known at the moment only for $F=\mathbb{Q}$ (Hida Inv. 64, 1981), but it is conjectured for any totally real field F.

To treat case (iii), we follow closely the argument on p. 264 of [**62**]: if there were infinitely many $\mathfrak{p}$ satisfying case (iii), then by using Cebotarev density theorem, one would find a set of positive density of v's satisfying $a_v^2=4\cdot Nv^{k_1-1}$. Since k_1 is odd, this condition implies that v ramifies in $\mathbb{Q}(f)$ or is degree 2 over $\mathbb{Q}$. This set has density zero in F. This is a contradiction. Thus, the set of p's in case (iii) must be finite.

(b) As in [**62**], we proceed in two steps:

1) We establish the equality $\overline{G} = \mathcal{H}^0$ for some prime $\mathfrak{p}_f$,

2) We deduce from 1) the existence of $\gamma \in \overline{G}$ as desired for almost all ordinary p's splitting in F.

Let p a rational prime, $\mathfrak{p}|p$ in $\overline{\mathbb{Q}}$ dividing $\mathfrak{p}_f$ and $\mathfrak{q}$. We assume that it satisfies (a), that it splits completely in $\mathbb{Q}(f)$ and that f and f_σ are ordinary at $\mathfrak{q}$. We assume furthermore that for any quadratic Dirichlet character χ mod. $\mathfrak{n}$, there exists v prime to Norm($\mathfrak{n}$) such that $a_v \not\equiv \chi(v) \cdot a_{v^\sigma}$ mod. $\mathfrak{p}_f$.

These conditions are satisfied if $\mathfrak{p}_f$ is prime to all congruence numbers for all pairs $f, f_\sigma \otimes \chi$ (for the Hecke algebra of level $\mathrm{Norm}(\mathfrak{n})^2$, generated by Hecke operators outside Norm($\mathfrak{n}$)); indeed the eigensystems of f and the $f_\sigma \otimes \chi$, for any χ mod. $\mathfrak{n}$ are mutually distinct. Indeed, if $a_v = a_{v^\sigma}\chi(v)$, for almost all vs, then χ descends to $\mathbb{Q}$. It defines a quadratic extension $F'/\mathbb{Q}$. Let $E = F \cdot F'$. Let f_E be the base change of f to E. If τ generates $\mathrm{Gal}(F'/\mathbb{Q})$, the weight of f_E is $k_1(1+\tau) + k_\sigma(\sigma + \sigma\tau)$. The assumption implies that $f_E = (f_\sigma)_E = (f_E)_\sigma$; hence f_E should descend to F'. This is absurd since its weight is not invariant by $\mathrm{Gal}(E/F) = \{1, \sigma\}$. So these congruence numbers are not zero, and thus can be avoided.

Claim. — *For such p, $\overline{G} = \overline{\mathcal{H}}^0$.*

Proof. — If, not, Th. 3.8 of [**62**] (or rather, its proof) implies that there exists a quadratic character χ of conductor dividing $\mathfrak{n} \cdot p$ such that

$$\overline{\rho}_f \sim \overline{\rho}_{f_\sigma} \otimes \chi.$$

This implies first $a_v \equiv \chi(v) \cdot a_{v^\sigma}$ mod. $\mathfrak{p}$ for all v's prime to Norm($\mathfrak{n}$)p. Moreover, by ordinarity of the Galois representations at p (existence of an unramified line), it also implies that χ is unramified at p. Since χ is unramified at p, this is a contradiction by the choice of $\mathfrak{p}$.

In fact, for p as above and splitting totally in $\mathbb{Q}(f)$, we even have as in Lemma 5.4 of [**62**], a stronger result:

Let

$$\overline{\mathcal{H}}^0 = \{(g, g') \in \mathrm{GL}_2(I/pI) \times \mathrm{GL}_2(I/pI); \det(g) = \det(g') \in X\}$$

and

$$\overline{\mathbf{G}} = \mathrm{Im}(\mathrm{Gal}(\overline{F}/F) \longrightarrow \overline{\mathcal{H}}^0$$

Then,

$$(*) \qquad \overline{\mathbf{G}} = \overline{\mathcal{H}}^0.$$

2) Let p_0 be a prime satisfying the conditions of 1 and splitting totally in $\mathbb{Q}(f)$, so that $(*)$ holds. There exists $x \in \overline{\mathcal{H}}^0$ such that $\mathrm{Tr}(x)^2$ generates $I/p_0I \times I/p_0I$ over $\mathbb{F}_{p_0}$. Therefore, by Cebotarev density theorem, there are infinitely many finite places v such that the image of $(a_v^2, a_{v^\sigma}^2) \in I \times I$ in $I/p_0I \times I/p_0I$ generates this ring. For any such v, by Nakayama's lemma, $(a_v^2, a_{v^\sigma}^2)$ generates the ring $I_{(p_0)} \times I_{(p_0)}$ over $\mathbb{Z}_{(p_0)}$,

hence $\mathbb{Q}(f)\times\mathbb{Q}(f)$ over $\mathbb{Q}$. Fix such a v; let $J=I[(a_v^2,a_{v^\sigma}^2)]$; it is of finite index in $I\times I$. for any prime $\mathfrak{p}$ not dividing the index of J in $I\times I$, we put $\gamma=\overline{\rho}_{\pi,\mathfrak{p}}(\mathrm{Fr}_v)$; it belongs to $\overline{G}$ and $\mathrm{Tr}(\gamma)^2$ generates $k'\times k'$ over $\mathbb{F}_p$ (for $k'=I/\mathfrak{p}$). For those $\mathfrak{p}$'s, we conclude that $\overline{G}=\mathcal{H}^0$. QED.

Remark. — Simplifications of this proof and sharper bounds for the prime p can be found in Dimitrov's thesis [**19**].

In the $(2,2)$-CM case, let $f=\theta(\phi)$. For any p and any p-adic field K (with valuation ring $\mathcal{O}$ and residue field k) containing the field $\mathbb{Q}(\phi)$ of values of ϕ, we still denote by $\phi=\phi^{\mathrm{gal}}:\mathrm{Gal}(\overline{M}/M)\to K^\times$ the p-adic Galois avatar of the Hecke character ϕ. Thus, we have

$$\rho_\pi=\mathrm{Ind}_{\mathbb{Q}}^{M}(\phi).$$

Let $T\subset G=\mathrm{GSp}_4\subset\mathrm{GL}(4)$ be the standard torus of G; the homomorphism $\psi:\mathrm{Gal}(\overline{M}/M)\to\mathrm{GL}_4(\mathcal{O})$ given by $\psi=\mathrm{diag}(\phi,\phi^\sigma,\phi^{\sigma\tau},\phi^\tau)$ takes values in $T(\mathcal{O})$ by $(**)$. We have $\rho_\pi|_M\cong\psi$. Let I_ϕ be the ring of integers of $\mathbb{Q}(\phi)$; denote by k' the subfield of $k=\mathcal{O}/(\varpi)$ image of I_ϕ by the reduction map $\mathcal{O}\to k$.

We claim that for almost all p's which split totally in M, the image Ψ of ψ contains a subgroup of index $\leqslant n_1\cdot n_\sigma$ of $A=\{t\in T(k')\mid\nu(t)\in X\}$.

Observe that $\Psi\subset A$ and $\nu(\Psi)=\nu(A)$. Moreover, since the conductor $\mathfrak{f}$ of ϕ is prime to p, we see by class-field theory that the restriction of ψ to the compositum of inertia subgroups above p contains all $\mathrm{diag}(a^{n_1},b^{n_\sigma},a^{n_1}\cdot b^{-n_\sigma},1)$ with $a,b\in k'^\times$. Since $k'^\times$ is cyclic, we conclude. QED

Remark. — Note that in the $(2,2)$-CM case, p is ordinary for f and f_σ at $\mathfrak{p}$ if and only if p splits in $M=E\cdot F$.

Corollary 2. — *If $p\notin S$, splits in F, is ordinary for f and f_σ (at some $\mathfrak{p}_f|p$), and is greater than* $\max(5,\boldsymbol{w}+1)$*, (π,p) satisfies all the assumptions of Theorems 1 and 2.*

Calculations communicated to us by H. Yoshida [**91**] establish that the unique level one Hilbert cusp form over $F=\mathbb{Q}(\sqrt{5})$ of weight $(14,2)$ (hence $m=6$) admits a non-zero cuspidal theta lift π which is a classical holomorphic Siegel cusp form of level 5 and weight 8 (that is, $a=b=5$, $c=10$). The motive associated to π is rank four with Hodge weights $0,6,7,13$.

– The field $\mathbb{Q}(f)$ is equal to F and the order I_f is maximal.
– The prime 31 is greater than the motivic weight $\boldsymbol{w}=13$;
– it splits in F:

$$(31)=\mathfrak{p}\mathfrak{p}^\sigma,\quad\mathfrak{p}=\Big(\frac{13+3\sqrt{5}}{2}\Big),$$

– $\mathfrak{p}$ is ordinary for f and f_σ,
– the image Ψ of $\overline{\rho}_\pi$ is equal to

$$\{1,\sigma\}\propto\{(g,g')\in\mathrm{GL}_2(\mathbb{F}_{31})\times\mathrm{GL}_2(\mathbb{F}_{31})\mid\det g=\det g'\in(\mathbb{F}_{31}^\times)^{13}\}.$$

The verification of this last point uses Th. 3.1 of [**62**]; the main points are

– to show, for $\mathbb{F}_{31} = I_f/\mathfrak{p}$ that:

$$\Psi_f = \operatorname{Im}\overline{\rho}_f = \{g \in \mathrm{GL}(2, \mathbb{F}_{31}) \mid \det g \in (\mathbb{F}_{31}^\times)^{13}\}.$$

Indeed, Ψ_f contains a unipotent element: consider the degree 2 prime $\lambda = (3)$ in F; the number $a_\lambda^2 - 4N(\lambda)^2$ has order one at $\mathfrak{p}$. By [**68**] Lemma 1, this ensures the existence of a unipotent element. Ψ_f is not contained in a Borel: there is a prime $\mathfrak{q}$ above 11 such that $\overline{\rho}_f(\mathrm{Fr}_\mathfrak{q})$ is elliptic.

– To find a $\gamma \in \Psi$ such that $\mathrm{Tr}(\gamma)^2$ generates $I_f/(31)$ over $\mathbb{F}_{31}$. Take for that the prime $\mathfrak{q}$ above 11 as above and

$$\gamma = (\overline{\rho}_f(\mathrm{Fr}_\mathfrak{q}), \overline{\rho}_f(\mathrm{Fr}_{\mathfrak{q}^\sigma}) \in \mathrm{GL}_2(\mathbb{F}_{31}) \times \mathrm{GL}_2(\mathbb{F}_{31}).$$

One has $\mathrm{Tr}(\gamma)^2 = (28, 1) \in \mathbb{F}_{31} \times \mathbb{F}_{31}$, which generates $\mathbb{F}_{31} \times \mathbb{F}_{31}$ over $\mathbb{F}_{31}$.

This provides therefore an explicit example of a couple (π, p) satisfying all our assumptions. Other potential examples for the same F and f are $p = 19, 29$; indeed, they satisfy all the conditions above, except that non-trivial unipotent elements have not been found in the limit of the calculations of $a_\lambda^2 - 4N(\lambda)^2$ (namely, λ dividing at most 31).

Yoshida [**91**] also found that for $F = \mathbb{Q}(\sqrt{13})$, the unique level one Hilbert cusp form of weight $(10, 2)$ lifts to a nonzero holomorphic scalar-valued Siegel cuspform of level 13, weight $(6, 6)$ $(a = b = 3)$ with $\mathbb{Q}(f) = F$, and I_f maximal. The rank 4 motive associated to π has Hodge weights $0, 4, 5, 9$. The primes $p = 17$ and 29 are greater than $\boldsymbol{w} = 9$, split in F; they are ordinary for f and f_σ. The image of Galois contains $\{(x, y) \in \mathbb{F}_{p^2} \times \mathbb{F}_{p^2} \mid N(x) = N(y) \in \mathbb{F}_p^9\} \propto \{1, \sigma\}$. However, in the limit of the calculations (λ dividing at most 61) no unipotent has been found in the image for those primes. It would be interesting to find examples of cusp forms f of the minimal possible weight, namely $(4, 2)$. The theta lift π would then occur in middle degree cohomology with constant coefficients: $a = b = 0$, and the Hodge-Tate weights of ρ_π would be $0, 1, 2, 3$.

8. Proof of Theorem 2

The main tool in the proof of Th. 2 is the minimal compactification $j : X \hookrightarrow X^*$ (see 8.1 below). This compactification is far from being smooth (for $g > 1$), but it has some advantages over toroidal compactifications; namely, the strata have a very simple combinatoric and, as a consequence, the Hecke correspondences extend canonically to the boundary. Let us consider the long exact sequence of the boundary:

$$\begin{tikzcd}
\cdots\ H^d_c(S_U, V_\lambda(\mathcal{O})) \arrow[r] \arrow[d, equal] & H^d(S_U, V_\lambda(\mathcal{O})) \arrow[r] \arrow[d, equal] & H^d_\partial(S_U, V_\lambda(\mathcal{O}))\ \cdots \arrow[d, equal] \\
\cdots\ H^d_{\text{ét}}(X^*_{\overline{\mathbb{Q}}}, j_! V_\lambda(\mathcal{O})) \arrow[r] & H^d_{\text{ét}}(X^*_{\overline{\mathbb{Q}}}, Rj_* V_\lambda(\mathcal{O})) \arrow[r] & H^d_{\text{ét}}(\partial X^*_{\overline{\mathbb{Q}}}, Rj_* V_\lambda(\mathcal{O}))\ \cdots
\end{tikzcd}$$

In this section, we shall repeatedly use the standard spectral sequence for an étale sheaf $\mathcal{F}$ on X^*, and a diagram $j : X \hookrightarrow X^* \hookleftarrow Y : i$

$$H^\bullet(Y, i^* R^\bullet j_* \mathcal{F}) \Longrightarrow H^\bullet(Y, i_* Rj_* \mathcal{F}).$$

It will allow us to study (localization at $\mathfrak{m}$ of) $H^\bullet(Y, i^* R^\bullet j_* \mathcal{F})$, rather than the hypercohomology of the complex $i^* Rj_* \mathcal{F})$.

We will thus be left with the study of the Galois action on the boundary cohomology group

$$H^\bullet_{\text{ét}}(\partial X^*_{\overline{\mathbb{Q}}}, R^\bullet j_* V_\lambda(\mathcal{O}))$$

in order to show that its localization at $\mathfrak{m}$ vanishes. First, let us recall the description of $X^*_{\overline{\mathbb{Q}}}$ and the form of the spectral sequence attached to its stratification.

8.1. The minimal compactification. — The arithmetical minimal compactification $X^* = X^*_g$ of $X = X_g$ is defined in non-adelic terms in Th. 2.3 of Chapter V of [**13**]. It is a normal projective scheme over $\mathbb{Z}[1/N]$. We are only interested in the generic fiber $X^*_{\mathbb{Q}} = X^* \otimes \mathbb{Q}$. In this setting, an adelic definition can be found in [**58**] or [**59**] Sect. 3 for a general reductive group G; let us describe the strata adelically for $G = \mathrm{GSp}(2g)$. We need some notations. For $r = 1, \dots, g$, let $P_r = M_r \cdot U_{P_r}$ be the standard maximal parabolic of G associated to the simple root α_{g-r+1} (see Sect. 3.2.2). Its Levi group M_r is isomorphic to $\mathrm{GL}(r) \times \mathrm{GSp}(2g-2r)$ (recall that $\mathrm{GSp}(0) = \mathbb{G}_m$ by convention). We decompose it accordingly into a product of group schemes over $\mathbb{Z}$: $M_r = M_{r,\ell} \times M_{r,h}$, where the index ℓ, resp. h, denotes the linear, resp. hermitian part of M_r. Thus, $M_{r,h} \cong \mathrm{GSp}(2g-2r)$ admits a Shimura variety, which is a Siegel variety of genus $g - r$, while $M_{r,\ell}$ does not. Let $\kappa_r : P_r \to M_r = P_r/U_{P_r}$ and let $P_{r,h}$ be the inverse image of $M_{r,h}$ by κ_r. Let $K_{r,h}$ be the standard maximal compact times center in $M_{r,h}(\mathbb{R})$, and $\mathcal{Z}_{g-r} = M_{r,h}(\mathbb{R})/K_{r,h}$ be the Siegel space of genus $g - r$ (it has two connected components $\mathcal{Z}^\pm_{g-r}$); then the compactified symmetric space $\mathcal{Z}^*_g$ can be described set-theoretically as:

$$\mathcal{Z}^*_g = \bigsqcup_{r=0}^{g} G(\mathbb{Q}) \times^{P_r(\mathbb{Q})} \mathcal{Z}_{g-r}$$

therefore,

$$S^*_U = G(\mathbb{Q}) \backslash \mathcal{Z}^*_g \times G(\mathbb{A}_f)/U.$$

For any subgroup $V_r \subset P_r(\mathbb{A})$, let us denote by $V_{r,h}$ its projection to $M_{r,h}(\mathbb{A}) = P_r(\mathbb{A})/M_{r,\ell}(\mathbb{A}) \cdot U_{P_r}(\mathbb{A})$. Then, by simple manipulations we obtain

$$S_U^* = \bigsqcup_{r=0}^{g} \bigsqcup_{\dot{x}} S_{g-r,{}^xU_{r,h}} \tag{8.1.1}$$

where

– $\dot{x}$ runs over the finite set $P_r(\mathbb{Q})P_{r,h}(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/U$, and x denotes an arbitrary representative of $\dot{x}$ in $G(\mathbb{A}_f)$; for later use, we may and do choose x so that its p-component x_p is trivial;
– we have put ${}^xU_r = x \cdot U \cdot x^{-1} \cap P_r(\mathbb{A})$,
– we have

$$S_{g-r,{}^xU_r} = M_{r,h}(\mathbb{Q})\backslash M_{r,h}(\mathbb{A})/{}^xU_{r,h} = M_{r,h}(\mathbb{Q})\backslash \mathcal{Z}_{g-r} \times M_{r,h}(\mathbb{A}_f)/{}^xU_{r,h}.$$

Note that the disjoint union is set-theoretic, not topological; see below though.

For each $\dot{x}$, a standard application of ths Strong Approximation Theorem shows that the connected components of $S_{g-r,{}^xU_{r,h}}$ are indexed by a system $\{m_{f,h}\}$ of representatives in $M_{r,h}(\mathbb{A}_f)$ of the (finite) set of double cosets $M_{r,h}(\mathbb{Q})\backslash M_{r,h}(\mathbb{A})/{}^xU_{r,h} \cdot M_{r,h}(\mathbb{R})^+$, where $M_{r,h}(\mathbb{R})^+$ denotes the subgroup of $M_{r,h}(\mathbb{R})$ of elements with positive similitude factor. Recall that we have assumed that U is good; the condition $\nu(U) = \widehat{\mathbb{Z}}^\times$ implies that for any $r \geqslant 1$, the set $M_{r,h}(\mathbb{Q})\backslash M_{r,h}(\mathbb{A})/{}^xU_{r,h} \cdot M_{r,h}(\mathbb{R})^+$ has only one element. That is, $S_{g-r,{}^xU_{r,h}}$ is connected.

Let

$$\Gamma_{M_{r,h}}(x) = M_{r,h}(\mathbb{Q}) \cap ({}^xU_{r,h} \times M_{r,h}(\mathbb{R})^+),$$

then, we have a canonical identification

$$S_{g-r,{}^xU_{r,h}} = \Gamma_{M_{r,h}}(x)\backslash \mathcal{Z}_{g-r}^+$$

this is a Siegel variety of genus $g-r$.

By [**58**] Sect. 12.3, the decomposition (8.1.1) of S_U^* into locally closed subsets canonically descends to $\mathbb{Q}$ into a stratification of $X_{\mathbb{Q}}^*$. We have

$$\partial X_{\mathbb{Q}}^* = X_1 \sqcup \cdots \sqcup X_g$$

where the stratum X_r is defined over $\mathbb{Q}$. Actually,

$$X_r = \bigsqcup_{\dot{x}} X_{r,x} \tag{8.1.2}$$

with $\dot{x} \in P(\mathbb{Q})P_{r,h}(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/U$ and where $X_{r,x}$ is the canonical descent to $\mathbb{Q}$ of $S_{g-r,{}^xU_{r,h}}$. (8.1.2) is a disjoint union in the Zariski topology.

Recall For the Zariski topology of X^*, one has $\overline{X}_i \supset X_j$ for $i < j$ and

$$\overline{X}_i - \overline{X}_{i+1} = X_i.$$

8.2. Spectral sequence associated to the stratification. — To the stratification $\partial X^*_{\overline{\mathbb{Q}}} = \overline{X}_1 \supset \cdots \overline{X}_g \supset \overline{X}_{g+1} = \varnothing$ is associated a spectral sequence in Betti or étale cohomology

$$(8.2.1) \quad E_1^{p-1,q} = H_c^{p-1+q}(\overline{X}_p - \overline{X}_{p+1}, k_p^* Rj_* V_\lambda(k)) \Longrightarrow H^{p-1+q}(\partial X^*_{\overline{\mathbb{Q}}}, Rj_* V_\lambda(k))$$

where $k_r : X_r \hookrightarrow \partial X^*$ denotes the locally closed embedding of $X_r = \overline{X}_r - \overline{X}_{r+1}$. It is compatible with algebraic correspondences preserving the stratification. It is mentioned as a remark in Milne, Etale Coh. Chap. III, Remark 1.30. We don't know a complete reference for it, hence we sketch the proof: Given a stratification on a scheme Y, by closed subsets $Y = Y_0 \supset Y_1 \supset \cdots \supset Y_{n+1} = \varnothing$, given a complex of etale sheaves $\mathbf{V}$ on Y with constructible cohomology, we consider for $p < q$ the open immersion $j_{pq} : Y_p - Y_q \hookrightarrow Y_p$ and the closed immersion $i_{pq} : Y_q \hookrightarrow Y_p$. Let $\mathbf{V}_p = i_{0p}^* \mathbf{V}$; we have $\mathbf{V}_q = i_{pq}^* \mathbf{V}_p$ for any $p < q$. We have short exact sequences

$$0 \longrightarrow j_{pq,!} \mathbf{V}_p|_{Y_p - Y_q} \longrightarrow \mathbf{V}_p \longrightarrow i_{pq,*} i_{pq}^* \mathbf{V}_p \longrightarrow 0$$

This yields a stratification on the complex $\mathbf{V}$:

$$0 \subset j_{01\,!}(\mathbf{V}|_{Y-Y_1}) \subset j_{02\,!}(\mathbf{V}|_{Y-Y_2}) \subset \cdots \subset j_{0p\,!}(\mathbf{V}|_{Y-Y_p}) \subset \cdots \mathbf{V}.$$

Note that for any $p \geqslant 1$:

$$j_{0p\,!}(\mathbf{V}|_{Y-Y_p})/j_{0,p-1\,!}(\mathbf{V}|_{Y-Y_{p-1}}) \cong i_{0,p-1\,*} j_{p-1,p\,!} \mathbf{V}_{p-1}|_{Y_{p-1}-Y_p},$$

hence,

$$E_1^{p-1,q} = H_c^{p-1+q}(Y_{p-1} - Y_p, (i_{0,p-1}^* \mathbf{V})|_{Y_{p-1}-Y_p})$$

as desired.

Let us apply this sequence to our stratification. We have for any $r \geqslant 1$:

$$\overline{X}_r - \overline{X}_{r+1} = \bigsqcup_{\dot{x}} X_{r,x}.$$

So,

$$(8.2.2) \qquad E_1^{r-1,s} = \bigoplus_{\dot{x}} H_c^{r-1+s}(X_{r,x}, Rj_* V_\lambda(k)|_{X_{r,x}}).$$

By the standard spectral sequence

$$H_c^\bullet(X_{r,x}, R^\bullet j_* V_\lambda(k)|_{X_{r,x}}) \Longrightarrow H_c^\bullet(X_{r,x}, Rj_* V_\lambda(k)|_{X_{r,x}}).$$

We are left with the study of $R^\bullet j_* V_\lambda(k) = \mathrm{gr}^\bullet Rj_* V_\lambda(k)$.

8.3. The restriction of the higher direct image sheaf to the strata. — It is easy to determine the restriction mentioned above on the analytic site (in Betti cohomology). The details are in [**35**] Sect. 2.2.5. One finds that the sheaf $R^\bullet j_* V_\lambda(k)$ restricted to the stratum $S_{g-r,{}^xU_{r,h}}$ is the locally constant sheaf on $S_{g-r,{}^xU_{r,h}}$ associated to the $\Gamma_{M_{r,h}}(x)$-module:

$$H^\bullet(\Gamma_{M_{r,\ell}}(x), H^\bullet(\Gamma_{U_{P_r}}(x), V_\lambda(k)))$$

where

$$\Gamma_{M_{r,\ell}}(x) = M_{r,\ell}(\mathbb{Q}) \cap ({}^xU_{r,\ell} \times M_{r,\ell}(\mathbb{R})), \quad \text{for } {}^xU_{r,\ell} = \kappa_r({}^xU) \cap M_{r,\ell}(\mathbb{A}_f)$$

and

$$\Gamma_{U_{P_r}}(x) = U_{P_r}(\mathbb{Q}) \cap ({}^xU \cap U_{P_r}(\mathbb{A}_f) \times U_{P_r}(\mathbb{R})).$$

The main result of [**59**] is that, replacing the Betti site by the étale site, this result remains true. More precisely, by Th. (5.3.1) of [**59**], the sheaf $R^\bullet j_* V_\lambda(\mathbb{F}_p)$ over $X^*_{/\mathbb{Q}}$ restricted to $X_{r,x\,/\mathbb{Q}}$ is obtained by canonical construction from the representation of $M_{r,h} \otimes \mathbb{F}_p$ on

$$H^\bullet(\Gamma_{M_{r,\ell}}(x), H^\bullet(\Gamma_{U_{P_r}}(x), V_\lambda(\mathbb{F}_p))).$$

(and similarly for k instead of $\mathbb{F}_p$). We then mention a mod. p version of Kostant decomposition theorem. Recall we have chosen the representatives $x \in G(\mathbb{A}_f)$ so that $x_p = 1$. This implies in particular that $\Gamma_{U_{P_r}}(x)$ is dense in $U_{P_r}(\mathbb{Z}_p)$. For any reductive subgroup $M \subset G$, and any $(M, B \cap M)$-dominant weight μ of $T \cap M$, let $V_{M,\mu}$ be the Weyl $\mathbb{Z}_p$-module of highest weight μ for M.

Lemma 14. — *Assuming $p - 1 > |\lambda + \rho|$, then, for any $r \geqslant 1$, the semisimplification of the $\mathbb{F}_p\Gamma_{M_r}(x)$-module*

$$H^q(\Gamma_{U_{P_r}}(x), V_\lambda(\mathbb{F}_p))$$

is an $M_r(\mathbb{F}_p)$-module whose decomposition into irreducible M_r-modules is given by:

$$H^q(\Gamma_{U_{P_r}}(x), V_\lambda(\mathbb{F}_p))^{ss} = \bigoplus_{\substack{w" \in W^{P_r} \\ \ell(w")=q}} V_{M_r, w"(\lambda+\rho)-\rho}$$

Proof. — Over $\mathbb{Q}_p$, the module itself is semisimple and the decomposition is given by Kostant's theorem. By Theorem C of [**61**], for p as stated,

$$H^\bullet(\Gamma_{U_{P_r}}(x), V_\lambda(\mathbb{Z}_p))$$

is torsion-free. Therefore $H^\bullet(\Gamma_{U_{P_r}}(x), V_\lambda(\mathbb{Z}_p))$ is a stable lattice in

$$H^\bullet(\Gamma_{U_{P_r}}(x), V_\lambda(\mathbb{Q}_p))$$

Then, the determination of its composition factors as $\mathbb{Z}_p[M_r(\mathbb{F}_p)]$-module, for p as stated, is the content of Cor. 3.8 of [**61**].

Recall that $M_r = M_{r,\ell} \times M_{r,h}$. Let $T_\ell = T \cap M_{r,\ell}$ and $T_h = T \cap M_{r,h}$; note that T_ℓ consists in the $t \in T$ of the form

$$\mathrm{diag}(t_g, \dots, t_{g-r+1}, 1 \dots, 1, t_{g-r+1}^{-1}, \dots, t_1^{-1}),$$

while the maximal torus T_h of $M_{r,h}$ consists in the elements

$$t = \mathrm{diag}(t_g, \dots, t_1, \nu \cdot t_1^{-1}, \cdots, \nu \cdot t_g^{-1})) \in T$$

such that $t_g = \cdots = t_{g-r+1} = 1$. For $\mu_{w"} = w"(\lambda+\rho) - \rho \in X^*(T)$, we denote the restrictions to T_ℓ resp. T_h by $\mu_{w",\ell} = \mu_{w"}|_{T_\ell}$, and $\mu_{w",h} = \mu_{w"}|_{T_h}$; since $\mu_{w"}$ is dominant for $(M, B \cap M)$, $\mu_{w",\ell}$, resp. $\mu_{w",h}$, is dominant for $(M_\ell, B \cap M_\ell)$, resp. $(M_h, B \cap M_h)$.

By Theorem 1 of [**61**], it follows from $p-1 > |\lambda+\rho|$, that the irreducible $M_{r/\mathbb{Z}_p}$-module $V_{M_r,\mu_{w"}}$ can be decomposed as a tensor product of irreducible $\mathbb{Z}_p$-modules over $M_{r,\ell}$ resp. $M_{r,h}$:

$$V_{M_r,\mu_{w"}} = V_{M_{r,h},\mu_{w",h}} \otimes V_{M_{r,\ell},\mu_{w",\ell}}.$$

Therefore, as $M_{r,h}$-module, we have

$$\begin{aligned}(8.3.1)\quad & H^\bullet(\Gamma_{M_{r,\ell}}(x), H^\bullet(\Gamma_{U_{P_r}}(x), V_\lambda(\mathbb{F}_p)) \\ & \qquad = \bigoplus_{w''\in W^{P_r}} H^\bullet(\Gamma_{M_{r,\ell}}(x), V_{M_{r,\ell},\mu_{w'',\ell}}) \otimes V_{M_{r,h},\mu_{w'',h}}.\end{aligned}$$

Thus, the étale sheaf on $X_{r,x\,/\mathbb{Q}}$ associated to this representation of $M_{r,h}$ is

$$(8.3.2)\qquad \bigoplus_{w''\in W^{P_r}} H^\bullet(\Gamma_{M_{r,\ell}}(x), V_{M_{r,\ell},\mu_{w'',\ell}}) \otimes V_{M_{r,h},\mu_{w'',h}}(\mathbb{F}_p).$$

In particular, the Galois action on the étale cohomology over $X_{r,x}\otimes\overline{\mathbb{Q}}$ of this sheaf arises only from the second factors of each summand.

8.4. "Hodge-Tate weights" of the E_1-terms. — Recall that $x_p = 1$, hence ${}^xU_{r,h}$ is of level prime to p, so that $X_{r,x}$ has good reduction at p. For each $r \geqslant 1$, and each $w'' \in W^{P_r}$, let us determine the Hodge filtration of the crystalline representations

$$H_c^\bullet(X_{r,x}\otimes\overline{\mathbb{Q}}_p, V_{M_{r,h},\mu_{w'',h}}(\mathbb{F}_p)).$$

We have $\dim X_{r,x} = d_r = (g-r)(g-r+1)/2$. Since $d_r + |\mu_{w'',h}| < p-1$, Faltings' comparison Th. 5.3 of [**22**] applies. Again, as in Sect. 7.2, one determines the weights using the modulo p BGG complex (quasi-isomorphic to de Rham by Cor. 1 to Th. 6). Let $Q(G_{g-r})$ be the Siegel parabolic of $G_{g-r} = M_{r,h}$ and $M(G_{g-r})$ its standard Levi subgroup. The weights are given by

$$-(w'(\mu_{w''}+\rho_h)-\rho_h)(H_h) = -w'(w''(\lambda+\rho)-\rho+\rho_h)-\rho_h)(H_h)$$

where $w' \in W_{G_{g-r}}^{M(G_{g-r})}$. By the description of T_h given above, we see that $H_h = H$ and $w'(-\rho+\rho_h) = -\rho+\rho_h$, hence, the weights are

$$(8.3.1)\qquad p(w) = -(w(\lambda+\rho)-\rho)(H) \quad \text{for}\, w = w'\circ w''$$

Claim. — *For $r \geqslant 1$ and $w'' \in W^{P_r}$, let*

$$W_G(w'') = \{w \in W_G \mid w = w'\circ w'',\ \textit{for}\ w' \in W_{G_{g-r}}^{M(G_{g-r})}\}.$$

Then, the function $W_G(w'') \to \mathbb{N}$, $w \mapsto p(w)$ cannot take both values 0 and $\boldsymbol{w}$.

Proof. — As already observed, the function $w \mapsto p(w)$ factors through $W_M \backslash W_G$. We see that $p(w) = 0$ if and only if $w \in W_M$ and $p(w) = \boldsymbol{w}$ if and only if $w \in W_M w_0$ where w_0 is the longest length element of W_G. Recall that $p(w) = j_B = \sum_{i \in B}(a_i + i)$ where B denotes the subset of $[1, g]$ corresponding to the $\{\pm 1\}^g$-component of the Weyl group as in Sect. 2.3.1. The point is to verify that $|p(w'w) - p(w)| < \boldsymbol{w}$ for $w' \in W_{G_{g-r}}^{M(G_{g-r})}$. We have the compatible identifications

$$\begin{aligned} W_G &\cong \mathfrak{S}_g \propto \{\pm 1\}^g \\ W_M &\cong \mathfrak{S}_g \\ W_{P_r} &\cong \mathfrak{S}_r \times (\mathfrak{S}_{g-r} \propto \{\pm 1\}^{g-r}) \\ W_{G_{g-r}} &\cong \mathfrak{S}_{g-r} \propto \{\pm 1\}^{g-r} \end{aligned}$$

By definition of the semidirect product, we have:

$$w'w = (\sigma, w_B)(\sigma', w_{B'}) = (\sigma\sigma', {\sigma'}^{-1}(B) \Delta B')$$

where $C \Delta C'$ denotes the symmetric difference of subsets C, C' of $[1, g]$. Since the elements w' being in $W_{G_{g-r}}$, the cardinality of B is at most $g - r$, hence the same holds for ${\sigma'}^{-1}(B)$. In particular, if $w \in W_M$, *i.e.* $B' = \varnothing$, then for any B, ${\sigma'}^{-1}(B)\Delta B' \neq [1, g]$ and similarly if $w \in W_M w_0$, *i.e.* $B' = [1, g]$, then for any B, ${\sigma'}^{-1}(B)\Delta B' \neq \varnothing$, as desired.

8.5. Hecke algebras for strata. — Let S be a finite set of primes containing the level of all strata but not containing p. Let $\mathcal{H}(G_g)^S = \bigotimes_{\ell \notin S} \mathcal{H}(G_g)_\ell$, resp. $\mathcal{H}(M(G_g))^S = \bigotimes_{\ell \notin S} \mathcal{H}(M(G_g))_\ell$ be the abstract Hecke algebras generated over $\mathbb{Z}$ by double classes at all primes $\ell \notin S$, for $G_g = G$ resp. the Levi $M(G_g)$ of the Siegel parabolic $Q(G_g)$. For each $r \geqslant 1$, we fix $M_r = \mathrm{GL}(r) \times G_{g-r}$, $\mathrm{diag}(A, B, \nu \cdot {}^tA^{-1}) \mapsto (A, B)$, where $\nu = \nu(B)$. By this identification, we can decompose $\mathcal{H}(M_r) = \mathcal{H}(\mathrm{GL}(r)) \otimes \mathcal{H}(G_{g-r})$; we introduce also $\mathcal{H}(M(G_{g-r}))$. For each prime $q \notin S$, by Satake isomorphism, we see that the fraction fields of the q-local Hecke algebras over $\mathbb{R}$ fit in a diagram of finite field extensions:

$$\begin{tikzcd} \mathrm{Fr}(\mathcal{H}(M(G_g))_q)_\mathbb{R} \arrow[r] & \mathrm{Fr}(\mathcal{H}(\mathrm{GL}(r))_q \otimes \mathcal{H}(M(G_{g-r}))_q)_\mathbb{R} \\ \mathrm{Fr}(\mathcal{H}(G_g)_q)_\mathbb{R} \arrow[r] \arrow[u] & \mathrm{Fr}(\mathcal{H}(\mathrm{GL}(r))_q \otimes \mathcal{H}(G_{g-r})_q)_\mathbb{R} \arrow[u] \end{tikzcd}$$

It corresponds (see [**13**] Sect. VII.1 p. 246) by Galois correspondence to the diagram of subgroups of $\mathfrak{S}_g \propto \{\pm 1\}^g$:

$$\begin{tikzcd} & \mathfrak{S}_r \times \mathfrak{S}_{g-r} \arrow[dl] \arrow[dr] & \\ \mathfrak{S}_g \arrow[dr] & & \mathfrak{S}_r \times (\mathfrak{S}_{g-r} \propto \{\pm 1\}^{g-r}) \arrow[dl] \\ & \mathfrak{S}_g \propto \{\pm 1\}^g & \end{tikzcd}$$

The diagram of fields can be descended from $\mathbb{R}$ to $\mathbb{Q}$ by using twisted action of the Weyl groups as in Sect. VII.1 p. 246 of [**13**]. In particular, $\mathcal{H}(M(G_g))_q$ and $\mathcal{H}(\mathrm{GL}(r))_q \otimes \mathcal{H}(G_{g-r})_q$ are linearly disjoint over $\mathcal{H}(G_g)_q$:

$$\begin{aligned}(8.5.1)\quad & \mathrm{Fr}(\mathcal{H}(\mathrm{GL}(r))_q \otimes \mathcal{H}(M(G_{g-r}))_q) \\ &\qquad = \mathrm{Fr}(\mathcal{H}(\mathrm{GL}(r))_q \otimes \mathcal{H}(G_{g-r})_q) \cdot \mathrm{Fr}(\mathcal{H}(M(G_g))_q).\end{aligned}$$

On the other hand, as a consequence of Satake isomorphism, the Hecke-Frobenius element

$$U_{q,G} = K_g \operatorname{diag}(q \cdot 1_g, 1_g) K_g$$

where K_g denotes the standard hyperspecial maximal compact subgroup of $M(G_g)$, resp.

$$U_{q,G_{g-r}} = K_{g-r} \operatorname{diag}(q \cdot 1_{g-r}, 1_{g-r}) K_{g-r}$$

(with a similar definition for K_{g-r}), generates $\mathrm{Fr}(\mathcal{H}(M(G_g))_q)$ over $\mathrm{Fr}(\mathcal{H}(G_g)_q)$, resp. $\mathrm{Fr}(\mathcal{H}(M(G_{g-r}))_q)$ over $\mathrm{Fr}(\mathcal{H}(G_{g-r})_q)$ (see Sect. VII.1 of [**13**]). For $r = g$, note that we define G_0 as $\mathbb{G}_m$ and $U_q = [q]$. Then, for any $r = 1, \dots, g$, we have

$$U_{q,G} = 1_{\mathcal{H}(\mathrm{GL}(r)} \otimes U_{q,G_{g-r}}$$

From (8.5.1), we see that the minimal polynomial $\mathrm{Irr}(X, U_{q,G}, \mathcal{H}(G_g))$ is divisible by $\mathrm{Irr}(X, 1_{\mathcal{H}(\mathrm{GL}(r))} \otimes U_{q,G_{g-r}}, \mathcal{H}(\mathrm{GL}(r))_q \otimes \mathcal{H}(G_{g-r})_q)$.

