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p-ADIC HODGE THEORY AND VALUES OF ZETA FUNCTIONS OF MODULAR FORMS

by

Kazuya Kato

Abstract. — If f is a modular form, we construct an Euler system attached to f from which we deduce bounds for the Selmer groups of f. An explicit reciprocity law links this Euler system to the p-adic zeta function of f which allows us to prove a divisibility statement towards Iwasawa's main conjecture for f and to obtain lower bounds for the order of vanishing of this p-adic zeta function. In particular, if f is associated to an elliptic curve E defined over $\mathbb{Q}$, we prove that the p-adic zeta function of f has a zero at $s = 1$ of order at least the rank of the group of rational points on E.

Contents

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Introduction

One of the most fascinating subjects in number theory is the study of mysterious relations between zeta functions and “arithmetic groups”. Here “arithmetic groups” include ideal class groups of number fields, Mordell-Weil groups of abelian varieties over number fields, Selmer groups associated to Galois representations of number fields, etc., which play important roles in number theory. Among such relations, we have Iwasawa theory (relation between zeta functions and ideal class groups) which is a refinement in 20th century of the class number formula in 19th century, Birch Swinnerton-Dyer conjectures (relation between zeta functions and Mordell-Weil groups), etc., and much of such relations are still conjectural. When we study such relations, a big difficulty is that zeta functions and arithmetic groups are too much different in nature; zeta functions are analytic and arithmetic groups are algebraic and it is very difficult to understand why they are closely related.

After Kolygavin, it was recognized that zeta functions have not only the usual analytic shapes (Euler products), but also arithmetic shapes (Euler systems), and that it is useful to consider these arithmetic shapes for the study of relations between zeta functions and arithmetic groups; it is more easy to understand the relation between the arithmetic shapes of zeta functions and arithmetic groups which are not far in nature, than the relation of analytic shapes and arithmetic groups.

$$\begin{array}{ccc} \text{zeta function} & \xleftrightarrow{\text{too far}} & \text{arithmetic groups} \\ \text{(analytic)} & & \text{(algebraic)} \\ \text{Euler systems} & \xleftrightarrow{\text{near}} & \text{arithmetic groups} \\ \text{(= arithmetic shapes} & & \text{(algebraic)} \\ \text{of zeta functions)} & & \end{array}$$

In this paper, by considering the Euler systems of Beilinson elements in K_2 of modular curves, which are regarded as "arithmetic shapes" of zeta functions of elliptic modular forms, and by using p-adic Hodge theory, we obtain results on the relations between zeta functions of elliptic modular forms and Selmer groups associated to modular forms, and results in Iwasawa theory of modular forms.

Since it is now known that all elliptic curves over $\mathbb{Q}$ are modular ([**Wi**] [**BCDT**]), this gives also results on Birch Swinnerton-Dyer conjectures for elliptic curves over $\mathbb{Q}$.

The main results of this paper are the following. (Please see the text for the precise statements.)

Theorem. — *Let f be an eigen cusp form for $\Gamma_1(N)$ of weight $k \geqslant 2$.*

(1) *(Thm. 14.2) Let $r \in \mathbb{Z}$, $1 \leqslant r \leqslant k-1$, and assume $r \neq k/2$. Then for any finite abelian extension* K *of* $\mathbb{Q}$*, the Selmer group* $\mathrm{Sel}(\mathrm{K}, f, r)$ *of f over* K *with r twist is a finite group.*

(2) *(Thm. 14.2) Assume k is even. Let K be a finite abelian extension of $\mathbb{Q}$. Let $\chi : \mathrm{Gal}(K/\mathbb{Q}) \to \mathbb{C}^\times$ be a character, and assume $\mathrm{L}(f, \chi, k/2) \neq 0$. Then the χ-part* $\mathrm{Sel}(\mathrm{K}, f, k/2)^{(\chi)}$ *of* $\mathrm{Sel}(\mathrm{K}, f, k/2)$ *is a finite group.*

(3) *(Thm. 18.4) Assume k is even. Then*

$$p\text{-adic corank of } \mathrm{Sel}\left(\mathrm{K}, f, k/2\right) \leqslant \mathrm{ord}_{s=k/2} \quad (p\text{-adic zeta function of } f).$$

(4) *(Thm. 17.4) Assume f is good ordinary at p. Then*

$$\mathfrak{X} \underset{\mathrm{def}}{=} \mathrm{Hom}(\varinjlim_n \mathrm{Sel}(\mathbb{Q}(\zeta_{p^n}), f, r), (\mathbb{Q}_p/\mathbb{Z}_p)(r))$$

for $1 \leqslant r \leqslant k-1$ is independent of r and the characteristic ideal of $\mathfrak{X}$ divides p^n times the p-adic zeta function of f for some $n \geqslant 0$.

In some cases, we can drop p^n in (4) (Thm 17.4 (3)). This (4) is a partial answer to a conjecture of Greenberg ([**Gr1**], the case of elliptic curves was conjectured by Mazur [**Ma1**]) who predicts the equality in place of divisibility in (4). We also obtain results on "Iwasawa main conjecture for modular forms without p-adic zeta functions" (Thm. 12.5) and results on Tamagawa number conjectures ([**BK2**]) for modular forms (Thm. 14.5).

There are already many results on these subjects (for example, [**BD**], [**CW**], [**Ru2**], [**Ko**], [**Ne**], ...). Most of former works use elliptic units and Heegner points as "arithmetic shapes of zeta functions", whereas we use Beilinson elements instead. The part of the above Theorem concerning eigen cusp forms f with complex multiplication depends on results of [**Ru2**] on main conjectures of imaginary quadratic fields.

The plan of this paper is as follows. In Chapter I, we define Euler systems of Beilinson elements in K_2 of modular curves (§ 2) and also Euler systems in the spaces of modular forms (§ 4). The former Euler systems are related to $\lim_{s\to 0} s^{-1}\,\mathrm{L}(f,s)$ for cusp forms f of weight 2 by the theory of Beilinson, and the latter Euler systems are related to the zeta values $\mathrm{L}(f,r)$ ($r \in \mathbb{Z}$, $1 \leqslant r \leqslant k-1$) of cusp forms f of weight

$k \geqslant 2$ by the theory of Shimura. In Chapter 2, by using the above Euler systems in K_2 of modular forms, we define p-adic Euler systems in the Galois cohomology of p-adic Galois representations associated to eigen cusp forms of weight $\geqslant 2$ (§ 8). We prove that via p-adic Hodge theory, these p-adic Euler systems are closely related to the Euler systems in the space of modular forms (§ 9), and hence closely related to the zeta values $\mathrm{L}(f,r)$ ($r \in \mathbb{Z}$, $1 \leqslant r \leqslant k-1$) for cusp forms of weight $k \geqslant 2$. In chapter III and Chapter IV, by using this relation of our p-adic Euler systems with zeta values, and by using the general theory of Euler systems in Galois cohomology, we obtain our main results.

A large part of results of this paper in the case of modular forms of weight 2 were introduced in Scholl [**Sc2**] and Rubin [**Ru3**].

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CHAPTER I

EULER SYSTEMS IN K_2 OF MODULAR CURVES AND EULER SYSTEMS IN THE SPACES OF MODULAR FORMS

In this Chapter I, we consider Euler systems in K_2 of modular curves (§ 2) and Euler systems in the spaces of modular forms (§ 4). The former (resp. latter) come from the work of Beilinson [**Be**] (resp. Shimura [**Sh**]) and are related to the zeta values $\lim_{s\to 0} s^{-1}L(f,s)$ (resp. $L(f,r)$ ($1 \leqslant r \leqslant k-1$)) for cusp forms f of weight 2 (resp. k), by the theory in [**Be**] (resp. [**Sh**]).

§ 1 is a review on Siegel units (resp. Eisenstein series) and is a preparation for § 2 (resp. § 4).

1. Siegel units

We review the theory of Siegel units which are functions on modular curves having zeros and poles only on cusps. Cf. [**KL**].

1.1. For $N \geqslant 3$, let $Y(N)$ be the modular curve over $\mathbb{Q}$ of level N without cusps, which represents the functor

$$S \longmapsto \left\{ \begin{array}{l} \text{the set of isomorphism classes of triples } (E, e_1, e_2) \text{ where } E \text{ is an elliptic} \\ \text{curve over } S \text{ and } (e_1, e_2) \text{ is a pair of sections of } E \text{ over } S \text{ which forms a} \\ \mathbb{Z}/N\text{-basis of } {}_N E = \operatorname{Ker}(N : E \to E). \end{array} \right.$$

Cf. [**DR**].

$Y(N)$ is a smooth irreducible affine curve. The total constant field of $Y(N)$ (the field of all algebraic numbers in the affine ring $\mathcal{O}(Y(N))$ is not $\mathbb{Q}$, but is generated over $\mathbb{Q}$ by a primitive N-th root of 1. Let $X(N)$ be the smooth compactification of $Y(N)$.

If $N, N' \geqslant 3$ and $N \mid N'$, we have a finite étale surjective morphism $Y(N') \to Y(N)$ which represents $(E, e_1, e_2) \mapsto (E, (N'/N)\, e_1, (N'/N)\, e_2)$. We regard $\mathcal{O}(Y(N))$ as a subring of $\mathcal{O}(Y(N'))$ via the pull back.

1.2. The aim of § 1 is to review basic facts about the Siegel units

$${}_c g_{\alpha,\beta} \in \bigcup_N \mathcal{O}(Y(N))^\times, \quad g_{\alpha,\beta} \in \bigcup_N \mathcal{O}(Y(N))^\times \otimes \mathbb{Q}$$

where $(\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2 \smallsetminus \{(0,0)\}$ and c is an integer which is prime to 6 and to the orders of α, β. These elements satisfy

$${}_c g_{\alpha,\beta} \in \mathcal{O}(Y(N))^\times, \quad g_{\alpha,\beta} \in \mathcal{O}(Y(N))^\times \otimes \mathbb{Q} \ \text{ if } \ N\alpha = N\beta = 0,$$

$${}_c g_{\alpha,\beta} = (g_{\alpha,\beta})^{c^2} (g_{c\alpha,c\beta})^{-1} \ \text{ in } \ \mathcal{O}(Y(N))^\times \otimes \mathbb{Q}.$$

We introduce Siegel units by using the following proposition.

Proposition 1.3. — *Let E be an elliptic curve over a scheme S. Let c be an integer which is prime to* 6. *Then:*

(1) *There exists a unique element ${}_c\theta_E$ of $\mathcal{O}(E \smallsetminus {}_cE)^\times$ satisfying the following conditions* (i) (ii).

(i) *${}_c\theta_E$ has the divisor $c^2(0) - {}_cE$ on E, where* (0) *denotes the zero section of E regarded as a Cartier divisor on E and ${}_cE = \operatorname{Ker}(c : E \to E)$ is also regarded as a Cartier divisor on E.*

(ii) *$N_a({}_c\theta_E) = {}_c\theta_E$ for any integer a which is prime to c, where N_a is the norm map $\mathcal{O}(E \smallsetminus {}_{ac}E)^\times \to \mathcal{O}(E \smallsetminus {}_cE)^\times$ associated to the pull back homomorphism $\mathcal{O}(E \smallsetminus {}_cE) \to \mathcal{O}(E \smallsetminus {}_{ac}E)$ by the multiplication $a : E \smallsetminus {}_{ac}E \to E \smallsetminus {}_cE$.*

(2) *If d is also an integer which is prime to 6, we have an equality in $\mathcal{O}(E \smallsetminus {}_{cd}E)^\times$*

$$({}_d\theta_E)^{c^2}(c^*({}_d\theta_E))^{-1} = ({}_c\theta_E)^{d^2}(d^*({}_c\theta_E))^{-1}$$

where c^* (resp. d^*) denotes the pull back by the multiplication c (resp. d): $E \to E$.

(3) *Let $\mathfrak{H} = \{\tau \in \mathbb{C};\ \mathrm{Im}(\tau) > 0\}$ be the upper half plane. For $\tau \in \mathfrak{H}$ and $z \in \mathbb{C} \smallsetminus c^{-1}(\mathbb{Z}\tau + \mathbb{Z})$, let ${}_c\theta(\tau, z)$ be the value at z of ${}_c\theta$ of the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ over $\mathbb{C}$. Then,*

$${}_c\theta(\tau, z) = q^{\frac{1}{12}(c^2-1)}(-t)^{\frac{1}{2}(c-c^2)} \cdot \gamma_q(t)^{c^2}\gamma_q(t^c)^{-1}$$

where $q = e^{2\pi i\tau}$, $t = e^{2\pi iz}$ and

$$\gamma_q(t) = \prod_{n\geqslant 0}(1 - q^n t)\prod_{n\geqslant 1}(1 - q^n t^{-1}).$$

(4) *If $h : E \to E'$ is an isogeny of elliptic curves over S whose degree is prime to c, then the norm map h_* sends ${}_c\theta_E$ to ${}_c\theta_{E'}$.*

The proof of Prop. 1.3 is given in 1.10 later.

1.4. We define Siegel units.

In 1.3, consider the case where E is the universal elliptic curve over $Y(N)$, $N \geqslant 3$. We define ${}_cg_{\alpha,\beta}$. Take $N \geqslant 1$ such that $N\alpha = N\beta = 0$, and write $(\alpha, \beta) = (a/N, b/N) \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2 \smallsetminus \{(0,0)\}$ $(a, b \in \mathbb{Z})$, and define

$${}_cg_{\alpha,\beta} = \iota^*_{\alpha,\beta}({}_c\theta_E) \in \mathcal{O}(Y(N))^\times$$

where

$$\iota_{\alpha,\beta} = ae_1 + be_2 : Y(N) \longrightarrow E \smallsetminus {}_cE.$$

Here $\iota^*_{\alpha,\beta}({}_c\theta_E)$ is defined since the image of $ae_1 + be_2$ does not intersect with the divisor ${}_cE$ by the assumptions that c is prime to the orders of α, β and $(\alpha, \beta) \neq (0, 0)$.

By taking c such that $(c, 6) = 1$, $c \equiv 1 \bmod N$ and $c \neq \pm 1$, let

$$g_{\alpha,\beta} = {}_cg_{\alpha,\beta} \otimes (c^2 - 1)^{-1} \in \mathcal{O}(Y(N))^\times \otimes \mathbb{Q}.$$

Then it is seen from 1.3 (2) that $g_{\alpha,\beta}$ is independent of the choice of such c, and

$${}_cg_{\alpha,\beta} = (g_{\alpha,\beta})^{c^2}(g_{c\alpha,c\beta})^{-1} \quad \text{in} \quad \mathcal{O}(Y(N))^\times \otimes \mathbb{Q}$$

for any integer c such that $(c, 6N) = 1$.

As elements of $\cup_N \mathcal{O}(Y(N))^\times$ (resp. $\cup_N \mathcal{O}(Y(N))^\times \otimes \mathbb{Q}$), ${}_cg_{\alpha,\beta}$ (resp. $g_{\alpha,\beta}$) do not depend on the choice of N above.

Remark 1.5. — To have a perspective view on Siegel units, a good way would be to find some "truth" in the following wrong statement.

"If E is an elliptic curve over a scheme S, there exists an element θ_E of $\mathcal{O}(E \smallsetminus (0))^\times$ which has the divisor (0) on E and $N_a(\theta_E) = \theta_E$ for any non-zero integer a. We have ${}_c\theta_E = (\theta_E)^{c^2} \cdot (c^*(\theta_E))^{-1}$. In the case E is the universal elliptic curve over $Y(N)$, we have $g_{\alpha,\beta} = \iota^*_{\alpha,\beta}(\theta_E)$."

Though the existence of such θ_E would nicely explain properties of ${}_c\theta_E$ and of Siegel units, θ_E does not exist in fact since the degree of a principal divisor should be 0 but (0) has degree 1.

1.6. The group $\mathrm{GL}_2(\mathbb{Z}/N)$ acts on $Y(N)$ from the left in the following way. An element $\sigma = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/N)$ sends (E, e_1, e_2) to (E, e_1', e_2') where

$$\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

The induced action by σ on the total constant field sends a primitive N-th root α of 1 to $\alpha^{\det(\sigma)}$.

Lemma 1.7

(1) *For $\sigma \in \mathrm{GL}_2(\mathbb{Z}/N)$ and for $(\alpha, \beta) \in \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2 \smallsetminus \{(0,0)\}$, we have*

$$\sigma^*({}_c g_{\alpha,\beta}) = {}_c g_{\alpha',\beta'}$$
$$\sigma^*(g_{\alpha,\beta}) = g_{\alpha',\beta'}$$

where c is any integer which is prime to N and (α', β') is defined by $(\alpha', \beta') = (\alpha, \beta)\sigma$.

(2) *(Distribution property.) Let $(\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2 \smallsetminus \{(0,0)\}$, and let a be a non-zero integer. Then*

$$ {}_c g_{\alpha,\beta} = \prod_{\alpha',\beta'} {}_c g_{\alpha',\beta'} \quad \text{in} \quad \bigcup_N \mathcal{O}(Y(N))^\times$$
$$ g_{\alpha,\beta} = \prod_{\alpha',\beta'} g_{\alpha',\beta'} \quad \text{in} \quad \bigcup_N \mathcal{O}(Y(N))^\times \otimes \mathbb{Q}$$

where c is any integer which is prime to a and to the orders of α, β, and α' (resp. β') ranges over all elements of $\mathbb{Q}/\mathbb{Z}$ such that $a\alpha' = \alpha$ (resp. $a\beta' = \beta$).

1.7 (1) is proved easily. 1.7 (2) is deduced from $N_a({}_c\theta_E) = {}_c\theta_E$.

1.8. Let $Y(N)(\mathbb{C})$ be the set of $\mathbb{C}$-valued points of the $\mathbb{Q}$-scheme $Y(N)$. We have a canonical map

$$\nu : \mathfrak{H} \longrightarrow Y(N)(\mathbb{C})\,; \quad \tau \longmapsto (\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), \tau/N, 1/N).$$

Via ν, we often regard elements of $\mathcal{O}(Y(N))$ as functions on $\mathfrak{H}$. The standard action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{H}$ and the above action of $\mathrm{GL}_2(\mathbb{Z}/N)$ on $Y(N)(\mathbb{C})$ are compatible via ν. We have an isomorphism of analytic spaces

$$(\mathbb{Z}/N)^\times \times \Gamma(N)\backslash\mathfrak{H} \xrightarrow{\sim} Y(N)(\mathbb{C})\,; \quad (a, \tau) \longmapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \nu(\tau)$$

where $\Gamma(N) = \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N))$.

The $\mathbb{C}$-valued points $\nu(\tau)$ of $Y(N)$ for $\tau \in \mathfrak{H}$ give a common homomorphism from the total constant field of $Y(N)$ into $\mathbb{C}$. We always regard the total constant field of

$Y(N)$ as a subfield of $\mathbb{C}$ via this homomorphism. In this paper, ζ_N denotes $e^{2\pi i/N}$ which generates this total constant field.

1.9. The pull back of ${}_cg_{\alpha,\beta}$ ($(\alpha,\beta) \in \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2 \smallsetminus \{(0,0)\}$) under $\nu : \mathfrak{H} \to Y(N)(\mathbb{C})$ is the function ${}_c\theta_{\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})}(\alpha\tau + \beta \bmod \mathbb{Z}\tau + \mathbb{Z})$ in $\tau \in \mathfrak{H}$. From 1.3 (3), we can deduce that the pull back of $g_{a/N,b/N}$ on $\mathfrak{H}$ for a, $b \in \mathbb{Z}$, $0 \leqslant a < N$, $(a \bmod N, b \bmod N) \neq (0,0)$ is equal to

$$q^w \cdot \prod_{n\geqslant 0}(1 - q^n q^{a/N}\zeta_N^b) \cdot \prod_{n>0}(1 - q^n q^{-a/N}\zeta_N^{-b})$$

where $w = 1/12 - a/2N + (1/2)(a/N^2)$. Here q^α for $\alpha \in \mathbb{Q}$ means $e^{2\pi i\alpha\tau}$.

1.10. We prove Prop. 1.3. (See [**Sc2**, § 1.2] for another proof.)

We prove 1.3 (1). We first prove the uniqueness of ${}_c\theta_E$. Let f and g be elements of $\mathcal{O}(E \smallsetminus {}_cE)^\times$ having the properties (i) (ii) of ${}_c\theta_E$. Then $g = uf$ for some invertible constant $u \in \mathcal{O}(S)^\times$. For an integer a which is prime to c, we have $g = N_a(g) = N_a(uf) = u^{a^2}f$ (since $a : E \to E$ is of degree a^2) and hence $u^{a^2-1} = 1$. By taking $a = 2$ (resp. $a = 3$), we have $u^3 = 1$ (resp. $u^8 = 1$). Hence $u = 1$.

Next we prove the existence of ${}_c\theta_E$. Since we have already the uniqueness, we can work locally on S. First we show that locally on S, $c^2(0) - {}_cE$ is a principal divisor. For this, by "Abel's theorem" it is sufficient to prove that the image of $c^2(0) - {}_cE$ under the isomorpism of Abel $\mathrm{Pic}(E)^{\deg=0} \to E(S)$ is zero. For any integer a which is prime to c, the image of the divisor $c^2(0) - {}_cE$ under the multiplication $a : E \to E$ is $c^2(0) - {}_cE$ itself. Since the map a_* on $\mathrm{Pic}(E)^{\deg=0}$ is compatible with the multiplication by a on $E(S)$ via the isomorphism of Abel, the image of $c^2(0) - {}_cE$ in $E(S)$ is invariant under the multiplication by a. By taking $a = 2$, we see that the image of $c^2(0) - {}_cE$ in $E(S)$ is zero. Now by what we have proved, there exists locally on S a function $f \in \mathcal{O}(E \smallsetminus {}_cE)^\times$ having the divisor $c^2(0) - {}_cE$. If a is an integer which is prime to c, the divisor of $N_a(f)$, which is the image of $c^2(0) - {}_cE$ under the multiplication by a, is equal to $c^2(0) - {}_cE$. Hence $N_a(f)$ has the same divisor as f and so $N_a(f) = u_af$ for some invertible constant u_a. If b is also an integer which is prime to c, $N_aN_b = N_bN_a$ shows that $u_a^{b^2-1} = u_b^{a^2-1}$. Hence if we put $g = u_2^{-3}u_3f$ we have

$$N_a(g) = u_2^{-3a^2}u_3^{a^2}u_af = u_2^{-3(a^2-1)}u_3^{a^2-1}u_ag = u_a^{-3(2^2-1)}u_a^{3^2-1}u_ag = g.$$

Hence g has the properties (i) (ii) of ${}_c\theta_E$. Since we have already the uniqueness, this local existence of ${}_c\theta_E$ proves the global existence of ${}_c\theta_E$.

We prove 1.3 (2). As is easily seen, both $({}_d\theta_E)^{c^2}(c^*({}_d\theta_E))^{-1}$ and $({}_c\theta_E)^{d^2}(d^*({}_c\theta_E))^{-1}$ have the same divisor $c^2d^2(0) - c^2{}_dE - d^2{}_cE + {}_{cd}E$. Let $u \in \mathcal{O}(S)^\times$ be the ratio of these two elements. Since these two elements are invariant under N_2 and N_3, we have by applying N_2 (resp. N_3) that $u^4 = u$ (resp. $u^9 = u$). Hence $u = 1$.

We prove 1.3 (3). Let

$$f(z) = q^{\frac{1}{12}(c^2-1)}(-t)^{\frac{1}{2}(c-c^2)} \cdot \gamma_q(t)^{c^2}\gamma_q(t^c)^{-1} \quad \text{where } t = e^{2\pi i z}.$$

Then it is directly checked that as a function of t, $f(z)$ is invariant under the transformation $t \mapsto qt$. Hence $f(z)$ depends only on z mod $\mathbb{Z}\tau + \mathbb{Z}$. Hence $f(z)$ is a meromorphic function on $E = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$. It is seen easily that this function has the characterizing properties (i) (ii) of ${}_c\theta_E$ in (1).

We prove 1.3 (4). For any integer a which is prime to c, we have

$$N_a h_*({}_c\theta_E) = h_* N_a({}_c\theta_E) = h_*({}_c\theta_E).$$

Since the degree of h is prime to c, $h_*({}_c\theta_E)$ has the divisor $c^2(0) -_c E'$. Hence by the uniqueness of ${}_c\theta_{E'}$, we have $h_*({}_c\theta_E) =_c \theta_{E'}$.

2. Euler systems in K_2 of modular curves

2.1. In this section, we consider "zeta elements (elements which are related to zeta functions)" in K_2 of the modular curves $Y(M, N)$.

For M, $N \geqslant 1$, the modular curves $Y(M, N)$ are defined as follows.

Take $L \geqslant 3$ such that $M \mid L$, $N \mid L$. Define

$$Y(M, N) = G\backslash Y(L)$$

where

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/L) \; ; \right.$$
$$\left. a \equiv 1 \bmod M, \quad b \equiv 0 \bmod M, \quad c \equiv 0 \bmod N, \quad d \equiv 1 \bmod N \right\}$$

Then $Y(M, N)$ is independent of the choice of L.

We have $Y(N, N) = Y(N)$ if $N \geqslant 3$.

Let $X(M, N)$ be the smooth compactification of $Y(M, N)$.

If $M + N \geqslant 5$, the $\mathbb{Q}$-scheme $Y(M, N)$ represents the functor

$$S \longmapsto \left\{ \begin{array}{l} \text{the set of isomorphism classes of triples } (E, e_1, e_2) \text{ where } E \text{ is an elliptic} \\ \text{curve over } S \text{ and } e_1 \text{ and } e_2 \text{ are sections of } E \text{ over } S \text{ such that } Me_1 = \\ Ne_2 = 0 \text{ and } \mathbb{Z}/M \times \mathbb{Z}/N \to E\,; \ (a, b) \mapsto ae_1 + be_2 \text{ is injective.} \end{array} \right.$$

The canonical morphism $Y(L) \to Y(M, N)$ $(M \mid L, N \mid L)$ represents $(E, e_1, e_2) \mapsto (E, (L/M)\, e_1, (L/N)\, e_2)$.

In the rest of this section, except in 2.8, we always assume

$$M, \ N \geqslant 2, \quad M + N \geqslant 5.$$

2.2. For integers c, d such that $(c, 6M) = 1$ and $(d, 6N) = 1$, we define elements ${}_{c,d}z_{M,N}$, which we call "zeta elements" by

$${}_{c,d}z_{M,N} = \{{}_cg_{1/M,0}, {}_dg_{0,1/N}\} \in \mathrm{K}_2(Y(M,N)).$$

Note ${}_cg_{1/M,0} \in \mathcal{O}(Y(M,1))^\times$ and ${}_dg_{0,1/N} \in \mathcal{O}(Y(1,N))^\times$ by 1.7 (1). We define an element $z_{M,N}$, which we call also a zeta element, by

$$z_{M,N} = \{g_{1/M,0}, g_{0,1/N}\} \in \mathrm{K}_2(Y(M,N)) \otimes \mathbb{Q}.$$

We have

$${}_{c,d}z_{M,N} = \left(c^2 - \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)^*\right)\left(d^2 - \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)^*\right) \cdot z_{M,N} \quad \text{in} \quad \mathrm{K}_2(Y(M,N)) \otimes \mathbb{Q}.$$

Here, for $a \in (\mathbb{Z}/M)^\times$ and $b \in (\mathbb{Z}/N)^\times$, $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)^*$ denotes the pull back by the action of $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ on $Y(M,N)$ which represents $(E, e_1, e_2) \mapsto (E, ae_1, be_2)$.

In 2.3 and 2.4 below, we consider the behavior of zeta elements under norm homomorphisms, and in 2.6, we consider the relation between zeta elements and zeta functions.

Proposition 2.3. — *Let M', $N' \geqslant 2$, and assume $M \mid M'$, $N \mid N'$. Assume further that*

$$\mathrm{prime}(M) = \mathrm{prime}(M'), \quad \mathrm{prime}(N) = \mathrm{prime}(N'),$$

where for an integer $a \geqslant 1$, prime(a) *denotes the set of all prime divisors of a. Then, the norm homomorphism*

$$\mathrm{K}_2(Y(M',N')) \longrightarrow \mathrm{K}_2(Y(M,N))$$

sends ${}_{c,d}z_{M',N'}$ to ${}_{c,d}z_{M,N}$ for any integers c, d such that $(c, 6M) = 1$ and $(d, 6N) = 1$. After $\otimes\, \mathbb{Q}$, it sends $z_{M',N'}$ to $z_{M,N}$.

Proposition 2.4. — *Let ℓ be a prime number which does not divide M. Let c, d be integers such that $(c, 6M\ell) = 1$ and $(d, 6N\ell) = 1$.*
Then the norm homomorphism

$$\mathrm{K}_2(Y(M\ell, N\ell)) \longrightarrow \mathrm{K}_2(Y(M,N))$$

sends ${}_{c,d}z_{M\ell,N\ell}$ to

$$\left(1 - T'(\ell)\left(\begin{smallmatrix} 1/\ell & 0 \\ 0 & 1 \end{smallmatrix}\right)^* + \left(\begin{smallmatrix} 1/\ell & 0 \\ 0 & 1/\ell \end{smallmatrix}\right)^* \cdot \ell\right) \cdot {}_{c,d}z_{M,N}$$

in the case ℓ does not divide N, and to

$$\left(1 - T'(\ell)\left(\begin{smallmatrix} 1/\ell & 0 \\ 0 & 1 \end{smallmatrix}\right)^*\right) \cdot {}_{c,d}z_{M,N}$$

in the case ℓ divides N. Here $T'(\ell)$ is the "dual Hecke operator" explained in 2.9 below. The similar statement holds for $z_{M\ell,N\ell}$ and $z_{M,N}$ (after $\otimes\, \mathbb{Q}$).

The proofs of 2.3 and 2.4 are given in 2.11–2.13 below.

2.5. We next describe how zeta elements are related to zeta functions.
We consider the operator-valued zeta function

$$Z_{M,N}(s) = \sum_{(n,M)=1} T'(n)\left(\begin{smallmatrix} 1/n & 0 \\ 0 & 1 \end{smallmatrix}\right)^* \cdot n^{-s}$$

($T'(n)$ is the dual Hecke operator explained in 2.9 below), acting on $\mathrm{H}^1(Y(M,N)(\mathbb{C}),\mathbb{C})$. (Here $Y(M,N)(\mathbb{C})$ denotes the set of $\mathbb{C}$-valued points of $Y(M,N)$ as a $\mathbb{Q}$-scheme.) This converges absolutely when $\mathrm{Re}(s) > 2$. This function $Z_{M,N}(s)$ has a presentation as an Euler product whose Euler factor at a prime number ℓ is

$$\Big(1 - T'(\ell)\left(\begin{smallmatrix} 1/\ell & 0 \\ 0 & 1 \end{smallmatrix}\right)^* \cdot \ell^{-s} + \left(\begin{smallmatrix} 1/\ell & 0 \\ 0 & 1/\ell \end{smallmatrix}\right)^* \cdot \ell^{1-2s}\Big)^{-1} \quad \text{if} \quad (\ell, MN) = 1,$$

$$\Big(1 - T'(\ell)\left(\begin{smallmatrix} 1/\ell & 0 \\ 0 & 1 \end{smallmatrix}\right)^* \cdot \ell^{-s}\Big)^{-1} \quad \text{if} \quad (\ell, M) = 1 \quad \text{and} \quad \ell \mid N,$$

$$1 \quad \text{if} \quad \ell \mid M.$$

The function $Z_{M,N}(s)$ has an analytic continuation to the whole $\mathbb{C}$ as an operator valued meromorphic function in s, and is holomorphic at $s \neq 2$. Furthermore, $Z_{M,N}(0) = 0$.

As is reviewed in 2.10 below, we have the regulator map

$$\mathrm{reg}_{M,N} : \mathrm{K}_2(Y(M,N)) \longrightarrow \mathrm{H}^1(Y(M,N)(\mathbb{C}), \mathbb{R}\cdot i).$$

As is explained in 2.7 below, we have a special element

$$\delta_{M,N} \in \mathrm{H}^1(Y(M,N)(\mathbb{C}), \mathbb{Z}).$$

Let

$$(\delta_{M,N})^{\pm} = \frac{1}{2} \cdot (\delta_{M,N} \pm \iota(\delta_{M,N})) \in \mathrm{H}^1(Y(M,N)(\mathbb{C}), \mathbb{Q})$$

where ι denotes the pull back by the complex conjugation on $Y(M,N)(\mathbb{C})$.

The following Thm. 2.6 is deduced from the work of Beilinson in [**Be**, § 5]. We will give the proof of Thm. 2.6 in § 7.

Theorem 2.6. — *Assume* prime$(M) \subset$ prime(N). *Then we have*

$$\mathrm{reg}_{M,N}(z_{M,N}) = \lim_{s\to 0} \frac{1}{s} \cdot Z_{M,N}(s) \cdot 2\pi i \cdot (\delta_{M,N})^{-}.$$

2.7. The definition of the special cohomology class $\delta_{M,N}$ is as follows.

By Poincaré duality, the canonical pairing

$$\mathrm{H}^1(Y(M,N)(\mathbb{C}), \mathbb{Z}) \times \mathrm{H}^1_c(Y(M,N)(\mathbb{C}), \mathbb{Z}) \longrightarrow \mathbb{Z}$$

(H^1_c means the compact support cohomology) induces isomorphisms

$$\begin{aligned} \text{(2.7.1)} \qquad \mathrm{H}^1(Y(M,N)(\mathbb{C}), \mathbb{Z}) &\cong \mathrm{Hom}\left(\mathrm{H}^1_c(Y(M,N)(\mathbb{C}), \mathbb{Z}), \mathbb{Z}\right) \\ &\cong \mathrm{H}_1(X(M,N)(\mathbb{C}), \{\text{cusps}\}, \mathbb{Z}), \end{aligned}$$

where

$$\{\text{cusps}\} = X(M,N)(\mathbb{C}) \smallsetminus Y(M,N)(\mathbb{C})$$

We define $\delta_{M,N} \in \mathrm{H}^1(Y(M,N)(\mathbb{C}),\mathbb{Z})$ to be the image of

$$\mathrm{class}(\varphi) \in \mathrm{H}_1(X(M,N)(\mathbb{C}),\{\mathrm{cusps}\},\mathbb{Z})$$

under (2.7.1), where φ is the continuous map

$$(0,\infty) \longrightarrow X(N)(\mathbb{C})\ ;\quad \varphi(y) = \nu(yi) \ \text{ for } \ 0 < y < \infty,$$

which is a route from a cusp to a cusp.

2.8. (In 2.8, we do not make the assumptions $M,\ N \geqslant 2,\ M+N \geqslant 5$.) We give a preliminary to introduce Hecke operators.

For $A \geqslant 1$, define $\mathbb{Q}$-schemes

$$Y(M,N(A)),\quad Y(M(A),N)$$

as follows. Take $L \geqslant 3$ such that $M \mid L$ and $AN \mid L$ (resp. $AM \mid L$ and $N \mid L$). Define $Y(M,N(A))$ (resp. $Y(M(A),N)$) to be the quotient of $Y(L)$ by the action of the subgroup of $\mathrm{GL}_2(\mathbb{Z}/L)$ consisting of $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ such that

$$\begin{aligned} &a \equiv 1 \bmod M,\quad b \equiv 0 \bmod M \quad (\text{resp. } AM),\\ &c \equiv 0 \bmod AN \quad (\text{resp. } N),\quad d \equiv 1 \bmod N.\end{aligned}$$

We have canonical projections

$$\begin{aligned} Y(M,AN) &\longrightarrow Y(M,N(A)) \longrightarrow Y(M,N),\\ Y(AM,N) &\longrightarrow Y(M(A),N) \longrightarrow Y(M,N).\end{aligned}$$

Now assume $M+N \geqslant 5$. Then the $\mathbb{Q}$-scheme $Y(M,N(A))$ (resp. $Y(M(A),N)$) represents the functor

$$S \longmapsto \left\{ \begin{array}{l} \text{the set of isomorphism classes of } (E,e_1,e_2,C) \text{ where } (E,e_1,e_2) \text{ gives an } S\text{-}\\ \text{valued point of } Y(M,N) \text{ and } C \text{ is a cyclic subgroup scheme of } E \text{ of order}\\ AN \text{ (resp. } AM) \text{ satisfying the following condition.}\end{array}\right.$$

The condition is that C contains the section e_2 (resp. e_1) and the homomorphism

$$\begin{aligned} &\mathbb{Z}/M \times C \longrightarrow E\ ;\quad (x,y) \longmapsto xe_1 + y\\ (\text{resp.}\quad &C \times \mathbb{Z}/N \longrightarrow E\ ;\quad (x,y) \longmapsto x + ye_2\,)\end{aligned}$$

is injective.

The canonical projections $Y(M,N(A)) \to Y(M,N)$ and $Y(M(A),N) \to Y(M,N)$ are given by $(E,e_1,e_2,C) \mapsto (E,e_1,e_2)$, and the canonical projection $Y(M,AN) \to Y(M,N(A))$ (resp. $Y(AM,N) \to Y(M(A),N)$) is given by $(E,e_1,e_2) \mapsto (E,Ae_1,e_2,\mathbb{Z}e_2)$ (resp. $(E,e_1,e_2) \mapsto (E,Ae_1,e_2,\mathbb{Z}e_1)$).

We have an isomorphism

$$\begin{aligned} \varphi_A : Y(M,N(A)) &\xrightarrow{\sim} Y(M(A),N)\\ (E,e_1,e_2,C) &\longmapsto (E',e_1',e_2',C')\end{aligned}$$

where $E' = E/NC$, e_1' is the image of e_1 in E', e_2' is the image of $A^{-1}e_2 \cap C$ in E', and C' is the image of $A^{-1}\mathbb{Z}e_1$ in E'. Here $A^{-1}e_2$ (resp. $A^{-1}\mathbb{Z}e_1$) denotes the inverse image of e_2 (resp. $\mathbb{Z}e_1$) under the multiplication by A. ($A^{-1}e_2 \cap C$ is just a closed subscheme of E, but the image of $A^{-1}e_2 \cap C$ in E' becomes a section of E' over S.) The inverse morphism of φ_A is given by

$$\begin{aligned} \varphi_A^{-1} : Y(M(A), N) &\xrightarrow{\simeq} Y(M, N(A)) \\ (E, e_1, e_2, C) &\longmapsto (E', e_1', e_2', C') \end{aligned}$$

where $E' = E/MC$, e_1' is the image of $A^{-1}e_1 \cap C$ in E', e_2' is the image of e_2 in E', and C' is the image of $A^{-1}\mathbb{Z}e_2$ in E'.

If we denote the canonical morphisms

$$\mathfrak{H} \longrightarrow Y(M, A(N))(\mathbb{C}) \quad \text{and} \quad \mathfrak{H} \longrightarrow Y(M(A), N)(\mathbb{C})$$

(induced by $\nu : \mathfrak{H} \to Y(AMN)(\mathbb{C})$) both by ν, φ_A is the unique morphism satisfying

$$\varphi_A(\nu(\tau)) = \nu(A\tau) \quad \text{for any} \quad \tau \in \mathfrak{H}.$$

2.9. The Hecke operators $T(n)$ and the dual Hecke operators $T'(n)$ ($n \geqslant 1$, $(n, M) = 1$) on $\mathrm{K}_2(Y(M, N))$ and on $\mathrm{H}^1(Y(M, N)(\mathbb{C}), \mathbb{Z})$ are defined as follows.

First, $T(1) = T'(1) = 1$.

Next, we give the definitions of $T(\ell)$ and $T'(\ell)$ for a prime number ℓ which does not divide M. Let

$$\mathrm{pr} : Y(M, N(\ell)) \longrightarrow Y(M, N) \quad \text{and} \quad \mathrm{pr}' : Y(M(\ell), N) \longrightarrow Y(M, N)$$

be the canonical projections. We define

$$T(\ell) = (\mathrm{pr}')_* \circ (\varphi_\ell^{-1})^* \circ \mathrm{pr}^*, \quad T'(\ell) = \mathrm{pr}_* \circ \varphi_\ell^* \circ (\mathrm{pr}')^*.$$

Here $(\ )^*$ means the pull back and $(\ )_*$ means the norm (or trace) homomorphism.

If ℓ does not divide N, we have

$$T'(\ell) = T(\ell)\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1/\ell \end{smallmatrix}\right)^*.$$

In the case n is a power ℓ^e ($e \geqslant 0$) of a prime number ℓ which does not divide M, $T(n)$ and $T'(n)$ are defined as follows. If $\ell \mid N$, $T(\ell^e) = T(\ell)^e$, $T'(\ell^e) = T'(\ell)^e$. If ℓ does not divide N, $T(\ell^e)$ and $T'(\ell^e)$ are defined inductively, by

$$\begin{aligned} T(\ell^{e+2}) &= T(\ell)T(\ell^{e+1}) + T(\ell^e)\left(\begin{smallmatrix} 1/\ell & 0 \\ 0 & \ell \end{smallmatrix}\right)^* \cdot \ell, \\ T'(\ell^{e+2}) &= T'(\ell)T'(\ell^{e+1}) + T'(\ell^e)\left(\begin{smallmatrix} 1/\ell & 0 \\ 0 & \ell \end{smallmatrix}\right)^* \cdot \ell. \end{aligned}$$

Finally, for $n = \prod_\ell \ell^{e(\ell)}$ ($e(\ell) \geqslant 0$) where ℓ ranges over all prime numbers which do not divide M, $T(n)$ and $T'(n)$ are defined by

$$T(n) = \prod_\ell T(\ell^{e(\ell)}), \quad T'(n) = \prod_\ell T'(\ell^{e(\ell)}).$$

Then, for any m, $n \geqslant 1$ such that $(mn, M) = 1$ and for any $a \in (\mathbb{Z}/M)^\times$, $b \in (\mathbb{Z}/N)^\times$, the operators $T(m)$, $T(n)$, $T'(m)$, $T'(n)$, $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)^*$ commute with each other.

In the Poincaré duality

$$\mathrm{H}^1(Y(M,N)(\mathbb{C}),\mathbb{Z}) \times \mathrm{H}^1_c(Y(M,N)(\mathbb{C}),\mathbb{Z}) \longrightarrow \mathbb{Z},$$

$T(n)$ and $T'(n)$ are transposes of each other.

2.10. Let Y be a smooth algebraic curve over $\mathbb{C}$. We review the definition of the regulator map ([**Be**])

$$\mathrm{K}_2(Y) \longrightarrow \mathrm{H}^1(Y(\mathbb{C}),\mathbb{R}\cdot i).$$

(In our application, Y is taken to be $Y(M,N)\otimes_{\mathbb{Q}}\mathbb{C}$.) Since Y is a disjoint union of smooth connected curves, the definition is reduced to the case Y is connected. Now assuming Y is connected, let K be the function field of Y. First, we define

$$\mathrm{K}_2(K) \longrightarrow \varinjlim_U \mathrm{H}^1(U(\mathbb{C}),\mathbb{R}\cdot i)$$

where U ranges over all non-empty Zariski open set of Y. For f, $g \in K^\times$, let U be a Zariski open set of Y such that f, $g \in \mathcal{O}(U)^\times$. Define a C^∞-differential form $\eta_{f,g}$ on $U(\mathbb{C})$ by

$$\eta_{f,g} = \log(|f|)\cdot \mathrm{dlog}(g|g|^{-1}) - \log(|g|)\cdot \mathrm{dlog}(f|f|^{-1}).$$

Then $\mathrm{d}\eta_{f,g} = 0$, and hence $\mathrm{class}(\eta_{f,g}) \in \mathrm{H}^1(U(\mathbb{C}),\mathbb{R}\cdot i)$ is defined. It can be shown that the map

$$K^\times \otimes K^\times \longrightarrow \varinjlim_U \mathrm{H}^1(U(\mathbb{C}),\mathbb{R}\cdot i)\ ;\quad f\otimes g \longmapsto \mathrm{class}(\eta_{f,g})$$

factors through the canonical surjection

$$K^\times \otimes K^\times \longrightarrow \mathrm{K}_2(K)\ ;\quad f\otimes g \longmapsto \{f,g\}.$$

We have a commutative diagram of exact sequences

$$\begin{tikzcd}
 & \mathrm{K}_2(Y) \arrow[r] & \mathrm{K}_2(K) \arrow[r, "\partial"] \arrow[d] & \bigoplus_{y\in Y(\mathbb{C})}\mathbb{C}^\times \arrow[d] \\
0 \arrow[r] & \mathrm{H}^1(Y(\mathbb{C}),\mathbb{R}\cdot i) \arrow[r] & \varinjlim_U \mathrm{H}^1(U(\mathbb{C}),\mathbb{R}\cdot i) \arrow[r, "\partial"] & \bigoplus_{y\in Y(\mathbb{C})}\mathbb{R}
\end{tikzcd}$$

where the right vertical arrow is $z \mapsto \log(|z|)$, the y-component of the upper ∂ for $y \in Y(\mathbb{C})$ is the tame symbol map

$$\{f,g\} \longmapsto \{(-1)^{mn} f^n g^{-m}\}(y) \quad (m = \mathrm{ord}_y(f),\quad n = \mathrm{ord}_y(g))$$

and the y-component of the lower ∂ is $(2\pi i)^{-1}$ times the evaluation at the homology class of a small loop around y with the standard orientation. This diagram defines the regulator map $\mathrm{K}_2(Y) \to \mathrm{H}^1(Y(\mathbb{C}),\mathbb{R}\cdot i)$.

2.11. We prove Prop. 2.3.

In general, if $f : U \to V$ is a morphism of schemes which is finite and locally free, and if $u \in \mathcal{O}(U)^\times$ and $v \in \mathcal{O}(V)^\times$, the norm map $f_* : \mathrm{K}_2(U) \to \mathrm{K}_2(V)$ satisfies the projection formula

$$f_*(\{u, v\}) = \{f_*(u), v\}, \quad f_*(\{v, u\}) = \{v, f_*(u)\}$$

where $f_*(u)$ denotes the image of u under the norm map $\mathcal{O}(U)^\times \to \mathcal{O}(V)^\times$.

Hence, it is enough to prove the case $M' = M$ and the case $N' = N$. Since the both cases are proved similarly, we assume $N = N'$. In this case, our task is to prove that if $M \mid M'$ and $\mathrm{prime}(M) = \mathrm{prime}(M')$, the norm map $\mathcal{O}(Y(M', N))^\times \to \mathcal{O}(Y(M, N))^\times$ sends ${}_cg_{1/M',0}$ to ${}_cg_{1/M,0}$.

Take an integer $L \geqslant 3$ such that $M' \mid L$ and $N \mid L$. Let $a = M'/M$, let G be the subgroup of $\mathrm{GL}_2(\mathbb{Z}/L)$ corresponding to $Y(M, N)$, and let H be the subgroup of G corresponding to $Y(M', N)$. For each $(x, y) \in (\mathbb{Z}/a)^2$, fix an element $s_{x,y}$ of $\mathrm{GL}_2(\mathbb{Z}/L)$ of the form $\left(\begin{smallmatrix} 1+Mu & Mv \\ 0 & 1 \end{smallmatrix}\right)$ such that $u \equiv x \bmod a$ and $v \equiv y \bmod a$. (This is possible because $\mathrm{prime}(M') = \mathrm{prime}(M)$.) Then $s_{x,y}$ for $(x, y) \in (\mathbb{Z}/a)^2$ form a system of representatives of $H\backslash G$. Hence the norm homomorphism $\mathcal{O}(Y(M', N))^\times \to \mathcal{O}(Y(M, N))^\times$ sends ${}_cg_{1/M',0}$ to

$$\prod_{(x,y)\in(\mathbb{Z}/a)^2} s^*_{x,y}({}_cg_{1/M',0}) = \prod_{(x,y)\in(\mathbb{Z}/a)^2} {}_cg_{(1/M')+(x/a),y/a} = {}_cg_{1/M,0}$$

where the first equation follows from 1.7 (1) and the second follows from 1.7 (2).

We give a preliminary lemma for the proof of Prop 2.4.

Lemma 2.12. — *Let $(\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2 \smallsetminus \{(0, 0)\}$. Let $A \geqslant 1$, and let c be an integer which is prime to $6A$ and to the orders of α, β. Then we have*

$$\varphi^*_A({}_cg_{\alpha,\beta}) = \prod_{\beta'} {}_cg_{\alpha,\beta'}$$

(i.e. ${}_cg_{\alpha,\beta}(A\tau) = \prod_{\beta'} {}_cg_{\alpha,\beta'}(\tau)$ as functions on $\mathfrak{H}$) where β' ranges over all elements of $\mathbb{Q}/\mathbb{Z}$ such that $A\beta' = \beta$.

This is proved by using the analytic presentation 1.3 (3) of Siegel units.

2.13. We prove Prop. 2.4.

Take $L \geqslant 3$ such that $M \mid L$, $N \mid L$.

The morphism $Y(M\ell, N\ell) \to Y(M, N)$ factors as

$$Y(M\ell, N\ell) \longrightarrow Y(M, N\ell) \longrightarrow Y(M, N(\ell)) \longrightarrow Y(M, N).$$

Let G_0, G_1, G_2, G_3 be the subgroups of $\mathrm{GL}_2(\mathbb{Z}/L\ell)$ corresponding to $Y(M\ell, N\ell)$, $Y(M, N\ell)$, $Y(M, N(\ell))$, $Y(M, N)$, respectively. (So, $G_0 \subset G_1 \subset G_2 \subset G_3$.)

Step 1.— First we show that the norm map $\mathrm{K}_2(Y(M\ell, N\ell)) \to \mathrm{K}_2(Y(M, N\ell))$ sends ${}_{c,d}z_{M\ell,N\ell}$ to $\{{}_cg_{1/M,0} \cdot \varphi_\ell^*({}_cg_{\alpha,0}^{-1}), {}_dg_{0,1/N\ell}\}$ where α denotes the unique element of $\frac{1}{M}\mathbb{Z}/\mathbb{Z}$ such that $\ell\alpha = 1/M$.

In fact, since ${}_dg_{0,1/N\ell} \in \mathcal{O}(Y(M, N\ell))^\times$, it is enough to show that the norm map $\mathcal{O}(Y(M\ell, N\ell))^\times \to \mathcal{O}(Y(M, N\ell))^\times$ sends ${}_cg_{1/M\ell,0}$ to ${}_cg_{1/M\ell,0} \cdot \varphi_\ell^*({}_cg_{\alpha,0}^{-1})$. For each $(x, y) \in (\mathbb{Z}/\ell)^\times \times \mathbb{Z}/\ell$, fix an element $s_{x,y}$ of $\mathrm{GL}_2(\mathbb{Z}/L\ell)$ of the form $\left(\begin{smallmatrix} u & v \\ 0 & 1 \end{smallmatrix}\right)$ such that $u \equiv 1 \bmod M$, $v \equiv 0 \bmod M$, $u \equiv Mx \bmod \ell$, $v \equiv My \bmod \ell$. Then $s_{x,y}$ for $(x, y) \in (\mathbb{Z}/\ell)^\times \times \mathbb{Z}/\ell$ form a system of representatives of $G_0\backslash G_1$. Hence the norm map $\mathcal{O}(Y(M\ell, N\ell))^\times \to \mathcal{O}(Y(M, N\ell))^\times$ sends ${}_cg_{1/M\ell,0}$ to

$$\begin{aligned}\prod_{(x,y)\in(\mathbb{Z}/\ell)^\times\times\mathbb{Z}/\ell} s_{x,y}^*({}_cg_{1/M\ell,0}) &= \prod_{(x,y)\in(\mathbb{Z}/\ell)^\times\times\mathbb{Z}/\ell} {}_cg_{\alpha+(x/\ell),y/\ell} \\ &= \Big(\prod_{(x,y)\in(\mathbb{Z}/\ell)^2} {}_cg_{\alpha+(x/\ell),y/\ell}\Big) \cdot \Big(\prod_{y\in\mathbb{Z}/\ell} {}_cg_{\alpha,y/\ell}\Big)^{-1}, \\ &= {}_cg_{1/M,0} \cdot \varphi_\ell^*({}_cg_{\alpha,0}^{-1}) \quad \text{by 1.7 (2) and 2.12.}\end{aligned}$$

Step 2.— We show that the norm map $\mathrm{K}_2(Y(M, N\ell)) \to \mathrm{K}_2(Y(M, N(\ell)))$ sends $\{{}_cg_{1/M,0} \cdot \varphi_\ell^*({}_cg_{\alpha,0}^{-1}), {}_dg_{0,1/N\ell}\}$ to

$$\begin{gathered}\{{}_cg_{1/M,0} \cdot \varphi_\ell^*({}_cg_{\alpha,0}^{-1}), \varphi_\ell^*({}_dg_{0,1/N})\} \\ (\text{resp.} \quad \{{}_cg_{1/M,0} \cdot \varphi_\ell^*({}_cg_{\alpha,0}^{-1}), \varphi_\ell^*({}_dg_{0,1/N}) \cdot {}_dg_{0,\beta}^{-1}\}\,)\end{gathered}$$

in the case ℓ divides N (resp. does not divide N, where β is the unique element of $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$ such that $\ell\beta = 1/N$).

For each $x \in \mathbb{Z}/\ell$ (resp. $(\mathbb{Z}/\ell)^\times$), fix an element s_x of $\mathrm{GL}_2(\mathbb{Z}/L\ell)$ of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right)$ such that $u \equiv 1 + Nx \bmod N\ell$ (resp. $u \equiv 1 \bmod N$ and $u \equiv Nx \bmod \ell$). Then s_x for $x \in \mathbb{Z}/\ell$ (resp. $(\mathbb{Z}/\ell)^\times$) form a system of representatives of $G_1\backslash G_2$. Hence the norm map $\mathcal{O}(Y(M, N\ell))^\times \to \mathcal{O}\big(Y(M, N(\ell))\big)^\times$ sends ${}_dg_{0,1/N\ell}$ to

$$\begin{aligned}\prod_x s_x^*{}_dg_{0,1/N\ell} &= \prod_x {}_dg_{0,(1/N\ell)+(x/\ell)} \quad (\text{resp.} \quad \prod_x {}_dg_{0,\beta+(x/\ell)}\,) \\ &= \varphi_\ell^*({}_dg_{0,1/N}) \quad (\text{resp.} \quad \varphi_\ell^*({}_dg_{0,1/N}) \cdot {}_dg_{0,\beta}^{-1}\,) \quad \text{by 2.12.}\end{aligned}$$

Since ${}_cg_{1/M,0} \cdot \varphi_\ell^*({}_cg_{\alpha,0}) \in \mathcal{O}(Y(M, N(\ell)))^\times$, this proves the above statement on the K_2-norm map $\mathrm{K}_2(Y(M, N\ell)) \to \mathrm{K}_2\big(Y(M, N(\ell))\big)$.

It remains to consider what happens in $\mathrm{K}_2\big(Y(M, N(\ell))\big) \to \mathrm{K}_2(Y(M, N))$.

Step 3.— We will prove the following (2.13.1) and (2.13.2) in Step 4. Let $\mathrm{pr} : Y(M, N(\ell)) \to Y(M, N)$ be the canonical projection.

(2.13.1) In the case $\ell \mid N$, $\mathrm{pr}_*\varphi_\ell^*({}_dg_{0,1/N}) = {}_dg_{0,1/N}$.

(2.13.2) In the case ℓ does not divide N, we have

$$\mathrm{pr}_*\varphi_\ell^*({}_cg_{\alpha,0}) = {}_cg_{1/M,0}\cdot({}_cg_{\alpha,0})^\ell$$
$$\mathrm{pr}_*\varphi_\ell^*({}_dg_{0,1/N}) = {}_dg_{0,1/N}\cdot({}_cg_{0,\beta})^\ell.$$

We prove Prop. 2.4 by using these (2.13.1), (2.13.2).

First assume $\ell \mid N$. Then, the norm map $\mathrm{K}_2(Y(M\ell,N\ell))\to \mathrm{K}_2(Y(M,N))$ sends ${}_{c,d}z_{M\ell,N\ell}$ to

$$\begin{aligned}&\mathrm{pr}_*\{{}_cg_{1/M,0}\cdot\varphi_\ell^*({}_cg_{\alpha,0}^{-1}),\varphi_\ell^*({}_dg_{0,1/N})\}\quad\text{(by Step 1 and Step 2)}\\ &\quad= \{{}_cg_{1/M,0},\mathrm{pr}_*\varphi_\ell^*({}_dg_{0,1/N})\} - T'(\ell)\{{}_cg_{\alpha,0},{}_dg_{0,1/N}\}\\ &\quad= {}_{c,d}z_{M,N} - T'(\ell)\begin{pmatrix}1/\ell & 0\\ 0 & 1\end{pmatrix}^*({}_{c,d}z_{M,N})\quad\text{(by (2.13.1)).}\end{aligned}$$

Next assume ℓ does not divide N. Then the norm map $\mathrm{K}_2(Y(M\ell,N\ell))\to \mathrm{K}_2(Y(M,N))$ sends ${}_{c,d}z_{M\ell,N\ell}$ to

$$\begin{aligned}&\mathrm{pr}_*\{{}_cg_{1/M,0}\cdot\varphi_\ell^*({}_cg_{\alpha,0}^{-1}),\varphi_\ell^*({}_dg_{0,1/N})\cdot({}_dg_{0,\beta}^{-1})\}\quad\text{(by Step 1 and Step 2)}\\ &\quad= \{{}_cg_{1/M,0},\mathrm{pr}_*\varphi_\ell^*({}_dg_{0,1/N})\} - T'(\ell)\{{}_cg_{\alpha,0},{}_dg_{0,1/N}\}\\ &\qquad - (\ell+1)\{{}_cg_{1/M,0},{}_dg_{0,\beta}\} + \{\mathrm{pr}_*\varphi_\ell^*({}_cg_{\alpha,0}),{}_dg_{0,\beta}\}\end{aligned}$$

(here $\ell+1$ appears because $Y(M,N(\ell))$ is of degree $\ell+1$ over $Y(M,N)$ in the case ℓ does not divide N)

$$= {}_{c,d}z_{M,N} - T'(\ell)\begin{pmatrix}1/\ell & 0\\ 0 & 1\end{pmatrix}^*{}_{c,d}z_{M,N} + \begin{pmatrix}1/\ell & 0\\ 0 & 1/\ell\end{pmatrix}^*\cdot\ell\cdot{}_{c,d}z_{M,N}\quad\text{by (2.13.2).}$$

Step 4.— We prove the statements (2.13.1), (2.13.2). Since the three equalities in (2.13.1), (2.13.2) are proved in the similar way, we give here only the proof of $\mathrm{pr}_*\varphi_\ell^*({}_dg_{0,1/N}) = {}_dg_{0,1/N}\cdot({}_cg_{0,\beta})^\ell$ in the case ℓ does not divide N. For each $x\in\mathbb{Z}/\ell$, fix an element s_x of $\mathrm{GL}_2(\mathbb{Z}/\ell)$ of the form $\left(\begin{smallmatrix}1&0\\u&1\end{smallmatrix}\right)$ such that $u\equiv 0 \bmod N$, $u\equiv x \bmod \ell$. Let σ be an element of $\mathrm{GL}_2(\mathbb{Z}/L\ell)$ whose image in $\mathrm{GL}_2(\mathbb{Z}/N)$ is $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and whose image in $\mathrm{GL}_2(\mathbb{Z}/\ell)$ is $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Then s_x for $x\in\mathbb{Z}/\ell$ and σ form a system of representatives of $G_2\backslash G_3$. Hence

$$\mathrm{pr}_*\varphi_\ell^*({}_dg_{0,1/N}) = \Big(\prod_{x\in\mathbb{Z}/\ell} s_x^*\varphi_\ell^*({}_dg_{0,1/N})\Big)\cdot\sigma^*\varphi_\ell^*({}_dg_{0,1/N}).$$

Since $\varphi_\ell^*({}_dg_{0,1/N}) = \prod_{y\in\mathbb{Z}/\ell}{}_dg_{0,\beta+(y/\ell)}$, we have

$$\prod_{x\in\mathbb{Z}/\ell} s_x^*\varphi_\ell^*({}_dg_{0,1/N}) = \prod_{x,y\in\mathbb{Z}/\ell}{}_dg_{xy/\ell,\beta+(y/\ell)} = \Big(\prod_{\substack{x,y\in\mathbb{Z}/\ell\\ y\neq 0}}{}_dg_{x/\ell,\beta+(y/\ell)}\Big)\cdot({}_dg_{0,\beta})^\ell.$$

On the other hand,

$$\sigma^*\varphi_\ell^*({}_dg_{0,1/N}) = \prod_{y\in\mathbb{Z}/\ell}{}_dg_{y/\ell,\beta}.$$

Hence

$$\mathrm{pr}_*\varphi_\ell^*({}_dg_{0,1/N}) = \Big(\prod_{x,y\in\mathbb{Z}/\ell} {}_dg_{x/\ell,\beta+(y/\ell)}\Big)\cdot({}_dg_{0,\beta})^\ell = {}_dg_{0,1/N}\cdot({}_dg_{0,\beta})^\ell \text{ by 1.7 (2).}$$

3. Eisenstein series

The aim of this section is to review the basic facts about Eisenstein series. See Weil [**We**], Katz [**KN**], Katz-Mazur [**KM**] for the proofs of the results introduced here.

Let $N \geqslant 3$, let $X(N)$ be the smooth compactification of $Y(N)$, and let $M_k(X(N))$ ($k \in \mathbb{Z}$) be the space of modular forms on $X(N)$ of weight k. (We review the definition of $M_k(X(N))$ in 3.1.) In this section, we introduce the following modular forms indexed by $(\alpha,\beta)\in(\mathbb{Q}/\mathbb{Z})^2$, called Eisenstein series.

(i) $E^{(k)}_{\alpha,\beta}\in\bigcup_N M_k(X(N))$ ($k\geqslant 1$, $k\neq 2$).

(ii) $\widetilde{E}^{(2)}_{\alpha,\beta}\in\bigcup_N M_2(X(N))$.

($\widetilde{E}^{(2)}_{\alpha,\beta}$ is something like "$E^{(2)}_{\alpha,\beta}-E^{(2)}_{0,0}$", but the modular forms $E^{(2)}_{\alpha,\beta}$ do not exist.)

(iii) $F^{(k)}_{\alpha,\beta}\in\bigcup_N M_k(X(N))$ ($k\geqslant 1$; we assume $(\alpha,\beta)\neq(0,0)$ in the case $k=2$).

If $N\alpha=N\beta=0$, the elements (i) (iii) belong to $M_k(X(N))$, and the elements (ii) belong to $M_2(X(N))$.

We define these elements algebraically in 3.2–3.6 first, and then give the analytic presentations of them in 3.8.

Eisenstein series are additive analogues of Siegel units (which are multiplicative elements).

3.1. We introduce algebraic definitions of $M_k(X(N))$ and of the subspace $S_k(X(N))$ of $M_k(X(N))$ consisting of cusp forms.

Let $\lambda : E\to Y(N)$ be the universal elliptic curve, and let

$$\overline{\lambda}:\overline{E}\longrightarrow X(N)$$

be the smooth Néron model of E over $X(N)$. Let

$$\mathrm{coLie}(\overline{E}) = \overline{\lambda}_*(\Omega^1_{\overline{E}/X(N)}).$$

Then $\mathrm{coLie}(\overline{E})$ is an invertible $\mathcal{O}_{X(N)}$-module, and

$$\Omega^1_{\overline{E}/X(N)} = \overline{\lambda}^*\,\mathrm{coLie}(\overline{E}).$$

Define

$$M_k(X(N)) = \Gamma\big(X(N),\mathrm{coLie}(\overline{E})^{\otimes k}\big). \tag{3.1.1}$$

We have another equivalent definition

$$M_k(X(N)) = \Gamma\big(X(N),\mathrm{coLie}(\overline{E})^{\otimes(k-2)}\otimes_{\mathcal{O}(X(N))}\Omega^1_{X(N)/\mathbb{Q}}(\log(\mathrm{cusps}))\big) \tag{3.1.2}$$

where $\Omega^1_{X(N)/\mathbb{Q}}(\log(\text{cusps}))$ denotes the sheaf of differential forms on $X(N)$ with log poles at cusps.

The equivalence of the two definitions (3.1.1) and (3.1.2) is given by the isomorphism (3.1.4) below. Denote $\lambda_*(\Omega^1_{E/Y(N)})$ by $\mathrm{coLie}(E)$. We have a canonical homomorphism

$$\mathrm{coLie}(E) \longrightarrow \Omega^1_{Y(N)/\mathbb{Q}} \otimes_{\mathcal{O}(Y(N))} \mathrm{R}^1\lambda_*(\mathcal{O}_E) \tag{3.1.3}$$

on $Y(N)$ as the connecting map of the exact sequence

$$0 \longrightarrow \lambda^*(\Omega^1_{Y(N)/\mathbb{Q}}) \longrightarrow \Omega^1_{E/\mathbb{Q}} \longrightarrow \Omega^1_{E/Y(N)} \longrightarrow 0.$$

By composing (3.1.3) with the Serre duality

$$\mathrm{R}^1\lambda_*(\mathcal{O}_E) \otimes_{\mathcal{O}(Y(N))} \mathrm{coLie}(E) = \mathrm{R}^1\lambda_*(\mathcal{O}_E) \otimes_{\mathcal{O}(Y(N))} \lambda_*(\Omega^1_{E/Y(N)}) \longrightarrow \mathcal{O}_{Y(N)},$$

we obtain the composite map

$$\mathrm{coLie}(E)^{\otimes 2} \longrightarrow \Omega^1_{Y(N)/\mathbb{Q}} \otimes_{\mathcal{O}(Y(N))} \mathrm{R}^1\lambda_*(\mathcal{O}_E) \otimes_{\mathcal{O}(Y(N))} \mathrm{coLie}(E) \longrightarrow \Omega^1_{Y(N)/\mathbb{Q}}.$$

It is known ([**KM**, 10.13]) that this composite map induces an isomorphism

$$\mathrm{coLie}(E)^{\otimes 2} \cong \Omega^1_{X(N)/\mathbb{Q}}(\log(\text{cusps})). \tag{3.1.4}$$

The space $S_k(X(N))$ of cusp forms on $X(N)$ of weight k is defined by

$$S_k(X(N)) = \Gamma\big(X(N), \mathrm{coLie}\,(\overline{E})^{\otimes(k-2)} \otimes_{\mathcal{O}_{X(N)}} \Omega^1_{X(N)/\mathbb{Q}}\big) \subset M_k(X(N)).$$

3.2. We define elements

$${}_cE^{(k)}_{\alpha,\beta} \in M_k(X(N)) \quad \text{for} \quad k \geqslant 1 \quad \text{and} \quad (\alpha,\beta) \in \big(\tfrac{1}{N}\mathbb{Z}/\mathbb{Z}\big)^2,$$

where c is an integer which is prime to $6N$. Once the elements $E^{(k)}_{\alpha,\beta}$ for $k \geqslant 1$, $k \neq 2$, and the elements $\widetilde{E}^{(2)}_{\alpha,\beta}$ are defined, ${}_cE^{(k)}_{\alpha,\beta}$ is expressed as

$$\begin{aligned} {}_cE^{(k)}_{\alpha,\beta} &= c^2 E^{(k)}_{\alpha,\beta} - c^k E^{(k)}_{c\alpha,c\beta} \quad \text{for} \quad k \geqslant 1, \quad k \neq 2, \\ {}_cE^{(2)}_{\alpha,\beta} &= c^2 \widetilde{E}^{(2)}_{\alpha,\beta} - c^2 \widetilde{E}^{(2)}_{c\alpha,c\beta}. \end{aligned}$$

But it is convenient to define ${}_cE^{(k)}_{\alpha,\beta}$ first, by using the theta function ${}_c\theta_E$ in Prop. 1.3.

Let the notation be as in 1.3.

Let c be an integer which is prime to 6, and consider the element

$$\mathrm{dlog}({}_c\theta_E) \in \Gamma(E \smallsetminus {}_cE, \Omega^1_{E/Y(N)}) = \Gamma(E \smallsetminus {}_cE, \lambda^*\,\mathrm{coLie}(E)).$$

For $r \in \mathbb{Z}$, let

$$D : \lambda^*\mathrm{coLie}(E)^{\otimes r} \longrightarrow \lambda^*\mathrm{coLie}(E)^{\otimes(r+1)}$$

be the map defined locally by

$$f \otimes \omega^{\otimes r} \longmapsto \frac{\mathrm{d}f}{\omega} \otimes \omega^{\otimes(r+1)},$$

where $f \in \mathcal{O}_E$, ω is a local basis of coLie(E), and $\mathrm{d}f/\omega \in \mathcal{O}_E$ is determined by

$$\mathrm{d}f = \Big(\frac{\mathrm{d}f}{\omega}\Big) \cdot \omega \quad \text{in} \quad \Omega^1_{E/Y(N)}.$$

For $k \geqslant 1$, we have an element

$$D^{k-1}\,\mathrm{dlog}({}_c\theta_E) \in \Gamma(E \smallsetminus {}_cE, \lambda^*\mathrm{coLie}(E)^{\otimes k}).$$

Now for $(\alpha, \beta) \in \big(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\big)^2 \smallsetminus \{(0,0)\}$, assuming $(c, 6N) = 1$, we define

$${}_cE^{(k)}_{\alpha,\beta} = \iota^*_{\alpha,\beta}(D^{k-1}\,\mathrm{dlog}({}_c\theta_E)) \in \Gamma(Y(N), \mathrm{coLie}(E)^{\otimes k})$$

where $\iota_{\alpha,\beta} : Y(N) \to E \smallsetminus {}_cE$ is as in 1.4. Then ${}_cE^{(k)}_{\alpha,\beta}$ belongs to $M_k(X(N)) \subset \Gamma(Y(N), \mathrm{coLie}(E)^{\otimes k})$.

We next define ${}_cE^{(k)}_{0,0}$ for $k \geqslant 1$ and for an integer c such that $(c, 6) = 1$. The following is deduced from $N_a({}_c\theta_E) = {}_c\theta_E$:

If $(\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2 \smallsetminus \{(0,0)\}$, a, $c \in \mathbb{Z} \smallsetminus \{0\}$, and if c is prime to $6a$ and to the orders of α, β, then

$$a^k\,{}_cE^{(k)}_{\alpha,\beta} = \sum_{\alpha',\beta'} {}_cE^{(k)}_{\alpha',\beta'}$$

where α' (resp. β') ranges over all of $\mathbb{Q}/\mathbb{Z}$ such that $a\alpha' = \alpha$ (resp. $a\beta' = \beta$).

This shows that the following element ${}_cE^{(k)}_{0,0}$ is independent of the choice of $a \neq \pm 1$ which is prime to c:

$${}_cE^{(k)}_{0,0} \underset{\mathrm{def}}{=} (a^k - 1)^{-1} \sum_{\alpha,\beta}{}' \, {}_cE^{(k)}_{\alpha,\beta}$$

where $\sum'$ is the sum over all non-zero elements (α, β) of $\big(\frac{1}{a}\mathbb{Z}/\mathbb{Z}\big)^2$.

3.3. We define $E^{(k)}_{\alpha,\beta} \in M_k(X(N))$ for $k \geqslant 1$, $k \neq 2$ and for $(\alpha, \beta) \in \big(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\big)^2$. From Prop. 1.3 (2), we obtain

$$c^2\,{}_dE^{(k)}_{\alpha,\beta} - c^k\,{}_dE^{(k)}_{c\alpha,c\beta} = d^2\,{}_cE^{(k)}_{\alpha,\beta} - d^k\,{}_cE^{(k)}_{d\alpha,d\beta}$$

for any integers c, d which are prime to $6N$. This shows that if we take c which is prime to $6N$ such that $c \equiv 1 \bmod N$ and $c \neq \pm 1$, then

$$E^{(k)}_{\alpha,\beta} \underset{\mathrm{def}}{=} (c^2 - c^k)^{-1}\,{}_cE^{(k)}_{\alpha,\beta}$$

is independent of the choice of such c. We have

$${}_cE^{(k)}_{\alpha,\beta} = c^2 E^{(k)}_{\alpha,\beta} - c^k E^{(k)}_{c\alpha,c\beta} \quad \text{for} \quad k \neq 2 \quad \text{and} \quad (\alpha,\beta) \in (\mathbb{Q}/\mathbb{Z})^2$$

where c is any integer which is prime to 6 and to the orders of α, β.

3.4. We define $\widetilde{E}^{(2)}_{\alpha,\beta} \in M_k(X(N))$ for $(\alpha,\beta) \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z}^2)$.

In general, if $f : E \to S$ is an elliptic curve over a scheme S on which 6 is invertible, there exists locally on S a pair (ω, x) satisfying the following conditions; ω is a basis of the invertible $\mathcal{O}_S$-module $\operatorname{coLie}(E)$, x is a section of I^{-2} where I denotes the invertible ideal of $\mathcal{O}_E$ defining the origin of E, the image of x in $f_*(I^{-2})/f_*(I^{-1})$ is a basis of the invertible $\mathcal{O}_S$-module $f_*(I^{-2})/f_*(I^{-1})$, and there exist sections a, b of $\mathcal{O}_S$ such that

$$\left(\frac{\mathrm{d}x}{\omega}\right)^2 = 4x^3 + ax + b.$$

Here $\frac{\mathrm{d}x}{\omega}$ is defined by $\mathrm{d}x = (\frac{\mathrm{d}x}{\omega})\omega$ in $I^{-3}\Omega^1_{E/S}$. Furthermore, $x \otimes \omega^2 \in I^{-2} \cdot f^* \operatorname{coLie}(E)^{\otimes 2}$ is independent of the choice of such pair (x,ω), and hence is defined globally on S. Take $Y(N)$ as S and the universal elliptic curve as E, and let

$$\wp \in \Gamma(E, I^{-2} \cdot f^*\operatorname{coLie}(E)^{\otimes 2})$$

be the section which is locally $x \otimes \omega^{\otimes 2}$ for a pair (x,ω) as above.

Now let $(\alpha,\beta) \in \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2 \smallsetminus \{(0,0)\}$. We define

$$\widetilde{E}^{(2)}_{\alpha,\beta} = \iota^*_{\alpha,\beta}(\wp) \in \Gamma(Y(N), \operatorname{coLie}(E)^{\otimes 2}).$$

Then $\widetilde{E}^{(2)}_{\alpha,\beta}$ belongs to $M_k(X(N)) \subset \Gamma(Y(N), \operatorname{coLie}(E)^{\otimes 2})$.

We define

$$\widetilde{E}^{(2)}_{0,0} = 0.$$

3.5. The constructions in 3.2–3.3 and those in 3.4 are related as follows. We have

$$D\,\mathrm{dlog}({}_c\theta_E) = c^2\wp - c^*\wp \tag{3.5.1}$$

(E is the universal elliptic curve) where c^* denotes the pull back by the multiplication by $c : E \to E$. (This can be proved for example, by using the analytic descriptions of $\wp$ and $D\,\mathrm{dlog}({}_c\theta_E)$ given in 3.8 below.) From this we have for any $k \geqslant 2$,

$$D^{k-1}\,\mathrm{dlog}({}_c\theta_E) = c^2 D^{k-2}\wp - c^* D^{k-2}\wp. \tag{3.5.2}$$

From (3.5.1) and (3.5.2), we have for $(\alpha,\beta) \in (\mathbb{Q}/\mathbb{Z})^2 \smallsetminus \{(0,0)\}$

$${}_cE^{(2)}_{\alpha,\beta} = c^2\widetilde{E}^{(2)}_{\alpha,\beta} - c^2\widetilde{E}^{(2)}_{c\alpha,c\beta}$$

(this formula holds also in the case $(\alpha,\beta) = (0,0)$) and

$$E^{(k)}_{\alpha,\beta} = \iota^*_{\alpha,\beta}(D^{k-2}\wp) \quad \text{for} \quad k \geqslant 3$$

(this formula presents another algebraic definition of $E^{(k)}_{\alpha,\beta}$ for $k \geqslant 3$).

3.6. Next we define the elements $F^{(k)}_{\alpha,\beta} \in M_k(X(N))$ for $k \geqslant 1$ (we assume $(\alpha,\beta) \neq (0,0)$ in the case $k=2$).

For $(\alpha,\beta) \in \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2$, writing $\alpha = a/N$, $\beta = b/N$, define

$$F^{(k)}_{\alpha,\beta} = N^{-k} \sum_{x,y\in\mathbb{Z}/N} E^{(k)}_{x/N,y/N} \cdot \zeta_N^{bx-ay} \in M_k(X(N)) \quad (\text{here } k \neq 2),$$

$$F^{(2)}_{\alpha,\beta} = N^{-2} \sum_{x,y\in\mathbb{Z}/N} \widetilde{E}^{(2)}_{x/N,y/N} \cdot \zeta_N^{bx-ay} \in M_2(X(N)) \quad (\text{here } (\alpha,\beta) \neq (0,0)).$$

By the "distribution property" (i) (ii) of 3.7 (2) below, $F^{(k)}_{\alpha,\beta}$ is independent of the choices of N, a, b such that $\alpha = a/N$ and $\beta = b/N$. We have

$$E^{(k)}_{a/N,b/N} = N^{k-2} \sum_{x,y\in\mathbb{Z}/N} F^{(k)}_{x/N,y/N} \cdot \zeta_N^{bx-ay} \quad \text{if} \quad k \neq 2.$$

In the case $k=2$, we have the formula for $\widetilde{E}^{(2)}_{a/N,b/N}$ of the same form, except that (x,y) ranges in this case over all elements of $(\mathbb{Z}/N)^2 \smallsetminus \{(0,0)\}$.

The following 3.7 is an additive analogue of 1.7.

Lemma 3.7. — *Let $k \geqslant 1$.*

(1) *Let $\sigma \in \mathrm{GL}_2(\mathbb{Z}/N)$, $(\alpha,\beta) \in \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2$, $(\alpha',\beta') = (\alpha,\beta)\cdot\sigma$. Then we have:*

(i) $\sigma^*(E^{(k)}_{\alpha,\beta}) = E^{(k)}_{\alpha',\beta'}$, *if $k \neq 2$.*

(ii) $\sigma^*(\widetilde{E}^{(2)}_{\alpha,\beta}) = \widetilde{E}^{(2)}_{\alpha',\beta'}$.

(iii) $\sigma^*(F^{(k)}_{\alpha,\beta}) = F^{(k)}_{\alpha',\beta'}$ *; here, we assume $(\alpha,\beta) \neq (0,0)$ in the case $k=2$.*

(2) *(Distribution property.) Let $(\alpha,\beta) \in (\mathbb{Q}/\mathbb{Z})^2$, $a \in \mathbb{Z}$, $a \neq 0$. Then we have the following equalities where α' (resp. β') ranges over all elements of $\mathbb{Q}/\mathbb{Z}$ such that $a\alpha' = \alpha$ and $a\beta' = \beta$.*

(i) $a^k E^{(k)}_{\alpha,\beta} = \sum_{\alpha',\beta'} E^{(k)}_{\alpha',\beta'}$, *if $k \neq 2$.*

(ii) $a^2 \widetilde{E}^{(2)}_{\alpha,\beta} = \sum_{\alpha',\beta'} \widetilde{E}^{(2)}_{\alpha',\beta'}$.

(iii) $a^{2-k} F^{(k)}_{\alpha,\beta} = \sum_{\alpha',\beta'} F^{(k)}_{\alpha',\beta'}$ *; here, we assume $(\alpha,\beta) \neq (0,0)$ in the case $k=2$.*

(1) is proved easily. (2) is deduced from the analytic descriptions of Eisenstein series given in 3.8 below.

3.8. We review the analytic theory of Eisenstein series.

Let $\lambda : E \to Y(N)$ be the universal elliptic curve. Then we have a cartesian diagram

$$\begin{tikzcd} (\mathfrak{H} \times \mathbb{C})/\sim \arrow[r] \arrow[d] & E(\mathbb{C}) \arrow[d, "\lambda"] \\ \mathfrak{H} \arrow[r, "\nu"] & Y(N)(\mathbb{C}) \end{tikzcd}$$

where $\sim$ is the equivalence relation defined as follows. For (τ, z), $(\tau', z') \in \mathfrak{H} \times \mathbb{C}$, $(\tau, z) \sim (\tau', z')$ if and only if $\tau = \tau'$ and $z - z' \in \mathbb{Z}\tau + \mathbb{Z}$.

Let (τ, z) be the standard coordinate of $\mathfrak{H} \times \mathbb{C}$. Then the pull back of $F \in M_k(X(N))$ to $\mathfrak{H}$ is written in the form

$$F = f(\tau) \otimes \left(\frac{\mathrm{d}t}{t}\right)^{\otimes k} \quad (t = e^{2\pi i z}) = f(\tau) \otimes (2\pi i \mathrm{d}z)^{\otimes k}$$

where $f(\tau)$ is a holomorphic function on $\mathfrak{H}$ satisfying

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N).$$

Classically, this f was called a modular form. We will identify an element F of $M_k(X(N))$ with the above holomorphic function $f(\tau)$.

It is known ([**KM**, 10.13]) that the isomorphism (3.1.4)

$$\mathrm{coLie}(\overline{E})^{\otimes 2} \cong \Omega^1_{X(N)/\mathbb{Q}}(\log(\text{cusps}))$$

sends $\left(\mathrm{d}t/t\right)^{\otimes 2}$ to $\mathrm{d}q/q$ $(q = e^{2\pi i \tau})$.

We now describe the pull backs on $\mathfrak{H} \times \mathbb{C}$ or on $\mathfrak{H}$, of the objects which appeared in this section.

First, $\wp$ in 3.4 is described as

$$\wp = \wp(\tau, z) \otimes (2\pi i \mathrm{d}z)^{\otimes 2}$$

where $\wp(\tau, z)$ is Weierstrass' $\wp$-function defined by

$$\wp(\tau, z) = (2\pi i)^{-2}\Big(z^{-2} + \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \{(z + m\tau + n)^{-2} - (m\tau + n)^{-2}\}\Big).$$

Define functions $E^{(k)}(\tau, z)$ on $\mathfrak{H} \times \mathbb{C}$ and functions $E^{(k)}_{\alpha,\beta}$ on $\mathfrak{H}$ ($k \geqslant 0$, $(\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2$; we assume $(\alpha, \beta) \neq (0, 0)$ in the case $k = 0$) as follows. (If $k \neq 0$, 2, as in (3.8.4) below, this notation $E^{(k)}_{\alpha,\beta}$ is compatible with the notation for the modular forms $E^{(k)}_{\alpha,\beta}$.) If $k \geqslant 3$, the definitions are simply

$$E^{(k)}(\tau, z) = (-1)^k \cdot (k-1)! \cdot (2\pi i)^{-k} \sum_{(m,n)\in\mathbb{Z}^2} (z + m\tau + n)^{-k}.$$

$$E^{(k)}_{\alpha,\beta}(\tau) = E^{(k)}\big(\tau, \widetilde{\alpha}\tau + \widetilde{\beta}\big) \quad \text{for} \quad (\alpha, \beta) \neq (0, 0)$$

where $\big(\widetilde{\alpha}, \widetilde{\beta}\big)$ is a lifting of (α, β) to $\mathbb{Q}^2$.

$$E^{(k)}_{(0,0)}(\tau) = (-1)^k \cdot (k-1)! \cdot (2\pi i)^{-k} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} (m\tau + n)^{-k}.$$

To include the cases $k = 0, 1, 2$, define for $k \geqslant 0$

$$E(k,\tau,z,s) = \sum_{(m,n)\in\mathbb{Z}^2} (z+m\tau+n)^{-k}|z+m\tau+n|^{-s}.$$

$$E_{(0,0)}(k,\tau,s) = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} (m\tau+n)^{-k}|m\tau+n|^{-s}.$$

These series converge absolutely when $k + \mathrm{Re}(z) > 2$. When k, τ, z are fixed, these functions in s have analytic continuations as meromorphic functions on the whole s-plane. These functions for $k \geqslant 1$ are holomorphic on the whole s-plane, and $E(0,\tau,z,s)$ has a zero at $s = 0$. Let

$$E^{(k)}(\tau,z) = (-1)^k\cdot(k-1)!\cdot(2\pi i)^{-k}E(k,\tau,z,0) \quad \text{for} \quad k\geqslant 1,$$

$$E^{(0)}(\tau,z) = \lim_{s\to 0} s^{-1}E(0,\tau,z,s),$$

$$E^{(k)}_{\alpha,\beta}(\tau) = E^{(k)}\big(\tau,\widetilde{\alpha}\tau+\widetilde{\beta}\big) \quad \text{for} \quad k\geqslant 0 \quad \text{and} \quad (\alpha,\beta)\neq(0,0)$$

where $(\widetilde{\alpha},\widetilde{\beta})$ is a lifting of (α,β) to $\mathbb{Q}^2$,

$$E^{(k)}_{0,0}(\tau) = (-1)^k\cdot(k-1)!\cdot(2\pi i)^{-k}E_{(0,0)}(k,\tau,0) \quad \text{for} \quad k\geqslant 1.$$

We have:

(3.8.1) For $k \geqslant 1$, $D^{k-1}\,\mathrm{dlog}({}_c\theta_E) = \big(c^2E^{(k)}(\tau,z) - c^kE^{(k)}(\tau,cz)\big)\otimes\big(\frac{\mathrm{d}t}{t}\big)^{\otimes k}$.

(3.8.2) $\log(|{}_c\theta_E|) = c^2E^{(0)}(\tau,z) - E^{(0)}(\tau,cz)$.

(3.8.3)
$$\begin{aligned}\wp &= (E^{(2)}(\tau,z) - E^{(2)}_{0,0}(\tau))\otimes\big(\tfrac{\mathrm{d}t}{t}\big)^{\otimes 2}.\\ D^{k-2}\wp &= E^{(k)}(\tau,z)\otimes\big(\tfrac{\mathrm{d}t}{t}\big)^{\otimes k} \quad \text{for } k\geqslant 3.\end{aligned}$$

(3.8.4) Let $(\alpha,\beta)\in(\mathbb{Q}/\mathbb{Z})^2$. Then:

(i) If $k\geqslant 1$, $k\neq 2$, the modular form $E^{(k)}_{\alpha,\beta}$ coincides with the function $E^{(k)}_{\alpha,\beta}$ defined analytically above.

(ii) $\widetilde{E}^{(2)}_{\alpha,\beta} = E^{(2)}_{\alpha,\beta} - E^{(2)}_{0,0}$.

(iii) $\log|g_{\alpha,\beta}| = E^{(0)}_{\alpha,\beta}$ for $(\alpha,\beta)\neq(0,0)$.

(iv) The functions $E^{(0)}_{\alpha,\beta}$ $((\alpha,\beta)\neq(0,0))$ and $E^{(2)}_{\alpha,\beta}$ are C^∞-functions on $\mathfrak{H}$ but not holomorphic.

3.9. We next consider q-expansions. It is known that the pull back of an element of $M_k(X(N))$ on $\mathfrak{H}$ has a presentation

$$\sum_{n\in\mathbb{Z},n\geqslant 0} a_nq^{n/N} \quad (a_n\in\mathbb{Q}(\zeta_N))$$

$(q^{n/N} = e^{2\pi in\tau/N})$ called q-expansion. We give q-expansions of Eisenstein series.

For $\alpha \in \mathbb{Q}/\mathbb{Z}$, define $\zeta(\alpha, s)$ and $\zeta^*(\alpha, s)$ by

$$\zeta(\alpha, s) = \sum_{\substack{n\in\mathbb{Q}, n>0 \\ n \bmod \mathbb{Z}=\alpha}} n^{-s}, \quad \zeta^*(\alpha, s) = \sum_{n=1}^{\infty} e^{2\pi i \alpha n} \cdot n^{-s}.$$

Note $\zeta(0, s) = \zeta^*(0, s) = \zeta(s)$.

