ASTÉRISQUE 296

ANALYSE COMPLEXE, SYSTÈMES DYNAMIQUES, SOMMABILITÉ DES SÉRIES DIVERGENTES ET THÉORIES GALOISIENNES (I)

VOLUME EN L'HONNEUR DE JEAN-PIERRE RAMIS

édité par

Michèle Loday-Richaud

Société Mathématique de France 2004
Publié avec le concours du Centre National de la Recherche Scientifique

M. Loday-Richaud
Université d'Angers – UFR Sciences, LAREMA, UMR 6093, 2 boulevard Lavoisier, 49 045 Angers Cedex 01 (France).
E-mail : michele.loday@univ-angers.fr

***Classification mathématique par sujets* (2000). —** 01A70, 11R32, 11R39, 11S80, 12Hxx, 13Nxx, 17B15, 32G34, 32S65, 33E17, 34A30, 34A34, 34Mxx, 35A27, 35D10, 37F75, 39A13, 58B34, 81T15.

***Mots clefs.* —** Algèbre différentielle ; monodromie p-adique ; équations différentielles, équations aux différences, équations aux q-différences dans le champ complexe ; modules de Krichever ; descente ; pinceaux de feuilletages ; courbure ; D-modules holonomes ; b-fonctions ; intégrabilité ; géométrie non commutative (à la Connes) ; renormalisation ; algèbres de Hopf ; théories de Galois ; correspondance de Riemann-Hilbert ; déformations isomonodromiques.

ANALYSE COMPLEXE, SYSTÈMES DYNAMIQUES, SOMMABILITÉ DES SÉRIES DIVERGENTES ET THÉORIES GALOISIENNES (I)

VOLUME EN L'HONNEUR DE JEAN-PIERRE RAMIS

édité par Michèle Loday-Richaud

***Résumé.* —** Cet ouvrage en deux volumes rassemble les actes du colloque *Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes* organisé à Toulouse du 22 au 26 septembre 2003 à l'occasion du soixantième anniversaire de Jean-Pierre Ramis.

En introduction, le premier volume propose deux textes de souvenirs et trois textes de synthèse des travaux de J.-P. Ramis en analyse complexe, en théorie des équations différentielles linéaires et en théorie des équations différentielles non-linéaires. Suivent des textes essentiellement consacrés aux théories galoisiennes, à l'arithmétique et à l'intégrabilité : analogies entre théories différentielles et théories arithmétiques, équations aux q-différences classiques ou p-adiques, problème de Riemann-Hilbert et renormalisation, b-fonctions, problèmes de descente, modules de Krichever, lieu d'intégrabilité, théorie de Drach et équation de Painlevé VI.

Le deuxième volume rassemble des textes plutôt liés à des questions d'analyse et de géométrie : stabilité de Lyapounov, analyse asymptotique et dynamique pour des pinceaux de trajectoires, analyse WKB et géométrie de Stokes, équations de Painlevé I et II, formes normales des singularités de type nœud-col, tores invariants d'équations aux dérivées partielles.

© Astérisque 296, SMF 2004

***Abstract* (Complex analysis, dynamical systems, summability of divergent series and Galois theories (I), Volume in honor of Jean-Pierre Ramis)**

These two bound volumes present the proceedings of the conference *Complex Analysis, Dynamical Systems, Summability of Divergent Series and Galois Theories* held in Toulouse from September 22$^{\text{nd}}$ to September 26$^{\text{th}}$ 2003, on the occasion of J.-P. Ramis' 60$^{\text{th}}$ birthday.

The first volume opens with two texts composed of recollections and three texts on J.-P. Ramis' works on Complex Analysis and Ordinary Differential Equations Theory, both linear and non-linear. This introduction is followed by papers concerned with Galois Theories, Arithmetic or Integrability: analogies between differential and arithmetical theories, q-difference equations, classical or p-adic, the Riemann-Hilbert problem and renormalisation, b-functions, descent problems, Krichever modules, the set of integrability, Drach theory and the VI$^{\text{th}}$ Painlevé equation.

The second volume contains papers dealing with analytical or geometrical aspects: Lyapunov stability, asymptotic and dynamical analysis for pencils of trajectories, monodromy in moduli spaces, WKB analysis and Stokes geometry, first and second Painlevé equations, normal forms for saddle-node type singularities, invariant tori for PDEs.

TABLE DES MATIÈRES

Résumés des articles xiii

Abstracts xvii

Avant-Propos xxi

B. Malgrange — *Les premiers travaux de Jean-Pierre Ramis* 1

G. Ruget — *Témoignage* 7

D. Bertrand — *Travaux de J.-P. Ramis sur les équations différentielles linéaires* 11
Acte I. Filtrations Gevrey *Dijon, 1976* ([R1]) 11
Acte II. (Re)sommation et groupes de Galois différentiels *Les Houches (1979)* ([R2]), *Rio (1985)* ([R4]), *Strasbourg (1991)* ([Mr-R6]) 13
Acte III. La conjecture d'Abhyankar différentielle *Toulouse (Nuit de la musique, 1993)* ([R10]) 16
Références 19

D. Cerveau — *Travaux de J.-P. Ramis sur les équations différentielles non linéaires* 21
1. Systèmes hamiltoniens (Morales-Ramis) 21
2. Perturbations singulières 24
3. Feuilletages holomorphes 26
Références 31

M. Loday-Richaud — *Souvenirs strasbourgeois* 33

Y. André — *Galois representations, differential equations, and q-difference equations : sketch of a p-adic unification* 43
Introduction 43
1. A mysterious analogy : linear complex differential equations and coverings in characteristic p, tame and wild 44

2. The p-adic analog of this analogy. An equivalence of categories. 46
3. Another analogy : linear differential equations and q-difference equations ; confluence 50
4. The p-adic analog of this analogy. Another equivalence of categories [**AdV**] 51
References 53

Y. André & L. Di Vizio — *q-difference equations and p-adic local monodromy* 55
Introduction 55
Part I. Rank 1 57
1. Generalities on p-adic q-difference equations of rank 1 57
2. An example : the q-exponential function 61
3. Solvability (at the boundary) 65
4. A characterization of solvability 67
5. Reduction to the case of q-difference equations with polynomial coefficient 70
6. Frobenius structure in rank 1 : existence criterion 73
7. q-deformation of differential equations with strong Frobenius structure ... 78
8. The group of isomorphism classes of q-difference equations of rank 1 admitting a Frobenius structure 80
Appendices to part I 82
9. Frobenius structure of $d_q y(x) = \pi_q y(x)$ 82
10. p-adic q-exponential and Koblitz' Gamma function 84
Part II. Higher rank 87
11. Preliminaries : unramified extensions of $\mathcal{E}^\dagger$ 87
12. q-difference modules and Frobenius structures 89
13. "Unit-root" q-difference modules 98
14. Quasi-unipotence 104
15. Applications 108
References 110

A. Connes — *Renormalisation et ambiguïté galoisienne* 113
1. Introduction 114
2. Renormalisation, position du problème 116
3. Structure algébrique des graphes de Feynman 122
4. Renormalisation et problème de Riemann-Hilbert 127
5. Le groupe de renormalisation 132
6. Le groupe G et les difféomorphismes formels 134
7. Le groupe de renormalisation et la théorie de Galois 136
Références 141

Y. Laurent — *b-functions and integrable solutions of holonomic $\mathcal{D}$-module* .. 145
Introduction 145
1. V-filtration and b-functions 146
2. Reductive Lie algebras 153
References 164

A. Lins Neto — *Curvature of pencils of foliations* 167
1. Introduction 167
2. Proofs 172
References 189

M. van der Put — *Skew differential fields, differential and difference equations* 191
Introduction 191
1. The construction of skew differential fields 192
2. Skew differential fields over $\mathbf{R}(\{x\})$ 192
3. Descent for q-difference equations 200
4. Descent for ordinary difference equations 202
References 205

M. van der Put & M. Reversat — *Krichever modules for difference and differential equations* 207
Introduction 207
1. Abelian differential equations 208
2. Abelian difference equations 217
References 225

J. Sauloy — *Algebraic construction of the Stokes sheaf for irregular linear q-difference equations* 227
1. Introduction and general conventions 227
2. Local analytic classification 231
3. Algebraic summation 240
4. The q-Gevrey filtration on the Stokes sheaf 247
References 250

H. Umemura — *Monodromy preserving deformation and differential Galois group I* 253
1. Introduction 253
2. Review of R. Fuchs' paper 256
3. Infinite dimensional differential Galois theory 257
4. Proof of Theorem 1.1 260
5. Framework of proving Theorem 1.3 264
6. Questions 266
7. Appendix : Extract from a letter of B. Malgrange to D. Bertrand 267
References 269

TABLE DES MATIÈRES DU VOLUME II

Résumés des articles xiii

Abstracts xvii

Avant-Propos xxi

F. Cano, R. Moussu & F. Sanz — *Pinceaux de courbes intégrales d'un champ de vecteurs analytique* 1
0. Introduction 1
1. Enlacement asymptotique et pinceau intégral 4
2. Pinceau intégral hyperbolique 13
3. Pinceau intégral central de type I 15
4. Pinceau intégral final de type II 20
5. Le cas général 28
Références 33

B. Dubrovin — *On analytic families of invariant tori for PDEs* 35
1. Introduction 35
2. Can one see the shape of a Riemann surface looking at the water waves? . 42
3. Infinite genus theta-functions of Riemann surfaces without Riemann surfaces 56
References 63

N. Joshi, K. Kajiwara & M. Mazzocco — *Generating Function Associated with the Determinant Formula for the Solutions of the Painlevé II Equation* ... 67
1. Introduction 67
2. Hankel Determinant Formula and Isomonodromic Problem 69
3. Solutions of Isomonodromic Problems and Determinant Formula 73
4. Summability of the generating function 76
References 77

V. Kaloshin, J.N. Mather & E. Valdinoci — *Instability of resonant totally elliptic points of symplectic maps in dimension* 4 79
1. Introduction 79
2. Suspension of a symplectic map near totally elliptic points of a time periodic fiber-convex Hamiltonian 84
3. Scheme of construction of diffusing trajectories using Mather action functional 87
4. Mather diffusion theorem 91
5. Averaged mechanical systems corresponding to single and double resonances 93
6. Definition of $U^s_{\delta(\ell_0,\Gamma)}$ 95
7. Definition of $W^s_{\delta(\ell_0,\Gamma)}$ using type 2 non-degeneracy (of Barrier functions) ..101
8. Variational principle and restatement of Mather diffusion theorem 105
9. Application 106
Appendix A. Mather minimal sets 108
Appendix B. Proofs of auxiliary lemmas 111
References 114

T. Kawai, T. Koike, Y. Nishikawa & Y. Takei — *On the Stokes geometry of higher order Painlevé equations* 117
0. Introduction 118
1. P_J-hierarchy with a large parameter ($J =$ I, II-1 or II-2) 122
2. Relations between the Stokes geometry of the (P_J)-hierarchies and that of their underlying Lax pairs 129
3. The inevitability of the Nishikawa phenomenon 140
4. Introduction of a new Stokes curve to explain the Nishikawa phenomenon 144
5. Examples of Stokes geometry 154
Appendix A. Some properties of $\mathcal{K}_j$ and K_j 160
Appendix B. Another formulation of the $P_{\rm I}$-hierarchy 163
References 165

F. Loray — *Versal deformation of the analytic saddle-node* 167
Introduction and results 167
1. Martinet-Ramis' invariants 170
2. Proof of Theorem 1 173
3. Gluing Lemmae 175
4. Proof of Theorem 2 178
5. Proof of Theorem 4 184
References 186

C. Simpson — *Asymptotics for general connections at infinity* 189
1. Introduction 189
2. The compactified moduli space of connections 193
3. Curves going to infinity 194
4. Genericity results for the spectral data 196
5. Pullback to a ramified covering and gauge transformations 198

6. Laplace transform of the monodromy operators 202
7. Analytic continuation of the Laplace transform 206
8. Description of cells using trees 210
9. Remoteness of points 213
10. Calculations of gradient flows 214
11. Choice of the vector fields W_{ij} 217
12. Results on the dynamics of our flowing maps 219
13. Proofs 225
14. Conclusion 227
References 229

RÉSUMÉS DES ARTICLES

Les premiers travaux de Jean-Pierre Ramis
Bernard Malgrange 1

Exposé des travaux de Jean-Pierre Ramis avant 1978-80, travaux portant sur les espaces analytiques banachiques, la dualité en géométrie analytique et les $\mathcal{D}$-modules.

Témoignage
Gabriel Ruget 7

Ce texte est une évocation des années 1962-1971 passées avec Jean-Pierre Ramis à l'École Normale Supérieure et à l'université entre Paris et Orsay, puis à Tunis.

Travaux de J.-P. Ramis sur les équations différentielles linéaires
Daniel Bertrand 11

L'article donne un survol des travaux de J-P. Ramis sur les singularités irrégulières : théorèmes d'indice Gevrey, dévissage du groupe de Galois analytique, problème inverse.

Travaux de J.-P. Ramis sur les équations différentielles non linéaires
Dominique Cerveau 21

On survole trois articles de J.-P. Ramis traitant de phénomènes non linéaires.

Souvenirs strasbourgeois
Michèle Loday-Richaud 33

Ce texte est une évocation des années 1971-1987 que j'ai passées à Strasbourg dans le groupe de travail organisé par Jean-Pierre Ramis sur les équations fonctionnelles et, en particulier, différentielles.

Galois representations, differential equations, and q-difference equations : sketch of a p-adic unification
Yves André 43

Ce texte est une introduction développée à l'article suivant, plus technique [**AdV**]. Nous expliquons comment [**AdV**] est lié à deux thèmes majeurs de l'œuvre de J.-P. Ramis, et comment ceux-ci trouvent leur unification en passant au monde p-adique.

q-difference equations and p-adic local monodromy
Yves André & Lucia Di Vizio 55

Nous présentons une théorie p-adique des équations aux q-différences sur des couronnes arbitrairement minces de rayon extérieur 1. Après une étude détaillée des équations de rang 1, nous nous penchons sur le cas de rang supérieur et nous démontrons un théorème de monodromie locale (un q-analogue de la conjecture de quasi-unipotence de Crew). Cela nous permet de définir, dans ce contexte, un foncteur canonique de « confluence » des équations aux q-différences vers les équations différentielles, qui s'avère être une équivalence de catégories (en présence de structures de Frobenius).

Renormalisation et ambiguïté galoisienne
Alain Connes 113

Cet article donne un exposé détaillé de mes résultats sur la renormalisation en collaboration avec D. Kreimer. Le premier résultat essentiel est l'identité entre le procédé récursif utilisé par les physiciens pour éliminer les divergences en théorie des champs quantiques et la décomposition de Birkhoff des lacets à valeurs dans un groupe pro-unipotent. Le groupe impliqué dans la renormalisation est celui des difféographismes, construit à partir des graphes de Feynman. Le deuxième résultat important est la construction d'une action de ce groupe des difféographismes sur les constantes de couplage sans dimension de la théorie physique. Le lien précis entre mon travail avec Kreimer et la correspondance de Riemann-Hilbert a été obtenu en collaboration avec M. Marcolli et est explicité à la fin de ce texte. Nous établissons une correspondance de Riemann-Hilbert entre connexions plates équisingulières et représentations d'un groupe de Galois « motivique » explicite U^*. Ce groupe joue dans le contexte de la renormalisation un rôle analogue à celui du tore exponentiel de Jean-Pierre Ramis dans la théorie locale des singularités irrégulières des équations différentielles et il apporte une réponse satisfaisante à la recherche proposée par P. Cartier d'un groupe de Galois « cosmique » qui sous-tend la renormalisation.

b-functions and integrable solutions of holonomic $\mathcal{D}$-module
Yves Laurent 145

Un célèbre théorème de Harish-Chandra montre que les distributions invariantes propres sur un groupe de Lie semi-simple sont des fonctions localement intégrables. Nous donnons ici une version algébrique de ce théorème en termes de polynômes associés à un $\mathcal{D}$-module holonome.

Curvature of pencils of foliations
Alcides Lins Neto 167

Nous nous intéressons au *pinceau de feuilletages* $\mathcal{P} = \mathcal{P}(\mathcal{F}, \mathcal{G})$ engendré par deux feuilletages $\mathcal{F}$ et $\mathcal{G}$ holomorphes singuliers distincts sur une surface complexe compacte M et appartenant à la même classe, *i.e.*, $N_{\mathcal{F}} = N_{\mathcal{G}}$. La forme de courbure du pinceau $\mathcal{P}$ est une 2-forme $\Theta = \Theta(\mathcal{P})$ qui coïncide avec

la courbure de Chern (*cf.* [**Ch**]) ; lorsque $\Theta(\mathcal{P}) \equiv 0$ on dit que le pinceau est *plat*. Dans cet article, nous donnons des conditions suffisantes de platitude d'un pinceau (Théorème 2). Nous regardons comment se traduit la platitude dans le pseudo-groupe d'holonomie des feuilletages de $\mathcal{P}$ et, en particulier, nous étudions dans certains cas l'ensemble $\{\mathcal{H} \in \mathcal{P} \mid \mathcal{H}$ admet une intégrale première$\}$ (Théorème 1).

Skew differential fields, differential and difference equations
MARIUS VAN DER PUT 191

On étudie la descente sur un corps k d'une équation différentielle ou aux différences donnée sur un corps K et qui est isomorphe à toutes ses conjuguées sous l'action du groupe de Galois $\mathrm{Gal}(K/k)$ de K sur k. On traite le cas de plusieurs classes d'équations.

Krichever modules for difference and differential equations
MARIUS VAN DER PUT & MARC REVERSAT 207

On compare explicitement les modules de Krichever différentiels ou aux différences de rang 1 aux extensions abéliennes différentielles ou aux différences. Ceci permet de donner, dans ces situations, un analogue du théorème de Kronecker-Weber.

Algebraic construction of the Stokes sheaf for irregular linear q-difference equations
JACQUES SAULOY 227

La classification analytique locale des équations aux q-différences linéaires irrégulières a été récemment réalisée par J.-P. Ramis, J. Sauloy et C. Zhang. Leur description fait intervenir un q-analogue du faisceau de Stokes et des théorèmes de type Malgrange-Sibuya et elle s'appuie sur la sommation discrète de C. Zhang. Nous montrons ici comment retrouver une partie de ces résultats par voie algébrique et nous décrivons le dévissage q-Gevrey du q-faisceau de Stokes par des fibrés vectoriels holomorphes sur une courbe elliptique.

Monodromy preserving deformation and differential Galois group I
HIROSHI UMEMURA 253

En 1914, J. Drach interpréta le travail de R. Fuchs sur les déformations isomonodromiques et la sixième équation de Painlevé en termes de sa théorie de Galois de dimension infinie. La note de Drach contient une idée très originale mais difficile à comprendre. Nous analysons sa note en appliquant notre théorie de Galois différentielle de dimension infinie. Cela nous donne des exemples non triviaux dont nous pouvons calculer notre groupe de Galois.

ABSTRACTS

Les premiers travaux de Jean-Pierre Ramis
Bernard Malgrange 1

Exposition of Jean-Pierre Ramis' works before 1978-80, works devoted to Banach analytic spaces, duality in complex analysis and $\mathcal{D}$-modules.

Témoignage
Gabriel Ruget 7

We relate memories of the years 1962-1971 spent at the École Normale Supérieure in Paris, at the Universities of Paris and Orsay and in Tunis with Jean-Pierre Ramis.

Travaux de J.-P. Ramis sur les équations différentielles linéaires
Daniel Bertrand 11

We survey the work of J-P. Ramis on irregular singularities, with special emphasis on his Gevrey theorems, his extension of Schlesinger's density theorem, and the analogies his theory provides between analytic differential Galois groups and fundamental groups in positive characteritic.

Travaux de J.-P. Ramis sur les équations différentielles non linéaires
Dominique Cerveau 21

About three papers of J.-P. Ramis related to non linear phenomena.

Souvenirs strasbourgeois
Michèle Loday-Richaud 33

This paper deals with memories of the time (1971-1987) I spent in Strasbourg as a member of the working group on differential or functional equations around Jean-Pierre Ramis.

Galois representations, differential equations, and q-difference equations : sketch of a p-adic unification
Yves André 43

This is a broad introduction to the following, more technical, paper [**AdV**]. We explain how [**AdV**] relates to two major themes of J.-P. Ramis' work, which eventually become unified in the p-adic world.

q-difference equations and p-adic local monodromy
Yves André & Lucia Di Vizio 55

We present a p-adic theory of q-difference equations over arbitrarily thin annuli of outer radius 1. After a detailed study of rank one equations, we consider higher rank equations and prove a local monodromy theorem (a q-analog of Crew's quasi-unipotence conjecture). This allows us to define, in this context, a canonical functor of "confluence" from q-difference equations to differential equations, which turns out to be an equivalence of categories (in the presence of Frobenius structures).

Renormalisation et ambiguïté galoisienne
Alain Connes 113

This paper contains a detailed exposition of my joint work with Kreimer on renormalization. The first key result is the identity between the recursive process used by physicists to remove the divergencies in quantum field theory and the Birkhoff decomposition of loops with values in a pro-unipotent Lie group. The relevant group for renormalization is the group of diffeographisms which is constructed from Feynman graphs. The second key result is the construction of an action of the group of diffeographisms on the dimensionless coupling constants of the theory. The precise link between my work with Kreimer and the Riemann-Hilbert correspondence was obtained in collaboration with M. Marcolli and is explained briefly at the end of the paper. We construct a Riemann-Hilbert correspondence between flat equisingular connections and representations of a specific motivic Galois group U^*. This group is the analogue in renormalization of the exponential torus of Ramis in the local theory of irregular singular differential equations. Our work gives a natural candidate for the « cosmic Galois group » envisaged by Cartier as the symmetry underlying renormalization.

b-functions and integrable solutions of holonomic $\mathcal{D}$-module
Yves Laurent 145

A famous theorem of Harish-Chandra shows that all invariant eigendistributions on a semi-simple Lie group are locally integrable functions. We give here an algebraic version of this theorem in terms of polynomials associated with a holonomic $\mathcal{D}$-module.

Curvature of pencils of foliations
Alcides Lins Neto 167

Let $\mathcal{F}$ and $\mathcal{G}$ be two distinct singular holomorphic foliations on a compact complex surface M, in the same class, that is $N_{\mathcal{F}} = N_{\mathcal{G}}$. In this case, we can define the *pencil* $\mathcal{P} = \mathcal{P}(\mathcal{F}, \mathcal{G})$ *of foliations generated by* $\mathcal{F}$ *and* $\mathcal{G}$. We can associate to a pencil $\mathcal{P}$ a meromorphic 2-form $\Theta = \Theta(\mathcal{P})$, the form of curvature of the pencil, which is in fact the Chern curvature (*cf.* [**Ch**]). When $\Theta(\mathcal{P}) \equiv 0$ we will say that the pencil is *flat*. In this paper we give some sufficient conditions

for a pencil to be flat. (Theorem 2). We will see also how the flatness reflects in the pseudo-group of holonomy of the foliations of $\mathcal{P}$. In particular, we will study the set $\{\mathcal{H} \in \mathcal{P} \mid \mathcal{H}$ has a first integral$\}$ in some cases (Theorem 1).

Skew differential fields, differential and difference equations
MARIUS VAN DER PUT 191

The central question is : Let a differential or difference equation over a field K be isomorphic to all its Galois twists w.r.t. the group $\mathrm{Gal}(K/k)$. Does the equation descend to k ? For a number of categories of equations an answer is given.

Krichever modules for difference and differential equations
MARIUS VAN DER PUT & MARC REVERSAT 207

In order to understand an analogue of Kronecker-Weber and some abelian Langlands theory for function fields, an explicit comparison between Krichever differential (or difference) modules of rank one and abelian differential (or difference) equations is carried out.

Algebraic construction of the Stokes sheaf for irregular linear q-difference equations
JACQUES SAULOY 227

The local analytic classification of irregular linear q-difference equations has recently been obtained by J.-P. Ramis, J. Sauloy and C. Zhang. Their description involves a q-analog of the Stokes sheaf and theorems of Malgrange-Sibuya type and is based on a discrete summation process due to C. Zhang. We show here another road to some of these results by algebraic means and we describe the q-Gevrey devissage of the q-Stokes sheaf by holomorphic vector bundles over an elliptic curve.

Monodromy preserving deformation and differential Galois group I
HIROSHI UMEMURA 253

In 1914, J. Drach interpreted in terms of his infinite dimensional differential Galois theory R. Fuchs' work on the monodromy preserving deformation and the sixth Painlevé equation. This note of Drach contains a quite original idea but it is difficult to understand. We analyze his note by our infinite dimensional differential Galois theory. We get non-trivial examples of which we can calculate our Galois group.

AVANT-PROPOS

Ce colloque a bénéficié du soutien financier des organismes suivants :
- Centre National de la Recherche Scientifique,
- Ministère de l'Education Nationale,
- Ministère des Affaires Étrangères,
- Conseil Régional Midi-Pyrénées,
- Ville de Toulouse,
- Université d'Angers,
- Université de Bourgogne,
- Université Rennes I,
- Université de La Rochelle,
- Université Strasbourg I,
- Université Toulouse I,
- Université Toulouse III,
- Laboratoire Paul Painlevé (Lille I),
- IRMA (Strasbourg I),
- Laboratoire GRIMM (Toulouse II),
- Institut de Mathématique (Toulouse III),
- PICS France-Mexique,

du soutien matériel de l'Institut de Mathématique de Toulouse III
et du patronage de la Société Mathématique de France.

Comité Scientifique : Daniel Bertrand, Dominique Cerveau, Alain Chenciner, Bernard Malgrange, Jean-Christophe Yoccoz.

Comité d'organisation : Anne Duval, Frédéric Fauvet, Martine Klughertz, Michèle Loday-Richaud, Emmanuel Paul, Laurent Stolovitch.

Nous adressons nos remerciements à tous ceux qui ont contribué au succès du colloque : donateurs, conférenciers, participants,... Nous formulons une mention particulière à l'intention des personnels administratifs et techniques de l'Université Toulouse III qui nous ont apporté leur aide avec dévouement et compétence.

Jean-Pierre Ramis

Jean-Pierre Ramis, né le 26 mars 1943 à Montpellier, a été élève de l'École Normale Supérieure d'octobre 1962 à septembre 1966. Assistant à la Faculté des Sciences de Paris puis attaché de recherches au CNRS il a soutenu, en mars 1969, un doctorat d'état préparé sous la direction de Henri Cartan (*Sous-ensembles analytiques d'une variété analytique banachique*). Maître de conférences à la Faculté des Sciences de Paris en 1968-69 puis à Tunis d'octobre 1969 à septembre 1971, il est nommé ensuite à l'Université Louis Pasteur à Strasbourg où il reste jusqu'en 1994, d'abord, comme maître de conférences puis, rapidement, comme professeur. Il est, depuis septembre 1994, professeur à l'Université Paul Sabatier à Toulouse et, depuis 1996, membre de l'Institut Universitaire de France. Il a reçu le prix Doisteau-Blutel de l'Académie des Sciences en 1982 et le prix A. Joannidès de l'Académie des Sciences en 2002. Il a été nommé chevalier des palmes académiques en 1999.



Strasbourg, novembre 2004

Liste des participants

Rachel ABABOU	École Militaire de Saint-Cyr
Jonathan AÏDAN	Université Paris 6
Luca AMODEI	Université Toulouse 3
Yves ANDRÉ	É.N.S., Paris
Xavier ANTOINE	Université Toulouse 3
Claude ARNOLD	Université Toulouse 3
José Manuel AROCA	Universidad de Valladolid (Espagne)
Alain BACCINI	Université Toulouse 3
Moulay BARKATOU	Université de Limoges
François BERTELOOT	Université Toulouse 3
Michel BERTHIER	Université de La Rochelle
Daniel BERTRAND	Université Paris 6
Michel BOILEAU	Université Toulouse 3
Jean-Benoît BOST	Université Paris 11
Delphine BOUCHER	Université Rennes 1
Boele L. J. BRAAKSMA	University of Groningen (Pays-Bas)
Manuel BRONSTEIN	I.N.R.I.A. Sophia Antipolis
Xavier BUFF	Université Toulouse 3
Mireille CANALIS-DURAND	Université Aix-Marseille 3
Felipe CANO	Universidad de Valladolid (Espagne)
Jose CANO	Universidad de Valladolid (Espagne)
Pierre CARTIER	C.N.R.S. Paris
Guy CASALE	Université Toulouse 3
Francisco J. CASTRO-JIMENEZ	Universidad de Sevilla (Espagne)
Dominique CERVEAU	Université Rennes 1
Alain CHENCINER	I.M.C.C.E. et Université Paris 7
Alain CONNES	Collège de France
Ovidiu COSTIN	Rutgers University (U.S.A.)
Laure COUTIN	Université Toulouse 3
Sadja CHETTAB	Université de La Rochelle
Benoît CHEVALLIER	Université Toulouse 2
Élie COMPOINT	Université Lille 1
Robert CONTE	C.E.A. de Saclay
Eduardo COREL	Universidad de Valladolid (Espagne)
Nuria CORRAL	Université Paris 7
Teresa CRESPO	Universitat de Barcelona (Espagne)
Jacky CRESSON	Université de Franche-Comté (Besançon)
Éric DELABAERE	Université d'Angers
Jesus Manuel DEL BLANCO	Universidad de Valladolid (Espagne)
Thierry DELMOTTE	Université Toulouse 3
Florian DELOUP	Université Toulouse 3
Lucia DI VIZIO	Université Toulouse 3
Raouf DRIDI	Université de Annaba (Algérie)
Anne DUVAL	Université Lille 1
Guillaume DUVAL	Universidad de Valladolid (Espagne)

Philippe EYSSIDIEUX	Université Toulouse 3
Frédéric FAUVET	Université Strasbourg 1
Bastien FERNANDEZ	C.P.T. Université Aix-Marseille
Étienne FIEUX	Université Toulouse 3
Thomas FORGET	Université de La Rochelle
Laurence FOURRIER	Université Toulouse 3
Jean-Pierre FRANÇOISE	Université Paris 6
Philippe GAILLARD	Université Rennes 1
Lubomir GAVRILOV	Université Toulouse 3
Roger GAY	Université Bordeaux 1
Luis GIRALDO	Universidad de Cádiz (Espagne)
Alexey GLUTSYUK	E.N.S., Lyon
Joseph GRIFONE	Université Toulouse 3
John GUASCHI	Université Toulouse 3
Vincent GUEDJ	Université Toulouse 3
Michel GUILLEMOT	Université Toulouse 3
Vincent GUIRARDEL	Université Toulouse 3
Zbigniew HAJTO	Universitat de Barcelona (Espagne)
Charlotte HARDOUIN	Université Paris 6
Julia HARTMANN	Universität Heidelberg (Espagne)
Herwig HAUSER	Universität Innsbruck (Autriche)
Alain HENAULT	Université Bordeaux 1
Yulij IL'YASHENKO	Cornell University (U.S.A.) et Steklov Institute (Moscou)
Geertrui Klara IMMINK	Rijksuniversiteit Groningen (Pays-Bas)
Christian KASSEL	Université Strasbourg 1
Takahiro KAWAI	R.I.M.S. Kyoto University (Japon)
Vadim KALOSHIN	Caltech (U.S.A.)
Christophe KAPOUDJAN	Université Toulouse 3
Tatsuya KOIKE	Kyoto University (Japon)
Anatoly N. KOCHUBEI	National Academy of Sciences of Ukraine
Martine KLUGHERTZ	Université Toulouse 3
Patrick LABORDE	Université Toulouse 3
Yves LAURENT	Université Grenoble 1
Michel LEDOUX	Université Toulouse 3
Antonius H. M. LEVELT	Université de Nimègue (Pays-Bas)
Paulette LIBERMANN	Université Paris 7
Alcides LINS NETO	I.M.P.A. Rio de Janeiro (Brésil)
Michèle LODAY-RICHAUD	Université d'Angers
Frank LORAY	Université Rennes 1
Andrzej J. MACIEJEWSKI	University of Zielona Gora (Pologne)
Stéphane MALEK	Universität Ulm (Allemagne)
Bernard MALGRANGE	Université Grenoble 1
Fabienne MAROTTE	Université de La Rochelle
Patrick MARTINEZ	Université Toulouse 3
Mohamed MASMOUDI	Université Toulouse 3
Jean-François MATTEI	Université Toulouse 3
Éric MATZINGER	Université Strasbourg 1
Marta MAZZOCCO	Cambridge University (Grande-Bretagne)
Greg MAC SHANE	Université Toulouse 3

Frédéric MENOUS	Université Paris 11
Françoise MICHEL	Université Toulouse 3
Claude MITSCHI	Université Strasbourg 1
Juan MORALES RUIZ	Univ. Politecnica de Catalonia, Barcelona (Espagne)
Giovanni MORANDO	Università de Padua (Italie)
Sergiu MOROIANU	Université Toulouse 3
Jean MOULIN-OLLAGNIER	École Polytechnique, Paris
Claire MOURA	Université Toulouse 3
Robert MOUSSU	Université de Bourgogne (Dijon)
Jorge MOZO FERNANDEZ	Universidad de Valladolid (Espagne)
Tien Zung NGUYEN	Université Toulouse 3
Stéphanie NIVOCHE	Université Toulouse 3
Yozuke OHYAMA	University of Osaka (Japon)
Kazuo OKAMOTO	University of Tokyo (Japon)
Umar ORTIKOV	Samarkand State University (Ouzbekistan)
Emmanuel PAUL	Université Toulouse 3
Serpil PEHLIVAN	Suleyman Demirel University (Turquie)
Frédéric PHAM	Université de Nice Sophia-Antipolis
Monique PONTIER	Université Toulouse 3
Georgi POPOV	Université de Nantes
Gulnora POSHOHODJAEVA	Samarkand State University (Ouzbekistan)
Igor POTEMINE	Université Toulouse 3
Jean PRADINES	Université Toulouse 3
Maria PRZYBYLSKA	INRIA Sophia-Antipolis
Andrea PULITA	Université Paris 6
Marius van der PUT	University of Groningen (Pays-Bas)
Zhongmin QIAN	Université Toulouse 3
Georges RACINET	Max-Planck Institut für Mathematik, Bonn (Allemagne)
Jean-Pierre RAMIS	Université Toulouse 3
Jean-Marc RASOAMANANA	Université d'Angers
Marc REVERSAT	Université Toulouse 3
Jose Luis MARTINS	Fraunhofer Gesellschaft (Allemagne)
Javier RIBON	U.C.L.A. (U.S.A.)
Claude André ROCHE	Université Toulouse 3
Jean-Philippe ROLIN	Université de Bourgogne, Dijon
Pierre ROUCHON	École Nationale Supérieure des Mines de Paris
Robert ROUSSARIE	Université de Bourgogne, Dijon
Claude SABBAH	École Polytechnique, Paris
Paulo SAD	I.M.P.A., Rio de Janeiro (Brésil)
Masa-hiko SAITO	Kobe University (Japon)
Fernando SANZ	Universidad de Valladolid (Espagne)
Jacques SAULOY	Université Toulouse 3
David SAUZIN	I.M.C.C.E., Observatoire de Paris
Jean-Marc SCHLENKER	Université Toulouse 3
Reinhard SCHÄFKE	Université Strasbourg 1
Ahmed SEBBAR	Université Bordeaux 1
Carlos SIMPSON	Université de Nice Sophia-Antipolis
Michael SINGER	North Carolina State University (U.S.A.)
Juan SOUTO	Universität Bonn (Allemagne)
Maryvonne SPIESSER	Université Toulouse 3

Mark SPIVAKOVSKI	Université Toulouse 3
Catherine STENGER	Université de La Rochelle
Laurent STOLOVITCH	Université Toulouse 3
Jean-Marie STRELCYN	Université de Rouen
Joseph TAPIA	Université Toulouse 3
Loïc TEYSSIER	Université Rennes 1
Muriel TEISSEYRE	G.R.E.Q.A.M., Université Aix-Marseille 3
Jean THOMANN	Université Strasbourg 1
Pascal THOMAS	Université Toulouse 3
Sébastien TRILLES	Université Toulouse 3
Felix ULMER	Université Rennes 1
Liane VALÈRE	Université de Savoie, Chambéry
Judith VANCOSTENOBLE	Université Toulouse 3
Veeravalli VARADARAJAN	U.C.L.A. (U.S.A.)
Alberto VERJOVSKY	I.M.U., Cuernavaca (Mexique)
Jean-François VIAUD	Université de La Rochelle
Juan Ignacio VILLALBA SANCHEZ	Universidad de Alcala (Espagne)
Guy WALLET	Université de La Rochelle
Jacques-Arthur WEIL	Université de Limoges
Min WU	I.N.R.I.A. Sophia-Antipolis
Jean-Claude YAKOUBSOHN	Université Toulouse 3
Alain YGER	Université Bordeaux 1
Changgui ZHANG	Université Lille 1
Ahmed ZERIANI	Université Toulouse 3

Conférences non reproduites dans ce volume

Les conférences suivantes n'ont pas fait l'objet d'un texte rédigé dans ces volumes :

- Jean-Benoît BOST : *Arithmetic properties of foliations and complex analysis*
- Boele BRAAKSMA : *Resurgence relations for some differential and difference equations*
- Ovidiu COSTIN : *Analyzability of solutions of complex PDEs and of functions near singularity barriers*
- Yulij IL'YASHENKO : *Persistence theorems and problems for dynamical systems*
- Bernard MALGRANGE : *Déformations isomonodromiques et fonctions τ*
- Juan MORALES RUIZ : *A Galois differential approach to perturbation of asymptotic surfaces in Hamiltonian systems : statements and examples*
- Kazuo OKAMOTO : *Painlevé systems : from Strasbourg to Toulouse*
- David SAUZIN : *Gevrey examples of Arnold diffusion*
- Alberto VERJOVSKY : *Quadrics, orthogonal actions and involutions in complex projective spaces*

Boris DUBROVIN et Hiroshi UMEMURA n'ont pas pu venir présenter leur conférence.



Toulouse, septembre 2003
Archives du colloque

Astérisque
296, 2004, p. 1–6

LES PREMIERS TRAVAUX DE JEAN-PIERRE RAMIS

par

Bernard Malgrange

***Résumé.* —** Exposé des travaux de Jean-Pierre Ramis avant 1978-80, travaux portant sur les espaces analytiques banachiques, la dualité en géométrie analytique et les $\mathcal{D}$-modules.

***Abstract* (Early works of Jean-Pierre Ramis). —** Exposition of Jean-Pierre Ramis' works before 1978-80, works devoted to Banach analytic spaces, duality in complex analysis and $\mathcal{D}$-modules.

Avant d'aborder mon sujet proprement dit, j'aimerais rappeler quelques souvenirs personnels. Je ne me souviens plus quand j'ai rencontré Jean-Pierre pour la première fois ; peut-être était-ce aux exposés qu'il avait faits en 1967 sur sa thèse au Séminaire Lelong. Nos relations sont devenues plus suivies lors du séminaire que j'avais organisé l'année suivante à Orsay, à l'usage des jeunes chercheurs. Pour éviter la dérive mondaine qui guette trop souvent les séminaires, il y avait deux règles :

– Premièrement, il ne figurait pas sur la liste officielle (cette liste était diffusée à l'époque par Belgodère, le bibliothécaire de l'Institut Henri Poincaré ; plus tard, cette publication a été remplacée par «l'officiel» qui sévit encore aujourd'hui) ; d'où son nom de «séminaire clandestin» auquel Jean-Pierre fait allusion dans quelques unes de ses publications.

– Deuxièmement, il y avait un droit d'entrée : chaque auditeur devait s'engager, au moins en principe, à faire un exposé.

Les sujets abordés : entre autres, théorème de dualité et de convexité pour les équations aux dérivées partielles et en analyse complexe, sujets que nous allons retrouver un peu plus loin. Jean-Pierre et Gabriel Ruget étaient parmi les participants les plus assidus et les plus actifs. Ce séminaire a bien marché ; j'en vois la meilleure preuve

***Classification mathématique par sujets* (2000). —** 01A70.
***Mots clefs.* —** Travaux, Ramis.

© Astérisque 296, SMF 2004

dans le fait qu'il a continué sans problème en mon absence, en janvier-février 68 où j'étais à Princeton (et aussi, je crois en 68–69, où j'étais parti à Tunis ; en tout état de cause, l'inutilité de son organisateur est certainement la meilleure preuve de son succès).

Nos relations ont été interrompues en 68–69, où Jean-Pierre terminait sa thèse à Paris pendant que j'étais à Tunis comme je viens de le dire ; puis les deux années suivantes, où Jean-Pierre et Gabriel Ruget étaient eux à Tunis, comme coopérants militaires, pendant que, de mon côté, j'étais parti à Grenoble.

À son retour de service, en septembre 71, Jean-Pierre était nommé à Strasbourg, où il est resté jusqu'en 91, puis est allé à Toulouse. Depuis 71, et jusqu'à maintenant, nos relations mathématiques ont été constantes, soit par des visites de l'un ou de l'autre, soit par correspondance. Ma correspondance avec Jean-Pierre est, de très loin, la plus abondante et la plus suivie de celles que j'ai eues ; toutefois ces relations ne se sont traduites qu'une fois par un article en collaboration.

J'en viens à mon sujet. Le premier travail de Jean-Pierre, qui a fait l'objet de sa thèse, est relatif aux espaces de Banach. L'idée d'étudier les espaces analytiques banachiques venait de la thèse de Douady ; ils y servent à fabriquer des espaces de modules, et précisément l'analogue analytique des schémas de Hilbert qui paramètrent les sous-schémas propres d'un schéma donné. Mais le point de vue est ici totalement différent : chez Douady, ils servent d'intermédiaire pour fabriquer en fin de compte des espaces de dimension finie. Chez Jean-Pierre, suivant je crois, la suggestion d'Henri Cartan, son directeur de thèse, il s'agit d'étudier les germes de sous-ensembles analytiques d'un espace de Banach ; incidemment, les espaces analytiques banachiques généraux n'ont guère d'intérêt : par exemple, n'importe quel compact K peut être muni d'une structure d'espace analytique banachique, en le considérant comme l'ensemble des caractères de l'espace des fonctions continues $\mathcal{C}(K)$.

Les ingrédients essentiels de Ramis sont ; d'une part, l'extension au cas banachique du théorème de préparation de Weierstrass ; d'autre part, le résultat suivant, beaucoup plus surprenant : les anneaux de germes de fonctions holomorphes sur un Banach sont factoriels, contrairement à ce qu'aurait pu laisser penser la démonstration usuelle, par récurrence sur le nombre de variables.

À partir de là, Ramis montre que, pour les germes analytiques de codimension finie, *i.e.*, définissables par un nombre fini d'équations, on a des propriétés presque aussi bonnes qu'en dimension finie : décomposition en un nombre fini de composantes irréductibles, théorème des zéros, propriétés des lieux singuliers, généralisation des théorèmes de Remmert-Stein et de Chow, etc. ; les idéaux correspondants sont les idéaux réduits (= égaux à leur racine) qui ont un système fini de générateurs en tant qu'idéaux réduits (pas en tant qu'idéaux tout court) [Il est amusant de noter l'analogie

avec la situation que l'on rencontre en algèbre différentielle.] Cette thèse a fait l'objet d'un livre, paru aux Ergebnisse en 1970.

Ses résultats sont utilisés dans la thèse de Ruget, consacrée aux cycles analytiques de dimension infinie. Je ne sais pas s'ils ont été souvent utilisés depuis ; c'est en tout cas une théorie bien au point, à la disposition des amateurs éventuels.

J'en viens maintenant au second sujet : la dualité en analyse complexe. La situation était la suivante : après le théorème de dualité de Serre, relatif aux fibrés vectoriels sur une variété analytique complexe lisse, on savait traiter le cas d'un faisceau cohérent sur une variété (j'avais donné une première méthode, utilisant la division des distributions ; puis Grothendieck m'en avait indiqué une autre, plus élémentaire en dualisant sur un recouvrement de Stein). D'autre part, Grothendieck avait développé une théorie générale en géométrie algébrique englobant non seulement les variétés singulières, mais aussi les situations relatives (*i.e.*, les morphismes $X \to Y$, le cas absolu étant celui où $Y =$ un point). Le problème se posait donc d'établir des résultats analogues en géométrie analytique, et aussi d'en tirer des conséquences pour les espaces p-convexes ou p-concaves.

C'est ce programme que Ramis va réaliser dans une série de 4 articles, la plupart en collaboration avec G. Ruget, et l'un, en outre, en collaboration avec J.-L. Verdier.

Le premier article, consacré au cas absolu, définit le complexe dualisant $K_X^{\cdot}$ d'un espace analytique X général (complexe qui, dans le cas lisse, se réduit à Ω^n décalé de n, $n = \dim X$). La construction est directement inspirée de celle de Grothendieck en géométrie algébrique : en gros, on considère l'anneau des germes en un point comme un schéma affine, auquel on applique la construction algébrique. Il faut ensuite se recoller d'un point aux points voisins.

On emploie alors la méthode de Grothendieck des recouvrements de Stein mentionnée ci-dessus ; contrairement au cas des schémas où l'on a un énoncé purement algébrique, on obtient alors un énoncé vectoriel topologique d'un type qui avait été dégagé par Martineau, et qui est le suivant : on a deux complexes vectoriels topologiques (définis à quasi-isomorphisme topologique près), en dualité, représentant respectivement $R\Gamma(X, \mathcal{F})$ et $R\operatorname{Hom}_c(X; \mathcal{F}, K_X^{\cdot})$ [ou aussi $R\Gamma_c(X, \mathcal{F})$ et $R\operatorname{Hom}(X; \mathcal{F}, K_X)$]. Ce théorème ne nécessite pratiquement pas d'hypothèses ; mais ils ne donne de résultats sur les groupes de cohomologie que si l'on sait que ceux-ci sont *séparés* (dans la situation qui nous occupe, un lemme de Schwartz affirme qu'il suffit pour cela qu'ils soient de dimension finie) ; on pourra alors déduire d'informations sur un complexe des informations sur l'autre. Dans un autre article, Ramis utilise ces résultats dans le cas (p, q) convexe-concave pour étendre des résultats d'Andreotti-Grauert.

Un autre article, avec Ruget et Verdier, étend ces résultats aux morphismes propres ; bien sûr, le théorème des images directes de Grauert qui affirme la cohérence des images directes $R^k f_* \mathcal{F}$ d'un faisceau cohérent par une application propre joue ici un rôle essentiel. C'est le moment de signaler que le théorème de Grauert, longtemps

obscur, avait reçu entre 71 et 74 trois démonstrations agréables, dues respectivement à Forster-Knorr, Kiehl-Verdier, et Houzel.

Le dernier article de Ramis et Ruget, intitulé «Résidus et dualité» (par analogie avec le livre de Hartshorne), se propose, lui, l'objectif beaucoup plus ambitieux et difficile que les précédents d'avoir un énoncé de dualité tout à fait général, et d'en déduire des résultats sur les espaces relativement p-convexes ou p-concaves. Il fait appel pour cela à toute une série de techniques.

D'une part, l'algèbre homologique évétique reprise de Kiehl-Verdier (et développée) ; ici, il ne suffit pas de mettre des topologies sur des $R\,\mathrm{Hom}$, mais il faut définir des $R\,\mathrm{Homtop}$, des Tortop, etc.

D'autre part, les résolutions semi-simpliciales de Forster-Knorr, qui permettent de donner, en un certain sens, une trivialisation des morphismes, et aussi de donner des résolutions semi-simpliciales libres des faisceaux cohérents.

Bien sûr, pour énoncer les théorèmes généraux de dualité, outre les ingrédients précédents, on a besoin du complexe dualisant $K_X^{\cdot}$; mais, pour représenter l'image directe à support propre $Rf_!K_X^{\cdot}$, on a encore besoin d'un autre ingrédient remarquable et qui m'a beaucoup surpris à l'époque : il s'agit de la possibilité de plonger quasi-isomorphiquement $K_X^{\cdot}$ dans le complexe de Dolbeault-de Rham des courants ${}'\mathcal{D}^{n,\cdot}$ (décalé de n ; ${}'\mathcal{D}^{p,q}$ désigne les courants de bidegré p, q) ; et ceci, d'une manière compatible aux immersions. Ceci se fait à partir de la théorie des valeurs principales et résidus de Herrera.

Je ne sais pas si cette construction, que je trouve un peu mystérieuse, a été reprise et développée ultérieurement ; si ce n'est pas le cas, elle mériterait de l'être, et aussi d'être plus connue, indépendamment de son application à la dualité.

Je ne donnerai pas l'énoncé lui-même des deux théorèmes de dualité «abstraits» obtenus à partir de là (en gros, il s'agit d'énoncés disant que la dualité échange image directe et image directe à support propre). Ramis et Ruget les utilisent dans les cas p-convexe et p-concave, par exemple pour obtenir le résultat suivant : soit $f : X \to Y$, fortement p-concave (avec X dénombrable à l'infini, de dimension bornée) ; si G est $\mathcal{O}_X$-cohérent, $R^k f_* G$ est cohérent pour $k \leq \mathrm{proj}\, G - p - 2 - \dim Y$. Ils donnent aussi des exemples pour montrer que cette borne est la meilleure possible.

Outre le théorème de dualité, la démonstration utilise un résultat de rétrécissement démontré par Ramis dans le second article cité ci-dessus, et aussi la méthode d'Houzel de démonstration de la cohérence (ce qui fait qu'on utilise finalement les méthodes des trois articles relatifs au théorème de Grauert mentionnés plus haut !).

Finalement, leurs résultats, avec ceux de Siegfried dans le cas p-convexe relatif, sont optimaux dans les cas p-convexe et p-concave relatifs ; par contre en 1994 Ramis notait que, dans le cas convexe-concave, on n'a pas jusqu'à présent réussi à étendre à la situation relative les résultats optimaux connus dans le cas absolu ; je ne sais pas si la situation a changé depuis.

J'en viens maintenant au troisième sujet dont je voulais parler, celui de la contribution de Ramis à la théorie des modules holonomes. Je serai ici assez bref ; d'une part, parce que sa contribution se limite pour l'essentiel à deux articles, et qu'il n'a pas poursuivi dans cette direction ; d'autre part, parce que ce sujet est, dans une grande mesure, un passage intermédiaire entre ses préoccupations antérieures, et celles qui vont suivre à partir de 1977, et qui sont toujours les siennes à l'heure actuelle.

Je dirai juste un mot du second article « Dimension cohomologique des modules fuchsiens », datant de 1980, et qui examine les relations entre une notion de dimension homologique locale des modules holonomes réguliers, et une notion de profondeur du complexe de leurs solutions ; ces travaux sont liés à des résultats de Barth et Ogus, et les généralisent.

J'en dirai un peu plus de l'autre article « Variations sur le thème GAGA », paru au Séminaire Lelong-Skoda 76–77. Plus que du thème GAGA proprement dit, il s'agit plutôt de variations sur le théorème de comparaison des cohomologies de de Rham algébrique et transcendante de Grothendieck. Cet article consiste essentiellement à démontrer et à faire un certain nombre de commentaires sur le théorème suivant :

Soit M un module holonome sur X lisse, et soit Y une hypersurface de X ; les propriétés suivantes sont équivalentes :

i) *L'application naturelle* $R\operatorname{Hom}_{\mathcal{D}_X}(M,\mathcal{O}_X) \mid Y \to R\operatorname{Hom}_{\mathcal{D}_X}(M,\mathcal{O}_{\widehat{X}|Y})$ *est un quasi-isomorphisme (*$\mathcal{O}_{\widehat{X}|Y}$ *désigne le complété formel de X le long de Y).*

ii) *L'application naturelle* $\mathrm{DR}(R\Gamma_{[Y]}M) \to R\Gamma_Y \mathrm{DR}\, M$ *est un quasi-isomorphisme (*DR = *complexe de de Rham ;* $R\Gamma_{[Y]}M$ = *cohomologie à support « modéré » dans Y).*

iii) *L'application naturelle*

$$\mathrm{DR}\, R\Gamma_{[Y]}M \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{D}_X^\infty \longrightarrow R\Gamma_Y \mathrm{DR}\, M$$

est un quasi-isomorphisme.

Ramis propose de définir la régularité des modules holonomes par le fait que ces propriétés sont vraies pour tout Y ; par exemple, avec cette terminologie, le théorème de Grothendieck signifie essentiellement que $\mathcal{O}_X$ est régulier. (En dimension un, il était connu, par des calculs de Deligne et de l'auteur, que ces propriétés sont équivalentes à la régularité à la Fuchs.)

L'intérêt de cette définition est qu'elle permet facilement d'établir des propriétés de conservation par image directe propre, ou par image réciproque ; son inconvénient est d'être implicite, et pas directement vérifiable sur les équations définissant un D-module.

Cette approche de Ramis est voisine de celle de Mebkhout, et elles se recoupent largement. (Incidemment, je signale aussi un autre point de contact ultérieur entre eux, à savoir l'extension à plusieurs variables par Laurent-Mebkhout de la filtration-Gevrey de l'irrégularité, due en dimension un à Ramis ; mais ici, je sors de mon sujet.)

De leur côté, Kashiwara et Kawai ont pris une approche différente : ils partent d'une définifion microlocale de la régularité, qui généralise essentiellement la définition de Fuchs, et ils montrent son équivalence avec la définition ci-dessus.

Kashiwara et Mebkhout ont continué leurs travaux sur ce sujet, leurs résultats culminant dans ce qu'il est convenu aujourd'hui d'appeler « correspondance de Riemann-Hilbert ». De son côté, Jean-Pierre n'a pas poursuivi et s'est tourné vers les singularités irrégulières et tous les problèmes qu'elles posent : classes de Gevrey, sommabilité, phénomène de Stokes, relations avec Galois différentiel, etc. Je laisse aux autres orateurs le soin de vous en parler.

B. Malgrange, Université de Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, UFR de Mathématiques, B.P. 74, 38402 St Martin d'Hères Cedex (France)
E-mail : Bernard.Malgrange@ujf-grenoble.fr

Astérisque
296, 2004, p. 7–9

TÉMOIGNAGE

par

Gabriel Ruget

Dans la feuille de patate douce
Elle enveloppe sa vie
La goutte d'eau.
Kikaku

***Résumé.* —** Ce texte est une évocation des années 1962-1971 passées avec Jean-Pierre Ramis à l'École Normale Supérieure et à l'université entre Paris et Orsay, puis à Tunis.

***Abstract* (Recollection). —** We relate memories of the years 1962-1971 spent at the École Normale Supérieure in Paris, at the Universities of Paris and Orsay and in Tunis with Jean-Pierre Ramis.

Au regard d'une vie professionnelle, ce n'est pas un si long parcours que j'ai partagé avec Jean-Pierre, mais c'était à une époque de la vie que l'on aime bien se remémorer ; à l'âge de dix-huit ans ou presque et à Paris, avant que ce ne soit en Afrique (la nôtre n'était pas la corne, c'était le Maghreb). C'est la période de 1969 à 1971 qui nous a rendus, aux yeux de quelques mathématiciens, presque indiscernables (genre salles U ou V ?) : postdocs en quelque sorte, nous habitions deux petites maisons construites dans une nouvelle banlieue de Tunis pour y être louées à des coopérants, face à face, de part et d'autre d'une rue en terre battue, la rue Sanhaja. Pendant six mois par an, nous allions avec nos grosses Citroën à la même station service pour la même corvée de mazout : il fallait nourrir les poêles que nous avions dû installer dès qu'ils n'avaient plus « manqués » au Magasin Général ; les autres six mois nous fréquentions les mêmes plages, lieux idéaux pour mûrir au soleil quelques preuves mathématiques. Nous nous étions rencontrés sept ans plus tôt à l'entrée à l'École Normale où nous allions tous deux trouver nos épouses. Un observateur attentif aurait tout de même pu noter quelques différences : parisien versus provincial, normalien et mathématicien de

***Classification mathématique par sujets* (2000). —** Primaire 01A70 ; Secondaire 32-03.
***Mots clefs.* —** Jean-Pierre Ramis, Espaces analytiques complexes.

Texte lu par Jacques Sauloy.