The Hecke algebra $\mathcal{H}(G_g)^S$ acts on each stratum $X_r = \bigsqcup_{\dot{x}} X_{r,x}$ by $\mathbb{Q}$-rational algebraic correspondences. Indeed, there is a surjective homomorphism of $\mathbb{Z}$-algebras

$$\phi_{g-r} : \mathcal{H}(M(G_g))^S \longrightarrow \mathcal{H}(M(G_{g-r}))^S,$$

$$\begin{aligned}&[G_g(\mathbb{Z}_q) \cdot \operatorname{diag}(a_r, b_{2g-2r}, c_r) \cdot G_g(\mathbb{Z}_q)] \longmapsto \\ &\qquad \begin{cases} [G_{g-r}(\mathbb{Z}_q) \cdot \operatorname{diag}(b_{2g-2r}) \cdot G_{g-r}(\mathbb{Z}_q)] & \text{if } a_r \in T_{\mathrm{GL}(r)}(\mathbb{Z}_q) \\ 0 & \text{if not.} \end{cases}\end{aligned}$$

See [**26**], Sect. IV.3.

On $S_{g-r}, {}^xU_{r,h}$, we let the double class $[U\alpha U]$ act by the algebraic correspondence associated to $\phi_{g-r}([U\alpha U])$. By the theory of canonical models, since $\nu(U) = \widehat{\mathbb{Z}}^\times$, these correspondences are defined over $\mathbb{Q}$.

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{H}(G_g)$ associated to $\overline{\theta}_\pi$. Let

$$W^{r,s} = E_{1\,\mathfrak{m}}^{r-1,s} = (\bigoplus_{\dot{x}} H_c^{r-1+s}(X_{r,x} \otimes \overline{\mathbb{Q}}, R^\bullet j_* V_\lambda(k)|_{X_{r,x}})[\mathfrak{m}]$$

Lemma 15. — *For any $q \notin S$, the characteristic polynomial of $\overline{\rho}_\pi$ annihilates the action of the geometric Frobenius Fr_q on $W^{r,s}$.*

Proof. — By Theorem 4.2, Chap. VIII of [**13**], we know that

$$\mathrm{Irr}(X, U_{q,G_{g-r}}, \mathcal{H}(G_{g-r})_q)$$

annihilates Fr_q on $W^{r,s}$. By the divisibility relation obtained above, we also have $\mathrm{Irr}(X, U_{q,G}, \mathcal{H}(G_g))|_{X=\mathrm{Fr}_q} = 0$ on $W^{r,s}$. By definition of $\overline{\rho}_\pi$, we have $\mathrm{char}(\overline{\rho}_\pi(\mathrm{Fr}_q)) = \mathrm{Irr}(X, U_{q,G}, \mathcal{H}(G_g))$, as desired.

8.6. End of the proof. — By the previous lemma, we can apply Lemma 13 to $W^{r,s}$ (for $r \geqslant 1$): if $W^{r,s} \neq 0$, both characters 1 and $\omega^{-\boldsymbol{w}}$ occur in $W^{r,s}|_{I_p}$. This contradicts the Claim in Sect. 8.4. Thus, we have for any $s \geqslant 0$, $E_{1\,\mathfrak{m}}^{r-1,s} = 0$. By (8.2.1) and (8.2.2), we conclude that for any $r \geqslant 1$ and any $s \geqslant 0$, $H^{r-1+s}(\partial X^*, R^\bullet j_* V_\lambda(k))_\mathfrak{m} = 0$ as desired. By the long exact sequence of cohomology of the boundary, we obtain $H^d_c(X, V_\lambda(\mathcal{O}))_\mathfrak{m} = H^d(X, V_\lambda(\mathcal{O}))_\mathfrak{m}$. We deduce the corollary:

Corollary 3. — *For (π, p) as in Th. 1, the natural maps induce an isomorphism*

$$H^d_c(X, V_\lambda(\mathcal{O}))_\mathfrak{m} = H^d(X, V_\lambda(\mathcal{O}))_\mathfrak{m}.$$

This is the first part of theorem 2.

8.7. Intersection cohomology. — For the minimal compactification $j : X \hookrightarrow X^*$ and an etale sheaf $\mathcal{F}$ over X, we consider the intermediate extension $j_{!,*}\mathcal{F}$. By [**2**], prop. 2.1.11, we have the following description of this complex:

$$j_{!,*}\mathcal{F} = \tau_{<c_g} Rj_{g,*} \tau_{<c_{g-1}} Rj_{g-1,*} \cdots \tau_{<c_1} Rj_{1,*}\mathcal{F}$$

where for $U_r = \coprod_{0 \leqslant i \leqslant r} X_i$, we put $j_r : U_{r-1} \hookrightarrow U_r$, $r = 1, \dots, g$, c_r is the codimension of the stratum X_r in $\overline{X}_{r-1}$, and the truncation $\tau_{<c}$ is the canonical truncation; it is characterized by $\mathcal{H}^j(\tau_{<c}\mathcal{K}) = \mathcal{H}^j(\mathcal{K})$ if $j < c$, and $\mathcal{H}^j(\tau_{<c}\mathcal{K}) = 0$ if $j \geqslant c$.

We have

$$\begin{array}{ccccccc}
\cdots & H^d_c(S_U, V_\lambda(\mathcal{O})) & \longrightarrow & IH^d(S_U, V_\lambda(\mathcal{O})) & \longrightarrow & IH^d_\partial(S_U, V_\lambda(\mathcal{O})) & \cdots \\
& \Vert & & \Vert & & \Vert & \\
\cdots & H^d_{\text{ét}}(X^*_{\overline{\mathbb{Q}}}, j_! V_\lambda(\mathcal{O})) & \longrightarrow & H^d_{\text{ét}}(X^*_{\overline{\mathbb{Q}}}, j_{!,*} V_\lambda(\mathcal{O})) & \longrightarrow & H^d_{\text{ét}}(\partial X^*_{\overline{\mathbb{Q}}}, j_{!,*} V_\lambda(\mathcal{O})) & \cdots
\end{array}$$

Proposition 6. — $IH^\bullet_\partial(S_U, V_\lambda(\mathcal{O}))_\mathfrak{m} = 0$.

The proof will be similar to the usual cohomology case: it relies on Pink's theorem, lemma 13 and a variant of Claim 8.4. Some more induction is needed though, due to the successive truncations involved in defining $j_{!,*}V_\lambda$.

By the spectral sequence (Sect. 8.2) associated to our stratification, we are reduced to show

$$H^\bullet_{c,\text{ét}}(X_{r,x}, j_{!,*}V_\lambda(k))_\mathfrak{m} = 0.$$

Lemma 16. — *$H^\bullet_{c,\text{ét}}(X_{r,x}, j_{!,*}V_\lambda(\mathbb{F}_p))$ admits a filtration stable by Galois and Hecke actions and whose successive quotients are Galois and Hecke subquotients of*

$$H^\bullet(X_{r,x}, R^\bullet j_{g,*} \circ R^\bullet j_{g-1,*} \circ \cdots \circ R^\bullet j_{1,*}V_\lambda(\mathbb{F}_p))$$

where the $\bullet$ denote unspecified given integers.

Proof. — We write the argument for $g = 1$ and 2. For $g = 1$, it follows directly from the second spectral sequence associated to the complex $\tau_{<c_1}Rj_{1,*}V_\lambda(\mathbb{F}_p)$:

$$H^\bullet_c(X_r, \tau_{<c_1}R^\bullet j_{1,*}V_\lambda(\mathbb{F}_p)) \Longrightarrow H^\bullet_c(X_r, \tau_{<c_1}Rj_{1,*}V_\lambda(\mathbb{F}_p)).$$

In this notation, $\tau_{<c_1}R^\bullet j_{1,*}V_\lambda(\mathbb{F}_p)$ denotes $R^\bullet j_{1,*}V_\lambda(\mathbb{F}_p)$ if the unspecified integer $\bullet$ is $< c_1$, and is zero if not.

For $g = 2$, applying this "second spectral sequence" to

$$\tau_{<c_2}Rj_{2,*}(\tau_{<c_1}Rj_{1,*}V_\lambda(\mathbb{F}_p))),$$

The group $H^d_{c,\text{ét}}(X_{r,x}, j_{!,*}V_\lambda(\mathbb{F}_p))$ admits a dévissage by subquotients of

$$H^\bullet_c(X_r, \tau_{<c_2}R^\bullet j_{2,*}\tau_{<c_1}Rj_{1,*}V_\lambda(\mathbb{F}_p)).$$

(with similar convention concerning $\tau_{<c_2}R^\bullet j_{2,*}(\dots)$). The complex inside the cohomology is filtered, hence the cohomology itself is filtered and its graded pieces are subquotients of

$$H^\bullet_c(X_r, \tau_{<c_2}R^\bullet j_{2,*}\tau_{<c_1}R^\bullet j_{1,*}V_\lambda(\mathbb{F}_p))$$

by the formalism of spectral sequences.

Let

$$W(r) = \prod_{s=0}^{r} W^{P_s}_{G_{g-s}}$$

(so, $W(0) = \{1\}$). For $w(r) = (w_r, \dots, w_1) \in W(r)$, the symbol $w(r) \cdot (\lambda + \rho(r))$ is defined by induction by

$$w(r+1) \cdot (\lambda + \rho(r+1)) = w_{r+1} \cdot (w(r) \cdot (\lambda + \rho(r)) + \rho_{r+1}).$$

(recall that ρ_r denotes the half-sum of positive roots of G_{g-r} for the order deduced from (G_g, B_g, T_g)) and $w_r \cdot (\lambda + \rho_r) = w_r(\lambda + \rho_r) - \rho_r$. One sees by induction on r that $|\lambda + \rho| < p - 1$ implies $|w(r) \cdot (\lambda + \rho(r))|_r < p - 1$ for any $r \geqslant 0$.

Definition 8. — Let λ be a p-small dominant weight of $G = G_g$. For any integer $r \in [1, g]$, we say that a locally constant sheaf on the stratum X_r is a Kostant sheaf of type λ if it comes by the canonical construction from a $\mathbb{F}_p\Gamma_{M_r}(x)$-module whose semisimplification is a direct sum of irreducible $M_r(\mathbb{F}_p)$-modules $V_{w(r)\cdot(\lambda+\rho(r))}$ for some $w(r)$'s of $W(r)$.

Remark. — The category of Kostant sheaves of type λ on X_r is abelian and stable by extension. However, it is probably not be semisimple.

Lemma 17. — *The sheaf $R^{\alpha_g} j_{g,*} \circ \cdots \circ R^{\alpha_1} j_{1,*} V_\lambda(\mathbb{F}_p)$ is constructible finite étale; for $r = 0, \dots, g$, its restriction to the stratum X_r is a Kostant sheaf of type λ which is* 0 *unless $\alpha_{r+1} = \cdots = \alpha_g = 0$.*

Proof. — For this proof, some more notations are needed. Let $j_{pq} : U_p \hookrightarrow U_q$ for $p < q$; thus, $j = j_{0,g} = \cdots = j_r \circ j_{0,r}$. Let $i_{p,q} : X_p \hookrightarrow U_q$ denotes the locally closed immersion of X_p in U_q (composition of the closed immersion $i_p : X_p \hookrightarrow U_p$ followed by $j_{p,q}$). Note that $j_{0,r} = i_{0,r}$.

For each r, we consider the abelian category $\mathcal{C}_r$ of constructible étale sheaves in $\mathbb{F}_p$-vector spaces over U_r; let $\mathcal{A}_r$ be the (full) abelian subcategory of $\mathcal{C}_r$ generated by the $j_{s,r,!} i_{s,s,*} F_s$ $(0 \leqslant s \leqslant r)$ where F_s is a Kostant sheaf of type λ on X_s. Since these sheaves are supported by the strata X_s and since there are no non-zero morphisms between sheaves with disjoint support, $\mathcal{A}_r$ consists exactly in the objects mentioned.

Let $\mathcal{B}_r$ be the (full) abelian subcategory of $\mathcal{C}_r$ stable by extension generated by $\mathcal{A}_r$. It coincides with the subcategory of $\mathcal{C}_r$ of sheaves whose restriction to each stratum X_r is Kostant of type λ.

Let us first prove that the sheaves of the form $G = j_{r-i,r,!} i_{s,r-i,*} F_s$, $0 \leqslant s \leqslant r - i$ are objects of $\mathcal{B}_r$.

Indeed, we have the short exact sequence:

$$0 \longrightarrow j_{r-i-1,r,!} i_{s,r-i-1,*} F_s \longrightarrow G \longrightarrow j_{r-i,r,!} i_{r-i,r-i,*} i^*_{r-i,r-i} G \longrightarrow 0$$

We show first that the right member of this short exact sequence belongs to $\mathcal{A}_r$. We recall that the closure of X_s in X^* coincides with the minimal compactification X^*_s of X_s. So, we can apply the main result of [**59**] to the open (in X^*_s) immersion $i_{s,r-i}$ in order to compute the restrictions to the stratum X_{r-i} of the sheaf $i_{s,r-i,*} F_s$. This yields the formula

$$i^*_{r-i,r-i} G = i^*_{r-i,r-i} i_{s,r-i,*} F_s = F_{r-i}$$

for a locally constant sheaf F_{r-i}. Therefore,

$$j_{r-i,r,!} i_{r-i,r-i,*} i^*_{r-i,r-i} G = j_{r-i,r,!} i_{r-i,r-i,*} F_{r-i}$$

is in $\mathcal{B}_r$.
On the other hand, by decreasing induction on i, the sheaf $j_{r-i-1,r,!} i_{s,r-i-1,*} F_s$ on the left is in $\mathcal{B}_r$ (the first step of the induction is true since for $i = r - s$, we have $j_{s,r,!} i_{s,s,*} F_s \in \mathcal{B}_r$). In particular, the sheaves $i_{s,r,*} F_s$ are objects of $\mathcal{B}_r$.

Remark. — If any finite $\mathbb{F}_p \Gamma_{M_r}(x)$-module with p-small highest weight (in the set-theoretic sense: that is, for the action of $T(\mathbb{Z}/p\mathbb{Z})$) were algebraic with p-small weight in the schematic sense, it would follow from [**61**] Lemma 1.11 that it would be semisimple. This statement however, is false as shown by the example $V = \mathrm{Sym}^p \mathbb{F}_p^2$ for GL_2 and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Thus, $\mathcal{A}_r$ and $\mathcal{B}_r|_{X_r}$ are not semisimple. Fortunately, this semisimplicity won't be used in the sequel.

Let us return to the proof of Lemma 17. We proceed by induction on g. It is clear for $g = 1$. Assume the result is true for $g - 1$.

It is enough to show by induction on $r \geqslant 0$ the following statement

$$(P_r) \qquad R^{\alpha_r} j_{r-1,r,*} \circ \cdots \circ R^{\alpha_1} j_{0,1,*} V_\lambda(\mathbb{F}_p) \in \mathcal{B}_r.$$

(P_r) is obvious for $r = 0$. For $r = 1$, let $\mathbb{R}_1 = R^{\alpha_1} j_{0,1,*} V_\lambda$; we know that $\mathbb{R}_1|_{X_1}$ is a Kostant sheaf by Lemma 14. Therefore, we have an exact sequence on U_1:

$$0 \longrightarrow j_{0,1,!} V_\lambda \otimes T_0 \longrightarrow \mathbb{R}_1 \longrightarrow i_{1,1,*} i_{1,1}^* \mathbb{R}_1 \longrightarrow 0$$

for some multiplicity vector space T_0 (with $T_0 = \mathbb{F}_p$ if $\alpha_1 = 0$ and 0 otherwise).

Induction step. — Assume that (P_{r-1}) holds. Note that $R^\bullet j_{r-1,r,*}$ preserves $\mathcal{C}_r$. Let

$$\mathbb{R}_{r-1} = R^{\alpha_{r-1}} j_{r-2,r-1,*} \circ \cdots \circ R^{\alpha_1} j_{0,1,*} V_\lambda(\mathbb{F}_p).$$

By assumption there is a filtration $F^\bullet \mathbb{R}_{r-1}$ whose graded pieces are in $\mathcal{A}_{r-1}$.

Hence, since $\mathcal{B}_r$ is abelian, $R^\bullet j_{r-1,r,*} \mathbb{R}_{r-1}$ will be in $\mathcal{B}_r$ if for each s between 0 and $r - 1$:

$$(8.6.1) \qquad R^\bullet j_{r-1,r,*} j_{s,r-1,!} i_{s,s,*} F_s \text{ is in } \mathcal{B}_r$$

for any Kostant sheaf F_s of type λ.

We can assume $s = 0$ (by replacing X by the Siegel variety X_s), and we have to prove that $R^\bullet j_{r-1,r,*} j_{0,r-1,!} F_0 \in \mathcal{B}_r$. We prove in the Appendix that such a sheaf is constructible with respect to the natural stratification of X^*. Therefore, it remains only to show that for each $s \leqslant r$, the locally constant sheaf

$$R^\bullet j_{r-1,r,*} j_{0,r-1,!} F_0|_{X_s}$$

is Kostant of type λ.

For this purpose, it will be enough to show that $R^\bullet j_{r-1,r,*} j_{0,r-1,*} F_0$ is constructible and Kostant on each stratum X_s ($s \leqslant r$). Indeed, let us consider the short exact sequences

$$0 \longrightarrow j_{t,r-1,!} j_{0,t,*} F_0 \longrightarrow j_{t+1,r-1,!} j_{0,t+1,*} F_0 \longrightarrow j_{t+1,r-1,!} i_{t+1,t+1,*} F_{t+1} \longrightarrow 0$$

where $t = 0, \ldots, r-2$ and $F_t = i_t^*(j_{0,t,*} F_0)$. Note that by the induction hypothesis (for the Siegel variety X_{t+1}) $R^\bullet j_{r-1,r,*} j_{t+1,r-1,!} i_{t+1,t+1,*} F_{t+1} \in \mathcal{B}_r$. Therefore, by considering long exact sequences for $R j_{r-1,r,*}$ associated to these short exact sequences, we see that $R^\bullet j_{r-1,r,*} j_{0,r-1,!} F_0 \in \mathcal{B}_r$ if and only if $R^\bullet j_{r-1,r,*} j_{0,r-1,*} F_0 \in \mathcal{B}_r$.

This sheaf is the $E_2^{\bullet,0}$-term in the spectral sequence of composition of two functors abutting at

$$R^\bullet j_{0,r,*} F_0$$

By Sublemma 1 below, this abutment is of type $\mathcal{B}_r$. Let us check that for $q > 0$,

$$E_2^{p,q} = R^p j_{r-1,r,*} R^q j_{0,r-1,*} F_0$$

belongs to $\mathcal{B}_r$.

We notice that for any $q > 0$, $R^q j_{0,r-1,*}F_0$ is supported on $X_1 \cup \cdots \cup X_{r-1}$, hence we can apply the induction assumption to X_1^* which has a stratification of length $g-1$; we obtain

$$\text{If } q > 0, \quad E_2^{p,q} \in \mathcal{B}_r.$$

The conclusion follows then from sublemma 2.

Sublemma 1. — *Let X^* be a space with a stratification Σ of length g. For each $r = 0, \ldots, g$, let $\mathcal{A}_r$ be an abelian subcategory of locally constant sheaves on X_r; assume that for any $s \leqslant r \leqslant g$, $i_r^* R^\bullet i_{s,*}$ sends $\mathcal{A}_s$ to $\mathcal{A}_r$. Let $\mathcal{B}$ be the smallest abelian category of Σ-constructible étale sheaves on X^* which is stable by extensions (that is, which is thick) and contains $j_{s,!}i_{s,s,*}F_s$ (for $s = 0, \ldots, g$). Then $R^\bullet j_*$ sends $\mathcal{A}_0$ to $\mathcal{B}$.*

Proof. — Let $V_0 \in \mathcal{A}_0$ and $F = R^\bullet j_* V_0$. Consider the filtration

$$F_g = j_! F|_{U_0} \subset \cdots \subset F_r = j_{r,!}F|_{U_{g-r}} \subset \cdots \subset F_0 = F$$

The successive quotients are given by

$$F_{i-1}/F_i \cong j_{g-i+1,!}i_{g-i+1,*}i_{g-i+1}^* F_{i-1}.$$

Note that $i_{g-i+1}^* F_{i-1} = i_{g-i+1}^* F$ belongs to $\mathcal{B}$ by assumption.

We conclude by the following trivial lemma.

Sublemma 2. — *Let $\mathcal{B}$ be a full thick abelian subcategory of an abelian category $\mathcal{C}$ which is stable by subobjects and quotients. Let $E_2^{p,q} \Rightarrow H^{p+q}$ in $\mathcal{C}$ be a spectral sequence concentrated in $p, q \geqslant 0$. Assume that $E_2^{p,q} \in \mathcal{B}$ for any $E_2^{p,q}$, $q \neq q_0$, and $E_\infty^{p,q} \in \mathcal{B}$ for any p, q, then $E_2^{p,q_0} \in \mathcal{B}$.*

Proof. — By decreasing induction on the r of the spectral sequence $E_r^{p,q}$.

From these two lemmata, th. 2.(ii) will follow if we show

Lemma 18. — *For any $s = 1, \ldots, g$, we have*

$$\mathbf{H} = H_c^\bullet(X_s, V_{w(s)\cdot(\lambda+\rho(s))})_{\mathfrak{m}} = 0.$$

Proof. — As in Section 8.4, we see that the Hodge-Tate weights occuring in $\mathbf{H}$ are

$$-w_s' \cdot w_s'' \cdots w_1' \cdot w_1'' \cdot (\lambda + \rho(s))(H)$$

that is,

$$p(w) = -(w(\lambda + \rho) - \rho)(H) \quad \text{for } w = w_s' \circ w_s'' \circ \cdots w_1' \circ w_1''$$

As in 8.4, since $s \geqslant 1$, 0 and $\boldsymbol{w}$ cannot occur simultaneously as weights for this cohomology group. On the other hand, by the Galois-theoretic argument 8.6 they should, if $\mathbf{H} \neq 0$ by Lemma 13. We conclude $\mathbf{H} = 0$.

It is maybe useful to state in a single result an outcome of our proof of Theorems 1 and 2:

Corollary 4. — *Under the assumptions for $\pi, p, \mathfrak{m}$ as before, we have:*

$$H_c^\bullet(S_U, V_\lambda(\mathcal{O}))_\mathfrak{m} = IH^\bullet(S_U, V_\lambda(\mathcal{O}))_\mathfrak{m} = H^\bullet(S_U, V_\lambda(\mathcal{O}))_\mathfrak{m} = H^d(S_U, V_\lambda(\mathcal{O}))_\mathfrak{m}.$$

Comment. — This corollary requires **(RLI)**, but does not require the regularity of λ. When λ is regular, we have already mentioned that

$$H^\bullet_{\mathrm{cusp}}(S_U, V_\lambda(\mathbb{C})) = IH^\bullet(S_U, V_\lambda(\mathbb{C})) = H_!^\bullet(S_U, V_\lambda(\mathbb{C})) = H_!^d(S_U, V_\lambda(\mathbb{C})).$$

moreover, it seems plausible that for such a λ, for any $q < d$, $H^q(S_U, V_\lambda(\mathbb{C})) = 0$. It might result from Franke spectral sequence. It does indeed for $g = 2$ (see Appendix A of [**77**]). If it were true, harmonic analysis would provide a complex version of our theorem, without localization:

For, $q < d$,

$$H^q_{\mathrm{cusp}}(S_U, V_\lambda(\mathbb{C})) = IH^q(S_U, V_\lambda(\mathbb{C})) = H_!^q(S_U, V_\lambda(\mathbb{C})) = H^q(S_U, V_\lambda(\mathbb{C})) = 0$$

and

$$H^d_{\mathrm{cusp}}(S_U, V_\lambda(\mathbb{C})) = IH^d(S_U, V_\lambda(\mathbb{C})) = H_!^d(S_U, V_\lambda(\mathbb{C})).$$

But of course

$$H_!^d(S_U, V_\lambda(\mathbb{C})) \neq H^d(S_U, V_\lambda(\mathbb{C})).$$

9. Application to a control theorem

In this section, we want to apply Theorem 1 for improving upon Theorem 6.2 of [**77**]. More precisely, we want to replace the non effective assumption on the prime p there, (namely, p prime to the order of the torsion subgroups of $H^q(S_U, V_\lambda(\mathbb{Z}))$ for $q = 1, 2, 3$) by an "effective" assumption $p-1 > \max(a_2+a_1+3, 4)$ which in particular is independent of the level (however, we need to assume the mod. p non-Eisensteiness condition **(RLI)** which is far from being effective, but depends only on $\overline{\rho}_\pi$). Note however that we need to localize the Hecke algebra at the maximal ideal given by θ_π modulo ϖ. This is innocuous for questions of congruences between θ_π and characters coming from other representations occuring in H^3.

We prefer to treat axiomatically the general case $G = \mathrm{GSp}(2g)_\mathbb{Q}$ of an arbitrary genus g, assuming conjectures (which are proven for $g = 2$). Most notations in this section follow those of Section 7 of [**77**]. Let $\lambda = (a_g, \ldots, a_1; c)$ be a dominant regular weight (*i.e.* $a_g > \cdots > a_1 > 0$) and π a cuspidal representation of level U occuring in $H^d(S_U, V_\lambda(\mathbb{C}))$. Recall that B denotes the standard Borel subgroup B of G and B^+ its unipotent radical. Let p be a prime not dividing N. for any $n \geqslant 1$, let

$$U_0(p^n) = \{g \in U \mid g \text{ mod. } p^n \in B(\mathbb{Z}/p^n\mathbb{Z})\}$$

resp.

$$U_1(p^n) = \{g \in U \mid g \text{ mod. } p^n \in B^+(\mathbb{Z}/p^n\mathbb{Z})\}$$

The p-component of $U_0(p^n)$ resp. $U_1(p^n)$ is the Iwahori subgroup (resp. strict Iwahori subgroup) of level p^n; it is denoted by $I_n \subset G(\mathbb{Z}_p)$, resp. $J_n \subset G(\mathbb{Z}_p)$. Let $S_1(p^n)$ resp. $S_0(p^n)$ be the Siegel variety associated to $U_1(p^n)$ resp. to $U_0(p^n)$. For each $n \geqslant 1$, let

$$\mathcal{W}^q_{\lambda,n} = H^q(S_1(p^n), V'_\lambda(K/\mathcal{O}))$$

where V'_λ denotes the Iwahoric induction of λ that is the lattice in $V_\lambda(K)$ consisting in λ^{-1}-equivariant rational functions f on G/B^+ taking integral values on the Iwahori subgroup I_1 of $G(\mathbb{Z}_p)$. Thus V'_λ is I_1-stable (hence J_n-stable for any $n \geqslant 1$). Note that it contains the $G(\mathbb{Z}_p)$-stable lattice V_λ defined similarly, but with the stronger condition $f(G(\mathbb{Z}_p)) \subset \mathcal{O}$. Let $\mathcal{W}^q_\lambda$ be the inductive limit over $n \geqslant 1$ of the $\mathcal{W}^q_{\lambda,n}$.

Let $\mathcal{W}^\bullet_{\lambda,n} = \bigoplus \mathcal{W}^q_{\lambda,n}$, resp. $\mathcal{W}^\bullet_\lambda = \bigoplus \mathcal{W}^q_\lambda$. We introduce several abstract Hecke algebras: Let

$$D_p = \{d \in T(\mathbb{Q}_p) \cap M_{2g}(\mathbb{Z}_p)^{\mathrm{prim}} \mid \mathrm{ord}_p(\alpha(d)) \leqslant 0 \text{ for any positive root } \alpha\}$$

where $M_{2g}(\mathbb{Z}_p)^{\mathrm{prim}}$ denotes the set of integral matrices with relatively prime entries. D_p is a semigroup. Let $\mathcal{H}^N$, resp. $\mathcal{H}^{N,I_n}$, resp. $\mathcal{H}^{N,J_n}$ be the abstract Hecke $\mathcal{O}$-algebra outside N and integral at p, resp. integral at p of type I_n, resp. integral at p of type J_n:

$$\begin{aligned}
\mathcal{H}^N &= \bigotimes_{\ell \text{ prime to } Np} \mathcal{O}[G(\mathbb{Q}_\ell)//G(\mathbb{Z}_\ell)] \otimes \mathcal{O}[U_pD_pU_p//U_p],\\
\mathcal{H}^{N,I_n} &= \bigotimes_{\ell \text{ prime to } Np} \mathcal{O}[G(\mathbb{Q}_\ell)//G(\mathbb{Z}_\ell)] \otimes \mathcal{O}[I_nD_pI_n//I_n],\\
\mathcal{H}^{N,J_n} &= \bigotimes_{\ell \text{ prime to } Np} \mathcal{O}[G(\mathbb{Q}_\ell)//G(\mathbb{Z}_\ell)] \otimes \mathcal{O}[J_nD_pJ_n//J_n].
\end{aligned}$$

For any $n \geqslant 1$, there is a natural surjective homomorphism $\mathcal{H}^{N,J_n} \to \mathcal{H}^{N,I_n}$, but that there is no homomorphism $\mathcal{H}^{N,I_1}(\mathcal{O}) \to \mathcal{H}^N$. Assume that π satisfies the condition **(AO)** of automorphic ordinarity at p (see introduction). Let us recall how one can transfer the character $\theta_\pi : \mathcal{H}^N \to \mathcal{O}$ to a character $\theta'_\pi : \mathcal{H}^{N,I_1} \to \mathcal{O}$. The inclusion of lattices $V_\lambda \subset V'_\lambda$, together with the finite morphis $S_0(p) \to S_U$ give rise to a morphism of sheaves $(S_U, V_\lambda(\mathcal{O})) \to (S_0(p), V'_\lambda)$, hence a morphism on cohomology

$$\iota : H^\bullet_*(S_U, V_\lambda(\mathcal{O})) \longrightarrow H^\bullet_*(S_0(p), V'_\lambda(\mathcal{O})).$$

Moreover, the Hecke operators $T_{p,i}$, $i = 1, \dots, g$, defining the condition **(AO)** act on these cohomology groups. Observe however that for each i, $T_{p,i}$ act differently in prime-to-p level (e. g. on S_U), and in level p (e. g. on $S_0(p)$). They define idempotents on these cohomology groups; let $e_0 = \lim_{n\to\infty}(\prod_{i=1}^g T_{p,i})^{n!}$ be the idempotent defined on $H^\bullet_*(S_U, V_\lambda(\mathcal{O}))$, and $e = \lim_{n\to\infty}(\prod_{i=1}^g T_{p,i})^{n!}$ defined on $H^\bullet_*(S_0(p), V'_\lambda(\mathcal{O}))$ by the same formula (with a different meaning though).

***Lemma 19* (Hida's stabilization lemma)**. — *If λ is regular, the homomorphism*

$$H^\bullet_*(S_U, V_\lambda(\mathcal{O})) \longrightarrow H^\bullet_*(S_0(p), V'_\lambda(\mathcal{O})), \quad x \longmapsto e \cdot \iota(x)$$

induced by the diagram

$$\begin{array}{ccc} H^\bullet_*(S_U, V_\lambda(\mathcal{O})) & \longrightarrow & H^\bullet_*(S_0(p), V'_\lambda(\mathcal{O})) \\ \bigcup & & e\big\downarrow \\ e_0 \cdot H^\bullet_*(S_U, V_\lambda(\mathcal{O})) & & e \cdot H^\bullet_*(S_0(p), V'_\lambda(\mathcal{O})) \end{array}$$

is an isomorphism sending an eigenclass for $\mathcal{H}^N$ to an eigenclass for $\mathcal{H}^{N,I_1}$.

Proof. — See Prop. 3.2 of [**77**] (proven there for GSp(4) over a totally real field: it generalizes directly to arbitrary g).

Denote by $\boldsymbol{h}_\lambda(U;\mathcal{O})$, resp. $\boldsymbol{h}_\lambda(U_1(p^n);\mathcal{O})$, resp. $\boldsymbol{h}_\lambda(U_0(p^n);\mathcal{O})$, the image of $\mathcal{H}^N$ in $\mathrm{End}_\mathcal{O}(H^\bullet(S_U, V_\lambda(\mathcal{O})))$, resp. of $\mathcal{H}^{N,J_n}$ in $\mathrm{End}_\mathcal{O}(\mathcal{W}^\bullet_n)$, resp. $\mathcal{H}^{N,J_n}$ in $\mathrm{End}_\mathcal{O}(H^\bullet(S_0(p^n), V'_\lambda(\mathcal{O})))$. By the lemma above for $* = \varnothing$, the character $\theta_\pi : \boldsymbol{h}_\lambda(U;\mathcal{O}) \to \mathcal{O}$ induces a character $\theta'_\pi : \boldsymbol{h}_\lambda(U_0(p);\mathcal{O}) \to \mathcal{O}$; hence (compatible) characters of $\boldsymbol{h}_\lambda(U_1(p^n);\mathcal{O})$ for any $n \geqslant 1$. Let

$$\boldsymbol{h}_\lambda = \varprojlim_n \boldsymbol{h}_\lambda(U_1(p^n);\mathcal{O}).$$

Note that $\boldsymbol{h}_\lambda$ acts faithfully on $\mathcal{W}^\bullet$. Let $\mathfrak{m}' = \mathrm{Ker}\,\overline{\theta}'_\pi$ be the maximal ideal of $\boldsymbol{h}_\lambda$ associated to π. The localization $\mathcal{W}^q_\lambda(\mathfrak{m}')$ of $\mathcal{W}^q_\lambda$, resp. $\mathcal{V}^q_\lambda$ at $\mathfrak{m}'$ is contained in the ordinary part $e \cdot \mathcal{W}^q_\lambda$ and is therefore a localization of this ordinary part. Note that $T(\mathbb{Z}_p) \subset D_p$; by action on $\mathcal{W}^q_{\lambda,n}$, we obtain (compatible) group homomorphisms

$$\langle\,\rangle_\lambda : T(\mathbb{Z}_p) \longrightarrow \boldsymbol{h}_\lambda(U_1(p^n);\mathcal{O}).$$

By linearization, we obtain a continuous $\mathcal{O}$-algebra homomorphism from the completed group algebra $\mathcal{O}[[T(\mathbb{Z}_p)]]$ to $\boldsymbol{h}_\lambda$. For any discrete $\mathcal{O}[[T(\mathbb{Z}_p)]]$-module $\mathcal{W}$, the Pontryagin dual $\mathcal{W}^\star = \mathrm{Hom}(\mathcal{W}, K/\mathcal{O})$ is a compact topological $\mathcal{O}[[T(\mathbb{Z}_p)]]$-module. Let

$$T_1 = \mathrm{Ker}(T(\mathbb{Z}_p) \longrightarrow T(\mathbb{F}_p)) \quad \text{and}\, \Lambda = \mathcal{O}[[T_1]]$$

Λ is an Iwasawa algebra in $(g+1)$-variables. Recall that an arithmetic character $\chi : T(\mathbb{Z}_p) \to \mathcal{O}^\times$ is a product $\chi = \varepsilon\mu$ where ε is of finite order, factoring through, say, $T(\mathbb{Z}/p^n\mathbb{Z})$ and $\mu \in X^\star(T)$ is algebraic. If $\chi \equiv 1$ mod. ϖ, it can be identified to a character of T_1. It induces canonically an $\mathcal{O}$-algebra homomorphism $\chi : \Lambda \to \mathcal{O}$. Its kernel P_χ is a prime ideal of Λ called an arithmetic prime. We say that $\chi = \mu\varepsilon$ is dominant regular if μ is.

Theorem 9. — *Given a π cuspidal of level N; let p be a prime not dividing N such that the conditions* **(Gal)**, **(RLI)**, **(AO)** *and* **(GO)** *hold, and that $p - 1 > \max(a_1 + \cdots + a_g + d, 4)$; then*

(i) $\mathcal{W}^\bullet_\lambda(\mathfrak{m}') = \mathcal{W}^d_\lambda(\mathfrak{m}')$ *and* $\mathcal{W}^d_\lambda(\mathfrak{m}')^\star$ *satisfies the exact control theorem: for any regular dominant arithmetic χ, there is a canonical isomorphism*

$$H^d(S_0(p^n), V'_{\lambda\otimes\chi}(K/\mathcal{O}))_{\mathfrak{m}'} \longrightarrow \mathcal{W}^d_\lambda(\mathfrak{m}')[\chi]$$

Same result for the compactly supported version $\mathcal{CW}_\lambda(\mathfrak{m}')$ *of* $\mathcal{W}_\lambda(\mathfrak{m}')$ *and for its image* $\mathcal{W}^d_{!,\lambda}(\mathfrak{m}')$ *in* $\mathcal{W}_\lambda(\mathfrak{m}')$.

(ii) *The inclusion* $\mathcal{W}^d_{!,\lambda}(\mathfrak{m}') \subset \mathcal{W}^d_\lambda(\mathfrak{m}')$ *is an equality.*

(iii) $\mathcal{W}^d_\lambda(\mathfrak{m}')^\star$ *is free of finite rank over* Λ.

Proof

(i) The proof makes use of Hida's Exact Control criterion (Lemma 7.1 of [**42**]) together with the calculations of Section 3 of [**77**] which generalize readily to $\mathrm{GSp}(2g)_{\mathbb{Q}}$. We prove $\mathcal{W}^q_\lambda(\mathfrak{m}') = 0$ and $\mathcal{CW}^q_\lambda(\mathfrak{m}') = 0$ by induction on $q < d$. For that, by Theorem 3.2(ii) and isomorphism (3.16) of [**77**], it is enough to show that $H^q(S_0(p), V'_\lambda(K/\mathcal{O}))_{\mathfrak{m}'} = 0$. By Proposition 3.2 of [**77**] and its proof (relating $\mathfrak{m}'$ and $\mathfrak{m}$), this amounts to see $H^q(S_U, V_\lambda(K/\mathcal{O}))_{\mathfrak{m}} = 0$. This is precisely what is stated in Theorem 1 in the introduction, under our assumptions. Thus, exactly as in the proof of Theorem 3.2 of [**77**], we obtain (i) for $\mathcal{W}^q$. In an exactly similar manner, we show the control for the compact support analogue, based on the Exact Control criterion for compactly supported cohomology.