Proposition 3.10. — *Let $k \geqslant 1$ and α, $\beta \in \mathbb{Q}/\mathbb{Z}$*

(1) *Assume $k \neq 2$. Write*

$$E^{(k)}_{\alpha,\beta} = \sum_{n\in\mathbb{Q}, n\geqslant 0} a_n q^n.$$

Then a_n for $n > 0$ are given by

$$\sum_{n\in\mathbb{Q}, n>0} a_n n^{-s} = \zeta^*(\alpha, s)\zeta^*(\beta, s-k+1) + (-1)^k \zeta(-\alpha, s)\zeta^*(-\beta, s-k+1).$$

In the case $k \neq 1$, $a_0 = 0$ if $\alpha \neq 0$, and $a_0 = \zeta^*(\beta, 1-k)$ if $\alpha = 0$.
In the case $k = 1$, $a_0 = \zeta(\alpha, 0)$ if $\alpha \neq 0$, and

$$a_0 = \frac{1}{2}(\zeta^*(\beta, 0) - \zeta^*(-\beta, 0)) \quad \textit{if} \quad \alpha = 0.$$

(2) *Write*

$$\widetilde{E}^{(2)}_{\alpha,\beta} = \sum_{n\in\mathbb{Q}, n\geqslant 0} a_n q^n.$$

Then a_n for $n > 0$ are given by

$$\sum_{n\in\mathbb{Q}, n>0} a_n n^{-s} = \zeta(\alpha, s)\zeta^*(\beta, s-1) + \zeta(-\alpha, s)\zeta^*(-\beta, s-1) - 2\zeta(s)\zeta(s-1).$$

We have $a_0 = 0$ if $\alpha \neq 0$, and $a_0 = \zeta^*(\beta, -1) - \zeta(-1)$ if $\alpha = 0$.

(3) *Here, we assume $(\alpha, \beta) \neq (0, 0)$ in the case $k = 2$. Write*

$$F^{(k)}_{\alpha,\beta} = \sum_{n\in\mathbb{Q}, n\geqslant 0} a_n q^n.$$

Then a_n for $n > 0$ are given by

$$\sum_{n\in\mathbb{Q}, n>0} a_n n^{-s} = \zeta(\alpha, s-k+1)\zeta^*(\beta, s) + (-1)^k \zeta(-\alpha, s-k+1)\zeta^*(-\beta, s).$$

In the case $k \neq 1$, $a_0 = \zeta(\alpha, 1-k)$.
In the case $k = 1$, $a_0 = \zeta(\alpha, 0)$ if $\alpha \neq 0$, and

$$a_0 = \frac{1}{2}(\zeta^*(\beta, 0) - \zeta^*(-\beta, 0)) \quad \textit{if} \quad \alpha = 0.$$

By Prop. 3.10 and by the "q-presentation" of $g_{\alpha,\beta}$ in 1.9 (note that the number $1/12 - a/2N + (1/2)(a/N^2)$ $(0 \leqslant a < N)$ which appeared in 1.9 coincides with $-\zeta(a/N, -1)$), we have

Proposition 3.11

(1) $E^{(1)}_{\alpha,\beta} = F^{(1)}_{\alpha,\beta}$ *for any* $(\alpha,\beta) \in (\mathbb{Q}/\mathbb{Z})^2$.

(2) $\mathrm{dlog}(g_{\alpha,\beta}) = -F^{(2)}_{\alpha,\beta}$ *in* $\Gamma(X(N), \Omega^1_{X(N)/\mathbb{Q}}(\log(\mathrm{cusps})))$ *for any* $(\alpha,\beta) \in (\mathbb{Q}/\mathbb{Z})^2 \setminus \{(0,0)\}$.

4. Euler systems in the space of modular forms

4.1. For $k \in \mathbb{Z}$ and a curve X over $\mathbb{Q}$ of the form $G\backslash X(N)$ with $N \geqslant 3$ and G a subgroup of $\mathrm{GL}_2(\mathbb{Z}/N)$, we define the space $M_k(X)$ (resp. $S_k(X)$) of modular forms (resp. cusp forms) of weight k on X to be the G-fixed part of $M_k(X(N))$ (resp. $S_k(X(N))$). (This definition makes sense since in the case N, $N' \geqslant 3$ and $N \mid N'$, $M_k(X(N))$ (resp. $S_k(X(N))$) coincides with the $\mathrm{Gal}(X(N')/X(N))$-fixed part of $M_k(X(N'))$ (resp. $S_k(X(N'))$).)

In the rest of this section, fix $k \geqslant 2$, M, $N \geqslant 1$, such that $M + N \geqslant 5$.

4.2. We define elements

$$_{c,d}z_{M,N}(k,r,r') \in M_k(X(M,N))$$

which we call "zeta elements" or "zeta modular forms", for integers r, r', c, d under the following assumptions (4.2.1) (4.2.2).

$$\text{(4.2.1)} \qquad \begin{array}{l} 1 \leqslant r \leqslant k-1, \ 1 \leqslant r' \leqslant k-1, \ \text{at least one of } r, r' \text{ is } k-1. \\ \text{If } r = k-2 \text{ and } r' = k-1, \text{ then } M \geqslant 2. \end{array}$$

$$\text{(4.2.2)} \qquad (c, M) = 1, \quad (d, N) = 1.$$

We also define elements

$$z_{M,N}(k,r,r') \in M_k(X(M,N))$$

which we call also zeta elements, for integer r, r' under the following assumption (4.2.3).

$$\text{(4.2.3)} \qquad (r,r') \text{ are as in (4.2.1), and } (r,r') \neq (2,k-1), (k-1,2), (k-1,k-2).$$

Under the assumption (4.2.3), we define $z_{M,N}(k,r,r')$ to be

$$(-1)^r \cdot (r-1)!^{-1} \cdot M^{k-r-2} N^{-r} \cdot F^{(k-r)}_{1/M,0} \cdot E^{(r)}_{0,1/N} \ \text{ if } \ r' = k-1,$$
$$(-1)^{r'} \cdot (k-2)!^{-1} \cdot M^{r'-k} N^{-r'} \cdot E^{(k-r')}_{1/M,0} \cdot E^{(r')}_{0,1/N} \ \text{ if } \ r = k-1.$$

(In the case $r = r' = k-1$, these definitions are compatible because $F^{(1)}_{\alpha,\beta} = E^{(1)}_{\alpha,\beta}$.) Under the assumptions (4.2.1) and (4.2.2), we define $_{c,d}z_{M,N}(k,r,r')$ by putting c, d to the above definitions. That is, we define $_{c,d}z_{M,N}(k,r,r')$ to be

$$(-1)^r \cdot (r-1)!^{-1} \cdot M^{k-r-2} N^{-r} \cdot {}_cF^{(k-r)}_{1/M,0} \cdot {}_dE^{(r)}_{0,1/N} \ \text{ if } \ r' = k-1,$$
$$(-1)^{r'} \cdot (k-2)!^{-1} \cdot M^{r'-k} N^{-r'} \cdot {}_cE^{(k-r')}_{1/M,0} \cdot {}_dE^{(r')}_{0,1/N} \ \text{ if } \ r = k-1.$$

Here

$$\begin{aligned}
{}_cF^{(h)}_{\alpha,\beta} &= c^2\, F^{(h)}_{\alpha,\beta} - c^{2-h}\, F^{(h)}_{c\alpha,c\beta} \quad (h \geqslant 1, \ \ (\alpha,\beta) \neq (0,0) \ \text{ if } \ h=2),\\
{}_cE^{(h)}_{\alpha,\beta} &= c^2 E^{(h)}_{\alpha,\beta} - c^h E^{(h)}_{c\alpha,c\beta} \quad (h \geqslant 1, \ \ h \neq 2),\\
{}_cE^{(2)}_{\alpha,\beta} &= c^2\, \widetilde{E}^{(2)}_{\alpha,\beta} - c^2\, \widetilde{E}^{(2)}_{c\alpha,c\beta}
\end{aligned}$$

(c is an integer which is prime to the orders of α, β).

Under the assumptions (4.2.2) and (4.2.3), we have

$$ {}_{c,d}z_{M,N}(k,r,r') = \Big(c^2 - c^u \cdot \big(\begin{smallmatrix} c & 0\\ 0 & 1\end{smallmatrix}\big)^*\Big)\Big(d^2 - d^v \cdot \big(\begin{smallmatrix} 1 & 0\\ 0 & d\end{smallmatrix}\big)^*\Big) \cdot z_{M,N}(k,r,r') $$

where

$$(u,v) = \begin{cases} (r+2-k, r) & \text{if } r' = k-1,\\ (k-r', r') & \text{if } r = k-1.\end{cases} \tag{4.2.4}$$

Proposition 4.3. — *Let M', $N' \geqslant 1$ and assume $M \mid M'$, $N \mid N'$. Assume further that*

$$\text{prime}(M) = \text{prime}(M'), \quad \text{prime}(N) = \text{prime}(N').$$

Then the trace map

$$M_k(X(M',N')) \longrightarrow M_k(X(M,N))$$

sends ${}_{c,d}z_{M',N'}(k,r,r')$ (resp. $z_{M',N'}(k,r,r')$) to ${}_{c,d}z_{M,N}(k,r,r')$ (resp. $z_{M,N}(k,r,r')$) for any r, r', c, d (resp. r, r') satisfying (4.2.1) *and* (4.2.2) *(resp.* (4.2.3)*).*

Proposition 4.4. — *Let ℓ be a prime number which does not divide M. Then the trace map*

$$M_k(X(M\ell,N\ell)) \longrightarrow M_k(X(M,N))$$

sends ${}_{c,d}z_{M\ell,N\ell}(k,r,r')$ to

$$\Big(1 - T'(\ell)\big(\begin{smallmatrix} 1/\ell & 0\\ 0 & 1\end{smallmatrix}\big)^* \cdot \ell^{-r} + \big(\begin{smallmatrix} 1/\ell & 0\\ 0 & 1/\ell\end{smallmatrix}\big)^* \cdot \ell^{k-1-2r}\Big) \cdot {}_{c,d}z_{M,N}(k,r,r')$$

in the case ℓ does not divide N, and to

$$\Big(1 - T'(\ell)\big(\begin{smallmatrix} 1/\ell & 0\\ 0 & 1\end{smallmatrix}\big)^* \cdot \ell^{-r}\Big) \cdot {}_{c,d}z_{M,N}(k,r,r')$$

in the case ℓ divides N, for any r, r', c, d satisfying (4.2.1), (4.2.2) *and* $(cd,\ell)=1$. *Here $T'(\ell)$ is the "dual Hecke operator" explained in 4.9 below. We have the result of the same form for $z_{M\ell,N\ell}(k,r,r')$ and $z_{M,N}(k,r,r')$ for any r, r' satisfying* (4.2.3).

The proofs of Prop. 4.3 and Prop. 4.4 will be given in 4.11–4.13.

4.5. We describe how these zeta elements are related to zeta functions.

Let $\lambda : E \to Y(M,N)$ be the universal elliptic curve, and define a local system $\mathcal{H}^1$ on $Y(M,N)(\mathbb{C})$ by

$$\mathcal{H}^1 = \mathrm{R}^1\lambda_*(\mathbb{Z}).$$

Then $\mathcal{H}^1$ is locally isomorphism to $\mathbb{Z}^2$. The cohomology group

$$\mathrm{H}^1\big(Y(M,N)(\mathbb{C}), \mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1)\big)$$

will be important for us. For any commutative ring A, let

$$\begin{aligned} V_{k,A}(Y(M,N)) &= \mathrm{H}^1\big(Y(M,N)(\mathbb{C}), \mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1) \otimes_{\mathbb{Z}} A\big), \\ V_{k,A}(X(M,N)) &= \mathrm{H}^1\big(X(M,N)(\mathbb{C}), j_* \,\mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1) \otimes_{\mathbb{Z}} A\big). \end{aligned} \tag{4.5.1}$$

where j is the inclusion map $Y(M,N) \to X(M,N)$. We consider the operator-valued zeta function

$$z_{M,N}(k,s) = \sum_{(n,M)=1} T'(n)\begin{pmatrix} 1/n & 0 \\ 0 & 1 \end{pmatrix}^* \cdot n^{-s}$$

($T'(n)$ is the dual Hecke operator explained in 4.9 below), acting on $V_{k,\mathbb{C}}(Y(M,N))$. (In the case $k=2$, $Z_{M,N}(k,s)$ coincides with $Z_{M,N}(s)$ in 2.5.) This converges absolutely when $\mathrm{Re}(s) > k$. The function $Z_{M,N}(k,s)$ has a presentation as an Euler product whose Euler factor at a prime number ℓ is

$$\Big(1 - T'(\ell)\begin{pmatrix} 1/\ell & 0 \\ 0 & 1 \end{pmatrix}^* \cdot \ell^{-s} + \begin{pmatrix} 1/\ell & 0 \\ 0 & 1/\ell \end{pmatrix}^* \cdot \ell^{k-1-2s}\Big)^{-1} \quad \text{if } (\ell, MN) = 1,$$

$$\Big(1 - T'(\ell)\begin{pmatrix} 1/\ell & 0 \\ 0 & 1 \end{pmatrix}^* \cdot \ell^{-s}\Big)^{-1} \quad \text{if } (\ell, M) = 1 \text{ and } \ell \mid N,$$

$$1 \text{ if } \ell \mid M.$$

The function $Z_{M,N}(k,s)$ has an analytic continuation to the whole $\mathbb{C}$ as an operator-valued meromorphic function in s and is holomorphic at $s \neq k$.

As is reviewed in 4.10 below, we have the period map

$$\mathrm{per}_{M,N} : M_k(X(M,N)) \longrightarrow V_{k,\mathbb{C}}(Y(M,N)).$$

As is explained in 4.7 below, we have special elements

$$\delta_{M,N}(k,j) \in V_{k,\mathbb{Z}}(Y(M,N)) \quad (1 \leqslant j \leqslant k-1).$$

(The element $\delta_{M,N} \in \mathrm{H}^1(Y(M,N)(\mathbb{C}), \mathbb{Z})$ defined in 2.7 coincides with $\delta_{M,N}(2,1)$.)

Let $\iota : V_{k,\mathbb{Z}}(Y(M,N)) \to V_{k,\mathbb{Z}}(Y(M,N))$ be the map induced by the complex conjugation on $Y(M,N)(\mathbb{C})$ and on $E(\mathbb{C})$, and we denote the $\mathbb{C}$-linear automorphism of $V_{k,\mathbb{C}}(Y(M,N))$ induced by ι by the same letter ι. For an element x of $V_{k,\mathbb{C}}(Y(M,N))$, let

$$x^{\pm} = \frac{1}{2}(1 \pm \iota)(x).$$

The following Thm. 4.6, which relates zeta elements to zeta values $Z_{M,N}(k,r)$ $(1 \leqslant r \leqslant k-1)$, is deduced from the work of Shimura [**Sh**]. We give the proof of Thm. 4.6 in § 7.

Theorem 4.6. — *Assume* $\mathrm{prime}(M) \subset \mathrm{prime}(N)$. *Then we have*

$$\left(\mathrm{per}_{M,N}(z_{M,N}(k,r,r'))\right)^{\pm} = Z_{M,N}(k,r)\cdot(2\pi i)^{k-r-1}\cdot\delta_{M,N}(k,r')^{\pm}$$

for any r, r' *satisfying* (4.2.3) *and for* $\pm = (-1)^{k-r-1}$, *and*

$$\left(\mathrm{per}_{M,N}({}_{c,d}z_{M,N}(k,r,r'))\right)^{\pm} = Z_{M,N}(k,r)\cdot(2\pi i)^{k-r-1}\cdot\gamma^{\pm}$$

for any r, r', c, d *satisfying* (4.2.1) *and* (4.2.2) *and for* $\pm = (-1)^{k-r-1}$, *where*

$$\gamma = \left(c^2 - c^u\cdot\left(\begin{smallmatrix} c & 0\\ 0 & 1\end{smallmatrix}\right)^*\right)\left(d^2 - d^v\cdot\left(\begin{smallmatrix} 1 & 0\\ 0 & d\end{smallmatrix}\right)^*\right)\delta_{M,N}(k,r')$$

with u, v *as in 4.2.*

4.7. The definition of the special cohomology class $\delta_{M,N}(k,j)$ $(1 \leqslant j \leqslant k-1)$ is as follows.

We have canonical isomorphisms

$$\begin{aligned}\mathrm{H}^1(Y(M,N)(\mathbb{C}), \mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1)) \quad &(= V_{k,\mathbb{Z}}(Y(M,N)))\\ &\cong \mathrm{H}_1(X(M,N)(\mathbb{C}), \{\mathrm{cusps}\}, \mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1))\\ &\cong \mathrm{H}_1(X(M,N)(\mathbb{C}), \{\mathrm{cusps}\}, \mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}_1))\end{aligned}$$

where $\mathcal{H}_1 = \mathcal{H}om(\mathcal{H}^1, \mathbb{Z})$, the first isomorphism is by Poincaré duality and the second isomorphism comes from the canonical isomorphism $\mathcal{H}^1 \cong \mathcal{H}_1$. Here we used the relative homology with coefficients which may be not a well known object. The definition is explained below.

We define $\delta_{M,N}(k,j) \in V_{k,\mathbb{Z}}(Y(M,N))$ to be the image of

$$\mathrm{class}(\varphi,\alpha) \in \mathrm{H}_1\big(X(M,N)(\mathbb{C}), \{\mathrm{cusps}\}, \mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}_1)\big)$$

under the above composite isomorphism, where φ is the continuous map

$$(0,\infty) \longrightarrow X(M,N)(\mathbb{C})\ ; \quad \varphi(y) = \nu(yi) \ \text{ for } \ 0 < y < \infty,$$

which is a route from a cusp to a cusp, and α is the following element of $\Gamma((0,\infty), \varphi^{-1}(\mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}_1))$. The stalk of $\varphi^{-1}(\mathcal{H}_1)$ at $y \in (0,\infty)$ is identified with $\mathrm{H}_1(\mathbb{C}/(\mathbb{Z}yi+\mathbb{Z}),\mathbb{Z}) = \mathbb{Z}yi + \mathbb{Z}$. The sheaf $\varphi^{-1}(\mathcal{H}_1)$ on $(0,\infty)$ is a constant sheaf of rank 2 with basis e_1, e_2, where the stalk of e_1 (resp. e_2) at $y \in (0,\infty)$ is yi (resp. 1) $\in \mathbb{Z}yi + \mathbb{Z}$. We define

$$\alpha = e_1^{j-1}e_2^{k-j-1}.$$

If M', $N' \geqslant 1$ and $M \mid M'$, $N \mid N'$, the trace map $V_{k,\mathbb{Z}}(Y(M',N')) \to V_{k,\mathbb{Z}}(Y(M,N))$ sends $\delta_{M',N'}(k,j)$ to $\delta_{M,N}(k,j)$.

In the above, the relative homology with coefficients is defined as follows. If X is a topological space, C is a closed subset of X, and $\mathcal{F}$ is a locally constant sheaf of finitely generated $\mathbb{Z}$-modules on $U = X \smallsetminus C$, $H_m(X,C,\mathcal{F})$ is the cohomology in degree $-m$ of the complex $R\mathrm{Hom}(R\Gamma(X, j_!\mathcal{H}om(\mathcal{F},\mathbb{Z})),\mathbb{Z})$, where j is the inclusion map $U \to X$. If we have a pair (φ,α) of a continuous map $\varphi : [0,1] \to X$ such that $\varphi^{-1}(U) = (0,1)$

and an element α of $\Gamma((0,1),\varphi^*\mathcal{F})$, we have $\mathrm{class}(\varphi,\alpha)\in H_1(X,C,\mathcal{F})$ to be the image of α under the composition

$$\Gamma((0,1),\varphi^*\mathcal{F})\simeq H_1([0,1],\{0,1\},\varphi^*\mathcal{F})\longrightarrow H_1(X,C,\mathcal{F}).$$

4.8. We give a preliminary to introduce Hecke operators.

Let $A\geqslant 1$. We define isomorphisms

$$\varphi_A^*=(\varphi_A^{-1})_*: M_k(X(M(A),N))\xrightarrow{\sim} M_k(X(M,N(A)))$$
$$(\varphi_A)_*=(\varphi_A^{-1})^*: M_k(X(M,N(A)))\xrightarrow{\sim} M_k(X(M(A),N))$$

and homomorphisms

$$\varphi_A^*=(\varphi_A^{-1})_*: V_{k,\mathbb{Z}}(Y(M(A),N))\xrightarrow{\sim} V_{k,\mathbb{Z}}(Y(M,N(A)))$$
$$(\varphi_A)_*=(\varphi_A^{-1})^*: V_{k,\mathbb{Z}}(Y(M,N(A)))\xrightarrow{\sim} V_{k,\mathbb{Z}}(Y(M(A),N)).$$

Here $X(M,N(A))$ (resp. $X(M(A),N)$) denotes the smooth compactification of the curve $Y(M,N(A))$ (resp. $Y(M(A),N)$) in 2.8. $V_{k,\mathbb{Z}}(Y(M(A),N))$ and $V_{k,\mathbb{Z}}(Y(M,N(A)))$ are defined in the evident way.

Let E_1 be the universal elliptic curve over $Y(M,N(A))$ and let E_2 be the universal elliptic curve over $Y(M(A),N)$. Then we have canonical homomorphism

$$E_1\longrightarrow\varphi_A^*(E_2),\quad E_2\longrightarrow(\varphi_A^{-1})^*(E_1)\tag{4.8.1}$$

which are isogenies of degree A. Hence the pull back by φ_A followed by the pull back by $E_1\to\varphi_A^*(E_2)$ (resp. the push down by $E_2\to(\varphi_A^{-1})^*E_1$ followed by the push down by φ_A^{-1}) gives homomorphisms

$$M_k(X(M(A),N))\longrightarrow M_k(X(M,N(A)))$$

and

$$V_{k,\mathbb{Z}}(Y(M(A),N))\longrightarrow V_{k,\mathbb{Z}}(Y(M,N(A)))$$

which we denote by φ_A^* (resp. $(\varphi_A^{-1})_*$). We have $\varphi_A^*=(\varphi_A^{-1})_*$. Similarly, the pull back by φ_A^{-1} followed by the pull back by $E_2\to(\varphi_A^{-1})^*(E_1)$ (resp. the push down by $E_1\to\varphi_A^*E_2$ followed by the push down by φ_A) gives homomorphisms

$$M_k(X(M,N(A)))\longrightarrow M_k(X(M(A),N))$$

and

$$V_{k,\mathbb{Z}}(Y(M,N(A)))\longrightarrow V_{k,\mathbb{Z}}(Y(M(A),N))$$

which we denote by $(\varphi_A^{-1})^*$ (resp. $(\varphi_A)_*$). We have $(\varphi_A^{-1})^*=(\varphi_A)_*$. Here for the pull back and push down for on the spaces M_k by

$$E_1\longrightarrow\varphi_A^*(E_2)\quad\text{and}\quad E_2\longrightarrow(\varphi_A^{-1})^*(E_1),$$

we use the definitions (3.1.2) of M_k, not the definition (3.1.1) of M_k.

We have also

$$\varphi_A^* \circ (\varphi_A^{-1})^* = (\varphi_A^{-1})_* \circ (\varphi_A)_* = A^{k-2}$$
$$(\varphi_A^{-1})^* \circ \varphi_A^* = (\varphi_A)_* \circ (\varphi_A^{-1})_* = A^{k-2}.$$

We have

$$(\varphi_A^* f)(\tau) = A^{k-1} f(A\tau) \quad \text{for any } f \in M_k(X(M(A), N)), \tag{4.8.2}$$

$$((\varphi_A^{-1})^* f)(\tau) = A^{-1} f(A^{-1}\tau) \quad \text{for any } f \in M_k\big(X(M, N(A))\big). \tag{4.8.3}$$

4.9. The Hecke operators $T(n)$ and the dual Hecke operators $T'(n)$ on $M_k(X(M,N))$ and on $V_{k,\mathbb{Z}}(Y(M,N))$ are defined for integers $n \geqslant 1$ which are prime to M as follows. (In the case $k = 2$, these operators on $V_{2,\mathbb{Z}}(Y(M,N))$ were already given in 2.9.)

First, $T(1) = T'(1) = 1$.

Next we give the definitions of $T(\ell)$ and $T'(\ell)$ for a prime number ℓ which does not divide M. Let $\mathrm{pr} : X(M, N(\ell)) \to X(M,N)$ and $\mathrm{pr}' : X(M(\ell), N) \to X(M,N)$ be the canonical projections. We define

$$T(\ell) = (\mathrm{pr}')_* \circ (\varphi_\ell^{-1})^* \circ \mathrm{pr}^*, \quad T'(\ell) = \mathrm{pr}_* \circ \varphi_\ell^* \circ \mathrm{pr}'^*.$$

If ℓ does not divide N, we have $T'(\ell) = T(\ell)\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1/\ell \end{smallmatrix}\right)^*$.

In the case n is a power ℓ^e ($e \geqslant 0$) of a prime number ℓ which does not divide M, $T(n)$ and $T'(n)$ are defined as follows. If $\ell \mid N$, $T(\ell^e) = T(\ell)^e$, $T'(\ell^e) = T'(\ell)^e$. If ℓ does not divide N, $T(\ell^e)$ and $T'(\ell^e)$ are defined inductively, by

$$T(\ell^{e+2}) = T(\ell)T(\ell^{e+1}) + \left(\begin{smallmatrix} 1/\ell & 0 \\ 0 & \ell \end{smallmatrix}\right)^* T(\ell^e) \cdot \ell^{k-1},$$
$$T'(\ell^{e+2}) = T'(\ell)T'(\ell^{e+1}) + \left(\begin{smallmatrix} 1/\ell & 0 \\ 0 & \ell \end{smallmatrix}\right)^* T'(\ell^e) \cdot \ell^{k-1}.$$

Finally, for $n = \prod_\ell \ell^{e(\ell)}$ ($e(\ell) \geqslant 0$) where ℓ ranges over all prime numbers which do not divide M, $T(n)$, $T'(n)$ are defined by

$$T(n) = \prod_\ell T(\ell^{e(\ell)}), \quad T'(n) = \prod_\ell T'(\ell^{e(\ell)}).$$

For any m, $n \geqslant 1$ which are prime to M, and for any $a \in (\mathbb{Z}/M)^\times$, $b \in (\mathbb{Z}/N)^\times$, the operators $T(m)$, $T(n)$, $T'(m)$, $T'(n)$, $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)^*$ commute with each other.

The similar definition gives Hecke operators on the compact support cohomology group

$$V_{k,A,c}(Y(M,N)) = \mathrm{H}^1_c(Y(M,N)(\mathbb{C}), \mathrm{Sym}^{k-2}_{\mathbb{Z}}(\mathcal{H}^1) \otimes_{\mathbb{Z}} A)$$

(A a commutative ring). In the Poincaré duality

$$V_{k,\mathbb{Q}}(Y(M,N)) \times V_{k,\mathbb{Q},c}(Y(M,N)) \longrightarrow \mathbb{Q}$$

induced by the canonical pairing $\mathcal{H}^1 \times \mathcal{H}^1 \to \mathbb{Z}$, $T(n)$ and $T'(n)$ are transposes of each other.

The operators $T(n)$ $((n,M)=1)$ on $M_k(X(M,N))$ are described also by using q-expansions. Let ℓ be a prime number which does not divide M. Let $f \in M_k(X(M,N)) \otimes \mathbb{C}$. Then the pull back of f on $\mathfrak{H}$ has the form

$$f = \sum_{n\in\mathbb{Z},n\geqslant 0} a_n q^{n/M} \quad (a_n \in \mathbb{C}).$$

Write

$$T(\ell)f = \sum_{n\in\mathbb{Z},n\geqslant 0} b_n q^{n/M}.$$

Then:

(4.9.1) *$b_n = a_{n\ell}$ if ℓ divides N or if ℓ does not divide n.*

(4.9.2) *Assume ℓ does not divide N and ℓ divides n, and assume $\left(\begin{smallmatrix} 1/\ell & 0 \\ 0 & \ell \end{smallmatrix}\right)^* f = \varepsilon(\ell) f$ for some $\varepsilon(\ell) \in \mathbb{C}^\times$. Then $b_n = a_{n\ell} + \varepsilon(\ell)\ell^{k-1}a_{n/\ell}$.*

In the case $M \geqslant 2$, the definitions of the operators $T(n)$ and $T'(n)$ in this paper differ from the definitions in some literatures. In the case $M = N$, the operators $T(n)\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)^* = T'(n)\left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right)^*$ $((n,N)=1)$ in our notation are the Hecke operators in Deligne [**De1**]. The advantages and the disadvantages of the operators $T(n)$, $T'(n)$, $T(n)\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)^*$, $T(n)\left(\begin{smallmatrix} 1/n & 0 \\ 0 & 1 \end{smallmatrix}\right)^*$ $((n,M)=1)$ are:

(4.9.3) *In the case $M = N$, $T(n)\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)^*$ and $T'(n)\left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right)^*$ commute with the action of $\mathrm{GL}_2(\mathbb{Z}/N)$. But they do not preserve the direct summands of $\mathrm{H}^1(Y(M,N)(\mathbb{C}),\mathbb{Z})$ corresponding to connected components of $Y(M,N)(\mathbb{C})$.*

(4.9.4) *$T(n)$ and $T'(n)$ preserve the direct summands of $\mathrm{H}^1(Y(M,N)(\mathbb{C}),\mathbb{Z})$ corresponding to connected components. But in the case $M = N$, they do not commute with the action of $\mathrm{GL}_2(\mathbb{Z}/N)$.*

4.10. We review the definition of the period map

$$M_k(X(M,N)) \longrightarrow V_{k,\mathbb{C}}(Y(M,N)).$$

We also review the period map

$$S_k(X(M,N)) \longrightarrow V_{k,\mathbb{C},c}(Y(M,N))$$

which we will use in § 7. We denote $X = X(M,N)$, $Y = Y(M,N)$. Let $j : Y(\mathbb{C}) \to X(\mathbb{C})$ be the inclusion map. Let $\overline{\lambda} : \overline{E} \to X$ be the smooth Néron model of the universal elliptic curve $\lambda : E \to Y$. We denote by $(\ )^{\mathrm{an}}$ the analytic objects associated to algebraic objects. For example, $\mathcal{O}_X^{\mathrm{an}}$ denotes the sheaf of holomorphic functions on $X(\mathbb{C})$. We have a homomorphism

$$\mathrm{coLie}(\overline{E})^{\mathrm{an}} \longrightarrow \mathcal{O}_X^{\mathrm{an}} \otimes_{\mathbb{Z}} j_*(\mathcal{H}^1) \tag{4.10.1}$$

on $X(\mathbb{C})$, as the connecting homomorphism of the exact sequence

$$0 \longrightarrow (\overline{\lambda}^{\mathrm{an}})^{-1}(\mathcal{O}_X^{\mathrm{an}}) \longrightarrow \mathcal{O}_{\overline{E}}^{\mathrm{an}} \xrightarrow{\mathrm{d}} (\Omega^1_{\overline{E}/X})^{\mathrm{an}} \longrightarrow 0$$

on $\overline{E}(\mathbb{C})$. (4.10.1) induces

$$(4.10.2) \qquad \left(\mathrm{coLie}(\overline{E})^{\otimes(k-2)}\right)^{\mathrm{an}} \longrightarrow \mathcal{O}_X^{\mathrm{an}} \otimes_{\mathbb{Z}} j_*(\mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1)).$$

On the other hand, consider the exact sequence

$$0 \longrightarrow j_!\mathbb{C} \longrightarrow \mathcal{O}_X^{\mathrm{an}}(-\mathrm{cusps}) \xrightarrow{\mathrm{d}} (\Omega^1_{X/\mathbb{Q}})^{\mathrm{an}} \longrightarrow 0$$

on $X(\mathbb{C})$ where $\mathcal{O}_X^{\mathrm{an}}(-\mathrm{cusps})$ denotes the subsheaf of $\mathcal{O}_X^{\mathrm{an}}$ consisting of functions which have zero at all cusps. This exact sequence tensored with $j_*\,\mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1)$ gives the connecting maps

$$(4.10.3) \quad \mathrm{H}^0\big(X(\mathbb{C}), (\Omega^1_{X/\mathbb{Q}})^{\mathrm{an}} \otimes_{\mathbb{Z}} j_*\,\mathrm{Sym}^{k-2}(\mathcal{H}^1)\big) \\ \longrightarrow \mathrm{H}^1(X(\mathbb{C}), j_!\,\mathrm{Sym}^{k-2}(\mathcal{H}^1)) \otimes_{\mathbb{Z}} \mathbb{C} = V_{k,\mathbb{C},c}(Y),$$

$$(4.10.4) \quad \mathrm{H}^0\big(Y(\mathbb{C}), (\Omega^1_{Y/\mathbb{Q}})^{\mathrm{an}} \otimes_{\mathbb{Z}} \mathrm{Sym}^{k-2}(\mathcal{H}^1)\big) \\ \longrightarrow \mathrm{H}^1(Y(\mathbb{C}), \mathrm{Sym}^{k-2}(\mathcal{H}^1)) \otimes_{\mathbb{Z}} \mathbb{C} = V_{k,\mathbb{C}}(Y).$$

The period maps are defined as the composite maps

$$\begin{aligned} S_k(X(M,N)) &= \mathrm{H}^0\big(X, (\Omega^1_{X/\mathbb{Q}}) \otimes_{\mathcal{O}_X} \mathrm{coLie}(\overline{E})^{\otimes(k-2)}\big) \\ &\xrightarrow{(4.10.2)} \mathrm{H}^0(X(\mathbb{C}), (\Omega^1_{X/\mathbb{Q}})^{\mathrm{an}} \otimes_{\mathbb{Z}} j_*\,\mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1)) \otimes_{\mathbb{Z}} \mathbb{C} \\ &\xrightarrow{(4.10.3)} V_{k,\mathbb{C},c}(Y). \end{aligned}$$

$$\begin{aligned} M_k(X(M,N)) &\subset \mathrm{H}^0\big(Y, (\Omega^1_{Y/\mathbb{Q}}) \otimes_{\mathcal{O}_Y} \mathrm{coLie}(E)^{\otimes(k-2)}\big) \\ &\xrightarrow{(4.10.2)} \mathrm{H}^0(Y(\mathbb{C}), (\Omega^1_{Y/\mathbb{Q}})^{\mathrm{an}} \otimes_{\mathbb{Z}} \mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1)) \otimes_{\mathbb{Z}} \mathbb{C} \\ &\xrightarrow{(4.10.4)} V_{k,\mathbb{C}}(Y). \end{aligned}$$

4.11. We prove Prop. 4.3

It is enough to prove the case $M' = M$ and the case $N' = N$. Since both cases are proved similarly, we give only the proof of the case $N = N'$. In this case, our task is to prove that if $M \mid M'$ and $\mathrm{prime}(M) = \mathrm{prime}(M')$, the trace map $M_h(Y(M',N)) \to M_h(Y(M,N))$ ($h \geqslant 1$) sends $(M')^{h-2}F^{(h)}_{1/M',0}$ to $M^{h-2}F^{(h)}_{1/M,0}$ (we assume $M \geqslant 2$ in the case $h = 2$) and $(M')^{-h}E^{(h)}_{1/M',0}$ to $M^{-h}E^{(h)}_{1/M,0}$ (in the case $h = 2$, we replace E by $\widetilde{E}$). These are proved by the same arguments as in 2.11 (we use Lemma 3.7 in place of Lemma 1.7.)

We give a preliminary lemma for the proof of prop. 4.4.

Lemma 4.12. — *Let $(\alpha,\beta) \in (\mathbb{Q}/\mathbb{Z})^2$, and let $A \geqslant 1$, $h \geqslant 1$. Then:*

(1) $\varphi_A^*(E^{(h)}_{\alpha,\beta}) = \sum_{\substack{\beta' \in \mathbb{Q}/\mathbb{Z} \\ A\beta' = \beta}} A^{-1}E^{(h)}_{\alpha,\beta'}$. *Here in the case $h = 2$, we replace E by $\widetilde{E}$.*

(2) $\varphi_A^*(F^{(h)}_{\alpha,\beta}) = \sum_{\substack{\beta'\in\mathbb{Q}/\mathbb{Z}\\ A\beta'=\beta}} A^{h-2}F^{(h)}_{\alpha,\beta'}$. *Here in the case* $h = 2$, *we assume* $(\alpha,\beta)\neq(0,0)$.

Proof. — We prove (1). From the analytic descriptions of Eisenstein series in 3.8, we obtain

$$E^{(h)}_{\alpha,\beta}(A\tau) = A^{-h}\sum_{\substack{\beta'\in\mathbb{Q}/\mathbb{Z}\\ A\beta'=\beta}} E^{(h)}_{\alpha,\beta'}(\tau) \quad (\tau\in\mathfrak{H}).$$

(1) follows from this and from (4.8.2). (2) follows from (1). □

4.13. We prove Prop. 4.4. Since the proof for zeta elements with c, d is similar to that without c, d, we give here only the proof for zeta elements without c, d. Let $1\leqslant j\leqslant k-1$. Assume that $j\neq 2$ and that $M\geqslant 2$ in the case $j=k-2$ (resp. Assume that $j\neq 2$, $j\neq k-2$). By 4.12 and by similar arguments as in 2.13, we have that the trace map $M_k(M\ell,N\ell)\to M_k(X(M\ell,N))$ sends

$$F^{(k-j)}_{1/M\ell,0}\cdot E^{(j)}_{0,1/N\ell} \quad (\text{resp. } E^{(k-j)}_{1/M\ell,0}\cdot E^{(j)}_{0,1/N\ell})$$

to

$$(1) \qquad (\ell^{j+2-k}F^{(k-j)}_{1/M,0} - \ell^{j+2-k}\varphi_\ell^* F^{(k-j)}_{\alpha,0}\ \ )\cdot E^{(j)}_{0,1/N\ell}$$
$$(\text{resp. } (\ell^{k-j}E^{(k-j)}_{1/M,0} - \ell\varphi_\ell^* E^{(k-j)}_{\alpha,0}\ \ )\cdot E^{(j)}_{0,1/N\ell}).$$

Here α is the unique element of $\frac{1}{M}\mathbb{Z}/\mathbb{Z}$ such that $\ell\alpha = 1/M$. The trace map $M_k(X(M,N\ell))\to M_k\big(X(M,N(\ell))\big)$ sends the element (1) to

$$(2) \qquad (\ell^{j+2-k}F^{(k-j)}_{1/M,0} - \ell^{j+2-k}\varphi_\ell^* F^{(k-j)}_{\alpha,0}\ \ )\cdot \ell\varphi_\ell^* E^{(j)}_{0,1/N}$$
$$(\text{resp. } (\ell^{k-j}E^{(k-j)}_{1/M,0} - \ell\varphi_\ell^* E^{(k-j)}_{\alpha,0}\ \ )\cdot \ell\varphi_\ell^* E^{(j)}_{0,1/N})$$

in the case $\ell\mid N$, and to

$$(2') \qquad (\ell^{j+2-k}F^{(k-j)}_{1/M,0} - \ell^{j+2-k}\varphi_\ell^* F^{(k-j)}_{\alpha,0}\ \ )\cdot (\ell\varphi_\ell^* E^{(j)}_{0,1/N} - E^{(j)}_{0,\beta})$$
$$(\text{resp. } (\ell^{k-j}E^{(k-j)}_{1/M,0} - \ell\varphi_\ell^* E^{(k-j)}_{\alpha,0}\ \ )\cdot (\ell\varphi_\ell^* E^{(j)}_{0,1/N} - E^{(j)}_{0,\beta})$$

in the case $(\ell,N)=1$ where β denotes the unique element of $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$ such that $\ell\beta = 1/N$. Concerning the trace map $\mathrm{pr}_*: M_h\big(X(M,N(\ell))\big)\to M_h(X(M,N))$ $(h\geqslant 1)$, we have

(4.13.1) In the case $\ell\mid N$, we have for $h\geqslant 1$

$$\ell\,\mathrm{pr}_*\varphi_\ell^* E^{(h)}_{0,1/N} = \ell^h E^{(h)}_{0,1/N}$$

(we replace $E^{(2)}$ by $\widetilde{E}^{(2)}$ in the case $h=2$).

(4.13.2) In the case ℓ does not divide N, we have for $h \geqslant 1$

$$\ell^{2-h}\mathrm{pr}_*\varphi_\ell^* F_{\alpha,0}^{(h)} = \ell^{2-h} F_{1/M,0}^{(h)} + \ell F_{\alpha,0}^{(h)}$$
$$\ell \mathrm{pr}_*\varphi_\ell^* E_{\alpha,0}^{(h)} = \ell^h E_{1/M,0}^{(h)} + \ell E_{\alpha,0}^{(h)}$$

(we replace E by $\widetilde{E}$ in the case $h = 2$),

$$\ell\, \mathrm{pr}_*\varphi_\ell^* E_{0,1/N}^{(h)} = \ell^h E_{0,1/N}^{(h)} + \ell E_{0,\beta}^{(h)}$$

(we replace E by $\widetilde{E}$ in the case $h = 2$).

(These (4.13.1) and (4.13.2) are proved by the arguments similar to the proofs of (2.13.1) and (2.13.2).) By (4.8.2), we have

$$\varphi_\ell^*(fg) = \ell \varphi_\ell^*(f) \cdot \varphi_\ell^*(g) \tag{4.13.3}$$

for $f \in M_h(X(M(\ell), N))$, $g \in M_{h'}(X(M(\ell), N))$ (h, $h' \geqslant 1$). By using (4.13.1)–(4.13.3), we obtain the following: in the case $\ell \mid N$, the trace map $M_k\big(X(M, N(\ell))\big) \to M_k(X(M, N))$ sends the element (2) to

$$\begin{aligned}
&\ell^{j+2-k} F_{1/M,0}^{(k-j)} \cdot \ell\, \mathrm{pr}_*\varphi_\ell^* E_{0,1/N}^{(j)} - \ell^{j+2-k} T'(\ell)\big(F_{\alpha,0}^{(k-j)} \cdot E_{0,1/N}^{(j)}\big) \\
&\quad = (-1)^j \cdot (j-1)! \cdot (M\ell)^{j+2-k} \cdot (N\ell)^j \\
&\qquad\qquad \big(z_{M,N}(k, j, k-1) - \ell^{-j} T'(\ell)\big(\begin{smallmatrix} 1/\ell & 0 \\ 0 & 1 \end{smallmatrix}\big)^* z_{M,N}(k, k-1, j)\big)
\end{aligned}$$

$$\begin{aligned}
&(\text{resp. } \ell^{k-j} E_{1/M,0}^{(k-j)} \cdot \ell\, \mathrm{pr}_*\varphi_\ell^* E_{0,1/N}^{(j)} - \ell^j T'(\ell)\big(E_{\alpha,0}^{(k-j)} \cdot E_{0,1/N}^{(j)}\big) \\
&\quad = (-1)^j \cdot (k-2)! \cdot (M\ell)^{k-j} \cdot (N\ell)^j \\
&\qquad\qquad \big(z_{M,N}(k, k-1, j) - \ell^{1-k} T'(\ell)\big(\begin{smallmatrix} 1/\ell & 0 \\ 0 & 1 \end{smallmatrix}\big)^* z_{M,N}(k, k-1, j)\big)\ ).
\end{aligned}$$

In the case ℓ does not divide N, the trace map $M_k\big(X(M, N(\ell))\big) \to M_k(X(M, N))$ sends the element (2') to

$$\begin{aligned}
&\ell^{j+2-k} F_{1/M,0}^{(k-j)} \cdot \ell \mathrm{pr}_*\varphi_\ell^* E_{0,1/N}^{(j)} - \ell^{j+2-k} T'(\ell)\big(F_{\alpha,0}^{(k-j)} \cdot E_{0,1/N}^{(j)}\big) \\
&\qquad\qquad - (\ell+1)\ell^{j+2-k} F_{1/M,0}^{(k-j)} \cdot E_{0,\beta}^{(j)} + \big(\ell^{j+2-k}\mathrm{pr}_*\varphi_\ell^* F_{\alpha,0}^{(k-j)}\big) \cdot E_{0,\beta}^{(j)} \\
&\quad = (-1)^j \cdot (j-1)! \cdot (M\ell)^{j+2-k} \cdot (N\ell)^j \\
&\qquad \big(z_{M,N}(k, j, k-1) - \ell^{-j} T'(\ell)\big(\begin{smallmatrix} 1/\ell & 0 \\ 0 & 1 \end{smallmatrix}\big)^* z_{M,N}(k, j, k-1) \\
&\qquad\qquad + \big(\begin{smallmatrix} 1/\ell & 0 \\ 0 & 1/\ell \end{smallmatrix}\big)^* \cdot \ell^{k-1-2j} \cdot z_{M,N}(k, j, k-1)\big)
\end{aligned}$$

$$\begin{aligned}(\text{resp.}\quad & \ell^{k-j}E^{(k-j)}_{1/M,0}\cdot \ell\mathrm{pr}_*\varphi_\ell^* E^{(j)}_{0,1/N} - \ell T'(\ell)\big(E^{(k-j)}_{\alpha,0}\cdot E^{(j)}_{0,1/N}\big)\\ &\qquad - (\ell+1)\ell^{k-j}E^{(k-j)}_{1/M,0}\cdot E^{(j)}_{0,\beta} + \big(\ell\mathrm{pr}_*\varphi_\ell^* E^{(k-j)}_{\alpha,0}\big)\cdot E^{(j)}_{0,\beta}\\ &= (-1)^j\cdot(k-2)!\cdot(M\ell)^{k-j}\cdot(N\ell)^j\\ &\qquad \big(z_{M,N}(k,k-1,j) - \ell^{1-k}T'(\ell)\left(\begin{smallmatrix}1/\ell & 0\\ 0 & 1\end{smallmatrix}\right)^* z_{M,N}(k,k-1,j)\\ &\qquad\qquad + \left(\begin{smallmatrix}1/\ell & 0\\ 0 & 1/\ell\end{smallmatrix}\right)^*\cdot\ell^{1-k}\cdot z_{M,N}(k,k-1,j)\big)\,.\end{aligned}$$

This proves Prop. 4.4.

5. Euler systems on $X_1(N)\otimes\mathbb{Q}(\zeta_m)$.

Let

$$X_1(N) = X(1,N),\quad Y_1(N) = Y(1,N).$$

The total constant fields of these curves are $\mathbb{Q}$.

We will always identify $X_1(N)\otimes\mathbb{Q}(\zeta_m)$ ($m\geqslant 1$) with the quotient of $X(L)$ ($m\mid L$, $N\mid L$, $L\geqslant 3$) by the action of the group

$$\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(\mathbb{Z}/L)\,;\quad c\equiv 0,\quad d\equiv 1 \bmod N,\quad ad-bc\equiv 1 \bmod m\right\}.$$

Hence $Y_1(N)\otimes\mathbb{Q}(\zeta_m)$ is regarded as a quotient of $Y(m,L)$ for any $L\geqslant 1$ such that $m\mid L$, $N\mid L$.

Fix N, $m\geqslant 1$.

In this section, we define zeta elements in $\mathrm{K}_2(Y_1(N)\otimes\mathbb{Q}(\zeta_m))\otimes\mathbb{Q}$ and zeta elements in $M_k(X_1(N))\otimes\mathbb{Q}(\zeta_m)$ ($k\geqslant 2$).

5.1. Let ξ and S be as in the following (5.1.1).

(5.1.1) *Either ξ is a symbol $a(A)$ where a, $A\in\mathbb{Z}$, $A\geqslant 1$ and S is a non-empty finite set of primes containing* prime(mA), *or ξ is an element of* $\mathrm{SL}_2(\mathbb{Z})$ *and S is a non-empty finite set of prime numbers containing* prime(mN).

We define zeta elements

$$z_{1,N,m}(\xi,S)\in \mathrm{K}_2(Y_1(N)\otimes\mathbb{Q}(\zeta_m))\otimes\mathbb{Q}$$

as follows.

First we define $z_{1,N,m}(\xi,S)$ in the case ξ is a symbol $a(A)$ for a, $A\in\mathbb{Z}$, $A\geqslant 1$. Take $M\geqslant 1$, $L\geqslant 4$ such that

$$mA\mid M,\quad N\mid L,\quad M\mid L,\quad \mathrm{prime}(M)=S,\quad \mathrm{prime}(L)=S\cup\mathrm{prime}(N).$$

As is explained below, there exists a unique morphism of schemes

$$Y(M,L)\longrightarrow Y_1(N)\otimes\mathbb{Q}(\zeta_m)\tag{5.1.2}$$

which is compatible with $\mathfrak{H} \to \mathfrak{H}$; $\tau \mapsto A^{-1}(\tau + a)$. Let

$$t_{m,a(A)} : \mathrm{K}_2(Y(M,L)) \longrightarrow \mathrm{K}_2(Y_1(N) \otimes \mathbb{Q}(\zeta_m))$$

be the norm map associated to (5.1.2). We define

$$z_{1,N,m}(a(A),S) = t_{m,a(A)}(z_{M,L}).$$

By Prop. 2.3, $z_{1,N,m}(a(A),S)$ is independent of the choices of M, L.

The morphism (5.1.2) is obtained as the composition

$$Y(M,L) \xrightarrow{\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)} Y(M,L) \longrightarrow Y(m(A),L) \xrightarrow[\sim]{\varphi_A^{-1}} Y(m,L(A)) \longrightarrow Y_1(N) \otimes \mathbb{Q}(\zeta_m)$$

where $\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)$ denotes the automorphism of $Y(M,L)$ induced by the automorphism $\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)$ of $Y(L)$.

As we will see in 5.10, $z_{1,N,m}(a(A),S)$ does not change when we replace $a(A)$ by $a'(A)$ for any integer a' such that $a \equiv a' \bmod A$.

Next we define $z_{1,N,m}(\xi,S)$ in the case $\xi \in \mathrm{SL}_2(\mathbb{Z})$. Take $L \geqslant 3$ such that

$$m \mid L, \quad N \mid L, \quad \mathrm{prime}(L) = S.$$

We define $z_{1,N,m}(\xi,S)$ to be the image of $\xi^*(z_{L,L})$ under the norm map $\mathrm{K}_2(Y(L)) \to \mathrm{K}_2(Y_1(N) \otimes \mathbb{Q}(\zeta_m))$ associated to the canonical projection $Y(L) \to Y_1(N) \otimes \mathbb{Q}(\zeta_m)$. By Prop. 2.3, $z_{1,N,m}(\xi,S)$ is independent of the choice of L.

We have also the "with c,d-version" of the above zeta elements, which belong to $\mathrm{K}_2(Y_1(N) \otimes \mathbb{Q}(\zeta_m))$ without $\otimes\mathbb{Q}$, by replacing $z_{M,L}$ in the above definitions by ${}_{c,d}z_{m,L}$. But we will not discuss about these elements, for we will not use them in this paper. Zeta elements with $\xi = a(A)$ and zeta elements with $\xi \in \mathrm{SL}_2(\mathbb{Z})$ will play different roles; see the end of 8.1.

5.2. Let $k \geqslant 2$. Let ξ, S be as in (5.1.1). We define the following elements of $M_k(X_1(N) \otimes \mathbb{Q}(\zeta_m)) = M_k(X_1(N)) \otimes \mathbb{Q}(\zeta_m)$;

$${}_{c,d}z_{1,N,m}(k,r,r',\xi,S)$$

for integers r, r', c, d satisfying (5.2.1) and (5.2.2) below, and elements

$$z_{1,N,m}(k,r,r',\xi,S)$$

for integers r, r' satisfying (5.2.3) below.

(5.2.1) $1 \leqslant r \leqslant k-1$, $1 \leqslant r' \leqslant k-1$, at least one of r, r' is $k-1$.

(5.2.2) $\mathrm{prime}(cd) \cap S = \varnothing$, and $(d,N) = 1$.

(5.2.3) r and r' are as in (5.2.1) and satisfy

$$(r,r') \neq (2,k-1),\ (k-1,2),\ (k-1,k-2).$$

First we define these elements in the case $\xi = a(A)$.

Take $M \geqslant 1$, $L \geqslant 4$ such that

$$mA \mid M, \quad N \mid L, \quad M \mid L, \quad \mathrm{prime}(M) = S, \quad \mathrm{prime}(L) = S \cup \mathrm{prime}(N).$$

Let

$$t_{m,a(A)} : M_k(X(M,L)) \longrightarrow M_k(X_1(N) \otimes \mathbb{Q}(\zeta_m))$$

be the composite map

$$M_k(X(M,L)) \xrightarrow{\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)_*} M_k(X(M,L)) \xrightarrow{\text{trace}}$$
$$M_k(X(m(A),L)) \xrightarrow[\sim]{(\varphi_A^{-1})_*} M_k\big(X(m,L(A))\big) \xrightarrow{\text{trace}} M_k(X_1(N) \otimes \mathbb{Q}(\zeta_m)).$$

We define

$${}_{c,d}z_{1,N,m}(k,r,r',a(A),S) = t_{m,a(A)}({}_{c,d}z_{m,L}(k,r,r')),$$

for r, r', c, d as in (5.2.1) (5.2.2), and

$$z_{1,N,m}(k,r,r',a(A),S) = t_{m,a(A)}(z_{m,L}(k,r,r')).$$

for r, r' as in (5.2.3). By Prop. 4.3, these elements are independent of the choices of M, L, as above.

As we will see in 5.10, these zeta elements do not change if we replace $a(A)$ by $a'(A)$ for any integer a' such that $a \equiv a'$ mod A.

Next we define the zeta elements in the case $\xi \in \mathrm{SL}_2(\mathbb{Z})$. Take $L \geqslant 3$ such that

$$m \mid L, \quad N \mid L, \quad \text{prime}(L) = S.$$

We define ${}_{c,d}z_{1,N,m}(k,r,r',\xi,S)$ for r, r', c, d as in (5.2.1) (5.2.2) (resp. $z_{1,N,m}(k,r,r',\xi,S)$ for r, r' as in (5.2.3)) to be the image of $\xi^*({}_{c,d}z_{L,L}(k,r,r'))$ (resp. $\xi^*(z_{L,L}(k,r,r'))$) under the trace map

$$M_k(X(L)) \longrightarrow M_k(X_1(N) \otimes \mathbb{Q}(\zeta_m)).$$

By Prop. 4.3, these elements are independent of the choice of L as above.

Let $N' \geqslant 1$ be a multiple of N. In the case $\xi \in \mathrm{SL}_2(\mathbb{Z})$, assume $\text{prime}(N') \subset S$. Then the trace map

$$M_k(X_1(N') \otimes \mathbb{Q}(\zeta_m)) \longrightarrow M_k(X_1(N) \otimes \mathbb{Q}(\zeta_m))$$

sends ${}_{c,d}z_{1,N',m}(k,r,r',\xi,S)$ with $(d,N') = 1$ (resp. $z_{1,N',m}(k,r,r',\xi,S)$) to ${}_{c,d}z_{1,N,m}(k,r,r',\xi,S)$ (resp. $z_{1,N,m}(k,r,r',\xi,S)$).

Proposition 5.3. — *Let ξ and S be as in* (5.1.1), *let $m' \geqslant 1$, and let S' be a finite set of prime numbers such that $S \cup \text{prime}(m') \subset S'$.*

(1) *The norm map*

$$\mathrm{K}_2(Y_1(N) \otimes \mathbb{Q}(\zeta_{m'})) \otimes \mathbb{Q} \longrightarrow \mathrm{K}_2(Y_1(N) \otimes \mathbb{Q}(\zeta_m)) \otimes \mathbb{Q}$$

sends $z_{1,N,m'}(\xi,S')$ to

$$\Big(\prod_{\ell \in S'-S} (1 - T'(\ell)\sigma_\ell^{-1} + \Delta'(\ell)\sigma_\ell^{-2} \cdot \ell)\Big) \cdot z_{1,N,m}(\xi,S),$$

where $\sigma_\ell \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$; $\sigma_\ell(\zeta_m) = \zeta_m^\ell$, $\Delta'(\ell)$ *denotes* $\left(\begin{smallmatrix}\ell & 0\\ 0 & 1/\ell\end{smallmatrix}\right)^*$ *in the case* ℓ *does not divide* N, *and* $\Delta'(\ell) = 0$ *in the case* ℓ *divides* N.

(2) *Let* $k \geqslant 2$, *and let* r, r', c, d *be as in* (5.2.1) (5.2.2), *and assume* $\mathrm{prime}(cd) \cap S' = \varnothing$. *Then the trace map*

$$M_k(X_1(N) \otimes \mathbb{Q}(\zeta_{m'})) \longrightarrow M_k(X_1(N) \otimes \mathbb{Q}(\zeta_m))$$

sends ${}_{c,d}z_{1,N,m'}(k,r,r',\xi,S')$ *to*

$$\Big(\prod_{\ell \in S'-S} (1 - T'(\ell)\sigma_\ell^{-1} \cdot \ell^{-r} + \Delta'(\ell)\sigma_\ell^{-2} \cdot \ell^{k-1-2r})\Big) \cdot {}_{c,d}z_{1,N,m}(k,r,r',\xi,S)$$

where $\Delta'(\ell)$ *denotes* $\left(\begin{smallmatrix}\ell & 0\\ 0 & 1/\ell\end{smallmatrix}\right)^*$ *in the case* ℓ *does not divide* N, *and* $\Delta'(\ell) = 0$ *in the case* ℓ *divides* N. *We have the result of the same form for* $z_{1,N,m'}(k,r,r',\xi,S')$ *and* $z_{1,N,m}(k,r,r',\xi,S)$ *for any integers* r, r' *satisfying* (5.2.3).

For the proof, see 5.7.

5.4. In 4.5, we defined the space $V_{k,\mathbb{Q}}(Y_1(N))$ ($k \geqslant 2$) in the case $N \geqslant 4$. We extend the definition including the cases $N = 1, 2, 3$. In general, for a curve Y of the form $G\backslash Y(N)$ with $N \geqslant 3$ and G a subgroup of $\mathrm{GL}_2(\mathbb{Z}/N)$, we define the space $V_{k,\mathbb{Q}}(Y)$ to be the G-fixed part of $V_{k,\mathbb{Q}}(Y(N))$. (This definition makes sense since in the case N, $N' \geqslant 3$ and $N \mid N'$, $V_{k,\mathbb{Q}}(Y(N))$ coincides with the $\mathrm{Gal}(Y(N')/Y(N))$-fixed part of $V_{k,\mathbb{Q}}(Y(N'))$.) In the case $k = 2$, $V_{2,\mathbb{Q}}(Y)$ is simply $\mathrm{H}^1(Y(\mathbb{C}), \mathbb{Q})$. For a commutative ring A over $\mathbb{Q}$, let

$$V_{k,A}(Y) = V_{k,\mathbb{Q}}(Y) \otimes_{\mathbb{Q}} A.$$

We used dual Hecke operators in 5.3, but in fact we have explained them in § 2, § 4 only in the case $N \geqslant 4$. Including the cases $N = 1, 2, 3$, the operators $T(n)$, $T'(n)$ ($n \geqslant 1$) on $M_k(X_1(N))$ ($k \in \mathbb{Z}$), $V_{k,\mathbb{Q}}(Y_1(N))$ ($k \geqslant 2$), $\mathrm{K}_2(Y_1(N) \otimes K)) \otimes \mathbb{Q}$ for a field $K \supset \mathbb{Q}$ are defined as follows. Take M, $L \geqslant 1$, such that $M + L \geqslant 5$, $(n, M) = 1$, $N \mid L$, $M \mid L$, $L \mid MN$. Then

$$X(M,L) \longrightarrow X(1,N) = X_1(N)$$

is a Galois covering allowing ramification. The operators $T(n)\left(\begin{smallmatrix}n & 0\\ 0 & 1\end{smallmatrix}\right)$ and $T'(n)\left(\begin{smallmatrix}1/n & 0\\ 0 & 1\end{smallmatrix}\right)$ on $M_k(X(M,L))$, $V_{k,\mathbb{Q}}(Y(M,L))$, $\mathrm{K}_2(Y(M,L) \otimes K) \otimes \mathbb{Q}$ are invariant under the action of $\mathrm{Gal}(X(M,L)/X_1(N))$, and hence induce operators $T(n)$ and $T'(n)$ on the $\mathrm{Gal}(X(M,L)/X_1(N))$-invariant parts $M_k(X_1(N))$, $V_{k,\mathbb{Q}}(Y_1(N))$, $\mathrm{K}_2(Y_1(N) \otimes K) \otimes \mathbb{Q}$, respectively. These last operators $T(n)$ and $T'(n)$ are independent of the choices of M, L. If $N \geqslant 4$, these $T(n)$, $T'(n)$ coincide with the ones given in § 2 and § 4.

For a subfield K of $\mathbb{C}$, the regulator map

$$\mathrm{reg}_{1,N} : \mathrm{K}_2(Y_1(N) \otimes K) \otimes \mathbb{Q} \longrightarrow \mathrm{H}^1(Y_1(N)(\mathbb{C}), \mathbb{R} \cdot i)$$

and the period map

$$\mathrm{per}_{1,n} : M_k(X_1(N)) \otimes K \longrightarrow V_{k,\mathbb{C}}(Y_1(N)) \quad (k \geqslant 2)$$

are defined also for any $N \geqslant 1$; They are induced from those of $Y_1(L)$, $N \mid L$, $L \geqslant 4$. They commute with $T(n)$, $T'(n)$ $(n \geqslant 1)$.

5.5. We define special elements

$$\delta_{1,N}(k,j,a(A)), \quad \delta_{1,N}(k,j,\alpha) \in V_{k,\mathbb{Q}}(Y_1(N))$$

$(1 \leqslant j \leqslant k-1,\ a,\ A \in \mathbb{Z},\ A \geqslant 1,\ \alpha \in \mathrm{SL}_2(\mathbb{Z}))$.

First we define $\delta_{1,N}(k,j,a(A))$. Take $L \geqslant 4$ such that $N \mid L$. Consider the continuous map

$$\varphi_{a(A)} : (0,\infty) \longrightarrow Y_1(L)(\mathbb{C})\,; \quad y \longmapsto \nu(A^{-1}(yi+a)).$$

Then $\varphi_{a(A)}$ is a route from a cusp to a cusp. The stalk of $\varphi_{a(A)}^{-1}(\mathcal{H}_1)$ at $y \in (0,\infty)$ is identified with $\mathbb{Z}\cdot A^{-1}(yi+a)+\mathbb{Z}$. Let β_1 (resp. β_2) be the element of $\Gamma((0,\infty),\varphi_{a(A)}^{-1}(\mathcal{H}_1))$ whose stalk at $y \in (0,\infty)$ is yi (resp. 1) $\in \mathbb{Z}\cdot A^{-1}(yi+a)+\mathbb{Z}$. Then $\delta_{1,N}(k,j,a(A))$ is defined to be the image of the class$(\varphi_{a(A)}, \beta_1^{j-1}, \beta_2^{k-j-1})$ under

$$\mathrm{H}_1(X_1(L)(\mathbb{C}),\{\text{cusps}\},\mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}_1)) \cong \mathrm{H}^1(X_1(L)(\mathbb{C}),\mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1)) \xrightarrow{\text{trace}} V_{k,\mathbb{Q}}(Y_1(N)).$$

Then $\delta_{1,N}(k,j,a(A))$ is independent of the choice of L. In particular, $\delta_{1,N}(2,1,a(A)) \in \mathrm{H}^1(Y_1(N)(\mathbb{C}),\mathbb{Q})$ is the image of the class of the route from a cusp to a cusp $(0,\infty) \to Y_1(N),(\mathbb{C})$; $y \mapsto \nu(A^{-1}(yi+a))$ in $\mathrm{H}_1(X_1(N)(\mathbb{C}),\{\text{cusps}\},\mathbb{Z})$.

For $L \geqslant 4$ such that $A \mid L$, $N \mid L$, $\delta_{1,N}(k,j,a(A))$ coincides with the image of $\delta_{A,L}(k,j) \in V_{k,\mathbb{Z}}(Y(A,L))$ under the composite map

$$V_{k,\mathbb{Q}}(Y(A,L)) \xrightarrow{\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)_*} V_{k,\mathbb{Q}}(Y(A,L)) \xrightarrow{\text{trace}}$$
$$V_{k,\mathbb{Q}}(Y(1(A),L)) \xrightarrow[\sim]{(\varphi_A^{-1})_*} V_{k,\mathbb{Q}}\big(Y(1,L(A))\big) \xrightarrow{\text{trace}} V_{k,\mathbb{Q}}(Y_1(N)).$$

Next we define $\delta_{1,N}(k,j,\alpha)$ for $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Take $L \geqslant 3$ such that $N \mid L$. We define $\delta_{1,N}(k,j,\alpha)$ to be the image of $\alpha^*(\delta_{L,L}(k,j))$ under the trace map $V_{k,\mathbb{Q}}(Y(L)) \to V_{k,\mathbb{Q}}(Y_1(N))$. Then $\delta_{1,N}(k,j,\alpha)$ is independent of the choice of L.

In the following Thm. 5.6, for $k \geqslant 2$ and a finite set S of prime numbers such that $\mathrm{prime}(m) \subset S$, and for a character $\chi : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$, let

$$Z_{1,N,S}(k,\chi,s) = \sum_{(n,S)=1} \chi(n)T'(n)n^{-s}$$

which acts on $V_{k,\mathbb{C}}(Y_1(N))$, where $(n,S)=1$ means that n ranges over all positive integers such that $\mathrm{prime}(n) \cap S = \varnothing$. Let

$$\sigma_b \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \quad (b \in (\mathbb{Z}/m)^\times)$$

be the element which sends ζ_m to ζ_m^b.

As in § 4, let $x^\pm = (1/2)(1 \pm \iota)(x)$ for $x \in V_{k,\mathbb{C}}(Y_1(N))$.

Theorem 5.6. — *Let ξ and S be as in* (5.1.1) *and let $\chi : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$ be a character.*

(1) *Let $\pm = -\chi(-1)$. Then we have*

$$\sum_{b\in(\mathbb{Z}/m)^\times} \chi(b)\operatorname{reg}_{1,N}\left(\sigma_b(z_{1,N,m}(\xi,S))\right) = \lim_{S\to 0} s^{-1} Z_{1,N,S}(2,\chi,s)\cdot 2\pi i\cdot \delta_{1,N}(2,1,\xi)^{\pm}$$

(2) *Let $k \geqslant 2$. Then*

$$\sum_{b\in(\mathbb{Z}/m)^\times} \chi(b)\operatorname{per}_{1,N}\left(\sigma_b(z_{1,N,m}(k,r,r',\xi,S))\right)^{\pm}$$
$$= Z_{1,N,S}(k,\chi,r)\cdot(2\pi i)^{k-r-1}\cdot\delta_{1,N}(k,r',\xi)^{\pm}$$

for any integers r, r' satisfying (5.2.3) *and for $\pm = (-1)^{k-r-1}\chi(-1)$. Next let r, r', c, d be integers satisfying* (5.2.1) (5.2.2). *In the case $\xi \in \mathrm{SL}_2(\mathbb{Z})$, assume $c \equiv d \equiv 1 \bmod N$. Let $\pm = (-1)^{k-r-1}\chi(-1)$ and let u, $v \in \mathbb{Z}$ be as in 4.2. Then*

$$\sum_{b\in(\mathbb{Z}/m)^\times} \chi(b)\operatorname{per}_{1,N}\left(\sigma_b({}_{c,d}z_{1,N,m}(k,r,r',\xi,S))\right)^{\pm} = Z_{1,N,S}(k,\chi,r)\cdot(2\pi i)^{k-r-1}\cdot\gamma^{\pm}$$

where

$$\begin{aligned}\gamma = c^2d^2\delta_{1,N}(k,r',a(A)) &- c^ud^2\overline{\chi}(c)\delta_{1,N}(k,r',ac(A))\\ &- c^2d^v\overline{\chi}(d)\cdot\left(\begin{smallmatrix}1/d & 0\\ 0 & d\end{smallmatrix}\right)^*\delta_{1,N}(k,r',\text{“}a/d\text{”}(A))\\ &\qquad - c^ud^v\overline{\chi}(cd)\cdot\left(\begin{smallmatrix}1/d & 0\\ 0 & d\end{smallmatrix}\right)^*\delta_{1,N}(k,r',\text{“}ac/d\text{”}(A))\end{aligned}$$

in the case $\xi = a(A)$, where “e/d” for $e \in \mathbb{Z}$ means any integer such that $d\cdot$ “e/d” $\equiv e \bmod N$, and

$$\gamma = (c^2 - c^u\overline{\chi}(c))(d^2 - d^v\overline{\chi}(d))\delta_{1,N}(k,r',\xi)$$

in the case $\xi \in \mathrm{SL}_2(\mathbb{Z})$.

5.7. Prop. 5.3 is deduced from the propositions 2.3, 2.4, 4.3, 4.4 and Thm. 5.6 is deduced from the theorems 2.6, 4.6, by the following (5.7.1)–(5.7.3). (We use (5.7.1) and (5.7.2) (resp. (5.7.1) and (5.7.3)) in the case $\xi = a(A)$ (resp. $\xi \in \mathrm{SL}_2(\mathbb{Z})$).)

(5.7.1) *On $\mathrm{K}_2(Y_1(N)\otimes\mathbb{Q}(\zeta_m))$, $M_k(X_1(N)\otimes\mathbb{Q}(\zeta_m))$ ($k \in \mathbb{Z}$), and $V_{k,\mathbb{Q}}(Y_1(N)\otimes\mathbb{Q}(\zeta_m))$ ($k \geqslant 2$), we have*

$$\begin{pmatrix} b & 0\\ 0 & 1\end{pmatrix}^* = \sigma_b \quad \textit{for}\quad b \in (\mathbb{Z}/m)^\times.$$

(5.7.2) *Let $a \in \mathbb{Z}$, A, $M \geqslant 1$, $L \geqslant 4$, $mA \mid M$, $N \mid L$, $M \mid L$. Let*

$$\begin{aligned} &t_{m,a(A)} : \mathrm{K}_2(Y(M,L)) \longrightarrow \mathrm{K}_2(Y_1(N)\otimes\mathbb{Q}(\zeta_m))\\ \textit{(resp.}\quad &t_{m,a(A)} : M_k(X(M,L)) \longrightarrow M_k(Y_1(N))\otimes\mathbb{Q}(\zeta_m)\\ \textit{resp.}\quad &t_{m,a(A)} : V_{k,\mathbb{Q}}(Y(M,L)) \longrightarrow V_{k,\mathbb{Q}}(Y_1(N))\otimes\mathbb{Q}(\zeta_m))\end{aligned}$$

be the map defined in 5.1 (resp. defined in 5.2, resp. defined in the same way as $t_{m,a(A)}$ *for* M_k *in 5.2). Then we have*

$$t_{m,an(A)} \circ T(n) = T(n) \circ t_{m,a(A)}, \quad t_{m,a(A)} \circ T'(n) = T'(n) \circ t_{m,an(A)}$$

for any $n \geqslant 1$ *such that* $(n, M) = 1$, *and*

$$t_{m,av(A)} \circ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}^* = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}^* \circ t_{m,au(A)}$$

for any integers u, v *such that* $(u, M) = 1$, $(v, L) = 1$. *In particular,* $T'(n)\left(\begin{smallmatrix} 1/n & 0 \\ 0 & 1 \end{smallmatrix}\right)^*$ $(n \geqslant 1,\ (n, M) = 1)$ *and* $\left(\begin{smallmatrix} v & 0 \\ 0 & v \end{smallmatrix}\right)$ $(v \in \mathbb{Z},\ (v, L) = 1)$ *commute with* $t_{m,a(A)}$.

(5.7.3) *Let* $L \geqslant 3$. *Then for* $n \geqslant 1$ *such that* $(n, L) = 1$, *the operator* $T'(n)\left(\begin{smallmatrix} 1/n & 0 \\ 0 & 1 \end{smallmatrix}\right)^* = T(n)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/n \end{smallmatrix}\right)^*$ *on* $\mathrm{K}_2(Y(L))$ *(resp.* $M_k(X(L))$, *resp.* $V_{k,\mathbb{Z}}(Y(L))$*) commutes with the action of* $\mathrm{GL}_2(\mathbb{Z}/L)$.

The proofs of (5.7.1)–(5.7.3) are easy and hence omitted.

We give explicit presentations of the zeta elements of this section in some special cases.

Proposition 5.8. — *Let* $a \in \mathbb{Z}$, $A \geqslant 1$, *and assume*

$$\mathrm{prime}(A) \subset \mathrm{prime}(m), \quad N \geqslant 4, \quad mA \mid N,$$

Let $S = \mathrm{prime}(m)$.

(1) $$z_{1,N,m}(a(A), S) = \left\{ \textstyle\prod_x g_{1/m,x}, g_{0,1/n} \right\}$$
where x *ranges over all elements of* $\mathbb{Q}/\mathbb{Z}$ *such that* $mx = -a/A$.

(2) *Let* $k \geqslant 2$. *Assume*

$$\mathrm{prime}(A) \subset \mathrm{prime}(m), \quad N \geqslant 4, \quad mA \mid N, \quad S = \mathrm{prime}(m).$$

Then for integers r, r', c, d *as in* (5.2.1) (5.2.2) *(resp. for integers* r, r' *as in* (5.2.3)*),* ${}_{c,d}z_{1,N,m}(k, r, r', a(A), S)$ *(resp.* $z_{1,N}(k, r, r', a(A), S)$*) is equal to*

$$A^{r'-1} \cdot (-1)^r \cdot (r-1)!^{-1} m^{k-r-2} N^{-r} \cdot \sum_x {}_cF^{(k-r)}_{1/m,x} \cdot {}_dE^{(r)}_{0,1/N}$$

$$(\textit{resp.}\quad A^{r'-1} \cdot (-1)^r \cdot (r-1)!^{-1} m^{k-r-2} N^{-r} \cdot \sum_x F^{(k-r)}_{1/m,x} \cdot E^{(r)}_{0,1/N}\ )$$

if $r' = k-1$, *and to*

$$A^{r'-1} \cdot (-1)^{r'} \cdot (k-2)!^{-1} m^{r'-k} N^{-r'} \cdot \sum_x {}_cE^{(k-r')}_{1/m,x} \cdot {}_dE^{(r')}_{0,1/N}$$

$$(\textit{resp.}\quad A^{r'-1} \cdot (-1)^{r'} \cdot (k-2)!^{-1} m^{r'-k} N^{-r'} \cdot \sum_x E^{(k-r')}_{1/m,x} \cdot E^{(r')}_{0,1/N}\ )$$

if $r = k-1$, *where* x *ranges over all elements of* $\mathbb{Q}/\mathbb{Z}$ *such that* $mx = -a/A$.

Proof. — Concerning (2), the proof for zeta elements with c, d is similar to that without c, d. So we give the proof of (2) for zeta elements without c, d.

By the arguments as in the proof of Prop. 2.3 (resp. 4.3), we can proceed as follows. The element

$$\{g_{1/mA,0}, g_{0,1/N}\} \in \mathrm{K}_2(Y(mA, N)) \otimes \mathbb{Q}$$
$$(\text{resp.}\quad (!)^{(k-j)}_{1/mA,0} \cdot E^{(j)}_{0,1/N} \in M_k(X(mA, N))$$
$$\text{where } (!) = F \text{ or } E \text{ and } 1 \leqslant j \leqslant k-1)$$

is sent to

$$\{g_{1/mA,-a/mA}, g_{0,1/N}\} \in \mathrm{K}_2(Y(mA, N)) \otimes \mathbb{Q}$$
$$(\text{resp.}\quad (!)^{(k-j)}_{1/mA,-a/mA} \cdot E^{(j)}_{0,1/N} \in M_k(X(mA, N)))$$

under $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)_* = \left(\begin{smallmatrix} 1 & -a \\ 0 & 1 \end{smallmatrix}\right)^*$, and then to

$$\Big\{\prod_x g_{x,-a/mA}, g_{0,1/N}\Big\} \in \mathrm{K}_2(Y(m(A), N)) \otimes \mathbb{Q}$$
$$(\text{resp.}\quad \sum_x (!)^{(k-j)}_{x,-a/mA} \cdot E^{(j)}_{0,1/N} \in M_k(X(m(A), N)))$$

by norm (resp. trace), where x ranges over all elements of $\mathbb{Q}/\mathbb{Z}$ such that $Ax = 1/m$. By using 2.12 and then 2.7 (2) (resp. 4.12 and then 4.7 (2)), we see that this element is sent by $(\varphi_A^{-1})_* = \varphi_A^*$ to

$$\Big\{g_{1/m,-a/mA}, \prod_y g_{0,y}\Big\} \in \mathrm{K}_2(Y(m, N(A))) \otimes \mathbb{Q}$$
$$(\text{resp.}\quad A^s \cdot (!)^{(k-j)}_{1/mA,-a/mA} \cdot \sum_y E^{(j)}_{0,y} \in M_k\big(X(m, N(A))\big)$$
$$(s = 0 \text{ if } (!) = F, \text{ and } s = k-j-1 \text{ if } (!) = E))$$

where y ranges over all elements of $\mathbb{Q}/\mathbb{Z}$ such that $Ay = 1/N$. By using 2.7 (2) (resp. 4.7 (2)), we see that that this element is sent by norm (resp. trace) to

$$\{g_{1/m,-a/mA}, g_{0,1/N}\} \in \mathrm{K}_2(Y(m, N)) \otimes \mathbb{Q}$$
$$(\text{resp.}\quad A^t \cdot (!)^{(k-j)}_{1/m,-a/mA} \cdot E^{(j)}_{0,1/N} \in M_k(X(m, N))$$
$$(t = j \text{ if } (!) = F, \text{ and } t = k-1 \text{ if } (!) = E)).$$

This element is sent by norm (resp. trace) to

$$\{\prod_x g_{1/m,x}, g_{0,1/N}\} \in \mathrm{K}_2(Y_1(N) \otimes \mathbb{Q}(\zeta_m)) \otimes \mathbb{Q}$$
$$(\text{resp.}\quad A^t \cdot \sum_x (!)^{(k-j)}_{1/m,x} \cdot E^{(j)}_{0,1/N} \in M_k(X_1(N) \otimes \mathbb{Q}(\zeta_m))).$$

where x ranges over all elements of $\mathbb{Q}/\mathbb{Z}$ such that $mx = -A/a$. This proves 5.8 (1) (resp. 5.8 (2) for zeta elements without c, d). □

Remark 5.9. — Let $k \geqslant 2$, let ξ, S be as in (5.1.1) and assume $\xi = a(A)$. Then Prop. 5.8 shows that the zeta element ${}_{c,d}z_{1,N,m}(k,r,r',\xi,S)$ for r, r', c, d as in (5.2.1) (5.2.2) (resp. $z_{1,N,m}(k,r,r',\xi,S)$ for r, r' as in (5.2.3)) is defined also as follows avoiding the strange map $t_{m,a(A)}$. Take $M \geqslant 4$ such that $\mathrm{prime}(M) = S$, and let $m' = mM$, $N' = AmMN$. Then this zeta element is the image of the zeta element ${}_{c,d}z_{1,N',m'}(k,r,r',\xi,S)$ (resp. $z_{1,N',m'}(k,r,r',\xi,S)$) under the trace map

$$M_k(X_1(N') \otimes \mathbb{Q}(\zeta_{m'})) \longrightarrow M_k(X_1(N) \otimes \mathbb{Q}(\zeta_m)),$$

and the latter zeta element is given by Prop. 5.8 since

$$\mathrm{prime}(A) \subset \mathrm{prime}(m'), \quad N' \geqslant 4, \quad m'A \mid N', \quad S = \mathrm{prime}(m').$$

A similar remark works for elements $z_{1,N,m}(a(A),S)$ in $\mathrm{K}_2 \otimes \mathbb{Q}$.

By Prop. 5.8 and by Rem. 5.9, we have

Corollary 5.10. — *Let a, $A \in \mathbb{Z}$, $A \geqslant 1$, and let $\xi = a(A)$. Then the zeta elements in 5.1 (resp. 5.2) including the symbol ξ do not change when we replace ξ by $a'(A)$ for any integer a' such that $a' \equiv a \bmod A$, and they do not change (resp. they are multiplied by $b^{r'-1}$) when we replace ξ by $ab(Ab)$ for a positive integer b.*

6. Projections to eigen cusp forms

In this section, we consider zeta elements associated to each newform.

Fix $k \geqslant 2$, $N \geqslant 1$.

6.1. We fix a normalized newform ([**AL**], [**De2**])

$$f = \sum_{n \geqslant 1} a_n q^n \in S_k(X_1(N)) \otimes \mathbb{C}$$

of weight k and of level N. We have

$$\begin{aligned} &a_1 = 1 \\ &T(n)f = a_n f, \quad T'(n)f = \bar{a}_n f, \quad \text{for any} \quad n \geqslant 1. \end{aligned}$$

6.2. The zeta function

$$\mathrm{L}(f,s) = \sum_{n \geqslant 1} a_n n^{-s}$$

can be written in the form of the Euler product

$$\prod_{\ell} (1 - a_\ell \ell^{-s} + \varepsilon(\ell)\ell^{k-1-2s})^{-1}$$

where ℓ ranges over all prime numbers and

$$\varepsilon : (\mathbb{Z}/N)^\times \longrightarrow \mathbb{C}^\times$$

is a homomorphism characterized by

$$\begin{pmatrix} 1/d & 0 \\ 0 & d \end{pmatrix}^* f = \varepsilon(d) f \quad \text{for any} \quad d \in (\mathbb{Z}/N)^\times.$$

(If ℓ divides N, $\varepsilon(\ell)$ means 0.)

For $m \geqslant 1$ and for a finite set S of prime numbers such that $\operatorname{prime}(m) \subset S$, and for a character $\chi : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$, let

$$\begin{aligned} \mathrm{L}_S(f, \chi, s) &= \sum_{(n,S)=1} a_n \chi(n) n^{-s} \\ &= \prod_{\ell \notin S} \left(1 - a_\ell \chi(\ell) \ell^{-s} + \varepsilon(\ell) \chi^2(\ell) \ell^{k-1-2s}\right)^{-1} \end{aligned}$$

These zeta functions converge absolutely when $\operatorname{Re}(s) > (k+1)/2$, and are extended as holomorphic functions to the whole s-plane.

6.3. Let

$$F = \mathbb{Q}(a_n \, ; \quad n \geqslant 1) \subset \mathbb{C}.$$

Then F is a finite extension of $\mathbb{Q}$ and is stable under the complex conjugation. The above ε has values in F.

We will define a quotient $\mathbb{Q}$-vector space $S(f)$ of $M_k(X_1(N))$ and a quotient $\mathbb{Q}$-vector space $V_F(f)$ of $V_{k,\mathbb{Q}}(Y_1(N))$, corresponding to f. These spaces will have structures of F-linear spaces, and $\dim_F(S(f)) = 1$, $\dim_F(V_F(f)) = 2$. In the case $k = 2$, we will define a quotient $\mathbb{Q}$-vector space $\mathrm{K}_2(f, K)$ of $\mathrm{K}_2(Y_1(N) \otimes K) \otimes \mathbb{Q}$ for a field $K \supset \mathbb{Q}$, which also has a structure of an F-linear space.

Let $m \geqslant 1$ and let ξ, S be as in (5.1.1). For integers r, r', c, d satisfying (5.2.1) (5.2.2) (resp. integers r, r' satisfying (5.2.3)) we define the zeta element

$${}_{c,d}z_m(f, r, r', \xi, S) \quad (\text{resp. } z_m(f, r, r', \xi, S)\,) \quad \in S(f) \otimes \mathbb{Q}(\zeta_m)$$

as the image of the element ${}_{c,d}z_{1,N,m}(k, r, r', \xi, S)$ (resp. $z_{1,N,m}(k, r, r', \xi, S)$) of $M_k(X_1(N)) \otimes \mathbb{Q}(\zeta_m)$. In the case $k = 2$, we define the zeta element

$$z_m(f, \xi, S) \in \mathrm{K}_2(f, \mathbb{Q}(\zeta_m))$$

as the image of $z_{1,N,m}(\xi, S)$.

We define $S(f)$ to be the quotient of $M_k(X_1(N)) \otimes_{\mathbb{Q}} F$ by the F-submodule generated by the images of the operators $T(n) \otimes 1 - 1 \otimes a_n$ for all $n \geqslant 1$. Hence as a $\mathbb{Q}$-vector space, $S(f)$ is a quotient of $M_k(X_1(N))$. Furthermore, $S(f)$ is a one dimensional F-vector space, and $S(f) \otimes_F \mathbb{C}$ is generated as a $\mathbb{C}$-vector space by the image of f under the canonical map $S_k(X_1(N)) \otimes_{\mathbb{Q}} \mathbb{C} \to S(f) \otimes_F \mathbb{C}$.

On $S(f)$, $T(n)$ acts by a_n and $T'(n)$ acts by $\overline{a}_n$.

We define $V_F(f)$ to be the quotient of $V_{k,F}(Y_1(N))$ by the F-submodule generated by the images of $T(n) \otimes 1 - 1 \otimes a_n$ for all $n \geqslant 1$. As a $\mathbb{Q}$-vector space, $V_F(f)$ is a quotient of $V_{k,\mathbb{Q}}(Y_1(N))$. On $V_F(f)$, $T(n)$ acts by a_n and $T'(n)$ acts by $\overline{a}_n$.

For a commutative ring A over F, define

$$V_A(f) = V_F(f) \otimes_F A.$$

The complex conjugation ι on $V_{k,\mathbb{Q}}(Y_1(N))$ induces an A-linear map $\iota : V_A(f) \to V_A(f)$. Hence we obtained an A-linear action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $V_A(f)$.

We have

$$\dim_F(V_F(f)) = 2, \quad \dim_F(V_F(f)^+) = \dim_F(V_F(f)^-) = 1$$

where $(\ )^{\pm}$ means the part on which ι acts as ± 1, respectively.

The period map $\mathrm{per}_{1,N} : M_k(X_1(N)) \to V_{k,\mathbb{C}}(Y_1(N))$ induces an F-linear map

$$\mathrm{per}_f : S(f) \longrightarrow V_{\mathbb{C}}(f)$$

which we call the period map of f.

For $1 \leqslant j \leqslant k-1$ and for $\xi = a(A)$ with $a \in \mathbb{Z}$, $A \geqslant 1$ (resp. for $\xi \in \mathrm{SL}_2(\mathbb{Z})$), let

$$\delta(f, j, \xi) \in V_F(f)$$

be the image of $\delta_{1,N}(k, j, \xi)$.

In the case $k = 2$, we define $\mathrm{K}_2(f, K)$ for a field $K \supset \mathbb{Q}$, to be the quotient of $\mathrm{K}_2(Y_1(N) \otimes K) \otimes_{\mathbb{Z}} F$ by the F-submodule generated by the images of $T(n) \otimes 1 - 1 \otimes a_n$ and $T'(n) \otimes 1 - 1 \otimes \bar{a}_n$ for all $n \geqslant 1$. As a $\mathbb{Q}$-vector space, $\mathrm{K}_2(f, K)$ is a quotient of $\mathrm{K}_2(Y_1(N) \otimes K) \otimes \mathbb{Q}$. On $\mathrm{K}_2(f, K)$, $T(n)$ acts by a_n and $T'(n)$ acts by $\bar{a}_n$. In the case $K \subset \mathbb{C}$, the regulator map $\mathrm{reg}_{1,N} : \mathrm{K}_2(Y_1(N) \otimes K) \to V_{k,\mathbb{C}}(Y_1(N))$ induces an F-linear map

$$\mathrm{reg}_f : \mathrm{K}_2(f, K) \longrightarrow V_{\mathbb{C}}(f)$$

which we call the regulator map of f.