© Astérisque 296, SMF 2004

naissance versus normalien et mathématicien de passage. Jean-Pierre était bien plus à son aise dans ce milieu intellectuel de la Montagne Sainte-Geneviève, à la fois « établi » et mobile, avec l'exquise courtoisie de qui est incontestable. Il savait qu'il était bâti pour ce type de course de fond (il y en a d'autres) que sont les mathématiques. Il lisait Lautréamont et Pieyre de Mandiargues quand j'en étais à Sartre et Camus, voire à Anouilh et Giraudoux ! Il programmait au ciné-club des séries de comédies musicales américaines, alors que je croyais la Dolce Vita ou Bergman modernes. Il recherchait sur le boulevard Saint-Germain (du bon côté) des cabinets italiens cependant que la maison Knoll, sur le même boulevard, me paraissait le parangon du bon goût mobilier. Bref, une culture capitale, plutôt que les échos qu'on peut en recevoir au ras des rayonnages d'une librairie auvergnate (puis-je préciser que la sociologie de l'ENS n'a pas beaucoup changée aujourd'hui). Il était plus intransigeant en mathématiques qu'en politique, sans atteindre toutefois la radicalité de nos condisciples Pierre Berthelot et Jacques Lafontaine. Il était entendu que les mathématiques étaient de l'art pour l'art. Elie Cartan brillait au firmament bien plus fort que Henri — auquel nous allions pourtant devoir beaucoup dans nos débuts — mais aussi que Henri Poincaré, dont nous ne découvririons l'énormité du génie que plus tard. La sécheresse (au moins apparente) d'André Weil et de Jean-Pierre Serre était un modèle indépassable de style. Les oukases tombaient dru sur X qui était allé se commettre dans un cabinet ministériel, Y qui, bien qu'ancien bourbakiste, ne savait plus distinguer SGA et formalismes vides ; même l'arrivée de Pierre Deligne nous a surpris, non parce qu'il n'était pas normalien, mais : pouvait-on vraiment être un mathématicien génial en étant gentil ?

Printemps 69, l'appétit de vivre de Jean-Pierre est un moment freiné par une hépatite. C'est ce qui suscitera notre première vraie collaboration : au lieu de potasser la deuxième thèse que Michel Demazure m'avait choisie, je mets au point au plus vite les technicités des complexes dualisants en géométrie analytique, sujet que Bernard Malgrange nous avait proposé. Le premier papier aux Publications Mathématiques de l'IHES sera suivi de plusieurs autres aux Inventiones mathematicae, conçus à Tunis en même temps que Bruno et Nathalie. Les bébés étaient inspirés (et plus !) par Monique et Claudine. Le programme de travail sur la dualité devait beaucoup à Jean-Louis Verdier, mais aussi au merveilleux marché de Tunis et à ses arrivages de gambas royales, aux escapades vers le Sud, aux restaurants maloufs et aux délicieuses pâtisseries des terrasses de Sidi Bou Saïd, à quelques couples de coopérants déjà installés qui avaient su créer un climat d'amitié généreuse, et enfin, à l'extraordinaire vitalité du département de mathématiques de l'Université de Tunis à l'époque où Mohamed Baouendi, en particulier, venait de le rejoindre. Nos styles de travail étaient un peu différents : tracer les avenues, déplacer quelques rochers encombrants, organiser la plantation des arbres ne relève pas forcément des mêmes talents.

Été 71 : nous avons failli partir tous les deux à Strasbourg ; j'irai finalement à Paris mais, c'est le plus souvent à Strasbourg que nous nous reverrons, avec de nouveaux amis très accueillants dont le regretté Raymond Gérard, qui a contribué aux choix mathématiques ultérieurs de Jean-Pierre. Il était cependant écrit que cette période d'exercices de poids et haltères sur les fondements de la géométrie complexe devait se clore peu après que se serait fermé pour nous l'accès comme quotidien à la Méditerranée. Chacun devait trouver sa voie, et ces voies ont été d'abord assez éloignées, puis très éloignées ; nos rencontres ont été trop brèves pour être significatives. Nous avions l'un et l'autre touché terre, mais dans deux continents différents. Reste une appréciation mutuelle, illuminée par le souvenir de ce que l'on sait, bien après, avoir été le plus bel âge de votre vie.

G. Ruget, École Normale Supérieure, 45 rue d'Ulm, 75005 Paris (France)
E-mail : Gabriel.Ruget@ens.fr

Astérisque
296, 2004, p. 11–20

TRAVAUX DE J.-P. RAMIS SUR LES ÉQUATIONS DIFFÉRENTIELLES LINÉAIRES

par

Daniel Bertrand

***Résumé.* —** L'article donne un survol des travaux de J-P. Ramis sur les singularités irrégulières : théorèmes d'indice Gevrey, dévissage du groupe de Galois analytique, problème inverse.

***Abstract* (On the work of J-P. Ramis on linear differential equations). —** We survey the work of J-P. Ramis on irregular singularities, with special emphasis on his Gevrey theorems, his extension of Schlesinger's density theorem, and the analogies his theory provides between analytic differential Galois groups and fundamental groups in positive characteritic.

Plutôt qu'une description linéaire des travaux en question, j'ai choisi ici de dégager trois moments marquants dans les recherches de J-P. Ramis sur les singularités irrégulières dans le champ complexe : les premiers pas, essentiellement formels, du « dévissage Gevrey » ; l'approche progressive du théorème de densité, menant à la construction du π_1 sauvage ; enfin, la solution de l'analogue différentiel de la conjecture d'Abhyankar. Comme on le verra, cette progression n'est pas le fait du hasard, chaque étape portant en germe les fondements de la suivante.

Je renvoie à l'exposé de J. Sauloy [**Sa**] au colloque pour les travaux plus récents de Ramis sur les problèmes (tout aussi linéaires) d'équations aux q-différences.

Acte I.
Filtrations Gevrey
Dijon, 1976 ([R1])

Rappelons tout d'abord le théorème d'indice de Malgrange. Soient $\mathcal{O} = \mathbf{C}\{z\}$ l'anneau des germes de fonctions holomophes en 0 et $\widehat{\mathcal{O}} = \mathbf{C}[[z]]$ son complété formel pour la valuation z-adique v. Soit, par ailleurs, $L = \sum_{i=0,\dots,n} b_i(d/dz)^i$ un opérateur

***Classification mathématique par sujets* (2000). —** Primary 34A30, Secondary 12H05.
***Mots clefs.* —** Équations différentielles linéaires, singularités irrégulières, théorie de Galois différentielle, groupe fondamental.

© Astérisque 296, SMF 2004

différentiel à coefficients dans $\mathcal{O}$. On attache à L son polygone de Newton $\mathcal{N}(L)$ (en 0), enveloppe convexe supérieure des droites d'appui de pentes $\geqslant 0$ des points du plan de coordonnées $(i, v(b_i) - i)$, $i = 0, \dots, n$.

On vérifie facilement que L, vu comme un endomorphisme de $\widehat{\mathcal{O}}/\mathcal{O}$, est surjectif. D'après le théorème de Malgrange, L est de Fredholm, et son indice $\chi(L, \widehat{\mathcal{O}}/\mathcal{O})$, qui coïncide donc avec la dimension de son noyau, est égal à la hauteur $(v(b_n) - n) - \inf_{i=0,\dots,n}(v(b_i) - i)$ dont s'élève le polygone de Newton entre les points d'abscisse $i = 0$ et $i = n$. En particulier, $L \in \mathcal{D} = \mathcal{O}[1/z][d/dz]$ vérifie la condition de Fuchs (c'est-à-dire $\mathcal{N}(L)$ est plat) si et seulement si toute extension de $\mathcal{D}/\mathcal{D}L$ par l'objet trivial $\mathcal{D}/\mathcal{D}d/dz$ qui se trivialise après extension des scalaires à $\widehat{\mathcal{O}}[1/z]$ se trivialise déjà dans $\mathcal{D}$.

Comme l'a montré Deligne, l'irrégularité de Malgrange fournit un analogue différentiel du conducteur de Swan. Il est donc naturel de tenter de l'exprimer au moyen d'une filtration. Celle que propose Ramis dans [**R1, R3**], consiste à dévisser $\widehat{\mathcal{O}}$ au moyen des espaces de séries Gevrey définis, pour tout nombre réel s, par

$$\widehat{\mathcal{O}}_s = \big\{ \textstyle\sum_{n\geqslant 0} a_n z^n \in \widehat{\mathcal{O}},\ \sum_{n\geqslant 0} \dfrac{a_n}{(n!)^s} z^n \in \mathcal{O} \big\}$$

$$\widehat{\mathcal{O}}_{(s)} = \big\{ \textstyle\sum_{n\geqslant 0} a_n z^n \in \widehat{\mathcal{O}},\ \sum_{n\geqslant 0} \dfrac{a_n}{(n!)^s} z^n \text{ converge sur } \mathbf{C} \big\}.$$

Que des conditions Gevrey gouvernent les solutions de L en une singularité irrégulière n'est bien entendu pas une idée nouvelle : il y a tout juste un siècle, Maillet avait remarqué que toute solution formelle d'une équation différentielle algébrique est une série Gevrey, et Perron avait considérablement précisé ce résultat dans le cas linéaire. Ramis montre que les endomorphismes que L induit sur $\widehat{\mathcal{O}}_s$ et $\widehat{\mathcal{O}}_{(s)}$, $s \geqslant 0$, sont de Fredholm, et il relie leurs indices aux invariants suivants, récemment introduits dans la théorie par Gérard et Levelt.

Pour tout nombre réel $k \geqslant 0$, soient $i_k(L)$ et $I_k(L)$ la plus petite et la plus grande des abscisses des points de contacts avec $\mathcal{N}(L)$ de la droite d'appui de pente k. Par exemple, $i_0(L) = 0$, et $I_0(L)$ est le degré du polynôme indiciel de L. Les opposées des ordonnées de ces points de contact fournissent, pour $s = 1/k$, les indices recherchés. On en déduit que $\chi(L, \widehat{\mathcal{O}}/\widehat{\mathcal{O}}_s) = \dim \ker(L, \widehat{\mathcal{O}}/\widehat{\mathcal{O}}_s)$ est la hauteur dont s'élève le polygone de Newton entre les points d'abscisse $i = 0$ et $i = i_k(L)$. Idem, avec $i = I_k(L)$, pour $\widehat{\mathcal{O}}_{(s)}$.

Je ne déclinerai pas les nombreux corollaires de ce résultat, et renvoie à [**LR-P**] pour une preuve faisceautique inspirée de Deligne et Malgrange, incluant un raffinement à double précision de la filtration Gevrey, qu'on trouve aussi en théorie des équations aux différences [**Du**]. Mais revenons à [**R1**], et à ses dernières pages.

Ramis y remarque qu'en échelonnant suivant l'indice k la transformation de Laplace inverse formelle, la méthode de Borel permet, sous une hypothèse de croissance exponentielle de la transformée le long d'une direction, de sommer les solutions Gevrey de L en de vraies solutions, avec « une assertion d'unicité dans un certain secteur ».

C'est sa première rencontre avec la notion de k-sommabilité, promise, comme on le sait, à un brillant avenir (*cf.* § 2). Comme me l'apprend Malgrange, la sommabilité de Borel ($k = 1$) était réapparue, peu auparavant, dans les travaux de physiciens comme J-J. Loeffel et A. Sokal. Nul doute que les rencontres de la RCP Maths-Physique de Strasbourg, qu'a lontemps organisées Ramis, ont ici joué un rôle.

Quant au dernier paragraphe de [**R1**], il résout la moitié d'un problème classique de la théorie des E-fonctions de Siegel. Une E-fonction est une série formelle holonome $f = \sum_{n\geqslant 0} a_n z^n$ à coefficients dans un corps de nombres F, dont les images par tout plongement complexe de F appartiennent à $\cap_{\varepsilon>0}\widehat{\mathcal{O}}_{-1+\varepsilon}$, et telles que les dénominateurs des a_n vérifient une condition du même type. Si $L \in \mathbf{C}(z)[d/dz]$, les résultats de Ramis valent aussi pour les ordres $s < 0$, pour peu qu'on complète le polygone de Newton par la contribution du point ∞. Ramis en déduit que les images en question appartiennent alors à $\widehat{\mathcal{O}}_{-1}$, et ne sont dans $\widehat{\mathcal{O}}_{(-1)}$ que si f est un polynôme. L'analogue de ce résultat pour les dénominateurs des a_n reste ouvert, mais la notion de « série Gevrey de type arithmétique » à laquelle les travaux de Ramis ont conduit Y. André est devenue centrale en théorie des E-fonctions (voir [**A1**]).

Pour clore ce chapitre sur le même registre[(1)], je mentionnerai le dévissage sous-jacent aux théorèmes de Bézivin et Robba [**B-R**] sur les opérateurs de Polya : on filtre cette fois l'anneau des séries formelles $F[[z]]$ par son sous-espace des fonctions globalement bornées, et par le localisé de $F[z]$ en 0.

Acte II.
(Re)sommation et groupes de Galois différentiels
Les Houches (1979) ([**R2**]), *Rio (1985)* ([**R4**]), *Strasbourg (1991)* ([**Mr-R6**])

Comme on l'a vu, on décèle la présence d'une singularité irrégulière par un défaut de convergence. La théorie de Galois différentielle exhibe une autre anomalie : contrairement au cas régulier (théorème de densité de Schlesinger), le groupe de Galois différentiel d'une équation irrégulière n'est en général pas engendré par sa monodromie. Les travaux de Ramis des années 80 conduisent à une magnifique jonction de ces deux thèmes.

Soient K le corps des fractions de $\mathcal{O}$, et $\widehat{K}$ son complété formel. Il sera ici plus commode d'exprimer les résultats en termes d'un système différentiel

$$DY := \frac{d}{dz}Y - BY = 0, \quad B \in gl_n(K)$$

(1) Au lecteur qui trouverait ces digressions arithmétiques oiseuses, je dirai qu'elles apparaissent en filigrane dans de nombreux travaux de Ramis. Souvenirs d'un amour de jeunesse ? (Voir [**Se**], bas de la page 5.)

équivalent sur K à l'équation $Ly = 0$. D'après le théorème de Thomae-Fabry-Poincaré-Hukuhara-Turrittin-Levelt, il existe, après passage à une extension finie de $\widehat{K}$ (que nous supposerons triviale dans ce qui suit), une matrice diagonale $Q = \mathrm{diag}(q_1, \dots, q_n)$ formée de polynômes en $\frac{1}{z}$ sans termes constants, une matrice constante Λ commutant à Q, et un élément $\widehat{P}$ de $GL_n(\widehat{K})$ tels que $\widehat{P}e^Q z^\Lambda$ soit une matrice fondamentale de solutions « formelles » de D. Les degrés des q_i non nuls sont les pentes $k_r > \cdots > k_1 > 0$ du polygone de Newton de L, et le nombre de q_i de degré k vaut $I_k(L) - i_k(L)$. Appliquant les résultats du § 1 au système différentiel $\mathrm{Hom}(D, D_0)$, où $D_0 = \frac{d}{dz} - Q' - \frac{1}{z}\Lambda$, on voit que $\widehat{P}$ est constitué de séries Gevrey d'indice $\leqslant 1/\kappa_1$, où κ_1 est le plus petit des degrés des polynômes $q_i - q_j$ non nuls. L'« extraordinaire » [**D1**] théorème des développements asymptotiques énonce que, pour tout élément θ du cercle S^1 des directions autour de 0, il existe un germe de secteur U de bissectrice θ, et une matrice P_U holomorphe sur U, y admettant un développement asymptotique égal à $\widehat{P}$, tels que $\mathcal{Y}_U = P_U e^Q z^\Lambda$ soit une matrice fondamentale de solutions de D holomorphes sur U. Voir [**R5-S**] pour une extension de ce théorème au cas non linéaire, et avec des conditions Gevrey.

Notons respectivement $\Sigma(D_0)$ et $\widetilde{\Sigma}(D_0)$ l'ensemble des lignes de Stokes et des directions singulières de $\mathrm{Hom}(D, D_0)$ (ou de $\mathrm{End}(D_0)$, cela revient au même) : ce sont les θ le long desquels, au voisinage de 0, l'un au moins des $e^{q_i - q_j}$ change de comportement asymptotique, ou au contraire admet une décroissance maximale. La formule précédente montre que pour tout secteur ouvert U' tel que $U' - U$ ne rencontre pas $\Sigma(D_0)$, le prolongement analytique à U' de P_U reste asymptotique à $\widehat{P}$. D'où, un recouvrement de S^1 par des ouverts d'angles π/κ_s, $\kappa_s = \max \deg(q_i - q_j)$, sur lesquels D admet des solutions fondamentales du type précédent. Mais cette collection n'est justifiable d'aucune « assertion d'unicité » ; de ce fait, les matrices de connexion $C_{i,j} = \mathcal{Y}_{U_j}^{-1}\mathcal{Y}_{U_i} \in GL_n(\mathbf{C})$ qui relient ces solutions à la traversée d'une ligne de Stokes n'ont pas d'interprétation canonique.

C'est la vérification de conditions Gevrey sur les *restes* des développements asymptotiques qui permet à Ramis d'y accéder. Pour tout secteur U, soit $\mathcal{A}_s(U)$ l'anneau des fonctions f holomorphes sur U, y admettant un développement asymptotique dont le n-ième reste est majoré sur U par $r_n|z|^n$, avec $\sum_{n\geqslant 0} r_n z^n \in \widehat{\mathcal{O}}_s$. Une série formelle $\widehat{f} \in \widehat{\mathcal{O}}_{1/k}$ est dite k-sommable dans la direction θ si elle est développement asymptotique, sur un ouvert U_θ contenant l'arc $[\theta - \frac{\pi}{2k}, \theta + \frac{\pi}{2k}]$, d'une fonction $f \in \mathcal{A}_{1/k}(U_\theta)$, alors unique, qu'on appelle la k-somme canonique $\sigma_\theta(\widehat{f})$ de $\widehat{f}$ le long de θ, et qui se calcule au moyen du procédé de Borel. Ramis introduit dans [**R2**] le sous-anneau $\mathcal{O}_{1/k} \subset \widehat{\mathcal{O}}_{1/k}$ des *séries k-sommables*, c'est-à -dire, k-sommables dans toute direction sauf un nombre fini, et montre que lorsque $End(D_0)$ ne présente qu'une pente $\kappa_1 = \kappa_s = k > 0$, les coefficients de la matrice $\widehat{P}$ supra sont k-sommables, d'ensemble de directions exceptionnelles $\widetilde{\Sigma}(D_0)$. D'où, pour toute direction singulière d, deux resommations canoniques $\sigma_{d_\pm}(\widehat{P})$ de $\widehat{P}$ le long de directions séparées par d, et une matrice de connexion *naturelle* C_d reliant les solutions $\mathcal{Y}_{d_\pm}$ correspondantes.

En reflet de ce qu'« *en géométrie algébrique, un point est beaucoup plus petit qu'en géométrie analytique* » [**D2**], la k-sommabilité n'est pas une notion purement locale, et l'extension de ces résultats au cas de plusieurs pentes est délicate. Ramis l'aborde au moyen de son théorème taubérien [**R4**] : pour $k_2 > k_1$, on a bien sûr $\widehat{\mathcal{O}}_{1/k_2} \subset \widehat{\mathcal{O}}_{1/k_1}$, mais $\widehat{\mathcal{O}}_{1/k_2} \cap \mathcal{O}_{1/k_1}$ est réduit à $\mathcal{O}$. La forme définitive de la théorie fait appel à la notion de multisommabilité d'Écalle [**Ec**], dont on trouvera dans [**Ml-R7**] une présentation particulièrement agréable, ne nécessitant pas de référence aux faisceaux $\mathcal{A}_s$; voir aussi [**LR**], [**LR-P**] pour une traduction imagée de cette notion en termes des gros points de la géométrie analytique de Deligne. La conclusion est que les dernières lignes de l'alinéa précédent restent valables dans le cas général.

Quel est l'enjeu d'un tel travail ? Outre le renouveau d'intérêt qu'il a suscité pour les séries divergentes et auquel le joli opuscule [**R8**] n'a pas peu contribué, c'est en théorie de Galois différentielle que Ramis en tire la conséquence la plus frappante. Soit $G_{D,f} = \mathrm{Gal}_\partial(\widehat{K}(e^Q z^\Lambda)/\widehat{K})$ le groupe de Galois différentiel formel de D : il s'identifie à conjugaison près, à un sous-groupe fermé du groupe de Galois différentiel G_D de D sur K. Par ailleurs, pour tout $\theta \notin \widetilde{\Sigma}(D_0)$, les procédés de resommation $\sigma_\theta : \mathcal{O}_{1/k} \to \mathcal{A}_{1/k}(U_\theta)$ et leurs extensions au cas multisommable sont des morphismes d'algèbres différentielles. Les matrices C_d représentent donc des éléments St_d de G_D, d'ailleurs unipotents. La normalité des extensions de Picard-Vessiot entraîne alors la généralisation suivante du *théorème de densité* (voir [**R4**], [**Mr-R6**], et pour une approche tannakienne, [**LR**]) : le groupe de Galois G_D est engendré topologiquement par $G_{D,f}$ et par la collection des automorphismes de Stokes St_d, $d \in \widetilde{\Sigma}(D_0)$. L'anomalie signalée plus haut a ainsi disparu : la taille des points analytiques permet de voir ces derniers comme de nouveaux automorphismes de monodromie.

Parmi les nombreuses applications de ce théorème, je citerai le calcul du groupe de Galois différentiel des équations de Hamburger (voir [**Du-Mi**], [**Mi**], pour le cas hypergéométrique), et la vérification, à partir du théorème de Morales-Ramis, de la non intégrabilité de certains systèmes de Hénon-Heiles (voir [**Mo-R11**] et l'exposé de D. Cerveau au colloque). Dans chaque cas, on joue sur deux tableaux : le groupe de Stokes est suffisamment local pour permettre les calculs, mais l'est sufisamment peu pour contrôler des propriétés globales. Le théorème d'isomorphisme de Birkhoff-Sibuya-Malgrange ([**Ml**], [**BV**]) fournit d'ailleurs une approche des St_d, et leur calcul, sans recours à la multisommabilité : voir [**Ju**], [**Ba**], [**LR**].

Terminons par une conséquence plus théorique, qui va jouer un rôle crucial au § 3. La catégorie des systèmes différentiels sur K est tannakienne, et donc équivalente à celle des représentations de dimension finie d'un groupe pro-algébrique $\pi_{1,s}$, que la théorie de la ramification suggère d'appeler le π_1 sauvage. C'est une description explicite de ce groupe (autrement dit, une correspondance de Riemann-Hilbert pour toutes les équations différentielles sur K) qu'en collaboration avec Martinet, Ramis déduit dans [**Mr-R6**] de son théorème de densité.

Tout d'abord, $\pi_{1,s}$ admet comme quotient le groupe modéré $\pi_{1,m}$ attaché à la sous-catégorie pleine des systèmes différentiels réguliers ; d'après Schlesinger, ce dernier est isomorphe à l'enveloppe algébrique $\mathbf{Z}^{env}$ de $\mathbf{Z}$, produit du groupe additif $\mathbf{G}_a$ (action sur $\mathrm{Log}(z)$) par un groupe de type multiplicatif, de groupe de caractères $\mathbf{C}^*$ (action sur les z^λ). Le groupe $\pi_{1,sf}$ attaché de même aux systèmes différentiels sur $\widehat{K}$ est le produit semi-direct de $\pi_{1,m}$ par un pro-tore « exponentiel » $\mathcal{T}$ (action sur les e^q), de groupe des caractères isomorphe au $\mathbf{Z}$-module $\mathcal{E}$ des polynômes de Puiseux en $1/z$ sans termes constants, sur lesquels le générateur canonique γ de $\pi_{1,m}$ agit naturellement. En décomposant l'action adjointe de $\mathcal{T}$ sur les logarithmes des automorphismes de Stokes, Ramis construit alors un « groupe de résurgence » $\mathcal{R}$, d'algèbre de Lie engendrée librement par des symboles $\Delta_{d,q}$, où d parcourt le revêtement universel de S^1, et $q \in \mathcal{E}$ admet le long de d une décroissance maximale à l'approche de 0 ; ces symboles correspondent aux dérivations étrangères d'Écalle. Les relations $\mathrm{Ad}(\tau)(\Delta_{d,q}) = \chi_q(\tau)\Delta_{d,q}, \mathrm{Ad}(\gamma)\Delta_{d,q} = \Delta_{d-2\pi,q}$ munissent $\mathcal{R}$ d'une action de $\pi_{1,sf}$, et $\pi_{1,s}$ *est le produit semi-direct de* $\pi_{1,sf}$ *par* $\mathcal{R}$ *relativement à cette action.* Enfin, la sous-catégorie pleine des systèmes différentiels de pentes $\leqslant k$ fournit une filtration de $\mathcal{R}$ et de $\mathcal{T}$ par des sous-groupes $\mathcal{R}^{>k}$ (*cf.* [**R4**]), $\mathcal{T}^{>k}$ (*cf.* [**Ka**]), et ces filtrations sont compatibles à l'action de $\mathcal{T}$ sur $\mathcal{R}$.

La dernière colonne de la figure 1 ci-contre résume cette description de $\pi_{1,s}$, la troisième, celle de $\pi_{1,sf}$. Le groupe de Galois absolu de $\widehat{K}$ que décrit la seconde (théorème de Puiseux) s'identifie au groupe des composantes connexes de $\pi_{1,sf}$, donc aussi de $\pi_{1,s}$, en vertu d'un lemme de Gabber que la connexité de $\mathcal{R}$ et $\mathcal{T}$ redonne directement (voir [**Mr-R6**], p. 397). Enfin, la première colonne est reprise de Katz [**Ka**], à l'addition près de l'extension abélienne intermédiaire, que m'a suggérée P. Colmez.

L'analogie entre ces colonnes est indéniable, mais doit être maniée avec précaution. Par exemple, comme tout pro-p-groupe, $D\mathcal{P}$ est nilpotent, alors que $\mathcal{R}$ n'est pas unipotent (même si ses générateurs canoniques se trouvent l'être). Pour une critique plus élaborée, voir [**A3**].

Acte III.
La conjecture d'Abhyankar différentielle
Toulouse (Nuit de la musique, 1993) ([**R10**])

Soient $\mathcal{K}$ un corps différentiel, disons pour simplifier de corps de constantes $\mathbf{C}$, et G un groupe algébrique linéaire sur $\mathbf{C}$. Le problème inverse de la théorie de Galois différentielle consiste à rechercher un système différentiel $D = \dfrac{d}{dz} - B$, $B \in gl_n(\mathcal{K})$, de groupe de Galois différentiel isomorphe à G, à en donner un critère d'existence en termes de la structure de G, et quand ce dernier est rempli, à minimiser le type des singularités des systèmes réalisant G.

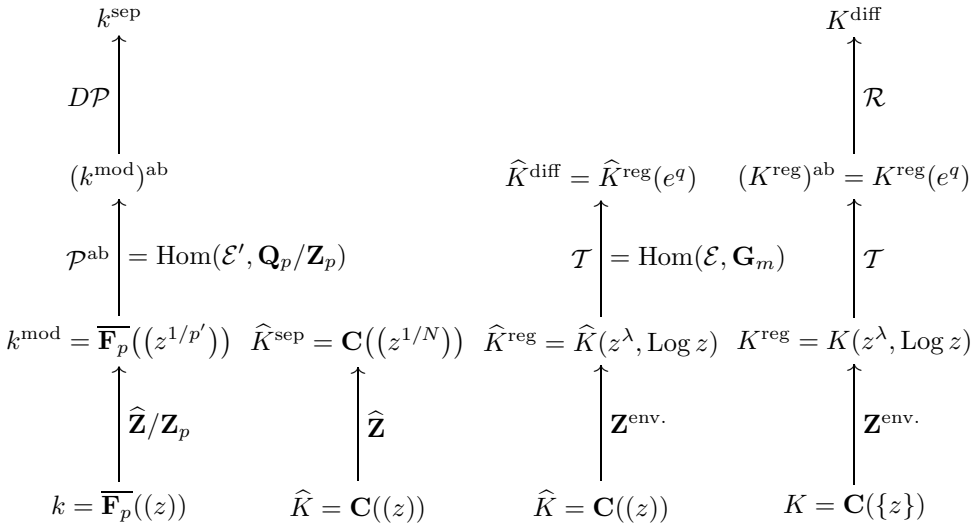


FIGURE 1. i) dans la colonne de droite, $\mathcal{R}$ est le groupe dérivé de $\mathrm{Gal}_\partial(K^{\mathrm{diff}}/K^{\mathrm{reg}})$ et $(K^{\mathrm{reg}})^{\mathrm{ab}}$ désigne le compositum des extensions de Picard-Vessiot de K^{reg} qui sont abéliennes *et* contenues dans la clôture de Picard-Vessiot K^{diff} de K. Rappelons qu'une extension de Picard-Vessiot de K^{reg} n'admet en général pas de plongement dans K^{diff}.
ii) dans la colonne de gauche, $D\mathcal{P}$ est le groupe dérivé du groupe d'inertie sauvage $\mathcal{P} := \mathrm{Gal}(k^{\mathrm{sep}}/k^{\mathrm{mod}})$. La description du quotient $\mathcal{P}^{\mathrm{ab}}$ est fournie par la théorie d'Artin-Schreier : considérer les équations $\wp(X) = q$, où $\wp(X) = \mathrm{Frob}_p(X) - X$, et q est un vecteur de Witt de k^{mod}. On a posé $\mathcal{E}' = W(k^{\mathrm{mod}})/\wp W(k^{\mathrm{mod}}) \otimes \mathbf{Q}_p/\mathbf{Z}_p$ (voir [**Bo**], p. 49, ex. 21).

La réponse dépend bien entendu de $\mathcal{K}$. Par exemple, elle est toujours positive si $\mathcal{K} = \mathbf{C}(z)$, et je renvoie à [**Si**] pour un historique du sujet. En revanche, hormis un travail de Kovacic pour les groupes résolubles, rien n'était connu dans le cas local $\mathcal{K} = K$, où le problème revient à décrire tous les groupes algébriques quotients de $\pi_{1,s}$. C'est bien ce point de vue qui permet à Ramis de le résoudre, dans [**R9**] pour les groupes semi-simples, puis en toute généralité. Mais la formulation définitive a nécessité plusieurs étapes. Pour utiliser une métaphore du rugby [(2)],

(i) le premier essai de Ramis lui est quasiment dicté par sa description du $\pi_{1,s}$: G est groupe de Galois différentiel sur K si et seulement s'il existe un tore $T \subset G$, un élément $a \in N_G(T)$ engendrant G modulo sa composante neutre G^0, et une algèbre de Lie $\mathrm{Lie}(N) \subset \mathrm{Lie}(G)$ de dimension $\leqslant 1$ commutant à T et a, tels que $\mathrm{Lie}(G)$ soit

[(2)]Nous sommes maintenant à Toulouse.

engendrée par $\mathrm{Lie}(T), \mathrm{Lie}(N)$ et les espaces de poids non nuls pour l'action de $\mathrm{Lie}(T)$ sur $\mathrm{Lie}(G)$;

(ii) la transformation est effectuée par C. Mitschi et M. Singer [**Mi-Si**] , qui réécrivent ce critère dans le style de Kovacic, en termes du radical unipotent R_u de G : le groupe G/G^0 doit être cyclique, et agir trivialement sur le groupe $R_u/[R_u, G^0]$, dont la dimension doit être $\leqslant 1$;

(iii) la partie est enfin gagnée dans [**R10**] avec la traduction finale suivante, inspirée de la conjecture d'Abhyankar sur le problème inverse de la théorie de Galois classique pour les corps *globaux* de caractéristique p. Rappelons tout d'abord que ce problème ne pose aucune difficulté pour les corps locaux comme $k = \overline{\mathbf{F}}_p((z))$: un groupe fini G est groupe de Galois sur k si et seulement s'il admet un p-Sylow *unique*, donc normal, à quotient *cyclique*. Pour tout groupe fini G, soit alors $p(G)$ le sous-groupe engendré par tous les p-Sylows de G. Le critère proposé par Abhyankar, et établi peu auparavant par Raynaud, pour que G soit groupe de Galois d'une extension galoisienne du corps $\overline{\mathbf{F}}_p(z)$ non ramifiée hors de ∞, est que $p(G) = G$. Partant maintenant d'un groupe algébrique G, Ramis introduit le sous-groupe $L(G)$ de G engendré par ses tores maximaux, et déduit de (i) et (ii) que *G est groupe de Galois différentiel sur le corps K si et seulement si $G/L(G)$ est topologiquement cyclique.*

L'analogie se resserre lorsqu'en caractéristique nulle aussi, on passe aux corps globaux. Soient $\mathcal{K}_p$ (resp. $\mathcal{K}$) un corps de fonctions d'une variable sur $\overline{\mathbf{F}}_p$ (resp. $\mathbf{C}$), de genre g, S un ensemble fini de places de $\mathcal{K}_{(p)}$, de cardinal $s \geqslant 1$, et σ un point de S. D'après Harbater, un groupe fini G est groupe de Galois d'une extension galoisienne de $\mathcal{K}_p$, non ramifiée hors de S, et sans ramification sauvage hors de σ, si et seulement si $G/p(G)$ admet $2g + s - 1$ générateurs. Par recollement, Ramis déduit de (iii) qu'un groupe algébrique G est groupe de Galois d'un module différentiel sur $\mathcal{K}$, sans singularités hors de S, et sans singularité irrégulière hors de σ, si et seulement si $G/L(G)$ admet $2g + s - 1$ générateurs topologiques.

C'est donc bien le caractère semi-local du groupe de résurgence qui limite les comparaisons inspirées par notre tableau du § 2. Il y manque d'ailleurs une cinquième colonne : celle des équations différentielles p-adiques, pour laquelle je renvoie à l'exposé d'Y. André au colloque [**A3**], et à [**A2**].

Il est devenu banal de ne plus distinguer l'arithmétique de la géométrie. Le voyage auquel nous a convié Jean-Pierre Ramis, depuis ses théorèmes d'indice formels jusqu'à cette solution du problème inverse, offre un bel éclairage sur les liens plus mystérieux que tissent ces disciplines avec l'analyse.

Remerciements. — Je remercie vivement B. Malgrange, M. Loday-Richaud, C. Mitschi et M. Matignon pour leurs commentaires sur une version préliminaire de ce texte, et avec quelque retard, mais tout aussi vivement, B. Teissier pour m'avoir jadis assigné la lecture de [**R1**] comme sujet de « 2e thèse ».

Références

Travaux de J-P. Ramis cités dans le texte

[R1] J.-P. Ramis – « Dévissage Gevrey », in *Journées Singulières de Dijon*, Astérisque, vol. 59-60, Société Mathématique de France, 1978, p. 173–204.

[R2] ———, « Les séries k-sommables et leurs applications », in *Complex analysis, microlocal calculus and relativistic quantum theory*, Lecture Notes in Phys., vol. 126, Springer, Berlin-New York, 1980, p. 178–199.

[R3] ———, *Théorèmes d'indices Gevrey pour les équations différentielles ordinaires*, vol. 48, Mem. Amer. Math. Soc., no. 296, American Mathematical Society, 1984.

[R4] ———, « Filtrations Gevrey sur le groupe de Picard-Vessiot d'une ED irrégulière », Prép. IMPA, 45, 1985, p. 1–38.

[R5-S] J.-P. Ramis & Y. Sibuya – « Hukuhara's domains and fundamental existence and uniqueness theorems for asymptotic solutions of Gevrey type », *Asymptot. Anal.* **2** (1989), p. 39–94.

[Mr-R6] J. Martinet & J.-P. Ramis – « Elementary acceleration and multisummability », *Ann. Inst. H. Poincaré. Phys. Théor.* **54** (1991), p. 331–401.

[Ml-R7] B. Malgrange & J.-P. Ramis – « Fonctions multisommables », *Ann. Inst. Fourier (Grenoble)* **42** (1992), p. 353–368.

[R8] J.-P. Ramis – *Séries divergentes et théories asymptotiques*, Panoramas et Synthèses, Société Mathématique de France, 1994, (Journées X-UPS, 1991).

[R9] ———, « About the solution of some inverse problems in differential Galois theory by Hamburger equations », in *Differential equations, dynamical systems, and control science*, Lecture Notes in Pure and Appl. Math., vol. 152, Dekker, New York, 1994, p. 277–299.

[R10] ———, « About the inverse problem in differential Galois theory : the differential Abhyankar conjecture », in *The Stokes phenomenon and Hilbert's 16th problem (Groningen, 1995)* (Braaksma et al., éds.), World Sci. Publishing, River Edge, NJ, 1996, p. 261–278.

[Mo-R11] J. Morales Ruiz & J.-P. Ramis – « Galoisian obstructions to integrability of Hamiltonian systems I, II », *Methods Appl. Anal.* **8** (2001), p. 33–95 & 97–111.

Autres références

[A1] Y. André – « Séries Gevrey de type arithmétique. I, II », *Ann. of Math.* **151** (2000), p. 705–740 et 741–756.

[A2] ———, « Filtrations de type Hasse-Arf et monodromie p-adique », *Invent. Math.* **148** (2002), p. 258–317.

[A3] ———, « Galois representations, differential equations, and q-difference equations : sketch of a p-adic unification », ce volume, p. 43–53.

[BV] D. Babbitt & V. Varadarajan – *Local moduli for meromorphic differential equations*, Astérisque, vol. 169-170, Société Mathématique de France, 1989.

[Ba] W. Balser – « Zur Einzigkeitssatz in der Invariantentheorie meromorpher Differentialgleichungen », *J. reine angew. Math.* **318** (1979), p. 53–82.

[B-R] J.-P. Bézivin & P. Robba – « Rational solutions of linear differential equations », *J. Austral. Math. Soc. Ser. A* **46** (1989), p. 184–196.

[Bo] N. Bourbaki – *Algèbre commutative*, Masson, Paris, 1983, Chap. IX.

[D1] P. Deligne – Lettre à B. Malgrange, Août 1977.

[D2] _____, Lettre à J.-P. Ramis, Janvier 1986[(3)]

[Du] A. Duval – « Opérateurs aux différences », Thèse, Univ. Strasbourg, 1984, voir aussi *Ann. Inst. Fourier* **37** (1987), p. 45–80.

[Du-Mi] A. Duval & C. Mitschi – « Matrices de Stokes et groupes de Galois des équations hypergéométriques confluentes généralisées », *Pacific J. Math.* **138** (1989), p. 25–56.

[Ec] J. Écalle – *Les fonctions résurgentes*, Publ. Math., Univ. Orsay, 1985.

[Ju] W. Jurkat – *Meromorphe Differentialgleichungen*, Lect. Notes in Math., vol. 637, Springer, 1978.

[Ka] N. Katz – « On the calculation of some differential groups », *Invent. Math.* **87** (1987), p. 13–61.

[LR] M. Loday-Richaud – « Classification méromorphe locale des systèmes différentiels linéaires méromorphes : phénomène de Stokes et applications », Thèse, Univ. Orsay, 1991, voir aussi *Ann. Inst. Fourier* **44** (1994), p. 849–906.

[LR-P] M. Loday-Richaud & G. Pourcin – « On index theorems for linear ordinary differential operators », *Ann. Inst. Fourier (Grenoble)* **47** (1997), p. 1379–1424.

[Ml] B. Malgrange – *Équations différentielles à coefficients polynomiaux*, Progress in Math., Birkhäuser, 1991.

[Mi] C. Mitschi – « Groupes de Galois différentiels et G-fonctions », Thèse, Univ. Strasbourg, 1989, voir aussi *Pac. J. Math* **176** (1996), p. 365–405.

[Mi-Si] C. Mitschi & M. Singer – « On Ramis's solution of the local inverse problem of differential Galois theory », *J. Pure Appl. Algebra* **110** (1996), p. 185–194.

[Sa] J. Sauloy – « Algebraic construction of the Stokes sheaf for irregular linear q-difference equations », ce volume, p. 227–251.

[Se] J.-P. Serre – *Cours d'arithmétique*, Coll. Sup., PUF, 1970.

[Si] M. Singer – « Direct and inverse problems in differential Galois theory », in *Selected Works of E. Kolchin*, American Mathematical Society, 1999, p. 527–554.

D. Bertrand, Université de Paris 6, Institut de Mathématiques, Case 247, 4, Place Jussieu, 75252 Paris Cedex 05 • *E-mail :* bertrand@math.jussieu.fr

[(3)]Les lettres entre Deligne, Malgrange et Ramis sur le sujet contiennent un fourmillement d'idées, qui n'ont pas toutes été reprises dans les articles ultérieurs. Avec la permission de leurs auteurs, elles mériteraient d'être rassemblées, par exemple, en un volume de documents mathématiques.

Astérisque
296, 2004, p. 21–31

TRAVAUX DE J.-P. RAMIS SUR LES ÉQUATIONS DIFFÉRENTIELLES NON LINÉAIRES

par

Dominique Cerveau

À Jean et Jean-Pierre

***Résumé.* —** On survole trois articles de J.-P. Ramis traitant de phénomènes non linéaires.

***Abstract* (Works of J.-P. Ramis on non-linear differential equations). —** About three papers of J.-P. Ramis related to non linear phenomena.

Ce texte retrace à peu près mot pour mot ma conférence en l'honneur de Jean-Pierre... les plaisanteries (on ne cherchait pas... Hortense... mais Marta) et l'émotion en moins, car nous pensions tous à Jean, enfin tous ceux qui l'ont connu. J'ai choisi délibérement de ne pas parler de tous les travaux de Jean-Pierre concernant le non-linéaire, mais d'en extraire trois thèmes qui, j'espère ne pas me tromper, lui sont chers... L'exposé est plutôt d'aspect informel, plutôt destiné aux étudiants..., pas toujours très précis, mais on pourra consulter les vrais articles, bien sûr...

C'était la fête de Jean-Pierre, alors j'ai oublié beaucoup de monde de par mes choix... ils m'en excuseront.

1. Systèmes hamiltoniens (Morales-Ramis)

Dans cette partie, il m'arrive de plagier le remarquable Bourbaki de Michèle Audin sur le sujet (Sém. Bourbaki, Exp. n° 884, mars 2001)... elle ne m'en voudra pas non plus.

On se donne une variété complexe M de dimension $2n$ munie d'une forme symplectique disons méromorphe. Le corps $\mathcal{M}(M)$ des fonctions méromorphes sur M se

***Classification mathématique par sujets* (2000). —** Primary 34A34 ; Secondary 34Mxx.
***Mots clefs.* —** Systèmes hamiltoniens, théorie de Galois différentielle, feuilletages holomorphes, perturbations singulières.

© Astérisque 296, SMF 2004

laisse munir du crochet de Poisson $\{\,,\,\}$ hérité naturellement du crochet de Lie sur les champs via la correspondance

$$\text{fonction (ou hamiltonien) } H \longleftrightarrow \text{ champ hamiltonien } X_H$$

définie par la formule

$$dH = i_{X_H}\omega.$$

L'exemple le plus simple est $M = \mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$ muni des coordonnées (x, y) avec $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. Bien sûr, on a

$$X_H = \sum_i \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i} - \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i}$$

et l'équation qui détermine le flot de X_H n'est autre que celle de Hamilton.

Le premier exemple cette fois explicite que l'on peut donner est celui correspondant à la description des petits mouvements autour d'une position d'équilibre ; ici

$$H = \frac{1}{2}(y_1^2 + \cdots + y_n^2) + \sum \lambda_i x_i^2$$

le premier terme correspondant à l'énergie cinétique (comme toujours $m = 1$!), le second la troncature de l'énergie potentielle en un point critique. On a

$$X_H = \sum_{i=1}^n \Big(2\lambda_i x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}\Big) = \sum_{I=1}^n X_{H_i}$$

avec $H_i = y_i^2 + 2\lambda_i x_i^2$; comme les X_{H_i} commutent (les variables sont indépendantes) bien sûr $\{H_i, H_j\} = 0$. On découvre ainsi n-intégrales premières de X_H : $X_H(H_i) = 0$, en involution (c'est la condition de commutation) et indépendantes, aux points génériques, au sens du calcul différentiel. Il se trouve, et c'est un exercice, que n est le maximum d'intégrales premières comme ci-dessus que l'on puisse espérer.

On dit qu'un système hamiltonien (*i.e.* la donnée de H ou de X_H) est intégrable si l'on peut trouver n-intégrales premières indépendantes H_i et en involution ; suivant le contexte les H_i sont holomorphes, ici méromorphes souvent au voisinage d'une trajectoire particulière ; en réel, *i.e.* lorsque M est une variété réelle on demande aux H_i d'être C^∞ ou analytiques réelles. Justement en réel, lorsque M est compacte, si X_H est intégrable les X_{H_i} sont complets et produisent une action de $\mathbb{R}^n$ sur M. Les orbites génériques de cette action sont des tores $\mathbb{T}^n$ sur lesquels le flot X_H est équivalent à un flot linéaire, c'est le théorème d'Arnold-Liouville. Bien sûr, en complexe, on perdra ce caractère de complétude dans le cas où M est non compacte ou bien lorsque les X_{H_i} sont méromorphes, mais on comprend bien l'enjeu.

Expérimentalement, l'intégrabilité est rare et en tout cas difficile à prouver. Les travaux de Moralès-Ramis à la suite de ceux de Ziglin produisent des critères effectifs et efficaces de non intégrabilité. Pour comprendre celà il faut rappeler ce qu'est l'équation aux variations le long d'une solution particulière, concept qui remonte au

moins à Poincaré. Soit Γ une trajectoire de X_H paramétrée localement par le flot $t \rightsquigarrow x(t)$, *i.e.*

$$\dot{x}(t) = X_H(x(t)).$$

Si Y est un champ de vecteurs (local) commutant à $X = X_H$, l'écriture explicite de la commutation avec des notations évidentes donne le long de Γ :

$$0 = \sum_1 \Big(\sum_k X_k(x(t)) \frac{\partial Y_i}{\partial x_k}(x(t)) - Y_k(x(t)) \frac{\partial X_i}{\partial x_k}(x(t)) \Big) \frac{\partial}{\partial x_i}.$$

On reconnaît immédiatement que

$$\frac{\partial}{\partial t}\big(Y_i(x(t))\big) - \sum_k \frac{\partial X_i}{\partial x_k}(x(t)) \cdot Y_k(x(t)) = 0.$$

Ainsi les composantes $Y_1, \ldots, Y_n$ de Y le long de Γ sont solutions de l'équation linéaire, dite équation variationnelle,

$$\dot{y}(t) = \frac{\partial X}{\partial x}(x(t)) \cdot y(t). \qquad (E.V.)$$

Bien sûr, ceci est écrit en coordonnées locales, mais les spécialistes diront que l'on définit ainsi et cette fois globalement une connexion linéaire sur le fibré i^*TM où i est l'inclusion de Γ dans M. Evidemment l'intégrabilité de X_H, qui se traduit en termes de champs commutants, va donner des propriétés spéciales à $(E.V.)$. N'oublions pas que, comme X_H est méromorphe, $(E.V.)$ peut présenter en certains points de Γ des singularités régulières ou non, c'est en fait le piment de l'histoire. Voici un certain nombre de faits, certains élémentaires qui vont de façon naturelle nous emmener au résultat.

Fait 1. — Si f est intégrale première de X_H et Γ une trajectoire de X_H (le long de laquelle X_H n'est pas complètement singulier), alors la partie initiale de f le long de Γ :

$$\mathrm{In}_\Gamma f : i^*TM \longrightarrow \mathbb{C}$$

qui localement s'écrit :

$$Y = (y_1, \ldots, y_{2n}) \rightsquigarrow \mathrm{In}_\Gamma f(y_1, \ldots, y_{2n})$$

est intégrale première de $(E.V.)$. Si f est holomorphe ; $\mathrm{In}_\Gamma f$ est essentiellement le premier jet non nul de f le long de Γ et si $f = P/Q$ est méromorphe $\mathrm{In}_\Gamma f = \mathrm{In}_\Gamma P / \mathrm{In}_\Gamma Q$.

Fait 2. — Ce fait est dû à Ziglin dans le cas méromorphe et — comme l'a signalé Alain Chenciner — est connu de Poincaré dans le cas holomorphe. Il dit que si $f_1, \ldots, f_m$ sont des fonctions méromorphes indépendantes à l'origine de $\mathbb{C}^n$ (*i.e.* si $df_1 \wedge \cdots \wedge df_m \not\equiv 0$) alors, il existe des polynômes $P_i \in \mathbb{C}[t_1, \ldots, t_i]$ tels que les $F_i = P_i(f_1, \ldots, f_i)$ aient leurs parties initiales $\mathrm{In}_0 F_i$ indépendantes.

Fait 3. — Il relie le linéaire au non-linéaire, mais, on l'avait deviné dès le discours sur $(E.V.)$, le groupe de Galois $\mathrm{Gal}_\Gamma(E.V.)$ de l'équation linéaire $(E.V.)$ laisse invariantes les parties initiales $\mathrm{In}_\Gamma F$ des intégrales premières F de X_H (définies au voisinage de Γ).

Fait 4. — qui provient naturellement de la nature symplectique du problème : le groupe $\mathrm{Gal}_\Gamma(E.V.)$ s'identifie à un sous-groupe du groupe symplectique $\mathrm{Sp}(T_{x_0}M)$, $x_0 \in \Gamma$ point de base générique.

Fait 5. — Si $g_1, \ldots, g_n$ sont des fonctions indépendantes (locales sur $\mathbb{C}^{2n}$) et en involution (relativement au crochet de Poisson standard) et si de plus les g_i sont annulées par les éléments d'une sous-algèbre de Lie $\mathcal{L} \subset sp(n)$, alors $\mathcal{L}$ est abélienne (on voit ici les éléments de $\mathcal{L}$ comme des dérivations).

Ces cinq faits donnent grosso modo l'idée du théorème de Ramis-Morales (de l'un des...).

Théorème 1. — *Soit $H \in \mathcal{M}(M)$; si X_H est intégrable le long de Γ, courbe intégrale non stationnaire, alors* $\mathrm{Gal}_\Gamma(E.V.)$ *est virtuellement abélien (i.e. la composante neutre du groupe algébrique* $\mathrm{Gal}_\Gamma(E.V.)$ *est abélienne).*

C'est évidemment le fait 5 qui se passe au niveau des algèbres qui introduit le virtuellement.

Dans le cas où $(E.V.)$ est de type Fuchs (penser à Γ comme une courbe rationnelle) le groupe de $\mathrm{Gal}_\Gamma(E.V.)$ est l'adhérence de Zariski du groupe de monodromie de $(E.V.)$; on retrouve donc dans ce cas un résultat de Ziglin. Mais le critère est beaucoup plus fort ; on peut avoir par exemple $\Gamma \cong \mathbb{CP}(1)$ et $(E.V.)$ avec un seul point singulier, irrégulier. Alors la monodromie est bien sûr triviale, alors que $\mathrm{Gal}_\Gamma(E.V.)$ peut être énorme du fait de la complexité de l'irrégularité : intervention des phénomènes de Stokes, du tore exponentiel (*cf.* l'exposé de Daniel Bertrand). D'autre part, ce critère se prête bien au calcul effectif, ce qui n'est pas pour déplaire à Jean-Pierre. J'oubliais de mentionner que de nombreux exemples explicites sont maintenant traités (*cf.* Bourbaki de M. Audin pour une bibliographie et les articles originaux), à la fois par Morales et Ramis et d'autres auteurs.

2. Perturbations singulières

Dans cette partie je m'appuie sur l'un des travaux en collaboration avec R. Schäfke de Strasbourg. Il y a beaucoup d'autres articles concernant ce thème... mais comme non spécialiste j'ai choisi de présenter celui qui m'a posé... le moins de difficultés, en espérant, et je pense ne pas trop me tromper, qu'il s'agit bien là de l'un des articles importants.

Poincaré voit les sources de divergence dans les problèmes d'équations différentielles tantôt venant de la présence de « petits diviseurs » (intégration) tantôt de celle de

« grands multiplicateurs » (différentiation). C'est sans doute dans cette seconde philosophie qu'il faut rattacher les travaux de Jean-Pierre. D'ailleurs, l'une des philosophies de Jean-Pierre dans les problèmes de « normalisations » et de « perturbations » est sans doute ce lien diffus que je vais tenter d'expliquer entre « les phénomènes de divergence avec estimations Gevrey » et ce que l'on a coutume d'appeler les « corrections exponentiellement petites ».

Nous allons nous placer dans le cadre des systèmes lents-rapides. De quoi s'agit-il ? On considère une équation différentielle, finalement ordinaire,

$$\begin{cases} \dot{x} = \varepsilon g(u, x, \varepsilon) \\ \dot{u} = f(u, x, \varepsilon) \end{cases} \tag{1}$$

où ε est un petit paramètre (que l'on pourrait appeler $\hbar$ dans d'autres contextes), mais qui a un rôle particulier. On s'autorise des changements de variables, mais surtout on ne touche pas à $\varepsilon \ldots$ Sans trop être précis disons que f et g sont analytiques réelles et que u et v sont des variables vectorielles. On devine que x s'appelle variable lente et u variable rapide, il suffit de contempler l'équation. Dans cet article les auteurs présentent un cas spécial, et je vais faire comme eux, très lié à la mécanique ; précisément,

$$\begin{cases} \dot{x} = \varepsilon F(u, \varphi, \varepsilon) \\ \dot{\varphi} = \omega(x) + \varepsilon G(x, \varphi, \varepsilon) \end{cases} \tag{2}$$

La notation φ indique qu'il s'agit d'une variable angulaire, $\varphi \in \mathbb{R}^m/2\pi\mathbb{Z}^m$, les F et G sont analytiques et périodiques en φ. Il s'agit donc d'une petite perturbation du système $\dot{\varphi} = \omega(x)$.

Ce que l'on voudrait faire c'est en quelque sorte découpler le système, *i.e.* par un changement de variable ad-hoc, enlever l'intervention de la variable rapide dans l'équation du mouvement perturbé. On veut trouver une « transformation », contentons-nous pour l'instant de l'appeler « raisonnable » qui conjugue le système perturbé à un nouveau système... lui-même raisonnable

$$\begin{cases} \dot{y} = \varepsilon \widetilde{F}(y, \varepsilon) \\ \dot{\psi} = \omega(y) + \varepsilon \widetilde{G}(y, \varepsilon) \end{cases} \tag{3}$$

que je présente donc dans le cas spécial, sans trop préciser pour l'instant qui sont les $\widetilde{F}$, $\widetilde{G}$.

Il se trouve que l'on peut fabriquer un changement de coordonnées formel en ε, noté $\widetilde{T}$ qui transforme le système perturbé en (3) avec $\widetilde{F}$, $\widetilde{G}$ formels en ε. Plus précisément, $\widetilde{T}$ est du type suivant :

$$\begin{aligned} x &= y + \varepsilon U_1(y, \psi) + \varepsilon^2 U_2(y, \psi) + \cdots \\ \varphi &= \psi + \varepsilon V_1(y, \psi) + \varepsilon^2 V_2(y, \psi) + \cdots \end{aligned}$$

où les U_i, V_i sont analytiques sur un domaine commun et périodiques en la variable angulaire. Ce changement de coordonnées n'est pas unique mais il le devient si l'on convient de normaliser les U_i, V_i par $U_n(y,0) = V_n(y,0)$; modulo cette convention on obtient le très joli

Théorème 2. — *$\widetilde{T}$ et les $\widetilde{F}$ et $\widetilde{G}$ correspondants sont* 1*-Gevrey en la variable ε.*

On voit donc apparaître les phénomènes de divergence avec estimations Gevrey. Pour les relier aux « corrections exponentiellement petites » mentionnons un résultat essentiel de Neishstadt, que les auteurs redémontrent en substituant à la technique de « sommation au plus petit terme » une technique de transformation de « Laplace incomplète » ajoutée à des estimations de type Gevrey. L'énoncé dit que si dans l'équation (2) on a une minoration $|\omega(x)| > c > 0$ alors le système se réduit à

$$\begin{cases} \dot{y} = \varepsilon \overline{F}(y,\varepsilon) + \alpha(y,\varphi,\varepsilon) \\ \dot{\varphi} = \omega(x) + \varepsilon \overline{G}(y,\varepsilon) + \beta(y,\varphi,\varepsilon) \end{cases} \tag{4}$$

où les α et β satisfont à

$$|\alpha| + |\beta| < Ce^{-\rho/|\varepsilon|}.$$

Si on néglige α et β on a la normalisation souhaitée. Il se trouve que la transformation T qui réduit le système (2) à (4) ainsi que les $\overline{F}$ et $\overline{G}$ sont des « réalisations » analytiques des transformations formelles initiales $\widetilde{T}$, $\widetilde{F}$ et $\widetilde{G}$ précédentes (on pense à une incarnation sur des secteurs complexes contenant les axes réels).

Ce travail en connexion avec les résultats de D. Sauzin, P. Lochack, etc. a de nombreuses applications : systèmes hamiltoniens, invariants adiabatiques, etc. Il a été généralisé et précisé par Jean-Pierre et d'autres collaborateurs.

3. Feuilletages holomorphes

Il s'agit de présenter en particulier deux travaux remarquables de Jean-Pierre et Jean Martinet sur les singularités résonnantes et les « nœuds-cols ». C'est là où je suis censé être le plus compétent ; c'est là où j'ai le plus de mal. Comme je l'ai dit, tout le monde dans la salle pense à Jean, ceux qui l'ont connu en tout cas et puis c'est techniquement difficile.