(ii) Similarly, the degree d boundary cohomology is controlled, and vanishes in weight λ (*i.e.* $\chi = 1$) by our Main Th. 2. Therefore, by Nakayama's lemma, it vanishes Λ-adically, and $\mathcal{W}^d_{!,\lambda}(\mathfrak{m}') = \mathcal{W}^d_\lambda(\mathfrak{m}')$.

(iii) We use the following criterion: a discrete Λ-module $\mathcal{W}$ is Λ-cofree of corank $r < \infty$ if and only if there exists an infinite set of arithmetic characters χ such that $\bigcap_\chi P_\chi = 0$ in Λ, and for which $\mathcal{W}[\chi]$ is $\mathcal{O}$-divisible, cofree of constant corank r. We take the set of algebraic dominant characters $\chi = \mu\lambda^{-1}$ with μ regular dominant and congruent to λ mod. p, and apply the control formula stated in (i). We need to see that $H^d(S_0(p), V'_\mu(K/\mathcal{O}))_{\mathfrak{m}'}$ is p-divisible (and furthermore, of constant corank). The long exact sequence

$$H^d(S_0(p), V'_\mu(K))_{\mathfrak{m}'} \longrightarrow H^d(S_0(p), V'_\mu(K/\mathcal{O}))_{\mathfrak{m}'} \longrightarrow H^{d+1}_c(S_0(p), V'_\mu(\mathcal{O}))_{\mathfrak{m}'}$$

shows it is enough to verify that the H^{d+1} is torsion-free. By Poincaré-duality (Th. 6.4 of [**77**]), it amounts to see that $H^{d-1}_c(S_0(p), V'_{\widehat{\mu}}(K/\mathcal{O}))_{\mathfrak{m}'}$ is divisible; in fact it is null because by (i), since $\widehat{\mu}$ is regular dominant, one knows that $\mathcal{CW}^{d-1}_{\widehat{\lambda}}(\mathfrak{m}')$ is zero and that it is controlled:

$$H^{d-1}_c(S_0(p), V'_{\widehat{\mu}}(K/\mathcal{O}))_{\mathfrak{m}'} = \mathcal{CW}^{d-1}_{\widehat{\lambda}}(\mathfrak{m}')[\widehat{\chi}] = 0.$$

This shows the divisibility of $\mathcal{W}^d_\lambda(\mathfrak{m}')[\chi]$ for all μ's as above. The corank $r(\chi)$ can be read off from the dimension over the residue field k of the ϖ-torsion. Note that in Λ, $P_\chi + (\varpi)$ is the maximal ideal, hence does not depend on χ. Thus $r(\chi) = \dim_k \mathcal{W}^d_\lambda(\mathfrak{m}')[\mathfrak{m}_\Lambda]$ is independent of χ. QED.

Let $\boldsymbol{h}_{\mathfrak{m}} = \boldsymbol{h}_\lambda(U;\mathcal{O})(\mathfrak{m}')$ be the localization of $\boldsymbol{h}_\lambda$ at $\mathfrak{m}'$. It acts faithfully on $\mathcal{W}^\bullet_\lambda(\mathfrak{m}') = \mathcal{W}^d_\lambda(\mathfrak{m}')$.

Theorem 10. — *Under the same assumptions,*

(i) $\boldsymbol{h}_{\mathfrak{m}}$ *is a finite torsion-free* Λ*-algebra,*

(ii) *there exists a finite integrally closed extension* $\mathbf{I}$ *of* Λ *and a* Λ*-algebra homomorphism* $\Theta : \boldsymbol{h}_m \to \mathbf{I}$ *such that for any* $\mu \in X$ *such that* $\mu \equiv \lambda$ *mod.* p *and* $\phi = \mu\lambda^{-1}$ *is dominant regular, for* P *a prime in* $\mathbf{I}$ *above* P_ϕ *and* $\mathcal{O}' = \mathbf{I}/P$*, there is a commutative diagram*

$$\begin{tikzcd} \boldsymbol{h}_{\mathfrak{m}}/P_\phi \boldsymbol{h}_{\mathfrak{m}} \arrow[r] \arrow[d] & \mathcal{O}' \\ \boldsymbol{h}_\mu(U;\mathcal{O})_{\mathfrak{m}} \arrow[ur] & \end{tikzcd}$$

where the horizontal arrow is $\Theta \otimes \mathrm{Id}_{\mathbf{I}/P}$ *and the oblique arrow is* θ_{π_P} *for some cuspidal automorphic representation* π_P *occuring in* $H^d(S_U, V_\mu(\mathbb{C}))$*. For* $\mu = \lambda$*, one has* $\theta_{\pi_P} = \theta_\pi$ *on* $\mathcal{H}^N$*.*

(iii) *If* π' *is another cuspidal representation occuring in* $H^d(S_U, V_\lambda(\mathbb{C}))$*, if* $\theta_\pi \equiv \theta_{\pi'}$ *mod.* $\max(\overline{\mathbb{Z}}_p)$*, there exists another finite integrally closed extension* $\mathbf{I}'$ *of* Λ *and a* Λ*-algebra homomorphism* $\Theta' : \boldsymbol{h}_{\mathfrak{m}} \to \mathbf{I}'$ *lifting* $\theta_{\pi'}$ *and for any* μ *and any arithmetic ideal* P'' *in the compositum* $\mathbf{I}\cdot\mathbf{I}'$*; let* $P = P'' \cap \mathbf{I}$ *and* $P' = P'' \cap \mathbf{I}'$*; we have*

$$\theta_{\pi_P} \equiv \theta_{\pi'_{P'}} \ \textit{mod.}\ \max(\overline{\mathbb{Z}}_p).$$

Comments

1) We call Θ a Hida family in $(g+1)$-variables lifting θ_π. Statement (iii) means that congruences to θ_π (outside N) can be lifted to families of congruences.

2) Statement (i) implies that $\boldsymbol{h}_{\mathfrak{m}}$ is flat of relative dimension $(g+1)$ over $\mathcal{O}$; this was predicted by calculations in Sect. 9, Example 2, and Sect. 10.5.3, Conjecture I, of [**76**]; it was already proven $g = 2$ in [**77**] under stronger assumptions on p.

3) The representations π_P occuring in the family whose existence is stated in (iii) are cuspidal because $\boldsymbol{h}_{\mathfrak{m}}$ is cuspidal: by Th. 9(ii), $\mathcal{W}^d_{!,\lambda}(\mathfrak{m}') = \mathcal{W}^d_\lambda(\mathfrak{m}')$ for any μ as in the theorem, $H^d(S_U, V_\mu(\mathcal{O}))_{\mathfrak{m}} \subset H^d_{\mathrm{cusp}}(S_U, V_\mu(\mathbb{C}))$ by our Th. 2 and the considerations at the end of Sect. 2.1.

Proof. — It results from the previous one as in Corollary 7.5-7.7 of [**77**].

10. Application to Taylor-Wiles' systems

In this section, we apply Theorem 1 to show that some cohomology group M_Q is free over a finite group algebra $\mathcal{O}[\Delta_Q]$ (this is the non-trivial condition to be verified for having a Taylor-Wiles' system: Condition (TW3) of Definition 1.1 in [**29**], see also Proposition 1 of [**73**]. More precisely, let us fix as above a cuspidal stable representation π whose finite part π_f occurs in $H^d(S_U, V_\lambda(\mathbb{C}))$, for a regular dominant weight λ. Let p be a prime at which the level group K is unramified. Let $r \geqslant 1$. We consider sets $Q = \{q_1, \ldots, q_r\}$ consisting of primes q which are congruent to 1 mod. p and such that the four roots of $\overline{\theta}_\pi(P_q(X)$ are distinct and belong to k. For each $q \in Q$, we fix one of these roots and denote it by α_q. Let $(\mathbb{Z}/q\mathbb{Z})^\times = \Delta_q \times (\mathbb{Z}/q\mathbb{Z})^{(p)}$ where Δ_q is the p-Sylow subgroup and $(\mathbb{Z}/q\mathbb{Z})^{(p)}$ the non-p-part of $(\mathbb{Z}/q\mathbb{Z})^\times$. Let $\Delta_Q = \prod_{q\in Q} \Delta_q$. We put

$$U_Q = \Big\{ g \in U \mid \text{ for any } q \in Q,\, g \equiv \begin{pmatrix} u & * & * & * \\ 0 & * & * & * \\ 0 & 0 & u^{-1} & * \\ 0 & 0 & 0 & * \end{pmatrix} \text{ mod. } q, \quad u \in (\mathbb{Z}/q\mathbb{Z})^{(p)} \Big\}$$

and

$$U_0(Q) = \big\{ g \in U_Q \mid \text{ for any } q \in Q,\, g \text{ mod. } q \in B(\mathbb{Z}/q\mathbb{Z}) \big\}$$

Let $\mathcal{H}_Q$ be the abstract Hecke algebra for U_Q generated over $\mathcal{O}$ by

– Hecke operators T's outside

$$S_Q = \mathrm{Ram}(U) \cup \{p\} \cup Q$$

– the U_q's for each $q \in Q$:

$$U_q = U_Q \cdot \mathrm{diag}(1, \ldots, 1, q, \ldots, q) \cdot U_Q$$

– and by the normal action of $\Delta_Q = K_0(Q)/K_Q$.

$\theta_\pi : \mathcal{H}_Q \to \mathcal{O}$ resp. $\overline{\theta}_\pi : \mathcal{H}_Q \to k$ define $\mathcal{O}$-algebra homomorphisms. Let

$$\mathfrak{m}_Q = \big\langle \varpi, T - \theta_\pi(T), (T \,\mathrm{outside}\, S_Q), U_q - \alpha_q, (q \in Q) \big\rangle.$$

It is a maximal ideal of $\mathcal{H}_Q$. Consider the following "d-th homology module":

$$M_Q = H^d(S_{U_Q}, V_\lambda(K/\mathcal{O}))^\star_{\mathfrak{m}_Q}$$

It has a natural action of the ring $\mathcal{O}[\Delta_Q]$. This ring is a complete intersection noetherian local ring.

Theorem 11. — *Assume that* **(Gal)**, **(RLI)**, **(GO)** *hold, and* $p - 1 > \max(|\lambda + \rho|, 4)$*; then, for any* Q *as above* M_Q *is free over* $\mathcal{O}[\Delta_Q]$.

Proof. — By Theorem 1, we know that M_Q is free as $\mathcal{O}$-module. Hence, it is enough to show that $\overline{M}_Q = M_Q/\varpi \cdot M_Q$ is free over $\Lambda_Q = k[\Delta_Q]$. By Pontryagin duality, $\overline{M}_Q$ is the k-dual of the ϖ-torsion submdodule N_Q of $H^d(S_{U_Q}, V_\lambda(K/\mathcal{O}))_{\mathfrak{m}_Q}$. By the long exact sequence for

$$0 \longrightarrow V_\lambda(\varpi^{-1}\mathcal{O}/\mathcal{O}) \longrightarrow V_\lambda(K/\mathcal{O}) \longrightarrow V_\lambda(K/\mathcal{O}) \longrightarrow 0$$

and the vanishing of $H^{d-1}({}_{U_Q}S, V_\lambda(K/\mathcal{O}))_{\mathfrak{m}_Q}$, we see that

$$N_Q = H^d(S_{U_Q}, V_\lambda(k))_{\mathfrak{m}_Q}.$$

Moreover, Λ_Q is complete intersection, hence is a Frobenius algebra: the freeness of $\overline{M}_Q$ is equivalent to that of N_Q.

To show that N_Q is free, we follow Fujiwara's approach (Sect. 3 of [**28**]). Since Λ_Q is artinian local, freeness is equivalent to flatness: $\mathrm{Tor}_j^{\Lambda_Q}(N_Q, k) = 0$ for $j > 0$. For any ℓ prime to N, consider the sub-semigroup $D'_{Q,\ell}$ of $T(\mathbb{Q}_\ell)\cap M_{2g}(\mathbb{Z}_\ell)_{\mathrm{prim}}$ consisting in t's such that $\mathrm{ord}_\ell(\alpha(t) \leqslant 0$ for any positive root α of (G,B,T). Let $D_{Q,\ell} = U_{Q,\ell}\cdot D'_{Q,\ell}\cdot U_{Q,\ell}$. For $q\in Q$, the local Hecke algebra $\mathcal{H}_{Q,q} = \mathbb{Z}[U_{Q,q}\backslash D_{Q,q}/U_{Q,q}]$ is generated by

$$\Delta_q \text{ and } \mathrm{diag}(1,q^{a_2},\dots,q^{a_g},q^{c-a_g},\dots,q^{c-a_2},q^c), \quad \text{for } 0\leqslant a_2\leqslant\dots\leqslant a_g\leqslant c/2.$$

Note that

$$\mathcal{H}_Q = \bigotimes_{\ell\notin S_Q}\mathcal{H}_\ell^{\mathrm{unr}}\otimes\Big(\bigotimes_{q\in Q}\mathcal{H}_{Q,q}\Big)$$

We view $V_\lambda(k)$ as an étale sheaf over $X_Q = S_{U_Q}\otimes\mathbb{Q}$. For $t\in T(\mathbb{A}^N)$ and $t_\ell\in D'_{Q,\ell}$, the Hecke correspondence $[U_QtU_Q]$ acts on $(X_Q,V_\lambda(k))$ via the diagram

(10.1)
$$\begin{tikzcd}[column sep=small] & S_{U_Q\cap t^{-1}U_Qt} \cong S_{U_Q\cap tU_Qt^{-1}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S_{U_Q} & & S_{U_Q}\end{tikzcd}$$

where π_1 and π_2 are the canonical coverings induced by the inclusions of the level groups, the horizontal isomorphism is induced by right multiplication by t^{-1}. The action on the sheaf $V_\lambda(k)$ is via $\pi_{1,*}\circ[t^{-1}]\circ\pi_2^*$, where $[t^{-1}]:\pi_2^*V_\lambda(k)\to\pi_1^*V_\lambda(k)$ is induced by a right action of the p-component t^{-1} on the representation V_λ which preserves integrality: see for instance [**77**] Section 3.5.

We can form a complex $C^\bullet$ representing $R\Gamma(X,V_\lambda(k))$ endowed with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\times\mathcal{H}_Q$. One can take for instance the global sections $C^\bullet(X_Q,V_\lambda(k))$ of the étale Godement resolution

$$\mathcal{C}^\bullet(X,V_\lambda(k))$$

of $V_\lambda(k)$ (see [**27**] Sect. 12, p. 129, and Section 3.4 [**29**]) whose terms are acyclic. More precisely, by functoriality of the construction, the diagrams (10.1) still operate on $(X_Q,\mathcal{C}^\bullet)$ and induce endomorphisms $[U_QtU_Q]$ of $C^\bullet$. The diagrams (10.1) are defined over $\mathbb{Q}$, hence the action of Galois by transport of structure commutes to these endomorphisms. The main property that we shall use for the Godement resolution is the following. Let $f:X\to Y$ be a finite étale Galois covering with Galois group G, let $\mathcal{G}$ be an étale sheaf on Y, let $\mathcal{C}^\bullet(Y,\mathcal{G}))$, resp. $\mathcal{C}^\bullet(X,f^*(\mathcal{G})$ be the Godement resolution of $\mathcal{G}$ resp. $f^*\mathcal{G}$ on Y resp. X. G acts on $f_*\mathcal{C}^\bullet(X,f^*(\mathcal{G})$ and the adjunction map $a:\mathcal{G}\to f_*f^*\mathcal{G}$ induces an isomorphism

$$(f_*\mathcal{C}^\bullet(X,f^*\mathcal{G}))^G = \mathcal{C}^\bullet(Y,\mathcal{G}).$$

In particular for $q\in Q$ and $G=\Delta_q$, we shall make use of the formula

$$(C^\bullet(X_Q,V_\lambda(k)))^{\Delta_q} = C^\bullet(X_Q/\Delta_q,V_\lambda(k)). \tag{10.2}$$

The hypercohomology spectral sequence applied to $C^\bullet\otimes_{\Lambda_Q}k$ gives rise to the Tor-spectral sequence:

$$E_2^{i,j} = \mathrm{Tor}_{-i}^{\Lambda_Q}(H^j(C^\bullet),k)\longrightarrow H^{i+j}(C^\bullet\otimes k)$$

All the maps involved are $k[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})] \times \mathcal{H}_Q$-linear. Let us tensor this spectral sequence with the localized Hecke algebra $\mathcal{H}_{Q,\mathfrak{m}_Q}$. We get

$$E_2^{i,j}(\mathfrak{m}_Q) = \mathrm{Tor}_{-i}^{\Lambda_Q}(H^j(C^\bullet)_{\mathfrak{m}_Q}, k) \longrightarrow H^{i+j}(C^\bullet \otimes k)_{\mathfrak{m}_Q}$$

Fact. — *$H^j(C^\bullet)_{\mathfrak{m}_Q} = 0$ for any $j \neq d$.*

Proof. — By Theorem 1, we know that

$$H^j(S_{U_Q} \otimes \overline{\mathbb{Q}}, V_\lambda(k))_{\mathfrak{m}_Q} = 0 \quad \text{for } j > d.$$

This fact implies that the spectral sequence is concentrated on $E_2^{i,d}(\mathfrak{m}_Q) = \mathrm{Tor}_{-i}^{\Lambda_Q}(N_Q, k)$ and therefore degenerates:

$$H^{i+d}(C^\bullet \otimes_{\Lambda_Q} k)_{\mathfrak{m}_Q} = E_2^{i,d}(\mathfrak{m}_Q).$$

It remains to see that $H^{i+d}(C^\bullet \otimes_{\Lambda_Q} k)_{\mathfrak{m}_Q} = 0$ unless $i = 0$.

For this purpose, we consider the exact sequence of complexes

$$(10.3) \qquad 0 \longrightarrow \prod_{q\in Q}(C^\bullet)^{\Delta_q} \longrightarrow (C^\bullet)^{\oplus Q} \longrightarrow (C^\bullet)^{\oplus Q} \longrightarrow C^\bullet \longrightarrow C^\bullet \otimes_{\Lambda_Q} k \longrightarrow 0$$

where for each $q \in Q$, the q-th component of the middle arrow is the multiplication by $\delta_q - 1$ on $C^\bullet$, for δ_q a generator of Δ_q. By Theorem 1 of this paper and by (10.2), we see that the first four complexes of (10.3) have no $\mathfrak{m}_Q$-localized cohomology in degree $> d$. By considering long exact sequences, and by exactness of $\mathfrak{m}_Q$-localization, this implies that the same holds for the complex of Δ_Q-coinvariants $C^\bullet_{\Delta_Q} = C^\bullet \otimes_{\Lambda_Q} k$. This concludes the proof.

11. Appendix I: On the constructibility of certain étale sheaves

Let X^* be the minimal compactification over $\mathbb{Q}$ of the Siegel variety X over the rationals. Let Σ be the standard stratification on X^*; the strata have dimension $c_r = r(r+1)/2$, $r = g, g-1, \ldots, 0$. Let $r \geqslant 0$ and U_r be the union of the strata of dimension greater than c_r; we write Σ_r for the stratification on U_r induced by Σ. Let $j_r : U_r \hookrightarrow X^*$ be the natural open immersion. The goal of this appendix is to provide a proof for the following proposition which is used in Sect. 8.7 for proving Lemma 18.

Proposition 7. — *For any Σ_r-constructible torsion étale sheaf V on U_r, for any $i \geqslant 0$, $R^i j_{r,*}V$ is Σ-constructible.*

Proof. — Since r is fixed, we abbreviate $j_r = j$. We use a smooth toroidal compactification of X. Let U be the level group of our Siegel variety. Let $\mathbf{S} = (\mathbf{S}_\xi)_\xi$ be a U-admissible regular rational polyhedral cone decomposition of $S^2(\mathbb{Z}^g)$ (see [**13**] Chap. IV, Th. 6.7 and [**58**] Sect. 12.4); in the above notation, ξ runs over the set of rational boundary components in the minimal compactification X^* and $\mathbf{S}_\xi$ is a polyhedral cone decomposition of $S^2(N_\xi)$ for a quotient N_ξ of $\mathbb{Z}^g$ of rank r_ξ, depending only on ξ (here, r_ξ is the genus of the Siegel variety ξ). Let $X_\mathbf{S}$ be the corresponding toroidal compactification of X over $\mathbb{Q}$. It is smooth and $X_{\mathbf{S}} - X$ is a divisor with normal crossings, whose irreducible components are smooth; it is endowed with a proper morphism $\pi : X_{\mathbf{S}} \to X^*$ defined over $\mathbb{Q}$, inducing the identity on X. The toroidal stratification $\{Z(\sigma)\}_{\sigma\in\mathbf{S}/\,\mathrm{GL}(X)}$ is compatible to (and finer than) the inverse image $\pi^{-1}(\Sigma)$ of the stratification Σ (see Th. 6.7 of [**13**]). By [**13**] Chap. IV.3 or [**59**] 3.10, the restriction π_ξ of π above any rational boundary component ξ of X^* is a proper morphism with singularities of smooth dnc type: let $F_\xi = X_{\mathbf{S}} \times_{X^*} \xi$, then, locally for the étale topology, we have $\mathcal{O}_{F_\xi} \cong \mathcal{O}_\xi[T_1,\dots,T_m]/(T_1\cdots T_n)$. More precisely, F_ξ is a disjoint union

$$F_\xi = \bigcup_{\sigma\in\mathbf{T}_\xi} Z(\sigma)$$

where

– $\mathbf{T}_\xi$ is the set of cones $\sigma \in \mathbf{S}_\xi$ whose elements are all definite positive on N_ξ,
– $Z(\sigma) = \Xi_\xi \times^{E_\xi} Z_\xi(\sigma)$ (in the notations of [**13**] p. 106) are the toroidal strata.

Note that $\mathbf{T}_\xi$ has the property that any cone of $\mathbf{S}_\xi$ containing a cone in $\mathbf{T}_\xi$ is in $\mathbf{T}_\xi$; therefore, F_ξ is closed in the toric immersion $\Xi_{\xi,\mathbf{S}_\xi}$. Moreover, the $Z(\sigma)$ are smooth as well as their closures; thus, F_ξ is étale-locally the boundary of a toric immersion of E_ξ for T_ξ, of smooth dnc type, as desired.

Let $U_{r,\mathbf{S}}$ be the inverse image of U_r by π, and $j_{\mathbf{S}} : U_{r,\mathbf{S}} \hookrightarrow X_{\mathbf{S}}$ the corresponding open immersion. We have $\pi\circ j_{\mathbf{S}} = j\circ\pi$. Similarly, let $k : X \hookrightarrow U_r$ resp $k_{\mathbf{S}} : X \hookrightarrow U_{r,\mathbf{S}}$. By a simple dévissage, one can assume that our étale sheaf is of the form $V = k_!W$ for a locally constant sheaf W on X. Then, we have

$$k_!W = \pi_* \circ k_{\mathbf{S},!}W$$

Let $V_{\mathbf{S}} = k_{\mathbf{S},!}W$. We have $R^q\pi_*V_{\mathbf{S}} = 0$ if $q > 0$, by proper base change. Hence, $R^ij_* \circ \pi_*V_{\mathbf{S}} = R^i(j_*\circ\pi_*)V_{\mathbf{S}} = R^i(\pi_*\circ j_{\mathbf{S},*})V_{\mathbf{S}}$ which is the abutment of a spectral sequence whose E_2-term is $R^p\pi_*\circ R^qj_{\mathbf{S},*}V_{\mathbf{S}}$.

We show now that the sheaves $R^qj_{\mathbf{S},*}V_{\mathbf{S}}$ are constructible for the natural toroidal stratification. By compatibility of the toroidal stratification of $X_{\mathbf{S}}$ with that of the toric immersion of $E = \mathrm{Hom}(S^2(\mathbb{Z}^g), \mathbb{G}_m)$, we can view $X \hookrightarrow U_{r,\mathbf{S}} \hookrightarrow X_{\mathbf{S}}$, local-etally as $E \hookrightarrow E_r(\sigma) \hookrightarrow E(\sigma)$ where $E = \mathbb{G}_m^N$, $E_r(\sigma) = \mathbb{G}_m^{(N-n)} \times \mathbb{A}^n$ and $E(\sigma) = \mathbb{A}^N$. We are now in a cartesian product situation, and therefore, by Künneth formula, we are left with the one-dimensional case $\mathbb{G}_m \overset{k'}{\hookrightarrow} \mathbb{G}_m \overset{j'}{\hookrightarrow} \mathbb{A}^1$ or $\mathbb{G}_m \overset{k'}{\hookrightarrow} \mathbb{A}^1 \overset{j'}{\hookrightarrow} \mathbb{A}^1$. It is easy then to see that $R^ij'_*$ of $k'_!V$ is constructible.

By Lemma 20 below, the higher direct images $R^p\pi_*(R^qj_{\mathbf{S},*}V_{\mathbf{S}})$ are Σ-constructible. In the spectral sequence

$$E_2^{p,q} = R^p\pi_*(R^qj_{\mathbf{S},*}V_{\mathbf{S}}) \Longrightarrow R^{p+q}j_{r,*}V$$

all the terms $E_2^{p,q}$ are Σ-constructible. Since the full subcategory of Σ-constructible étale sheaves inside the category of constructible is abelian, it follows that the abutment is Σ-constructible.

Lemma 20. — *Let Y be an integral scheme over $\mathbb{Q}$ and $f : X \to Y$ be a proper morphism of smooth dnc type. Let $T = (X_0, X_1, \dots, X_n)$ be the stratification of X defined by $X_0 = X^{\mathrm{smooth}}$, $X_{i+1} = (\overline{X_i} - X_i)^{\mathrm{smooth}}$. Let $\mathcal{F}$ be a T-constructible torsion étale sheaf on X. Then $R^i f_* \mathcal{F}$ is locally constant.*

Proof. — By properness of f, we know that $R^i f_* \mathcal{F}$ is constructible on Y with finite fibers. To check it is locally constant we proceed by induction on dimension of X; the maps

$$X_0 \xhookrightarrow{j} X \xhookleftarrow{i} X_1$$

provide a dévissage:

$$0 \longrightarrow j_! \mathcal{F}|_{X_0} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow 0$$

By stability of locally constant sheaves by kernels and extensions, we are left with the case of

$$R^i f_* j_! \mathcal{F}|_{X_0}.$$

By a theorem of M. Artin (exposé XII [**74**], see also Illusie's Appendix, p. 252-261 in [**75**]) this sheaf is locally constant (in general, we would need that $\mathcal{F}|_{X_0}$ is tamely ramified along the divisor with normal crossings $\overline{X_0} - X_0$ for a smooth compactification $X_0 \hookrightarrow \overline{X_0}$ over Y, but it is automatic here, since we are in characteristic 0).

12. Appendix II: An explicit construction of the log crystal $\overline{\mathcal{V}}_\lambda$

In this appendix, we use Weyl's invariant theory to construct automorphic vector bundles over $\mathbb{Z}_p$, associated to dominant weights of the symplectic group $G = GSP_{2g}$ and of the Levi M of the Siegel parabolic of G. The defect of this method (comparing with that of section 5.2) is the lack of functoriality. The advantage is to show clearly how the Hodge structure is obtained by plethysms from that of $R^1 f_* \Omega^\bullet_{A/X}$.

As before, X is the natural smooth model of S_U over $\mathbb{Z}_{(p)}$, $\overline{X}$ is a toroidal compactification over $\mathbb{Z}_{(p)}$. It is projective smooth and its divisor at infinity D has normal crossings. Let $f : A \to X$ be the universal principally polarized g-dimensional abelian variety over X; let $Y = A \times_X \cdots \times_X A$ be the fiber product of A by itself s-times above X and $f_s : Y = A^s \to X$ its structural map. Let us recall some facts on algebraic correspondences.

II.1. Correspondences over $\mathbb{Z}_{(p)}$. — We view $f : A \to X$ over $\mathbb{Z}_{(p)}$ for a prime p not dividing N. Let $s \geqslant 1$. Let $Z^\bullet(Y/X)$ be the free abelian group generated by irreducible closed X-subschemes $Z \subset Y \times_X Y$, flat over X. It is graded by the relative codimension of cycles. Its quotient $A^\bullet(Y \times_X Y/X)$ by the submodule of cycles on $Y \times_X Y$ rationally equivalent to zero is denoted by $\mathrm{Corr}^\bullet(Y/X)$ and is called the group of correspondences on Y relative to X ([**31**] Section 20.1). By smoothness of

$f_s : Y \to X$ and of X over $\mathbb{Z}_{(p)}$, the group $\mathrm{Corr}^\bullet(Y/X)$ carries a natural structure of graded ring (see Ex. 20.1.1 (c) and Ex. 20.2.3 of [**31**]).

Let $C^\bullet(Y/X)_{(p)} = C^\bullet(Y/X) \otimes \mathbb{Z}_{(p)}$.

A correspondence $Z \in \mathrm{Corr}^r(Y/X)_{(p)}$ gives rise (because of the smoothness of the base X over $\mathbb{Z}_{(p)}$) to a cohomology class

$$\mathrm{Cl}(Z) \in R^{2r}(f_s \times f_s)_* \Omega^\bullet_{Y\otimes Y/X}$$

defined by the relative cycle map (See [**20**] Chap. IV). Let $\delta = g \cdot s = \dim Y$.

We follow [**51**], Sect 3 in a relative setting: by Künneth formula and Poincaré duality, we have

$$R^{2r}(f_s \times f_s)_* \Omega^\bullet_{Y\otimes Y/X} = \bigoplus_{0\leqslant m\leqslant 2r} \mathrm{Hom}_{\mathcal{O}_X}(R^{m+2\delta-2r} f_{s,*}\Omega^\bullet_{Y/X}, R^m f_{s,*}\Omega^\bullet_{Y/X})$$

We can therefore view the m-th component of $\mathrm{Cl}(Z)$ as a degree $2r - 2\delta$ endomorphism of $R^\bullet f_{s,*}\Omega^\bullet_{Y/X}$. This defines a homomorphism

$$\mathrm{Corr}^\bullet(Y/X)_{(p)} \longrightarrow \mathrm{End}_{\mathcal{O}_X} R^\bullet f_{s\,*}\Omega^\bullet_{Y/X}$$

which corresponds to letting a cycle Z act by "$pr_{1\,*} \circ pr_2^*$" on the sheaf $R^\bullet f_{s\,*}\Omega^\bullet_{Y/X}$. More precisely, we have:

Lemma 21. — *Let $u \in R^*(f_s \times f_s)_*\Omega^\bullet_{Y\otimes Y/X}$, then $u(x) = pr_{1\,*}(pr_2^*(x) \cup u)$.*

Proof. — [**51**] Sect. 3.

This homomorphism sends cycles Z of relative codimension $\delta + r$ ($-\delta \leqslant r \leqslant \delta$) to degree $2r$ endomorphisms. We denote by

$$\mathcal{C}(Y/X) = \bigoplus_{-\delta\leqslant r\leqslant\delta} \mathcal{C}^{2r}(Y/X)$$

the graded algebra generated by the cycle classes of correspondences; it is a finite free $\mathbb{Z}_{(p)}$-algebra.

In particular, we can view cycles D of Y as cycles in $Y \times_X Y$ via the diagonal immersion $Y \hookrightarrow Y \times_X Y$ (the two resulting projections $pr_i : D \to Y$ are equal). This yields

$$A^r(Y/X) \longrightarrow \mathrm{Corr}^{r+\delta}(Y/X)_{(p)} \longrightarrow \mathrm{End}_{\mathcal{O}_X} R^\bullet f_{s\,*}\Omega^\bullet_{Y/X}.$$

Write $D \mapsto [D]$ for this homomorphism. On the other hand, the action of the cycle D by $-\cup \mathrm{Cl}(D)$ yields another homomorphism

$$A^r(Y/X) \longrightarrow \mathrm{End}_{\mathcal{O}_X} R^\bullet f_{s\,*}\Omega^\bullet_{Y/X}$$

Lemma 22. — *Let $\iota : Y \to Y \times_X Y$ be the diagonal immersion and Δ its image. Then for any cycle D of Y, we have*

$$\begin{aligned}\mathrm{Cl}_{Y\times Y}(\iota_* D) = \iota_* \mathrm{Cl}_Y(D) &= pr_1^*(\mathrm{Cl}_Y(D)) \cup \mathrm{Cl}_{Y\times Y}(\Delta)\\ &= pr_2^*(\mathrm{Cl}_Y(D)) \cup \mathrm{Cl}_{Y\times Y}(\Delta)\end{aligned}$$

Proof. — By the functoriality of the cycle class map we have the following commutative diagram:

$$\begin{tikzcd} A^r(Y/X) \arrow[r] \arrow[d, "\iota_*"'] & R^{2r} f_{s\,*}\Omega^\bullet_{Y/X} \arrow[d, "\iota_*"] \\ \mathrm{Corr}^{r+\delta}(Y/X) \arrow[r] & R^{2r+2\delta} f_{s\,*}\Omega^\bullet_{Y\times_X Y/X} \end{tikzcd}$$

where the horizontal arrows are the cycle maps, the left vertical arrow exists by properness of ι and the right vertical one is the Poincaré dual of ι^*. It remains to check that the ι_* on the right satisfies

$$\iota_*(x) = pr_1^*(x) \cup \mathrm{Cl}_{Y\times Y}(\Delta) = pr_2^*(x) \cup \mathrm{Cl}_{Y\times Y}(\Delta).$$

By definition of the Poincaré duality, it amounts to

$$\mathrm{Tr}_{Y\times Y}(x \cup \iota^*(y)) = \mathrm{Tr}_Y(pr_1^*(x) \cup \mathrm{Cl}_{Y\times Y}(\Delta) \cup y)$$

One has $\Delta = \iota_*(Y)$, therefore by using Poincaré duality, we can rewrite the right hand side as $\mathrm{Tr}_{Y\times Y}(\iota^* \circ pr_1^*(x) \cup \iota^*(y))$, or $\mathrm{Tr}_{Y\times Y}(x \cup \iota^*(y))$, as desired. same for pr_2.

Corollary 5. — *We have*

$$[D] = - \cup \mathrm{Cl}_Y(D).$$

Proof. — We apply the two previous lemmata, noticing that

$$\begin{aligned} pr_1^*(pr_2^*(x \cup \mathrm{Cl}_Y(D)) \cup \mathrm{Cl}_{Y\times Y}(\Delta)) &= pr_1^*(pr_1^*(x \cup \mathrm{Cl}_Y(D)) \cup \mathrm{Cl}_{Y\times Y}(\Delta)) \\ &= x \cup \mathrm{Cl}_Y(D). \end{aligned}$$

Another particular correspondences used in the next, are given by cycles of the form $D \times_X Y$ in $Y \times_X Y$ where D is a relativ cycle in Y of relative codimension r. The action of such correspondence is given by the following diagram:

$$\begin{tikzcd} R^m f_{s,*}\Omega^\bullet_{Y/X} \arrow[r, "{[D\times Y]}"] \arrow[d] & R^{m-2r} f_{s,*}\Omega^\bullet_{Y/X} \\ R^{2\delta-m} f_{s,*}\Omega^\bullet_{Y/X} \arrow[r, "-\cup D"] & R^{2\delta-m+2r} f_{s,*}\Omega^\bullet_{Y/X} \arrow[u] \end{tikzcd}$$

where the vertical maps are given by the polarization of the abelian scheme Y wich identifie each cohomology space with it's dual and by Poincaré duality.

II.2. The $\mathbb{Z}_{(p)}$-schematic version of Construction 5.1. — In this section, we consider dominant weights λ for (G, B, T) such that $s = |\lambda|$ satisfies $s + d < p - 1$. We attach to such weights λ a vector bundle $\mathcal{V}_\lambda$ with connection. Note that because of the need of compatibility with the transcendental construction over $\mathbb{C}$ (using the restriction of the G-representation on V_λ to the Siegel parabolic), the definition will involve duals. We define first the vector bundle $\mathcal{V}_1$ associated to the standard representation V_1 of G as

$$\mathcal{V}_1^\vee = R^1 f_* \Omega^\bullet_{A/X},$$

endowed with the Gauss-Manin connection.

We now use the sheaf-theoretic analogue of Construction 5.1 to define the dual of $\mathcal{V}_\lambda$ over X and X_n as a direct factor in $R^\bullet f_{s\,*}\Omega^\bullet_{Y/X}$ cut out by algebraic correspondences over $\mathbb{Z}_{(p)}$. More precisely, we find an idempotent e_λ in $\mathcal{C}(Y/X)_{(p)}$ realizing this cut out:

$$\mathcal{V}_\lambda^\vee = e_\lambda \cdot R^\bullet f_{s\,*}\Omega^\bullet_{Y/X}$$

The construction is in four steps:

1) Project $R^\bullet f_{s\,*}\Omega^\bullet_{Y/X}$ to $(\mathcal{V}_1^\vee)^{\otimes s}$. This is realized by the Liebermann trick. Since Y is an abelian scheme, we have

$$R^\bullet f_{s\,*}\Omega^\bullet_{Y/X} = \bigwedge^\bullet R^1 f_{s\,*}\Omega^\bullet_{Y/X}$$

Moreover, by Künneth formula, one has

$$R^1 f_{s\,*}\Omega^\bullet_{Y/X} = (\mathcal{V}_1^\vee)^{\oplus s}$$

Therefore,

$$R^\bullet f_{s\,*}\Omega^\bullet_{Y/X} = \bigoplus_{0\leqslant j_1\leqslant 2g,\ldots,0\leqslant j_s\leqslant 2g} \bigwedge^{j_1}\mathcal{V}_1^\vee \otimes \cdots \otimes \bigwedge^{j_s}\mathcal{V}_1^\vee$$

The summand corresponding to $(j_1,\ldots,j_s)$ in the decomposition above is the kernel of the correspondences on Y given by $[m_1]^*\times\cdots\times[m_s]^* - m_1^{j_1}\cdots m_s^{j_s}$ for all $m_1,\ldots,m_s\in\mathbb{Z}$. Recall that we assumed also $p>5$, hence $\max(d,4)<p-1$ implies for any $g\geqslant 1$ that $2g<p-1$. Hence for any $\alpha=1,\ldots,s$, we have $j_\alpha<p-1$. Therefore by choosing $(m_1,\ldots,m_s)$ suitably (that is, with coordinates generating $(\mathbb{Z}/p\mathbb{Z})^\times$), we can construct an idempotent e_1 in $\mathcal{C}(Y/X)_{(p)}$ (of degree 0) such that $e_1\cdot R^\bullet f_*\Omega^\bullet_{A/X} = \mathcal{V}_1^{\vee\,\otimes s}$.