Proposition 6.4. — *Let ξ, S be as in* (5.1.1). *Let $m' \geqslant 1$, $m \mid m'$, and let S' be a finite set of prime numbers such that $S \cup \mathrm{prime}(m') \subset S'$.*

(1) *Let r, r', c, d be integers satisfying* (5.2.1) (5.2.2) *and* $\mathrm{prime}(cd) \cap S' = \varnothing$. *Then the trace map*

$$S(f) \otimes \mathbb{Q}(\zeta_{m'}) \longrightarrow S(f) \otimes \mathbb{Q}(\zeta_m)$$

sends ${}_{c,d}z_{m'}(f, r, r', \xi, S')$ to

$$\Big(\prod_{\ell \in S' - S} \big(1 - \bar{a}_\ell \sigma_\ell^{-1} \cdot \ell^{-r} + \bar{\varepsilon}(\ell) \sigma_\ell^{-2} \cdot \ell^{k-1-2r}\big)\Big) \cdot {}_{c,d}z_m(f, r, r', \xi, S).$$

We have the result of the same form for $z_{m'}(f, r, r', \xi, S')$ and $z_m(f, r, r', \xi, S)$ for any integers r, r' satisfying (5.2.3).

(2) *Assume $k = 2$. Then the norm map*

$$\mathrm{K}_2(f, \mathbb{Q}(\zeta_{m'})) \longrightarrow \mathrm{K}_2(f, \mathbb{Q}(\zeta_m))$$

sends $z_{m'}(f, \xi, S')$ to

$$\Big(\prod_{\ell \in S' - S} \big(1 - \bar{a}_\ell \sigma_\ell^{-1} + \bar{\varepsilon}(\ell) \sigma_\ell^{-2} \cdot \ell\big)\Big) \cdot z_m(f, \xi, S).$$

This follows from Prop. 5.3.

6.5. We introduce the dual cusp form f^* of f. Let

$$f^* = \sum_{n\geqslant 1} \overline{a}_n q^n.$$

Then $f^* \in M_k(X_1(N)) \otimes \mathbb{C}$, and f^* is an eigen newform of level N.

Theorem 6.6. — *Let ξ, S be as in* (5.1.1). *Let $\chi : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$ be a character.*

(1) *For integers r, r' satisfying* (5.2.3) *and for $\pm = (-1)^{k-r-1}\chi(-1)$, we have*

$$\sum_{b\in(\mathbb{Z}/m)^\times} \chi(b)\,\mathrm{per}_f\left(\sigma_b(z_m(f,r,r',\xi,S))\right)^\pm = \mathrm{L}_S(f^*,\chi,r)\cdot(2\pi i)^{k-r-1}\cdot\delta(f,r',\xi)^\pm.$$

Next let r, r', c, d be integers satisfying (5.2.1) (5.2.2). *In the case $\xi \in \mathrm{SL}_2(\mathbb{Z})$, assume $c \equiv d \equiv 1 \bmod N$. Let $\pm = (-1)^{k-r-1}\chi(-1)$ and let u, $v \in \mathbb{Z}$ be as in* (4.2.4). *Then*

$$\sum_{b\in(\mathbb{Z}/m)^\times} \chi(b)\,\mathrm{per}_f\left(\sigma_b({}_{c,d}z_m(f,r,r',\xi,S))\right)^\pm = \mathrm{L}_S(f^*,\chi,r)\cdot(2\pi i)^{k-r-1}\cdot\gamma^\pm$$

where

$$\gamma = c^2d^2\delta(f,r,r',a(A)) - c^u d^2\overline{\chi}(c)\delta(f,r',ac(A)) \\ - c^2d^v\overline{\chi}(d)\varepsilon(d)\delta(f,r',\text{“}a/d\text{”}(A)) + c^u d^v\overline{\chi}(cd)\varepsilon(d)\delta(f,r',\xi)$$

in the case $\xi = a(A)$, and

$$\gamma = (c^2 - c^u\overline{\chi}(c))(d^2 - d^v\overline{\chi}(d))\delta(f,r',\xi)$$

in the case $\xi \in \mathrm{SL}_2(\mathbb{Z})$.

(2) *Assume $k = 2$. Let $\pm = -\chi(-1)$. Then we have*

$$\sum_{b\in(\mathbb{Z}/m)^\times} \chi(b)\,\mathrm{reg}_f\left(\sigma_b(z_{1,N,m}(\xi,S))\right) = \lim_{s\to 0} s^{-1}\,\mathrm{L}_S(f^*,\chi,s)\cdot 2\pi i\cdot\delta(f,1,\xi)^\pm.$$

This follows from Thm. 5.6

7. The proofs of the zeta value formulas

In this section, we give the proofs of Thm. 2.6 and Thm. 4.6.

In 7.1–7.17, we fix $k \geqslant 2$, $N \geqslant 4$, and $m \geqslant 1$ such that $m \mid N$, and consider $Y_1(N) \otimes \mathbb{Q}(\zeta_m)$. Thm. 2.6 and Thm. 4.6 are statements for $Y(M,N)$, but as is explained in 7.18–7.20, we can reduce them to a result (Prop. 7.12) on $Y_1(N)\otimes\mathbb{Q}(\zeta_m)$.

We will identify $Y_1(N)(\mathbb{C})$ with $\Gamma_1(N)\backslash\mathfrak{H}$ via $\nu : \mathfrak{H} \to Y(N)(\mathbb{C}) \to Y_1(N)(\mathbb{C})$ (1.8), where

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\ ;\quad c \equiv 0,\quad d \equiv 1 \bmod N \right\}.$$

We will regard modular forms as functions on $\mathfrak{H}$ as in 3.8.

We start with the following result of Shimura.

Proposition 7.1 **([Sh]).** — *Let $0 \leqslant j \leqslant k-1$, and let*

$$f = \sum_{n \geqslant 1} a_n q^n \in S_k(X_1(N)) \otimes \mathbb{C}, \quad g = \sum_{n \geqslant 0} b_n q^n \in M_{k-j}(X_1(N)) \otimes \mathbb{C}.$$

Assume $f \neq 0$, $g \neq 0$ and that f and g are eigen forms of $T(n)$ for any $n \geqslant 1$ such that $(n, N) = 1$. Then

$$\int_{\Gamma_1(N)\backslash\mathfrak{H}} \overline{f(\tau)} g(\tau) \cdot N^{-j-2(s-k+1)} E\big(j, \tau, 1/N, 2(s-k+1)\big) y^{s-1}\, \mathrm{d}x \wedge \mathrm{d}y$$
$$= (4\pi)^{-s} \Gamma(s) D(f, g, s)$$

($\tau = x + iy \in \mathfrak{H}$, x, $y \in \mathbb{R}$), where $E\big(j, \tau, 1/N, s\big)$ is as in 3.8 and $D(f, g, s)$ is a zeta function defined as in 7.2 below.

7.2. For f, g as in Prop. 7.1, the function $D(f, g, s)$ is defined as follows. Write

$$T(n)f = \lambda(n)f, \quad T(n)g = \eta(n)g \quad ((n, N) = 1, \quad \lambda(n),\ \eta(n) \in \mathbb{C}).$$

Then the Dirichlet series

$$\sum_{(n,N)=1} \lambda(n)n^{-s} \quad \text{and} \quad \sum_{(n,N)=1} \eta(n)n^{-s}$$

are expressed as Euler products. For a prime number ℓ which does not divide N, let $\{(1-\alpha_1\ell^{-s})(1-\alpha_2\ell^{-s})\}^{-1}$ be the Euler factor of $\sum_{(n,N)=1} \lambda(n)n^{-s}$ at ℓ, let $\{(1-\beta_1\ell^{-s})(1-\beta_2\ell^{-s})\}^{-1}$ be the Euler factor of $\sum_{(n,N)=1} \eta(n)n^{-s}$ at ℓ, and define a polynomial $P_\ell(u)$ by

$$P_\ell(u) = (1 - \overline{\alpha}_1\beta_1 u)(1 - \overline{\alpha}_1\beta_2 u)(1 - \overline{\alpha}_2\beta_1 u)(1 - \overline{\alpha}_2\beta_2 u).$$

Define $L_{(N)}(\overline{\lambda} \otimes \eta, s)$ by

$$L_{(N)}(\overline{\lambda} \otimes \eta, s) = \prod_\ell P_\ell(\ell^{-s})^{-1}$$

where ℓ ranges over all prime numbers which do not divide N. Let

$$S(N) = \{n \geqslant 1\,;\ \ \mathrm{prime}(n) \subset \mathrm{prime}(N)\}.$$

We define

$$D(f, g, s) = L_{(N)}(\overline{\lambda} \otimes \eta, s) \cdot \sum_{n \in S(N)} \overline{a}_n b_n n^{-s}.$$

7.3. In 7.3–7.6, we fix some notation.

Let $\chi : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$ be a character. For $h \geqslant 1$, we define elements

(7.3.1) $\quad F_\chi^{(h)}$ (here we assume $m \geqslant 2$ in the case $h = 2$),

(7.3.2) $\quad E_\chi^{(h)}$ (here we assume $\chi \neq 1$ in the case $h = 2$)

of $M_h(X_1(N)) \otimes \mathbb{C}$ as follows:

$$F_\chi^{(h)} = m^{h-2} \sum_{a,b} \chi(a) F_{a/m,b/m}^{(h)}$$

$$E_\chi^{(h)} = m^{-h} \sum_{a,b} \chi(a) E_{a/m,b/m}^{(h)} \quad \text{for} \quad h \neq 2,$$

$$E_\chi^{(2)} = m^{-2} \sum_{a,b} \chi(a) \widetilde{E}_{a/m,b/m}^{(2)}$$

where a ranges over all elements of $(\mathbb{Z}/m)^\times$ and b ranges over all elements of $\mathbb{Z}/m$. Since $\sum_b F_{a/m,b/m}^{(h)}$, $\sum_b E_{a/m,b/m}^{(h)}$ $(h \neq 2)$, and $\sum_b \widetilde{E}_{a/m,b/m}^{(h)}$ $(h = 2)$ belong to $M_h(X_1(N) \otimes \mathbb{Q}(\zeta_m)) = M_h(X_1(N)) \otimes \mathbb{Q}(\zeta_m)$, (7.3.1), (7.3.2) are defined as elements of $M_h(X_1(N)) \otimes \mathbb{C}$. These elements are zero unless $\chi(-1) = (-1)^h$ because $F_{-\alpha,-\beta}^{(h)} = (-1)^h F_{\alpha,\beta}^{(h)}$ and the similar formulas hold for $E_{\alpha,\beta}^{(h)}$ $(h \neq 2)$ and $\widetilde{E}_{\alpha,\beta}^{(2)}$.

7.4. Let $\chi : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$ be a character. Let r, r' be integers satisfying the following (7.4.1).

$$\begin{array}{l} 0 \leqslant r \leqslant k-1, \ 1 \leqslant r' \leqslant k-1, \ \text{and at least one of } r, r' \text{ is } k-1. \\ \text{Furthermore } m \geqslant 2 \text{ in the case } (r,r') = (k-2,k-1), \\ \text{and } \chi \neq 1 \text{ in the case } (r,r') = (k-1,k-2). \end{array} \tag{7.4.1}$$

We define a function $z_\chi(r,r')$ on $\mathfrak{H}$ by

$$z_\chi(r,r') = \begin{cases} (-1)^r \cdot (r-1)!^{-1} \cdot N^{-r} \cdot F_\chi^{(k-r)} \cdot E_{0,1/N}^{(r)} & \text{if } r' = k-1, \\ (-1)^{r'} \cdot (k-2)!^{-1} \cdot N^{-r'} \cdot E_\chi^{(k-r')} \cdot E_{0,1/N}^{(r')} & \text{if } r = k-1 \end{cases} \tag{7.4.2}$$

For an integer d which is prime to N, we define a function ${}_d z_\chi(r,r')$ on $\mathfrak{H}$ by

$${}_d z_\chi(r,r') = \begin{cases} (-1)^r \cdot (r-1)!^{-1} \cdot N^{-r} \cdot F_\chi^{(k-r)} \cdot {}_d E_{0,1/N}^{(r)} & \text{if } r' = k-1, \\ (-1)^{r'} \cdot (k-2)!^{-1} \cdot N^{-r'} \cdot E_\chi^{(k-r')} \cdot {}_d E_{0,1/N}^{(r')} & \text{if } r = k-1 \end{cases} \tag{7.4.3}$$

These functions with $r' = k-1$ (resp. $r = k-1$) are zero unless $\chi(-1) = (-1)^{k-r}$ (resp. $\chi(-1) = (-1)^{k-r'}$).

The functions in (7.4.2) with $(r,r') \neq (0,k-1)$, $(2,k-1)$, $(k-1,2)$, and the elements (7.4.3) with $r \neq 0$ are elements of $M_k(X_1(N)) \otimes \mathbb{C}$. In general, functions in (7.4.2) (7.4.3) are C^∞-functions on $\mathfrak{H}$, but not necessarily holomorphic.

The following is clear from the definition.

Lemma 7.5. — *Let r, r' be as in* (7.4.1) *and assume $r \neq 0$. Assume $m \geqslant 2$. Then we have*

$$\sum_{a \in (\mathbb{Z}/m)^\times} \chi(a) \sigma_a\big(z_{1,N,m}(k,r,r',0(1),\mathrm{prime}(m))\big) = z_\chi(r,r'),$$

if $(r,r') \neq (2,k-1)$, $(k-1,2)$ $(k-1,k-2)$, *and*

$$\sum_{a\in(\mathbb{Z}/m)^\times} \chi(a)\sigma_a\big({}_{c,d}z_{1,N,m}(k,r,r',0(1),\mathrm{prime}(m))\big) = (c^2 - c^u\overline{\chi}(c))_d z_\chi(r,r')$$

for any integers c,d *such that* $(c,m) = (d,mN) = 1$, *where* $u = r+2-k$ *if* $r' = k-1$, *and* $u = k-r'$ *if* $r = k-1$.

7.6. Define zeta functions $Z^T_{1,N}(k,s)$ and $Z^T_{1,N}(k,\chi,s)$, whose values are operators acting on $V_{k,\mathbb{C}}(Y_1(N))$, by

$$Z^T_{1,N}(k,s) = \sum_{n\geqslant 1} T(n)n^{-s}, \quad Z^T_{1,N}(k,\chi,s) = \sum_{n\geqslant 1} \chi(n)T(n)n^{-s}.$$

(The letter T means that these zeta functions are defined by using $T(n)$ not $T'(n)$.)

For $f = \sum_{n\geqslant 1} a_n q^n \in S_k(X_1(N))\otimes\mathbb{C}$, define

$$[f]_1 = a_1.$$

We have

$$a_n = [T(n)f]_1$$

for all $n \geqslant 1$. Define

$$\mathrm{L}(f,s) = \sum_{n\geqslant 1} a_n n^{-s} = \big[Z^T_{1,N}(k,s)f\big]_1.$$

For an integer $j \geqslant 1$, define

$$\begin{aligned}\Omega(f,j) &= (2\pi i)^{k-1}\int_0^\infty f(iy)\cdot(iy)^{j-1}\cdot \mathrm{d}(iy),\\ \overline{\Omega}(f,j) &= \text{ the complex conjugate of } \Omega(f,j).\end{aligned}$$

Then we have

$$\Omega(f,j) = (2\pi i)^{k-j-1}\cdot(-1)^j\cdot(j-1)!\cdot \mathrm{L}(f,j) \tag{7.6.1}$$

for any $j \geqslant 1$. (This follows from the well known fact

$$\int_0^\infty f(iy)y^{s-1}\,\mathrm{d}y = (2\pi)^{-s}\Gamma(s)\,\mathrm{L}(f,s)\,.)$$

Proposition 7.7. — *Let* $\chi : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$ *be a character, and let* $f \in S_k(X_1(N))\otimes\mathbb{C}$. *Let* r, r' *be as in* (7.4.1), *and assume* $\chi(-1) = (-1)^{k-r}$ *(resp.* $(-1)^{k-r'}$*) if* $r' = k-1$ *(resp.* $r = k-1$*). Then*

$$(-8\pi^2 i)^{k-1}\cdot\int_{\Gamma_1(N)\backslash\mathfrak{H}} \overline{f(\tau)}\cdot\frac{1}{2}\cdot z_\chi(r,r')\cdot y^{k-2}\mathrm{d}x\wedge\mathrm{d}y$$

is equal to

$$(2\pi i)^{k-r-1}\cdot\overline{\Omega}(Z^T_{1,N}(k,\overline{\chi},r)f,r')$$

in the case $r \neq 0$, and to

$$\frac{1}{2} \cdot (2\pi i)^{k-1} \cdot \overline{\Omega}((\lim_{s\to 0} s^{-1} Z^T_{1,N}(k, \overline{\chi}, s))f, r')$$

in the case $r = 0$.

7.8. The proof of Prop. 7.7 is given in 7.8–7.10. To prove Prop. 7.7 in the case $(r, r') = (j, k-1)$ (resp. $(r, r') = (k-1, j)$), we will apply Prop. 7.1 to the case $g = 1/2 \cdot F^{(k-j)}_{\chi}$ (resp. $1/2 \cdot E^{(k-j)}_{\chi}$) with $\chi(-1) = (-1)^{k-j}$ (we assume $m \geqslant 2$ (resp. $\chi \neq 1$) in the case $j = k-2$). If we write

$$g = \sum_{n\geqslant 0} b_n q^n,$$

we have by 3.10

$$\sum_{n\geqslant 1} b_n n^{-s} = \mathrm{L}(\chi, s-k+r+1)\zeta(s-k+r'+1) \tag{7.8.1}$$

(Here $\mathrm{L}(\chi, s) = \sum_{n\geqslant 1,(n,N)=1} \chi(n)n^{-s}$.) From (7.8.1), we have

$$T(n)g = b_n g \quad \text{for all} \quad n \geqslant 1 \quad \text{such that} \quad (n, N) = 1. \tag{7.8.2}$$

Lemma 7.9. — *If f is as in Prop. 7.1 and g is as above, $D(f, g, s)$ is equal to the complex conjugate of*

$$\left[Z^T_{1,N}(k, \overline{\chi}, \overline{s} - k + r + 1)Z^T_{1,N}(k, \overline{s} - k + r' + 1)f\right]_1.$$

Proof. — By (7.8.1) and (7.8.2), we have

$$\mathrm{L}_{(N)}(\overline{\lambda} \otimes \eta, s) = \mathrm{L}_{(N)}(\overline{\lambda}, \chi, s-k+r+1) \cdot \mathrm{L}_{(N)}(\overline{\lambda}, s-k+r'+1). \tag{7.9.1}$$

Here $\mathrm{L}_{(N)}(\overline{\lambda}, \chi, s) = \sum_{n\geqslant 1\ (n,N)=1} \overline{\lambda}_n \chi(n)n^{-s}$. On the other hand, $\sum_{n\in S(N)} \overline{a}_n b_n n^{-s}$ is the complex conjugate of $\left[\sum_{n\in S(N)} T(n)\overline{b}_n n^{-s} f\right]_1$, and by the fact

$$T(nn') = T(n)T(n') \quad \text{for } n, n' \in S(N),$$

this $\sum_{n\in S(N)} T(n)\overline{b}_n n^{-s}$ is expressed as:

(7.9.2)
$$\sum_{n\in S(N)} T(n)\overline{b}_n n^{-s} = \Big(\sum_{n\in S(N)} T(n)\overline{\chi}(n)n^{-(s-k+r+1)}\Big)\Big(\sum_{n\in S(N)} T(n)n^{-(s-k+r'+1)}\Big).$$

Now 7.9 follows from 7.9.1, 7.9.2, and the fact

$$T(nn') = T(n)T(n') \ \text{ if } \ (n, N) = 1 \ \text{ and } n' \in S(N). \qquad \square$$

7.10. We prove Prop. 7.7. We may assume that f is an eigen form of $T(n)$ for any $n \geqslant 1$ such that $(n, N) = 1$, since $S_k(X_1(N)) \otimes \mathbb{C}$ is generated over $\mathbb{C}$ by such forms. For such f, by (7.6.1) and Lemma 7.9, the case $r \neq 0$ (resp. $r = 0$) of Prop. 7.7 is obtained by putting $s = k-1$ in 7.1 (resp. by taking $\lim_{s\to k-1}$ of $(s-k+1)^{-1}$ times the both sides of 7.1).

7.11. For M, $L \geqslant 1$ such that $M+L \geqslant 5$, we defined in 4.5 a $\mathbb{C}$-linear operator

$$\iota : V_{k,\mathbb{C}}(Y(M,L)) \longrightarrow V_{k,\mathbb{C}}(Y(M,L)).$$

Now we define another operator

$$\iota' : V_{k,\mathbb{C}}(Y(M,L)) \longrightarrow V_{k,\mathbb{C}}(Y(M,L)),$$

which is anti-$\mathbb{C}$-linear, by

$$\iota'(x \otimes y) = x \otimes \overline{y} \quad (x \in V_{k,\mathbb{R}}(Y(M,L)), y \in \mathbb{C}).$$

Proposition 7.12. — *Let $\chi : (\mathbb{Z}/m)^{\times} \to \mathbb{C}^{\times}$ be a character, and let*

$$Z_{1,N}(k,\chi,s) = \sum_{n\geqslant 1} \chi(n) T'(n) n^{-s}$$

which is an operator-valued function acting on $V_{k,\mathbb{C}}(Y_1(N))$.

(1) *Assume $\chi(-1) = (-1)^{k-r}$ (resp. $(-1)^{k-r'}$) if $r' = k-1$ (resp. $r = k-1$). Then*

$$\text{(7.12.1)} \quad \mathrm{per}_{1,N}\big(\frac{1}{2}\cdot z_{\chi}(r,r')\big) + (-1)^{k-r-1}\iota'\big(\mathrm{per}_{1,N}\big(\frac{1}{2}\cdot z_{\overline{\chi}}(r,r')\big)\big) = Z_{1,N}(k,\chi,r)\cdot(2\pi i)^{k-r-1}\cdot\delta_{1,N}(k,r')$$

in $V_{k,\mathbb{C}}(Y_1(N))$ if $(r,r') \neq (0,k-1)$, $(2,k-1)$, $(k-1,2)$, and

$$\text{(7.12.2)} \quad \mathrm{per}_{1,N}\big(\frac{1}{2}\cdot {}_{d}z_{\chi}(r,r')\big) + (-1)^{k-r-1}\iota'\big(\mathrm{per}_{1,N}\big(\frac{1}{2}\cdot {}_{d}z_{\overline{\chi}}(r,r')\big)\big) = Z_{1,N}(k,\chi,r)\cdot(2\pi i)^{k-r-1}\cdot\Big(d^2 - d^j\overline{\chi}(d)\cdot\left(\begin{smallmatrix}1/d & 0\\ 0 & d\end{smallmatrix}\right)^*\Big)\cdot\delta_{1,N}(k,r')$$

in $V_{k,\mathbb{C}}(Y_1(N))$ if $r \neq 0$.

(2) *Assume $m \geqslant 2$ and $\chi(-1) = 1$. Then*

$$\text{(7.12.3)} \quad \sum_{a\in\mathbb{Z}/m^{\times}} \chi(a)\cdot\mathrm{reg}_{1,N}\Big(\prod_{b\in\mathbb{Z}/m}\{g_{a/m,b/m}, g_{0,1/N}\}\Big) = \lim_{s\to 0} s^{-1} Z_{1,N}(2,\chi,s)\cdot 2\pi i\cdot\delta_{1,N}(2,1)$$

in $V_{2,\mathbb{C}}(Y_1(N))$.

The proof of Prop. 7.12 is given in 7.13–7.17.

7.13. In this 7.13, by using Poincaré duality, we reduce Prop. 7.12 to Prop. 7.7 and to a statement (Prop. 7.14) concerning the "boundary" of zeta elements at cusps.

The canonical pairing $\langle\,,\,\rangle : \mathcal{H}^1 \times \mathcal{H}^1 \to \mathbb{Z}$ on $Y_1(N)(\mathbb{C})$ induces the pairing

$$\mathrm{Sym}^{k-2}_{\mathbb{Z}}(\mathcal{H}^1) \times \mathrm{Sym}^{k-2}_{\mathbb{Z}}(\mathcal{H}^1) \longrightarrow \mathbb{Q}\,;$$

$$(x_1\ldots x_{k-2}, y_1\ldots y_{k-2}) \longmapsto \frac{1}{(k-2)!}\cdot\sum_{\sigma\in\mathfrak{S}_{k-2}}\prod_{j=1}^{k-2}\langle x_j, y_{\sigma(j)}\rangle$$

$(x_1, \ldots, x_{k-2}, y_1, \ldots, y_{k-2} \in \mathcal{H}^1)$. This induces a perfect duality of finite dimensional $\mathbb{Q}$-vector spaces

(7.13.1) $$\langle\ ,\ \rangle : V_{k,\mathbb{Q}}(Y_1(N)) \times V_{k,\mathbb{Q},c}(Y_1(N)) \longrightarrow \mathbb{Q}$$

and also a perfect self-duality of the finite dimensional $\mathbb{Q}$-vector space

$$V_{k,\mathbb{Q}}(X_1(N)) \underset{\mathrm{def}}{=} \mathrm{H}^1(X_1(N)(\mathbb{C}), j_* \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1)) \otimes \mathbb{Q}$$

where j denotes the inclusion map $Y_1(N)(\mathbb{C}) \to X_1(N)(\mathbb{C})$. (We regard $V_{k,\mathbb{Q}}(X_1(N))$ as a subspace of $V_{k,\mathbb{Q}}(Y_1(N))$ and at the same time as a quotient of $V_{k,\mathbb{Q},c}(Y_1(N))$.) Furthermore the period map (4.10)

$$\mathrm{per} : S_k(X_1(N)) \otimes \mathbb{C} \longrightarrow V_{k,\mathbb{C},c}(Y_1(N))$$

induces the isomorphism of Shimura

(7.13.2) $$\begin{aligned}(S_k(X_1(N)) \otimes \mathbb{C})^2 \xrightarrow{\sim} V_{k,\mathbb{C}}(X_1(N)) &\underset{\mathrm{def}}{=} V_{k,\mathbb{Q}}(X_1(N)) \otimes \mathbb{C} \\ (f,g) \longmapsto \mathrm{per}(f) &+ \iota' \mathrm{per}(g).\end{aligned}$$

Consider the exact sequence

(7.13.3) $$\begin{aligned}0 \longrightarrow V_{k,\mathbb{C}}(X_1(N)) \longrightarrow &V_{k,\mathbb{C}}(Y_1(N)) \\ &\xrightarrow{\partial} \oplus_x (\mathrm{R}^1 j_* \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1))_x \otimes \mathbb{C}\end{aligned}$$

where x ranges over all cusps in $X_1(N)(\mathbb{C})$ and $(\ )_x$ means the stalk at x.

By (7.13.2) and (7.13.3) and by the self-duality of $V_{k,\mathbb{C}}(X_1(N))$, we have

(7.13.4) *For $z \in V_{k,\mathbb{C}}(Y_1(N))$, $z = 0$ if and only if z satisfies the following conditions: $\langle z, \mathrm{per}(f)\rangle = \langle z, \iota' \mathrm{per}(f)\rangle = 0$ for all $f \in S_k(X_1(N)) \otimes \mathbb{C}$, and $\partial(z) = 0$.*

Concerning the Poincaré duality (7.13.1), the following (7.13.5)–(7.13.7) hold.

(7.13.5) *For $f \in S_k(X_1(N)) \otimes \mathbb{C}$, $g \in M_k(X_1(N)) \otimes \mathbb{C}$, we have*

$$\langle \mathrm{per}(g), \mathrm{per}(f)\rangle = 0, \quad \langle \iota' \mathrm{per}(g), \iota' \mathrm{per}(f)\rangle = 0,$$
$$\langle \mathrm{per}(g), \iota' \mathrm{per}(f)\rangle = (-8\pi^2 i)^{k-1} \int_{\Gamma_1(N)\backslash \mathfrak{H}} \overline{f(\tau)} g(\tau) y^{k-2} \mathrm{d}x \wedge \mathrm{d}y$$
$(\tau = x + iy,\ x,\ y \in \mathbb{R})$.
$\langle \iota' \mathrm{per}(g), \mathrm{per}(f)\rangle$ = the complex conjugate of $\langle \mathrm{per}(g), \iota' \mathrm{per}(f)\rangle$.

(7.13.6) *For $z \in S_k(X_1(N)) \otimes \mathbb{C}$, and for $1 \leqslant j \leqslant k-1$, we have*

$$\langle \delta_{1,N}(k,j), \mathrm{per}(f)\rangle = \Omega(f,j),$$
$$\langle \delta_{1,N}(k,j), \iota' \mathrm{per}(f)\rangle = \overline{\Omega}(f,j).$$

(7.13.7) *Let $f \in S_2(X_1(N))$, and let w be a closed C^∞-differential form on $Y_1(N)(\mathbb{C})$ which has at each cusp the growth $\mathrm{O}(r^{-1} \log(r)^c)$ $(r \to 0)$ for some $c > 0$ where r*

denotes the distance from the cusp (the metric is defined by fixing an analytic isomorphism between an open neighbourhood of the cusp and an open set of $\mathbb{C}$). Then

$$\langle \mathrm{class}(w), \mathrm{per}(f)\rangle = \int_{\Gamma_1(N)\backslash\mathfrak{H}} w \wedge f \cdot \mathrm{dlog}(q)$$

$$\langle \mathrm{class}(w), \iota' \,\mathrm{per}(f)\rangle = \int_{\Gamma_1(N)\backslash\mathfrak{H}} w \wedge \overline{f} \cdot \mathrm{dlog}(\overline{q}).$$

($(-8\pi^2 i)^{k-1}$ in (7.13.5) comes from

$$\mathrm{dlog}(q) \wedge \mathrm{dlog}(\overline{q}) = -8\pi^2 \cdot \mathrm{d}x \wedge \mathrm{d}y$$

and from

$$\int_{\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})} \mathrm{dlog}(t) \wedge \mathrm{dlog}(\overline{t}) = -8\pi^2 i \cdot \mathrm{Im}(\tau),$$

where $t = e^{2\pi i z}$.)

We reduce Prop. 7.12 (1) to Prop. 7.7 and Prop. 7.14 below. Let $f \in S_k(X_1(N)) \otimes \mathbb{C}$. By (7.13.5)–(7.13.7), we have

$$\langle \text{l.h.s. of } (7.12.1), \iota' \,\mathrm{per}(f)\rangle = (-8\pi^2 i)^{k-1} \int_{\Gamma_1(N)\backslash\mathfrak{H}} \overline{f(\tau)} \cdot \frac{1}{2} \cdot z_\chi(r,r') \cdot y^{k-2}\mathrm{d}x \wedge \mathrm{d}y,$$

$$\langle \text{r.h.s. of } (7.12.1), \iota' \,\mathrm{per}(f)\rangle = (2\pi i)^{k-r-1}\, \overline{\Omega}(Z^T_{1,N}(k,\overline{\chi},r) \cdot f, r')$$

(since $T'(n)$ and $T(n)$ are the adjoints of each other in the Poincaré duality). Next let d be an integer which is prime to N, and let $h = \left(d^2 - d^j\chi(d) \cdot \left(\begin{smallmatrix} d & 0 \\ 0 & 1/d \end{smallmatrix}\right)^*\right) f$. Then by (7.13.5)–(7.13.7), we have

$$\begin{aligned}\langle \text{l.h.s. of } (7.12.2), \iota' \,\mathrm{per}(f)\rangle &= (-8\pi^2 i)^{k-1} \int_{\Gamma_1(N)\backslash\mathfrak{H}} \overline{f(\tau)} \cdot \frac{1}{2} \cdot {}_d z_\chi(r,r') \cdot y^{k-2}\mathrm{d}x \wedge \mathrm{d}y \\ &= (-8\pi^2 i)^{k-1} \int_{\Gamma_1(N)\backslash\mathfrak{H}} \overline{h(\tau)} \cdot \frac{1}{2} \cdot z_\chi(r,r') \cdot y^{k-2}\mathrm{d}x \wedge \mathrm{d}y,\end{aligned}$$

$$\langle \text{r.h.s. of } (7.12.2), \iota' \,\mathrm{per}(f)\rangle = (2\pi i)^{k-r-1}\, \overline{\Omega}(Z^T_{1,N}(k,\overline{\chi},r) \cdot h, r').$$

Hence by 7.7, we have for $e = 1, 2$,

$$\langle \text{l.h.s. of } (7.12.\mathrm{e}), \iota'\mathrm{per}(f)\rangle = \langle \text{r.h.s. of } (7.12.\mathrm{e}), \iota'\mathrm{per}(f)\rangle.$$

By taking the complex conjugate, we have for $e = 1, 2$,

$$\begin{aligned}&\langle \text{l.h.s. of } (7.12.\mathrm{e}), \mathrm{per}(f)\rangle \\ &\quad= \text{the complex conj. of } (-1)^{k-r-1} \cdot \langle \text{l.h.s. of } (7.12.\mathrm{e}) \text{ for } \overline{\chi}, \iota' \,\mathrm{per}(f)\rangle \\ &\quad= \text{the complex conj. of } (-1)^{k-r-1} \cdot \langle \text{r.h.s. of } (7.12.\mathrm{e}) \text{ for } \overline{\chi}, \iota' \,\mathrm{per}(f)\rangle \\ &\quad= \langle \text{r.h.s. of } (7.12.\mathrm{e}), \mathrm{per}(f)\rangle.\end{aligned}$$

Hence by (7.13.4), Prop. 7.12 (1) is reduced to Prop. 7.7 and to Prop. 7.14 below.

We reduce Prop. 7.12 (2) to Prop. 7.7 and to Prop. 7.14 below. For g, $h \in \mathcal{O}(Y_1(N))^{\times} \otimes \mathbb{Q}$, the differential form

$$\eta_{g,h} = \log(|g|) \cdot \mathrm{dlog}(h|h|^{-1}) - \log(|h|) \cdot \mathrm{dlog}(g|g|^{-1})$$

in 2.10 is written also as

$$\begin{aligned}\eta_{g,h} &= \log(|g|) \cdot \mathrm{dlog}(h) + \log(|h|) \cdot \mathrm{dlog}(\overline{g}) - \mathrm{d}\{\log(|g|) \cdot \log(|h|)\} \\ &= -\log(|g|) \cdot \mathrm{dlog}(\overline{h}) - \log(|h|) \cdot \mathrm{dlog}(g) + \mathrm{d}\{\log(|g|) \cdot \log(|h|)\}.\end{aligned}$$

Since $\mathrm{class}(\mathrm{d}\{\log(|g|) \cdot \log(|h|)\}) = 0$, we have by (7.13.7)

$$\begin{aligned}\langle \mathrm{class}(\eta_{g,h}), \iota' \,\mathrm{per}(f)\rangle &= \int_{\Gamma_1(N)\backslash\mathfrak{H}} \log(|g|) \cdot \mathrm{dlog}(h) \wedge \overline{f} \cdot \mathrm{dlog}(\overline{q}) \\ \langle \mathrm{class}(\eta_{g,h}), \mathrm{per}(f)\rangle &= -\int_{\Gamma_1(N)\backslash\mathfrak{H}} \log(|g|) \cdot \mathrm{dlog}(\overline{h}) \wedge f \cdot \mathrm{dlog}(q) \\ &= -\text{ complex conj. of } \langle \mathrm{class}(\eta_{g,h}), \iota' \,\mathrm{per}(f)\rangle\end{aligned}$$

for any $f \in S_2(X_1(N)) \otimes \mathbb{C}$. We have also

$$\log(|g_{\alpha,\beta}|) = -E^{(0)}_{\alpha,\beta}(\tau), \quad \mathrm{dlog}(g_{\alpha,\beta}) = -F^{(2)}_{\alpha,\beta}$$

((3.8.4) (iii), 3.11 (2)). These imply, for any $f \in S_2(X_1(N)) \otimes \mathbb{C}$,

$$\begin{aligned}\langle \text{l.h.s. of (7.12.3)}, \iota' \,\mathrm{per}(f)\rangle &= \int_{\Gamma_1(N)\backslash\mathfrak{H}} \overline{f(\tau)} \cdot z_{\chi}(0,1)(\tau) \cdot \mathrm{dlog}(q) \wedge \mathrm{dlog}(\overline{q}) \\ &= (-8\pi^2 i) \cdot \int_{\Gamma_1(N)\backslash\mathfrak{H}} \overline{f(\tau)} \cdot z_{\chi}(0,1)(\tau) \cdot \mathrm{d}x \wedge \mathrm{d}y \\ \langle \text{r.h.s. of (7.12.3)}, \iota' \,\mathrm{per}(f)\rangle &= (2\pi i) \cdot \overline{\Omega}(\lim_{s\to 0} s^{-1} Z^{T}_{1,N}(2, \overline{\chi}, s) \cdot f, 1)\end{aligned}$$

(since $T'(n)$ and $T(n)$ are the adjoints of each other in the Poincaré duality). By the case $k = 2$ and $r = 0$ of Prop. 7.7, we have

$$\langle \text{l.h.s. of (7.12.3)}, \iota' \mathrm{per}(f)\rangle = \langle \text{r.h.s. of (7.12.3)}, \iota' \mathrm{per}(f)\rangle.$$

By taking the complex conjugate, we have

$$\begin{aligned}&\langle \text{l.h.s. of (7.12.3)}, \mathrm{per}(f)\rangle \\ &\quad = -\text{ the complex conj. of } \langle \text{l.h.s of (7.12.3) for } \overline{\chi}, \iota' \,\mathrm{per}(f)\rangle \\ &\quad = -\text{ the complex conj. of } \langle \text{r.h.s of (7.12.3) for } \overline{\chi}, \iota' \,\mathrm{per}(f)\rangle \\ &\quad = \langle \text{r.h.s. of (7.12.3)}, \mathrm{per}(f)\rangle.\end{aligned}$$

Hence by (7.13.4), Prop. 7.12 (2) is reduced to Prop. 7.7 and Prop. 7.14.

Proposition 7.14. — ∂(l.h.s. of (7.12.e)) $= \partial$(r.h.s. of (7.12.e)) *for* $e = 1, 2, 3$, *where* ∂ *is as in* (7.13.3) *(we take* $k = 2$ *in the case* $e = 3$*).*

7.15. We give preliminaries for the proof of Prop. 7.14.

Let $\infty \in X_1(N)(\mathbb{C})$ (resp. $\widetilde{\infty} \in X(N)(\mathbb{C})$) be the standard cusp, which is the limit point of the image of $yi \in \mathfrak{H}$ $(y \to \infty)$.

The cusps of $X_1(N)(\mathbb{C})$ are described as follows. Let Σ be the set of pairs (v, w) such that $v \in \mathbb{Z}/N$ and $w \in ((\mathbb{Z}/N)/(v))^\times$. Let $\Sigma/\pm 1$ be the quotient of Σ by the equivalence $(v, w) \sim (-v, -w)$. Then there is a unique bijection

$$\{\text{cusps of } X_1(N)(\mathbb{C})\} \longrightarrow \Sigma/\pm 1$$

which sends the image of $\left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right)\widetilde{\infty} \in X(N)(\mathbb{C})$ in $X_1(N)(\mathbb{C})$ to the class of $(v, w \bmod v)$ in $\Sigma/\pm 1$ for any $\left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N)$.

We define a canonical homomorphism

$$R : (\mathrm{R}^1 j_* \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1))_\infty \longrightarrow \mathbb{Z}$$

as follows. Take $c \gg 0$, and let $U = \{\tau \in \mathfrak{H};\ \operatorname{Im}(\tau) > c\}$. Then the map $\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)\backslash U \to \Gamma_1(N)\backslash\mathfrak{H} \cong Y_1(N)(\mathbb{C})$ is an open immersion, and the image of this map has the form

$$(\text{an open neibourhood of } \infty \text{ in } X_1(N)(\mathbb{C})) - \{\infty\}.$$

This map induces

$$\begin{aligned}
(\mathrm{R}^1 j_* \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1))_\infty &\cong \mathrm{H}^1\left(\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)\backslash U, \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1)\right) \\
&\cong \mathrm{H}^0(U, \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1))\Big/\left(1 - \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^*\right)\mathrm{H}^0(U, \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1)) \\
&\cong \mathrm{H}^0(U, \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}_1))\Big/\left(1 - \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)_*\right)\mathrm{H}^0(U, \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}_1)).
\end{aligned}$$

The pull back of $\mathcal{H}_1$ on U is a constant sheaf whose stalk at $\tau \in U$ is identified with $\mathbb{Z}\tau + \mathbb{Z}$. Let e_1 (resp. e_2) be the section of $\mathcal{H}_1$ on U whose stalk at $\tau \in U$ is τ (resp. 1). Then $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)_*(e_1) = e_1 + e_2$. From this we see that

$$\mathrm{H}^0(U, \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}_1))/(1 - \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)_*)\,\mathrm{H}^0(U, \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}_1)) \otimes \mathbb{Q}$$

is a one dimensional $\mathbb{Q}$-vector space generated by the class of e_1^{k-2}, and the classes of $e_1^s e_2^t$ $(s \geqslant 0,\ t \geqslant 1,\ s + t = k - 2)$ in this space are zero. Hence there is a unique homomorphism

$$\mathrm{H}^0(U, \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}_1))/(1 - \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)_*)\,\mathrm{H}^0(U, \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}_1)) \longrightarrow \mathbb{Z}$$

which sends the class of e_1^{k-2} to 1 and the classes of $e_1^s e_2^t$ $(s \geqslant 0,\ t \geqslant 1,\ s + t = k - 2)$ to 0. This is the definition of R.

We define a homomorphism

$$R : (\mathrm{R}^1 j_* \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1))_{\widetilde{\infty}} \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$$

(we use the same letter R) to be the composite

$$(\mathrm{R}^1 j_* \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1))_{\widetilde{\infty}} \otimes \mathbb{Q} \xleftarrow{\sim} (\mathrm{R}^1 j_* \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1))_\infty \otimes \mathbb{Q} \xrightarrow[\sim]{R} \mathbb{Q}$$

We have a commutative diagram

$$\begin{array}{ccc} M_k(X(N)) & \longrightarrow & \mathbb{C} \\ {\scriptstyle \mathrm{per}_{N,N}}\big\downarrow & & \big\downarrow{\scriptstyle \times(2\pi i)^{k-1}} \\ V_{k,\mathbb{C}}(Y(N)) & \xrightarrow{R} & \mathbb{C} \end{array} \tag{7.15.1}$$

where the upper horizontal arrow is $\sum_{n=0}^{\infty} a_n q^{n/N} \mapsto a_0$. We have also a commutative diagram

$$\begin{array}{ccc} \mathrm{K}_2(Y(N)) & \longrightarrow & \mathbb{R} \\ {\scriptstyle \mathrm{reg}_{N,N}}\big\downarrow & & \big\downarrow{\scriptstyle \times 2\pi i} \\ V_{2,\mathbb{C}}(Y(N)) & \xrightarrow{R} & \mathbb{C} \end{array} \tag{7.15.2}$$

where the upper horizontal arrow is induced by

$$\mathrm{K}_2(\mathbb{C}((q^{1/N}))) \longrightarrow \mathbb{R}\,; \quad \{q^{\alpha}au, q^{\beta}bv\} \longmapsto \beta\cdot\log(|a|) - \alpha\log(|b|)$$

$(\alpha,\, \beta \in (1/N)\,\mathbb{Z},\ a,\, b \in \mathbb{C}^{\times},\ u,\, v \in 1 + q^{1/N}\mathbb{C}[[q^{1/N}]])$.

Let

$$B = \oplus_x (\mathrm{R}^1 j_* \operatorname{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1))_x \otimes \mathbb{Q}$$

where x ranges over all cusps of $X_1(N)(\mathbb{C})$. For $\left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N)$, let

$$R \circ \left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right)^* : B \otimes \mathbb{C} \longrightarrow \mathbb{C}$$

be the homomorphism induced by pulling back to $X(N)$, then pulling back by $\left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right)$, and then taking R of the $\widetilde{\infty}$-component. This map depends only on $(v, w \bmod v) \in \Sigma$.

For $z \in B \otimes \mathbb{C}$, $z = 0$ if and only if $R \circ \left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right)^*(z) = 0$ for all $\left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N)$. By this, Prop. 7.14 follows from Lemma 7.16, 7.17 below.

Lemma 7.16. — *Let $\chi : (\mathbb{Z}/m)^{\times} \to \mathbb{C}^{\times}$ be a homomorphism, and let $\left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N)$.*

(1) *Let r, r' be as in (7.4.1), and assume $\chi(-1) = (-1)^{k-r}$ (resp. $(-1)^{k-r'}$) in the case $r' = k-1$ (resp. $r = k-1$). Let*

$$z_{\chi} = z_{\chi}(r, r')$$

assuming $(r, r') \neq (0, k-1),\ (2, k-1),\ (k-1, 2)$

$$(\textit{resp. } z_{\chi} = {}_d z_{\chi}(r, r')$$

for an integer d such that $(d, N) = 1$ assuming $r \neq 0$), and let

$$B_{\chi} = R \circ \left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right)^* \circ \partial \circ \mathrm{per}_{1,N}(z_{\chi}), \quad P = \frac{1}{2}(B_{\chi} + (-1)^{k-r-1} \cdot \overline{B_{\overline{\chi}}}).$$

(i) *Assume* $k \geqslant 3$, $r' = k-1$. *Then, in the case* $v \neq 0$,

$$P = 0.$$

In the case $v = 0$, *if* $Q(w)$ *denotes*

$$\mathrm{L}(\chi, r+1-k) \cdot \chi(w) \cdot \Big\{\zeta\big(w/N, s\big) + (-1)^r \zeta\big(-w/N, s\big)\Big\}_{s=r} \cdot (2\pi i)^{k-r-1} N^{-r}$$

(here $\zeta(\alpha, s)$, $(\alpha \in \mathbb{Q}/\mathbb{Z})$ *is as in 3.9), we have*

$$P = Q(w) \quad (\textit{resp.} \quad P = d^2 Q(w) - d^j \overline{\chi}(d) Q(dw)\,).$$

(ii) *Assume* $k \geqslant 3$, $2 \leqslant r' \leqslant k-2$. *Then,*

$$P = 0.$$

(iii) *Let* $k \geqslant 3$, $(r, r') = (k-1, 1)$. *Then if* $Q(v)$ *denotes*

$$-\mathrm{L}(\chi, r) N^{-1} \chi(v) \Big(\zeta\big(v/N, 0\big) - \zeta\big(-v/N, 0\big)\Big),$$

we have

$$P = Q(v) \quad (\textit{resp.}\ P = d^2 Q(v) - d^j \overline{\chi}(d) Q(dv)).$$

(iv) *Assume* $k = 2$ *(so* $r = r' = 1$*). Then in the case* $v \neq 0$, *if* $Q(v)$ *denotes*

$$-\mathrm{L}(\chi, 1)\chi(v)\Big(\zeta\big(v/N, 0\big) - \zeta\big(-v/N, 0\big)\Big) \cdot N^{-1},$$

we have

$$P = Q(v) \quad (\textit{resp.}\ P = d^2 Q(v) - d^j \overline{\chi}(d) Q(dv)).$$

In the case $v = 0$, *if* $Q(w)$ *denotes*

$$\mathrm{L}(\chi, 0)\chi(w)\Big\{\zeta\big(w/N, s\big) - \zeta\big(-w/N, s\big)\Big\}_{s=1} \cdot N^{-1},$$

we have

$$P = Q(w) \quad (\textit{resp.}\ P = d^2 Q(w) - d^j \overline{\chi}(d) Q(dw)).$$

(2) *Let* $m \geqslant 2$, *and assume* $\chi(-1) = 1$. *Let*

$$P = \sum_{a \in (\mathbb{Z}/m)^\times} \chi(a) R \circ \left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right)^* \circ \partial \circ \mathrm{reg}_{1,N} \Big(\Big\{ \prod_{b \in \mathbb{Z}/m} g_{a/m, b/m}, g_{0,1/N} \Big\}\Big).$$

Then, in the case $v \neq 0$,

$$P = -\lim_{s \to 0} s^{-1}\, \mathrm{L}(\chi, s) \cdot \chi(v)\Big(\zeta\big(v/N, -1\big) + \zeta\big(-v/N, -1\big)\Big) \cdot 2\pi i.$$

In the case $v = 0$,

$$P = \mathrm{L}(\chi, -1)\chi(w) \lim_{s \to 0} s^{-1}\Big(\zeta\big(w/N, s\big) + \zeta\big(-w/N, s\big)\Big) \cdot 2\pi i.$$

Proof. — By using the commutative diagram (7.15.1) (resp. (7.15.2)), 7.16 (1) (resp. 7.16 (2)) follows from the expressions of the constant terms of Eisenstein series in 3.10 (resp. from the "q-presentations" of Siegel units in 1.9) and the following well known equalities (7.16.1) (resp. (7.16.2)) and (7.16.3) for $\zeta(\alpha, s)$ and $\zeta^*(\alpha, s)$.

(7.16.1) *For $\alpha \in \mathbb{Q}/\mathbb{Z}$ and for any integer $r \geqslant 1$,*

$$\frac{(-1)^r}{(r-1)!} \cdot \frac{1}{2}\big(\zeta^*(\alpha, 1-r) + (-1)^r \zeta^*(-\alpha, 1-r)\big) = \\ (2\pi i)^{-r}\big\{\zeta(\alpha, s) + (-1)^r \zeta(\alpha, s)\big\}_{s=r}.$$

(7.16.2) *For $\alpha \in \mathbb{Q}/\mathbb{Z} \smallsetminus \{0\}$, if we denote $\exp(2\pi i\alpha)$ by η, we have*

$$\log|1-\eta| = \lim_{s\to 0} s^{-1}(\zeta(\alpha, s) + \zeta(-\alpha, s)).$$

(7.16.3) *For any integer $r \leqslant -1$, $\zeta(\alpha, r) = (-1)^{r-1}\zeta(-\alpha, r)$.* □

Lemma 7.17. — *Let $\chi : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$ be a character, and let $\left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N)$. Let j be any integer such that $1 \leqslant j \leqslant k-1$. Let*

$$P(s) = R \circ \left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right)^* \circ \partial(Z_{1,N}(k, \chi, s) \cdot \delta_{1,N}(k, j)).$$

(1) *Assume $k \geqslant 3$, $j = k-1$. Then, in the case $v \neq 0$,*

$$P(s) = 0.$$

In the case $v = 0$,

$$P(s) = \mathrm{L}(\chi, s-k+1)\Big(\chi(w)\zeta(w/N, s) + (-1)^k \chi(-w)\zeta(-w/N, s)\Big) \cdot N^{-s}.$$

(2) *Assume $k \geqslant 3$, $2 \leqslant j \leqslant k-2$. Then*

$$P(s) = 0.$$

(3) *Assume $k \geqslant 3$, $j = 1$. Then*

$$P(s) = -\mathrm{L}(\chi, s)\Big(\chi(v)\zeta(v/N, s-k+1) + (-1)^k \chi(-v)\zeta(-v/N, s-k+1)\Big) \cdot N^{-s+k-2}.$$

(4) *Assume $k = 2$ (so $j = 1$). Then, in the case $v \neq 0$,*

$$P(s) = -\mathrm{L}(\chi, s)\Big(\chi(v)\zeta(v/N, s-1) + \chi(-v)\zeta(-v/N, s-1)\Big) \cdot N^{-s}.$$

In the case $v = 0$ and $m \geqslant 2$,

$$P(s) = \mathrm{L}(\chi, s-1)\Big(\chi(w)\zeta(w/N, s) + \chi(-w)\zeta(-w/N, s)\Big) \cdot N^{-s}.$$

In the case $v = 0$ and $m = 1$,

$$P(s) = \zeta(s-1)\Big(\zeta(w/N, s) + \zeta(-w/N, s) - 2\zeta(s)\Big) \cdot N^{-s}.$$

Proof. — Let Σ and B be as in 7.15. For $(v,w)\in\Sigma$, let

$$b(v,w)\in B$$

be the element characterized by the following properties: $b(v,w)$ is supported on the cusp corresponding to the element (v,w) mod $\pm$ of $\Sigma/\pm$, and $R\circ\left(\begin{smallmatrix} t & u\\ v & w\end{smallmatrix}\right)^* b(v,w)=1$ for any t, $u\in\mathbb{Z}/N$ such that $\left(\begin{smallmatrix} t & u\\ v & w\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z}/N)$. Then for any $\left(\begin{smallmatrix} t & u\\ v & w\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z}/N)$ and any $(v',w')\in\Sigma$, we have

$$R\circ\left(\begin{smallmatrix} t & u\\ v & w\end{smallmatrix}\right)^* b(v',w')=\begin{cases}1 & \text{if } v=v',\ w\equiv w' \bmod (v)\\ (-1)^k & \text{if } v=-v' \text{ and } w\equiv -w' \bmod (v)\\ 0 & \text{otherwise.}\end{cases}\tag{7.17.1}$$

For $n\geqslant 1$, let $T'(n): B\to B$ be the dual Hecke operator. Let ℓ be a prime number, let $\mathrm{ord}_\ell(N)$ be the ℓ-adic order of N, and let $\mathrm{ord}_\ell(N,v)$ be the ℓ-adic order of the order of the ring $(\mathbb{Z}/N)/(v)$. By considering the dual Hecke operators locally at cusps, we have

$$T'(\ell)b(v,w)=b(v,\ell w)+\ell^{k-1}b(\ell v,w)$$

if $\mathrm{ord}_\ell(N)=0$,

$$T'(\ell)b(v,w)=b(v,\ell w)+\sum_{w'}\ell^{k-1}b(\ell v,w')$$

if $\mathrm{ord}_\ell(N)>\mathrm{ord}_\ell(N,v)=0$, where w' ranges over all elements of $((\mathbb{Z}/N)/(\ell v))^\times$ whose image in $((\mathbb{Z}/N)/(v))^\times$ coincides with w,

$$T'(\ell)b(v,w)=\ell^{k-1}\sum_{w'}b(\ell v,w')$$

if $\mathrm{ord}_\ell(N)>\mathrm{ord}_\ell(N,v)>0$, where w' ranges over all elements of $((\mathbb{Z}/N)/(\ell v))^\times$ whose image in $((\mathbb{Z}/N)/(v))^\times$ coincides with w,

$$T'(\ell)b(v,w)=\ell^{k-1}b(\ell v,w)$$

if $\mathrm{ord}_\ell(N)=\mathrm{ord}_\ell(N,v)>0$. From this we have

$$\begin{aligned}&Z_{1,N}(k,\chi,s)\cdot b(0,1)=\\&\qquad \mathrm{L}(\chi,s-k+1)\sum_{y\in(\mathbb{Z}/N)^\times}\chi(y)\zeta(y/N,s)\cdot N^{-s}\cdot\left(\begin{smallmatrix} y & 0\\ 0 & 1/y\end{smallmatrix}\right)^*\cdot b(0,1)\end{aligned}\tag{7.17.2}$$

$$Z_{1,N}(k,\chi,s)\cdot b(1,0)=\mathrm{L}(\chi,s)\sum_{n\geqslant 1}\chi(n)n^{-(s-k+1)}\cdot\Big(\sum_{y\in(\mathbb{Z}/(N,n))^\times}b(n,y)\Big).\tag{7.17.3}$$

On the other hand,

$$\partial(\delta_{1,N}(k,k-1)) = b(0,1) \quad \text{if} \quad k \geqslant 3, \tag{7.17.4}$$

$$\partial(\delta_{1,N}(k,j)) = 0 \quad \text{if} \quad 2 \leqslant j \leqslant k-2, \tag{7.17.5}$$

$$\partial(\delta_{1,N}(k,1)) = -N^{-1}b(1,0) \quad \text{if} \quad k \geqslant 3, \tag{7.17.6}$$

$$\partial(\delta_{1,N}(2,1)) = b(0,1) - N^{-1}b(1,0). \tag{7.17.7}$$

Lemma 7.17 follows from these (7.17.2)–(7.17.7). □

7.18. In the rest of this section, we deduce Thm. 2.6 and Thm. 4.6 from Prop. 7.12. Let M, $N \geqslant 1$, $M+N \geqslant 5$.

First we remark that Thm. 2.6 and Thm. 4.6 can be formulated without using the operator ι, but using ι' instead.

The image of the period map

$$M_k(X(M,N)) \longrightarrow V_{k,\mathbb{C}}(Y(M,N))$$

and the image of the regulator map

$$\mathrm{K}_2(Y(M,N)) \longrightarrow V_{2,\mathbb{C}}(Y(M,N))$$

are contained in the fixed part of the operator $\iota \circ \iota' = \iota' \circ \iota$.

Hence

$$\iota\, \mathrm{per}_{M,N}(x) = \iota'\, \mathrm{per}_{M,N}(x) \quad \text{for any} \quad x \in M_k(X(M,N)), \tag{7.18.1}$$

$$\iota\, \mathrm{reg}_{M,N}(x) = \iota'\, \mathrm{reg}_{M,N}(x) \quad \text{for any} \quad x \in \mathrm{K}_2(Y(M,N)) \otimes \mathbb{Q}. \tag{7.18.2}$$

Thus, the left hand sides of Thm. 2.6 and Thm. 4.6 are rewritten in the forms using ι' instead of ι. On the other hand, the right hand sides of Thm. 2.6 and Thm. 4.6 can be rewritten also in the form without using ι by the following lemma.

Lemma 7.19. — $\iota(\delta_{M,N}(k,j)) = (-1)^{k-j} \cdot \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)^* \cdot \delta_{M,N}(k,j)$ *$(1 \leqslant j \leqslant k-1)$.*

Proof. — For $y > 0$, let E_y be the elliptic curve over $\mathbb{R}$ defined by the equation

$$Y^2 = 4X^3 - 10 \cdot E_{0,0}^{(4)}(yi) \cdot X - \frac{7}{6} \cdot E_{0,0}^{(6)}(yi).$$

(Since $E_{0,0}^{(h)}$ for $h \geqslant 1$, $h \neq 2$ has a q-expansion with rational coefficients, and since $q = e^{2\pi i \tau} \in \mathbb{R}$ if $\tau = yi$, $E_{0,0}^{(h)}(yi)$ belongs to $\mathbb{R}$.) We have an isomorphism

$$e_y : \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) \xrightarrow{\sim} E_y(\mathbb{C})\,;$$

$$e_y(z) = \Big(\wp(yi,z), \big(\frac{\mathrm{d}}{2\pi i \mathrm{d}z}\big)\wp(yi,z)\Big)$$

where $\wp(\tau,z)$ ($\tau \in \mathfrak{H}$, $z \in \mathbb{C}$) is as in 3.8. The map $\varphi : (0,\infty) \to Y(M,N)(\mathbb{C})$ in 4.7 is written as

$$y \longmapsto \big(E_y, e_y(yi/M), e_y(1/N)\big).$$

By the definition of $\wp(\tau,z)$ in 3.8, we have

$$\iota(e_y(z)) = e_y(-\overline{z}) \quad \text{for} \quad z \in \mathbb{C} \tag{7.19.1}$$

where ι denotes the complex conjugation $E_y(\mathbb{C}) \to E_y(\mathbb{C})$. From (7.19.1), we have

$$\iota\big(e_y(yi/M)\big) = e_y(yi/M), \quad \iota\big(e_y(1/N)\big) = -e_y(1/N).$$

Hence the complex conjugation $\iota : Y(M,N)(\mathbb{C}) \to Y(M,N)(\mathbb{C})$ satisfies

$$\iota(\varphi(y)) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \cdot \varphi(y).$$

Furthermore (7.19.1) also shows that $\iota : \mathrm{H}_1(E_y(\mathbb{C}),\mathbb{Z}) \to \mathrm{H}_1(E_y(\mathbb{C}),\mathbb{Z})$ satisfies

$$\iota(e_1) = e_1, \quad \iota(e_2) = -e_2$$

where e_1, e_2 are as in 4.7, and hence we have

$$\iota(\mathrm{class}(\varphi, e_1^{j-1}e_2^{k-j-1})) = (-1)^{k-j-1} \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)_* \mathrm{class}(\varphi, e_1^{j-1}e_2^{k-j-1})$$

where ι denotes the automorphism of $\mathrm{H}_1(X(M,N)(\mathbb{C}), \{\mathrm{cusps}\}, \mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}_1))$ induced by the complex conjugation of $X(M,N)(\mathbb{C})$ and that of $E(\mathbb{C})$ with E the universal elliptic curve over $Y(M,N)$. Via the isomorphism

$$\mathrm{H}^1(Y(M,N)(\mathbb{C}), \mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}^1)) \cong \mathrm{H}_1(X(M,N)(\mathbb{C}), \{\mathrm{cusps}\}, \mathrm{Sym}_{\mathbb{Z}}^{k-2}(\mathcal{H}_1)),$$

the complex conjugation ι on the l.h.s corresponds to $(-1)^{k-1}\iota$ of the r.h.s, and hence we obtain

$$\begin{aligned} \iota\big(\delta_{M,N}(k,j)\big) &= (-1)^{k-1} \cdot (-1)^{k-j-1} \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)_* \delta_{M,N}(k,j) \\ &= (-1)^j \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)^* \delta_{M,N}(k,j). \end{aligned}$$

Since $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)^*$ acts on $(-1)^k$ on $V_{k,\mathbb{Z}}(Y(M,N))$, this implies

$$\iota\big(\delta_{M,N}(k,j)\big) = (-1)^{k-j} \cdot \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)^* \delta_{M,N}(k,j). \qquad \square$$

7.20. Now we reduce Thm. 2.6 and Thm. 4.6 to Prop. 7.12.

We first prove Thm. 4.6. Let M, $N \geqslant 1$, $M+N \geqslant 5$, and assume $\mathrm{prime}(M) \subset \mathrm{prime}(N)$. The proof for zeta elements with c,d and that for those without c,d are given in the same way, and so we give here the proof with c,d. We will apply Prop. 7.12 by taking (MN, M) as (N, m) of 7.12. Take $M' \geqslant 1$ and $L \geqslant 4$ such that

$$M^2 \mid M', \quad MN \mid L, \quad \mathrm{prime}(M') = \mathrm{prime}(M), \quad \mathrm{prime}(L) = \mathrm{prime}(MN)$$

and let

$$Y(M',L) \longrightarrow Y_1(MN) \otimes \mathbb{Q}(\zeta_M) \tag{7.20.1}$$

be the composite

$$Y(M',L) \longrightarrow Y(M(M),L) \xrightarrow[\sim]{\varphi_M^{-1}} Y(M,L(M)) \longrightarrow Y_1(MN) \otimes \mathbb{Q}(\zeta_M)$$

which is used to define ${}_{c,d}z_{1,MN,M}(k,r,r',0(M),\mathrm{prime}(M))$ (5.2). Then it can be shown that the canonical projection $Y(M',L) \to Y(M,N)$ factors through (7.20.1). Hence the trace maps

$$M_k(X(M',L)) \longrightarrow M_k(X(M,N)) \quad \text{and} \quad V_{k,\mathbb{Q}}(Y(M',L)) \longrightarrow V_{k,\mathbb{Q}}(Y(M,N))$$

factor through the surjections

$$\begin{aligned} t_{M,0(M)} &: M_k(X(M',L)) \longrightarrow M_k(X_1(MN)\otimes\mathbb{Q}(\zeta_M)), \\ t_{M,0(M)} &: V_{k,\mathbb{Q}}(Y(M',L)) \longrightarrow V_{k,\mathbb{Q}}(Y_1(MN)\otimes\mathbb{Q}(\zeta_M)), \end{aligned}$$

respectively.

The trace map $M_k(X(M',L)) \to M_k(X(M,N))$ sends ${}_{c,d}z_{M',L}(k,r,r')$ to ${}_{c,d}z_{M,N}(k,r,r')$ by Prop. 4.3, and $t_{M,0(M)}$ sends ${}_{c,d}z_{M',L}(k,r,r')$ to

$$z_{1,MN,M}(k,r,r',0(M),\mathrm{prime}(M)) = M^{r'-1}z_{1,MN,M}(k,r,r',0(1),\mathrm{prime}(M))$$

(Prop. 5.10). The trace map $V_{k,\mathbb{Q}}(Y(M',L)) \to V_{k,\mathbb{Q}}(Y(M,N))$ sends $\delta_{M',L}(k,r')$ to $\delta_{M,N}(k,r')$, and $t_{M,0(M)}$ sends $\delta_{M',L}(k,r')$ to

$$\delta_{1,MN}(k,r',0(M)) = A^{r'-1}\delta_{1,MN}(k,r') \in V_{k,\mathbb{Q}}(Y_1(MN)) \subset V_{k,\mathbb{Q}}(Y_1(MN)\otimes\mathbb{Q}(\zeta_M)).$$

(Here we regard $V_{k,\mathbb{Q}}(Y_1(MN))$ as a direct summand of $V_{k,\mathbb{Q}}(Y_1(MN)\otimes\mathbb{Q}(\zeta_M))$ in the canonical way.) By these facts and by (7.18.1), 7.19, Thm. 4.6 is reduced to the special case $\xi = 0(1)$, $S = \mathrm{prime}(m)$, $m \mid N$, $N \geqslant 4$ of Thm. 5.6 (2). By Lemma 7.5, this case follows from Prop. 7.12 (1).

In the similar way, Thm. 2.6 is reduced to the special case $\xi = 0(1)$, $S = \mathrm{prime}(m)$, $m \mid N$, $N \geqslant 4$ of Thm. 5.6 (1), and this case follows from Prop. 7.12 (2).

CHAPTER II
p-ADIC EULER SYSTEMS

In this Chapter II, we define Euler systems in the Galois cohomology groups related to cusp forms. We define them in § 8 by using Euler systems in K_2 of modular curves. A mysterious fact is that, via p-adic Hodge theroy, they are related to the Euler systems in the spaces of modular forms (see § 9) and hence to the zeta values $L(f,r)$ ($1 \leqslant r \leqslant k-1$) for cusp forms f of weight k (§ 9). We will deduce this fact from a generalized explicit reciprocity law in [**KK3**].

In Chapter II, we fix a prime number p.

We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

8. Definitions of p-adic Euler systems

8.1. Fix $k \geqslant 2$.

In this section, we construct the following "p-adic zeta element" (8.1.1)–(8.1.3) basing on zeta elements in K_2 of modular curves constructed in § 2. The right hand sides of (8.1.1)–(8.1.3) are étale cohomology groups, and $(k-r)$ means the Tate twist, as explained in 8.1–8.3. Our method to define p-adic zeta elements by using Beilinson elements is the "modular curve version" of the method of Soulé [**So**] in the cyclotomic theory in which he defined various p-adic cyclotomic elements by using cylotomic units. In the next section, these p-adic zeta elements will be related to zeta elements in the spaces of modular forms considered in § 4.

$$(8.1.1)\qquad {}_{c,d}z^{(p)}_{M,N}(k,r,r') \in \mathrm{H}^1\big(\mathbb{Z}[1/p], V_{k,\mathbb{Z}_p}(Y(M,N))(k-r)\big)$$

where M, $N \geqslant 1$, $M+N \geqslant 5$, and r, r', c, d are integers satisfying

$$1 \leqslant r' \leqslant k-1, \quad (c, 6pM) = 1, \quad (d, 6pN) = 1.$$

$$(8.1.2)\qquad {}_{c,d}z^{(p)}_{1,N,m}(k,r,r',\xi,S) \in \mathrm{H}^1\big(\mathbb{Z}[1/p,\zeta_m], V_{k,\mathbb{Z}_p}(Y_1(N))(k-r)\big)$$

where N, $m \geqslant 1$, ξ, S are as in (5.1.1), $p \in S$, and r, r', c, d are integers satisfying

$$1 \leqslant r' \leqslant k-1, \quad \mathrm{prime}(cd) \cap S = \varnothing, \quad (cd, 6) = 1, \quad (d, N) = 1.$$

$$(8.1.3)\qquad {}_{c,d}z^{(p)}_{m}(f,r,r',\xi,S) \in \mathrm{H}^1\big(\mathbb{Z}[1/p,\zeta_m], V_{\mathrm{O}_\lambda}(f)(k-r)\big)$$

where $m \geqslant 1$, $f = \sum_{n\geqslant 1} a_n q^n$ is a normalized newform in $M_k(X_1(N)) \otimes \mathbb{C}$ ($N \geqslant 1$), ξ, S, r, r', c, d are as in (8.1.2), λ is a finite place of $F = \mathbb{Q}(a_n;\ n \geqslant 1)$, and O_λ is the valuation ring of λ.

The p-adic zeta elements (8.1.1) (resp. (8.1.2), (8.1.3)) will be defined in 8.4 (resp. 8.9, resp. 8.11).

In this paper, the p-adic zeta elements in (8.1.2), (8.1.3) and zeta elements in § 5 and § 6 with $\xi = a(A)$ will take care of zeta values with bad Euler factors (Euler factors at primes which divide N). Those with $\xi \in \mathrm{SL}_2(\mathbb{Z})$ can not take care of bad Euler factors, but will take care of delicate integrality.

8.2. We fix notation concerning étale cohomology.

We denote the étale cohomology group $\mathrm{H}^q_{\text{ét}}$ just by H^q. Furthermore, for a ring R, we denote the étale cohomology group $\mathrm{H}^q_{\text{ét}}(\mathrm{Spec}(R),\ )$ simply by $\mathrm{H}^q(R,\ )$. In the case R is an integral domain with field of fractions K, we denote $\mathrm{H}^q(R, j_*(\mathfrak{A}))$ simply by $\mathrm{H}^q(R,\mathfrak{A})$ for a sheaf of abelian groups $\mathfrak{A}$ on $\mathrm{Spec}(K)_{\text{ét}}$, where $j : \mathrm{Spec}(K) \to \mathrm{Spec}(R)$ is the inclusion morphism.

For a field K with a fixed separable closure $\overline{K}$, we identify a sheaf $\mathfrak{A}$ on $\mathrm{Spec}(K)_{\text{ét}}$ with the corresponding $\mathrm{Gal}(\overline{K}/K)$-set.

Let K be a finite extension of $\mathbb{Q}_\ell$ for a prime number ℓ (resp. K be a finite extension of $\mathbb{Q}$). Let $R = K$ (resp. R be a ring of the form $\mathrm{O}_K[a^{-1}]$ for some $a \in \mathrm{O}_K \smallsetminus \{0\}$

such that p is invertible in R). For a finitely generated $\mathbb{Z}_p$-module T endowed with a continuous action of $\mathrm{Gal}(\overline{K}/K)$ (resp. a continuous action of $\mathrm{Gal}(\overline{K}/K)$ which is unramified at almost all finite places of K), we denote

$$\mathrm{H}^q(R,T) \underset{\mathrm{def}}{=} \varprojlim_n \mathrm{H}^q(R,T/p^n).$$

It is known that $\mathrm{H}^q(R,T)$ is a finitely generated $\mathbb{Z}_p$-module, and is zero if $q > 2$ (resp. if $q > 2$ and p is odd or if $q > 2$ and K is totally imaginary). For a finite dimensional $\mathbb{Q}_p$-vector space V endowed with a continuous action of $\mathrm{Gal}(\overline{K}/K)$, we denote

$$\mathrm{H}^q(R,V) \underset{\mathrm{def}}{=} \mathrm{H}^q(R,T)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$$

where T is a $\mathrm{Gal}(\overline{K}/K)$-stable $\mathbb{Z}_p$-lattice in V. (Such T exists, and the r.h.s is independent of the choice of T.)

If K is a finite extension of $\mathbb{Q}_\ell$ for a prime number ℓ, $\mathrm{H}^q(K,T)$ and $\mathrm{H}^q(K,V)$ coincide with the continuous Galois cohomology groups $\mathrm{H}^q(\mathrm{Gal}(\overline{K}/K),T)$ and $\mathrm{H}^q(\mathrm{Gal}(\overline{K}/K),V)$, respectively.

8.3. For M, $N \geqslant 1$ such that $M+N \geqslant 5$, define a smooth $\mathbb{Z}_p$-sheaf $\mathcal{H}^1_p$ on $Y(M,N)_{\text{ét}}$ as follows. Let $\lambda : E \to Y(M,N)$ be the universal elliptic curve. We define

$$\mathcal{H}^1_p = \mathrm{R}^1\lambda_*(\mathbb{Z}_p).$$

The $\mathbb{Z}_p$-sheaf on $Y(M,N)(\mathbb{C})$ associated to $\mathcal{H}^1_p$ coincides with $\mathcal{H}^1\otimes\mathbb{Z}_p$, and the étale cohomology group $\mathrm{H}^1\big(Y(M,N)\otimes\overline{\mathbb{Q}},\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p)\otimes_{\mathbb{Z}_p}A\big)$ for $A=\mathbb{Z}_p$, $\mathbb{Q}_p$ or $\mathbb{Z}/p^n$ ($n \geqslant 1$) is identified with $V_{k,A}(Y(M,N)) = \mathrm{H}^1\big(Y(M,N)(\mathbb{C}),\mathrm{Sym}^{k-2}_{\mathbb{Z}}(\mathcal{H}^1)\otimes A\big)$ (4.5.1). Thus, for such A, $V_{k,A}(Y(M,N))$ is endowed with a canonical action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This action is unramified at any prime number which does not divide pMN. This explains the notation on the r.h.s of (8.1.1).

For any curve Y of the form $G\backslash Y(N)$ with $N \geqslant 3$ and G a subgroup of $\mathrm{GL}_2(\mathbb{Z}/N)$, $V_{k,\mathbb{Q}_p}(Y)$ for $k \geqslant 2$ (5.4) is endowed with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is induced from the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_{k,\mathbb{Q}_p}(Y(N))$.

We have defined $V_{k,\mathbb{Z}_p}(Y_1(N))$ for $N \geqslant 4$ (the case $M = 1$ of (4.5.1)). In the case $N = 1$, 2, 3, we define $V_{k,\mathbb{Z}_p}(Y_1(N))$ as follows (in an ad hoc way). Let $1 \leqslant N \leqslant 3$. Take $L \geqslant 4$ such that $N \mid L$, and let $V_{k,\mathbb{Z}_p}(Y_1(N))$ be the image of the trace map $V_{k,\mathbb{Z}_p}(Y_1(L)) \to V_{k,\mathbb{Q}_p}(Y_1(N))$. Then $V_{k,\mathbb{Z}_p}(Y_1(N))$ is independent of the choice of L. The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_{k,\mathbb{Q}_p}(Y_1(N))$ induces the actions of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_{k,\mathbb{Z}_p}(Y_1(N))$. This explains the notations on the r.h.s of (8.1.2).

Finally the notation on the r.h.s of (8.1.3) is as follows. For a finite place λ of $F = \mathbb{Q}(a_n\,;\ n \geqslant 1)$, let

$$F_\lambda,\quad \mathrm{O}_\lambda$$

be the local field of F at λ, and the valuation ring of F_λ, respectively. Let $V_{\mathrm{O}_\lambda}(f)$ be the O_λ-submodule of $V_{F_\lambda}(f)$ (6.3) generated by the image of $V_{k,\mathbb{Z}_p}(Y_1(N))$. Then

$V_{O_\lambda}(f)$ is a free O_λ-module of rank 2. The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_{\mathbb{Q}_p}(Y_1(N))$ induces an F_λ-linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_{F_\lambda}(f)$ and an O_λ-linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_{O_\lambda}(f)$.

8.4. We define the p-adic zeta elements (8.1.1).

By 2.3, we have an element

$$({}_{c,d}z_{Mp^n,Np^n})_{n\geqslant 1} \in \varprojlim_n \mathrm{K}_2(Y(Mp^n,Np^n)).$$

where the inverse limit is taken with respect to the norm map. For r, $r' \in \mathbb{Z}$ such that $1 \leqslant r' \leqslant k-1$, we define below a canonical homomorphism

$$\mathrm{Ch}_{M,N}(k,r,r') : \varprojlim_n \mathrm{K}_2(Y(Mp^n,Np^n)) \longrightarrow \mathrm{H}^1\big(\mathbb{Z}[1/p], V_{k,\mathbb{Z}_p}(Y(M,N))(k-r)\big).$$

We define

$${}_{c,d}z^{(p)}_{Mp^n,Np^n}(k,r,r') = \mathrm{Ch}_{M,N}(k,r,r')\big(({}_{c,d}z_{Mp^n,Np^n})_{n\geqslant 1}\big).$$

The definition of $\mathrm{Ch}_{M,N}(k,r,r')$ is as follows.

Let E be the universal elliptic curve over $Y(M,N)$, and let T_pE be the p-adic Tate module of E regarded as a p-adic smooth sheaf on $Y(M,N)_{\text{ét}}$. Poincaré duality gives a canonical isomorphism

$$T_pE \cong \mathcal{H}^1_p(1) \tag{8.4.1}$$

where (1) means the Tate twist, and this induces

$$\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p) \cong (\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(T_pE))(2-k). \tag{8.4.2}$$

Define $\mathrm{Ch}_{M,N}(k,r,r')$ to be the composite map

(8.4.3)

$$\begin{aligned}
\varprojlim_n \mathrm{K}_2(Y(Mp^n,Np^n)) &\longrightarrow \varprojlim_n \mathrm{H}^2(Y(Mp^n,Np^n),(\mathbb{Z}/p^n)(2))\\
&\longrightarrow \varprojlim_n \mathrm{H}^2(Y(Mp^n,Np^n),(\mathrm{Sym}^{k-2}(T_pE/p^n))(2-r))\\
&\xrightarrow{\sim} \varprojlim_n \mathrm{H}^2(Y(Mp^n,Np^n),(\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p)/p^n)(k-r))\\
&\longrightarrow \varprojlim_n \mathrm{H}^2(Y(M,N),(\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p)/p^n)(k-r))\\
&\longrightarrow \varprojlim_n \mathrm{H}^1(\mathbb{Q},V_{k,\mathbb{Z}/p^n}(Y(M,N))(k-r))
\end{aligned}$$

where:

The first arrow is the Chern character map. (For a scheme X on which p is invertible and for f, $g \in \mathcal{O}(X)^\times$, the Chern character map $\mathrm{K}_2(X) \to \mathrm{H}^2(X,(\mathbb{Z}/p^n)(2))$ sends $\{f,g\}$ to $h(f)\cup h(g)$, where h is the connecting map $\mathcal{O}(X)^\times \to \mathrm{H}^1(X,(\mathbb{Z}/p^n)(1))$ of the Kummer sequence

$$0 \longrightarrow (\mathbb{Z}/p^n)(1) \longrightarrow \mathcal{O}_X^\times \xrightarrow{p^n} \mathcal{O}_X^\times \longrightarrow 0,$$

and $\cup$ is the cup product.) The second arrow is the product with

$$e_{1,n}^{\otimes(r'-1)} \otimes e_{2,n}^{\otimes(k-r'-1)} \otimes (\zeta_{p^n})^{\otimes(-r)}$$

where $(e_{1,n}, e_{2,n})$ is the basis of T_pE/p^n over $Y(Mp^n, Np^n)$. The third arrow is by (8.4.2). The fourth arrow is the trace map. The last arrow is defined by the spectral sequence

$$E_2^{a,b} = \mathrm{H}^a(\mathbb{Q}, \mathrm{H}^b(Y(M,N) \otimes \overline{\mathbb{Q}},\ )) \Longrightarrow \mathrm{H}^{a+b}(Y(M,N),\ )$$

and by the fact $\mathrm{H}^b(Y(M,N) \otimes \overline{\mathbb{Q}},\ ) = 0$ for $b \geqslant 2$. (The last fact is because $Y(M,N) \otimes \overline{\mathbb{Q}}$ is an affine curve over an algebraically closed field.) By the following 8.5, 8.6, the image of $\mathrm{Ch}_{M,N}(k,r,r')$ is contained in the image of the canonical injection

$$\begin{aligned}\mathrm{H}^1(\mathbb{Z}[1/p], V_{k,\mathbb{Z}_p}(Y(M,N))(k-r)) \\ &= \varprojlim_n \mathrm{H}^1(\mathbb{Z}[1/p], V_{k,\mathbb{Z}/p^n}(Y(M,N))(k-r)) \\ &\longrightarrow \varprojlim_n \mathrm{H}^1(\mathbb{Q}, V_{k,\mathbb{Z}/p^n}(Y(M,N))(k-r))\end{aligned}$$

Lemma 8.5 (**[Pe0**, 2.2.4], **[Ru4**, B3.3]). — *Let K be a finite extension of $\mathbb{Q}$, let O_K be the ring of integers of K, and let T be a finite $\mathbb{Z}_p$ module endowed with a continuous action of* $\mathrm{Gal}(\overline{K}/K)$*. Then:*

(1) *For any set S of finite places of K containing all places lying over p, the canonical map $\mathrm{H}^1(\mathrm{O}_K[S^{-1}], T) \to \mathrm{H}^1(K,T)$ is injective.*

(2) *The image of $\varprojlim_n \mathrm{H}^1(K(\zeta_{p^n}), T) \to \mathrm{H}^1(K,T)$ is contained in the image of $\mathrm{H}^1(\mathrm{O}_K[1/p], T) \to \mathrm{H}^1(K,T)$.*

Proof. — We have an exact sequence

$$\oplus_v \mathrm{H}^1_v(\mathrm{O}_K[S^{-1}], T) \longrightarrow \mathrm{H}^1(\mathrm{O}_K[S^{-1}], T) \longrightarrow \mathrm{H}^1(K,T) \longrightarrow \oplus_v \mathrm{H}^2_v(\mathrm{O}_K[S^{-1}], T)$$

where v ranges over all maximal ideals of $\mathrm{O}_K[S^{-1}]$ and H^1_v means the cohomology with support in v. For each v, we have an exact sequence

$$\begin{aligned}\mathrm{H}^0(\mathrm{O}_v, T) \xrightarrow{\sim} \mathrm{H}^0(K_v, T) \longrightarrow \mathrm{H}^1_v(\mathrm{O}_K[S^{-1}], T) \\ \longrightarrow \mathrm{H}^1(\mathrm{O}_v, T) \longrightarrow \mathrm{H}^1(K_v, T) \longrightarrow \mathrm{H}^2_v(\mathrm{O}_K[S^{-1}], T)\end{aligned}$$

where K_v is the local field of K at v and O_v is the valuation ring of K_v. It is sufficient to prove that for each maximal ideal v of $\mathrm{O}_K[1/p]$, $\mathrm{H}^1(\mathrm{O}_v, T) \to \mathrm{H}^1(K_v, T)$ is injective and that the image of $\varprojlim_n \mathrm{H}^1(K_v(\zeta_{p^n}), T) \to \mathrm{H}^1(K_v, T)$ is contained in the image of $\mathrm{H}^1(\mathrm{O}_v, T)$. We have

$$\mathrm{H}^1(\mathrm{O}_v, T) = \mathrm{H}^1(\mathrm{Gal}(K_v^{\mathrm{ur}}/K_v), \mathrm{H}^0(K_v^{\mathrm{ur}}, T))$$

where K_v^{ur} denotes the maximal unramified extension of K_v, and hence $\mathrm{H}^1(\mathrm{O}_v, T) \to \mathrm{H}^1(K_v, T)$ is injective. The cokernel of $\mathrm{H}^1(\mathrm{O}_v, T) \to \mathrm{H}^1(K_v, T)$ is isomorphic to

$$
\begin{aligned}
&\mathrm{H}^0(\mathrm{Gal}(K_v^{\mathrm{ur}}/K_v), \mathrm{H}^1(K_v^{\mathrm{ur}}, T)) \\
&\qquad \xrightarrow{\sim} \{\mathrm{H}^1(\mathrm{Gal}(K_v^{\mathrm{ur}}/K_v), \mathrm{H}^0(K_v^{\mathrm{ur}}, T^\vee(1)))\}^\vee \quad ((\ )^\vee = \mathrm{Hom}(\ , \mathbb{Q}_p/\mathbb{Z}_p)) \\
&\qquad \xrightarrow{\sim} \{\mathrm{H}^1(\mathbb{F}_v, \mathrm{H}^0(K_v^{\mathrm{ur}}, T^\vee(1)))\}^\vee
\end{aligned}
$$

where $\mathbb{F}_v$ denotes the residue field of v, and the composite map

$$\varprojlim_n \mathrm{H}^1(K_v(\zeta_{p^n}), T) \longrightarrow \mathrm{H}^1(K_v, T) \longrightarrow \{\mathrm{H}^1(\mathbb{F}_v, \mathrm{H}^0(K_v^{\mathrm{ur}}, T^\vee(1)))\}^\vee$$

factors through $\{\varinjlim_n \mathrm{H}^1(\mathbb{F}_v(\zeta_{p^n}), \mathrm{H}^0(K_v^{\mathrm{ur}}, T^\vee(1)))\}^\vee$. Hence we are reduced to showing $\varinjlim_n \mathrm{H}^1(\mathbb{F}_v(\zeta_{p^n}), \mathrm{H}^0(K_v^{\mathrm{ur}}, T^\vee(1))) = 0$ and hence to the fact that the p-cohomological dimension of the field $\bigcup_{n\geqslant 1} \mathbb{F}_v(\zeta_{p^n})$ is zero. □

8.6. The projections $Y(Mp^n, Np^n) \to Y(M, N)$ factor canonically as

$$Y(Mp^n, Np^n) \longrightarrow Y(M, N) \otimes \mathbb{Q}(\zeta_{p^n}) \longrightarrow Y(M, N),$$

and hence the image of $\mathrm{Ch}_{M,N}(k, r, r')$ is contained in the image of

$$
\begin{aligned}
&\varprojlim_{m,n} \mathrm{H}^1\Big(\mathbb{Q}, \mathrm{H}^1(Y(M, N) \otimes \mathbb{Q}(\zeta_{p^n}) \otimes \overline{\mathbb{Q}}, \mathrm{Sym}^{k-2}_{\mathbb{Z}/p^m}(\mathcal{H}_p^1/p^m))\Big) \\
&\qquad\qquad\qquad\qquad = \varprojlim_{m,n} \mathrm{H}^1\big(\mathbb{Q}(\zeta_{p^n}), V_{k,\mathbb{Z}/p^m}(Y(M, N))\big).
\end{aligned}
$$

By this and by 8.5, we obtain the last comment in 8.4.

The following 8.7 is deduced from the norm properties 2.3, 2.4 of zeta elements in K_2 by Lemma 8.8 below.

Proposition 8.7. — *Let the notation be as in* (8.1.1).

(1) *Let M', $N' \geqslant 1$ and assume*

$$
\begin{gathered}
M \mid M', \quad N \mid N', \quad (c, M') = (d, N') = 1, \\
\mathrm{prime}(Mp) = \mathrm{prime}(M'p), \quad \mathrm{prime}(Np) = \mathrm{prime}(N'p).
\end{gathered}
$$

Then the norm map

$$\mathrm{H}^1\Big(\mathbb{Z}[1/p], V_{k,\mathbb{Z}_p}(Y(M', N'))(k-r)\Big) \longrightarrow \mathrm{H}^1\Big(\mathbb{Z}[1/p], V_{k,\mathbb{Z}_p}(Y(M, N))(k-r)\Big)$$

sends ${}_{c,d}z^{(p)}_{M',N'}(k, r, r')$ *to* ${}_{c,d}z^{(p)}_{M,N}(k, r, r')$.