Commençons par un état des lieux dans les années 75. Bien qu'une spécialité de l'école française au XIX$^{\text{e}}$ et au début du XX$^{\text{e}}$ les équations différentielles holomorphes sont un peu délaissées, sans doute du fait de l'activité intense sur les feuilletages ordinaires réels. Bien sûr, on connaît les théorèmes de Poincaré (que j'énonce en dimension 2) ; si

$$\omega = adx_1 + bdx_2 = \lambda_1 x_1 dx_2 + \lambda_2 x_2 dx_1 + \text{h.o.t}$$

est une 1-forme holomorphe et si le quotient λ_1/λ_2 n'est pas dans l'ensemble $\mathbb{R}_{>0} \cup \{-2,-3,\ldots\} \cup \{-1/2,-1/3,\ldots\}$ alors le feuilletage associé $\mathcal{F}_\omega$ est linéarisable

holomorphiquement, en fait ω elle-même l'est... Ainsi que le théorème de Poincaré-Dulac. Bien sûr aussi, il y a le magnifique théorème de Siegel complété par Brjuno : lorsque λ_1/λ_2 est irrationnel « loin » de $\mathbb{Q}$ alors ω est linéarisable. Via le théorème de Kupka-Reeb ces énoncés passent en dimension quelconque pour les feuilletages de codimension un, je n'en dirai pas plus. Mille neuf cent soixante quinze c'est à peu près l'époque où Bernard Malgrange publie ses deux Frobenius singuliers : grosso modo petitesse du lieu singulier implique présence d'intégrales premières. À l'IMPA on commence à étudier la topologie des singularités. Jean et Jean-Pierre vont s'intéresser au chaînon manquant, *i.e.* la classification des germes de feuilletages de $\mathbb{C}^2$ en 0 résonnants ($\lambda_1/\lambda_2 \in \mathbb{Q} > 0$) et des nœuds cols $\lambda_1 \neq 0$ (disons égal à 1) et $\lambda_2 = 0$. Ils vont établir une correspondance de type Riemann-Hilbert :

$$\text{Feuilletages} \longleftrightarrow \text{Difféomorphismes}$$

que je vais esquisser très grossièrement.

Il faut avant tout rappeler le théorème de Briot et Bouquet qui produit des courbes lisses invariantes ; plus précisément, dans le cas résonnant il y a deux courbes lisses transverses et invariantes, nous les prendrons comme axes de coordonnées x_1, x_2, et dans le cas nœud col il y a toujours deux courbes comme ci-dessus mais l'une formelle et l'autre convergente : ce sera l'axe des x_1.

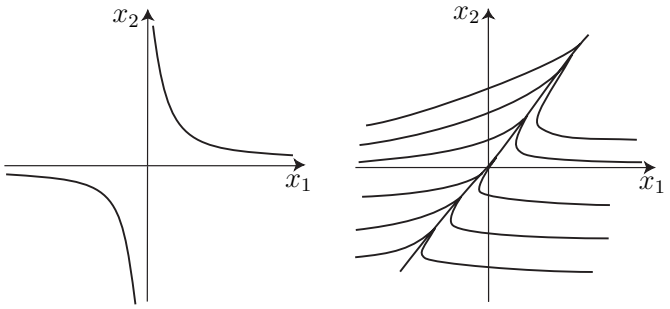


FIGURE 1

Introduisons le concept d'holonomie d'une singularité ayant une courbe lisse invariante (figure 2).

La courbe lisse invariante est ici l'axe des x_1 ; près d'un point m de cet axe et loin de la singularité la feuille $\mathcal{L}_m$ par m est quasi horizontale en m ; si bien que la restriction à $\mathcal{L}_m$ de la projection $p : (x_1, x_2) \rightsquigarrow x_1$ est un revêtement tout du moins près de l'axe des x_1. On peut donc relever les chemins tracés dans $(x_2 = 0)$ et situés un peu loin de la singularité dans les feuilles $\mathcal{L}_m$ proches de $x_2 = 0$. Plus précisément, supposons notre feuilletage défini dans le polydisque $\mid x_i \mid < 2$ et considérons la droite $(1, x_2)$ alors le lacet $\gamma : [0, 1] \ni \theta \rightsquigarrow (e^{2i\pi\theta}, 0)$ se relève en un chemin $\widetilde{\gamma}(\theta)$ avec $\overline{\gamma}(0) = (1, x_2)$, $\widetilde{\gamma}(1) = (1, h(x_2))$.

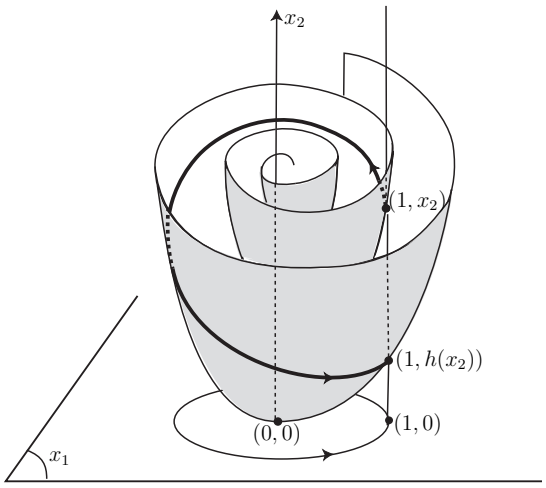


FIGURE 2

L'application $x_2 \rightsquigarrow h(x_2)$ est holomorphe ; c'est l'holonomie du lacet γ ou simplement l'holonomie. J.-F. Mattei et R. Moussu ont démontré que deux feuilletages résonnants sont conjugués si et seulement si leurs holonomies le sont. Pour parler de la classification il faut d'abord parler de la classification formelle. Présentons-la lorsque $\lambda_1 = \lambda_2 = 1$, c'est plus facile, *i.e.* $\omega = d(x_1x_2) + \cdots$. Bien sûr, on suppose notre feuilletage non linéarisable, ce qui est équivalent à dire qu'il n'a pas d'intégrale première holomorphe. Il y a un difféomorphisme formel $\widehat{\phi}$ tel que, à unité multiplicative près, on ait

$$\widehat{\phi}^*\omega = \Omega_{k,\lambda} = (x_2dx_1 + x_1dx_2) + (x_1x_2)^k\{(\lambda-1)x_2dx_1 + \lambda x_1dx_2\}.$$

L'entier k et le nombre complexe λ sont les deux invariants formels de notre feuilletage. On remarque que la forme

$$\frac{\Omega_{k,\lambda}}{(x_1x_2)^k} = -\frac{1}{k}d\Big(\frac{1}{(x_1x_2)^k}\Big) + (\lambda-1)\frac{dx_1}{x_1} + \lambda\frac{dx_2}{x_2}$$

est fermée et l'on constate que la fonction multivaluée $x_1(x_1x_2)^{-\lambda}\exp 1/kx_1x_2$ est intégrale première de $\Omega_{k,\lambda}$. Via cette intégrale première on calcule facilement l'holonomie du modèle formel : $x_2 \rightsquigarrow \widetilde{h}(x_2)$ avec

$$\widetilde{h} = \exp X_{k,\lambda}$$

où $X_{k,\lambda}$ est le champ de vecteur

$$X_{k,\lambda} = 2i\pi\frac{x_2^{k+1}}{1+\lambda x_2^k}\,\frac{\partial}{\partial x_2}.$$

On démontre que l'holonomie h du feuilletage initial est formellement conjuguée à $\widetilde{h}$. Lorsque $k=1$ et $\lambda=0$ on a

$$\widetilde{h} : z \rightsquigarrow \frac{z}{1+2i\pi z}$$

qui, dans ce cas, est un automorphisme global de $\mathbb{CP}(1)$. Il laisse invariant la famille des cercles passant par 0 et tangents à l'axe imaginaire (figure 3).

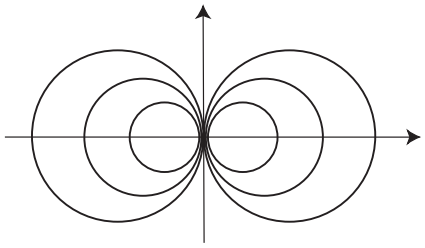


FIGURE 3

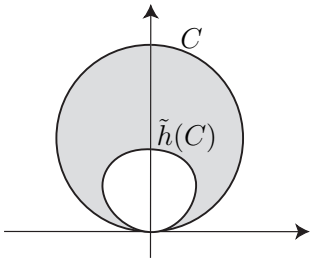


FIGURE 4

Pour décrire l'espace des orbites de cet $\widetilde{h}$ on choisit un domaine fondamental compris entre un cercle C tangent à l'axe réel et son image $\widetilde{h}(C)$ (figure 4).

Outre l'orbite $\{0\}$ qui est spéciale, toutes les autres passent dans le domaine fondamental ; on obtient l'espace des orbites en identifiant C et $\widetilde{h}(C)$ par $\widetilde{h}$. On obtient une sphère S^2 où l'on identifie pôle nord et pôle sud (figure 5).

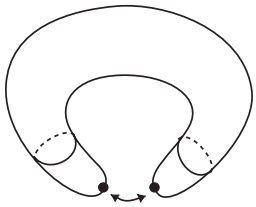


FIGURE 5

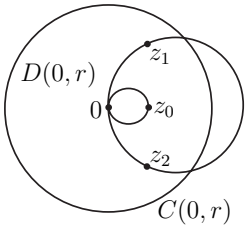


FIGURE 6

Il y a une difficulté à expliquer ce qu'est l'espace des orbites pour un germe. Reprenons l'exemple précédent et dessinons un disque $D(0,r)$ centré en l'origine de bord $C(0,r)$. Disons que l'orbite de z_0 par h dans $D(0,r)$ est cette fois l'ensemble des $\widetilde{h}^n(z_0)$ tels que les $\widetilde{h}(z_0), \widetilde{h}^2(z_0), \ldots, \widetilde{h}^n(z_0)$ soient dans $D(0,r)$ auxquels on ajoute les $\widetilde{h}^{-m}(z_0)$, tels que $\widetilde{h}^{-(m-1)}(z_0), \ldots, \widetilde{h}^{-1}(z_0)$ soient encore dans $D(0,r)$ (figure 6).

On voit que l'orbite relative à $D(0,r)$ du point z_0 est la même que dans le cas global, et que celle des points z_1 et z_2 (situés dans une même orbite globale) différent. Pour repérer les orbites il faut « deux domaines fondamentaux », l'un dessus et l'autre

dessous (figure 7), et une identification (figure 8) qui va dire que certaines orbites passent dans ces deux domaines...

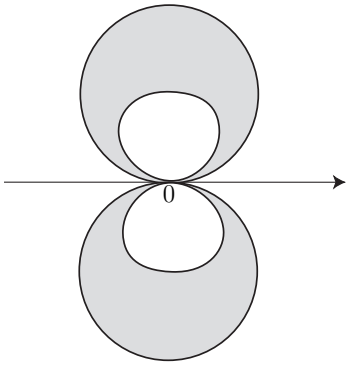


FIGURE 7

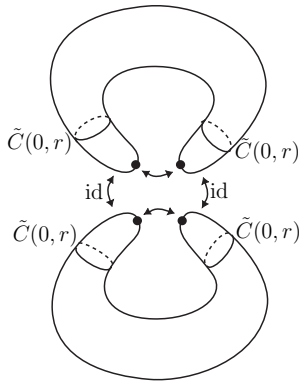


FIGURE 8

où $\widetilde{C}(0,r)$ est l'image de $C(0,r)$ après identification... Puis ensuite, il faut faire tendre r vers 0... Etudier les espaces d'orbites des difféomorphismes holomorphes h conjugués formellement à $\widetilde{h}$ revient grosso modo à perturber les identifications id précédentes. Ceci se fait via l'analyse Gevrey et la théorie de la sommation. On commence par dessiner, toujours sur notre exemple, un secteur S^+ qui voit le domaine fondamental du haut

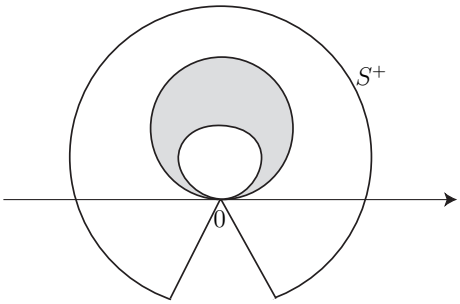


FIGURE 9

et un secteur S^- qui voit le domaine fondamental du bas. Il y a des conjugaisons sectorielles holomorphes φ^+ et φ^- asymptotes à la conjugaison formelle entre h et $\widetilde{h}$ qui produisent deux cocyles sur les intersections des secteurs. Essentiellement ces cocycles correspondent aux deux identifications précédentes. En parallèle de cette analyse sur les orbites des difféomorphismes, Martinet et Ramis mènent une analyse similaire pour les espaces de feuilles des feuilletages résonnants. Cette analyse parallèle

leur permet, outre la classification explicite dont je ne parle même pas, le très joli résultat que je donne toujours dans le cas spécial :

Théorème 3. — *L'espace des feuilletages résonnants formellement conjugués à* $\Omega_{1,0}$ *modulo conjugaison holomorphe s'identifie à l'espace des difféomorphismes formellement conjugués à* $x \rightsquigarrow x/(1+2i\pi x)$*, modulo conjugaison holomorphe.*

Ceci montre que la correspondance d'holonomie dans cette situation est bijective. Ce résultat a été par la suite généralisé au cas hyperbolique, de façon directe par J.-C. Yoccoz et R. Perez Marco.

Le cas des nœuds col est plus subtil ; outre la classification des feuilletages Martinet et Ramis montrent que les holonomies classifient, mais que tous les difféomorphismes (susceptibles de l'être) ne sont pas des holonomies de nœud-col ; ici la correspondance d'holonomie est seulement injective...

Remerciements. — Ils s'adressent à M.-F. Chériaux, M.-A. Paulmier, F. Loray et surtout à M. Loday qui a bien voulu faire les figures.

Références

[1] J. Martinet & J.-P. Ramis – « Problèmes de modules pour des équations différentielles non linéaires du premier ordre », *Publ. Math. Inst. Hautes Études Sci.* **55** (1982), p. 63–164.

[2] ———, « Classification analytique des équations différentielles non linéaires du premier ordre », *Ann. scient. Éc. Norm. Sup.* 4^{e} *série* **16** (1983), p. 571–621.

[3] J. Morales Ruiz & J.-P. Ramis – « Galoisian obstructions to integrability of Hamiltonian systems I & II », *Methods Appl. Anal.* **8** (2001), no. 1, p. 33–95 & 97–111.

[4] J.-P. Ramis & R. Schäfke – « Gevrey separation of fast and slow variable », *Nonlinearity* **9** (1996), no. 2, p. 353–384.

D. Cerveau, Université de Rennes, UFR de mathématiques, Campus de Beaulieu, 35042 Rennes (France) • *E-mail :* dominique.cerveau@univ-rennes1.fr

Astérisque
296, 2004, p. 33–42

SOUVENIRS STRASBOURGEOIS

par

Michèle Loday-Richaud

« La vie n'est pas ce que l'on a vécu ; mais
ce dont on se souvient et comment on s'en souvient. »
Gabriel García Márquez (*Vivre pour la raconter*)

***Résumé.* —** Ce texte est une évocation des années 1971-1987 que j'ai passées à Strasbourg dans le groupe de travail organisé par Jean-Pierre Ramis sur les équations fonctionnelles et, en particulier, différentielles.

***Abstract* (Memories of Strasbourg). —** This paper deals with memories of the time (1971-1987) I spent in Strasbourg as a member of the working group on differential or functional equations around Jean-Pierre Ramis.

Cher Jean-Pierre, chère Monique, chers amis,[(1)]

On m'a confié la tâche délicate d'évoquer les années strasbourgeoises. Ce sera, bien sûr, une évocation partielle de souvenirs qui me concernent également ; qu'on m'en excuse.

Votre arrivée à Strasbourg, Jean-Pierre, Monique, fut un événement à plusieurs titres. En 1971 et pour la première fois, le Département de Mathématique recrutait en même temps et sur un même thème deux Maîtres de Conférences (ancienne dénomination), Jean-Pierre Ramis et Gabriel Ruget, pour développer une nouvelle équipe. Comme vous le savez, Gabriel Ruget fut détourné vers Paris durant l'été et 2 s'est changé en 1. D'autre part, c'est le couple Jean-Pierre et Monique Ramis que le Département de Mathématique recrutait. Un exemple unique là encore. Monique nous a quittés un an plus tard pour aller enseigner en classe préparatoire.

***Classification mathématique par sujets* (2000). —** Primaire 01A70 ; Secondaire 34-03.
***Mots clefs.* —** Jean-Pierre Ramis, séminaire de Strasbourg, équations différentielles.
(1) Allocution prononcée lors de la Conférence de Toulouse à l'occasion du soixantième anniversaire de Jean-Pierre Ramis. À la demande de plusieurs collègues le texte ci-dessous apporte davantage de précisions, en particulier, sur le contenu scientifique du séminaire de Strasbourg.
Je tiens à remercier C. Mitschi et D. Bertrand pour leurs commentaires sur la première version de ce texte.

© Astérisque 296, SMF 2004

J'ai des souvenirs un peu vagues des premières années. Peu avant l'arrivée de Jean-Pierre je m'étais intéressée aux fonctions différentiables. Anne Duval et moi qui suivions des voies parallèles pensions travailler avec notre regretté collègue Georges Glaeser lui aussi récemment recruté à Strasbourg, mais il nous laissa orphelines en déclarant brusquement que, désormais, il se consacrerait à la didactique des mathématiques. Notre intérêt pour le nouveau venu, Jean-Pierre Ramis, s'en trouva renforcé.

Que se passait-il à Strasbourg à l'époque ?

Un séminaire d'algèbre singulièrement misogyne. Il n'en fut d'emblée pas question. L'année était mal choisie pour le séminaire de probabilités avec lequel Pierre Cartier était venu nous allécher à l'École Normale : notre, lui aussi regretté, collègue Paul-André Meyer s'apprêtait à effectuer un séjour de longue durée à l'étranger et ne voulait pas accueillir de jeunes. Il y avait bien un séminaire sur les groupes de Lie mais il alternait entre Strasbourg et Nancy, voire Heidelberg. Quant au séminaire dit trajectorien, nous le trouvions alors un peu sectaire et mondain.

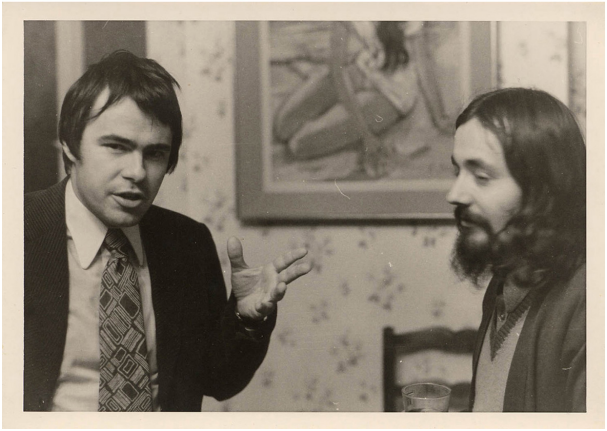
Jean-Pierre a tout de suite pris en main le séminaire d'analyse réelle et complexe alors moribond. Au programme, il mit des mathématiques difficiles et techniques : « Résidu et dualité » bien évidemment ! Un souvenir fort est que l'accueil y était chaleureux et bienveillant. Je veux insister sur la diversité des publics auxquels ce séminaire s'adressait. Chacun, mathématicien du plus brillant au plus modeste, fortement ou non impliqué dans le sujet, y avait une place et s'y sentait bien. Comme se sentaient bien les élèves de Jean-Pierre dans les longues discussions, quelquefois passionnées et toujours passionnantes, qu'il leur accordait.

Jean-Pierre est passé rapidement du statut de Maître de Conférences ancienne formule à celui de Professeur, devenant ainsi le plus jeune professeur strasbourgeois et pour longtemps. Il n'y avait pas, comme aujourd'hui, de blocage de carrières mais la période Giscard nous a crédités de 0 poste pendant de longues années ; nous sommes donc restés jeunes très longtemps. Je voudrais ici raconter une anecdote. Avec les galons sont arrivées les tâches administratives. C'est ainsi que Jean-Pierre s'est vu chargé de représenter les mathématiques dans une commission pluridisciplaire de l'Université. Sitôt la réunion terminée, le Directeur du Département de Mathématique, Daniel Bernard sauf erreur, recevait un coup de fil du Directeur de Sciences Économiques demandant quel était ce petit jeune sans cravate qu'on leur envoyait. N'avait-on pas de Professeur ? On aurait pu penser qu'à la suite de cet incident Jean-Pierre investisse dans l'achat d'une cravate. Pas du tout. Sa solution : demander que désormais soit mentionné sur la convocation « Monsieur Jean-Pierre Ramis, Professeur de 1^e classe ». En toute circonstance il a su trouver des solutions simples et efficaces...

Tout en arborant un style administratif parfois brouillon et décontracté, Jean-Pierre a géré efficacement et humainement de nombreuses commissions ou organismes dont

je vous épargnerai une liste exhaustive. Devant l'affiche annoncant la manifestation qui nous réunit aujourd'hui un collègue me disait « J'ai connu Ramis au CNU[2] ; c'est quelqu'un qui portait des jugements très modérés, n'est-ce-pas ? ». Je le crois volontiers, « modérés et justes ».

La RCP[3] qu'il a gérée pendant de longues années a alors pleinement joué son rôle d'interface entre mathématiques et physique théorique. On sait que Jean-Pierre s'intéresse aux sujets les plus variés. Je crois qu'il a trouvé là une occasion d'apprendre beaucoup de physique. En tout cas, dans le séminaire, nous avons souvent bénéficié de motivations et d'interprétations issues de la physique.



En réalité, Jean-Pierre possédait une cravate...
(ici en compagnie de Paul Girault)

Un domaine dans lequel Jean-Pierre a également joué un rôle important et que beaucoup d'entre vous ignorent sans doute est le calcul formel. Jean-Pierre a été l'un des moteurs essentiels du groupe de calcul formel sur les équations différentielles de l'Institut IMAG[4] et du réseau européen CATHODE[5]. À Strasbourg même, il a su rallier à son projet, alors qu'il était ingénieur au Centre de Calcul de l'IN2P3[6], Jean Thomann qui n'a jamais ménagé sa peine dans une bonne humeur égale. C'est l'un des aspects du travail de Jean-Pierre qui me séduit beaucoup. Il ne se contente pas de savoir qu'une famille d'objets est classifiée par un ensemble de cohomologie plus

[2] Conseil National des Universités.
[3] Recherche Coopérative sur Programme : réunion bi-annuelle entre mathématiciens et physiciens théoriciens à Strasbourg.
[4] Institut d'Informatique et de Mathématiques Appliquées de Grenoble.
[5] Computer Algebra Tools for Handling Ordinary Differential Equations.
[6] Institut National de Physique Nucléaire et de Physique des Particules situé à Cronenbourg en banlieue nord de Strasbourg alors que le Département de Mathématique se trouve au centre ville.

ou moins tordu ; il s'efforce de rechercher des algorithmes permettant à un utilisateur potentiel, mathématicien, physicien, chimiste ou autre, de savoir à quoi ressemble l'objet particulier qui le préoccupe, lui. Jean-Pierre pousse (ou encourage d'autres à pousser) cette démarche jusqu'au développement de logiciels informatiques.

En 76-77 je suis partie aux États-Unis, enceinte puis en congé de maternité, ce qui a provoqué une sorte de scandale local. Jean-Pierre alors président de la Commission des Enseignants m'a épaulée de sa bienveillance et de sa complicité comme il l'a toujours fait et le fait encore avec tous ses collaborateurs.

À mon retour, quelques mois plus tard, il parlait japonais ou, plus précisément, il avait engagé le séminaire dans l'étude de « SKK », Sato, Kashiwara, Kawai. Pour moi, c'était du chinois et, à nouveau, je me demandais dans quelle direction j'allais continuer. Mais, par chance, Monique a obtenu à ce moment-là une classe de Mathématiques Spéciales et, pour occuper de longs week-ends studieux, Jean-Pierre a commencé à s'amuser avec des équations différentielles ordinaires. L'équation $y' = y$ lui a servi de premier terrain d'expérimentations qu'il effectuait sur une petite calculatrice de poche. On connaît la suite.

Je ne saurais dire combien cette époque a été passionnante ; une de ces périodes dont on sait qu'elles sont magiques quand elles se terminent. Très vite Jean Martinet a rejoint l'équipe ; il a lui aussi grandement ajouté à la magie par sa gentillesse, son enthousiasme et sa joie de vivre. Tout était bonheur. Jean-Pierre nous a fait découvrir des merveilles. Nous vivions en direct son activité de chercheur dont il parlait beaucoup — sur place, là où on le rencontrait, dans les couloirs, dans les escaliers — mais que nous avions souvent bien du mal à suivre. Oui, c'était souvent bien hermétique ! Alors il répétait inlassablement, sans jamais montrer le moindre agacement, les mêmes explications qui, miraculeusement, grâce peut-être à de tout petits changements, finissaient par devenir claires.

Ce fut d'abord l'époque des théorèmes d'indice et de la sommabilité ; au début, c'est de k-sommabilité simple qu'il s'agissait. Les théorèmes d'indice plaçaient les recherches de Jean-Pierre dans la continuité de ses activités antérieures : Bernard Malgrange venait de démontrer qu'un opérateur différentiel linéaire (ordinaire) admet un indice à la fois quand on le fait agir sur l'espace des séries formelles (il détermine noyau et conoyau à la main) et quand on le fait agir sur l'espace des séries convergentes (il s'appuie dans ce cas sur le théorème d'Ascoli). La différence entre ces deux indices mesure l'irrégularité de l'opérateur. Jean-Pierre étend ces théorèmes aux espaces de séries Gevrey reliant ainsi par une famille à un paramètre séries formelles et séries convergentes. Dans l'esprit de ce que fait B. Malgrange dans le cas convergent il procède par perturbations compactes. Il troque ainsi les limites inductives ou projectives d'espaces de Fréchet nucléaires qui faisaient ses délices depuis plusieurs années contre des limites inductives ou projectives d'espaces de Banach mais, grosso modo, il joue toujours avec le même meccano.

Ces premiers résultats sur les indices et la k-sommabilité ont été consignés dans un article assez condensé à « Astérisque » *Dévissage Gevrey*, qu'entre nous, nous appelions *Le sein nu* par référence à la citation de Valéry que Jean-Pierre y avait mise en exergue. Ce papier a connu immédiatement un grand succès. Les collègues ne s'y étaient pas trompés : c'était le début de quelque chose d'important. Étonnamment les théorèmes d'indice n'ont jamais été mis au programme du séminaire (pas plus, d'ailleurs, que ne le seront les perturbations singulières). C'est en 1984-85 seulement que Jean-Pierre les présentera en détail dans un cours de DEA.

À partir de 1978-79, j'ai suivi le séminaire ED-EF (Equations Différentielles-Equations Fonctionnelles) régulièrement. Le programme s'est d'abord réparti à peu près équitablement entre EDO et EDF (Equations Différentielles Ordinaires et Equations aux Différences Finies). Nous apprenions les rudiments : travaux de Hukuhara, Perron, Nörlund, Pincherle, la prépublication — malheureusement jamais publiée — de Bernard Malgrange sur la classification formelle des $\mathcal{D}$-modules,... Puis Jean-Pierre a commencé à parler de k-sommabilité : développements asymptotiques avec conditions Gevrey, théorèmes de Phragmen-Lindelöf, de Watson, de Nevanlinna... La version classique (due à Bernard Malgrange) du théorème d'isomorphisme infinitésimal permet de voir les séries formelles modulo celles qui sont convergentes comme étant les éléments du premier groupe de cohomologie $H^1(S^1, \mathcal{A}^0)$ du faisceau $\mathcal{A}^0$ des germes de fonctions analytiques plates au-dessus du cercle S^1 des directions autour de l'origine. Jean-Pierre en cherche une version Gevrey mais la méthode utilisée par B. Malgrange qui s'appuie sur une résolution de Dolbeault et le théorème de prolongement de Whitney sur des fermés, résiste ; j'en reparlerai. Ce théorème simple jouera un rôle fondamental dans la suite. Sous sa forme Gevrey, il porte en germe la superbe théorie de la sommabilité (simple et multiple) telle qu'on la trouve dans l'unique article qu'ont publié en commun B. Malgrange et J.-P. Ramis. Ce dernier travail tire profit du lemme suivant dû à Jean-Pierre (et indépendamment sous une forme plus faible à Y. Sibuya) et dont il utilisera le concept ultérieurement sous la dénomination de k-quasi-fonction : considérons un 0-cocycle $(f_j)_{j\in J}$ sur S^1 à valeurs dans le faisceau $\mathcal{A}$ (faisceau des germes de fonctions analytiques admettant un développement asymptotique en 0) tel que les différences deux à deux (le 1-cobord) soient exponentiellement plates d'ordre k ; alors, la série asymptotique commune aux f_j est de type Gevrey d'ordre k et, surtout, l'asymptoticité des f_j satisfait elle-aussi à des conditions de type Gevrey d'ordre k. La réciproque est élémentaire. On va ainsi pouvoir caractériser l'infinité de conditions horribles que sont les conditions d'asymptoticité Gevrey par un nombre fini de conditions sympathiques.

Dès cette époque Jean-Pierre, avec son sens très méditerranéen de la convivialité, a su intéresser aux équations différentielles de nombreux collègues, participants du séminaire ou collaborateurs. Outre Jean Martinet qui jouera un rôle de tout premier plan dans le domaine, je voudrais citer Gérard Schiffmann qui nous a longuement



Les familiers reconnaîtront le théorème de factorisation.

parlé d'EDF, Jean-Pierre Jouanolou et Vazgain Avanissian ; Raymond Gérard, bien sûr, avec qui il a entretenu des échanges soutenus jusqu'à ce que la querelle autour de l'analyse non-standard ne distande leurs relations. Pendant plusieurs années — fin des années 70, début des années 80 — le séminaire « Équations différentielles et équations fonctionnelles » de Jean-Pierre Ramis et le séminaire « Systèmes de Pfaff » de Raymond Gérard fonctionnaient en parallèle avec, à peu près, les mêmes participants. Une année, par égard pour nos collègues messins, le séminaire « Systèmes de Pfaff » a migré à Metz. C'est Kazuo Okamoto qui inaugura le séminaire de Metz par un exposé sur l'équation de Toda. Les petits cadeaux que nous faisaient dans le train (nous les femmes en tout cas) les nombreux invités japonais de Raymond Gérard restent au nombre des souvenirs agréables.

Parmi les collaborateurs de Jean-Pierre, Bernard Malgrange, quoique non-strasbourgeois, faisait de fréquentes visites à Strasbourg et était un participant assidu des RCP. J'ai plaisir à évoquer ma première rencontre avec lui. Jean-Pierre avait eu l'idée de redémontrer le théorème d'isomorphisme de B. Malgrange en utilisant une intégrale de Cauchy (désormais appelée de Cauchy-Heine dans ce contexte). Y ajouter des conditions Gevrey devenait un jeu d'enfant. C'est au rez-de-chaussée du bâtiment de Mathématique, devant la bibliothèque, à la croisée de deux couloirs où Jean-Pierre et Bernard s'étaient rencontrés qu'ils discutaient aussitôt de la question. Passant par là au bon moment j'ai été invitée à bénéficier des nouvelles toutes fraîches. C'était tellement simple ! Comment avait-on pu l'ignorer ? Rappelez-vous : le théorème d'isomorphisme en question établissait une bijection abstraite entre l'espace des séries formelles modulo les convergentes et un H^1. La formule de Cauchy-Heine donne une formule intégrale explicite et particulièrement simple pour la flèche inverse. Si j'en ai tout de suite vu la beauté je n'en ai pas tout de suite compris la portée ; pourtant j'en ferai amplement usage par la suite.

Au début des années 80 le séminaire a été surtout consacré aux équations différentielles non linéaires du premier ordre et, nécessairement aussi, aux difféomorphismes de $\mathbb{C}$. Avec la participation de Jean Martinet à plein temps dans le domaine, séminaire ED-EF et séminaire trajectorien s'étaient beaucoup rapprochés, parfois même confondus. Jean-Pierre et Jean alternaient les exposés sur leurs tout nouveaux travaux : problèmes de modules (publié aux Publications Scientifiques de l'I.H.E.S), classification analytique locale (publié aux Annales de l'E.N.S), et les applications qui en découlent à l'étude du noeud-col et des champs de vecteurs résonants, cols et foyers réels. Les équations ou champs de vecteurs considérés sont supposés de partie linéaire non nulle. À forme normale fixée, la classification consiste à donner une version non linéaire du théorème de Malgrange-Sibuya. La non-linéarité les contraint à une approche géométrique basée sur le théorème de structure de Newlander-Nirenberg ; cet ingrédient essentiel qui leur a été fourni par Bernard Malgrange sera à nouveau utilisé pour les équations aux q-différences. L'espace classifiant est décrit par les « défauts de sommation » tenant lieu d'automorphismes de Stokes des applications normalisantes.

En janvier 1982, Jean-Pierre qui venait de découvrir les travaux de Jean Écalle et les analogies avec les siens propres a invité ce dernier à passer quelques jours à Strasbourg. Dès son arrivée, j'ai eu le privilège avec Anne Duval de partager l'interminable conversation qu'ils ont eue dans le bureau de Jean-Pierre. C'était surréaliste ! Le bureau de Jean-Pierre à l'époque, c'était 80% de l'espace occupé par des documents. Galanterie oblige, Jean-Pierre nous a proposé les deux fauteuils bas pendant que lui-même et Jean Écalle s'asseyaient sur des chaises de taille normale de l'autre côté de son bureau. Je voyais alternativement l'un et l'autre entre d'énormes piles de documents posés sur le bureau et leur conversation se faisait dans une langue extraterrestre ! À la suite de cet épisode, Jean Écalle nous a raconté sans fin, matin et après-midi pendant 4 jours, l'état de ses travaux : la théorie de la résurgence, classification et synthèse, calcul moulien,... J'ai pris des notes de moins en moins denses au fil du temps et qui se terminent par « Je n'y comprends plus rien ».

Ce fut le début de l'épisode Dulac et d'une collaboration entre Jean-Pierre, Jean Écalle, Jean Martinet et Robert Moussu, qui fut par moments plutôt houleuse. Ces questions n'ont pratiquement pas été évoquées au séminaire.

La théorie de Galois différentielle est devenue le thème majeur du séminaire dès la fin de l'année 1985. C'est une vision géométrique qu'en a Jean-Pierre, en osmose avec l'analyse des singularités et les problèmes de sommation et qu'il concrétise dans sa théorie de Cauchy sauvage. Là encore, il ne recule pas devant les calculs nécessaires au développement d'exemples et il ne perd pas de vue le calcul effectif. C'est à cette époque que Claudine Mitschi a rejoint le groupe. Nous avons eu un peu de mal à la convaincre mais je crois pouvoir dire qu'une fois sa décision prise, elle ne l'a jamais regrettée.

Puis, ou plutôt simultanément, est venue l'époque de la multisommabilité. Jean Martinet n'a malheureusement pas eu le temps de mener à bien sa propre vision de

la théorie basée sur une analogie non-standard. Nous avons tous été très affectés par sa disparition mais c'est de son enthousiasme sur ces questions balbutiantes tel qu'il nous l'a fait partager à Saint-Hugues ou à Luminy dont je veux me souvenir.

À partir de 1987 ce sont les équations aux q-différences qui furent au centre des préoccupations de l'équipe mais, là encore j'avais quitté Strasbourg, définitivement cette fois-ci pour un poste à Orsay, et je n'ai pas eu le plaisir de voir se développer cette thématique.

Pour terminer cette évocation du séminaire je voudrais encore préciser comment il fonctionnait. Au début de l'année, réunion d'organisation et distribution du travail : chacun son exposé, quelques journées spéciales, la rédaction d'un ouvrage, le développement d'algorithmes et des logiciels correspondants, etc. La première année, je sortis de la réunion un peu paniquée parce que je devais faire le troisième exposé sur un sujet qui m'était totalement inconnu (Travaux de Nevanlinna... me semble-t-il) et rédiger rapidement un chapitre du livre en projet. Heureusement, j'ai eu la bonne idée de demander tout de suite à Jean-Pierre s'il croyait vraiment que nous allions pouvoir faire tout ça en un an. Non ! me dit-il, mais si on veut avancer il faut avoir des projets. Moi qui suis naturellement angoissée, cela m'a rassurée. D'ailleurs, les deux premiers exposés ne se sont terminés qu'en février et j'ai donc eu largement le temps de préparer le mien. Je crois pouvoir dire aussi — j'espère que tu approuveras, Jean-Pierre — que, 25 ans après, ce programme, bien que largement avancé, n'est toujours pas bouclé.

J'ai eu beaucoup de plaisir à travailler dans l'équipe qu'a rassemblée Jean-Pierre autour de lui et nous avons été nombreux dans ce cas. Ce n'est pas un hasard si Jean-Pierre a eu tant d'élèves : j'en ai compté 25 — mais j'ai bien dû en oublier quelques uns — dont 10 femmes collectivement désignées sous le patronyme de « Ramissettes » parmi lesquelles on distingue parfois les « Ramissettes historiques » de celles qui ont suivi... J'ajoute mes remerciements à ceux qu'a présentés Claudine Mitschi tout à l'heure parce que tu as su, Jean-Pierre, nous orienter tous vers des sujets passionnants, connexes mais bien distincts de sorte que nous avons pu échanger sans jamais nous sentir en concurrence.

Au bonheur quotidien sont venus s'ajouter de nombreux voyages pour des conférences en des lieux magiques : le VVF[(7)] du Col de Saales où nous avons rencontré l'équipe de calcul formel de Grenoble pour la première fois, le Schloß Ringberg en Bavière, Guanajuato et Cuernavaca au Mexique, Medina del Campo en Espagne, Saint-Hugues en Chartreuse, Luminy de nombreuses fois...

[(7)] Village Vacances Familles devenu aujourd'hui Village de Vacances en France.



1990 – Ithaca.

Toute une entreprise...



1992 – Luminy.

J.-P. Ramis et quelques élèves : C. Zhang, F. Fauvet, M. Toulouse-Boucher, G. Chen.



1997 – Cuernavaca. *Après le travail, la tequila... en compagnie de E. Salem, M. Klughertz, C. Mitschi, A. Duval, M. Loday, M. Flexor, M. Canalis-Durand.*



1991 – Guanajuato. *Des sosies de Jean-Pierre Ramis et Robert Moussu.*

Enfin, je ne peux terminer cette évocation sans souligner que la large culture mathématique de Jean-Pierre n'est que l'une des facettes de sa grande érudition. Ceux d'entre nous qui le connaissent un peu mieux ne se sont jamais lassés de l'entendre parler avec passion de littérature surtout, mais aussi de musique, de peinture, d'art ou de planche à voile ! Roger Gay qui nous fait l'amitié de sa présence aujourd'hui pourrait tout particulièrement en témoigner. C'est Jean-Pierre qui m'a fait découvrir des auteurs comme André Pieyre de Mandiargues ou Pierre Louÿs. Quand je ne savais que lire il avait toujours un bonne référence à me suggérer non sans une pointe de malice quelquefois. C'est lui qui m'a conseillé la lecture de *Angels and Insects* de A.S. Byatt pour me montrer jusqu'où peut aller la perfidie des femmes. Excuse-moi, Jean-Pierre, tu as encore dû te tromper de signe mais c'est surtout la perfidie des hommes que j'y ai vue.

Jean-Pierre a rejoint Toulouse, la ville de son enfance, en 1994. De nouveaux horizons se sont ouverts...

M. Loday-Richaud, Université d'Angers – UFR Sciences, UMR 6093, 2 boulevard Lavoisier, 49 045 Angers Cedex 01 (France) • *E-mail :* michele.loday@univ-angers.fr

Astérisque
296, 2004, p. 43–53

GALOIS REPRESENTATIONS, DIFFERENTIAL EQUATIONS, AND q-DIFFERENCE EQUATIONS : SKETCH OF A p-ADIC UNIFICATION

by

Yves André

***Abstract.* —** This is a broad introduction to the following, more technical, paper [**AdV**]. We explain how [**AdV**] relates to two major themes of J.-P. Ramis' work, which eventually become unified in the p-adic world.

***Résumé* (Représentations galoisiennes, équations différentielles et aux q-différences: esquisse d'une unification p-adique)**

Ce texte est une introduction développée à l'article suivant, plus technique [**AdV**]. Nous expliquons comment [**AdV**] est lié à deux thèmes majeurs de l'œuvre de J.-P. Ramis, et comment ceux-ci trouvent leur unification en passant au monde p-adique.

Introduction

Two remarkable analogies haved played an important role in Jean-Pierre Ramis' work:

– the analogy between linear complex differential equations and coverings in characteristic p (reported in D. Bertrand's contribution to this volume),

– the analogy between linear differential equations and q-difference equations (reported in J. Sauloy's contribution).

Our aim is to explain the analogs of these analogies in the p-adic world. We will see that once transposed into that context, these analogies become much more precise, and eventually lead to some equivalences of categories!

2000 *Mathematics Subject Classification.* — Primary 12H25; Secondary 34A30, 11S80, 14H30, 39A13, 11S15.

***Key words and phrases.* —** Differential equations, q-difference equations, coverings, wild singularities, local Galois representation, overconvergence, p-adic local monodromy.

© Astérisque 296, SMF 2004

1. A mysterious analogy: linear complex differential equations and coverings in characteristic p, tame and wild

1.1. A dictionary. — This analogy grew out of discussions between J.-P. Ramis and M. Raynaud during the "Nuit de la Musique 1993"[(1)]. Let us recall it in the form of a "dictionary":

Differential side	Characteristic-p side
$X = \overline{X} \smallsetminus S$ affine curve / $\mathbb{C}$ ($\overline{X}$ complete)	$X = \overline{X} \smallsetminus S$ affine curve / $k \subset \overline{\mathbf{F}}_p$ ($\overline{X}$ complete)
differential module / X	unramified Galois covering of X
singular point (in S)	branch point (in S)
regular singular point	tame branch point, *i.e.,* the ramification index at s is prime to p
irregular singular point	wild branch point
local differential Galois group at $s \in S$	inertia group at $s \in S$
(global) differential Galois group G (a linear alg. group / $\mathbb{C}$)	covering group G (a finite group)
torus in G	*p-Sylow subgroup of* G
$L(G)$: normal subgroup generated by all tori	$p(G)$: normal subgroup generated by all p-Sylow's
monodromy map $\mu : \pi_1(X) \to G/L(G)$	monodromy map $\mu : \pi_1^{(p')}(X) \to G/p(G)$
μ has Zariski-dense image (Ramis condition for the existence of a diff. module on X with diff. Galois group G, all singularities $s \in S$ being regular but one)	μ is surjective (Harbater condition for the existence of an unramified G-covering of X, all branch points $s \in S$ being tame but one).

Comment. — In the right-hand column, $\overline{\mathbf{F}}_p$ denotes a fixed algebraic closure of the field $\mathbf{F}_p$ with p-elements, and $\pi_1^{(p')}(X)$ denotes the profinite group which classifies unramified coverings of X of degree prime to p, *i.e.,* the prime-to-p quotient of Grothendieck's algebraic fundamental group $\pi_1(X)$ of X. According to Grothendieck,

(1) Older sources, in the ℓ-adic context, will be evoked in the next section.

$\pi_1^{(p')}(X)$ is a free prime-to-p profinite group on $2g+|S|-1$ generators (g denotes the genus of $\overline{X}$ and S is assumed to be non-empty).[(2)]

1.2. ℓ-adic linearized variant ($\ell \neq p$). — There is a somewhat older and more standard version of this dictionary (*cf. e.g.* the end of [**K**]) in which objects in the right-hand column are replaced by more linear ones (in fact $\mathbb{Z}_\ell$-linear[(3)] ones, for some fixed (but arbitrary) prime number $\ell \neq p$). It consists essentially in considering at once the whole tower of unramified coverings of X of degree a power of ℓ. In that way, finite groups are replaced, in the right-hand column, by ℓ-adic Lie groups, or even by algebraic groups over $\mathbb{Q}_\ell$ (by taking a suitable algebraic envelope).

Differential side	Characteristic-p side
$X = \overline{X} \smallsetminus S$ affine curve / $\mathbb{C}$ ($\overline{X}$ complete)	$X = \overline{X} \smallsetminus S$ affine curve / $k \subset \overline{\mathbf{F}}_p$ ($\overline{X}$ complete)
differential module M on X	lisse ℓ-adic sheaf $\mathcal{L}$ on X (ℓ-adic continuous representation of $\pi_1(X)$)
differential Galois group (an algebraic group / $\mathbb{C}$)	monodromy group (image of $\pi_1(X)$ or its Zariski closure, an algebraic group / $\mathbb{Q}_\ell$)
local differential Galois group	image of inertia group $\mathcal{I}$ (or its Zariski closure)
de Rham cohomology groups $H^i_{\mathrm{dR}}(X, M)$	étale cohomology groups $H^i_{\text{ét}}(X, \mathcal{L})$
$\chi(M) = \sum(-1)^i \dim H^i_{\mathrm{dR}}(X, M)$	$\chi(\mathcal{L}) = \sum(-1)^i \dim H^i_{\text{ét}}(X, \mathcal{L})$
Deligne-Malgrange irregularity $\mathrm{irr}(M, s)$ *at* s	*Swan conductor* $\mathrm{sw}(M, s)$ *at* $s \in S$
Deligne's formula for $\chi(M)$ in terms of $\mathrm{rk}\, M$ and irregularities	Grothendieck's formula for $\chi(\mathcal{L})$ in terms of $\mathrm{rk}\, M$ and Swan conductors.

[(2)] Referee's remark. Earlier presentations of the Ramis-Raynaud dictionary can be found in M. van der Put's Bourbaki talk: Recent work on differential Galois theory (Exposé 849, Astérisque 252 (1998), 341-367), as well as in van der Put and Singer's book *Galois Theory of Linear Differential Equations*, Springer-Verlag (2003).

[(3)] Recall that the ring of ℓ-adic integers $\mathbb{Z}_\ell$ is the limit of the system $\cdots \to \mathbb{Z}/\ell^{n+1}\mathbb{Z} \to \mathbb{Z}/\ell^n\mathbb{Z} \to \cdots \mathbb{Z}/\ell\mathbb{Z} = \mathbf{F}_\ell$, so that any ℓ-adic integer can be expressed as a series $\sum_0^\infty a_n \ell^n$ where $a_n \in \{0, 1, \ldots, \ell-1\}$. The field of fractions of $\mathbb{Z}_\ell$ is $\mathbb{Q}_\ell = \mathbb{Z}_\ell[\frac{1}{\ell}]$. In the sequel, we denote by $\overline{\mathbb{Q}}_\ell$ a fixed algebraic closure of $\mathbb{Q}_\ell$.

1.3. The ℓ-adic local monodromy theorem ($\ell \neq p$). — Let us recall the structure of the absolute Galois groups which play a role in the "characteristic-p side". We now assume that $k = \mathbf{F}_{p^n} \subset \overline{k} = \overline{\mathbf{F}}_p$ is the field with p^n elements. Then,

$$G_k := \mathrm{Gal}(\overline{k}/k) = \widehat{\mathbb{Z}} = \prod_{\ell'} \mathbb{Z}_{\ell'} \quad \text{and}$$

$G_{k((x))} := \mathrm{Gal}\left(k((x))^{\mathrm{sep}}/k((x))\right)$ can be unscrewed via two exact sequences:

$$1 \longrightarrow \mathcal{I} \longrightarrow G_{k((x))} \longrightarrow G_k \longrightarrow 1$$

and

$$1 \longrightarrow \mathcal{P} \longrightarrow \mathcal{I} \longrightarrow \mathbb{Z}_\ell \times \prod_{\ell' \neq p,\ell} \mathbb{Z}_{\ell'} \longrightarrow 1$$

where $\mathcal{I} = G_{\overline{k}((x))}$ is the *inertia group*, and $\mathcal{P}$ is a pro-p-group called the *wild inertia group*.

This reflects the fact that in contrast to the char. 0 case, the algebraic closure of $\overline{k}((x))$ contains many more elements than just Puiseux series. For instance, roots z of the Artin-Schreier equation $z^{-p} - z^{-1} = x^{-1}$ cannot be expressed as Puiseux series.

Correspondingly, one has a tower of Galois extensions

$$k((x)) \subset \overline{k}((x)) \overset{\text{tame}}{\subset} \bigcup_{p \nmid n} \overline{k}((x^{1/n})) \overset{\text{wild}}{\subset} (k((x)))^{\mathrm{sep}},$$

with respective Galois groups $G_k, \mathcal{I}/\mathcal{P}$ and $\mathcal{P}$.

***Theorem 1.1* (Grothendieck [G])**. — *Every ℓ-adic representation of $G_{k((x))}$ is* quasi-unipotent*, i.e., a suitable open subgroup of $\mathcal{I}$ acts (through its quotient in $\mathbb{Z}_\ell$) by unipotent matrices.*

This can also be formulated, in the "Tannakian vein", as an equivalence of $\otimes$-categories

$$Rep_{\overline{\mathbb{Q}}_\ell}(\mathcal{I} \times \mathbb{G}_a) \xrightarrow{\sim} \{\text{continuous } \overline{\mathbb{Q}}_\ell\text{-reps. of } \mathcal{I} \text{ which extend to reps. of } G_F\}$$

where $\mathcal{I}$ appears in the left-hand side as a constant group-scheme (and representations are understood in the scheme-theoretic sense), and in the right-hand side as a profinite topological group.

2. The p-adic analog of this analogy. An equivalence of categories.

2.1. Some motivation. Frobenius and overconvergence.— At least two aspects of the above dictionary 1.2 are unsatisfactory: the arbitraryness of the auxiliary prime number ℓ, and the very different natures of the cohomologies occurring in the left-hand (De Rham) and right-hand (étale) columns.

Both drawbacks would disappear, and the analogy would become much closer, if one could replace ℓ by p, and étale cohomology by some appropriate cohomology of De Rham type.

It turns out that this is indeed possible, provided one lifts the geometric situation from characteristic p to characteristic 0 (p-adic lifting), and replaces ℓ-adic sheaves by some analytic differential modules. The basic ideas here are due to Dwork[(4)].

i) The relevant "lifting" of X is constructed as follows: one fixes a p-adic field K (a finite extension of $\mathbb{Q}_p$) with residue field $k = \mathbf{F}_{p^n}$, and one fixes a smooth complete curve $\overline{\mathcal{X}}/K$ whose reduction modulo p is our given complete curve $\overline{X}/k$. Removing the open disks of radius 1 in $\overline{\mathcal{X}}$ which reduce mod. p to the points in S, one gets a p-adic analytic curve $\mathcal{X}$ which lifts X.

As was pointed out by Dwork, one should actually remove "infinitesimally more": the relevant space is the limit $\mathcal{X}^\dagger$ of $\overline{\mathcal{X}}$ deprived from disks of radius $1-\varepsilon$ around the singularities, when $\varepsilon \to 1$ ($\mathcal{X}^\dagger$ is a pro-ringed space[(5)]).

ii) According to Dwork again, the relevant p-adic differential modules should have two features:

1) *Frobenius structure*: after change of variable $x \mapsto x^{p^n}$, the new differential module is isomorphic to the old one,

2) *Overconvergence*: the differential module (and its Frobenius structure) should be also defined in some annulus inside each singular disk; in other words, should be defined over $\mathcal{X}^\dagger$.

Examples

a) $\overline{X} = \mathbb{P}^1_k$, $S = \{0\}$, $\mathcal{X}$ = outer unit disk $|x| \geqslant 1$. Let π be Dwork's constant, *i.e.,* a fixed root of the equation $\pi^{p-1} = -p$ in $\overline{\mathbb{Q}}_p$. Then the differential equation $y' = -\pi/x^2 y$ has the required properties (overconvergence of Frobenius means that $y(x^p)/y(x) = e^{(\pi/x^p)-(\pi/x)}$ has a p-adic radius of convergence > 1 in $1/x$).

b) Let us consider the differential module $x^{-\alpha}e^{1/x}\mathcal{O}$ (endowed with the derivation where xd/dx) where $\mathcal{O}$ is some ring of analytic functions away from the origin.

Let us first consider the complex case. The corresponding "period" is

$$\int_\gamma x^{-\alpha} e^{1/x} \frac{dx}{x} = -\frac{2\pi i}{\Gamma(1-\alpha)} \sim \Gamma(\alpha).\,(6)$$

According to Ramis' precise Gevrey theory, an optimal choice for $\mathcal{O}$ is

$$\mathbb{C}[[1/x]]_{-1,1^-} := \left\{ \sum a_n x^{-n} \mid \exists\, \kappa > 0,\, \exists\, r \in]0,1[\,;\ |a_n| \leqslant \kappa r^n/n! \right\}.\,(7)$$

(4) We refer to [**R**] for a nice introduction to this circle of ideas.

(5) Working with $\mathcal{X}$ instead of $\mathcal{X}^\dagger$ would provide unwanted infinite-dimensional cohomology spaces in general.

(6) Here π is the usual one! The symbol $\sim$ means equality up to some factor in $\overline{\mathbb{Q}}^*$. The chosen loop comes from $-\infty$ and returns to $-\infty$ after turning once counterclockwise around the origin.

(7) $(-1, 1^-)$ is a characteristic index for which $\dim H^1 = 1$.

Let us now turn to the p-adic case. Overconvergence is satisfied after rescaling $1/x \mapsto \pi/x$. However strange this condition may look at first sight, it is nothing else than a Gevrey condition: indeed, in the p-adic case,

$$\left\{ \sum a_n x^{-n} \mid \exists\, \kappa > 0,\ \exists\, r \in]0,1[\,;\ |a_n| \leqslant \kappa r^n/|n!| \right\}$$

is precisely the ring of *analytic functions* on a disk of radius $> |\pi|$, which gives the ring of analytic functions on $\mathcal{X}^\dagger$ after rescaling $1/x \mapsto \pi/x$. On the other hand, one can evaluate Frobenius, and it turns out that its eigenvalues in some appropriate sense are, up to some algebraic factor, special values $\sim \Gamma_p(\alpha)$ of Morita's p-adic gamma function.

This is a general phenomenon, and the new version of the dictionary, with p-adic right-hand column, now looks as follows:

Differential side	Characteristic-p side
$X = \overline{X} \smallsetminus S$ affine curve / $\mathbb{C}$	$X = \overline{X} \smallsetminus S$ affine curve / $k \subset \overline{\mathbf{F}}_p$ overconvergent lifting $\mathcal{X}^\dagger$ of X over a p-adic field
differential module M on X	differential module $\mathcal{M}^\dagger$ on $\mathcal{X}^\dagger$ (admitting a Frobenius structure)
$H^i_{\mathrm{dR}}(X, M)$	$H^i_{\mathrm{dR}}(X^\dagger, \mathcal{M}^\dagger)$
periods	*eigenvalues of Frobenius.*

Remark. — At the referee's suggestion, let us mention that there exists a completely different approach to the Ramis-Raynaud analogies, also involving p-adic differential equations, namely the theory of iterative differential Galois groups developped by H. Matzat and M. van der Put ([**MvP1, MvP2**]). This variant of differential Galois theory applies to function fields of characteristic p, and relies on the notion of an iterative derivation (Hasse, F.-K. Schmidt). The differential Galois groups attached to iterative differential modules are linear algebraic groups (not necessarily finite) over the field of (iterative) constants k, and one has a Galois correspondence. In special cases, Matzat and van der Put establish an iterative differential analog of the Abhyankar conjecture.

Iterative differential modules can be lifted to global p-adic differential equations of a very special kind, but the relationship with the above theory remains unclear.

2.2. The p-adic local monodromy theorem (Crew's conjecture). — Let again K be a finite extension of $\mathbb{Q}_p$, with residue field $k = \mathbf{F}_{p^n}$.

The lifting process associates to X over the finite field k the p-adic pro-space $\mathcal{X}^\dagger$ over K. In this process, localization around a point $s \in S$ becomes localization on an "infinitely thin annulus".