Then, we realize the contractions $\phi_{i,j}$'s and their duals $\psi_{i,j}$'s defined in Sect. 5.1.1, as algebraic correspondences in $\mathcal{C}(Y/X)_{(p)}$.

2) The $\psi_{i,j}$'s:

For any $t\geqslant 1$, let $Y_t = A\times_X\cdots\times_X A$, t times, and $f_t : Y_t\to X$ the corresponding structural map. We abbreviate $Y_s=Y$. Let $p_{i,j}: Y\to A\times A$ be the projection to the ith and jth components. Consider the Poincaré divisor P in $A\times_X A$ (corresponding to the Poincaré bundle).

Definition 9. — The de Rham polarisation $\Psi_P\in\mathcal{V}_1^{\vee\,\otimes 2}$ is defined as the projection of $\mathrm{Cl}_{A\times A}(P)\in R^2 f_{2,*}\Omega^\bullet_{A^2/X}$ to $(Rf_*\Omega^\bullet_{A/X})^{\otimes 2}$ given by the Künneth formula.

Consider the pull-back of P by $p_{i,j}$; it is a divisor $P_{i,j}$ in Y. By 5.2.1, it defines a degree 2 endomorphism $[P_{i,j}]$ of $R^\bullet f_{s*}\Omega^\bullet_{Y/X}$. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{V}_1^{\vee\otimes s-2} \arrow[r, hook] \arrow[d, "\Psi_{P,i,j}"'] & R^{s-2}f_{s-2}\Omega^\bullet_{Y_{s-2}/X} \arrow[r, hook] & R^{s-2}f_{s,*}\Omega^\bullet_{Y/X} \arrow[d, "-\cup \mathrm{Cl}(P_{i,j})"] \\
\mathcal{V}_1^{\vee\otimes s} \arrow[rr, hook] & & R^s f_{s,*}\Omega^\bullet_{Y/X}
\end{tikzcd}$$

where the horizontal arrows are given by Künneth formula, and $\Psi_{P,i,j}$ consists in inserting Ψ_P at ith and jth indexes. Therefore, the morphism $\Psi_{P,i,j}$ is induced by the divisor $P_{i,j}$.

3) The $\phi_{i,j}$'s: Consider the self-intersection $2g-1$ times of P; it is a 1-cycle on $A\times A$. Take its pull-back to Y by the projection $p_{i,j}: Y\to A\times A$ and again to $Y\times_X Y$ by the first projection $p_1 : Y\times_X Y\to Y$. Then, intersect this with the pull-back of the diagonal Δ_{s-2} in the self-product of the remaining $s-2$ copies of A in Y. The resulting cycle $Z_{P,i,j}$ is codimension $\delta-1$ in $Y\times_X Y$; therefore, it gives rise to a degree -2 endomorphism of the cohomology.

Definition 10. — Let $\Phi_P : \mathcal{V}_1^{\vee\otimes 2}\to\mathcal{O}_X$ be the linear dual of the projection to $(R^{2g-1}f_*\Omega^\bullet_{A/X})^{\otimes 2}$ by Künneth formula of $cl(P^{2g-1})\in R^{4g-4}(f\times f)_*\Omega^\bullet_{A\times A/X}$.

Consider the contraction $\Phi_{P,i,j}:\mathcal{V}_1^{\vee\otimes s}\to\mathcal{V}_1^{\vee\otimes s-2}$ by Φ_P at indexes i and j. We have a commutative diagram:

$$\begin{tikzcd}
\mathcal{V}_1^{\vee\otimes s} \arrow[r, hook] \arrow[d, "\Phi_{P,i,j}"'] & R^s f_{s,*}\Omega^\bullet_{Y/X} \arrow[d, "Z_{P,i,j}"] \\
\mathcal{V}_1^{\vee\otimes s-2} \arrow[r, hook] & R^{s-2}f_{s,*}\Omega^\bullet_{Y/X}
\end{tikzcd}$$

Thus, $\Phi_{P,i,j}$ is given by the correspondence $Z_{P,i,j}$.

4) Apply the Young symmetrizer c_λ to $\mathcal{V}_1^{\vee\langle s\rangle}$. This projector has $\mathbb{Z}_{(p)}$-coefficients and belongs to a group algebra of automorphisms of f_s, hence defines an element of $\mathcal{C}(Y/X)$ as in 5.2.1.

Let us summarize the above constructions. For any dominant weight λ of G such that $|\lambda|<p$, we associate a coherent locally free $\mathcal{O}_X$-module $\mathcal{V}_\lambda$ such that

– $\mathcal{V}_1^\vee = R^1f_*\Omega^\bullet_{A/X}$ is associated to the standard representation.

– $\mathcal{V}_\lambda^\vee\otimes_{\mathbb{Z}_{(p)}}\mathbb{C}$ is the classical complex automorphic bundle associated to λ (see for example [**13**] p. 222).

– Let us consider the additive functor $V\to\mathcal{V}^\vee$ from the semisimple category of G-representations over $\mathbb{Z}_{(p)}$ of p-small weights to the category of coherent locally free $\mathcal{O}_X$-modules defined as above for simple objects. It is a functor of abelian categories which commutes with tensor products and duality. This functor sends the $\phi_{i,j}$'s resp. $\psi_{i,j}$ of Sect. 5.1.1 to the $\Phi_{i,j}$'s resp. $\Psi_{i,j}$ of the present section.

II.3. The Gauss-Manin connection. — Over $\mathbb{C}$, the automorphic vector bundle $\mathcal{V}_\lambda(\mathbb{C})$ over S_U carries a natural integrable connection given by the monodromy action $G(\mathbb{Q}) \to \mathrm{Aut}(V_\lambda)$, $g \mapsto (v \mapsto g \cdot v)$, where V_λ est the irreducible $G(\mathbb{C})$-representation of highest weight λ. We call this connection the monodromy connection. To get an algebraic connection on the algebraic locally free $\mathcal{O}_X$-module $\mathcal{V}_\lambda^\vee$, we first note that the sheaves $\mathcal{H}^m_{\mathrm{dR}}(Y/X) = R^m f_{s,*}\Omega^\bullet_{Y/X}$ are naturally endowed with the Gauss-Manin connection ([**49**]). We claim that this connection induces after analytification, the monodromy connection. Indeed, we have just to verify this compatibility on $\mathcal{H}^1_{\mathrm{dR}}(A/X) = R^1 f_*\Omega^\bullet_{A/X}$. This implies in particular that the Gauss-Manin connection commute to the idempotent used to define $\mathcal{V}(\mathbb{C})$.

Corollary 6. — *Over $\mathbb{Z}_{(p)}$, the Gauss-Manin connection on $\mathcal{V}_1^\vee$ commutes to algebraic correspondences and therefore induces an integrable connection on $\mathcal{V}_\lambda$ ($|\lambda| < p$).*

Proof. — Note that $\mathcal{H}^i_{\mathrm{dR}}$ is locally free, hence commutes to base-change: Cor. 2 Chap. 2.5 of [**55**]. We may replace $\mathbb{Z}_p$ by $\mathbb{C}$ and the assertion follows from the discussion above.

II.4. Canonical extension to toroidal compactification over $\mathbb{Z}_{(p)}$. — In the complex setting, Mumford ([**56**], see also [**13**], section VI.4) define a canonical extension $\overline{\mathcal{V}}_\lambda(\mathbb{C})$ over $\overline{X}(\mathbb{C})$ of the automorphic vector bundle $\mathcal{V}_\lambda(\mathbb{C})$. As explained by Harris ([**37**], (4.2.2)), this canonical extension is the extension provided by Deligne's existence theorem. As the toroidal extension is defined over $\mathbb{Q}$, we deduce that the extension is also defined over $\mathbb{Q}$, we denote by $\overline{\mathcal{V}}_{\lambda,\mathbb{Q}}$ this extension over $\mathbb{Q}$, viewed as a coherent locally free module over $\overline{X}_\mathbb{Q} = \overline{X} \otimes_{\mathbb{Z}_p} \mathbb{Q}$. To extend this automorphic sheaves to $\mathbb{Z}_{(p)}$, we proceed as follows.

First, consider

$$\begin{array}{ccc} A & \hookrightarrow & \overline{A} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ X & \hookrightarrow & \overline{X} \end{array}$$

(for the construction of $\overline{A}$ over $\mathbb{Z}[1/N]$, see Th. 1.1 of IV.1 [**13**]) then, the canonical extension $\overline{\mathcal{V}_1}^\vee$ of the standard sheaf $\mathcal{V}_1 = R^1 f_*\Omega^\bullet_{A/X}$ to $\overline{X}$ is

$$\overline{\mathcal{V}}_1^\vee = R^1\overline{f}_*\Omega^\bullet_{\overline{A}/\overline{X}}(\log\infty_{\overline{A}/\overline{X}})$$

(where $\Omega^\bullet_{\overline{A}/\overline{X}}(\log\infty_{\overline{A}/\overline{X}})$ denotes the complex of relative differentials with relative logarithmic poles as defined in section 4.3).

For $s < p$, let $\overline{f}_s : \overline{Y} \to \overline{X}$ be a toroidal compactification of $f_s : Y \to X$. Consider the coherent sheaf $R^s\overline{f}_{s\,*}\Omega^\bullet_{\overline{Y}/\overline{X}}(\log\infty)$; by [**44**] Cor. 2.4, the assumption $s < p$ implies that it is locally free. Moreover, by Step 1 of Section II.2 in this Appendix, its restriction to X is associated to the representation $\bigwedge^s(V_{\mathrm{st}}^{\oplus s})$. By the unicity of the canonical extension, $R^s\overline{f}_{s\,*}\Omega^\bullet_{\overline{Y}/\overline{X}}(\log\infty)$ coincides with the image of this representation by the functor $\overline{V}_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ defined in section 5.2.3.

Then, for a dominant weight λ such that $|\lambda| = s < p$, the representation V_λ is a direct factor of $\bigwedge^s(V_{\mathrm{st}}^{\oplus s})$ (see Cor. 1 of Sect. 5.1.1). Therefore its image by the functor $\overline{V}_{\mathbb{Z}_p}$ is a direct factor in $R^s\overline{f}_{s\,*}\Omega^\bullet_{\overline{Y}/\overline{X}}(\log\infty)$ which is locally free. This shows that the canonical extension $\overline{\mathcal{V}}_\lambda^\vee$ is locally free.

By the calculations of Section II.2 of this Appendix, we see moreover that $\overline{\mathcal{V}}_\lambda^\vee$ can also be defined as

$$\overline{\mathcal{V}}_\lambda^\vee = j_*\overline{\mathcal{V}}_{\lambda,\mathbb{Q}}^\vee \cap R^s\overline{f}_{s\,*}\Omega^\bullet_{\overline{Y}/\overline{X}}(\log\infty)$$

where $j : \overline{X}_\mathbb{Q} \to \overline{X}$ is the open immersion of the generic fiber $\overline{X}_\mathbb{Q}$ in $\overline{X}$.

$\overline{\mathcal{V}}_\lambda^\vee$ is a coherent locally free $\mathcal{O}_{\overline{X}}$-module, direct factor of $R^s\overline{f}_{s\,*}\Omega^\bullet_{\overline{Y}/\overline{X}}(\log\infty)$ and $\overline{\mathcal{V}}_\lambda^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q} = \overline{\mathcal{V}}_{\lambda,\mathbb{Q}}^\vee$. Moreover the Gauss-Manin connexion induces an integrable connection on $\overline{\mathcal{V}}_\lambda^\vee$. Note that this definition is legitimate by the semisimplicity of the category of G-represention over $\mathbb{Z}_{(p)}$ with p-small weight (Lemma 7 of Sect. 5.1.1 with G instead of M).

Remark. — A better way to extend this automorphic sheaves is to extend the idempotents e_λ to the toroidal compactification: if $\overline{Y}$ is a scheme and Y is an open subscheme, then there is an exact sequence ([**31**] I.1.8):

$$A_\bullet(\overline{Y} - Y) \longrightarrow A_\bullet(\overline{Y}) \longrightarrow A_\bullet(Y) \longrightarrow 0$$

The natural way to extend a cycle of Y to $\overline{Y}$ is to take it's closure. In the case of a toroidal imbedding, Lemma 3.1. of [**37**] suggest to consider the normalization of the closure. So we obtain correspondances $\overline{e}_\lambda$ over $\overline{Y}$. Unfortunately, we can not see that $\overline{e}_\lambda$ is an idempotent. The problem is that the closure of the intersection of two cycles is not equal, in general, to the intersection of the closure of this cycles.

II.5. Automorphic bundles for the Levi M. — To every B_M-dominant weight μ, one can also associate $\mathcal{W}_{\mu,n}$, a locally free $\mathcal{O}_{X_n}$-module; it is called the automorphic bundle attached to μ. The construction is similar to the one sketched above. Consider the semiabelian scheme $f_\mathcal{G} : \mathcal{G} \to \overline{X}$ associated to our fixed toroidal compactification (see Th. 5.7, Chap. IV of [**13**]), which extends the universal abelian surface $f : A \to X$. Then, the automorphic bundle on X_n associated to the standard representation W_1 is $\mathbf{Lie}(A/X_n)^\vee$, and by part (3) of Theorem 5.7 of [**13**] mentioned above, its canonical extension $\overline{\mathcal{W}}_{\mu,n}$ is $\mathbf{Lie}(\mathcal{G}/\overline{X}_n)^\vee$. Then one uses the same trick as above to construct $\overline{\mathcal{W}}_{\mu,n}$ from the tensor product of $\mathbf{Lie}(\mathcal{G}/\overline{X}_n)^\vee$ by itself s-times. We note here that we can use the result of Harris ([**37**], Th. 4.2) to recover the rationality of the canonical extension of such automorphic vector bundles.

References

[1] A. Ash, D. Mumford, M. Rapoport, Y.S. Tai: Smooth compactification of locally symmetric varieties, Math. Sci. Press., Brookline, 1975.

[2] A. A. Beilinson, J. Bernstein, P. Deligne: *Faisceaux pervers*, Astérisque 100 (1982).

[3] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand: *Differential operators on the base affine space and a study of the* **g***-modules*, in Lie Groups and their Representations, ed. I.M. Gelfand, Proc. Conf. Budapest 1971, Adam Hilger Publ., London, 1975.

[4] P. Berthelot, A. Ogus: Notes on crystalline cohomology, P.U.P., Princeton, 1978.

[5] A. Borel: *Stable real cohomology of arithmetic groups*, Ann. Sci. Éc. Norm. Sup. (4) 7 (1974) 235-272.

[6] A. Borel: *Stable and L^2-cohomology of arithmetic groups*, Bull. Am. Math. Soc., (N.S.) 3, (1980), 1025-1027.

[7] A. Borel: *Regularization theorems in Lie algebra cohomology, applications*, Duke Math. J. 50 (1983), 605-623.

[8] A. Borel, N. Wallach: Continuous cohomology, discrete subgroups, and representations of reductive groups, P.U.P. 1980.

[9] D. Blasius, M. Harris, D. Ramakrishnan: *Coherent cohomology, limits of discrete series and Galois conjugation*, Duke Math. J. 73, 1994, 647-684.

[10] D. Blasius, J.D. Rogawski: *Motives for Hilbert modular forms*, Inv. Math. 114, (1993) 55-87.

[11] C. Breuil: *Une remarque sur les représentations locales p-adiques et les congruences entre formes modulaires de Hilbert*, Bull. Soc. Math. France 127 (1999), no. 3, 459-472.

[12] N. Bourbaki: Groupes et Algèbres de Lie, Chap. 7, 8, Hermann 1975.

[13] C.L. Chai, G. Faltings: Degeneration of Abelian Varieties, Erg. Math. Wiss. 3. folge, 22, Springer Verlag, 1990.

[14] L. Clozel: *Motifs et formes automorphes: application du principe de fonctorialité*, *in* Proc. Ann Arbor Conf. (Automorphic Forms, Shimura Varieties and L-functions, vol. I and II, Eds. L. Clozel, J.S. Milne), vol. I, 77-159., Persp. in Math., Acad. Press 1990

[15] P. Deligne: *Travaux de Griffiths*, Sém. Bourbaki exp. 376, LN 180, Springer Verlag 1972.

[16] F. Diamond: *The Taylor-Wiles construction and multiplicity one*, Inv. Math. 128, (1997) 379-391.

[17] M. Demazure, A. Grothendieck: Schémas en groupes III (SGA 3, 1962/64), Springer Lect. Notes 153, Springer Verlag, 1970.

[18] R. Deheuvels: Formes quadratiques et groupes classiques, PUF, Paris 1981.

[19] M. Dimitrov: Valeur critique de la fonction L adjointe d'une forme modulaire de Hilbert et arithmétique du motif correspondant, thèse Paris 13, 2003.

[20] F. El Zein: Complexe dualisant et applications à la classe fondamentale d'un cycle, Mémoire de la SMF 58, 1978.

[21] G. Faltings: *On the cohomology of locally symmetric hermitian spaces*, in Séminaire d'Algèbre, pp. 349-366, Springer Lecture Notes 1029, 1983.

[22] G. Faltings: *Crystalline cohomology and p-adic Galois representations*, in Algebraic Analysis, ed. J.-I. Igusa, Proc JAMI inaugural Conference, the Johns Hopkins Univ. Press, 1990.

[23] G. Faltings: *F-isocrystals on open varieties. Results and conjectures*, *in* Grothendieck's Festschrift, Birkhäuser, 1990, Boston. pp. 219-248.

[24] J.-M. Fontaine, G. Laffaille: *Construction de représentations p-adiques*, Ann. Sci. Éc. Norm. Sup. (4), 15 (1982) 547-608.

[25] J. Franke: *Harmonic analysis in weighted L^2-spaces*, Ann. Sci. Éc. Norm. Sup. (4), 31 (1998) 181-279.

[26] E. Freitag: Siegelsche Modulfunktionen, Grundl. Math. Wiss. 254, Springer Verlag 1983.

[27] E. Freitag, R. Kiehl: Etale cohomology and the Weil Conjecture, Springer Verlag 1988.

[28] K. Fujiwara: *Level optimization in the totally real case*, preprint 1999.

[29] K. Fujiwara: *Deformation rings and Hecke algebras in the totally real case*, version 2.0, preprint 1999.

[30] K. Fujiwara: *Arithmetical compactifications of Shimura varieties*, preprint.

[31] W. Fulton: Intersection Theory, Erg. Math., Springer Verlag 1984.

[32] W. Fulton, J. Harris: Representation Theory, Springer Verlag 1991.

[33] A. Grothendieck, J. Dieudonné: *Eléments de Géométrie Algébrique IV, Etude locale des schémas et des morphismes de schémas (seconde partie)*, Pub. Math. IHES 24 (1965).

[34] G. Harder: unpublished notes, 1992.

[35] G. Harder: Eisensteinkohomologie und die Konstruktion gemischter motive, SLN 1562, Springer Verlag 1993.

[36] M. Harris: *Automorphic forms of $\overline{\partial}$-cohomology as coherent cohomology classes*, J. Diff. Geom. 32, (1990) 1-63.

[37] M. Harris: *Functorial properties of toroidal compactifications of locally symmetric varieties*, Proc. London Math. Soc 59, (1989) 1-22.

[38] M. Harris, R. Taylor: *Deformations of automorphic Galois representations*, preprint 1998.

[39] M. Harris, S. Zucker: *Boundary Cohomology of Shimura varieties: Eisenstein classes in coherent cohomology*, Ann. Sci. Éc. Norm. Sup. (4), 27 (1994), no. 3, 249-344.

[40] H. Hida: *On p-adic Hecke algebras for* GL_2 *over totally real fields*, Ann. of Math. 128 (1988), 295-384.

[41] H. Hida: *Nearly ordinary Hecke algebras and Galois representations of several variables*, in Algebraic Analysis, ed. J.-I. Igusa, Proc JAMI inaugural Conference, the Johns Hopkins Univ. Press, 1990.

[42] H. Hida: *Control Theorems of p-nearly Ordinary Cohomology Groups*, Bull. Soc. Math. France 123, (1995) 425-475.

[43] H. Hida: *Control theorems for coherent sheaves on Shimura varieties of PEL type*, preprint.

[44] L. Illusie: *Réduction semi-stable et décomposition de complexes de de Rham à coefficients*, Duke Math. J. 60, (1990) 139-185.

[45] L. Illusie: *Cohomologie de de Rham et cohomologie étale p-adique (d'après G. Faltings, J.-M. Fontaine et al.)*, Séminaire Bourbaki, Exp. No. 726, vol. 1989/90. Astérisque No. 189-190, (1990) 325-374.

[46] J.C. Jantzen: Representations of algebraic groups, Acad. Press, 1987.

[47] J.C. Jantzen: *letter to the authors*, May 11, 1998.

[48] K. Kato: *Logarithmic structures of Fontaine-Illusie*, Algebraic Analysis, ed. J.-I. Igusa, Proc JAMI inaugural Conference, the Johns Hopkins Univ. Press, 1990.

[49] N. Katz: *Nilpotent connections and the monodromy theorem: applications of a result of Turrittin*, Publ. Math. I.H.E.S. 39 (1971), 175-232.

[50] N. Katz, W. Messing: *Some consequences of the Riemann hypothesis for varieties over finite fields*, Inv. Math. 23, (1974) 73-77.

[51] S. Kleiman: *The Standard Conjectures*, p. 3-20, in Motives, vol. I, PSPM 55, AMS 1994.

[52] G. Laumon: *Sur la cohomologie à supports compacts des variétés de Shimura pour* $\mathrm{GSp}(4)_{/\mathbb{Q}}$, Comp. Math. 105, (1996) 267-359.

[53] G. Laumon: *Fonctions zeta des variétés de Siegel de dimension trois*, to appear.

[54] B. Mazur, J. Tilouine: *Représentations galoisiennes, différentielles de Kähler et conjectures principales*, Publ. Math. IHES 71, 1990.

[55] D. Mumford: Abelian Varieties, Oxford Univ. Press 1970.

[56] D. Mumford: *Hirzebruch's proportionality theorem in the non-compact case*, Inv. Math. 42 (1977), 239-272.

[57] A. Mokrane: *Cohomologie cristalline des variétés ouvertes*, Maghreb Math. Rev., 2, (1993) 161-175.

[58] R. Pink: Arithmetical compactification of mixed Shimura varieties, Dissertation, Bonn 1989.

[59] R. Pink: *On ℓ-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification*, Math. Ann. 292, (1992) 197-240.

[60] B. Perrin-Riou: *Representations p-adiques ordinaires, in* Périodes p-adiques, Astérisque 223, 1994.

[61] P. Polo, J. Tilouine: *Bernstein-Gelfand-Gelfand complexes and cohomology of nilpotent groups over $\mathbb{Z}_p$, this volume.*

[62] K. Ribet: *On ℓ-adic representations attached to modular forms*, Inv. Math. 28 (1975) 245-275.

[63] B. Roberts: *Non-vanishing of global theta-lifts from orthogonal groups*, preprint, 1999.

[64] B. Roberts: *letter to the authors, 21/07/99.*

[65] B. Roberts: *Global L-packets for* GSp(2) *and theta lifts*, Documenta Mathematica, 6 (2001), 247-314.

[66] T. Saito: *Hilbert modular forms and p-adic Hodge theory*, preprint, April 1999.

[67] J. Schwermer: *On Euler products and residual Eisenstein cohomology classes for Siegel modular varieties*, Forum Math. 7 (1995), 1-28.

[68] G. Shimura: *A reciprocity law in non-solvable extensions*, J. reine ang. Math., 221 (1966), 209-220.

[69] G. Shimura: *On the Eisenstein series of Hilbert Modular Groups*, Rev. Matematica Iberoamericana, 1 , (1985 no. 3, 1-42.

[70] T.A. Springer: *Reductive groups*, in Automorphic Forms, Representations, and L-functions, Corvallis, PSPM XXXIII, t. 1, AMS 1979.

[71] R. Taylor: *Galois representations associated to Siegel modular forms of low weight*, Duke Math. J. 63, (1991) 281-332.

[72] R. Taylor: *On the ℓ-adic cohomology of Siegel threefolds*, Inv. Math. 114, (1993) 289-310.

[73] R. Taylor, A. Wiles: *Ring-theoretic properties of certain Hecke algebras*, Ann. Math. 141, (1995) 553-572.

[74] SGA4: M. Artin, A. Grothendieck, J.-L. Verdier: Théorie des topos et cohomologie étale des schémas, Lect. Notes in Math. 305, Springer Verlag 1972-73.

[75] SGA4$\frac{1}{2}$: P. Deligne et al.: Cohomologie Étale, Lect. Notes in Math. 569, Springer Verlag 1977.

[76] J. Tilouine: Deformations of Galois representations and Hecke algebras, Narosa Publ., New Delhi 1996.

[77] J. Tilouine, E. Urban: *Several variable p-adic families of Siegel-Hilbert cusp eigensystems and their Galois representations*, Ann. Sci. Ec. Norm. Sup. (4), 32, (1999) 499-574.

[78] T. Tsuji: *Syntomic complexes and p-adic vanishing cycles*, J. reine angew. Math. 472, (1996) 69-138.

[79] E. Urban: *On the torsion of the cohomology of the Siegel threefold*, preprint 1998.

[80] E. Urban: *Sur les représentations p-adiques associées aux représentations cuspidales de* $\mathrm{GSp}(4)_{\mathbb{Q}}$, preprint 1998.

[81] M.-F. Vignéras: *Correspondances entre représentations automorphes de* GL(2)*sur une extension quadratique et* GSp(4) *sur* $\mathbb{Q}$*, Conjecture locale de Langlands pour* GSp(4) in Proc. Bowdoin Conf. (1984): "The Selberg Trace Formula and Related Topics", eds. D. Hejhal, P. Sarnak, A. Terras, Cont. Math. Ser. 53, 1986, AMS Publ., Providence RI.

[82] D. Vogan, G. Zuckermann: *Unitary representations with non-zero cohomology*, Comp. Math. 53 (1984), 51-90.

[83] N. Wach: *Représentations cristallines de torsion*, Comp. Math. 108, (1997) 185-240.

[84] J.-L. Waldspurger: *Cohomologie des espaces de formes automorphes, d'après J. Franke*, Séminaire Bourbaki, Exp. 809, Astérisque 241, 1997.

[85] N. R. Wallach: *On the constant term of a square integrable automorphic form*, *in* Proc. Conf on Operator Algebras and Group Representations, held in Neptun (Romania), Eds. Arsene, Stratila, Verona, Voiculescu; Monographs and Studies in Mathematics, Pitman Publ., London 1984.

[86] R. Weissauer: *An application of the hard Lefschetz theorem*, preprint 1996.

[87] R. Weissauer: *Four dimensional Galois representations*, preprint, 1996.

[88] A. Wiles: *On p-adic representations for totally real fields*, Ann. Math. 123, (1986), 407-456.

[89] A. Wiles: *On λ-adic representations associated to modular forms*, Inv. Math. 94, (1988), 529-573.

[90] H. Yoshida: *Siegel's modular forms and the arithmetic of quadratic forms*, Inv. Math. 60, (1980) 193-248.

[91] H. Yoshida: *Letter to the authors*, August 3, 1999.

Astérisque
280, 2002, p. 97–135

BERNSTEIN-GELFAND-GELFAND COMPLEXES AND COHOMOLOGY OF NILPOTENT GROUPS OVER $\mathbb{Z}_{(p)}$ FOR REPRESENTATIONS WITH p-SMALL WEIGHTS

by

Patrick Polo & Jacques Tilouine

Abstract. — Given a connected reductive group defined and split over $\mathbb{Z}_{(p)}$, we study Bernstein-Gelfand-Gelfand complexes over $\mathbb{Z}_{(p)}$ and prove a $\mathbb{Z}_{(p)}$-analogue of Kostant's theorem computing the $\mathfrak{n}$-homology of the Weyl module $V(\lambda)$, when λ belongs to the closure of the fundamental p-alcove.

***Résumé* (Complexes de Bernstein-Gelfand-Gelfand et cohomologie de groupes nilpotents sur $\mathbb{Z}_{(p)}$ pour les représentations de poids p-petits)**

Étant donné un groupe réductif connexe défini et déployé sur $\mathbb{Z}$, nous étudions certains complexes de Bernstein-Gelfand-Gelfand sur $\mathbb{Z}_{(p)}$ et établissons un analogue sur $\mathbb{Z}_{(p)}$ d'un théorème de Kostant, en calculant la $\mathfrak{n}$-homologie du module de Weyl $V(\lambda)$ lorsque λ appartient à l'adhérence de la p-alcôve fondamentale.

Introduction

Let G be a connected reductive linear algebraic group defined and split over $\mathbb{Z}$, let T be a maximal torus, W the Weyl group, R the root system, $R^\vee$ the set of coroots, R^+ a set of positive roots, and ρ the half-sum of the elements of R^+. Let $X = X(T)$ be the character group of T and let X^+ be the set of those $\lambda \in X$ such that $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for all $\alpha \in R^+$.

For any $\lambda \in X^+$, let $V_{\mathbb{Z}}(\lambda)$ be the Weyl module for G over $\mathbb{Z}$ with highest weight λ (see 1.3) and, for any commutative ring A, let $V_A(\lambda) = V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} A$.

Let p be a prime integer and let

$$\overline{C}_p := \{\nu \in X \mid 0 \leqslant \langle \nu + \rho, \beta^\vee \rangle \leqslant p, \quad \forall \beta \in R^+\},$$

the closure of the fundamental p-alcove.

The aim of this paper is to prove that several results about $V_{\mathbb{Q}}(\lambda)$, due to Kostant [**33**], Bernstein-Gelfand-Gelfand [**3**], Lepowsky [**37**], Rocha [**46**], and Pickel [**43**], hold

2000 ***Mathematics Subject Classification.*** — 17B50, 17B56, 20G30, 20J06, 57T15.
Key words and phrases. — BGG complexes, (co)homology of nilpotent groups, Weyl modules.

true over $\mathbb{Z}_{(p)}$ when $\lambda \in X^+ \cap \overline{C}_p$: this is the precise meaning of the notion of p-smallness mentioned in the title.

In more details, let B be the Borel subgroup corresponding to R^+, let P be a standard parabolic subgroup containing B, let P^- be the opposed parabolic subgroup containing T, let U_P^- be its unipotent radical, and let $L = P \cap P^-$, a Levi subgroup. Let R_L be the root system of L, let $R_L^+ = R_L \cap R^+$, and

$$X_L^+ := \{\xi \in X \mid \langle \xi, \alpha^\vee \rangle \geqslant 0, \quad \forall \alpha \in R_L^+\}.$$

For any $\xi \in X_L^+$ and any commutative ring A, let $V_A^L(\xi)$ be the Weyl module for L over A with highest weight ξ.

Let $\mathfrak{g}, \mathfrak{p}, \mathfrak{u}_P^-$ be the Lie algebras over $\mathbb{Z}$ of G, P, U_P^-, respectively, and let $U(\mathfrak{g})$ and $U(\mathfrak{p})$ be the enveloping algebras of $\mathfrak{g}$ and $\mathfrak{p}$. For $\xi \in X_L^+$, consider the generalized Verma module

$$M_{\mathfrak{p}}^{\mathbb{Z}}(\xi) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_{\mathbb{Z}}^L(\xi).$$

For any commutative ring A, let $M_{\mathfrak{p}}^A(\xi) = M_{\mathfrak{p}}^{\mathbb{Z}}(\xi) \otimes_{\mathbb{Z}} A$.

Let $N = |R^+|$ and, for $i = 0, 1, \ldots, N$, let $W(i) := \{w \in W \mid \ell(w) = i\}$, where ℓ denotes the length function on W relative to B. Further, let

$$W^L = \{w \in W \mid wX^+ \subseteq X_L^+\} \quad \text{and} \quad W^L(i) := W^L \cap W(i).$$

After several recollections in Section 1, we prove in Section 2 the following Theorem (under certain restrictions on G and p, see 2.8).

Theorem A. — *Let $\lambda \in X^+ \cap \overline{C}_p$. There exists an exact sequence of $U(\mathfrak{g})$-modules:*

$$0 \longrightarrow D_N(\lambda) \longrightarrow \cdots \longrightarrow D_0(\lambda) \longrightarrow V_{\mathbb{Z}_{(p)}}(\lambda) \longrightarrow 0,$$

where each $D_i(\lambda)$ admits a finite filtration of $U(\mathfrak{g})$-submodules with associated graded

$$\operatorname{gr} D_i(\lambda) \cong \bigoplus_{w \in W^L(i)} M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(w(\lambda + \rho) - \rho).$$

That is, following the terminology introduced in [**46**], $V_{\mathbb{Z}_{(p)}}(\lambda)$ admits a weak generalized Bernstein-Gelfand-Gelfand resolution. From this, one obtains immediately the following (see 2.1 and 2.9).

***Theorem B* (Kostant's theorem over $\mathbb{Z}_{(p)}$)**. — *Let $\lambda \in X^+ \cap \overline{C}_p$. For each i, there is an isomorphism of L-modules:*

$$H_i(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w \in W^L(i)} V_{\mathbb{Z}_{(p)}}^L(w(\lambda + \rho) - \rho).$$

Let $\Gamma := U_P^-(\mathbb{Z})$ be the group of $\mathbb{Z}$-points of U_P^-, it is a finitely generated, torsion free, nilpotent group. By a result of Pickel [**43**], there is a natural isomorphism $H_*(\mathfrak{u}_P^-, V_{\mathbb{Q}}(\lambda)) \cong H_*(\Gamma, V_{\mathbb{Q}}(\lambda))$. In Section 3, we prove a slightly weaker version of this result over $\mathbb{Z}_{(p)}$ when λ is p-small (see 3.8).

Theorem C. — *Let $\lambda \in X^+ \cap \overline{C}_p$. For each $n \geqslant 0$, $H_n(U_P^-(\mathbb{Z}), V_{\mathbb{Z}_{(p)}}(\lambda))$ has a natural $L(\mathbb{Z})$-module filtration such that*

$$\operatorname{gr} H_n(U_P^-(\mathbb{Z}), V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w \in W^L(n)} V^L_{\mathbb{Z}_{(p)}}(w(\lambda+\rho)-\rho).$$

The proof of this result has two parts. In the first, we develop certain general results valid for any finitely generated, torsion free, nilpotent group Γ. In particular, using a beautiful theorem of Hartley [**22**], we obtain in an algebraic manner a spectral sequence relating the homology of a certain graded, torsion-free, Lie ring $\operatorname{gr}_{\mathrm{isol}} \Gamma$ associated with Γ to the homology of Γ itself, the coefficients being a Γ-module with a "nilpotent" filtration and its associated graded (see Theorem 3.5). This gives a purely algebraic, homological version (with coefficients) of a cohomological spectral sequence obtained, using methods of algebraic topology, by Cenkl and Porter [**9**]. In fact, our methods also have a cohomological counterpart. This will be developped in a subsequent paper [**44**].

In the second part of the proof, we first show that in our case where $\Gamma = U_P^-(\mathbb{Z})$, one has $\operatorname{gr}_{\mathrm{isol}} \Gamma \cong \mathfrak{u}_P^-$, and then deduce from the truth of Kostant's theorem over $\mathbb{Z}_{(p)}$ that the spectral sequence mentioned above degenerates at E_1.

Next, in Section 4, we obtain a result à la Bernstein-Gelfand-Gelfand concerning now the distribution algebras $\operatorname{Dist}(G)$ and $\operatorname{Dist}(P)$. In this case, there exists a standard *complex* (not a resolution!)

$$\mathcal{S}_\bullet(G,P,\lambda) = \operatorname{Dist}(G) \otimes_{\operatorname{Dist}(P)} (\Lambda^\bullet(\mathfrak{g}/\mathfrak{p}) \otimes V_{\mathbb{Z}}(\lambda)).$$

For $\xi \in X_L^+$, consider the generalized Verma module (for $\operatorname{Dist}(G)$ and $\operatorname{Dist}(P)$)

$$\mathcal{M}_P^{\mathbb{Z}}(\xi) := \operatorname{Dist}(G) \otimes_{\operatorname{Dist}(P)} V_{\mathbb{Z}}^L(\xi),$$

and, for any commutative ring A, set $\mathcal{S}_\bullet^A(G,P,\lambda) = \mathcal{S}_\bullet(G,P,\lambda) \otimes_{\mathbb{Z}} A$ and $\mathcal{M}_P^A(\xi) = \mathcal{M}_P^{\mathbb{Z}}(\xi) \otimes_{\mathbb{Z}} A$.

Under the assumption that $\mathfrak{u}_P^-$ is abelian, we obtain, by using an idea borrowed from [**16**, § VI.5] plus arguments from Section 2, the following result (see 4.3). Let $\mathcal{D}G$ denote the derived subgroup of G.

Theorem D. — *Assume that $\mathcal{D}G$ is simply-connected, that $X(T)/\mathbb{Z}R$ has no p-torsion and that $\mathfrak{u}_P^-$ is abelian. Let $\lambda \in X^+ \cap \overline{C}_p$. Then the standard complex $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)$ contains as a direct summand a subcomplex $\mathcal{C}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)$ such that, for every $i \geqslant 0$,*

$$\mathcal{C}_i^{\mathbb{Z}_{(p)}}(G,P,\lambda) \cong \bigoplus_{w \in W^L(i)} \mathcal{M}_P^{\mathbb{Z}_{(p)}}(w(\lambda+\rho)-\rho).$$

Presumably, the hypothesis that $\mathfrak{u}_P^-$ be abelian can be removed, but the proof would then require considerably more work. Since the abelian case is sufficient for the applications in the companion paper by A. Mokrane and J. Tilouine [**39**], we content ourselves with this result. We hope to come back to the general case later.

To conclude this introduction, let us mention that the results of this text are used in [**39**] in the case where G is the group of symplectic similitudes. In this case, $\mathcal{D}G$ is simply-connected and $\mathbb{Z}R$ is a direct summand of $X(T)$. When P is the Siegel parabolic, Theorem D occurs in [**39**, § 5.4] as an important step to establish a modulo p analogue of the Bernstein-Gelfand-Gelfand complex of [**16**, Chap.VI, Th. 5.5], while Theorem C (in its cohomological form) is used in [**39**, § 8.3] to study mod. p versions of Pink's theorem on higher direct images of automorphic bundles.

The notations of [**39**] follow those of [**16**] and are therefore different from the ones used in the present paper, which are standard in the theory of reductive groups. A dictionary is provided in the final section of this text.

Acknowledgements. — We thank the referee for his/her thoughtful reading of the manuscript and for several valuable suggestions (including references [**4**] and [**29**]). We also thank Jens Carsten Jantzen for providing us with a copy of his article [**29**].