(2) *Let ℓ be a prime number which is prime to $Mpcd$. Then the norm map*

$$\mathrm{H}^1\Big(\mathbb{Z}[1/p], V_{k,\mathbb{Z}_p}(Y(M\ell, N\ell))(k-r)\Big) \longrightarrow \mathrm{H}^1\Big(\mathbb{Z}[1/p], V_{k,\mathbb{Z}_p}(Y(M, N))(k-r)\Big)$$

sends ${}_{c,d}z^{(p)}_{M\ell,N\ell}(k, r, r')$ *to* $\big(1 - T'(\ell)\left(\begin{smallmatrix}1/\ell & 0\\ 0 & 1\end{smallmatrix}\right)^* \cdot \ell^{-r} + \left(\begin{smallmatrix}1/\ell & 0\\ 0 & 1/\ell\end{smallmatrix}\right)^* \cdot \ell^{k-1-2r}\big) \cdot {}_{c,d}z^{(p)}_{M,N}(k, r, r')$ *in the case ℓ does not divide N, and to* $\big(1 - T'(\ell)\left(\begin{smallmatrix}1/\ell & 0\\ 0 & 1\end{smallmatrix}\right)^* \cdot \ell^{-r}\big) \cdot {}_{c,d}z^{(p)}_{M,N}(k, r, r')$ *in the case ℓ divides N.*

Lemma 8.8. — *The homomorphism* $\mathrm{Ch}_{M,N}(k,r,r')$ *in 8.4 has the following properties.*

(1) $T'(n) \circ \mathrm{Ch}_{M,N}(k,r,r') = n^{r'-1}\,\mathrm{Ch}_{M,N}(k,r,r') \circ T'(n)$ *for any integer* n *which is prime to* Mp.

(2) $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)^* \circ \mathrm{Ch}_{M,N}(k,r,r') = a^{r'-1}b^{k-r'-1}(ab)^{-r}\,\mathrm{Ch}_{M,N}(k,r,r') \circ \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)^*$ *for any integers* a, b *such that* $(a, Mp) = 1$ *and* $(b, Np) = 1$.

This lemma is proved easily.

8.9. We define p-adic zeta elements (8.1.2).

Assume first $\xi = a(A)$. Take $M \geqslant 1$, $L \geqslant 4$ such that

$$mA \mid M, \quad N \mid L, \quad M \mid L, \quad \mathrm{prime}(M) = S, \quad \mathrm{prime}(L) = S \cup \mathrm{prime}(N).$$

Let

$$t_{m,a(A)} : V_{k,\mathbb{Q}_p}(Y(M,L)) \longrightarrow V_{k,\mathbb{Q}_p}(Y_1(N) \otimes \mathbb{Q}(\zeta_m))$$

be the homomorphism defined in the same way as the map $t_{m,a(A)} : M_k(X(M,L)) \to M_k(Y_1(N) \otimes \mathbb{Q}(\zeta_m))$ in 5.2. Let $V_{k,\mathbb{Z}_p}(Y_1(N) \otimes \mathbb{Q}(\zeta_m))$ be the $\mathbb{Z}_p$-lattice of $V_{k,\mathbb{Q}_p}(Y_1(N) \otimes \mathbb{Q}(\zeta_m))$ defined to be the image of the canonical injection

$$\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})] \otimes V_{k,\mathbb{Z}_p}(Y_1(N)) \longrightarrow V_{k,\mathbb{Q}_p}(Y_1(N) \otimes \mathbb{Q}(\zeta_m)).$$

Then $t_{m,a(A)}$ induces a homomorphism

$$V_{k,\mathbb{Z}_p}(Y(M,L)) \longrightarrow V_{k,\mathbb{Z}_p}(Y_1(N) \otimes \mathbb{Q}(\zeta_m)).$$

Hence we have a homomorphism

$$\begin{aligned} t_{m,a(A)} : \mathrm{H}^1\big(\mathbb{Z}[1/p], V_{k,\mathbb{Z}_p}(Y(M,L))\big) &\longrightarrow \mathrm{H}^1\big(\mathbb{Z}[1/p], V_{k,\mathbb{Z}_p}(Y_1(N) \otimes \mathbb{Q}(\zeta_m))\big) \\ &\cong \mathrm{H}^1\big(\mathbb{Z}[1/p, \zeta_m], V_{k,\mathbb{Z}_p}(Y_1(N))\big). \end{aligned}$$

We define

$${}_{c,d}z^{(p)}_{1,N,m}(k,r,r',\xi,S) = t_{m,a(A)}\big({}_{c,d}z^{(p)}_{M,L}(k,r,r')\big).$$

By Prop. 8.7 (1), $z^{(p)}_{1,N,M}(k,r,r',\xi,S)$ is independent of the choices of M, L.

Next assume $\xi \in \mathrm{SL}_2(\mathbb{Z})$. Take $L \geqslant 3$ such that

$$m \mid L, \quad N \mid L, \quad \mathrm{prime}(L) = S.$$

The trace map $V_{k,\mathbb{Q}_p}(Y(L)) \to V_{k,\mathbb{Q}_p}(Y_1(N) \otimes \mathbb{Q}(\zeta_m))$ induces a homomorphism

$$V_{k,\mathbb{Z}_p}(Y(L)) \longrightarrow V_{k,\mathbb{Z}_p}(Y_1(N) \otimes \mathbb{Q}(\zeta_m)).$$

Hence we have a homomorphism

$$\begin{aligned} \mathrm{H}^1\big(\mathbb{Z}[1/p], V_{k,\mathbb{Z}_p}(Y(L))\big) &\longrightarrow \mathrm{H}^1\big(\mathbb{Z}[1/p], V_{k,\mathbb{Z}_p}(Y_1(N) \otimes \mathbb{Q}(\zeta_m))\big) \\ &\cong \mathrm{H}^1\big(\mathbb{Z}[1/p, \zeta_m], V_{k,\mathbb{Z}_p}(Y_1(N))\big). \end{aligned}$$

We define ${}_{c,d}z^{(p)}_{1,N,m}(k,r,r',\xi,S)$ to be the image of $\xi^*({}_{c,d}z^{(p)}_{L,L}(k,r,r'))$ under this homomorphism. By prop. 8.7 (1), this element is independent of the choice of L.

Proposition 8.10. — *Let the notation be as in* (8.1.2). *Let* $m' \geqslant 1$, $m \mid m'$, *and let* S' *be a finite set of prime numbers such that* $S \cup \mathrm{prime}(m') \subset S'$ *and* $\mathrm{prime}(cd) \cap S' = \varnothing$. *Then the norm map*

$$\mathrm{H}^1\big(\mathbb{Z}[\zeta_{m'}, 1/p], V_{k,\mathbb{Z}_p}(Y_1(N))\big) \longrightarrow \mathrm{H}^1\big(\mathbb{Z}[\zeta_m, 1/p], V_{k,\mathbb{Z}_p}(Y_1(N))\big)$$

sends ${}_{c,d}z^{(p)}_{1,N,m'}(k, r, r', \xi, S')$ *to*

$$\Big(\prod_{\ell \in S'-S} \big(1 - T'(\ell)\sigma_\ell^{-1} \cdot \ell^{-r} + \Delta'(\ell)\sigma_\ell^{-2} \cdot \ell^{k-1-2r}\big)\Big) \cdot {}_{c,d}z^{(p)}_{1,N,m}(k, r, r', \xi, S)$$

where $\Delta'(\ell)$ *denotes* $\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1/\ell \end{smallmatrix}\right)^*$ *in the case* ℓ *does not divide* N *and* $\Delta'(\ell) = 0$ *in the case* ℓ *divides* N.

This follows from 8.7, in the same way as 5.3 followed from 4.3, 4.4.

8.11. We define p-adic zeta elements ${}_{c,d}z^{(p)}_m(f, r, r', \xi, S)$ in (8.1.3) to be the image of the p-adic zeta element ${}_{c,d}z^{(p)}_{1,N,m}(k, r, r', \xi, S)$.

Proposition 8.12. — *Let the notation be as in* (8.1.3). *Let* $m' \geqslant 1$, $m \mid m'$, *let* S' *be a finite set of prime numbers such that* $S \cup \mathrm{prime}(m') \subset S'$ *and* $\mathrm{prime}(cd) \cap S' = \varnothing$. *Then the norm map*

$$\mathrm{H}^1\big(\mathbb{Z}[\zeta_{m'}, 1/p], V_{O_\lambda}(f)\big) \longrightarrow \mathrm{H}^1\big(\mathbb{Z}[\zeta_m, 1/p], V_{O_\lambda}(f)\big)$$

sends ${}_{c,d}z^{(p)}_{m'}(f, r, r', \xi, S')$ *to*

$$\Big(\prod_{\ell \in S'-S} \big(1 - \overline{a}_\ell\sigma_\ell^{-1} \cdot \ell^{-r} + \overline{\varepsilon}(\ell)\sigma_\ell^{-2} \cdot \ell^{k-1-2r}\big)\Big) \cdot {}_{c,d}z^{(p)}_m(f, r, r', \xi, S).$$

This follows from 8.10.

9. Relation with Euler systems in the spaces of modular forms

In this section, we state that the p-adic zeta elements in § 8 are related to the zeta modular forms in § 4, via the p-adic Hodge theory (Thm. 9.5, 9.6, 9.7). The proof of Thm. 9.5 is given in § 10, § 11. Thm. 9.6 and Thm. 9.7 follow from Thm. 9.5 easily.

First in 9.1–9.3, we review necessary things from p-adic Hodge theory. See Faltings [**Fa1**] [**Fa2**] [**Fa3**], Fontaine [**Fo1**] [**Fo2**] [**Fo3**], Fontaine-Messing [**FM**], Tsuji [**TT**], ...

9.1. Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic p. Let B_{dR} be the "field of p-adic periods" associated to K, defined by Fontaine ([**Fo2**], [**Fo3**]). B_{dR} is a complete discrete valuation field whose valuation ring contains the algebraic closure $\overline{K}$ of K. The action of $\mathrm{Gal}(\overline{K}/K)$ on $\overline{K}$ is extended to a canonical action of $\mathrm{Gal}(\overline{K}/K)$ on B_{dR}, and

$$\mathrm{H}^0(K, B_{\mathrm{dR}}) = K.$$

$(\mathrm{H}^0(K,\ ) = \mathrm{H}^0(\mathrm{Gal}(\overline{K}/K),\ ) =$ the $\mathrm{Gal}(\overline{K}/K)$-fixed part.)

9.2. We review de Rham representations.

For a finite dimensional $\mathbb{Q}_p$-vector space V endowed with a continuous action of $\mathrm{Gal}(\overline{K}/K)$, define a K-vector space $D_{\mathrm{dR}}(V)$ by

$$D_{\mathrm{dR}}(V) = \mathrm{H}^0(K, B_{\mathrm{dR}} \otimes V).$$

(We denote $D_{\mathrm{dR}}(V)$ by $D_{\mathrm{dR}}(K,V)$ in the case we need to make K explicit.) Then $D_{\mathrm{dR}}(V)$ has a descending filtration $(D^i_{\mathrm{dR}}(V))_{i\in\mathbb{Z}}$ defined by

$$D^i_{\mathrm{dR}}(V) = \mathrm{H}^0(K, B^i_{\mathrm{dR}} \otimes V)$$

where B^i_{dR} denotes the subset of B_{dR} consisting of elements whose normalized valuation is $\geqslant i$.

In general, $\dim_K(D_{\mathrm{dR}}(V)) \leqslant \dim_{\mathbb{Q}_p}(V)$, and we say V is a de Rham representation of $\mathrm{Gal}(\overline{K}/K)$ when the equality holds here. De Rham representations are stable under taking direct sum, tensor products, duals, and under taking subquotients. If L is a finite extension of K in $\overline{K}$, V is de Rham if and only if it is de Rham as a representation of $\mathrm{Gal}(\overline{K}/L)$, and we have $L \otimes_K D_{\mathrm{dR}}(K,V) \xrightarrow{\sim} D_{\mathrm{dR}}(L,V)$ if V is de Rham.

The following (9.2.1) and (9.2.2) provide important examples of de Rham representations ([**Fa2**], [**Fa3**], [**TT**]).

(9.2.1) *Let X be a proper smooth scheme over K, let $m \in \mathbb{Z}$, and let*

$$V = \mathrm{H}^m(X \otimes_K \overline{K}, \mathbb{Q}_p).$$

Then V is a de Rham representation of $\mathrm{Gal}(\overline{K}/K)$, and $D_{\mathrm{dR}}(V)$ is identified with the de Rham cohomology $\mathrm{H}^m_{\mathrm{dR}}(X/K)$. The filtration on $D_{\mathrm{dR}}(V)$ coincides with the Hodge filtration on $\mathrm{H}^m_{\mathrm{dR}}(X/K)$.

(9.2.2) *Let Y be a curve of the form $G\backslash Y(N)$ with $N \geqslant 3$ and with a subgroup G of $\mathrm{GL}_2(\mathbb{Z}/N)$. Let $k \geqslant 2$, and let*

$$V = V_{k,\mathbb{Q}_p}(Y) \quad (8.3).$$

Then V is a de Rham representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and

$$D^i_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V) \quad \text{for} \quad i \leqslant 0, \quad D^i_{\mathrm{dR}}(V) = 0 \quad \text{for} \quad i \geqslant k,$$
$$D^i_{\mathrm{dR}}(V) = M_k(X) \otimes \mathbb{Q}_p \quad \text{for} \quad 1 \leqslant i \leqslant k-1,$$

where X is a smooth compactification of Y. (See § 11.)

9.3. For a de Rham representation V, we have a canonical homomorphism

$$\exp^* : \mathrm{H}^1(K,V) \longrightarrow D^0_{\mathrm{dR}}(V)$$

called the dual exponential map ([**BK2**], [**KK2**]). This is defined as the composite

$$\mathrm{H}^1(K,V) \longrightarrow \mathrm{H}^1(K, B^0_{\mathrm{dR}} \otimes V) \xleftarrow{\sim} \mathrm{H}^0(K, B^0_{\mathrm{dR}} \otimes V) = D^0_{\mathrm{dR}}(V)$$

where the middle isomorphism is the product with the element

$$\log(\chi_{\mathrm{cyclo}}) \in \mathrm{H}^1(K, \mathbb{Z}_p) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\overline{K}/K), \mathbb{Z}_p)$$

defined as follows. Here $\mathrm{H}^1(K,\ )$ are the continuous Galois cohomology groups $\mathrm{H}^1(\mathrm{Gal}(\overline{K}/K),\ )$. As a homomorphism $\mathrm{Gal}(\overline{K}/K) \to \mathbb{Z}_p$, $\log(\chi_{\mathrm{cyclo}})$ is the composite of the cyclotomic character $\chi_{\mathrm{cyclo}} : \mathrm{Gal}(\overline{K}/K) \to \mathbb{Z}_p^\times$ and the logarithm $\mathbb{Z}_p^\times \to \mathbb{Z}_p$.

9.4. Let Y and $X \supset Y$ be as in (9.2.2). For $k \geqslant 2$ and $1 \leqslant i \leqslant k-1$, consider the dual exponential maps

$$\exp^* : \mathrm{H}^1(\mathbb{Q}_p, V_{k,\mathbb{Q}_p}(Y)(i)) \longrightarrow M_k(X) \otimes \mathbb{Q}_p. \tag{9.4.1}$$

Here to define (9.4.1), we used the fact

$$D^0_{\mathrm{dR}}(V_{k,\mathbb{Q}_p}(Y)(i)) = D^i_{\mathrm{dR}}(V_{k,\mathbb{Q}_p}(Y)) = M_k(X) \otimes \mathbb{Q}_p$$

(Tate twist shifts the filtration of D_{dR}).

Let f be a normalized newform in $M_k(X_1(N)) \otimes \mathbb{C}$ of weight k and of level N, and let F, λ, F_λ be as in 8.3. Then by (9.2.2), $V_{F_\lambda}(f)$ is a de Rham representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$,

$$D^i_{\mathrm{dR}}(V_{F_\lambda}(f)) = D_{\mathrm{dR}}(V_{F_\lambda}(f)) \ \text{ for } \ i \leqslant 0, \quad D^i_{\mathrm{dR}}(V_{F_\lambda}(f)) = 0 \ \text{ for } \ i \geqslant k,$$

and as a quotient of $D^i_{\mathrm{dR}}(V_{k,\mathbb{Q}_p}(Y_1(N))) = M_k(X_1(N)) \otimes \mathbb{Q}_p$ $(1 \leqslant i \leqslant k-1)$, we have

$$D^i_{\mathrm{dR}}(V_{F_\lambda}(f)) = S(f) \otimes_F F_\lambda \ \text{ for } \ 1 \leqslant i \leqslant k-1.$$

Here $S(f)$ is as in 6.3. We have the dual exponential map

$$\exp^*_f : \mathrm{H}^1(\mathbb{Q}(\zeta_m) \otimes \mathbb{Q}_p, V_{F_\lambda}(f)(i)) \longrightarrow S(f) \otimes_F F_\lambda \otimes \mathbb{Q}(\zeta_m)$$

for $1 \leqslant i \leqslant k-1$.

Theorem 9.5. — *Let the notation be as in* (8.1.1). *Assume* $1 \leqslant r \leqslant k-1$, *at least one of* r, r' *is* $k-1$, *and* $\mathrm{prime}(M) \subset \mathrm{prime}(N)$. *Assume further that* $M \geqslant 2$ *in the case* $(r,r') = (k-2,k-1)$. *Then the dual exponential map* (9.4.1) *with* $Y = Y(M,N)$ *and* $i = k-r$ *sends the image of* ${}_{c,d}z^{(p)}_{M,N}(k,r,r')$ *in* $\mathrm{H}^1(\mathbb{Q}_p, V_{k,\mathbb{Q}_p}(Y(M,N))(k-r))$ *to the following element of* $M_k(X(M,N)) \subset M_k(X(M,N)) \otimes \mathbb{Q}_p$:

$${}_{c,d}z_{M,N}(k,r,r') \ \text{ if } \ p \ \text{ divides } \ M,$$

$$\Big(1 - T'(p)\big(\begin{smallmatrix} 1/p & 0 \\ 0 & 1 \end{smallmatrix}\big)^* \cdot p^{-r}\Big) \cdot {}_{c,d}z_{M,N}(k,r,r') \ \text{ if } \ (p,M) = 1 \ \text{ and } \ p \mid N,$$

$$\Big(1 - T'(p)\big(\begin{smallmatrix} 1/p & 0 \\ 0 & 1 \end{smallmatrix}\big)^* \cdot p^{-r} + \big(\begin{smallmatrix} 1/p & 0 \\ 0 & 1/p \end{smallmatrix}\big)^* \cdot p^{k-1-2r}\Big) \cdot {}_{c,d}z_{M,N}(k,r,r') \ \text{ if } \ (p,N) = 1.$$

See § 10, § 11 for the proof of Thm. 9.5.

Theorem 9.6. — *Let the notation be as in* (8.1.2). *Assume* $1 \leqslant r \leqslant k-1$ *and at least one of* r, r' *is* $k-1$. *Then the dual exponential map* (9.4.1) *with* $Y = Y_1(N) \otimes \mathbb{Q}(\zeta_m)$ *and* $i = k-r$ *sends the image of* ${}_{c,d}z^{(p)}_{1,N,m}(k,r,r',\xi,S)$ *in* $\mathrm{H}^1\big(\mathbb{Q}(\zeta_m) \otimes \mathbb{Q}_p, V_{k,\mathbb{Q}_p}(Y_1(N))\big)$ *to*

$${}_{c,d}z_{1,N,m}(k,r,r',\xi,S) \in M_k(X_1(N)) \otimes \mathbb{Q}(\zeta_m) \subset M_k(X_1(N)) \otimes \mathbb{Q}(\zeta_m) \otimes \mathbb{Q}_p.$$

This follows from Thm. 9.5 and Prop. 4.4.

Theorem 9.7. — *Let the notation be as in* (8.1.3)*. Assume* $1 \leqslant r \leqslant k-1$*, and at least one of* r*,* r' *is* $k-1$*. Let* F*,* λ*,* F_λ *be as in 8.3. Then the map*

$$\exp_f^* : \mathrm{H}^1(\mathbb{Q}(\zeta_m) \otimes \mathbb{Q}_p, V_{F_\lambda}(f)(k-r)) \longrightarrow S(f) \otimes_F F_\lambda \otimes \mathbb{Q}(\zeta_m)$$

sends the image of ${}_{c,d}z_m^{(p)}(f, r, r', \xi, S)$ *to*

$${}_{c,d}z_m(f, r, r', \xi, S) \in S(f) \otimes \mathbb{Q}(\zeta_m) \subset S(f) \otimes_F F_\lambda \otimes \mathbb{Q}(\zeta_m).$$

This follows from Thm. 9.6.

10. Generalized explicit reciprocity laws

In this section, we deduce Thm. 9.5 from a generalized explicit reciprocity law proved in [**KK3**]. In this proof, we use the compatibility of two dual exponential maps (10.9.5) and this compatibility is proved in § 11. Generalized explicit reciprocity laws in [**KK3**] are related to generalized explicit reciprocity laws of Vostokov ([**Vo**]).

10.1. In 10.1–10.5, we review the theory of B_{dR} not assuming the residue field is perfect. The theory of B_{dR} in this general case was studied in the unpublished work of Tsuzuki [**TN**] and is explained in [**KK3**, § 2].

In 10.1–10.5, let $\mathcal{K}$ be a complete discrete valuation field of characteristic 0 with residue field $\mathfrak{K}$ of characteristic p. We assume $[\mathfrak{K} : \mathfrak{K}^p] < \infty$. In our application, $\mathcal{K}$ will be a p-adic completion of the function field of a modular curve.

We define the ring B_{dR} over $\mathcal{K}$, and we define an action of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$, a filtration, and a connection on B_{dR}, as follows.

Let $\overline{\mathcal{K}}$ be the algebraic closure of $\mathcal{K}$. For a subfield K of $\mathcal{K}$, and for $n \geqslant 1$, let

$$B_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K) = \mathrm{H}^0\Big(\big(\operatorname{Spec}(\mathrm{O}_{\overline{\mathcal{K}}}/p^n)/\operatorname{Spec}(\mathrm{O}_K/p^n)\big)_{\mathrm{crys}}, \mathcal{O}_{\mathrm{crys}}\Big)$$

where O_K is the discrete valuation ring $\mathrm{O}_{\mathcal{K}} \cap K$, $(\ )_{\mathrm{crys}}$ means the crystalline site with respect to the standard divided power structure on the ideal (p) of O_K/p^n, and $\mathcal{O}_{\mathrm{crys}}$ is the structure sheaf of the crystalline site. (The case $K = \mathcal{K}$ and the case $K = \mathbb{Q}_p$ will be important for us.) Let $J_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K)$ be the kernel of the canonical surjection $B_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K) \to \mathrm{O}_{\overline{\mathcal{K}}}/p^n$, and for $q \geqslant 0$, let $J_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K)^{[q]}$ be the q-th divided power of $J_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K)$. Define

$$\begin{aligned} B_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K) &= \varprojlim_n B_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K), \\ J_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K) &= \varprojlim_n J_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K), \\ J_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K)^{[q]} &= \varprojlim_n J_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K)^{[q]}, \\ B^+_{\mathrm{dR},\overline{\mathcal{K}}/K} &= \varprojlim_q \big(B_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K)/J_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K)^{[q]} \otimes \mathbb{Q}\big). \end{aligned}$$

Then $B^+_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p}$ is a discrete valuation ring. Let $B_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p}$ be the field of fractions of $B^+_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p}$ and define

$$B_{\mathrm{dR},\overline{\mathcal{K}}/K} = B^+_{\mathrm{dR},\overline{\mathcal{K}}/K} \otimes_{B^+_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p}} B_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p}$$

We denote

$$B_{\mathrm{dR}} = B_{\mathrm{dR},\overline{\mathcal{K}}/\mathcal{K}}.$$

The inclusion map $\mathrm{O}_{\mathcal{K}}/p^n \hookrightarrow B_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_{\mathcal{K}})$ induce

$$\mathcal{K} \hookrightarrow B_{\mathrm{dR}}.$$

In the case $\mathfrak{K}$ is perfect, $B_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p} \xrightarrow{\sim} B_{\mathrm{dR},\overline{\mathcal{K}}/\mathcal{K}}$ and this is the "usual" B_{dR} which appeared in § 9.

$\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ acts naturally on $B_{\mathrm{dR},\overline{\mathcal{K}}/K}$.

We define a filtration on $B_{\mathrm{dR},\overline{\mathcal{K}}/K}$ as follows. Let

$$(\mathbb{Z}/p^n)(1) \longrightarrow J_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathbb{Z}_p)$$

be the homomorphism $\alpha \mapsto \log(y^{p^n})$ where α is a p^n-th root of 1 and $y \in B_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathbb{Z}_p)$ is a lifting of the image of α in $\mathrm{O}_{\overline{\mathcal{K}}}/p^n$. (Then y^{p^n} is independent of the choice of y, and belongs to $1 + J_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathbb{Z}_p)$ so that $\log(y^{p^n}) \in J_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathbb{Z}_p)$ is defined.) We obtain a homomorphism

$$\mathbb{Z}_p(1) \longrightarrow J_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathbb{Z}_p)$$

by passing to the inverse limit. The image of a generator t of $\mathbb{Z}_p(1)$ in the discrete valuation ring $B_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p}$ is a prime element. Hence

$$B_{\mathrm{dR},\overline{\mathcal{K}}/K} = B^+_{\mathrm{dR},\overline{\mathcal{K}}/K}[t^{-1}].$$

For $i \geqslant 0$, define

$$J^{[i]}_{\mathrm{dR},\overline{\mathcal{K}}/K} = \varprojlim_q \left(J_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K)^{[i]}/J_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_K)^{[q]} \otimes \mathbb{Q}\right) \subset B^+_{\mathrm{dR},\overline{\mathcal{K}}/K}.$$

We define the filtration on $B_{\mathrm{dR},\overline{\mathcal{K}}/K}$ by

$$B^i_{\mathrm{dR},\overline{\mathcal{K}}/K} = \bigcup_{j\geqslant 0} t^{-j} J^{[i+j]}_{\mathrm{dR},\overline{\mathcal{K}}/K}.$$

In the case $K = \mathbb{Q}_p$, this filtration coincides with the filtration given by the normalized discrete valuation of $B_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p}$.

We denote $B^i_{\mathrm{dR},\overline{\mathcal{K}}/\mathcal{K}}$ simply by B^i_{dR}.

We discuss about the canonical connection on B_{dR}. Let

$$\widehat{\Omega}^1_{\mathcal{K}} = (\varprojlim_n \Omega^1_{\mathrm{O}_{\mathcal{K}}/\mathbb{Z}}/p^n) \otimes \mathbb{Q}.$$

We have

$$\dim_{\mathcal{K}}(\widehat{\Omega}^1_{\mathcal{K}}) = \log_p([\mathfrak{K} : \mathfrak{K}^p]).$$

Let $\widehat{\Omega}^q_{\mathcal{K}}$ be the q-th exterior power of $\widehat{\Omega}^1_{\mathcal{K}}$ over $\mathcal{K}$. Then there exists a unique $B_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p}$-linear map

$$d : B_{\mathrm{dR}} \longrightarrow \widehat{\Omega}^1_{\mathcal{K}} \otimes_{\mathcal{K}} B_{\mathrm{dR}}$$

satisfying the following conditions (i)–(iii).

(i) $d(ab) = a\,db + b\,da$ for $a, b \in \mathcal{K}$.

(ii) The restriction of d to $\mathcal{K}$ coincides with $\mathcal{K} \to \widehat{\Omega}^1_{\mathcal{K}};\ a \mapsto da$.

(iii) $d(J^{[q]}_{\mathrm{dR},\overline{\mathcal{K}}/\mathcal{K}}) \subset \widehat{\Omega}^1_{\mathcal{K}} \otimes_{\mathcal{K}} J^{[q-1]}_{\mathrm{dR},\overline{\mathcal{K}}/\mathcal{K}}$ for any $q \geqslant 1$.

Furthermore, for any $i \in \mathbb{Z}$, we have an exact sequence

$$(10.1.1)\quad 0 \longrightarrow B^i_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p} \longrightarrow B^i_{\mathrm{dR}} \xrightarrow{\ d\ } \widehat{\Omega}^1_{\mathcal{K}} \otimes_{\mathcal{K}} B^{i-1}_{\mathrm{dR}} \xrightarrow{\ d\ } \widehat{\Omega}^2_{\mathcal{K}} \otimes_{\mathcal{K}} B^{i-2}_{\mathrm{dR}} \xrightarrow{\ d\ } \cdots$$

where $d : \widehat{\Omega}^q_{\mathcal{K}} \otimes_{\mathcal{K}} B_{\mathrm{dR}} \to \widehat{\Omega}^{q+1}_{\mathcal{K}} \otimes_{\mathcal{K}} B_{\mathrm{dR}}$ is defined by $d(x \otimes y) = dx \otimes y + (-1)^q x \otimes dy$.

10.2. As in the perfect residue field case, we have the functor D_{dR} and the notion "de Rham representation", defined as follows.

For a finite dimensional $\mathbb{Q}_p$-vector space V endowed with a continuous action of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$, let

$$D_{\mathrm{dR}}(V) = \mathrm{H}^0(\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K}), B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V).$$

Then $D_{\mathrm{dR}}(V)$ is a $\mathcal{K}$-vector space endowed with a filtration $(D^i_{\mathrm{dR}}(V))_{i\in\mathbb{Z}}$ defined by

$$D^i_{\mathrm{dR}}(V) = \mathrm{H}^0(\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K}), B^i_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V),$$

and with a connection

$$\nabla : D_{\mathrm{dR}}(V) \longrightarrow \widehat{\Omega}^1_{\mathcal{K}} \otimes_{\mathcal{K}} D_{\mathrm{dR}}(V)$$

induced by $d : B_{\mathrm{dR}} \to \widehat{\Omega}^1_{\mathcal{K}} \otimes_{\mathcal{K}} B_{\mathrm{dR}}$ and by the identity map of V.

We have always

$$\dim_{\mathcal{K}}(D_{\mathrm{dR}}(V)) \leqslant \dim_{\mathbb{Q}_p}(V).$$

We say V is a de Rham representation of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ if the equality holds here. If V is a de Rham representation, we have an isomorphism

$$B_{\mathrm{dR}} \otimes_{\mathcal{K}} D_{\mathrm{dR}}(V) \xrightarrow{\ \sim\ } B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$$

which sends $\sum_{i+j=n} B^i_{\mathrm{dR}} \otimes_{\mathcal{K}} D^j_{\mathrm{dR}}(V)$ onto $B^n_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ for each $n \in \mathbb{Z}$. De Rham representations are stable under direct sums, tensor products, Tate twists, duals, and under taking subquotients. For de Rham representations, the functor D_{dR} commutes with the operations direct sum, tensor product, dual and Tate twists. If $\mathcal{L}$ is a finite extension of $\mathcal{K}$, V is de Rham if and only if it is de Rham as a representation of $\mathrm{Gal}(\overline{\mathcal{L}}/\mathcal{L})$, and we have $\mathcal{L} \otimes_{\mathcal{K}} D_{\mathrm{dR}}(V) \xrightarrow{\ \sim\ } D_{\mathrm{dR}}(\mathcal{L}, V)$ if V is de Rham where $D_{\mathrm{dR}}(\mathcal{L}, V)$ denotes D_{dR} of V as a representation of $\mathrm{Gal}(\overline{\mathcal{L}}/\mathcal{L})$.

10.3. Let G be a p-divisible group over $O_{\mathcal{K}}$, let T_pG be the Tate module of G, and let $V_pG = \mathbb{Q} \otimes T_pG$. Then V_pG is a de Rham representation of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$, and there exists a canonical isomorphism

$$D_{\mathrm{dR}}(V_pG) = \mathcal{K} \otimes_{O_{\mathcal{K}}} D(G)$$

preserving the filtrations and the connections, where $D(G)$ is the covariant Dieudonné module of G (the $O_{\mathcal{K}}$-dual of the contravariant Dieudonné module of G in [**BBM**]). (As in [**BBM**], $D(G)$ has a canonical filtration such that

$$\begin{aligned} &D^{-1}(G) = D(G), \quad D^1(G) = 0, \\ &D^0(G) = \mathrm{coLie}(G^*), \quad D(G)/D^0(G) = \mathrm{Lie}(G), \end{aligned}$$

where G^* denotes the dual p-divisible group of G.)

10.4. For a de Rham representation V of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ and for any $i, j \in \mathbb{Z}$ such that $i \leqslant j$, we have (the meaning of the notation [ ] below is explained soon later):

$$\mathrm{H}^q(\mathcal{K}, [(B^i_{\mathrm{dR}}/B^j_{\mathrm{dR}}) \otimes_{\mathbb{Q}_p} V]) = 0 \quad \text{for} \quad q \geqslant 2, \tag{10.4.1}$$

$$\begin{aligned} [D^i_{\mathrm{dR}}(V)/D^j_{\mathrm{dR}}(V)] &\xrightarrow{\sim} \mathrm{H}^0(\mathcal{K}, [(B^i_{\mathrm{dR}}/B^j_{\mathrm{dR}}) \otimes_{\mathbb{Q}_p} V]) \\ &\xrightarrow{\sim} \mathrm{H}^1(\mathcal{K}, [(B^i_{\mathrm{dR}}/B^j_{\mathrm{dR}}) \otimes_{\mathbb{Q}_p} V]) \end{aligned} \tag{10.4.2}$$

where the last isomorphism is given by the product with $\log(\chi_{\mathrm{cyclo}}) \in \mathrm{H}^1(\mathcal{K}, \mathbb{Z}_p)$.

The meanings of [ ] are as follows. Let (Ab_p) be the category of abelian groups A satisfying the following condition (i).

(i) A is killed by some power of p.

Let $(\mathrm{Gal}_{\mathcal{K},p})$ be the category of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$-modules A satisfying (i) and the following condition (ii).

(ii) For any $x \in A$, the stabilizer of x in $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ is open in $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$.

Then the functor

$$\mathrm{H}^q(\mathcal{K}, \ ) : (\mathrm{Gal}_{\mathcal{K},p}) \longrightarrow (\mathrm{Ab}_p)$$

induces

$$\mathrm{H}^q(\mathcal{K}, \ ) : \mathrm{ind}(\mathrm{pro}(\mathrm{Gal}_{\mathcal{K},p})) \longrightarrow \mathrm{ind}(\mathrm{pro}(\mathrm{Ab}_p))$$

where pro( ) means the category of pro-objects and ind( ) means the category of ind-objects. We define an object $[D^i_{\mathrm{dR}}(V)/D^j_{\mathrm{dR}}(V)]$ of $\mathrm{ind}(\mathrm{pro}(\mathrm{Ab}_p))$ and an object $[(B^i_{\mathrm{dR}}/B^j_{\mathrm{dR}}) \otimes_{\mathbb{Q}_p} V]$ of $\mathrm{ind}(\mathrm{pro}(\mathrm{Gal}_{\mathcal{K},p}))$, as follows. The equality (10.4.1) and the isomorphisms in (10.4.2) are considered in the category $\mathrm{ind}(\mathrm{pro}(\mathrm{Ab}_p))$.

For a finitely generated $O_{\mathcal{K}}$-module M, we denote by $[M]$ the object "$\varprojlim_n$" M/p^n of $\mathrm{pro}(\mathrm{Ab}_p)$. For a finite dimensional $\mathcal{K}$-vector space N, we denote by $[N]$ the object "$\varinjlim$" $[M]$ of $\mathrm{ind}(\mathrm{pro}(\mathrm{Ab}_p))$ where M ranges over all finitely generated $O_{\mathcal{K}}$-submodules of N. This defines the object $[D^i_{\mathrm{dR}}(V)/D^j_{\mathrm{dR}}(V)]$ of $\mathrm{ind}(\mathrm{pro}(\mathrm{Ab}_p))$.

On the other hand, let M be a finitely generated $B_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathbb{Z}_p)$-module endowed with an action of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$, satisfying the following conditions (a) (b) (c).

(a) M is killed by $J_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathbb{Z}_p)^{[q]}$ for some $q \geqslant 0$. (Then we have $M \xrightarrow{\sim} \varprojlim_n M/p^n$.)

(b) $\sigma(ax) = \sigma(a)\sigma(x)$ for any $\sigma \in \mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$, $a \in B_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathbb{Z}_p)$, $x \in M$.

(c) The action of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ on M is continuous with respect to the p-adic topology on M.

Then we denote by $[M]$ the object "$\varprojlim$"M/p^n of $\mathrm{pro}(\mathrm{Gal}_{\mathcal{K},p})$. Next let N be a $B^+_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p}$-module endowed with an action of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ satisfying the following condition $(\star)$.

$(\star)$ N is the union of all finitely generated $B_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathbb{Z}_p)$-submodules M of N which are stable under the action of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ and which satisfy the above conditions (a) (b) (c).

Then we denote by $[N]$ the object "$\varinjlim$"$[M]$ of $\mathrm{ind}(\mathrm{pro}(\mathrm{Gal}_{\mathcal{K},p}))$.

For example, for a finite dimensional $\mathbb{Q}_p$-vector space V endowed with a continuous action of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$, and for $i \leqslant j$, $(B^i_{\mathrm{dR}}/B^j_{\mathrm{dR}})\otimes_{\mathbb{Q}_p} V$ satisfies the condition $(\star)$. Hence the object $[(B^i_{\mathrm{dR}}/B^j_{\mathrm{dR}}) \otimes_{\mathbb{Q}_p} V]$ of $\mathrm{ind}(\mathrm{pro}(\mathrm{Gal}_{\mathcal{K},p}))$ is defined.

10.5. Let $e \geqslant 0$ be the integer defined by $[\mathfrak{K} : \mathfrak{K}^p] = p^e$. Then, for a de Rham representation V of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$, we define a homomorphism

$$\exp^* : \mathrm{H}^{e+1}(\mathcal{K}, [V]) \longrightarrow \mathrm{Coker}\left([\widehat{\Omega}^{e-1}_{\mathcal{K}} \otimes_{\mathcal{K}} D^{1-e}_{\mathrm{dR}}(V)] \xrightarrow{\nabla} [\widehat{\Omega}^{e}_{\mathcal{K}} \otimes_{\mathcal{K}} D^{-e}_{\mathrm{dR}}(V)]\right)$$

in $\mathrm{ind}(\mathrm{pro}(\mathrm{Ab}_p))$, called the dual exponential map. Here, $[V]$ denotes the object of $\mathrm{ind}(\mathrm{pro}(\mathrm{Gal}_{K,p}))$ defined to be "$\varinjlim$"$[T]$ in which T ranges over all $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$-stable $\mathbb{Z}_p$-lattices in V and $[T] \underset{\mathrm{def}}{=}$ "$\varprojlim_n$" T/p^n. Take a sufficiently large $q \geqslant 0$ and consider the exact sequence

$$\begin{aligned}0 \longrightarrow [B^0_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p}/B^q_{\mathrm{dR},\mathcal{K}/\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V] \longrightarrow [B^0_{\mathrm{dR}}/B^q_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V] \xrightarrow{d}\\ [\widehat{\Omega}^1_{\mathcal{K}} \otimes_{\mathcal{K}} B^{-1}_{\mathrm{dR}}/B^{q-1}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V] \xrightarrow{d} [\widehat{\Omega}^2_{\mathcal{K}} \otimes_{\mathcal{K}} B^{-2}_{\mathrm{dR}}/B^{q-2}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V] \xrightarrow{d} \cdots\end{aligned}$$

(10.1.1). By (10.4.1) and (10.4.2), this exact sequence induces

$$\begin{aligned}&\mathrm{H}^{e+1}(\mathcal{K}, [B^0_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p}/B^q_{\mathrm{dR},\mathcal{K}/\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V])\\ &\quad\cong \mathrm{Coker}\Big(\mathrm{H}^1(\mathcal{K}, [\widehat{\Omega}^{e-1}_{\mathcal{K}} \otimes_{\mathcal{K}} B^{1-e}_{\mathrm{dR}}/B^{q+1-e}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V])\\ &\qquad\qquad \longrightarrow \mathrm{H}^1(\mathcal{K}, [\widehat{\Omega}^{e}_{\mathcal{K}} \otimes_{\mathcal{K}} B^{-e}_{\mathrm{dR}}/B^{q-e}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V])\Big)\\ &\quad\cong \mathrm{Coker}\left([\widehat{\Omega}^{e-1}_{\mathcal{K}} \otimes_{\mathcal{K}} D^{1-e}_{\mathrm{dR}}(V)] \xrightarrow{\nabla} [\widehat{\Omega}^{e}_{\mathcal{K}} \otimes_{\mathcal{K}} D^{-e}_{\mathrm{dR}}(V)]\right).\end{aligned}$$

We define $\exp^*$ to be the composite

$$\mathrm{H}^{e+1}(\mathcal{K},[V]) \longrightarrow \mathrm{H}^{e+1}(\mathcal{K},[B^0_{\mathrm{dR},\overline{\mathcal{K}}/\mathbb{Q}_p}/B^q_{\mathrm{dR},\mathcal{K}/\mathbb{Q}_p}\otimes_{\mathbb{Q}_p} V])$$
$$\xrightarrow{\sim} \mathrm{Coker}\left([\widehat{\Omega}^{e-1}_{\mathcal{K}}\otimes_{\mathcal{K}} D^{1-e}_{\mathrm{dR}}(V)] \xrightarrow{\nabla} [\widehat{\Omega}^{e}_{\mathcal{K}}\otimes_{\mathcal{K}} D^{-e}_{\mathrm{dR}}(V)]\right).$$

10.6. Now we apply the above general theory to the following field $\mathcal{K}$ related to a modular curve.

Fix M, $N \geqslant 1$ such that $M+N \geqslant 5$, $(MN,p)=1$. Take a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_N]$ lying over p, and let $\mathbb{Z}[\zeta_N]_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completion of $\mathbb{Z}[\zeta_N]$. Let

$$\mathcal{K} = \left(\varprojlim_n (\mathbb{Z}[\zeta_N]_{\mathfrak{p}}[[q^{1/M}]][q^{-1}])/p^n\right)[1/p]).$$

This field $\mathcal{K}$ is a complete discrete valuation field of mixed characteristics with valuation ring

$$\mathrm{O}_{\mathcal{K}} = \varprojlim_n \left(\mathbb{Z}[\zeta_N]_{\mathfrak{p}}[[q^{1/M}]][q^{-1}]/p^n\right)$$

and with residue field

$$\mathfrak{K} = \mathbb{F}_p(\zeta_N)((q^{1/M}))$$

which satisfies $[\mathfrak{K}:\mathfrak{K}^p]=1$.

Fix an algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ and fix an embedding

$$\bigcup_{m,n\geqslant 1} \mathbb{Z}[\zeta_m][[q^{1/n}]] \longrightarrow \overline{\mathcal{K}}$$

($\zeta_m = \exp(2\pi i/m)$, $q^{1/n} = \exp(2\pi i\tau/n)$ as before) over $\mathbb{Z}[\zeta_N][[q^{1/M}]]$.

10.7. Let $\mathfrak{E}$ be the elliptic curve over $\mathrm{O}_{\mathcal{K}}$ which is obtained from the Tate curve over $\mathbb{Z}[[q]][q^{-1}]$ of q-invariant q ([**DR**, VII.1]) via the embedding $\mathbb{Z}[[q]][q^{-1}] \to \mathrm{O}_{\mathcal{K}}$. Let $E = \mathfrak{E}\otimes_{\mathrm{O}_{\mathcal{K}}}\mathcal{K}$.

Define a p-divisible group G over $\mathrm{O}_{\mathcal{K}}$ by

$$G = \bigcup_n \mathrm{Ker}(p^n : \mathfrak{E} \longrightarrow \mathfrak{E}).$$

By the theory of Tate curves, the torsion part of $E(\overline{\mathcal{K}})$ is identified with the torsion part of $(\mathrm{O}_{\overline{\mathcal{K}}})^\times/q^{\mathbb{Z}}$ as a $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$-module, and we have an exact sequence of p-divisible groups over $\mathrm{O}_{\mathcal{K}}$

$$0 \longrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)(1) \longrightarrow G \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$$

whose $\overline{\mathcal{K}}$-valued points coincide with

$$0 \longrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)(1) \longrightarrow \bigcup_{n\geqslant 1} {}_{p^n}\left((\mathrm{O}_{\overline{\mathcal{K}}})^\times/q^{\mathbb{Z}}\right) \xrightarrow{v} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$$

where v sends q^{1/p^n} mod $q^{\mathbb{Z}}$ to $1/p^n$. We have canonical isomorphisms

$$\mathrm{coLie}(E) \cong \mathcal{K}\otimes_{\mathrm{O}_{\mathcal{K}}} \mathrm{coLie}(G). \tag{10.7.1}$$

$$\mathrm{coLie}(G) \cong \mathrm{coLie}((\mathbb{Q}_p/\mathbb{Z}_p)(1)) \cong \mathrm{O}_{\mathcal{K}}, \tag{10.7.2}$$

where the first isomorphism in (10.7.2) is induced from the embedding $(\mathbb{Q}_p/\mathbb{Z}_p)(1)\to G$.

10.8. We introduce finite extensions $\mathcal{K}_m$ of $\mathcal{K}$ and schemes Y_m over $\mathcal{K}$ $(m \geqslant 0)$.

For $m \geqslant 0$, let

$$\mathcal{K}_m = \mathcal{K}(q^{1/p^m}, \zeta_{p^m}) = \big(\varprojlim_n (\mathbb{Z}[\zeta_{Np^m}]_{\mathfrak{p}_m}[[q^{1/Mp^m}]][q^{-1}])/p^n\big)\,[1/p]$$

where $\mathfrak{p}_m$ is the unique prime ideal of $\mathbb{Z}[\zeta_{Np^m}]$ lying over $\mathfrak{p}$. We have a morphism

$$\text{(10.8.1)} \qquad \operatorname{Spec}(\mathcal{K}_m) \longrightarrow Y(Mp^m, Np^m)$$

corresponding to the triple $(E \otimes_{\mathcal{K}} \mathcal{K}_m, q^{1/Mp^m} \bmod q, \zeta_{Np^m})$ over $\mathcal{K}_m$. For $k \in \mathbb{Z}$, this morphism induces

$$\text{(10.8.2)} \qquad M_k(X(Mp^m, Np^m)) \longrightarrow \mathcal{K}_m \otimes_{\mathcal{K}} \operatorname{coLie}(E)^{\otimes k} = \mathcal{K}_m$$

where the last identification is by (10.7.1), (10.7.2). This map (10.8.2) coincides with the q-expansion ([**DR**]). That is, the q-expansion

$$M_k(X(Mp^m, Np^m)) \longrightarrow \mathbb{C}[[q^{1/Mp^m}]]$$

(4.9) has the image in $\mathbb{Z}[\zeta_{Np^m}][[q^{1/Mp^m}]] \otimes \mathbb{Q}$, and the induced map

$$M_k(X(Mp^m, Np^m)) \longrightarrow \mathcal{K}_m$$

coincides with (10.8.2).

Let

$$Y_m = Y(Mp^m, Np^m) \otimes_{Y(M,N)} \operatorname{Spec}(\mathcal{K}) \quad \text{for} \quad m \geqslant 0$$

where $\operatorname{Spec}(\mathcal{K}) \to Y(M,N)$ is the case $m = 0$ of (10.8.1). Then Y_m is a finite étale Galois covering of $\operatorname{Spec}(\mathcal{K})$ with Galois group $\mathrm{GL}_2(\mathbb{Z}/p^m)$. The morphism (10.8.1) induces an open immersion

$$i_m : \operatorname{Spec}(\mathcal{K}_m) \longrightarrow Y_m,$$

and the image of the open immersion

$$\left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right) \circ i_m : \operatorname{Spec}(\mathcal{K}_m) \longrightarrow Y_m \quad \text{for} \quad \left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/p^m)$$

depends only on the pair (u, w). If we denote the image by $U(u, w)$,

$$Y_m = \coprod_{(u,w) \in A_m} U(v, w)$$

where

$$A_m = \{(u, w) \in (\mathbb{Z}/p^m)^2\,;\ \ u \ \text{ and } \ w \ \text{ generate } \ \mathbb{Z}/p^m\}.$$

10.9. We will consider the de Rham representations

$$V = \mathcal{H}_p^1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

and $\mathrm{Sym}^{k-2} V$ of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ for $k \geqslant 2$, where $\mathcal{H}_p^1$ here denotes the pull back of the $\mathcal{H}_p^1$ on $Y(M,N)$ via $\mathrm{Spec}(\mathcal{K}) \to Y(M,N)$. (So, $\mathcal{H}_p^1 \cong T_p(E)(-1) = T_p(G)(-1)$.) For $1 \leqslant i \leqslant k-1$, we have a homomorphism

$$\exp^* : \mathrm{H}^2\big(Y_m, [(\mathrm{Sym}^{k-2} V)(i)]\big) \longrightarrow [\mathcal{O}(Y_m)]. \tag{10.9.1}$$

induced from the dual exponential map for $\mathrm{Sym}^{k-2}(V)(i)$ in 10.5 and from the isomorphism

$$\begin{aligned}\mathrm{Coker}([\mathcal{O}(Y_m) \otimes_{\mathcal{K}} D^i_{\mathrm{dR}}(\mathrm{Sym}^{k-2} V)] \\ \longrightarrow [\widehat{\Omega}^1_{\mathcal{K}} \otimes_{\mathcal{K}} \mathcal{O}(Y_m) \otimes_{\mathcal{K}} D^{i-1}_{\mathrm{dR}}(\mathrm{Sym}^{k-2} V)]) \cong [\mathcal{O}(Y_m)]\end{aligned} \tag{10.9.2}$$

which is obtained as follows.

First we define a canonical element q_∞ of $B_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathbb{Z}_p)^\times$ as follows. For $n \geqslant 1$, let $q_n = y^{p^n} \in B_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathbb{Z}_p)^\times$ where $y \in B_n(\mathrm{O}_{\overline{\mathcal{K}}}/\mathbb{Z}_p)$ is any lifting of the image of q^{1/p^n} in $\mathrm{O}_{\overline{\mathcal{K}}}/p^n$. Then q_n is independent of the choice of y. Let

$$q_\infty = (q_n)_{n \geqslant 1} \in B_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathbb{Z}_p)^\times.$$

Next we define a canonical element

$$\xi \in D^1_{\mathrm{dR}}(V)$$

as follows. Let ξ_1, ξ_2 be the basis of T_pG defined by

$$\xi_1 = (q^{1/p^n} \bmod q)_n, \quad \xi_2 = (\zeta_{p^n})_n.$$

By writing the group law of T_pG additively, let

$$\begin{aligned}\xi = t \otimes \xi_1 \otimes (\zeta_{p^n})_n^{\otimes(-1)} + \log(q/q_\infty) \otimes \xi_2 \otimes (\zeta_{p^n})_n^{\otimes(-1)} \\ \in \mathrm{H}^0(\mathcal{K}, J_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_{\mathcal{K}}) \otimes_{\mathbb{Z}} T_pG \otimes \mathbb{Z}_p(-1))\end{aligned}$$

where t is the element of $J_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_{\mathbb{Z}_p}) \subset J_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_{\mathcal{K}})$ corresponding to the basis $(\zeta_{p^n})_n$ of $\mathbb{Z}_p(1)$ (10.1). Here, since the elements q and q_∞ of $B_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_{\mathcal{K}})^\times$ have the same image in $\widehat{\mathrm{O}}_{\overline{\mathcal{K}}}$, we can take $\log(q/q_\infty) \in J_\infty(\mathrm{O}_{\overline{\mathcal{K}}}/\mathrm{O}_{\mathcal{K}})$.

Finally, for $k \geqslant 2$, let

$$\xi^{k-2} \in D^{k-2}_{\mathrm{dR}}(\mathrm{Sym}^{k-2} V)$$

be the $(k-2)$-fold product of ξ. We define

$$\mathcal{O}(Y_m) \longrightarrow \widehat{\Omega}^1_{\mathcal{K}} \otimes_{\mathcal{K}} \mathcal{O}(Y_m) \otimes_{\mathcal{K}} D^{i-1}_{\mathrm{dR}}(\mathrm{Sym}^{k-2} V) \tag{10.9.3}$$

to be the $\mathcal{K}_m$-linear map which sends 1 to $\mathrm{dlog}(q) \otimes \xi^{k-2}$. Then it can be shown ([**KK3**, §4.2]) that (10.9.3) induces an isomorphism (10.9.2).

For $m \geqslant 0$, $k \geqslant 2$, r, $r' \in \mathbb{Z}$ such that $1 \leqslant r' \leqslant k-1$, let

$$\mathrm{Ch}_m(k,r,r') : \varprojlim_n \mathrm{K}_2(Y_n) \longrightarrow \varprojlim_n \mathrm{H}^2\big(Y_m, (\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p)/p^n)(k-r)\big)$$

be the composite homomorphism

$$\begin{aligned}\varprojlim_n \mathrm{K}_2(Y_n) &\longrightarrow \varprojlim_n \mathrm{H}^2(Y_n, (\mathbb{Z}/p^n)(2))\\ &\longrightarrow \varprojlim_n \mathrm{H}^2\big(Y_n, (\mathrm{Sym}^{k-2}(T_pE)/p^n)(2-r)\big)\\ &\xrightarrow{\sim} \varprojlim_n \mathrm{H}^2\big(Y_n, (\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p)/p^n)(k-r)\big)\\ &\longrightarrow \varprojlim_n \mathrm{H}^2\big(Y_m, (\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p)/p^n)(k-r)\big)\end{aligned}$$

where:

The first arrow is the Chern character map. The second arrow is the product with

$$e_{1,n}^{\otimes(r'-1)} \otimes e_{2,n}^{\otimes(k-r'-1)} \otimes (\zeta_{p^n})^{\otimes(-r)},$$

where $e_{1,n}$, $e_{2,n}$ is the canonical basis of $(T_pE)/p^n$ over Y_n (that is, the pull back of the canonical basis of T_pE/p^n over $Y(p^n)$), and the last arrow is the trace map associated to $Y_n \to Y_m$.

As is easily seen, the following diagram is commutative

(10.9.4)

$$\begin{array}{ccc}\varprojlim_n \mathrm{K}_2(Y(Mp^n, Np^n)) & \longrightarrow & \varprojlim_n \mathrm{H}^2\big(Y(Mp^m, Np^n), (\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p)/p^n)(k-r)\big)\\ \big\downarrow & & \big\downarrow\\ \varprojlim \mathrm{K}_2(Y_n) & \longrightarrow & \varprojlim_n \mathrm{H}^2\big(Y_m, (\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p)/p^n)(k-r)\big)\end{array}$$

Here the upper horizontal arrow is the composite of the first four arrows in the definition of $\mathrm{Ch}_{Mp^m,Np^m}(k,r,r')$ in 8.4, and the lower horizontal arrow is $\mathrm{Ch}_m(k,r,r')$. Furthermore, as we will see in § 11, the following diagram is commutative if $1 \leqslant r' \leqslant k-1$.

(10.9.5)

$$\begin{array}{ccc}\varprojlim_n \mathrm{H}^2\big(Y(Mp^m, Np^m), (\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p)/p^n)(k-r)\big) & \longrightarrow & M_k(X(Mp^m, Np^m)) \otimes \mathbb{Q}_p\\ \big\downarrow & & \big\downarrow\\ \varprojlim_n \mathrm{H}^2\big(Y_m, (\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p)/p^n)(k-r)\big) & \longrightarrow & \mathcal{O}(Y_m)\end{array}$$

Here the upper horizontal arrow is induced from the dual exponential map

$$\exp^* : \mathrm{H}^1\big(\mathbb{Q}_p, V_{k,\mathbb{Q}_p}(Y(Mp^m, Np^m))(k-r)\big) \longrightarrow M_k(X(Mp^m, Np^m)) \otimes \mathbb{Q}_p$$

in (9.4.1), and the lower horizontal arrow is the dual exponential map (10.9.1).

As is explained in 10.11 below, Thm. 9.5 is reduced to

Proposition 10.10. — *Let the notation be as in* (8.1.1). *Assume* $1 \leqslant r \leqslant k-1$, *at least one of* r, r' *is* $k-1$, *and* $(MN, p) = 1$. *Then*

$$\exp^* \circ \mathrm{Ch}_m(k,r,r')\big(({}_{c,d}z_{Mp^n,Np^n})_{n\geqslant 1}\big) = {}_{c,d}z_{Mp^m,Np^m}(k,r,r')$$

for $m \geqslant 1$.

10.11. We show that Thm. 9.5 follows from Prop. 10.10 (assuming the commutativity of the diagram (10.9.5) which will be shown in § 11). In this 10.11, we do not assume $(MN, p) = 1$. By 4.3, 4.4 and 8.7, we may replace (M, N) by (M', N') for any M', $N' \geqslant 1$ such that $M \mid M'$, $N \mid N'$ prime$(M') =$ prime(Mp), prime$(N') =$ prime(Np). Hence we may assume the following: $M = M_0 p^m$, $N = N_0 p^m$ for some M_0, N_0, $m \geqslant 1$ such that $(M_0 N_0, p) = 1$ and $M_0 + N_0 \geqslant 5$. Hence by taking M_0 and N_0 as M and N in Prop. 10.10, Thm. 9.5 is reduced to Prop. 10.10 by the commutativity of the diagrams (10.9.4) and (10.9.5) and by the injectivity of

$$M_k(X(Mp^m, Np^m)) \otimes \mathbb{Q}_p \longrightarrow \prod_{\mathfrak{p}} \mathcal{O}(Y_m)^{(\mathfrak{p})}$$

where $\mathfrak{p}$ ranges over all prime ideals of $\mathbb{Z}[\zeta_N]$ lying over p and $\mathcal{O}(Y_m)^{(\mathfrak{p})}$ denotes $\mathcal{O}(Y_m)$ defined by using $\mathfrak{p}$ as in 10.6–10.8.

In the rest of this section, we prove Prop. 10.10 by using the following result which is a special case of the generalized explicit reciprocity law [**KK3**, Thm. 4.3.1].

Proposition 10.12. — *Let $\mathcal{O}(G) = \varprojlim_n \mathcal{O}({}_{p^n}G)$. Let $\theta_{1,n}$ and $\theta_{2,n}$ $(n \geqslant 1)$ be elements of $\mathcal{O}(G)^\times$, and assume*

$$N_p(\theta_{1,n+1}) = \theta_{1,n}, \quad N_p(\theta_{2,n+1}) = \theta_{2,n} \quad \textit{for all} \quad n \geqslant 1$$

where N_p is the norm map $\mathcal{O}(G)^\times \to \mathcal{O}(G)^\times$ associated to the pull back homomorphism $\mathcal{O}(G) \to \mathcal{O}(G)$ by the multiplication by p on G. Let

$$u_n = \{\theta_{1,n}(e_{1,n}), \theta_{2,n}(e_{2,n})\} \in \mathrm{K}_2(Y_n) \quad \textit{for} \quad n \geqslant 1$$

where $(e_{1,n}, e_{2,n})$ is the canonical basis of ${}_{p^n}G$ over Y_n. Then

$$(u_n)_{n \geqslant 1} \in \varprojlim_n \mathrm{K}_2(Y_n).$$

Furthermore, for $k \geqslant 2$, $1 \leqslant j \leqslant k-1$, $m \geqslant 1$, the homomorphism

$$\exp^* \circ \mathrm{Ch}_m(k, k-1, j) : \ \varprojlim_n \mathrm{K}_2(Y_n) \longrightarrow \mathcal{O}(Y_m)$$

sends $(u_n)_{n \geqslant 1}$ to

$$\frac{p^{-km}}{(k-2)!} \cdot \Big(\big(\frac{\mathrm{d}}{\omega}\big)^{k-j} \log(\theta_{1,m}) \Big)(e_{1,m}) \cdot \Big(\big(\frac{\mathrm{d}}{\omega}\big)^{j} \log(\theta_{2,m}) \Big)(e_{2,m})$$

where ω is the canonical O_K-basis of coLie(G) *($\cong$ coLie$((\mathbb{Q}_p/\mathbb{Z}_p)(1))$).*

Here for $f \in \mathcal{O}(G)^\times$ and $i \geqslant 1$, $\big(\frac{d}{\omega}\big)^i \log(f)$ means $\big(\frac{d}{\omega}\big)^{i-1} (\mathrm{d}f/f\omega)$ with $\big(\frac{d}{\omega}\big)^{i-1}$ the $(i-1)$-fold iteration of $\frac{d}{\omega}$: $\mathcal{O}(G) \to \mathcal{O}(G)$, and $\big(\big(\frac{d}{\omega}\big)^i \log(f)\big)(e_{h,m}) \in \mathcal{O}(Y_m)$ means the value of $\big(\frac{d}{\omega}\big)^i \log(f)$ at $e_{h,m}$ for $h = 1, 2$.

10.13. In this 10.13, we prove Prop. 10.10 in the case $r = k-1$. In 10.14–10.17, we will prove Prop. 10.10 in the case $r' = k-1$. Assume $r = k-1$. In 10.12, take

$$\theta_{1,n},\quad \theta_{2,n} \in \mathcal{O}(G)^{\times}$$

as follows. Let α_n (resp. β_n) be the unique M (resp. N)-torsion point of $E(\mathcal{K})$ such that $p^n\alpha_n$ (resp. $p^n\beta_n$) is equal to $q^{1/M} \bmod q^{\mathbb{Z}}$ (resp. $\zeta_N \bmod q^{\mathbb{Z}}$). Let $\theta_{1,n}$ (resp. $\theta_{2,n}$) be the unique element of $\mathcal{O}(G)^{\times}$ whose image under the pull back $\mathcal{O}(G) \xrightarrow{\sim} \mathcal{O}(G)$ by the multiplication by M (resp. N) : $G \xrightarrow{\sim} G$ coincides with the pull back of ${}_c\theta_{\mathfrak{E}}$ (resp. ${}_d\theta_{\mathfrak{E}}$) by the addition $\mathfrak{E} \to \mathfrak{E}$; $x \mapsto x + \alpha_n$ (resp. $x + \beta_n$). Then

$$N_p(\theta_{1,n+1}) = \theta_{1,n},\quad N_p(\theta_{2,n+1}) = \theta_{2,n} \quad \text{for all} \quad n \geqslant 1$$

by the characterizing properties of ${}_c\theta_{\mathfrak{E}}$ and ${}_d\theta_{\mathfrak{E}}$ in 1.3 (1). Furthermore,

$$\theta_{1,n}(e_{1,n}) = {}_c g_{1/Mp^n,0},\quad \theta_{2,n}(e_{2,n}) = {}_d g_{0,1/Np^n}.$$

Hence for $m \geqslant 1$, Prop. 10.12 shows that the image of $({}_{c,d}z_{Mp^n,Np^n})_{n\geqslant 1}$ under $\exp^* \circ \mathrm{Ch}_m(k, k-1, j)$ coincides with

$$\begin{aligned}
&\frac{p^{-km}}{(k-2)!} \cdot \Big(\big(\frac{\mathrm{d}}{\omega}\big)^{k-j} \log(\theta_{1,m})\Big)(\xi_{1,m}) \cdot \Big(\big(\frac{\mathrm{d}}{\omega}\big)^{j} \log(\theta_{2,m})\Big)(\xi_{2,m}) \otimes \omega^{\otimes k} \\
&\qquad = \frac{p^{-km}}{(k-2)!} \cdot M^{j-k}\, {}_cE^{(k-j)}_{1/Mp^m,0} \cdot N^{-j}\, {}_dE^{(j)}_{0,1/Np^m} \\
&\qquad = {}_{c,d}z_{Mp^m,Np^m}(k, k-1, j).
\end{aligned}$$

10.14. We prove Prop. 10.10 in the case $r' = k-1$. We reduce 10.10 for the triple $(k, r, r') = (k, r, k-1)$ to 10.10 for the triple $(r+1, r, r)$.

Let (ξ_1, ξ_2) be the basis of T_pG as in 10.9. Then there exists a unique $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$-homomorphism

$$\text{(10.14.1)} \qquad \mathrm{Sym}^{k-2}(T_pG) \longrightarrow \frac{1}{(k-2)!} \cdot \mathrm{Sym}^{r-1}(T_pG) \subset \mathbb{Q} \otimes \mathrm{Sym}^{r-1}(T_pG)$$

which sends ξ_1^{k-2} to ξ_1^{r-1}. This homomorphism is described as follows. Let $f(X,Y)$ be a homogeneous polynomial over $\mathbb{Z}_p$ of degree $k-2$, and let $g(X,Y) = \big(\frac{\partial}{\partial X}\big)^{k-r-1} f(X,Y)$. Then we define the image of $f(\xi_1, \xi_2)$ under (10.14.1) to be $((r-1)!/(k-2)!) \cdot g(\xi_1, \xi_2)$.

Since

$$\mathrm{Sym}^{k-2}(\mathcal{H}^1_p)(k-r) = \{\mathrm{Sym}^{k-2}(T_pG)\}(2-r)$$

and

$$\mathrm{Sym}^{r-1}(\mathcal{H}^1_p)(1) = \{\mathrm{Sym}^{r-1}(T_pG)\}(2-r)$$

as smooth $\mathbb{Z}_p$-sheaves over $\mathrm{Spec}(K)$, the homomorphism (10.14.1) induces a homomorphism

$$\eta : \mathrm{Sym}^{k-2}(\mathcal{H}^1_p)(k-r) \longrightarrow \frac{1}{(k-2)!} \cdot \mathrm{Sym}^{r-1}(\mathcal{H}^1_p)(1) \quad \text{over} \quad \mathrm{Spec}(\mathcal{K}).$$

We have a commutative diagram

$$\begin{array}{ccc} \varprojlim_n \mathrm{H}^2\big(Y_m, \mathrm{Sym}^{k-2}(\mathcal{H}^1_p)(k-r)/p^n\big) & \xrightarrow{\exp^*} & \mathcal{O}(Y_m) \\ \eta\big\downarrow & & \big\| \\ \varprojlim_n \mathrm{H}^2\big(Y_m, \big(\frac{1}{(k-2)!}\cdot \mathrm{Sym}^{r-1}(\mathcal{H}^1_p)(1)\big)/p^n\big) & \xrightarrow{\exp^*} & \mathcal{O}(Y_m) \end{array} \tag{10.14.2}$$

Let $\mathcal{O}(Y_m)^{\mathrm{int}}$ be the subring of $\mathcal{O}(Y_m)$ consisting of all elements whose restrictions to each point y of Y_m belong to the valuation ring of the field $\mathcal{O}(y)$. Since

$$\exp^* : \varprojlim_n \mathrm{H}^2(Y_m, \mathrm{Sym}^{r-1}(\mathcal{H}^1_p)(1)/p^n) \longrightarrow \mathcal{O}(Y_m)$$

comes from the morphism

$$\mathrm{H}^2(Y_m, [\mathrm{Sym}^{r-1}(\mathcal{H}^1_p)(1)]) \longrightarrow [\mathcal{O}(Y_m)]$$

in $\mathrm{ind}(\mathrm{pro}(\mathrm{Ab}_p))$, where $[\mathrm{Sym}^{r-1}(\mathcal{H}^1_p)(1)]$ is the object "$\varprojlim_n$" $\mathrm{Sym}^{r-1}(\mathcal{H}^1_p)(1)/p^n$ of $\mathrm{pro}(\mathrm{Gal}_{K,p})$, the image of $\varprojlim_n \mathrm{H}^2\big(Y_m, ((1/(k-2)!)\cdot \mathrm{Sym}^{r-1}(\mathcal{H}^1_p)(1))/p^n\big)$ in $\mathcal{O}(Y_m)$ under $\exp^*$ is contained in $C\cdot \mathcal{O}(Y_m)^{\mathrm{int}}$ for some non-zero integer C.

For $n \geqslant 0$ and for $(u,w)\in A_n$ (10.8), the restriction of η to $U(u,w)\subset Y_n$ sends $e_{1,n}^{\otimes(k-2)}$ to $u^{k-r-1}e_{1,n}^{\otimes(r-1)}$. Hence for $n\geqslant m$, the projection of $\eta\circ \mathrm{Ch}_m(k,r,k-1)$ to $\mathrm{H}^2\big(Y_m, ((1/(k-2)!)\cdot \mathrm{Sym}^{r-1}(\mathcal{H}^1_p)(1))/p^n\big)$ is induced from the composition of $\mathrm{Ch}_m(r+1,r,r)$ with

$$\begin{aligned} \mathrm{H}^2\big(Y_n, (\mathrm{Sym}^{r-1}(\mathcal{H}^1_p/p^n))(1)\big) \xrightarrow{(\star)} \mathrm{H}^2\big(Y_n, (\mathrm{Sym}^{r-1}(\mathcal{H}^1_p/p^n))(1)\big)& \\ \xrightarrow{\mathrm{trace}} \mathrm{H}^2\big(Y_m, (\mathrm{Sym}^{r-1}(\mathcal{H}^1_p/p^n))(1)\big)& \end{aligned}$$

where $(\star)$ multiplies each $U(u,w)$-component by u^{k-r-1}. Hence by (10.14.2), for any $n\geqslant m$, $\exp^*\circ \mathrm{Ch}_m(k,r,k-1)$ coincides modulo $p^nC\cdot \mathcal{O}(Y_m)^{\mathrm{int}}$ with the composite of $\exp^*\circ \mathrm{Ch}_n(r+1,r,r)$ with

$$\mathcal{O}(Y_n) \xrightarrow{(\star)} \mathcal{O}(Y_n) \xrightarrow{\mathrm{trace}} \mathcal{O}(Y_m) \tag{10.14.3}$$

for any $n\geqslant m$, where $(\star)$ multiplies each $U(u,w)$-component by $\widetilde{u}^{k-r-1}$ with $\widetilde{u}$ a lifting of u to $\mathbb{Z}$. Hence for the proof of 10.10 for the triple $(k,r,r')=(k,r,k-1)$, since we have proved 10.10 for the triple $(r+1,r,r)$ in 10.13, it is sufficient to show

Lemma 10.15. — *There exists a non-zero integer C such that the image of*

$${}_{c,d}z_{Mp^n,Np^n}(r+1,r,r)$$

under the map (10.14.3) is congruent to ${}_{c,d}z_{Mp^m,Np^m}(k,r,k-1)$ modulo $p^nC^{-1}\cdot \mathcal{O}(Y_m)^{\mathrm{int}}$ for any $n\geqslant m$.

10.16. For $\left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/p^m)$,

$$i_m^* \circ \left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right)^* \text{(the image of } {}_{c,d}z_{Mp^n,Np^n}(r+1,r,r) \text{ under (10.14.3))}$$
$$= \sum (\widetilde{u}')^{k-r-1} \cdot i_n^* \circ \left(\begin{smallmatrix} u' & v' \\ w' & x' \end{smallmatrix}\right)^* ({}_{c,d}z_{Mp^n,Np^n}(r+1,r,r))$$

where $\left(\begin{smallmatrix} u' & v' \\ w' & x' \end{smallmatrix}\right)$ ranges over all elements of $\mathrm{GL}_2(\mathbb{Z}/p^n)$ whose images in $\mathrm{GL}_2(\mathbb{Z}/p^m)$ coincide with $\left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right)$. Hence for the proof of Lemma 10.15, it is sufficient to show that there exists a non-zero integer C having the following property: For any $\left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/p^m)$ and for any $n \geqslant m$,

(10.16.1) $\quad i_m^* \circ \left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right)^* {}_{c,d}z_{Mp^m,\,Np^m}(k,r,k-1)$
$$\equiv \sum (\widetilde{u}')^{k-r-1} \cdot i_n^* \circ \left(\begin{smallmatrix} u' & v' \\ w' & x' \end{smallmatrix}\right)^* ({}_{c,d}z_{Mp^n,Np^n}(r+1,r,r)) \bmod p^n \cdot C^{-1} \cdot \mathrm{O}_{\mathcal{K}_n}.$$

For each $n \geqslant 0$ and $a \in \mathbb{Z}/p^n$, fix a lifting $\widetilde{a}$ of a to $\mathbb{Z}$ such that $a \equiv 1 \bmod M$, and a lifting $\widehat{a}$ of a to $\mathbb{Z}$ such that $a \equiv 0 \bmod N$. For $n \geqslant m$, let

$$f_n = \sum_{(u',v')} (\widetilde{u}')^{k-r-1} \circ i_n^* \circ (Mp^n)^{-1} \cdot {}_cF^{(1)}_{\widetilde{u}'/Mp^n,\widehat{v}'/Np^n}$$

where (u',v') ranges over all elements of $(\mathbb{Z}/p^n)^2$ whose images in $(\mathbb{Z}/p^m)^2$ coincide with (u,v). Then by the distribution property 3.7 (2) of Eisenstein series, the right hand side of (10.16.1) is equal to

$$f_n \cdot (-1)^r \cdot (r-1)!^{-1} \cdot i_m^* \circ \left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right)^* E^{(r)}_{0,1/Np^m}.$$

Since the left hand side of (10.16.1) is

$$i_m^* \circ (Mp^m)^{k-r-2} \cdot {}_cF^{(k-r)}_{\widetilde{u}/Mp^m,\widehat{v}/Np^m} \cdot (-1)^r \cdot (r-1)!^{-1} \cdot \left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right)^* E^{(r)}_{0,1/Np^m},$$

we are reduced to

Lemma 10.17. — *There exists a non-zero integer C such that*

$$i_m^* \circ (Mp^m)^{k-r-2} \cdot {}_cF^{(k-r)}_{\widetilde{u}/Mp^m,\widehat{v}/Np^m} \equiv f_n \bmod p^nC^{-1} \cdot \mathrm{O}_{\mathcal{K}_n}$$

for any $n \geqslant m$.

Proof. — In fact, we can take $C = p^m$. This congruence is a consequence of the theory of p-adic Eisenstein series in [**KN**].

It is sufficient to show that the q-expansions of both sides of 10.17 coincide mod p^{n-m}.

Let A (resp. B) be the "without c version" of the left (resp. right) hand side of 10.17. Write

$$A = \sum_{r \in \mathbb{Q}_{\geqslant 0}} a_r q^r, \quad B = \sum_{r \in \mathbb{Q}_{\geqslant 0}} b_r q^r.$$

It is sufficient to prove the following (1) and (2).

(1) $a_r \equiv b_r \bmod p^{n-m}$ for any $r \in \mathbb{Q}_{>0}$.

(2) The constant terms of the q-expansions of both sides of 10.17 coincide mod p^{n-m}.

We prove (1). By 3.10, $\sum_{r\in\mathbb{Q}_{>0}} b_r r^{-s}$ is equal to

$$(Mp^n)^{-1}\sum_{u',v'}(\widetilde{u}')^{k-r-1}(\zeta(\widetilde{u}'/Mp^n,s)\zeta^*(\widehat{v}'/Np^n,s)-\zeta(-\widetilde{u}'/Mp^n,s)\zeta^*(-\widehat{v}'/Np^n,s)).$$

We have easily

$$\sum_{v'}\zeta^*(\widehat{v}'/Np^n,s)=p^{n-m-s}\zeta^*(\widehat{v}/Np^m,s)$$

and the formula with v' (resp. v) relaced by $-v'$ (resp. $-v$). Hence $\sum_{r\in\mathbb{Q}_{>0}} b_r r^{-s}$ is equal to

$$(Mp^m)^{-1}\sum_{u'}(\widetilde{u}')^{k-r-1}(\zeta(\widetilde{u}'/Mp^m,s)\zeta^*(\widehat{v}/Np^n,s)-\zeta(-\widetilde{u}'/Mp^m,s)\zeta^*(-\widehat{v}/Np^n,s)).$$

On the other hand, by 3.10,

$$\sum_{r\in\mathbb{Q}_{>0}} a_r r^{-s}$$
$$=(Mp^m)^{k-r-2}\big(\zeta(\widetilde{u}/Mp^m,s)\zeta^*(\widehat{v}/Np^m,s)-\zeta(-\widetilde{u}/Mp^m,s)\zeta^*(-\widehat{v}/Np^m,s)\big).$$

Hence we are reduced to the elementary fact that the coefficients of the Dirichlet series $\sum_{u'}(\widetilde{u}')^{k-r-1}\zeta(\widetilde{u}'/Mp^m,s)$ and $(Mp^m)^{k-r-1}\zeta(\widetilde{u}/Mp^m,s)$ coincide mod p^n and the same holds when u' (resp. u) is replaced by $-u'$ (resp. $-u$).

Next we prove (2). By 3.10, the constant term of the right hand side of 10.17 is equal to

$$(Mp^n)^{-1}\sum_{u',v'}(\widetilde{u}')^{k-r-1}\big(c^2\zeta(\widetilde{u}'/Mp^n,0)-c\zeta(c\widetilde{u}'/Mp^n,0)\big)$$
$$=(Mp^m)^{-1}\sum_{u'}(\widetilde{u}')^{k-r-1}\big(c^2\zeta(\widetilde{u}'/Mp^n,0)-c\zeta(c\widetilde{u}'/Mp^n,0)\big).$$

On the other hand, the constant term of the left hand side of 10.17 is

$$(Mp^m)^{-1}\big(c^2\zeta(\widetilde{u}/Mp^m,1-k+r)-c^{2-k+r}\zeta(c\widetilde{u}/Mp^m,1-k+r)\big).$$

That these are congruent mod p^{n-m} is a consequence of the theory of p-adic Riemann zeta function of Kubota-Leopoldt. □

11. Modular forms and $\mathrm{B_{dR}}$

In this section, we review p-adic Hodge theory of modular forms, and prove the commutativity of the diagram (10.9.5) (the compatibility of the two dual exponential maps, one is $\exp^*$ for the Galois representations of the local field $\mathbb{Q}_p$ associated to modular forms, and the other is $\exp^*$ of the big local field $\mathcal{K}$ which is a "local field of the field of modular functions").

In p-adic Hodge theory, we follow the method of Tsuji in [**TT**] who uses the syntomic cohomology (the method started by Fontaine-Messing [**FM**]). (In p-adic Hodge theory, there is another method by Faltings who uses almost étale extensions [**Fa1**]–[**Fa4**].)

11.1. Following Scholl [**Sc1**], we identify various cohomology groups associated to modular forms with direct summands of cohomology groups of Kuga-Sato varieties. We review Kuga-Sato varieties.

Let $k \geqslant 2$, $N \geqslant 3$.

Let $E \to Y(N)$ be the universal elliptic curve, and let $E^{(k-2)}$ (= "the open Kuga-Sato variety" of weight k) be the $(k-2)$-fold fiber product of E over $Y(N)$.

Let $\widehat{E} \to X(N)$ be the universal generalized elliptic curve with level N structure [**DR**]. Let $\widehat{E}^{(k-2)} \to X(N)$ be the $(k-2)$-fold fiber product of $\widehat{E}$ over $X(N)$ and let $\mathrm{KS}_k = \mathrm{KS}_k(N)$ (the Kuga-Sato variety of weight k) be the canonical desingularization of $\widehat{E}^{(k-2)}$ constructed by Deligne in [**De1**, Lemma 5.4, Lemma 5.5].

Then $E^{(k-2)}$ is open in KS_k, and the complement of $E^{(k-2)}$ in KS_k, which coincides with the inverse image of the set of cusps of $X(N)$, is a divisor with normal crossings in KS_k.

11.2. We review the relation of Kuga-Sato varieties with modular forms.

Define a finite group G which acts on $E^{(k-2)}$ and a finite group $\widetilde{G}$ which acts on KS_k as follows.

Let $\mathfrak{S}_{k-2}$ be the symmetric group of degree $k-2$. Let G be the semi-direct product of the two groups $\mathfrak{S}_{k-2}$ and $\{\pm 1\}^{k-2}$ in which $\{\pm 1\}^{k-2}$ is normal and the action of $\mathfrak{S}_{k-2}$ on $\{\pm 1\}^{k-2}$ by inner automorphisms is given by the canonical action of $\mathfrak{S}_{k-2}$ on $\{\pm 1\}^{k-2}$ by permutations. Then G acts on $E^{(k-2)}$ as follows: $\{\pm 1\}$ acts on E by multiplication, and hence $\{\pm 1\}^{k-2}$ acts on $E^{(k-2)}$, and $\mathfrak{S}_{k-2}$ acts on $E^{(k-2)}$ by permuting the factors of the fiber product.

On the other hand, let $\widetilde{G}$ be the semi-direct product of G and $((\mathbb{Z}/N)^2)^{k-2}$ defined as follows. $((\mathbb{Z}/N)^2)^{k-2}$ is normal in $\widetilde{G}$, the action of $\{\pm 1\}^{k-2}$ on $((\mathbb{Z}/N)^2)^{k-2}$ by inner automorphisms is the one induced by the natural multiplicative action of $\{\pm 1\}$ on $(\mathbb{Z}/N)^2$, and the action of $\mathfrak{S}_{k-2}$ on $((\mathbb{Z}/N)^2)^{k-2}$ by inner automorphisms is by permutations. Then $\widetilde{G}$ acts on KS_k as follows: $\mathfrak{S}_{k-2}$ acts by permutation, the action of $\{\pm 1\}^{k-2}$ is induced by the action of $\{\pm 1\}$ on $\widehat{E}$ by multiplication, and the action of $((\mathbb{Z}/N)^2)^{k-2}$ is induced by the translation on $\widehat{E}$ by $(\mathbb{Z}/N)^2$ ($(\mathbb{Z}/N)^2$ is embedded in the smooth part of $\widehat{E}$ by the N-level structure of $\widehat{E}$).

Let $\varepsilon : G \to \{\pm 1\}$ be the homomorphism whose restriction to $\{\pm 1\}^{k-2}$ (resp. $\mathfrak{S}_{k-2}$) is the product map (resp. the sign function).

Let $\widetilde{\varepsilon} : \widetilde{G} \to \{\pm 1\}$ be the composition $\widetilde{G} \to G \xrightarrow{\varepsilon} \{\pm 1\}$. For a G-module (resp. $\widetilde{G}$-module) M, let

$$M(\varepsilon) \text{ (resp. } M(\widetilde{\varepsilon})\text{)} = \{x \in M\,;\, \sigma x = \varepsilon(\sigma)x \text{ (resp. } \widetilde{\varepsilon}(\sigma)x\text{) for all } \sigma \in G\}.$$

Then we have the following (11.2.1)-(11.2.6). (11.2.1), (11.2.3), (11.2.5) are shown in Scholl [**Sc1**], and (11.2.2), (11.2.4), (11.2.6) are shown in the same way but more easily.

We have canonical isomorphisms of $\mathbb{Q}[\mathrm{Gal}(\mathbb{C}/\mathbb{R})]$-modules

$$V_{k,\mathbb{Q}}(X(N)) \cong \mathrm{H}^{k-1}(\mathrm{KS}_k(\mathbb{C}), \mathbb{Q})(\widetilde{\varepsilon}), \tag{11.2.1}$$

$$V_{k,\mathbb{Q}}(Y(N)) \cong \mathrm{H}^{k-1}(E^{(k-2)}(\mathbb{C}), \mathbb{Q})(\varepsilon), \tag{11.2.2}$$

which induce isomorphisms of $\mathbb{Q}_p[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-modules

$$V_{k,\mathbb{Q}_p}(X(N)) \cong \mathrm{H}^{k-1}(\mathrm{KS}_k \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)(\widetilde{\varepsilon}), \tag{11.2.3}$$

$$V_{k,\mathbb{Q}_p}(Y(N)) \cong \mathrm{H}^{k-1}(E^{(k-2)} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)(\varepsilon). \tag{11.2.4}$$

($\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $V_{k,\mathbb{Q}_p}(X(N)) = V_k(X(N)) \otimes_{\mathbb{Q}} \mathbb{Q}_p$, since it is identified with the étale cohomology group $\mathrm{H}^1(X(N) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, j_* \operatorname{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ where j is the inclusion map $Y(N) \to X(N)$.)

Let $\Omega^\bullet_{\mathrm{KS}_k/\mathbb{Q}}(\log)$ be the de Rham complex on KS_k with log pole outside $E^{(k-2)}$, and denote for $m \geqslant 0$

$$\mathrm{H}^m_{\log-\mathrm{dR}}(\mathrm{KS}_k) = \mathrm{H}^m(\mathrm{KS}_k, \Omega^\bullet_{\mathrm{KS}_k/\mathbb{Q}}(\log)),$$
$$\mathrm{fil}^i \mathrm{H}^m_{\log-\mathrm{dR}}(\mathrm{KS}_k) = \mathrm{H}^m(\mathrm{KS}_k, \Omega^{\geqslant i}_{\mathrm{KS}_k/\mathbb{Q}}(\log)).$$

Then (H^*_{dR} denotes the usual de Rham cohomology),

$$\mathrm{fil}^i \mathrm{H}^{k-1}_{\mathrm{dR}}(\mathrm{KS}_k)(\widetilde{\varepsilon}) = \begin{cases} \mathrm{H}^{k-1}_{\mathrm{dR}}(\mathrm{KS}_k)(\widetilde{\varepsilon}) & \text{if } i \leqslant 0, \\ S_k(X(N)) & \text{if } 1 \leqslant i \leqslant k-1, \\ 0 & \text{if } i \geqslant k. \end{cases} \tag{11.2.5}$$

$$\mathrm{fil}^i \mathrm{H}^{k-1}_{\log-\mathrm{dR}}(\mathrm{KS}_k)(\widetilde{\varepsilon}) = \begin{cases} \mathrm{H}^{k-1}_{\log-\mathrm{dR}}(\mathrm{KS}_k)(\widetilde{\varepsilon}) & \text{if } i \leqslant 0, \\ M_k(X(N)) & \text{if } 1 \leqslant i \leqslant k-1, \\ 0 & \text{if } i \geqslant k. \end{cases} \tag{11.2.6}$$

(The action of $\widetilde{G}$ on $\mathrm{H}^*_{\log-\mathrm{dR}}(\mathrm{KS}_k)$ factors through G. This follows from (11.3.2) below.)

11.3. By comparison theorems of Betti cohomology and de Rham cohomology over $\mathbb{C}$, we have for any m

$$\mathrm{H}^m(\mathrm{KS}_k(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C} \simeq \mathrm{H}^m_{\mathrm{dR}}(\mathrm{KS}_k) \otimes_{\mathbb{Q}} \mathbb{C} \tag{11.3.1}$$

$$\mathrm{H}^m(E^{(k-2)}(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C} \simeq \mathrm{H}^m_{\log-\mathrm{dR}}(\mathrm{KS}_k) \otimes_{\mathbb{Q}} \mathbb{C} \tag{11.3.2}$$

The homomorphism

$$M_k(X(N)) \otimes \mathbb{C} \longrightarrow V_k(Y(N)) \otimes \mathbb{C}$$

induced by (11.3.2) (we take $m = k-1$) via (11.2.2) and (11.2.6) coincides with the period map in 4.10.