One is thus led to consider the so-called *Robba ring*, *i.e.,* the ring of K-analytic functions[8] on arbitrarily thin annuli $\mathcal{A}_{]1-\varepsilon,1[}$: $1-\varepsilon<|x|<1$ (x denotes a local coordinate in the singular unit disk above s).

$$\mathcal{R}=\mathcal{R}_K=\bigcup_{\varepsilon}\{K\text{-analytic functions on } \mathcal{A}_{]1-\varepsilon,1[}\}.$$

The subring $\mathcal{E}^\dagger$ of *bounded* functions also plays an important role, because it turns out to be an *henselian field* with residue field $k((x))$; in other words, giving a finite separable extension of $k((x))$ amounts to giving a finite unramified extension $\mathcal{E}^{\dagger\prime}$ of $\mathcal{E}^\dagger$ (and this provides a finite etale extension $\mathcal{R}'$ of $\mathcal{R}$).

Theorem 2.1. — *Every differential module over $\mathcal{R}$ which admits a Frobenius structure is* quasi-unipotent*, i.e., has a basis of solutions in $\mathcal{R}'[\log x]$, where $\mathcal{R}'$ is the finite etale extension of $\mathcal{R}$ attached to a finite separable extension of $k((x))$.*

This is the p-adic (deeper) analog of Grothendieck's ℓ-adic local monodromy theorem 1.1, as the following "Tannakian formulation" puts in evidence:

$$Rep_{\overline{\mathbb{Q}}_p}(\mathcal{I}\times\mathbb{G}_a)\xrightarrow{\sim}\{\text{differential modules}/\mathcal{R}_{\overline{\mathbb{Q}}_p}\text{ admitting a Frobenius structure}\}.$$

Example. — The 1-dimension representation of $\mathcal{I}=G_{\overline{k}((x))}$ attached to Artin-Scheier equation $z^{-p}-z^{-1}=x^{-1}$ corresponds to the differential module $y'=-\frac{\pi}{x^2}y$.

2.3. Hasse-Arf filtrations. — There are several approaches to Thm. 2.1. My own approach [**A2**] is based on the notion of *Hasse-Arf filtration* in a Tannakian category $\mathcal{T}$ [9].

Data. — For every object $M\in\mathcal{T}$, a separated decreasing filtration $(F^{>\lambda}M)_{\lambda\geqslant 0}$ functorial and exact in M, satisfying

$$F^{>\lambda}(\mathbf{1})=0,\ F^{>\lambda}(M)=F^{>\lambda}(N)=0\quad\Longrightarrow\quad F^{>\lambda}(M\otimes N)=F^{>\lambda}(M^\vee)=0.$$

One shows that every M then admits a canonical finite decomposition $M=\oplus\mathrm{gr}^{\lambda_i}M$, where $\mathrm{gr}^{\lambda_i}M$ is "of pure slope λ_i". This allows to attach to M its Newton polygon $NP(M)$ following the usual recipe. The "height" of $NP(M)$ is denoted by $h(M)$.

We say that the functorial filtration $(F^{>\lambda})$ is a *Hasse-Arf filtration* if $\forall\, M$, $h(M)\in\mathbb{N}$ (equivalently, if all Newton polygons have integral vertices).

Examples

1) The oldest example comes from arithmetic. $\mathcal{T}=Rep\,G_{k((x))}$ [10]. The classical theory of ramification provides a non-decreasing sequence of normal subgroups $G^\mu\subset G_{k((x))}$ (the higher ramification groups), and one defines a filtration as follows:

$$F^{>\lambda}M=0\iff G^\mu\text{ acts trivially on }M\ \forall\,\mu>\lambda.$$

[8] We refer to [**R'**] for a nice introduction to the theory of p-adic analytic functions.
[9] See also [**A4**] for a more detailed introduction to this topic.
[10] One could replace $k((x))$ by any local field.

In this context, $h(M)$ is called the Swan conductor of M and is denoted by $\mathrm{sw}(M)$. Its integrality is a classical theorem by Hasse and Arf.

2) $\mathcal{T} = \{\text{differential modules } M/\mathbb{C}[[x]]\}$. Every object is endowed with the Turrittin-Levelt slope filtration. In this context, $h(M)$ is the irregularity $irr(M)$, and its integrality follows from the definition.

3) $\mathcal{T} = \{\text{differential modules } M/\mathcal{R} \text{ admitting a Frobenius structure}\}$. Looking at the growth of solutions toward the *outer boundary* of $\mathcal{A}_{]1-\varepsilon,1[}$, Christol and Mebkhout have defined the (analytic) *filtration by p-adic slopes* of M. In this context, $h(M)$ is called the p-adic irregularity of M and is denoted by $\mathrm{irr}_p(M)$. Christol and Mebkhout have shown that $\mathrm{irr}_p(M)$ is always an integer [**CM**].

It turns out that, despite their very different natures, examples 1) and 3) correspond to each other via the equivalence of categories 2.1:

***Theorem 2.2* (Matsuda, Tsuzuki; Crew [C])**. — *The canonical $\otimes$-equivalence:*

$$Rep_{\overline{\mathbb{Q}}_p}(\mathcal{I} \times \mathbb{G}_a) \xrightarrow{\sim} \{\textit{differential modules } /\mathcal{R}_{\overline{\mathbb{Q}}_p} \textit{ admitting a Frobenius structure}\}$$

is compatible with the canonical filtrations (Hasse-Arf on the L.H.S., by p-adic slopes on the R.H.S.).

This can be summarized by the slogan

$$\boxed{\mathrm{sw} = \mathrm{irr}_p}$$

3. Another analogy: linear differential equations and q-difference equations; confluence

3.1. The q-world. — Let us provisionally abandon p in favor of q. The q-calculus has a long history (Euler, Gauss, Jacobi, Heine, ..., Ramanujan, ...[(11)]). It is based on the replacement of ordinary integers n by their q-analogs

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

The usual derivation d/dx is then replaced by the q-derivation

$$d_q : f(x) \longmapsto \frac{f(x) - f(qx)}{(1-q)x}$$

which sends x^n to $[n]_q x^{n-1}$ (and the q-exponential $e_q^x = \sum x^n/[n]_q^!$ to itself).

Differential equations are thus replaced by q-difference equations. The phenomenon of *confluence* arises when $q \to 1$: then $n_q \rightsquigarrow n$, $d_q \rightsquigarrow d/dx$, and q-difference equations tend to differential equations under appropriate convergence conditions. Conversely, for q close to 1, q-difference equations may be considered as deformations of differential equations.

(11) See some highlights in [**E**].

3.2. Non-commutative connections and q-deformations. — Let R be some "ring of functions" stable under the dilatation $\sigma_q : f(x) \mapsto f(qx)$.

Recall that a more intrinsic version of differential equations is provided by differential modules, or even better, by connections. Similarly, a more intrinsic version of q-differences is provided by *q-difference modules*: R-module M (projective of finite rank) + σ_q-linear automorphism.

This setting has one drawback: it does not allow to obtain the limit differential module in the case of confluence ($q \to 1$). However, one can also present differential modules as *connections*:

$$\nabla : M \longrightarrow \Omega^1_q \otimes M, \quad (\nabla(rm) = r\nabla(m) + dr \otimes m),$$

Ω^1_q = *non-commutative* bimodule of rank one: $f.\omega = \omega \cdot \sigma_q(f)$, $d : R \to \Omega^1_q$, $f \mapsto \omega \cdot d_q(f)$.

This gives rise to a unified theory of Galois differential groups in the differential and q-difference cases [**A1**], [**A3**], and a relevant setting for the algebraic study of confluence.

3.3. Analytic theory. — Here, for many reasons (*e.g.,* to avoid difficult problems of small divisors), one assumes $|q| \neq 1$.

The analytic theory of q-difference equations has been initiated by Adams, Birkhoff etc., and was revived by J.-P. Ramis in the early 90's. The analogy with differential equations is often straightforward at the formal/combinatorial level, but rather subtle at the analytic level, especially when wild phenomena or confluence are involved, *cf.* [**S**].

4. The p-adic analog of this analogy. Another equivalence of categories [AdV]

4.1. Frobenius structure. — We fix $n > 0$, and a prime p. Recall that a Frobenius structure on a differential module is an isomorphism between M and its "pull-back" by the change of variable $\phi : x \mapsto x^{p^n}$.[(12)]

In the q-difference case, one has the relation

$$\boxed{\sigma_q \, \phi = \phi \, \sigma_{q^{p^n}}},$$

so that the pull-back of a σ_q-module M is a priori a $\sigma_{q^{1/p^n}}$-module. In order to make a σ_q-module out of it, it suffices however to iterate p^n times the action of $\sigma_{q^{1/p^n}}$. We denote by $\phi^! M$ this new σ_q-module M. A Frobenius structure may then be defined to be an isomorphism between $\phi^! M$ and M.[(13)]

[(12)] Actually, one also has to twist the coefficients by some power of the so-called Frobenius automorphism of K, but we neglect this fact here.

[(13)] In [**AdV**], another notion of Frobenius structure is also considered.

4.2. p-adic confluence and canonical q-deformation. — We come back to the p-adic situation: K is p-adic field with residue field $k \subset \mathbf{F}_{p^n}$. We fix $q \in K$, $q \neq$ root of unity. We are interested in "confluence" (note that here $|q-1| < 1 \ \rightarrow \ |q| = 1$, in contrast to the usual postulate in the complex case). Actually, it simplifies matters to assume

$$\boxed{|1-q| < p^{-\frac{1}{p-1}}}.$$

Let M be a σ_q-module over the Robba ring $\mathcal{R} = \mathcal{R}_K$.

Theorem 4.1. — *There is a canonical "functor of confluence"*

$$\{q\text{-difference modules } /\mathcal{R}_{\overline{K}} \text{ admitting a Frobenius structure}\} \longrightarrow \{\text{differential modules } /\mathcal{R}_{\overline{K}} \text{ admitting a Frobenius structure}\}$$

which is an equivalence of tannakian categories.

Its construction uses *quasi-unipotence (q-analog of Crew's conjecture), cf.* [**AdV**]. In fact for any q-difference module over $\mathcal{R}_K$ admitting a Frobenius structure, there is a canonical sequence of $q^{p^{in}}$-difference structures on the same underlying $\mathcal{R}$-module (with $i \to \infty$, so that $q^{p^{in}} \to 1$) which converges to a differential structure.

4.3. Another Hasse-Arf filtration?— This subsection is tentative. By looking at the growth of solutions toward the *outer boundary* of $\mathcal{A}_{]1-\varepsilon,1[}$, it seems (not all details have been checked) that one can define, à la Christol and Mebkhout, a *filtration by p-adic slopes* on M, whence a notion of p-adic q-irregularity $q\text{-irr}_p(M)$, and that one has the following q-analog of 2.2:

Conjecture 4.2. — *The canonical $\otimes$-equivalence:*

$$Rep_{\overline{\mathbb{Q}}_p}(\mathcal{I} \times \mathbb{G}_a) \xrightarrow{\sim} \{q\text{-difference modules } /\mathcal{R}_{\overline{\mathbb{Q}}_p} \text{ admitting a Frobenius structure}\}$$

is compatible with the canonical filtrations (Hasse-Arf on the L.H.S., by p-adic slopes on the R.H.S.).

This can be summarized by the slogan

$$\boxed{\mathrm{sw} = q\text{-irr}_p}.$$

Remark. — In this context, it is interesting to notice that the formal slope filtration for complex q-difference modules with $|q| \neq 1$ (*cf.* [**S**]) is *not* a Hasse-Arf filtration (in contrast both to the differential case and to the p-adic case), since negative slopes may occur, for instance. Once again, we see that the q-analogy is much tighter in the p-adic case.

References

[A1] Y. André – "Différentielles non commutatives et théorie de Galois différentielle ou aux différences", *Ann. Sci. École Norm. Sup.* **5** (2001), p. 1–55.

[A2] ———, "Filtrations de type Hasse-Arf et monodromie p-adique", *Invent. Math.* **148** (2002), no. 2, p. 285–317.

[A3] ———, "On Galois theory of q-deformations of differential equations", in *Deformations of differential equations and asymtotic analysis (Kyoto, 2002)* (Haraoka, ed.), RIMS report, vol. 1296, 2002, p. 35–46.

[A4] ———, "Singularities of p-adic differential equations; toward p-adic Stokes phenomena?", in *Differential equations and the Stokes phenomenon* (Braaksma, Immink, van der Put & Top, eds.), World scientific, 2003, p. 1–13.

[AdV] Y. André & L. Di Vizio – "q-difference equations and p-adic local monodromy", this volume.

[B] D. Bertrand – "Travaux de J.-P. Ramis sur les équations différentielles linéaires", this volume.

[CM] G. Christol & Z. Mebkhout – "Sur le théorème de l'indice des équations différentielles p-adiques III", *Ann. of Math.* **151** (2000), p. 385–457.

[C] R. Crew – "Canonical extensions, irregularities, and the Swan conductor", *Math. Ann.* **316** (2000), p. 19–37.

[E] T. Ernst – "The history of q-calculus and a new method", Ph.D. Thesis, University of Uppsala, 2001.

[G] A. Grothendieck – "Good reduction of abelian varieties", *Ann. of Math. (2)* **88** (1968), p. 492–517, appendix to J.P. Serre & J. Tate.

[K] N. Katz – "On the calculation of some differential Galois groups", *Invent. Math.* **87** (1987), p. 13–61.

[MvP2] B. Matzat & M. van der Put – "Constructive differential Galois theory", in *Galois Groups and Fundamental Groups* (L. Schneps, ed.), MSRI Publications, vol. 41, Cambridge University Press, 2003, p. 425–467.

[MvP1] ———, "Iterative differential equations and the Abhyankar conjecture", *J. reine angew. Math.* **257** (2003), p. 1–52.

[R] P. Robba – "Une introduction naïve aux cohomologies de Dwork", in *Introductions aux cohomologies p-adiques* (D. Barsky & P. Robba, eds.), Mém. Soc. math. France (N.S.), vol. 23, Société Mathématique de France, 1986, p. 61–105.

[R'] A. Robert – *A course in p-adic analysis*, Graduate Texts in Math., vol. 198, Springer-Verlag, New York, 2000.

[S] J. Sauloy – "Algebraic construction of the Stokes sheaf for irregular linear q-difference equations", this volume.

Y. André, École Normale Supérieure, 45 rue d'Ulm, 75005 Paris (France)
E-mail : yves.andre@ens.fr

Astérisque
296, 2004, p. 55–111

q-DIFFERENCE EQUATIONS AND p-ADIC LOCAL MONODROMY

by

Yves André & Lucia Di Vizio

***Abstract.* —** We present a p-adic theory of q-difference equations over arbitrarily thin annuli of outer radius 1. After a detailed study of rank one equations, we consider higher rank equations and prove a local monodromy theorem (a q-analog of Crew's quasi-unipotence conjecture). This allows us to define, in this context, a canonical functor of "confluence" from q-difference equations to differential equations, which turns out to be an equivalence of categories (in the presence of Frobenius structures).

***Résumé* (Équations aux q-différences et monodromie p-adique). —** Nous présentons une théorie p-adique des équations aux q-différences sur des couronnes arbitrairement minces de rayon extérieur 1. Après une étude détaillée des équations de rang 1, nous nous penchons sur le cas de rang supérieur et nous démontrons un théorème de monodromie locale (un q-analogue de la conjecture de quasi-unipotence de Crew). Cela nous permet de définir, dans ce contexte, un foncteur canonique de « confluence » des équations aux q-différences vers les équations différentielles, qui s'avère être une équivalence de catégories (en présence de structures de Frobenius).

Introduction

In the context of p-adic differential equations, the expression "local theory" occurs in two different senses. In the naive sense, it refers to the study of the behaviour of solutions in a small punctured disk around a given singularity. This theory has been reasonably well-understood for a long time(1).

On the other hand, according to some insights of Dwork and Grothendieck, the geometrically relevant p-adic differential equations are those which admit analytic solutions in all non-singular open unit disks, and which extend a little inside the singular disks. They should be understood as objects (cohomological coefficients) belonging to geometry in characteristic p. It is then consistent with this viewpoint

2000 *Mathematics Subject Classification.* — Primary 39A13; Secondary 33D05, 12H50.
***Key words and phrases.* —** p-adic monodromy, q-difference equations, p-adic representation.
(1) Although by no means completely understood, *cf.* for instance the problems raised by Ramero's theory [**Ra98**] in its differential variant.

© Astérisque 296, SMF 2004

to call "local theory" the study of the behaviour of solutions in arbitrarily thin annuli with outer radius 1 contained in singular open unit disks[(2)].

In this sense, the local theory of p-adic differential equations has been developped first by Robba (in rank one), then by Christol and Mebkhout (in arbitrary rank), and has recently reached full maturity with the proof of the so-called local monodromy theorem (Crew's quasi-unipotence conjecture) which provides a bridge toward the theory of p-adic Galois representations.

The objective of this paper is to set up a local theory of p-adic q-difference equations, parallel to the differential theory, and to put a link forward between the two theories.

* * *

In the history of the theory of p-adic differential equations, going from the rank 1 case to arbitrary rank has been a difficult step. This is due in part to the fact that the study of rank 1 p-adic differential equations indulges fairly down-to-earth methods (*cf.* for instance [**R85**], [**CC96**]). In the first part of the paper we develop an analogous theory for p-adic q-difference equations of rank 1. The techniques employed are inspired by the differential case and, due to their explicit and direct nature, bring to the fore the relationship with differential equations. In fact, we construct a *canonical deformation functor* from the category of p-adic differential equations of rank 1 to the category of p-adic q-difference equations, which we describe explicitly.

The first part is organized as follows. In § 1 we recall some basic facts of p-adic q-difference algebra proved in [**DV03**]. In § 2 we prove some properties of the q-exponential function which play a significant role in the sequel. Sections § 3 and § 4 contain a q-analog of Dwork-Robba's criterion of solvability and its application to q-difference equations of rank 1 with meromorphic coefficient. The results in § 4 are used in the next section to show that one can actually reduce the study of rank-one q-difference equations analytic over an arbitrary thin annulus of outer radius 1, to the study of rank 1 q-difference equations with polynomial coefficient. This reduction is crucial for the characterization of q-difference equations with Frobenius structure (*cf.* § 6). We finish the first part by proving that for a q-difference equation having a Frobenius structure is equivalent to being a "deformation" of a differential equation with strong Frobenius structure (*cf.* § 7). From there, we obtain the p-adic monodromy theorem in the rank 1 case and the deformation functor (*cf.* § 8).

There are two appendices, the first one being devoted to the Frobenius structure of the q-exponential series. In the second one, we give a q-analog of Dwork's approach to the p-adic gamma function via the Frobenius structure of so-called exponential modules.

* * *

(2) See the previous paper [**A**] for more detail and perspective, and for the apparatus of analogies which motivates the present paper.

In the second part, we consider q-difference modules M of arbitrary rank over the "Robba ring" $\mathcal{R}$ of analytic functions on an arbitrarily thin annulus of outer radius 1. We prove the local monodromy theorem for those q-difference modules which admit a Frobenius structure: there exists a finite étale extension $\mathcal{R}'/\mathcal{R}$ coming from characteristic p, such that $M \otimes_{\mathcal{R}} \mathcal{R}'[\log x]$ becomes a trivial q-difference module (*cf.* § 14.2, § 14.3 for various equivalent precise statements). We follow K. Kedlaya's approach to the p-adic local monodromy theorem in the differential case, proving along the way a q-analog of Tsuzuki's theorem on unit-root objects.

This second part is organized as follows. We first discuss finite étale extensions $\mathcal{R}'/\mathcal{R}$ coming from characteristic p, and how the q-difference operator d_q extends to $\mathcal{R}'$ (the lack of an explicit expression for this extended operator leads to many technical difficulties in the sequel). We then introduce and investigate two notions of Frobenius structures for q-difference modules: the strong Frobenius structure (analogous to its differential counterpart), and the confluent weak Frobenius structure (which yields a sequence of q^{p^n}-difference modules converging to a differential module with Frobenius structure).

In § 13, we analyse q-difference modules over $\mathcal{R}$ with overconvergent (strong) Frobenius structure of slope 0. As in Tsuzuki's theorem, they arise from finite p-adic representations of the inertia group of a local field of characteristic p.

We then prove three versions of the theorem of quasi-unipotence for q-difference modules over $\mathcal{R}$ which admit a strong Frobenius structure. We also show that such q-difference modules have a confluent weak Frobenius structure.

This gives rise to a canonical functor of "confluence" between such q-difference modules (M, Σ_q), and differential modules over $\mathcal{R}$ which admit a strong Frobenius structure, which has a canonical quasi-inverse (15.1, 15.2). More precisely, for any such (M, Σ_q), there is a canonical sequence of $q^{p^{is}}$-difference structures on the $\mathcal{R}$-module M (for fixed s and with $i \to \infty$, so that $q^{p^{is}} \to 1$), related to each other by Frobenius, and which converges to a differential structure on M.

PART I
RANK 1

1. Generalities on p-adic q-difference equations of rank 1

1.1. The q-difference algebra of analytic functions over an annulus

Let K be a field of characteristic zero, complete with respect to a non archimedean absolute value $|\ |$, with residue field k of characteristic $p > 0$. We denote by $\mathcal{O}_K$ its ring of integers and we assume that the absolute value is normalized by $|p| = p^{-1}$.

For any interval $I \subset \mathbb{R}_{\geqslant 0}$ we consider the K-algebra $\mathcal{A}_K(I)$ of analytic functions with coefficients in K on the annulus $\mathcal{C}_K(I) = \{x \in K : |x| \in I\}$:

$$\mathcal{A}_K(I) = \left\{ \textstyle\sum_{n\in\mathbb{Z}} a_n x^n : a_n \in K; \lim_{n\to\pm\infty} |a_n|\rho^n = 0 \ \forall\, \rho \in I \right\}.$$

We denote by $\mathcal{M}_K(I)$ its field of fractions (the field of meromorphic functions on $\mathcal{C}_K(I)$), and by $\mathcal{B}_K(I)$ the subring of bounded elements of $\mathcal{A}_K(I)$. The theory of Newton polygons shows that every invertible analytic function on $\mathcal{C}(I)$ is bounded, so that $\mathcal{A}_K(I)^* = \mathcal{B}_K(I)^*$. We will omit the subscript K when there is no ambiguity.

We fix once and for all an element $q \in K$, such that $|1-q| < 1$ and that q is not a root of unity. The algebra $\mathcal{A}(I)$ has a natural structure of a q-difference algebra. This means that the homeomorphism

$$\begin{aligned} \mathcal{C}(I) &\longrightarrow \mathcal{C}(I) \\ x &\longmapsto qx \end{aligned}$$

induces a K-algebra isomorphism

$$\begin{aligned} \sigma_q : \mathcal{A}(I) &\longrightarrow \mathcal{A}(I) \\ f(x) &\longmapsto f(qx) \end{aligned}$$

Similarly for $\mathcal{M}(I)$ and $\mathcal{B}(I)$.

1.2. The q-derivation. — To the operator σ_q one associates a "twisted derivation" d_q defined by

$$d_q(f)(x) = \frac{f(qx) - f(x)}{(q-1)x},$$

which satisfies the twisted Leibniz Formula:

$$d_q(fg)(x) = f(qx)d_q(g)(x) + d_q(f)(x)g(x). \tag{1}$$

For any pair of integers $n \geqslant i \geqslant 1$ and any $f, g \in \mathcal{M}(I)$ the q-derivation d_q verifies:

$$d_q x^n = [n]_q x^{n-1}, \text{ where } [n]_q = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q-1}; \tag{2}$$

$$\frac{d_q^n}{[n]_q^!} x^i = \binom{n}{i}_q x^{n-i}, \text{ where } [0]_q^! = 1,\ [n]_q^! = [n]_q [n-1]_q^! \text{ and } \binom{n}{i}_q = \frac{[n]_q^!}{[i]_q^! [n-i]_q^!}; \tag{3}$$

$$d_q^n(fg)(x) = \sum_{j=0}^{n} \binom{n}{j}_q d_q^{n-j}(f)(q^j x) d_q^j(g)(x). \tag{4}$$

1.3. q-difference equations. — Let us now consider a *q-difference equation of rank* 1 *with coefficients in* $\mathcal{M}(I)$:

$$y(qx) = a(x)y(x), \quad a(x) \in \mathcal{M}(I). \tag{5}$$

We shall often write (5) in the form

$$d_q y(x) = g(x)y(x), \text{ with } g(x) = \frac{a(x)-1}{(q-1)x}. \tag{6}$$

For any $u \in \mathcal{M}(I)^*$, $z(x) = u(x)^{-1}y(x)$ is a solution of the q-difference equation

$$z(qx) = \left[u(qx)^{-1}a(x)u(x)\right] z(x) \tag{7}$$

or equivalently of

$$d_q z(x) = \left[\frac{u(x)}{u(qx)}g(x) + \frac{d_q u(x)}{u(qx)}\right] z(x). \tag{8}$$

We shall say that equations (5) and (7) (or (6) and (8)) are $\mathcal{M}(I)$-equivalent[(3)].

From (6), one derives the following sequence of equations

$$d_q^n(y)(x) = g_n(x)y(x),$$

with $g_1(x) = g(x)$, $g_{n+1}(x) = g_n(qx)g_1(x) + d_q g_n(x)$. It is convenient to set $g_0(x) = 1$.

If $g(x)$ is analytic at 0, then $\sum_{n\geqslant 0} \frac{g_n(0)}{[n]_q^!}x^n$ is a formal solution of $y(qx) = a(x)y(x)$.

1.4. Generic points. — In order to apply the technique of generic points, we shall have to use an auxiliary extension of normed fields Ω/K, with the following properties (for the construction of such a field, see for instance [**Ro00**, § 3, 2]):

1) the field Ω is complete and algebraically closed;
2) the set of values of Ω is $\mathbb{R}_{\geqslant 0}$;
3) the residue field of Ω is a transcendental extension of the residue field of K.

For any $\rho \in \mathbb{R}_{\geqslant 0}$ the field Ω contains an element t_ρ, called a *generic point (at distance ρ from 0)*, such that t_ρ is transcendent over K and $|t_\rho| = \rho$, so that the norm induced over $K(t_\rho) \subset \Omega$ is defined by

$$\left|\frac{\sum a_i t_\rho^i}{\sum b_j t_\rho^j}\right| = \frac{\sup_i |a_i|\rho^i}{\sup_j |b_j|\rho^j}.$$

Definition 1.1. — For any $\rho \in I$, we call the number

$$R_\rho(\sigma_q - a(x)) = \inf\left(\rho, \liminf_{n\to\infty}\left|\frac{g_n(t_\rho)}{[n]_q^!}\right|^{-1/n}\right)$$

the generic radius of convergence of $y(qx) = a(x)y(x)$ at t_ρ. We will write simply R_ρ when no confusion is possible.

(3) We shall also use a similar terminology for other rings of functions.

1.5. Properties of the generic radius of convergence. — The following proposition summarizes some facts about the generic radius of convergence, which are proven in [**DV03**].

Proposition 1.2

1) **(Twisted Taylor expansion)** *Let $d_q y(x) = g(x)y(x)$ be a q-difference equation with coefficient $g(x) \in \mathcal{M}(I)$ and let $\xi \in \mathcal{C}(I)$. Suppose that $g(x)$ does not have any pole in $q^{\mathbb{N}}\xi$. Then $d_q y(x) = g(x)y(x)$ has an analytic solution in a neighborhood of ξ if and only if*

$$R := \liminf_{n\to\infty} \left|\frac{g_n(\xi)}{[n]_q^!}\right|^{-1/n} > |(q-1)\xi|.$$

In that case, the unique analytic solution $y(x)$ of $y(qx) = a(x)y(x)$ in the open disk $D(\xi, R^-)$ verifying $y(\xi) = 1$ coincides with the sum of the series

$$\sum_{n\geqslant 0} \frac{g_n(\xi)}{[n]_q^!}(x,\xi)_{n,q}, \quad \text{where } (x,\xi)_{n,q} = (x-\xi)(x-q\xi)\cdots(x-q^{n-1}\xi).$$

2) *Let $b(x) = u(qx)^{-1}a(x)u(x)$, with $u(x) \in \mathcal{M}(I)^*$. Then $R_\rho(\sigma - a(x)) = R_\rho(\sigma - b(x))$ for any $\rho \in I$, i.e., the generic radius of convergence is invariant under $\mathcal{M}(I)$-equivalence.*

3) **(q-analog of the Dwork-Robba effective bound)** *If $R_\rho > |q-1|\rho$, then*

$$\left|\frac{g_n(t_\rho)}{[n]_q^!}\right| \leqslant \frac{1}{R_\rho^n}, \text{ for any } n \geqslant 1.$$

4) **(Transfer to an ordinary disk)** *Let $g(x)$ be analytic over $D(\xi, \rho^-)$, with $\xi \in K$ and $|\xi| \leqslant \rho$, and let $R_\rho > |q-1|\rho$. Then $d_q y(x) = g(x)y(x)$ has an analytic solution in the disk $D(\xi, R_\rho^-)$. Moreover, the equation $d_q y(x) = g(x)y(x)$ has an analytic solution in the disk $D(\xi, \rho^-)$ if and only if $R_\rho = \rho$.*

5) **(Transfer to a regular singular disk)** *Let $a(x) \in \mathcal{A}(]0,1[)$ and $u(x) \in K[\![x]\!]$ be a formal power series with coefficients in K such that $u(qx)^{-1}a(x)u(x) \in K$. If $R_\rho = \rho$, the series $u(x)$ converges for $|x| < \rho$.*

Corollary 1.3. — *Let $y(qx) = a(x)y(x)$ be a q-difference equation with $a(x) \in \mathcal{A}([0,1[)$ (resp. $\mathcal{A}([0,1[) \cap \mathcal{M}([0,1])$). Then $y(qx) = a(x)y(x)$ has a solution $y(x)$ analytic and bounded in $\mathcal{C}([0,1[)$ if and only if $\lim_{\rho\to 1} R_\rho = 1$ (resp. $R_1 = 1$).*

Proof. — Let $a(x) \in \mathcal{A}([0,1[)$. It follows from the assertions in 4) of the previous proposition that the existence of a solution $y(x)$ analytic and bounded over $\mathcal{C}([0,1[)$ implies $R_\rho = \rho$ for any $\rho \in]0,1[$. Hence we conclude that $\lim_{\rho\to 1} R_\rho = 1$.

On the other hand, suppose $\lim_{\rho\to 1} R_\rho = 1$. Again, by the assertions of 4), that the formal solution

$$y(x) = \sum_{n\geqslant 0} \frac{g_n(0)}{[n]_q^!}x^n$$

of $y(qx) = a(x)y(x)$ converges in $D(0, R_\rho^-)$ for any $\rho \in]0,1[$. This proves that $y(x) \in \mathcal{A}([0,1[)$. Finally statement 3. implies that

$$\left|\frac{g_n(0)}{[n]_q^!}\right| \leqslant \left|\frac{g_n(t_\rho)}{[n]_q^!}\right| \leqslant \frac{1}{R_\rho^n}. \tag{9}$$

By letting ρ tend to 1, one proves that $y(x)$ is bounded.

If $a(x) \in \mathcal{A}([0,1[) \cap \mathcal{M}([0,1])$, the generic radius of convergence R_1 is defined. Assertion 4) of (1.2) states that $y(qx) = a(x)y(x)$ has a solution $y(x) \in \mathcal{A}([0,1[)$ if and only if $R_1 = 1$. Moreover the existence of the analytic solution $y(x) \in \mathcal{A}([0,1[)$ implies that $R_\rho = \rho$ for any $\rho \in]0,1[$, therefore the inequality (9) allows to conclude that $y(x)$ is bounded. □

It is customary in the theory of q-difference equations to assume that the coefficient a is invertible[(4)]. We shall follow this tradition, and consider mostly q-difference equations $y(qx) = a(x)y(x)$ with $a(x) \in \mathcal{A}(I)^* = \mathcal{B}(I)^*$. Written in the form $d_q(y) = gy$, this implies that $g \in \mathcal{B}(I)$. Actually most of the time, we shall not only have $g \in \mathcal{B}(I)$ as in the differential case (the logarithmic differential of any element of $\mathcal{A}(I)$ belongs to $\mathcal{B}(I)$), but also $|(q-1)g|_{\mathcal{B}(I)} < 1$, $|a|_{\mathcal{B}(I)} = 1$.

2. An example: the q-exponential function

2.1. The q-exponential $e_q(x)$. — The power series

$$e_q(x) = \sum_{n \geqslant 0} \frac{x^n}{[n]_q^!}$$

is a q-deformation of the exponential series and satisfies the q-difference equation

$$d_q e_q = e_q,$$

that is to say

$$e_q(qx) = (1 + (q-1)x)e_q(x).$$

Proposition 2.1. — *The series $e_q(x)$ has radius of convergence $\prod_{i \geqslant 0} \left|[p]_{q^{p^i}}\right|^{1/p^{i+1}}$.*

Proof. — Every positive integer n can be uniquely written in the form $n = p^s m + k$, where $m, s, k \in \mathbb{Z}$, m and p are coprime and $0 \leqslant k \leqslant p-1$. If $k > 0$ then $|[n]_q| = \left|q^{p^s m}[k]_q + [p^s m]_q\right| = 1$, in fact $|[k]_q| = 1$ and $|[p^s m]_q| = |[m]_{q^{p^s}}[p^s]_q| < 1$. Therefore

$$\left|[n]_q^!\right| = \prod_{i=1}^{[n/p]} |[p]_q[i]_{q^p}| = \prod_{i \geqslant 0} \left|[p]_{q^{p^i}}^{[n/p^{i+1}]}\right|,$$

(4) This convention is also in use in the higher rank case, where $a(x)$ is a matrix; it allows to define the dual system.

where the product on the right is actually finite. It follows that

$$\limsup_{n\to\infty} \left|[n]_q^!\right|^{1/n} = \prod_{i\geqslant 0} \left|[p]_{q^{p^i}}^{1/p^{i+1}}\right|. \qquad \square$$

Proposition 2.2. — *If*

$$\operatorname{dist}(aq^{\mathbb{Z}_p}, bq^{\mathbb{Z}_p}) = \min_{\alpha\in\mathbb{Z}_p} |a - q^\alpha b| < \textit{radius of convergence of } e_q(x), \tag{10}$$

the analytic function $e_q(ax)/e_q(bx)$, with $a, b \in K$, $a \neq 0 \neq b$, is overconvergent, i.e., it has a radius of convergence > 1.

Remark 2.3. — Notice that $\exp(ax)/\exp(bx)$, with $a, b \in K$, is overconvergent if and only if $|a - b| < |\pi|$, hence the inequality above is actually a q-deformation of the analogous condition in the differential framework.

Proof. — The series $e_q(ax)/e_q(bx)$ is solution of the q-difference equation

$$y(qx) = \frac{1 + (q-1)ax}{1 + (q-1)bx}\, y(x).$$

Notice that $\min_{\alpha\in\mathbb{Z}_p} |q^\alpha a - b|$ is realized for $\alpha \in \mathbb{Z}$, $\alpha \geqslant 0$, hence, by multiplying $e_q(ax)/e_q(bx)$ by $(1, (q-1)ax)_{\alpha,q}$, we can assume that $\min_{\alpha\in\mathbb{Z}_p} |a - q^\alpha b| = |a - b|$. Observing that for any integer $n \geqslant 1$

$$d_q(1, (q-1)bx)_{n,q} = (q^n - 1)b(1, (q-1)bqx)_{n-1,q},$$

one verifies by induction that the series $e_q(ax)/e_q(bx)$ is solution of

$$d_q^n y(x) = \frac{(a,b)_{n,q}}{(1, (q-1)bx)_{n,q}} y(x).$$

This proves that

$$\frac{e_q(ax)}{e_q(bx)} = \sum_{n\geqslant 0} \frac{(a,b)_{n,q}}{[n]_q^!} x^n.$$

Let $r(e_q(x))$ be the radius of convergence of $e_q(x)$. The condition $|a - b| < r(e_q(x))$ implies that

$$\limsup_{n\to\infty} \left|\frac{(a,b)_{n,q}}{[n]_q^!}\right|^{1/n} = \frac{1}{r(e_q(x))} \limsup_{n\to\infty} \left|\prod_{i=0}^{n-1} (a - bq^i)\right|^{1/n} < 1. \qquad \square$$

2.2. The analytic function $\log e_q(x)$. — If $|e_q(x) - 1| < 1$, it makes sense to consider the analytic function $L_q(x) = \log e_q(x)$. From the q-difference equation satisfied by e_q, one derives immediately the equation:

$$L_q(qx) = L_q(x) + \log(1 + (q-1)x),$$

which can be rewritten in the form

$$d_q L_q(x) = \sum_{n\geqslant 0} (-1)^n (q-1)^n \frac{x^n}{n+1}.$$

We find that the analytic function $L_q(x)$ has the (well-known[5]) expansion (*cf.* [**HL46**, § 2], [**Q**]):

$$L_q(x) = \sum_{n\geqslant 1} \frac{(-1)^{n-1}(q-1)^{n-1}}{[n]_q n}\, x^n.$$

Proposition 2.4

1) *The series $L_q(x)$ converges for $|x| < |q-1|^{-1}$.*
2) *If $|x| < \dfrac{|p|^{1/(p-1)}}{|q-1|} \sup\left(|p|^{1/(p-1)}, |q-1|\right)$, then $|L_q(x)| < |x|$.*

Proof

1) It is enough to notice that

$$\liminf_{n\to\infty} \left|\frac{(q-1)^{n-1}}{[n]_q n}\right|^{-1/n} = \frac{1}{|q-1|} \liminf_{n\to\infty} \left|n^2 \log q\right|^{1/n} = \frac{1}{|q-1|}.$$

2) Since $L_q(x) = x\left(1 + \sum_{n\geqslant 1} \dfrac{(-1)^n (q-1)^n}{[n+1]_q (n+1)}\, x^n\right)$, we have to prove that $|x| < \frac{|p|^{1/(p-1)}}{|q-1|} \sup\left(|p|^{1/(p-1)}, |q-1|\right)$ implies

$$\sup_{n\geqslant 1} \left|\frac{(q-1)^n}{[n+1]_q (n+1)} x^n\right| < 1.$$

This follows from the inequalities $|p|^{1/(p-1)} \leqslant |n+1|^{1/n}$ and $\sup\left(|p|^{1/(p-1)}, |q-1|\right) \leqslant |[n+1]_q|^{1/n}$. □

In the sequel of this section, we assume that K *contains the p-th roots of unity.* It then also contains $p-1$ distinct non zero roots of the equation $\pi^p = -p\pi$. One picks one of them and denotes it by π (Dwork's constant). Notice that $|\pi| = |p|^{1/(p-1)}$.

Corollary 2.5. — *If $|q-1| < |\pi|$, the series $e_q(\pi x)/\exp(\pi x)$ is overconvergent.*

Proof. — Let us consider the series

$$L_q(\pi x) = \pi x + \sum_{n\geqslant 2} (-1)^{n-1} \frac{(1-q)^{n-1}}{[n]_q n} \pi^n x^n.$$

Notice that the assumption $|q-1| < |\pi|$ implies $|[n]_q| = |n|$. For $n = 2, \ldots, p-1$ and $|x| < |\pi(q-1)|^{-1/2}$ the following inequality holds:

$$\left|\frac{(1-q)^{n-1}}{[n]_q n} \pi^n x^n\right| < |\pi|.$$

On the other hand, for $n \geqslant p$ and $|x| < |q-1|^{-1}|\pi|^{1/2} \leqslant |q-1|^{-1}|\pi|^{-(n-3)/n}$ we have

$$\left|\frac{(1-q)^{n-1}}{[n]_q n} \pi^n x^n\right| = \left|\frac{(q-1)^n \pi^n}{n^2} x^n\right| < |\pi|.$$

[5] the complex analog is essentially the quantum dilogarithm

We conclude that there exists $\varepsilon > 0$ such that for $|x| < 1+\varepsilon$ we have $|L_q(\pi x)-\pi x| < |\pi|$. Hence the series $e_q(\pi x)/\exp(\pi x) = \exp\left(L_q(\pi x) - \pi x\right)$ is analytic for $|x| < 1 + \varepsilon$. □

2.3. Overconvergent solutions of ordinary q-difference equation at 0

In this subsection we are going to use the analytic function L_q to construct an overconvergent solution of a linear q-difference equation of rank 1, under a suitable hypothesis.

Lemma 2.6. — *Let $y(qx) = a(x)y(x)$ be a q-difference equation such that $a(x)$ is an analytic function at 0, with $a(0) = 1$. Then write $a(x)$ as an infinite product $\prod_{i\geqslant 1}\left(1+\mu_i x^i\right)$. If there exists $\varepsilon > 0$ such that*

$$\sup_{i\geqslant 1}\frac{|\mu_i|}{|q^i-1|}(1+\varepsilon)^i < |\pi|$$

then the infinite product

$$\prod_{i\geqslant 1} e_{q^i}\left(\frac{\mu_i}{q^i-1}x^i\right)$$

converges to an overconvergent solution of $y(qx) = a(x)y(x)$.

Proof. — Since

$$\frac{|\mu_i|}{|q^i-1|}(1+\varepsilon)^i < |\pi|\,\frac{\sup\left(|\pi|,\left|q^i-1\right|\right)}{|q^i-1|} \quad \text{for any } i \geqslant 1,$$

we have

$$\sup_{|x|<1+\varepsilon}\left|L_{q^i}\left(\frac{\mu_i x^i}{q^i-1}\right)\right| < \sup_{|x|<1+\varepsilon}\left|\frac{\mu_i x^i}{q^i-1}\right| < |\pi|.$$

It follows that

$$z(x) = \sum_{i\geqslant 1} L_{q^i}\left(\frac{\mu_i x^i}{q^i-1}\right)$$

is an analytic function for $|x| < 1+\varepsilon$ and that $\sup_{|x|<1+\varepsilon}|z(x)| < |\pi|$. We conclude that

$$\exp z(x) = \prod_{i\geqslant 1} e_{q^i}\left(\frac{\mu_i x^i}{q^i-1}\right)$$

is an overconvergent solution of $y(qx) = a(x)y(x)$. □

Proposition 2.7. — *Let $y(qx) = a(x)y(x)$ be a q-difference equation such that $a(x) = \prod_{i\geqslant 1}\left(1+\mu_i x^i\right)$ is an overconvergent analytic function. Then there exists a positive integer M and a positive real number ε such that $y(qx) = a(x)y(x)$ is $\mathcal{M}([0,1+\varepsilon[)$-equivalent to*

$$y(qx) = \prod_{i=1}^{M}\left(1+\mu_i x^i\right) y(x).$$

Proof. — Since $a(x)$ is overconvergent there exists a positive integer M and a real number $\varepsilon > 0$ such that $|\mu_i| > (1+\varepsilon)^i$ for any $i > M$. Let $0 < \varepsilon' < \varepsilon$. For any $|x| \leqslant 1+\varepsilon' < 1+\varepsilon$ we have

$$\lim_{i\to\infty} \frac{(1+\varepsilon)^i |x^i|}{|(q^i-1)|} = 0.$$

Let us fix an integer $M > 0$ such that

$$\sup_{i>M} \sup_{|x|<1+\varepsilon'} \frac{(1+\varepsilon)^i |x^i|}{|(q^i-1)|} < |\pi|$$

and set

$$\widetilde{a}(x) = \prod_{i>M} \left(1 + \frac{\mu_i}{x^i}\right).$$

It follows from the previous lemma that there exists $u(x) \in \mathcal{A}([0, 1+\varepsilon'[)$ such that $u(qx) = \widetilde{a}(x)u(x)$, *i.e.*, such that $\dfrac{u(qx)}{u(x)} a(x) = \prod_{i=1}^{M} \left(1 + \mu_i x^i\right)$. □

3. Solvability (at the boundary)

Recall that the Robba ring $\mathcal{R} = \mathcal{R}_x = \mathcal{R}_{K,x}$ is the ring

$$\mathcal{R} = \cup_{\varepsilon>0} \mathcal{A}(]1-\varepsilon, 1[)$$

of analytic functions on some thin annulus with the unit circle as outer boundary.

The subring

$$\mathcal{E}^\dagger = \mathcal{E}^\dagger_x = \mathcal{E}^\dagger_{K,x} = \cup_{\varepsilon>0} \mathcal{B}(]1-\varepsilon, 1[)$$

of bounded functions is endowed with the sup-norm $|\sum a_n x^n|_{\mathcal{E}^\dagger} = \sup |a_n|$ (caution: this is not a Banach ring). If the valuation of K is discrete, this is a Henselian field, with residue field $k((x))$.

We introduce the subrings

$$\mathcal{B} = \mathcal{B}_x = \mathcal{A}([0,1[) \cap \mathcal{E}^\dagger, \quad \mathcal{H}^\dagger = \mathcal{H}^\dagger_x = \cup_{\varepsilon>0} \mathcal{A}(]1-\varepsilon, \infty])$$

(on which the restriction of $|\ |_{\mathcal{E}^\dagger}$ is the sup-norm, according to the principle of the maximum).

In this section, we begin the study of q-difference equations $y(qx) = a(x)y(x)$ with $a(x) \in \mathcal{R}^* = (\mathcal{E}^\dagger)^*$.

By iteration of the operator d_q, we deduce from the q-difference equation $y(qx) = a(x)y(x)$ a sequence of equations

$$d_q^n y(x) = g_n(x) y(x),$$

with $g_n(x) \in \mathcal{E}^\dagger$ and $g_0(x) = 1$. Since $a(x) \in \mathcal{B}(]1-\varepsilon, 1[)$ for some $\varepsilon > 0$, it makes sense to consider $R_\rho(\sigma_q - a(x))$ for $\rho \in]1-\varepsilon, 1[$.

Definition 3.1. — The equation $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, is said to be *solvable*[(6)] (at the outer boundary) if

$$\lim_{\rho \to 1} R_\rho(\sigma_q - a(x)) = 1.$$

Remark 3.2. — It follows immediately from (1.2) that solvability is invariant under $\mathcal{E}^\dagger$-equivalence.

One can define the notion of solvability without using the generic radius of convergence:

Lemma 3.3. — $\lim_{\rho\to 1} R_\rho = \inf\left(1, \liminf_{n\to\infty} \left|\frac{g_n(x)}{[n]_q^!}\right|_{\mathcal{E}^\dagger}^{-1/n}\right)$.

Proof[(7)]. — Let us set

$$R_{\mathcal{E}^\dagger} = \inf\left(1, \liminf_{n\to\infty} \left|\frac{g_n(x)}{[n]_q^!}\right|_{\mathcal{E}^\dagger}^{-1/n}\right).$$

Notice that for any $\rho \in]1-\varepsilon, 1[$, we have

$$R_\rho = \liminf_{n\to\infty} \left(\sup_{0\leqslant s\leqslant n} \left|\frac{g_n(t_\rho)}{[n]_q^!}\right|\right)^{-1/n} \text{ and } R_{\mathcal{E}^\dagger} = \liminf_{n\to\infty} \left(\sup_{0\leqslant s\leqslant n} \left|\frac{g_n(x)}{[n]_q^!}\right|_{\mathcal{E}^\dagger}\right)^{-1/n}.$$

Moreover (*cf.* [**CD94**, 2.1]) $h_\rho(n) = \sup_{0\leqslant s\leqslant n} \left|g_n(t_\rho)/[n]_q^!\right|$ is a continuous function of ρ and

$$\lim_{\rho\to 1} h_\rho(n) = \sup_{0\leqslant s\leqslant n} \left|\frac{g_n(x)}{[n]_q^!}\right|_{\mathcal{E}^\dagger},$$

hence it is enough to prove the uniform convergence of the sequence $h_\rho(n)^{1/n}$.

The proof of [**DV02**, 4.2.7] actually shows that for any positive integers $N \geqslant n > s \geqslant 0$ such that $N = [N/n]\, n + s$, the $h_\rho(n)$'s verify the inequality

$$h_\rho(N)^{1/N} \leqslant h_\rho(n)^{(1/n)+(1/N)} \left|\frac{[N]_q!}{([n]_q!)^{[N/n]}\, [s]_q!}\right|^{-1/N}.$$

Hence, letting $N \to \infty$, we obtain

$$\frac{1}{R_\rho} \leqslant h_\rho(n)^{1/n} \left(\left|[n]_q^!\right|^{1/n} \lim_{N\to\infty} |[N]_q^!|^{-1/N}\right).$$

(6) This terminology is very unsatisfactory (solvable in what?) but has been of constant use in the theory of p-adic differential equations since Robba's studies. For want of a better word, we shall adopt it here.

(7) The continuity of the function R_ρ is proved in [**CD94**]. The proof that follows uses an argument of uniform convergence and it is a q-analog of an unpublished proof by F. Baldassarri and L. Di Vizio.

It follows from (1.2) that

$$\frac{1}{R_\rho}\left(\left|[n]_q^!\right|^{-1/n}\lim_{N\to\infty}|[N]_q^!|^{1/N}\right)\leqslant h_\rho(n)^{1/n}\leqslant\frac{1}{R_\rho},$$

which finishes the proof. □

4. A characterization of solvability

The following characterization of solvability is a q-analog of a result by Dwork and Robba [**DR77**, 5.4] and will be used to prove that solvable q-difference equations of rank 1 are $\mathcal{E}^\dagger$-equivalent to solvable q-difference equations with coefficients in $K\,[1/x]$.

Proposition 4.1. — *Let Ω/K be the extension introduced in subsection 1.4. The following assertions are equivalent:*

1) *The q-difference equation $y(qx)=a(x)y(x)$, $a(x)\in\mathcal{E}^\dagger$, is solvable.*
2) *There exists a sequence $R_n(x)\in\mathcal{E}_\Omega^\dagger$, such that*
$$\lim_{n\to\infty}\left|\frac{R_n(qx)}{R_n(x)}-a(x)\right|_{\mathcal{E}_\Omega^\dagger}=0.$$
3) *There exists a sequence $R_n(x)\in\mathcal{E}_\Omega^\dagger$ such that*
$$\lim_{n\to\infty}\left|\frac{d_q(R_n)(x)}{R_n(x)}-g(x)\right|_{\mathcal{E}_\Omega^\dagger}=0,\quad \textit{where } g(x)=\frac{a(x)-1}{(q-1)x}.$$

Proof. — The equivalence between *2.* and *3.* is straightforward.

Let us prove that *3.* implies *1.* We set $g_1(x)=g(x)$ and $g_{N+1}(x)=d_qg_N(x)+g_1(x)g_N(qx)$ and we fix $\varepsilon\in\,]0,1[$ and $n>0$. We claim that if the inequality

$$\left|\frac{d_q(R_n)(x)}{R_n(x)}-g(x)\right|_{\mathcal{E}_\Omega^\dagger}\leqslant\varepsilon$$

is satisfied then we have

$$\left|\frac{d_q^N(R_n)(x)}{R_n(x)}-g_N(x)\right|_{\mathcal{E}_\Omega^\dagger}\leqslant\varepsilon\text{ for any }N\geqslant1.$$

We prove our claim by induction. In fact, it follows from proposition 1.2 that

$$\left|\frac{d_q^N(R_n)(x)}{R_n(x)}\right|_{\mathcal{E}_\Omega^\dagger}\leqslant|[N]_q^!|$$

and hence that $|g_N(x)|_{\mathcal{E}^\dagger}\leqslant1$. Therefore, recursively, one obtains:

$$\begin{aligned}\left|d_q^{N+1}\right.&(R_n)(x)-g_{N+1}(x)R_n(x)|_{\mathcal{E}_\Omega^\dagger}\\&=\left|d_q\left(d_q^N(R_n)(x)-g_N(x)R_n(x)\right)+g_N(qx)\left(d_q(R_n)(x)-g_1(x)R_n(x)\right)\right|_{\mathcal{E}_\Omega^\dagger}\\&\leqslant\sup\left(\left|d_q^N(R_n)(x)-g_N(x)R_n(x)\right|_{\mathcal{E}_\Omega^\dagger},|g_N(qx)\left(d_q(R_n)(x)-g_1(x)R_n(x)\right)|_{\mathcal{E}_\Omega^\dagger}\right)\\&\leqslant\varepsilon\,|R_n(x)|_{\mathcal{E}_\Omega^\dagger}.\end{aligned}$$

Let us fix $N>0$ and let $\varepsilon<|[N]_q^!|$. Then there exists $n\gg 0$, depending on N, such that

$$\left|\frac{g_N(x)}{[N]_q^!}\right|_{\mathcal{E}_\Omega^\dagger}\leqslant \sup\left(\left|\frac{d_q^N(R_n)(x)}{[N]_q^!R_n(x)}-\frac{g_N(x)}{[N]_q^!}\right|_{\mathcal{E}_\Omega^\dagger},\left|\frac{d_q^NR_n(x)}{[N]_q^!R_n(x)}\right|_{\mathcal{E}_\Omega^\dagger}\right)\leqslant\sup\left(\frac{\varepsilon}{[N]_q^!},1\right)\leqslant 1,$$

which implies that $y(qx)=a(x)y(x)$ is solvable.

Let us now prove that *1.* implies *3.* Let us consider the sequence of elements of $\mathcal{E}^\dagger$:

$$\beta_0=1,\ \beta_1(x)=-g(x),\ \beta_{N+1}(qx)=d_q\beta_N(x)+\beta_1(x)\beta_N(x).$$

We choose a generic point $t_1\in\Omega$ such that $|t_1|=1$ and we set

$$R_n(x)=\sum_{N=0}^n\frac{\beta_N(x)}{[N]_q^!}(x,t_1)_{N,q}\in\mathcal{E}^\dagger[t_1]\subset\mathcal{E}_\Omega^\dagger,\quad\text{for any } n\geqslant 1.$$

Notice that $R_n(x)$ satisfies the inhomogeneous q-difference equation

$$\begin{aligned}d_qR_n(x)-g(x)R_n(x)&\\&=\sum_{N=0}^n\frac{d_q\beta_N(x)+\beta_1(x)\beta_N(x)}{[N]_q^!}(x,t_1)_{N,q}-\sum_{N=1}^n\frac{\beta_N(qx)}{[N-1]_q^!}(x,t_1)_{N-1,q}\\&=[n+1]_q\frac{\beta_n(x)}{[n]_q^!}(x,t_1)_{n-1,q}.\end{aligned}$$

The following lemma allows to conclude the proof by considering the subsequence $(R_{p^n-1}(x))_{n\in\mathbb{Z}_{>0}}$. □

Lemma 4.2. — $\left|\beta_n(x)/[n]_q^!\right|_{\mathcal{E}^\dagger}\leqslant|R_n(x)|_{\mathcal{E}_\Omega^\dagger}$.

Proof. — Consider the polynomial ring $\mathcal{E}^\dagger[z]$. We have an embedding of valued K-algebras $(\mathcal{E}^\dagger,|\ |_{\mathcal{E}^\dagger})\subset(\mathcal{E}^\dagger[z],|\ |_{\mathcal{E}^\dagger,z})$, where $|\ |_{\mathcal{E}^\dagger,z}$ is defined by

$$\left|\textstyle\sum_i a_i(x)z^i\right|_{\mathcal{E}^\dagger,z}=\sup_i|a_i(x)|_{\mathcal{E}^\dagger},\quad\text{for any }\textstyle\sum_i a_i(x)z^i\in\mathcal{E}^\dagger[z].$$

Let $d_{q,z}$ be a q-difference derivation acting on $\mathcal{E}^\dagger[z]$, in the following way

$$d_{q,z}\left(\textstyle\sum_{i\geqslant 0}a_i(x)z^i\right)=\sum_{i\geqslant 1}a_i(x)[i]_qz^{i-1},\quad\text{for any }\sum_{i\geqslant 0}a_i(x)z^i\in\mathcal{E}^\dagger[z].$$

Observe that

$$\left|\frac{d_{q,z}^n}{[n]_q^!}\left(\textstyle\sum_{i\geqslant 0}a_i(x)z^i\right)\right|_{\mathcal{E}^\dagger,z}\leqslant\left|\textstyle\sum_{i\geqslant 0}a_i(x)z^i\right|_{\mathcal{E}^\dagger,z}$$

and that

$$\left|\textstyle\sum_{i\geqslant 0}a_i(x)z^i\right|_{\mathcal{E}^\dagger,z}=\left|\textstyle\sum_{i\geqslant 0}a_i(x)t_1^i\right|_{\mathcal{E}_\Omega^\dagger}.$$

We set $R_n(x,z)=\sum_{N=0}^n\frac{\beta_N(x)}{[N]_q^!}(x,z)_{N,q}\in\mathcal{E}^\dagger[z]$. Since

$$\frac{d_{q,z}^n}{[n]_q^!}(x,z)_{N,q}=\begin{cases}0&\text{for any pair of integers } n>N>0,\\(-1)^nq^{n(n-1)/2}&\text{if } n=N,\end{cases}$$

we conclude that

$$\left|\frac{\beta_n(x)}{[n]_q^!}\right|_{\mathcal{E}^\dagger} = \left|\frac{d_{q,z}^n(R_n)}{[n]_q^!}(x,z)\right|_{\mathcal{E}^\dagger,z} \leqslant |R_n(x,z)|_{\mathcal{E}^\dagger,z} = |R_n(x,t_1)|_{\mathcal{E}_\Omega^\dagger}\,. \qquad \square$$

Remark 4.3. — Notice that if $g(x) \in K[x]$ then $R_n(x) \in \Omega[x]$.