1. Notation and preliminaries

1.1. Let G be a connected reductive linear algebraic group, defined and split over $\mathbb{Z}$. Let T be a maximal torus, W the Weyl group, R the root system and $R^\vee$ the set of coroots. Fix a set Δ of simple roots, let R^+ and R^- be the corresponding sets of positive and negative roots, and let B, B^- denote the associated Borel subgroups and U, U^- their unipotent radicals. (For all this, see, for example, [**11**] or [**28**, § II.1]).

Let $X = X(T)$ (resp. $X^\vee = X^\vee(T)$) be the group of characters (resp. cocharacters) of T, and denote by $\langle\ ,\ \rangle$ the natural pairing between them. Elements of X will be called weights, in accordance with the terminology in Lie theory. Let $\leqslant$ denote the partial order on X defined by the positive cone $\mathbb{N}R^+$, that is, $\mu \leqslant \lambda$ if and only if $\lambda - \mu \in \mathbb{N}R^+$. Let $\mathbb{Z}R \subset X$ be the root lattice and let ρ be the half-sum of the positive roots; it belongs to $X \otimes \mathbb{Z}[1/2]$. Define, as usual, the dot action of W on X by

$$w \cdot \lambda = w(\lambda + \rho) - \rho,$$

for $\lambda \in X, w \in W$. It is easy to see that $w\rho - \rho \in \mathbb{Z}R$: applying w to the equality $2\rho = \sum_{\beta \in R^+} \beta$ and substracting, one obtains the well-known formula

$$(*) \qquad \rho - w\rho = \sum_{\beta \in R^+, w^{-1}\beta \in R^-} \beta.$$

Therefore, denoting by $N(w)$ the term on the right hand-side of $(*)$, one may also define the dot action by the formula

$$w \cdot \lambda = w\lambda - N(w),$$

from which it is clear that $w \cdot \lambda$ does indeed belong to X.

Let X^+ be the set of dominant weights:

$$X^+ := \{\lambda \in X \mid \forall \alpha \in R^+, \quad \langle \lambda, \alpha^\vee \rangle \geqslant 0\},$$

where $\alpha^\vee$ denotes the coroot associated with α.

1.2. Enveloping and distribution algebras.— Let $\mathfrak{g} = \mathrm{Lie}(G)$ (resp. $\mathfrak{t} = \mathrm{Lie}(T)$) be the Lie algebra of G (resp. T); they are finite free $\mathbb{Z}$-modules. Let $U(\mathfrak{g})$ denote the enveloping algebra of $\mathfrak{g}$ over $\mathbb{Z}$, and let $\mathrm{Dist}(G)$ denote the algebra of distributions of G (see [**28**, Chap. I.7]). If G is semi-simple and simply-connected, $\mathrm{Dist}(G)$ coincides with the Kostant $\mathbb{Z}$-form of $U(\mathfrak{g})$ ([**34**]), see [**28**, § II.1.12] or [**5**, VIII, §§ 12.6–8]. We shall denote it by $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g})$ or simply $\mathcal{U}(\mathfrak{g})$; sometimes it will also be convenient to denote it by $\mathcal{U}_{\mathbb{Z}}(G)$.

Similarly, if H is a closed subgroup of G defined over $\mathbb{Z}$, we shall denote $\mathrm{Dist}(H)$ also by $\mathcal{U}_{\mathbb{Z}}(H)$.

By an H-module we shall mean a rational H-module, that is, a $\mathbb{Z}[H]$-comodule. More generally, for any commutative ring A, an H_A-module means an A-module with a structure of $A[H]$-comodule. If V is an H-module, then, as is well-known, V is also an $\mathcal{U}_{\mathbb{Z}}(H)$-module and a fortiori an $U(\mathrm{Lie}(H))$-module.

If M is a T-module, it is the direct sum of its weight spaces M_λ, for $\lambda \in X$, see, for example, [**28**, § I.2.11].

For future use, let us record here the following

Proposition. — *Let P be a standard parabolic subgroup of G, let V be a finite dimensional $P_{\mathbb{Q}}$-module and let M be a $\mathbb{Z}$-lattice in V. Then M is a P-submodule if and only if it is an $\mathcal{U}_{\mathbb{Z}}(P)$-submodule.*

Proof. — Without loss of generality we may assume that P contains B. Let P^- be the opposed standard parabolic subgroup and let U_P^- be its unipotent radical. By the Bruhat decomposition, the multiplication map induces an isomorphism of $U_P^- \times B$ onto an open subset of P, see, for example, [**28**, § II.1.10]. This implies that the arguments in [**28**, II.8.1] are valid for P, and the proposition then follows from [**28**, I.10.13].

1.3. Weyl modules.— For $\lambda \in X^+$, let $V_{\mathbb{Q}}(\lambda)$ denote the irreducible $G_{\mathbb{Q}}$-module with highest weight λ, and let $V_{\mathbb{Z}}(\lambda)$ be the corresponding Weyl module over $\mathbb{Z}$; that is,

$$V_{\mathbb{Z}}(\lambda) := \mathcal{U}_{\mathbb{Z}}(G)v_\lambda$$

is the $\mathcal{U}_{\mathbb{Z}}(G)$-submodule generated by a fixed vector $v_\lambda \neq 0$ of weight λ. It is a G-module by Proposition 1.2 above. Of course, up to isomorphism, $V_{\mathbb{Z}}(\lambda)$ does not depend on the choice of v_λ. For future use, let us also record the following (obvious) lemma.

Lemma. — *Let M be a $\mathbb{Z}$-free, G-module and $v \in M$ an element fixed by U and of weight λ. Then the submodule $\mathcal{U}_{\mathbb{Z}}(G)v$ is isomorphic to $V_{\mathbb{Z}}(\lambda)$.*

Proof. — The $\mathcal{U}_{\mathbb{Q}}(G)$-submodule of $M \otimes \mathbb{Q}$ generated by v is isomorphic to $V_{\mathbb{Q}}(\lambda)$.

1.4. Contravariant duals.— Let us fix an anti-involution τ of G which is the identity on T and exchanges B and B^- (see [**28**, II.1.16]). Then τ induces anti-involutions on $\mathcal{U}_{\mathbb{Z}}(G)$, on $\mathfrak{g}$ and on $U_{\mathbb{Z}}(\mathfrak{g})$, which we denote by the same letter τ.

For any ring A and G_A-module V, let us denote by V^τ the A-dual $\mathrm{Hom}_A(V, A)$, regarded as a G_A-module via τ. It may be called the "contravariant dual" of V, as for $V = V_{\mathbb{Z}}(\lambda)$ this is closely related to the so-called "contravariant form" on $V_{\mathbb{Z}}(\lambda)$; see [**28**, II.8.17] and the discussion in the next subsection 1.5.

Note that if V is a free A-module, the weights of T in V and V^τ are the same. In particular, the irreducible $G_{\mathbb{Q}}$-modules $V_{\mathbb{Q}}(\lambda)$ and $V_{\mathbb{Q}}(\lambda)^\tau$ are isomorphic.

1.5. Admissible lattices.— For use in the companion article by Mokrane and Tilouine [**39**] and also in the next subsection, let us discuss some properties of admissible lattices. Of course, this is fairly well-known to representation theorists, but we spell out the details for the convenience of readers with a different background.

As noted above, we may identify $V_{\mathbb{Q}}(\lambda) = V_{\mathbb{Q}}(\lambda)^\tau$. Under this identification, $V_{\mathbb{Q}}(\lambda)$ becomes equipped with a non-degenerate, G-invariant bilinear form $\langle\ ,\ \rangle$ such that

$$(*) \qquad \langle gv, v'\rangle = \langle v, \tau(g)v'\rangle \quad \text{and} \quad \langle Xv, v'\rangle = \langle v, \tau(X)v'\rangle,$$

for $v, v' \in V_{\mathbb{Q}}(\lambda)$, $g \in G$, $X \in \mathcal{U}_{\mathbb{Z}}(G)$. (This is the contravariant form mentioned in the previous subsection).

Let us fix, once for all, a non-zero vector $v_\lambda \in V_{\mathbb{Q}}(\lambda)_\lambda$. The identification $V_{\mathbb{Q}}(\lambda) = V_{\mathbb{Q}}(\lambda)^\tau$ may be chosen so that $\langle v_\lambda, v_\lambda\rangle = 1$.

Recall that a $\mathbb{Z}$-lattice $\mathcal{L} \subset V_{\mathbb{Q}}(\lambda)$ is called an admissible lattice if it is stable under $\mathcal{U}_{\mathbb{Z}}(G)$. By Proposition 1.2, this implies that $\mathcal{L}$ is a G-module and is therefore the direct sum of its T-weight spaces.

Let $\mathcal{E}(\lambda)$ denote the set of admissible lattices $\mathcal{L} \subset V_{\mathbb{Q}}(\lambda)$ such that $\mathcal{L} \cap V_{\mathbb{Q}}(\lambda)_\lambda = \mathbb{Z}v_\lambda$. Clearly, $V_{\mathbb{Z}}(\lambda) := \mathcal{U}_{\mathbb{Z}}(G)v_\lambda$ is the unique minimal element of $\mathcal{E}(\lambda)$.

For any $\mathcal{L} \in \mathcal{E}(\lambda)$, the dual G-module $\mathcal{L}^\tau$ identifies with

$$\{x \in V_{\mathbb{Q}}(\lambda) \mid \langle x, \mathcal{L}\rangle \subseteq \mathbb{Z}\}.$$

It follows from $(*)$ that $\mathcal{L}^\tau$ is an admissible lattice, and since $\langle v_\lambda, v_\lambda\rangle = 1$ it belongs to $\mathcal{E}(\lambda)$. Therefore, $\mathcal{L}^\tau \supseteq V_{\mathbb{Z}}(\lambda)$ and hence $\mathcal{L} \subseteq V_{\mathbb{Z}}(\lambda)^\tau$. Let us record this as the next

Lemma. — *The set of admissible lattices $\mathcal{L} \subset V_{\mathbb{Q}}(\lambda)$ such that $\mathcal{L} \cap V_{\mathbb{Q}}(\lambda)_\lambda = \mathbb{Z}v_\lambda$ contains a unique minimal element, $V_{\mathbb{Z}}(\lambda)$, and a unique maximal element, $V_{\mathbb{Z}}(\lambda)^\tau$.*

The above minimal and maximal lattices are denoted by $V(\lambda)_{\min}$ and $V(\lambda)_{\max}$ in [**39**] and in Section 5 below.

1.6. Weyl modules and induced modules.— Let us recall the definition of the induction functor $\mathrm{Ind}_{B^-}^G$. For any B^--module M,

$$\mathrm{Ind}_{B^-}^G(M) := (\mathbb{Z}[G] \otimes M)^{B^-},$$

where $\mathbb{Z}[G]$ is regarded as a $G\times B^-$-module via $((g,b)\phi)\,(g') = \phi(g^{-1}g'b)$, for $g, g' \in G$, $b \in B^-$ and where the invariants are taken with respect to the diagonal action of B^-; it is a left exact functor, see [**28**, § I.3.3]. As in [**28**, § II.2.1], we shall denote simply by $H^i(\ \ )$ the right derived functors $R^i \operatorname{Ind}_{B^-}^G(\ \ )$.

Let $\lambda \in X$; it may be regarded in a natural manner as a character of either B^- or B. Moreover, since τ is the identity on T, one has $\lambda(\tau(b)) = \lambda(b)$ for any $b \in B^-$.

For any ring A, let us denote by A_λ the free A-module of rank one on which B^- acts via the character λ. Then,

$$H^0(A_\lambda) \cong \{\phi \in A[G] \mid \phi(gb) = \lambda(b^{-1})\phi(g), \ \ \forall\, g \in G, b \in B^-\}.$$

Proposition. — *Let $\lambda \in X^+$.*

a) $H^0(\mathbb{Z}_\lambda) \cong V_{\mathbb{Z}}(\lambda)^\tau$.

b) *If k is a field, $H^0(k_\lambda) \cong H^0(\mathbb{Z}_\lambda) \otimes k \cong V_k(\lambda)^\tau$. Thus, in particular, $V_k(\lambda)$ is irreducible if and only if $H^0(k_\lambda)$ is so.*

Proof. — First, by flat base change ([**28**, I.3.5]), one has $H^0(\mathbb{Z}_\lambda) \otimes \mathbb{Q} \cong H^0(\mathbb{Q}_\lambda)$. Moreover, $H^0(\mathbb{Q}_\lambda) \cong V_{\mathbb{Q}}(\lambda)$, by the theorem of Borel-Weil-Bott (see, for example, [**28**, II.5.6]).

Further, since $\mathbb{Z}[G]$ is a free $\mathbb{Z}$-module (being a subring of $\mathbb{Z}[U] \otimes \mathbb{Z}[B^-]$), so is $H^0(\mathbb{Z}_\lambda)$. Therefore, $H^0(\mathbb{Z}_\lambda)$ may be identified with a G-submodule of $V_{\mathbb{Q}}(\lambda)$, and the identification may be chosen so that $H^0(\mathbb{Z}_\lambda) \cap V_{\mathbb{Q}}(\lambda)_\lambda = \mathbb{Z}v_\lambda$, *i.e.*, so that $H^0(\mathbb{Z}_\lambda)$ belongs to $\mathcal{E}(\lambda)$.

Now, there is a natural G-module map $\phi : V_{\mathbb{Z}}(\lambda)^\tau \to H^0(\mathbb{Z}_\lambda)$ defined by

$$x \longmapsto \left(g \mapsto \langle x, \tau(g^{-1})v_\lambda\rangle\right).$$

Moreover, since $V_{\mathbb{Z}}(\lambda)$ is generated by v_λ as a G-module, ϕ is injective. Since $V_{\mathbb{Z}}(\lambda)^\tau$ is the largest element of $\mathcal{E}(\lambda)$, this implies that ϕ induces an isomorphism $V_{\mathbb{Z}}(\lambda)^\tau \cong H^0(\mathbb{Z}_\lambda)$. This proves assertion a).

Let us prove assertion b). For each $i \geqslant 0$, there is an exact sequence

$$0 \longrightarrow H^i(\mathbb{Z}_\lambda) \otimes k \longrightarrow H^i(k_\lambda) \longrightarrow \operatorname{Tor}^{\mathbb{Z}}(H^{i+1}(\mathbb{Z}_\lambda), k) \longrightarrow 0,$$

see [**28**, I.4.18]. Next, by Kempf's vanishing theorem ([**28**, II.4.6]), one has $H^i(\mathbb{Z}_\lambda) = 0$ for $i \geqslant 1$. The first isomorphism of assertion b) follows. Finally, the second is a consequence of assertion a) and the natural isomorphisms

$$\operatorname{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}(\lambda), \mathbb{Z}) \otimes k \cong \operatorname{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}(\lambda), k) \cong \operatorname{Hom}_k(V_k(\lambda), k).$$

This completes the proof of the proposition.

1.7. Parabolic subgroups and unipotent radicals.— Now, let P be a standard parabolic subgroup of G containing B, let L be the Levi subgroup of P containing T, and let P^- be the standard parabolic subgroup opposed to P, that is, P^- is the unique parabolic subgroup containing B^- such that $P^- \cap P = L$.

Let U_P^- (resp. U_P) denote the unipotent radical of P^- (resp. P) and let $\mathfrak{u}_P^- = \mathrm{Lie}(U_P^-)$, $\mathfrak{u}_P = \mathrm{Lie}(U_P)$ and $\mathfrak{p} = \mathrm{Lie}(P)$. Then $\mathfrak{u}_P^-$, $\mathfrak{u}_P$ and $\mathfrak{p}$ are free $\mathbb{Z}$-modules and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}_P^-$. Thus, in particular, $\mathfrak{g}/\mathfrak{p}$ is a free $\mathbb{Z}$-module.

Further, if V is a P-module then, by standard arguments, the homology groups

$$H_i(\mathfrak{u}_P^-, V) := \mathrm{Tor}_i^{U(\mathfrak{u}_P^-)}(\mathbb{Z}, V)$$

carry a natural structure of L-modules. For example, they can be computed as the homology of the standard Chevalley-Eilenberg complex $\Lambda^\bullet(\mathfrak{u}_P^-) \otimes V$, which carries a natural action of L.

For any commutative ring A, we set $V_A(\lambda) := V_{\mathbb{Z}}(\lambda) \otimes A$ and $\mathfrak{g}_A := \mathfrak{g} \otimes A$. The enveloping algebra of $\mathfrak{g}_A$ identifies with $U_{\mathbb{Z}}(\mathfrak{g}) \otimes A$ and is denoted by $U_A(\mathfrak{g})$. One defines similarly $U_A(\mathfrak{u}_P^-)$ and $\mathcal{U}_A(\mathfrak{g})$, etc...

Since $U_{\mathbb{Z}}(\mathfrak{u}_P^-)$ is a free $\mathbb{Z}$-module, one has, for every $i \geqslant 0$,

$$\mathrm{Tor}_i^{U_A(\mathfrak{u}_P^-)}(A, V_A(\lambda)) \cong \mathrm{Tor}_i^{U_{\mathbb{Z}}(\mathfrak{u}_P^-)}(\mathbb{Z}, V_A(\lambda)).$$

We shall denote these groups simply by $H_i(\mathfrak{u}_P^-, V_A(\lambda))$; as noted above they are L_A-modules.

Our goal in Section 2 is to show that celebrated results of Kostant ([**33**, Cor. 8.1]) and Bernstein-Gelfand-Gelfand ([**3**, Th. 9.9]), which describe respectively, for any $\lambda \in X^+$, the L-module structure of $H_\bullet(\mathfrak{u}_P^-, V_{\mathbb{Q}}(\lambda))$ and a minimal $U_{\mathbb{Q}}(\mathfrak{u}_P^-)$-resolution of $V_{\mathbb{Q}}(\lambda)$, hold true when $\mathbb{Q}$ is replaced by $\mathbb{Z}_{(p)}$, for any prime integer p such that

$$p \geqslant \langle \lambda + \rho, \alpha^\vee \rangle, \quad \forall\, \alpha \in R^+.$$

1.8. Weyl modules for a Levi subgroup.— We need to introduce more notation. Let W_L and R_L denote the Weyl group and root system of L, and let $R_L^\pm := R_L \cap R^\pm$. Let X_L^+ denote the set of L-dominant weights:

$$X_L^+ := \{\lambda \in X \mid \forall \alpha \in R_L^+, \quad \langle \lambda, \alpha^\vee \rangle \geqslant 0\}.$$

Let $W^L := \{w \in W \mid wX^+ \subseteq X_L^+\}$. It is well-known, and easy to check, that W^L is also equal to $\{w \in W \mid w^{-1}R_L^+ \subseteq R^+\}$.

Let ℓ and $\leqslant$ denote the length function and Bruhat-Chevalley order on W associated with the set Δ of simple roots. Then, for $i \geqslant 0$, set

$$W(i) := \{w \in W \mid \ell(w) = i\} \quad \text{and} \quad W^L(i) := W^L \cap W(i).$$

For any $\xi \in X_L^+$, let $V_{\mathbb{Q}}^L(\xi)$ denote the irreducible $L_{\mathbb{Q}}$-module with highest weight ξ and let $V_{\mathbb{Z}}^L(\xi)$ be the corresponding Weyl module for L. Observe that $V_{\mathbb{Q}}^L(\xi)$ (and then $V_{\mathbb{Z}}^L(\xi)$) identifies with the $L_{\mathbb{Q}}$-submodule of $V_{\mathbb{Q}}(\xi)$ (resp. L-submodule of $V_{\mathbb{Z}}(\xi)$) generated by v_ξ.

More generally, one has the following

Lemma. — *Let M be a P-module which is $\mathbb{Z}$-free and let $v \in M$ be a non-zero element of weight ξ. Assume that v is U-invariant (this is the case, for instance, if ξ is a maximal weight of M). Then the $\mathcal{U}_{\mathbb{Z}}(P)$-submodule of M generated by v is isomorphic to $V^L_{\mathbb{Z}}(\xi)$.*

Proof. — Recall that $\mathcal{U}_{\mathbb{Z}}(P) \cong \mathcal{U}_{\mathbb{Z}}(L) \otimes \mathcal{U}_{\mathbb{Z}}(U_P)$ (see [**28**, § II.1.12]). Since v is fixed by U, it is annihilated by the augmentation ideal of $\mathcal{U}_{\mathbb{Z}}(U_P)$. Therefore, $\mathcal{U}_{\mathbb{Z}}(P)v = \mathcal{U}_{\mathbb{Z}}(L)v$ and, since M is $\mathbb{Z}$-free, the result follows from Lemma 1.3.

1.9. The fundamental p-alcove.— In this subsection and the next one, let p be a prime integer. The notion of p-smallness mentioned in the title of this article is defined as follows. We shall say that $\lambda \in X$ is p-small if it satisfies the condition:

$$(\dagger) \qquad \langle \lambda + \rho, \alpha^\vee \rangle \leqslant p, \quad \forall\, \alpha \in R.$$

An equivalent definition of p-smallness is as follows. Let W_p denote the affine Weyl group with respect to p. Recall that W_p is the subgroup of automorphisms of $X(T)\otimes\mathbb{R}$ generated by the reflections $s_{\beta,np}$, for $\beta \in R^+$, $n \in \mathbb{Z}$, where, for $\lambda \in X(T) \otimes \mathbb{R}$,

$$s_{\beta,np}(\lambda) = \lambda - (\langle \lambda, \beta^\vee \rangle - np)\beta,$$

and that W_p is the semi-direct product of W and the group $p\mathbb{Z}R$ acting by translations. We consider the dot action of W_p on $X(T) \otimes \mathbb{R}$, defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$.

The fundamental p-alcove C_p is defined by

$$C_p := \{\lambda \in X(T) \otimes \mathbb{R} \mid 0 < \langle \lambda + \rho, \beta^\vee \rangle < p,\ \ \forall\, \beta \in R^+\}.$$

Its closure

$$\overline{C}_p := \{\lambda \in X(T) \otimes \mathbb{R} \mid 0 \leqslant \langle \lambda + \rho, \beta^\vee \rangle \leqslant p,\ \ \forall\, \beta \in R^+\}$$

is a fundamental domain for the dot action of W_p on $X(T) \otimes \mathbb{R}$ (for all this, see for example [**28**, § II.6.1]).

Then, for $\lambda \in X^+$, the condition of p-smallness is equivalent to the requirement that λ belongs to $\overline{C}_p$. Thus, an arbitrary $\lambda \in X$ is p-small if and only if it belongs to $W \cdot \overline{C}_p$.

Let ρ_L be the half-sum of the elements of R^+_L. Note that $\langle \rho_L, \alpha^\vee \rangle = 1$ for any $\alpha \in \Delta \cap R_L$ and hence $\rho - \rho_L$ vanishes on R_L. Therefore, if a weight $\xi \in X^+_L$ is p-small, it is a fortiori p-small for L.

The fact that $V_{\mathbb{F}_p}(\lambda)$ is irreducible when λ is p-small is of course very well-known to representation-theorists; for the convenience of readers with a different background, we record this here as the next

Lemma. — *Let $\lambda \in X^+$ and $\xi \in X^+_L$. If λ (resp. ξ) is p-small, $V_{\mathbb{F}_p}(\lambda)$ (resp. $V^L_{\mathbb{F}_p}(\xi)$) is irreducible and self-dual for the contravariant duality.*

Proof. — The first assertion is a consequence of [**28**, II.8.3], combined with Proposition 1.6. Further, since irreducible $G_{\mathbb{F}_p}$-modules are determined by their highest weight, the second assertion follows from the first.

Corollary. — *If $\lambda \in X^+ \cap \overline{C}_p$ then, for any $\Lambda \in \mathcal{E}(\lambda)$, one has*

$$V_{\mathbb{Z}_{(p)}}(\lambda) = \Lambda \otimes \mathbb{Z}_{(p)} = V_{\mathbb{Z}_{(p)}}(\lambda)^\tau.$$

Proof. — By the previous lemma, one has $V_{\mathbb{F}_p}(\lambda) = V_{\mathbb{F}_p}(\lambda)^\tau$. The result then follows by Nakayama's lemma.

1.10. A vanishing result.— Let us record the following

Lemma. — *For all $\lambda, \mu \in X^+$, one has $\mathrm{Ext}^1_G(V_{\mathbb{F}_p}(\lambda), V_{\mathbb{F}_p}(\mu)^\tau) = 0$ and also*

$$\mathrm{Ext}^1_G(V_{\mathbb{Z}}(\lambda), V_{\mathbb{Z}}(\mu)^\tau) = 0 = \mathrm{Ext}^1_G(V_{\mathbb{Z}_{(p)}}(\lambda), V_{\mathbb{Z}_{(p)}}(\mu)^\tau).$$

Proof. — Since $V_{\mathbb{F}_p}(\mu)^\tau \cong H^0(\mu)$, by Proposition 1.6, the assertion over $\mathbb{F}_p$ is a consequence of [**28**, Prop. II.4.13]. The assertions over $\mathbb{Z}$ or $\mathbb{Z}_{(p)}$ then follow from a theorem of universal coefficients [**28**, Prop. I.4.18].

Corollary. — *Suppose that $\lambda, \mu \in X^+ \cap \overline{C}_p$. Then*

$$\mathrm{Ext}^1_G(V_{\mathbb{F}_p}(\lambda), V_{\mathbb{F}_p}(\mu)) = 0 = \mathrm{Ext}^1_G(V_{\mathbb{Z}_{(p)}}(\lambda), V_{\mathbb{Z}_{(p)}}(\mu)).$$

Proof. — By the results in 1.9, $V_{\mathbb{F}_p}(\mu)$ and $V_{\mathbb{Z}_{(p)}}(\mu)$ are self-dual. Thus, the corollary follows from the previous lemma.

1.11. We shall need later the following lemma. Recall that U_P denotes the unipotent radical of P and that one has $P = L \ltimes U_P$.

Lemma. — *Let M be a P-module, finite free over $\mathbb{Z}_{(p)}$. Assume that each weight ν of M satisfies $\langle \nu + \rho, \alpha^\vee \rangle \leqslant p$, for any $\alpha \in R_L$.*

a) *There exists a sequence of P-submodules $0 = M_0 \subset \cdots \subset M_r = M$ such that*

$$M_i/M_{i-1} \cong V^L_{\mathbb{Z}_{(p)}}(\xi_i), \text{ where } \xi_i \in X^+_L \text{ and } \xi_j \leqslant \xi_i \text{ if } j \geqslant i.$$

The set $\{\xi_1, \dots, \xi_r\}$ is uniquely determined by M; in fact the $V^L_{\mathbb{Q}}(\xi_i)$ are the irreducible composition factors of the $L_{\mathbb{Q}}$-module $M_{\mathbb{Q}}$.

b) *Moreover, there is an isomorphism of L-modules $M_{|L} \cong \bigoplus_{i=1}^r V^L_{\mathbb{Z}_{(p)}}(\xi_i)$. In particular, if U_P acts trivially on M, then $M \cong \bigoplus_{i=1}^r V^L_{\mathbb{Z}_{(p)}}(\xi_i)$.*

Proof. — Let us prove assertion a) by induction on the rank of M, following [**15**, Lemma 11.5.3]. There is nothing to prove if $M=0$. If $M\neq 0$, let ξ_1 be a maximal weight of M, let $v\in M$ be a primitive element of weight ξ_1 and denote by N the $\mathcal{U}_{\mathbb{Z}_{(p)}}(P)$-submodule generated by v. Then $N\cong V^L_{\mathbb{Z}_{(p)}}(\xi_1)$, by Lemma 1.8. By assumption, $\xi_1\in\overline{C}_p$ and hence $N_{\mathbb{F}_p}:=N\otimes\mathbb{F}_p$ is irreducible.

On the other hand, since M is free over $\mathbb{Z}_{(p)}$, one obtains an exact sequence of P-modules

$$0\longrightarrow \operatorname{Tor}_1^{\mathbb{Z}_{(p)}}(M/N,\mathbb{F}_p)\longrightarrow N_{\mathbb{F}_p}\xrightarrow{\ \phi\ } M_{\mathbb{F}_p},$$

and $\phi(v)\neq 0$, as v is a primitive element. Since $N_{\mathbb{F}_p}$ is irreducible, ϕ is injective. Thus, $\operatorname{Tor}_1^{\mathbb{Z}_{(p)}}(M/N,\mathbb{F}_p)=0$ and this implies that M/N is free over $\mathbb{Z}_{(p)}$. Since M/N has smaller rank than M, the first part of assertion a) follows by the inductive hypothesis. The second part is then clear.

Finally, the first part of assertion b) follows from Corollary 1.10, applied to L, and the last part is clear.

2. Kostant's theorem over $\mathbb{Z}_{(p)}$

2.1. Our goal in this section is to prove the following theorem. Recall from 1.9 the definition of $\overline{C}_p$, the closure of the fundamental p-alcove.

Theorem. — *Let $\lambda\in X^+$ and let p be a prime integer such that $\lambda\in\overline{C}_p$. Then, for each i, there is an isomorphism of L-modules*

$$H_i\big(\mathfrak{u}_P^-,V_{\mathbb{Z}_{(p)}}(\lambda)\big)\cong\bigoplus_{w\in W^L(i)}V^L_{\mathbb{Z}_{(p)}}(w\cdot\lambda).$$

By standard arguments, it suffices to prove the theorem in the case where G is semi-simple; one can further assume that G is simply-connected and, then, that the root system R is irreducible. Similarly, the result for SL_n is easily derived from the result for GL_n (for technical reasons, the latter is easier to handle, see below).

Therefore, while in 2.2–2.8 G still denotes an arbitrary connected reductive linear algebraic group, defined and split over $\mathbb{Z}$, we shall assume in subsection 2.9, where we prove Theorem 2.1, that G is either GL_n or almost simple and simply-connected of type $\neq A$.

Remark. — The hypothesis $\lambda\in X^+\cap\overline{C}_p$ implies that

$$(\dagger)\qquad p\geqslant\langle\lambda+\rho,\alpha^\vee\rangle\geqslant\langle\rho,\alpha^\vee\rangle,\quad\forall\,\alpha\in R^+.$$

Recall also that it is customary, in representation theory, to introduce the so-called Coxeter number of G, defined by

$$h:=1+\operatorname{Max}\{\langle\rho,\alpha^\vee\rangle,\ \alpha\in R^+\}.$$

Therefore, the condition $(\dagger)$ above implies that $p\geqslant h-1$, and reduces to this inequality when $\lambda=0$.

2.2. Standard resolutions for $U(\mathfrak{g})$.— Recall first the standard Koszul resolution of the trivial module:

$$\cdots \longrightarrow U(\mathfrak{g})\otimes\Lambda^2(\mathfrak{g}) \xrightarrow{d_2} U(\mathfrak{g})\otimes\mathfrak{g} \xrightarrow{d_1} U(\mathfrak{g}) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where each differential d_k is defined by the formula

$$\begin{aligned} d_k(u\otimes x_1\wedge\cdots\wedge x_k) := & \sum_{i=1}^{k}(-1)^{i-1}\, ux_i\otimes x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge x_k \\ & + \sum_{1\leqslant i<j\leqslant k}(-1)^{i+j}\, u\otimes[x_i,x_j]\wedge x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\widehat{x_j}\wedge\cdots\wedge x_k. \end{aligned}$$

Let $\pi_{\mathfrak{p}}$ denote the natural projection $\Lambda^\bullet(\mathfrak{g})\to\Lambda^\bullet(\mathfrak{g}/\mathfrak{p})$; it is a morphism of P-modules. Then, there is a surjective morphism of $U(\mathfrak{g})$-modules:

$$\begin{aligned} \phi_{\mathfrak{p}} : U(\mathfrak{g})\otimes\Lambda^\bullet(\mathfrak{g}) &\longrightarrow U(\mathfrak{g})\otimes_{U(\mathfrak{p})}\Lambda^\bullet(\mathfrak{g}/\mathfrak{p}) \\ u\otimes x &\longmapsto u\otimes_{U(\mathfrak{p})}\pi_{\mathfrak{p}}(x). \end{aligned}$$

It is well-known, and easy to check, that each d_k induces a map $d_k^{\mathfrak{p}}$ such that $\phi_{\mathfrak{p}}\circ d_k = d_k^{\mathfrak{p}}\circ\phi_{\mathfrak{p}}$. Thus, one obtains a complex of $U(\mathfrak{g})$-modules

$$\cdots \longrightarrow U(\mathfrak{g})\otimes_{U(\mathfrak{p})}\Lambda^2(\mathfrak{g}/\mathfrak{p}) \xrightarrow{d_2^{\mathfrak{p}}} U(\mathfrak{g})\otimes_{U(\mathfrak{p})}(\mathfrak{g}/\mathfrak{p}) \xrightarrow{d_1^{\mathfrak{p}}} U(\mathfrak{g})\otimes_{U(\mathfrak{p})}\mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

which is still exact, for it is easily seen that the proof of [**3**, Th.9.1] is valid over $\mathbb{Z}$. This complex is called the standard resolution of the trivial module $\mathbb{Z}$ relative to $U(\mathfrak{g})$ and $U(\mathfrak{p})$. We shall denote it by $S_\bullet(\mathfrak{g},\mathfrak{p},\mathbb{Z})$ or simply $S_\bullet(\mathfrak{g},\mathfrak{p})$.

Let V be a $\mathbb{Z}$-free $U(\mathfrak{g})$-module. Then $S_\bullet(\mathfrak{g},\mathfrak{p})\otimes V$, with the diagonal action of $\mathfrak{g}$, is an $U(\mathfrak{g})$-resolution of V by modules which are free over $U(\mathfrak{u}_P^-)$.

Further, recall the "tensor identity" [**19**, Prop. 1.7] : for any $U(\mathfrak{p})$-module E, there is a natural isomorphism of $U(\mathfrak{g})$-modules

$$(U(\mathfrak{g})\otimes_{U(\mathfrak{p})}E)\otimes V \cong U(\mathfrak{g})\otimes_{U(\mathfrak{p})}(E\otimes V_{|\mathfrak{p}}),$$

where $V_{|\mathfrak{p}}$ denotes V regarded as an $U(\mathfrak{p})$-module. Applying these isomorphisms to the terms of the resolution $S_\bullet(\mathfrak{g},\mathfrak{p})\otimes V$, one obtains an $U(\mathfrak{g})$-resolution

$$\begin{aligned} \cdots \longrightarrow U(\mathfrak{g})\otimes_{U(\mathfrak{p})}(\Lambda^2(\mathfrak{g}/\mathfrak{p})\otimes V|_{\mathfrak{p}}) \xrightarrow{d_2} U(\mathfrak{g})\otimes_{U(\mathfrak{p})}(\mathfrak{g}/\mathfrak{p}\otimes V|_{\mathfrak{p}}) \\ \xrightarrow{d_1} U(\mathfrak{g})\otimes_{U(\mathfrak{p})}V|_{\mathfrak{p}} \xrightarrow{\varepsilon} V \longrightarrow 0, \end{aligned}$$

where the differentials d_k are now given by

$$
\begin{aligned}
d_k(1 \otimes \overline{x}_1 \wedge \cdots \wedge \overline{x}_k \otimes v) := \sum_{i=1}^{k} (-1)^{i-1}\, x_i \otimes \overline{x}_1 \wedge \cdots \wedge \widehat{\overline{x}_i} \wedge \cdots \wedge \overline{x}_k \otimes v \\
+ \sum_{1 \leqslant i < j \leqslant k} (-1)^{i+j}\, 1 \otimes \pi_{\mathfrak{p}}([x_i, x_j]) \wedge \overline{x}_1 \wedge \cdots \wedge \widehat{\overline{x}_i} \wedge \cdots \widehat{\overline{x}_j} \wedge \cdots \wedge \overline{x}_k \otimes v \\
+ \sum_{i=1}^{k} (-1)^{i}\, 1 \otimes \overline{x}_1 \wedge \cdots \wedge \widehat{\overline{x}_i} \wedge \cdots \wedge \overline{x}_k \otimes x_i v,
\end{aligned}
$$

for $x_1, \ldots, x_k \in \mathfrak{g}$ and $v \in V$ (we have denoted $\pi_{\mathfrak{p}}(x_i)$ by $\overline{x}_i$). We shall call it the standard resolution of V relative to the pair $(U(\mathfrak{g}), U(\mathfrak{p}))$, and denote it by $S_\bullet(\mathfrak{g}, \mathfrak{p}, V)$. When $V = V_{\mathbb{Z}}(\lambda)$, we shall denote it by $S_\bullet(\mathfrak{g}, \mathfrak{p}, \lambda)$.

2.3. Let p be a prime integer and recall the notation of 1.9.

Lemma. — *Let $\lambda \in X^+ \cap \overline{C}_p$. Then all weights of $V_{\mathbb{Z}}(\lambda) \otimes \Lambda(\mathfrak{g}/\mathfrak{p})$ are p-small.*

Proof. — As T-module, $\Lambda(\mathfrak{g}/\mathfrak{p})$ identifies with $\Lambda(\mathfrak{u}_P^-)$ and hence is a submodule of $\Lambda(\mathfrak{u}^-)$, where $\mathfrak{u}^-$ is the Lie algebra of U^-.

By a result of Kostant ([**33**, Lemma 5.9]), there is a T-isomorphism

$$\rho \otimes \Lambda(\mathfrak{u}^-) \cong V_{\mathbb{Z}}(\rho).$$

Therefore, if ν is a weight of $V_{\mathbb{Z}}(\lambda) \otimes \Lambda(\mathfrak{g}/\mathfrak{p})$, then $\nu + \rho$ is a weight of $V_{\mathbb{Z}}(\lambda) \otimes V_{\mathbb{Z}}(\rho)$. This implies that $\langle \nu + \rho, \alpha^\vee \rangle \leqslant p$, for all $\alpha \in R$.

Indeed, let μ be the dominant W-conjugate of $\nu + \rho$, it is also a weight of $V_{\mathbb{Z}}(\lambda) \otimes V_{\mathbb{Z}}(\rho)$. Clearly, it suffices to prove that $\langle \mu, \alpha^\vee \rangle \leqslant p$, for all $\alpha \in R^+$. Further, since μ is dominant, it suffices to prove that $\langle \mu, \gamma^\vee \rangle \leqslant p$ when $\gamma^\vee$ is a maximal coroot. But it is well-known that a maximal coroot is a dominant coweight, i.e. satisfies $\langle \beta, \gamma^\vee \rangle \geqslant 0$ for all $\beta \in R^+$, see e.g. [**5**, VI,§ 1, Prop.8]. Finally, since $\mu = \lambda + \rho - \theta$ with $\theta \in \mathbb{N}R^+$, it follows that

$$\langle \mu, \gamma^\vee \rangle \leqslant \langle \lambda + \rho, \gamma^\vee \rangle \leqslant p.$$

This proves the lemma.