By Tsuji [**TT**], for any m, $\mathrm{H}^m(\mathrm{KS}_k \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_p}, \mathbb{Q}_p)$ is a de Rham representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and we have a canonical isomorphism preserving filtrations

$$\mathrm{D}_{\mathrm{dR}}(\mathrm{H}^m(\mathrm{KS}_k \otimes \overline{\mathbb{Q}_p}, \mathbb{Q}_p)) \simeq \mathrm{H}^m_{\mathrm{dR}}(\mathrm{KS}_k) \otimes \mathbb{Q}_p \tag{11.3.3}$$

which is the p-adic version of (11.3.1). Form $m = k-1$, this gives by (11.2.3) and (11.2.5)

$$\mathrm{D}^i_{\mathrm{dR}}(V_{k,\mathbb{Q}_p}(X(N))) = \begin{cases} \mathrm{D}_{\mathrm{dR}}(V_{k,\mathbb{Q}_p}(X(N))) & \text{if } i \leqslant 0, \\ S_k(X(N)) \otimes \mathbb{Q}_p & \text{if } 1 \leqslant i \leqslant k-1 \\ 0 & \text{if } i \geqslant k. \end{cases} \tag{11.3.4}$$

By [**TT**], we have also a canonical homomorphism

$$\mathrm{D}_{\mathrm{dR}}(\mathrm{H}^m(E^{(k-2)} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_p}, \mathbb{Q}_p)) \longrightarrow \mathrm{H}^m_{\mathrm{log-dR}}(\mathrm{KS}_k) \otimes \mathbb{Q}_p \tag{11.3.5}$$

which is compatible with (11.3.3) and with filtrations, and which is the p-adic version of (11.3.2). See 11.4 for the constructions of the isomorphism (11.3.3) and the homomorphism (11.3.5). Unfortunately, [**TT**]does not contain results which show that (11.3.5) is an isomorphism. In (11.10), we will show that the homomorphism (11.3.5) for $m = k-1$ induces an isomorphism

$$\mathrm{D}_{\mathrm{dR}}(\mathrm{H}^{k-1}(E^{(k-2)} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_p}, \mathbb{Q}_p))(\varepsilon) \xrightarrow{\sim} \mathrm{H}^{k-1}_{\mathrm{log-dR}}(\mathrm{KS}_k)(\varepsilon) \otimes \mathbb{Q}_p \tag{11.3.6}$$

preserving filtrations. (Hence by comparing the dimensions by (11.3.2), we have that $V_{k,\mathbb{Q}_p}(Y(N)) = \mathrm{H}^{k-1}(E^{(k-2)} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_p}, \mathbb{Q}_p))(\varepsilon)$ is a de Rham representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.) The isomorphism (11.3.6) gives by (11.2.4) and (11.2.6)

$$\mathrm{D}^i_{\mathrm{dR}}(V_{k,\mathbb{Q}_p}(Y(N))) = \begin{cases} \mathrm{D}_{\mathrm{dR}}(V_{k,\mathbb{Q}_p}(Y(N))) & \text{if } i \leqslant 0, \\ M_k(X(N)) \otimes \mathbb{Q}_p & \text{if } 1 \leqslant i \leqslant k-1 \\ 0 & \text{if } i \geqslant k. \end{cases} \tag{11.3.7}$$

In [**Fa2**, Thm 8.1], Faltings has a result "D_{dR} of the p-adic étale cohomology of an open variety is the de Rham cohomology with log poles" which gives a canonical isomorphism between the two groups which appear in (11.3.5). He uses the method of almost étale extensions. But we formulate in this section everything in the method of syntomic cohomology by Fontaine-Messing-Tsuji (this is because the author is familiar with syntomic cohomology and not so much with almost étale extensions). We use the definition of the map (11.3.5) given by the method of syntomic cohomology, and prove the bijectivity of (11.3.6) in 11.10 by the method of syntomic cohomology, and avoid long arguments to check the relation between the method of Fontaine-Messing-Tsuji and that of Faltings.

11.4. We review the methods in [**TT**] and see how the isomorphism (11.3.3) and the homomorphism (11.3.5) are constructed as special cases of general results in [**TT**].

Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with perfect residue field, and let X be a proper semi-stable scheme over O_K (that is, X is regular, the generic fiber X_K of X is smooth over K, and the special fiber of X is a divisor with normal crossings), and let U be a dense open subscheme of X_K such that $(X \smallsetminus U)_{\mathrm{red}}$ is a divisor of X with normal crossings. (In the application, X will be an integral model of a Kuga-Sato variety, and U will be a Kuga-Sato variety or an open Kuga-Sato variety.)

For integers m, r such that $0 \leqslant m \leqslant r$, we have a canonical isomorphism [**TT**, Thm. 0.5]

$$\text{(11.4.1)} \qquad \mathrm{H}^m(X \otimes_{O_K} O_{\overline{K}}, \mathcal{S}_{\mathbb{Q}_p}^{\log}(r)) \xrightarrow{\sim} \mathrm{H}^m(U \otimes_K \overline{K}, \mathbb{Q}_p)(r)$$

between the log syntomic cohomology of $X \otimes_{O_K} O_{\overline{K}}$ (the left hand side) and the p-adic étale cohomology of $U \otimes_K \overline{K}$ (the right hand side).

Here

$$\mathrm{H}^m(X \otimes_{O_K} O_{\overline{K}}, \mathcal{S}_{\mathbb{Q}_p}^{\log}(r)) \underset{\mathrm{def}}{=} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n \mathrm{H}^m(X \otimes_{O_K} O_{\overline{K}}, \widetilde{\mathcal{S}}_n(r)_{\overline{X}})$$

with $\widetilde{\mathcal{S}}_n(r)_{\overline{X}}$ the syntomic complex on $\overline{X} = X \otimes_{O_K} O_{\overline{K}}/p^n$ defined with respect to the canonical log structure of $X \otimes_{O_K} O_{\overline{K}}/p^n$ which is induced by the log structure of X associated to the divisor $(X \smallsetminus U)_{\mathrm{red}}$ and the canonical log structures of $\mathrm{Spec}(O_{\overline{K}})$ and $\mathrm{Spec}(O_K)$. On the other hand, for integers $m, r \geqslant 0$, we have a canonical homomorphism [**TT**, 4.8]

$$\text{(11.4.2)} \quad \mathrm{H}^m(X \otimes_{O_K} O_{\overline{K}}, \widetilde{\mathcal{S}}_n(r)_{\overline{X}}) \longrightarrow \mathrm{H}^m\left(((X \otimes_{O_K} O_{\overline{K}}/p^n)/(O_K/p^n))_{\mathrm{log-crys}}, J_{\mathrm{crys}}^{[r]}\right)$$

where log-crys means the log crystalline site with respect to above log structure of $X \otimes_{O_K} O_{\overline{K}}/p^n$ and the canonical log structure of the base $\mathrm{Spec}(O_K/p^n)$, and $J_{\mathrm{crys}}^{[r]}$ denotes the r-th divided power of the ideal $\mathrm{Ker}(\mathcal{O}_{\mathrm{crys}} \to \mathcal{O}_X)$ of the structure sheaf $\mathcal{O}_{\mathrm{crys}}$ of the log crystalline site. We have a canonical isomorphism [**TT**, 4.7.6]

$$\text{(11.4.3)} \quad \varprojlim_s \mathbb{Q} \otimes \varprojlim_n \mathrm{H}^m\left(((X \otimes_{O_K} O_{\overline{K}}/p^n)/(O_K/p^n))_{\mathrm{log-crys}}, \mathcal{O}_{\mathrm{crys}}/J_{\mathrm{crys}}^{[s]}\right) \simeq \mathrm{B}_{\mathrm{dR}}^+ \otimes_K \mathrm{H}_{\mathrm{log-dR}}^m(X_K/K) \qquad (\mathrm{B}_{\mathrm{dR}}^+ = \mathrm{B}_{\mathrm{dR},\overline{K}/K}^+ = \mathrm{B}_{\mathrm{dR},\overline{K}/\mathbb{Q}_p}^+).$$

By (11.4.1), (11.4.2), (11.4.3), we have a canonical homomorphism

$$\text{(11.4.4)} \quad \mathrm{H}^m(U \otimes_K \overline{K}, \mathbb{Q}_p) \simeq \mathrm{H}^m(X \otimes_{O_K} O_{\overline{K}}, \mathcal{S}_{\mathbb{Q}_p}^{\log}(r))(-r) \longrightarrow \mathrm{B}_{\mathrm{dR}} \otimes_K \mathrm{H}_{\mathrm{log-dR}}^m(X_K/K) \qquad (\mathrm{B}_{\mathrm{dR}} = \mathrm{B}_{\mathrm{dR},\overline{K}/K} = \mathrm{B}_{\mathrm{dR},\overline{K}/\mathbb{Q}_p}).$$

for $r \geqslant m$, and this map is independent of the choice of $r \geqslant m$ and commutes with the action of $\mathrm{Gal}(\overline{K}/K)$. By taking the $\mathrm{Gal}(\overline{K}/K)$-invariant part of the induced map

$$\text{(11.4.5)} \qquad \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathrm{H}^m(U \otimes_K \overline{K}, \mathbb{Q}_p) \longrightarrow \mathrm{B}_{\mathrm{dR}} \otimes_K \mathrm{H}_{\mathrm{log-dR}}^m(X_K/K),$$

we have a canonical homomorphism

$$\text{(11.4.6)} \qquad \mathrm{D}_{\mathrm{dR}}(\mathrm{H}^m(U \otimes_K \overline{K}, \mathbb{Q}_p)) \longrightarrow \mathrm{H}_{\mathrm{log-dR}}^m(X_K/K)$$

Tsuji proves in [**TT**] that in the case of $U = X_K$, the maps (11.4.5) and (11.4.6) are bijective and induce isomorphisms of filtrations.

The isomorphism (11.3.3) and the homomorphisms (11.3.5) are obtained from the above general theory as follows (we follow [**Sa1**]). Take a multiple N' of N such that the genus of $X(N')$ as a curve over $\mathbb{Q}(\zeta_{N'})$ is $\geqslant 2$. Let $\mathfrak{X}(N')$ be the fine moduli space over $\mathbb{Z}$ of generalized elliptic curves with N'-level structures [**KM**]. Then $\mathfrak{X}(N')$ is a proper flat curve over $\mathbb{Z}$ such that $\mathfrak{X}(N') \otimes_{\mathbb{Z}} \mathbb{Q} = X(N')$. Take a finite extension K of $\mathbb{Q}_p^{\mathrm{ur}}$ which is Galois over $\mathbb{Q}_p$ such that $X(N') \otimes_{\mathbb{Q}} K$ has semi-stable reduction. Let $\mathfrak{C}$ be the minimal semi-stable model of $X(N') \otimes_{\mathbb{Q}} K$ over O_K. By the assumption on the genus, $\mathfrak{C}$ is the minimal desingularization of the normalization of $\mathfrak{X}(N')$ in $X(N') \otimes_{\mathbb{Q}} K$. Let $\widehat{\mathfrak{E}} \to \mathfrak{C}$ be the base change of the universal generalized elliptic curve over $\mathfrak{X}(N)$, let X be the canonical desingularization by Deligne of the $k-2$ fold product of $\widehat{\mathfrak{E}}$ over $\mathfrak{C}$ (see [**Sa1**, p. 610]). Let $U = X_K$ (resp. $U = E^{(k-2)} \times_{Y(N)} Y(N') \otimes_{\mathbb{Q}} K$). Then, X is proper semi-stable over O_K, and $(X \smallsetminus U)_{\mathrm{red}}$ is a divisor of X with normal crossings. By applying the above general theory to (X, U), and then taking the $\mathrm{Gal}(K/\mathbb{Q}_p) \times \mathrm{Aut}(X(N')/X(N))$-invariant part of (11.4.6), we obtain the isomorphism (11.3.3) (resp. the homomorphism (11.3.5)).

11.5. The theory of Tsuji can be generalized to the case the residue field k of K is not necessarily perfect as follows, without essential changes.

Let K, X and U be as in 11.4 except that we do not assume here that the residue field k of K is perfect, but we assume $[k : k^p] < \infty$.

For integers m, r such that $0 \leqslant m \leqslant r$, we have a canonical isomorphism

$$\mathrm{H}^m(X \otimes_{O_K} O_{\overline{K}}, \mathcal{S}_{\mathbb{Q}_p}^{\log}(r)) \xrightarrow{\sim} \mathrm{H}^m(U \otimes_K \overline{K}, \mathbb{Q}_p)(r). \tag{11.5.1}$$

In fact, the canonical map from the left hand side to the right hand side is defined in the same way as in [**TT**], and the bijectivity of it is reduced to the perfect residue field case, for we have an intermediate field K' such that $K \subset K' \subset \overline{K}$ which is a henselian discrete valuation field with perfect residue field. (In fact take a lifting $(b_i)_i$ of a p-base of the residue field of K to O_K, and choose a p^n-th root $b_{i,n}$ of b_i in $\overline{K}$ for each i and $n \geqslant 0$ satisfying $b_{i,n+1}^p = b_{i,n}$ for all i and $n \geqslant 0$. Then $K' = K(b_{i,n}; \forall i, \forall n \geqslant 0)$ is such a field. The completion of a henselian discrete valuation field K' does not change the log syntomic cohomology and the p-adic cohomology for X, U over $O_{K'}$ as above.)

Remark. — The log crystalline site which is used here for the definition of $\widetilde{\mathcal{S}}_n(r)_{\overline{X}}$ is $((X \otimes_{O_K} O_{\overline{K}}/p^n)/(\mathbb{Z}/p^n))_{\mathrm{log-crys}}$ with respect to the canonical log structure of $X \otimes_{O_K} O_{\overline{K}}/p^n$ and the trivial log structure on the base $\mathrm{Spec}(\mathbb{Z}/p^n)$. In the perfect residue field case, Tsuji uses instead the log crystalline site $((X \otimes_{O_K} O_{\overline{K}}/p^n)/W_n)_{\mathrm{log-crys}}$ where W_n is the Witt ring of the residue field of K with length n, with respect to the canonical log structure of $X \otimes_{O_K} O_{\overline{K}}/p^n$ and the trivial log structure on the base $\mathrm{Spec}(W_n)$, but these two log crystalline sites give the same $\widetilde{\mathcal{S}}_n(r)_{\overline{X}}$.

We have canonical homomorphisms

$$(11.5.2)\quad \mathrm{H}^m(X \otimes_{O_K} O_{\overline{K}}, \widetilde{\mathcal{S}}_n(r)_{\overline{X}}) \longrightarrow \mathrm{H}^m((X \otimes_{O_K} O_{\overline{K}}/p^n)/(\mathbb{Z}/p^n))_{\mathrm{log-crys}}, J^{[r]}_{\mathrm{crys}}) \\ \longrightarrow \mathrm{H}^m((X \otimes_{O_K} O_{\overline{K}}/p^n)/(O_K/p^n))_{\mathrm{log-crys}}, J^{[r]}_{\mathrm{crys}})$$

for integers $m, r \geqslant 0$, and an isomorphism

$$(11.5.3)\quad \varprojlim_s \mathbb{Q} \otimes \varprojlim_n \mathrm{H}^m\Big(((X \otimes_{O_K} O_{\overline{K}}/p^n)/(O_K/p^n))_{\mathrm{log-crys}}, \mathcal{O}_{\mathrm{crys}}/J^{[s]}_{\mathrm{crys}}\Big) \\ \simeq \mathrm{B}^+_{\mathrm{dR}} \otimes_K \mathrm{H}^m_{\mathrm{log-dR}}(X_K/K) \qquad (\mathrm{B}^+_{\mathrm{dR}} = \mathrm{B}^+_{\mathrm{dR},\overline{K}/K}).$$

which are obtained in the same way as in the perfect residue fields case.

By (11.5.1), (11.5.2), (11.5.3), we have a canonical homomorphism

$$(11.5.4)\quad \mathrm{H}^m(U \otimes_K \overline{K}, \mathbb{Q}_p) \simeq \mathrm{H}^m(X \otimes_{O_K} O_{\overline{K}}, \mathcal{S}^{\log}_{\mathbb{Q}_p}(r))(-r) \\ \longrightarrow \mathrm{B}_{\mathrm{dR}} \otimes_K \mathrm{H}^m_{\mathrm{log-dR}}(X_K/K) \qquad (\mathrm{B}_{\mathrm{dR}} = \mathrm{B}_{\mathrm{dR},\overline{K}/K})$$

for $r \geqslant m$, and this map is independent of the choice of r and commutes with the action of $\mathrm{Gal}(\overline{K}/K)$. By taking the $\mathrm{Gal}(\overline{K}/K)$-invariant part of the induced map

$$(11.5.5)\qquad \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathrm{H}^m(U \otimes_K \overline{K}, \mathbb{Q}_p) \longrightarrow \mathrm{B}_{\mathrm{dR}} \otimes_K \mathrm{H}^m_{\mathrm{log-dR}}(X_K/K),$$

we have a canonical homomorphism

$$(11.5.6)\qquad \mathrm{D}_{\mathrm{dR}}(\mathrm{H}^m(U \otimes_K \overline{K}, \mathbb{Q}_p)) \longrightarrow \mathrm{H}^m_{\mathrm{log-dR}}(X_K/K).$$

Via (11.5.6), the connection

$$\nabla : \mathrm{D}_{\mathrm{dR}}(\mathrm{H}^m(U \otimes_K \overline{K}, \mathbb{Q}_p)) \longrightarrow \widehat{\Omega}^1_K \otimes_K \mathrm{D}_{\mathrm{dR}}(\mathrm{H}^m(U \otimes_K \overline{K}, \mathbb{Q}_p))$$

in 10.2 commutes with the Gauss-Manin connection

$$\mathrm{H}^m_{\mathrm{log-dR}}(X_K/K) \longrightarrow \widehat{\Omega}^1_K \otimes_K \mathrm{H}^m_{\mathrm{log-dR}}(X_K/K).$$

This is because the image of $\mathrm{H}^m(U \otimes_K \overline{K}, \mathbb{Q}_p)$ in $\mathrm{B}_{\mathrm{dR}} \otimes_K \mathrm{H}^m_{\mathrm{log-dR}}(X_K/K)$ is contained in the kernel of $\nabla \otimes 1 + 1 \otimes \nabla$ as is seen from the factorization (11.5.2).

By the same method as in [**TT**], we can prove that in the case $U = X_K$, the maps (11.5.5) and (11.5.6) are bijective and induce isomorphisms of filtrations. (We will use this fact only in the case X is a product of finite copies of an elliptic curve of good reduction over $O_{\overline{K}}$.)

11.6. We consider an interaction of 11.4 (the perfect residue field case) and the case $[k : k^p] = p$ in 11.5, which will be used for the proof of the commutativity of the diagram (10.9.5).

Let K, X, U be as in 11.4. We assume here that the residue field of K is algebraically closed. Let C be a proper normal curve over O_K, let $X \to C$ be a morphism over O_K, and let ν be a generic point of the special fiber of C. Then the local ring $\mathcal{O}_{C,\nu}$ is a discrete valuation ring whose residue field $\mathfrak{K}$ satisfies $[\mathfrak{K} : \mathfrak{K}^p] = p$. Let $\mathscr{K}$ be the field of fractions of the completion of $\mathcal{O}_{C,\nu}$. Assume that K is algebraically closed in $\mathscr{K}$, $X \otimes_C O_K$ is smooth over O_K, and that

$$X \otimes_C O_{\mathscr{K}} = U \otimes_C O_{\mathscr{K}}.$$

(In our application in 11.8, X will be an integral model of a Kuga-Sato variety, U will be an open Kuga-Sato variety, and C will be an integral model of a modular curve. In that case, $X \otimes_C O_{\mathscr{K}} = U \otimes_C O_{\mathscr{K}}$ and this scheme is the $(k-2)$-fold product over $O_{\mathscr{K}}$ of an elliptic curve over $O_{\mathscr{K}}$ of good reduction.)

For $m \geqslant 0$, let

$$\mathrm{H}_e^m = \mathrm{H}^m(U \otimes_K \overline{K}, \mathbb{Q}_p), \qquad \mathrm{H}_d^m = \mathrm{H}^m_{\log-\mathrm{dR}}(X_K/K),$$
$$\mathrm{H}_E^m = \mathrm{H}^m(U \otimes_C \overline{\mathscr{K}}, \mathbb{Q}_p), \qquad \mathrm{H}^m_{D/\mathscr{K}} = \mathrm{H}^m_{\mathrm{dR}}(X \otimes_C \mathscr{K}/\mathscr{K}).$$

By 11.5, H_E^m is a de Rham representation of $\mathrm{Gal}(\overline{\mathscr{K}}/\mathscr{K})$, and $\mathrm{D}_{\mathrm{dR}}(\mathrm{H}_E^m) = \mathrm{H}^m_{D/\mathscr{K}}$.

Fix an integer $\ell \geqslant 0$. Assume we are given a de Rham representation $\mathrm{H}^\ell_{e'}$ of $\mathrm{Gal}(\overline{K}/K)$ contained in $\mathrm{Ker}(\mathrm{H}_e^\ell \to \mathrm{H}_E^\ell)$. Let $\mathrm{H}^\ell_{d'} = \mathrm{D}_{\mathrm{dR}}(\mathrm{H}^\ell_{e'})$. (In our application in 11.8, we will take $\ell = k-1, \mathrm{H}^{k-1}_{e'} = \mathrm{H}^{k-1}_e(\varepsilon), \mathrm{H}^{k-1}_{d'} = \mathrm{H}^{k-1}_d(\varepsilon)$, where (ε) is as in 11.2.)

In the following, we express by [ ] an object of $\mathrm{ind}(\mathrm{pro}(\mathrm{Ab}_p))$ or of $\mathrm{ind}(\mathrm{pro}(\mathrm{Gal}_{\mathscr{K},p}))$ as in 10.4. In Prop 11.7 below, we will compare the two dual exponential maps :

$$\exp^* : \mathrm{H}^1(K, [\mathrm{H}^\ell_{e'}(r)]) \longrightarrow [\mathrm{fil}^r \mathrm{H}^\ell_{d'}],$$
$$\exp^* : \mathrm{H}^2(\mathscr{K}, [\mathrm{H}^{\ell-1}_E(r)]) \longrightarrow [\widehat{\Omega}^1_{\mathscr{K}} \otimes_{\mathscr{K}} \mathrm{fil}^{r-1}\mathrm{H}^{\ell-1}_{D/\mathscr{K}}]/\nabla[\mathrm{fil}^r\mathrm{H}^{\ell-1}_{D/\mathscr{K}}]$$

for $r \in \mathbb{Z}$.

Proposition 11.7. — *For any $r \in \mathbb{Z}$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{H}^1(K, [\mathrm{H}^\ell_{e'}(r)]) \arrow[r, "\exp^*"] \arrow[d, "\alpha"'] & {[\mathrm{fil}^r \mathrm{H}^\ell_{d'}]} \arrow[d, "\beta"] \\
\mathrm{H}^2(\mathscr{K}, [\mathrm{H}^{\ell-1}_E(r)]) \arrow[r, "\exp^*"] & {[\widehat{\Omega}^1_{\mathscr{K}} \otimes_{\mathscr{K}} \mathrm{fil}^{r-1}\mathrm{H}^{\ell-1}_{D/\mathscr{K}}]/\nabla[\mathrm{fil}^r\mathrm{H}^{\ell-1}_{D/\mathscr{K}}]}
\end{tikzcd}$$

Here the maps α and β are defined as follows.

First we define α. Since $\mathscr{K}$ is the completion of a henselian discrete valuation field which is of transcendence degree 1 over K, the composite field $\overline{K}\mathscr{K}$ in $\overline{\mathscr{K}}$ is of cohomological dimension 1 [**Se1**, Chap. II,§ 3.3]. Hence we have an exact sequence

$$0 \longrightarrow [\mathrm{H}^1(\overline{K}\mathscr{K}, \mathrm{H}_E^{\ell-1})] \longrightarrow [\mathrm{H}^\ell(U \otimes_K \overline{K}\mathscr{K}, \mathbb{Q}_p)] \longrightarrow [\mathrm{H}_E^\ell].$$

Since the image of $\mathrm{H}_{e'}^\ell$ in H_E^ℓ is zero by our assumption, we have a map $[\mathrm{H}_{e'}^\ell] \to [\mathrm{H}^1(\overline{K}\mathscr{K}, \mathrm{H}_E^{\ell-1})]$, and hence a map

$$\mathrm{H}^1(K, [\mathrm{H}_{e'}^\ell]) \longrightarrow \mathrm{H}^1(K, \mathrm{H}^1(\overline{K}\mathscr{K}, [\mathrm{H}_E^{\ell-1}(r)])).$$

Since the residue field of K is algebraically closed, the cohomological dimension of K is 1 [**Se1**, Chap. II, § 3.3] as well as that of $\overline{K}\mathscr{K}$, and hence we have for any m

$$\mathrm{H}^1(K, \mathrm{H}^1(\overline{K}\mathscr{K}, [\mathrm{H}_E^{m-1}(r)])) \xrightarrow{\sim} \mathrm{H}^2(\mathscr{K}, [\mathrm{H}_E^{m-1}(r)]).$$

Hence we have the map α.

The map β is defined as follows. For any m, let

$$\mathrm{H}_{D/K}^m = K \otimes_{O_K} \varprojlim_n \mathrm{H}^m(X \otimes_C O_{\mathscr{K}}, \Omega^\bullet_{(X\otimes_C O_{\mathscr{K}})/O_K}(\log)/p^n),$$
$$\mathrm{fil}^r\mathrm{H}_{D/K}^m = K \otimes_{O_K} \varprojlim_n \mathrm{H}^m(X \otimes_C O_{\mathscr{K}}, \Omega^{\geqslant r}_{(X\otimes_C O_{\mathscr{K}})/O_K}(\log)/p^n),$$

(note that the base of the differential here is O_K and not $O_{\mathscr{K}}$). Then we have long exact sequences

$$\cdots \longrightarrow \mathrm{H}_{D/K}^m \longrightarrow \mathrm{H}_{D/\mathscr{K}}^m \xrightarrow{\nabla} \widehat{\Omega}^1_{\mathscr{K}} \otimes_{\mathscr{K}} \mathrm{H}_{D/\mathscr{K}}^m$$
$$\longrightarrow \mathrm{H}_{D/K}^{m+1} \longrightarrow \mathrm{H}_{D/\mathscr{K}}^{m+1} \xrightarrow{\nabla} \widehat{\Omega}^1_{\mathscr{K}} \otimes_{\mathscr{K}} \mathrm{H}_{D/\mathscr{K}}^{m+1} \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \mathrm{fil}^r\mathrm{H}_{D/K}^{m-1} \longrightarrow \mathrm{fil}^r\mathrm{H}_{D/\mathscr{K}}^{m-1} \xrightarrow{\nabla} \widehat{\Omega}^1_{\mathscr{K}} \otimes_{\mathscr{K}} \mathrm{fil}^{r-1}\mathrm{H}_{D/\mathscr{K}}^{m-1}$$
$$\longrightarrow \mathrm{fil}^r\mathrm{H}_{D/K}^m \longrightarrow \mathrm{fil}^r\mathrm{H}_{D/\mathscr{K}}^m \xrightarrow{\nabla} \widehat{\Omega}^1_{\mathscr{K}} \otimes_{\mathscr{K}} \mathrm{fil}^{r-1}\mathrm{H}_{D/\mathscr{K}}^m \longrightarrow \cdots$$

Since the composition $\mathrm{H}_{d'}^\ell \to \mathrm{H}_d^\ell \to \mathrm{H}_{D/K}^\ell \to \mathrm{H}_{D/\mathscr{K}}^\ell$ is zero (for it is the D_{dR} of the zero map $\mathrm{H}_{e'}^\ell \to \mathrm{H}_E^\ell$), the last long exact sequence gives the map β.

11.8. Let $K, X, U = (E^{(k-2)} \times_{Y(N)} Y(N')) \otimes_{\mathbb{Q}} K$ be as at the end of 11.4, and let C be the scheme $\mathfrak{C}$ at the end of 11.4. Let $\ell = k-1$, $\mathrm{H}_{e'}^\ell = \mathrm{H}^\ell(U \otimes_K \overline{K}, \mathbb{Q}_p)(\varepsilon) \simeq V_{k,\mathbb{Q}_p}(Y(N'))$. Then Prop. 11.7 in this case proves the commutativity of the diagram (10.9.5).

11.9. We prove Prop. 11.7.

For an integer $s \geqslant r$ and for $m \geqslant 0$, let

$$\begin{aligned}
{}^r_s\mathrm{H}^m_d &= \mathrm{fil}^r\mathrm{H}^m_d/\mathrm{fil}^s\mathrm{H}^m_d,\\
{}^r_s\mathrm{H}^\ell_{d'} &= \mathrm{fil}^r\mathrm{H}^\ell_{d'}/\mathrm{fil}^s\mathrm{H}^\ell_{d'},\\
{}^r_s\mathrm{H}^m_b &= \mathrm{fil}^r(\mathrm{B}_{\mathrm{dR},\overline{K}/K}\otimes_K \mathrm{H}^m_d)/\mathrm{fil}^s(\mathrm{B}_{\mathrm{dR},\overline{K}/K}\otimes_K \mathrm{H}^m_d)\\
&= \varinjlim_j \mathbb{Q}\otimes\varprojlim_n \mathrm{H}^m\Big(((X\otimes_{O_K} O_{\overline{K}}/p^n)/(O_K/p^n))_{\mathrm{log-crys}}, J^{[r+j]}_{\mathrm{crys}}/J^{[s+j]}_{\mathrm{crys}}\Big)(-j),\\
{}^r_s\mathrm{H}^\ell_{b'} &= \mathrm{B}^r_{\mathrm{dR},\overline{K}/K}/\mathrm{B}^s_{\mathrm{dR},\overline{K}/K}\otimes_{\mathbb{Q}_p}\mathrm{H}^\ell_{e'},\\
&= \mathrm{fil}^r(\mathrm{B}_{\mathrm{dR},\overline{K}/K}\otimes_K \mathrm{H}^\ell_{d'})/\mathrm{fil}^s(\mathrm{B}_{\mathrm{dR},\overline{K}/K}\otimes_K \mathrm{H}^\ell_{d'}),\\
{}^r_s\mathrm{H}^m_{D/K} &= \mathrm{fil}^r\mathrm{H}^m_{D/K}/\mathrm{fil}^s\mathrm{H}^m_{D/K},\\
{}^r_s\mathrm{H}^m_{D/\mathscr{K}} &= \mathrm{fil}^r\mathrm{H}^m_{D/\mathscr{K}}/\mathrm{fil}^s\mathrm{H}^m_{D/\mathscr{K}},\\
{}^r_s\mathrm{H}^m_{B/K} &= \varinjlim_j \mathbb{Q}\otimes\varprojlim_n \mathrm{H}^m\Big(((X\otimes_C O_{\overline{\mathscr{K}}}/p^n)/(O_K/p^n))_{\mathrm{log-crys}}, J^{[r+j]}_{\mathrm{crys}}/J^{[s+j]}_{\mathrm{crys}}\Big)(-j),\\
{}^r_s\mathrm{H}^m_{B/\mathscr{K}} &= \mathrm{B}^r_{\mathrm{dR},\overline{\mathscr{K}}/\mathscr{K}}/\mathrm{B}^s_{\mathrm{dR},\overline{\mathscr{K}}/\mathscr{K}}\otimes_{\mathbb{Q}_p}\mathrm{H}^m_E,\\
&= \mathrm{fil}^r(\mathrm{B}_{\mathrm{dR},\overline{\mathscr{K}}/\mathscr{K}}\otimes_{\mathscr{K}} \mathrm{H}^m_{D/\mathscr{K}})/\mathrm{fil}^s(\mathrm{B}_{\mathrm{dR},\overline{\mathscr{K}}/\mathscr{K}}\otimes_{\mathscr{K}} \mathrm{H}^m_{D/\mathscr{K}}),\\
&= \varinjlim_j \mathbb{Q}\otimes\varprojlim_n \mathrm{H}^m\Big(((X\otimes_C O_{\overline{\mathscr{K}}}/p^n)/(O_{\mathscr{K}}/p^n))_{\mathrm{log-crys}}, J^{[r+j]}_{\mathrm{crys}}/J^{[s+j]}_{\mathrm{crys}}\Big)(-j).
\end{aligned}$$

For $s \gg 0$, the upper (resp. lower) $\exp^*$ in 11.7 is the composition of the upper (resp. lower) rows of the following diagram.

$$\begin{tikzcd}
\mathrm{H}^1(K,[\mathrm{H}^\ell_{e'}(r)]) \arrow[rr] \arrow[d, "\alpha"'] & & \mathrm{H}^1(K,[{}^r_s\mathrm{H}^\ell_{b'}]) \arrow[d] & {[{}^r_s\mathrm{H}^\ell_{d'}]} \arrow[l, "\sim"'] \arrow[d, "\beta"] \\
\mathrm{H}^2(\mathscr{K},[\mathrm{H}^{\ell-1}_E(r)]) \arrow[r] & \mathrm{H}^2(\mathscr{K},[{}^r_s\mathrm{H}^{\ell-1}_{B/K}]) & P \arrow[l, "\sim", "\delta"'] & Q \arrow[l, "\sim"']
\end{tikzcd} \tag{11.9.1}$$

Here

$$\begin{aligned}
P &= \mathrm{H}^1(K,[\widehat{\Omega}^1_{\mathscr{K}}\otimes_{\mathscr{K}} {}^{r-1}_{s-1}\mathrm{H}^{\ell-1}_{B/\mathscr{K}}])/\nabla(\mathrm{H}^1(\mathscr{K},[{}^r_s\mathrm{H}^{\ell-1}_{B/\mathscr{K}}])),\\
Q &= [\widehat{\Omega}^1_{\mathscr{K}}\otimes_{\mathscr{K}} {}^{r-1}_{s-1}\mathrm{H}^{\ell-1}_{D/\mathscr{K}}]/\nabla([{}^r_s\mathrm{H}^{\ell-1}_{D/\mathscr{K}}]),
\end{aligned}$$

the map δ is the connecting map of the exact sequence

$$0 \longrightarrow [{}^r_s\mathrm{H}^{\ell-1}_{B/K}] \longrightarrow [{}^r_s\mathrm{H}^{\ell-1}_{B/\mathscr{K}}] \xrightarrow{\ \nabla\ } [\widehat{\Omega}^1_{\mathscr{K}}\otimes_{\mathscr{K}} {}^{r-1}_{s-1}\mathrm{H}^{\ell-1}_{B/\mathscr{K}}] \longrightarrow 0, \tag{11.9.2}$$

the middle vertical arrow is induced by β, and the two horizontal isomorphisms on the extreme right are cup products with $\log(\chi_{\mathrm{cyclo}})$. For the proof of Prop. 11.7, it is sufficient to prove the commutativity of the two squares in (11.9.1). The commutativity of the right square is clear. The left square is divided into three squares as in

the following diagram.
(11.9.3)

$$\begin{tikzcd}
\mathrm{H}^1(K,[\mathrm{H}^{\ell}_{e'}(r)]) \arrow[rr] \arrow[d] & & \mathrm{H}^1(K,[{}^r_s\mathrm{H}^{\ell}_{b'}]) \arrow[d] \\
\mathrm{H}^1\Big(K,\mathrm{H}^1(\overline{K}\mathscr{K},[\mathrm{H}^{\ell-1}_E(r)])\Big) \arrow[r] \arrow[d, equal] & \mathrm{H}^1\Big(K,\mathrm{H}^1(\overline{K}\mathscr{K},[{}^r_s\mathrm{H}^{\ell-1}_{B/K}])\Big) \arrow[d, equal] & \mathrm{H}^1(K,R) \arrow[l, "\delta"'] \arrow[d] \\
\mathrm{H}^2(\mathscr{K},[\mathrm{H}^{\ell-1}_E(r)]) \arrow[r] & \mathrm{H}^2(\mathscr{K},[{}^r_s\mathrm{H}^{\ell-1}_{B/K}]) & P \arrow[l, "\delta"']
\end{tikzcd}$$

Here P and the lower δ are as above,

$$R=\mathrm{H}^0(\overline{K}\mathscr{K},[\widehat{\Omega}^1_{\mathscr{K}}\otimes_{\mathscr{K}}{}^{r-1}_{s-1}\mathrm{H}^{\ell-1}_{B/\mathscr{K}}])/\nabla(\mathrm{H}^0(\overline{K}\mathscr{K},[{}^r_s\mathrm{H}^{\ell-1}_{B/\mathscr{K}}])),$$

the upper δ is induced by the connecting map of (11.9.2), and the lower vertical arrow on the right hand side is defined as

$$\mathrm{H}^1(K,R)=\mathrm{H}^1(K,\mathrm{H}^0(\overline{K}\mathscr{K},[\widehat{\Omega}^1_{\mathscr{K}}\otimes_{\mathscr{K}}{}^{r-1}_{s-1}\mathrm{H}^{\ell-1}_{B/K}]))/\nabla\mathrm{H}^1(K,\mathrm{H}^0(\overline{K}\mathcal{K},[{}^r_sH^{\ell-1}_{B/K}]))\longrightarrow P.$$

The commutativities of the lower two squares in (11.9.3) are clear. It remains to prove the commutativity of the upper square of (11.9.3). This square is $\mathrm{H}^1(K,\ )$ of the following square

(11.9.4)
$$\begin{tikzcd}
{[\mathrm{H}^m_{e'}(r)]} \arrow[rr] \arrow[d] & & {[{}^r_s\mathrm{H}^m_{b'}]} \arrow[d] \\
\mathrm{H}^1(\overline{K}\mathscr{K},[\mathrm{H}^{m-1}_E(r)]) \arrow[r] & \mathrm{H}^1(\overline{K}\mathscr{K},[{}^r_s\mathrm{H}^{m-1}_{B/K}]) & R \arrow[l, "\delta"']
\end{tikzcd}$$

Let $f,g:[\mathrm{H}^m_{e'}(r)]\to\mathrm{H}^1(\overline{K}\mathscr{K},[{}^r_s\mathrm{H}^{m-1}_{B/K}])$ be the two morphisms defined by (11.9.4); f is the composition of the left vertical arrow and the lower horizontal arrow on the left hand side, and g is th composition of the other arrows. Let

$$h:\mathrm{H}^1(\overline{K}\mathscr{K},[{}^r_s\mathrm{H}^{m-1}_{B/K}])\longrightarrow\mathrm{H}^m(\overline{K}\mathscr{K},[{}^r_s\mathrm{R}\Gamma_{B/K}])$$

be the canonical injection, where

(11.9.5) $[{}^r_s\mathrm{R}\Gamma_{B/K}]$
$$=\text{“}\mathbb{Q}\otimes\text{”}\ \text{“}\varinjlim_j\text{”}\ \text{“}\varprojlim_n\text{”}\ \mathrm{R}\Gamma\Big(((X\otimes_C O_{\overline{\mathscr{K}}}/p^n)/(O_K/p^n))_{\log-\mathrm{crys}},J^{[r+j]}_{\mathrm{crys}}/J^{[s+j]}_{\mathrm{crys}}\Big)(-j),$$

("$\mathbb{Q}\otimes$" means "$\varinjlim_M$" $M^{-1}\mathbb{Z}\otimes\cdot$.) The composite $h\circ f$ ($h\circ g$) coincides with $d\circ b\circ i$ ($=c\circ a\circ i$) of the following commutative diagram

$$\begin{array}{ccccc} [\mathrm{H}^{\ell}_{e'}(r)] & \xrightarrow{\quad i\quad} & [\mathrm{H}^{\ell}_{e}(r)] & \xrightarrow{\qquad a\qquad} & [{}^{r}_{s}\mathrm{H}^{\ell}_{b}] \\ & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle c} \\ & & [\mathrm{H}^{\ell}(\overline{K}\mathscr{K},\mathrm{R}\Gamma_E)] & \xrightarrow{d} & [\mathrm{H}^{\ell}(\overline{K}\mathscr{K},\mathrm{R}\Gamma_{B/K})] \end{array}$$

where $\mathrm{R}\Gamma_E=$ "$\mathbb{Q}\otimes$" "$\varprojlim_n$"$\mathrm{R}\Gamma(U\otimes_C\mathscr{K},\mathbb{Z}/p^n)$. Hence $h\circ f=h\circ g$ and we have $f=g$ by the injectivity of h.

11.10. We give the proof of the bijectivity of (11.3.6). Let

$$D_k(X(N))=\mathrm{H}^{k-1}_{\mathrm{dR}}(\mathrm{KS}_k)(\widetilde{\varepsilon}),\qquad D_{k,\log}(X(N))=\mathrm{H}^{k-1}_{\log-\mathrm{dR}}(\mathrm{KS}_k)(\varepsilon).$$

Our task is to prove that the canonical map

$$\mathrm{D}_{\mathrm{dR}}(V_{k,\mathbb{Q}_p}(Y(N)))\longrightarrow D_{k,\log}(X(N))\otimes\mathbb{Q}_p \tag{11.10.1}$$

is bijective. By (11.3.3), we have

$$\mathrm{D}_{\mathrm{dR}}(V_{k,\mathbb{Q}_p}(X(N)))\xrightarrow{\sim}D_k(X(N))\otimes\mathbb{Q}_p.$$

By comparing the dimensions by (11.3.2), we see that it is sufficient to show that (11.10.1) is an injection.

We define a homomorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$-modules

$$R:V_{k,\mathbb{Q}_p}(Y(N))\otimes\mathbb{Q}\longrightarrow\mathbb{Q}_p(1-k)$$

and a homomorphism

$$\mathrm{Res}:D_{k,\log}(X(N))\longrightarrow\mathbb{Q}(\zeta_N)$$

as follows. First, R is the composition

$$\mathrm{H}^1(Y(N)\otimes\overline{\mathbb{Q}},\mathrm{Sym}^{k-2}(\mathcal{H}^1_p))\longrightarrow(\mathrm{R}^1j_*\,\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p))_{\overline{\infty}}\otimes\mathbb{Q}_p\xrightarrow{\sim}\mathbb{Q}_p(1-k)$$

where the last isomorphism is defined as follows. Let $L=\mathbb{Q}(\zeta_N)((q^{1/N}))$. Since the pull back E' of the universal elliptic curve over $Y(N)$ to $\mathrm{Spec}(L)$ is the pull back of the q-Tate curve over $\mathbb{Z}[[q]][q^{-1}]$, we have an exact sequence

$$0\longrightarrow\mathbb{Q}_p(1)\longrightarrow T_pE'\otimes\mathbb{Q}\longrightarrow\mathbb{Q}_p\longrightarrow0$$

of $\mathrm{Gal}(\overline{L}/L)$-modules in which $T_pE'\otimes\mathbb{Q}\to\mathbb{Q}_p$ sends $(q^{1/p^n}\bmod q^{\mathbb{Z}})_n$ to 1. This induces a homomorphism of $\mathrm{Gal}(\overline{L}/L)$-modules

$$\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(T_pE)\longrightarrow\mathbb{Q}_p,$$

and hence a homomorphism of $\mathrm{Gal}(\overline{L}/L)$-modules

$$\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p)\longrightarrow\mathbb{Q}_p(2-k).$$

This induces isomorphisms of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$-modules

$$\begin{aligned}(\mathrm{R}^1 j_* \,\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p))_{\widetilde{\infty}} \otimes \mathbb{Q} = \mathrm{H}^1(L^{\mathrm{ur}}, \mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p)) \otimes \mathbb{Q} \\ \xrightarrow{\sim} \mathrm{H}^1(L^{\mathrm{ur}}, \mathbb{Q}_p(2-k)) \xrightarrow{\sim} \mathbb{Q}_p(1-k)\end{aligned}$$

where the last map comes from the isomorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$-modules

$$\mathbb{Q}_p \xrightarrow{\sim} \mathrm{H}^1(L^{\mathrm{ur}}, \mathbb{Q}_p(1))$$

which sends 1 to the image of q under the symbol map (Kummer theory) $(L^{\mathrm{ur}})^\times \to \mathrm{H}^1(L^{\mathrm{ur}}, \mathbb{Q}_p(1))$. This gives the desired map

$$(\mathrm{R}^1 j_* \,\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\mathcal{H}^1_p))_{\widetilde{\infty}} \longrightarrow \mathbb{Q}_p(1-k).$$

Next Res is the composition

$$D_{k,\log}(X(N))/D_k(X(N)) \xleftarrow{\sim} M_k(X(N))/S_k(X(N)) \longrightarrow \mathbb{Q}(\zeta_N)$$

where the last arrow is defined by $\sum_{n\geqslant 0} a_n q^{n/N} \mapsto a_0$.

We will show that the following diagram is commutative.

$$\text{(11.10.2)}\qquad \begin{array}{ccc} V_{k,\mathbb{Q}_p}(Y(N)) & \xrightarrow{\ R\ } & \mathbb{Q}_p(1-k) \\ \downarrow & & \downarrow \\ \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}} D_{k,\log}(X(N)) & \xrightarrow{\ \mathrm{Res}\ } & \mathrm{B}_{\mathrm{dR}} \end{array}$$

(This is a p-adic analogue of the commutative diagram (7.15.1).) We prove the bijectivity of (11.3.6) admitting the commutativity of (11.10.2). We have an exact sequence of $\mathrm{Gal}(\mathbb{Q}_p/\mathbb{Q}(\zeta_N))$-modules

$$\text{(11.10.3)}\qquad 0 \longrightarrow V_{k,\mathbb{Q}_p}(X(N)) \longrightarrow V_{k,\mathbb{Q}_p}(Y(N)) \longrightarrow \bigoplus_\sigma \mathbb{Q}_p(1-k)$$

where σ ranges over all elements of $\mathrm{GL}_2(\mathbb{Z}/N)$ and the last arrow is $(R\circ\sigma^*)_\sigma$, and an exact sequence

$$\text{(11.10.4)}\qquad 0 \longrightarrow \mathrm{B}_{\mathrm{dR}} \otimes D_k(X(N)) \longrightarrow \mathrm{B}_{\mathrm{dR}} \otimes D_{k,\log}(X(N)) \longrightarrow \bigoplus_\sigma \mathrm{B}_{\mathrm{dR}}$$

where σ ranges over all elements of $\mathrm{GL}_2(\mathbb{Z}/N)$ and the last arrow is $(\mathrm{Res}\circ\sigma^*)_\sigma$. By the commutativity of (11.10.2), we have a homomorphism $\mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} (11.10.3) \to (11.10.4)$. By taking $\mathrm{H}^0(\mathbb{Q}(\zeta_N)\otimes\mathbb{Q}_p, ?)$ of this, we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathbb{Q}(\zeta_N)\otimes_{\mathbb{Q}}(1) & \longrightarrow & \mathbb{Q}(\zeta_N)\otimes_{\mathbb{Q}}(2) & \longrightarrow & \bigoplus_\sigma \mathbb{Q}(\zeta_N)\otimes\mathbb{Q}_p \\ & \downarrow & & \downarrow & & \| \\ 0 \longrightarrow & \mathbb{Q}(\zeta_N)\otimes_{\mathbb{Q}}(1)' & \longrightarrow & \mathbb{Q}(\zeta_N)\otimes_{\mathbb{Q}}(2)' & \longrightarrow & \bigoplus_\sigma \mathbb{Q}(\zeta_N)\otimes\mathbb{Q}_p \end{array}$$

where

$$(1) \ = \mathrm{D_{dR}}(V_{k,\mathbb{Q}_p}(X(N))), \qquad (2) \ = \mathrm{D_{dR}}(V_{k,\mathbb{Q}_p}(Y(N))),$$
$$(1)' = D_k(X(N)) \otimes \mathbb{Q}_p, \qquad (2)' = D_{k,\log}(X(N)) \otimes \mathbb{Q}_p.$$

Since $(1) \xrightarrow{\sim} (1)'$, this diagram proves the injectivity of $(2) \to (2)'$ as desired.

Now we prove the commutativity of (11.10.2).

Let $\mathbb{C}_p$ be the completion over $\overline{\mathbb{Q}_p}$. Since $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$ acts trivially on $(V_{k,\mathbb{Q}_p}(Y(N)))/(V_{k,\mathbb{Q}_p}(X(N)))(k-1)$ and since $\mathrm{Res} : B_{\mathrm{dR}} \otimes_{\mathbb{Q}} D_{k,\log}(X(N)) \to B_{\mathrm{dR}}$ kills the image of $V_{k,\mathbb{Q}_p}(X(N))$, the image of the composition

$$V_{k,\mathbb{Q}_p}(Y(N))(k-1) \longrightarrow \mathrm{B_{dR}} \otimes_{\mathbb{Q}} D_{k,\log}(X(N)) \longrightarrow \mathrm{B_{dR}}$$

is contained in $\mathrm{H}^0(\mathbb{Q}_p(\zeta_N), B_{\mathrm{dR}}) = \mathbb{Q}_p(\zeta_N)$. Since $B_{\mathrm{dR}} \to \mathbb{C}_p$ induces an injection $\mathbb{Q}_p(\zeta_N) \to \mathbb{C}_p$, it is sufficient to prove that the diagram

(11.10.5)
$$\begin{tikzcd}
(V_{k,\mathbb{Q}_p}(Y(N)))(k-1) \arrow[r, "R"] \arrow[d] & \mathbb{Q}_p \arrow[d, hook] \\
\mathrm{fil}^{k-1}\left(\mathrm{B}^+_{\mathrm{dR}} \otimes D_{k,\log}(X(N))\right) \arrow[r, "\mathrm{Res}"] & \mathbb{C}_p
\end{tikzcd}$$

is commutative. The lower horizontal arrow factors as

$$\mathrm{fil}^{k-1}\left(\mathrm{B}^+_{\mathrm{dR}} \otimes D_{k,\log}(X(N))\right) \longrightarrow \mathrm{gr}^{k-1}(\mathrm{B}^+_{\mathrm{dR}} \otimes D_{k,\log}(X(N)))$$
$$= \mathbb{C}_p \otimes_{\mathbb{Q}} M_k(X(N)) \xrightarrow{\mathrm{Res}} \mathbb{C}_p$$

where the last map Res is $\sum_{n\geqslant 0} a_n q^{n/N} \longmapsto a_0$.

We will define certain groups V, V', D, D' with commutative diagrams

(11.10.6)
$$\begin{tikzcd}
V_{k,\mathbb{Q}_p}(Y(N))(k-1) \arrow[r] \arrow[d] & V \arrow[r] \arrow[d, "\mathrm{Dlog}"'] & V' \arrow[d, "\mathrm{Dlog}"] \\
\mathbb{C}_p \otimes_{\mathbb{Q}} M_k(X(N)) \arrow[r] & D \arrow[r] & D'
\end{tikzcd}$$

(11.10.7)
$$\begin{tikzcd}
\mathbb{Q}_p \arrow[r, "v"] \arrow[d] & V' \arrow[d, "\mathrm{Dlog}"] \\
\mathbb{C}_p \arrow[r, "w"] & D'
\end{tikzcd}$$

such that the composition of the upper horizontal rows of (11.10.6) coincides with the composition

$$V_{k,\mathbb{Q}_p}(Y(N))(k-1) \xrightarrow{R} \mathbb{Q}_p \xrightarrow{v} V',$$

the composition of the lower horizontal row of (11.10.6) coincides with the composition

$$\mathbb{C}_p \otimes_{\mathbb{Q}} M_k(X(N)) \xrightarrow{\mathrm{Res}} \mathbb{C}_p \xrightarrow{w} D',$$

and w is injective. This will prove the commutativity of (11.10.5).

The groups V, V', D and D' are defined as follows. Let $N', K, X, U = E^{(k-1)} \times_{Y(N)} Y(N') \otimes_{\mathbb{Q}} K$, and $\mathfrak{C}$ be as at the end of 11.4. Let $\mathrm{Spec}(O_K) \to \mathfrak{C}$ be the standard cusp. Then the completion of the local ring of $\mathfrak{C}$ at the image of the closed point of $\mathrm{Spec}(O_K)$ under this map is identified with $O_K[[q^{1/N'}]]$. For any $n \geqslant 1$, the smooth part of the irreducible component of $X \times_{\mathfrak{C}} \mathrm{Spec}(O_K[[q^{1/N'}]]/(q^n))$ containing the origin of X is canonically identified with $\mathbb{G}^{k-2}_{m,O_K[[q^{1/N'}]]/(q^n)}$. Let $(t_j)_{1\leqslant j\leqslant k-2}$ be the standard coordinates of $\mathbb{G}_m^{k-2}$, and let R be the completion of the local ring of $O_K[t_1^{\pm}, \dots, t_{k-2}^{\pm}]$ at the prime ideal generated by the maximal ideal m_K of O_K. Then R is a discrete valuation ring. Let ν be the generic point of $\mathbb{G}^{k-2}_{m,O_K/m_K}$ and regard ν as a point of codimension two of X. Then the completion of the local ring $\mathcal{O}_{X,\nu}$ is identified with $R[[q^{1/N'}]]$. Let $\mathcal{O}^h_{X,\nu}$ be the henselization of $\mathcal{O}_{X,\nu}$. We define

$$\begin{aligned}
V &= \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n \mathrm{H}^{k-1}(\mathcal{O}^h_{X,\nu}[1/q] \otimes_{O_K} \overline{K}, (\mathbb{Z}/p^n)(k-1)),\\
V' &= \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n \mathrm{H}^{k-1}(\mathrm{frac}(R)((q^{1/N'})) \otimes_{O_K} \overline{K}, (\mathbb{Z}/p^n)(k-1)),\\
D &= \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n (\mathcal{O}_{\overline{K}} \otimes_{O_K} R[[q^{1/N'}]])/p^n,\\
D' &= \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n (\mathcal{O}_{\overline{K}} \otimes_{O_K} R)/p^n,
\end{aligned}$$

where $\mathrm{frac}(R)$ denotes the field of fractions of R.

By [**TT**, § 3], there is a unique homomorphism

$$\mathrm{Dlog} : V \longrightarrow D$$

satisfying

$$\mathrm{Dlog}(\{f_1, \dots, f_{k-1}\}) \cdot \mathrm{dlog}(t_1) \wedge \cdots \wedge \mathrm{dlog}(t_{k-2}) \wedge \mathrm{dlog}(q) = \mathrm{dlog}(f_1) \wedge \cdots \wedge \mathrm{dlog}(f_{k-1})$$

in

$$\begin{aligned}
&\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n \Big((\mathcal{O}_{\overline{K}} \otimes_{O_K} q^{-1/N'} \Omega^{k-1}_{R[[q^{1/N'}]]/O_K})/p^n \Big)\\
&\quad = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n \Big((\mathcal{O}_{\overline{K}} \otimes_{O_K} R[[q^{1/N'}]])/p^n \Big) \otimes \mathrm{dlog}(t_1) \wedge \cdots \wedge \mathrm{dlog}(t_{k-2}) \wedge \mathrm{dlog}(q)
\end{aligned}$$

for any $f_1, \dots, f_{k-1} \in (\mathcal{O}^h_{X,\nu} \otimes_{O_K} \overline{K})^\times$, where $\{f_1, \dots, f_{k-1}\}$ is the image of $f_1 \otimes \cdots \otimes f_{k-1}$ under the symbol map [**TT**, § 3.2]. The diagram

$$\begin{array}{ccc} V_{k,\mathbb{Q}_p}(Y(N))(k-1) & \longrightarrow & V \\ \downarrow & & \downarrow \mathrm{Dlog} \\ \mathbb{C}_p \otimes M_k(X(N)) & \longrightarrow & D \end{array}$$

is commutative. On the other hand, let

$$\mathrm{Dlog} : V' \longrightarrow D'$$

be the homomorphism characterized by

$$\mathrm{Dlog}(\{f_1, \dots, f_{k-2}, f_{k-1}q\}) \cdot \mathrm{dlog}(t_1) \wedge \cdots \wedge \mathrm{dlog}(t_{k-2}) = \mathrm{dlog}(f_1) \wedge \cdots \wedge \mathrm{dlog}(f_{k-2})$$

in

$$\begin{aligned} &\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n ((\mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \Omega^{k-2}_{R/\mathcal{O}_K})/p^n) \\ &\qquad = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n (O_{\overline{K}} \otimes_{\mathcal{O}_K} R/p^n) \otimes \mathrm{dlog}(t_1) \wedge \cdots \wedge \mathrm{dlog}(t_{k-2}) \end{aligned}$$

for any $f_1, \dots, f_{k-1} \in (\mathrm{frac}(R) \otimes_{O_K} \overline{K})^\times$. (The existence of this map Dlog follows from the bijectivity of the symbol map

(11.10.8)

$$K^M_{k-1}(\mathrm{frac}(R)(\!(q^{1/N'})\!) \otimes_K \overline{K})/p^n \xrightarrow{\sim} \mathrm{H}^{k-1}(\mathrm{frac}(R)(\!(q^{1/N'})\!) \otimes_K \overline{K}, (\mathbb{Z}/p^n)(k-1))$$

which follows from

(11.10.9) $$K^M_r(\mathrm{frac}(R) \otimes_K \overline{K})/p^n \xrightarrow{\sim} \mathrm{H}^r(\mathrm{frac}(R) \otimes_K \overline{K}, (\mathbb{Z}/p^n)(r))$$

for all r [**BK1**] because (11.10.8) is isomorphic to the direct sum of (11.10.9) for $r = k-1, k-2$. From the constructions of the maps Dlog, the diagram

$$\begin{array}{ccc} V & \longrightarrow & V' \\ \mathrm{Dlog} \downarrow & & \downarrow \mathrm{Dlog} \\ D & \longrightarrow & D' \end{array}$$

is commutative, where $D \to D'$ is given by $\sum_{n \geqslant 0} a_n q^{n/N'} \mapsto a_0$. We define the map $v : \mathbb{Q}_p \to V'$ by sending 1 to $\{t_1, \dots, t_{k-2}, q\}$ and the map $w : \mathbb{C}_p \to D'$ to be the inclusion map. The diagram

$$\begin{array}{ccc} \mathbb{Q}_p & \xrightarrow{v} & V' \\ \downarrow & & \downarrow \mathrm{Dlog} \\ \mathbb{C}_p & \hookrightarrow & D' \end{array}$$

is clearly commutative.

It remains to show that the compositions

$$V_{k,\mathbb{Q}_p}(Y(N))(k-1) \longrightarrow V \longrightarrow V'$$
$$V_{k,\mathbb{Q}_p}(Y(N))(k-1) \xrightarrow{R} \mathbb{Q}_p \xrightarrow{v} V'$$

coincide. This follows from the fact that if E' is the pull back of the q-Tate elliptic curve over $\mathbb{Q}((q))$, the following two compositions

$$T_pE' \xrightarrow{\sim} \mathrm{H}^1(E' \otimes_{\mathbb{Q}((q))} \overline{\mathbb{Q}((q))}, \mathbb{Z}_p(1)) \xrightarrow{a} \mathrm{H}^1(\mathbb{G}_{m,\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)),$$
$$T_pE' \xrightarrow{b} \mathbb{Z}_p \xrightarrow{c} \mathrm{H}^1(\mathbb{G}_{m,\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))$$

coincide. Here a is the restriction to the smooth part of the irreducible component of the special fiber of E' containing the origin (this part is isomorphic to $\mathbb{G}_m$), b is the homomorphism which kills $\mathbb{Z}_p(1)$ and sends $(q^{1/p^n} \bmod q^{\mathbb{Z}})_n$ to 1, and c is the map which sends 1 to the symbol $\{t\}$ where t is the standard coordinate of $\mathbb{G}_m$.

CHAPTER III
IWASAWA THEORY OF MODULAR FORMS (WITHOUT p-ADIC ZETA FUNCTIONS)

In this chapter (§ 12-§ 15), we study the following subjects:

(1) Analogue of Iwasawa main conjecture for modular forms (§ 12).
(2) The finiteness of Selmer groups (§ 14).
(3) The Tamagawa number conjecture [**BK2**] for modular forms (§ 14).

The "main conjecture" which we study in this chapter is not concerned with p-adic zeta functions of modular forms. In the next chapter, we will study the main conjecture involving p-adic zeta functions [**Ma1, Gr1**] for modular forms which are ordinary at p.

It has been known that once p-adic Euler systems as in Chapter II are constructed and the result in § 12 is obtained, then the rest of Chapter III and Chapter IV can be proved (see [**Pe1**], [**Pe3**], [**Ru4**]). The author gives in this paper all necessary arguments, for he thinks that is convenient for the reader, but many arguments in Chapter III and Chapter IV are already given in literatures. The author is thankful to the referee for pointing out the existence of several literatures.

In Chap. III, we fix $k \geqslant 2, N \geqslant 1$, and a normalized newform

$$f = \sum_{n \geqslant 1} a_n q^n \in S_k(X_1(N)) \otimes \mathbb{C}.$$

We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

12. The main conjecture, I

In this section, we state results concerning "the main conjecture without p-adic zeta functions" (Conj. 12.10) for modular forms. The proofs of the results are given in § 13,§ 15.

We fix a prime number p.

12.1. Let p be a prime number, and let

$$G_n = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \qquad \text{for } n \geqslant 0,$$
$$G_\infty = \varprojlim_n G_n = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) \quad \text{where } \mathbb{Q}(\zeta_{p^\infty}) = \cup_n \mathbb{Q}(\zeta_{p^n}).$$

Then the cyclotomic character gives an isomorphism

$$\kappa = \chi_{\text{cyclo}} : G_\infty \xrightarrow{\sim} \mathbb{Z}_p^\times.$$

For $c \in \mathbb{Z}_p^\times$, let $\sigma_c \in G_\infty$ be the unique element such that $\kappa(\sigma_c) = c$.

For a finite extension L of $\mathbb{Q}_p$, the ring

$$O_L[[G_\infty]] = \varprojlim_n O_L[G_n].$$

has the following structure, as is well known. Let Δ be the torsion part of G_∞, and let

$$G_\infty^1 = \begin{cases} \{\sigma \in G_\infty\,;\, \kappa(\sigma) \equiv 1 \bmod p\} & \text{in the case } p \neq 2, \\ \{\sigma \in G_\infty\,;\, \kappa(\sigma) \equiv 1 \bmod 4\} & \text{in the case } p = 2. \end{cases}$$

Then

$$G_\infty = G_\infty^1 \times \Delta, \quad G_\infty^1 \simeq \mathbb{Z}_p, \quad \Delta \simeq \begin{cases} \mathbb{Z}/(p-1) & \text{if } p \neq 2, \\ \mathbb{Z}/2 & \text{if } p = 2. \end{cases}$$

We have

$$\begin{aligned} O_L[[G_\infty]] &= O_L[[\Delta \times G_\infty^1]] = O_L[\Delta][[G_\infty^1]] \simeq O_L[\Delta][[\mathbb{Z}_p]] \\ &= \varprojlim_n O_L[\Delta][\mathbb{Z}/p^n] = \varprojlim_n O_L[\Delta][X]/(X^{p^n} - 1) = O_L[\Delta][[T]] \end{aligned}$$

($T = X - 1$). Hence we have:

(12.1.1) *$O_L[[G_\infty]]$ is a two dimensional complete semi-local ring.*

(12.1.2) *In the case $p \neq 2$ (resp. $p = 2$), for $j \in \mathbb{Z}/(p-1)$ (resp. $j \in \mathbb{Z}/2$), let $O_L[[G_\infty]]_j$ be the quotient of $O_L[[G_\infty]]$ divided by the ideal $(\sigma - \kappa(\sigma)^j; \sigma \in \Delta)$. Then*

$$O_L[[G_\infty]]_j \xleftarrow{\sim} O_L[[G_\infty^1]] \simeq O_L[[T]].$$

We have

$$O_L[[G_\infty]] \xrightarrow{\sim} \prod_{j \in \mathbb{Z}/(p-1)} O_L[[G_\infty]]_j \ \text{if } p \neq 2.$$

If $p = 2$, the canonical map $O_L[[G_\infty]] \to \prod_{j \in \mathbb{Z}/2} O_L[[G_\infty]]_j$ is injective and the cokernel is killed by 2.

(12.1.3) *$O_L[[G_\infty]] \otimes \mathbb{Q}$ is the product of the principal ideal domains $O_L[[G_\infty]]_j \otimes \mathbb{Q}$.*

(12.1.4) *If $\mathfrak{p}$ is a prime ideal of height one in $O_L[[G_\infty]]$, the local ring $O_L[[G_\infty]]_{\mathfrak{p}}$ is a discrete valuation ring except in the case $p = 2 \in \mathfrak{p}$.*

12.2. Let T be a finitely generated $\mathbb{Z}_p$-module endowed with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is unramified at almost all prime numbers. We denote for $q \in \mathbb{Z}$

$$\mathbf{H}^q(T) = \varprojlim_n \mathrm{H}^q(\mathbb{Z}[\zeta_{p^n}, 1/p], T)$$

where H^q is the étale cohomology as in 8.2, and the inverse limit is taken with respect to trace maps. The following are known:

(12.2.1) *$\mathbf{H}^q(T) = 0$ if $q \neq 1, 2$ and $\mathbf{H}^1(T)$ and $\mathbf{H}^2(T)$ are finitely generated $\mathbb{Z}_p[[G_\infty]]$-modules.*

(12.2.2) *For any prime ideal $\mathfrak{q}$ of $\mathbb{Z}_p[[G_\infty]]$ of height 0 (so $\mathbb{Z}_p[[G_\infty]]_{\mathfrak{q}}$ is a field), $\dim(\mathbf{H}^1(T)_{\mathfrak{q}}) - \dim(\mathbf{H}^2(T)_{\mathfrak{q}}) = \mathrm{rank}_{\mathbb{Z}_p}(T^-)$ where $T^\pm$ is the part of T on which the complex conjugation acts by ± 1. (*[**Ta2**, Thm. 2.2], [**Pe3**, § 1.3]*).*

It is conjectured that $\mathbf{H}^2(T)$ is always a torsion $\mathbb{Z}_p[[G_\infty]]$-module ([**Pe3**] Appendix 3: conjecture de Leopoldt faible), and hence (by (12.2.2)) that $\dim(\mathbf{H}^1(T)_{\mathfrak{q}}) = \mathrm{rank}_{\mathbb{Z}_p}(T^-)$ for any prime ideal $\mathfrak{q}$ of $\mathbb{Z}_p[[G_\infty]]$ of height 0.

On the other hand, for a finitely generated $\mathbb{Z}_p$-module T endowed with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, let

$$\mathbf{H}^q_{\mathrm{loc}}(T) = \varprojlim_n \mathrm{H}^q(\mathbb{Q}_p(\zeta_{p^n}), T).$$

Then the following are known:

(12.2.3) *$\mathbf{H}^q_{\mathrm{loc}}(T) = 0$ if $q \neq 1, 2$, $\mathbf{H}^1_{\mathrm{loc}}(T)$ and $\mathbf{H}^2_{\mathrm{loc}}(T)$ are finitely generated $\mathbb{Z}_p[[G_\infty]]$-modules, and*

$$\dim(\mathbf{H}^1_{\mathrm{loc}}(T)_{\mathfrak{q}}) = \mathrm{rank}_{\mathbb{Z}_p}(T), \quad \dim(\mathbf{H}^2_{\mathrm{loc}}(T)_{\mathfrak{q}}) = 0$$

for any prime ideal $\mathfrak{q}$ of $\mathbb{Z}_p[[G_\infty]]$ of height 0.

The structure of $\mathbf{H}^2_{\mathrm{loc}}(T)$ is well understood: By local Tate duality [**Se1**, Chap. II, § 5.2],

$$\mathbf{H}^2_{\mathrm{loc}}(T) \simeq \mathrm{Hom}\left(\mathrm{H}^0(\mathbb{Q}_p(\zeta_{p^\infty}), \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Q}/\mathbb{Z})), \mathbb{Q}/\mathbb{Z}\right)(-1).$$

Finally, for a finite dimensional $\mathbb{Q}_p$-vector space V endowed with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is unramified at almost all prime numbers (resp. endowed with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$), let

$$\mathbf{H}^q(V) = \mathbf{H}^q(T) \otimes \mathbb{Q} \qquad (\text{resp. } \mathbf{H}^q_{\mathrm{loc}}(V) = \mathbf{H}^q_{\mathrm{loc}}(T) \otimes \mathbb{Q})$$

where T is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (resp. $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$)-stable $\mathbb{Z}_p$-lattice of V. Then $\mathbf{H}^q(V)$ (resp. $\mathbf{H}^q_{\mathrm{loc}}(V)$) is independent of the choice of T.

12.3. Let k, N and $f = \sum_{n\geqslant 1} a_n q^n$ be as in the beginning of Chap. III, and let $F = \mathbb{Q}(a_n\,;\, n \geqslant 1)$ be as before. Let λ be a place of F lying over p, F_λ the local field of F at λ, O_λ the valuation ring of F_λ, m_λ the maximal ideal of O_λ, and let

$$\Lambda = O_\lambda[[G_\infty]].$$

Consider the two dimensional representation $V_{F_\lambda}(f)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over F_λ associated to f (8.3).

We will prove the following results Thm. 12.4, 12.5, 12.6.

Theorem 12.4. — *Take any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice T of V_{F_λ}. Then:*

(1) *$\mathbf{H}^2(T)$ is a torsion Λ-module.*

(2) *$\mathbf{H}^1(T)$ is a torsion free Λ-module, and $\mathbf{H}^1(T) \otimes \mathbb{Q} = \mathbf{H}^1(V_{F_\lambda}(f))$ is a free $\Lambda \otimes \mathbb{Q}$-module of rank 1.*

(3) *If $p \neq 2$ and if $T/m_\lambda T$ is irreducible as a two dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over O_λ/m_λ, $\mathbf{H}^1(T)$ is a free Λ-module of rank 1.*

Theorem 12.5

(1) *There exists a unique F_λ-linear map*

$$V_{F_\lambda}(f) \longrightarrow \mathbf{H}^1(V_{F_\lambda}(f))\,; \quad \gamma \longmapsto \mathbf{z}_\gamma^{(p)}$$

having the following property : Let $r \in \mathbb{Z}, 1 \leqslant r \leqslant k-1$, let $n \geqslant 0$, and let $\gamma \in V_F(f)$. Then the image of $\mathbf{z}_\gamma^{(p)}$ under the composite map

$$\mathbf{H}^1(V_{F_\lambda}(f)) \simeq \mathbf{H}^1(V_{F_\lambda}(f)(k-r)) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), V_{F_\lambda}(f)(k-r)) \xrightarrow{\exp^*} S(f) \otimes_F F_\lambda \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^n})$$

(the first isomorphism is the product with $(\zeta_{p^n})_{n\geqslant 1}^{\otimes(k-r)}$) belongs to $S(f) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^n})$, and the map

$$S(f) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^n}) \longrightarrow V_{\mathbb{C}}(f)^{\pm};$$
$$x \otimes y \longmapsto \sum_{\sigma \in G_n} \chi(\sigma)\sigma(y)\,\mathrm{per}_f(x)^{\pm},$$

where χ is any character $G_n \to \mathbb{C}^\times$ and $\pm = (-1)^{k-r-1}\chi(-1)$, sends the image of $\mathbf{z}_\gamma^{(p)}$ to

$$(2\pi i)^{k-r-1} \cdot L_{\{p\}}(f^*, \chi, r) \cdot \gamma^{\pm}.$$

We have

$$\mathbf{z}_{\iota(\gamma)}^{(p)} = -\sigma_{-1}(\mathbf{z}_\gamma^{(p)})$$

where $\iota : V_{F_\lambda}(f) \to V_{F_\lambda}(f)$ is the action of the complex conjugation.

(2) *Let $Z(f)$ be the $\Lambda \otimes \mathbb{Q}$-submodule of $\mathbf{H}^1(V_{F_\lambda}(f))$ generated by $\mathbf{z}_\gamma^{(p)}$ for all $\gamma \in V_{F_\lambda}(f)$. Then $\mathbf{H}^1(V_{F_\lambda}(f))/Z(f)$ is a torsion $\Lambda \otimes \mathbb{Q}$-module.*

(3) *Let $\mathfrak{p}$ be a prime ideal of Λ of height one which does not contain p. Then*

$$\mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^2(V_{F_\lambda}(f))_{\mathfrak{p}}) \\ \leqslant \mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^1(V_{F_\lambda}(f))_{\mathfrak{p}}/Z(f)_{\mathfrak{p}}) + \mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^2_{\mathrm{loc}}(V_{F_\lambda}(f))_{\mathfrak{p}}).$$

If $\mathbf{H}^2_{\mathrm{loc}}(V_{F_\lambda}(f))_{\mathfrak{p}} \neq 0$, then f and $\mathfrak{p}$ satisfy the following

(12.5.1) $k = 2$, *f is not potentially of good reduction at p* (12.7), *$\mathfrak{p}$ is the kernel of the ring homomorphism $\Lambda \to F_\lambda$ induced by $\kappa^{-2}\chi : G_\infty \to F_\lambda^\times$ for some homomorphism $\chi : G_\infty \to F_\lambda^\times$ of finite order, and*

$$\mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^2_{\mathrm{loc}}(V_{F_\lambda}(f))_{\mathfrak{p}}) = 1.$$

(4) *Let T be a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice of $V_{F_\lambda}(f)$, and let $Z(f,T)$ be the Λ-submodule of $\mathbf{H}^1(T) \otimes \mathbb{Q}$ generated by $\mathbf{z}_\gamma^{(p)}$ for all $\gamma \in T$. Assume $p \neq 2$, and assume that the following* (12.5.2) *is satisfied.*

(12.5.2) *There exists an O_λ-basis of T for which the image of the homomorphism*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^\infty})) \longrightarrow \mathrm{GL}_{O_\lambda}(T) \simeq \mathrm{GL}_2(O_\lambda)$$

contains $\mathrm{SL}_2(\mathbb{Z}_p)$. Here the last isomorphism is given by this basis of T.

Then,

$$Z(f,T) \subset \mathbf{H}^1(T) \quad \text{in } \mathbf{H}^1(T) \otimes \mathbb{Q}.$$

Furthermore,

$$\mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^2(V_{F_\lambda}(f))_{\mathfrak{p}}) \leqslant \mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^1(V_{F_\lambda}(f))_{\mathfrak{p}}/Z(f)_{\mathfrak{p}})$$

for any prime ideal of Λ of height one unless f and $\mathfrak{p}$ satisfy (12.5.1) *in (3).*

Theorem 12.6. — *Let $T = V_{O_\lambda}(f)$* (8.3). *Let Z be the Λ-submodule of $\mathbf{H}^1(V_{O_\lambda}(f))$ generated by the following elements (see* (8.1.3), (8.11)*).*

(1) ${}_{c,d}\mathbf{z}^{(p)}_{p^n}(f,k,j,a(A),\mathrm{prime}(pA))_{n\geqslant 1} \in \mathbf{H}^1(T)$

$(1 \leqslant j \leqslant k-1,\ a, A \in \mathbb{Z},\ A \geqslant 1,\ c,d \in \mathbb{Z},\ (c,6pA) = (d,6pN) = 1)$.

(2) ${}_{c,d}\mathbf{z}^{(p)}_{p^n}(f,k,j,\alpha,\mathrm{prime}(pN))_{n\geqslant 1} \in \mathbf{H}^1(T)$

$(1 \leqslant j \leqslant k-1,\ \alpha \in \mathrm{SL}_2(\mathbb{Z}),\ c,d \in \mathbb{Z},\ (cd,6pN) = 1,\ c \equiv d \equiv 1 \bmod N)$.

Then $Z \subset Z(f,T)$ and $Z(f,T)/Z$ is a finite group.

Remark 12.7. — In Thm. 12.5 (3), "f is potentially of good reduction at p" means that one of the following equivalent conditions (1), (2) is (hence both of them are) satisfied.

(1) There exists a finite extension K of $\mathbb{Q}_p$ having the following properties. For any finite place v of F which does not lie over p, the representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_{F_v}(f)$ is unramified. For any finite place v of F which lies over p, the representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_{F_v}(f)$ is crystalline.

(2) There exist at least one finite place v of F and a finite extension K of $\mathbb{Q}_p$ having the following property : Either v does not lie over p and the representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_{F_v}(f)$ is unramified, or v lies over p and the representation of $\mathrm{Gal}(\overline{K}/K)$ on $V_{F_v}(f)$ is crystalline.

The equivalence between these conditions follows from the works [**Ca, Sa1**]. These conditions are satisfied if p does not divide N.

Remark 12.8. — We say that f has complex multiplication (f has CM) if there is an imaginary quadratic field K of $\mathbb{Q}$ and a Hecke character ψ of K (that is, a continuous homomorphism $\psi : C_K \to \mathbb{C}^\times$ where C_K denotes the idele class group of K) such that

$$L(f,s) = L(\psi,s).$$

Here,

$$L(\psi,s) = \prod_v (1-\psi(v)N(v)^{-s})^{-1}$$

where v ranges over all finite places of K at which ψ is unramified, $\psi(v)$ is the image of prime elements of K_v under ψ, and $N(v)$ is the norm of v.

By Ribet [**Ri1, Ri3**] (generalization of Serre [**Se3**]), we have the following.

(12.8.1) *If f has no CM, then, for almost all finite places λ of F, there exist a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-stable O_λ-lattice T of $V_{F_\lambda}(f)$ which satisfies the condition* (12.5.2) *at Thm 12.5 (4).*

Note that if the condition (12.5.2) at Thm 12.5 (4) is satisfied for one $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice T of $V_{F_\lambda}(f)$, all $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattices of $V_{F_\lambda}(f)$ have the form aT for some $a \in F_\lambda^\times$ (see the proof of 14.7), and hence the condition (12.5.2) at Thm 12.5 (4) is satisfied for any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice of $V_{F_\lambda}(f)$.

(12.8.2) *If f has no CM, then, for any finite place λ of F, there exist a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-stable O_λ-lattice T of $V_{F_\lambda}(f)$ and an O_λ-basis of T such that the image of the action* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^\infty})) \to \mathrm{GL}_2(O_\lambda)$ *with respect to this basis contains an open subgroup of* $\mathrm{SL}_2(\mathbb{Z}_p)$.

The proofs of Thm. 12.4, 12.5, 12.6 are given in § 13 in the case f has no CM and in § 15 in the case f has CM. The proof in the case f has CM heavily depends on the work of Rubin [**Ru1**] on the main conjecture for imaginary quadratic fields.

12.9. Let $p \neq 2$.

We recall the classical Iwasawa theory. We have for $n \geqslant 0$,

$$\mathrm{H}^1(\mathbb{Z}[\zeta_{p^n}, 1/p], \mathbb{Z}_p(1)) \simeq \mathbb{Z}[\zeta_{p^n}, 1/p]^\times \otimes \mathbb{Z}_p,$$
$$\mathrm{H}^2(\mathbb{Z}[\zeta_{p^n}, 1/p], \mathbb{Z}_p(1)) \simeq \mathrm{Cl}(\mathbb{Q}(\zeta_{p^n})) \otimes \mathbb{Z}_p$$

where $\mathrm{Cl}(\mathbb{Q}(\zeta_{p^n}))$ denotes the ideal class group of $\mathbb{Q}(\zeta_{p^n})$. Let

$$\mathbb{Z}_p[[G_\infty]]_+ = \mathbb{Z}_p[[G_\infty]]/(\sigma_{-1}-1),$$

and let Z be the $\mathbb{Z}_p[[G_\infty]]_+$-submodule of $\mathbf{H}^1(\mathbb{Z}_p(1))^+$ generated by the image of

$$((1-\zeta_{p^n})(1-\zeta_{p^n}^{-1}))_{n\geqslant 1} \in \varprojlim_n (\mathbb{Z}[\zeta_{p^n}, 1/p]^\times)^+.$$

Then $\mathbf{H}^2(\mathbb{Z}_p(1))^+$ and $\mathbf{H}^1(\mathbb{Z}_p(1))^+/Z$ are torsion $\mathbb{Z}_p[[G_\infty]]_+$-modules. The classical Iwasawa main conjecture proved by Mazur-Wiles [**MW**] states

$$\mathrm{length}_{(\mathbb{Z}_p[[G_\infty]]_+)_\mathfrak{p}}(\mathbf{H}^2(\mathbb{Z}_p(1))^+_\mathfrak{p}) = \mathrm{length}_{(\mathbb{Z}_p[[G_\infty]]_+)_\mathfrak{p}}(\mathbf{H}^1(\mathbb{Z}_p(1))^+_\mathfrak{p}/Z_\mathfrak{p})$$

for any prime ideal $\mathfrak{p}$ of Λ of height one.

This Iwasawa main conjecture is generalized to "main conjectures of motives" as in [**KK2**, Chap. I, § 3.2] and [**Pe3**, § 4.4]. These main conjectures are specialized to the following "main conjecture for modular forms".

***Conjecture 12.10* (main conjecture).** — *Let T be a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-stable O_λ-lattice of $V_{F_\lambda}(f)$ and let $\mathfrak{p}$ be a prime ideal of Λ of height one. In the case $p=2$, assume $\mathfrak{p}$ does not contain* 2. *Then $Z(f,T)_\mathfrak{p} \subset \mathbf{H}^1(T)_\mathfrak{p}$ and*

$$\mathrm{length}_{\Lambda_\mathfrak{p}}(\mathbf{H}^2(T)_\mathfrak{p}) = \mathrm{length}_{\Lambda_\mathfrak{p}}(\mathbf{H}^1(T)_\mathfrak{p}/Z(f,T)_\mathfrak{p}).$$

13. The method of Euler systems

In this section, we give the proof of the theorems in § 12 by using the method of Euler systems in the case f has no CM. The proof of the CM case will be completed in § 15.

In this section, we fix a prime number p.

13.1. The method of Euler systems started by Kolyvagin bounds arithmetic groups by using a system of "zeta elements". (See [**Ko**]; a similar idea was found by Thaine independently [**Th**].) We use results on Euler systems in Perrin-Riou [**Pe4**], Rubin [**Ru4**], and [**KK4**].

Let L be a finite extension of $\mathbb{Q}_p$ and let T be a free O_L-module of finite rank endowed with a continuous O_L-linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is unramified at almost all prime numbers. Let Σ be a finite set of prime numbers containing p and all prime numbers at which the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on T ramifies, and let

$$\Xi = \{m \geqslant 1\,;\, \mathrm{prime}(m)\cap\Sigma = \{p\}\}.$$

For a prime number ℓ which is not contained in Σ, let

$$P_\ell(t) = \det_{O_L}(1 - \mathrm{Fr}_\ell\cdot t : T \to T) \in O_L[t]$$

where Fr_ℓ is the arithmetic Frobenius at ℓ.

By an Euler system for (T, L, Σ), we mean a system of elements $z_m \in \mathrm{H}^1(\mathbb{Z}[\zeta_m, 1/p], T)$ defined for $m \in \Xi$, satisfying the following condition.

(13.1.1) *For $m, m' \in \Xi$ such that $m \mid m'$, the norm map*

$$\mathrm{H}^1\Big(\mathbb{Z}[\zeta_{m'}, 1/p], T\Big) \longrightarrow \mathrm{H}^1\Big(\mathbb{Z}[\zeta_m, 1/p], T\Big)$$

sends $z_{m'}$ to $\Big(\prod_\ell P_\ell(\ell^{-1}\sigma_\ell^{-1})\Big) \cdot z_m$, where ℓ ranges over all prime numbers which divide m' but do not divide m, σ_ℓ is the arithmetic Frobenius of ℓ in $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, and we regard $P_\ell(\ell^{-1}\sigma_\ell^{-1})$ as an element of the group ring $O_L[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})]$ which acts naturally on $\mathrm{H}^1(\mathbb{Z}[\zeta_m, 1/p], T)$.

***Example 13.2* (classical example).** — Let $L = \mathbb{Q}_p$, $T = \mathbb{Z}_p(1)$, $\Sigma = \{p\}$. For $m \in \Xi$ (that is, for any $m \geqslant 1$ such that $p|m$), let $z_m \in \mathrm{H}^1(\mathbb{Z}[\zeta_m, 1/p], \mathbb{Z}_p(1))$ be the image of $(1-\zeta_m)(1-\zeta_m^{-1}) \in \mathbb{Z}[\zeta_m, 1/p]^\times$ under the isomorphism

$$\mathbb{Z}[\zeta_m, 1/p]^\times \otimes \mathbb{Z}_p \simeq \mathrm{H}^1\Big(\mathbb{Z}[\zeta_m, 1/p], \mathbb{Z}_p(1)\Big)$$

Then $(z_m)_m$ is an Euler system for $(\mathbb{Z}_p(1), \mathbb{Q}_p, \Sigma)$. In fact,

$$P_\ell(t) = 1 - \ell t, \quad P_\ell(\ell^{-1}\sigma_\ell^{-1}) = 1 - \sigma_\ell^{-1},$$

and (13.1.1) follows from the fact that for any $m \geqslant 2$ and for any prime number ℓ, the norm map

$$\mathbb{Q}(\zeta_{m\ell})^\times \longrightarrow \mathbb{Q}(\zeta_m)^\times$$

sends $1-\zeta_{m\ell}$ to $1-\zeta_m$ if ℓ divides m, and to $(1-\zeta_m)(1-\sigma_\ell^{-1}(\zeta_m))^{-1}$ if ℓ does not divide m.

***Example 13.3* (the crucial example for this paper).** — Let λ be a place of F lying over p, let $r \in \mathbb{Z}$, and let $T = V_{O_\lambda}(f)(k-r)$. Fix an integer j such that $1 \leqslant j \leqslant k-1$, and fix non-zero integers c, d. Let ξ be either a symbol of the form $a(A)$ ($a, A \in \mathbb{Z}$, $A \geqslant 1$) or an element of $\mathrm{SL}_2(\mathbb{Z})$. In the case, $\xi = a(A)$, we assume $(c, 6pA) = 1$ and $(d, 6pN) = 1$. In the case $\xi \in \mathrm{SL}_2(\mathbb{Z})$, we assume $(cd, 6pN) = 1$.

By fixing these, let $\Sigma = \mathrm{prime}(cdpAN)$ in the case $\xi = a(A)$, and let $\Sigma = \mathrm{prime}(cdpN)$ in the case $\xi \in \mathrm{SL}_2(\mathbb{Z})$. For $m \in \Xi$, define $z_m \in \mathrm{H}^1(\mathbb{Z}[\zeta_m, 1/p], T)$ by

$$z_m = \begin{cases} {}_{c,d}z_m^{(p)}(f, r, j, \xi, \mathrm{prime}(mA)) & \text{if } \xi = a(A), \\ {}_{c,d}z_m^{(p)}(f, r, j, \xi, \mathrm{prime}(mN)) & \text{if } \xi \in \mathrm{SL}_2(\mathbb{Z}). \end{cases}$$

Then $(z_m)_m$ is an Euler system for (T, F_λ, Σ).

In fact, for a prime number ℓ which does not divide Np, we have

$$\det_{O_L}(1 - \mathrm{Fr}_\ell^{-1} \cdot t; V_{F_\lambda}(f)) = 1 - a_\ell t + \varepsilon(\ell)\ell^{k-1}t^2,$$

and this polynomial has the form $(1-\alpha t)(1-\beta t)$, $\alpha, \beta \in \mathbb{C}$, $|\alpha| = |\beta| = \ell^{(k-1)/2}$. Hence

$$P_\ell(t) = \det_{O_L}(1 - \mathrm{Fr}_\ell \cdot t; T) = 1 - \overline{a}_\ell \ell^{1-r} t + \overline{\varepsilon}(\ell)\ell^{k+1-2r}t^2,$$

$$P_\ell(\ell^{-1}\sigma_\ell^{-1}) = 1 - \overline{a}_\ell \ell^{-r}\sigma_\ell^{-1} + \overline{\varepsilon}(\ell)\ell^{k-1-2r}\sigma_\ell^{-2}.$$

Hence $(z_m)_m$ is an Euler System for (T, F_λ, Σ) by Prop. 8.12.

By [**Pe4, Ru4, KK4**], we have

Theorem 13.4. — *Let (T, L, Σ) be as in 13.1, and let $(z_m)_m$ be an Euler system for (T, L, Σ). Let $\Lambda = O_F[[G_\infty]]$, let Z be the Λ-submodule of $\mathbf{H}^1(T)$ generated by $(z_{p^n})_n$, and let J be the ideal of Λ generated by $h(Z)$ for all Λ-homomorphisms $h : \mathbf{H}^1(T) \to \Lambda$. On the other hand, let*

$$\mathbf{H}^2(T)_0 \underset{\text{def}}{=} \mathrm{Ker}(\mathbf{H}^2(T) \to \mathbf{H}^2_{\mathrm{loc}}(T)).$$

Assume the following

(i) *$Z_{\mathfrak{q}} \neq 0$ for any prime ideal $\mathfrak{q}$ of Λ of height* 0.