4.1. Solvability of q-difference equations with constant coefficients

Corollary 4.4. — *The q-difference equation $y(qx) = ay(x)$, with constant coefficient $a \in K$, is solvable if and only if $a \in q^{\mathbb{Z}_p}$.*

Proof. — It follows from proposition 4.1 and remark 4.3 that $y(qx) = ay(x)$ is solvable if and only if a is the limit in $\mathcal{E}_\Omega^\dagger$ of a sequence $R_n(qx)R_n(x)^{-1}$, with $R_n(x) \in \Omega[x]$. Therefore if $y(qx) = ay(x)$ is solvable, the coefficient a is the limit of a sequence in $q^{\mathbb{Z}}$, *i.e.*, $a \in q^{\mathbb{Z}_p}$. Conversely, if $a = q^\alpha$ with $\alpha \in \mathbb{Z}_p$, then there exists a sequence of integers $\alpha_n \in \mathbb{Z}$ such that $\alpha_n \to \alpha$ and hence that $|\frac{(qx)^{\alpha_n}}{x^{\alpha_n}} - a|_{\mathcal{E}^\dagger} \to 0$. $\square$

Remark 4.5. — By induction on $n \geqslant 1$ (*cf.* [**DV03**, 1.2.4]), one can prove that the solutions of the equation $y(qx) = ay(x)$ are necessarily solutions of the sequence of equations:

$$\frac{d_q^n}{[n]_q^!}y(x) = \frac{(a-1)(a-q)\cdots(a-q^{n-1})}{(q-1)(q^2-1)\cdots(q^n-1)q^{n(n-1)/2}x^n}\,y(x).$$

Therefore the previous corollary implies that the series

$$\sum_{n\geqslant 0} \frac{(a-1)(a-q)\cdots(a-q^{n-1})}{(q-1)(q^2-1)\cdots(q^n-1)}\,x^n \tag{11}$$

has radius of convergence 1 if and only if $a \in q^{\mathbb{Z}_p}$, generalizing [**DV03**, 8.2], where the radius of convergence of (11) was calculated under the assumption $|1-q| < |p|^{1/(p-1)}$. This should allow to drop the assumption $|q-1| < |\pi|$ in [**DV03**, § 3].

4.2. Solvability of q-difference equations meromorphic at zero. —

The next corollary concerns rank 1 q-difference equations whose coefficient has at worst a pole at 0 and is analytic in $\mathcal{C}(]0,1[)$.

Corollary 4.6. — *Consider a solvable q-difference equation $y(qx) = a(x)y(x)$, with $x^N a(x) \in \mathcal{B}$ for some positive integer N. Let $a_\infty(x) \in K(x)$ be a rational function such that all the finite zeros and poles of $a_\infty(x)$ are in $\mathcal{C}([0,1[)$, and that $a(x)/a_\infty(x)$ is an invertible analytic function in $\mathcal{B}$ having value 1 at 0. Then the q-difference equations $y(qx) = a_\infty(x)y(x)$ and $y(qx) = \frac{a(x)}{a_\infty(x)}y(x)$ are both solvable.*

Proof. — It follows from (4.1) and its proof that there exists a sequence $R_n(x) \in \mathcal{E}_\Omega^\dagger \cap \mathcal{A}_\Omega(]0,1[)$ such that

$$\lim_{n\to\infty}\left|\frac{R_n(qx)}{R_n(x)} - a(x)\right|_{\mathcal{E}_\Omega^\dagger} = 0.$$

Let $G_n(x) \in \Omega(x)$ be a rational function such that $R_n(x)/G_n(x)$ is an invertible analytic function over $\mathcal{C}([0,1[)$, having value 1 at 0, and that all the poles and zeros of $G_n(x)$ are in $\mathcal{C}([0,1[) \cup \{\infty\}$. Hence the Taylor expansion at ∞ of $G_n(qx)/G_n(x)$ defines an invertible element of $\mathcal{H}^\dagger_\Omega$ and $|G_n(qx)/G_n(x)|_{\mathcal{E}^\dagger_\Omega} = 1$. Then,

$$\begin{aligned}
&\left|\frac{R_n(qx)}{R_n(x)} - a(x)\right|_{\mathcal{E}^\dagger_\Omega} \\
&= \left|\left(\frac{R_n(qx)G_n(x)}{R_n(x)G_n(qx)} - \frac{a(x)}{a_\infty(x)}\right)\frac{G_n(qx)}{G_n(x)} + \frac{a(x)}{a_\infty(x)}\left(\frac{G_n(qx)}{G_n(x)} - a_\infty(x)\right)\right|_{\mathcal{E}^\dagger_\Omega} \\
&= \left|\frac{a(x)}{a_\infty(x)}\right|_{\mathcal{E}^\dagger_\Omega} \left|\left(\frac{R_n(qx)G_n(x)}{R_n(x)G_n(qx)} - \frac{a(x)}{a_\infty(x)}\right)\frac{a_\infty(x)}{a(x)} + \frac{G_n(x)}{G_n(qx)}\left(\frac{G_n(qx)}{G_n(x)} - a_\infty(x)\right)\right|_{\mathcal{E}^\dagger_\Omega}.
\end{aligned}$$

Since $\left(\frac{R_n(qx)G_n(x)}{R_n(x)G_n(qx)} - \frac{a(x)}{a_\infty(x)}\right)\frac{a_\infty(x)}{a(x)} \in x\mathcal{B}$ and $\frac{G_n(x)}{G_n(qx)}\left(\frac{G_n(qx)}{G_n(x)} - a_\infty(x)\right) \in \mathcal{H}^\dagger$ we obtain the equality

$$\left|\frac{R_n(qx)}{R_n(x)} - a(x)\right|_{\mathcal{E}^\dagger_\Omega} = \sup\left(\left|\frac{R_n(qx)G_n(x)}{R_n(x)G_n(qx)} - \frac{a(x)}{a_\infty(x)}\right|_{\mathcal{E}^\dagger_\Omega}, \left|\frac{G_n(qx)}{G_n(x)} - a_\infty(x)\right|_{\mathcal{E}^\dagger_\Omega}\right).$$

By proposition 4.1 we conclude that both $y(qx) = a_\infty(x)y(x)$ and $y(qx) = \frac{a(x)}{a_\infty(x)}y(x)$ are solvable q-difference equations. □

5. Reduction to the case of q-difference equations with polynomial coefficient

In [**R85**], Robba has shown that any rank one differential equation over $\mathcal{E}^\dagger$ is equivalent to a differential equation with coefficient in $K[1/x]$[8]. His method uses a kind of additive decomposition of the coefficient (using logarithmic derivatives) and cannot be translates into the q-difference context. In this section we prove a q-analog of Robba's result using a kind of multiplicative decomposition of the coefficient.

Proposition 5.1. — *Any solvable q-difference equation $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, is $\mathcal{E}^\dagger$-equivalent to a solvable q-difference equation of the form*

$$y(qx) = q^{l_0}\prod_{i=1}^{M}\left(1 + \frac{\mu_i}{x^i}\right)y(x), \tag{12}$$

where $l_0 \in \mathbb{Z}_p$, M is a positive integer, $\mu_i \in K$ and $|\mu_i| \leqslant |q-1|$ for $i = 1, \ldots, M$.

Proof. — The proof is divided into several steps.

[8] Actually he considered only differential equations with rational coefficients, but his argument extends in general.

Step 0. — There exists an unique multiplicative decomposition

$$a(x) = \frac{\lambda}{x^N}\, l(x)\, m(x)$$

of $a(x)$ in $\mathcal{E}^\dagger$ such that
- $\lambda \in K$, $\lambda \neq 0$;
- $N \in \mathbb{Z}$;
- $l(x)$ is an invertible analytic function in $1 + x\mathcal{B}$;
- $m(x)$ is invertible function in $1 + \frac{1}{x}\mathcal{H}^\dagger$.

Proof of Step 0. — [**CM02**, 6.5] and [**C81a**]. □

Step 1. — The q-difference equation $y(qx) = a(x)y(x)$ is $\mathcal{H}^\dagger$-equivalent to a q-difference equation of the form

$$y(qx) = \frac{\lambda}{x^N}\, l(x) \prod_{i=1}^{M} \left(1 + \frac{\mu_i}{x^i}\right) y(x), \tag{13}$$

where $\lambda \in K$, $\lambda \neq 0$, $N, M \in \mathbb{Z}$, $M > 0$, $\mu_i \in K$ for any $i = 1, \ldots, M$ and $l(x)$ is an invertible analytic function in $1 + x\mathcal{B}$.

Proof of Step 1. — The analytic function $m(x) \in \mathcal{H}^\dagger$, considered in Step 0, can be uniquely written as a convergent infinite product

$$m(x) = \prod_{i=1}^{\infty} \left(1 + \frac{\mu_i}{x^i}\right). \tag{14}$$

It follows from proposition 2.7 that there exists $z(x) \in \mathcal{H}^\dagger$ such that $z(qx) = \widetilde{m}(x)z(x)$ and hence that $\frac{z(qx)}{z(x)}a(x)$ has the form (13). □

Step 2. — The equation (13) is $\mathcal{B}$-equivalent to the solvable q-difference equation

$$y(qx) = \frac{\lambda}{x^N} \prod_{i=1}^{M} \left(1 + \frac{\mu_i}{x^i}\right) y(x), \tag{15}$$

with $\lambda \in K$, $\lambda \neq 0$, $N, M \in \mathbb{Z}$, $M > 0$ and $|\mu_i| < 1$ for any $i = 1, \ldots, M$.

Proof of Step 2. — Let us write $l(x)$ as an infinite convergent product

$$l(x) = \prod_{i=1}^{\infty} \left(1 + \lambda_i x^i\right).$$

Since $l(x)$ does not have any zero in $\mathcal{C}([0, 1[)$, we have $|\lambda_i| \leqslant 1$ for any $i > 0$. As far as the μ_i's are concerned, it is enough to recall that the analytic function $m(x)$ (*cf.* (14)) is invertible in $\mathcal{H}^\dagger$ to conclude that $|\mu_i| < 1$ for any $i = 1, \ldots, M$. Hence it follows from (4.6) that the q-difference equations

$$y(qx) = \prod_{i=1}^{\infty} \left(1 + \lambda_i x^i\right) y(x) \tag{16}$$

and

$$y(qx) = \frac{\lambda}{x^N} \prod_{i=1}^{M} \Big(1 + \frac{\mu_i}{x^i}\Big)\, y(x)$$

are both solvable. Since $l(x) \in \mathcal{B}$, the equation (16) has a solution $u(x) \in \mathcal{B}$ (*cf.* 1.3), which establishes the $\mathcal{B}$-equivalence between (13) and (15). □

Step 3. — The solvability of (15) implies that $\lambda \in q^{\mathbb{Z}_p}$, $N = 0$ and $|\mu_i| < |q-1|$ for any $i = 1, \dots, M$.

Proof of Step 3. — Let $b(x) = (\lambda/x^N) \prod_{i=1}^{M} \big(1 + \mu_i/x^i\big)$. Since $|(b(x)-1)/(q-1)x|_{\mathcal{E}^\dagger} \leqslant 1$, necessarily $|b(x)|_{\mathcal{E}^\dagger} = 1$ and hence $|\lambda| = 1$.

Let us write inductively $d_q^n y(x) = g_n(x) y(x)$ for any $n \geqslant 1$. Writing $g_n(x)$ explicitely in terms of $b(x)$ (*cf.* [**DV03**, 1.2.4]) we obtain

$$\begin{aligned} \left|\frac{g_n}{[n]_q^!}\right|_{\mathcal{E}^\dagger} &= \left|\frac{(-1)^n}{[n]_q^!(q-1)^n x^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j}_{q^{-1}} q^{-j(j-1)/2} b(x) b(qx) \cdots b(q^{j-1}x)\right|_{\mathcal{E}^\dagger} \\ &\geqslant \left|\frac{(-1)^n}{[n]_q^!(q-1)^n} \frac{\lambda^n}{x^{(N+1)n}}\right|_{\mathcal{E}^\dagger} \qquad (17) \\ &> \frac{1}{|q-1|^n}. \end{aligned}$$

This shows that if $N \neq 0$, the equation $y(qx) = b(x)y(x)$ is not solvable, in contradiction to the hypothesis. Therefore it is enough to prove that $\lambda \in \mathbb{Z}_p$ assuming that $N = 0$.

Since $y(qx) = b(x)y(x)$ is solvable, there exists $R_n(x) \in \Omega\,[x, 1/x]$ (*cf.* Remark 4.3) such that

$$\left|\frac{R_n(qx)}{R_n(x)} - b(x)\right|_{\mathcal{E}_\Omega^\dagger} \longrightarrow 0,$$

hence λ is a limit of a sequence in $q^{\mathbb{Z}}$. This proves that $\lambda \in q^{\mathbb{Z}_p}$ and that the equation $y(qx) = b(x)\lambda^{-1}y(x)$ is also solvable.

So we are reduced to proving the statement for a solvable q-difference equation of the form

$$y(qx) = \prod_{i=1}^{M} \Big(1 + \frac{\mu_i}{x^{n_i}}\Big)\, y(x),$$

or equivalently of the form

$$d_q y(x) = \Big(\sum_{i=1}^{M} \frac{\mu_i (q-1)^{-1}}{x^{n_i+1}} \prod_{j>i} \Big(1 + \frac{\mu_j}{x^{n_j}}\Big) \Big) y(x).$$

Since $|\mu_i| < 1$, the solvability (*cf.* 1.2) implies that $|\mu_i| \leqslant |q-1|$ for any $i = 1, \dots, M$.

Step 3 finishes the proof. □

6. Frobenius structure in rank 1: existence criterion

From now on, we assume that the *residue field k of K is perfect.*

In [**CC96**, § 2], it is proven that a differential equation of the form

$$\frac{y'(x)}{y(x)} = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_m}{x^m} \in \frac{1}{x} K\left[\frac{1}{x}\right]$$

has a strong Frobenius structure if and only if it is solvable and there exists a positive integer s such that $(p^s - 1)a_1 \in \mathbb{Z}$. In this section we prove an analogous result for q-difference equations. Some steps of our proof use methods that can be adapted to the differential case, simplifying some technical details in [**CC96**].

Let us consider a Frobenius automorphism τ of K, *i.e.,* a continuous automorphism of the field K lifting the Frobenius automorphism of the residue field k. Let s be a positive integer and let us assume that q is τ^s-invariant. Usually one considers the semilinear endomorphim $\phi = \phi_s$ of $\mathcal{E}^\dagger$ defined by

$$\phi_s\left(\sum_{n\in\mathbb{Z}} a_n x^n\right) = \sum_{n\in\mathbb{Z}} \tau^s(a_n) x^{p^s n}.$$

An analogous endomorphism ϕ can be defined over the q-difference algebra of analytic functions over a disk or an annulus, centered at 0 or at ∞.

Definition 6.1. — We say that a q-difference equation $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, has a *(strong) Frobenius structure* if there exists $u(x) \in \left(\mathcal{E}^\dagger\right)^*$ such that

$$\frac{u(qx)}{u(x)} a(x) = a(x)^\phi a(qx)^\phi \cdots a(q^{p^s-1}x)^\phi \tag{18}$$

for a suitable choice of s.

Remark 6.2. — Notice that $v(x)^{\phi\sigma_q^{p^s}} = v(x)^{\sigma_q\phi}$ for any $v(x) \in \mathcal{E}^\dagger$ (here $v(x)^{\sigma_q\phi}$ means $\phi\circ\sigma_q(v(x))$). Let $y(x)$ be a solution of the equation $y(qx) = a(x)y(x)$ in an extension of $\mathcal{E}^\dagger$. If $y(qx) = a(x)y(x)$ has a Frobenius structure then $y(x)^\phi = u(x)y(x)$ with $u(x) \in \left(\mathcal{E}^\dagger\right)^*$ and (18) can be written in the following way:

$$y(x)^{\phi\sigma_q^{p^s}} = y(x)^{\phi\sigma_{q^{p^s}}} = y(x)^{\sigma_q\phi}.$$

Lemma 6.3. — *If a q-difference equation $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, has a Frobenius structure, then it is solvable.*

Proof. — Let $\varepsilon > 0$ be such that $a(x) \in \mathcal{A}(]1-\varepsilon, 1[)$ and let $z(x) = \sum_{n\geqslant 0} \alpha_n (x, t_\rho)_{n,q}$, with $\alpha_n \in K$, be a solution of $y(qx) = a(x)y(x)$ at the generic point t_ρ, for $\rho \in]1-\varepsilon, 1[$.

It follows from the assumption that there exists $u(x) \in \left(\mathcal{E}^\dagger\right)^*$ such that $z(x)^\phi = u(x)z(x)$ is a solution of $y(qx) = a_{p^s}(x)^\phi y(x)$ at $t_\rho^{\tau^s}$. Hence we obtain

$$R_{\rho^{p^s}}(\sigma_q - a_{p^s}(x)^\phi) \leqslant R_\rho(\sigma_q - a(x))^{p^s} \leqslant \rho^{p^s} \leqslant 1. \tag{19}$$

Since $y(qx) = a(x)y(x)$ and $y(qx) = a_{p^s}(x)^\phi y(x)$ are $\mathcal{E}^\dagger$-equivalent, we have

$$\lim_{\rho\to 1} R_\rho(\sigma_q - a(x)) = \lim_{\rho\to 1} R_\rho(\sigma_q - a_{p^s}(x)^\phi).$$

This forces $y(qx) = a(x)y(x)$ to be solvable. □

We recall (*cf.* proof of (5.1), Step 0) that any invertible $a(x) \in \mathcal{E}^\dagger$ can be uniquely written as a product $a(x) = (\lambda/x^N)\, l(x)\, m(x)$, with $\lambda \in K$, $N \in \mathbb{Z}$, $l(x) \in (1 + x\mathcal{B})^*$ and $m(x) \in \left(1 + \frac{1}{x}\mathcal{H}^\dagger\right)^*$. The main result of this section is:

Theorem 6.4. — *A q-difference equation of rank 1 with coefficient in $\mathcal{E}^\dagger$, i.e.,*

$$y(qx) = \frac{\lambda}{x^N}\, l(x)\, m(x)\, y(x),$$

has a Frobenius structure if and only if it is solvable and there exists a positive integer s such that $\lambda^{p^s-1} \in q^{\mathbb{Z}}$.

6.1. Idea of the proof of theorem 6.4. — It follows from (6.3) and (5.1) that it is enough to prove the statement:

Proposition 6.5. — *A q-difference equation*

$$y(qx) = q^{l_0} \prod_{i=1}^{M} \left(1 + (q-1)\frac{\mu_i}{x^i}\right) y(x), \tag{20}$$

with $l_0 \in \mathbb{Z}_p$ and $\mu_1, \dots, \mu_M \in K$, has a Frobenius structure if and only if it is solvable and there exists an integer $s \geqslant 0$ such that $l_0 \in \mathbb{Z}/(p^s - 1)$.

First of all let us remark that:

Lemma 6.6. — *A q-difference equation $y(qx) = q^{l_0} y(x)$, with $l_0 \in \mathbb{Z}_p$ (cf. (4.4)), has a Frobenius structure if and only if $l_0 \in \mathbb{Z}/(p^s - 1)$.*

Proof. — The equation $y(qx) = q^{l_0} y(x)$ is $\mathcal{E}^\dagger$-equivalent to $y(qx) = q^{l_0 p^s} y(x)$ if and only if $(p^s - 1)l_0 \in \mathbb{Z}$. □

Eventually, the proposition 6.5 is a consequence of the following proposition:

Proposition 6.7. — *A q-difference equation of the form*

$$y(qx) = \prod_{i=1}^{M} \left(1 + (q-1)\frac{\mu_i}{x^i}\right) y(x), \tag{21}$$

with $\mu_1, \dots, \mu_M \in K$, has a Frobenius structure if and only if it is solvable.

In fact:

Proof of proposition 6.5 (assuming proposition 6.7). — Suppose that (20) has a Frobenius structure, which implies that it is solvable. Since $l_0 \in \mathbb{Z}_p$, the equation $y(qx) = q^{l_0} y(x)$ is solvable. This implies that the equation

$$y(qx) = \prod_{i=1}^{M} \left(1 + (q-1)\frac{\mu_i}{x^i}\right) y(x)$$

is also solvable, and hence that it has a Frobenius structure. Moreover it proves that also $y(qx) = q^{l_0} y(x)$ has a Frobenius structure, *i.e.,* that $l_0 \in \mathbb{Z}/(p^s - 1)$ (*cf.* (6.6)).

On the other hand, if $l_0 \in \mathbb{Z}/(p^s - 1)$ and (20) is solvable, the equation $y(qx) = q^{l_0} y(x)$ has a Frobenius structure and $y(qx) = \prod_{i=1}^{M} \left(1 + (q-1)\frac{\mu_i}{x^i}\right) y(x)$ is solvable. Then it follows from (6.7) that (20) has a Frobenius structure. □

6.2. Proof of proposition 6.7. — First we prove a lemma, which is a fundamental step in the proof of (6.7). It is a q-analog of a particular case of [**Mo77**, Prop. 1]:

Lemma 6.8. — *Let $u(qx) = v(x)u(x)$ be a q-difference equation such that $v(x)$ is an analytic element on $\mathcal{C}([0,1])$, without zeros and poles in $\mathcal{C}([0,1])$ and $u(x)$ is a non zero analytic element on $\mathcal{C}([0,1[)$. Then $u(x)$ is an analytic element on $\mathcal{C}([0,1])$.*

Proof. — One has to show (*cf.* [**Mo77**, Th. 4]) that for any $a \in K$, $|a| = 1$, there exists an analytic element on $D(0,1^-) \cup D(a,1^-)$ whose restriction to $D(0,1^-)$ coincides with $u(x)$. Let us fix $a \in K$, $|a| = 1$. It follows from [**Mo77**, Th. 1] that there exists $g_a(x)$, called *the singular factor* of $u(x)$ with respect to $D(a,1^-)$, such that

1) $g_a(x)$ is an analytic element on $\mathbb{P}^1 \smallsetminus D(a,1^-)$, without zeros in $\mathbb{P}^1 \smallsetminus D(a,1^-)$;

2) there exists an integer m such that $\lim_{x\to\infty}(x-a)^m g_a(x) = 1$;

3) $h_a(x) = u(x)/g_a(x)$ is an analytic element on $D(0,1^-) \cup D(a,1^-)$, with no zeros in $D(a,1^-)$.

It follows from our hypothesis on

$$v(x) = \frac{u(qx)}{u(x)} = \frac{g_a(qx)}{g_a(x)}\frac{h_a(qx)}{h_a(x)}$$

and the properties of $g_a(x)$ and $h_a(x)$ that:

1) $g_a(qx)/g_a(x)$ is an analytic element on $\mathbb{P}^1 \smallsetminus D(a,1^-)$, without zeros in $\mathbb{P}^1 \smallsetminus D(a,1^-)$;

2) $\lim_{x\to\infty} g_a(qx)/g_a(x) = q^m$;

3) $h_a(qx)/h_a(x)$ is an analytic element without zeros on $D(a,1^-)$. Since both $u(qx)/u(x)$ and $g_a(qx)/g_a(x)$ are analytic elements on $D(0,1^-)$, without zeros in $D(0,1^-)$, the same is true for $h_a(qx)/h_a(x)$. This proves that $h_a(qx)/h_a(x)$ is an analytic element on $D(0,1^-) \cup D(a,1^-)$, with no zeros in $D(a,1^-)$.

We conclude that $q^{-m} g_a(qx)/g_a(x)$ is the singular factor of $v(x)$ with respect to $D(a,1^-)$. This implies that $g_a(qx)/g_a(x) = q^m$ and hence $g_a(x) = \lambda x^m$, for some $\lambda \in K$ and $m \in \mathbb{Z}$. By the definition of singular factor, $g_a(x)$ does not have any

zero in $\mathbb{P}^1 \smallsetminus D(a, 1^-)$, which implies that $m = 0$. Eventually, $u(x) = \lambda h_a(x)$, which means that $u(x)$ is the restriction of an analytic element on $D(0, 1^-) \cup D(a, 1^-)$, as claimed. □

Now we are ready to prove (6.7). Remark that one implication is a particular case of (6.3). So let us suppose that (21) is solvable and prove that it has a Frobenius structure. The proof is divided into steps:

Step 0. — It is enough to prove that there exists a solution $y(x) \in \mathcal{B}$ of the q-difference equation

$$y(qx) = \prod_{i=1}^{M} \left(1 + (q-1)\mu_i x^i\right) y(x), \tag{22}$$

with $|\mu_i| \leqslant 1$ for any $i = 1, \dots, M$ and $|q-1| < |\pi|$, and a Frobenius endomorphism ϕ such that $y(x)/y(x)^\phi$ is overconvergent.

Proof of Step 0. — Notice that the fact that (21) is solvable implies that $|\mu_i| \leqslant 1$ for any $i = 1, \dots, M$ (*cf.* (5.1)). Moreover (21) has a solution $y(x)$ analytic and bounded over the disk $\mathcal{C}(]1, \infty])$ (*cf.* (1.3)). We have to prove that there exists a positive integer s such that $y(x)/y(x)^\phi \in \mathcal{E}^\dagger$. By iteration we may replace q by an integral power of q and hence suppose that $|q-1| < |\pi|$: this doesn't change the solution $y(x)$, which is still solution of the iterated equation, and the inequality $|\mu_i| \leqslant 1$ is still verified after a reduction of the type (5.1, Step1), since the iterated equation is necessarily solvable. A change of variable of the form $x \mapsto 1/x$ allows to conclude. □

Step 1. — There exists $h > 0$ such that $y(x)^{p^h}$ is overconvergent, hence algebraic over the field E of analytic elements with coefficients in K, (*i.e.* the completion of the field of rational functions $K(x)$ with respect to the norm induced by $\mathcal{E}^\dagger$).

Proof of Step 1. — The solution $y(x)$ can be written as a product of q-exponentials:

$$y(x) = \prod_{i=1}^{M} e_{q^i}\left(\frac{\mu_i}{[i]_q} x^i\right).$$

The analytic function

$$\log y(x) = \sum_{i=1}^{M} L_{q^i}\left(\frac{\mu_i}{[i]_q} x^i\right)$$

converges for

$$|x| < \inf_{i=1,\dots,M} \left|\frac{[i]_q}{(q^i-1)\mu_i}\right|^{1/i} = \inf_{i=1,\dots,M} \left|\frac{1}{(q-1)\mu_i}\right|^{1/i}.$$

Therefore there exists $\varepsilon > 0$ such that $\log y(x)$ is analytic and bounded over the disk $|x| < 1+\varepsilon$. We deduce that there exists and integer $h > 0$ such that $|p^h \log y(x)| < |\pi|$ for any $|x| < 1 + \varepsilon$, and hence, taking the exponential of $\log y(x)^{p^h}$, that $y(x)^{p^h}$ converges for $|x| < 1 + \varepsilon$. □

Step 2. — There exists $s > 0$ such that $y(x)/y(x)^\phi \in E$.

Proof of Step 2. — It follows from Step 1. that the q-difference algebra $E[y(x)]$ is a finite extension of E. Since $y(qx)^i = a(x)^i y(x)^i$ for any $i \in \mathbb{N}$, the q-difference module $E[y(x)] = \oplus_{i=0}^d E\, y(x)^i$, with $d = \deg E[y(x)]/E$, is semisimple. Moreover the Frobenius ϕ_1 stabilizes $E[y(x)]$ as a subalgebra of $\mathcal{B}$ (*cf.* [**C86**, Th. 5.2]). It follows that $E\,\phi_s(y(x))$, $s \in \mathbb{N}$, is a finite family of sub-q-difference modules of rank 1 of $E[y(x)]$, and hence that there exists $s \in \mathbb{N}$ such that $E\,\phi_s(y(x)) = E\,y(x)$ (*cf.* [**C81b**, 10.1] and [**CC96**, proof of th. 2.3.1]). □

Step 3. — $y(x)/y(x)^\phi$ is an analytic element on $\mathcal{C}([0,1])$.

Proof of Step 3. — The analytic element $y(x)/y(x)^\phi$ is solution of the q-difference equation:

$$\frac{u(qx)}{u(x)} = v(x), \text{ with } v(x) = \prod_{i=1}^{M} \frac{1+(q-1)\mu_i x^i}{\prod_{j=0}^{p^s-1}(1+(q-1)\mu_i^{\tau^s} q^{ij} x^{p^s i})} \in K(x).$$

Since $|\mu_i| \leqslant 1$, $v(x)$ is an analytic element on $\mathcal{C}([0,1])$, without zeros and poles in $\mathcal{C}([0,1])$. We deduce that $u(x)$ is an analytic element over $\mathcal{C}([0,1])$ from lemma 6.8. □

Step 4. — $y(x)/y(x)^\phi$ is overconvergent.

Proof of Step 4. — This statement is proved in the second part for q-difference systems of any rank (*cf.* 13.3). We will give here a simplified proof under the assumption $|1-q| < |\pi|^{p^s} M$. This implies that there exists $\eta > |\pi|^{-1}$ such that

$$v(x) = \prod_{i=1}^{M} \frac{1+(q-1)\mu_i x^i}{\prod_{j=0}^{p^s-1}(1+(q-1)\mu_i^{\tau^s} q^{ij} x^{p^s i})}$$

is analytic for $|x| \leqslant \eta$. Let $d_q^n u(x) = g_n(x)u(x)$ for any $n \geqslant 1$, with $g_1(x) = v(x) - 1/(q-1)x$. We have

$$\sup_{|x|\leqslant\eta} |g_1(x)| \leqslant 1.$$

Therefore the recursive relation $g_{n+1}(x) = g_n(qx)g_1(x) + d_q g_n(x)$ implies that

$$\sup_{|x|\leqslant\eta} |g_{n+1}(x)| \leqslant \frac{1}{\eta^n}.$$

In particular $|g_{n+1}(0)| \leqslant \eta^{-n}$. Since $u(x) = 1 + \sum_{n\geqslant 1} \frac{g_n(0)}{[n]_q^!} x^n$, we conclude that $u(x)$ converges for $|x| < \eta|\pi|$, with $\eta|\pi| > 1$. The same is true for $y(x)/y(x)^\phi$ since $u(x)$ and $y(x)/y(x)^\phi$ coincide up to a non zero constant factor. □

6.3. Remark. — Notice that the Step 1 above combined with (5.1) proves following statement:

Corollary 6.9. — *Let $y(qx) = a(x)y(x)$ be a q-difference equation with Frobenius structure. Then, there exists a non negative integer h and a solution $y(x)$ of $y(qx) = a(x)y(x)$ in a finite extension of $\mathcal{E}^\dagger$ such that $y(x)^{p^h} \in \mathcal{E}^\dagger$.*

In the next sections we will show that the solution $y(x)$ is actually in an unramified extension of $\mathcal{E}^\dagger$, which is a much stronger statement.

7. q-deformation of differential equations with strong Frobenius structure

In the previous section we have given a *naïve* definition of the (strong) Frobenius structure for q-difference equations of rank 1. In the higher rank case we are going to consider another Frobenius structure that we call confluent weak Frobenius structure. Proposition 7.3 below establishes the equivalence between the two definitions for q-difference equations of rank 1.

Definition 7.1. — We say that $y(qx) = a(x)y(x)$ has *confluent weak Frobenius structure* if there exists a sequence of $q^{p^{sn}}$-difference equations $y(q^{p^{sn}}x) = a_n(x)y(x)$, with $q^{\tau^s} = q$ and $a_0(x) = a(x)$, such that

1) for any $n \geqslant 1$ the equations

$$y(q^{p^{s(n-1)}}x) = a_{n-1}(x)y(x) \text{ and } y(q^{p^{s(n-1)}}x) = a_n(x)^\phi y(x)$$

are $\mathcal{E}^\dagger$-equivalent via $u_n(x) \in \left(\mathcal{E}^\dagger\right)^*$;

2) the sequences $(a_n(x) - 1)/\left(q^{p^{sn}} - 1\right)x$ and $u_n(x)$ converge in $\mathcal{E}^\dagger$.

Remark 7.2. — In the notation of the previous definition, let $(a_n(x) - 1)/\left(q^{p^{sn}} - 1\right)x \to g(x)$ and $u_n(x) \to u(x)$. Then the differential equation $\dfrac{dy}{dx}(x) = g(x)y(x)$ has a strong Frobenius structure:

$$g(x) + \frac{u'(x)}{u(x)} = p^s x^{p^s - 1} g(x)^\phi.$$

In other words, there exists a discrete family of difference equations

$$d_{q^{p^{sn}}} y(x) = \frac{a_n(x) - 1}{(q^{p^{sn}} - 1)\,x} y(x)$$

with an action of the Frobenius, which "tends" to the differential equation $\dfrac{dy}{dx}(x) = g(x)y(x)$, having a strong Frobenius structure.

Proposition 7.3. — *For a q-difference equation $y(qx) = a(x)y(x)$, with $a(x) \in (\mathcal{E}^\dagger)^*$, it is equivalent to have a strong Frobenius structure or a confluent weak Frobenius structure.*

Proof. — Let us suppose that $y(qx) = a(x)y(x)$ has a Frobenius structure. Then the sequence of iterated difference equations

$$y(q^{p^{sn}}x) = a_n(x)y(x), \quad \text{with } a_n(x) = \prod_{i=0}^{p^{sn}-1} a(q^i x).$$

satisfies condition 1. Since $y(qx) = a(x)y(x)$ is solvable, the equation $y(q^{p^{sn}}x) = a_n(x)y(x)$ is solvable for any $n \geqslant 1$, hence

$$\left|\frac{a_n(x)-1}{(q^{p^{sn}}-1)\,x}\right|_{\mathcal{E}^\dagger} \leqslant 1.$$

This proves that $a_n(x) \to 1$ when $n \to \infty$. Moreover it follows from (6.9) that $y(qx) = a(x)y(x)$ admits a solution $y(x)$ such that $y(x)^{p^h} \in \mathcal{E}^\dagger$, for some $h \in \mathbb{Z}_{\geqslant 0}$. Since

$$\frac{d_{q^{p^{sn}}}\left(y(x)^{p^h}\right)}{y(x)^{p^h}} = \frac{a_n(x)^{p^h}-1}{(q^{p^{sn}}-1)x} = \frac{a_n(x)-1}{(q^{p^{sn}}-1)x} \prod_{\substack{\zeta^{p^h}=1 \\ \zeta\neq 1}} (a_n(x)-\zeta)\,,$$

we obtain

$$\begin{aligned}\frac{\frac{d}{dx}\left(y(x)^{p^h}\right)}{y(x)^{p^h}} &= \lim_{n\to\infty} \frac{d_{q^{p^{sn}}}\left(y(x)^{p^h}\right)}{y(x)^{p^h}} \\ &= \lim_{n\to\infty} \left(\frac{a_n(x)-1}{(q^{p^{sn}}-1)x} \prod_{\substack{\zeta^{p^h}=1 \\ \zeta\neq 1}} (a_n(x)-\zeta)\right) \\ &= p^h \lim_{n\to\infty} \frac{a_n(x)-1}{(q^{p^{sn}}-1)x},\end{aligned}$$

which proves that $(a_n(x)-1)/(q^{p^{sn}}-1)x$ has a limit $g(x)$ in $\mathcal{E}^\dagger$. The existence of the sequence $u_n(x)$ and its limit, as well as the strong Frobenius structure of $\frac{dy}{dx}(x) = g(x)y(x)$, is a consequence of the fact that $\frac{dy}{dx}(x) = g(x)y(x)$ and $y(x) = a(x)y(x)$ have the same solution in some finite extension of $\mathcal{E}^\dagger$.

Suppose we have a sequence of equations satisfying 1. and 2. Then clearly $\frac{dy}{dx}(x) = g(x)y(x)$ has a strong Frobenius structure, hence it is solvable.

We claim that every equation $y_n(q^{p^{sn}}x) = a_n(x)y_n(x)$ is solvable. We set

$$d^k_{q^{p^{sn}}} y_n(x) = h^k_n(x)y_n(x) \quad \text{and} \quad \frac{d^k y}{dx^k}(x) = g_k(x)y(x).$$

Then for any $n \gg 1$ we have $|h^1_n(x)|_{\mathcal{E}^\dagger} = |g_1(x)|_{\mathcal{E}^\dagger}$ and $|q^{p^{sn}}-1| < |\pi|$, which means that $|h^k_n(x)/[k]^!_{q^{p^{sn}}}|_{\mathcal{E}^\dagger} = |g_k(x)/k!|_{\mathcal{E}^\dagger}$. It follows from lemma 3.3 that $y_n(q^{p^{sn}}x) = a_n(x)y_n(x)$ is solvable for any $n \gg 1$. We deduce that $y_n(q^{p^{sn}}x) = a_n(x)y_n(x)$ is solvable for any $n \geqslant 0$ from the inequality (19), describing the action of the Frobenius on the generic radius of convergence.

For any $n \geqslant 0$ consider the decomposition of $a_n(x)$ (*cf.* (5.1), Step 0):

$$a_n(x) = \frac{\lambda_n}{x^{N_n}}\, l_n(x)\, m_n(x),$$

with $\lambda_n \in K$, $N_n \in \mathbb{Z}$, $l_n(x) \in (1 + x\mathcal{B})^*$ and $m_n(x) \in (1 + \frac{1}{x}\mathcal{H}^\dagger)^*$, and the analogous decomposition of $u_n(x)$:

$$u_n(x) = \frac{\lambda'_n}{x^{N'_n}}\, l'_n(x)\, m'_n(x).$$

By performing a gauge transformation we can assume that $N'_n = 0$ for any n. Moreover since $y_n(q^{p^{sn}}x) = a_n(x)y_n(x)$ is solvable for any $n \geqslant 0$, we necessarily have $N_n = 0$ and $\lambda_n \in q^{\mathbb{Z}_p}$ for any $n \geqslant 0$ (*cf.* the proof of (5.1)). Moreover condition 1. and the uniqueness of the above decompositions imply that

$$\begin{cases} \lambda_n^{\tau^s} = \lambda_{n-1}, \\ \dfrac{l'_n(q^{p^{s(n-1)}}x)}{l'_n(x)} l_n(x)^\phi = l_{n-1}(x), \\ \dfrac{m'_n(q^{p^{s(n-1)}}x)}{m'_n(x)} m_n(x)^\phi = m_{n-1}(x), \end{cases} \qquad \text{for any } n \geqslant 1.$$

The first equality means that $y(q^{p^{s(n-1)}}x) = \lambda_n^{\tau^s} y(x)$ and $y(q^{p^{s(n-1)}}x) = \lambda_{n-1}y(x)$ are $\mathcal{E}^\dagger$-equivalent. Since $q^{\tau^s} = q$ and $\lambda_n \in q^{\mathbb{Z}_p}$, λ_n is τ^s-invariant. Hence we have $\lambda_n = \lambda_{n-1} = \lambda_0$. The unicity of the decomposition above shows that λ_0 is the exponent of the limit differential equation $\frac{dy}{dx}(x) = g(x)y(x)$ and hence that $\lambda_o = q^{l_0}$, with $(p^n - 1)l_0 \in \mathbb{Z}$, for some positive integer n. We conclude from theorem 6.4 that $y(qx) = a(x)y(x)$ has a strong Frobenius structure. □

8. The group of isomorphism classes of q-difference equations of rank 1 admitting a Frobenius structure

We know that a solvable differential equation (*resp.* a differential equation with Frobenius structure) of rank one $y' = gy$ defined over $\mathcal{E}^\dagger$ has a solution of the form $x^\alpha u(x)$ (*resp.* $v(x)$), where $\alpha \in \mathbb{Z}_p$ and $u(x)$ (*resp.* $v(x)$) is an element of some finite unramified extension of $\mathcal{E}^\dagger$[(9)] (*cf.* [**Cr87**], [**T98b**], [**Co01**]). The same is true for q-difference equations:

Proposition 8.1. — *Let $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, be a solvable q-difference equation (resp. a q-difference equation with Frobenius structure). Then $y(qx) = a(x)y(x)$ has a solution of the form $x^\alpha u(x)$ (resp. $v(x)$), where $\alpha \in \mathbb{Z}_p$ and $u(x)$ (resp. $v(x)$) is an element of a finite unramified extension of $\mathcal{E}^\dagger$.*

(9) Following [**S68**], we call unramified extension of valued fields an extension of valued fields such that the ramification index is equal to 1 and the extension of the residue fields is separable.

Proof. — We know from (5.1), (6.7) and (6.9) that $y(qx) = a(x)y(x)$ has a solution of the form $x^\alpha u(x)$, where $\alpha \in \mathbb{Z}_p$ and $u(x)$ is an element of an extension of $\mathcal{E}^\dagger$, solution of a q-difference equation with Frobenius structure. Moreover we know that the Frobenius structure of $y(qx) = a(x)y(x)$ forces $\alpha \in \mathbb{Z}/(p^s - 1)$ for some positive integer s, which amounts to saying that $\alpha \in \mathbb{Z}_p \cap \mathbb{Q}$. It follows from (7.3) that $u(x)$ is solution of a differential equation over $\mathcal{E}^\dagger$ with strong Frobenius structure, hence that $u(x)$ (*resp.* $v = x^\alpha u$) is an element of a finite unramified extension of $\mathcal{E}^\dagger$. □

The $\mathcal{E}^\dagger$-equivalence classes of differential equations of rank one $y'(x) = g(x)y(x)$ form a group with respect to addition of the coefficient $g(x)$, and equivalence classes of equations with Frobenius structure form a subgroup, which we denote by $d\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger}$. Note that, for such equations, the space $K.y \in (\mathcal{E}^\dagger)'$ defines a character of the absolute Galois group $G_{k((x))}$, which depends only on the class of the equation, and this provides a homomorphism

$$d\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger} \longrightarrow X_K(G_{k((x))})$$

to the group of K-rational characters of $G_{k((x))}$. If the residue field k is algebraically closed, this is a bijection; the inverse map associates to the character χ the χ-eigenspace in any big enough finite unramified extension $(\mathcal{E}^\dagger)'$ of $\mathcal{E}^\dagger$ (this is an $\mathcal{E}^\dagger$-line), endowed with the natural derivation.

In the same vein, the $\mathcal{E}^\dagger$-equivalence classes of q-difference equations of rank one $y(qx) = a(x)y(x)$ form a group with respect to multiplication of the coefficient $a(x)$. Equations with strong Frobenius structure form a subgroup, which we denote by $\sigma_q\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger}$. Similarly, the space $K.y \in (\mathcal{E}^\dagger)'$ defines a character of the absolute Galois group $G_{k((x))}$, which depends only on the class of the equation, and this provides a homomorphism

$$\sigma_q\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger} \longrightarrow X_K(G_{k((x))})$$

to the character group of $G_{k((x))}$. If the residue field k is algebraically closed, this is a bijection; the inverse map associates to the character χ the χ^{-1}-eigenspace in any big enough finite unramified extension $(\mathcal{E}^\dagger)'$ of $\mathcal{E}^\dagger$ (this is an $\mathcal{E}^\dagger$-line), endowed with the natural σ_q-action.

On the other hand, proposition 7.3 associates by "confluence" to any element of $\sigma_q\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger}$ an element of $d\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger}$, and it is easy to see that they correspond to the same character of $G_{k((x))}$. One thus arrives at the following

Theorem 8.2. — *Let us assume that k is algebraically closed. There are canonical group isomorphisms*

$$\sigma_q\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger} \xrightarrow{\cong} X_K(G_{k((x))}) \xrightarrow{\cong} d\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger},$$

the composite being given by "confluence".

Remark 8.3. — The group of tame characters of $G_{k((x))}$ is canonically isomorphic to $(\mathbb{Z}_p \cap \mathbb{Q})/\mathbb{Z}$. For any $\alpha \in \mathbb{Z}_p \cap \mathbb{Q}$, the corresponding object of $d\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger}$ (*resp.* $\sigma_q\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger}$) is represented by $d/dx + \alpha/x$ (*resp.* $d_q + [\alpha]_q/x$) and depends only on the class of α mod $\mathbb{Z}$.

APPENDICES TO PART I

9. Frobenius structure of $d_q y(x) = \pi_q y(x)$

9.1. q-analog of the Dwork exponential. — Recall that the "Dwork exponential" $\theta(x) = \exp(\pi x - \pi x^p)$ expresses the Frobenius structure of the differential equation $y' = \pi y$, and has radius of convergence > 1.

In the same way one can define an element π_q such that $|\pi_q|$ is the radius of convergence of $e_q(x)$[(10)]. Consider the q-difference equation satisfied by $e_q(\pi_q x)$:

$$y(qx) = a(x)y(x) \quad \text{with } a(x) = (1 + (q-1)\pi_q x). \tag{23}$$

It is defined over $\mathcal{E}^\dagger_{1/x}$ and it has a Frobenius structure (*cf.* proposition 6.7). Equation (23) can be iterated in the following way:

$$y(q^{p^s}x) = a_{p^s}(x)y(x) \quad \text{with } a_{p^s}(x) = a(x)a(qx)\cdots a(q^{p^s-1}x).$$

Then the series $e_q(\pi_q x)^\phi = e_q(\pi_q^{\tau^s} x^{p^s})$ is solution of the *q-difference* equation

$$y(qx) = a_{p^s}(x)^\phi y(x). \tag{24}$$

The fact that (23) has a Frobenius structure means that (23) and (24) are $\mathcal{E}^\dagger_{1/x}$-equivalent, *i.e.,* that

$$\frac{e_q(\pi_q x)}{e_q(\pi_q x)^\phi} = \frac{e_q(\pi_q x)}{e_q(\pi_q^{\tau^s} x^{p^s})} \in \left(\mathcal{E}^\dagger_{1/x}\right)^*.$$

(10) The choice of such an element is not canonical. A possible choice would be the following: let n be the smallest positive integer such that $|q^{p^n} - 1| < |\pi|$; then one can choose π_q such that

$$\pi_q^{p^n} = (-[p]_q)^{p^{n-1}} (-[p]_{q^p})^{p^{n-2}} \cdots \left(-[p]_{q^{p^{n-1}}}\right) \pi.$$

In other words, we have chosen $\pi_{q^{p^n}} = \pi$ and we have set recursively $\pi^p_{q^{p^i}} = -[p]_{q^{p^i}} \pi_{q^{p^{i+1}}}$, for any $i = 0, \ldots, n-1$. This is a good choice, in fact if $|q^{p^n} - 1| < |\pi|$ then $|[p]_{q^{p^n}}| = |p|$ and hence

$$\begin{aligned} |\pi_q| &= \textstyle\prod_{i=0}^{n-1} \left|[p]_{q^{p^i}}\right|^{1/p^{i+1}} |\pi|^{1/p^n} \\ &= \textstyle\prod_{i=0}^{n-1} \left|[p]_{q^{p^i}}\right|^{1/p^{i+1}} |p\pi|^{1/p^{n+1}} \\ &= \textstyle\prod_{i=0}^{n} \left|[p]_{q^{p^i}}\right|^{1/p^{i+1}} |\pi|^{1/p^{n+1}}. \end{aligned}$$

This proves that $|\pi_q|$ is equal to the radius of convergence of $e_q(x)$ (*cf.* (2.1)).

Since τ is a Frobenius automorphism such that τ^s fixes q, the definition of π_q implies that $|\pi_q^\tau - \pi_q| < |\pi_q|$. Therefore the existence of a strong Frobenius structure for (23) is equivalent to the overconvergence of

$$\frac{e_q(\pi_q x)}{e_q(\pi_q x^{p^s})}.$$

A reasonable analog of Dwork's exponential $\theta(x)$ (from the viewpoint of strong Frobenius structures[(11)]) would be the following series

$$\frac{e_q(\pi_q x)}{e_q(\pi_q x^p)}.$$

Unfortunately we are not able to prove that this series is overconvergent, unless $|1-q| < |\pi|$. In fact if $|1-q| < |\pi|$ its overconvergence is a immediate consequence of the overconvergence of the Dwork's exponential $\theta(x)$ and (2.5).

9.2. q-analog of Artin-Hasse exponential series. — Another reasonable q-analog of the Dwork exponential, from the viewpoint of confluent weak Frobenius structures, is the modified series

$$\theta_q(x) = \frac{e_q(\pi_q x)}{e_{q^p}(\pi_{q^p} x^p)}.$$

If $|q-1| < |\pi|$, the overconvegence of $\theta_q(x)$ immediately follows from the overconvergence of $\theta(x)$ and (2.5). To prove the overconvergence of $\theta_q(x)$ under more general hypotheses, one could try to construct an analog of the Artin-Hasse series, but our proof, which is quite similar to the one in the differential case, works only under restrictive assumptions. Anyway we are going to sketch it.

Proposition 9.1. — *Suppose that there exists $\widetilde{q} \in \mathbb{Q}_p$ such that $\widetilde{q}^p = q$. Let*

$$E_q(x) = \prod_{i \geqslant 0} e_{q^{p^i}}\left(\frac{x^{p^i}}{[p^i]_q}\right) = e_q(x) e_{q^p}\left(\frac{x^p}{[p]_q}\right) e_{q^{p^2}}\left(\frac{x^{p^2}}{[p^2]_{q^p}[p]_q}\right)\cdots$$

Then $E_q(x)$ is analytic and bounded by 1 on the disk $D(0,1^-)$.

Sketch of the proof

Step 1. — We deduce from the formula

$$e_{\widetilde{q}}([p]_{\widetilde{q}} x) = e_q(x) e_q(\widetilde{q}x) \cdots e_q(\widetilde{q}^{p-1}x)$$

that

$$\frac{E_q(x) E_q(\widetilde{q}x) \cdots E_q(\widetilde{q}^{p-1}x)}{E_q(x^p)} = e_{\widetilde{q}}([p]_{\widetilde{q}} x) \in 1 + [p]_{\widetilde{q}} x \mathcal{O}_K[\![x]\!].$$

The hypothesis $\widetilde{q} \in \mathbb{Q}_p$ is used here to prove that $[p]_{\widetilde{q}}^n / [n]_{\widetilde{q}}^!$ is a p-adic integer. Such an estimate seems to be difficult when $|\widetilde{q}-1| \geqslant |\pi|$.

(11) not from the viewpoint of confluent weak Frobenius structures, *cf.* § 9.2, the next appendix, and 12.12.

Step 2 (q-analog of the Dieudonné's theorem). — One proves that for any $f(x) \in 1 + xK[\![x]\!]$ we have

$$f(x) \in 1 + \mathcal{O}_K[\![x]\!] \iff \frac{f(x)f(\widetilde{q}x)\cdots f(\widetilde{q}^{p-1}x)}{f^\tau(x^p)} \in 1 + [p]_{\widetilde{q}} x \mathcal{O}_K[\![x]\!].$$

One concludes the proof remembering that $q^\tau = q$, since $q \in \mathbb{Q}_p$, and hence that $E_q^\tau(x^p) = E_q(x^p)$. □

Corollary 9.2. — *The series*

$$e_q(\pi x)e_{q^p}(-\pi x^p) = \frac{e_q(\pi x)}{e_{q^{-p}}(\pi x^p)}$$

is overconvergent.

To prove the corollary one follows the proof in the differential case. One has to take into account that $q \in \mathbb{Q}_p$ implies $|[n]_q| = n$ for any integer n. Moreover $|p\pi - [p]_q\pi| < |\pi|$ implies that $e_{q^p}(\pi^p x^p/[p]_q)\, e_{q^p}(-\pi x^p)^{-1}$ is overconvergent (*cf.* corollary 2.2).

10. p-adic q-exponential and Koblitz' Gamma function

In this appendix, we leave our local framework and outline some *global aspects* of q-difference equations of rank one with overconvergent Frobenius structure. More precisely, we sketch a q-analog of part of Dwork's paper [**D83**], in which he related Morita's p-adic Gamma function Γ_p to the Frobenius structure of his exponential modules.

We shall recover in this way a function $\Gamma_{p,q}$ which is a p-adic analog of Jackson's Γ_q function as well as a q-analog of Morita's Γ_p function, and which had been previously introduced by N. Koblitz [**Ko80**][**Ko82b**] using Morita's approach[(12)].

10.1. Dwork's operator ψ. — In this appendix, the singular disk is the unit disk at infinity, so that the relevant ring of overconvergent functions is $\mathcal{H}^\dagger_{1/x} = \cup_{\varepsilon>0}\mathcal{A}([0, 1+\varepsilon[)$. We work over $K = \mathbb{Q}_p(\pi)$ and with $\tau = \mathrm{id}$ for simplicity.

Following Dwork, let us introduce the operator ψ defined by

$$\psi\Big(\sum a_n x^n\Big) = \sum a_{pn} x^n.$$

This is a left inverse of the Frobenius operator induced by $\phi : x \mapsto x^p$. It acts on $\mathcal{H}^\dagger$ and intertwines $\delta_q = xd_q$ and xd_{q^p} up to multiplication by $[p]_q$:

$$[p]_q \delta_{q^p}\psi = \psi\delta_q.$$

More generally, for any $a \in \mathbb{Z}_p$, ψ sends the space $x^a\mathcal{H}^\dagger$ to $x^b\mathcal{H}^\dagger$ where b is the so-called *successor* of $a \in \mathbb{Z}_p$, *i.e.,* the unique p-adic integer b such that $pb - a \in \mathbb{Z} \cap [0, p[$.

(12) In fact, one of us defined $\Gamma_{p,q}$ using q-exponential modules - à la Dwork -, and was told afterwards by F. Sullivan that this function had been defined earlier - à la Morita - by Koblitz.

10.2. q-Exponential modules. — The simplest examples of Dwork's "exponential modules" are of the form $x^a e^{\pi x}\mathcal{H}^\dagger_{1/x}$. They lack strong Frobenius structure (except when a is a rational number) but have a weak Frobenius structure: if b denotes the successor of a, $x^a e^{\pi x}\mathcal{H}^\dagger_{1/x}$ is isomorphic to the Frobenius inverse image of $x^b e^{\pi x}\mathcal{H}^\dagger_{1/x}$ (due to the overconvergence of Dwork's exponential).

A similar phenomenon occurs in the q-difference case. Proposition 2.1 suggests to replace $e^{\pi x}$ by $e_q(\pi_q x)$ for some constant π_q with the following property: defining the sequence $(\pi_q^{p^n})$ by $\pi_{q^{p^{n+1}}} = -\pi^p_{q^{p^n}}/[p]_{q^{p^n}}$, we assume that $|\pi_{q^{p^n}}| = |\pi|$ for n big enough[(13)]. Then proposition 2.1 shows that $e_q(\pi_q x)$ has radius of convergence 1.

However, we shall have to assume that $|q-1| < p^{-1/p-1}$ in order to ensure that the series

$$\theta_q(x) = \frac{e_q(\pi_q x)}{e_{q^p}(\pi_{q^p} x^p)}$$

is overconvergent. The equation $\pi_q^p = -[p]_q \pi_{q^p}$ is irrelevant at this point but will be used in the sequel.

We set

$$f_{q,a} = x^a e_q(\pi_q x),$$

and consider the $\mathcal{H}^\dagger_{1/x}$-module $f_{q,a}\mathcal{H}^\dagger_{1/x}$ equipped with the natural action of $\delta_q = x d_q$. A simple computation shows that $\delta_q f_{q,a} = (q^a \pi_q x + [a]_q) f_{q,a}$. Let us write the classes modulo $\operatorname{Im}\delta_q$ in braces. So,

$$\{f_{q,a+1}\} = -q^{-a}\pi_q^{-1}[a]_q\{f_{q,a}\} \text{ in } \frac{f_{q,a}\mathcal{H}^\dagger_{1/x}}{\delta_q\left(f_{q,a}\mathcal{H}^\dagger_{1/x}\right)},$$

from which one deduces that this cokernel has dimension 1 over K and is generated by $\{f_{q,a}\}$.

If b is the successor of a, then

$$\frac{f_{q,a}(x)}{f_{q^p,b}(x^p)} = \frac{e_q(\pi_q x)}{e_{q^p}(\pi_q x^p)}$$

is overconvergent, which expresses the fact that $f_{q,a}\mathcal{H}^\dagger_{1/x}$ is isomorphic to the Frobenius inverse image of $f_{q^p,b}\mathcal{H}^\dagger_{1/x}$. The same argument as in the differential case shows that ψ extends to a mapping $f_{q,a}\mathcal{H}^\dagger_{1/x} \to f_{q^p,b}\mathcal{H}^\dagger_{1/x}$ by the following formula

$$\psi(f_{q,a}.f) = f_{q^p,b}\psi(x^{a-pb}\theta_q(x)f).$$

(13) It is easy to see that such a π_q exists at least after replacing K by a finite extension. On the other hand, there does not seem to be a "uniform", canonical choice for such π_q, and one can actually show that the sequence $(\pi_{q^{p^n}})$ never converges.