2.4. Verma modules and filtrations.— For any $\xi \in X_L^+$, define the generalized Verma module (for $U(\mathfrak{g})$ and $U(\mathfrak{p})$)

$$M_{\mathfrak{p}}(\xi) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_{\mathbb{Z}}^L(\xi).$$

For any commutative ring A, set $M_{\mathfrak{p}}^A(\xi) := M_{\mathfrak{p}}(\xi) \otimes_{\mathbb{Z}} A$ and observe that it identifies with $U_A(\mathfrak{g}) \otimes_{U_A(\mathfrak{p})} V_A^L(\xi)$.

For $\lambda \in X^+$, we set also

$$S_\bullet^A(\mathfrak{g}, \mathfrak{p}, \lambda) := S_\bullet(\mathfrak{g}, \mathfrak{p}, \lambda) \otimes_{\mathbb{Z}} A.$$

Let us assume that $\lambda \in X^+ \cap \overline{C}_p$. Then, by Lemma 2.3, all weights of $V_{\mathbb{Z}}(\lambda) \otimes \Lambda^\bullet(\mathfrak{g}/\mathfrak{p})$ are p-small. Therefore, by Lemma 1.11, there exists, for each i, a P-module filtration

$$0 = F_0 \subset \cdots \subset F_r = \Lambda^i(\mathfrak{g}/\mathfrak{p}) \otimes V_{\mathbb{Z}_{(p)}}(\lambda)$$

such that each F_j/F_{j-1} is isomorphic to $V^L_{\mathbb{Z}_{(p)}}(\xi^i_j)$, for some $\xi^i_j \in X^+_L$ (not necessarily distinct). Let us denote by $\Omega^i_{\mathfrak{p}}(\lambda)$ the multiset of those ξ^i_j (each $\xi \in X^+_L$ occuring as many times as $V^L_{\mathbb{Z}_{(p)}}(\xi)$ occurs in the filtration).

Moreover, as $U(\mathfrak{g})$ is free over $U(\mathfrak{p})$, the functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} -$ is exact. Therefore, one obtains the

Lemma. — *Let $\lambda \in X^+ \cap \overline{C}_p$. Then each $S^{\mathbb{Z}_{(p)}}_i(\mathfrak{g}, \mathfrak{p}, \lambda)$ admits a finite filtration by $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})$-modules such that the successive quotients are the $M^{\mathbb{Z}_{(p)}}_{\mathfrak{p}}(\xi)$, for $\xi \in \Omega^i_{\mathfrak{p}}(\lambda)$.*

2.5. A conjugacy result in $\mathfrak{g}^*$.— We will need in the next subsection the following lemma. It is proved in [**29**, Lemma 6.6] under the assumption that $\mathfrak{g}_{\overline{\mathbb{F}}_p} \cong \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$ as G-modules, and in [**31**, Lemma 3.3] under the assumption that G is almost simple and distinct from SO_{2n+1} if $p = 2$. Let $\mathfrak{u}$ be the Lie algebra over $\mathbb{Z}$ of U and let $\mathfrak{u}_{\overline{\mathbb{F}}_p} = \mathfrak{u} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p$.

Lemma. — *Each $\chi \in \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$ is conjugate under G to an element χ' such that $\chi'(\mathfrak{u}_{\overline{\mathbb{F}}_p}) = 0$.*

Proof. — Let $\mathcal{B}$ denote the variety of Borel subgroups of G, let Z be the closed subvariety of $\mathcal{B} \times \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$ consisting of pairs (B', χ) such that χ vanishes on the derived subalgebra of $\operatorname{Lie} B'$, and let π denote the projection $Z \to \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$. Then, the lemma is equivalent to the surjectivity of π.

But, $\mathcal{B}$ being projective, $\pi(Z)$ is a closed subvariety and, since $\dim Z = \dim \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$, the surjectivity of π will follow if we show that the set of those $\chi \in \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$ such that $\pi^{-1}(\chi)$ is finite, is not empty. But this follows from an argument of Steinberg [**50**, Lemma 3.2] (one may also consult [**25**, Prop. 4.1]). Namely, for each $\beta \in R$, let X_β be a generator of $\mathfrak{g}_\beta$. We claim that if $\chi \in \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$ satisfies $\chi(\mathfrak{b}_{\overline{\mathbb{F}}_p}) = 0$ and $\chi(X_{-\alpha}) \neq 0$, for every $\alpha \in \Delta$, then $\pi^{-1}(\chi) = \{B\}$.

Indeed, let B' be a Borel subgroup such that χ vanishes on $\mathfrak{u}'$, the derived subalgebra of $\operatorname{Lie} B'$. Then $B' = g(B)$ for some $g \in G$ and, using the Bruhat decomposition, one may write $g = un_wb$ for some $w \in W$, $b \in B$ and $u \in U \cap n_w^{-1}(U)$. If $w \neq 1$, there exists a simple root $\alpha \in \Delta$ such that $w^{-1}\alpha \in R^-$. Let $\beta = -w^{-1}\alpha$, then $n_wX_\beta = cX_{-\alpha}$ for some non-zero $c \in \overline{\mathbb{F}}_p$. Set $x = b^{-1}c^{-1}X_\beta$. Then $x \in \mathfrak{u}_{\overline{\mathbb{F}}_p}$ and, by hypothesis,

$$0 = \chi(gx) = \chi(uX_{-\alpha}).$$

But $uX_{-\alpha} - X_{-\alpha}$ belongs to $\mathfrak{b}_{\overline{\mathbb{F}}_p}$ and hence the assumptions on χ imply that $\chi(uX_{-\alpha}) = \chi(X_{-\alpha}) \neq 0$, a contradiction. This contradiction shows that $w = 1$, whence $g \in B$ and $B' = B$. This completes the proof of the lemma.

2.6. The Harish-Chandra homomorphism

2.6.1. Let $\mathfrak{u}^- = \operatorname{Lie} U^-$ and let A be a commutative ring. By the PBW theorem, one has

$$U_A(\mathfrak{g}) = U_A(\mathfrak{t}) \oplus \big(\mathfrak{u}^- U_A(\mathfrak{g}) + U_A(\mathfrak{g})\mathfrak{u}\big)\,.$$

Let δ_A denote the A-linear projection from $U_A(\mathfrak{g})$ to $U_A(\mathfrak{t})$ defined by this decomposition.

Let $U_A(\mathfrak{g})^G \subset U_A(\mathfrak{g})^T$ be the subrings of G-invariant and T-invariant elements for the adjoint action. Observe that, since elements of $U_A(\mathfrak{g})^T$ have weight zero,

$$U_A(\mathfrak{g})^T \subseteq U_A(\mathfrak{t}) \oplus \mathfrak{u}^- U_A(\mathfrak{g})\mathfrak{u}.$$

The restriction of δ_A to $U_A(\mathfrak{g})^T$ is a ring homomorphism; indeed one sees easily that the arguments in the proof of [**13**, Lemme 7.4.2] or [**31**, Lemma 5.1] carry over in our case. Let θ_A denote the restriction of δ_A to $U_A(\mathfrak{g})^G$.

Lemma. — $\theta_{\overline{\mathbb{F}}_p} : U_{\overline{\mathbb{F}}_p}(\mathfrak{g})^G \to U_{\overline{\mathbb{F}}_p}(\mathfrak{t})$ *is injective.*

Proof. — Taking into account Lemma 2.5, the proof is exactly the same as the one of [**29**, Lemma 9.1]. For the convenience of the reader, we record it briefly. Let $U = U_{\overline{\mathbb{F}}_p}(\mathfrak{g})$, let $x \mapsto x^{[p]}$ denotes the p-th power map of $\mathfrak{g}_{\overline{\mathbb{F}}_p}$ and, for $\chi \in \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$, let U_χ denote the quotient of U by the two-sided ideal generated by the elements $x^p - x^{[p]} - \chi(x)$, for $x \in \mathfrak{g}_{\overline{\mathbb{F}}_p}$.

Let $u \in U^G$ with $\theta_{\overline{\mathbb{F}}_p}(u) = 0$. Then, $u \in \mathfrak{u}^- U\mathfrak{u}$ and, being G-invariant, $u = g(u)$ belongs to $g(\mathfrak{u}^-)Ug(\mathfrak{u})$, for every $g \in G$. Let L be a simple U-module. By Lemma 2.5 and, say, [**29**, 2.4], L is a $U_{g\chi}$-module, for some $g \in G$ and $\chi \in \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$ such that $\chi(\mathfrak{u}) = 0$. Then, one deduces from [**29**, § 6.7] or [**17**, Prop. 1.5] that L is generated by a vector v annihilated by $g(\mathfrak{u})$ (in [**17**], it is assumed that G is semi-simple and simply-connected but this hypothesis is not used in the proof of Prop. 1.5). Thus, $uv = g(u)v = 0$ and hence $uL = 0$. Therefore, u annihilates every simple U-module, that is, belongs to every maximal left ideal of U. Hence, $1 + u$ is a unit in U; but the only units in U are the non-zero scalars, and it follows that $u = 0$. (The last part of the argument is due to Curtis [**10**]).

Remark. — In [**31**, 9.4.d)], it is mistakenly asserted that $\theta_{\overline{\mathbb{F}}_p}$ is not injective in the case where $G = \mathrm{SO}(2n+1)$ and $p = 2$; but in fact the element q considered in [**31**, 9.1] is not G-invariant.

2.6.2. Note that $U_{\overline{\mathbb{F}}_p}(\mathfrak{t}) = S_{\overline{\mathbb{F}}_p}(\mathfrak{t})$ identifies with $\mathcal{P}(\mathfrak{t}^*_{\overline{\mathbb{F}}_p})$, the algebra of regular functions on

$$\mathfrak{t}^*_{\overline{\mathbb{F}}_p} := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{t}, \overline{\mathbb{F}}_p) \cong X(T) \otimes \overline{\mathbb{F}}_p.$$

The dot action of W on $U_{\overline{\mathbb{F}}_p}(\mathfrak{t})$ is defined, therefore, by $(w \cdot P)(\lambda) = P(w^{-1} \cdot \lambda)$, for $w \in W$, $P \in U_{\overline{\mathbb{F}}_p}(\mathfrak{t})$, $\lambda \in \mathfrak{t}^*_{\overline{\mathbb{F}}_p}$. For typographical reasons, let us denote by $U_{\overline{\mathbb{F}}_p}(\mathfrak{t})^{W\bullet}$ the subalgebra of invariants for this action. Then, as in [**31**, Lemma 5.2] or [**29**, 9.5], one obtains that $\theta_{\overline{\mathbb{F}}_p}(U_{\overline{\mathbb{F}}_p}(\mathfrak{g})^G) \subseteq U_{\overline{\mathbb{F}}_p}(\mathfrak{t})^{W\bullet}$. Moreover, under certain assumptions on G and p, this inclusion is an equality. Recall that a prime p is called *good* for R if it satisfies the following: for every $\gamma^\vee \in R^\vee$ expressed in terms of the simple coroots as

$$\gamma^\vee = \sum_{\alpha \in \Delta} n_\alpha(\gamma^\vee)\alpha^\vee,$$

one has $p > n_\alpha(\gamma^\vee)$ for all α. Then, one has the following mod. p analogue of Harish-Chandra's isomorphism. Let $\mathcal{D}G$ denote the derived subgroup of G, see [**28**, II.1.18].

Theorem ([29]). — *Assume that $\mathcal{D}G$ is simply-connected, that p is good for R, and that $X(T)/\mathbb{Z}R$ has no p-torsion. Then $\theta_{\overline{\mathbb{F}}_p}$ induces an isomorphim of algebras*

$$U_{\overline{\mathbb{F}}_p}(\mathfrak{g})^G \cong U_{\overline{\mathbb{F}}_p}(\mathfrak{t})^{W\bullet}.$$

Proof. — Under the stated assumptions, this is proved in [**29**, § 9.6]. For the convenience of the reader, let us outline the steps of the proof. Firstly, it is proved in [**29**, § 9.6] that it suffices to prove that the natural map

$$U_{\mathbb{Z}_{(p)}}(\mathfrak{t})^{W\bullet} \otimes \overline{\mathbb{F}}_p \longrightarrow U_{\overline{\mathbb{F}}_p}(\mathfrak{t})^{W\bullet}$$

is surjective. Secondly, since $\mathcal{D}G$ is simply-connected, $\{\alpha^\vee,\ \alpha \in \Delta\}$ is part of a basis of $X^\vee(T)$; see [**28**, II.1.18] or [**48**, Prop. 8.1.8.(iii)], and it follows that the previous map is surjective if and only if the analogous map $U_{\mathbb{Z}_{(p)}}(\mathfrak{t})^W \otimes \overline{\mathbb{F}}_p \to U_{\overline{\mathbb{F}}_p}(\mathfrak{t})^W$ is so. Finally, this surjectivity result follows, under the assumption that p is good and does not divide $|X(T)/\mathbb{Z}R|$, from [**12**], Cor. of Th. 2 (applied to the lattice $M = X^\vee(T) \cong \mathrm{Lie}\, T$ and the root system $R^\vee$).

Remark. — The theorem is proved by completely different methods in [**31**] in the case where G is almost simple and $p \neq 2$ if $G = \mathrm{SO}(2n+1)$; these methods can be extended to the case where G is reductive under the assumption that $p \neq 2$ if $\alpha^\vee/2 \in X^\vee(T)$, for some $\alpha \in R$. However, the version of the theorem given above is sufficient for our purposes.

2.6.3. Central characters. — For any $\mu \in X(T)$, its differential $d\mu$ induces an A-linear map $\mathfrak{t}_A \to A$ and hence an A-algebra morphism $U_A(\mathfrak{t}) \to A$, still denoted by $d\mu$. Thus, μ gives rise to an A-algebra morphism $\chi_{\mu,A} := d\mu \circ \theta_A$, from $U_A(\mathfrak{g})^G$ to A.

For any morphism of commutative rings $f : A \to B$, it is easily seen that the following diagram is commutative:

$$\begin{tikzcd} U_A(\mathfrak{g})^G \arrow[r, "\theta_A"] \arrow[d, "f"'] & U_A(\mathfrak{t}) \arrow[r, "d\mu"] \arrow[d, "f"] & A \arrow[d, "f"] \\ U_B(\mathfrak{g})^G \arrow[r, "\theta_B"] & U_B(\mathfrak{t}) \arrow[r, "d\mu"] & B. \end{tikzcd}$$

Thus, one has $\chi_{\mu,B} \circ f = f \circ \chi_{\mu,A}$.

Recall that $U_A(\mathfrak{g})^G \subseteq U_A(\mathfrak{t}) \oplus \mathfrak{u}^- U_A(\mathfrak{g})\mathfrak{u}$. Thus, if M is a $U_A(\mathfrak{g})$-module generated by an element v of weight μ annihilated by $\mathfrak{u}$, then $U_A(\mathfrak{g})^G$ acts on M by the character $\chi_{\mu,A}$ (see [**13**, Prop.7.4.4]).

Let π denote the morphism $\mathbb{Z}_{(p)} \to \overline{\mathbb{F}}_p$, let $\chi_{\mu,p} := \chi_{\mu,\mathbb{Z}_{(p)}}$ and $\overline{\chi}_{\mu,p} := \pi \circ \chi_{\mu,p} = \chi_{\mu,\overline{\mathbb{F}}_p}$, and set $J_{\mu,p} := \operatorname{Ker} \chi_{\mu,p}$. Then, one deduces immediately from the previous theorem the following

Corollary. — *Keep the hypotheses of the previous theorem. Let $\lambda, \mu \in X(T)$. If $\chi_{\lambda,\overline{\mathbb{F}}_p} = \chi_{\mu,\overline{\mathbb{F}}_p}$, there exists $w \in W$ such that $\mu - w \cdot \lambda \in pX(T)$.*

2.7. Decomposition w.r.t. central characters mod. p.— Let $\lambda \in X^+$ and let p be a prime integer such that $\lambda \in \overline{C}_p$. Recall the multisets $\Omega^i_{\mathfrak{p}}(\lambda)$ from 2.4 and let $\Omega^\bullet_{\mathfrak{p}}(\lambda)$ denote their disjoint union.

By Lemma 2.4, each $S_i^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)$ admits a finite $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})$-filtration, whose quotients are the $M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(\xi)$, where ξ runs through $\Omega^i_{\mathfrak{p}}(\lambda)$. It follows that $S_\bullet^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)$ is annihilated by the ideal

$$I := \prod_{\xi \in \Omega^\bullet_{\mathfrak{p}}(\lambda)} J_{\xi,p}$$

(each ξ being counted with its multiplicity).

The following lemma is straightforward.

Lemma. — *Let A be a commutative ring and let $P_1, \ldots, P_r$ be ideals of A such that $P_1 \cdots P_r = 0$ and $P_i + P_j = A$ if $j \neq i$. Then, for any A-module M, one has*

$$M = \bigoplus_{i=1}^{r} M^{P_i}, \quad \textit{where } M^{P_i} = \{m \in M \mid P_i\, m = 0\}.$$

Further, the assignment $M \mapsto M^{P_i}$ is an exact functor.

We shall apply the lemma to $A := U_{\mathbb{Z}_{(p)}}(\mathfrak{g})^G / I$. Note that A is a finite $\mathbb{Z}_{(p)}$-module. Moreover, it is easily seen that the maximal ideals of A are the $pA + J_{\xi,p} = \operatorname{Ker} \overline{\chi}_{\xi,p}$, for $\xi \in \Omega^\bullet_{\mathfrak{p}}(\lambda)$. (By abuse of notation, we still denote by $J_{\xi,p}$ the image of $J_{\xi,p}$ in A).

Let $\overline{\chi}_1, \ldots, \overline{\chi}_r$ be the distinct algebra homomorphisms $A \to \overline{\mathbb{F}}_p$, numbered so that $\overline{\chi}_1 = \overline{\chi}_{\lambda,p}$, and, for $i = 1, \ldots, r$, let

$$P_i := \prod_{\substack{\xi \in \Omega^{\bullet}_{\mathfrak{p}}(\lambda) \\ \overline{\chi}_{\xi,p} = \overline{\chi}_i}} J_{\xi,p}.$$

Clearly, $P_1 \cdots P_r = 0$ and $pA + P_i + P_j = A$ if $j \neq i$. Since A is a finite $\mathbb{Z}_{(p)}$-module, the latter implies, by Nakayama lemma, that $P_i + P_j = A$ if $j \neq i$.

Then, one deduces from the previous lemma that $S^{\mathbb{Z}_{(p)}}_{\bullet}(\mathfrak{g}, \mathfrak{p}, \lambda)$ is the direct sum of the $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})$-submodules corresponding to the characters $\overline{\chi}_1, \ldots, \overline{\chi}_r$, that is,

$$(*) \qquad S^{\mathbb{Z}_{(p)}}_{\bullet}(\mathfrak{g}, \mathfrak{p}, \lambda) = \bigoplus_{i=1}^{r} S^{\mathbb{Z}_{(p)}}_{\bullet}(\mathfrak{g}, \mathfrak{p}, \lambda)^{P_i}.$$

Moreover, since the differentials in the complex $S^{\mathbb{Z}_{(p)}}_{\bullet}(\mathfrak{g}, \mathfrak{p}, \lambda)$ are $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})$-equivariant, each $S^{\mathbb{Z}_{(p)}}_{\bullet}(\mathfrak{g}, \mathfrak{p}, \lambda)^{P_i}$ is a direct summand *subcomplex*. In particular, since $\overline{\chi}_1 = \overline{\chi}_{\lambda,p}$, this is true for

$$S^{\mathbb{Z}_{(p)}}_{\bullet}(\mathfrak{g}, \mathfrak{p}, \lambda)_{\overline{\chi}_{\lambda,p}} := S^{\mathbb{Z}_{(p)}}_{\bullet}(\mathfrak{g}, \mathfrak{p}, \lambda)^{P_1}.$$

Further, since $M \mapsto M_{\overline{\chi}_{\lambda,p}}$ is an exact functor and since

$$M^{\mathbb{Z}_{(p)}}_{\mathfrak{p}}(\xi)_{\overline{\chi}_{\lambda,p}} = \begin{cases} M^{\mathbb{Z}_{(p)}}_{\mathfrak{p}}(\xi) & \text{if } \overline{\chi}_{\xi,p} = \overline{\chi}_{\lambda,p}\,; \\ 0 & \text{otherwise,} \end{cases}$$

one obtains, as in [**3**, Lemma 9.7], the following

Corollary. — *$S^{\mathbb{Z}_{(p)}}_{\bullet}(\mathfrak{g}, \mathfrak{p}, \lambda)$ contains the subcomplex $S^{\mathbb{Z}_{(p)}}_{\bullet}(\mathfrak{g}, \mathfrak{p}, \lambda)_{\overline{\chi}_{\lambda,p}}$ as a direct summand. Moreover, for $i \geqslant 0$, each $S^{\mathbb{Z}_{(p)}}_{i}(\mathfrak{g}, \mathfrak{p}, \lambda)_{\overline{\chi}_{\lambda,p}}$ has a filtration whose quotients are the $M^{\mathbb{Z}_{(p)}}_{\mathfrak{p}}(\xi)$, for those $\xi \in \Omega^{i}_{\mathfrak{p}}(\lambda)$ (counted with multiplicities) such that $\overline{\chi}_{\xi,p} = \overline{\chi}_{\lambda,p}$.*

2.8. The main step towards the description of $S^{\mathbb{Z}_{(p)}}_{\bullet}(\mathfrak{g}, \mathfrak{p}, \lambda)_{\overline{\chi}_{\lambda,p}}$ is the following proposition.

Proposition. — *Assume that $\mathcal{D}G$ is simply-connected and $X(T)/\mathbb{Z}R$ has no p-torsion. Let $\lambda \in X^+ \cap \overline{C}_p$ and $\xi \in \Omega^{\bullet}_{\mathfrak{p}}(\lambda)$. If $\overline{\chi}_{\xi,p} = \overline{\chi}_{\lambda,p}$, then $\xi = w \cdot \lambda$ for some $w \in W^L$.*

Proof. — Let ξ be as in the proposition. Observe that, by 2.1(†), the assumption $X^+ \cap \overline{C}_p \neq \varnothing$ implies that p is good for R. Therefore, the hypotheses of Theorem 2.6.2 are satisfied. Thus, by Corollary 2.6.3, $\overline{\chi}_{\xi,p} = \overline{\chi}_{\lambda,p}$ implies that there exist $y \in W$ and $\nu \in X(T)$ such that $y \cdot \xi = \lambda + p\nu$. Moreover, since $y \cdot \xi$ is a weight of $\Lambda(\mathfrak{g}/\mathfrak{p}) \otimes V_{\mathbb{Z}}(\lambda)$, then $y \cdot \xi - \lambda \in \mathbb{Z}R$ and hence $p\nu \in \mathbb{Z}R \cap pX(T)$. Since $X(T)/\mathbb{Z}R$ has no p-torsion, it follows that $\nu \in \mathbb{Z}R$ and hence $\xi \in W_p \cdot \lambda$.

Now, let $w \in W$ such that $w^{-1}(\xi + \rho)$ is dominant and let $\xi^+ := w^{-1} \cdot \xi$. Then, by Lemma 2.3, $\xi^+ \in \overline{C}_p$. But $\xi^+ \in W_p \cdot \lambda$; since $\overline{C}_p$ is a fundamental domain for the dot action of W_p, it follows that $\xi^+ = \lambda$, and hence $\xi = w \cdot \lambda$.

Further, since $\xi \in \Omega_{\mathfrak{p}}^{\bullet}(\lambda) \subseteq X_L^+$, for any $\alpha \in R_L^+$ one has $\langle w \cdot \lambda, \alpha^\vee\rangle \geqslant 0$ and hence

$$\langle \lambda + \rho, w^{-1}\alpha^\vee\rangle \geqslant \langle \rho, \alpha^\vee\rangle > 0.$$

This implies that $w \in W^L$. The proposition is proved.

Remark. — In the first version of this paper, the previous proposition was stated under the assumption that G is either GL_n or almost simple and simply connected of type $\neq A$ and the proof relied on [**31**, Th. 1] in the second case and on results of Carter and Lusztig ([**8**], proof of Theorems 3.8 and 4.1) in the first case. We are indebted to the referee for pointing out that the result could be stated and proved in a uniform manner by using the version of Harish-Chandra's isomorphism given in [**29**, §9].

We can now prove the following analogue of [**3**, Th. 9.9] and [**37**, Th. 3.10], [**46**, Th. 7.11].

Theorem. — *Assume that $\mathcal{D}G$ is simply-connected, that $X(T)/\mathbb{Z}R$ has no p-torsion, and that $\lambda \in X^+ \cap \overline{C}_p$. Then $S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g},\mathfrak{p},\lambda)_{\overline{\chi}_{\lambda,p}}$ is an $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})$-resolution of $V_{\mathbb{Z}_{(p)}}(\lambda)$ and each $S_i^{\mathbb{Z}_{(p)}}(\mathfrak{g},\mathfrak{p},\lambda)_{\overline{\chi}_{\lambda,p}}$ with $i \geqslant 0$ has a filtration whose quotients are exactly the $M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(w\cdot\lambda)$, for $w \in W^L(i)$, each occuring once.*

Proof. — By Corollary 2.7 and the previous proposition, each $S_i^{\mathbb{Z}_{(p)}}(\mathfrak{g},\mathfrak{p},\lambda)_{\overline{\chi}_{\lambda,p}}$ with $i \geqslant 0$ has a filtration whose quotients are the $M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(\xi)$, for those $\xi \in \Omega_{\mathfrak{p}}^i(\lambda)$ (counted with multiplicities) such that $\xi = w\cdot\lambda$ for some $w \in W^L$.

Conversely, for $w \in W^L$, Kostant has showed that $V_{\mathbb{Q}}^L(w\cdot\lambda)$ occurs with multiplicity one in $\Lambda^{\bullet}(\mathfrak{g}/\mathfrak{p}) \otimes V_{\mathbb{Z}}(\lambda)$, in degree equal to $\ell(w)$, see [Ko1], Lemma 5.12 and end of proof of Th. 5.14. This completes the proof of the theorem.

2.9. Proof of theorem 2.1.— In this subsection, we assume that G is either GL_n or almost simple and simply-connected of type $\neq A$. As observed in 2.1, this assumption entails no loss of generality in the proof of Kostant's theorem over $\mathbb{Z}_{(p)}$. Keep the notation of 2.7-2.8. Note that $\mathbb{Z}R$ is a direct summand of $X(T)$ if $G = \mathrm{GL}_n$, while if G is almost simple of type $\neq A$, the assumption $X^+ \cap \overline{C}_p \neq \varnothing$ implies that $X(T)/\mathbb{Z}R$ has no p-torsion. Therefore, the hypotheses of Theorem 2.8 are satisfied.

Observe next that, as $U_{\mathbb{Z}_{(p)}}(\mathfrak{u}_P^-)$-module, any $M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(\xi)$ is isomorphic to $U_{\mathbb{Z}_{(p)}}(\mathfrak{u}_P^-) \otimes V_{\mathbb{Z}_{(p)}}^L(\xi)$, hence free. Thus, by Theorem 2.8, $S_i^{\mathbb{Z}_{(p)}}(\mathfrak{g},\mathfrak{p},\lambda)_{\overline{\chi}_{\lambda,p}}$ is a free $U_{\mathbb{Z}_{(p)}}(\mathfrak{u}_P^-)$-module, for each $i \geqslant 0$.

Therefore, $H_{\bullet}(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda))$ is the homology of the complex

$$C_{\bullet} := \mathbb{Z}_{(p)} \otimes_{U_{\mathbb{Z}_{(p)}}(\mathfrak{u}_P^-)} S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g},\mathfrak{p},\lambda)_{\overline{\chi}_{\lambda,p}}.$$

Further, by Theorem 2.8, again, for $i \geqslant 0$ each C_i has an L-module filtration whose successive quotients are the $V_{\mathbb{Z}_{(p)}}^L(w\cdot\lambda)$, for $w \in W^L(i)$.

By Corollary 1.10, applied to L, one obtains that these filtrations split, that is, for each $i \geqslant 0$ one has isomorphisms of L-modules

$$C_i \cong \bigoplus_{w \in W^L(i)} V^L_{\mathbb{Z}_{(p)}}(w \cdot \lambda).$$

Further, we claim that the differentials $d_i : C_i \to C_{i-1}$ are zero. Indeed, one has $H_i(C_\bullet) \otimes \mathbb{Q} \cong H_i(\mathfrak{u}_P^-, V_{\mathbb{Q}}(\lambda))$ and, by Kostant's theorem ([**33**, Cor. 8.1] or [**3**], Cor. of Th. 9.9), the latter is isomorphic to $C_i \otimes \mathbb{Q}$. It follows, for a reason of dimension, that $d_i \otimes 1 = 0$. Since C_{i-1} is torsion-free, this implies that $d_i = 0$.

Thus, we have obtained, for each $i \geqslant 0$, an isomorphism of L-modules

$$H_i(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w \in W^L(i)} V^L_{\mathbb{Z}_{(p)}}(w \cdot \lambda).$$

This completes the proof of Theorem 2.1.

2.10. Analogue in cohomology.— Recall the anti-involution τ from 1.4; it exchanges P^- and P and stabilizes L. Let $\lambda \in X^+ \cap \overline{C}_p$. Since $H_\bullet(\mathfrak{u}_P^-, V)$ is a free $\mathbb{Z}_{(p)}$-module, one obtains, by standard arguments, an isomorphism of L-modules

$$H_\bullet(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda))^\tau \cong H^\bullet(\mathfrak{u}_P, V_{\mathbb{Z}_{(p)}}(\lambda)^\tau).$$

Further, since $V_{\mathbb{Z}_{(p)}}(\lambda) = V_{\mathbb{Z}_{(p)}}(\lambda)^\tau$ and $V^L_{\mathbb{Z}_{(p)}}(w \cdot \lambda) = V^L_{\mathbb{Z}_{(p)}}(w \cdot \lambda)^\tau$, for $w \in W^L$, by Corollary 1.9, applied to G and L, one obtains the

Corollary. — *Let $\lambda \in X^+ \cap \overline{C}_p$. For each $i \geqslant 0$, there is an isomorphism of $L_{\mathbb{Z}_{(p)}}$-modules*

$$H^i(\mathfrak{u}_P, V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w \in W^L(i)} V^L_{\mathbb{Z}_{(p)}}(w \cdot \lambda).$$

3. Cohomology of the groups $U_P^-(\mathbb{Z})$

3.1. Let us recall several definitions and facts about finitely generated, torsion free, nilpotent groups. Let Γ be such a group, say of class c. Let $\mathcal{F}$ be a finite series

$$\Gamma = F^1\Gamma \supset F^2\Gamma \supset \cdots \supset F^{d+1}\Gamma = \{1\}$$

of normal subgroups of Γ. Following the terminology in Passman's book [**42**, p.85], let us say that $\mathcal{F}$ is an N-series if $(F^i\Gamma, F^j\Gamma) \subseteq F^{i+j}\Gamma$ for all i, j. Since every subgroup of Γ is finitely generated (see [**21**, Lemma 1.9] or [**42**, Chap. 3, Lemma 4.2]), each $F^i\Gamma/F^{i+1}\Gamma$ is then a finitely generated abelian group.

Let us denote temporarily by $r(\mathcal{F})$ the rank of $\bigoplus_{i=1}^d F^i\Gamma/F^{i+1}\Gamma$. This rank is in fact an invariant of Γ. Indeed, $\mathcal{F}$ can be refined to a sequence of normal subgroups

$$\Gamma = H^1 \supset H^2 \supset \cdots \supset H^{n+1} = \{1\}$$

such that each H^i/H^{i+1} is cyclic, and for any such refinement the number of infinite cyclic quotients equals $r(\mathcal{F})$. But, for any subnormal series $\Gamma = S^1 \triangleright S^2 \triangleright \cdots \triangleright S^{m+1} = \{1\}$ such that each quotient S^i/S^{i+1} is cyclic, the number of infinite cyclic quotients is an invariant called the rank, or Hirsch number, of Γ and denoted by $h(\Gamma)$; see the discussion before Lemma 10.2.10 in [**42**] or [**51**, Chap. 2, Th. 3.20]. Together, these arguments show that $r(\mathcal{F}) = h(\Gamma)$.

If $\mathcal{F}$ is an N-series, the associated graded abelian group

$$\mathrm{gr}_{\mathcal{F}}\,\Gamma := \bigoplus_{i\geqslant 1} F^i\Gamma/F^{i+1}\Gamma$$

has a natural structure of Lie algebra over $\mathbb{Z}$ (see, for example, [**36**, Chap. I, Th. 2.1]).

Further, $\mathcal{F}$ is called an N_0-series if it is an N-series and each $F^i\Gamma/F^{i+1}\Gamma$ is torsion-free. Such series exist, see [**30**, Th. 2.2] or [**42**, Chap. 11, Lemma 1.8], and in this case $\mathrm{gr}_{\mathcal{F}}\,\Gamma$ is a free $\mathbb{Z}$-module of rank $h(\Gamma)$.

Let $\{C^i(\Gamma)\}_{i\geqslant 1}$ denote the lower central series; as is well-known, it is the fastest descending N-series. We shall denote the corresponding graded Lie algebra simply by $\mathrm{gr}\,\Gamma$. Further, for each i, set

$$C^{(i)}(\Gamma) := \{x \in \Gamma \mid x^n \in C^i(\Gamma) \text{ for some } n > 0\}.$$

By [**42**, Lemma 11.1.8] (see also [**21**, § 4]), $\{C^{(i)}(\Gamma)\}_{i\geqslant 1}$ is an N_0-series. It is clearly the fastest descending N_0-series. Following [**21**, § 4], we will call it the isolated lower central series. We will denote by $\mathrm{gr}_{\mathrm{isol}}\,\Gamma$ the associated Lie algebra over $\mathbb{Z}$

$$\mathrm{gr}_{\mathrm{isol}}\,\Gamma := \bigoplus_{i\geqslant 1} C^{(i)}(\Gamma)/C^{(i+1)}(\Gamma);$$

it is a free $\mathbb{Z}$-module of rank $h(\Gamma)$. Clearly, there is an isomorphism of graded Lie algebras $\mathrm{gr}\,\Gamma\otimes\mathbb{Q} \cong \mathrm{gr}_{\mathrm{isol}}\,\Gamma\otimes\mathbb{Q}$.

Let I denote the augmentation ideal of the group ring $\mathbb{Z}\Gamma$ and, for $n \geqslant 0$, let $I^{(n)}$ denote the isolator of I^n, that is,

$$I^{(n)} := \{x \in \mathbb{Z}\Gamma \mid mx \in I^n \text{ for some } m > 0\}.$$

Equivalently, if $I_{\mathbb{Q}}$ denotes the augmentation ideal of $\mathbb{Q}\Gamma$, then $I^{(n)} = \mathbb{Z}\Gamma \cap I_{\mathbb{Q}}^n$.

Let us consider the graded rings

$$\mathrm{gr}_{\mathrm{isol}}\,\mathbb{Z}\Gamma := \bigoplus_{n\geqslant 0} I^{(n)}/I^{(n+1)} \quad \text{and} \quad \mathrm{gr}\,\mathbb{Q}\Gamma := \bigoplus_{n\geqslant 0} I_{\mathbb{Q}}^n/I_{\mathbb{Q}}^{n+1}.$$

The former is a subring of the latter and, by a result of Quillen ([**45**]), there is an isomorphism of graded Hopf algebras $U_{\mathbb{Q}}(\mathrm{gr}\,\Gamma\otimes\mathbb{Q}) \cong \mathrm{gr}\,\mathbb{Q}\Gamma$. Further, one has the following more precise result of Hartley :

Theorem **([23, Th. 2.3.3′]).** — *There is an isomorphism of graded Hopf algebras*

$$U_{\mathbb{Z}}(\mathrm{gr}_{\mathrm{isol}}\,\Gamma) \cong \mathrm{gr}_{\mathrm{isol}}\,\mathbb{Z}\Gamma.$$

3.2. Let A be a finitely generated subring of $\mathbb{Q}$ (thus, $A=\mathbb{Z}[1/m]$ for some m and A is a PID). Let $\mathfrak{u}$ be a nilpotent Lie algebra of class c over A, which is a finite free A-module, say of rank r. Let $\mathfrak{u}_{\mathbb{Q}}=\mathfrak{u}\otimes_A\mathbb{Q}$, then $U_{\mathbb{Q}}(\mathfrak{u}_{\mathbb{Q}})\cong U_A(\mathfrak{u})\otimes_A\mathbb{Q}$; we shall denote it by $U_{\mathbb{Q}}(\mathfrak{u})$. By the PBW theorem, $U_A(\mathfrak{u})$ is a subalgebra of $U_{\mathbb{Q}}(\mathfrak{u})$.

Let $\mathcal{F}$ be a finite sequence

$$\mathfrak{u}=F^1\mathfrak{u}\supset F^2\mathfrak{u}\supset\cdots\supset F^{d+1}\mathfrak{u}=\{0\}$$

of Lie ideals of $\mathfrak{u}$. As in the previous paragraph, let us say that $\mathcal{F}$ is an N-series if $[F^i\mathfrak{u},F^j\mathfrak{u}]\subseteq F^{i+j}\mathfrak{u}$, and is an N_0-series if further each $F^i\mathfrak{u}/F^{i+1}\mathfrak{u}$ (which is a finitely generated module over the PID A) is torsion free, and hence a free A-module.

Let $\{C^i(\mathfrak{u})\}_{i\geqslant1}$ denote the lower central series of $\mathfrak{u}$ and define the isolated lower central series $\{C^{(i)}(\mathfrak{u})\}_{i\geqslant1}$ by

$$C^{(i)}(\mathfrak{u}):=\{x\in\mathfrak{u}\mid nx\in C^i(\mathfrak{u})\text{ for some }n>0\}.$$

This is, clearly, the fastest descending N_0-series of $\mathfrak{u}$. Consider the graded Lie algebras

$$\operatorname{gr}_{\mathrm{isol}}\mathfrak{u}:=\bigoplus_{i\geqslant1}C^{(i)}(\mathfrak{u})/C^{(i+1)}(\mathfrak{u})\quad\text{and}\quad\operatorname{gr}\mathfrak{u}_{\mathbb{Q}}:=\bigoplus_{i\geqslant1}C^{i}(\mathfrak{u}_{\mathbb{Q}})/C^{i+1}(\mathfrak{u}_{\mathbb{Q}}).$$

Then $\operatorname{gr}_{\mathrm{isol}}\mathfrak{u}$ is a free A-module of rank r and there is an isomorphism of graded Lie algebras $(\operatorname{gr}_{\mathrm{isol}}\mathfrak{u})\otimes_A\mathbb{Q}\cong\operatorname{gr}\mathfrak{u}_{\mathbb{Q}}$.