(ii) $\mathrm{rank}_{O_L}(T^+) = \mathrm{rank}_{O_L}(T^-) = 1$.

(iii) *There exists an integer w such that for any prime number ℓ which is not contained in Σ, all eigenvalues of Fr_ℓ^{-1} on the $\overline{L}$-vector space $T \otimes_{O_L} \overline{L}$ are algebraic numbers whose all complex conjugates have absolute value $\ell^{w/2}$.*

(iv) *$T \otimes_{O_L} L$ is irreducible as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over L.*

(v) *There exists an element σ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^\infty}))$ such that*

$$\dim_L(\mathrm{Ker}(1 - \sigma; T \otimes_{O_L} L \to T \otimes_{O_L} L)) = 1.$$

Then we have:

(1) *$\mathbf{H}^2(T)$ is a torsion Λ-module.*

(2) *Let $\mathfrak{p}$ be a prime ideal of Λ of height one which does not contain p. Then*

$$\mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^2(T)_{0,\mathfrak{p}}) \leqslant \mathrm{length}_{\Lambda_{\mathfrak{p}}}(\Lambda_{\mathfrak{p}}/J_{\mathfrak{p}}).$$

(3) *Assume that there exists an element σ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^\infty}))$ such that*

$$\mathrm{Coker}(1 - \sigma : T \to T)$$

is a free O_L-module of rank 1, and assume that $T \otimes_{O_L} O_L/m_L$ is irreducible as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over O_L/m_L. Assume further $p \neq 2$. Then

$$\mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^2(T)_{0,\mathfrak{p}}) \leqslant \mathrm{length}_{\Lambda_{\mathfrak{p}}}(\Lambda_{\mathfrak{p}}/J_{\mathfrak{p}}).$$

for any prime ideal $\mathfrak{p}$ of Λ of height one.

In the case of 13.3, the conditions (ii), (iii), (iv) in Thm. 13.4 are satisfied ((iii) is due to Deligne [**De1**], and (iv) is due to Ribet [**Ri1**]). However if f has CM, this theorem is not applied because the condition (v) is not satisfied in the CM case. So the CM case will be discussed separately in § 15. Concerning the condition (i) in Thm. 13.4, we use the following 13.5 and 13.6.

Theorem 13.5

(1) *(Jacquet-Shalika [**JS**]). $L(f, s)$ has no zero on* $\mathrm{Re}(s) \geqslant \frac{k+1}{2}$.

(2) *(Rohrlich* [**Ro2**]*). Assume k is even. Let S be a finite set of prime numbers. Then the set*

$$\left\{\chi \in \bigcup_{\substack{m\geqslant 1\\ \mathrm{prime}(m)\subset S}} \mathrm{Hom}((\mathbb{Z}/m)^\times, \mathbb{C}^\times); L_S(f,\chi,k/2)=0\right\}$$

is finite.

(When $m|m'$, we regard $\mathrm{Hom}((\mathbb{Z}/m)^\times,\mathbb{C}^\times)\subset \mathrm{Hom}((\mathbb{Z}/m')^\times,\mathbb{C}^\times)$.)

Theorem 13.6 **(Ash-Stevens [AS]).** — *Let $L\geqslant 3$. Then $V_{k,\mathbb{Z}}(Y(L))$ is generated over $\mathbb{Z}$ by the elements*

$$\alpha^*\delta_{L,L}(k,j)\quad (\alpha\in \mathrm{GL}_2(\mathbb{Z}/L), 1\leqslant j\leqslant k-1).$$

Proposition 13.7. — *Define $Z\subset \mathbf{H}^1(V_{O_\lambda}(f))$ as in Thm. 12.6. Then $Z_{\mathfrak{q}}\neq 0$ for any prime ideal $\mathfrak{q}$ of Λ of height 0.*

Proof. — Let $\pm$ be $+$ (resp. $-$) if the image of $-\sigma_{-1}$ in Λ_q is 1 (resp. -1). By 13.6, for some $\alpha\in\mathrm{SL}_2(\mathbb{Z})$ and some integer j such that $1\leqslant j\leqslant k-1$, we have $\delta(f,j,\alpha)^\pm\neq 0$. Take such α, j and take integers c,d such that $(cd,6p)=1, c\equiv d\equiv 1 \bmod N$, and $c^2\neq 1, d^2\neq 1$. Then by Thm. 6.6, Thm. 9.7, and by Thm. 13.5, for almost all homomorphisms $\chi: G_\infty\to \overline{F}_\lambda^\times$ of finite order such that $\chi(-1)=\pm$, the element $({}_{c,d}\mathbf{z}^{(p)}_{p^n}(f,k,j,\alpha,\mathrm{prime}(pN)))_{n\geqslant 1}\in Z\subset \mathbf{H}^1(V_{O_\lambda}(f))$ (12.6) is sent to a non-zero element by the homomorphism

$$\begin{aligned}(13.7.1)\quad \mathbf{H}^1(V_{O_\lambda}(f))\simeq \mathbf{H}^1(V_{O_\lambda}(f)(1))\longrightarrow \mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), V_{F_\lambda}(f)(1))\\ \xrightarrow{\exp^*} S(f)\otimes_F F_\lambda \xrightarrow{\chi} S(f)\otimes_F \overline{F}_\lambda\end{aligned}$$

where $n\geqslant 0$ is an integer such that χ factors through $G_\infty\to G_n$, and the last arrow in (13.7.1) is $a\mapsto \sum_{\sigma\in G_n}\sigma(a)\otimes\chi(\sigma)$. The composite map (13.7.1) factors through $\mathbf{H}^1(V_{F_\lambda}(f))/\mathfrak{p}\mathbf{H}^1(V_{F_\lambda}(f))$ where $\mathfrak{p}$ is the kernel of the ring homomorphism $\Lambda\to\overline{F}_\lambda$ which sends $\sigma\in G_\infty$ to $\kappa(\sigma)^{-1}\chi(\sigma)^{-1}$. Hence for infinitely many prime ideals $\mathfrak{p}$ of Λ of height one such that $\mathfrak{p}\supset\mathfrak{q}$, the image of Z in $\mathbf{H}^1(V_{F_\lambda}(f))/\mathfrak{p}\mathbf{H}^1(V_{F_\lambda}(f))$ is not zero. This proves $Z_{\mathfrak{q}}\neq 0$. □

In the rest of § 13, we assume f has no CM and we prove Thm. 12.4, 12.5, 12.6 under this assumption. The proofs of these theorems in the case f has CM are given in § 15.

13.8. We prove Thm. 12.4

To prove 12.4 (1), we may assume $T=V_{O_\lambda}(f)$. In this case, the fact $\mathbf{H}^2(T)$ is a torsion Λ-module follows from (12.8.2), Thm. 13.4 (1), and 13.7.

We prove 12.4 (2). By (12.2.2) and by 12.4 (1), it is sufficient to prove that $\mathbf{H}^1(T)$ is a torsion free Λ-module. (This torsion free property is deduced also from a general result [**Pe3**, Lemme p. 27].)

Let x be a non-zero-divisor of Λ. We prove that $x : \mathbf{H}^1(T) \to \mathbf{H}^1(T)$ is injective. There exists a multiple of x of the form $p^n y$ where $n \geqslant 0$ and y is a non-zero-divisor of Λ such that $\Lambda/y\Lambda$ is p-torsion free. (In fact, the image of x under the norm map

$$\Lambda = O_\lambda[[G_\infty]] \longrightarrow O_\lambda[[G^1_\infty]] \simeq O_\lambda[[X]]$$

has the form $p^n y$ for $n \geqslant 0$ and y as above, and x divides $p^n y$ in Λ.) Hence we may assume $x = p$ or $\Lambda/x\Lambda$ is p-torsion free.

Let

$$j : \operatorname{Spec}(\mathbb{Z}[1/p]) \smallsetminus \operatorname{prime}(N) \longrightarrow \operatorname{Spec}(\mathbb{Z}[1/p])$$

be the inclusion map.

First consider the case $x = p$. By the exact sequence

$$0 \longrightarrow j_*T \xrightarrow{\ p\ } j_*T \longrightarrow j_*(T/p),$$

the injectivity of $p : \mathbf{H}^1(T) \to \mathbf{H}^1(T)$ is reduced to $\varprojlim_n \mathrm{H}^0(\mathbb{Q}(\zeta_{p^n}), T/p) = 0$. Take $m \geqslant 1$ such that $\mathrm{H}^0(\mathbb{Q}(\zeta_{p^n}), T/p) = \mathrm{H}^0(\mathbb{Q}(\zeta_{p^\infty}), T/p)$. Then, for $n \geqslant m$, the norm map

$$\mathrm{H}^0(\mathbb{Q}(\zeta_{p^{n+1}}), T/p) \longrightarrow \mathrm{H}^0(\mathbb{Q}(\zeta_{p^n}), T/p)$$

is the multiplication by $[\mathbb{Q}(\zeta_{p^{n+1}}) : \mathbb{Q}(\zeta_{p^n})] = p$, and hence is the zero map. Hence $\varprojlim_n \mathrm{H}^0(\mathbb{Q}(\zeta_{p^n}), T/p) = 0$.

Next we consider the case $\Lambda/x\Lambda$ is p-torsion free. For $n \geqslant 0$,

$$\mathrm{H}^q(\mathbb{Z}[\zeta_{p^n}, 1/p], T) \simeq \mathrm{H}^q(\mathbb{Z}[1/p], T \otimes_{O_\lambda} O_\lambda[G_n])$$

where $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the tensor product as follows : $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by $\sigma \otimes \sigma_n^{-1}$ where σ_n denotes the canonical image of σ in G_n. We have

$$\mathbf{H}^q(T) \simeq \varprojlim_n \mathrm{H}^q(\mathbb{Z}[1/p], T \otimes_{O_\lambda} O_\lambda[G_n]).$$

Hence we have an exact sequence

$$\mathrm{H}^0(\mathbb{Q}, T \otimes_{O_\lambda} \Lambda/x\Lambda) \longrightarrow \mathbf{H}^1(T) \xrightarrow{\ x\ } \mathbf{H}^1(T).$$

where $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act on the tensor product as follows : $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by $\sigma \otimes \sigma_\infty^{-1}$ where σ_∞ denotes the canonical image of σ in G_∞. We prove

$$\mathrm{H}^0(\mathbb{Q}, T \otimes_{O_\lambda} \Lambda/x\Lambda) = 0.$$

The set $\mathrm{H}^0(\mathbb{Q}, T \otimes_{O_\lambda} \Lambda/x\Lambda)$ is identified with the set of all $O_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-homomorphism $\mathrm{Hom}_{O_\lambda}(\Lambda/x\Lambda, O_\lambda) \to T$ ($\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\Lambda/x\Lambda$ here via $\sigma \mapsto \sigma_\infty^{-1}$). Since the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{Hom}_{O_\lambda}(\Lambda/x\Lambda, O_\lambda)$ is abelian and the representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_{F_\lambda}(f)$ is irreducible and is not abelian, there is no such non-trivial homomorphism.

We prove 12.4 (3). Let x, y be elements of Λ such that (x, y) is a maximal ideal of Λ. It is sufficient to prove that x and y form a regular sequence for $\mathbf{H}^1(T)$, that is

$$x : \mathbf{H}^1(T) \longrightarrow \mathbf{H}^1(T) \quad \text{and} \quad y : \mathbf{H}^1(T)/x\mathbf{H}^1(T) \longrightarrow \mathbf{H}^1(T)/x\mathbf{H}^1(T)$$

are injective. The former is already proved. We prove the latter. By

$$\mathbf{H}^1(T)/x\mathbf{H}^1(T) \subset \mathrm{H}^1(\mathbb{Z}[1/p], T\otimes_{O_\lambda}\Lambda/x\Lambda)$$

and by the exact sequence

$$0 \longrightarrow j_*(T\otimes_{O_\lambda}\Lambda/x\Lambda) \xrightarrow{\ y\ } j_*(T\otimes_{O_\lambda}\Lambda/x\Lambda) \longrightarrow j_*(T\otimes_{O_\lambda}\Lambda/(x,y)),$$

it is sufficient to prove $\mathrm{H}^0(\mathbb{Z}[1/p], T\otimes_{O_\lambda}\Lambda/(x,y)) = 0$. Here $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\Lambda/(x,y)$ via $\sigma\mapsto\sigma_\infty^{-1}$, and hence $\Lambda/(x,y)\simeq(O_\lambda/m_\lambda)(r)$ for some $r\in\mathbb{Z}$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module over O_λ ((r) means the Tate twist). By the assumption of irreducibility, the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-fixed part of $(T/m_\lambda T)(r)$ is trivial. □

13.9. In 13.9-13.11, we give preliminaries for the proofs of Thm. 12.5 and Thm. 12.6.

For a commutative ring R, let $Q(R)$ be the total quotient ring of R. That is,

$$Q(R) = \{ab^{-1}\,;\ a,b\in R, b \text{ is a non-zero-divisor}\}.$$

In this 13.9, we define the p-adic zeta element $\mathbf{z}_\gamma^{(p)}$ for $\gamma\in V_{F_\lambda}(f)$, which appears in Thm. 12.5, first as an element of $\mathbf{H}^1(V_{F_\lambda}(f))\otimes_\Lambda Q(\Lambda)$. We will see in 13.12 that $\mathbf{z}_\gamma^{(p)}$ belongs to $\mathbf{H}^1(V_{F_\lambda}(f))$.

Fix elements α_1,α_2 of $\mathrm{SL}_2(\mathbb{Z})$ and integers j_1,j_2 such that $1\leqslant j_i\leqslant k-1$ $(i=1,2)$ and such that $\delta(f,j_1,\alpha_1)^+\neq 0$, $\delta(f,j_2,\alpha_2)^-\neq 0$. (13.6).

Let $\gamma\in V_{F_\lambda}(f)$. We have

$$\gamma = b_1\delta(f,j_1,\alpha_1)^+ + b_2\delta(f,j_2,\alpha_2)^-$$

for some $b_1,b_2\in F_\lambda$. Fix $c,d\in\mathbb{Z}$ such that $(cd,6p)=1$, $c\equiv d\equiv 1 \bmod N$, $c^2\neq 1$, $d^2\neq 1$. Define $\mathbf{z}_\gamma^{(p)}\in\mathbf{H}^1(V_{F_\lambda}(f))\otimes_\Lambda Q(\Lambda)$ by

$$\begin{aligned}\mathbf{z}_\gamma^{(p)} = &\Big\{\mu(c,d,j_1)^{-1}\cdot b_1\cdot\Big({}_{c,d}\mathbf{z}_{p^n}^{(p)}(f,k,j_1,\alpha_1,\mathrm{prime}(pN))\Big)_{n\geqslant 1}\Big\}^-\\ &+\Big\{\mu(c,d,j_2)^{-1}\cdot b_2\cdot\Big({}_{c,d}\mathbf{z}_{p^n}^{(p)}(f,k,j_2,\alpha_2,\mathrm{prime}(pN))\Big)_{n\geqslant 1}\Big\}^+\end{aligned}$$

where

$$\mu(c,d,j) = (c^2-c^{k+1-j}\cdot\sigma_c)(d^2-d^{j+1}\cdot\sigma_d)\cdot\prod_\ell(1-\overline{a}_\ell\ell^{-k}\sigma_\ell^{-1})\in\Lambda$$

for $j\in\mathbb{Z}$ in which ℓ ranges over all prime numbers $\neq p$ which divide N. It is easy to see that $\mu(c,d,j)$ is a non-zero-divisor of Λ for any $j\in\mathbb{Z}$.

By Thm. 6.6 and Thm. 9.7, if $\gamma\in V_F(f)\subset V_{F_\lambda}(f)$ and χ is a homomorphism $G_\infty\to\overline{\mathbb{Q}}^\times$ of finite order such that $c^2-c^{k-j}\chi(c)\neq 0$, $d^2-d^j\chi(d)\neq 0$ for $j=j_1,j_2$ and $1-\overline{a}_\ell\ell^{1-k}\chi(\ell)\neq 0$ for any prime number $\ell\neq p$ which divides N, the homomorphism

$$\mathbf{H}^1(V_{F_\lambda}(f))\otimes_\Lambda\Lambda[\mu^{-1}]\longrightarrow S(f)\otimes_F\overline{F}_\lambda\qquad(\mu=\mu(c,d,j_1)\mu(c,d,j_2))$$

induced by (13.7.1) sends $\mathbf{z}_\gamma^{(p)}$ to an element of $S(f)\otimes_F\overline{\mathbb{Q}}$ whose image under $\mathrm{per}_f: S(f)\otimes_F\overline{\mathbb{Q}}\to V_F(f)\otimes_F\mathbb{C}$ coincides with $L_S(f^*,\chi,k-1)\cdot\gamma^{\pm}$ where $S=\mathrm{prime}(pN)$ and $\pm=\chi(-1)$.

Since $\mathbf{H}^1(V_{F_\lambda}(f))$ is a free $\Lambda[1/p]$-module of rank 1 (Thm.12.4 (2)), this shows that $\mathbf{z}_\gamma^{(p)}$ is independent of the choices of $\alpha_1, j_1, \alpha_2, j_2, c, d$ as above. This also shows

$$\mathbf{z}_{\iota(\gamma)}^{(p)}=-\sigma_{-1}(\mathbf{z}_\gamma^{(p)}).$$

since the action of σ_{-1} on $\mathbf{H}^1(V_{F_\lambda})$ commutes with the action of $-\chi(-1)$ on $S(f)\otimes_F\overline{F}_\lambda$ via the map (13.7.1).

We express the elements

$$\Big({}_{c,d}\mathbf{z}_{p^n}^{(p)}(f,k,j,a(A),\mathrm{prime}(pA))\Big)_{n\geqslant1}\quad\text{and}\quad\Big({}_{c,d}\mathbf{z}^{(p)}(f,k,j,\alpha,\mathrm{prime}(pN))\Big)_{n\geqslant1}$$

($\alpha\in\mathrm{SL}_2(\mathbb{Z})$) by using $\mathbf{z}_\gamma^{(p)}$.

Lemma 13.10. — *We have the following equalities in $\mathbf{H}^1(V_{F_\lambda}(f))\otimes_\Lambda Q(\Lambda)$.*

(1) *Let $1\leqslant j\leqslant k-1$, $a,A\in\mathbb{Z}$, $A\geqslant1$, and c,d be integers such that $(c,6pA)=(d,6pN)=1$. Then*

$$\Big({}_{c,d}\mathbf{z}_{p^n}^{(p)}(f,k,j,a(A),\mathrm{prime}(pA))\Big)_{n\geqslant1}=\Big\{\prod_\ell(1-\overline{a}_\ell\ell^{-k}\sigma_\ell^{-1}+\overline{\varepsilon}(\ell)\ell^{-k-1}\sigma_\ell^{-2})\Big\}\cdot$$
$$(c^2d^2\mathbf{z}_{\gamma_1}^{(p)}-c^{k+1-j}d^2\sigma_c\mathbf{z}_{\gamma_2}^{(p)}-c^2d^{j+1}\varepsilon(d)\sigma_d\mathbf{z}_{\gamma_3}^{(p)}+c^{k+1-j}d^{j+1}\varepsilon(d)\sigma_{cd}\mathbf{z}_{\gamma_4}^{(p)})$$

where ℓ ranges over all prime numbers $\neq p$ which divide A, and

$$\gamma_1=\delta(f,j,a(A)),\qquad \gamma_2=\delta(f,j,ac(A)),$$
$$\gamma_3=\delta(f,j,\text{“}a/d\text{”}\,(A)),\qquad \gamma_4=\delta(f,j,\text{“}ac/d\text{”}\,(A)),$$

Here “a/d” means any integer b such that $bd\equiv a \bmod A$, and “ac/d” means any integer b such that $bd\equiv ac \bmod A$.

(2) *Let $1\leqslant j\leqslant k-1$, $\alpha\in\mathrm{SL}_2(\mathbb{Z})$, and let c,d be integers such that $(cd,6p)=1$ and $c\equiv d\equiv1 \bmod N$. Then*

$$(13.10.1)\quad \Big({}_{c,d}\mathbf{z}_{p^n}^{(p)}(f,k,j,\alpha,\mathrm{prime}(pN))\Big)_{n\geqslant1}=$$
$$(c^2-c^{k+1-j}\sigma_c)(d^2-d^{j+1}\sigma_d)\Big\{\textstyle\prod_\ell(1-\overline{a}_\ell\ell^{-k}\sigma_\ell^{-1})\Big\}\cdot\mathbf{z}_{\delta(f,j,\alpha)}^{(p)}$$

where ℓ ranges over all prime numbers $\neq p$ which divide N.

Proof. — By Thm. 12.4 (2), this is obtained by computing the images of the elements in problem under the map (13.7.1) by using Thm. 6.6 and Thm. 9.7. □

Lemma 13.11. — *Let $A\geqslant1$ and let ν be a homomorphism $(\mathbb{Z}/A)^\times\to\overline{\mathbb{Q}}^\times$.*

(1) *For integers c, d such that $(c, 6Ap) = (d, 6ANp) = 1$, we have*

$$\sum_{a\in(\mathbb{Z}/A)^\times} \nu(a)\cdot\Big({}_{c,d}\mathbf{z}^{(p)}_{p^n}(f,k,k-1,a(A),\mathrm{prime}(pA))\Big)_{n\geqslant 1} =$$
$$\Big\{\prod_\ell (1-\overline{a}_\ell \ell^{-k}\sigma_\ell^{-1}+\overline{\varepsilon}(\ell)\ell^{-k-1}\sigma_\ell^{-2})\Big\}\cdot$$
$$(c^2-c^2\nu(c)^{-1}\sigma_c)(d^2-d^k\varepsilon(d)\nu(d)\sigma_d)\cdot \sum_{a\in(\mathbb{Z}/A)^\times}\nu(a)\mathbf{z}^{(p)}_{\delta(f,k-1,a(A))}$$

in $\mathbf{H}^1(V_{F_\lambda}(f))\otimes_\Lambda Q(\Lambda)\otimes_{O_\lambda}\overline{F}_\lambda$, where ℓ ranges over all prime numbers $\neq p$ which divide A.

(2) *The element $\sum_{a\in(\mathbb{Z}/A)^\times}\nu(a)\delta(f,k-1,a(A))$ of $V_F(f)\otimes\overline{\mathbb{Q}}$ is not zero if $L_{\mathrm{prime}(A)}(f^*,\nu^{-1},k-1)\neq 0$ and if the conductor of ν is A.*

Proof. — (1) follows from Lemma 13.10 (1). We prove (2). The canonical pairing (7.13.1)

$$\langle\,,\,\rangle : V_k(Y_1(N))\times V_{k,c}(Y_1(N))\longrightarrow\mathbb{Q}$$

induces

$$\langle\,,\mathrm{per}(f^*)\rangle : V_F(f)\longrightarrow\mathbb{C}.$$

We have for $A\geqslant 1$ and $a\in\mathbb{Z}$,

$$\langle\delta(f,k-1,a(A)),f^*\rangle = (-2\pi)^{k-1}\cdot A^{k-2}\cdot\int_0^\infty f^*(yi+a/A)y^{k-2}\,dy$$

From this, we obtain

$$\sum_{a\in(\mathbb{Z}/A)^\times}\nu(a)\langle\delta(f,k-1,a(A)),f^*\rangle$$
$$=(-1)^{k-1}A^{k-2}\cdot(k-2)!\cdot G(\nu,\zeta_A)\cdot L_{\mathrm{prime}(A)}(f^*,\nu^{-1},k-1).$$

where $G(\nu,\zeta_A)$ is the Gauss sum. This proves 13.11 (2). □

13.12. In this 13.12, we prove Thm. 12.5 (1) and Thm. 12.6.

Let $T=V_{O_\lambda}(f)$. Define $Z(f,T)$ to be the Λ-submodule of $\mathbf{H}^1(T)\otimes_\Lambda Q(\Lambda)$ generated by $\mathbf{z}^{(p)}_\gamma$ for all $\gamma\in T$. On the other hand, let $Z\subset\mathbf{H}^1(T)$ be the Λ-submodule in Thm. 12.6.

By Lemma 13.10, we have $Z\subset Z(f,T)$. We prove that $Z(f,T)/Z$ is a finite group. This will show that

$$Z(f,T)\subset\mathbf{H}^1(T)\otimes\mathbb{Q}=\mathbf{H}^1(V_{F_\lambda}(f)).$$

By 13.6 and by 13.10 (2), there is a non-zero-divisor μ of Λ such that $\mu\cdot Z(f,T)\subset Z$ and such that $\Lambda/\mu\Lambda$ is p-torsion free. Hence it is sufficient to show $Z(f,T)_\mathfrak{p}=Z_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of height one which does not contain p. Let $\mathfrak{p}$ be such a prime ideal, and let h be the map $\Lambda\to\Lambda/\mathfrak{p}$, and embed $\Lambda/\mathfrak{p}$ into $\overline{F}_\lambda$ over O_λ. Fix an embedding $\overline{\mathbb{Q}}\to\overline{F}_\lambda$ over F. Take integers c, d such that $(c,6p)=(d,6Np)=1$ and

$c^2 \neq 1$, $d^2 \neq 1$. By Thm. 13.5, there exist a power A of p and a homomorphism $\nu : (\mathbb{Z}/A)^\times \to \overline{\mathbb{Q}}^\times \subset \overline{F}_\lambda^\times$ of conductor A satisfying the following (13.12.1)-(13.12.4).

(13.12.1) $$L_{\{p\}}(f^*, \nu^{-1}, k-1) \neq 0$$

(13.12.2) $$c^2 - c^2\nu(c)^{-1}h(\sigma_c) \neq 0$$

(13.12.3) $$d^2 - d^k\varepsilon(d)\nu(d)h(\sigma_d) \neq 0$$

(13.12.4) *Let* $\pm$ *be* $+$ *(resp.* $-$*) if the image of* $-\sigma_{-1}$ *in* $\Lambda/\mathfrak{p}$ *is* 1 *(resp.* -1*). Then* $\nu(-1) = \pm$.

Let L be the subfield of $\overline{F}_\lambda$ generated over $\Lambda/\mathfrak{p}$ by the values of ν, and let $\mathfrak{p}'$ be the kernel of the homomorphism $O_L[[G_\infty]] \to L$ induced by the canonical map $\Lambda \to \Lambda/\mathfrak{p} \subset L$ by O_L-linearity.

Let $\gamma \in V_F(f)$. Let $\pm$ be as in (13.12.4). Then the image of $\mathbf{z}_\gamma^{(p)}$ in $\mathbf{H}^1(V_{F_\lambda}(f)) \otimes Q(\Lambda_\mathfrak{p})$ coincides with that of $\mathbf{z}_{\gamma^\pm}^{(p)}$. By 13.11 (2), (13.12.1), (13.12.4),

$$\gamma^\pm = b \cdot \sum_{a \in (\mathbb{Z}/A)^\times} \nu(a)\delta(f, k-1, a(A)) \quad \text{in } V_F(f) \otimes_F \overline{\mathbb{Q}}$$

for some $b \in \overline{\mathbb{Q}}$. We have

$$\mathbf{z}_\gamma^{(p)} = b \cdot \sum_{a \in (\mathbb{Z}/A)^\times} \nu(a)\mathbf{z}_{\delta(f,k-1,a(A))}^{(p)} \quad \text{in } \mathbf{H}^1(V_{F_\lambda}(f)) \otimes_\Lambda Q(\Lambda_\mathfrak{p}) \otimes_{F_\lambda} L.$$

By 13.11 (1), (13.12.2), (13.12.3),

$$\sum_{a \in (\mathbb{Z}/A)^\times} \nu(a) \cdot \left({}_{c,d}\mathbf{z}_{p^n}^{(p)}(f, k, k-1, a(A), \mathrm{prime}(pA))\right)_{n \geqslant 1} = \mu \cdot \sum_{a \in (\mathbb{Z}/A)^\times} \nu(a)\mathbf{z}_{\delta(f,k-1,a(A))}^{(p)}$$

for some $\mu \in (O_L[[G_\infty]]_{\mathfrak{p}'})^\times$, and hence $\sum_{a \in (\mathbb{Z}/A)^\times} \nu(a)\mathbf{z}_{\delta(f,k-1,a(A))}^{(p)}$ belongs to $Z \otimes_\Lambda O_L[[G_\infty]]_{\mathfrak{p}'}$ in $\mathbf{H}^1(V_{F_\lambda}(f)) \otimes_\Lambda Q(O_L[[G_\infty]]_{\mathfrak{p}'})$. Hence $\mathbf{z}_\gamma^{(p)}$ also belongs to $Z \otimes_\Lambda O_L[[G_\infty]]_{\mathfrak{p}'}$ in $\mathbf{H}^1(V_{F_\lambda}(f)) \otimes_\Lambda Q(O_L[[G_\infty]]_{\mathfrak{p}'})$. This proves $\mathbf{z}_\gamma^{(p)} \in Z_\mathfrak{p}$.

It remains to prove that $\mathbf{z}_\gamma^{(p)}$ ($\gamma \in V_{F_\lambda}(f)$) has the property stated in Thm. 12.5 (1). Let $r \in \mathbb{Z}$, $1 \leqslant r \leqslant k-1$, let $\chi : G_\infty \to \overline{\mathbb{Q}}^\times$ be a homomorphism of finite order, and consider the composite map

(13.12.5) $$\mathbf{H}^1(V_{F_\lambda}(f)) \simeq \mathbf{H}^1(V_{F_\lambda}(f)(k-r)) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), V_{F_\lambda}(f)(k-r)) \xrightarrow{\exp^*} S(f) \otimes_F F_\lambda \otimes_\mathbb{Q} \mathbb{Q}(\zeta_{p^n}) \xrightarrow{\chi} S(f) \otimes_F \overline{F}_\lambda$$

where the last arrow is

$$\omega \otimes a \otimes b \longmapsto \sum_{\sigma \in G_n} \sigma(b)\omega \otimes a\chi(\sigma).$$

Let $h : \Lambda \to \overline{F}_\lambda$ be the ring homomorphism induced by $\kappa^{r-k}\chi^{-1} : G_\infty \to \overline{F}_\lambda^\times$, and let $\mathfrak{p}$ be the kernel of h. Then the map (13.12.5) is a Λ-homomorphism with respect to the action of Λ on $S(f) \otimes_F \overline{F}_\lambda$ via h. Take integers c, d such that $(c, 6p) = (d, 6Np) = 1$

and $c^2 \neq 1$, $d^2 \neq 1$. By Thm. 13.5, there exist a power A of p and a homomorphism $\nu : (\mathbb{Z}/A)^\times \to \overline{\mathbb{Q}}^\times \subset \overline{F}_\lambda^\times$ satisfying (13.12.1)-(13.12.4). Let $\gamma \in V_F(f)$. Let $\pm$ be as in (13.12.4). Then the image of $\mathbf{z}_\gamma^{(p)}$ in $\mathbf{H}^1(V_{F_\lambda}(f))_\mathfrak{p}$ coincides with that of $\mathbf{z}_{\gamma^\pm}^{(p)}$. By 13.11 (2), (13.12.1), (13.12.4), $\gamma^\pm = b \cdot \sum_{a\in(\mathbb{Z}/A)^\times} \nu(a)\delta(f, k-1, a(A))$ for some $b \in \overline{\mathbb{Q}}$. We have

$$\mathbf{z}_\gamma^{(p)} = b \cdot \sum_{a\in(\mathbb{Z}/A)^\times} \nu(a)\mathbf{z}_{\delta(f,k-1,a(A))}^{(p)};$$

$$\sum_{a\in(\mathbb{Z}/A)^\times} \nu(a) \cdot \left({}_{c,d}\mathbf{z}_{p^n}^{(p)}(f, k, k-1, a(A), \mathrm{prime}(pA))\right)_{n\geqslant 1} = \mu \cdot \sum_{a\in(\mathbb{Z}/A)^\times} \nu(a)\mathbf{z}_{\delta(f,k-1,a(A))}^{(p)}$$

where $\mu = (c^2 - c^2\nu(c)^{-1}\sigma_c)(d^2 - d^k\varepsilon(d)\nu(d)\sigma_d)$. Hence the image of $\mathbf{z}_\gamma^{(p)}$ under (13.12.5) coincides with the image of

$$\sum_{a\in(\mathbb{Z}/A)^\times} \nu(a) \cdot \left({}_{c,d}\mathbf{z}_{p^n}^{(p)}(f, k, k-1, a(A), \mathrm{prime}(pA))\right)_{n\geqslant 1}$$
$$\cdot\, b \cdot (c^2 - c^2\nu(c)^{-1}\chi(c)^{-1})^{-1}(d^2 - d^k\varepsilon(d)\nu(d)\chi(d)^{-1})^{-1}.$$

By Thm. 6.6 and Thm. 9.7, this image is an element of $S(f)\otimes_F \overline{\mathbb{Q}}$ whose image under per_f coincides with $(2\pi i)^{k-r-1}L_{\{p\}}(f^*, \chi, r) \cdot \gamma^\pm$. □

13.13. We prove Thm. 12.5 (2) (3).

Thm. 12.5 (2) follows from Thm. 12.4 (2) and the fact that $Z(f)_\mathfrak{q} \neq 0$ for prime ideals $\mathfrak{q}$ of height 0 (13.7).

The inequality in Thm. 12.5 (3) is a consequence of the inequality in Thm. 13.4 (2).

It remains to prove the statement about the vanishing of $\mathbf{H}^2_{\mathrm{loc}}(V_{F_\lambda}(f))_\mathfrak{p}$ in Thm. 12.5 (3). Assume $\mathbf{H}^2_{\mathrm{loc}}(V_{F_\lambda}(f))_\mathfrak{p} \neq 0$. Let T be a $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-stable O_λ-lattice of $V_{F_\lambda}(f)$. By Tate's local duality, the Pontrjagin dual of $\mathbf{H}^2_{\mathrm{loc}}(T)$ is isomorphic to

$$\mathrm{H}^0(\mathbb{Q}_p(\zeta_{p^\infty}), \mathrm{Hom}_{O_\lambda}(T, F_\lambda/O_\lambda)(1))$$

Denote this O_λ-module by C. If $\mathbf{H}^2_{\mathrm{loc}}(V_{F_\lambda}(f))_\mathfrak{p} \neq 0$, $\mathbf{H}^2_{\mathrm{loc}}(T)$ is not finite, and hence C contains an O_λ-submodule which is isomorphic to F_λ/O_λ. Since

$$\mathrm{Hom}_{O_\lambda}(F_\lambda/O_\lambda, C) \subset \mathrm{H}^0(\mathbb{Q}_p(\zeta_{p^\infty}), \mathrm{Hom}_{F_\lambda}(V_{F_\lambda}(f), F_\lambda)(1)),$$

this means that the last space is not zero. Hence $V_{F_\lambda}(f)$ has a non-zero quotient representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over F_λ on which the action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ factors through the canonical projection $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to G_\infty$. Since $V_{F_\lambda}(f)$ is of Hodge-Tate as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ [**Fa1**], it follows from [**Se2**, Chap. III, Appendix] that this quotient representation has a non-zero quotient representation U of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over F_λ such that for some $n \geqslant 0$, the action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p(\zeta_{p^n}))$ on U is given by κ^r for some $r \in \mathbb{Z}$. (κ denotes the cyclotomic character as before.) Let $\mathbb{C}_p$ be the p-adic completion of $\overline{F}_\lambda$. Then

(13.13.1) $$V_{F_\lambda}(f) \otimes_{F_\lambda} \mathbb{C}_p \simeq \mathbb{C}_p \oplus \mathbb{C}_p(1-k) \qquad [\mathbf{Fa1}].$$

If $U = V_{F_\lambda}$, we have $U \otimes_{F_\lambda} \mathbb{C}_p \simeq \mathbb{C}_p(r)^{\oplus 2}$ which is a contradiction. Hence

$$\dim_{F_\lambda}(U) = 1 \quad \text{and} \quad U \otimes_{F_\lambda} \mathbb{C}_p \simeq \mathbb{C}_p(r).$$

By (13.13.1), we have $r \in \{0, 1-k\}$. On the other hand, the Frobenius operator on $\mathrm{D}_{\mathrm{crys}}(\mathbb{Q}_p(\zeta_{p^n}), U)$ ($\mathrm{D}_{\mathrm{crys}}(\mathbb{Q}_p(\zeta_{p^n}), \ )$ means the $\mathrm{D}_{\mathrm{crys}}$ for the local field $\mathbb{Q}_p(\zeta_{p^n})$) is the multiplication by p^{-2r}, but by [**Sa2**], the eigenvalues of the Frobenius operator on $\mathrm{D}_{\mathrm{crys}}(\mathbb{Q}_p(\zeta_{p^n}), U)$ must have complex absolute value $p^{(k-1)/2}$ (resp. $p^{k/2}$) if f is (resp. is not) potentially of good reduction at p. Hence $-2r \in \{k-1, k\}$. Since $r \in \{0, 1-k\}$ and $k \geqslant 2$, this means $k = 2$ and $r = -1$ and that f is not potentially of good reduction at p. Furthermore, the O_λ-module C has O_λ-corank 1. (If it has corank 2, the action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ on $V_{F_\lambda}(f)$ factors through G_∞, and then by Serre [**Se2**, Chap. III, Appendix], $V_{F_\lambda}(f)$ should be potentially of good reduction at p.) By duality, $\mathbf{H}^2_{\mathrm{loc}}(V_{F_\lambda}(f))$ is a one-dimensional F_λ-vector space, and is isomorphic to $U(-1)$ as a Λ-module. □

13.14. We prove Thm. 12.5 (4). Since $T = a \cdot V_{O_\lambda}(f)$ for some $a \in F_\lambda^\times$ under the assumption of Thm. 12.5 (4) (see 12.8), we may assume $T = V_{O_\lambda}(f)$. In this case, since $Z(f,T)/Z$ is a finite group, $Z(f,T)_\mathfrak{p} \subset \mathbf{H}^1(T)_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of Λ of height one. Since $\mathbf{H}^1(T)$ is a free Λ-module under the assumption of 12.5 (4) by 12.4 (2), this means $Z(f,T) \subset \mathbf{H}^1(T)$. The inequality in Thm. 12.5 (4) follows from Thm. 13.4 (3). □

14. Finiteness of Selmer groups and Tamagawa number conjectures

In this section, we prove results on finiteness of Selmer groups associated to modular forms (Thm. 14.2). The proof is given completely in this section in the case f has no CM and the proof for the CM case will be completed in § 15. Thm. 14.2 in the CM case has been proved in many cases (Rubin [**Ru2**], Guo [**Gu**], Han [**Ha**], Dee [**DJ**],...). We also consider in this section the Tamagawa number conjecture for modular forms (Thm. 14.5).

14.1. We define the Selmer groups of p-adic Galois representations of number fields by the method of [**BK2**], as follows.

Let K be a finite extension of $\mathbb{Q}$, let p be a prime number, and let T be a free $\mathbb{Z}_p$-module of finite rank endowed with a continuous actions of $\mathrm{Gal}(\overline{K}/K)$. We define the Selmer group $\mathrm{Sel}(K,T) \subset \mathrm{H}^1(K, T \otimes \mathbb{Q}/\mathbb{Z})$ by

$$\mathrm{Sel}(K,T) = \mathrm{Ker}(\mathrm{H}^1(K, T\otimes\mathbb{Q}/\mathbb{Z}) \longrightarrow \bigoplus_v \mathrm{H}^1(K_v, T\otimes\mathbb{Q}/\mathbb{Z})/\,\mathrm{Image}(\mathrm{H}^1_f(K_v, T\otimes\mathbb{Q})))$$

where v ranges over all places of K, and H^1_f is as in [**BK2**, § 3]. (The notation f in H^1_f has nothing to do with our cusp form f.) We review the definition of H^1_f. For a finite dimensional $\mathbb{Q}_p$-vector space V endowed with a continuous action of $\mathrm{Gal}(\overline{K}_v/K_v)$, the subspace $\mathrm{H}^1_f(K_v, V)$ of $\mathrm{H}^1(K_v, V)$ is defined by

$$\mathrm{H}^1_f(K_v, V) = \begin{cases} \mathrm{H}^1(K_v, V) = 0 & \text{if } v \text{ is archimedian,} \\ \mathrm{Ker}(\mathrm{H}^1(K_v, V) \to \mathrm{H}^1(K_v^{\mathrm{ur}}, V)) & \text{if } v \text{ is a finite place not lying over } p, \\ \mathrm{Ker}(\mathrm{H}^1(K_v, V) \to \mathrm{H}^1(K_v, \mathrm{B}_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)) & \\ & \text{if } v \text{ is a finite place lying over } p. \end{cases}$$

Here K_v^{ur} denotes the maximal unramified extension of F_v.

If A is an abelian variety over K, the usual Selmer group $\mathrm{Sel}(K, A)$ of A coincides with $\bigoplus_p \mathrm{Sel}(K, T_p(A))$ where p ranges over all prime numbers.

In the case $K = \mathbb{Q}$, we denote $\mathrm{Sel}(\mathbb{Q}, T)$ simply by $\mathrm{Sel}(T)$.

In this section, we prove

Theorem 14.2. — *Let K be a finite abelian extension of $\mathbb{Q}$.*

(1) *Let r be an integer such that $1 \leqslant r \leqslant k-1$ and $r \neq k/2$. Then for any finite place λ of F and any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice T of $V_{F_\lambda}(f)(r)$, $\mathrm{Sel}(K,T)$ is finite. For almost all finite places λ of F, $\mathrm{Sel}(K,T) = 0$ for any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice T of $V_{F_\lambda}(f)(r)$.*

(2) *Assume k is even. Let $\chi : \mathrm{Gal}(K/\mathbb{Q}) \to \mathbb{C}^\times$ be a character, and assume $L(f, \chi, k/2) \neq 0$. Then for any finite place λ of F and any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice T of $V_{F_\lambda}(f)(k/2)$, the "χ-part" $\mathrm{Sel}(K,T)^{(\chi)}$ of $\mathrm{Sel}(K,T)$ is finite. For almost all finite places λ of F, $\mathrm{Sel}(K,T)^{(\chi)} = 0$ for any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice T of $V_{F_\lambda}(f)(k/2)$.*

The above "χ-part" is defined as follows. Let $G = \mathrm{Gal}(K/\mathbb{Q})$, and let $I_\chi \subset \mathbb{Z}[G]$ be the kernel of the ring homomorphism $\mathbb{Z}[G] \to \overline{\mathbb{Q}}$ induced by χ. Then, for a G-module M, the χ-part $M^{(\chi)}$ of M is defined by

$$M^{(\chi)} = \{x \in M \,;\, I_\chi \cdot x = 0\}.$$

In 14.2 (2), $L(f, \chi, s)$ means $L_S(f, \chi, s)$ in which we identify χ with the composite homomorphism

$$(\mathbb{Z}/m)^\times \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \longrightarrow \mathrm{Gal}(K/\mathbb{Q}) \xrightarrow{\ \chi\ } \mathbb{C}^\times$$

for the smallest integer $m \geqslant 1$ such that $K \subset \mathbb{Q}(\zeta_m)$, and $S = \mathrm{prime}(m) = \{\text{primes which ramify in } K/\mathbb{Q}\}$.

Corollary 14.3. — *Let A be an abelian variety over $\mathbb{Q}$ such that there is a surjective homomorphism $J_1(N) \to A$ for some $N \geqslant 1$, where $J_1(N)$ denotes the Jacobian variety of $X_1(N)$. Let K be a finite abelian extension of $\mathbb{Q}$, let $\chi : \mathrm{Gal}(K/\mathbb{Q}) \to \mathbb{C}^\times$ be a character, and assume $L(A, \chi, 1) \neq 0$. Then:*

(1) *The χ-part $\mathrm{Sel}(K, A \otimes_{\mathbb{Q}} K)^{(\chi)}$ of $\mathrm{Sel}(K, A \otimes_{\mathbb{Q}} K)$ is finite.*

(2) *The χ-part of $A(K)^{(\chi)}$ is finite.*

In the case $J_1(N)$ is replaced by $J_0(N)$ and $K=\mathbb{Q}$, this result is contained in the work of Kolyvagin-Logachov [**KoL**]. In the case A is an elliptic curve with complex multiplication, this result is contained in the work of Rubin [**Ru1**] which generalized the work of Coates-Wiles [**CW**].

We can replace the "χ-parts" $\mathrm{Sel}(K,T)^{(\chi)}$, $\mathrm{Sel}(K,A\otimes_{\mathbb{Q}}K)^{(\chi)}$ and $A(K)^{(\chi)}$ in 14.2 and 14.3 by the "χ-quotients" $\mathrm{Sel}(K,T)_{(\chi)}$, $\mathrm{Sel}(K,A\otimes_{\mathbb{Q}}K)_{(\chi)}$ and $A(K)_{(\chi)}$, respectively, where $M_{(\chi)}=M/I_\chi M$ for a G-module M. This is because the kernel and the cokernel of the canonical map $M^{(\chi)}\to M_{(\chi)}$ are killed by some non-zero integer, and because for $M=\mathrm{Sel}(K,T),\mathrm{Sel}(K,A\otimes_{\mathbb{Q}}K)$ or $A(K)$, the kernel and the cokernel of $n:M\to M$ are finite for any non-zero integer n.

I learned from Professor John Coates that the following result is deduced from Cor. 14.3 by using the theorem of Rohrlich introduced in 13.5 (2).

Theorem 14.4. — *Let A be an abelian variety over $\mathbb{Q}$ such that there is a surjective homomorphism $J_1(N)\to A$ for some $N\geqslant 1$. Then for any $m\geqslant 1$, $\bigcup_n A(\mathbb{Q}(\zeta_{m^n}))$ is finitely generated as an abelian group.*

The argument to deduce 14.4 from 14.2 is given in Rohrlich [**Ro1**, § 3] where he considered the case A is an elliptic curve with complex multiplication by using the result of Rubin [**Ru1**].

The "anti-cyclotomic" analogues of Thm. 14.3, Thm. 14.4 were obtained by Bertolini and Darmon [**BD**].

The following theorem is related to the Tamagawa number conjecture in [**BK2**].

Theorem 14.5. — *Let $r\in\mathbb{Z}$, $1\leqslant r\leqslant k-1$. Let p be a prime number, λ a place of F lying over p, and let T be a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice of $V_{F_\lambda}(f)(r)$. In (1) (resp. (2) and (3)) below, we assume $L(f,k/2)\neq 0$ in the case $r=k/2$ (resp. $L_{\{p\}}(f,k-r)\neq 0$ in the case $r>k/2$).*

(1) $\mathrm{H}^2(\mathbb{Z}[1/p],T)$ *is finite and* $\mathrm{rank}_{O_\lambda}(\mathrm{H}^1(\mathbb{Z}[1/p],T))=1$.

(2) *Let γ be an element of $V_{F_\lambda}(f)$, and let z be the image of $\mathbf{z}_\gamma^{(p)}$ under*

$$\mathbf{H}^1(V_{F_\lambda}(f))\simeq\mathbf{H}^1(V_{F_\lambda}(f)(r))\longrightarrow\mathrm{H}^1(\mathbb{Z}[1/p],V_{F_\lambda}(f)(r)).$$

Let $\pm=(-1)^{r-1}$. Then, if $\gamma^\pm\neq 0$, z is an F_λ-basis of $\mathrm{H}^1(\mathbb{Z}[1/p],V_{F_\lambda}(f)(r))$.

(3) *Assume $p\neq 2$. Assume either $k\geqslant 3$ or f is potentially of good reduction at p, and assume further that the condition* (12.5.1) *in Thm. 12.5 (4) is satisfied. Let γ,z, and $\pm$ be as in (2), and assume that $\gamma^\pm$ is an O_λ-basis of $T(-r)^\pm$. Then we have*

$$\#(\mathrm{H}^2(\mathbb{Z}[1/p],T))\leqslant[\mathrm{H}^1(\mathbb{Z}[1/p],T):z].$$

Here, $[\mathrm{H}^1(\mathbb{Z}[1/p],T):z]$ is defined as follows. Let L be a finite extension of $\mathbb{Q}_p$ (we take F_λ as L in the above), let M be a finitely generated O_L-module such that $\dim_L(M\otimes\mathbb{Q})=1$, and let z be a non-zero element of $M\otimes\mathbb{Q}$. Take $y\in M$ and a non-zero integer c such that $z = c^{-1}y$ in $M \otimes \mathbb{Q}$. We define

$$[M : z] = [M : O_L y] \cdot [O_L : cO_L]^{-1}$$

(then this is independent of the choices of y and c).

The Tamagawa number conjecture in [**BK2**] generalized by [**FP**], [**KK2**] predicts

$$\#(\mathrm{H}^2(\mathbb{Z}[1/p], T) = [\mathrm{H}^1(\mathbb{Z}[1/p], T) : z]$$

in 14.5 (3).

The rest of § 14 is devoted to the proofs of Thm 14.2 and Thm 14.5.

14.6. In this 14.6, we show that for the proof of Thm. 14.2, we may assume $K = \mathbb{Q}$.

Let the notation be as in Thm. 14.2. Let $G = \mathrm{Gal}(K/\mathbb{Q})$. Then the following is proved without difficulty:

$$\mathrm{Sel}(K, T) = \mathrm{Sel}(T \otimes_{\mathbb{Z}} \mathbb{Z}[G]) \ (\underset{\text{def}}{=} \mathrm{Sel}(\mathbb{Q}, T \otimes_{\mathbb{Z}} \mathbb{Z}[G]))$$

where the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathbb{Z}[G]$ is as follows:

(14.6.1) *For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, σ acts on $\mathbb{Z}[G]$ as the multiplication by the image of σ^{-1} under the canonical map $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to G$.*

The normalization of $O_F[G]$ has the form $\prod_{i\in I} O_{L_i}$ where $(L_i)_{i\in I}$ is a finite family of finite extensions of $\mathbb{Q}$. Let c be a non-zero integer which kills $(\prod_{i\in I} O_{L_i})/O_F[G]$. Then the kernel and the cokernel of

$$\mathrm{Sel}(T \otimes_{\mathbb{Z}} \mathbb{Z}[G]) \longrightarrow \mathrm{Sel}\left(T \otimes_{O_F} \textstyle\prod_{i\in I} O_{L_i}\right) = \displaystyle\bigoplus_{i\in I} \bigoplus_{v} \mathrm{Sel}(T \otimes_{O_\lambda} O_v)$$

are killed by c, where for each i, v ranges over all places of L_i lying over λ and O_v denotes the valuation ring of v, and where $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on O_{L_i} and on O_v via (14.6.1). Hence $\mathrm{Sel}(K, T)$ is finite if each $\mathrm{Sel}(T \otimes_{O_\lambda} O_v)$ is finite, and $\mathrm{Sel}(K, T) = 0$ if each $\mathrm{Sel}(T \otimes_{O_\lambda} O_v)$ is zero and λ does not divide c. Let $m \geqslant 1$ and S be as in the remark after Thm.14.2, and for $i \in I$, let

$$\nu_i : (\mathbb{Z}/m)^\times \longrightarrow \mathbb{C}^\times$$

be the composite map:

$$(\mathbb{Z}/m)^\times \longrightarrow G \longrightarrow \mathbb{Z}[G]^\times \longrightarrow (O_{L_i})^\times \subset \overline{\mathbb{Q}}^\times \subset \mathbb{C}^\times,$$

and let $f_i = \sum_{n\geqslant 1} a_{n,i} q^n$ be the normalized newform such that $L_S(f_i, s) = L_S(f, \nu_i, s)$. (Hence $a_{n,i} = a_n \nu_i(n)$ if n is prime to m.) Let $F_i = \mathbb{Q}(a_{n,i}; n \geqslant 1) \subset L_i$. Let v be a place of L_i lying over λ, and let w be the place of F_i lying over v. Then by comparing the action of Frobenius substitutions of prime numbers which are prime to Nm, we see that $V_{F_\lambda}(f) \otimes_{F_\lambda} L_{i,v}$ with the action (14.6.1) of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $L_{i,v}$ is isomorphic to $V_{F_{i,w}}(f_i) \otimes_{F_{i,w}} L_{i,v}$ with the trivial action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $L_{i,v}$, as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over $L_{i,v}$. Take any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_w-lattice T' of $V_{F_{i,w}}(f_i)$. Then both $T \otimes_{O_\lambda} O_v$ (here $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on O_v via (14.6.1)) and $T' \otimes_{O_w} O_v$ (here $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on O_v trivially) are regarded as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_v-lattice of $V_{F_\lambda}(f) \otimes_{F_\lambda} L_{i,v}$. Since $\mathrm{Sel}(T' \otimes_{O_w} O_v) = \mathrm{Sel}(T') \otimes_{O_w} O_v$ (here $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on O_v trivially), the finiteness (resp. the vanishing) of $\mathrm{Sel}(T \otimes_{O_\lambda} O_v)$ (here $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on O_v via (14.6.1)) is reduced (resp. reduced for almost all λ by Lemma 14.7 below) to the finiteness (resp. the vanishing) of $\mathrm{Sel}(T')$. Concerning the χ-part, $\mathrm{Sel}(K,T)^{(\chi)} = \mathrm{Sel}(T \otimes_{\mathbb{Z}} \mathbb{Z}[G])^{(\chi)}$. The image of $\chi : \mathbb{Q}[G] \to \mathbb{C}$ is some of the number fields L_i which appeared when we took the normalization of $\mathbb{Z}[G]$ in the above argument. The kernel and the cokernel of

$$\mathrm{Sel}(T \otimes_{\mathbb{Z}[G]} O_{L_i})^{(\chi)} \longrightarrow \mathrm{Sel}(T \otimes_{\mathbb{Z}[G]} O_{L_i})$$

are killed by the non-zero integer c which appeared in the above argument. Hence the finiteness of $\mathrm{Sel}(K,T)^{(\chi)}$ (resp. the vanishing of $\mathrm{Sel}(K,T)^{(\chi)}$ for almost all λ) is reduced by the above argument to the finiteness (resp. vanishing) of $\mathrm{Sel}(T')$ where T' is as above.

Lemma 14.7. — *Almost all finite places λ of F satisfy the following condition: For any finite extension P of F_λ and for any two $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_P-lattices T, T' of $V_{F_\lambda}(f) \otimes_{F_\lambda} P$, there exists $a \in P^\times$ such that $T' = aT$.*

Proof. — Here we give the proof in the case f has no CM. The proof for the CM case will be given in 15.19.

Assume f has no CM. Then as in (12.8.1), for almost λ, there exists an F_λ-basis (e_1, e_2) of $V_{F_\lambda}(f)$ such that $O_\lambda e_1 + O_\lambda e_2$ is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the image of the homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(O_\lambda)$ associated to this basis contains $\mathrm{SL}_2(\mathbb{Z}_p)$. We show that such λ satisfies the condition stated in 14.7. Let T be a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_P-lattice of $V_{F_\lambda}(f) \otimes_{F_\lambda} P$. Let $a_1e_1 + a_2e_2 \in T (a_i \in O_P)$. Then by applying $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}_p)$ to $a_1e_1 + a_2e_2$, we obtain $a_2e_1, a_1e_2 \in T$. By applying $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ to a_2e_1, a_1e_2, we obtain $a_1e_1, a_2e_2 \in T$. This shows that $T = a(O_Pe_1 + O_Pe_2)$ where a is a generator of the fractional O_P-ideal generated by all $a_1, a_2 \in P$ such that $a_1e_1 + a_2e_2 \in T$. □

14.8. In general, for a finite extension K of $\mathbb{Q}$ and for a free $\mathbb{Z}_p$-module T of finite rank endowed with a continuous action of $\mathrm{Gal}(\overline{K}/K)$, let

$$\mathcal{S}(K,T) = \mathrm{Ker}(\mathrm{H}^1(O_K[1/p], T \otimes \mathbb{Q}/\mathbb{Z}) \longrightarrow \bigoplus_{v|p} \mathrm{H}^1(K_v, T \otimes \mathbb{Q}/\mathbb{Z}) / \mathrm{Image}(\mathrm{H}^1_f(K_v, T \otimes \mathbb{Q}))).$$

where v ranges over all places of K lying over p. Then $\mathrm{Sel}(K,T) \subset \mathcal{S}(K,T)$ and $\mathcal{S}(K,T)/\mathrm{Sel}(K,T)$ is a finite group which is embedded into the direct sum of

$$H^1(\mathbb{F}_v, H^0(K_v^{\mathrm{ur}}, T\otimes\mathbb{Q}/\mathbb{Z}))/\mathrm{Image}(H^1_f(K_v, T\otimes\mathbb{Q}))) = H^1(\mathbb{F}_v, H^0(K_v^{\mathrm{ur}}, T\otimes\mathbb{Q}/\mathbb{Z}))/(\mathrm{div})$$

where v ranges over all finite places of K not lying over p and div denotes the divisible part. (The last group is zero if T is unramified at v.).

In the case $K = \mathbb{Q}$, we denote $\mathcal{S}(\mathbb{Q}, T)$ by $\mathcal{S}(T)$.

For the proof of Thm 14.2, it is sufficient to prove the following : Assume $1 \leqslant r \leqslant k-1$. If k is even and $r = k/2$, assume $L(f,r) \neq 0$. Then $\mathcal{S}(T)$ is finite for any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice T of $V_{F_\lambda}(f)(r)$, and for almost all λ, $\mathcal{S}(T) = 0$ for any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice T of $V_{F_\lambda}(f)(r)$.

14.9. We give preliminaries on the duality theory of étale cohomology for global and local fields ([**FL**], [**FP**], [**KK1**]).

Let K be a finite extension of $\mathbb{Q}$, let L be a finite extension of $\mathbb{Q}_p$, and let T be a finitely generated O_L-module endowed with a continuous O_L-linear action of $\mathrm{Gal}(\overline{K}/K)$ which is unramified at almost all finite places of K.

By the duality theory of Poitou-Tate [**Ta1, Ma2**], we have sequences of O_L-modules

$$(14.9.1)\quad 0 \longrightarrow \mathrm{H}^0(O_K[1/p], T) \longrightarrow \mathrm{H}^0(K \otimes \mathbb{Q}_p, T)$$
$$\longrightarrow \{\mathrm{H}^2(O_K[1/p], T^*(1) \otimes \mathbb{Q}/\mathbb{Z})\}^\vee \longrightarrow \mathrm{H}^1(O_K[1/p], T) \longrightarrow \mathrm{H}^1(K \otimes \mathbb{Q}_p, T)$$
$$\longrightarrow \{\mathrm{H}^1(O_K[1/p], T^*(1) \otimes \mathbb{Q}/\mathbb{Z})\}^\vee \longrightarrow \mathrm{H}^2(O_K[1/p], T) \longrightarrow \mathrm{H}^2(K \otimes \mathbb{Q}_p, T)$$
$$\longrightarrow \{\mathrm{H}^0(O_K[1/p], T^*(1) \otimes \mathbb{Q}/\mathbb{Z})\}^\vee \longrightarrow 0$$

$$(14.9.2)\quad 0 \longrightarrow \mathrm{H}^0(O_K[1/p], T \otimes \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^0(K \otimes \mathbb{Q}_p, T \otimes \mathbb{Q}/\mathbb{Z})$$
$$\longrightarrow \{\mathrm{H}^2(O_K[1/p], T^*(1))\}^\vee \longrightarrow \mathrm{H}^1(O_K[1/p], T \otimes \mathbb{Q}/\mathbb{Z})$$
$$\longrightarrow \mathrm{H}^1(K \otimes \mathbb{Q}_p, T \otimes \mathbb{Q}/\mathbb{Z}) \longrightarrow \{\mathrm{H}^1(O_K[1/p], T^*(1))\}^\vee$$
$$\longrightarrow \mathrm{H}^2(O_K[1/p], T \otimes \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^2(K \otimes \mathbb{Q}_p, T \otimes \mathbb{Q}/\mathbb{Z})$$
$$\longrightarrow \{\mathrm{H}^0(O_K[1/p], T^*(1))\}^\vee \longrightarrow 0$$

($T^* = \mathrm{Hom}_{O_L}(T, O_L)$ endowed with the dual action of $\mathrm{Gal}(\overline{K}/K)$, and $\{\ \}^\vee = \mathrm{Hom}_{O_L}(\ , L/O_L)$), which are exact in the case $p \neq 2$, and exact upto $\times 2$ in the case $p = 2$. Here we say that a sequence of abelian groups

$$\dots \longrightarrow C_i \xrightarrow{f_i} C_{i+1} \xrightarrow{f_{i+1}} \dots$$

is exact upto $\times 2$ if $2 \cdot \mathrm{Image}(f_i) \subset \mathrm{Ker}(f_{i+1})$ and $2 \cdot \mathrm{Ker}(f_{i+1}) \subset \mathrm{Image}(f_i)$ for all i.

We have the local Tate duality

$$\{\mathrm{H}^q(K \otimes \mathbb{Q}_p, T^*(1) \otimes \mathbb{Q}/\mathbb{Z})\}^\vee \simeq \mathrm{H}^{2-q}(K \otimes \mathbb{Q}_p, T) \qquad (q \in \mathbb{Z}).$$

If v is a place of K lying over p and $V = T \otimes \mathbb{Q}$ is de Rham as a representation of $\mathrm{Gal}(\overline{K}_v/K_v)$, we have the duality ([**BK2**], § 3)

$$\{H^1(K_v, T^*(1) \otimes \mathbb{Q}/\mathbb{Z}) / \mathrm{Image}(\mathrm{H}^1_f(K_v, V^*(1))\}^\vee \simeq \mathrm{H}^1_f(K_v, T)$$

where $\mathrm{H}^1_f(K_v, T) \subset \mathrm{H}^1(K_v, V)$ denotes the inverse image of $\mathrm{H}^1_f(K_v, V) \subset \mathrm{H}^1(K_v, V)$. Hence, if V is de Rham as a representation of $\mathrm{Gal}(\overline{K}_v/K_v)$ for any place v of K lying over p, we obtain from (14.9.1) a sequence of O_L-modules

$$(14.9.3)\quad 0 \longrightarrow \mathrm{H}^1(O_K[1/p],T)/\mathrm{H}^1_f(O_K[1/p],T) \xrightarrow{a} \mathrm{H}^1(K\otimes\mathbb{Q}_p,T)/\mathrm{H}^1_f(K\otimes\mathbb{Q}_p,T) \longrightarrow \{\mathcal{S}(K,T^*(1))\}^\vee \longrightarrow \mathrm{H}^2(O_K[1/p],T) \longrightarrow \mathrm{H}^2(K\otimes\mathbb{Q}_p,T) \longrightarrow \{\mathrm{H}^0(O_K[1/p],T^*(1)\otimes\mathbb{Q}/\mathbb{Z})\}^\vee \longrightarrow 0$$

(here we define $\mathrm{H}^1_f(K\otimes\mathbb{Q}_p,\ )$ to be the direct sum of $\mathrm{H}^1_f(K_v,\ )$ for places v of K lying over p, and $\mathrm{H}^1_f(O_K[1/p],T)$ to be the inverse image of $\mathrm{H}^1_f(K\otimes\mathbb{Q}_p,V)\subset\mathrm{H}^1(K\otimes\mathbb{Q}_p,V)$ in $\mathrm{H}^1(O_K[1/p],T)$), and from (14.9.2) a sequence of O_L-modules

$$(14.9.4)\quad 0 \longrightarrow \mathcal{S}(K,T) \longrightarrow \mathrm{H}^1(O_K[1/p],T\otimes\mathbb{Q}/\mathbb{Z}) \xrightarrow{b} \mathrm{H}^1(K\otimes\mathbb{Q}_p,T\otimes\mathbb{Q}/\mathbb{Z})/\operatorname{Image}(\mathrm{H}^1_f(K\otimes\mathbb{Q}_p,V)) \longrightarrow \{\mathrm{H}^1_f(K\otimes\mathbb{Q}_p,T^*(1))\}^\vee \longrightarrow \mathrm{H}^2(O_K[1/p],T\otimes\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^2(K\otimes\mathbb{Q}_p,T\otimes\mathbb{Q}/\mathbb{Z}) \longrightarrow \{\mathrm{H}^0(O_K[1/p],T^*(1))\}^\vee \longrightarrow 0$$

which are exact in the case $p\neq 2$, and exact upto $\times 2$ in the case $p=2$.

We will use also the following results of Euler-Poincaré characteristics which are deduced from Tate [**Ta2**, Thm. 2.2]. Assume $K=\mathbb{Q}$ (we will need only this case).

$$(14.9.5)\qquad \sum_{q\in\mathbb{Z}}(-1)^q\operatorname{rank}_{O_L}(\mathrm{H}^q(\mathbb{Z}[1/p],T)) = -\operatorname{rank}_{O_L}(T^-).$$

$$(14.9.6)\qquad \sum_{q\in\mathbb{Z}}(-1)^q\operatorname{rank}_{O_L}(\mathrm{H}^q(\mathbb{Q}_p,T)) = -\operatorname{rank}_{O_L}(T).$$

14.10. We review some basic properties of the Galois representation $V_{F_\lambda}(f)$. The following are known (see [**Ca, Sa1, Sa2**]):

(14.10.1) *$V_{F_\lambda}(f^*)$ is isomorphic to $\operatorname{Hom}_{F_\lambda}(V_{F_\lambda}(f),F_\lambda)(1-k)$ as a representation of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over F_λ. Hence for $r\in\mathbb{Z}$ and for a $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable lattice T of $V_{F_\lambda}(f)(r)$, $T^*(1)$ is isomorphic to a $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice of $V_{F_\lambda}(f^*)(k-r)$ as a representation of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over O_λ.*

(14.10.2) *The action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\det_{F_\lambda}(V_{F_\lambda}(f))$ is given by $\kappa^{1-k}\varepsilon^{-1}$ where κ is the cyclotomic character and ε is regarded as the character of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by*

$$\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \operatorname{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \simeq (\mathbb{Z}/N)^\times \xrightarrow{\varepsilon} F^\times.$$

(14.10.3) *For any prime number ℓ which is not divided by λ,*

$$\det_{F_\lambda}(1-\mathrm{Fr}_\ell^{-1}u\,;\,\mathrm{H}^0(\mathbb{Q}_\ell^{\mathrm{ur}},V_{F_\lambda}(f))) = 1-a_\ell u+\varepsilon(\ell)\ell^{k-1}u^2.$$

Here Fr_ℓ is the arithmetic Frobenius of ℓ. ($\mathrm{H}^0(\mathbb{Q}_\ell^{\mathrm{ur}},\ )$ means the fixed part by the inertia subgroup of $\operatorname{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$.)

(14.10.4) *For the prime number p lying under λ,*

$$\det_{F_\lambda}(1-\varphi u\,;\,\mathrm{D}_{\mathrm{crys}}(\mathbb{Q}_p, V_{F_\lambda}(f))) = 1 - a_p u + \varepsilon(p)p^{k-1}u^2.$$

where φ is the Frobenius operator.

(14.10.5) *Let ℓ be a prime number and let α be an element of $\overline{\mathbb{Q}}$ such that $1-\alpha u$ divides $1 - a_\ell u + \varepsilon(\ell)\ell^{k-1}u^2$. Then $|\alpha| = \ell^{(k-1)/2}$ or $|\alpha| = \ell^{(k-2)/2}$. If ℓ does not divide N, then $|\alpha| = \ell^{(k-1)/2}$. In particular, Thm. 13.5 (1) implies that $L_S(f,s)$ has no zero on $\{s \in \mathbb{C}\,;\,\mathrm{Re}(s) \geqslant \frac{k+1}{2}\}$ for any finite set of primes S.*

Proposition 14.11. — *Let $r \in \mathbb{Z}$.*

(1) *For any finite place λ of F and for any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice T of $V_{F_\lambda}(f)(r)$, $\mathrm{H}^0(\mathbb{Q}, T\otimes\mathbb{Q}/\mathbb{Z})$ is finite.*

(2) *For almost all finite places λ of F, we have $\mathrm{H}^0(\mathbb{Q}, T\otimes\mathbb{Q}/\mathbb{Z}) = 0$ for any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice T of $V_{F_\lambda}(f)(r)$.*

Proof. — (1) follows from $\mathrm{H}^0(\mathbb{Q}, V_{F_\lambda}(f)(r)) = 0$. We prove (2). Let ℓ be a prime number which does not divide N and which λ does not divide. Then by (14.10.3), the action of $1 - a_\ell\mathrm{Fr}_\ell + \varepsilon(\ell)\ell^{k-1}\mathrm{Fr}_\ell^2$ is zero on $V_{F_\lambda}(f)$, and hence $1 - a_\ell\ell^{-r}\mathrm{Fr}_\ell + \varepsilon(\ell)\ell^{k-1-2r}\mathrm{Fr}_\ell^2 = 0$ on $T\otimes\mathbb{Q}/\mathbb{Z}$. Since Fr_ℓ acts trivially on $\mathrm{H}^0(\mathbb{Q}, T\otimes\mathbb{Q}/\mathbb{Z})$, we have that $1 - a_\ell\ell^{-r} + \varepsilon(\ell)\ell^{k-1-2r} = 0$ on $\mathrm{H}^0(\mathbb{Q}, T\otimes\mathbb{Q}/\mathbb{Z})$. If we can prove there exists a prime number ℓ which does not divide N and which satisfies $1 - a_\ell\ell^{-r} + \varepsilon(\ell)\ell^{k-1-2r} \neq 0$, then we have that $\mathrm{H}^0(\mathbb{Q}, T\otimes\mathbb{Q}/\mathbb{Z}) = 0$ in the case λ does not divide $\ell(1 - a_\ell\ell^{-r} + \varepsilon(\ell)\ell^{k-1-2r})$. Assume $1 - a_\ell\ell^{-r} + \varepsilon(\ell)\ell^{k-1-2r} = 0$ for all prime numbers ℓ which do not divide N. Then

$$1 - a_\ell\ell^{-s} + \varepsilon(\ell)\ell^{k-1-2s} = (1-\ell^{r-s})(1-\varepsilon(\ell)\ell^{k-1-r-s})$$

for all prime numbers ℓ which do not divide N, and hence we have

$$L_S(f,s) = \zeta_S(s-r)L_S(\varepsilon, s-k+1+r)$$

where $S = \mathrm{prime}(N)$, which is absurd. □

Proposition 14.12. — *Assume $1 \leqslant r \leqslant k-1$. Let p be the prime number lying under λ. Then:*

(1) $\dim_{F_\lambda}(\mathrm{H}^1(\mathbb{Q}_p, V_{F_\lambda}(f)(r))) = 2$.
(2) $\dim_{F_\lambda}(\mathrm{H}^1_f(\mathbb{Q}_p, V_{F_\lambda}(f)(r))) = 1$.
(3) $\mathrm{H}^q(\mathbb{Q}_p, V_{F_\lambda}(f)(r)) = 0$ *for* $q \neq 1$.

Proof. — By the result on $\varprojlim_n \mathrm{H}^2(\mathbb{Q}_p(\zeta_{p^n}), T_\lambda(f))$ in Thm. 12.5 (3), we have $\mathrm{H}^q(\mathbb{Q}_p, V_{F_\lambda}(f)(r)) = 0$ for $q \neq 1$.

By (14.9.6), we have $\dim_{F_\lambda}(\mathrm{H}^1(\mathbb{Q}_p, V_{F_\lambda}(f)(r))) = 2$. From the exact sequence

$$\begin{aligned}0 \longrightarrow \mathrm{H}^0(\mathbb{Q}_p, V_{F_\lambda}(f)(r)) &\longrightarrow \mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f)) \\ &\longrightarrow \mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f)) \oplus \mathrm{D}_{\mathrm{dR}}(V_{F_\lambda}(f))/\mathrm{D}^r_{\mathrm{dR}}(V_{F_\lambda}(f)) \\ &\longrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, V_{F_\lambda}(f)(r)) \longrightarrow 0\end{aligned}$$

[**BK2**, § 3], we have

$$\dim_{F_\lambda}(\mathrm{H}^1_f(\mathbb{Q}_p, V_{F_\lambda}(f)(r))) = \dim_{F_\lambda}(\mathrm{D}_{\mathrm{dR}}(V_{F_\lambda}(f))/\mathrm{D}^r_{\mathrm{dR}}(V_{F_\lambda}(f))) = 1. \qquad \square$$

14.13. Let $r \in \mathbb{Z}$, $1 \leqslant r \leqslant k-1$. In the case $r = k/2$, assume $L(f, k/2) \neq 0$. In this 14.13, we prove (1)(2) of Thm. 14.5 and the finiteness of $\mathcal{S}(T)$ for any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice T of $V_{F_\lambda}(f)(r)$, by using Thm. 12.5 (3) (the proof of Thm. 12.5 (3) in the CM case will be given in § 15).

Let $\pm = (-1)^{r-1}$, let γ be an element of $V_F(f)$ such that $\gamma^\pm \neq 0$, and let z be the image of $\mathbf{z}^{(p)}_\gamma$ under the composition

$$\mathbf{H}^1(V_{F_\lambda}(f)) \xrightarrow{\sim} \mathbf{H}^1(V_{F_\lambda}(f))(r) \longrightarrow \mathrm{H}^1(\mathbb{Z}[1/p], V_{F_\lambda}(f)(r)). \tag{14.13.1}$$

Then the image of z under

$$\mathrm{H}^1(\mathbb{Z}[1/p], V_{F_\lambda}(f)(r)) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, V_{F_\lambda}(f)(r))/\mathrm{H}^1_f(\mathbb{Q}_p, V_{F_\lambda}(f)(r)) \xrightarrow{\exp^*} S(f) \otimes_F F_\lambda$$

is an element of $S(f)$ whose image under $\mathrm{per}_f : S(f) \to V_{\mathbb{C}}(f)^\pm$ coincides with $L_{\{p\}}(f, k-r)\cdot\gamma^\pm$. ($\exp^*$ kills $\mathrm{H}^1_f(\mathbb{Q}_p,)$; [**BK2**, § 3]). This shows that if $L_{\{p\}}(f, k-r) \neq 0$, the image of $\mathbf{z}^{(p)}_\gamma$ in $\mathrm{H}^1(\mathbb{Q}_p, V_{F_\lambda}(f)(r))/\mathrm{H}^1_f(\mathbb{Q}_p, V_{F_\lambda}(f)(r))$ is not zero.

We prove the finiteness of $\mathrm{H}^2(\mathbb{Z}[1/p], T)$ in the case $r \leqslant k/2$. Let $\mathfrak{p}$ be the kernel of the O_λ-homomorphism

$$\Lambda \longrightarrow O_\lambda$$

which sends σ_c ($c \in \mathbb{Z}_p{}^\times$) to c^{-r}. Then the map (14.13.1) factors through $\mathbf{H}^1(V_{F_\lambda}(f))_{\mathfrak{p}}/\mathfrak{p}\mathbf{H}^1(V_{F_\lambda}(f))_{\mathfrak{p}}$. Since $L_{\{p\}}(f, k-r) \neq 0$ by 13.5 (1) and 14.10.5, we see that the image of $\mathbf{z}^{(p)}_\gamma$ in $\mathbf{H}^1(V_{F_\lambda}(f))$ is a $\Lambda_{\mathfrak{p}}$-basis of $\mathbf{H}^1(V_{F_\lambda}(f))_{\mathfrak{p}}$. Hence $\mathbf{H}^2(V_{F_\lambda}(f))_{\mathfrak{p}} = 0$ by Thm. 12.5 (3). Since

$$\mathrm{H}^2(\mathbb{Z}[1/p], V_{F_\lambda}(f)(r)) \simeq \mathbf{H}^2(V_{F_\lambda}(f))_{\mathfrak{p}}/\mathfrak{p}\mathbf{H}^2(V_{F_\lambda}(f))_{\mathfrak{p}}(r),$$

we have $\mathrm{H}^2(\mathbb{Z}[1/p], V_{F_\lambda}(f)(r)) = 0$. This proves the finiteness of $\mathrm{H}^2(\mathbb{Z}[1/p], T)$.

We prove $\mathrm{rank}_{O_\lambda}(\mathrm{H}^1(\mathbb{Z}[1/p], T)) = 1$ in the case $r \leqslant k/2$. This follows from the finiteness of $\mathrm{H}^2(\mathbb{Z}[1/p], T)$ and $\mathrm{H}^0(\mathbb{Z}[1/p], T) = 0$ by (14.9.5).

We prove the finiteness of $\mathcal{S}(T)$ in the case $r \geqslant k/2$. By the duality (14.10.1), it is sufficient to prove the finiteness of $\mathcal{S}(T^*(1))$ in the case $r \leqslant k/2$. Consider the sequence (14.9.3) (we put $K = \mathbb{Q}$). Since the image of $z \in \mathrm{H}^1(\mathbb{Z}[1/p], V_{F_\lambda}(f)(r))$ in the one dimensional F_λ-vector space $\mathrm{H}^1(\mathbb{Q}_p, V_{F_\lambda}(f)(r))/\mathrm{H}^1_f(\mathbb{Q}_p, V_{F_\lambda}(f))$ is not zero, the cokernel of

$$\mathrm{H}^1(\mathbb{Z}[1/p], T) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, T)/\mathrm{H}^1_f(\mathbb{Q}_p, T))$$

is finite. Furthermore, $\mathrm{H}^2(\mathbb{Z}[1/p], T)$ is finite as we have already seen. Hence by the sequence (14.9.3), we obtain the finiteness of $\mathcal{S}(T^*(1))$.

We prove the finiteness of $\mathcal{S}(T)$ in the case $r \leqslant k/2$. Consider the part

$$(14.13.2)\quad 0 \longrightarrow \mathcal{S}(T) \longrightarrow \mathrm{H}^1(\mathbb{Z}[1/p], T\otimes\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, T\otimes\mathbb{Q}/\mathbb{Z})/\operatorname{Image}(\mathrm{H}^1_f(\mathbb{Q}_p, V))$$

of (14.9.4) (we put $K = \mathbb{Q}$). Since $\mathrm{H}^1(\mathbb{Z}[1/p], T)$ has O_λ-rank 1 as we have seen, $\mathrm{H}^1(\mathbb{Z}[1/p], T\otimes\mathbb{Q}/\mathbb{Z})$ has O_λ-corank 1. The last group in (14.13.2) also has O_λ-corank 1 by 14.12 (1) (2). By the fact that the image of z in

$$\mathrm{H}^1(\mathbb{Q}_p, V_{F_\lambda}(f)(r))/\mathrm{H}^1_f(\mathbb{Q}_p, V_{F_\lambda}(f)(r))$$

is not zero, the last arrow in (14.13.2) has finite cokernel. Hence we have the finiteness of $\mathcal{S}(T)$.

We prove the finiteness of $\mathrm{H}^2(\mathbb{Z}[1/p], T)$ in the case $r \geqslant k/2$. By the sequence (14.9.3), this follows from the finiteness of $\mathcal{S}(T^*(1))$ and the finiteness of $\mathrm{H}^2(\mathbb{Q}_p, T)$.

We prove $\operatorname{rank}_{O_\lambda}(\mathrm{H}^1(\mathbb{Z}[1/p], T)) = 1$ in the case $r \geqslant k/2$. This follows from the finiteness of $\mathrm{H}^2(\mathbb{Z}[1/p], T)$ and $\mathrm{H}^0(\mathbb{Z}[1/p], T) = 0$ by (14.9.5). $\square$

14.14. We prove Thm. 14.5 (3).

Let $\mathfrak{p}$ be the kernel of the O_λ-homomorphism $\Lambda \to O_\lambda$ which sends G_∞ to 1. Since Λ is a finite product of regular local rings, $\mathfrak{p}$ is a principal ideal. Let a be a generator of $\mathfrak{p}$. By the argument as in 13.8, we have an exact sequence

$$(14.14.1)\qquad 0 \longrightarrow \mathbf{H}^1(T)/a\mathbf{H}^1(T) \longrightarrow \mathrm{H}^1(\mathbb{Z}[1/p], T) \longrightarrow {}_a\mathbf{H}^2(T) \longrightarrow 0$$

and an isomorphism

$$(14.14.2)\qquad \mathbf{H}^2(T)/a\mathbf{H}^2(T) \simeq \mathrm{H}^2(\mathbb{Z}[1/p], T).$$

By Lemma 14.15 below which we apply by taking Λ as A and $\mathbf{H}^2(T)$ and $\mathbf{H}^1(T)/Z(f,T)$ as M, we obtain from 12.5 (4)

$$\#(\mathbf{H}^2(T)/a\mathbf{H}^2(T)) \cdot \#({}_a\mathbf{H}^2(T))^{-1} \leqslant [\mathbf{H}^1(T)/a\mathbf{H}^1(T) : z]$$

where ${}_a(\ )$ denotes $\operatorname{Ker}(a)$ and z is the image of $\mathbf{z}_\gamma^{(p)}$ under (14.13.1) for an O_λ-basis γ of $T(-r)^-$. Hence

$$\begin{aligned}
\#(\mathrm{H}^2(\mathbb{Z}[1/p], T)) &= \#(\mathbf{H}^2(T)/a\mathbf{H}^2(T)) && (14.14.2)\\
&\leqslant \#({}_a\mathbf{H}^2(T)) \cdot [\mathbf{H}^1(T)/a\mathbf{H}^1(T) : z] &&\\
&= [\mathrm{H}^1(\mathbb{Z}[1/p], T) : z] && (14.14.1).
\end{aligned}$$

Lemma 14.15. — *Let A be a Noetherian commutative ring, let C be the category of finitely generated A-modules M such that the support of M in* $\operatorname{Spec}(A)$ *is of codimension $\geqslant 2$, and let $G(C)$ be the Grothendieck group of the abelian category C. Let M be a finitely generated A-module whose support is of codimension $\geqslant 1$, let $a \in A$, and* *assume that* $M_{\mathfrak{p}} = 0$ *for any prime ideal* $\mathfrak{p}$ *of height one which contains* a. *Then* M/aM *and* ${}_aM = \mathrm{Ker}(a : M \to M)$ *belongs to* C, *and we have*

$$[M/aM] - [{}_aM] = \sum_{\mathfrak{q}} \mathrm{length}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}).[A/(\mathfrak{q} + aA)]$$

in $G(C)$, *where* $\mathfrak{q}$ *ranges over all prime ideals of* A *of height one which do not contain* a, *and where* $[\,]$ *denotes the class in* $G(C)$.

Proof. — The A-module M has a finite filtration whose graded quotients satisfy the following (i) or (ii) :

(i) It belongs to C.

(ii) It is isomorphic to $A/\mathfrak{q}$ for a prime ideal $\mathfrak{q}$ of A of height one which does not contain a.

Hence we are reduced to the case M itself satisfies (i) or (ii). □

***Proposition 14.16* (See Flach [Fla] for a more general study).** — *Let* $r \in \mathbb{Z}$, $1 \leqslant r \leqslant k-1$. *In the case* $r = k/2$, *assume* $L(f, k/2) \neq 0$. *Let* T *be a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-stable* O_λ*-lattice of* $V_{F_\lambda}(f)(r)$. *Then*

(1) $\#(\mathcal{S}(T)) = \#(\mathcal{S}(T^*(1)))$

(2) *Assume* $r \leqslant k/2$, *let* $\gamma \in V_{F_\lambda}(f)$, *and let* $\pm = (-1)^{r-1}$, *and assume* $\gamma^{\pm}$ *is an* O_λ*-basis of* $T(-r)^{\pm} = T^{-}(-r)$ *and let* $z \in \mathrm{H}^1(\mathbb{Z}[1/p], V_{F_\lambda}(f)(r))$ *be the image of* $\mathbf{z}_\gamma^{(p)}$ *under* (14.13.1). *Then*

$$\#(\mathcal{S}(T)) = \mu^{-1} \cdot \nu \cdot \#(\mathrm{H}^0(\mathbb{Q}, T \otimes \mathbb{Q}/\mathbb{Z})) \cdot \#(\mathrm{H}^0(\mathbb{Q}, T^*(1) \otimes \mathbb{Q}/\mathbb{Z}))$$

where

$$\mu = [\mathrm{H}^1(\mathbb{Z}[1/p], T) : z] \cdot \#(\mathrm{H}^2(\mathbb{Z}[1/p], T))^{-1},$$
$$\nu = [\mathrm{H}^1(\mathbb{Q}_p, T)/\mathrm{H}^1_f(\mathbb{Q}_p, T) : z] \cdot \#(\mathrm{H}^2(\mathbb{Q}_p, T))^{-1}.$$

Here in the definition of ν, *we denote the image of* z *in* $\mathrm{H}^1(\mathbb{Q}_p, T)/\mathrm{H}^1_f(\mathbb{Q}_p, T)$ *by the same letter* z.

Proof. — By the duality (14.10.1), we may assume $r \leqslant k/2$ in the proof of (1). Hence we assume $r \leqslant k/2$.

Consider the exact sequence (14.9.3) (we put $K = \mathbb{Q}$). Since $\mathrm{H}^1(\mathbb{Z}[1/p], T)$ and $\mathrm{H}^1(\mathbb{Z}[1/p], T)/\mathrm{H}^1_f(\mathbb{Z}[1/p], T)$ are of rank 1 over O_λ and the latter is torsion free, we have that $\mathrm{H}^1_f(\mathbb{Z}[1/p], T)$ coincides with the torsion part $\mathrm{H}^0(\mathbb{Q}, T \otimes \mathbb{Q}/\mathbb{Z})$ of $\mathrm{H}^1(\mathbb{Z}[1/p], T)$. Hence by the exact sequence (14.9.3), we obtain

$$\#(\mathcal{S}(T^*(1))) = \mu^{-1} \cdot \nu \cdot \#(\mathrm{H}^0(\mathbb{Q}, T \otimes \mathbb{Q}/\mathbb{Z})) \cdot \#(\mathrm{H}^0(\mathbb{Q}, T^*(1) \otimes \mathbb{Q}/\mathbb{Z}))$$

For a homomorphism h of abelian groups whose kernel and the cokernel are finite, let

$$[h] = \#(\mathrm{Coker}(h)) \cdot \#(\mathrm{Ker}(h))^{-1}.$$

Let a, b be the arrows as in (14.9.3), (14.9.4). Then the kernels and the cokernels of a, b are finite as we have seen. By the exact sequences

$$0 \longrightarrow \mathrm{H}^1(\mathbb{Z}[1/p], T)/\mathrm{H}^1_f(\mathbb{Z}[1/p], T) \otimes \mathbb{Q}/\mathbb{Z} \\ \longrightarrow \mathrm{H}^1(\mathbb{Z}[1/p], T) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{H}^2(\mathbb{Z}[1/p], T) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, T)/\mathrm{H}^1_f(\mathbb{Q}_p, T) \otimes \mathbb{Q}/\mathbb{Z} \\ \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, T \otimes \mathbb{Q}/\mathbb{Z})/\operatorname{Image}(\mathrm{H}^1_f(\mathbb{Q}_p, V)) \longrightarrow \mathrm{H}^2(\mathbb{Q}_p, T) \longrightarrow 0,$$

we have

$$[a] \cdot [b] = \#(\mathrm{H}^2(\mathbb{Q}_p, T)) \cdot \#(\mathrm{H}^2(\mathbb{Z}[1/p], T))^{-1}. \tag{14.16.1}$$

Consider the sequence (14.9.4). Since $\mathrm{H}^2(\mathbb{Z}[1/p], T)$ is finite, and since the p-cohomological dimension of $\mathrm{Spec}(\mathbb{Z}[1/p])$ is 2, we have

$$\mathrm{H}^2(\mathbb{Z}[1/p], T \otimes \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{H}^2(\mathbb{Z}[1/p], T) \otimes \mathbb{Q}/\mathbb{Z} = 0.$$

By the finiteness of $\mathcal{S}(T^*(1))$, we have that $\mathrm{H}^1_f(\mathbb{Z}[1/p], T^*(1))$ is finite and hence

$$\mathrm{H}^1_f(\mathbb{Z}[1/p], T^*(1)) \simeq \mathrm{H}^0(\mathbb{Z}[1/p], T^*(1) \otimes \mathbb{Q}/\mathbb{Z}).$$

Hence by the exact sequence (14.9.4), we obtain

$$\#(\mathcal{S}(T)) = [b]^{-1} \cdot \#(\mathrm{H}^0(\mathbb{Z}[1/p], T^*(1) \otimes \mathbb{Q}/\mathbb{Z})). \tag{14.16.2}$$

On the other hand, by the exact sequence (14.9.3), we have

(14.16.3)

$$\#(\mathcal{S}(T^*(1))) = [a] \cdot \#(\mathrm{H}^2(\mathbb{Z}[1/p], T)) \cdot \#(\mathrm{H}^2(\mathbb{Q}_p, T))^{-1} \cdot \#(\mathrm{H}^0(\mathbb{Z}[1/p], T^*(1) \otimes \mathbb{Q}/\mathbb{Z})).$$

By (14.16.1), (14.16.2) and (14.16.3), we have

$$\#(\mathcal{S}(T)) = \#(\mathcal{S}(T^*(1))). \qquad \square$$

14.17. We next relate the number ν in Prop. 14.16 to the λ-adic absolute value of (zeta value)/(period), by the method in [**BK2**, § 4] basing on the theory of Fontaine-Lafaille in [**FL**].