10.3. $\Gamma_{p,q}$**.** — Since ψ intertwines δ_q and δ_{q^p} up to a factor $[p]_q$, it passes to the "cohomology":

$$\psi : \frac{f_{q,a}\mathcal{H}^\dagger_{1/x}}{\delta_q\big(f_{q,a}\mathcal{H}^\dagger_{1/x}\big)} \longrightarrow \frac{f_{q^p,b}\mathcal{H}^\dagger_{1/x}}{\delta_{q^p}\big(f_{q^p,b}\mathcal{H}^\dagger_{1/x}\big)}$$

and one can define an element $\Gamma_{p,q}(a) \in K^*$ by

$$\psi\{f_{q,a}\} = \pi_q^{pb-a}\Gamma_{p,q}(a)\{f_{q^p,b}\}.$$

One then computes:

– $\psi\{f_{q,0}\} = \Gamma_{p,q}(0)\{f_{q^p,0}\} = \{f_{q^p,0}\psi(\theta_q(x))\} = \{f_{q^p,0}\}$.

– If a is a unit, b is a successor of $a+1$, and one has

$$\begin{aligned}\Gamma_{p,q}(a+1)&\{f_{q^p,b}\}\\ &= \pi_q^{a-pb+1}\psi\{f_{q,a+1}\} = -q^{-a}\pi_q^{a-pb}[a]_q\psi\{f_{q,a}\} = -q^{-a}[a]_q\Gamma_{p,q}(a)\{f_{q^p,b}\}\\ &= [-a]_q\Gamma_{p,q}(a)\{f_{q^p,b}\}.\end{aligned}$$

– If a is divisible by p, then $a = pb$ and $b+1$ is a successor of $a+1$, whence:

$$\begin{aligned}\Gamma_{p,q}(a+1)&\{f_{q^p,b+1}\}\\ &= \pi_q^{1-p}\psi\{f_{q,a+1}] = -\pi_q^{-p}q^{-a}[a]_q\psi\{f_{q,a}\} = -\pi_q^{-p}q^{-a}[a]_q\Gamma_{p,q}(a)\{f_{q^p,b}\}\\ &= -\Gamma_{p,q}(a)\{f_{q^p,b+1}\},\end{aligned}$$

where

$$-\frac{\pi_q^{-p}\pi_{q^p}[a]_q}{[b]_{q^p}} = -\pi_q^{-p}\pi_{q^p}[p]_q = 1$$

by assumption.

Therefore, as a function of $a \in \mathbb{Z}_p$, $\Gamma_{p,q}(a)$ satisfies the functional equations of Koblitz' function of the same name:

$$\Gamma_{p,q}(0) = 1, \quad \frac{\Gamma_{p,q}(\alpha+1)}{\Gamma_{p,q}(a)} = \begin{cases}[-a]_q & \text{if } a \text{ is a unit,}\\ -1 & \text{if } |\alpha|_p < 1.\end{cases}$$

In order to check that $\Gamma_{p,q}(a)$ is Koblitz' function, it remains to prove its continuity. In fact, we shall prove the so-called "Boyarsky principle" for q-exponential modules $f_{q,a}\mathcal{H}^\dagger_{1/x}$, *i.e.,* the analyticity of $\Gamma_{p,q}$ on each disk $D(-k,|p|^+)$, $k = 0,1,\dots,p-1$.

Let us expand $\theta_q(x) = \sum \varepsilon_n x^n$. For every $a \in \mathbb{Z}_p \cap D(-k,1^-)$, one has

$$\begin{aligned}\psi\{f_{a,q}\} &= \pi_q^k\Gamma_{p,q}(a)\{f_{q^p,b}\} = \{f_{q^p,b}\psi(x^{-k}\theta_q(x))\} = \{\textstyle\sum \varepsilon_{pn+k}f_{q^p,b+n}\}\\ &= \{\textstyle\sum \varepsilon_{pn+k}(-1)^n q^{-npb-n(n-1)p/2}\pi_{q^p}^{-n}[b]_{q^p}[b+1]_{q^p}\dots[b+n-1]_{q^p}f_{q^p,b}\}\end{aligned}$$

hence

$$\begin{aligned}\Gamma_{p,q}(a) = \sum \varepsilon_{pn+k}(-1)^n q^{(-n(a+k)-n(n-1)p)/2}&\pi_{q^p}^{-n-k}\\ &\cdot \left[\frac{a+k}{p}\right]_{q^p}\left[\frac{a+k}{p}+1\right]_{q^p}\dots\left[\frac{a+k}{p}+n-1\right]_{q^p}\end{aligned}$$

which is analytic in $D(-k,|p|^+)$, since $|1-q|<|\pi|$, as follows from the expansion

$$[c]_{q^p} = -1 + \sum_{m\geqslant 0} \binom{c}{m} (q^p-1)^{m-1}.$$

It would be interesting to extend this approach "à la Boyarsky" to other q-difference modules with p-adic parameters, notably to Koblitz' p-adic hypergeometric q-difference equations [**Ko82b**].

PART II
HIGHER RANK

11. Preliminaries: unramified extensions of $\mathcal{E}^\dagger$

In this part, it will be essential to deal not only with $\mathcal{E}^\dagger_x$ but also with its unramified extensions as well. Whereas any such extension is of the form $\mathcal{E}^\dagger_{x'}$ for some new variable x', the q-difference operator d_q and Frobenius fail to act on x' as simply as they do on x. Thus the relatively down-to-earth methods of part I do not apply to the case of $\mathcal{E}^\dagger_{x'}$. In this preliminary section, we give some tools to handle this issue.

11.1. Topologies. — Let K be a complete non-archimedean field of characteristic 0, with residue field k of characteristic $p>0$. We keep the notation of part I for rings of analytic functions. For any interval I, the ring $\mathcal{B}(I)$ of bounded analytic functions on the annulus $\mathcal{C}(I)$ is endowed with the topology given by the sup-norm $|\ |_I$, for which it is complete. It is also endowed with a coarser topology (strictly coarser if I is not closed): the Frechet topology defined by the norms $|\ |_J$ for all closed $J\subset I$. Its Frechet completion is $\mathcal{A}(I)$.

Similarly, the topology of the Robba ring $\mathcal{R}$ is the finest for which the injections $\mathcal{A}(]1-\varepsilon,1[)\hookrightarrow\mathcal{R}$, are continuous, *cf. e.g.,* [**Cr98**, 4], and the induced topology on $\mathcal{E}^\dagger$ is coarser than the topology defined by the p-adic norm $|\ |=|\ |_{\mathcal{E}^\dagger}$ [(14)].

The truncation $\gamma_{x>0}$ in positive degrees is continuous, both as an operator $\mathcal{R}\to x\mathcal{A}([0,1[)$ and as an operator $\mathcal{E}^\dagger\to x\mathcal{B}([0,1[)$.

If K is discretely valued and J is closed, $\mathcal{A}(J)$ is a Banach space, and the Banach norm coincides with the sup-norm on $\mathcal{C}(J)$ if the endpoints of J lie in $\sqrt{|K^*|}$ ([**Cr98**, 4.2]).

11.2. Unramified extensions and absolute values. — We assume in the rest of this section that k is perfect and that K is discretely valued. Then $\mathcal{E}^\dagger_{K,x}$ is the field of fractions of the henselian ring $\mathcal{O}_{\mathcal{E}^\dagger_{K,x}}$ with residue field $k((x))$. Any finite unramified

(14) In [**Ke**], $\mathcal{R}$ is denoted by $\Gamma^{k((x))}_{\mathrm{an,con}}$ and $\mathcal{E}^\dagger$ by $\Gamma^{k((x))}_{\mathrm{con}}[1/p]$.

extension of $\mathcal{E}^\dagger_{K,x}$ is of the form $\mathcal{E}^\dagger_{K',x'}$, for a finite unramified extension K'/K and a new variable x' algebraic over $\mathcal{E}^\dagger_{K,x}$ [**Ma95**].

Among the finite étale extensions of the Robba ring $\mathcal{R}_{K,x}$, those of the form $\mathcal{R}_{K',x'} = \mathcal{R}_{K,x} \otimes_{\mathcal{E}^\dagger_{K,x}} \mathcal{E}^\dagger_{K',x'}$ play a distinguished role in the local theory of p-adic differential equations (as was first emphasized in the work of R. Crew [**Cr98**]). We shall see that they play a similar role in the local theory of p-adic q-difference equations. The most suitable way to deal with all these extensions is to embed them (up to isomorphism) in a fixed canonical "big Robba ring" $\widetilde{\mathcal{R}}$, as was done by K. Kedlaya [**Ke**][(15)] (the field of constants of $\widetilde{\mathcal{R}}$ is $\widetilde{K} := K \otimes_{W(k)} W(\overline{k})$).

For any element $y \in \mathcal{R}_{K,x}$ and any $r \in]0,1[$, one defines $|y|_{|x|=r}$ (also written, abusively, $|y|_r$) to be the sup-norm of y on the circle $|x| = r$ if $y \in \mathcal{A}([r,1[)$, ∞ otherwise. Similarly for $\mathcal{R}_{K',x'}$. Note that if $y \in \mathcal{E}^\dagger_{K,x}$, $|y|_r = |y|$ for any r close enough to 1.

The drawback of this notion is its dependence on the choice of x'. Kedlaya has shown how to bypass it by introducing (partially defined) canonical absolute values $|\,|_{r,\mathrm{can}}$ on $\widetilde{\mathcal{R}}$[(16)]. He proves that for any $y \in \mathcal{R}_{K',x'} \subset \widetilde{\mathcal{R}}$ and any r close enough to 1, $|y|_{r,\mathrm{can}}$ coincides with the naive absolute value $|y|_{|x'|=r}$ [**Ke**, 3.7].

On the other hand, let τ denote a Frobenius endomorphism of K. It extends uniquely to $\widetilde{K}$. Let us fix a positive integer s and consider the endomorphism $\phi_s = \phi$ of $\mathcal{R}_{K,x}$ given by $\phi(\sum a_n x^n) = \sum \tau^s(a_n) x^{p^s n}$. Then ϕ extends canonically to a τ^s-semilinear endomorphism of $\widetilde{\mathcal{R}}$, and one has the formula $|y^\phi|_{r^{1/p^s},\mathrm{can}} = |y|_{r,\mathrm{can}}$ [**Ke**, following prop. 3.11]. Hence for r close enough to 1,

$$|y^\phi|_{|x'|=r^{1/p^s}} = |y|_{|x'|=r}.$$

11.3. Good coordinates. — Because $\mathcal{E}^\dagger_x$ is henselian, ϕ lifts uniquely to any finite unramified extension of $\mathcal{E}^\dagger_x$. By definition $\phi(x) = x^p$. For a finite unramified extension of $\mathcal{E}^\dagger_x$ with tame residual extension of degree say n, one can take $x' = x^{1/n}$ so that again $\phi(x') = (x')^p$. However, in the case of a wild totally ramified residual extension (say of degree p^m), it is not possible to choose x' such that $\phi(x') = (x')^p$, although $\phi(x') \equiv (x')^p \bmod p$.

In order to get some control on the Frobenius action in the course of computations, it is important to choose x' carefully. For this purpose, one can use so-called Katz-Gabber extensions. According to Katz-Gabber [**Ka86**, 1.4.2], one can choose x' in such a way that x' is algebraic of degree p^m over $k(x)$ and such that the extension $k(x,x')/k(x)$ defines (via x) a finite covering $\overline{f}_k : \overline{C}_k \to \mathbf{P}^1_k$ unramified above $\mathbf{P}^1 \smallsetminus 0$ and totally ramified at $x' = x = 0$. We denote by $f_k : C_k \to \mathbf{P}^1 \smallsetminus 0$ the restriction

(15) denoted by $\Gamma^{\overline{k((x))}}_{an,con}$ or $\Gamma^{alg}_{an,con}$ in *loc. cit.*

(16) Those attached to his partial valuations $w_{\frac{\log r^{-1}}{\log p}}$.

of $\overline{f}_k$ above $\mathbf{P}^1 \smallsetminus 0$. Extending k if necessary, we assume that one of points, say P_k, of $\overline{C}_k$ with $x' = \infty$ is k-rational.

Let $\overline{f} : \overline{C} \to \mathbf{P}^1_{\mathcal{O}_K}$ be a lifting of this finite covering, with $\overline{C}$ projective and flat over $\mathcal{O}_K$[17]. We denote again by the letter x (*resp.* x') a lifting of the first (*resp.* second) coordinate to $\overline{C}$ (we refer to such a coordinate x' as a "*good coordinate*" for this finite extension of $\mathcal{E}^\dagger_x$).

We denote by $f : C \to \mathbf{P}^1 \smallsetminus 0$ the restriction of $\overline{f}$ above $\mathbf{P}^1 \smallsetminus 0$. The finite covering $f_K : \overline{C}_K \to \mathbf{P}^1_K \smallsetminus 0$ is unramified above the disk $D([1, \infty])$.

Let $\mathcal{O}^\dagger(C)$ denote the (p-adic) weak completion of the affine algebra $\mathcal{O}(C)$. Via f^*, this an étale extension of $\mathcal{O}_{\mathcal{H}^\dagger_x}$. It is known that $\mathcal{O}^\dagger(C)$ is henselian (*cf. e.g.,* [**E02**, th. 3]), from which it follows that the endomorphism ϕ of $\mathcal{E}^\dagger_{x'}$ preserves $\mathcal{O}^\dagger(C)$. On the other hand, $\mathcal{O}_K[[1/x]]$ and $\mathcal{O}_K[[1/x']]$ are both ϕ-equivariantly isomorphic to the completion of $\mathcal{O}(C)$ at P_k.

We now fix an element $q \in K^*$ (not a root of unity) satisfying $|q| = 1$, and fixed under τ^s. The homothety $x \mapsto qx$ extends uniquely to an automorphism of each of the topological K-algebras $\mathcal{E}^\dagger_x, \mathcal{E}^\dagger_{x'}, \mathcal{R}_x, \mathcal{R}_{x'}, \widetilde{\mathcal{R}}$, and also of $\mathcal{O}^\dagger(C)$ (the latter ring being henselian). We denote all these extensions somewhat abusively by the symbol σ_q. Of course, in general $\sigma_q(x')$ is not proportional to x'.

The assumption $\tau^s(q) = q$ ensures that σ_q and ϕ generate a twisted polynomial ring $K[\sigma_q, \phi]$ of endomorphisms of any of the previous rings:

$$\sigma_q \, \phi = \phi \, \sigma_q^{p^s} = \phi \, \sigma_{q^{p^s}}.$$

12. q-difference modules and Frobenius structures

12.1. σ-modules. — Let R be a commutative integral $\mathbb{Q}$-algebra and let σ be an injective endomorphism of R.

Definition 12.1. — A *σ-module*, or *σ-difference module* over R is a free R-module M of finite rank equipped with an R-linear *isomorphism*

$$\Sigma : \sigma_* M := M \otimes_{R,\sigma} R \longrightarrow M.$$

The σ-modules over R form a category in an obvious way[18], which is linear over the fixed ring of R under σ.

In terms of a basis $(m_i)_{i=1,\dots,\mu}$ of M, the datum of Σ is equivalent to the datum of an invertible matrix, which it is convenient to define as the *inverse* $A \in GL_\mu(R)$ of the matrix of Σ. With this convention, the σ-difference matrix system

$$\sigma(Y) = AY$$

is equivalent to condition that $\sum_i Y_{ij} m_i$ is fixed by Σ.

[17] For a conceptual proof of the existence of such a lifting, *cf. e.g.,* [**Cr98**, 8.3]; here, it suffices to lift to characteristic 0 an equation relating x and x'.

[18] and even a monoïdal symmetric rigid category

If (m'_j) is another basis, with $m'_j = \sum (H^{-1})_{ij} m_i$, then the inverse A' of the matrix of Σ in this new basis is $H^\sigma A H^{-1}$.

Remark 12.2. — σ-modules can also be understood in terms of non-commutative connections, *cf.* [**A01**].

In the sequel, we consider the case where R is $\mathcal{E}_x^\dagger$ or $\mathcal{R}_x$, or (if k is perfect and K is discretely valued) one of their finite extensions $\mathcal{E}_{x'}^\dagger, \mathcal{R}_{x'}$. In such cases, we consider σ-modules as topological modules. When $\sigma = \sigma_q$, the notion of σ-module amounts to that of q-difference module over R, our main object of study. When $\sigma = \phi$, we denote the isomorphism defining the structure of ϕ-module by

$$\Phi : \phi_* M := M \otimes_{R,\phi} R \longrightarrow M.$$

This example is studied at length in [**Ke**].

Remark 12.3. — If K is discretely valued, $\mathcal{E}^\dagger$ is a field and $\mathcal{R}$ is a *Bézout ring*: every finitely generated ideal is principal (as was remarked by several authors (*cf. e.g.,* [**Cr98**, 4.9]), this follows from Lazard's theory of principal parts). Any Bézout ring R is integrally closed and coherent, and any finitely generated R-module is a direct sum of its torsion submodule and of a free module, *cf. e.g.,* [**Cr98**, 4.9]. Thus, in that case, one could replace "free" by "locally free" in the definition of σ-modules.

If R is Bézout, and the subring of σ_q-constants is a field K, then the category of σ_q-modules is tannakian over K.

12.2. Strong Frobenius structure. — We assume that q is fixed under τ^s. We shall be interested in situations where the given module M is at the same time a σ_q-module and a ϕ-module.

Definition 12.4. — A *strong Frobenius structure* on a q-difference module (M, Σ_q) over R is the datum of a structure of ϕ-module Φ on M, Σ_q and Φ being subject to the following "integrability condition":

$$\Sigma_q \, \Phi = \Phi \, (\Sigma_q)^{p^s},$$

to be understood in the sense that the following diagram commutes

$$\begin{array}{ccc} \phi_* \sigma_{q^{p^s},*} M = \sigma_{q,*} \phi_* M & \xrightarrow{\phi_* \Sigma_{q^{p^s}} = \phi_* \Sigma_q^{p^s}} & \phi_* M \\ \sigma_{q,*}\Phi \downarrow & & \downarrow \Phi \\ \sigma_{q,*} M & \xrightarrow{\Sigma_q} & M. \end{array}$$

We also say that (M, Σ_q, Φ) is a σ_q-ϕ-module over R.

If Σ_q and Φ are represented by the matrices A_q^{-1} and F^{-1} respectively in a given basis (m_i) of M, the integrability condition translates into

$$F^{\sigma_q} A_q = A_{q^{p^s}}^{\phi} F, \quad \text{where } A_{q^{p^s}} := A_q^{\sigma_{q^s}} A_q^{\sigma_{q^{s-1}}} \cdots A_q.$$

Remark 12.5. — Let us consider the q-difference system $Y^{\sigma_q} = AY$ attached to $(M, \Sigma_q, (m_i))$. Then $Y^{\phi} = FY$ satisfies the q^{1/p^s}-difference system with matrix A_q^{ϕ}, which can be turned, after p^s iterations, into the q-difference system $(Y^{\phi})^{\sigma_q} = A_{q^{p^s}} Y^{\phi}$.

We denote by $\sigma_q\text{-}\,Mod_R^{\phi}$ (*resp.* $\sigma_q\text{-}\,Mod_R^{(\phi)}$) the category of σ_q–modules over R endowed with a specified (*resp.* an unspecified) strong Frobenius structure. If R is Bézout and if the subring of σ_q-constants is a field K, then $\sigma_q\text{-}\,Mod_R^{(\phi)}$ is tannakian over K while $\sigma_q\text{-}\,Mod_R^{\phi}$ is tannakian over the field K_s of elements of K fixed under τ^s.

For any ring homomorphism $R \to R'$ compatible with σ_q, τ[19], and for any replacement of ϕ by some power ϕ^i, there is an obvious functor between the corresponding categories of q-difference modules with strong Frobenius structures.

Example 12.6 **(q-Exponential)**. — In the case of a q-difference module of rank $\mu = 1$, identified, after the choice of a basis, with a difference equation $y(qx) = a(x)y(x)$ with $a(x) \in R^*$, the strong Frobenius structure is unique if it exists and is given by y^{ϕ}/y provided it belongs to R (otherwise, there is no strong Frobenius structure).

This is the case of the q-difference equation satisfied by $e_q(\pi_q x)$ as in appendix 9.

Example 12.7 **(q-Logarithm)**. — The equation $d_q \ell_q(x) = 1/x$, or equivalently

$$\ell_q(qx) = \ell_q(x) + q - 1$$

has the obvious solution ("q-logarithm")

$$\ell_q(x) = \frac{q-1}{\log q} \log x.$$

One has

$$\ell_q(x)^{\phi} = p^s \ell_q(x).$$

Consider the linear system of order 2

$$\sigma_q Y(x) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} Y(x), \tag{25}$$

having $\left(\begin{smallmatrix} q-1 \\ \ell_q(x) \end{smallmatrix}\right)$ as solution. Then $\left(\begin{smallmatrix} q-1 \\ \ell_q(x)^{\phi} \end{smallmatrix}\right) = \left(\begin{smallmatrix} q-1 \\ p^s \ell_q(x) \end{smallmatrix}\right)$ is a solution of

$$\sigma_q Y(x) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{p^s} Y(x). \tag{26}$$

[19] satisfying the usual relation $\sigma_q \, \phi = \phi \, \sigma_q^{p^s}$

Since

$$\begin{pmatrix} q-1 \\ p^s \ell_q(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^s \end{pmatrix} \begin{pmatrix} q-1 \\ \ell_q(x) \end{pmatrix}$$

we conclude that (25) and (26) are $\mathcal{E}^\dagger$-equivalent.

The differential Galois group is obviously the additive group $\mathbb{G}_a$, and one has a canonical fully faithful $\otimes$-functor

$$Rep_K \mathbb{G}_a \longrightarrow \sigma_q\text{-}\, Mod^{(\phi)}_{\mathcal{R}_x}$$

which sends the standard two-dimensional representation to the q-difference module attached to ℓ_q.

***Example 12.8* (q-Difference modules arising from Galois representations)**

Let us assume that k is perfect and K is discretely valued. Let $k'((x'))$ be a finite Galois extension of $k((x))$, and let $\mathcal{E}^\dagger_{K',x'}$ be the corresponding finite unramified Galois extension of $\mathcal{E}^\dagger_{K,x}$. Then $\mathcal{E}^\dagger_{K',x'}$ has a canonical structure of σ_q-ϕ-module over $\mathcal{E}^\dagger_{K,x}$, given by the canonical extension of σ_q and ϕ to $\mathcal{E}^\dagger_{K',x'}$.

We denote by $d\text{-}\, Mod^{\phi}_R$ (*resp.* $d\text{-}\, Mod^{(\phi)}_R$) the category of differential modules (free) over R endowed with a specified (*resp.* an unspecified) strong Frobenius structure. If R is Bézout and if the subring of σ_q-constants is a field K, then $d\text{-}\, Mod^{(\phi)}_R$ is tannakian over K while $d\text{-}\, Mod^{\phi}_R$ is tannakian over the field K_s of elements of K fixed under τ^s.

Let $G_{k((x))}$ be the absolute Galois group of $k((x))$. There are well-known $\otimes$-functors

$$D^\phi_d : Rep_{K_s} G_{k((x))} \longrightarrow d\text{-}\, Mod^{\phi}_{\mathcal{R}_x},$$
$$D^{(\phi)}_d : Rep_{K} G_{k((x))} \longrightarrow d\text{-}\, Mod^{(\phi)}_{\mathcal{R}_x},$$

given by

$$V \longmapsto (V \otimes \mathcal{R}_{K',x'})^{G_{k((x))}}$$

for suitable $\mathcal{R}_{K',x'}$ (depending on V), *cf.* [**T98b**] (here $G_{k((x))}$ is considered as a constant profinite group-scheme, and representations are understood in the algebraic sense, *i.e.,* as representations of a group scheme; in particular, representations of $G_{k((x))}$ in this sense have finite image).

This carries over to the q-difference case. For any $V \in Rep_K G_{k((x))}$ of dimension μ, there is a finite unramified extension $\mathcal{E}^\dagger_{K',x'}$ (say in $\widetilde{\mathcal{R}}$) such that $(V \otimes_K \mathcal{E}^\dagger_{K',x'})^{G_{k((x))}}$ (*resp.* $(V \otimes_K \mathcal{R}_{K',x'})^{G_{k((x))}}$) is a $\mathcal{E}^\dagger_{K,x}$-module (*resp.* $\mathcal{R}_{K,x}$-module) M of rank μ. This module inherits a natural q-difference structure and strong Frobenius structure from $\mathcal{E}^\dagger_{K',x'}$ (*resp.* $\mathcal{R}_{K',x'}$). If $K = K'$ (*e.g.,* if k is algebraically closed), one recovers V from M by the following recipe "à la Fontaine": $V \cong (M \otimes \mathcal{E}^\dagger_{K',x'})^{\Sigma_q}$ (*resp.* $\cong V(M) := (M \otimes \mathcal{R}^\dagger_{K',x'})^{\Sigma_q}$).

Proposition 12.9. — *There is a canonical fully faithful K_s-linear $\otimes$-functor*

$$D^\phi_{\sigma_q} : Rep_{K_s} G_{k((x))} \longrightarrow \sigma_q\text{-}\, Mod^{\phi}_{\mathcal{R}_x},$$

and a canonical fully faithful K-linear $\otimes$-functor

$$D^{(\phi)}_{\sigma_q} : Rep_K\, G_{k((x))} \longrightarrow \sigma_q\text{-}\, Mod^{(\phi)}_{\mathcal{R}_x}$$

given by

$$V \longmapsto (V \otimes \mathcal{R}_{K',x'})^{G_{k((x))}}$$

for suitable $\mathcal{R}_{K',x'}$ (depending on V).

The fact that the functor is full is seen as usual using internal $\underline{Hom}$; it reduces to the fact that $D^{(\phi)}_{\sigma_q}(V^{G_{k((x))}})$ is the largest trivial subobject of $D^{(\phi)}_{\sigma_q}(V)$.

Remark 12.10. — Combining the last two examples, one finds a canonical fully faithful K-linear $\otimes$-functor

$$D^{(\phi)}_{\sigma_q} : Rep_K(G_{k((x))} \times \mathbb{G}_a) \longrightarrow \sigma_q\text{-}\, Mod^{(\phi)}_{\mathcal{R}_x}$$

given by

$$V \longmapsto D^{(\phi)}_{\sigma_q}(V) := (V \otimes \mathcal{R}_{K',x'}[\log x])^{G_{k((x))} \times \mathbb{G}_a(K)}$$

for suitable $\mathcal{R}_{K',x'}$.

12.3. Confluent weak Frobenius structure. — Let us remark that if M is a $q^{p^{(i+1)s}}$-difference module, then $\phi_* M$ has a natural structure of $q^{p^i s}$-difference module (in a given basis they are "given by" matrices $A_{q^{p^{is}}}$ and $A^{\phi}_{q^{p^{is}}}$ respectively).

On the other hand, $q^{p^{is}} \to 1$ when $i \to \infty$, and the phenomenon of confluence occurs in this natural way, in the p-adic setting.

Combining these two remarks, this suggests to introduce another type of Frobenius structure, which seems to have no counterpart in the differential case: a sequence of Frobenius predecessors ($q^{p^{is}}$-difference modules) which for $i \to \infty$ converge to a strong Frobenius structure on the limit differential module. More precisely:

Definition 12.11. — A *confluent weak Frobenius structure* on a q-difference module (M, Σ_q) over R is a sequence $(M_i = (M, \Sigma_{q^{p^{is}}}))_{i \geqslant 0}$ of $q^{p^{is}}$-difference structures on M, and isomorphisms

$$\Phi_i : \phi_* M_{i+1} \longrightarrow M_i$$

of $q^{p^{is}}$-difference modules, such that

1) the operators $\Delta_i = \frac{1}{(q^{p^{is}}-1)}(\Sigma_{q^{p^{is}}} - \mathrm{Id})$ converge to a derivation Δ_∞ on M,
2) Φ_i converges to a strong Frobenius Φ_∞ for the differential module (M, Δ_∞).

In terms of associated matrices, this amounts to the data of a sequence of matrices F_i, $A_{q^{p^{is}}} \in GL_\mu(R)$, $i \geqslant 0$, related by the following relations:

$$F_i^{\sigma_{q^{p^{is}}}} A_{q^{p^{is}}} = A^{\phi}_{q^{p^{(i+1)s}}} F_i,$$

and such that the sequences (F_i) and $(G_i := (q^{p^i}-1)^{-1}(A_{q^{p^i}}-I))$ both converge in $M_\mu(R)$ to some limits F_∞ and G_∞ respectively (F_i represents the "quotient" $Y_{i+1}^{\phi}Y_i^{-1}$ where Y_i is a fundamental solution of the system $\sigma_{q^{p^{is}}}Y_i = A_{q^{p^{is}}}Y_i$).

We denote by $\sigma_q\text{-}\,Mod_R^{\mathrm{conf}\,\phi}$ the category of σ_q–modules over R endowed with a specified confluent weak Frobenius structure.

We denote by $\sigma_q\text{-}\,Mod_R^{\mathrm{conf}(\phi)}$ the analogous category where one leaves the morphisms Φ_i unspecified: objects consist of a sequence $(M, \Sigma_{q^{p^{is}}}))_{i\geqslant 0}$ of $q^{p^{is}}$-difference structures on M converging to a differential module structure on M, these data being part of an unspecified confluent weak Frobenius structure.

If R is Bézout and if the subring of σ_q-constants is a field K, then $\sigma_q\text{-}\,Mod_R^{\mathrm{conf}(\phi)}$ is tannakian over K while $\sigma_q\text{-}\,Mod_R^{\mathrm{conf}\,\phi}$ is tannakian over K_s.

One has a canonical $\otimes$-functors "limit differential module":

$$\mathrm{Lim}_\infty^{\phi} : \sigma_q\text{-}\,Mod_R^{\mathrm{conf}\,\phi} \longrightarrow d\text{-}\,Mod_R^{\phi}, \qquad (M, \Sigma_q, (M_i, \Phi_i)) \longmapsto (M_\infty, \Delta_\infty, \Phi),$$
$$\mathrm{Lim}_\infty^{(\phi)} : \sigma_q\text{-}\,Mod_R^{\mathrm{conf}(\phi)} \longrightarrow d\text{-}\,Mod_R^{(\phi)}, \qquad (M, \Sigma_q, (M_i,)) \longmapsto (M_\infty, \Delta_\infty).$$

***Example 12.12* (*q*-Exponential)**. — We assume that $\pi^\tau = \pi$ and that $|q-1| < p^{-1/p-1}$. Let us consider the q-difference module $(M = \mathcal{R}, \Sigma_q)$ of rank $\mu = 1$ attached to q-difference system $d_q y = \pi y$ (with solution $e_q(\pi x)$); explicitly,

$$\Sigma_q(1) = (1 + (q-1)\pi x)^{-1}.$$

Then corollary 2.5 shows that $(\Sigma_{q^{p^{is}}})$ together with

$$(\Phi_i, \Phi_i(1) = e_{q^{p^i}}(\pi x^{p^i})/e_{q^{p^{(i+1)}}}(\pi x^{p^{i+1}}))$$

form a confluent weak Frobenius structure with limit differential module $(\mathcal{R}, \Delta_\infty = x\frac{d}{dx} + \pi x)$.

***Example 12.13* (*q*-Logarithm)**. — The sequence of matrices $A_{q^{p^{is}}} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $F_i = I$, defines a confluent weak Frobenius structure for the q-logarithm (in this case $Y_i = \left(\begin{smallmatrix} (q-1)[p^{is}]_q & 0 \\ (q-1)\frac{[p^{is}]_q}{p^{is}} \log x & 1 \end{smallmatrix}\right)$). One has a canonical fully faithful $\otimes$-functor

$$Rep_K\, \mathbb{G}_a \longrightarrow \sigma_q\text{-}\,Mod_{\mathcal{R}_x}^{(\mathrm{conf}\,\phi)}$$

which sends the standard two-dimensional representation to the q-difference module attached to ℓ_q.

***Example 12.14* (q-Difference modules which arise from Galois representations)**

Let us assume that k is perfect and K is discretely valued. Let $k'((x'))$ be a finite Galois extension of $k((x))$, let $\mathcal{E}^{\dagger}_{K',x'}$ be the corresponding finite unramified Galois extension of $\mathcal{E}^{\dagger}_{K,x}$, and let $\mathcal{R}_{K',x'}$ be the corresponding finite étale extension of $\mathcal{R}_{K,x}$.

Remark 12.15. — $(\mathcal{E}^\dagger_{K',x'},\sigma_q)$ has a canonical confluent weak Frobenius structure, given by $\Sigma_{q^{p^{is}}} = \sigma_q^{p^{is}}$ and $\Phi_i = \phi$. One has $\delta_{q^{p^{is}}} = \frac{1}{(q^{p^{is}}-1)}(\sigma_q^{p^{is}} - \mathrm{Id}) \to xd/dx$ on $\mathcal{E}^\dagger_{K,x}$, and we are about to see that the same holds on $\mathcal{E}^\dagger_{K',x'}$.

We may assume that $K' = K$. Let ϖ be a uniformizer of $\mathcal{O}_K$, and let us write $ord_\varpi(q^{p^{is}} - 1) = n_i$ ($n_i \to \infty$ with i). Then $\sigma_q^{p^{is}}$ is identity on the henselian ring $\mathcal{O}_{\mathcal{E}^\dagger_x}/\varpi^{n_i}$, hence also on its étale extension $\mathcal{O}_{\mathcal{E}^\dagger_{x'}}/\varpi^{n_i}$. This means that all $\delta_{q^{p^{is}}}$ extend to $\mathcal{O}_K$-linear endomorphisms of $\mathcal{O}_{\mathcal{E}^\dagger_{x'}}$. The formula $\delta_{q^{p^{is}}}(ab) - a\delta_{q^{p^{is}}}(b) - \delta_{q^{p^{is}}}(a)b = (q^{p^{is}} - 1)\delta_{q^{p^{is}}}(a)\delta_{q^{p^{is}}}(b)$ shows that $\delta_{q^{p^{is}}}$ induces a derivation of $\mathcal{O}_{\mathcal{E}^\dagger_{x'}}/\varpi^{n_i}$. Since its restriction to $\mathcal{O}_{\mathcal{E}^\dagger_x}/\varpi^{n_i}$ is xd/dx, it is xd/dx. Therefore $\delta_{q^{p^{is}}} \to xd/dx$ on $\mathcal{E}^\dagger_{x'}$.

Remark 12.16. — For r close enough to 1, one has $|\sigma_q(x')|_{|x'|=r'} = |\sigma_q(x')|_{r',\mathrm{can}} = r'$ for every $r' \in [r,1[$, hence $\delta_q(\mathcal{A}_{x'}([r,r'])) \subset (\mathcal{A}_{x'}([r,r']))$. Arguing as above, one shows that $\delta_{q^{p^{is}}} \to xd/dx$ on $\mathcal{A}_{x'}([r,r'])$. Similarly, $\delta_{q^{p^{is}}} \to xd/dx$ on $\mathcal{R}_{K',x'}$, and $\mathcal{R}_{K',x'}$ has a canonical confluent weak Frobenius structure, given by $\Sigma_{q^{p^{is}}} = \sigma_q^{p^{is}}$ and $\Phi_i = \phi$.

One can then play the game of (12.8) with confluent weak Frobenius structures instead of strong Frobenius structures, and get:

Proposition 12.17. — *There is a canonical fully faithful K_s-linear $\otimes$-functor*

$$D^{\mathrm{conf}\,\phi}_{\sigma_q} : Rep_{K_s} G_{k((x))} \longrightarrow \sigma_q\text{-}\,Mod^{\mathrm{conf}\,\phi}_{\mathcal{R}_x},$$

and a canonical fully faithful K-linear $\otimes$-functor

$$D^{(\mathrm{conf}\,\phi)}_{\sigma_q} : Rep_K G_{k((x))} \longrightarrow \sigma_q\text{-}\,Mod^{(\mathrm{conf}\,\phi)}_{\mathcal{R}_x}$$

given by

$$V \longmapsto (V \otimes \mathcal{R}_{K',x'})^{G_{k((x))}}$$

for suitable $\mathcal{R}_{K',x'}$ (depending on V).

Remark 12.18. — Combining the last two examples, one finds a canonical fully faithful K-linear $\otimes$-functor

$$D^{(\mathrm{conf}\,\phi)}_{\sigma_q} : Rep_K(G_{k((x))} \times \mathbb{G}_a) \longrightarrow \sigma_q\text{-}\,Mod^{(\mathrm{conf}\,\phi)}_{\mathcal{R}_x}$$

given by

$$V \longmapsto D^{(\mathrm{conf}\,\phi)}_{\sigma_q}(V) := (V \otimes \mathcal{R}_{K',x'}[\log x])^{G_{k((x))} \times \mathbb{G}_a(K)}$$

for suitable $\mathcal{R}_{K',x'}$, as well as a canonical fully faithful K_s-linear $\otimes$-functor

$$D^{\mathrm{conf}\,\phi}_{\sigma_q} : Rep_{K_s}(G_{k((x))} \times \mathbb{G}_a) \longrightarrow \sigma_q\text{-}\,Mod^{\mathrm{conf}\,\phi}_{\mathcal{R}_x}.$$

Composing the former with the limit functor $\mathrm{Lim}_\infty{}^{(\phi)}$, one gets the $\otimes$-functor

$$D^{(\phi)}_d : Rep_K(G_{k((x))} \times \mathbb{G}_a) \longrightarrow d\text{-}\,Mod^{(\phi)}_{\mathcal{R}_x}.$$

This is easily checked on the regular representation of a finite Galois quotient G of $G_{k((x))}$, for which $(V \otimes \mathcal{R}_{K',x'})^{G_{k((x))}}$ is nothing but the Galois extension $\mathcal{R}'/\mathcal{R}$ in $\widetilde{\mathcal{R}}$ with Galois group G.

12.4. Solvability at the outer boundary. — We generalize definition 3.1 to the higher rank case. As in § 1.4, let t_r be a generic point of absolute value $r \in]|q-1|, 1[$ in some complete extension Ω of K. For any $\rho \in]|q-1|, r]$, let $\mathcal{A}_\Omega(t_r, \rho)$ be the ring of analytic functions in the Ω-disk $|x - t_r| < \rho$. This is a q-difference ring in a canonical way, and the canonical embedding $\mathcal{A}_K([r, 1[) \hookrightarrow \mathcal{A}_\Omega(t_r, \rho)$ is compatible with σ_q.

Let r' be in $]|q-1|, 1[$.

Definition 12.19. — A q-difference module M over $\mathcal{A}_K([r', 1[)$ is solvable (at the outer boundary) if there is a function

$$r \in]r', 1[\longmapsto \rho(r) \in]|q-1|, r]$$

such that $\lim_{r\to 1} \rho(r) = 1$ and $M \otimes_{\mathcal{A}_K([r,1[)} \mathcal{A}_\Omega(t_r, \rho(r))$ has a basis of elements fixed under Σ_q.

Since any q-difference module M over $\mathcal{E}^\dagger_x$ or $\mathcal{R}_x$ is "defined over $\mathcal{A}_K([r, 1[)$" for some r close enough to 1, this provides a definition of solvability (at the outer boundary) for such modules. We say that a q-difference module M over some finite unramified extension $\mathcal{E}^\dagger_{x'}$ (*resp.* $\mathcal{R}_{x'}$) is solvable if the underlying q-difference module over $\mathcal{E}^\dagger_x$ (*resp.* $\mathcal{R}_x$) is.

Let us choose a basis of M and denote by A_q the inverse of the matrix of Σ_q in this basis. Let us define a sequence of matrices

$$G_{q,0} = I, \quad G_{q,1} = \frac{1}{q-1}(A_q - I), \quad G_{q,m+1} = x \cdot d_q(G_{q,m}) + G^{\sigma_q}_{q,m}(G_{q,1} - m_q \cdot q^{-m}.I).$$

The q-difference system $Y^{\sigma_q} = A_q Y$ then gives rise to the sequence of systems

$$x^m d_q^m Y = G_{q,m} Y.$$

For the solution around t_r normalized by $Y(t_r)$, one has the twisted Taylor expansion (*cf.* (1.2), [**DV02**, 4], [**DV03**, 3])

$$Y(x) = \sum_{m \geqslant 0} t_r^{-m} \frac{G_{q,m}(t_r)}{[m]^!_q} (x, t_r)_{q,m}.$$

From this formula, denoting by $|\pi_q| < 1$ the limit of $|[m]^!_q|^{1/m}$, the following lemma follows immediately.

Lemma 12.20. — *M is solvable if and only if*

$$\limsup_{r\to 1} \limsup_m |G_{q,m}|_r^{1/m} \leqslant |\pi_q|.$$

Proposition 12.21. — *Any q-difference module M over $\mathcal{R}_{K',x'}$ with a strong Frobenius structure is solvable.*

Indeed, the strong Frobenius structure M induces a strong Frobenius structure on the underlying q-difference module over $\mathcal{R}_{K,x}$, hence we may assume that $\mathcal{R}_{K',x'} = \mathcal{R}_{K,x}$. Dwork's well-known argument applies (if M is defined over $\mathcal{A}_K([r^p, 1[)$ and

$M \otimes_{\mathcal{A}_K([r^p,1[)} \mathcal{A}_\Omega(t_{r^p},\rho^p)$ has a basis of elements fixed under Σ_q, and if r and ρ are close enough to 1, then $\phi_* M$ is defined over $\mathcal{A}_K([r,1[)$ and $\phi_* M \otimes_{\mathcal{A}_K([r,1[)} \mathcal{A}_\Omega(t_r,\rho)$ has a basis of elements fixed under Σ_q).

Proposition 12.22. — *For any solvable q-difference module over $\mathcal{R}_{K',x'}$ (resp. $\mathcal{E}^\dagger_{K',x'}$),* $\lim_{i\to\infty}(\Sigma_q)^{p^i} = I$ *in* $GL(M)$.

Proof. — Let us first consider the case of $\mathcal{R}_{K',x'}$. This is a problem about the underlying q-difference module over $\mathcal{R}_{K,x}$, hence we may assume that $\mathcal{R}_{K',x'} = \mathcal{R}_{K,x}$. Let us take a basis of M, and consider matrices A_q and $G_{q,m}$ as before, and the sequence $A_{q^n} = A^{\sigma_q^{n-1}} \cdots A^{\sigma_q} A$. We have to show that $\lim_{i\to\infty} A_{q^{p^i}} = I$ in $GL_\mu(\mathcal{R})$. The relations between the iterates of σ_q and of d_q are given by the formula (*cf.* [**DV02**, 1.1.11], [**DV03**, 1.2])

$$\sigma_q^n = \sum_{m=0}^{n} \binom{n}{m}_q (q-1)^m q^{m(m-1)/2} x^m d_q^m.$$

It implies the following relation

$$A_{q^n} = \sum_{m=0}^{n} \binom{n}{m}_q (q-1)^m q^{m(m-1)/2} G_{q,m}.$$

Let us now take $n = p^i$, and let i tend to ∞. Let us cut the previous sum into three pieces $I + \sum_{m<p^{i/2}} + \sum_{m\geqslant p^{i/2}}$ and write, for any fixed r close enough to 1,

$$|A_{q^n}|_r \leqslant \max\left(1,\ \sup_{0<m<p^{i/2}} \left|\binom{p^i}{m}_q\right| |q-1|^m |G_{q,m}|_r, \sup_{m\geqslant p^{i/2}} \left|\binom{p^i}{m}_q\right| |q-1|^m |G_{q,m}|_r\right).$$

The quantity $\sup_{0<m<p^{i/2}} \left|\binom{p^i}{m}_q\right| |q-1|^m |G_{q,m}|_r$ tends to 0 because the factors $\binom{p^i}{m}_q$ tend uniformly to 0. The quantity $\sup_{m\geqslant p^{i/2}} \left|\binom{p^i}{m}_q\right| |q-1|^m |G_{q,m}|_r$ tends to 0 because the factors $|G_{q,m}|_r$ tend uniformly to 0 due to the solvability condition.

Let us now assume that M is defined over $\mathcal{E}^\dagger_{K,x}$. In order to get $\lim_{i\to\infty} A_{q^{p^i}} = I$ in $GL_\mu(\mathcal{E}^\dagger)$, one needs some uniformity in r in the previous estimates. This is provided by the effective bound à la Dwork-Robba [**DV03**, 5]: for $\rho(r)$ as in (12.19),

$$|G_{q,m}|_r \leqslant \left(\sup_{m^{(1)}<\cdots<m^{(\mu-1)}\leqslant m} \left|\frac{[m]_q^!}{\prod m_q^{(i)}}\right|\right) \left(\sup_{n<\mu} |G_{q,n}|_r\right) \left(\frac{r}{\rho(r)}\right)^m,$$

which is valid for any r close enough to 1, and gives

$$|G_{q,m}| \leqslant \left(\sup_{m^{(1)}<\cdots<m^{(\mu-1)}\leqslant m} \left|\frac{[m]_q^!}{\prod m_q^{(i)}}\right|\right) \left(\sup_{n<\mu} |G_{q,n}|\right),$$

at the limit $r = 1$ by solvability (at the outer boundary). □

13. "Unit-root" q-difference modules

In this section, we study unit-root q-difference modules over $\mathcal{R}$, that is σ_q-ϕ-modules M over $\mathcal{R}$ for which there exists a $\mathcal{O}_{\mathcal{E}^\dagger}$-lattice $\mathcal{M}$ in M such that Φ induces an isomorphism $\phi_*\mathcal{M} \cong \mathcal{M}$.

We prove the q-analog of Tsuzuki's theorem (in the differential case): after passing to a finite separable extension of $k((x))$ and to the corresponding finite étale extension of $\mathcal{R}$, M admits a basis of vectors fixed by Σ_q and Φ simultaneously.

We follow Christol's approach [**C01**] of that theorem, which is more analytic than Tsuzuki's proof. The fact that what follows looks more involved than [**C01**] is not due to pecularities of the q-difference theory, but to the fact that we had to fill two gaps in Christol's paper[(20)].

13.1. Overconvergence of solutions. — We start with a complete non-archimedean field of characteristic 0, with residue field k of characteristic $p > 0$. We don't assume K discretely valued nor k perfect, but we assume that K contains the p-th roots of unity (so that Dwork's constant π belongs to K), and we fix $q \in K$ such that $|q-1| < |\pi|$.

Let α and η be real numbers such that $0 < \alpha < 1 < \eta$. Define

$$\beta = p\frac{\log \alpha}{\log \alpha + \log |p|},$$

so that

$$\alpha^{1-\log r/p \log \eta} p^{-\log r/p \log \eta}$$

takes the value 1 at $r = \eta^\beta$, and

$$\alpha < \pi \Longleftrightarrow \beta > 1.$$

Notice that $\beta < p$.

Lemma 13.1. — *Let $Y \in GL_\mu(\mathcal{B}([0,1[))$ be such that*

$$|Y - I|_1 \leqslant \alpha \quad \textit{and} \quad G := x \cdot d_q Y \cdot Y^{-1} \in M_\mu(\mathcal{O}_{\mathcal{B}([0,\eta[)}).$$

Then there exists $Y' \in GL_\mu(\mathcal{B}([0,1[))$ such that

$$|Y' - I|_1 \leqslant \alpha \quad \textit{and} \quad G' := x \cdot d_q Y' \cdot Y'^{-1} \in M_\mu(\mathcal{O}_{\mathcal{B}([0,\eta^\beta[)})$$

and such that

$$Y'(x^p) \cdot Y^{-1} \in GL_\mu(\mathcal{B}([0,\eta[)).$$

(20) The first gap lies in [**C01**, prop. 13] where two different computations are made in $\mathcal{B}([0,1[)$ and $\mathcal{E}^\dagger_{1/x}$ respectively, and are subsequently compared in the "intersection $\mathcal{B}([0,1[) \cap \mathcal{E}^\dagger_{1/x}$", which is meaningless. The second gap occurs at the end of the proof (th. 17), where one is supposed to redo the argument of prop. 13 over a finite unramified extension $\mathcal{E}^\dagger_{1/x'}$ of $\mathcal{E}^\dagger_{1/x}$. But the argument of prop. 13 makes strong use of the explicit form $x \to x^p$ of Frobenius, and breaks down for $\mathcal{E}^\dagger_{1/x'}$.

Proof. — (The proof is entirely parallel to that given in [**C01**, Lemma 1] in the differential case.) There are three steps.

Step 1. — It is straightforward to check that the matrix G_m defined inductively by

$$G_0 = I, \quad G_{m+1} = x \cdot d_q(G_m) + G_m(qx)(G(x) - m_q \cdot q^{-m} \cdot I)$$

satisfies

$$x^m d_q^m Y = G_m Y.$$

From these formulas, taking into account the fact that the operator $d_q^m/[m]_q^!$ does not increase the sup-norm on $\mathcal{B}([0,1[)$ (*cf.* [**DV02**, 4.2.1], [**DV03**, 2.1]), one derives the following estimates:

$$|G_m|_1 \leqslant \left|[m]_q^! \cdot \frac{d_q^m}{[m]_q^!}(Y - I)\right| \leqslant \left|[m]_q^!\right| \cdot \alpha, \quad |G_m|_\eta \leqslant 1.$$

Since $\log r \mapsto \log |G_m|_r$ is a convex function, one obtains by interpolation

$$|G_m|_r \leqslant \left(\left|[m]_q^!\right| \cdot \alpha\right)^{1-\log r/\log \eta}$$

for every r between 1 and η. In particular, $|G_m|_r < 1$ and $\lim_m |G_m|_r = 0$ if $r < \eta$.

Step 2. — We shall see that

$$Y' := \frac{1}{p} \sum_{y^p = x} Y(y)$$

satisfies the conditions of the lemma. Since the m-th Taylor coefficient of Y' is nothing but the pm-th Taylor coefficient of Y, it is clear that $|Y' - I|_1 \leqslant \alpha$.

Moreover, considering the q-analog of the Taylor expansion, one finds

$$\begin{aligned} Y'(x^p)Y(x)^{-1} &= \frac{1}{p} \sum_{\zeta^p = 1} Y(\zeta x)Y(x)^{-1} \\ &= I + \sum_1^\infty G_m(x)\left(\frac{1}{p} \sum_{\zeta^p = 1} \frac{(\zeta - 1)\dots(\zeta - q^{m-1})}{[m]_q^!}\right). \end{aligned}$$

Since $|q - 1| < |\pi|$, the coefficient of G_m is in $\mathcal{O}_K$, and it easily follows that $H := Y'(x^p) \cdot Y^{-1} \in GL_\mu(\mathcal{B}([0,\eta[))$.

Step 3. — One has

$$p_q G'(x^p) = (x \cdot d_q)(Y(x^p))Y(x^p)^{-1} = x d_q H \cdot H^{-1}(qx) + HGH^{-1}(qx)$$

which lies in $GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta[)})$, hence $G' \in M_\mu(\mathcal{B}([0,\eta^p[))$ and $|G'|_{\eta^p} \leqslant |p_q|^{-1} = |p|^{-1}$.

On the other hand, $|G'|_1 = |d_q Y' . (Y')^{-1}|_1 \leqslant \alpha$. By log-convex interpolation, one finds

$$|G'|_r \leqslant \alpha^{1-\log r/p\log \eta} p^{-\log r/p \log \eta}$$

for every r between 1 and η^p. In particular, $|G'|_{\eta^\beta} \leqslant 1$ by the choice of β, whence the lemma. □

Assume moreover that $\alpha < |\pi|$ and let n be a non-negative integer such that $\eta^{\beta^n} \cdot |\pi| > 1$. Then the real number $\eta' := (\eta^{\beta^n} \cdot |\pi|)^{p^{-n}}$ lies in the interval $]1, \eta[$.

Proposition 13.2. — *Under the assumption of the lemma, $Y \in GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta'[)})$.*

Proof (cf. [**C01**, Prop. 3]*).* — One has $|x^{-m}(G_m/[m]_q^!)|_\eta \leqslant |1/[m]_q^!|\eta^{-m} \leqslant (|\pi|\eta)^{-m}$. If $n = 0$, this is $\leqslant (\eta')^{-m}$, and by the q-analog of the Taylor expansion of Y, one concludes that $Y \in GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta'[)})$.

One then argues by induction on n. By the previous lemma, there is Y' which satisfies our assumptions with n replaced by $n-1$ and η replaced by η^β. By induction, $Y' \in GL_\mu(\mathcal{O}_{\mathcal{B}([0,(\eta')^p[)})$. Hence $Y'(x^p) \in GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta'[)})$, and since $H \in GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta[)}) \subset GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta'[)})$, one concludes that $Y = H^{-1}Y'(x^p) \in GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta'[)})$ as well. □

Corollary 13.3. — *Let $Y \in GL_\mu(\mathcal{B}(]1,\infty])$ be such that $|Y - I| < |\pi|$ and $Y^{\sigma_q}Y^{-1} \in M_\mu(\mathcal{H}_x^\dagger)$. Then $Y \in GL_\mu(\mathcal{H}_x^\dagger)$.*

Indeed, the assumption implies $(xd_qY) \cdot Y^{-1} \in M_\mu(\mathcal{H}_x^\dagger)$. In fact, since actually $Y \in GL_\mu(\mathcal{O}_{\mathcal{B}(]1,\infty])})$, $(xd_qY) \cdot Y^{-1} \in M_\mu(\mathcal{O}_{\mathcal{H}_x^\dagger})$. After change of variable $x \mapsto 1/x$, and for η close enough to 1, one is in the situation of the previous proposition.

13.2. q-analog of Tsuzuki's theorem: the case when F is close to I

We assume that K *is endowed with a Frobenius automorphism τ and that k is algebraically closed.* We fix a positive integer s and consider the Frobenius $\phi : \sum a_n x^n \mapsto \sum \tau^s(a_n)x^{p^s n}$ on $\mathcal{E}_x^\dagger$. We *assume that* $q^{\tau^s} = q$.

Lemma 13.4. — *For any $F_0 \in GL_\mu(\mathcal{O}_K)$ such that $|F_0 - I| < 1$, there exists $H_0 \in GL_\mu(\mathcal{O}_K)$ such that $H_0^{\tau^s} F_0 H_0^{-1} = I$ and $|H_0 - I| = |H_0^{-1} - I| \leqslant |F_0 - I|$.*

For $s = 1$, this is part of [**C01**, Lemma 12]. The same proof works for any $s > 0$.