Let $J_{\mathbb{Q}}$ denote the augmentation ideal of $U_{\mathbb{Q}}(\mathfrak{u})$. Then the graded algebra

$$\operatorname{gr}U_{\mathbb{Q}}(\mathfrak{u}):=\bigoplus_{n\geqslant0}J_{\mathbb{Q}}^n/J_{\mathbb{Q}}^{n+1}$$

is a primitively generated, graded Hopf algebra; it is isomorphic to $U_{\mathbb{Q}}(\operatorname{gr}\mathfrak{u}_{\mathbb{Q}})$, by [**32**] or [**52**, Prop. 1]. In fact, as in the case of group rings, a little more is true. For $n\geqslant1$, let $J^{(n)}=U_A(\mathfrak{u})\cap J_{\mathbb{Q}}^n$. Then the graded ring

$$\operatorname{gr}_{\mathrm{isol}}U_A(\mathfrak{u}):=\bigoplus_{n\geqslant0}J^{(n)}/J^{(n+1)}$$

identifies with a subring of $\operatorname{gr}U_{\mathbb{Q}}(\mathfrak{u})$. Further, one deduces from the proof of [**52**, Prop. 1] the following result. Let $X_1,\dots,X_r$ be an A-basis of $\mathfrak{u}$ compatible with the filtration $\{C^{(i)}(\mathfrak{u})\}_{i=1}^c$, *i.e.*, such that for $s=1,\dots,c$, the X_j with $j>r-\dim C^s(\mathfrak{u}_{\mathbb{Q}})$ form an A-basis of $C^{(s)}(\mathfrak{u})$, and, for each i, let $\mu(i)$ be the largest integer k such that $X_i\in C^{(k)}(\mathfrak{u})$.

Proposition

a) *The ordered monomials $X_1^{n_1}\cdots X_r^{n_r}$ with $\sum_{i=1}^r n_i\,\mu(i)\geqslant n$ form an A-basis of $J^{(n)}$, for any $n\geqslant0$.*

b) *There is an isomorphism of graded Hopf algebras $U_A(\operatorname{gr}_{\mathrm{isol}}\mathfrak{u})\cong\operatorname{gr}_{\mathrm{isol}}U_A(\mathfrak{u})$.*

3.3. Let Γ be, as in 3.1, a finitely generated, torsion free, nilpotent group of class c and let $\Gamma = H^1 \supset \cdots \supset H^{r+1} = \{1\}$ be a refinement of the isolated lower central series such that each H^i/H^{i+1} is an infinite cyclic group, generated by the image of an element g_i of H^i. Then, $r = h(\Gamma)$ and $\{g_1, \ldots, g_r\}$ is called a system of canonical parameters (or canonical basis) of Γ; it induces a bijection $\mathbb{Z}^r \cong \Gamma$, given by $(n_1, \ldots, n_r) \mapsto g_1^{n_1} \cdots g_r^{n_r}$; we will denote the R.H.S. simply by $g(n_1, \ldots, n_r)$. Let $\{e_1, \ldots, e_r\}$ be the standard basis of $\mathbb{Z}^r$; then $g(e_i) = g_i$.

Let $\mathcal{P}_{r,r}$ denote the subring of the polynomial ring $\mathbb{Q}[\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_r]$ consisting of those polynomials which take integral values on $\mathbb{Z}^r \times \mathbb{Z}^r$. By a result of Ph. Hall [**21**, Th. 6.5], there exist polynomials $P_1, \ldots, P_r \in \mathcal{P}_{r,r}$ such that

$$(\star) \qquad g(x_1, \ldots, x_r)\, g(y_1, \ldots, y_r) = g(P_1(x, y), \ldots, P_r(x, y)),$$

for any $x, y \in \mathbb{Z}^r$.

Therefore, there exists an algebraic unipotent group scheme U, defined over a finitely generated subring A of the rationals, and whose underlying scheme is affine space $\mathbb{A}^r_A$, such that Γ identifies with the subgroup $\mathbb{Z}^r$ of $U(A) = A^r$.

Remark. — If Γ is of class c, one may take $A = \mathbb{Z}[1/c!]$; this can be deduced, for example, from the Campbell-Hausdorff formula.

Let $k \in \{1, \ldots, r\}$. Since $P_k(x, 0) = x_k$ and $P_k(0, y) = y_k$ for every $x, y \in \mathbb{Z}^r$, the part of degree $\leqslant 1$ of P_k is $\xi_k + \eta_k$ and its part of degree 2, call it b_k, is bilinear in the ξ_i and the η_j. Thus, one has

$$P_k(\xi, \eta) = \xi_k + \eta_k + \sum_{i,j=1}^{r} b_k(e_i, e_j)\, \xi_i \eta_j + \text{terms of degree} > 2.$$

Let $\mathfrak{m}$ denote the ideal $(\xi_1, \ldots, \xi_r)$ of $A[U] = A[\xi_1, \ldots, \xi_r]$, let

$$\mathfrak{u} := \operatorname{Hom}_A(\mathfrak{m}/\mathfrak{m}^2, A)$$

be the Lie algebra of U over A, and let $\{v_1, \ldots, v_r\}$ be the A-basis of $\mathfrak{u}$ dual to the basis $\{\overline{\xi}_1, \ldots, \overline{\xi}_r\}$. Then, the Lie brackets are given by

$$(1) \qquad [v_i, v_j] = \sum_{k=1}^{r} (b_k(e_i, e_j) - b_k(e_j, e_i))\, v_k,$$

see, for example, [**35**, § 1] or [**9**, § 1].

Proposition. — *There is an isomorphism of graded Lie algebras over A*

$$\operatorname{gr}_{\mathrm{isol}} \Gamma \otimes_{\mathbb{Z}} A \cong \operatorname{gr}_{\mathrm{isol}} \mathfrak{u},$$

under which each $\overline{g}_i$ corresponds to $\overline{v}_i$.

Proof. — First, for each i, let $\nu(i)$ denote the largest integer n such that $g_i \in C^{(n)}(\Gamma)$. Denote by $\overline{g}_i$ the image of g_i in $\mathrm{gr}_{\mathrm{isol}}^{\nu(i)}\Gamma$; then $\{\overline{g}_1, \ldots, \overline{g}_r\}$ is a $\mathbb{Z}$-basis of $\mathrm{gr}_{\mathrm{isol}}\Gamma$.

For $k = 1, \ldots, r$, let $Q_k := P_k - \xi_k - \eta_k$ be the part of P_k of degree > 1. Recall that, for $x_1, \ldots, x_r \in \mathbb{Z}$, $g(\sum_{i=1}^r x_i\, e_i)$ denotes the element $g_1^{x_1} \cdots g_r^{x_r}$ of Γ.

Let $i, j \in \{1, \ldots, r\}$ be arbitrary with $i < j$. Then, for every $x, y \in \mathbb{Z}^r$, one has $g(xe_i)g(ye_j) = g(xe_i + ye_j)$ and hence $Q_k(xe_i, ye_j) = 0 = b_k(xe_i, ye_j)$ for any k. In particular, $b_k(e_i, e_j) = 0$.

On the other hand, since $g_j^x \in C^{(\nu(j))}(\Gamma)$ and $g_i^y \in C^{(\nu(i))}(\Gamma)$ one has,

$$g_j^x\, g_i^y \equiv g_i^y\, g_j^x\, g\Bigg(\sum_{\substack{k\\ \nu(k)=\nu(i)+\nu(j)}} Q_k(x,y)e_k\Bigg) \quad \text{mod. } C^{(\nu(i)+\nu(j)+1)}(\Gamma).$$

Further, since the commutator induces a bilinear map on $\mathrm{gr}_{\mathrm{isol}}\Gamma$, one has, when $\nu(k) = \nu(i) + \nu(j)$,

$$Q_k(xe_j, ye_i) = xyQ_k(e_j, e_i) = xyb_k(e_j, e_i).$$

Then, an easy computation shows that

$$g_i^x\, g_j^y\, g_i^{-x}\, g_j^{-y} \equiv g\Bigg(\sum_{\substack{k\\ \nu(k)=\nu(i)+\nu(j)}} -xyb_k(e_j, e_i)\, e_k\Bigg) \quad \text{mod. } C^{(\nu(i)+\nu(j)+1)}(\Gamma).$$

Using the fact that $b_k(e_i, e_j) = 0$, one deduces that the Lie bracket on $\mathrm{gr}_{\mathrm{isol}}\Gamma$ is given by

$$(2) \qquad [\overline{g}_i, \overline{g}_j] = \sum_{\substack{k\\ \nu(k)=\nu(i)+\nu(j)}} (b_k(e_i, e_j) - b_k(e_j, e_i))\, \overline{g}_k.$$

The proposition is then a consequence of the following claim.

Claim. — *For $\ell = 1, \ldots, c$, $C^{(\ell)}(\mathfrak{u})$ is the A-span of those v_k such that $\nu(k) \geqslant \ell$.*

Indeed, using (1), the claim implies that $\mathrm{gr}_{\mathrm{isol}}\,\mathfrak{u}$ is the Lie algebra having an A-basis $\{\overline{v}_1, \ldots, \overline{v}_r\}$ and brackets given by

$$(3) \qquad [\overline{v}_i, \overline{v}_j] = \sum_{\substack{k\\ \nu(k)=\nu(i)+\nu(j)}} (b_k(e_i, e_j) - b_k(e_j, e_i))\, \overline{v}_k.$$

Comparing with (2), one obtains that $\mathrm{gr}_{\mathrm{isol}}\Gamma \otimes_{\mathbb{Z}} A \cong \mathrm{gr}_{\mathrm{isol}}\,\mathfrak{u}$.

Let us now prove the claim by induction on $r + \ell$. Recall that c denotes the class of Γ. By induction, we may reduce to the case where $C^{(c)}(\Gamma) = \mathbb{Z}g_r$.

Since $\mathrm{gr}_{\mathrm{isol}}\Gamma \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{gr}\,\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated in degree 1, there exist $s < t < r$ such that $\nu(t) = c - 1$ and $[\overline{g}_s, \overline{g}_t] = n\overline{g}_r$, for some non-zero integer n. Then, $(g_s, g_t) = g_r^n$ and hence, by the previous calculations, one has $b_r(e_t, e_s) = -n$, while $b_r(e_s, e_t) = 0$. Therefore, by (1), $[v_s, v_t] = nv_r$.

For any $k < r$, the image of v_k in $\mathfrak{u}/Av_r$ belongs to $C^{(\nu(k))}(\mathfrak{u}/Av_r)$, by induction hypothesis. Thus, there exist a positive integer m_k and $a_k \in A$ such that

$$m_k v_k - a_k v_r \in C^{\nu(k)}(\mathfrak{u}). \tag{4}$$

Applying this to $k = t$ and using the fact that v_r is central, one obtains that

$$m_t\, n\, v_r = [v_s, m_t v_t - a_t v_r]$$

belongs to $C^c(\mathfrak{u})$, and hence $v_r \in C^{(c)}(\mathfrak{u})$. In turn, this implies, by (4), that $v_k \in C^{(\nu(k))}(\mathfrak{u})$, for each $k < r$. This proves the claim and completes the proof of the proposition.

3.4. Filtered Noetherian rings with the AR-property.— Let us recall several results about the homology of filtered Noetherian rings with the Artin-Rees property. Some basic references for this material are [**47**], [**6**], [**20**]; see also [**40**, Chap. I] and [**14**, § 1]. (Note, however, that in [**20**] the assertions in lines 8-12 of 2.8 and assertion (ii) of Theorem 3.3 are not correct; it is not difficult to provide counter-examples).

Let S be a left Noetherian ring. A sequence $\mathcal{I} := \{I_1, I_2, \ldots\}$ of two-sided ideals is said to be *admissible* if $I_1 \supseteq I_2 \supseteq \cdots$ and $I_j I_k \subseteq I_{j+k}$ for $j, k \geqslant 0$ (where one sets $I_0 = S$). Given such a sequence, let

$$\operatorname{gr} S := \bigoplus_{n \geqslant 0} I_n / I_{n+1} \quad \text{and} \quad \widehat{S} := \operatorname*{proj.lim.}_{n \geqslant 0} S/I_n$$

be the associated graded ring and completion, respectively.

Let S-filt denote the category of $\mathbb{N}$-filtered left S-modules: objects are left S-modules M equipped with a decreasing filtration $M = F^0 M \supseteq F^1 M \supseteq \cdots$ such that $I_n F^k M \subseteq F^{n+k} M$, and a morphism $f : M \to N$ between two such objects is an S-morphism which preserves the filtrations. Then f induces a morphism of $\operatorname{gr} S$-modules $\operatorname{gr} f : \operatorname{gr} M \to \operatorname{gr} N$ and this defines a functor gr from S-filt to the category of $\mathbb{N}$-graded $\operatorname{gr} S$-modules. Further, f is called *strict* if one has $f(M) \cap F^k N = f(F^k M)$ for any k.

An object M of S-filt is called *separated* if $\bigcap_{n \geqslant 0} F^n M = \{0\}$, and *discrete* if $F^n M = \{0\}$ for some $n \geqslant 0$.

The category S-filt is equipped with shift functors s^n, for $n \geqslant 0$, defined as follows. If M is an object of S-filt, $s^n M = M$ as S-module but $F^p(s^n M) = F^{p-n} M$ for $p \geqslant 0$, with the convention that $F^k M = M$ if $k < 0$. If M is an $\mathbb{N}$-graded S-module, the shifted module $s^n M$ is defined in an analogous manner.

An object L of S-filt is called *filt-free* if it a direct sum of shifted modules $s^{d(\lambda)} S$, for λ running in some index set Λ. Then, $\operatorname{gr} L \cong \bigoplus_{\lambda \in \Lambda} s^{d(\lambda)} \operatorname{gr} S$.

Let M be an object of S-filt. Then a *strict filt-free resolution* of M is an S-module resolution

$$\cdots \longrightarrow L_1 \xrightarrow{f_1} L_0 \xrightarrow{f_0} M \longrightarrow 0 \tag{$\mathcal{E}$}$$

such that every L_n is filt-free and every f_n is a strict morphism in S-filt. By [**47**, Lemmas 1,2], the associated graded complex $(\operatorname{gr}\mathcal{E})$ is then a free $\operatorname{gr} S$-resolution of $\operatorname{gr} M$ and, conversely, if S is complete with respect to $\mathcal{I}$, any free $\operatorname{gr} S$-resolution of $\operatorname{gr} M$ can be obtained in this manner.

Let us consider also the category filt-S of $\mathbb{N}$-filtered *right* S-modules. All notions introduced previously for S-filt have, of course, their right-handed analogues. Now, if N (resp. M) is an object of filt-S (resp. S-filt), the abelian group $N \otimes_S M$ has a natural $\mathbb{N}$-filtration, defined by

$$F^n(N \otimes_S M) := \operatorname{Im}\Big(\sum\nolimits_{p+q=n} F^p N \otimes_S F^q M \longrightarrow N \otimes_S M\Big).$$

Moreover, it is easily seen that if either of N or M is a filt-free object, then the natural map $\operatorname{gr} N \otimes_{\operatorname{gr} S} \operatorname{gr} M \to \operatorname{gr}(N \otimes_S M)$ is an isomorphism.

Therefore, if one considers a strict filt-free resolution $L_\bullet$ of, say, M, the filtration on $N \otimes_S L_\bullet$ induces a natural spectral sequence with E_1-term (in cohomological notation)

$$E_1^{p,-q} = H^{p-q}(\operatorname{gr} N \otimes_S \operatorname{gr} L_\bullet)_p = \operatorname{Tor}_{q-p}^{\operatorname{gr} S}(\operatorname{gr} N, \operatorname{gr} M)_p.$$

Moreover, certain finiteness conditions ensure that this spectral sequence converges finitely to $\operatorname{Tor}_*^S(N, M)$. Firstly, by [**47**, Lemma 2.(g)] or [**20**, Th. 2.9], one has the following

Proposition (C). — *Assume that S is complete with respect to the filtration $\mathcal{I}$ and that $\operatorname{gr} S$ is left Noetherian. Let M, N be objects of S-filt and filt-S, respectively, such that M is separated and $\operatorname{gr} M$ finitely generated over $\operatorname{gr} S$, while N is discrete. Then the spectral sequence above converges finitely to $\operatorname{Tor}_*^S(N, M)$.*

Proof. — By the references cited above, any resolution of $\operatorname{gr} M$ by free $\operatorname{gr} S$-modules can be lifted to a strict filt-free resolution of M. Since $\operatorname{gr} M$ is finitely generated over $\operatorname{gr} S$, which is left Noetherian, one deduces that M admits a strict filt-free resolution $L_\bullet \to M \to 0$ such that each L_n is finitely generated. As N is assumed to be discrete, the filtration on $N \otimes_S L_\bullet$ is then discrete (and exhaustive) in each degree, and the proposition follows.

Secondly, the assumption that S be complete can be relaxed if one assumes that the sequence $\mathcal{I} = \{I = I_1 \supseteq I_2 \supseteq \cdots\}$ has the left Artin-Rees property, *i.e.*, that $\mathcal{I}$ satisfies the following : for any finitely generated left S-module M, any submodule $N \subseteq M$ and any $n \geqslant 0$, there exists $n' \geqslant n$ such that $N \cap I_{n'} M \subseteq I_n N$.

For any left S-module M, let us denote by $\widehat{M}$ its completion with respect to the filtration $\{I_n M\}$; it is an $\widehat{S}$-module and there is a natural morphism of $\widehat{S}$-modules $\tau_M : \widehat{S} \otimes_S M \to \widehat{M}$. As observed in [**6**, Prop. 3], one has the following proposition, which is proved exactly as in the commutative I-adic case (see [**2**, Chap. 10]).

Proposition (AR). — *Assume that S is left Noetherian and that $\mathcal{I}$ satisfies the left AR-property. Then, τ_M is an isomorphism for any finitely generated left S-module M and, therefore,*

a) *$\widehat{S}$ is flat as right S-module,*

b) *for each n, $\widehat{S}I_n = \operatorname{Ker}(\widehat{S} \to S/I_n)$ is a two-sided ideal and hence $\{\widehat{S}I_n\}$ is an admissible sequence in $\widehat{S}$,*

c) *the associated graded $\operatorname{gr}\widehat{S}$ is isomorphic to $\operatorname{gr} S$.*

Thus, in particular, if $P_\bullet \to S/I \to 0$ is a resolution of S/I by free S-modules, then $\widehat{S} \otimes_S P_\bullet$ is a free $\widehat{S}$-resolution of

$$\widehat{S} \otimes_S (S/I) = \widehat{S/I} = S/I.$$

Thus, for any right $\widehat{S}$-module N, there is a natural isomorphism

$$\operatorname{Tor}_\bullet^{\widehat{S}}(N, S/I) \cong \operatorname{Tor}_\bullet^{S}(N, S/I).$$

This is the case, in particular, if N is a right S-module with a discrete filtration. Therefore, one obtains the following theorem, which is essentially contained in [**20**, Th. 3.3′.(i)].

Theorem 3.4.1. — *Let S be a left Noetherian ring, $\mathcal{I}$ an admissible sequence of ideals. Suppose that $\mathcal{I}$ satisfies the left AR property and that $\operatorname{gr} S$ is left Noetherian. Let N be a right S-module with a discrete filtration. Then there is a finitely convergent spectral sequence*

$$E_1^{p,-q} = \operatorname{Tor}_{q-p}^{\operatorname{gr} S}(\operatorname{gr} N, S/I)_p \Longrightarrow \operatorname{Tor}_{q-p}^{\widehat{S}}(N, S/I) \cong \operatorname{Tor}_{q-p}^{S}(N, S/I).$$

For future use, let us derive the following equivariant version of the theorem. Let Λ be a group of automorphisms of S preserving the sequence $\mathcal{I}$. Let $S\Lambda$ denote the smash product $S\#\mathbb{Z}\Lambda$, that is, $S\Lambda = S \otimes_{\mathbb{Z}} \mathbb{Z}\Lambda$ as $(S, \mathbb{Z}\Lambda)$-bimodule, the multiplication being defined by

$$(s \otimes \lambda)(s' \otimes \lambda') = s\lambda(s') \otimes \lambda\lambda'.$$

Similarly, denote by $\widehat{S}\Lambda$ the smash product $\widehat{S}\#\mathbb{Z}\Lambda$. Observe that an $S\Lambda$-module is the same thing as an S-module M equipped with an action of Λ such that $\lambda sm = \lambda(s)\lambda m$, for $m \in M$, $s \in S$, $\lambda \in \Lambda$.

For every $n \geqslant 0$, let I'_n (resp. $\widehat{I'_n}$) denote the left ideal of $S\Lambda$ (resp. $\widehat{S}\Lambda$) generated by I_n; they are two-sided ideals and form an admissible sequence in $S\Lambda$ (resp. $\widehat{S}\Lambda$). In both cases, the associated graded is isomorphic to $(\operatorname{gr} S)\Lambda := (\operatorname{gr} S)\#\Lambda$.

Theorem 3.4.2. — *With notation as above, let N be a discrete object of $S\Lambda$-filt. There is a finitely convergent spectral sequence of Λ-modules*

$$E_1^{p,-q} = \operatorname{Tor}_{q-p}^{\operatorname{gr} S}(\operatorname{gr} N, S/I)_p \Longrightarrow \operatorname{Tor}_{q-p}^{S}(N, S/I).$$

Proof. — First, $I' := (S\Lambda)I$ is a two-sided ideal of $S\Lambda$, and $S\Lambda \otimes_S (S/I) \cong S\Lambda/I'$. Then, by standard arguments, it suffices to prove that: *i*) $\widehat{S}\Lambda$ is flat as right $S\Lambda$-module, and: *ii*) $\widehat{S}\Lambda \otimes_S (S/I) \cong S\Lambda/I'$.

But $\widehat{S}\Lambda$ is isomorphic to $\widehat{S} \otimes_S S\Lambda$ as $(\widehat{S}, S\Lambda)$-bimodule, and to $S\Lambda \otimes_S \widehat{S}$ as $(S\Lambda, \widehat{S})$-bimodule. This implies *i*) and *ii*).

3.5. Let us return to the finitely generated, torsion free, nilpotent group Γ and the associated unipotent algebraic group U_A. Recall the notation of subsections 3.1–3.3.

It is known that $\mathbb{Z}\Gamma$ and $U_A(\mathfrak{u})$ are left and right Noetherian and have the left and right AR-property with respect to the filtration by the powers of the augmentation ideal, see, for example, [**42**, Th. 2.7 & § 11.2], [**41**] and [**6**, Th. 1].

Further, by [**22**, Cor. 3.5], one has $I^{(cn)} \subseteq I^n$, where c is the class of Γ (and also the class of $\mathfrak{u}$), and a similar argument, using Proposition 3.2.a) shows that $J^{(cn)} \subseteq J^n$. From this one deduces easily that the sequences $\{I^{(n)}\}$ and $\{J^{(n)}\}$ also have the left and right AR-property. In the sequel, we equip $\mathbb{Z}\Gamma$ and $U_A(\mathfrak{u})$ with these sequences, which we call $\mathcal{I}$ and $\mathcal{J}$ respectively. By Theorem 3.1 and Proposition 3.2, the associated graded rings are left and right Noetherian.

Let V be an U_A-module. Then V is in a natural manner a representation of the Lie algebra $\mathfrak{u}$ and of the abstract group Γ. Let $\mathcal{F}$ be a finite sequence $V = F^0V \supset \cdots \supset F^{s+1}V = \{0\}$ of U_A-submodules. Let us say that $\mathcal{F}$ is an *admissible filtration* of V if it is an $\mathcal{I}$ (resp. $\mathcal{J}$) filtration of V regarded as $\mathbb{Z}\Gamma$ (resp. $U_A(\mathfrak{u})$) module, *i.e.*, if for any $i, n \geqslant 0$, both $I^{(n)}(F^iV)$ and $J^{(n)}(F^iV)$ are contained in $F^{i+n}V$.

Lemma. — *If V is an U_A-module which is finite free over A, it admits an admissible filtration.*

Proof. — By the theorem of Lie-Kolchin applied to $V_{\mathbb{Q}}$, one obtains that V^U, the submodule of invariants, is non-zero. Since

$$V^U = \{x \in V \mid \Delta_V(x) = x \otimes \varepsilon\},$$

where Δ_V is the coaction defining the comodule structure and ε is the augmentation of $A[U]$, and since $V \otimes_A A[U]$ is a free A-module, one sees that V/V^U is torsion-free, hence a free A-module.

Therefore, if one sets $F_0V = 0$ and defines inductively F_kV as the inverse image in V of the U-invariants in $V/F_{k-1}V$, the sequence $\{F_kV\}$ is increasing strictly, as long as $F_kV \neq V$, and each V/F_kV, if non-zero, is a finite free A-module. Since V is a Noetherian A-module, $F_NV = V$ for some N. Setting $F^iV = F_{N-i}V$, it is easily seen that, for any $i, n \geqslant 0$, both $I^n(F^iV)$ and $J^n(F^iV)$ are contained in $F^{i+n}V$. Further, since every $F^iV/F^{i+n}V$ is torsion-free, one obtains that $\{F^iV\}_{i=0}^N$ is an admissible filtration of V.

Then, one deduces from the results of 3.4 the following theorem. There are, obviously, equivariant versions; we leave their formulation to the reader.

Theorem. — *Let V be an U_A-module which is finite free over A and let $\mathcal{F}$ be any admissible filtration on V. Then there are two finitely convergent spectral sequences:*

i) $$E_1^{p,-q} = H_{q-p}(\mathrm{gr}_{\mathrm{isol}}\,\Gamma, \mathrm{gr}_{\mathcal{F}}\, V)_p \Longrightarrow H_{q-p}(\Gamma, V),$$

ii) $$E_1^{p,-q} = H_{q-p}(\mathrm{gr}_{\mathrm{isol}}\,\mathfrak{u}, \mathrm{gr}_{\mathcal{F}}\, V)_p \Longrightarrow H_{q-p}(\mathfrak{u}, V).$$

3.6. Finally, let us return to the setting of Sections 1 and 2. The unipotent group U_P^- is defined over $\mathbb{Z}$. Let $\Gamma := U_P^-(\mathbb{Z})$; it is, clearly, a torsion-free nilpotent group.

For each $\beta \in R$, let U_β be the corresponding root subgroup, let X_β be a generator of $\mathfrak{g}_\beta = \mathrm{Lie}\, U_\beta$, and let θ_β be the isomorphism $\mathbb{G}_a \to U_\beta$ such that $d\theta_\beta(1) = X_\beta$. Set $I := \Delta \setminus R_L^+$ and let $f_I : \mathbb{Z}R \to \mathbb{Z}$ be the additive function which coincides on the basis Δ with the negative of the characteristic function of I. That is,

$$f_I(\alpha) = \begin{cases} -1 \text{ if } \alpha \in I; \\ 0 \quad \text{if } \alpha \in \Delta \cap R_L^+. \end{cases}$$

Choose a numbering $\alpha_1, \ldots, \alpha_r$ of the elements of $R^- \setminus R_L^-$ such that $f_I(\alpha_i) \leqslant f_I(\alpha_j)$ if $i \leqslant j$. The multiplication map induces an isomorphism of $\mathbb{Z}$-schemes

$$U_{\alpha_1} \times \cdots \times U_{\alpha_r} \xrightarrow{\cong} U_P^-.$$

Moreover, it follows from the commutation formulas in [**49**, Lemma 15] or [**7**, 3.2.3–3.2.5] that, for any $s = 1, \ldots, r$, $U_{\alpha_s} \cdots U_{\alpha_r}$ is a closed, normal subgroup of U_P^-. One deduces that the $g_i := \theta_{\alpha_i}(1)$ generate Γ and, moreover, form a system of canonical parameters, that U_P^- is the algebraic group associated in 3.3 to Γ, and that the basis $\{v_1, \ldots, v_r\}$ of $\mathfrak{u}_P^-$ identifies with $\{X_{\alpha_1}, \ldots, X_{\alpha_r}\}$.

Lemma. — *One has* $\mathfrak{u}_P^- \cong \mathrm{gr}_{\mathrm{isol}}\, \mathfrak{u}_P^-$.

Proof. — Since T acts on $\mathfrak{u}_P^-$ by Lie algebra automorphisms, $\mathfrak{u}_P^-$ has a structure of graded Lie algebra given by the function f_I. That is, if one sets, for $i \geqslant 1$,

$$\mathfrak{u}_P^-(i) := \bigoplus_{\substack{\alpha \in R^- \\ f_I(\alpha) = i}} \mathfrak{g}_\alpha,$$

then

$$\mathfrak{u}_P^- = \bigoplus_{i \geqslant 1} \mathfrak{u}_P^-(i) \quad \text{and} \quad [\mathfrak{u}_P^-(i), \mathfrak{u}_P^-(j)] \subseteq \mathfrak{u}_P^-(i+j).$$

Therefore, the lemma will follow if we show that $C^{(i)}(\mathfrak{u}_P^-) = \mathfrak{u}_P^-(\geqslant i)$, where $\mathfrak{u}_P^-(\geqslant i)$ is defined in the obvious manner. Clearly, $C^i(\mathfrak{u}_P^-) \subseteq \mathfrak{u}_P^-(\geqslant i)$ and, since $\mathfrak{u}_P^-/\mathfrak{u}_P^-(\geqslant i)$ is torsion-free, one obtains that $C^{(i)}(\mathfrak{u}_P^-) \subseteq \mathfrak{u}_P^-(\geqslant i)$.

The converse inclusion $\mathfrak{u}_P^-(\geqslant i) \subseteq C^{(i)}(\mathfrak{u}_P^-)$ follows from an argument in the proof of [**4**, Prop. 4.7.(iii)]. For the convenience of the reader, let us recall here this short argument. Using induction on i, it suffices to prove that for any $\beta \in R^-$ such that $f_I(\beta) = i \geqslant 2$, there exists $\alpha \in R^+$ such that $f_I(-\alpha) = 1$ and $\langle \beta, \alpha^\vee \rangle < 0$, since then $\beta + \alpha \in R^- \setminus R_L^-$ and $[X_{-\alpha}, X_{\beta+\alpha}] = mX_\beta$ for some non-zero integer m.

As f_I is constant on orbits of $W_{\Delta\setminus I}$, we may assume that β belongs to X_L^+. Then, since $\beta \in R^-$ whilst dominant roots are positive, there exists $\alpha \in I$ such that $\langle \beta, \alpha^\vee \rangle < 0$. This completes the proof of the lemma.

Remark. — Our original proof of the inclusion $\mathfrak{u}_P^-(\geqslant i) \subseteq C^{(i)}(\mathfrak{u}_P^-)$ relied on the fact that, by [**33**, Cor. 8.1], $H_1(\mathfrak{u}_{P,\mathbb{Q}}^-, \mathbb{Q}) \cong \bigoplus_{\alpha \in I} V_{\mathbb{Q}}^L(-\alpha)$. We are indebted to the referee for pointing out the simple, direct argument in [**4**].

Recall the integers $\nu(i)$ introduced in the proof of Proposition 3.3. From this proposition and the previous lemma (and their proofs), one deduces the following

Corollary. — *There is an isomorphism of graded Hopf algebras* $\mathrm{gr}_{\mathrm{isol}}\, \mathbb{Z}\Gamma \cong U(\mathfrak{u}_P^-)$, *under which each* $\overline{g_i - 1}$ *corresponds to* X_{α_i}. *Further, for* $i = 1, \dots, r$, *one has* $\nu(i) = f_I(\alpha_i)$.

3.7. For any $\lambda \in X^+$, set
$$V_{\mathbb{Z}}(\lambda)(i) := \bigoplus_{\substack{\mu \in X \\ f_I(\mu - \lambda) = i}} V_{\mathbb{Z}}(\lambda)_\mu,$$
where the subscript μ denotes the μ-weight space. Then, each $V_{\mathbb{Z}}(\lambda)(i)$ is an L-submodule and there is an isomorphism of L-modules
$$V_{\mathbb{Z}}(\lambda) \cong \bigoplus_{i \geqslant 0} V_{\mathbb{Z}}(\lambda)(i).$$
Set $F^k V_{\mathbb{Z}}(\lambda) := \bigoplus_{i \geqslant k} V_{\mathbb{Z}}(\lambda)(i)$; this defines a filtration $\mathcal{F}$ of $V_{\mathbb{Z}}(\lambda)$ by P^--submodules, such that the associated graded is isomorphic to $V_{\mathbb{Z}}(\lambda)$ as L-module.

Proposition. — *One has* $I^{(n)} F^k V_{\mathbb{Z}}(\lambda) \subseteq F^{n+k} V_{\mathbb{Z}}(\lambda)$, *and* $\mathrm{gr}_{\mathcal{F}} V_{\mathbb{Z}}(\lambda) \cong V_{\mathbb{Z}}(\lambda)$ *as representations of* $\mathrm{gr}_{\mathrm{isol}}\, \mathbb{Z}\Gamma \cong U_{\mathbb{Z}}(\mathfrak{u}_P^-)$.

Proof. — Following [**22**], set, for $i = 1, \dots, r$ and $n \geqslant 0$,
$$u_i^{(n)} := g_i^{-[(n+1)/2]}\, (g_i - 1)^n,$$
where $[x]$ denotes the greatest integer not greater than x, and observe that $u_i^{(n)} \equiv (g_i - 1)^n$ modulo I^n. Further, for $\boldsymbol{j} \in \mathbb{N}^r$, set
$$u(\boldsymbol{j}) := u_1^{(j_1)} \cdots u_r^{(j_r)} \quad \text{and} \quad \nu(\boldsymbol{j}) = \sum_i j_i\, \nu(i).$$
Then, by [**22**, Theorem 3.2 (i) and Lemma 3.1], the elements $u(\boldsymbol{j})$ satisfying $\nu(\boldsymbol{j}) \geqslant n$ form a $\mathbb{Z}$-basis of $I^{(n)}$, for every $n \geqslant 0$.

From this one deduces that, in order to prove the proposition, it suffices to prove that, for any $v \in F^k V_{\mathbb{Z}}(\lambda)$ and $i = 1, \dots, r$, one has
$$(*) \qquad (g_i - 1)v - X_{\alpha_i} v \in F^{k+\nu(i)+1} V_{\mathbb{Z}}(\lambda).$$

The distribution algebra $\mathrm{Dist}(U_P^-)$ has a $\mathbb{Z}$-basis formed by the ordered products
$$X_{\alpha_1}^{(m_1)} \cdots X_{\alpha_r}^{(m_r)}, \quad \text{for } (m_1, \dots, m_r) \in \mathbb{N}^r,$$

where the elements $X_\beta^{(m)}$ satisfy $X_\beta^m = m!\,X_\beta^{(m)}$ for every $m \geqslant 0$. Further, the structure of $\mathbb{Z}[G]$-comodule on $V_\mathbb{Z}(\lambda)$ is such that, for any ring Ω, any $t \in \Omega$ and $v \in V_\Omega(\lambda)$, and any root α, one has

$$\theta_\alpha(t)v = \sum_{m \geqslant 0} t^m X_\alpha^{(m)} v,$$

where the R.H.S. is in fact a finite sum. Since $g_i = \theta_{\alpha_i}(1)$ and since each $X_{\alpha_i}^{(m)}$ has weight $m\alpha_i$ for the adjoint action of T, this immediately implies formula $(*)$. The proposition is proved.

3.8. We can now prove Theorem C of the Introduction. The discrete group $\Lambda = L(\mathbb{Z})$ normalizes $\Gamma = U_P^-$ and, hence, preserves the isolated powers of the augmentation ideal of $\mathbb{Z}\Gamma$. Therefore, by the equivariant version of Theorem 3.5 *i*), combined with Proposition 3.7, there is a finitely convergent spectral sequence of $L(\mathbb{Z})$-modules

$$H_*(\mathfrak{u}_P^-, V_\mathbb{Z}(\lambda)) \cong H_*(\mathrm{gr}_{\mathrm{isol}}\,\mathbb{Z}\Gamma, V_\mathbb{Z}(\lambda)) \Longrightarrow H_*(\Gamma, V_\mathbb{Z}(\lambda)). \tag{1}$$

It is, clearly, compatible with flat base change. Thus, for any prime integer p, one has a finitely convergent spectral sequence

$$H_*(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda)) \cong H_*(\mathrm{gr}_{\mathrm{isol}}\,\mathbb{Z}\Gamma, V_{\mathbb{Z}_{(p)}}(\lambda)) \Longrightarrow H_*(\Gamma, V_{\mathbb{Z}_{(p)}}(\lambda)). \tag{2}$$

Moreover, it is not difficult to check, by standard arguments, that the natural structure of $L(\mathbb{Z})$-module on $H_*(\mathrm{gr}_{\mathrm{isol}}\,\mathbb{Z}\Gamma, V_{\mathbb{Z}_{(p)}}(\lambda))$ considered in Theorem 3.4.2 is the restriction to $L(\mathbb{Z})$ of the natural structure of L-module on $H_*(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda))$. Therefore, if λ is p-small then, by Theorem 2.1, one obtains an isomorphism of $L(\mathbb{Z})$-modules

$$H_i(\mathrm{gr}_{\mathrm{isol}}\,\mathbb{Z}\Gamma, V_{\mathbb{Z}_{(p)}}(\lambda)) \cong H_i(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w \in W^L(i)} V^L_{\mathbb{Z}_{(p)}}(w \cdot \lambda),$$

for every $i \geqslant 0$. In particular, $H_*(\mathrm{gr}_{\mathrm{isol}}\,\mathbb{Z}\Gamma, V_{\mathbb{Z}_{(p)}}(\lambda))$ is a free $\mathbb{Z}_{(p)}$-module.

Finally, it is well-known that $\mathfrak{u}_P^- \otimes \mathbb{Q}$ is isomorphic to the Malcev-Jennings Lie algebra of Γ; this follows, for example, from the proof of [**35**, Lemma 1.9]. Therefore, by a result of Pickel [**43**, Th. 10], there is an isomorphism of graded vector spaces

$$H_\bullet(\mathfrak{u}_P^-, V_\mathbb{Q}(\lambda)) \cong H_\bullet(\Gamma, V_\mathbb{Q}(\lambda)).$$

This implies that the abutment of the spectral sequence in (2) has the same rank over $\mathbb{Z}_{(p)}$ as the E_1-term. Since the latter is a free $\mathbb{Z}_{(p)}$-module, one deduces that the spectral sequence degenerates at E_1. Therefore, we have obtained the following

Theorem. — *Let $\lambda \in X^+ \cap \overline{C}_p$. Then, for each $n \geqslant 0$, $H_n(U_P^-(\mathbb{Z}), V_{\mathbb{Z}_{(p)}}(\lambda))$ has a finite, natural $L(\mathbb{Z})$-module filtration such that*

$$\mathrm{gr}\, H_n(U_P^-(\mathbb{Z}), V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w \in W^L(n)} V^L_{\mathbb{Z}_{(p)}}(w \cdot \lambda).$$

By the universal coefficient theorem, one then obtains a similar result over $\mathbb{F}_p$. Finally, by an argument similar to the one in 2.10, one obtains the following analogue in cohomology.