We review necessary things in [**FL**].

A filtered Dieudonné module D (called simply "filtered module" by Fontaine, and called "Fontaine module" by Ogus) over $\mathbb{Z}_p$ is a $\mathbb{Z}_p$-module of finite type endowed with

- a decreasing filtration $(D^i)_{i\in\mathbb{Z}}$ where the D^i are direct summands of D,
- a family of homomorphisms $\varphi_i : D^i \to D$,

satisfying the following (i)-(iii).

(i) $D^i = D$ for $i \ll 0$ and $D^i = 0$ for $i \gg 0$.
(ii) $\varphi_i|_{D^{i+1}} = p\varphi_{i+1}$.
(iii) $D = \sum_{i\in\mathbb{Z}} \varphi_i(D^i)$.

The category of filtered Dieudonné modules over $\mathbb{Z}_p$ is abelian.

The condition (ii) shows that there is a unique homomorphism

$$\varphi : D \otimes \mathbb{Q} \longrightarrow D \otimes \mathbb{Q}$$

over $\mathbb{Q}_p$ such that φ_i on $D^i \otimes \mathbb{Q}$ coincides with the restriction of $p^{-i}\varphi$. This map φ is bijective.

To a filtered Dieudonné module D satisfying the condition

(14.17.1) $D^i = D$ *and* $D^j = 0$ *for some integers* i, j *such that* $j - i < p$,

Fontaine and Lafaille [**FL**] associated a finitely generated $\mathbb{Z}_p$-module $T(D)$ endowed with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, as follows. Take $r \in \mathbb{Z}$ such that $D^{r+1} = 0$ and $D^{r+2-p} = D$. Define

$$T(D) = \mathrm{Ker}(1 - \varphi_r : \mathrm{fil}^r(B_\infty(\overline{\mathbb{Z}_p}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} D) \to B_\infty(\overline{\mathbb{Z}_p}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} D) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-r)$$

where $\overline{\mathbb{Z}_p}$ is the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}_p}$,

$$\mathrm{fil}^r(B_\infty(\overline{\mathbb{Z}_p}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} D) = \sum_{i \geqslant 0} J(\overline{\mathbb{Z}_p}/\mathbb{Z}_p)^{[i]} \otimes D^{r-i} \subset B_\infty(\overline{\mathbb{Z}_p}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} D$$

(with notation as in 10.1) and φ_r is the unique homomorphism which coincides with $p^{-i}\varphi \otimes \varphi_{r-i}$ on $J(\overline{\mathbb{Z}_p}/\mathbb{Z}_p)^{[i]} \otimes D^{r-i}$ for $0 \leqslant i \leqslant p-2$. (For $0 \leqslant i \leqslant p-1$, since $\varphi(J(\overline{\mathbb{Z}_p}/\mathbb{Z}_p)^{[i]}) \subset p^i B_\infty(\overline{\mathbb{Z}_p}/\mathbb{Z}_p)$ and $B_\infty(\overline{\mathbb{Z}_p}/\mathbb{Z}_p)$ is torsion free [**FM**], $p^{-i}\varphi : J(\overline{\mathbb{Z}_p}/\mathbb{Z}_p)^{[i]} \to B_\infty(\overline{\mathbb{Z}_p}/\mathbb{Z}_p)$ is defined.) Then $T(D)$ is independent of the choice of r as above; for two choices r, r' such that $r \leqslant r'$, the canonical map $x \mapsto t^{r'-r}x \otimes t^{r-r'}$ with t a basis of $\mathbb{Z}_p(1)$ is an isomorphism from $T(D)$ defined by using r onto $T(D)$ defined by using r'.

By [**FL**], we have:

(14.17.2) *The functor* $D \mapsto T(D)$ *for* D *satisfying* (14.17.1) *with fixed* i *and* j *is exact and fully faithful. If* D *satisfies* (14.17.1) *for* (i, j) *and* D' *satisfies* (14.17.1) *for* (i', j') $(j-i < p, j'-i' < p)$, *and if* $(j+j')-(i+i') < p$ *(resp.* $(j'-i)-(i'+j-1) < p$*), then*

$$T(D \otimes_{\mathbb{Z}_p} D') \simeq T(D) \otimes_{\mathbb{Z}_p} T(D')$$
$$T(\mathrm{Hom}_{\mathbb{Z}_p}(D, D')) \simeq \mathrm{Hom}_{\mathbb{Z}_p}(T(D), T(D'))$$

Here the filtered Dieudonné module $D \otimes_{\mathbb{Z}_p} D'$ is defined in the evident way and the filtered Dieudonné module $\mathrm{Hom}_{\mathbb{Z}_p}(D, D')$ is defined by

$$\mathrm{Hom}_{\mathbb{Z}_p}(D, D')^i = \{h \in \mathrm{Hom}_{\mathbb{Z}_p}(D, D') \,;\, h(D^j) \subset D^{i+j} \text{ for all } j\}$$

$$\varphi_i(h)\ (h \in \mathrm{Hom}_{\mathbb{Z}_p}(D, D')^i) : \sum_j \varphi_j(x_j)\ (x_j \in D^j) \longmapsto \sum_j \varphi_{i+j} h(x_j).$$

(We will use these for Tate twists (a special case of tensor products) and for the dual $\mathrm{Hom}_{\mathbb{Z}_p}(\ , \mathbb{Z}_p)$.)

(14.17.3) *$T(D)\otimes\mathbb{Q}$ is a de Rham representation of* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ *and the canonical map $T(D)\to \mathrm{B}_{\mathrm{dR}}\otimes D$ induces an isomorphism*

$$\mathrm{D}_{\mathrm{dR}}(T(D)\otimes\mathbb{Q}) \xrightarrow{\sim} D\otimes\mathbb{Q}$$

which gives an isomorphism of filtrations.

In the following 14.18, for finitely generated $\mathbb{Z}_p$-modules M_1, M_2 and for an isomorphism of $\mathbb{Q}_p$-modules

$$h: M_1\otimes\mathbb{Q}\longrightarrow M_2\otimes\mathbb{Q},$$

let $[h]$ (denoted also by $[h:\ M_1 \dashrightarrow M_2\ ]$) be the number defined as follows. Take a $\mathbb{Z}_p$-lattice M_3 in $M_2\otimes\mathbb{Q}$ such that the image of M_1 in $M_2\otimes\mathbb{Q}$ and the canonical image of M_2 in $M_2\otimes\mathbb{Q}$ are contained in M_3. Define

$$\begin{aligned}[h] = \#(\mathrm{Coker}(h: M_1\to M_3))\cdot\#(\mathrm{Ker}(h: M_1\to M_3))^{-1}\\ \cdot(\mathrm{Coker}(M_2\to M_3))^{-1}\cdot\#(\mathrm{Ker}(M_2\to M_3)).\end{aligned}$$

Then $[h]$ is independent of the choice of M_3. In the case h comes from a $\mathbb{Q}_p$-homomorphism $\widetilde{h}: M_1\to M_2$ whose kernel and cokernel are finite, $[h]=\#(\mathrm{Coker}(\widetilde{h}))\cdot \#(\mathrm{Ker}(\widetilde{h}))^{-1} = [\widetilde{h}]$ with $[\widetilde{h}]$ as in the proof of Prop. 14.16.

Lemma 14.18. — *Let D be a torsion free filtered Dieudonné module over $\mathbb{Z}_p$, and let D' be the filtered Dieudonné module* $\mathrm{Hom}_{\mathbb{Z}_p}(D,\mathbb{Z}_p)(1)$. *Assume the following (i) and (ii).*

(i) *There are integers i, j such that $i\leqslant 0\leqslant j$, $D^i = D, D^j = 0$ and $j-i<p$.*
(ii) *The map $1-\varphi: D'\otimes\mathbb{Q}\to D'\otimes\mathbb{Q}$ is bijective*

Then

$$\exp^*: \mathrm{H}^1(\mathbb{Q}_p, T(D)\otimes\mathbb{Q})/\mathrm{H}^1_f(\mathbb{Q}_p, T(D)\otimes\mathbb{Q})\longrightarrow D^0\otimes\mathbb{Q}$$

is bijective, $\mathrm{H}^2(\mathbb{Q}_p, T(D))$ *is finite, and*

$$\begin{aligned}[\exp^*:\ \mathrm{H}^1(\mathbb{Q}_p, T(D))/\mathrm{H}^1_f(\mathbb{Q}_p, T(D)) \dashrightarrow D^0]\ \cdot\#(\mathrm{H}^2(\mathbb{Q}_p, T(D)))\\ = [1-\varphi:\ D'\dashrightarrow D']\ .\end{aligned}$$

(The condition on $1-\varphi$ of $D'\otimes\mathbb{Q}$ in Lemma 14.18 (ii) is equivalent to the bijectivity of $1-p^{-1}\varphi^{-1}: D\otimes\mathbb{Q}\to D\otimes\mathbb{Q}$. We have

$$[1-\varphi:\ D'\dashrightarrow D']\ = [1-p^{-1}\varphi^{-1}:\ D\dashrightarrow D]\ .)$$

Proof. — Lemma 14.18 is the dual formulation of [**BK2**, Thm 4.1] ; the map $\exp^*$ in 14.18 is the $\mathbb{Q}_p$-dual of

$$\exp: D'/(D')^0\otimes\mathbb{Q}\longrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, T(D'))\otimes\mathbb{Q},$$

and hence is bijective by [**BK2**, Thm 4.1], and $\mathrm{H}^2(\mathbb{Q}_p, T(D)) \otimes \mathbb{Q}$ is the $\mathbb{Q}_p$-dual of $\mathrm{H}^0(\mathbb{Q}_p, T(D')) \otimes \mathbb{Q} = 0$. Furthermore [**BK2**, Thm 4.1 (iii)] shows

$$[\exp:\ D'/(D')^0 \dashrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, T(D'))] = [1-\varphi:\ D' \dashrightarrow D'] .$$

Let

$$P = \mathrm{H}^1(\mathbb{Q}_p, T(D) \otimes \mathbb{Q}/\mathbb{Z}) / \operatorname{Image}(\mathrm{H}^1(\mathbb{Q}_p, T(D) \otimes \mathbb{Q})).$$

Then we have an exact sequence

$$0 \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, T(D))/\mathrm{H}^1_f(\mathbb{Q}_p, T(D)) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow P \longrightarrow \mathrm{H}^2(\mathbb{Q}_p, T(D)) \longrightarrow 0.$$

By taking $\operatorname{Hom}(\ ,\mathbb{Q}/\mathbb{Z})$ of this exact sequence and by using the duality

$$\mathrm{H}^1_f(\mathbb{Q}_p, T(D')) \simeq \operatorname{Hom}(P, \mathbb{Q}/\mathbb{Z})$$

[**BK2**, Prop. 3.8], we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}(\mathrm{H}^2(\mathbb{Q}_p, T(D)), \mathbb{Q}/\mathbb{Z}) &\longrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, T(D')) \\ &\longrightarrow \operatorname{Hom}(\mathrm{H}^1(\mathbb{Q}_p, T(D))/\mathrm{H}^1_f(\mathbb{Q}_p, T(D)), \mathbb{Z}_p) \longrightarrow 0. \end{aligned}$$

From this, we have

$$\begin{aligned} &[\exp^*:\ \mathrm{H}^1(\mathbb{Q}_p, T(D))/\mathrm{H}^1_f(\mathbb{Q}_p, T(D)) \dashrightarrow D^0] \\ &= [\ D'/(D')^0 \dashrightarrow \operatorname{Hom}(\mathrm{H}^1(\mathbb{Q}_p, T(D))/\mathrm{H}^1_f(\mathbb{Q}_p, T(D)), \mathbb{Z}_p)] \\ &= [\exp^*:\ D'/(D')^0 \dashrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, T(D'))]\ \cdot \#(\mathrm{H}^2(\mathbb{Q}_p, T(D)))^{-1} \\ &= [1-\varphi:\ D' \dashrightarrow D']\ \cdot \#(\mathrm{H}^2(\mathbb{Q}_p, T(D)))^{-1}. \end{aligned}$$

□

Correction 14.19. — *In this opportunity, we correct a mistake in* [**BK2**, §4] *: All* H^1_e *in Lemma 4.5 should be corrected as* H^1_f.

14.20. Let p be a prime number which does not divide N. Then for any place λ of F lying over p, $V_{F_\lambda}(f)$ is a crystalline representation of $\operatorname{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Hence by [**LG**], there exists an O_λ-lattice D of $\mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f)) = \mathrm{D}_{\mathrm{dR}}(V_{F_\lambda}(f))$ which satisfies $\varphi(D^i) \subset p^i D^i$ for all $i \in \mathbb{Z}$ ($D^i \underset{\mathrm{def}}{=} D \cap \mathrm{D}^i_{\mathrm{dR}}(V_{F_\lambda}(f))$) and $(D, (D^i)_i, (p^{-i}\varphi \text{on } D^i)_i)$ is filtered Dieudonné module (Such D is called a strongly divisible O_λ-lattice in $\mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f))$). If furthermore $p > k$, then $T(D) \subset V_{F_\lambda}(f)$ is defined.

Proposition 14.21. — *Let $r \in \mathbb{Z}$, $1 \leqslant r \leqslant k/2$. In the case $r = k/2$, assume $L(f, k/2) \neq 0$. Let p be a prime number which does not divide N satisfying $p > k$, and let λ be a place of F lying over p. Let $D \subset \mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f))$ be a strongly divisible O_λ-lattice, and let $T = T(D)(r) \subset V_{F_\lambda}(f)(r)$. Let $\pm = (-1)^{r-1}$, take $\omega \in S_F(f)$ and $\gamma \in V_F(f)$ such that ω is an O_λ-basis of $D^1 (= D^{k-1})$ and $\gamma^\pm$ is an O_λ-basis of $T(-r)^\pm = T^-(-r)$, and define $\Omega \in \mathbb{C}^\times$ by $\operatorname{per}_f(\omega)^\pm = \Omega \cdot \gamma^\pm$. Let μ and ν be as in Prop. 14.16 (2) (defined with respect to T and γ). Then*

(1) $\nu = [O_\lambda : (2\pi i)^{r-1} L(f^*, k-r)/\Omega]$.

(2) $\#(\mathcal{S}(T(r))) = \mu^{-1} \cdot [O_\lambda : (2\pi i)^{r-1}L(f^*, k-r)/\Omega] \cdot \#(\mathrm{H}^0(\mathbb{Q}, T \otimes \mathbb{Q}/\mathbb{Z})) \cdot \#(\mathrm{H}^0(\mathbb{Q}, T^*(1) \otimes \mathbb{Q}/\mathbb{Z}))$.

Proof. — (2) follows from (1) and Prop. 14.16. We prove (1). We apply Lemma 14.18 to the filtered Dieudonné module $D(r)$. Let $m = k - r$. Then the filtered Dieudonné module $D(r)' = \mathrm{Hom}_{\mathbb{Z}_p}(D(r), \mathbb{Z}_p)(1)$ has the property that $D(r)' \otimes \mathbb{Q}$ is isomorphic to $\mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f^*)(m))$ as a $\mathbb{Q}_p$-vector space with an operator φ. Hence we have

$$\begin{aligned}[1-\varphi:\ D' \dashrightarrow D'] &= [O_\lambda : \det_{F_\lambda}\left(1-p^{-m}; \mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f^*))\right)] \\ &= [O_\lambda : 1 - \overline{a}_p p^{-m} + \overline{\varepsilon}(p)p^{k-1-2m}].\end{aligned}$$

By this and by

$$[\mathrm{H}^1(\mathbb{Q}_p, T)/\mathrm{H}^1_f(\mathbb{Q}_p, T) : z] \cdot [\exp^* :\ \mathrm{H}^1(\mathbb{Q}_p, T)/\mathrm{H}^1_f(\mathbb{Q}_p, T) \dashrightarrow D^r] = [D^r : \exp^*(z)].$$

we have

$$[D^r : \exp^*(z)] = \nu \cdot [O_\lambda : 1 - \overline{a}_p p^{-m} + \overline{\varepsilon}(p)p^{k-1-2m}].$$

On the other hand, since

$$\mathrm{per}(\exp^*(z))^{\pm} = (1 - \overline{a}_p p^{-m} + \overline{\varepsilon}(p)p^{k-1-2m}) \cdot L(f^*, m) \cdot (2\pi i)^{r-1}\gamma^{\pm},$$

we have

$$\exp^*(z) = (1 - \overline{a}_p p^{-m} + \overline{\varepsilon}(p)p^{k-1-2m}) \cdot (((2\pi i)^{r-1}L(f^*, m))/\Omega) \cdot \omega.$$

This shows

$$[D^r : \exp^*(z)] = [O_\lambda : (1 - \overline{a}_p p^{-m} + \overline{\varepsilon}(p)p^{k-1-2m}) \cdot (((2\pi i)^{r-1}L(f^*, m))/\Omega)].$$

By comparing those two expressions of $[D^r : \exp^*(z)]$, we obtain Prop. 14.21 (1). $\square$

14.22. We complete the proof of Thm. 14.2. Let r be an integer such that $1 \leqslant r \leqslant k-1$. In the case $r = k/2$, assume $L(f, k/2) \neq 0$. We have already shown that $\mathcal{S}(T)$ is finite for any finite place λ of F and any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice of $V_{F_\lambda}(f)(r)$. It remains to show that $\mathcal{S}(T)$ is zero for almost all finite places λ of F and for any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice T of $V_{F_\lambda}(f)(r)$.

By 14.16 (1) and by the duality (14.10.1), we may assume $r \leqslant k/2$. Let $\pm = (-1)^{r-1}$, take a non-zero element ω of $S_F(f)$ and an element γ of $V_F(f)$ such that $\gamma^{\pm} \neq 0$ and define $\Omega \in \mathbb{C}^{\times}$ by $\mathrm{per}_f(\omega)^{\pm} = \Omega \cdot \gamma^{\pm}$.

Take a multiple N' of N such that $N' \geqslant 3$, and let X be a proper smooth scheme over $\mathbb{Z}[1/N']$ such that $X \otimes \mathbb{Q} \simeq \mathrm{KS}_k(N')$.

By Fontaine-Messing [**FM**], if p does not divide N' and $p > k$, $\mathrm{H}^{k-1}(X, \Omega^{\bullet}_{X/\mathbb{Z}}) \otimes \mathbb{Z}_p$ has a structure of a filtered Dieudonné module whose filtration is given by $(\mathrm{H}^{k-1}(X, \Omega^{\geqslant i}_{X/\mathbb{Z}}) \otimes \mathbb{Z}_p)_i$, and $\mathrm{H}^{k-1}(\mathrm{KS}_k(N') \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_p}, \mathbb{Z}_p)$ is identified with $T(H^{k-1}(X, \Omega^{\bullet}_{X/\mathbb{Z}}) \otimes \mathbb{Z}_p)$ as a finitely generated $\mathbb{Z}_p$-module with an action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. For such p and for a place λ lying over p, let D_λ be the image of

$$(\mathrm{H}^{k-1}(X, \Omega^{\bullet}_{X/\mathbb{Z}}) \otimes O_\lambda)(\widetilde{\varepsilon}) \longrightarrow \mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f)).$$

($(\widetilde{\varepsilon})$ is as in § 11) Then D_λ is a strongly divisible O_λ-lattice, $D_\lambda^1 = D_\lambda^{k-1}$ is generated over O_λ by the image of

$$\mathrm{H}^{k-1}(X, \Omega_{X/\mathbb{Z}}^{\geqslant k-1})(\widetilde{\varepsilon}) \longrightarrow S(f).$$

and $T(D_\lambda)$ coincides with image of

$$(\mathrm{H}^{k-1}(\mathrm{KS}_k(N')(\mathbb{C}), \mathbb{Z}) \otimes O_\lambda)(\widetilde{\varepsilon}) \simeq (\mathrm{H}^{k-1}(\mathrm{KS}_k(N') \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} O_\lambda)(\widetilde{\varepsilon}) \longrightarrow V_{F_\lambda}(f).$$

Hence for almost all λ, ω is an O_λ-basis of $D_\lambda^1 = D_\lambda^{k-1}$, and $\gamma^\pm$ is an O_λ-basis of $T(D_\lambda)^\pm$. For such λ, we have

$$\begin{aligned}\#(\mathcal{S}(T(D_\lambda)(r))) &= \mu^{-1} \cdot [O_\lambda : (2\pi i)^{r-1} \cdot L(f^*, k-r)/\Omega] \\ &\quad \cdot \#(\mathrm{H}^0(\mathbb{Q}, T(F_\lambda)(r) \otimes \mathbb{Q}/\mathbb{Z})) \cdot \#(\mathrm{H}^0(\mathbb{Q}, T(F_\lambda)^*(1-r) \otimes \mathbb{Q}/\mathbb{Z}))\end{aligned}$$

by Prop. 14.21, where μ is the number in Prop. 14.16 defined with respect to $T(D_\lambda)(r)$ and γ and Ω is defined with respect to ω and γ. We have

$$[O_\lambda : (2\pi i)^{r-1} \cdot L(f^*, k-r)/\Omega] = 1$$

for almost all λ. We have

$$\mathrm{H}^0(\mathbb{Q}, T(F_\lambda)(r) \otimes \mathbb{Q}/\mathbb{Z}) = \mathrm{H}^0(\mathbb{Q}, T(F_\lambda)^*(1-r) \otimes \mathbb{Q}/\mathbb{Z}) = 0$$

for almost all λ by 14.11 (2), and we have $\mu \geqslant 1$ for almost all λ by 14.5 (3) in the non-CM case (see 15.23 for the CM-case). Hence $\mathcal{S}(T(D_\lambda)) = 0$ for almost all λ. By 14.7, this shows that $\mathcal{S}(T) = 0$ for almost all λ and for all $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattices T of $V_{F_\lambda}(f)(r)$. □

15. The case of complex multiplication

In this section, we prove the theorems 12.4, 12.5, 12.6, 14.2, 14.5 in the case f has complex multiplication. We deduce them from the work of Rubin on the main conjecture for quadratic imaginary fields.

In this section, we fix an imaginary quadratic field K. We fix also an embedding $K \to \mathbb{C}$.

15.1. We first review the work of Rubin on the main conjecture of imaginary quadratic fields.

By the fixed embedding $K \subset \mathbb{C}$, $\overline{\mathbb{Q}}$ becomes the algebraic closure of K in $\mathbb{C}$. Let K^{ab} be the maximal abelian extension of K in $\mathbb{C}$, and for a non-zero ideal $\mathfrak{f}$ of O_K, let $K(\mathfrak{f}) \subset K^{ab}$ be the ray class field of conductor $\mathfrak{f}$. Fix a prime number p and a non-zero ideal $\mathfrak{f}$ of O_K, and let

$$K(p^\infty \mathfrak{f}) = \bigcup_n K(p^n \mathfrak{f}), \quad G_{p^\infty \mathfrak{f}} = \mathrm{Gal}(K(p^\infty \mathfrak{f})/K).$$

Then

$$G_{p^\infty \mathfrak{f}} \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times (\text{a finite abelian group}).$$

Let

$$\mathfrak{h}^1 = \varprojlim_{K'} O_{K'}[1/p]^\times \otimes \mathbb{Z}_p, \quad \mathfrak{h}^2 = \varprojlim_{K'} \mathrm{Cl}(K')\{p\}$$

where K' ranges over all finite extensions of K contained in $K(p^\infty\mathfrak{f})$, $\mathrm{Cl}(K')\{p\}$ denotes the p-primary part of the ideal class group $\mathrm{Cl}(K')$ of K', and the inverse limits are taken with respect to norm maps. Then $\mathfrak{h}^q$ $(q = 1, 2)$ are finitely generated modules over the three dimensional semi-local ring $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$, $\mathfrak{h}^1$ is a torsion free $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$-module having the property that $(\mathfrak{h}^1)_\mathfrak{q}$ is of dimension one for any prime ideals $\mathfrak{q}$ of $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$ of height 0 ($\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]_\mathfrak{q}$ is a field for such $\mathfrak{q}$), and $\mathfrak{h}^2$ is a torsion $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$-module (that is, it is killed by a non-zero-divisor of $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$). Let $\mathfrak{z} \subset \mathfrak{h}^1$ be the "part of elliptic units" whose definition is reviewed in 15.5 below. Then $\mathfrak{h}^1/\mathfrak{z}$ is a torsion $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$-module.

The following Theorem is contained in the works of Rubin in [**Ru2, Ru4**].

Theorem 15.2 **(Rubin).** — *Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$ of height one. Consider the following conditions (a) (b) (c).*

(a) *$\mathfrak{p}$ does not contain p.*

(b) *p does not divide the order of the group of all the roots of 1 in the Hilbert class field of K, and p does not divide the order of the torsion part of $G_{p^\infty\mathfrak{f}}$.*

(c) *p splits in K.*

We have:

(1) *If either the condition (a) or (b) is satisfied, we have*

$$\mathrm{length}_{\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]_\mathfrak{p}}((\mathfrak{h}^2)_\mathfrak{p}) \leqslant \mathrm{length}_{\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]}((\mathfrak{h}^1/\mathfrak{z})_\mathfrak{p}).$$

(2) *If both the conditions (b) and (c) are satisfied, then*

$$\mathrm{length}_{\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]_\mathfrak{p}}((\mathfrak{h}^2)_\mathfrak{p}) = \mathrm{length}_{\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]}((\mathfrak{h}^1/\mathfrak{z})_\mathfrak{p}).$$

In [**Ru2**], Rubin has an equality (not only inequality) even in the case p does not split in K under a certain condition, but we do not use it in this paper.

15.3. We review here the theory of complex multiplication.

Let $\mathfrak{f}$ be a non-zero ideal of O_K such that $O_K^\times \to (O_K/\mathfrak{f})^\times$ is injective.

For a field K' over K, by a CM-pair with modulus $\mathfrak{f}$ over K', we mean a pair (E, α) where E is an elliptic curve over K' endowed with an isomorphism $O_K \xrightarrow{\sim} \mathrm{End}(E)$ such that the composite map

$$O_K \longrightarrow \mathrm{End}(E) \longrightarrow \mathrm{End}_{K'}(\mathrm{Lie}(E)) = K'$$

coincides with the inclusion map, and α is a torsion point in $E(K')$ such that the annihilator of α in O_K coincides with $\mathfrak{f}$.

Note that if (E, α) and (E', α') are CM-pairs of modulus $\mathfrak{f}$ over K' and if they are isomorphic, the isomorphism $(E, \alpha) \to (E', \alpha')$ is unique by the injectivity of $O_K^\times \to (O_K/\mathfrak{f})^\times$.

The following (15.3.1)-(15.3.3) summarize a central part of the theory of complex multiplication.

(15.3.1) *There exist a CM-pair of modulus* $\mathfrak{f}$ *over* $K(\mathfrak{f})$ *which is isomorphic to* $(\mathbb{C}/\mathfrak{f}, 1 \bmod \mathfrak{f})$ *over* $\mathbb{C}$. *This CM-pair of modulus* $\mathfrak{f}$ *over* $K(\mathfrak{f})$ *is unique upto isomorphism (and hence unique upto unique isomorphism by the above remark).*

We call the above CM-pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})$ the canonical CM-pair over $K(\mathfrak{f})$.

(15.3.2) *Let* K' *be a field over* K *and let* (E, α) *be a CM-pair of modulus* $\mathfrak{f}$ *over* K'. *Then there exist a unique homomorphism* $K(\mathfrak{f}) \to K'$ *for which* (E, α) *is obtained from the canonical CM-pair over* $K(\mathfrak{f})$ *by the base change.*

(15.3.3) *(relation with class field theory) Let* K' *be a finite abelian extension of* K, *let* $\mathfrak{a}$ *be a non-zero ideal of* O_K *whose all prime divisors are unramified in* K', *and let* $\sigma = (\mathfrak{a}, K'/K) \in \mathrm{Gal}(K'/K)$ *be the Artin symbol. On the other hand, let* (E, α) *be a CM-pair of modulus* $\mathfrak{f}$ *over* K' *and let* $(E^{(\sigma)}, \sigma(\alpha))$ *be the CM-pair of modulus* $\mathfrak{f}$ *over* K' *obtained form* (E, α) *by the base change* $\sigma : K' \to K'$. *Then* $(E^{(\sigma)}, \sigma(\alpha))$ *is isomorphic to* $(E/{}_{\mathfrak{a}}E, \alpha \bmod {}_{\mathfrak{a}}E)$ *where* ${}_{\mathfrak{a}}E$ *is the part of* E *annihilated by* $\mathfrak{a}$.

We will denote the unique isomorphism in (15.3.3) as

$$\eta_{\mathfrak{a}} : (E/{}_{\mathfrak{a}}E, \alpha \bmod {}_{\mathfrak{a}}E) \xrightarrow{\sim} (E^{(\sigma)}, \sigma(\alpha)).$$

15.4. We give a refinement of Prop. 1.3 in the case of complex multiplication. Let K' be a field over K and let E be an elliptic curve over K' such that $\mathrm{End}(E) \simeq O_K$. We normalize this isomorphism in the way that the composite map

$$O_K \xrightarrow{\sim} \mathrm{End}(E) \longrightarrow \mathrm{End}_{K'}(\mathrm{Lie}(E)) = K'$$

is the inclusion map.

Then for an ideal $\mathfrak{a}$ of O_K which is prime to 6, there is a unique element ${}_{\mathfrak{a}}\theta_E$ of $\mathcal{O}(E \smallsetminus {}_{\mathfrak{a}}E)^\times$ having the following properties (i) (ii).

(i) The divisor of ${}_{\mathfrak{a}}\theta$ is $N(\mathfrak{a}) \cdot (0) - {}_{\mathfrak{a}}E$.

(ii) $N_a({}_{\mathfrak{a}}\theta_E) = {}_{\mathfrak{a}}\theta_E$ for any integer a which is prime to $\mathfrak{a}$.

The unique existence of ${}_{\mathfrak{a}}\theta$ and the following properties (15.4.1)-(15.4.3) of ${}_{\mathfrak{a}}\theta$ are proved in the same way as the proof of Prop. 1.3.

(15.4.1) *If* $\mathfrak{a} = (c)$ *for an integer* c, ${}_{\mathfrak{a}}\theta = {}_c\theta$.

(15.4.2) *If* $E \to E'$ *is an isogeny between elliptic curves over* K' *such that* $\mathrm{End}(E) \simeq O_K$ *and* $\mathrm{End}(E') \simeq O_K$, *the norm map sends* ${}_{\mathfrak{a}}\theta_E$ *to* ${}_{\mathfrak{a}}\theta_{E'}$.

(15.4.3) *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are ideals of* O_K *which are prime to* 6,

$$({}_{\mathfrak{b}}\theta_E)^{N(\mathfrak{a})} \cdot (pr_1^*({}_{\mathfrak{b}}\theta_{E_1}))^{-1} = ({}_{\mathfrak{a}}\theta_E)^{N(\mathfrak{b})} \cdot (pr_2^*({}_{\mathfrak{a}}\theta_{E_2}))^{-1}$$

where $E_1 = E/{}_{\mathfrak{a}}E, E_2 = E/{}_{\mathfrak{b}}E$, *and for* $j = 1, 2$, pr_j *is the canonical projection* $E \to E_j$.

Now let $\mathfrak{f}$ be a non-zero ideal of O_K such that $O_K^\times \to (O_K/\mathfrak{f})^\times$ is injective, and let (E, α) be a CM-pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})$. Then for ideals $\mathfrak{a}, \mathfrak{b}$ of O_K which are prime to $6\mathfrak{f}$, we have (denote $(\mathfrak{a}, K(\mathfrak{f})/K)$ by $\tau_\mathfrak{a}$ and $(\mathfrak{b}, K(\mathfrak{f})/K)$ by $\tau_\mathfrak{b}$)

$$({}_\mathfrak{b}\theta_E(\alpha))^{N(\mathfrak{a})}\tau_\mathfrak{a}({}_\mathfrak{b}\theta_E(\alpha))^{-1} = ({}_\mathfrak{a}\theta_E(\alpha))^{N(\mathfrak{b})}\tau_\mathfrak{b}({}_\mathfrak{a}\theta_E(\alpha))^{-1} \tag{15.4.4}$$

In fact, in (15.4.3), we have $(pr_1^*({}_\mathfrak{b}\theta_{E_1}))(\alpha) = {}_\mathfrak{b}\theta_{E_1}(pr_1(\alpha)) = \tau_\mathfrak{a}({}_\mathfrak{b}\theta_E(\alpha))$ because $(E_1, pr_1(\alpha)) \simeq (E^{(\sigma)}, \sigma(\alpha))$ with $\sigma = \tau_\mathfrak{a}$ (15.3.3), and the similar thing holds when we replace $pr_1, \mathfrak{b}, E_1$ by $pr_2, \mathfrak{a}, E_2$, respectively. Hence we obtain (15.4.4) from (15.4.3).

15.5. We review the definition of the "part of elliptic units" $\mathfrak{z} \subset \mathfrak{h}^1$ in 15.1

Let $\mathfrak{f}$ be an ideal of O_K such that $O_K^\times \to (O_K/\mathfrak{f})^\times$ is injective, let (E, α) be the canonical CM-pair over $K(\mathfrak{f})$, and $\mathfrak{a}$ be an ideal of O_K which is prime to $6\mathfrak{f}$. Then the element

$$ {}_\mathfrak{a}z_\mathfrak{f} = {}_\mathfrak{a}\theta_E(\alpha)^{-1} \in K(\mathfrak{f})^\times$$

(the standard elliptic unit of modulus $\mathfrak{f}$ and of twist $\mathfrak{a}$) has the following relation with L-functions (Kronecker's limit formula): For any homomorphism $\chi : \mathrm{Gal}(K(\mathfrak{f})/K) \to \mathbb{C}^\times$, we have

$$\sum_\sigma \chi(\sigma) \log |\sigma({}_\mathfrak{a}z_\mathfrak{f})| = (N(\mathfrak{a}) - \chi(\mathfrak{a})^{-1}) \cdot \lim_{s\to 0} s^{-1} L_{K,\mathfrak{f}}(\chi, s) \tag{15.5.1}$$

where $L_{K,\mathfrak{f}}(\chi, s)$ denotes $\sum_\mathfrak{b} \chi(\mathfrak{b})N(\mathfrak{b})^{-s}$ in which $\mathfrak{b}$ ranges all ideals of O_K which are prime to $\mathfrak{f}$ and $\chi(\mathfrak{b})$ denotes $\chi((\mathfrak{b}, K(\mathfrak{f})/K))$. (This (15.5.1) is deduced from (3.8.2) by taking a suitable element of K as τ in (3.8.2).)

The element ${}_\mathfrak{a}\theta_E(\alpha)$ is a $\mathfrak{p}$-unit for any prime ideal $\mathfrak{p}$ of O_K which is prime to $\mathfrak{f}$, and is a unit if $\mathfrak{f}$ has at least two prime divisors.

Now we define $\mathfrak{z} \subset \mathrm{H}^1$. Let $\mathfrak{f}$ be a non-zero ideal of O_K. Then for $n \geqslant 1$ such that $O_K^\times \to (O_K/p^n\mathfrak{f})^\times$ is injective and for an ideal $\mathfrak{a}$ of O_K which is prime to $6p\mathfrak{f}$, the norm map of $K(p^{n+1}\mathfrak{f})/K(p^n\mathfrak{f})$ sends ${}_\mathfrak{a}z_{p^{n+1}\mathfrak{f}}$ to ${}_\mathfrak{a}z_{p^n\mathfrak{f}}$. (This is deduced from (15.4.2).) We define $\mathfrak{z}$ to be the $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$-module of $\mathfrak{h}^1$ generated by the elements $({}_\mathfrak{a}z_{p^n\mathfrak{f}})_{n\geqslant 1}$ where $\mathfrak{a}$ ranges over all ideals of O_K which are prime to $6p\mathfrak{f}$.

15.6. We rewrite Thm. 15.2 in the form using Galois cohomology.

For a finitely generated $\mathbb{Z}_p$-module T endowed with a continuous action of $\mathrm{Gal}(\overline{K}/K)$ which is unramified at almost all finite places of K, let

$$\mathbf{H}^q_{p^\infty\mathfrak{f}}(T) = \varprojlim_{K'} \mathrm{H}^q(O_{K'}[1/p], T) \qquad (q \in \mathbb{Z})$$

where K' ranges over all finite extensions of K contained in $K(p^\infty\mathfrak{f})$. Then $\mathbf{H}^q_{p^\infty\mathfrak{f}}(T)$ is a finitely generated $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$-module, and $\mathbf{H}^q_{p^\infty\mathfrak{f}}(T) = 0$ if $q \neq 1, 2$.

In the case $T = \mathbb{Z}_p(1)$, we have:

$$\mathfrak{h}^1 \simeq \mathbf{H}^1_{p^\infty\mathfrak{f}}(\mathbb{Z}_p(1)). \tag{15.6.1}$$

(15.6.2) *There exists a homomorphism of* $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$*-modules*

$$\mathfrak{h}^2 \longrightarrow \mathbf{H}^2_{p^\infty\mathfrak{f}}(\mathbb{Z}_p(1))$$

whose kernel and cokernel are finitely generated $\mathbb{Z}_p$*-modules. In particular,*

$$(\mathfrak{h}^2)_\mathfrak{p} \xrightarrow{\sim} \mathbf{H}^2_{p^\infty\mathfrak{f}}(\mathbb{Z}_p(1))_\mathfrak{p}$$

for any prime ideal $\mathfrak{p}$ *in* $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$ *of height one.*

((15.6.2) is proved as follows. From the Kummer sequence

$$0 \longrightarrow \mathbb{Z}/p^n\mathbb{Z}(1) \longrightarrow \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \longrightarrow 0,$$

we obtain exact sequences

$$0 \to \mathrm{Pic}(O_{K'}, [1/p])\{p\} \to \mathrm{H}^2(O_{K'}[1/p], \mathbb{Z}_p(1)) \to \varprojlim_n \mathrm{Ker}(p^n; \mathrm{Br}(O_{K'}[1/p])) \to 0$$

for extensions K' of $\mathbb{Q}$. We have an isomorphism

$$\varprojlim_n \mathrm{Ker}(p^n; \mathrm{Br}(O_{K'}[1/p])) = \mathrm{Ker}(\mathrm{sum} : \mathbb{Z}_p^\Sigma \longrightarrow \mathbb{Z}_p)$$

where Σ is the set of prime ideals of $O_{K'}$ lying over p, and we have a surjection $\mathrm{Cl}(K')\{p\} \to \mathrm{Pic}(O_{K'}, [1/p])$ whose kernel is generated by the classes of elements in Σ. By taking $\varprojlim_{K'}$ for finite extensions K' of K contained in $K(p^\infty\mathfrak{f})$, and by the finiteness of the number of places of $K(p^\infty\mathfrak{f})$ lying over p, we obtain (15.6.2).)

Let $Q(\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]])$ be the total quotient ring of $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$. We define an element

$$z_{p^\infty\mathfrak{f}} \in \mathbf{H}^1_{p^\infty\mathfrak{f}}(\mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]} Q(\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]])$$

(without ${}_\mathfrak{a}(\ )$) by

$$\begin{aligned} z_{p^\infty\mathfrak{f}} &= (N(\mathfrak{a}) - (\mathfrak{a}, K(p^\infty\mathfrak{f})/K))^{-1} \cdot ({}_\mathfrak{a}z_{p^n\mathfrak{f}})_n \\ &\in \mathfrak{h}^1 \otimes_{\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]} Q(\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]) \simeq \mathbf{H}^1_{p^\infty\mathfrak{f}}(\mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]} Q(\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]) \end{aligned}$$

where $\mathfrak{a}$ is any ideal of O_K which is prime to $6p\mathfrak{f}$ such that $\mathfrak{a} \neq O_K$ (then $N(\mathfrak{a}) - (\mathfrak{a}, K(p^\infty\mathfrak{f})/K)$ is a non-zero-divisor of $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$). Then $z_{p^\infty\mathfrak{f}}$ is independent of the choice of $\mathfrak{a}$, for

$$(N(\mathfrak{b}) - (\mathfrak{b}, K(p^\infty\mathfrak{f})/K)) \cdot ({}_\mathfrak{a}z_{p^n\mathfrak{f}})_n = (N(\mathfrak{a}) - (\mathfrak{a}, K(p^\infty\mathfrak{f})/K)) \cdot ({}_\mathfrak{b}z_{p^n\mathfrak{f}})_n$$

for any ideals $\mathfrak{a}, \mathfrak{b}$ of O_K which are prime to $6p\mathfrak{f}$ by (15.4.4).

We have :

(15.6.3) $(N(\mathfrak{a}) - (\mathfrak{a}, K(p^\infty\mathfrak{f})/K)) \cdot z_{p^\infty\mathfrak{f}} \in \mathbf{H}^1_{p^\infty\mathfrak{f}}(\mathbb{Z}_p(1))$ *for any ideal* $\mathfrak{a}$ *of* O_K *which is prime to* $6p\mathfrak{f}$.

The ideal I of $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$ generated by $N(\mathfrak{a}) - (\mathfrak{a}, K(p^\infty\mathfrak{f})/K)$ for all ideals $\mathfrak{a}$ of O_K which are prime to $6p\mathfrak{f}$ satisfies $I_\mathfrak{p} = \mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$ of height one. Hence we have

(15.6.4) $\mathfrak{z}_\mathfrak{p} = \mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]_\mathfrak{p} \cdot z_{p^\infty\mathfrak{f}}$ *for any prime ideal* $\mathfrak{p}$ *of* $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$ *of height one.*

By (15.5.1), we have

(15.6.5) *Let K' be a finite extension of K contained in $K(p^\infty\mathfrak{f})$, and let $z_{K'}$ be the image of $z_{p^\infty\mathfrak{f}}$ under $\mathcal{S}^{-1}\mathbf{H}^1_{p^\infty\mathfrak{f}}(\mathbb{Z}_p(1)) \to \mathrm{H}^1(O_{K'}[1/p], \mathbb{Q}_p(1))$ where $\mathcal{S}$ denotes the multiplicative subset of $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$ consisting of non-zero-divisors whose images in $\mathbb{Q}_p[\mathrm{Gal}(K'/K)]$ are invertible. (Note $N(\mathfrak{a}) - (\mathfrak{a}, K(p^\infty\mathfrak{f})/K) \in \mathcal{S}$ for any ideal $\mathfrak{a}$ of O_K which is prime to $p\mathfrak{f}$ such that $\mathfrak{a} \neq O_K$.) Then $z_{K'}$ belongs to the image of the canonical injection*

$$(O_{K'}[1/p])^\times \otimes \mathbb{Q} \longrightarrow \mathrm{H}^1(O_{K'}[1/p], \mathbb{Q}_p(1)).$$

If we regard $z_{K'}$ as an element of $(O_{K'}[1/p])^\times \otimes \mathbb{Q}$, we have

$$\sum_{\sigma\in\mathrm{Gal}(K'/K)} \chi(\sigma)\log(|\sigma(z_{K'})|) = \lim_{s\to 0} s^{-1}L_{K,p\mathfrak{f}}(\chi, s)$$

for any non-trivial homomorphism $\chi : \mathrm{Gal}(K'/K) \to \mathbb{C}^\times$.

Now Thm. 15.2 is reformulated as follows: For a prime ideal $\mathfrak{p}$ of $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$ of height one, let

$$\begin{aligned}\ell^1_{\mathfrak{p}} &= \mathrm{length}_{\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]_{\mathfrak{p}}}\big(\mathbf{H}^1_{p^\infty\mathfrak{f}}(\mathbb{Z}_p(1))_{\mathfrak{p}}/(\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]_{\mathfrak{p}}\cdot z_{p^\infty\mathfrak{f}})\big)\\ \ell^2_{\mathfrak{p}} &= \mathrm{length}_{\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]_{\mathfrak{p}}}(\mathbf{H}^2_{p^\infty\mathfrak{f}}(\mathbb{Z}_p(1))_{\mathfrak{p}}).\end{aligned}$$

Then, if either the conditions (a) or (b) in 15.2 is satisfied, we have

$$\ell^2_{\mathfrak{p}} \leqslant \ell^1_{\mathfrak{p}}.$$

If both the conditions (b) and (c) in 15.2 are satisfied, then

$$\ell^2_{\mathfrak{p}} = \ell^1_{\mathfrak{p}}.$$

15.7. We review basic facts about Hecke characters of K.

Let $\widehat{O_K} = \varprojlim_I O_K/I$ where I ranges over all non-zero ideals of O_K, and let $\widehat{K} = \widehat{O_K} \otimes \mathbb{Q}$. Then the adele ring of K is $\mathbb{C} \times \widehat{K}$, and the idele class group C_K of K is $(\mathbb{C}^\times \times \widehat{K}^\times)/K^\times$.

If ψ is a Hecke character of K (i.e. a continuous homomorphism $C_K \to \mathbb{C}^\times$), the homomorphism $\widehat{O_K}^\times \to \mathbb{C}^\times$ induced by ψ factors through the projection $\widehat{O_K}^\times \to (O_K/I)^\times$ for some non-zero ideal I of O_K, and there is a non-zero ideal which is the largest among such ideals I, called the conductor of ψ. If $\mathfrak{a}$ is an ideal of O_K which is prime to the conductor of ψ, we define $\psi(\mathfrak{a})$ to be $\psi(1, a)$ where 1 is the unit element of $\mathbb{C}^\times$ and a an element of $\widehat{K}^\times \cap \widehat{O_K}$ such that $\widehat{O_K}\mathfrak{a} = \widehat{O_K}\cdot a$ and such that the image of a under $\widehat{O_K} \to \widehat{O_K}/\mathfrak{f}$ is 1 (then $\psi(1, a)$ is independent of the choice of a). The L-function $L(\psi, s)$ of ψ is expressed as

$$L(\psi, s) = \sum_{\mathfrak{a}} \psi(\mathfrak{a})N(\mathfrak{a})^{-s}$$

where $\mathfrak{a}$ ranges over all ideals of O_K which are prime to the conductor of ψ.

For integers m, n and for a Hecke character ψ of K, we say that ψ is of type (m, n) if the restriction of ψ to the archimedean part $\mathbb{C}^\times$ of the idele group of K has the form $z \mapsto z^m \cdot \overline{z}^n$. It is known that if ψ is a Hecke character of type (m, n) for some integers m, n then the subfield of $\mathbb{C}$ generated by $\psi(\widehat{K}^\times)$ is a finite extension of $\mathbb{Q}$.

15.8. Let $r \geqslant 1$, and let ψ be a Hecke character of K of type $(-r, 0)$. We review the Galois representation, the period map, etc. associated to ψ.

Let L be the subfield of $\mathbb{C}$ generated by $\psi(\widehat{K}^\times)$ over K, which is a finite extension of K. In this 15.8, we define a one dimensional L-vector spaces $V_L(\psi)$ and $S(\psi)$, and a continuous L_λ-linear action of $\mathrm{Gal}(K^{ab}/K)$ on $V_{L_\lambda}(\psi) = V_L(\psi) \otimes_L L_\lambda$ for each finite place λ of L. We define also an L-linear map

$$\mathrm{per}_\psi : S(\psi) \longrightarrow V_{\mathbb{C}}(\psi) = V_L(\psi) \otimes_L \mathbb{C},$$

called the period map, and an isomorphism

$$S(\psi) \otimes_L L_\lambda \simeq \mathrm{D}^1_{\mathrm{dR}}(K \otimes \mathbb{Q}_p, V_{L_\lambda}(\psi)) = \mathrm{D}^r_{\mathrm{dR}}(K \otimes \mathbb{Q}_p, V_{L_\lambda}(\psi)) \tag{15.8.1}$$

for each finite place λ of L where p is the prime number lying under λ.

Take a non-zero ideal $\mathfrak{f}$ of O_K contained in the conductor of ψ such that $O_K^\times \to (O_K/\mathfrak{f})^\times$ is injective, and let (E, α) be the canonical CM-pair over $K(\mathfrak{f})$ (15.3).

We define

$$V_L(\psi) = \mathrm{H}^1(E(\mathbb{C}), \mathbb{Q})^{\otimes r} \otimes_K L$$

where $\otimes r$ is taken over K.

For a finite place λ of L, we have a canonical identification

$$V_{L_\lambda}(\psi) = \mathrm{H}^1_{\mathrm{\acute{e}t}}(E \otimes_{K(\mathfrak{f})} \overline{\mathbb{Q}}, \mathbb{Q}_p)^{\otimes r} \otimes_{K \otimes \mathbb{Q}_p} L_\lambda$$

where $V_{L_\lambda}(\psi) = V_L(\psi) \otimes_L L_\lambda$, p is the prime number lying under λ and $\otimes r$ is taken over $K \otimes \mathbb{Q}_p$. Hence we have an L_λ-linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K(\mathfrak{f}))$ on $V_{L_\lambda}(\psi)$. We extend this action to an L_λ-linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on $V_{L_\lambda}(\psi)$ as follows. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, we define the action of σ on $V_{L_\lambda}(\psi)$ to be the composite

$$V_{L_\lambda}(\psi) \xrightarrow{\sigma} \mathrm{H}^1_{\mathrm{\acute{e}t}}(E^{(\sigma)} \otimes_{K(\mathfrak{f})} \overline{\mathbb{Q}}, \mathbb{Q}_p)^{\otimes r} \otimes_{K \otimes \mathbb{Q}_p} L_\lambda \xrightarrow[\sim]{\eta_\psi} V_{L_\lambda}(\psi)$$

where η_ψ is the following isomorphism. Take an ideal $\mathfrak{a}$ of O_K which is prime to $\mathfrak{f}$ such that $(\mathfrak{a}, K(\mathfrak{f})/K)$ coincides with the restriction of σ to $K(\mathfrak{f})$. We define $\eta_\psi = \psi(\mathfrak{a})^{-1}(\eta_{\mathfrak{a}}^*)^{\otimes r}$ in which $\eta_{\mathfrak{a}}^*$ denotes the pull back on $\mathrm{H}^1_{\mathrm{\acute{e}t}}$ by

$$E \longrightarrow E/_{\mathfrak{a}}E \xrightarrow[\sim]{\eta_{\mathfrak{a}}} E^{(\sigma)} \quad (15.3.3).$$

Then η_ψ is independent of the choice of $\mathfrak{a}$.

This action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on $V_{L_\lambda}(\psi)$ is abelian, and described by class field theory as follows. Via the reciprocity map $\widehat{K}^\times/K^\times \simeq \mathrm{Gal}(K^{ab}/K)$ of class field theory, the image of $a \in \widehat{K}^\times$ in $\mathrm{Gal}(K^{ab}/K)$ acts on $V_{L_\lambda}(\psi)$ as multiplication by $a_v^r \psi(a)^{-1}$, where v is the place of K lying under λ and a_v denotes the v-component of a. Another class field theoretic description of this action by using Artin symbols is the following. The action of $\mathrm{Gal}(K^{ab}/K)$ on $V_{L_\lambda}(\psi)$ factors through $\mathrm{Gal}(K(p^\infty \mathfrak{f})/K)$, and for an ideal $\mathfrak{a}$

of O_K which is prime to $p\mathfrak{f}$, $(a, K(p^\infty\mathfrak{f})/K)$ acts on $V_{L_\lambda}(\psi)$ as the multiplication by $\psi(\mathfrak{a})^{-1}$.

Next we define $S(\psi)$ as an L-subspace of $\mathrm{coLie}(E)^{\otimes r}\otimes_K L$ such that $S(\psi)\otimes_K K(\mathfrak{f}) = \mathrm{coLie}(E)^{\otimes r}\otimes_K L$, in the following way, where $\otimes r$ is taken over $K(\mathfrak{f})$.

We define an L-linear action of $\mathrm{Gal}(K(\mathfrak{f})/K)$ on $\mathrm{coLie}(E)^{\otimes r}\otimes_K L$ such that $\sigma(ax) = \sigma(a)\sigma(x)$ for $a \in K(\mathfrak{f})$ and $x \in \mathrm{coLie}(E)^{\otimes r}\otimes_K L$ by the following rule : $\sigma \in \mathrm{Gal}(K(\mathfrak{f})/K)$ acts as the composite

$$\mathrm{coLie}(E)^{\otimes r}\otimes_K L \xrightarrow{\ \sigma\otimes 1\ } \mathrm{coLie}(E^{(\sigma)})\otimes_K L \xrightarrow[\sim]{\ \eta_\psi\ } \mathrm{coLie}(E)\otimes_K L$$

where η_ψ is the following isomorphism. Take an ideal $\mathfrak{a}$ of O_K which is prime to $\mathfrak{f}$ such that $\sigma = (\mathfrak{a}, K(\mathfrak{f})/K)$. We define

$$\eta_\psi = \psi(\mathfrak{a})^{-1}(\eta_{\mathfrak{a}}^*)^{\otimes r} : \mathrm{coLie}(E^{(\sigma)})^{\otimes r}\otimes_K L \xrightarrow{\sim} \mathrm{coLie}(E/_{\mathfrak{a}}E)^{\otimes r}\otimes_K L = \mathrm{coLie}(E)^{\otimes r}\otimes_K L$$

where the last identification is by the canonical isogeny $E \to E/_{\mathfrak{a}}E$. Then η_ψ is independent of the choice of $\mathfrak{a}$. Now we define $S(\psi)$ to be the fixed part in $\mathrm{coLie}(E)^{\otimes r}\otimes_K L$ under this action of $\mathrm{Gal}(K(\mathfrak{f})/K)$.

We define the period map

$$\mathrm{per}_\psi : S(\psi) \longrightarrow V_{\mathbb{C}}(\psi)$$

as the map induced by the period map $\mathrm{coLie}(E) \to \mathrm{H}^1(E(\mathbb{C}), \mathbb{C})$.

Finally we define the isomorphism (15.8.1). For any finite place v of K lying under λ, $V_{L_\lambda}(\psi)$ is de Rham as a representation of $\mathrm{Gal}(\overline{K}_v/K_v)$ because it is de Rham as a representation of $\mathrm{Gal}(\overline{K(\mathfrak{f})}_w/K(\mathfrak{f})_w)$ for any finite place w of $K(\mathfrak{f})$ lying over v. We have

$$\mathrm{D}_{\mathrm{dR}}(K(\mathfrak{f})\otimes\mathbb{Q}_p, \mathrm{H}^1_{\text{ét}}(E\otimes_{K(\mathfrak{f})}\overline{K(\mathfrak{f})}_w, \mathbb{Q}_p)) \simeq \mathrm{H}^1_{\mathrm{dR}}(E/K(\mathfrak{f}))\otimes\mathbb{Q}_p$$

and this induces

$$\mathrm{D}_{\mathrm{dR}}(K(\mathfrak{f})\otimes\mathbb{Q}_p, V_{L_\lambda}(\psi)) \simeq \mathrm{H}^1_{\mathrm{dR}}(E/K(\mathfrak{f}))^{\otimes r}\otimes_K L_\lambda$$

where $\otimes r$ is the r-fold tensor power as an invertible module over $K\otimes K(\mathfrak{f})$ (K acts via $K \simeq \mathrm{End}(E)\otimes\mathbb{Q}$ and $K(\mathfrak{f})$ acts because it is the base field). This induces

$$\mathrm{D}^j_{\mathrm{dR}}(K(\mathfrak{f})\otimes\mathbb{Q}_p, V_{L_\lambda}(\psi)) \simeq \mathrm{coLie}(E)^{\otimes r}\otimes_K L_\lambda \qquad \text{for } 1 \leqslant j \leqslant r$$

(the left hand side does not depend on j such that $1 \leqslant j \leqslant r$) where $\otimes r$ is taken over $K(\mathfrak{f})$ and $\mathrm{coLie}(E)$ is regarded as an $K(\mathfrak{f})$-subspace of $\mathrm{H}^1_{\mathrm{dR}}(E/K(\mathfrak{f}))$. By taking the $\mathrm{Gal}(K(\mathfrak{f})/K)$-invariant part of the both sides of the last isomorphism, we have

$$\mathrm{D}^j_{\mathrm{dR}}(K\otimes\mathbb{Q}_p, V_{L_\lambda}(\psi)) \simeq S(\psi)\otimes_L L_\lambda \quad \text{for } 1 \leqslant j \leqslant r.$$

In the above constructions of $V_L(\psi)$, $S(\psi)$, etc., we fixed $\mathfrak{f}$. However every construction does not depend of the choice of $\mathfrak{f}$ in the following sense. If $\mathfrak{f}'$ is another ideal of O_K which is contained in the conductor ψ and (E', α') (resp. (E'', α'')) denotes the canonical CM-pair over $K(\mathfrak{f}')$ (resp. $K(\mathfrak{f}\mathfrak{f}')$), E is identified with $E''/_{\mathfrak{f}'}E''$ and E'

is identified with $E''/_{\mathfrak{f}}E''$. Via the isomorphisms induced by the canonical isogenies $E'' \to E$ and $E'' \to E'$, we can identify constructions using $\mathfrak{f}$ and constructions $\mathfrak{f}'$.

Proposition 15.9. — *Let $r \geqslant 1$ and let ψ be a Hecke character of K of type $(-r, 0)$. Let p be a prime number, let $\mathfrak{f}$ be a non-zero ideal of O_K contained in the conductor of ψ, let K' be a finite extension of K contained in $K(p^\infty\mathfrak{f})$, and let $\gamma \in V_L(\psi)$. Then the image of $z_{p^\infty\mathfrak{f}}$ under*

$$\mathbf{H}^1_{p^\infty\mathfrak{f}}(\mathbb{Z}_p(1)) \xrightarrow{\gamma} \mathbf{H}^1_{p^\infty\mathfrak{f}}(\mathbb{Z}_p(1)) \otimes V_{L_\lambda}(\psi) \xrightarrow{\sim} \mathbf{H}^1_{p^\infty\mathfrak{f}}(V_{L_\lambda}(\psi)(1)) \longrightarrow$$

$$\mathrm{H}^1(O_{K'}[1/p], V_{L_\lambda}(\psi)(1)) \xrightarrow{\exp^*} \mathrm{D}^1_{\mathrm{dR}}(K' \otimes \mathbb{Q}_p, V_{L_\lambda}(\psi)) \underset{(15.8.1)}{\simeq} (S(\psi) \otimes_L L_\lambda) \otimes_K K'$$

is an element of $S(\psi) \otimes_K K'$ whose image under

$$\sum_{\sigma \in \mathrm{Gal}(K'/K)} \chi(\sigma)\, \mathrm{per}_\psi \circ \sigma : S(\psi) \otimes_K K' \longrightarrow V_{\mathbb{C}}(\psi)$$

coincides with $L_{p\mathfrak{f}}(\overline{\psi}, \chi, r) \cdot \gamma$ for any homomorphism $\chi : \mathrm{Gal}(K'/K) \to \mathbb{C}^\times$. Here $L_{p\mathfrak{f}}(\overline{\psi}, \chi, s)$ denotes $\sum_{\mathfrak{a}} \overline{\psi}(\mathfrak{a})\chi(\mathfrak{a})N(\mathfrak{a})^{-s}$ in which $\mathfrak{a}$ ranges over all ideals of O_K which are prime to $p\mathfrak{f}$.

Proof. — This 15.9 is proved in [**KK2**, Chap. III, § 1] under certain assumptions. The proof of 15.9 here follows the method there.

By using the trace maps, we see that we may replace K' by any finite extension of K' contained in $K(p^\infty\mathfrak{f})$. Hence we may assume $K' = K(\mathfrak{g})$, where $\mathfrak{g}$ is an ideal of O_K having the form $p^m\mathfrak{f}_0$, where $m \geqslant 1$ and $\mathfrak{f}_0$ is the smallest ideal of O_K which divides $\mathfrak{f}$ and which is prime to p, and such that $\mathfrak{g} \subset \mathfrak{f}$ and $O_K^\times \to (O_K/\mathfrak{g})^\times$ is injective.

Let (E, α) be the canonical CM-pair over $K' = K(\mathfrak{g})$. Let $\beta \in \mathrm{H}_1(E(\mathbb{C}), \mathbb{Q})$ be the image of $1 \in K$ under the canonical K-isomorphism $K \simeq \mathrm{H}_1(\mathbb{C}/\mathfrak{g}, \mathbb{Q})$ which is induced from the canonical isomorphism $\mathfrak{g} \simeq \mathrm{H}_1(\mathbb{C}/\mathfrak{g}, \mathbb{Z})$. By the K-linearity, it is sufficient to prove 15.9 in the case $\gamma = \beta^{\otimes(-r)}$.

The image of $p^m\beta$ in $\mathrm{H}_1(E(\mathbb{C}), \mathbb{Q}_p) = T_pE \otimes \mathbb{Q}$ is an $O_K \otimes \mathbb{Z}_p$-basis of T_pE which we denote by $\xi = (\xi_n)_{n \geqslant 0}$ (ξ_n is the O_K/p^n-basis of ${}_{p^n}E$ corresponding to ξ). We define torsion points α_n and ν_n ($n \geqslant 0$) on E as follows. For $n \geqslant 0$, let $\alpha_n \in E(K(p^n\mathfrak{f}_0))$ be the image of $p^{m-n}\beta$ in $\mathrm{H}_1(E(\mathbb{C}), \mathbb{Q}/\mathbb{Z}) \simeq E(\mathbb{C})_{\mathrm{tor}}$. So, $\alpha = \alpha_m$. For $n \geqslant 0$, let $\nu_n = \xi_n - \alpha_n$. Then ν_n is killed by $\mathfrak{f}_0$ and belongs to $E(K')$. Let $\theta_n(z) = {}_{\mathfrak{a}}\theta(z + \nu_n)$. We have $\theta_n(\xi_n) = {}_{\mathfrak{a}}\theta(\alpha_n)$. Furthermore $N_p(\theta_{n+1}) = \theta_n$ for $n \geqslant 1$.

By the assumption $\mathfrak{g} \subset \mathfrak{f}$, the action of $\mathrm{Gal}(K^{ab}/K')$ on $T_\ell E$ for any prime ℓ is unramified at any finite place of K' not lying over ℓ. Hence [**ST**], E is of good reduction at any finite place of K'.

We apply the generalized explicit reciprocity law [**KK3**, Thm. 4.3.1]. (We apply the case of height $h = 1$ of this theorem, though we applied the case $h = 2$ of this theorem in § 10.) For a place v of K' lying over p and for a prime ideal $\mathfrak{p}$ of O_K lying over p, let $G(v, \mathfrak{p})$ be the part of the Néron model of E over O_v killed by some power of $\mathfrak{p}$. Then $G(v,\mathfrak{p})$ is a p-divisible group over O_v and the triple $(K'_v, G(v,\mathfrak{p}), O_{\mathfrak{p}})$ satisfies the assumption of the triple (K, G, Λ) of [**KK3**, Thm. 4.3.1] with $h = 1$. By applying this theorem to $(K'_v, G(v,\mathfrak{p}), O_{\mathfrak{p}})$ for all v and $\mathfrak{p}$ as above, we obtain the following result: The image of

$$\xi^{\otimes(-r)} \otimes (\theta_n(\xi_n))_n \in (T_pE)^{\otimes(-r)} \otimes \varprojlim_n \mathrm{H}^1(O_{K(p^n\mathfrak{f})}[1/p], \mathbb{Z}_p(1))$$

under the map

$$\begin{aligned}(T_pE)^{\otimes(-r)} \otimes \varprojlim_n \mathrm{H}^1(O_{K(p^n\mathfrak{f})}[1/p], \mathbb{Z}_p(1)) &\longrightarrow \varprojlim_n \mathrm{H}^1(O_{K(p^n\mathfrak{f})}[1/p], (T_pE)^{\otimes(-r)}(1)) \\ &\longrightarrow \mathrm{H}^1(K' \otimes \mathbb{Q}_p, (T_pE)^{\otimes(-r)}(1)) \xrightarrow{\exp^*} \mathrm{coLie}(E)^{\otimes r} \otimes \mathbb{Q}_p\end{aligned}$$

coincides with

$$p^{-mr} \cdot (r-1)!^{-1} \cdot \Big(\big(\frac{d}{\omega}\big)^r \log(\theta_m)\Big)(\xi_m) \otimes \omega^{\otimes r}$$

where ω is a K'-basis of $\mathrm{coLie}(E)$. This means that the image of $\xi^{\otimes(-r)} \otimes (\theta_n(\xi_n))_n$ under

$$\begin{aligned}V_{L_\lambda}(\psi) \otimes \mathbf{H}^1_{p^\infty\mathfrak{f}}(\mathbb{Z}_p(1)) &\longrightarrow \mathbf{H}^1_{p^\infty\mathfrak{f}}(V_{L_\lambda}(\psi)(1)) \longrightarrow \mathrm{H}^1(O_{K'}[1/p], V_{L_\lambda}(\psi)(1)) \\ &\xrightarrow{\exp^*} \mathrm{D}^1_{\mathrm{dR}}(K' \otimes \mathbb{Q}_p, V_{L_\lambda}(\psi)) \simeq (S(\psi) \otimes_L L_\lambda) \otimes_K K'\end{aligned}$$

coincides with $p^{-mr} \cdot (r-1)!^{-1} \cdot ((\frac{d}{\omega})^r \log(\theta_m))(\xi_m) \otimes \omega^{\otimes r}$. Hence the image of

$$\beta^{\otimes(-r)} \otimes ({}_{\mathfrak{a}}\theta(\alpha_n))_n = p^{mr} \cdot \xi^{\otimes -r} \otimes (\theta_n(\xi_n))_{n \geqslant 1}$$

in $(S(\psi) \otimes_K K') \otimes_L L_\lambda$ is equal

$$\begin{aligned}&(r-1)!^{-1} \cdot ((\frac{d}{\omega})^r \log(\theta_m))(\xi_m) \otimes \omega^{\otimes r} \\ &\qquad = (r-1)!^{-1} \cdot \Big(\big(\frac{d}{\omega}\big)^r \log({}_{\mathfrak{a}}\theta)\Big)(\alpha_m) \otimes \omega^{\otimes r} \in S(\psi) \otimes_K K'.\end{aligned}$$

Hence we are reduced to the following (15.9.1). □

(15.9.1) *Let*

$${}_{\mathfrak{a}}z = (r-1)!^{-1} \cdot \Big(\big(\frac{d}{\omega}\big)^r \log({}_{\mathfrak{a}}\theta)\Big)(\alpha_m) \otimes \omega^{\otimes r} \in S(\psi) \otimes_K K'.$$

Then

$$\sum_\sigma \chi(\sigma)\, \mathrm{per}_\psi(\sigma({}_{\mathfrak{a}}z)) = (N(\mathfrak{a}) - \psi(\mathfrak{a})\chi(\mathfrak{a})^{-1}) L_{p\mathfrak{f}}(\overline{\psi}, \chi, r) \beta^{\otimes(-r)}$$

(σ ranges over $\mathrm{Gal}(K'/K)$*).*

(This (15.9.1) shows

(15.9.2) *For* $z = (N(\mathfrak{a}) - \psi(\mathfrak{a})\sigma_{\mathfrak{a}})^{-1}{}_{\mathfrak{a}}z$, $\sum_\sigma \chi(\sigma)\, \mathrm{per}_\psi(\sigma(z)) = L_{p\mathfrak{f}}(\overline{\psi}, \chi, r)\beta^{\otimes(-r)}$.)

Proof of (15.9.1). — Let $\sigma \in \mathrm{Gal}(K'/K)$ and fix an ideal $\mathfrak{b}$ of O_K which is prime to $\mathfrak{g}$ such that $(\mathfrak{b}, K'/K) = \sigma$. Then

$$\sigma({}_{\mathfrak{a}}z) = \psi(\mathfrak{b})^{-1}\Big(\big(\frac{d}{\omega}\big)^r \log {}_{\mathfrak{a}}\theta_{E/{}_{\mathfrak{b}}E}\Big)(\alpha \bmod {}_{\mathfrak{b}}E) \otimes \omega^{\otimes r}.$$

By the analytic theory of Eisenstein series in 3.8, this is equal to

$$N(\mathfrak{a})\psi(\mathfrak{b})^{-1} \sum_{c \in P(\mathfrak{b})} (c\beta)^{\otimes(-r)}|c|^{-s}\Big|_{s=0} - \psi(\mathfrak{a})\psi(\mathfrak{a}\mathfrak{b})^{-1} \sum_{c \in P(\mathfrak{a}\mathfrak{b})} (c\beta)^{\otimes(-r)}|c|^{-s}\Big|_{s=0}$$

where $P(J) = \{c \in K\,;\, c \equiv 1 \bmod J^{-1}\mathfrak{g}\}$. Let $Q(J)$ be the set of ideals I of O_K such that I is prime to $\mathfrak{g}$ and such that $(I, K'/K) = (J, K'/K)$. Then as is easily seen, we have a bijection $c \mapsto cJ$ from $P(J)$ to $Q(J)$, and $\psi(cJ) = c^r\psi(J)$ for $c \in P(J)$. Hence

$$\begin{aligned}\sigma({}_{\mathfrak{a}}z) &= N(\mathfrak{a}) \sum_{I \in Q(\mathfrak{b})} \psi(I)^{-1}N(I)^{-s}\beta^{\otimes(-r)}\Big|_{s=0} \\ &\qquad - \psi(\mathfrak{a}) \sum_{I \in Q(\mathfrak{a}\mathfrak{b})} \psi(I)^{-1}N(I)^{-s}\beta^{\otimes(-r)}\Big|_{s=0} \\ &= N(\mathfrak{a}) \sum_{I \in Q(\mathfrak{b})} \overline{\psi}(I)N(I)^{-s}\beta^{\otimes(-r)}\Big|_{s=r} \\ &\qquad - \psi(\mathfrak{a}) \sum_{I \in Q(\mathfrak{a}\mathfrak{b})} \overline{\psi}(I)N(I)^{-s}\beta^{\otimes(-r)}\Big|_{s=r}\end{aligned}$$

(since $\psi(I)\overline{\psi}(I) = N(I)^r$). This proves (15.9.1). □

15.10. In the rest of § 15, assume f has CM. Then $L(f, s) = L(\psi, s)$ for a Hecke character ψ of an imaginary quadratic field K of type $(1-k, 0)$ whose conductor divides N. We denote the conductor of ψ by $\mathfrak{f}$.

The field $F = \mathbb{Q}(a_n\,;\, n \geqslant 1)$ is contained in $L = K(\psi(K^\times))$ as is seen from $L(f, s) = L(\psi, s)$.

Let λ be a finite place of L. Then, as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over L_λ, $V_{L_\lambda}(f)$ is isomorphic to the representation

$$V_{L_v}^{\sim}(\psi) = V_{L_v}(\psi) \oplus \iota V_{L_v}(\psi)$$

induced from the representation $V_{L_v}(\psi)$ of the subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Here $\iota \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denotes the complex conjugation, and the action of $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_{L_v}^{\sim}(\psi)$ sends $(x, \iota y)$ $(x, y \in V_{L_v}(\psi))$ to $(\sigma(x), \iota(\iota\sigma\iota)(y))$ if σ belongs to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$, and to $((\iota\tau\iota)(y), \iota\tau(x))$ if $\sigma = \iota\tau$ with $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. This can be seen by comparing the eigenpolynomial of Frobenius of each prime number which is prime to N. (See [**Ri2**].)

Lemma 15.11. — *Fix an isomorphism of one dimensional L-vector spaces*

$$S(\psi) \xrightarrow{\sim} S(f) \otimes_F L. \qquad (15.11.1)$$

Then:

(1) *For each finite place* λ *of* L, *there exists a unique isomorphism of representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *over* L_λ

$$V_{L_v}^{\sim}(\psi) \xrightarrow{\sim} V_{L_v}(f) \tag{15.11.2}$$

such that the composition

$$\begin{aligned} S(\psi)\otimes_L L_\lambda \xrightarrow[\sim]{(15.8.1)} \mathrm{D}^1_{\mathrm{dR}}(K\otimes\mathbb{Q}_p, V_{L_\lambda}(\psi)) = \mathrm{D}^1_{\mathrm{dR}}(V_{L_v}^{\sim}(\psi)) \\ \xrightarrow[\sim]{(15.11.2)} \mathrm{D}^1_{\mathrm{dR}}(V_{L_v}(f)) \simeq S(f)\otimes_F L_\lambda \end{aligned}$$

coincides with the isomorphism induced by (15.11.1).

(2) *Let*

$$V_L^{\sim}(\psi) = V_L(\psi)\oplus \iota V_L(\psi) \qquad (\iota = \textit{the complex conjugation})$$

be the representation of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ *over* L *induced from the trivial representation* $V_L(\psi)$ *of the subgroup* $\{1\}$ *of* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, *and denote the composite* $S(\psi)\to V_L(\psi)\hookrightarrow V_L^{\sim}(\psi)$ *also by* per_ψ. *Then there exists a unique isomorphism of representations of* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ *over* L

$$V_L^{\sim}(\psi) \xrightarrow{\sim} V_L(f) \tag{15.11.3}$$

for which the diagram

$$\begin{tikzcd} S(\psi) \arrow[r, "\mathrm{per}_\psi"] \arrow[d, "(15.11.1)"'] & V_{\mathbb{C}}^{\sim}(\psi) = V_L^{\sim}(\psi)\otimes_L \mathbb{C} \arrow[d, "(15.11.3)"] \\ S(f)\otimes_F L \arrow[r, "\mathrm{per}_\psi"] & V_{\mathbb{C}}(f) \end{tikzcd}$$

is commutative

Proof.

(1) Take any isomorphism $h : V_{L_\lambda}^{\sim}(\psi) \xrightarrow{\sim} V_{L_\lambda}(f)$ of representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over L_λ. Then h induces

$$\mathrm{D}^1_{\mathrm{dR}}(V_{L_\lambda}^{\sim}(\psi)) \xrightarrow{\sim} \mathrm{D}^1_{\mathrm{dR}}(V_{L_\lambda}(f))$$

and hence

$$S(\psi)\otimes_L L_\lambda \xrightarrow{\sim} S(f)\otimes_F L_\lambda$$

which is c times the isomorphism induced by (15.11.1) for some $c\in L_\lambda^\times$. The isomorphism $c^{-1}h$ is the desired one. The uniqueness follows from the irreducibility of $V_{L_\lambda}^{\sim}(\psi)$ as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over L_λ.

(2) Fix a sign $\pm$. Identify $S(\psi)$ with $S(f)\otimes_F L$ via (15.11.1). It is sufficient to show that there exists an isomorphism of one dimensional L-vector spaces

$$h : V_L^{\sim}(\psi)^{\pm} \xrightarrow{\sim} V_L(\psi)^{\pm}$$

such that $h \circ \mathrm{per}^{\pm}_{\psi} = \mathrm{per}^{\pm}_{f}$. Here

$$\mathrm{per}^{\pm}_{\psi} = \frac{1 \pm \iota}{2} \circ \mathrm{per}_{\psi}, \quad \mathrm{per}^{\pm}_{f} = \frac{1 \pm \iota}{2} \circ \mathrm{per}_{f}.$$

By 6.6, 13.5 and (15.9.2), there exist $m \geqslant 1$, elements

$$z \in S(\psi) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_m), \quad z' \in S(f) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_m),$$

and non-zero elements

$$\gamma \in V^{\sim}_{L}(\psi)^{\pm}, \quad \gamma' \in V_F(f)^{\pm},$$

satisfying the following (i) (ii).

(i) For any character $\chi : \mathrm{Gal}(\mathbb{Q}(\zeta_m)/Q) \to \mathbb{C}^{\times}$ such that $\chi(-1) = \pm$,

$$\sum_{\sigma} \chi(\sigma) \,\mathrm{per}^{\pm}_{\psi}(\sigma(z)) = L_S(\psi, \chi, k-1)\gamma,$$

$$\sum_{\sigma} \chi(\sigma) \,\mathrm{per}^{\pm}_{f}(\sigma(z')) = L_S(f, \chi, k-1)\gamma',$$

where σ ranges over $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ and $S = \mathrm{prime}(mN)$.

(ii) There exists a character $\chi : \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \to \mathbb{C}^{\times}$ such that $\chi(-1) = \pm$ and such that $L_S(\psi, \chi, k-1) = L_S(f, \chi, k-1)$ is not zero.

By (ii), there exists $\sigma_0 \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ such that the image of $\sigma_0(z)$ under

$$S(\psi) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_m) \longrightarrow S(\psi) \otimes_L L(\zeta_m)$$

is not zero. Let b be the element of $L(\zeta_m)$ such that the image of $\sigma_0(z')$ in

$$S(f) \otimes_F L(\zeta_m) = S(\psi) \otimes_L L(\zeta_m)$$

is b times the image of $\sigma_0(z)$. We will show that b is a non-zero element of L. The L-linear map

$$h : V^{\sim}_{L}(\psi)^{\pm} \longrightarrow V_L(f)^{\pm}$$

which sends γ to $b^{-1}\gamma'$ satisfies $h \circ \mathrm{per}^{\pm}_{\psi} = \mathrm{per}^{\pm}_{f}$. Now we prove $b \in L^{\times}$. From (i) (by taking $\sum_{\chi} \chi(\tau)^{-1}$ ((i) for χ) where χ ranges over all characters $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \to \mathbb{C}^{\times}$ such that $\chi(-1) = \pm$), we see that for each $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, there exists $c_{\sigma} \in \mathbb{C}$ such that

$$\mathrm{per}^{\pm}_{\psi}(\sigma(z) \pm \sigma\iota(z)) = c_{\sigma}\gamma, \qquad \mathrm{per}^{\pm}_{f}(\sigma(z') \pm \sigma\iota(z')) = c_{\sigma}\gamma',$$

where ι is the complex conjugation. For $\sigma = \tau\sigma_0$ with $\tau \in \mathrm{Gal}(L(\zeta_m)/L)$, we have $c_{\sigma} \neq 0$ and

$$\tau(b) \cdot \mathrm{per}^{\pm}_{f}(\sigma(z) \pm \sigma\iota(z)) = c_{\sigma}\gamma'.$$

Hence $b \neq 0$, and $\tau(b)$ is independent of $\tau \in \mathrm{Gal}(L(\zeta_m)/L)$. This shows $b \in L^{\times}$. □

In the rest of § 15, we fix an isomorphism (15.11.1). In what follows, we identify $S(\psi)$ and $S(f) \otimes_F L$ via (15.11.1). We also identify the representations $V^{\sim}_{L_\lambda}(\psi)$ and $V_{L_\lambda}(f)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via (15.11.2) for any finite place λ of L.

The "philosophy of motif" tells that the isomorphism (15.11.3) should be compatible with our identification (15.11.2), but I can not prove it. (The problem is that in the case $k \geqslant 3$, it is not known that the motif associated to f [**Sc1**] and the motif associated to ψ coincide. Here, the former motif is a direct summand of the motif associated to the Kuga-Sato variety, and the latter motif is obtained from the $(k-1)$-fold tensor power of the motif associated to an elliptic curve with complex multiplication.) So to avoid the confusion, we will never use (15.11.3) as identification.

15.12. Let λ be a finite place of L. Then we have a canonical homomorphism of $O_\lambda[[G_{p^\infty\mathfrak{f}}]]$-modules

$$\mathbf{H}^1_{p^\infty\mathfrak{f}}(V_{L_\lambda}(\psi)) \longrightarrow \mathbf{H}^1(V_{L_\lambda}(f)) \tag{15.12.1}$$

since

$$\mathbf{H}^1(V_{L_\lambda}(f)) = \mathbf{H}^1(V_{L_\lambda}^{\sim}(\psi)) = \big(\varprojlim\nolimits_n \mathrm{H}^1(O_{K\otimes\mathbb{Q}(\zeta_{p^n})}[1/p], T)\big) \otimes \mathbb{Q}$$

where T is any $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-stable O_λ-lattice of $V_{L_\lambda}(\psi)$, and since we have a canonical homomorphism

$$K \otimes \mathbb{Q}(\zeta_{p^\infty}) \longrightarrow K(p^\infty\mathfrak{f}).$$

By 15.9, we have:

(15.12.2) *Let $\gamma \in V_L(\psi)$ and let $\gamma' \in V_L(f)$ be the image of γ under the isomorphism* (15.11.3). *For $n \geqslant 0$, consider the composite map*

$$\begin{aligned}\mathbf{H}^1_{p^\infty\mathfrak{f}}(V_{L_\lambda}(\psi)(1)) &\xrightarrow{(15.12.1)} \mathbf{H}^1(V_{L_\lambda}(f)(1)) \\ &\longrightarrow \mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), V_{L_\lambda}(f)(1)) \\ &\xrightarrow{\exp^*} \mathrm{D}^1_{\mathrm{dR}}(V_{L_\lambda}(f)) \otimes \mathbb{Q}(\zeta_{p^n}) \\ &\simeq S(f) \otimes_F L_\lambda \otimes \mathbb{Q}(\zeta_{p^n}).\end{aligned}$$

Let $\mathcal{S}$ be the set of non-zero-divisors of $\mathbb{Z}_p[[G_{p^\infty\mathfrak{f}}]]$ whose images in $\mathbb{Q}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})]$ are invertible. Then the induced map

$$\mathcal{S}^{-1}\mathbf{H}^1_{p^\infty\mathfrak{f}}(V_{L_\lambda}(\psi)(1)) \longrightarrow S(f) \otimes_F L_\lambda \otimes \mathbb{Q}(\zeta_{p^n})$$

sends $z_{p^\infty\mathfrak{f}} \otimes \gamma$ to an element of

$$S(f) \otimes_F L \otimes \mathbb{Q}(\zeta_{p^n})$$

whose image under $\sum_\sigma \chi(\sigma)\,\mathrm{per}_f \circ \sigma$, where χ is any homomorphism

$$\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \longrightarrow \mathbb{C}^\times$$

and σ ranges over $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$, coincides with

$$L_{\{p\}}(f^*, \chi, k-1) \cdot (\gamma')^\pm, \qquad \pm = \chi(-1).$$

Lemma 15.13. — *Let p be a prime number and assume K is not contained in $\mathbb{Q}(\zeta_{p^\infty})$. Let $\mathfrak{f}$ be the conductor of ψ, Δ the torsion part of $G_{p^\infty\mathfrak{f}}$, λ a finite place of L lying over p, $\mathfrak{p}$ a prime ideal of $O_\lambda[[G_\infty]]$, and $\mathfrak{q}$ the inverse image of $\mathfrak{p}$ in $O_\lambda[[G_{p^\infty\mathfrak{f}}]]$, under the surjection $O_\lambda[[G_{p^\infty\mathfrak{f}}]] \to O_\lambda[[G_\infty]]$. Assume that 2 and the order of Δ are invertible in the residue field of $\mathfrak{p}$. Then:*

(1) *$O_\lambda[[G_\infty]]_\mathfrak{p}$ and $O_\lambda[[G_{p^\infty\mathfrak{f}}]]_\mathfrak{q}$ are regular rings, and the kernel of $O_\lambda[[G_{p^\infty\mathfrak{f}}]]_\mathfrak{q} \to O_\lambda[[G_\infty]]_\mathfrak{p}$ is a principal ideal.*

(2) *Let a be a generator of the principal ideal in (1). Let T be a $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-stable O_λ-lattice of $V_{L_\lambda}(\psi)$ and let $T^\sim = T \oplus \iota T \subset V_{L_\lambda}^\sim(\psi)$.*

Then we have

$$\mathbf{H}^2_{p^\infty\mathfrak{f}}(T)_\mathfrak{q}/a\mathbf{H}^2_{p^\infty\mathfrak{f}}(T)_\mathfrak{q} \simeq \mathbf{H}^2(T^\sim)_\mathfrak{p} \tag{15.13.1}$$

and an exact sequence

$$0 \longrightarrow \mathbf{H}^1_{p^\infty\mathfrak{f}}(T)_\mathfrak{q}/a\mathbf{H}^1_{p^\infty\mathfrak{f}}(T)_\mathfrak{q} \longrightarrow \mathbf{H}^1(T^\sim)_\mathfrak{p} \longrightarrow \mathrm{Ker}(a; \mathbf{H}^2_{p^\infty\mathfrak{f}}(T))_\mathfrak{q} \longrightarrow 0. \tag{15.13.2}$$

Proof. — Consider the exact sequence of representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$

$$0 \longrightarrow T \otimes_{O_\lambda} O_\lambda[[G_{p^\infty\mathfrak{f}}]] \xrightarrow{a} T \otimes_{O_\lambda} O_\lambda[[G_{p^\infty\mathfrak{f}}]] \longrightarrow T \otimes_{O_\lambda} O_\lambda[[G_\infty]] \longrightarrow 0$$

where $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts on $O_\lambda[[G_{p^\infty\mathfrak{f}}]]$ (resp. $O_\lambda[[G_\infty]]$) by the multiplication by the image of σ^{-1} in $G_{p^\infty\mathfrak{f}}$ (resp. G_∞). Let v be a finite place of K which does not lie over p, and let $I_v \subset \mathrm{Gal}(\overline{K}_v/K_v)$ be the inertia subgroup. Then the cokernel of

$$\mathrm{H}^0(I_v, T \otimes_{O_\lambda} O_\lambda[[G_{p^\infty\mathfrak{f}}]]) \longrightarrow \mathrm{H}^0(I_v, T \otimes_{O_\lambda} O_\lambda[[G_\infty]])$$

is killed by the order of Δ. This is because the action of I_v on $T \otimes_{O_\lambda} O_\lambda[[G_{p^\infty\mathfrak{f}}]]$ factors through a homomorphism $I_v \to \Delta \subset G_{p^\infty\mathfrak{f}}$. Hence if j denotes the inclusion map $\mathrm{Spec}(O_K[1/p\mathfrak{f}]) \to \mathrm{Spec}(O_K[1/p])$, the cokernel of the last arrow of the exact sequence

$$0 \longrightarrow j_*(O_\lambda[[G_{p^\infty\mathfrak{f}}]]) \xrightarrow{a} j_*(O_\lambda[[G_{p^\infty\mathfrak{f}}]]) \longrightarrow j_*(T \otimes_{O_\lambda} O_\lambda[[G_\infty]])$$

is killed by the order of Δ. Hence this exact sequence induces an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbf{H}^1_{p^\infty\mathfrak{f}}(T)_\mathfrak{q} &\xrightarrow{a} \mathbf{H}^1_{p^\infty\mathfrak{f}}(T)_\mathfrak{q} \longrightarrow \mathbf{H}^1(T^\sim)_\mathfrak{p} \\ \longrightarrow \mathbf{H}^2_{p^\infty\mathfrak{f}}(T)_\mathfrak{q} &\xrightarrow{a} \mathbf{H}^2_{p^\infty\mathfrak{f}}(T)_\mathfrak{q} \longrightarrow \mathbf{H}^2(T^\sim)_\mathfrak{p} \longrightarrow 0. \end{aligned}$$ □

15.14. In the case $K \subset \mathbb{Q}(\zeta_{p^\infty})$, Lemma 15.13 is modified as follows. Let $G'_\infty = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/K) \subset G_\infty$, and let $\mathbf{H}^q(T) = \varprojlim_n \mathrm{H}^q(\mathbb{Z}[\zeta_{p^n}, 1/p], T)$. Then $\mathbf{H}^q(T^\sim) \simeq \mathbf{H}^q(T) \otimes_{O_\lambda[[G'_\infty]]} O_\lambda[[G_\infty]]$. Lemma 15.13 holds when we replace G_∞ by G'_∞, and $\mathbf{H}^q(T^\sim)$ by $\mathbf{H}^q(T)$.