Lemma 13.5. — *For any $F \in GL_\mu(\mathcal{E}_x^\dagger)$ such that $|F - I| < 1$, there exists $H \in GL_\mu(\mathcal{O}_K[[x]]) \subset GL_\mu(\mathcal{O}_{\mathcal{E}_x^\dagger})$ such that $H^\phi F H^{-1} \in GL_\mu(\mathcal{H}_x^\dagger))$ and $|H - I| = |H^{-1} - I| \leqslant |F - I|$.*

Proof. — For $s = 1$, this is part of [**C01**, Lemma 8]. The proof for any $s > 0$ is similar. One starts by noticing that the endomorphism $id - \phi$ of $xK[[x]]$ has an inverse: $\psi = \sum_1^\infty \phi^n$, and that this inverse stabilizes the subring $\gamma_{x>0}\,\mathcal{E}$. Let $\mathcal{U}$ be the closed subspace of $M_\mu(\mathcal{O}_K[[x]])$ consisting of matrices H satisfying $|H - I| = |H^{-1} - I| \leqslant |F - I|$, and let us consider the following endomorphism f of $\mathcal{U}$:

$$f(H) = H + (\psi \circ \gamma_{x>0})(H^\phi F H^{-1})$$

which can also be written as

$$f(H) = I + (\psi \circ \gamma_{x>0})(H^\phi F H^{-1} + H - H^\phi).$$

Let us check that f is a contraction; one has

$$\begin{aligned} f(H)-f(H') &= (\psi\circ\gamma_{x>0})(H^{\phi}(FH^{-1}-I)-(H')^{\phi}(F(H')^{-1}-I)+H-H') \\ &= (\psi\circ\gamma_{x>0})\big((H^{\phi}-(H')^{\phi})(FH^{-1}-I)\big) \\ &\qquad -\big((H')^{\phi}F(H')^{-1}(H-H')H^{-1}+H-H'\big)\big), \end{aligned}$$

and it is clear that the norm of both terms in the difference is $<|H-H'|$. Thus f has a fixed point H, and since ψ is invertible, $\gamma_{x>0}(H^{\phi}FH^{-1})=0$ as wanted. □

Proposition 13.6. — *Let $F\in GL_{\mu}(\mathcal{E}_x^{\dagger})$ satisfy $|F-I|<|\pi|$, and let A_q, $A_{q^{p^s}}\in GL_{\mu}(\mathcal{R}_x)$ be such that*

$$F^{\sigma_q}A_q=A_{q^{p^s}}^{\phi}F,\quad A_{q^{p^s}}=A_q^{\sigma_{q^{p^s-1}}}\cdots A_q.$$

Then $A_q\in GL_{\mu}(\mathcal{O}_{\mathcal{E}_x^{\dagger}})$ and there exists $Y\in GL_{\mu}(\mathcal{O}\mathcal{E}_x^{\dagger})$ such that

$$Y^{\sigma_q}=A_qY,\quad Y^{\phi}=FY.$$

Proof. — Using the two previous lemmas, one reduces to the case when $F\in GL_{\mu}(\mathcal{O}_{\mathcal{H}_x^{\dagger}})$ and $F(0)=I$ (still with $|F-I|<|\pi|$). Let us define a sequence of matrices $Y_m\in GL_{\mu}(\mathcal{O}_{\mathcal{H}_x^{\dagger}})$ by

$$Y_m=F^{-1}(F^{\phi})^{-1}\cdots(F^{\phi^m})^{-1}.$$

As above, let us set, for any positive integer n, $A_{q^n}=A_q^{\sigma_{q^{n-1}}}\cdots A_q$. From the relation $A_{q^{p^s}}^{\phi}=F^{\sigma_q}A_qF^{-1}$, one derives by induction on n the following relation in $GL_{\mu}(\mathcal{R}_x)$:

$$A_{q^{np^s}}^{\phi}=F^{\sigma_{q^n}}A_{q^n}F^{-1}.$$

Applying this to the powers of p^s, one then computes

$$(Y_m^{\sigma_q})^{-1}A_qY_m=(F^{\phi^m})^{\sigma_q}\cdots(F^{\phi})^{\sigma_q}F^{\sigma_q}A_qF^{-1}(F^{\phi})^{-1}\cdots(F^{\phi^m})^{-1}=A_{q^{p^{ms}}}^{\phi^m}.$$

We now use the fact (12.21, 12.22) that the sequence $A_{q^{p^{ms}}}$, hence also $A_{q^{p^{ms}}}^{\phi^m}$, tends to I in $GL_{\mu}(\mathcal{R}_x)$, and derive that $A_q\in GL_{\mu}(\mathcal{H}_x^{\dagger})$. Indeed, since $\gamma_{x>0}Y_m=\gamma_{x>0}Y_m^{\sigma_q}=0$, $\gamma_{x>0}A_q=Y_m^{\sigma_q}(\gamma_{x>0}(A_{q^{p^{ms}}}^{\phi^m}))Y_m^{-1}$; one has $\gamma_{x>0}(A_{q^{p^{ms}}}^{\phi^m})\to\gamma_{x>0}I=0$, whence $\gamma_{x>0}A_q=0$, and it follows that $A_q\in GL_{\mu}(\mathcal{H}_x^{\dagger})$.

By (12.21), (12.22) again, $|A_{q^{p^{ms}}}^{\phi^m}-I|_{\mathcal{E}^{\dagger}}\to 0$, hence $(Y_m^{\sigma_q})^{-1}A_qY_m$ tends to I also in $GL_{\mu}(\mathcal{B}(]1,\infty]))$. But Y_m has a limit Y in $GL_{\mu}(\mathcal{O}_{\mathcal{B}(]1,\infty]})$, hence $Y^{\sigma_q}Y^{-1}=A_q\in GL_{\mu}(\mathcal{B}(]1,\infty]))$. Applying (13.3), one concludes that $Y\in GL_{\mu}(\mathcal{H}_x^{\dagger})$, and that $Y^{\sigma_q}Y^{-1}=A_q$. Since $Y\in GL_{\mu}(\mathcal{O}_{\mathcal{B}(]1,\infty]})$, the entries of A_q and Y actually lie in $\mathcal{O}_{\mathcal{H}_x^{\dagger}}$.

On the other hand, it is obvious that $Y^{\phi}=FY$ in $GL_{\mu}(\mathcal{O}_{B(]1,\infty])})$, hence also in $GL_{\mu}(\mathcal{H}_x^{\dagger})$. □

Corollary 13.7. — *Let (M,Σ_q) be a q-difference module over $\mathcal{R}_x$ with a strong Frobenius structure Φ. Let us assume that there is a basis of M in which the Frobenius*

matrix F has entries in $\mathcal{E}^\dagger$ and satisfies $|F-I|<|\pi|$. Then there is a basis of M which is fixed under both Σ_q and Φ.

13.3. Going up in finite unramified extensions of $\mathcal{E}^\dagger$. — We now assume in addition that K *is discretely valued* (and again that k is algebraically closed). Hence K is a finite extension of the field of fractions of the Witt ring $W(k)$.

We are now looking for generalizations of the previous results when $\mathcal{E}^\dagger_{K,x}$ is replaced by a finite unramified extension $\mathcal{E}^\dagger_{K,x'}$ (x' being a good coordinate as in § 11.3). The difficulty lies of course in the fact that one cannot assume that $\sigma_q(x')$ is proportional to x', nor that $\phi(x')=(x')^p$.

We use again the notations $f: C \xrightarrow{x} \mathbf{P}^1_{\mathcal{O}_K}\smallsetminus 0$, etc... of § 11.3. Let d be the degree of f. Let us first generalize corollary 13.3.

Proposition 13.8. — *Let $Y'\in GL_\mu(B_{x'}(]1,\infty]))$ be such that $|Y'-I|<|\pi|$ and $(Y')^{\sigma_q}(Y')^{-1}\in M_\mu(\mathcal{O}^\dagger(C)_K)$. Then $Y'\in GL_\mu(\mathcal{O}^\dagger(C)_K)$.*

Proof. — We first notice that it is sufficient to prove that $Y'\in GL_\mu(\mathcal{O}^\dagger(C)[1/h])$ for some $h\in\mathcal{H}^\dagger_x$ of norm 1: the fact that $A'_q:=(Y')^{\sigma_q}(Y')^{-1}$ has no pole at $h=0$ will imply that Y' has no pole as well at $h=0$. After such a localization, we may assume that $\mathcal{O}^\dagger(C)[1/h]$ is a free $\mathcal{O}_{\mathcal{H}^\dagger_x}[1/h]$-module of rank d.

In the second place, we notice that $A'_q\in M_\mu(\mathcal{O}^\dagger(C))$. Let us endow the free $\mathcal{O}^\dagger(C)[1/h]$-module of rank μ, say $M'=\oplus_{i=1}^{i=\mu}\mathcal{O}^\dagger(C)[1/h]m'_i$, with the σ_q-linear endomorphism Σ_q defined by the matrix $(A'_q)^{-1}$ in the basis (m_i). Via f, M' gives rise to an "underlying" q-difference module M over $\mathcal{O}_{\mathcal{H}^\dagger_x}[1/h]$. We endow it with the basis (m_j) constructed via (m'_i) and a fixed basis of $\mathcal{O}^\dagger(C)[1/h]$ over $\mathcal{O}_{\mathcal{H}^\dagger_x}[1/h]$. In this basis, the q-difference system has a solution $Y\in GL_{\mu d}(\mathcal{B}_x(]1,\infty])$ with $Y(0)=I$. Let ϖ be a uniformizing parameter for $\mathcal{O}_K$, and let n be such that $|\pi|=|\varpi|^{n-1}$. The condition $|Y'-I|<|\pi|$ translates into: the image of the vectors m'_i in M'/ϖ^n are fixed under Σ_q. Since $\sigma_q\equiv \mathrm{id}\bmod\varpi^n$, this implies that the image of the vectors m_j in M/ϖ^n are fixed under Σ_q. Hence $|Y-I|<|\pi|$, and by (13.3), $Y\in GL_{\mu d}(\mathcal{O}_{\mathcal{H}^\dagger_x}[1/h])$, which implies that $Y'\in GL_\mu(\mathcal{O}^\dagger(C)[1/h])$. □

Let us now generalize lemma 13.5:

Lemma 13.9. — *For any $F'\in GL_\mu(\mathcal{E}^\dagger_{x'})$ such that $|F'-I|<1$, there exists $H\in GL_\mu(\mathcal{O}_K[[x']])\subset GL_\mu(\mathcal{O}_{\mathcal{E}^\dagger_{x'}})$ such that $H^\phi F'H^{-1}\in GL_\mu(\mathcal{O}^\dagger(C)_K)$ and $|H-I|=|H^{-1}-I|\leqslant|F'-I|$.*

Proof. — One can even require that $H^\phi F'H^{-1}\in GL_\mu(\mathcal{H}^\dagger_{x'})$. The proof is almost the same as in (13.5), except that one has to deal with two Frobenius endomorphisms at the same time: ϕ, and ϕ' which raises to x' to the power p^s. One introduces the

inverse $\psi' = \sum_1^\infty (\phi')^n$ of $id - \phi'$, and the space $\mathcal{U}'$ analogous to $\mathcal{U}$ with x' in place of x. One checks that the formula

$$f(H) = H + (\psi \circ \gamma_{(x')>0})(H^\phi F' H^{-1})$$

(with ϕ, not ϕ'!) defines an endomorphism of $\mathcal{U}'$. One checks as in (13.5) that f is a contraction by writing $f(H)$ in the form $f(H) = I + (\psi \circ \gamma_{(x')>0})(H^\phi(F'H^{-1} - I) + H + (H^\phi - H^{\phi'}))$ and using the fact that $|H^\phi - (H')^\phi - H^{\phi'} + (H')^{\phi'}| < |H - H'|$. □

At last, let us generalize proposition 13.6:

Proposition 13.10. — *Let $F' \in GL_\mu(\mathcal{E}^\dagger_{x'})$ satisfy $|F' - I| < |\pi|$, and let A'_q, $A'_{q^{p^s}} \in GL_\mu(\mathcal{R}_{x'})$ be such that*

$$(F')^{\sigma_q} A'_q = (A'_{q^{p^s}})^\phi F', \quad A'_{q^{p^s}} = (A'_q)^{\sigma_{q^{p^s}-1}} \cdots A'_q.$$

Then $A'_q \in GL_\mu(\mathcal{O}_{\mathcal{E}^\dagger_{x'}})$ there exists $Y' \in GL_\mu(\mathcal{O}_{\mathcal{E}^\dagger_{x'}})$ such that

$$(Y')^{\sigma_q} = A'_q Y', \quad (Y')^\phi = F'Y'.$$

Proof. — Using (13.8) and (13.9) in the place of (13.3) and (13.5) respectively, the argument is the same as in (13.6), except that one cannot directly apply the truncation operator $\gamma_{x'>0}$ since σ_q does not commute with $\gamma_{x'>0}$. We are in the situation where F', hence Y'_m, is in $GL_\mu(\mathcal{O}^\dagger(C)_K)$, and where $((Y'_m)^{\sigma_q})^{-1} A'_q Y'_m = (A'_{q^{p^{ms}}})^{\phi^m} \to I$ in $GL_\mu(\mathcal{R}_{x'})$, and we have to derive that $A'_q \in GL_\mu(\mathcal{O}^\dagger(C)_K)$.

We first notice that $\mathcal{O}^\dagger_K$ is a free $\mathcal{H}^\dagger_x$-module of rank d (it comes from a locally free $\mathcal{A}([r,\infty])$-module of rank d ($r < 1$), which is automatically free since $\mathcal{A}([r,\infty])$ is principal).

Let us endow the free $\mathcal{O}^\dagger(C)_K$-module of rank μ, say $M' = \oplus_{i=1}^{i=\mu} \mathcal{O}^\dagger(C)_K m'_i$, with a ϕ-linear endomorphism Φ defined by the matrix $(F')^{-1}$ in the basis (m_i), and endow $M' \otimes \mathcal{E}^\dagger_{x'}$ with the σ_q-linear endomorphism Σ_q defined by the matrix $(A'_q)^{-1}$ in the same basis. Via f, M' gives rise to an "underlying" ϕ-difference module M over $\mathcal{H}^\dagger_x$, and $M \otimes \mathcal{E}^\dagger_x$ becomes a σ_q-ϕ-module over $\mathcal{E}^\dagger_x$. We endow M with the basis (m_j) constructed via (m'_i) and a fixed basis of $\mathcal{O}^\dagger_K$ over $\mathcal{H}^\dagger_x$, and we denote by A_q and F respectively the inverse matrices of Σ_q and Φ in this basis. Then the argument of (13.6) applies and shows that $A_q \in GL_\mu(\mathcal{H}^\dagger_x)$, which implies that $A'_q \in GL_\mu(\mathcal{O}^\dagger(C)_K)$. □

13.4. q-analog of Tsuzuki's theorem: the general case. — In this section K is a complete discrete valuation field of characteristic 0, with *algebraically closed residue field* k of characteristic p. We assume that $\pi \in K$, that $|q-1| < |\pi|$. K is endowed with a Frobenius τ, and we assume that, for a given $s > 0$, $\tau^s(q) = q$.

As before, ϕ is the τ^s-linear endomorphism of $\mathcal{E}^\dagger$ (or $\mathcal{R}$) which sends x to x^{p^s}.

Theorem 13.11. — *Let (M, Σ_q) be a q-difference module over $\mathcal{R}_x$ with a strong Frobenius structure Φ. Let us assume that there is a $\mathcal{O}_{\mathcal{E}_x^\dagger}$-lattice $\mathcal{M}$ in M such that Φ induces an isomorphism*

$$\phi_*\mathcal{M} \xrightarrow{\cong} \mathcal{M}.$$

Then there is a finite unramified extension $\mathcal{E}_{x'}^\dagger$ of $\mathcal{E}_x^\dagger$ and a basis of $\mathcal{M} \otimes_{\mathcal{O}_{\mathcal{E}_x^\dagger}} \mathcal{O}_{\mathcal{E}_{x'}^\dagger}$ which is fixed under both Σ_q and Φ.

In particular, $(M \otimes_{\mathcal{R}_x} \mathcal{R}_{x'}, \Sigma_q)$ is a trivial q-difference module.

This follows from the previous proposition and the following lemma of N. Tsuzuki [**T98b**, 5.2.2] (*cf.* also [**C01**, Lemma 16]):

Lemma 13.12. — *For any $F \in GL_\mu(\mathcal{O}_{\mathcal{E}_x^\dagger})$, there exists a finite unramified extension $\mathcal{E}_{x'}^\dagger$ and $H \in GL_\mu(\mathcal{O}_{\mathcal{E}_{x'}^\dagger})$ such that $|H^\phi F H^{-1} - I| < |\pi|$.*

This is proven in [**T98b**, 5.1.1], [**C01**, lemme 16] for $s = 1$, but the proof works for any $s > 0$.

14. Quasi-unipotence

In this section K is assumed to be complete, discretely valued, of characteristic 0, with residue field k perfect of characteristic $p > 0$. K is endowed with a Frobenius τ. As before, ϕ is the τ^s-linear endomorphism of $\mathcal{E}^\dagger$ (or $\mathcal{R}$) which sends x to x^{p^s}.

We shall prove the quasi-unipotence of q-difference modules over $\mathcal{R}$ which admit a strong Frobenius structure, using Kedlaya's structure theorem for ϕ-modules over $\mathcal{R}$.

Another path toward quasi-unipotence seems possible, through the structure theorem of [**A02**] for tannakian categories with a Hasse-Arf filtration. However, this would require a q-analog of Christol-Mebkhout theory of exponents and analytic slopes.

14.1. Kedlaya's filtration. —

We shall rely on the following theorem [**Ke**, 6.10]:

Theorem 14.1. — *Let (M, Φ) be a ϕ-module over $\mathcal{R}$. Then there exists a unique finite ascending filtration (M_j) by saturated ϕ-modules such that*

i) *the quotients M_j/M_{j+1} have a single Φ-slope s_j,*

ii) *$j \mapsto s_j$ is increasing,*

iii) *each quotient M_j/M_{j+1} comes from a unique ϕ-module (N_j, Φ) over $\mathcal{E}^\dagger$, and $N_j \otimes_{\mathcal{E}^\dagger} \mathcal{E}$ has a unique slope in the sense of Dieudonné-Manin, which is s_j.*

Here, $\mathcal{E}$ is the usual notation for the p-adic completion of $\mathcal{E}^\dagger$. "M_j saturated" means as usual that M/M_j is torsion-free over $\mathcal{R}$, which implies that it is free, since it is also finitely presented and $\mathcal{R}$ is a Bézout ring. Notice that $M_1 = N_1 \otimes_{\mathcal{E}^\dagger} \mathcal{R}$.

The definition of Φ-slopes[21] involves the "big Robba ring" $\widetilde{\mathcal{R}}$ (*cf.* § 11.2), whose precise definition will not matter here. A ϕ-module L over $\mathcal{R}$ is said to have a single slope s if there exists a basis of eigenvectors for Φ in $L \otimes_{\mathcal{R}} \widetilde{\mathcal{R}}$ with eigenvalues in some finite extension of $\mathcal{O}_K$ of valuation equal to s.

We shall also use the following result:

Proposition 14.2. — *Let (M, Φ) be a ϕ-module over $\mathcal{R}$. Then up to replacing $k((x))$ by a finite separable extension (and $\mathcal{R}$ by the corresponding étale extension) and K by a finite extension,*

i) *there exists a basis $\underline{m}$ of M in which the inverse matrix F of Φ can be written as $F = D^{-1}U$, where D is a diagonal matrix with diagonal entries D_i in K and of non-decreasing valuations, and with $U \in GL_\mu(\mathcal{O}_{\mathcal{E}^\dagger})$, $|U - I| < 1$,*

ii) *the Φ-slopes of $N := \oplus \mathcal{E}^\dagger m_i$ coincide with the slopes of $N \otimes_{\mathcal{E}^\dagger} \mathcal{E}$ in the sense of Dieudonné-Manin,*

iii) *N contains N_1 (in the notation of § 14.1),*

iv) *the valuations of D_i are equal for $i < \operatorname{rk} N$, and are strictly smaller than the valuations of D_j for $j > \operatorname{rk} N$,*

v) *N is the direct sum of N_1 and $N_1' := \oplus_{i > \operatorname{rk} N} \mathcal{E}^\dagger m_i$,*

vi) *moreover, one may choose $\underline{m}$ in such a way that $N = \oplus_{i \leqslant \operatorname{rk} N} \mathcal{E}^\dagger m_i$, so that U may be written in block form as $U = \left(\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right)$, where $X \in GL_{\operatorname{rk} N}(\mathcal{O}_{\mathcal{E}^\dagger})$, $Z \in GL_{\mu - \operatorname{rk} N}(\mathcal{O}_{\mathcal{E}^\dagger})$, and Y has entries in $\mathcal{O}_{\mathcal{E}}^\dagger$.*

Proof. — Multiplying F by some constant and replacing K by a finite extension, one may assume that the Φ-slopes belong to the value group of $\mathcal{O}$.

For i) and ii), see [**Ke**, 6.9], where it is shown that the Dieudonné-Manin slopes of $N \otimes_{\mathcal{E}^\dagger} \mathcal{E}$ are the valuations of the D_i's.

The construction and characterization of N_1 in [**Ke**, 6.10] also shows iii). On the other hand, setting $\widetilde{\mathcal{E}} = K \otimes_{\mathbb{Z}_p} W(\overline{k((x))})$, it is shown in [**Ke**, 5.9] that Φ can be put into diagonal form D by some change of basis via a matrix $V \in GL_\mu(\mathcal{O}_{\widetilde{\mathcal{E}}})$ (applied to $\underline{m}$), with $|V - I| < 1$. Points iv), v) and v) follow easily. □

14.2. The local monodromy theorem. — We assume that $|q-1| < |\pi|$ and that $\tau^s(q) = q$.

Theorem 14.3. — *Any q-difference module (M, Σ_q) over $\mathcal{R}$ which admits a strong Frobenius structure is quasi-unipotent: after replacing $k((x))$ by a finite separable extension (and $\mathcal{R}$ by the corresponding étale extension), it admits a filtration by saturated q-difference submodules, with trivial quotients*[22].

[21] called special slopes in [**Ke**]; we call them Φ-slopes here in order to prevent any confusion with the (analytic) slopes à la Christol-Mebkhout, whose definition does not make use of any Frobenius structure, and which will appear in the sequel.

[22] as q-difference modules, *i.e.,* they admit a basis which is fixed under Σ_q.

Proof. — Multiplying F by some constant and replacing K by a finite extension, one may assume that the Φ-slopes belong to the value group of $\mathcal{O}$, and that the least Φ-slope is 0.

We may freely replace $k((x))$ by a finite separable extension and extend the discretely valued field K, so that we may assume that we are in the situation of the previous proposition, that $\pi \in K$ and that k is algebraically closed.

We shall prove that M_1 is stable under Σ_q. Applying (13.11), one deduces that it is trivialized, as a q-difference module, in some finite extension of $\mathcal{R}$ coming from a finite separable extension of $k((x))$. The same will be true for the M_j by induction, whence the quasi-unipotence of (M, Σ_q).

We rely of course on the integrability condition $F^{\sigma_q} A_q = A_{q^{p^s}}^{\phi^s} F$, in the basis provided by (14.2) vi). We write $A_q, A_{q^{p^s}}$ in block form

$$A_q = \begin{pmatrix} P_q & Q_q \\ R_q & S_q \end{pmatrix}, \quad A_{q^{p^s}} = \begin{pmatrix} P_{q^{p^s}} & Q_{q^{p^s}} \\ R_{q^{p^s}} & S_{q^{p^s}} \end{pmatrix}.$$

We have to show that $R_q = 0$.

The lower left corner of the integrability condition gives rise to the equation

$$R_q = (Z^{\sigma_q})^{-1} D'' R_{q^{p^s}}^{\phi} (D')^{-1} X,$$

where D' (*resp.* D'') is the diagonal matrix with diagonal entries D_i, $i \leqslant \operatorname{rk} M_1$ (*resp.* $i > \operatorname{rk} M_1$).

Let us fix r sufficiently close to 1 (so that our q-difference module with Frobenius structure is defined over $\mathcal{A}([r, 1[)$). Then the canonical absolute values $|\ |_{r^{1/p^s},\mathrm{can}}$ are defined for each of the matrices entering the last displayed equation. Moreover

$$|(Z^{\sigma_q})^{-1} D''|_{r^{1/p^s},\mathrm{can}} \cdot |(D')^{-1} X|_{r^{1/p^s},\mathrm{can}} \leqslant 1,$$

and $|R_{q^{p^s}}^{\phi}|_{r^{1/p^s},\mathrm{can}} = |R_{q^{p^s}}|_{r,\mathrm{can}}$. Thus $|R_q|_{r^{1/p^s},\mathrm{can}} \leqslant |R_{q^{p^s}}|_{r,\mathrm{can}}$. Now we may replace ϕ by a large power, *i.e.,* s is replaced by a large multiple. When $s \to \infty$, we know that $A_{q^{p^s}} \to I$ in $\mathcal{R}$ (by (12.21), (12.22)), hence $|R_{q^{p^s}}|_{r,\mathrm{can}} \to 0$. We conclude that $R_q = 0$. □

14.3. Logarithmic variant

Theorem 14.4. — *For any q-difference module (M, Σ_q) over $\mathcal{R}_x$ which admits a strong Frobenius structure, there is a finite étale extension $\mathcal{R}_{x'}/\mathcal{R}_x$ coming from a finite separable extension $k'((x'))/k((x))$, such that $M \otimes_{\mathcal{R}_x} \mathcal{R}_{x'}[\log x]$ has a basis fixed by Σ_q.*

This follows from the previous theorem and the following lemma:

Lemma 14.5. — *$d_q : \mathcal{R}_{x'}[\log x] \to \mathcal{R}_{x'}[\log x]$ is surjective.*

Proof. — It is easy straightforward that $\delta_q = x d_q = (q-1)^{-1}(\sigma_q - id)$ induces a K-linear isomorphism $\mathcal{R}_x/K \to \mathcal{R}_x/K$. Besides, note that $\delta_q(x^n log^k x)$ is a polynomial of degree $\leqslant k$ in $\log x$, with coefficients in $K[x, 1/x]$. From there, using the twisted

Leibniz rule, it is not difficult to conclude by induction on the degree of log that $\delta_q : \mathcal{R}_x[\log x] \to \mathcal{R}_x[\log x]$ is surjective.

Let us now turn to the etale extension $\mathcal{R}_{K',x'}[\log x]$. We may assume that $K = K'$. Again, by induction on the degree of log, one reduces the statement to the surjectivity of $\delta_q : \mathcal{R}_{x'}/K \to \mathcal{R}_{x'}/K$.

Recall (*cf.* (12.16)) that if I is a closed interval contained in $]0,1[$ and close enough to the point 1, one has $\delta_q(\mathcal{A}_{x'}(I)) \subset (\mathcal{A}_{x'}(I))$, and that $\delta_{q^{p^{is}}} \to xd/dx$ when $i \to \infty$.

One has

$$\delta_q(f) = g \Longleftrightarrow \delta_{q^{p^{is}}} f = \psi_i(g) := ([p^{is}]_q)^{-1}(g^{\sigma_{q^{p^{is}}-1}} + \cdots + g)\,.$$

It is easy to see that for given $g \in \mathcal{A}_{x'}(I)$, $\psi_i(g)$ forms a Cauchy sequence (for $j < i$, write $\psi_i(g)$ as $([p^{js}]_q)([p^{is}]_q)^{-1} \sum_{m=0}^{m=p^{(i-j)s}-1} \psi_j(g^{\sigma_{(q^{p^{js}})^m}})$ and compare each of the $p^{(i-j)s}$ terms to $\psi_j(g)$). Hence it converges in the Banach space $\mathcal{A}_{x'}(I)$.

On the other hand, although $x'd/dx' : \mathcal{A}_{x'}(I)/K \to \mathcal{A}_{x'}(I)/K$ is *not* surjective, every $h \in \mathcal{A}_{x'}(I)/K$ has a $(x'd/dx')$-preimage in $\mathcal{A}_{x'}(I')/K$, for any closed interval I' contained in the interior of I. For I close enough to 1, the same is true for xd/dx, which is a multiple of $x'd/dx'$ by a unit in $\mathcal{A}_{x'}(I)$: there exists $f \in \mathcal{A}_{x'}(I')/K$ such that $xd/dx(f) = h$. In the case $h = \lim \psi_i(g)$, one finds $\delta_q(f) = g$ in $\mathcal{A}_{x'}(I)/K$. Whence the result. □

As in the differential case (*cf.* [**A02**]), theorem 14.4 can be expressed in the following "tannakian way":

Theorem 14.6. — *Let us assume that k is algebraically closed. Then the canonical $\otimes$-functor*

$$D^{(\phi)}_{\sigma_q} : Rep_K(G_{k((x))} \times \mathbb{G}_a) \longrightarrow \sigma_q\text{-}\,Mod^{(\phi)}_{\mathcal{R}_x}$$

is an equivalence of categories. In fact, there is a canonical quasi-inverse $V^{(\phi)}_{\sigma_q}$ and a canonical isomorphism $D^{(\phi)}_{\sigma_q} \circ V^{(\phi)}_{\sigma_q} \cong \mathrm{Id}$.

This quasi-inverse is given by $V^{(\phi)}_{\sigma_q}(M) := (M \otimes_{\mathcal{R}} \mathcal{R}'[\log x])^{\Sigma_q}$ for $\mathcal{R}'$ big enough (in $\widetilde{\mathcal{R}}$), and the isomorphism $M \xrightarrow{\cong} D^{(\phi)}_{\sigma_q} V^{(\phi)}_{\sigma_q}(M)$ is induced by the canonical isomorphism

$$(M \otimes_{\mathcal{R}} \mathcal{R}'[\log x])^{\Sigma_q} \otimes_K \mathcal{R}'[\log x] \xrightarrow{\cong} M \otimes_{\mathcal{R}} \mathcal{R}'[\log x].$$

Proof. — We know that this functor is fully faithful, and its essential surjectivity is ensured by (14.4). The rest is formal and left to the reader. □

Remark 14.7. — This functor actually comes from a fully faithful $\otimes$-functor

$$Rep_K(G_{k((x))} \times \mathbb{G}_a) \longrightarrow \sigma_q\text{-}\,Mod^{(\phi)}_{\mathcal{E}^\dagger_x},$$

but the latter is *not* essentially surjective. For instance

$$\mathrm{Ext}_{\sigma_q\text{-}\,Mod^{(\phi)}_{\mathcal{R}}}(\mathcal{R},\mathcal{R}) = \mathrm{Ext}_{Rep\,\mathbb{G}_a}(K,K) = K,$$

while $\mathrm{Ext}_{\sigma_q\text{-}\,Mod^{(\phi)}_{\mathcal{E}^\dagger}}(\mathcal{E}^\dagger,\mathcal{E}^\dagger)$ is a K-space of infinite dimension.

Corollary 14.8. — *There is a canonical K-linear fully faithful $\otimes$-functor*

$$\sigma_q\text{-}\,Mod_{\mathcal{R}}^{(\phi)} \longrightarrow \sigma_q\text{-}\,Mod_{\mathcal{R}}^{\mathrm{conf}(\phi)}$$

given by $D_{\sigma_q}^{\mathrm{conf}(\phi)} \circ V_{\sigma_q}^{(\phi)}$(23).

Is it an equivalence of categories? This is likely. One way to tackle the question would be to prove directly the essential surjectivity of $D_{\sigma_q}^{(\mathrm{conf}\,\phi)}$ along the above lines; the major technical problem in this direction is to control the variation of Kedlaya's filtration attached to the confluent sequence of $q^{p^{is}}$-difference modules when $i \to \infty$.

15. Applications

15.1. Confluence. — We assume again that k is algebraically closed. We assume that $\pi \in K$, that $|q-1| < |\pi|$. K is endowed with a Frobenius τ, and we assume that, for a given $s > 0$, $\tau^s(q) = q$.

As before, ϕ is the τ^s-linear endomorphism of $\mathcal{R}$ which sends x to x^{p^s}.

We call the composed $\otimes$-functor

$$\mathrm{Conf} : \sigma_q\text{-}\,Mod_{\mathcal{R}}^{(\phi)} \longrightarrow \sigma_q\text{-}\,Mod_{\mathcal{R}}^{\mathrm{conf}(\phi)} \longrightarrow d\text{-}\,Mod_{\mathcal{R}}^{(\phi)},$$

where the first functor is $D_{\sigma_q}^{\mathrm{conf}(\phi)} \circ V_{\sigma_q}^{(\phi)}$ and the second is $\mathrm{Lim}_{\infty}^{(\phi)}$, "*the p-adic confluence functor*".

Theorem 15.1. — *One has* $\mathrm{Conf} \circ D_{\sigma_q}^{(\phi)} = D_d^{(\phi)}$. *In particular,* Conf *is an equivalence.*

Proof. — This is clear from remark 12.18, and from the fact that $D_{\sigma_q}^{(\phi)}$ and $D_d^{(\phi)}$ are equivalences, by the quasi-unipotence theorems. □

Remark 15.2. — It is clear that the restriction of these functors to rank one objects gives rise to the group isomorphisms of (8.2).

15.2. Canonical q-deformation. — We call the composed $\otimes$-functor

$$q\text{-Def} = \mathrm{D}_{\sigma_q}^{(\phi)} \circ \mathrm{V}_{\mathrm{d}}^{(\phi)} : d\text{-}\,Mod_{\mathcal{R}}^{(\phi)} \longrightarrow \sigma_q\text{-}\,Mod_{\mathcal{R}}^{(\phi)}$$

"*the p-adic q-deformation functor*".

Theorem 15.3. — q-Def *is a quasi-inverse of* Conf. *Moreover, it is canonically isomorphic to* $\phi_* \circ (q^{p^s}\text{-Def})$.

Proof. — By the quasi-unipotence theorems, it is clear that q-Def is an equivalence. In order to prove the first assertion, it is thus enough to see that q-Def is left quasi-inverse to Conf, which follows immediately from the fact that $\mathrm{Conf} \circ D_{\sigma_q}^{(\phi)} = D_d^{(\phi)}$.

For the second assertion, one remarks that q-Def(M) (*resp.* q^{p^s}-Def(M)) is the first (*resp.* the second) term in the sequence of modules defined by $D_{\sigma_q}^{\mathrm{conf}(\phi)} \circ V_d^{(\phi)}(M)$. □

(23) using tacitly the canonical isomorphism $D_{\sigma_q}^{(\phi)} \circ V_{\sigma_q}^{(\phi)} \cong \mathrm{Id}$.

15.3. Analytic slopes and exponents. — Let r' be in $]|q-1|,1[$, and λ be a non negative real number. By analogy with the differential case (*cf.* [**CM02**]), we introduce the following

Definition 15.4. — A q-difference module M over $\mathcal{A}_K([r',1[)$ has (analytic) *slopes* $\leqslant \lambda$ if there is a function

$$r \in]r',1[\longmapsto \rho(r) \in]r^{1+\lambda}, r]$$

such that $M \otimes_{\mathcal{A}_K([r,1[)} \mathcal{A}_\Omega(t_r, \rho(r))$ has a basis of elements fixed under Σ_q.

It is clear that this implies solvability in the sense of (12.19).

Since any q-difference module M over $\mathcal{R}_x$ is "defined over $\mathcal{A}_K([r,1[)$" for some r close enough to 1, this provides a definition of "having slopes $\leqslant \lambda$" for such modules. This property is stable by passage to subquotients, dual, tensor products and extensions. One checks exactly as in the differential case that for n prime to p, the base-change by $x \mapsto x^n$ of M is a $q^{1/n}$-difference module of slopes $\leqslant n\lambda$.

It is likely that the Christol-Mebkhout theory of slope filtrations carries over to the q-difference case, but we haven't checked all details.

We now assume for simplicity that the residue field k is algebraically closed.

***Example 15.5* (q-difference equations of rank 1 with Frobenius structure)**

We have seen in (7.3) that any q-difference equation over $\mathcal{R}$ with Frobenius structure has a non-zero solution in some finite unramified extension $\mathcal{E}^\dagger_{x'}$ of $\mathcal{E}^\dagger_x$, and that $y'/y \in \mathcal{E}^\dagger_x$. Thus the group isomorphism $\sigma\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger} \xrightarrow{\cong} d\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger}$ from (8.2) preserves the property of "having slopes $\leqslant \lambda$".

In particular, q-difference equations of rank 1 with Frobenius structure and slope 0 (*i.e.,* slope $\leqslant 0$) correspond to tame characters of $G_{k((x))}$ (via (8.2)), hence are of the form $d_q - [\alpha]_q/x$ with $\alpha \in \mathbb{Z}_p \cap \mathbb{Q}$.

Theorem 15.6. — *Any $M \in \sigma_q\text{-}Mod^{(\phi)}_{\mathcal{R}}$ with slope 0 has a filtration with graded pieces of rank one.*

Proof. — By quasi-unipotence, we may assume, after taking subquotients, that M corresponds to a representation V of $G_{k((x))}$ with finite image G (which is the semi-direct product of a cyclic group of order prime to p by a p-group, *cf.* [**S68**]). Representations of G then correspond to q-difference modules in the tannakian category generated by M, which have slope 0. By base-change by $x \mapsto x^n$ for suitable n prime to p, we may assume that G is a p-group, *i.e.,* that with are in the purely wild case. By the previous example, there is no one-dimensional representation of G which corresponds to a q-difference module of slope 0. Since G is a p-group, we conclude that it is trivial. □

It follows from that M is an iterated extension of q-difference modules of type $\mathcal{R}[d_q]/(d_q + [\alpha_i]_q)$, $i = 1, \dots, \mu = \operatorname{rk} M$, with $\alpha_i \in \mathbb{Z}_p \cap \mathbb{Q}$ well defined mod. $\mathbb{Z}$ and up

to permutation. Let us call these p-adic numbers (mod. $\mathbb{Z}$ and up to permutation) the *exponents* of M. Taking into account remark 8.3, we have:

Proposition 15.7. — *The functors* Conf *and* q-Def *preserve objects of (analytic) slope* 0 *and their exponents.*

References

[A01] Y. André – "Différentielles non commutatives et théorie de Galois différentielle ou aux différences", *Ann. Sci. École Norm. Sup.* **5** (2001), p. 1–55.

[A02] ———, "Filtrations de type Hasse-Arf et monodromie p-adique", *Invent. Math.* **148** (2002), no. 2, p. 285–317.

[A] ———, "Galois representations, differential equations and q-difference equations: sketch of a p-adic unification", this volume.

[CC96] B. Chiarellotto & G. Christol – "On overconvergent isocrystals and F-isocrystals of rank one", *Compositio Math.* **100** (1996), p. 77–99.

[C81a] G. Christol – "Décomposition des matrices en facteurs singuliers. Applications aux équations différentielles", in *Groupe d'étude sur l'analyse ultramétrique. 7^e-8^e années (Paris, 1979/1981)*, vol. 5, Secrétariat Math., Paris, 1981.

[C81b] ———, "Systèmes différentiels linéaires p-adiques, structure de Frobenius faible", *Bull. Soc. math. France* **109** (1981), no. 1, p. 83–122.

[C86] ———, "Fonctions et éléments algébriques", *Pacific J. Math.* **125** (1986), p. 1–37.

[C01] ———, "About a Tsuzuki theorem", in *p-adic functional analysis (Ioannina, 2000)*, Lecture Notes in Pure and Appl. Math., vol. 222, Dekker, New York, 2001, p. 63–74.

[CD94] G. Christol & B. Dwork – "Modules différentiels sur des couronnes", *Ann. Inst. Fourier (Grenoble)* **44** (1994), no. 3, p. 663–701.

[CM02] G. Christol & Z. Mebkhout – "Équations différentielles p-adiques et coefficients p-adiques sur les courbes", in *Cohomologies p-adiques et applications arithmétiques, II*, Astérisque, vol. 279, Société Mathématique de France, 2002, p. 125–183.

[Co01] P. Colmez – "Les conjectures de monodromie p-adiques", in *Sém. Bourbaki 2001/2002*, Astérisque, vol. 290, Société Mathématique de France, 2003, exp. n° 897, p. 53–101.

[Cr87] R. Crew – "F-isocrystals and p-adic representations", in *Algebraic geometry, Bowdoin, 1985 , Part 2 (Brunswick, Maine, 1985)*, Proc. Sympos. Pure Math., vol. 46, American Mathematical Society, Providence, RI, 1987, p. 111–138.

[Cr98] ———, "Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve", *Ann. Sci. École Norm. Sup.* **31** (1998), p. 717–763.

[DV02] L. Di Vizio – "Arithmetic theory of q-difference equations: the q-analog of Grothendieck-Katz's conjecture on p-curvatures", *Invent. Math.* **150** (2002), no. 3, p. 517–578.

[DV03] ———, "Introduction to p-adic q-difference equations (weak Frobenius structure and transfer theorems)", in *Geometric Aspects of Dwork's Theory*, vol. 2, de Gruyter, 2004, p. 615–675.

[D83] B. Dwork – "On the Boyarsky principle", *Amer. J. Math.* **285** (1983), p. 115–156.

[DR77] B. Dwork & P. Robba – "On ordinary linear p-adic differential equations", *Trans. Amer. Math. Soc.* **231** (1977), no. 1, p. 1–46.

[E02] J.-Y. Étesse – "Relèvement de schémas et algèbres de Monsky-Washnitzer: théorèmes d'équivalence et de pleine fidélité", *Rend. Sem. Mat. Univ. Padova* **107** (2002), p. 111–138.

[HL46] G.H. Hardy & J.E. Littlewood – "Notes on the theory of series XXIV. A curious power-series", *Math. Proc. Cambridge Philos. Soc.* **42** (1946), p. 85–90.

[Ka86] N. Katz – "Local-to-global extensions of representations of fundamental groups", *Ann. Inst. Fourier (Grenoble)* **36** (1986), p. 69–106.

[Ke] K. Kedlaya – "A p-adic monodromy theorem", to appear in *Ann. of Math.*

[Ko80] N. Koblitz – "q-extension of the p-adic gamma function", *Trans. Amer. Math. Soc.* **260** (1980), no. 2, p. 449–457.

[Ko82a] ———, "q-extension of the p-adic gamma function. II", *Trans. Amer. Math. Soc.* **273** (1982), no. 1, p. 111–129.

[Ko82b] ———, "p-adic analog of Heine's hypergeometric q-series", *Pacific J. Math.* **102** (1982), no. 2, p. 373–383.

[Ma95] S. Matsuda – "Local indices of p-adic differential operators corresponding to Artin-Schreier-Witt coverings", *Duke Math. J.* **77** (1995), p. 607–625.

[Mo77] E. Motzkin – "La décomposition d'un élément analytique en facteurs singuliers", *Ann. Inst. Fourier (Grenoble)* **27** (1977), p. 67–82.

[Q] C. Quesne – "Disentangling q-exponentials: A general approach", arXiv: `math-ph/0310038`.

[Ra98] L. Ramero – "On a class of étale analytic sheaves", *J. Algebraic Geom.* **7** (1998), no. 3, p. 405–504.

[R85] P. Robba – "Indice d'un opérateur différentiel p-adique IV. Cas des systèmes. Mesure de l'irrégularité dans un disque", *Ann. Inst. Fourier (Grenoble)* **35** (1985), p. 13–55.

[RC94] P. Robba & G. Christol – *Équations différentielles p-adiques. Applications aux sommes exponentielles*, Actualités Mathématiques, Hermann, Paris, 1994.

[Ro00] A. Robert – *A course in p-adic analysis*, Graduate Texts in Math., vol. 198, Springer-Verlag, New York, 2000.

[S68] J.-P. Serre – *Corps locaux*, Hermann, Paris, 1968.

[T98a] N. Tsuzuki – "The local index and the Swan conductor", *Compositio Math.* **111** (1998), p. 245–288.

[T98b] ———, "Finite local monodromy of overconvergent unit-root F-isocrystals on a curve", *Amer. J. Math.* **120** (1998), p. 1165–1190.

Y. André, École Normale Supérieure, 45 rue d'Ulm, 75005 Paris (France)
E-mail : `yves.andre@ens.fr`

L. Di Vizio, Laboratoire Émile Picard, Université Paul Sabatier, U.F.R. M.I.G., 118, route de Narbonne, 31062 Toulouse cedex 4 (France) • *E-mail :* `divizio@picard.ups-tlse.fr`

Astérisque
296, 2004, p. 113–143

RENORMALISATION ET AMBIGUÏTÉ GALOISIENNE

par

Alain Connes

***Résumé.* —** Cet article donne un exposé détaillé de mes résultats sur la renormalisation en collaboration avec D. Kreimer. Le premier résultat essentiel est l'identité entre le procédé récursif utilisé par les physiciens pour éliminer les divergences en théorie des champs quantiques et la décomposition de Birkhoff des lacets à valeurs dans un groupe pro-unipotent. Le groupe impliqué dans la renormalisation est celui des difféographismes, construit à partir des graphes de Feynman. Le deuxième résultat important est la construction d'une action de ce groupe des difféographismes sur les constantes de couplage sans dimension de la théorie physique. Le lien précis entre mon travail avec Kreimer et la correspondance de Riemann-Hilbert a été obtenu en collaboration avec M. Marcolli et est explicité à la fin de ce texte. Nous établissons une correspondance de Riemann-Hilbert entre connexions plates équisingulières et représentations d'un groupe de Galois « motivique » explicite U^*. Ce groupe joue dans le contexte de la renormalisation un rôle analogue à celui du tore exponentiel de Jean-Pierre Ramis dans la théorie locale des singularités irrégulières des équations différentielles et il apporte une réponse satisfaisante à la recherche proposée par P. Cartier d'un groupe de Galois « cosmique » qui sous-tend la renormalisation.

***Abstract* (Renormalisation and Galois Ambiguity). —** This paper contains a detailed exposition of my joint work with Kreimer on renormalization. The first key result is the identity between the recursive process used by physicists to remove the divergencies in quantum field theory and the Birkhoff decomposition of loops with values in a pro-unipotent Lie group. The relevant group for renormalization is the group of diffeographisms which is constructed from Feynman graphs. The second key result is the construction of an action of the group of diffeographisms on the dimensionless coupling constants of the theory. The precise link between my work with Kreimer and the Riemann-Hilbert correspondence was obtained in collaboration with M. Marcolli and is explained briefly at the end of the paper. We construct a Riemann-Hilbert correspondence between flat equisingular connections and representations of a specific motivic Galois group U^*. This group is the analogue in renormalization of the exponential torus of Ramis in the local theory of irregular singular differential equations. Our work gives a natural candidate for the "cosmic Galois group"envisaged by Cartier as the symmetry underlying renormalization.

***Classification mathématique par sujets* (2000). —** Primary 58B34 ; Secondary 81T15, 11R32.
***Mots clefs.* —** Renormalisation, Galois, Hopf.

© Astérisque 296, SMF 2004

1. Introduction

La renormalisation est sans doute l'un des procédés les plus élaborés pour obtenir des quantités numériques signifiantes à partir d'expressions mathématiques a priori dépourvues de sens. A ce titre, elle est fascinante autant pour le physicien que pour le mathématicien. La profondeur de ses origines en théorie des champs et la précision avec laquelle elle est corroborée par l'expérience en font l'un des joyaux de la physique théorique. Pour le mathématicien épris de sens, mais non corseté par la rigueur, les explications données jusqu'à présent butaient toujours sur le sens conceptuel de la partie proprement calculatoire, celle qui est utilisée par exemple en électrodynamique quantique et ne tombe pas sous la coupe des « théories asymptotiquement libres » auxquelles la théorie constructive peut prétendre avoir donné un statut mathématique satisfaisant. Cet état de fait a changé récemment et cet exposé se propose de donner la signification conceptuelle des calculs effectués par les physiciens dans la théorie de la renormalisation grâce à mon travail sur la renormalisation en collaboration avec Dirk Kreimer et la relation que nous avons établie entre renormalisation et problème de Riemann-Hilbert.

Le résultat clé est l'identité entre le procédé récursif utilisé par les physiciens et les formules mathématiques qui résolvent une application $\gamma : C \mapsto G$ d'un cercle $C \subset S^2$ à valeurs dans un groupe pronilpotent G en un rapport d'applications holomorphes $\gamma_\pm : C_\pm \mapsto G$ des composantes connexes du complémentaire de C dans S^2. La signification géométrique de cette décomposition (de Birkhoff ou Wiener-Hopf) provient directement de la théorie des fibrés holomorphes de groupe structural G sur la sphère de Riemann S^2.

Dans la renormalisation perturbative, les points de la sphère S^2 sont les dimensions complexes parmi lesquelles la dimension D de l'espace-temps est un point privilégié. Le problème étant que dans les théories physiquement intéressantes les quantités à calculer conspirent pour diverger précisément au point D. On peut organiser ces quantités comme élément $g \in G$ d'un groupe pronilpotent G par analogie avec le développement de Taylor d'un difféomorphisme et donner un sens à $g = g(z)$ en remplaçant dans les formules la dimension D par une valeur complexe $z \neq D$. Le procédé de renormalisation acquiert alors la signification suivante : la valeur cherchée $g \in G$ n'est autre que la valeur $g_+(D)$ en D de la partie holomorphe de la décomposition de Riemann-Hilbert $g(z) = g_-^{-1}(z)\, g_+(z)$ du lacet $g(z)$.

La nature exacte du groupe G impliqué dans la renormalisation a été clarifiée par les étapes essentielles suivantes.

La première est la découverte due à Dirk Kreimer de la structure d'algèbre de Hopf secrètement présente dans les formules récursives de Bogoliubov, Parasiuk, Hepp et Zimmermann.

La seconde (qui est le point de départ de notre collaboration) est la similitude entre l'algèbre de Hopf des arbres enracinés de Dirk et une algèbre de Hopf que

j'avais introduite avec Henri Moscovici pour organiser les calculs très complexes de géométrie noncommutative. Ceci nous a conduit avec Dirk à définir une algèbre de Hopf directement en termes de graphes de Feynman et à lui appliquer le théorème de Milnor-Moore pour en déduire une algèbre de Lie et un groupe de Lie pronilpotent G, analogue du groupe des difféomorphismes formels.

La troisième étape est l'identification de la recette combinatoire de Bogoliubov-Parasiuk-Hepp-Zimmermann avec la formule mathématique qui donne, par récurrence, la décomposition de Birkhoff d'un lacet à valeurs dans un groupe de Lie simplement connexe pronilpotent.

Enfin, la dernière étape est la construction d'une action du groupe G sur les constantes de couplage de la théorie physique. Ceci permet de relever le groupe de renormalisation comme un sous-groupe à un paramètre du groupe G et de montrer directement que les développements polaires des divergences sont entièrement déterminés par leur résidu.

En fait, ce groupe G est intimement relié au groupe des difféomorphismes formels des constantes de couplage sans dimension de la théorie physique. L'un des résultats essentiels de notre collaboration avec Dirk est en effet la construction, à partir de la formule qui donne la valeur *effective* d'une telle constante de couplage, d'un homomorphisme de G dans le groupe des difféomorphismes formels tangents à l'identité. Cela éclaire la nature du groupe G qu'il serait naturel d'appeler le groupe de *difféographismes* de la théorie. Cela permet aussi de formuler un corollaire indépendant du groupe G,

Théorème 1.1 **([14])**. — *Considérons la constante de couplage effective non renormalisée* $g_{\mathrm{eff}}(\varepsilon)$ *comme une série formelle en* g *et soit* $g_{\mathrm{eff}}(\varepsilon) = g_{\mathrm{eff}_+}(\varepsilon)\,(g_{\mathrm{eff}_-}(\varepsilon))^{-1}$ *sa décomposition de Birkhoff (opposée) dans le groupe des difféomorphismes formels. Alors le lacet* $g_{\mathrm{eff}_-}(\varepsilon)$ *est la constante de couplage nue et* $g_{\mathrm{eff}_+}(0)$ *la constante de couplage renormalisée.*

Il est naturel d'interpréter en termes galoisiens l'ambiguïté que le groupe de renormalisation introduit dans les théories physiques. La formulation mathématique du groupe de renormalisation comme un sous-groupe à un paramètre du groupe des difféographismes dans la section 5 permet de préciser cette question.

Nous montrerons le rôle que le groupe de renormalisation devrait jouer pour comprendre la composante connexe du groupe des classes d'idèles de la théorie du corps de classe comme un groupe de Galois. Cette idée s'appuie à la fois sur l'analogie entre la théorie des facteurs et la théorie de Brauer pour un corps local et sur la présence implicite en théorie des champs d'un « corps de constantes » plus élaboré que le corps $\mathbb{C}$ des nombres complexes. En fait, les calculs des physiciens regorgent d'exemples de « constantes » telles les constantes de couplage g des interactions (électromagnétiques, faibles et fortes) qui n'ont de « constantes » que le nom. Elles dépendent, en réalité, du niveau d'énergie μ auquel les expériences sont réalisées et sont des fonctions $g(\mu)$,

de sorte que les physiciens des hautes énergies étendent implicitement le « corps des constantes » avec lequel ils travaillent, passant du corps $\mathbb{C}$ des scalaires à un corps de fonctions $g(\mu)$. Le groupe d'automorphismes de ce corps engendré par $\mu\partial/\partial\mu$ est le groupe d'ambiguïté de la théorie physique.

Je terminerai cet article par un bref exposé de résultats récents obtenus en collaboration avec M. Marcolli ([**17**]) qui établissent enfin un lien précis entre renormalisation et théorie de Galois, en expliquant le rôle de la décomposition de Birkhoff en renormalisation par une correspondance de Riemann-Hilbert entre connexions plates *équisingulières* et représentations d'un *groupe de Galois motivique* explicite U^*. Ce groupe joue dans le contexte de la renormalisation un rôle analogue à celui du tore exponentiel dû à Jean-Pierre Ramis ([**35, 33**]) dans la théorie locale des singularités irrégulières des équations différentielles.

L'apparition naturelle de ce groupe comme enveloppe du groupe de renormalisation, et sa détermination comme produit semi-direct du groupe multiplicatif $\mathbb{G}_m$ par le groupe pro-unipotent U dont l'algèbre de Lie est l'algèbre libre

$$\mathrm{Lie}(U) = \mathcal{F}(1, 2, 3, \dots).$$

ayant un générateur en chaque degré positif, ouvrent la porte vers des relations très précises entre la théorie des motifs, le groupe de Grothendieck-Teichmuller et la renormalisation, relations suggérées par P. Cartier dans [**4**].

2. Renormalisation, position du problème

2.1. Motivation physique. — L'idée physique de la renormalisation est très claire et remonte aux travaux de Green au dix-neuvième siècle sur l'hydrodynamique. Pour prendre un exemple simple (voir le cours de théorie des champs de Sidney Coleman), si l'on calcule l'accélération initiale d'une balle de ping-pong plongée à quelques mètres sous l'eau, l'on obtient en appliquant la loi de Newton $F = m\,a$ à la poussée d'Archimède $F = (M - m)\,g$, où m est la masse inerte, et M la masse d'eau occupée par la balle, une accélération initiale de l'ordre de $11\,g$! (la balle pèse $m = 2,7$ grammes et a un diamètre de 4 cm de sorte que $M = 33,5$ grammes). En réalité, si l'on réalise l'expérience, l'accélération est de l'ordre de $2g$. En fait, comme le montre Green [**27**] la présence du fluide autour de la balle oblige à corriger la valeur m de la masse inerte dans la loi de Newton et à la remplacer par une « masse effective » qui en l'occurrence vaut $m + \frac{1}{2}\,M$. Dans cet exemple, l'on peut, bien sûr, déterminer la masse nue m en pesant la balle de ping-pong hors de l'eau, mais il n'en va pas de même pour un électron dans le champ électromagnétique, dont il est impossible de l'extraire. De plus, le calcul montre que pour une particule ponctuelle, comme le demande la relativité, la correction qui valait $\frac{1}{2}\,M$ ci-dessus est infinie.

2.2. Champs quantiques. — Vers 1947 les physiciens ont réussi à utiliser la distinction entre les deux masses qui apparaissent ci-dessus et, plus généralement, le concept de quantité physique « effective » pour éliminer les quantités infinies qui apparaissent en théorie des champs quantiques (voir [**21**] pour un aperçu historique).

Une théorie des champs en D dimensions est donnée par une fonctionnelle d'action classique

$$S(A) = \int \mathcal{L}(A)\, d^D x \tag{1}$$

où $A = A(x) = A(x^\mu)$ désigne un champ classique et le lagrangien est de la forme,

$$\mathcal{L}(A) = (\partial A)^2/2 - \frac{m^2}{2}\, A^2 - \mathcal{L}_{\text{int}}(A) \tag{2}$$

où $(\partial A)^2 = (\partial_0 A)^2 - \sum_{\mu\neq 0}(\partial_\mu A)^2$ et $\mathcal{L}_{\text{int}}(A)$ est un polynôme en A.

Le passage de la théorie classique à la théorie quantique est simple à décrire a priori. Il consiste d'abord à remplacer la notion classique de probabilité par celle (quantique) d'*amplitude de probabilité*.

En théorie des champs quantiques l'amplitude de probabilité d'une configuration classique A est donnée par la formule de Dirac et Feynman,

$$e^{i\, S(A)/\hbar} \tag{3}$$

où $\hbar$ est l'unité d'action de sorte que l'exposant $i\, S(A)/\hbar$ est sans dimension.

Cela permet alors de *définir* la valeur quantique d'une observable classique (*i.e.,* d'une fonctionnelle $\mathcal{O}$ sur les champs classiques) par

$$\langle \mathcal{O} \rangle = \mathcal{N} \int \mathcal{O}(A) e^{i\, S(A)/\hbar}\, D[A], \tag{4}$$

où $\mathcal{N}$ est un facteur de normalisation et où l'intégrale (de Feynman) n'a qu'un sens formel mais qui suffit largement, dans le cas où l'espace des champs classiques A est un espace linéaire, pour formuler sans difficulté les termes des développements perturbatifs qui font apparaître le problème de la renormalisation.