Corollary. — *Let $\lambda \in X^+ \cap \overline{C}_p$. Then, for each $n \geqslant 0$, $H^n(U_P(\mathbb{Z}), V_{\mathbb{F}_p}(\lambda))$ has a finite, natural $L(\mathbb{Z})$-module filtration such that*

$$\operatorname{gr} H^n(U_P(\mathbb{Z}), V_{\mathbb{F}_p}(\lambda)) \cong \bigoplus_{w \in W^L(n)} V^L_{\mathbb{F}_p}(w \cdot \lambda).$$

3.9. Let us derive in this subsection a corollary about the p-Lie algebra associated with the p-lower central series of Γ. (This result will not be used in the sequel).

Let $\mathcal{F}$ be a decreasing sequence $\Gamma = F^1\Gamma \supseteq F^2\Gamma \supseteq \cdots$ of normal subgroups of Γ. It is called an N_p-sequence if it is an N-sequence and $x \in F^i\Gamma$ implies that $x^p \in F^{pi}\Gamma$. In this case, $\operatorname{gr}_{\mathcal{F}} \Gamma$ is a graded p-Lie algebra, see [**36**, Chap. I, Cor. 6.8] or [**5**, Chap. II, § 5, Ex. 10].

For our purposes, it is convenient to define the p-lower central series $\{F^n_p\Gamma\}_{n\geqslant 1}$ as follows. Denoting by $I_{\mathbb{F}_p}$ the augmentation ideal of $\mathbb{F}_p\Gamma$, set

$$F^n_p\Gamma := \{x \in \Gamma \mid x - 1 \in I^n_{\mathbb{F}_p}\}.$$

This is an N_p-sequence (see [**42**, Lemma 3.3.1]), and we denote the associated graded p-Lie algebra by $\operatorname{gr}^\bullet_p \Gamma$.

The n-th term $F^n_p\Gamma$ of the p-lower central series is sometimes defined as the subgroup of Γ generated by all elements x^{p^s} satisfying $p^s\omega(x) \geqslant n$, where $\omega(x)$ denotes the largest integer i such that $x \in C^i(\Gamma)$. That the two definitions agree is due to Lazard [**36**, Chap. I, Th. 5.6 & 6.10] and Quillen [**45**], see also [**42**, § 11.1].

Let us denote by $\mathcal{L}ie_{\mathbb{F}_p}$ the category of Lie algebras over $\mathbb{F}_p$, by p-$\mathcal{L}ie_{\mathbb{F}_p}$ the subcategory of p-Lie algebras, and by gr-$\mathcal{L}ie_{\mathbb{F}_p}$ and p-gr-$\mathcal{L}ie_{\mathbb{F}_p}$, respectively, the subcategories of graded and graded p-Lie algebras over $\mathbb{F}_p$. The forgetful functor p-$\mathcal{L}ie_{\mathbb{F}_p} \to \mathcal{L}ie_{\mathbb{F}_p}$ has a left adjoint, denoted by p-$\mathcal{L}$; it takes gr-$\mathcal{L}ie_{\mathbb{F}_p}$ to p-gr-$\mathcal{L}ie_{\mathbb{F}_p}$.

Corollary. — *Let Γ be a finitely generated, torsion-free, nilpotent group, say of class c. Suppose that $\bigoplus_{i=1}^c C^{(i)}(\Gamma)/C^i(\Gamma)$ has no p-torsion. Then, there is an isomorphism of graded p-Lie algebras*

$$\operatorname{gr}^\bullet_p \Gamma \cong p\text{-}\mathcal{L}(\operatorname{gr}\Gamma \otimes \mathbb{F}_p).$$

Proof. — The hypothesis implies easily that $\operatorname{gr}\Gamma \otimes \mathbb{F}_p \cong \operatorname{gr}_{\mathrm{isol}} \Gamma \otimes \mathbb{F}_p$. Moreover, it follows from the proof of [**22**, Th. 3.2 (i)] that every $I^{(n)}/I^n$ has no p-torsion. This implies that, inside $\mathbb{F}_p\Gamma$, one has the identifications $I^{(n)} \otimes \mathbb{F}_p = I^n \otimes \mathbb{F}_p = I^n_{\mathbb{F}_p}$. One deduces from this, coupled with Theorem 3.1, the isomorphisms

$$\operatorname{gr}\mathbb{F}_p\Gamma \cong (\operatorname{gr}_{\mathrm{isol}} \mathbb{Z}\Gamma) \otimes \mathbb{F}_p \cong U_{\mathbb{Z}}(\operatorname{gr}_{\mathrm{isol}} \Gamma) \otimes \mathbb{F}_p \cong U_{\mathbb{F}_p}(\operatorname{gr}_{\mathrm{isol}} \Gamma \otimes \mathbb{F}_p) \cong U_{\mathbb{F}_p}(\operatorname{gr}\Gamma \otimes \mathbb{F}_p).$$

On the other hand, by Quillen [**45**], $\operatorname{gr}\mathbb{F}_p\Gamma$ is isomorphic as graded Hopf algebra to $U^{res}_{\mathbb{F}_p}(\operatorname{gr}^\bullet_p \Gamma)$, the restricted enveloping algebra of the p-Lie algebra $\operatorname{gr}^\bullet_p \Gamma$.

Recall that $U^{res}_{\mathbb{F}_p}$, the restricted enveloping algebra functor, is left adjoint to the forgetful functor $\mathcal{A}s_{\mathbb{F}_p} \to p\text{-}\mathcal{L}ie_{\mathbb{F}_p}$, where $\mathcal{A}s_{\mathbb{F}_p}$ denotes the category of associative $\mathbb{F}_p$-algebras (with unit), while the usual enveloping algebra functor is left adjoint to the forgetful functor $\mathcal{A}s_{\mathbb{F}_p} \to \mathcal{L}ie_{\mathbb{F}_p}$. Thus, since the adjoint of a composite is the composite of the adjoints, one has $U_{\mathbb{F}_p}(L) \cong U^{res}_{\mathbb{F}_p}(p\text{-}\mathcal{L}(L))$, for any $\mathbb{F}_p$-Lie algebra L.

Therefore, one obtains an isomorphism of graded Hopf algebras

$$U^{res}_{\mathbb{F}_p}(p\text{-}\mathcal{L}(\operatorname{gr}\Gamma \otimes \mathbb{F}_p)) \cong U^{res}_{\mathbb{F}_p}(\operatorname{gr}^\bullet_p \Gamma).$$

Taking primitive elements, this gives, by the theorem of Milnor-Moore [**38**, Th. 6.11], an isomorphism of graded p-Lie algebras $p\text{-}\mathcal{L}(\operatorname{gr}\Gamma \otimes \mathbb{F}_p) \cong \operatorname{gr}^\bullet_p \Gamma$. The corollary is proved.

Remark. — It is easy to see that the torsion primes in $\bigoplus_{i=1}^c C^{(i)}(\Gamma)/C^i(\Gamma)$ and in $\operatorname{gr}\Gamma$ are the same. Presumably, it should not be difficult to extract from the proof of Proposition 3.3 that the torsion primes in $\operatorname{gr}\mathfrak{u}$ are also the same.

4. Standard and BGG complexes for distribution algebras

4.1. As in subsection 2.2, there is defined a complex

$$\cdots \longrightarrow \mathcal{U}(G) \otimes_{\mathcal{U}(P)} \Lambda^2(\mathfrak{g}/\mathfrak{p}) \xrightarrow{d_2^{\mathfrak{p}}} \mathcal{U}(G) \otimes_{\mathcal{U}(P)} (\mathfrak{g}/\mathfrak{p}) \xrightarrow{d_1^{\mathfrak{p}}} \mathcal{U}(G) \otimes_{\mathcal{U}(P)} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

the differentials being defined by the same formula as in 2.2. Note, however, that this complex is *not* exact. We shall denote it by $\mathcal{S}_\bullet(G,P)$.

More generally, let V be a G-module and let $V_{|P}$ denote V regarded as an $\mathcal{U}(P)$-module. Then one obtains, as in 2.2, a complex of $\mathcal{U}(G)$-modules

$$\cdots \longrightarrow \mathcal{U}(G) \otimes_{\mathcal{U}(P)} (\Lambda^2(\mathfrak{g}/\mathfrak{p}) \otimes V|_P) \xrightarrow{d_2} \mathcal{U}(G) \otimes_{\mathcal{U}(P)} (\mathfrak{g}/\mathfrak{p} \otimes V|_P)$$
$$\xrightarrow{d_1} \mathcal{U}(G) \otimes_{\mathcal{U}(P)} V|_P \xrightarrow{\varepsilon} V \longrightarrow 0.$$

We shall call it the standard complex of V relative to the pair $(\mathcal{U}(G),\mathcal{U}(P))$, and denote it by $\mathcal{S}_\bullet(G,P,V)$. When $V = V_{\mathbb{Z}}(\lambda)$, we shall denote it simply by $\mathcal{S}_\bullet(G,P,\lambda)$.

Further, as in 2.4, let us define, for any $\xi \in X_L^+$, the generalized Verma module (for $\mathcal{U}(G)$ and $\mathcal{U}(P)$)

$$\mathcal{M}_P(\xi) := \mathcal{U}(G) \otimes_{\mathcal{U}(P)} V^L_{\mathbb{Z}}(\xi).$$

Set $\mathcal{M}_P^{\mathbb{Z}_{(p)}}(\xi) := \mathcal{M}_P(\xi) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ and $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda) := \mathcal{S}_\bullet(G,P,\lambda) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, for any $\lambda \in X^+$.

4.2. For the rest of this section, let us fix $\lambda \in X^+$ and a prime integer p such that $\lambda \in \overline{C}_p$. Recall from 2.4 that $\Omega^i_{\mathfrak{p}}(\lambda)$ denotes the multiset of those $\xi \in X_L^+$ such that $V^L_{\mathbb{Q}}(\xi)$ is a composition factor of $\Lambda^i(\mathfrak{g}/\mathfrak{p})_{\mathbb{Q}}$.

Since $\mathcal{U}(G)$ is free over $\mathcal{U}(P)$ (see, for example, [**28**, § II.1.12]), one obtains exactly as in 2.4 the following

Lemma. — *Let $\lambda \in X^+ \cap \overline{C}_p$. Then each $\mathcal{S}_i^{\mathbb{Z}_{(p)}}(G,P,\lambda)$ admits a finite filtration by $\mathcal{U}_{\mathbb{Z}_{(p)}}(G)$-modules such that the successive quotients are the $\mathcal{M}_P^{\mathbb{Z}_{(p)}}(\xi)$, for $\xi \in \Omega_{\mathfrak{p}}^i(\lambda)$.*

Next, since $U_{\mathbb{Z}_{(p)}}(\mathfrak{g}) \subset \mathcal{U}_{\mathbb{Z}_{(p)}}(G) \subset U_{\mathbb{Q}}(\mathfrak{g})$, one deduces that $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})^G$ is contained in the center of $\mathcal{U}_{\mathbb{Z}_{(p)}}(G)$. Recall the characters $\chi_{\mu,p}$ and $\overline{\chi}_{\mu,p} = \pi \circ \chi_{\mu,p}$, where π is the morphism $\mathbb{Z}_{(p)} \to \overline{\mathbb{F}}_p$, introduced in 2.6.3. If M is a $\mathcal{U}_{\mathbb{Z}_{(p)}}(G)$-module generated by an element of weight μ annihilated by $\mathfrak{u}$, then $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})^G$ acts on M by the character $\chi_{\mu,p}$ (see 2.6.3).

Let $I = \prod_{\xi \in \Omega_{\mathfrak{p}}^{\bullet}(\lambda)} \operatorname{Ker} \chi_{\xi,p}$ (each ξ being counted with its multiplicity). It follows from the previous lemma that $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)$ is a module over the ring $A := U_{\mathbb{Z}_{(p)}}(\mathfrak{g})^G/I$, which is a finite $\mathbb{Z}_{(p)}$-module. Let $\overline{\chi}_1, \dots, \overline{\chi}_r$ be the distinct algebra homomorphisms $A \to \overline{\mathbb{F}}_p$, numbered so that $\overline{\chi}_1 = \overline{\chi}_{\lambda,p}$, and, for $j = 1, \dots, r$, set

$$\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_j} := \Big\{ x \in \mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda) \mid \textstyle\prod_{\substack{\xi \in \Omega_{\mathfrak{p}}^{\bullet}(\lambda) \\ \overline{\chi}_{\xi,p} = \overline{\chi}_j}} (\operatorname{Ker} \chi_{\xi,p}) x = 0 \Big\}.$$

These are, clearly, subcomplexes of $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)$. Then, exactly as in 2.7($*$), one obtains the

Corollary. — *One has $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda) = \bigoplus_{j=1}^r \mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_j}$, a direct sum of complexes.*

4.3. Our aim in this section is to prove the following theorem.

Theorem. — *Assume that $\mathcal{D}G$ is simply-connected, that $X(T)/\mathbb{Z}R$ has no p-torsion and that $\mathfrak{u}_P^-$ is abelian. Let $\lambda \in X^+ \cap \overline{C}_p$. Consider the direct summand subcomplex $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_{\lambda,p}}$ defined in 4.2. Then, for each $i \geqslant 0$, one has*

$$\mathcal{S}_i^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_{\lambda,p}} \cong \bigoplus_{w \in W^L(i)} \mathcal{M}_P^{\mathbb{Z}_{(p)}}(w \cdot \lambda).$$

As in [**16**, VI.5], we treat first the case $\lambda = 0$ and then derive from it the general case.

4.4. The case $\lambda = 0$.— Since $\mathfrak{u}_P^-$ is abelian, the differentials in the standard Koszul complex computing $H_\bullet(\mathfrak{u}_P^-)$ are all zero and hence $H_\bullet(\mathfrak{u}_P^-) \cong \Lambda^\bullet(\mathfrak{u}_P^-)$. Therefore, by a result of Kostant [**33**, § 8.2], the composition factors of $\Lambda^i(\mathfrak{g}/\mathfrak{p})_{\mathbb{Q}}$ are exactly the $V_{\mathbb{Q}}^L(w \cdot 0)$, for $w \in W^L(i)$, each occuring with multiplicity one.

Moreover, as easily seen, the assumption that $\mathfrak{u}_P^-$ is abelian is equivalent to the fact that if $\alpha, \beta \in R^+ \setminus R_P^+$ then $\alpha + \beta \notin R$, which is also equivalent to the fact that U_P acts trivially on $\mathfrak{g}/\mathfrak{p}$. Therefore, by Corollary 1.10.b), each $\Lambda^i(\mathfrak{g}/\mathfrak{p})_{\mathbb{Z}_{(p)}}$ is the direct sum of the Weyl modules $V_{\mathbb{Z}_{(p)}}^L(w \cdot 0)$, for $w \in W^L(i)$. It follows that

$$(*) \qquad \mathcal{S}_i^{\mathbb{Z}_{(p)}}(G,P) \cong \bigoplus_{w \in W^L(i)} \mathcal{M}_P^{\mathbb{Z}_{(p)}}(w \cdot 0),$$

and, therefore, $\mathcal{S}_i^{\mathbb{Z}_{(p)}}(G,P) = \mathcal{S}_i^{\mathbb{Z}_{(p)}}(G,P)_{\overline{\chi}_{\lambda,p}}$ in this case. This proves the sought-for result when $\lambda = 0$ and $p \geqslant h-1$. (Note that no further assumption on G and p is needed in this case).

4.5. The general case.— Now, let $\lambda \in X^+ \cap \overline{C}_p$ be arbitrary. First, since $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda) = \mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P) \otimes V(\lambda)$, one deduces from 4.4 $(*)$ and the tensor identity ([**19**, Prop. 1.7]) that, for $i \geqslant 0$,

$$(1)\qquad \mathcal{S}_i^{\mathbb{Z}_{(p)}}(G,P,\lambda) \cong \bigoplus_{w\in W^L(i)} \mathcal{U}_{\mathbb{Z}_{(p)}}(G) \otimes_{\mathcal{U}_{\mathbb{Z}_{(p)}}(P)} \Big(V^L_{\mathbb{Z}_{(p)}}(w\rho-\rho) \otimes V_{\mathbb{Z}_{(p)}}(\lambda)\Big).$$

Let $\mathcal{S}_w^{\mathbb{Z}_{(p)}}(G,P,\lambda)$ denote the summand corresponding to w on the R.H.S. Then

$$(2)\qquad \mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda) = \bigoplus_{w\in W^L} \mathcal{S}_w^{\mathbb{Z}_{(p)}}(G,P,\lambda),$$

each $\mathcal{S}_w^{\mathbb{Z}_{(p)}}(G,P,\lambda)$ occuring in degree $\ell(w)$.

Therefore, by 4.2, one obtains that

$$(3)\qquad \mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_{\lambda,p}} \cong \bigoplus_{w\in W^L} \mathcal{S}_w^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_{\lambda,p}}.$$

Lemma. — *Assume further that $\mathcal{D}G$ is simply-connected and that $X(T)/\mathbb{Z}R$ has no p-torsion. Then, for every $w \in W^L$,*

$$\mathcal{S}_w^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_{\lambda,p}} \cong \mathcal{M}_P^{\mathbb{Z}_{(p)}}(w\cdot\lambda).$$

Proof. — First, exactly as in 2.7, one obtains that each $\mathcal{S}_w^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_{\lambda,p}}$ has a filtration whose quotients are the $\mathcal{M}_P^{\mathbb{Z}_{(p)}}(\xi)$ for those ξ such that $V^L_{\mathbb{Q}}(\xi)$ is a composition factor of the $L_{\mathbb{Q}}$-module $V^L_{\mathbb{Q}}(w\cdot 0)\otimes V_{\mathbb{Q}}(\lambda)$ (counted with multiplicities) and such that $\overline{\chi}_{\xi,p} = \overline{\chi}_{\lambda,p}$.

Moreover, under the assumptions of the lemma, one obtains, exactly as in the proof of Proposition 2.8, that any such ξ has the form $y\cdot\lambda$, for some $y \in W^L$.

But then the assumption that $V^L_{\mathbb{Q}}(y\cdot\lambda)$ occurs as a composition factor of $V^L_{\mathbb{Q}}(w\cdot 0)\otimes V_{\mathbb{Q}}(\lambda)$ implies that $y = w$ and that the multiplicity is one. This may be deduced, for example, from [**27**, Satz 2.25]. For the convenience of the reader, let us record a proof. Firstly, it is well-known that any composition factor of the $L_{\mathbb{Q}}$-module $V^L_{\mathbb{Q}}(w\cdot 0)\otimes V_{\mathbb{Q}}(\lambda)$ has the form $V^L_{\mathbb{Q}}(w\cdot 0+\nu)$, for some weight ν of $V_{\mathbb{Q}}(\lambda)$ and occurs with a multiplicity at most equal to $\dim V_{\mathbb{Q}}(\lambda)_\nu$, see, for example, [**24**, § 24, Ex. 12] or, better, the proof of Cor. 4.7 in [**1**]. Secondly, for such a ν, suppose that $w\cdot 0+\nu = y\cdot\lambda$, for some $y \in W$. Then,

$$y^{-1}w\rho - \rho = \lambda - y^{-1}\nu.$$

Let θ denote this weight. Since $y^{-1}w\rho$ (resp. $y^{-1}\nu$) is a weight of $V_{\mathbb{Q}}(\rho)$ (resp. $V_{\mathbb{Q}}(\lambda)$), one has $\theta \in -\mathbb{N}R^+$ (resp. $\theta \in \mathbb{N}R^+$) and, therefore, $\theta = 0$. Thus, since the stabilizer of ρ in W is trivial, $y = w$. Finally, $\nu = w\lambda$ has multiplicity one in $V_{\mathbb{Q}}(\lambda)$. This completes the proof of the lemma and, therefore, of Theorem 4.3.

5. Dictionary

Let $G = \mathrm{GSp}(2g)_{\mathbb{Z}}$ be the split reductive Chevalley group over $\mathbb{Z}$ defined by ${}^tXJX = \nu \cdot J$ where J is given by $g \times g$-blocks

$$J = \begin{pmatrix} & 0_g & & \cdot^{\cdot^{\cdot}} \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ -1 & & 0_g & \end{pmatrix}.$$

Let $B = TN$, resp. $Q = MU$, be the Levi decomposition of the upper triangular subgroup of G, resp. of the Siegel parabolic, *i.e.*, the maximal parabolic associated to α, the longest simple root for (G, B, T), so $M = L_I$ where $I = \Delta \backslash \{\alpha\}$. Note that $\mathcal{D}G = \mathrm{Sp}(2g)$ is simply-connected and that the unipotent radical of Q is abelian.

The group of characters X of T is identified to the sublattice

$$\{(a_g, \cdots, a_1; c) \in \mathbb{Z}^g \times \mathbb{Z} \mid c \equiv a_g + \cdots + a_1 \text{ mod. } 2\}$$

of $\mathbb{Z}^{g+1}$ in the following manner. The character $(a_g, \cdots, a_1; c)$ is defined by

$$\mathrm{diag}(t_g, \ldots, t_1, x \cdot t_1^{-1}, \ldots, x \cdot t_g^{-1}) \longmapsto t_g^{a_g} \cdot \cdots \cdot t_1^{a_1} \cdot x^{(c - a_1 - \cdots - a_g)/2}.$$

The weight lattice $P(R)$ coincides with X, and the root lattice $\mathbb{Z}R$ is the intersection of X with the hyperplane $\{c = 0\}$. In particular, $X/\mathbb{Z}R$ is torsion free. The cone $X^+ \subset X$ of dominant weights of G is given by the conditions $a_g \geqslant \cdots \geqslant a_1 \geqslant 0$. The half-sum ρ of the positive roots of G is then $\rho = (g, \ldots, 1; 0)$.

If $(\varepsilon_g, \ldots, \varepsilon_1)$ is the canonical basis of $\mathbb{Z}^g$, the highest coroot $\gamma^\vee$ of G is $\varepsilon_g + \varepsilon_{g-1}$. The condition $\langle \lambda + \rho, \gamma^\vee \rangle \leqslant p$ reads, therefore,

$$a_g + a_{g-1} + g + (g-1) \leqslant p.$$

For a character $\phi = (a_g, \ldots, a_1; c)$ we define its degree as $|\phi| = \sum_{i=1}^g a_i$; the dual character $\widehat{\phi} = (a_g, \ldots, a_1; -c)$ of ϕ has the same degree. Note that $|\rho| = g(g+1)/2$. So,

$$\langle \lambda + \rho, \gamma^\vee \rangle \leqslant |\lambda + \rho|$$

with equality for $g \leqslant 2$.

Let $\mathbf{V} = \langle e_g, \ldots, e_1, e_1^*, \ldots, e_g^* \rangle$ be the standard $\mathbb{Z}$-lattice on which G acts; given two vectors $v, w \in \mathbf{V}$, we write $\langle v, w \rangle_J = {}^tvJw$ for their symplectic product. Then Q is the stabilizer of the standard lagrangian lattice $\mathbf{W} = \langle e_g, \ldots, e_1 \rangle$; we have $\mathbf{V} = \mathbf{W} \oplus \mathbf{W}^*$; $M = L_I$ is the stabilizer of the decomposition $(\mathbf{W}, \mathbf{W}^*)$; one has $M \cong \mathrm{GL}(g) \times \mathbb{G}_m$. Let $B_M = B \cap M$ be the standard Borel of M.

Recall from 1.5 the definition of admissible lattices and, for $\lambda \in X^+$, the $\mathbb{Z}$-lattices $V(\lambda)_{\min}$ and $V(\lambda)_{\max}$.

Let $\lambda \in X^+$ and $n = |\lambda|$; for any (i, j) with $1 \leqslant i < j \leqslant n$, let $\phi_{i,j} : \mathbf{V}^{\otimes n} \to \mathbf{V}^{\otimes(n-2)}$ denote the contraction given by

$$v_1 \otimes \cdots \otimes v_n \longmapsto \langle v_i, v_j \rangle_J \; v_1 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes \widehat{v_j} \otimes \cdots \otimes v_n,$$

and let $\mathbf{V}^{\langle n\rangle}$ be the submodule of $\mathbf{V}^{\otimes n}$ defined as intersection of the kernels of the $\phi_{i,j}$. By applying the Young symmetrizer $c_\lambda = a_\lambda \cdot b_\lambda$ (see [**18**], 15.3 and 17.3) to $\mathbf{V}^{\langle n\rangle}$, one obtains an admissible $\mathbb{Z}$-lattice $V(\lambda)_{\mathrm{Young}}$ in $V_{\mathbb{Q}}(\lambda)$.

Then, by Corollary 1.9, one has the

Corollary. — *For any p-small weight $\lambda \in X^+$, one has canonically*

$$V(\lambda)_{\min} \otimes \mathbb{Z}_{(p)} = V(\lambda)_{\mathrm{Young}} \otimes \mathbb{Z}_{(p)} = V(\lambda)_{\max} \otimes \mathbb{Z}_{(p)}.$$

Moreover, a similar result holds for a weight $\mu \in X_M^+$ of M, provided it is p-small for M.

References

[1] H. H. Andersen, A new proof of old character formulas, pp.193-207 in: *Invariant theory*, Contemporary Math., Vol. 88 (R. Fossum, W. Haboush, M. Hochster, V. Lakshmibai eds.), Amer. Math. Soc., 1989.

[2] M. Atiyah, I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.

[3] I. N. Bernstein, I. M. Gelfand, S. I. Gelfand, Differential operators on the base affine space and a study of **g**-modules, in *Lie Groups and their Representations*, ed. I. M. Gelfand, Proc. Conf. Budapest 1971, Adam Hilger Publ., 1975.

[4] A. Borel, J. Tits, Homomorphismes "abstraits" de groupes algébriques simples, *Annals Math.* 97 (1973), 499-571.

[5] N. Bourbaki, *Groupes et Algèbres de Lie*, Chap.VII–VIII, Hermann, 1975.

[6] K. S. Brown, E. Dror, The Artin-Rees property and homology, *Israel J. Math.*, 22 (1975), 93-109.

[7] F. Bruhat, J. Tits, Groupes réductifs sur un corps local II, *Publ. Math. IHES*, 60 (1984), 5-184.

[8] R. Carter, G. Lusztig, On the modular representations of the general linear and symmetric groups, *Math. Z.*, 136 (1974), 193-242.

[9] B. Cenkl, R. Porter, Nilmanifolds and associated Lie algebras over the integers, *Pacific J. Math.*, 193 (2000), 5-29.

[10] C. W. Curtis, Noncommutative extensions of Hilbert rings, *Proc. Amer. Math. Soc.*, 4 (1953), 945-955.

[11] M. Demazure, Schémas en groupes réductifs, *Bull. Soc. math. France*, 93 (1965), 369-413.

[12] M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion, *Invent. math.*, 21 (1973), 287-301.

[13] J. Dixmier, *Algèbres enveloppantes*, Gauthier-Villars, 1974.

[14] S. Donkin, On the Noetherian property in endomorphism rings of certain comodules, *J. Algebra*, 70 (1981), 394-419.

[15] S. Donkin, *Rational representations of algebraic groups*, Lecture Notes in Maths. 1140, Springer-Verlag, 1985.

[16] G. Faltings, C. L. Chai, *Degeneration of abelian varieties*, Springer-Verlag, 1990.

[17] E. M. Friedlander, B. J. Parshall, Modular representation theory of Lie algebras, *Amer. J. Math.*, 110 (1988), 1055-1094.

[18] W. Fulton, J. Harris, *Representation theory (a first course)*, Springer-Verlag, 1994.

[19] H. Garland, J. Lepowsky, Lie algebra homology and the MacDonald-Kač formulas, *Invent. math.*, 34 (1976), 37-76.

[20] L. Grünenfelder, On the homology of filtered and graded rings, *J. Pure Appl. Algebra*, 14 (1979), 21-37.

[21] Ph. Hall, Nilpotent groups, Canadian Math. Congress, Edmonton 1957, reprinted pp.417-462 in *Collected works of Philip Hall*, Oxford Univ. Press, 1988.

[22] B. Hartley, Powers of the augmentation ideal in group rings of infinite nilpotent groups, *J. London Math. Soc.* (2), 25 (1982), 43-61.

[23] B. Hartley, Topics in the theory of nilpotent groups, pp. 61-120 in *Group Theory. Essays for Philip Hall*, Academic Press, 1984.

[24] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, 1972.

[25] J. E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Amer. Math. Soc., 1995.

[26] J. C. Jantzen, Darstellungen halbeinfacher Gruppen und kontravariante Formen, *J. Reine Ang. Math.*, 290 (1977), 117-141.

[27] J. C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Maths. 750, Springer-Verlag, 1979.

[28] J. C. Jantzen, *Representations of algebraic groups*, Academic Press, 1987.

[29] J. C. Jantzen, Representations of Lie algebras in prime characteristic, pp.185-235 in: *Representation theories and algebraic geometry* (Montréal 1997), A.Broer ed., Kluwer, 1998.

[30] S. A. Jennings, The group ring of a class of infinite nilpotent groups, *Can. J. Math.*, 7 (1955), 169-187.

[31] V. Kac, B. Weisfeiler, Coadjoint action of a semi-simple algebraic group and the center of the enveloping algebra in characteristic p, *Indag. Math.*, 38 (1976), 136-151.

[32] M. A. Knus, On the enveloping algebra and the descending central series of a Lie algebra, *J. Algebra*, 12 (1969), 335-338.

[33] B. Kostant, Lie algebra cohomology and the generalized Borel-Weil theorem, *Ann. Math.*, 74 (1961), 329-387.

[34] B. Kostant, Groups over $\mathbb{Z}$, pp.90-98 in: *Algebraic groups and their discontinuous subgroups*, Proc. Symp. Pure Math. IX (A. Borel, G.D. Mostow eds.), Amer. Math. Soc., 1966.

[35] L. A. Lambe, S. B. Priddy, Cohomology of nilmanifolds and torsion-free nilpotent groups, *Trans. A.M.S.*, 273 (1982), 39-55.

[36] M. Lazard, Sur les groupes nilpotents et les anneaux de Lie, *Ann. Éc. Norm. Sup.* (3ème série), 71 (1954), 101-190.

[37] J. Lepowsky, A generalization of the Bernstein-Gelfand-Gelfand resolution, *J. Algebra*, 49 (1977), 496-511.

[38] J. W. Milnor, J. C. Moore, On the structure of Hopf algebras, *Ann. Math.*, 81 (1965), 211-264.

[39] A. Mokrane, J. Tilouine, Cohomology of Siegel varieties with p-adic integral coefficients and applications, *this volume*.

[40] C. Năstăsescu, F. Van Oystaeyen, *Graded and filtered rings and modules*, Lecture Notes in Maths. 758, Springer-Verlag, 1979.

[41] Y. Nouazé, P. Gabriel, Idéaux premiers de l'algèbre enveloppante d'une algèbre de Lie nilpotente, *J. Algebra*, 6 (1967), 77-99.

[42] D. S. Passman, *The algebraic structure of group rings*, Wiley, 1977.

[43] P. F. Pickel, Rational cohomology of nilpotent groups and Lie algebras, *Comm. Algebra*, 6 (1978), 409-419.

[44] P. Polo, paper in preparation.

[45] D. Quillen, On the associated graded ring of a group ring, *J. Algebra*, 10 (1968), 411-418.

[46] A. Rocha-Caridi, Splitting criteria for $\mathfrak{g}$-modules induced from a parabolic and the Bernstein-Gelfand-Gelfand resolution of a finite dimensional, irreducible $\mathfrak{g}$-module, *Trans. A.M.S.*, 262 (1980), 335-366.

[47] G. Sjödin, On filtered modules and their associated graded modules, *Math. Scand.*, 33 (1973), 229-249.

[48] T. A. Springer, *Linear algebraic groups* (2nd edition), Birkhäuser, 1998.

[49] R. Steinberg, *Lectures on Chevalley groups*, Yale University lecture notes, 1968.

[50] R. Steinberg, Regular elements of semi-simple algebraic groups, *Publ. Math. IHES*, 25 (1965), 49-80.

[51] M. Suzuki, *Group Theory* I, Springer-Verlag, 1982.

[52] M. Vergne, Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes, *Bull. Soc. math. France*, 98 (1970), 81-116.

ASTÉRISQUE

2002

280. A. MOKRANE, P. POLO, J. TILOUINE – *Cohomology of Siegel varieties*
279. Cohomologies p-adiques et applications arithmétiques (II), P. BERTHELOT, J.-M. FONTAINE, L. ILLUSIE, K. KATO, M. RAPOPORT, éditeurs
278. Cohomologies p-adiques et applications arithmétiques (I), P. BERTHELOT, J.-M. FONTAINE, L. ILLUSIE, K. KATO, M. RAPOPORT, éditeurs
277. B. RÉMY – *Groupes de Kac-Moody déployés et presque déployés*
276. SÉMINAIRE BOURBAKI, volume 1999/2000, exposés 865-879

2001

275. J.-M. BISMUT, S. GOETTE – *Families torsion and Morse functions*
274. A. BONNET, G. DAVID – *Cracktip is a global Mumford-Shah minimizer*
273. K. NISHIYAMA, H. OCHIAI, K. TANIGUCHI, H. YAMASHITA, S. KATO – *Nilpotent orbits, associated cycles and Whittaker models for highest weight representations*
272. M. BOILEAU, J. PORTI – *Geometrization of 3-orbifolds of cyclic type (avec la collaboration de M. HEUSENER)*
271. M. KASHIWARA, P. SCHAPIRA – *Ind-Sheaves*
270. M. BONK, J. HEINONEN, P. KOSKELA – *Uniformizing Gromov hyperbolic spaces*
269. J.-L. WALDSPURGER – *Intégrales orbitales nilpotentes et endoscopie pour les groupes classiques non ramifiés*

2000

268. J. FRANCHETEAU, G. MÉTIVIER – *Existence de chocs faibles pour des systèmes quasi-linéaires hyperboliques multidimensionnels*
267. R. CERF – *Large Deviations for three Dimensional Supercritical Percolation*
266. SÉMINAIRE BOURBAKI, volume 1998/1999, exposés 850-864
265. O. BIQUARD – *Métriques d'Einstein asymptotiquement symétriques*
264. L. LIPSHITZ, Z. ROBINSON – *Rings of Separated Power Series and Quasi-Affinoid Geometry*
263. C. SABBAH – *Équations différentielles à points singuliers irréguliers et phénomène de Stokes en dimension 2*
262. A. VASY – *Propagation of singularities in three-body scattering*
261. Géométrie complexe et systèmes dynamiques, colloque en l'honneur d'Adrien Douady, Orsay 1995, M. FLEXOR, P. SENTENAC et J.-C. YOCCOZ, éditeurs

1999

260. S. D. CUTKOSKY – *Local monomialization and factorization of morphisms*
259. R. KRIKORIAN – *Réductibilité des systèmes produits-croisés à valeurs dans des groupes compacts*
258. Structure Theory of Set Addition, J.-M. DESHOUILLERS, B. LANDREAU and A.A. YUDIN, editors
257. J.-P. LABESSE – *Cohomologie, stabilisation et changement de base (avec la collaboration de L. BREEN et L. CLOZEL)*
256. F. MOREL – *Théorie homotopique des schémas*
255. R.E. KOTTWITZ, D. SHELSTAD – *Foundations of twisted endoscopy*
254. C.J. BUSHNELL, G. HENNIART – *Local tame lifting for $\mathrm{GL}(n)$ II : wildly ramified supercuspidals*
253. B. MAGNERON – *Involutions complexes et vecteurs sphériques associés pour les groupes de Lie nilpotents réels*

1998

252. SÉMINAIRE BOURBAKI, volume 1997/1998, exposés 835-849
251. Nombre et répartition de points de hauteur bornée, E. PEYRE, éditeur
250. C. BONATTI, R. LANGEVIN – *Difféomorphismes de Smale des surfaces (avec la collaboration de E. JEANDENANS)*
249. P. AUSCHER, P. TCHAMITCHIAN – *Square root problem for divergence operators and related topics*
248. P. COLMEZ – *Intégration sur les variétés p-adiques*
247. G. PISIER – *Non-commutative vector valued L_p-spaces and completely p-summing maps*

1997

246. V. TARASOV, A. VARCHENKO – *Geometry of q-hypergeometric functions, quantum affine algebras and elliptic quantum groups*
245. SÉMINAIRE BOURBAKI, volume 1996/1997, exposés 820-834
244. J.-M. BISMUT – *Holomorphic families of immersions and higher analytic torsion forms*
243. L. LAFFORGUE – *Chtoucas de Drinfeld et conjecture de Ramanujan-Petersson*
242. N. BURQ – *Pôles de diffusion engendrés par un coin*
241. SÉMINAIRE BOURBAKI, volume 1995/1996, exposés 805-819

1996

240. A. SÀ BARETTO, R. B. MELROSE, M. ZWORSKI – *Semilinear diffraction of conormal waves*
239. J.-L. VERDIER – *Des catégories dérivées des catégories abéliennes*
238. A. BROISE, F. DAL'BO, M. PEIGNÉ – *Méthodes des opérateurs de transfert : transformations dilatantes de l'intervalle et dénombrement de géodésiques fermées*
237. SÉMINAIRE BOURBAKI, volume 1994/1995, exposés 790-804
236. Hommage à P. A. MEYER et J. NEVEU
235. J.-P. OTAL – *Le théorème d'hyperbolisation pour les variétés fibrées de dimension 3*
234. A. GENESTIER – *Espaces symétriques de Drinfeld*

1995

233. I. KRIZ, J.P. MAY – *Operads, algebras modules and motives*
232. Recent advances in operator algebras (Orléans, 1992)
231. J.-C. YOCCOZ – *Petits diviseurs en dimension 1*
230. J.-Y. CHEMIN – *Fluides parfaits incompressibles*
229. B. PERRIN-RIOU – *Fonctions L p-adiques des représentations p-adiques*
228. Columbia University number theory seminar (New-York, 1992)
227. SÉMINAIRE BOURBAKI, volume 1993/1994, exposés 775-789

1994

226. K-theory (Strasbourg, 1992)
225. L. BREEN – *On the classification of 2-gerbes and 2-stacks*
224. P. SCHAPIRA, J.-P. SCHNEIDERS – *Index theorem for elliptic pairs*
223. Périodes p-adiques (séminaire de Bures, 1988)
222. Complex analytic methods in dynamical systems (IMPA, janvier 1992)
221. A. OGUS – *F-Crystals, Griffiths transversality, and the Hodge decomposition*
220. H.H. ANDERSEN, J.C. JANTZEN, W. SOERGEL – *Représentations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p : independence of p*
219. H. RUBENTHALER – *Les paires duales dans les algèbres de Lie réductives*