15.15. We prove Thm. 12.4 for f.

Once we prove Thm. 12.4 (1) (that $\mathbf{H}^2(V_{F_\lambda}(f))$ is a torsion $O_\lambda[[G_\infty]]$-module), we can deduce Thm. 12.4 (2) (3) from it by the argument in 13.8. We prove Thm. 12.4 (1). Let v be a place of L lying over λ. Since $\mathbf{H}^2(V_{F_\lambda}(f)) \otimes_{O_\lambda} O_v \simeq \mathbf{H}^2(V_{L_v}^{\sim}(\psi))$, it is sufficient to prove that $\mathbf{H}^2(V_{L_v}^{\sim}(\psi))$ is a torsion $O_v[[G_\infty]]$-module. Now by writing v as λ, let λ be a place of L lying over a prime number p.

We first assume K is not contained in $\mathbb{Q}(\zeta_{p^\infty})$. In lemma 15.13, let $\mathfrak{p}$ be a prime ideal of height 0 of $O_\lambda[[G_\infty]]$. Then $\mathfrak{q}$ is a prime ideal of height 1 of $O_\lambda[[G_{p^\infty\mathfrak{f}}]]$. By 13.5 and (15.12.2), the image of $z_{p^\infty\mathfrak{f}} \otimes V_{L_\lambda}(\psi)(-1)$ under

$$\mathbf{H}^1_{p^\infty\mathfrak{f}}(V_{L_\lambda}(\psi))_\mathfrak{q} \longrightarrow \mathbf{H}^1(V_{L_\lambda}(\psi))_\mathfrak{p}$$

is not zero. Since $\mathbf{H}^1_{p^\infty\mathfrak{f}}(V_{L_\lambda}(\psi))_\mathfrak{q}$ is a free $O_\lambda[[G_{p^\infty\mathfrak{f}}]]_\mathfrak{q}$-module of rank 1, this shows that the $O_\lambda[[G_{p^\infty\mathfrak{f}}]]_\mathfrak{q}$-module $\mathbf{H}^1_{p^\infty\mathfrak{f}}(V_{L_\lambda}(\psi))_\mathfrak{q}$ is generated by the image of $z_{p^\infty\mathfrak{f}} \otimes V_{L_\lambda}(\psi)$. Hence by the theorem 15.2 by Rubin, we have $\mathbf{H}^2_{p^\infty\mathfrak{f}}(V_{L_\lambda}(\psi))_\mathfrak{q} = 0$. Hence by (15.13.1), $\mathbf{H}^2(V_{L_\lambda}^{\sim}(\psi))_\mathfrak{p} = 0$. This shows that $\mathbf{H}^2(V_{L_\lambda}^{\sim}(\psi))$ is a torsion $O_\lambda[[G_\infty]]$-module.

The proof for the case $K \subset \mathbb{Q}(\zeta_{p^\infty})$ goes similarly by using 15.14 instead of 15.13. □

15.16. We prove Thm. 12.5 and Thm. 12.6 for f in the case K is not contained in $\mathbb{Q}(\zeta_{p^\infty})$. Since we have already proved Thm. 12.4 for f, Thm. 12.5 (1) (2) and Thm. 12.6 are proved by the same arguments in 13.9-13.13.

Next we prove Thm. 12.5 (3) (Thm. 12.5 (4) does not exist in the case with complex multiplication). By (15.12.2) and Thm. 12.4 (2), we have

(15.16.1) *Let $\gamma \in V_L(\psi)$ and let γ' be the image of γ in $V_L(f)$ under* (15.11.3). *Then the homomorphism* (15.12.1) *sends $z_{p^\infty} \otimes \gamma \otimes (\zeta_{p^n})_n^{\otimes(-1)}$ to $\mathbf{z}_{\gamma'}^{(p)}$.*

Hence 12.5 (3) is reduced to

Proposition 15.17. — *Let λ be a finite place of L lying over a prime number p, let $\mathfrak{p}$ be a prime ideal of $O_\lambda[[G_\infty]]$ of height* 1, *and assume either (a) or (b) in Thm. 15.2 is satisfied. Let T be a $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-stable O_λ-lattice in $V_{L_\lambda}(\psi)$ and let $Z(\psi, T)_\mathfrak{p}$ be the $\mathbb{Z}_p[[G_\infty]]_\mathfrak{p}$-submodule of $\mathbf{H}^1(T^\sim)_\mathfrak{p}$ generated by the image of $z_{p^\infty\mathfrak{f}} \otimes T(-1)$ under $\mathbf{H}^1_{p^\infty\mathfrak{f}}(T) \to \mathbf{H}^1(T^\sim)_\mathfrak{p}$. Then*

$$\mathrm{length}_{O_\lambda[[G_\infty]]_\mathfrak{p}}(\mathbf{H}^2(T^\sim)_\mathfrak{p}) \leqslant \mathrm{length}_{O_\lambda[[G_\infty]]_\mathfrak{p}}(\mathbf{H}^1(T^\sim)_\mathfrak{p}/Z(\psi, T)_\mathfrak{p}).$$

Proof. — Assume first K is not contained in $\mathbb{Q}(\zeta_{p^\infty})$. Let $\mathfrak{q}$ be the inverse image of $\mathfrak{p}$ in $O_\lambda[[G_{p^\infty\mathfrak{f}}]]$. We apply Lemma 14.12 to $A = O_\lambda[[G_{p^\infty\mathfrak{f}}]]_\mathfrak{q}$ and to the A-modules $\mathbf{H}^2_{p^\infty\mathfrak{f}}(T)_\mathfrak{q}$ and $\mathbf{H}^1_{p^\infty\mathfrak{f}}(T)_\mathfrak{q}/A\cdot(z_{p^\infty\mathfrak{f}} \otimes T(-1))$. Then by (15.13.1) and (15.13.2), Prop. 15.17 follows from theorem 15.2 of Rubin.

The proof for the case $K \subset \mathbb{Q}(\zeta_{p^\infty})$ goes similarly by using 15.14 instead of 15.13. □

15.18. We can prove Thm. 14.5 (1)(2) for f by using Thm. 12.5 (3) in the same way as in § 14. (Thm. 14.5 (3) does not exist in the case with complex multiplication.)

15.19. Here we give the proof of Lemma 14.7 in the case f has complex multiplication.

The following fact is proved easily : For a finite dimensional vector space V over a complete discrete valuation field P and for a finite extension P' of P, the canonical map

$$\{O_P\text{-lattices in } V\}/\sim \longrightarrow \{O_{P'}\text{-lattices in } P' \otimes_P V\}/\sim$$

is injective, where the first $\sim$ (resp. the second $\sim$) is the equivalence by multiplications by $P^\times$ (resp. $(P')^\times$).

Take an ideal $\mathfrak{a}$ of O_K which is prime to N such that $\psi(\mathfrak{a}) \neq \psi(\overline{\mathfrak{a}})$. We show that the condition in 14.7 is satisfied by all finite places λ of F which do not divide $(\psi(\mathfrak{a}) - \psi(\overline{\mathfrak{a}}))N(\mathfrak{a})$. By the above remark, we may assume $P \supset L_v$ for some place v of L lying over λ. Let e be a P-basis of $V_P(\psi)$, and let T be a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_P-lattice of $V_P^\sim(\psi)$ ($\simeq V_{F_\lambda}(f) \otimes_{F_\lambda} P$). We show that $T = a \cdot (O_P e + O_P \iota e)$ for some $a \in P^\times$. Let $a_1 e + a_2 \iota e \in T$ ($a_i \in P$). Let p be the prime number lying under λ. By applying $(\mathfrak{a}, K(p^\infty \mathfrak{f})/K)$ to $a_1 e + a_2 \iota e$ and by using $\iota(\mathfrak{a}, K(p^\infty \mathfrak{f})/K)\iota = (\overline{\mathfrak{a}}, K(p^\infty \mathfrak{f})/K)$, we have $\psi(\mathfrak{a})^{-1} a_1 e + \psi(\overline{\mathfrak{a}})^{-1} a_2 \iota e \in T$. (Note $\psi(\mathfrak{a}), \psi(\overline{\mathfrak{a}}) \in O_P^\times$ as is easily seen.) By $\psi(\mathfrak{a}) - \psi(\overline{\mathfrak{a}}) \in O_P^\times$, we have $a_1 e \in T$. By applying ι, we have $a_1 \iota e \in T$. Similarly we have $a_2 e, a_2 \iota e \in T$. This shows that $T = a \cdot (O_P e + O_P \iota e)$ where a is a generator of the fractional O_P-ideal generated by all $a_1, a_2 \in P$ such that $a_1 e + a_2 \iota e \in T$.

Lemma 15.20. — *Almost all finite places λ of F have the following property: For any* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-stable O_λ-lattice T of $V_{F_\lambda}(f)$, $T/m_\lambda T$ is irreducible as a representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Proof. — By 14.7, it is sufficient to prove that for almost all finite places λ of L and for any $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-stable lattice T of $V_{L_\lambda}(\psi)$, $T^\sim/m_\lambda T^\sim$ is irreducible as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Take an ideal $\mathfrak{a}$ of O_K which is prime to N such that $\psi(\mathfrak{a}) \neq \psi(\overline{\mathfrak{a}})$. By the similar argument as in 15.19, we can see that any finite place λ of L which does not divide $(\psi(\mathfrak{a}) - \psi(\overline{\mathfrak{a}}))N(\mathfrak{a})$ has this property. □

Proposition 15.21. — *Almost all finite places λ of F have the following property: For any* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-stable O_λ lattice T of $V_{F_\lambda}(f)$,*

$$Z(f, T) \subset \mathbf{H}^1(T) \quad \textit{in } \mathbf{H}^1(T) \otimes \mathbb{Q},$$

and

$$\mathrm{length}_{\Lambda_\mathfrak{p}}(\mathbf{H}^2(T)_\mathfrak{p}) \leqslant \mathrm{length}_{\Lambda_\mathfrak{p}}(\mathbf{H}^1(T)_\mathfrak{p}/Z(f,T)_\mathfrak{p})$$

for any prime ideal $\mathfrak{p}$ of Λ of height one.

Proof. — The first part is proved in the same way as in the non-CM case in 13.14 by 12.4 (1), 12.6, 14.7, 15.20.

We prove the second part (the property about length). If p is an odd prime number which does not divide the order of $\mathrm{Gal}(K(\mathfrak{f})/K)$ and which is unramified in K, then p does not divide the order of the torsion part of $G_{p^\infty\mathfrak{f}}$ as is easily seen. Hence by Prop. 15.17 and lemma 14.7, we are reduced to

Lemma 15.22. — *Let γ be a non-zero element of $V_L(\psi)$ and let γ' be an element of $V_L(f)$ such that $(\gamma')^+ \neq 0, (\gamma')^- \neq 0$. Then for almost all finite places λ of L, $\gamma'^+ = u\gamma^+$ and $\gamma'^- = v\gamma^-$ in $V_{L_\lambda}^-(\psi) = V_{L_\lambda}(f)$ for some $u, v \in O_\lambda^\times$.*

Proof. — Fix an L-basis ω of $S(\psi) = S(f) \otimes_F L$. For almost all finite places λ of L, the O_λ lattices

$$T_\lambda \underset{\text{def}}{=} O_\lambda\gamma^+ + O_\lambda\gamma^- \quad \text{and} \quad T'_\lambda \underset{\text{def}}{=} O_\lambda \cdot (\gamma')^+ + O_\lambda \cdot (\gamma')^-$$

of $V_{L_\lambda}^\sim(\psi) = V_{L_\lambda}(f)$ are $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable. For almost all λ, $T_\lambda = T(D_\lambda)$ and $T'_\lambda = T(D'_\lambda)$ for strongly divisible lattices D_λ and D'_λ of $\mathrm{D_{crys}}(V_{L_\lambda}^\sim(\psi)) = \mathrm{D_{crys}}(V_{L_\lambda}(f))$ such that ω is an O_λ-basis of $D_\lambda^1 = D_\lambda^{k-1}$ and also is an O_λ-basis of $(D'_\lambda)^1 = (D'_\lambda)^{k-1}$. For almost all λ, $T'_\lambda = a_\lambda \cdot T_\lambda$ for some $a_\lambda \in L_\lambda^\times$ (14.7) and this implies $D'_\lambda = a_\lambda D_\lambda$ and hence $(D'_\lambda)^1 = a_\lambda \cdot D_\lambda^1$. For almost all λ, since ω is an O_λ-basis of D_λ^1 and also an O_λ-basis of $(D'_\lambda)^1$, we have $a_\lambda \in O_\lambda^\times$ and hence $T'_\lambda = T_\lambda$. This proves Lemma 15.22. □

15.23. From 15.22, we can deduce the following result by the argument in § 14: *Let $r \in \mathbb{Z}$, $1 \leqslant r \leqslant k/2$. In the case $r = k/2$, assume $L(f, k/2) \neq 0$. Then for almost all places λ of F and for all* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-stable O_λ-lattices T of $V_{F_\lambda}(f)(r)$, the number μ in Prop. 14.16 (2) satisfies*

$$\mu \geqslant 1.$$

By this, the proof of Thm. 14.2 goes in the same way as in the non-CM-case.

CHAPTER IV
IWASAWA THEORY FOR MODULAR FORMS (WITH p-ADIC ZETA FUNCTIONS)

In this chapter, we study the Iwasawa theory concerning p-adic zeta functions of modular forms, and p-adic Birch and Swinnerton-Dyer conjectures for modular forms.

As in Chap. III, we fix $k \geqslant 2, N \geqslant 1$, and a normalized newform

$$f = \sum_{n \geqslant 1} a_n q^n \in S_k(X_1(N)) \otimes \mathbb{C}.$$

We also fix a prime number p and a place λ of $F = \mathbb{Q}(a_n\,;\, n \geqslant 1)$ lying over p.

We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We fix an algebraic closure $\overline{F}_\lambda$ of F_λ and an embedding $\overline{\mathbb{Q}} \to \overline{F}_\lambda$ over F_λ.

16. The p-adic zeta function

In this section, we review the theory of p-adic zeta function of f, and then show (Thm. 16.6) that the p-adic zeta function of f is the image of the p-adic zeta element of $f^* = \sum_{n\geqslant 1} \overline{a}_n q^n$ under a homomorphism of Perrin-Riou. This in fact provides a new construction of the p-adic zeta function of f.

16.1. We assume that there exists $\alpha \in (\overline{F}_\lambda)^\times$ such that

$$1 - \alpha u | 1 - a_p u + \varepsilon(p) p^{k-1} u^2 \qquad \text{in } \overline{F}_\lambda[u]$$

and

$$\mathrm{ord}_p(\alpha) < k - 1$$

where ord_p is the additive valuation of $\overline{F}_\lambda$ normalized by $\mathrm{ord}_p(p) = 1$. The p-adic zeta function of f is defined after we fix such α.

The p-adic zeta function of f corresponding to α lives in a certain ring $\mathscr{H}_{\infty,L}$ with $L = F_\lambda(\alpha)$ which contains $O_L[[G_\infty]]$ as a subring. We introduce the ring $\mathscr{H}_{\infty,L}$.

Let $G_n = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}), G_\infty = \varprojlim_n G_n$, and let $G_\infty = \Delta \times G^1_\infty$ be the decomposition in 12.1. Let u be a topological generator of G^1_∞. Then, for a finite extension L of $\mathbb{Q}_p$, $O_L[[G_\infty]]$ is identified with the ring $O_\lambda[\Delta][[u-1]]$ of formal power series over the group ring $O_\lambda[\Delta]$ in one variable $u - 1$.

For $h \geqslant 1$, let

$$\mathscr{H}_{h,L} = \Big\{ \sum_{\substack{n\geqslant 0\\ \sigma\in\Delta}} c_{n,\sigma} \cdot \sigma \cdot (u-1)^n \in L[\Delta][[u-1]];\ \lim_{n\to\infty} |c_{n,\sigma}|_p \cdot n^{-h} = 0 \text{ for all } \sigma \in \Delta \Big\}$$

where $|\ |_p$ denotes the multiplicative valuation of L normalized by $|p|_p = \frac{1}{p}$. Then

$$O_L[[G_\infty]] \subset \mathscr{H}_{1,L} \subset \mathscr{H}_{2,L} \subset \mathscr{H}_{3,L} \subset \cdots$$

Define

$$\mathscr{H}_{\infty,L} = \bigcup_h \mathscr{H}_{h,L}.$$

Then $\mathscr{H}_{\infty,L}$ is a ring since $\mathscr{H}_{i,L} \cdot \mathscr{H}_{j,L} \subset \mathscr{H}_{i+j,L}$ for any $i, j \geqslant 1$. We defined $\mathscr{H}_{h,L}$ and $\mathscr{H}_{\infty,L}$ by fixing u, but they are in fact independent of the choice of u in the following sense.

Let

$$X(G_\infty) = \mathrm{Hom}_{\mathrm{cont}}(G_\infty, \overline{L}^\times). \tag{16.1.1}$$

For $\chi \in X(G_\infty)$, we have a ring homomorphism

$$\begin{aligned} \mathscr{H}_{\infty,L} &\longrightarrow \overline{L} \\ \mu &\longmapsto \mu(\chi) \\ \sum_{\substack{n\geqslant 0 \\ \sigma\in\Delta}} c_{n,\sigma}\cdot\sigma\cdot(u-1)^n &\longmapsto \sum_{\substack{n\geqslant 0 \\ \sigma\in\Delta}} c_{n,\sigma}\cdot\chi(\sigma)\cdot(\chi(u)-1)^n. \end{aligned}$$

The homomorphism $\mathscr{H}_{\infty,L} \to \mathrm{Map}(X(G_\infty),\overline{L})$; $\mu \mapsto (\chi \mapsto \mu(\chi))$ is injective, and $\mathscr{H}_{\infty,L}$ is identified with a subring of $\mathrm{Map}(X(G_\infty),\overline{L})$. This subring is independent of the choice of u, and for $h \geqslant 1$, $\mathscr{H}_{h,L}$ regarded as a subset of $\mathrm{Map}(X(G_\infty),\overline{L})$ is independent of the choice of u.

***Theorem 16.2* (Amice-Vélu [AV], Vishik [Vi]).** — *Fix $\alpha \in (\overline{F}_\lambda)^\times$ as above. Fix also a non-zero element ω of $S(f^*)$, and a non-zero element $\gamma \in V_F(f^*)$ such that $\gamma^+ \neq 0$, $\gamma^- \neq 0$. Define $\Omega_+,\Omega_- \in \mathbb{C}^\times$ by*

$$\mathrm{per}(\omega) = \Omega_+\gamma^+ + \Omega_-\gamma^-.$$

Then there exist a unique element

$$L_{p\text{-adic},\alpha,\omega,\gamma}(f) \in \mathscr{H}_{k-1,F_\lambda(\alpha)}$$

(which we denote $L_{p\text{-adic},\alpha}(f)$ for simplicity) having the following properties (i) (ii) for any integer r such that $1 \leqslant r \leqslant k-1$.

(i) *Let $n \geqslant 1$, let $\chi : G_n \to \overline{F}_\lambda^\times$ be a homomorphism which does not factor through G_{n-1}, and regard $\kappa^r\chi$ as an element of $X(G_\infty)$* (16.1.1). *Then*

$$L_{p\text{-adic},\alpha}(f)(\kappa^r\chi^{-1}) = (r-1)!\cdot p^{nr}\alpha^{-n}\cdot G(\chi,\zeta_{p^n})^{-1}\cdot(2\pi i)^{k-r-1}\cdot\frac{1}{\Omega_\pm}\cdot L_{\{p\}}(f,\chi,r)$$

(both sides belong to $\overline{\mathbb{Q}}$, and the equality holds in $\overline{\mathbb{Q}}$) where $\pm = (-1)^{k-r-1}\chi(-1)$, χ is regarded here as a character of $(\mathbb{Z}/p^n)^\times$ via the cyclotomic character $G_n \simeq (\mathbb{Z}/p^n)^\times$, and $G(\chi,\zeta_{p^n})$ means the Gauss sum $\sum_b \chi(b)\zeta_{p^n}^b$ where b ranges over all elements of $(\mathbb{Z}/p^n)^\times$.

(ii)

$$L_{p\text{-adic},\alpha}(f)(\kappa^r) = (r-1)!\cdot(2\pi i)^{k-r-1}\cdot\frac{1}{\Omega_\pm}\cdot(1-p^{r-1}\alpha^{-1})(1-\varepsilon(p)p^{k-r-1}\alpha^{-1})\cdot L(f,r)$$

where $\pm = (-1)^{k-r-1}$.

Remark 16.3

(1) Let $h \geqslant 1$ and let $\mu,\mu' \in \mathscr{H}_{h,L}$. Let $r(1),\dots,r(h)$ be distinct h integers, and assume

$$\mu(\kappa^{r(i)}\chi) = \mu'(\kappa^{r(i)}\chi)$$

for any $i = 1,\dots,h$ and for almost all elements $\chi \in X(G_\infty)$ of finite orders. Then $\mu = \mu'$. Hence the property (i) in Thm. 16.2 characterizes the element $L_{p\text{-adic},\alpha}(f)$ of $\mathscr{H}_{k-1,F_\lambda(\alpha)}$.

(2) For $a,b \in F^\times$, we have $L_{p\text{-adic},\alpha,a\omega,b\gamma}(f) = a^{-1}b\cdot L_{p\text{-adic},\alpha,\omega,\gamma}(f)$.

The following theorems follows from the work of Perrin-Riou [**Pe2**].

Theorem 16.4 **(Perrin-Riou).** — *Let L be a finite extension of $\mathbb{Q}_p$ and let V be a finite dimensional L-vector space endowed with a continuous L-linear action of* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. *Assume V is a de Rham representation of* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, *and assume*

$$(16.4.1) \qquad \mathrm{D}_{\mathrm{crys}}(V^*(1)) \subset \mathrm{D}^0_{\mathrm{dR}}(V^*(1)) \quad \textit{in } \mathrm{D}_{\mathrm{dR}}(V^*(1))$$

where $V^* = \mathrm{Hom}_L(V, L)$ *endowed with the dual action of* $\mathrm{Gal}(\overline{L}/L)$. *Let* $\eta \in \mathrm{D}_{\mathrm{crys}}(V^*(1))$. *Then there exists a unique homomorphism*

$$\mathfrak{L}_\eta : \mathbf{H}^1_{\mathrm{loc}}(V) \longrightarrow \mathscr{H}_{\infty,L}$$

having the following properties (i) (ii) for any integer $r \geqslant 1$.

(i) *Let $n \geqslant 1$, and let $\chi : G_n \to \overline{L}^\times$ be a homomorphism which does not factor through G_{n-1}. Then for any $x \in \mathbf{H}^1_{\mathrm{loc}}(V)$, we have*

$$\mathfrak{L}_\eta(x)(\kappa^r \chi^{-1}) = (r-1)! \cdot G(\chi, \zeta_{p^n})^{-1} \sum_{\sigma \in G_n} \chi(\sigma) \langle \sigma(\exp^*(x_{-r,n})), (p^{-r}\varphi)^{-n}(\eta) \rangle.$$

Here $x_{-r,n}$ denotes the image of x under the composite

$$\mathbf{H}^1_{\mathrm{loc}}(V) \xrightarrow{\sim} \mathbf{H}^1_{\mathrm{loc}}(V(-r)) \xrightarrow{\mathrm{proj}} \mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), V(-r))$$

where the first arrow is the product with $((\zeta_{p^j})^{\otimes(-r)})_{j \geqslant 1}$ and the second arrow is the canonical projection (so $\exp^*(x_{-r,n})$ *is an element of* $\mathbb{Q}(\zeta_{p^n}) \otimes \mathrm{D}_{\mathrm{dR}}(V(-r)) = \mathbb{Q}_p(\zeta_{p^n}) \otimes \mathrm{D}_{\mathrm{dR}}(V)$), $\langle\ ,\ \rangle$ *is the canonical pairing*

$$(\mathbb{Q}(\zeta_{p^j}) \otimes \mathrm{D}_{\mathrm{dR}}(V)) \times \mathrm{D}_{\mathrm{crys}}(V^*(1)) \longrightarrow \overline{L}$$

induced by $\mathrm{D}_{\mathrm{dR}}(V) \times \mathrm{D}_{\mathrm{crys}}(V^*(1)) \to L$, *and φ is the Frobenius.*

(ii) *Assume* $\eta = (1 - p^{-r}\varphi)\eta'$ *with* $\eta' \in \mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f))$. *Then for any* $x \in \mathbf{H}^1_{\mathrm{loc}}(V)$,

$$\mathfrak{L}_\eta(x)(\kappa^r) = (r-1)! \cdot \langle \exp^*(x_{-r,0}), (1 - p^{r-1}\varphi^{-1})\eta' \rangle.$$

This map $\mathfrak{L}_\eta$ is Λ-linear, and the map $\eta \mapsto \mathfrak{L}_\eta$ is L-linear. Furthermore, if $r \geqslant 1$ and if η belongs to an L-subspace of $\mathrm{D}_{\mathrm{crys}}(V^*(1))$ *on which the slope of the Frobenius is $< h$, then* $\mathrm{Image}(\mathfrak{L}_\eta) \subset \mathscr{H}_{h,L}$.

Remark 16.5

(1) In the paper [**Pe2**], Perrin-Riou in fact defined a canonical homomorphism

$$\mathscr{H}_{\infty,L} \longrightarrow \mathscr{H}_{\infty,L} \otimes_\Lambda \Big(\varprojlim_n \mathrm{H}^1\big(\mathbb{Z}[\zeta_{p^n}, 1/p], T^*(1)\big)\Big)/(\text{some small thing}),$$

associated to $\eta \in \mathrm{D}_{\mathrm{crys}}(V^*(1))$. The homomorphism in 16.4 is obtained from this homomorphism by taking $\mathrm{Hom}_{\mathscr{H}_{\infty,L}}(\ , \mathscr{H}_{\infty,L})$.

(2) In the paper [**Pe2**], the assumption V is crystalline appears to have a map in the above (1) for $\eta \in \mathrm{D}_{\mathrm{crys}}(V^*(1))$. Kurihara,Tsuji and I checked that this assumption is not necessary [**KKT**]. See also [**CP**], [**Pe5**].

We apply 16.4 by taking $V_{F_\lambda}(f^*)(k)$ as V in 16.4. Since $(V_{F_\lambda}(f^*)(k))^*(1) = V_{F_\lambda}(f^*)^*(1-k)$ is isomorphic to $V_{F_\lambda}(f)$ as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (14.10.1), the condition (16.4.1) is satisfied.

Theorem 16.6. — *Let α, ω, γ be as in 16.2, and let*

$$\mathbf{z}_\gamma^{(p)}(f^*)(k) \in \mathbf{H}^1_{\mathrm{loc}}(V_{F_\lambda}(f^*)(k))$$

be the image of the p-adic zeta element

$$\mathbf{z}_\gamma^{(p)}(f^*) \in \mathbf{H}^1(V_{F_\lambda}(f^*))$$

(12.5 (1)) under the product with $(\zeta_{p^n})_{n\geqslant 1}^{\otimes k}$. Then:

(1) *There exists an element η of $F_\lambda(\alpha) \otimes_{F_\lambda} \mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f^*)^*(1-k))$ such that*

$$\varphi(\eta) = \alpha\eta \quad \textit{and} \quad \langle \omega, \eta\rangle = 1.$$

(2) *For η as in (1), we have*

$$L_{p\text{-adic},\alpha,\omega,\gamma}(f) = \mathfrak{L}_\eta(\mathbf{z}_\gamma^{(p)}(f^*)(k)).$$

Here ω is regarded as an element of $\mathrm{D}_{\mathrm{dR}}(V_{F_\lambda}(f^*))$ via the embedding $S(f^*) \subset \mathrm{D}_{\mathrm{dR}}(V_{F_\lambda}(f^*))$.

Proof. — We prove (1). Since $V_{F_\lambda}(f^*)^*(1-k) \simeq V_{F_\lambda}(f)$ and

$$1 - a_p u + \varepsilon(p)p^{k-1}u^2 = \det_{F_\lambda}(1 - \varphi u \text{ on } \mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f)))$$

by [**Sa1**], there exists a non-zero element η of $F_\lambda(\alpha) \otimes_{F_\lambda} \mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f^*)^*(1-k))$ such that $\varphi(\eta) = \alpha\eta$. It is sufficient to prove $\langle\omega,\eta\rangle \neq 0$. Since the annihilator of $S(f^*)$ in $F_\lambda(\alpha) \otimes_{F_\lambda} \mathrm{D}_{\mathrm{dR}}(V_{F_\lambda}(f^*)^*(1-k))$ is $F_\lambda(\alpha) \otimes_{F_\lambda} \mathrm{D}^{k-1}_{\mathrm{dR}}(V_{F_\lambda}(f^*)^*(1-k))$, it is sufficient to prove that in $F_\lambda(\alpha) \otimes_{F_\lambda} \mathrm{D}_{\mathrm{dR}}(V_{F_\lambda}(f^*)^*(1-k))$, η does not belong to $F_\lambda(\alpha) \otimes_{F_\lambda} \mathrm{D}^{k-1}_{\mathrm{dR}}(V_{F_\lambda}(f^*)^*(1-k))$. Hence it is sufficient to show that a non-zero element η in $F_\lambda(\alpha) \otimes_{F_\lambda} \mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f^*)^*(1-k))$ such that $\varphi(\eta) = \alpha\eta$ does not belong to $F_\lambda(\alpha) \otimes_{F_\lambda} \mathrm{D}^{k-1}_{\mathrm{dR}}(V_{F_\lambda}(f^*)^*(1-k))$. Consider the $F_\lambda(\alpha)$-subspace $F_\lambda(\alpha)\cdot\eta$ of $F_\lambda(\alpha) \otimes_{F_\lambda} \mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f^*)^*(1-k))$. This subspace is stable under the action of the Frobenius φ. The Newton polygon of this subspace has slope $\mathrm{ord}_p(\alpha) < k-1$, and if $\eta \in F_\lambda(\alpha) \otimes_{F_\lambda} \mathrm{D}^{k-1}_{\mathrm{dR}}(V_{F_\lambda}(f^*)^*(1-k))$, the Hodge polygon of this subspace has slope $k-1$ [**Fo3**, § 4.4]. By [**Fo3**, § 4.4], this contradicts the result of Tsuji and Faltings that $V_{F_\lambda}(f)$ is potentially semi-stable (as explained in 11.4).

Next we prove (2). Let $r \in \mathbb{Z}$, $1 \leqslant r \leqslant k-1$ and let n and χ be as in 16.4 (i). Write

$$\exp^*(\mathbf{z}_\gamma^{(p)}(f^*)(k)_{-r,n}) = c_{-r,n}\omega \qquad \text{with } c_{-r,n} \in \mathbb{Q}(\zeta_{p^n}) \otimes F.$$

Then by 16.4

$$\mathfrak{L}_\eta(\mathbf{z}_\gamma^{(p)}(f^*)(k))(\kappa^r\chi^{-1}) = (r-1)! \cdot p^{nr}\alpha^{-n} \cdot G(\chi,\zeta_{p^n})^{-1} \sum_{\sigma\in G_n} \chi(\sigma)\sigma(c_{-r,n}).$$

Let $\pm = (-1)^{k-r-1}\chi(-1)$. By the characterizing property of the p-adic zeta element $\mathbf{z}_\gamma^{(p)}(f^*)$ in 12.5, the image of $\sum_{\sigma\in G_n}\chi(\sigma)\sigma(c_{-r,n})\cdot\omega$ under

$$\mathbb{Q}(\zeta_{p^n})\otimes S(f^*) \longrightarrow \mathbb{C}\otimes_F V_F(f^*) \longrightarrow \mathbb{C}\otimes_F V_F(f^*)^{\pm}$$

is $(2\pi i)^{k-r-1}L_{\{p\}}(f,\chi,r)\cdot\gamma^{\pm}$. This shows

$$\Big(\sum_{\sigma\in G_n}\chi(\sigma)\sigma(c_{-r,n})\Big)\cdot\Omega_{\pm} = (2\pi i)^{k-r-1}L_{\{p\}}(f,\chi,r).$$

This proves $\mathfrak{L}_\eta(\mathbf{z}_\gamma^{(p)}(f^*)(k)) = L_{p\text{-adic},\alpha,\omega,\gamma}(f)$ by 16.3 (1). □

17. The main conjecture, II

In § 17 and § 18, we consider the relation between the p-adic zeta function of f and the Selmer groups associated to f. In this section, we consider the analogue of Iwasawa main conjecture in the case f is of good ordinary reduction.

The following is known:

Proposition 17.1. — *The following three conditions (i)-(iii) are equivalent.*

(i) *p does not divide N and $a_p \in O_\lambda^\times$.*

(ii) *p does not divide N and there exists an element α of $O_\lambda^\times$ such that $1-\alpha u$ divides the polynomial $1-a_pu+\varepsilon(p)p^{k-1}u^2$.*

(iii) *$V_{F_\lambda}(f)$ is crystalline as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, and there exists a one dimensional F_λ-subspace $V'_{F_\lambda}(f)$ of $V_{F_\lambda}(f)$ which is stable under the action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and is unramified as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.*

Furthermore, if these equivalent conditions are satisfied, the element α in (ii) is unique, and the subspace $V'_{F_\lambda}(f)$ in (iii) is unique, and if we put $V''_{F_\lambda}(f) = V_{F_\lambda}(f)/V'_{F_\lambda}(f)$, $V''_{F_\lambda}(f)(k-1)$ is unramified as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

We give the proof of 17.1 in 17.7.

We say that f has good ordinary reduction at λ if the equivalent conditions in 17.1 are satisfied.

If f has good ordinary reduction at λ, f^* also has good ordinary reduction at λ. This follows from $V_{F_\lambda}(f^*) \simeq \mathrm{Hom}_{F_\lambda}(V_{F_\lambda}(f),F_\lambda)(1-k)$ by using the above condition (iii).

Proposition 17.2. — *Assume f has good ordinary reduction at λ. Let $V'_{F_\lambda}(f)$ and $V''_{F_\lambda}(f)$ be as in 17.1, let T be a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice in $V_{F_\lambda}(f)$, and let*

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$$

be the exact sequence defined by $T' = T\cap V'_{F_\lambda}(f), T'' = T/T' \subset V''_{F_\lambda}(f)$. Then for $1\leqslant r\leqslant k-1$, the subgroup

$$\varinjlim_n \mathrm{Sel}(\mathbb{Q}(\zeta_{p^n}),T(r))(-r)$$

of $\varinjlim_n \mathrm{H}^1(\mathbb{Z}[\zeta_{p^n}, 1/p], T \otimes \mathbb{Q}/\mathbb{Z})$ *coincides with the kernel of*

$$\varinjlim_n \mathrm{H}^1(\mathbb{Z}[\zeta_{p^n}, 1/p], T \otimes \mathbb{Q}/\mathbb{Z}) \longrightarrow \varinjlim_n \mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), T'' \otimes \mathbb{Q}/\mathbb{Z})$$

and hence is independent of r.

The proof of 17.2 is given in 17.10.

17.3. Assume f has good ordinary reduction at λ. For a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice T of $V_{F_\lambda}(f)$, let

$$\mathrm{Sel}_\infty(T) = \varinjlim_n \mathrm{Sel}(\mathbb{Q}(\zeta_{p^n}), T(r))(-r) \qquad (1 \leqslant r \leqslant k-1)$$

which is independent of r, and let

$$\mathfrak{X}(T) = \mathrm{Hom}_{O_\lambda}(\mathrm{Sel}_\infty(T), F_\lambda/O_\lambda).$$

We regard $\mathfrak{X}(T)$ as a module over $\Lambda = O_\lambda[[G_\infty]]$ in the natural way. It is easily seen that $\mathfrak{X}(T)$ is a finitely generated Λ-module.

The aim of this section is to prove the following Thm. 17.4.

Theorem 17.4. — *Assume f has good ordinary reduction at λ. Let T be a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-*stable O_λ-lattice of $V_{F_\lambda}(f)$.*

(1) $\mathfrak{X}(T)$ *is a torsion Λ-module.*

(2) *Let α be as in 17.1, let ω be a non-zero element of $S(f^*)$, and let γ be an element of $V_F(f^*)$ such that $\gamma^+ \neq 0$ and $\gamma^- \neq 0$. Then, $L_{p\text{-adic},\alpha,\omega,\gamma}(f) \in \Lambda \otimes \mathbb{Q}$, and we have*

$$\mathrm{length}_{\Lambda_\mathfrak{p}}(\mathfrak{X}(T)_\mathfrak{p}) \leqslant \mathrm{ord}_\mathfrak{p}(L_{p\text{-adic},\alpha,\omega,\gamma}(f))$$

for any prime ideal $\mathfrak{p}$ in Λ of height one which does not contain p.

(3) *Let α, ω, γ be as in (2), and assume that both ω and γ are good for some* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-*stable O_λ-lattice of $V_{F_\lambda}(f)$ in the sense of 17.5 below. Assume further $p \neq 2$ and that the condition 12.5.2 in 12.5 (4) is satisfied. Then $L_{p\text{-adic},\alpha,\omega,\gamma}(f)$ belongs to Λ and*

$$\mathrm{length}_{\Lambda_\mathfrak{p}}(\mathfrak{X}(T)_\mathfrak{p}) \leqslant \mathrm{ord}_\mathfrak{p}(L_{p\text{-adic},\alpha,\omega,\gamma}(f))$$

for any prime ideal $\mathfrak{p}$ of Λ of height one.

17.5. Assume f has good ordinary reduction at λ. Then as we will see in 17.8, the composite map

$$S(f) \otimes_F F_\lambda \hookrightarrow \mathrm{D}_{\mathrm{dR}}(V_{F_\lambda}(f)) \longrightarrow \mathrm{D}_{\mathrm{dR}}(V''_{F_\lambda}(f)) \tag{17.5.1}$$

is an isomorphism. (Here $V''_{F_\lambda}(f)$ is as in 17.1.)

Let T be a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice of $V_{F_\lambda}(f)$. Then $\mathrm{H}^0(\mathbb{Q}_p, \widehat{\mathbb{Z}}_p^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} T''(k-1))$ is an O_λ-lattice of the one dimensional F_λ-vector space $\mathrm{D}_{\mathrm{dR}}(V''_{F_\lambda}(f))$, where T'' is the image of T in $V''_{F_\lambda}(f)$ and $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ denotes the p-adic completion of the valuation ring of $\mathbb{Q}_p^{\mathrm{ur}}$.

We say an element ω of $S(f)$ is good for T if the image of ω in $\mathrm{D_{dR}}(V''_{F_\lambda}(f))$ under the map (17.5.1) is an O_λ-basis of $\mathrm{H}^0(\mathbb{Q}_p, \widehat{\mathbb{Z}}_p^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} T''(k-1))$. Note that a good ω for T exists. In the case f of weight 2 and $T = (T_pE)(-1)$ for an elliptic curve E over $\mathbb{Q}$, ω is good for T if and only if ω is a $\mathbb{Z}_p$-basis of $\mathrm{coLie}(\overline{E}) \otimes \mathbb{Z}_p$ where $\overline{E}$ is the Néron model of E.

Recall 14.18 that we say an element γ of $V_{F_\lambda}(f)$ is good for T if γ^+ is an O_λ-basis of T^+ and γ^- is an O_λ-basis of T^-.

The following is an old conjecture due to Mazur and Greenberg.

***Conjecture 17.6* (main conjecture).** — *Assume f has good ordinary reduction at λ, and let $T, \mathfrak{X}(T), \alpha, \omega, \gamma$ be as in Thm. 17.4 (2). Then*

$$\mathrm{length}_{\Lambda_\mathfrak{p}}(\mathfrak{X}(T)_\mathfrak{p}) = \mathrm{ord}_\mathfrak{p}(L_{p\text{-adic},\alpha,\omega,\gamma}(f))$$

for any prime ideal $\mathfrak{p}$ of height one of Λ which does not contain p. If furthermore $p \neq 2$ and if ω and γ are good in the sense of 17.5 for some $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice of $V_{F_\lambda}(f^*)$ which is isomorphic to $T^*(1-k)$ as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over O_λ, then

$$\mathrm{length}_{\Lambda_\mathfrak{p}}(\mathfrak{X}(T)_\mathfrak{p}) = \mathrm{ord}_\mathfrak{p}(L_{p\text{-adic},\alpha,\omega,\gamma}(f))$$

for any prime ideal $\mathfrak{p}$ of height one.

See Greenberg [**Gr1**], [**Gr2**], Schneider [**Scp**] for more general aspects for motives.

As we will see in 17.13, the main conjecture 12.10 implies the main conjecture 17.6. The arguments to do this are similar to the arguments in deducing the classical Iwasawa main conjecture for the p-adic Riemann zeta function from the Iwasawa main conjecture of the form 12.9, and are well known to experts.

17.7. We prove 17.1.

Assume (i). We prove (ii). Write $1 - a_pu + \varepsilon(p)p^{k-1}u^2$ in the form $(1-\alpha u)(1-\beta u)$ with $\alpha, \beta \in \overline{F}_\lambda^\times$. If α does not belong to F_λ, we have $\mathrm{ord}_p(\alpha) = \mathrm{ord}_p(\beta)$ because α and β are conjugate over F_λ. Since $\alpha\beta = \varepsilon(p)p^{k-1}$, $\mathrm{ord}_p(\alpha) = \mathrm{ord}_p(\beta) = \frac{k-1}{2} > 0$ and this contradicts $\alpha + \beta = a_p \in O_\lambda^\times$. Hence $\alpha \in F_\lambda$. This shows $\alpha, \beta \in O_\lambda$. By $\alpha + \beta \in O_\lambda^\times$, one of α, β belongs to $O_\lambda^\times$. Hence (i) implies (ii).

Assume (ii). We prove (iii). Since p does not divide N, $V_{F_\lambda}(f)$ is a crystalline representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. The space $\mathrm{D_{crys}}(V_{F_\lambda}(f))$ has a Frobenius φ and a filtration $(\mathrm{D}^i_{\mathrm{dR}}(V_{F_\lambda}(f)))_{i\in\mathbb{Z}}$ via the identification $\mathrm{D_{crys}}(V_{F_\lambda}(f)) = \mathrm{D_{dR}}(V_{F_\lambda}(f))$. We have

$$\mathrm{D}^i_{\mathrm{dR}}(V_{F_\lambda}(f)) = \begin{cases} \mathrm{D_{dR}}(V_{F_\lambda}(f)) & \text{for } i \leqslant 0, \\ S(f) \otimes_F F_\lambda & \text{for } 1 \leqslant i \leqslant k-1, \\ 0 & \text{for } i \geqslant k. \end{cases}$$

Let α be as in condition (ii). Since

$$\det_{F_\lambda}\big(1 - \varphi u\,;\, \mathrm{D_{crys}}(V_{F_\lambda}(f))\big) = 1 - a_pu + \varepsilon(p)p^{k-1}u^2$$

[**Sa1**], $\mathrm{D_{crys}}(V_{F_\lambda}(f))_\alpha = \{x \in \mathrm{D_{crys}}(V_{F_\lambda}(f))\,;\, \varphi(x) = \alpha x\}$ is a one dimensional F_λ-vector space.

Let U be a one dimensional F_λ-vector space which is endowed with an unramified action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ such that the arithmetic Frobenius acts by α^{-1}. Then the Frobenius on $\mathrm{D_{crys}}(U)$ coincides with the multiplication by α, $\mathrm{D}^i_{\mathrm{dR}}(U) = \mathrm{D_{dR}}(U)$ for $i \leqslant 0$, and $\mathrm{D}^i_{\mathrm{dR}}(U) = 0$ for $i \geqslant 1$. Let $h : \mathrm{D_{crys}}(U) \to \mathrm{D_{crys}}(V_{F_\lambda}(f))$ be an injective F_λ-linear map whose image is $\mathrm{D_{crys}}(V_{F_\lambda}(f))_\alpha$. Then h preserves the Frobenius operators and the filtrations, and hence comes from an $F_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)]$-homomorphism $U \to V_{F_\lambda}(f)$ [**Fo1**]. Let $V'_{F_\lambda}(f)$ be the image of this homomorphism. Then $V'_{F_\lambda}(f)$ is unramified.

It is easy to see that (ii) implies (i).

We prove that (iii) implies (ii). Since $V_{F_\lambda}(f)$ is crystalline as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, p does not divide N by [**Ca**] and [**Sa1**]. Since

$$1 - \overline{a}_p u + \varepsilon(p)u^2 = \det(1 - \varphi u\,;\, \mathrm{D_{crys}}(V_{F_\lambda}(f))) \quad (14.10.4)$$

and since the slope of the unramified representation $V'_{F_\lambda}(f)$ is zero, we have (ii).

Thus we have proved that the conditions (i)-(iii) are equivalent. Assume now that these equivalent conditions are satisfied. Then α in (ii) is unique as is easily seen. Let $V''_{F_\lambda}(f) = V_{F_\lambda}(f)/V'_{F_\lambda}(f)$. Since

$$V'_{F_\lambda}(f) \otimes_{F_\lambda} V''_{F_\lambda}(f)(k-1) \simeq \big(\det{}_{F_\lambda}(V_{F_\lambda}(f))\big)(k-1)$$

is unramified (14.10.2), we have that $V''_{F_\lambda}(f)(k-1)$ is unramified. Since $V'_{F_\lambda}(f)$ is not isomorphic to $V''_{F_\lambda}(f)$, we see that $V'_{F_\lambda}(f)$ is unique.

This completes the proof of Prop. 17.1. $\square$

17.8. We prove the bijectivity of the composite map (17.5.1). For $i \in \mathbb{Z}$, we have an exact sequence

$$0 \longrightarrow \mathrm{D}^i_{\mathrm{dR}}(V'_{F_\lambda}(f)) \longrightarrow \mathrm{D}^i_{\mathrm{dR}}(V_{F_\lambda}(f)) \longrightarrow \mathrm{D}^i_{\mathrm{dR}}(V''_{F_\lambda}(f)) \longrightarrow 0.$$

For i such that $1 \leqslant i \leqslant k-1$,

$$\mathrm{D}^i_{\mathrm{dR}}(V'_{F_\lambda}(f)) = 0,\ \mathrm{D}^i_{\mathrm{dR}}(V_{F_\lambda}(f)) = S_F(f) \otimes_F F_\lambda \text{ and } \mathrm{D}^i_{\mathrm{dR}}(V''_{F_\lambda}(f)) = \mathrm{D}^i_{\mathrm{dR}}(V_{F_\lambda}(f)).$$

Hence $S_F(f) \otimes_F F_\lambda \to \mathrm{D}^i_{\mathrm{dR}}(V''_{F_\lambda}(f))$ is an isomorphism.

Lemma 17.9. — *Assume f has good ordinary reduction at λ, let T, T', T'' be as in 17.2, and let r be an integer such that $1 \leqslant r \leqslant k-1$. Then the image of $\varprojlim_n \mathrm{H}^1_f(\mathbb{Q}_p(\zeta_{p^n}), T(r))$ in $\mathbf{H}^1_{\mathrm{loc}}(T(r))$ coincides with the image of $\mathbf{H}^1_{\mathrm{loc}}(T'(r))$.*

Proof. — By Prop. 14.12, we have

$$\begin{aligned}
\mathrm{H}^0(\mathbb{Q}_p, V) = \mathrm{H}^2(\mathbb{Q}_p, V) = 0 \quad &\text{for } V = V'_{F_\lambda}(f)(r), V_{F_\lambda}(f)(r), V''_{F_\lambda}(f)(r),\\
\dim_{F_\lambda} \mathrm{H}^1(\mathbb{Q}_p, V_{F_\lambda}(f)(r)) &= 2,\\
\dim_{F_\lambda} \mathrm{H}^1_f(\mathbb{Q}_p, V_{F_\lambda}(f)(r)) &= 1,\\
\dim_{F_\lambda} \mathrm{H}^1(\mathbb{Q}_p, V) = 1 \quad &\text{for } V = V'_{F_\lambda}(f)(r), V''_{F_\lambda}(f)(r),\\
\mathrm{H}^1_f(\mathbb{Q}_p, V'_{F_\lambda}(f)(r)) &= \mathrm{H}^1(\mathbb{Q}_p, V'_{F_\lambda}(f)(r)),\\
\mathrm{H}^1_f(\mathbb{Q}_p, V''_{F_\lambda}(f)(r)) &= 0.
\end{aligned}$$

We have also $\mathrm{H}^m(\mathbb{Q}_p(\zeta_{p^n}), V_{F_\lambda}(f)(r)) = \oplus_i \mathrm{H}^m(\mathbb{Q}_p, V_{L_i}(f \otimes \chi_i)(r))$ and similar facts for $V'_{F_\lambda}(f)(r)$, $V''_{F_\lambda}(f)(r)$, where L_i are finite extensions of $\mathbb{Q}_p$ satisfying $F_\lambda[(\mathbb{Z}/p^n)^\times] = \prod_i L_i$ and where $\chi_i : (\mathbb{Z}/p^n)^\times \to L_i^\times$ are induced homomorphisms. These show that $\mathrm{H}^1_f(\mathbb{Q}_p(\zeta_{p^n}), V_{F_\lambda}(f)(r))$ coincides with the image of $\mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), V'_{F_\lambda}(f)(r))$ in $\mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), V_{F_\lambda}(f)(r))$. Hence $\mathrm{H}^1_f(\mathbb{Q}_p(\zeta_{p^n}), T(r))$ contains the image of the injection $\mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), T'(r)) \to \mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), T(f)(r))$, and

$$\mathrm{H}^1_f(\mathbb{Q}_p(\zeta_{p^n}), T(r))/\mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), T'(r))$$

is a finite group and is embedded into the torsion part of $\mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), T''(r))$. Since the torsion part of $\mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), T''(r))$ is the image of $\mathrm{H}^0(\mathbb{Q}_p(\zeta_{p^n}), T''(r) \otimes \mathbb{Q}/\mathbb{Z})$, it is sufficient to prove that $\varprojlim_n \mathrm{H}^0(\mathbb{Q}_p(\zeta_{p^n}), T''(r) \otimes \mathbb{Q}/\mathbb{Z})$ is zero. But this follows from the finiteness of $\mathrm{H}^0(\mathbb{Q}_p(\zeta_{p^\infty}), T''(r) \otimes \mathbb{Q}/\mathbb{Z})$ (13.13). □

17.10. We prove Prop. 17.2. Let $1 \leqslant i \leqslant k-1$. For each prime number ℓ, let

$$\begin{aligned}
A^n_\ell &= \mathrm{H}^1(\mathbb{Q}_\ell \otimes \mathbb{Q}(\zeta_{p^n}), T(r) \otimes \mathbb{Q}/\mathbb{Z}),\\
B^n_\ell &= \mathrm{Image}\left(\mathrm{H}^1_f(\mathbb{Q}_\ell \otimes \mathbb{Q}(\zeta_{p^n}), T(r) \otimes \mathbb{Q}) \to A^n_\ell\right).
\end{aligned}$$

For each prime number $\ell \neq p$, let

$$C^n_\ell = \mathrm{H}^1(\mathbb{Z}_\ell \otimes \mathbb{Z}[\zeta_{p^n}], T(r) \otimes \mathbb{Q}/\mathbb{Z}) \subset A^n_\ell.$$

Then for any prime number $\ell \neq p$, B^n_ℓ coincides with the biggest divisible subgroup of C^n_ℓ. We have

$$\begin{aligned}
\mathrm{Sel}(\mathbb{Q}(\zeta_{p^n}), T(r)) &= \mathrm{Ker}\Big(\mathrm{H}^1(\mathbb{Q}(\zeta_{p^n}), T(r) \otimes \mathbb{Q}/\mathbb{Z}) \to A^n_p/B^n_p \oplus \big(\bigoplus_{\ell \neq p} A^n_\ell/B^n_\ell\big)\Big)\\
\mathcal{S}(\mathbb{Q}(\zeta_{p^n}), T_\lambda(f)(r)) &= \mathrm{Ker}\Big(\mathrm{H}^1(\mathbb{Q}(\zeta_{p^n}), T(r) \otimes \mathbb{Q}/\mathbb{Z}) \to A^n_p/B^n_p \oplus \big(\bigoplus_{\ell \neq p} A^n_\ell/C^n_\ell\big)\Big)\\
\mathrm{H}^1(\mathbb{Z}[\zeta_{p^n}, 1/p], T(r) \otimes \mathbb{Q}/\mathbb{Z}) &= \mathrm{Ker}\Big(\mathrm{H}^1(\mathbb{Q}(\zeta_{p^n}), T(r) \otimes \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{\ell \neq p} A^n_\ell/C^n_\ell\Big).
\end{aligned}$$

We have $\varinjlim_n C^n_\ell = 0$ since

$$\mathrm{H}^1(\mathbb{Z}_\ell \otimes \mathbb{Z}[\zeta_{p^n}], \;) = \mathrm{H}^1(\mathbb{F}_\ell \otimes \mathbb{Z}[\zeta_{p^n}], \;)$$

and the p-cohomological dimension of $\bigcup_n \mathbb{F}_\ell(\zeta_{p^n})$ is zero. Hence we have also $\varinjlim_n B_\ell^n = 0$, and

$$\begin{aligned}\varinjlim_n \mathrm{Sel}(\mathbb{Q}(\zeta_{p^n}), T(r)) &= \varinjlim_n \mathcal{S}(\mathbb{Q}(\zeta_{p^n}), T(r)) \\ &= \varinjlim_n \mathrm{Ker}\left(\mathrm{H}^1(\mathbb{Z}[\zeta_{p^n}, 1/p], T(r)\otimes\mathbb{Q}/\mathbb{Z}) \to A_p^n/B_p^n\right).\end{aligned} \tag{17.10.1}$$

Hence it is sufficient to prove that B_p^n is equal to the kernel of

$$\varinjlim_n A_p^n \longrightarrow \varinjlim_n \mathrm{H}^1(\mathbb{Q}_p(\zeta_{p^n}), T''(r)\otimes\mathbb{Q}/\mathbb{Z}).$$

But this follows from Lemma 17.9 by duality [**BK2**, Prop. 3.8].

The following Prop. 17.11 and Lemma 17.12 are preliminaries for the proof of Thm. 17.4.

Proposition 17.11. — *Assume f has good ordinary reduction at λ, let T be a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-stable O_λ-lattice of $V_{F_\lambda}(f^*)$, let $T' = T\cap V'_{F_\lambda}(f^*)$, $T'' = T/T' \subset V''_{F_\lambda}(f^*)$. Let η be a basis of the invertible O_λ-module*

$$\mathrm{H}^0(\mathbb{Q}_p, \widehat{\mathbb{Z}}_p^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}\mathrm{Hom}_{O_\lambda}(T'', O_\lambda)(1-k)) \subset \mathrm{D}_{\mathrm{crys}}(V_{F_\lambda}(f^*)^*(1-k)).$$

Then the homomorphism

$$\mathfrak{L}_\eta : \mathbf{H}^1_{\mathrm{loc}}(T(k)) \longrightarrow \mathscr{H}_{\infty,F_\lambda} \quad (16.4)$$

induces an injection

$$\mathbf{H}^1_{\mathrm{loc}}(T(k))/\mathbf{H}^1_{\mathrm{loc}}(T'(k)) \longrightarrow \Lambda,$$

whose cokernel is a finite group.

Proof. — Since $\eta \in \mathrm{D}_{\mathrm{crys}}(V''_{F_\lambda}(f^*)^*(1-k))$, the map $\mathfrak{L}_\eta$ factors through $\mathbf{H}^1_{\mathrm{loc}}(T(k)) \to \mathbf{H}^1_{\mathrm{loc}}(T''(k))$. Since $\mathbf{H}^2_{\mathrm{loc}}(T'(k))$ is a finite group, the cokernel of the injective homomorphism

$$\mathbf{H}^1_{\mathrm{loc}}(T(k))/\mathbf{H}^1_{\mathrm{loc}}(T'(k)) \longrightarrow \mathbf{H}^1_{\mathrm{loc}}(T''(k))$$

is a finite group. It remains to prove that $\mathfrak{L}_\eta$ induces an injective homomorphism $\mathbf{H}^1_{\mathrm{loc}}(T''(k)) \to \Lambda$ with finite cokernel. Hence we are reduced to the following lemma 17.12 which we apply by taking $T''(k-1)$ as T in Lemma 17.12. □

Lemma 17.12. — *Let L be a finite extension of $\mathbb{Q}_p$, and let T be an invertible O_L-module endowed with a continuous unramified O_L-linear action of* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ *such that $\mathrm{H}^0(\mathbb{Q}_p, T) = 0$. Let η be an O_L-basis of the invertible O_L-module*

$$\mathrm{H}^0(\mathbb{Q}_p, \widehat{\mathbb{Z}}_p^{\mathrm{ur}}\otimes_{\mathbb{Z}_p} T^*).$$

Then $\mathfrak{L} : \mathbf{H}^1_{\mathrm{loc}}(T(1)) \to \mathscr{H}_{\infty,L}$ induces an injection $\mathbf{H}^1_{\mathrm{loc}}(T(1)) \to O_L[[G_\infty]]$ whose cokernel is a finite group.

Proof. — In this case, the homomorphism $\mathfrak{L}_\eta$ is expressed by using the theory of Coleman power series.

Let $\overline{\mathbb{F}}_p$ be the residue field of $\mathbb{Q}_p^{\mathrm{ur}}$, which is an algebraic closure of $\mathbb{F}_p$. Since T is unramified, T is regarded as $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-module. The map $\mathfrak{L}_\eta$ is written as

$$\mathbf{H}^1_{\mathrm{loc}}(T(1)) \xrightarrow{a} \mathrm{H}^0\Big(\mathbb{F}_p, \varprojlim_n \mathrm{H}^1(\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\zeta_{p^n}), \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} T\Big)$$
$$\xrightarrow{b} \mathrm{H}^0(\mathbb{Q}_p, \widehat{\mathbb{Z}}_p^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} T) \otimes_{O_L} O_L[[G_\infty]] \xrightarrow[\sim]{\eta} O_L[[G_\infty]]$$

where a is the evident map and b is defined as follows. Let

$$P = \varprojlim_n (\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\zeta_{p^n})^\times)\widehat{\ }, \quad Q = \varprojlim_n (\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[\zeta_{p^n}]^\times)\widehat{\ }$$

where $(\ )\widehat{\ }$ denotes the p-adic completion $\varprojlim_n (\ )/p^n(\ )$. Then

$$P \xrightarrow{\sim} \varprojlim_n \mathrm{H}^1(\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\zeta_{p^n}), \mathbb{Z}_p(1)),$$
$$P/Q \xrightarrow{\sim} \mathbb{Z}_p \qquad \text{by the additive valuation.}$$

By Coleman [**CR1, CR2**], we have an exact sequence of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-modules

$$(17.12.1) \qquad 0 \longrightarrow \mathbb{Z}_p \longrightarrow Q \xrightarrow{c} \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]] \longrightarrow 0,$$

where the map $\mathbb{Z}_p \to Q$ sends $1 \in \mathbb{Z}_p$ to $(-\zeta_{p^n})_{n\geqslant 1} \in Q$ and c is defined as follows. Let t be an indeterminate, and let

$$\varphi \text{ (resp. } \phi) : \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[t-1]] \longrightarrow \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[t-1]]$$

be the unique continuous ring homomorphism such that $\varphi(t) = t^p$ ($\phi(t) = t$) and such that the restriction of φ (resp. ϕ) to $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ is the Frobenius automorphism of $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. Since G_∞ acts on $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[t-1]]$ as $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$-linear continuous ring automorphisms in the way that $\sigma \in G_\infty$ sends t to $t^{\kappa(\sigma)}$, we can regard $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[t-1]]$ as a $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]$-module. For each $u = (u_n)_n \in \varprojlim_n \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[\zeta_{p^n}]^\times \subset Q$, by the theory of Coleman, there exists a unique element $\widetilde{u} = \widetilde{u}(t)$ of $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[t-1]]^\times$ (the Coleman power series associated to u) such that $u_n = \phi^{-n}(\widetilde{u})(\zeta_{p^n})$ for all $n \geqslant 1$. Furthermore, there exists a unique element μ_u of $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]$ such that

$$\frac{1}{p} \cdot \log((\widetilde{u})^p \varphi(\widetilde{u})^{-1}) = \mu_u \cdot t \text{ in } \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[t-1]].$$

Here $\mu_u \cdot t$ is defined by the above $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]$-module structure of $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[t-1]]$. The map c is defined by $c(u) = \mu_u$. Now the definition of b is as follows. Since

$$\mathrm{H}^0(\mathbb{F}_p, P/Q \otimes_{\mathbb{Z}_p} T) \simeq \mathrm{H}^0(\mathbb{Q}_p, T) = 0,$$

we have $\mathrm{H}^0(\mathbb{F}_p, P\otimes_{\mathbb{Z}_p} T) = \mathrm{H}^0(\mathbb{F}_p, Q\otimes_{\mathbb{Z}_p} T)$. The map b is defined to be the composite

$$\begin{aligned}\mathrm{H}^0(\mathbb{F}_p, P\otimes_{\mathbb{Z}_p} T) = \mathrm{H}^0(\mathbb{F}_p, Q\otimes_{\mathbb{Z}_p} T) &\xrightarrow{c} \mathrm{H}^0(\mathbb{F}_p, \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]\otimes_{\mathbb{Z}_p} T)\\ &= \mathrm{H}^0(\mathbb{Q}_p, \widehat{\mathbb{Z}}_p^{\mathrm{ur}}\otimes_{\mathbb{Z}_p} T)\otimes_{O_L} O_L[[G_\infty]].\end{aligned}$$

It remains to prove that the map a is bijective and that the map b is injective and the cokernel of b is finite. The bijectivity of a follows from $\mathrm{H}^1(\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\zeta_{p^n}), \mathbb{Z}_p(1)) = \mathrm{H}^1(\mathbb{Q}_p^{\mathrm{ur}}(\zeta_{p^n}), \mathbb{Z}_p(1))$ and from the spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(\mathbb{F}_p, \mathrm{H}^j(\mathbb{Q}_p^{\mathrm{ur}}(\zeta_{p^n}),\ )) \Longrightarrow E_\infty^{i+j} = \mathrm{H}^{i+j}(\mathbb{Q}_p(\zeta_{p^n}),\ ).$$

Next by the exact sequence (17.12.1), $\mathrm{Ker}(b) = \mathrm{H}^0(\mathbb{F}_p, T)$ and $\mathrm{Coker}(b) \subset \mathrm{H}^1(\mathbb{F}_p, T)$. Hence the injectivity of b follows from the vanishing of $\mathrm{H}^0(\mathbb{F}_p, T) = \mathrm{H}^0(\mathbb{Q}_p, T)$, and the finiteness of the cokernel of b is reduced to the finiteness of $\mathrm{H}^1(\mathbb{F}_p, T) = \mathrm{Coker}(\mathrm{Frob}_p - 1; T)$, which follows $\mathrm{Ker}(\mathrm{Frob}_p - 1; T) = \mathrm{H}^0(\mathbb{F}_p, T) = 0$. □

17.13. We prove Thm. 17.4.

First the property $L_{p\text{-adic},\alpha,\omega,\gamma}(f) \in \Lambda\otimes\mathbb{Q}$ (resp. $L_{p\text{-adic},\alpha,\omega,\gamma}(f) \in \Lambda$) in 17.4 (2) (resp. 17.4 (3)) is known but follows also from 17.11 and 16.6 (resp. 17.11, 12.5 (4) and 16.6).

Let T be a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice of $V_{F_\lambda}(f^*)$. Take any integer r such that $1 \leqslant r \leqslant k-1$. By taking $\varprojlim_n$ of the sequence (14.9.3) for $K = \mathbb{Q}(\zeta_{p^n})$ and for $T(r)$ (we use $T(r)$ as T in (14.9.3)), and by using (17.10.1), we obtain a sequence of Λ-modules

$$\begin{aligned}(17.13.1)\quad 0 \longrightarrow \mathbf{H}^1(T(k))/(\varprojlim_n \mathrm{H}^1_f(\mathbb{Z}[\zeta_{p^n}, 1/p], T(r))(k-r))\\ \longrightarrow \mathbf{H}^1_{\mathrm{loc}}(T(k))/(\varprojlim_n \mathrm{H}^1_f(\mathbb{Q}_p(\zeta_{p^n}), T(r))(k-r))\\ \longrightarrow \mathfrak{X}(T^*(1-k)) \longrightarrow \mathbf{H}^2(T(k)) \longrightarrow \mathbf{H}^2_{\mathrm{loc}}(T(k))\end{aligned}$$

($T^* = \mathrm{Hom}_{O_\lambda}(T, O_\lambda)$) which is exact if $p \neq 2$, and is exact upto $\times 2$ in the case $p = 2$.

We show first

$$(17.13.2)\qquad \varprojlim_n \mathrm{H}^1_f(\mathbb{Z}[\zeta_{p^n}, 1/p], T(r)) = 0.$$

For this, since $\mathbf{H}^1(T(k))$ has no Λ-torsion and is of Λ-rank 1 (12.4), it is sufficient to show that the image of $\mathbf{H}^1(T(k))$ in $\mathbf{H}^1_{\mathrm{loc}}(T(k))/(\varprojlim_n \mathrm{H}^1_f(\mathbb{Q}_p(\zeta_{p^n}), T(r))(k-r))$ is of Λ-rank 1. This fact is shown by observing that the image of $Z(f)(k)$ (12.5 (2)) is already of Λ-rank 1.

Next by 17.9, we have

$$(17.13.3)\qquad \mathbf{H}^1_{\mathrm{loc}}(T)/(\varprojlim_n \mathrm{H}^1_f(\mathbb{Q}_p(\zeta_{p^n}), T(r))(k-r)) = \mathbf{H}^1_{\mathrm{loc}}(T(k))/\mathbf{H}^1_{\mathrm{loc}}(T'(k))$$

where $T' = T \cap V'_{F_\lambda}(f^*)$. Furthermore by the latter half of Thm. 12.5 (3),

(17.13.4) $\mathbf{H}^2_{\mathrm{loc}}(T(k))$ *is a finite group.*

Let $\mathfrak{p}$ be a prime ideal of Λ of height one. In the case $\mathfrak{p}$ contains p, we assume $p \neq 2$ and that the condition (12.5.2) in 12.5 (4) is satisfied. By (17.13.2)-(17.13.4), we obtain from (17.13.1) an exact sequence

$$0 \longrightarrow \mathbf{H}^1(T(k))_{\mathfrak{p}} \longrightarrow \mathbf{H}^1_{\mathrm{loc}}(T(k))_{\mathfrak{p}}/\mathbf{H}^1_{\mathrm{loc}}(T'(k))_{\mathfrak{p}} \\ \longrightarrow \mathfrak{X}(T^*(1-k))_{\mathfrak{p}} \longrightarrow \mathbf{H}^2(T(k))_{\mathfrak{p}} \longrightarrow 0$$

Let ω be an element of $S(f^*)$ which is good for T, and let γ be an element of $V_F(f^*)$ which is good for T, in the sense of 17.5. By 17.11 and by 12.5, 16.6, we have an isomorphism $\mathbf{H}^1_{\mathrm{loc}}(T(k))_{\mathfrak{p}}/\mathbf{H}^1_{\mathrm{loc}}(T'(k))_{\mathfrak{p}} \simeq \Lambda_{\mathfrak{p}}$ which sends the image of $Z(f,T)(k)_{\mathfrak{p}}$ (12.5 (4)) onto $\Lambda_{\mathfrak{p}} \cdot L_{p\text{-adic},\alpha,\omega,\gamma}(f)$. Hence we obtain an exact sequence

$$0 \longrightarrow \mathbf{H}^1(T(k))_{\mathfrak{p}}/Z(f,T)(k)_{\mathfrak{p}} \longrightarrow \Lambda_{\mathfrak{p}}/(\Lambda_{\mathfrak{p}} \cdot L_{p\text{-adic},\alpha,\omega,\gamma}(f)) \\ \longrightarrow \mathfrak{X}(T^*(1-k))_{\mathfrak{p}} \longrightarrow \mathbf{H}^2(T(k))_{\mathfrak{p}} \longrightarrow 0$$

Hence $\mathfrak{X}(T^*(1-k))_{\mathfrak{p}}$ is a torsion $\Lambda_{\mathfrak{p}}$-module, and

$$\mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathfrak{X}(T^*(1-k))_{\mathfrak{p}}) - \mathrm{length}_{\Lambda_{\mathfrak{p}}}(\Lambda_{\mathfrak{p}}/(L_{p\text{-adic},\alpha,\omega,\gamma})) \\ = \mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^2(T(k))) - \mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^1(T(k))_{\mathfrak{p}}/Z(f,T)(k)_{\mathfrak{p}}).$$

Hence Thm. 17.4 (resp. Conj. 17.6) becomes a consequence of Thm. 12.5 (resp. Conj. 12.10).

18. p-adic Birch Swinnerton-Dyer conjectures

In this section, we assume k is even.

Let T be a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice of $V_{F_\lambda}(f)(k/2)$. We consider the relation between the corank of the Selmer group $\mathrm{Sel}(T)$ and the order of the p-adic zeta function of f at $s = k/2$.

18.1. For the order of the complex zeta function $L(f,s)$, the "modular form version" of the Birch Swinnerton-Dyer conjecture says

$$\mathrm{corank}_{O_\lambda}(\mathrm{Sel}(T)) = \mathrm{ord}_{s=k/2}(L(f,s)).$$

A p-adic analogue of this is formulated in [**MTT**].

Assume that there is $\alpha \in \overline{F}_\lambda$ such that $1 - \alpha u$ divides $1 - a_p u + \varepsilon(p)p^{k-1}u^2$ and such that $\mathrm{ord}_p(\alpha) < k-1$. Let ω be a non-zero element of $S(f^*)$, and let γ be an element of $V_F(f^*)$ such that $\gamma^+ \neq 0$ and $\gamma^- \neq 0$. Write $L_{p\text{-adic},\alpha,\omega,\gamma}(f)$ simply as $L_{p\text{-adic},\alpha}(f)$ (the choices of ω and γ are not important in the following). We denote $\mathscr{H}_{\infty,F_\lambda(\alpha)}$ simply by $\mathscr{H}_\infty$.

For an element μ of $\mathscr{H}_\infty$ and for $r \in \mathbb{Z}$, we define the order of μ at $s = r$, denoted by $\mathrm{ord}_{s=r}(\mu)$, as follows. Let $\mathfrak{p}$ be the kernel of the ring homomorphism $\mathcal{H}_\infty \to \overline{F}_\lambda(\alpha); \mu \mapsto \mu(\kappa^r)$ induced by the group homomorphism $\kappa^r : G_\infty \to \mathbb{Z}_p^\times \subset \overline{F}_\lambda^\times$.

Then as we will see in 18.6, the local ring $\mathscr{H}_{\infty,\mathfrak{p}}$ is a discrete valuation ring. We define $\mathrm{ord}_{s=r}(\mu) = \mathrm{length}_{\mathscr{H}_{\infty,\mathfrak{p}}}(\mathscr{H}_{\infty,\mathfrak{p}}/\mu\mathscr{H}_{\infty,\mathfrak{p}})$.

Conjecture 18.2 (**[MTT]**). — *If $\alpha \neq p^{(k-2)/2}$, we have*

$$\mathrm{corank}_{O_\lambda}(\mathrm{Sel}(T)) = \mathrm{ord}_{s=k/2}(L_{p\text{-adic},\alpha}(f)).$$

If $\alpha = p^{(k-2)/2}$, we have

$$\mathrm{corank}_{O_\lambda}(\mathrm{Sel}(T)) = \mathrm{ord}_{s=k/2}(L_{p\text{-adic},\alpha}(f)) - 1.$$

Remark 18.3. — If $\alpha = p^{(k-2)/2}$, then $L_{p\text{-adic},\alpha}(f, k/2) = 0$ (16.2 (ii)) and hence

$$\mathrm{ord}_{s=k/2}(L_{p\text{-adic},\alpha}(f)) \geqslant 1.$$

If $\alpha = p^{(k-2)/2}$, then p divides N. In the case $k = 2$ and f corresponds to an elliptic curve E over $\mathbb{Q}$, $\alpha = 1$ if and only if $E \otimes \mathbb{Q}_p$ is a Tate curve.

Theorem 18.4. — *Let T be $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice of $V_{F_\lambda}(f)(k/2)$. Then we have $\leqslant$ in place of $=$ in Conj. 18.2. That is,*

$$\mathrm{corank}_{O_\lambda}(\mathrm{Sel}(T)) \leqslant \begin{cases} \mathrm{ord}_{s=k/2}(L_{p\text{-adic},\alpha}(f)) & \text{if } \alpha \neq p^{(k-2)/2}, \\ \mathrm{ord}_{s=k/2}(L_{p\text{-adic},\alpha}(f)) - 1 & \text{if } \alpha = p^{(k-2)/2}. \end{cases}$$

In particular, if $k = 2, F = \mathbb{Q}$, and f corresponds to an elliptic curve E over $\mathbb{Q}$, we have

$$\mathrm{rank}(E(\mathbb{Q})) \leqslant \mathrm{ord}_{s=1}(L_{p\text{-adic},\alpha}(f))$$

if E is not a Tate curve, and

$$\mathrm{rank}(E(\mathbb{Q})) \leqslant \mathrm{ord}_{s=1}(L_{p\text{-adic},\alpha}(f)) - 1$$

if E is a Tate curve.

The arguments in the proof below for the case $\alpha \neq p^{(k-2)/2}$ is given in Perrin-Riou [**Pe1**], [**Pe3**]. The proof for the case $\alpha = p^{(k-2)/2}$ will be given in [**KKT**], and we give below the outline of it.

18.5. We prove that for $\mathfrak{p}$ as in 18.1, the local ring $\mathscr{H}_{\infty,\mathfrak{p}}$ is a discrete valuation ring. This is reduced to the case $r = 0$, and then to the following fact. Let L be a complete discrete valuation field of mixed characteristic $(0, p)$, and let

$$A = \left\{ \textstyle\sum_{n\geqslant 0} a_n X^n \,;\, |a_n|_p \cdot n^{-h} \to 0 \text{ for some } h \geqslant 1 \right\} \subset L[[X]].$$

Here $|\ |_p$ denotes the multiplicative valuation of L normalized by $|p|_p = p^{-1}$. Let $\mathfrak{p}$ be the prime ideal $\{\sum_{n\geqslant 0} a_n X^n \in A\,;\, a_0 = 0\}$ of A. Then the local ring $A_\mathfrak{p}$ is a discrete valuation ring. We prove this. If $\mu = \sum_{n\geqslant 0} a_n X^n \in A$, $\mu \neq 0$, and $m = \min\{n \geqslant 0\,;\, a_n \neq 0\}$, then $\mu X^{-m} = \sum_{n\geqslant 0} a_{n+m} X^n$ belongs to A. (In fact, $|a_n|_p \cdot n^{-h} \to 0$ for some $h \geqslant 1$, and hence $\lim_{n\to\infty} |a_{n+m}|_p \cdot n^{-h} = \lim_{n\to\infty} |a_n|_p \cdot (n-m)^{-h} = 0$.) Hence any non-zero element of A can be written in the form $X^m \mu$ for some $m \geqslant 0$

and for some $\mu \in A \setminus \mathfrak{p}$. This shows that all ideals of $A_{\mathfrak{p}}$ are given by (X^n) $(n \geqslant 0)$ and (0). Hence $A_{\mathfrak{p}}$ is a discrete valuation ring.

Proposition 18.6. — *Let T be a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-stable O_λ-lattice of $V_{F_\lambda}(f)(k/2)$, and let j be the canonical map*

$$\mathrm{H}^1(\mathbb{Z}[1/p], T^*(1)) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, T^*(1))/\mathrm{H}^1_f(\mathbb{Q}_p, T^*(1)).$$

Then

$$\mathrm{corank}_{O_\lambda}(\mathrm{Sel}(T)) = \begin{cases} \dim_{F_\lambda} \mathrm{H}^2(\mathbb{Z}[1/p], V_{F_\lambda}(f^*)(k/2)) & \text{if } j \otimes \mathbb{Q} \neq 0, \\ \dim_{F_\lambda} \mathrm{H}^2(\mathbb{Z}[1/p], V_{F_\lambda}(f^*)(k/2)) - 1 & \text{if } j \otimes \mathbb{Q} = 0. \end{cases}$$

This follows from the exact sequence in the category {abelian groups}/{finite groups}

$$\begin{aligned} \mathrm{H}^1(\mathbb{Z}[1/p], T^*(1)) \xrightarrow{\ j\ } \mathrm{H}^1(\mathbb{Q}_p, T^*(1))/\mathrm{H}^1_f(\mathbb{Q}_p, T^*(1)) \\ \longrightarrow \mathrm{Sel}(T)^\vee \longrightarrow \mathrm{H}^2(\mathbb{Z}[1/p], T^*(1)) \longrightarrow 0 \end{aligned}$$

where $(\ )^\vee = \mathrm{Hom}_{O_\lambda}(\ , K_\lambda/O_\lambda)$. (see (14.9.4). Here we used the fact $\mathrm{H}^2(\mathbb{Q}_p, T^*(1))$ is finite (14.12), and the fact that $T^*(1)$ is isomorphic to a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice of $V_{F_\lambda}(f^*)(k/2)$ as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over O_λ.

Lemma 18.7. — *Let $\mathfrak{p}$ be the ring homomorphism $\Lambda \to O_\lambda$ which sends G_∞ to 1. Then we have*

$$\dim_{F_\lambda}(\mathrm{H}^2(\mathbb{Z}[1/p], V_{F_\lambda}(f^*)(k/2))) \leqslant \mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^1(V_{F_\lambda}(f^*))(k/2)_{\mathfrak{p}}/Z(f^*)(k/2)_{\mathfrak{p}}).$$

Proof. — By

$$\mathbf{H}^2(V_{F_\lambda}(f^*))(k/2)_{\mathfrak{p}}/\mathfrak{p}\mathbf{H}^2(V_{F_\lambda}(f^*))(k/2)_{\mathfrak{p}} \simeq \mathrm{H}^2(\mathbb{Z}[1/p], V_{F_\lambda}(f^*)(k/2)),$$

we have

$$\begin{aligned} &\dim_{F_\lambda}(\mathrm{H}^2(\mathbb{Z}[1/p], V_{F_\lambda}(f^*)(k/2))) \\ &\qquad = \mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^2(V_{F_\lambda}(f^*))(k/2)_{\mathfrak{p}}/\mathfrak{p}\mathbf{H}^2(V_{F_\lambda}(f^*))(k/2)_{\mathfrak{p}}) \\ &\qquad \leqslant \mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^2(V_{F_\lambda}(f^*))(k/2)_{\mathfrak{p}}) \\ &\qquad \leqslant \mathrm{length}_{\Lambda_{\mathfrak{p}}}(\mathbf{H}^1(V_{F_\lambda}(f^*))(k/2)_{\mathfrak{p}}/Z(f^*)(k/2)_{\mathfrak{p}}), \end{aligned}$$

where the last $\leqslant$ follows from Thm. 12.5 (3). □

18.8. Let $\mathfrak{p}$ be as in Lemma 18.7. Then the kernel of the ring homomorphism

$$\mathrm{triv} : \mathscr{H}_\infty \longrightarrow F_\lambda(\alpha)$$

which sends G_∞ to 1 coincides with $\mathfrak{p}\mathscr{H}_\infty$.

For $r \in \mathbb{Z}$, let $\tau_r : \mathscr{H}_\infty \to \mathscr{H}_\infty$ be the ring isomorphism induced by

$$G_\infty \longrightarrow \mathscr{H}_\infty, \quad \sigma \longmapsto \kappa(\sigma)^r \sigma.$$

Let η be as in Thm. 16.6. Then the Λ-homomorphism

$$\mathfrak{L}_\eta : \mathbf{H}^1_{\mathrm{loc}}(V_{F_\lambda}(f^*)(k)) \longrightarrow \mathscr{H}_\infty$$

induces a Λ-homomorphism

$$\begin{aligned}\mathfrak{L}_{\eta,k/2} \cdot \mathbf{H}^1_{\mathrm{loc}}(V_{F_\lambda}(f^*)(k/2)) &\longrightarrow \mathscr{H}_\infty \\ x &\longmapsto \tau_{k/2}\mathfrak{L}_\eta(x \otimes (\zeta_{p^n})_{n\geqslant 1}^{\otimes(k/2)}).\end{aligned}$$

Let I be the ideal of $\mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty}$ generated by the image of $Z(f^*)(k/2)$ under $\mathfrak{L}_{\eta,k/2}$. Then I coincides with the ideal generated by $\tau_{k/2}L_{p\text{-adic},\alpha}(f)$. Hence

$$\begin{aligned}\text{(18.8.1)}\qquad \mathrm{length}_{\mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty}}(\mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty}/I) &= \mathrm{ord}_{s=0}(\tau_{k/2}L_{p\text{-adic},\alpha}(f)) \\ &= \mathrm{ord}_{s=k/2}(L_{p\text{-adic},\alpha}(f)).\end{aligned}$$

Lemma 18.9

(1) *We have*

$$\mathrm{length}_{\Lambda_\mathfrak{p}}(\mathbf{H}^1(V_{F_\lambda}(f^*))(k/2)_\mathfrak{p}/Z(f^*)(k/2)_\mathfrak{p}) \leqslant \mathrm{ord}_{s=k/2}(L_{p\text{-adic},\alpha}(f)).$$

(2) *If* $j \otimes \mathbb{Q} = 0$,

$$\mathrm{length}_{\Lambda_\mathfrak{p}}(\mathbf{H}^1(V_{F_\lambda}(f^*))(k/2)_\mathfrak{p}/Z(f^*)(k/2)_\mathfrak{p}) \leqslant \mathrm{ord}_{s=k/2}(L_{p\text{-adic},\alpha}(f)) - 1.$$

Proof. — (1) is clear from (18.8.1). Assume $j \otimes \mathbb{Q} = 0$. Since the composite

$$\text{(18.9.1)}\qquad \mathbf{H}^1(V_{F_\lambda}(f^*))(k/2) \xrightarrow{\mathfrak{L}_{\eta,k/2}} \mathscr{H}_\infty \xrightarrow{\mathrm{triv}} F_\lambda(\alpha)$$

coincides with $(1-p^{(k-2)/2}\alpha^{-1})(1-p^{(k-2)/2}\alpha)^{-1}$ times the composite

$$\begin{aligned}&\mathbf{H}^1(V_{F_\lambda}(f^*))(k/2) \longrightarrow \mathrm{H}^1(\mathbb{Z}[1/p], V_{F_\lambda}(f^*)) \xrightarrow{j\otimes\mathbb{Q}} \\ &\quad \mathrm{H}^1(\mathbb{Q}_p, V_{F_\lambda}(f^*)(k/2))/\mathrm{H}^1_f(\mathbb{Q}_p, V_{F_\lambda}(f^*)(k/2)) \xrightarrow{\exp^*} S(f^*)\otimes_F F_\lambda \xrightarrow{\eta} F_\lambda(\alpha)\end{aligned}$$

(16.4 (ii)), the map (18.9.1) is the zero map. Hence the image of

$$\mathfrak{L}_{\eta,k/2} : \mathbf{H}^1(V_{F_\lambda}(f^*))(k/2) \longrightarrow \mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty}$$

is contained in $\mathfrak{p}\mathscr{H}_\infty$. Hence we have the $\leqslant$ in

$$\begin{aligned}\mathrm{length}_{\Lambda_\mathfrak{p}}(\mathbf{H}^1(V_{F_\lambda}(f^*))(k/2)_\mathfrak{p}/Z(f^*)(k/2)_\mathfrak{p}) &\leqslant \mathrm{length}_{\mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty}}(\mathfrak{p}\mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty}/I) \\ &= \mathrm{length}_{\mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty}}(\mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty}/I) - 1. \quad \square\end{aligned}$$

18.10. Now the case $\alpha \neq p^{(k-2)/2}$ of Thm. 18.4 follows from Lemma 18.6, 18.7, 18.9.

18.11. Finally we give the outline of the proof of the case $\alpha = p^{(k-2)/2}$ of Thm. 18.4. In the case, by [**KKT**], the image of

$$\mathfrak{L}_{\eta,k/2} : \mathbf{H}^1_{\mathrm{loc}}(V_{F_\lambda}(f^*)(k/2)) \longrightarrow \mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty}$$

is contained in $\mathfrak{p}\mathscr{H}_\infty$, and the composite map

$$\mathfrak{L}_{\eta,k/2} : \mathbf{H}^1_{\mathrm{loc}}(V_{F_\lambda}(f^*)(k/2)) \longrightarrow \mathfrak{p}\mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty} \longrightarrow \mathfrak{p}\mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty}/\mathfrak{p}^2\mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty}$$

factors through the canonical projection

$$\mathbf{H}^1_{\mathrm{loc}}(V_{F_\lambda}(f^*)(k/2)) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, V_{F_\lambda}(f^*)(k/2))/\mathrm{H}^1_f(\mathbb{Q}_p, V_{F_\lambda}(f^*)(k/2)).$$

Hence in the case $\alpha = p^{(k-2)/2}$ and $j \otimes \mathbb{Q} = 0$, the image of

$$\mathfrak{L}_{\eta,k/2} : \mathbf{H}^1(V_{F_\lambda}(f^*)(k/2)) \longrightarrow \mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty}$$

is contained in $\mathfrak{p}^2\mathscr{H}_{\infty,\mathfrak{p}\mathscr{H}_\infty}$. These show that

$$\mathrm{length}_{\Lambda_\mathfrak{p}}(\mathbf{H}^1(V_{F_\lambda}(f^*))(k/2)_\mathfrak{p}/Z(f^*)(k/2)_\mathfrak{p}) \leqslant \mathrm{ord}_{s=k/2}(L_{p\text{-adic},\alpha}(f)) - 1$$

in the case $\alpha = p^{(k-2)/2}$ and $j \otimes \mathbb{Q} \neq 0$, and

$$\mathrm{length}_{\Lambda_\mathfrak{p}}(\mathbf{H}^1(V_{F_\lambda}(f^*))(k/2)_\mathfrak{p}/Z(f^*)(k/2)_\mathfrak{p}) \leqslant \mathrm{ord}_{s=k/2}(L_{p\text{-adic},\alpha}(f)) - 2$$

in the case $\alpha = p^{(k-2)/2}$ and $j \otimes \mathbb{Q} = 0$. This and Lemma 18.6, 18.7 prove the case $\alpha = p^{(k-2)/2}$ of Thm. 18.4.

Table of special Notation

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