On peut, par exemple, décrire la théorie par les fonctions de Green,

$$G_N(x_1, \ldots, x_N) = \langle\, 0\, |T\, \phi(x_1) \ldots \phi(x_N)|\, 0\, \rangle \tag{5}$$

où le symbole T signifie que les champs quantiques $\phi(x_j)$ sont écrits à temps croissant de droite à gauche et si l'on pouvait ignorer les problèmes de renormalisation, l'on pourrait calculer les fonctions de Green grâce à la formule

$$G_N(x_1, \ldots, x_N) = \mathcal{N} \int e^{i\, S(A)/\hbar}\, A(x_1) \cdots A(x_N)\, [dA] \tag{6}$$

où $\mathcal{N}$ est un facteur de normalisation requis par la normalisation de l'état de vide,

$$\langle\, 0 \mid 0\, \rangle = 1. \tag{7}$$

L'on pourrait alors calculer l'intégrale fonctionnelle (6) en théorie des perturbations en traitant le terme $\mathcal{L}_{\text{int}}$ comme une perturbation, le lagrangien libre étant

$$\mathcal{L}_0(A) = (\partial A)^2/2 - \frac{m^2}{2}\, A^2, \tag{8}$$

de sorte que

$$S(A) = S_0(A) + S_{\text{int}}(A) \tag{9}$$

où l'action libre S_0 définit une mesure gaussienne (en prenant $\hbar = 1$ pour l'instant)

$$\exp(i\, S_0(A))\, [dA] = d\mu. \tag{10}$$

On obtient alors le développement perturbatif des fonctions de Green sous la forme,

$$G_N(x_1, \dots, x_N) = \left(\sum_{n=0}^{\infty} i^n/n! \int A(x_1) \cdots A(x_N)\, (S_{\text{int}}(A))^n\, d\mu \right) \left(\sum_{n=0}^{\infty} i^n/n! \int S_{\text{int}}(A)^n\, d\mu \right)^{-1}. \tag{11}$$

2.3. Diagrammes de Feynman. — Les termes du développement (11) sont, par construction, donnés par des intégrales de polynômes sous la gaussienne $d\mu$ de la forme,

$$\int A(x_1) \cdots A(x_N)\, (S_{\text{int}}(A))^n\, d\mu \tag{12}$$

Ils s'obtiennent en intégrant par parties sous la gaussienne $d\mu$ et cela engendre un grand nombre de termes $U(\Gamma)$. Les graphes de Feynman Γ sont des données combinatoires qui servent à indicer les termes qui apparaissent dans le calcul de ces intégrales de polynômes sous une gaussienne. Ce sont des graphes dont les sommets peuvent être de plusieurs types correspondant aux termes du lagrangien de la théorie, et que nous définirons plus en détail ci-dessous. Les valeurs des termes $U(\Gamma)$ (que l'on appelle valeurs non-renormalisées du graphe Γ) sont données par des intégrales en un nombre fini de variables d'espace-temps.

L'on commence par simplifier notablement la combinatoire des graphes par des définitions convenables de fonctions génératrices. La fonctionnelle génératrice des fonctions de Green est donnée par la transformée de Fourier,

$$\begin{aligned} Z(J) &= \mathcal{N} \int e^{i\frac{S(A)+\langle J,A\rangle}{\hbar}}\, [dA] \\ &= \sum_{n=0}^{\infty} \frac{i^N}{N!} \int J(x_1) \cdots J(x_N) G_N(x_1, \dots, x_N) dx_1 \cdots dx_N \end{aligned} \tag{13}$$

où la *source* J est un élément du dual de l'espace linéaire des champs classiques A.

La zoologie des diagrammes de la théorie perturbative se simplifie d'abord en passant au logarithme de $Z(J)$ qui donne la fonctionnelle génératrice pour les fonctions de Green connexes G_c,

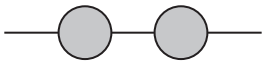
$$iW(J) = \mathrm{Log}(Z(J)) = \sum_{n=0}^{\infty} \frac{i^N}{N!} \int J(x_1) \cdots J(x_N)\, G_{N,c}(x_1, \ldots, x_N)\, dx_1 \cdots dx_N \tag{14}$$

Au niveau combinatoire formel, alors que la somme initiale (13) impliquait tous les graphes y compris ceux qui ne sont pas connexes, le log dans (14) pour $W(J)$ supprime tous les graphes non-connexes. De plus, le facteur de normalisation $\mathcal{N}$ dans (13) élimine tous les graphes sans arêtes externes. Enfin, le nombre L de boucles dans un graphe connexe détermine la puissance $\hbar^{L-1}$ de l'unité d'action qui multiplie ce terme et fait apparaître (14) comme un développement semi-classique.

Le pas suivant dans la simplification de la zoologie des graphes consiste à passer à l'*action effective* $S_{\mathrm{eff}}(A)$. Par définition, $S_{\mathrm{eff}}(A)$ est la transformée de Legendre de $W(J)$.

L'action effective donne les corrections quantiques de l'action classique. Par sa définition comme transformée de Legendre on voit que les calculs au niveau arbre (par la méthode de la phase stationnaire) à partir de $S_{\mathrm{eff}}(A)$ donnent les mêmes résultats que le calcul complet à partir de $S(A)$. Il en résulte que le calcul de l'action effective est un pas crucial dans l'analyse d'une théorie donnée.

Comme ci-dessus, l'action effective admet un développement formel en termes de graphes. En fait, cela élimine tous les graphes qui deviennent non-connexes quand on enlève une arête convenable. Les graphes restant sont appelés *Une Particule Irréductible* (1PI). Le graphe suivant n'est donc pas (1PI) :



La contribution de chacun de ces graphes Γ à la fonctionnelle non-linéaire $S_{\mathrm{eff}}(A)$ est donnée explicitement de la manière suivante, où N désigne le nombres d'arêtes externes de Γ :

$$\Gamma(A) = \frac{1}{N!} \int_{\sum p_j = 0} \widehat{A}(p_1) \cdots \widehat{A}(p_N)\, U(\Gamma(p_1, \ldots, p_N))\, dp_1 \cdots dp_N. \tag{15}$$

Ici $\widehat{A}(p)$ est la transformée de Fourier de $A(x)$ et la valeur *non-renormalisée*

$$U(\Gamma(p_1, \ldots, p_N)) \tag{16}$$

du graphe est définie en appliquant des règles simples (celles de Feynman) qui remplacent chaque arête *interne*[(1)] par un propagateur, *i.e.,* un terme de la forme

$$\frac{1}{k^2 - m^2}\,, \tag{17}$$

[(1)]Les propagateurs des arêtes externes sont éliminés pour les graphes (1PI)

où k est le moment le long de l'arête. Il n'y a rien de mystérieux dans l'apparition du propagateur (17) qui joue le rôle de l'inverse de la forme quadratique S_0 et provient simplement de la règle d'intégration par parties

$$\int f(A)\,\langle J, A\rangle\,\exp(i\,S_0(A))\,[dA] = \int \partial_X f(A)\,\exp(i\,S_0(A))\,[dA]$$

pourvu que

$$-i\,\partial_X S_0(A) = \langle J, A\rangle.$$

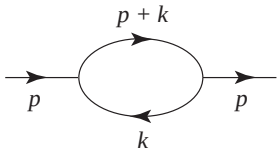
On intègre alors sur les moments k qui subsistent après avoir appliqué à chaque sommet la règle de conservation des moments (la somme des moments entrants est égale à zéro). Le nombre de variables d'intégration restantes est le nombre de boucles L du graphe Γ.

L'action effective est alors donnée par la série formelle

$$S_{\text{eff}}(A) = S_0(A) + \sum_{\Gamma\in 1PI} \frac{\Gamma(A)}{S(\Gamma)}, \tag{18}$$

où le facteur $S(\Gamma)$ est l'ordre du groupe de symétrie du graphe.

2.4. Divergences. — En règle générale, les intégrales $U(\Gamma(p_1,\dots,p_N))$ sont divergentes. La plus simple (avec le graphe correspondant) est de la forme (en ignorant les puissances de 2π et après rotation de Wick aux variables euclidiennes),

(19) 

$$= \int \frac{1}{k^2+m^2}\,\frac{1}{((p+k)^2+m^2)}\,d^Dk$$

Elle diverge en dimension $D = 4$. En général, les divergences les plus importantes sont causées par la présence, dans le domaine d'intégration, de moments de taille arbitrairement grande. La technique de renormalisation consiste d'abord à « régulariser » ces intégrales divergentes, par exemple, en introduisant un paramètre de « cutoff » Λ et en se restreignant à la portion correspondante du domaine d'intégration. Les intégrales sont alors finies, mais continuent, bien entendu, à diverger quand $\Lambda \to \infty$. On établit ensuite une dépendance entre les termes du lagrangien et Λ pour que les choses s'arrangent et que les résultats ayant un sens physique deviennent finis ! Cela est parfaitement justifié du point de vue de la physique par la distinction nécessaire entre les paramètres *nus* qui ne sont pas observables et les valeurs qui sont mesurées dans les expériences. En ajustant les paramètres *nus* au fur et à mesure que l'on enlève le « cutoff » on peut alors espérer éliminer les divergences. Par exemple, pour la théorie ϕ^3 de lagrangien

$$\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{6}\phi^3, \tag{20}$$

qui est suffisamment générique pour illustrer le procédé, le lagrangien va dépendre du « cutoff » comme

$$\frac{1}{2}(\partial_\mu\phi)^2(1-\delta Z(\Lambda)) - \left(\frac{m^2+\delta m^2(\Lambda)}{2}\right)\phi^2 - \frac{g+\delta g(\Lambda)}{6}\phi^3. \tag{21}$$

Dans le cas particulier des théories asymptotiquement libres, la forme explicite de la dépendance entre les constantes nues et le paramètre de régularisation Λ a permis dans des cas très importants ([**24**], [**22**]) de mener à bien le programme de la théorie constructive des champs ([**25**]).

Décrivons maintenant en détail la technique de renormalisation perturbative. Pour faire les choses systématiquement, on rajoute un « contre-terme » $C(\Gamma)$ au lagrangien de départ $\mathcal{L}$, chaque fois que l'on rencontre un diagramme 1PI qui est divergent, dans le but d'annuler la divergence correspondante. Pour les théories « renormalisables », les contre-termes $C(\Gamma)$ dont on a besoin sont tous déjà des termes du lagrangien $\mathcal{L}$ et ces contorsions peuvent s'interpréter à partir de l'inobservabilité des quantités numériques qui apparaissent dans $\mathcal{L}$, par opposition aux quantités physiques qui, elles, doivent rester finies.

La méthode du « cutoff » n'est pas agréable à mettre en pratique et les physiciens lui préfèrent la *régularisation dimensionnelle*, qui consiste simplement à définir l'intégration en D-dimension lorsque $D \notin \mathbb{N}$ n'est plus un entier. Ceci est très simple en se ramenant à des intégrales gaussiennes qui en dimension entière sont données par des expressions simples (23) qui continuent à avoir un sens en dimension arbitraire.

On utilise donc d'abord le passage aux *paramètres de Schwinger* que nous explicitons dans l'exemple ci-dessus :

$$\frac{1}{k^2+m^2}\,\frac{1}{(p+k)^2+m^2} = \int_{s>0\,t>0} e^{-s(k^2+m^2)-t((p+k)^2+m^2)}\,ds\,dt \tag{22}$$

puis, après avoir diagonalisé la forme quadratique qui apparaît en exposant, la formule suivante pour l'intégrale d'une gaussienne en dimension D,

$$\int e^{-\lambda\,k^2}\,d^Dk = \pi^{D/2}\,\lambda^{-D/2}, \tag{23}$$

ce qui donne la valeur non renormalisée du graphe (19) en dimension D sous la forme

$$\begin{aligned}\int_0^1\int_0^\infty e^{-(y(x-x^2)p^2+y\,m^2)}\int e^{-y\,k^2}\,d^Dk\,y\,dy\,dx \\ = \pi^{D/2}\int_0^1\int_0^\infty e^{-(y(x-x^2)p^2+y\,m^2)}\,y^{-D/2}\,y\,dy\,dx \\ = \pi^{D/2}\,\Gamma(2-D/2)\int_0^1\big((x-x^2)p^2+\,m^2\big)^{D/2-2}\,dx.\end{aligned} \tag{24}$$

L'intégrale restante se calcule en termes de fonctions hypergéométriques, mais le point essentiel est la singularité de la fonction Γ pour $D \in 4+2\,\mathbb{N}$. Notons que l'on a alors $D/2-2 \in \mathbb{N}$ de sorte que le coefficient du pôle est un polynôme en p et sa

transformée de Fourier est un terme *local*. Cela ne suffit pas pour que la théorie soit renormalisable ; si, par exemple $D = 8$, le coefficient du pôle est de degré 4 et la théorie n'est pas renormalisable. Quand $D = 6$, par contre, le coefficient du pôle est de degré 2 ; c'est un terme du lagrangien ce qui permet d'éliminer la divergence en ajoutant des contre-termes convenables $\delta Z(\varepsilon)$ et $\delta m^2(\varepsilon)$ au lagrangien de départ.

2.5. Sous divergences. — La principale complication dans la mise en œuvre de la procédure ci-dessus vient de l'existence de nombreux graphes Γ pour lesquels les divergences de $U(\Gamma)$ ne sont pas locales. Ceci se produit dès que l'on rencontre des pôles d'ordre > 1 dans la régularisation dimensionnelle, ce qui fait apparaître comme coefficient du terme en $1/\varepsilon$ les dérivées en D d'expressions comme $\int_0^1((x - x^2)p^2 + m^2)^{D/2-2}\,dx$ qui ne sont plus polynomiales en p même pour les valeurs entières de $D/2 - 2$ mais qui impliquent des termes comme $\log(p^2 + 4m^2)$. Un autre problème est que ces graphes possèdent déjà des sous-graphes dont les divergences doivent normalement être prises en compte avant d'aller plus loin. On a ainsi deux difficultés,

– Les divergences de $U(\Gamma)$ ne sont plus données par des termes locaux.

– Les corrections précédentes (celles des sous-graphes) doivent être prises en compte de manière cohérente.

Ces deux problèmes sont résolus par une méthode combinatoire précise, due à Bogoliubov-Parasiuk-Hepp et Zimmermann ([**2**]) et qui consiste d'abord à « préparer » le graphe Γ en remplaçant $U(\Gamma)$ par l'expression formelle,

$$\overline{R}(\Gamma) = U(\Gamma) + \sum_{\gamma\subset\Gamma} C(\gamma)U(\Gamma/\gamma) \tag{25}$$

où γ varie parmi tous les sous-graphes divergents. On montre alors que le calcul des divergences du graphe « préparé » ne donne que des expressions locales, qui pour les théories renormalisables se trouvent déjà dans le lagrangien $\mathcal{L}$.

3. Structure algébrique des graphes de Feynman

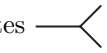
Dirk Kreimer a eu l'idée remarquable en 97 ([**31**]) d'interpréter la combinatoire des sous-divergences en terme d'une algèbre de Hopf $\mathcal{H}_K$ construite à partir des arbres « enracinés ».

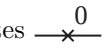
J'avais à la même époque, dans les calculs de Géométrie Noncommutative de l'indice transverse pour les feuilletages, montré, avec Henri Moscovici, ([**19**]) que la complexité extrême de ces calculs conduisait à introduire une algèbre de Hopf $\mathcal{H}_{cm}$, qui n'est ni commutative ni cocommutative mais qui est intimement reliée au groupe des difféomorphismes, dont l'algèbre de Lie apparaît en appliquant le théorème de Milnor-Moore à une sous-algèbre commutative.

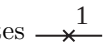
Après l'exposé de Dirk à l'IHES en février 98, nous avons tous les deux été intrigués par la similarité apparente entre ces deux algèbres de Hopf $\mathcal{H}_K$ et $\mathcal{H}_{cm}$ et notre collaboration a commencé par l'application du théorème de Milnor-Moore à l'algèbre de Hopf $\mathcal{H}_K$ et sa caractérisation en termes de cohomologie de Hochschild ([**10**]).

3.1. L'algèbre de Hopf des graphes de Feynman. — On peut, en fait ([**13**]), associer directement une algèbre de Hopf $\mathcal{H}$ à toute théorie renormalisable en utilisant la formule (25) pour définir le coproduit. Pour fixer les idées et simplifier les notations nous l'expliciterons pour la théorie φ^3 en $D = 6$ dimensions.

Les graphes sont construits à partir de sommets de trois types correspondants aux trois termes du lagrangien [(2)]

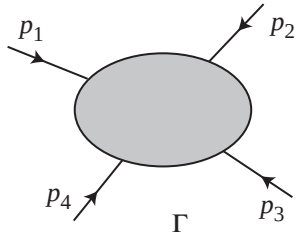
– Sommet à trois arêtes  associé au terme φ^3.

– Sommet à deux arêtes  associé au terme φ^2.

– Sommet à deux arêtes  associé au terme $(\partial\varphi)^2$.

Le nombre d'arêtes partant d'un sommet est le degré du monôme correspondant du lagrangien. Toute arête joint deux sommets distincts (arête interne) ou n'aboutit qu'à un seul sommet (arête externe).

Comme nous l'avons vu plus haut la valeur numérique $U(\Gamma(p_1, \ldots, p_N))$ d'un graphe dépend de la valeur des moments p_j *entrants* attachés aux arêtes externes du graphe



qui vérifient la règle de conservation des moments,

$$\sum p_i = 0.$$

Par définition, comme nous l'avons vu plus haut, un graphe Γ est (1PI) s'il est connexe et le reste quand on supprime l'une quelconque de ses arêtes internes.

En tant qu'algèbre, l'algèbre de Hopf $\mathcal{H}$ des graphes de Feynman est l'algèbre commutative libre engendrée par les graphes $\Gamma(p_1, \ldots, p_N)$ qui sont (1PI). Il est, en

[(2)] Pour la théorie de masse nulle le terme en φ^2 est absent ainsi que le type de sommet correspondant.

fait, plus pratique par linéarité d'autoriser comme structure externe une distribution arbitraire $\sigma : C^\infty(E_\Gamma) \to \mathbb{C}$ définie sur l'espace des fonctions C^∞ sur

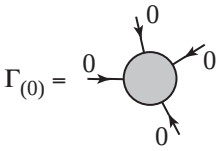
$$E_\Gamma = \left\{ (p_i)_{i=1,\dots,n} \; ; \; \sum p_i = 0 \right\}. \tag{26}$$

L'algèbre $\mathcal{H}$ est alors l'algèbre symétrique sur l'espace linéaire somme directe des espaces de distributions $C_c^{-\infty}(E_\Gamma)$; en d'autres termes,

$$\mathcal{H} = S(C_c^{-\infty}(\cup E_\Gamma)). \tag{27}$$

Nous utiliserons la notation $\Gamma_{(0)}$ pour indiquer la structure externe sur le graphe Γ donnée par la masse de Dirac en $0 \in E_\Gamma$,[3]

$$\Gamma_{(0)} = (\Gamma(p))_{p=0} \tag{28}$$



Pour les graphes à deux arêtes externes Γ nous noterons

$$\Gamma_{(1)} = \left(\frac{\partial}{\partial\, p^2} \, \Gamma(p) \right)_{p=0} \tag{29}$$

où $\pm p$ est le moment externe dont le signe est sans importance.

Pour définir le coproduit,

$$\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H} \tag{30}$$

il suffit de le donner sur les graphes 1PI ; on a

$$\Delta\Gamma = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma_{(i)} \otimes \Gamma / \gamma_{(i)}. \tag{31}$$

Ici, γ est un sous-ensemble non vide et de complémentaire non-vide $\gamma \subset \Gamma^{(1)}$ de l'ensemble $\Gamma^{(1)}$ des arêtes internes de Γ dont les composantes connexes γ' sont 1PI et telles que l'ensemble $\varepsilon(\gamma')$ des arêtes de Γ qui rencontrent γ' sans appartenir à γ' ait deux ou trois éléments [**13**]. On note $\gamma'_{(i)}$ le graphe qui admet γ' comme ensemble d'arêtes internes et $\varepsilon(\gamma')$ comme ensemble d'arêtes externes avec pour structure externe (28) pour $i = 0$ et (29) pour $i = 1$.

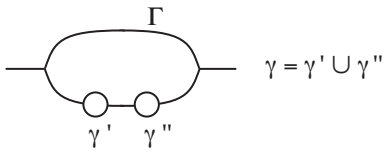
La somme (31) porte sur tous les multi-indices i de valeurs indépendantes sur les composantes connexes de γ. Ceux-ci prennent les valeurs 0 ou 1 pour les graphes à deux arêtes externes et la valeur 0 pour les graphes à trois arêtes externes. On note $\gamma_{(i)}$ le produit des graphes $\gamma'_{(i)}$ associés aux composantes connexes de γ. Le graphe

[3] pour la théorie de masse nulle, on prend $p^2 = \mu^2$ au lieu de $p = 0$.

$\Gamma/\gamma_{(i)}$ est obtenu en remplaçant chacune des composantes connexes γ' de γ par le sommet correspondant de type (i) pour $\gamma'_{(i)}$. On vérifie que $\Gamma/\gamma(i)$ est un graphe 1PI.

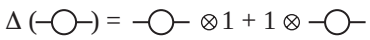
On notera que, bien que les composantes γ' de γ soient disjointes par construction, les graphes $\gamma'_{(i)}$ ne le sont plus nécessairement car les composantes peuvent avoir des arêtes externes communes, $\varepsilon(\gamma') \cap \varepsilon(\gamma'') \neq \varnothing$.

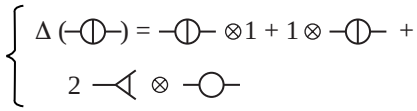
C'est le cas dans l'exemple suivant,

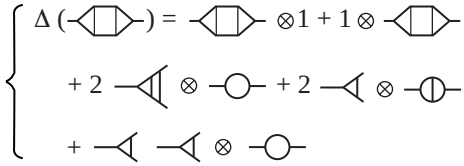


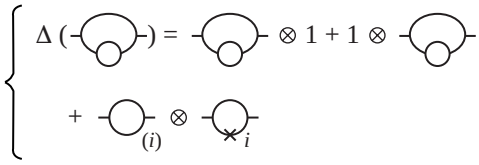
La structure externe du graphe $\Gamma/\gamma_{(i)}$ est identique à celle de Γ.

Voici quelques exemples de coproduit [**13**] :









On notera que le coproduit (31) vérifie par construction la propriété suivante de « linéarité à droite » qui signale sa similitude avec le coproduit donné par la composition des séries formelles :

Proposition 3.1. — *Soit $\mathcal{H}_{(1)}$ le sous-espace de $\mathcal{H}$ engendré par* 1 *et les graphes 1PI. Alors pour tout $\Gamma \in \mathcal{H}_{(1)}$ on a,*

$$\Delta(\Gamma) \in \mathcal{H} \otimes \mathcal{H}_{(1)}.$$

Le coproduit (31) se prolonge en un unique homomorphisme d'algèbre $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ et l'on a ([**13**])

Théorème 3.2. — *Le couple* $(\mathcal{H}, \Delta)$ *est une algèbre de Hopf.*

On notera que la structure externe des graphes joue un rôle mineur dans le coproduit car la structure externe de chacun des graphes $\Gamma/\gamma_{(i)}$ est identique à celle de Γ.

3.2. L'algèbre de Lie des graphes, le groupe G et sa structure. — L'algèbre de Hopf $\mathcal{H}$ admet plusieurs graduations naturelles. Il suffit pour définir une graduation de donner le degré des graphes 1PI puis de poser, en général,

$$\deg(\Gamma_1 \dots \Gamma_e) = \sum \deg(\Gamma_j), \quad \deg(1) = 0. \tag{32}$$

On doit vérifier que

$$\deg(\gamma) + \deg(\Gamma/\gamma) = \deg(\Gamma) \tag{33}$$

pour tout sous-graphe admissible γ.

Les deux graduations les plus naturelles sont

$$I(\Gamma) = \text{nombre d'arêtes internes de } \Gamma \tag{34}$$

et

$$v(\Gamma) = V(\Gamma) - 1 = \text{nombre de sommets de } \Gamma - 1. \tag{35}$$

On a aussi la combinaison importante

$$L = I - v = I - V + 1 \tag{36}$$

qui est le nombre de boucles du graphe.

Le théorème de Milnor-Moore montre que l'algèbre de Hopf $\mathcal{H}$ est duale de l'algèbre enveloppante d'une algèbre de Lie graduée $\underline{G}$ dont une base est donnée par les graphes 1-particule irréductibles. Le crochet de Lie de deux graphes est obtenu par insertion d'un graphe dans l'autre. Le groupe de Lie correspondant G est le groupe des caractères de $\mathcal{H}$.

Nous avons ensuite analysé le groupe G et montré qu'il est produit semi-direct d'un groupe abélien par un groupe très relié au groupe des difféomorphismes des constantes de couplage sans dimension de la théorie des champs (voir section 6).

Nous appliquons alors le théorème de Milnor-Moore à l'algèbre de Hopf bigraduée $\mathcal{H}$.

Ce théorème donne une structure d'algèbre de Lie sur,

$$\oplus_\Gamma C^\infty(E_\Gamma) = L \tag{37}$$

où, pour chaque graphe 1PI Γ, on définit E_Γ comme ci-dessus. Soit $X \in L$ et soit Z_X la forme linéaire sur $\mathcal{H}$ donnée, sur les monômes Γ, par

$$\langle \Gamma, Z_X \rangle = 0 \tag{38}$$

sauf si Γ est connexe et 1PI et, dans ce cas, par

$$\langle \Gamma, Z_X \rangle = \langle \sigma_\Gamma, X_\Gamma \rangle \tag{39}$$

où σ_Γ est la distribution qui donne la structure externe de Γ et X_Γ la composante correspondante de X. Par construction, Z_X est un caractère infinitésimal de $\mathcal{H}$ ainsi que les commutateurs,

$$[Z_{X_1}, Z_{X_2}] = Z_{X_1}\, Z_{X_2} - Z_{X_2}\, Z_{X_1}. \tag{40}$$

Le produit étant obtenu par transposition du coproduit de $\mathcal{H}$, *i.e.,* par

$$\langle Z_1\, Z_2, \Gamma \rangle = \langle Z_1 \otimes Z_2, \Delta\, \Gamma \rangle. \tag{41}$$

Soient Γ_j, $j = 1, 2$ des graphes 1PI et $\varphi_j \in C^\infty(E_{\Gamma_j})$ les fonctions test correspondantes.

Pour $i \in \{0, 1\}$, soit $n_i(\Gamma_1, \Gamma_2; \Gamma)$ le nombre de sous-graphes de Γ isomorphes à Γ_1 et tels que

$$\Gamma / \Gamma_1(i) \simeq \Gamma_2. \tag{42}$$

Soit (Γ, φ) l'élément de L associé à $\varphi \in C^\infty(E_\Gamma)$, le crochet de Lie de (Γ_1, φ_1) et (Γ_2, φ_2) est donné par,

$$\sum_{\Gamma, i} \sigma_i(\varphi_1)\; n_i(\Gamma_1, \Gamma_2; \Gamma)\, (\Gamma, \varphi_2) - \sigma_i(\varphi_2)\; n_i(\Gamma_2, \Gamma_1; \Gamma)\, (\Gamma, \varphi_1). \tag{43}$$

où les σ_i sont les structures externes spécifiées dans (28) et (29).

***Théorème 3.3* ([13])**. — *L'algèbre de Lie L est produit semi-direct d'une algèbre de Lie Abélienne L_0 par L_c où L_c admet une base canonique indéxée par les graphes $\Gamma_{(i)}$ avec*

$$[\Gamma, \Gamma'] = \sum_v \Gamma \circ_v \Gamma' - \sum_{v'} \Gamma' \circ_{v'} \Gamma \tag{44}$$

où $\Gamma \circ_v \Gamma'$ est obtenu en greffant Γ' sur Γ en v.

Nous noterons G_c le groupe des caractères de la sous-algèbre de Hopf associée à L_c.

4. Renormalisation et problème de Riemann-Hilbert

La clé de notre travail avec Dirk Kreimer réside dans l'identité entre les formules qui gouvernent la recette combinatoire des pas successifs de la renormalisation perturbative et celles qui calculent la décomposition de Birkhoff (ou de Wiener-Hopf) d'un lacet à valeurs dans un groupe pronilpotent simplement connexe.

4.1. Problème de Riemann-Hilbert. — L'origine de la décomposition de Birkhoff (ou de Wiener-Hopf) est le problème de Riemann-Hilbert qui vient du 21$^{\text{e}}$ problème de Hilbert qu'il formulait ainsi :

« Montrer qu'il existe toujours une équation différentielle linéaire fuchsienne de singularités et monodromies données. »

Sous cette forme, il admet une réponse positive due à Plemelj et Birkhoff (*cf.* [**1**] pour un exposé détaillé). Quand on le reformule pour les systèmes linéaires de la forme,

$$y'(z) = A(z)\, y(z), \quad A(z) = \sum_{\alpha \in S} \frac{A_\alpha}{z - \alpha}, \tag{45}$$

où S est l'ensemble fini donné des singularités, $\infty \notin S$, et les A_α sont des matrices complexes telles que

$$\sum A_\alpha = 0 \tag{46}$$

pour éviter les singularités à ∞, la réponse n'est pas toujours positive [**3**], mais la solution existe quand les matrices de monodromie M_α sont suffisamment proches de 1. On peut alors l'écrire explicitement sous la forme d'une série de polylogarithmes [**32**].

4.2. Décomposition de Birkhoff. — Soit $C \subset P_1(\mathbb{C})$ une courbe simple contenant l'ensemble S des singularités et $\gamma : C \mapsto \mathrm{GL}(n, \mathbb{C})$ l'application constante par morceaux sur $C \backslash S$ produit des matrices de monodromie M_α rencontrées le long du parcours.

Une solution $A(z)$ du problème de Riemann-Hilbert serait obtenue en posant

$$A(z)dz = d\gamma_\pm(z)\, \gamma_\pm(z)^{-1},$$

si l'on disposait de la *décomposition de Birkhoff* du lacet γ,

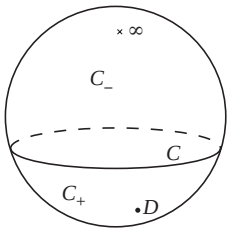
$$\gamma(z) = \gamma_-(z)^{-1}\, \gamma_+(z), \qquad \forall\, z \in C, \tag{47}$$

où les trois applications γ et $\gamma_\pm$ sont à valeurs dans $\mathrm{GL}_n(\mathbb{C})$, et $\gamma_\pm$ sont les valeurs au bord d'applications holomorphes

$$\gamma_\pm : C_\pm \longrightarrow \mathrm{GL}_n(\mathbb{C}), \tag{48}$$

avec C_- la composante connexe du complément de C contenant $\infty \notin C$ et C_+ la composante bornée. La condition $\gamma_-(\infty) = 1$ assure l'unicité de la décomposition si elle existe.

(49)



En général, pour une courbe simple C et un lacet régulier

$$\gamma : C \longmapsto \mathrm{GL}(n, \mathbb{C})$$

l'existence de la décomposition de Birkhoff (47) est intimement reliée à la classification des fibrés vectoriels holomorphes sur la sphère de Riemann $P_1(\mathbb{C})$, et équivalente à l'annulation,

$$c_1(L_j) = 0 \tag{50}$$

des nombres de Chern $n_j = c_1(L_j)$ des fibrés en droites holomorphes de la décomposition de Birkhoff-Grothendieck

$$E = \oplus L_j \tag{51}$$

où E est le fibré vectoriel holomorphe sur $P_1(\mathbb{C})$ associé à γ, *i.e.,* d'espace total

$$(C_+ \times \mathbb{C}^n) \cup_\gamma (C_- \times \mathbb{C}^n). \tag{52}$$

La discussion ci-dessus pour $G = \mathrm{GL}_n(\mathbb{C})$ s'étend aux groupes de Lie complexes arbitraires en remplaçant $\mathrm{GL}_n(\mathbb{C})$ par G dans les formules ci-dessus.

4.3. Décomposition de Birkhoff pour les groupes pronilpotents. — Quand G est un groupe de Lie complexe et simplement connexe l'existence (et l'unicité) de la décomposition de Birkhoff (47) est vraie pour tout γ.

Quand le lacet $\gamma : C \to G$ se prolonge en un lacet holomorphe : $C_+ \to G$, la décomposition de Birkhoff est donnée par $\gamma_+ = \gamma$, $\gamma_- = 1$. En général, pour $z_0 \in C_+$ l'évaluation,

$$\gamma \longrightarrow \gamma_+(z_0) \in G \tag{53}$$

donne un principe naturel pour extraire une valeur finie à partir de l'expression singulière $\gamma(z_0)$.

Soit C un cercle de rayon infinitésimal centré en z_0. Cette extraction de partie finie est alors une division par la partie polaire pour un lacet méromorphe γ. En effet, γ_- est alors essentiellement la partie polaire de γ en z_0. Explicitons maintenant les formules qui donnent cette extraction de partie polaire pour un groupe de Lie complexe pro-nilpotent G dont l'algèbre de Hopf de coordonnées $\mathcal{H}$ est graduée.

La table ci-dessous donne la traduction entre langages algébriques et géométriques, en désignant par $\mathcal{R}$ l'anneau des germes de fonctions méromorphes dans un voisinage de z_0, $\mathcal{R}_- \subset \mathcal{R}$ le sous-anneau des polynômes en $(z - z_0)^{-1}$ et $\mathcal{R}_+ \subset \mathcal{R}$ celui des fonctions régulières en z_0.

Pour $X \in \mathcal{H}$, notons le coproduit sous la forme

$$\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X'' \tag{54}$$

où, pour X homogène de degré n, tous les termes X' et X'' sont de degré $< n$. La décomposition de Birkhoff s'obtient de manière récursive grâce au théorème suivant :

Lacet γ de C à valeurs dans G	Homomorphisme $\phi : \mathcal{H} \to \mathcal{R}$
γ est régulier en z_0	$\phi(\mathcal{H}) \subset \mathcal{R}_+$
γ se prolonge en une application holomorphe de $\mathbb{P}_1(\mathbb{C}) \setminus \{z_0\} \to G$ et $\gamma(\infty) = 1$	$\phi(\mathcal{H}) \subset \mathcal{R}_-$
$\gamma(z) = \gamma_1(z)\gamma_2(z),\ \forall\, z \in C$	$\phi = \phi_1 \star \phi_2$
$z \to \gamma(z)^{-1}$	$\phi \circ S$

***Théorème 4.1* ([13])**. — *Soit $\phi : \mathcal{H} \to \mathcal{R}$ un homomorphisme d'algèbres et $\gamma : C \mapsto G$ le lacet correspondant. Sa décomposition de Birkhoff est donnée de manière récursive par les égalités*

$$\phi_-(X) = -T\left(\phi(X) + \sum \phi_-(X')\phi(X'')\right),$$
$$\phi_+(X) = \phi(X) + \phi_-(X) + \sum \phi_-(X')\phi(X'')$$

où T désigne la projection sur $\mathcal{R}_-$ parallèlement à $\mathcal{R}_+$.

4.4. Formules de BPHZ et décomposition de Birkhoff. — La clé de notre travail avec Dirk Kreimer réside dans l'identité entre les formules du théorème 4.1 et celles qui gouvernent la combinatoire des calculs de graphes. Nous avons déjà vu la formule qui définit la préparation d'un graphe,

$$\overline{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma)U(\Gamma/\gamma) \tag{55}$$

Celle qui donne le contre-terme $C(\Gamma)$ est alors,

$$C(\Gamma) = -T(\overline{R}(\Gamma)) = -T\left(U(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma)U(\Gamma/\gamma)\right) \tag{56}$$

et celle qui donne la valeur renormalisée du graphe est,

$$R(\Gamma) = \overline{R}(\Gamma) + C(\Gamma) = U(\Gamma) + C(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma)U(\Gamma/\gamma) \tag{57}$$

Il est alors clair, en posant $\phi = U$, $\phi_- = C$, $\phi_+ = R$, que ces équations sont identiques à celles du théorème 4.1 donnant la construction récursive de la décomposition de Birkhoff.

Décrivons plus en détails ce résultat. Étant donnée une théorie renormalisable en dimension D la théorie non-renormalisée donne, en utilisant la régularisation dimensionnelle, un lacet γ d'éléments du groupe G associé à la théorie dans la section 3. Le paramètre z du lacet $\gamma(z)$ est une variable complexe et $\gamma(z)$ est méromorphe dans un

voisinage de D. Notre résultat principal est que la théorie renormalisée est donnée par l'évaluation à $z = D$ de la partie non-singulière γ_+ de la décomposition de Birkhoff,

$$\gamma(z) = \gamma_-(z)^{-1}\,\gamma_+(z) \tag{58}$$

de γ.

Les règles de Feynman et la régularisation dimensionnelle associent un nombre,

$$U_\Gamma(p_1,\dots,p_N) = \int d^d\,k_1 \cdots d^d\,k_L\ I_\Gamma(p_1,\dots,p_N,k_1,\dots,k_L) \tag{59}$$

à chaque graphe Γ. Nous les utilisons en métrique euclidienne pour éviter les facteurs imaginaires.

Pour respecter les dimensions physiques des quantités impliquées quand on écrit ces règles en dimension d, il faut introduire une unité de masse μ et remplacer partout la constante de couplage par $\mu^{3-d/2}\,g$. On normalise ainsi les calculs par,

$$U(\Gamma) = g^{(2-N)}\,\mu^{-B}\,\langle\sigma, U_\Gamma\rangle \tag{60}$$

où $B = B(d)$ est la dimension de $\langle\sigma, U_\Gamma\rangle$.

On étend la définition aux réunions disjointes de graphes 1PI Γ_j par

$$U(\Gamma = \cup\,\Gamma_j) = \Pi\,U(\Gamma_j). \tag{61}$$

Le résultat principal est alors le suivant :

***Théorème 4.2* ([13])**

a) *Il existe une unique application méromorphe $\gamma(z) \in G$, $z \in \mathbb{C}$, $z \neq D$ dont les Γ-coordonnées sont données par $U(\Gamma)_{d=z}$.*

b) *La valeur renormalisée d'une observable physique $\mathcal{O}$ est obtenue en remplaçant $\gamma(D)$ dans le développement perturbatif de $\mathcal{O}$ par $\gamma_+(D)$ où*

$$\gamma(z) = \gamma_-(z)^{-1}\,\gamma_+(z)$$

est la décomposition de Birkhoff du lacet $\gamma(z)$ relativement à un cercle infinitésimal autour de D.

En d'autres termes, la théorie renormalisée est obtenue en évaluant en $z = D$ la partie holomorphe γ_+ de la décomposition de Birkhoff du lacet γ associé à la théorie non-renormalisée. La renormalisation apparaît ainsi comme un cas particulier d'un procédé général d'extraction de parties finies de nature multiplicative, basé sur la décomposition de Birkhoff.

Il est également remarquable que la partie γ_- de la décomposition de Birkhoff donne les contre-termes et que la même expression formelle déjà utilisée pour l'action effective (18) donne l'action avec ses contre-termes,

$$S_c(A) = S_0(A) + \sum_{\Gamma\in 1PI} \frac{C(\Gamma)(A)}{S(\Gamma)}. \tag{62}$$

5. Le groupe de renormalisation

Montrons comment le groupe de renormalisation apparaît très simplement de notre point de vue. Comme nous l'avons vu ci-dessus, la régularisation dimensionnelle implique le choix arbitraire d'une unité de masse μ et l'on constate d'abord que la partie singulière de la décomposition de Birkhoff de γ est, en fait, indépendante de ce choix. Il en résulte une contrainte très forte sur cette partie singulière et le groupe de renormalisation s'en déduit immédiatement. Nous en déduisons également une formule explicite pour l'action nue en terme du résidu.

5.1. Stabilité de γ_-. — On montre d'abord, en se limitant à la théorie φ_6^3 pour simplifier les notations, que, bien que le lacet $\gamma(d)$ dépende du choix de l'unité de masse μ,

$$\mu \longrightarrow \gamma_\mu(d), \tag{63}$$

la partie singulière γ_{μ^-} de sa décomposition de Birkhoff

$$\gamma_\mu(d) = \gamma_{\mu^-}(d)^{-1}\,\gamma_{\mu^+}(d) \tag{64}$$

est, en fait, indépendante de μ,

$$\frac{\partial}{\partial\mu}\,\gamma_{\mu^-}(d) = 0. \tag{65}$$

Cet énoncé découle immédiatement de l'analyse dimensionnelle.

Soit

$$\theta_t \in \operatorname{Aut} G, \quad t \in \mathbb{R}, \tag{66}$$

le groupe à un paramètre d'automorphismes du groupe de Lie G qui est associé à la graduation de l'algèbre de Hopf $\mathcal{H}$ donnée par le nombre de boucles

$$L(\Gamma) = \text{nombre de boucles } \Gamma \tag{67}$$

pour tout graphe 1PI Γ.

On a, par construction, l'égalité

$$\gamma_{e^t\mu}(d) = \theta_{t\varepsilon}(\gamma_\mu(d)) \qquad \forall\, t \in \mathbb{R},\ \varepsilon = D - d \tag{68}$$

Il en résulte que les lacets $\gamma = \gamma_\mu$ associés à la théorie non-renormalisée satisfont à la propriété suivante :

Proposition 5.1. — *La partie singulière γ_- de la décomposition de Birkhoff*

$$\gamma(d) = \gamma_-(d)^{-1}\,\gamma_+(d)$$

est inchangée par l'opération

$$\gamma(d) \longrightarrow \theta_{t\varepsilon}(\gamma(d)), \quad \varepsilon = D - d.$$

Comme $\gamma_-(d)\,\gamma(d)$ et $\gamma_-(d)\,\theta_{t\varepsilon}(\gamma(d))$ sont réguliers en $\varepsilon = 0$, il en est de même pour $\theta_{-t\varepsilon}(\gamma_-(d))\,\gamma(d)$ de sorte que l'on obtient :

Corollaire 5.2. — *La partie singulière $\gamma_-(d)$ est indépendante de μ et vérifie*

$$\gamma_-(d)\,\theta_{t\varepsilon}(\gamma_-(d)^{-1}) \text{ est régulier en } \varepsilon = 0. \tag{69}$$

Nous donnerons ci-dessous une caractérisation complète des lacets $\gamma_-(\varepsilon) \in G$ vérifiant cette propriété.

5.2. Le groupe F_t. — Il est facile de voir que la limite de (69) pour $\varepsilon \to 0$ définit un sous-groupe à un paramètre,

$$F_t \in G,\ t \in \mathbb{R}, \quad F_{t_1+t_2} = F_{t_1}\,F_{t_2}, \quad \forall\, t_j \in \mathbb{R}.$$

La signification de ce groupe est donnée par

Proposition 5.3. — *La valeur finie $\gamma_\mu^+(D)$ de la décomposition de Birkhoff vérifie pour tout $t \in \mathbb{R}$*

$$\gamma^+_{e^t\,\mu}(D) = F_t\,\gamma_\mu^+(D).$$

En effet, $\gamma_\mu^+(D)$ est la valeur régulière de $\gamma_-(d)\,\gamma_\mu(d)$ en $\varepsilon = 0$ et $\gamma^+_{e^t\,\mu}(D)$ celle de $\gamma_-(d)\,\theta_{t\varepsilon}(\gamma_\mu(d))$ ou, ce qui revient au même, celle de $\theta_{-t\varepsilon}(\gamma_-(d))\,\gamma_\mu(d)$ en $\varepsilon = 0$. Mais $\theta_{-t\varepsilon}(\gamma_-(d))\,\gamma_-(d)^{-1} \to F_t$ quand $\varepsilon \to 0$, d'où le résultat.

Le groupe F_t gouverne donc le comportement de la théorie renormalisée lorsque l'on modifie le choix arbitraire de l'échelle de masse μ. Il apparaît ainsi comme un groupe d'ambiguïté de la théorie physique. C'est cette ambiguïté que nous rapprocherons plus loin de l'ambiguïté entre les racines d'une équation algébrique inhérente à la théorie de Galois.

L'on peut réécrire l'énoncé de la proposition 5.3 en terme du générateur infinitésimal $\beta = \left(\frac{\partial}{\partial t}\,F_t\right)_{t=0}$ sous la forme

$$\mu\partial/\partial\mu\,\gamma_\mu^+(D) = \beta\,\gamma_\mu^+(D). \tag{70}$$

5.3. Le résidu et la fonction β. — Le corollaire 5.2 permet de montrer ([**14**]) que le générateur $\beta = \left(\frac{\partial}{\partial t}\,F_t\right)_{t=0}$ de ce sous-groupe est relié au *résidu* de γ

$$\mathrm{Res}_{\varepsilon=0}\,\gamma = -\left(\frac{\partial}{\partial u}\,\gamma_-\left(\frac{1}{u}\right)\right)_{u=0} \tag{71}$$

par l'équation

$$\beta = Y\ \mathrm{Res}\,\gamma \tag{72}$$

où $Y = \left(\frac{\partial}{\partial t}\,\theta_t\right)_{t=0}$ est la graduation associée au nombre de boucles des graphes.

Ceci est immédiat mais notre résultat ([**14**]) donne la formule explicite suivante (74) qui exprime $\gamma_-(\varepsilon)$ en fonction de β. Introduisons le produit semi-direct de l'algèbre de Lie G par la graduation. On a donc un élément Z_0 tel que

$$[Z_0, X] = Y(X) \qquad \forall\, X \in \text{Lie } G. \tag{73}$$

et le groupe G^* produit semi-direct de G par le groupe à un paramètre θ_t. On a alors ([**14**])

Théorème 5.4. — *L'application $\gamma_-(\varepsilon)$ est entièrement déterminée par son résidu par la formule*

$$\gamma_-(\varepsilon) = \lim_{t\to\infty} e^{-t\left(\frac{\beta}{\varepsilon}+Z_0\right)}\, e^{tZ_0}. \tag{74}$$

Les deux facteurs du terme de droite appartiennent au groupe G^* mais leur rapport appartient au groupe G.

Cette formule montre que toute la structure des divergences est uniquement déterminée par le résidu et donne une forme forte des relations de 't Hooft [**28**].

6. Le groupe G et les difféomorphismes formels

Bien entendu, on pourrait facilement objecter aux développements précédents en arguant que le mystère de la renormalisation n'est pas complètement éclairci car le groupe G construit à partir des graphes de Feynman apparaît également mystérieux. Cette critique est complètement levée par la merveilleuse relation, basée sur la physique, entre les algèbres de Hopf $\mathcal{H}$ des graphes de Feynman et celle, $\mathcal{H}_{\rm diff}$, des difféomorphismes formels.

6.1. L'action de G sur les constantes de couplage. — Nous montrons, dans le cas de masse nulle, que la formule qui donne la constante de couplage effective

$$g_{\rm eff} = \left(g + \sum_{\text{—}\bigcirc\text{—}} g^{2\ell+1}\,\frac{\Gamma}{S(\Gamma)}\right)\left(1 - \sum_{\text{—}\bigcirc\text{—}} g^{2\ell}\,\frac{\Gamma}{S(\Gamma)}\right)^{-3/2} \tag{75}$$

considérée comme une série formelle dans la variable g d'éléments de l'algèbre de Hopf $\mathcal{H}$, définit, en fait, un homomorphisme d'algèbres de Hopf de l'algèbre de Hopf $\mathcal{H}_{\rm diff}$ des coordonnées sur le groupe des difféomorphismes formels de $\mathbb{C}$ tels que

$$\varphi(0) = 0,\ \varphi'(0) = \mathrm{id} \tag{76}$$

vers l'algèbre de Hopf $\mathcal{H}$ de la théorie de masse nulle.

Soit $\mathcal{H}_{\rm diff}$ l'algèbre de Hopf des coordonnées du groupe des difféomorphismes formels vérifiant (76). Prenons les générateurs a_n de $\mathcal{H}_{\rm diff}$ donnés par

$$\varphi(x) = x + \sum_{n\geqslant 2} a_n(\varphi)\, x^n. \tag{77}$$

Le coproduit dans $\mathcal{H}_{\rm diff}$ est défini par l'égalité

$$\langle \Delta a_n,\, \varphi_1 \otimes \varphi_2\rangle = a_n(\varphi_2 \circ \varphi_1) \tag{78}$$

On utilise alors la formule (75) pour définir un homomorphisme d'algèbres

$$\Phi : \mathcal{H}_{\rm diff} \longrightarrow \mathcal{H} \tag{79}$$

de la manière suivante. On développe d'abord g_{eff} en série de puissances de g sous la forme

$$g_{\text{eff}} = g + \sum_{2}^{\infty} \alpha_n \, g^n \tag{80}$$

où tous les coefficients α_{2n} d'indice pair sont nuls et les coefficients α_{2n+1} sont des combinaisons linéaires finies de produits de graphes, de sorte que

$$\alpha_{2n+1} \in \mathcal{H} \qquad \forall\, n \geqslant 1. \tag{81}$$

L'on définit alors Φ par

$$\Phi(a_n) = \alpha_n. \tag{82}$$

C'est, par construction, un morphisme d'algèbres. Le résultat essentiel est que Φ est un morphisme d'algèbres de Hopf.

Théorème 6.1 **([14])**

$$(\Phi \otimes \Phi)\, \Delta\, x = \Delta\, \Phi(x) \qquad \forall\, x \in \mathcal{H}_{\text{diff}}.$$

Il en résulte, en transposant un homomorphisme de groupes,

$$\rho : G_c \longrightarrow G_2. \tag{83}$$

où G_2 est le groupe des caractères de $\mathcal{H}_{\text{diff}}$, *i.e.,* le groupe opposé du groupe des difféomorphismes formels. Cela donne, en particulier, une action formelle du groupe G_c sur la constante de couplage.

6.2. La décomposition de Birkhoff de la constante de couplage. — Nous montrons, en particulier, que l'image par ρ de $\beta = Y$ Res γ est la fonction β de la constante de couplage g.

Nous obtenons ainsi un corollaire du théorème principal qui se formule sans faire intervenir ni le groupe G ni l'algèbre de Hopf $\mathcal{H}$.

Théorème 6.2 **([14])**. — *Considérons la constante de couplage effective non-renormalisée $g_{\text{eff}}(\varepsilon)$ comme une série formelle en g et soit $g_{\text{eff}}(\varepsilon) = g_{\text{eff}_+}(\varepsilon)\,(g_{\text{eff}_-}(\varepsilon))^{-1}$ sa décomposition de Birkhoff (opposée) dans le groupe des difféomorphismes formels. Alors, le lacet $g_{\text{eff}_-}(\varepsilon)$ est la constante de couplage nue et $g_{\text{eff}_+}(0)$ la constante de couplage renormalisée.*

Comme la décomposition de Birkhoff d'un lacet à valeurs dans le groupe des difféomorphismes (formels) est évidemment reliée à la classification des fibrés (non-linéaires) holomorphes, ce résultat suggère qu'un tel fibré ayant pour base un voisinage de la dimension D de l'espace-temps et pour fibre les valeurs (complexifiées) des constantes de couplage devrait donner une interprétation entièrement géométrique de l'opération de renormalisation.

Il faut tout de même noter que la décomposition de Birkhoff a lieu ici relativement à un cercle infinitésimal autour de d et qu'il s'agit de difféomorphismes formels.

Les résultats ci-dessus montrent qu'au niveau des développements perturbatifs le procédé de renormalisation admet une interprétation géométrique simple grâce au groupe G et à la décomposition de Birkhoff. Le problème essentiel consiste à passer du développement perturbatif à la théorie non-perturbative, ce qui revient en termes de difféomorphismes à passer du développement de Taylor à la formule globale.

Il est très naturel de postuler que le passage de la théorie non-renormalisée à la théorie renormalisée dans le cadre non-perturbatif sera toujours donné par la décomposition de Birkhoff. Il est crucial pour appliquer cette idée de disposer de la décomposition de Birkhoff dans ces situations non-perturbatives, *i.e.,* en ce qui concerne les difféomorphismes pour des groupes de difféomorphismes dont le développement de Taylor vérifie des conditions de type Gevrey. Un résultat important dans cette direction a été récemment obtenu par F. Menous [**34**].

7. Le groupe de renormalisation et la théorie de Galois

Cette section consiste en la juxtaposition des quatre éléments d'un puzzle à la recherche d'une théorie de Galois satisfaisante aux places archimédiennes d'un corps de nombres. Le quatrième élément du puzzle est l'apparition dans ma collaboration récente avec M. Marcolli de la théorie de Galois motivique dans la reformulation de la renormalisation comme une correspondance de Riemann-Hilbert. Notre motivation était d'obtenir l'analogue du tore exponentiel dans la théorie développée par Jean-Pierre Ramis pour étendre la correspondance de Riemann-Hilbert au cadre irrégulier ([**33, 35**]).

7.1. Groupe de Weil. — La généralisation conceptuelle de la notion de corps de nombres est celle de corps global. Un corps k est *global* si c'est un sous-corps discret cocompact d'un anneau localement compact (non discret) semi-simple et commutatif A (*cf.* [**29**]). L'anneau topologique A est alors canoniquement associé à k et s'appelle l'anneau des Adèles de k ; on a

$$A = \prod_{\text{res}} k_v, \tag{84}$$

où le produit est le produit restreint des corps locaux k_v indéxés par les places de k. Les k_v sont les corps localement compacts obtenus comme complétions de k de même que l'on obtient les nombres réels en complétant les rationnels.

Quand la caractéristique de k est $p > 1$, *i.e.,* quand k est un corps de fonctions sur le corps fini $\mathbb{F}_q$, on a

$$k \subset k_0 \subset k_{\text{ab}} \subset k_{\text{sep}} \subset \overline{k}, \tag{85}$$

où $\overline{k}$ désigne une clôture algébrique de k, k_{sep} la clôture séparable dans $\overline{k}$, k_{ab} l'extension abélienne maximale et k_0 l'extension obtenue en adjoignant à k les racines de l'unité d'ordre premier à p.

On définit le groupe de Weil W_k comme le sous-groupe de $\text{Gal}(k_{\text{ab}} : k)$ formé par les automorphismes de $(k_{\text{ab}} : k)$ qui induisent sur k_0 une puissance entière de l'automorphisme de « Frobenius » θ :

$$\theta(\mu) = \mu^q \qquad \forall\, \mu \text{ racine de l'unité d'ordre premier à } p.$$

Le résultat principal de la théorie du corps de classe global est l'existence d'un isomorphisme canonique

$$W_k \simeq C_k = GL_1(A)/GL_1(k),$$

de groupes localement compacts.

Quand k est de caractéristique nulle, *i.e.,* un corps de nombres, on a un isomorphisme canonique

$$\text{Gal}(k_{\text{ab}} : k) \simeq C_k/D_k$$

où D_k désigne la composante connexe de l'élément neutre dans le groupe des classes d'idèles $C_k = GL_1(A)/GL_1(k)$; mais à cause des places archimédiennes de k, l'on n'a pas d'interprétation de C_k analogue au cas des corps de fonctions. Citons A. Weil [**37**] :

« La recherche d'une interprétation pour C_k si k est un corps de nombres, analogue en quelque manière à l'interprétation par un groupe de Galois quand k est un corps de fonctions, me semble constituer l'un des problèmes fondamentaux de la théorie des nombres à l'heure actuelle ; il se peut qu'une telle interprétation renferme la clé de l'hypothèse de Riemann... ».

Cela signifie qu'aux places archimédiennes (*i.e.,* aux complétions de k qui donnent soit les nombres réels soit les nombres complexes), il devrait y avoir un groupe continu de symétries secrètement présent.

7.2. Classification des facteurs. — Mon intérêt pour ce problème vient de mon travail sur la classification des facteurs qui indiquait clairement que l'on possédait là l'analogue de la théorie de Brauer qui est l'une des clés de la théorie du corps de classe local.

Les groupes de Galois sont par construction des limites projectives de groupes finis attachés à des extensions finies. Pour obtenir des groupes connexes il faut évidemment relaxer cette condition de finitude qui est la même que la restriction en théorie de Brauer aux algèbres simples centrales de dimension finie. Comme ce sont les places archimédiennes de k qui sont à l'origine de la composante connexe D_k, il est naturel de considérer la question préliminaire suivante :

« Existe-t-il une théorie de Brauer non-triviale d'algèbres simples centrales sur $\mathbb{C}$? »

J'ai montré dans [**7**] que la classification des facteurs *approximativement finis* sur $\mathbb{C}$ donnait une réponse satisfaisante à cette question. Ils sont classifiés par leur module

$$\mathrm{Mod}(M) \underset{\sim}{\subseteq} \mathbb{R}_+^*,$$

qui est un sous-groupe (virtuel) fermé de $\mathbb{R}_+^*$.

Ce groupe joue un rôle analogue dans le cas archimédien au module des algèbres simples centrales sur un corps local nonarchimédien. Dans ce dernier cas le module se définit très simplement par l'action du groupe multiplicatif d'une algèbre simple centrale sur la mesure de Haar du groupe additif. La définition de $\mathrm{Mod}(M)$ pour les facteurs est beaucoup plus élaborée, mais reste basée sur l'action du groupe $\mathbb{R}_+^*$ de changement d'échelle.

Pour poursuivre l'analogie avec la théorie de Brauer où le lien avec le groupe de Galois s'obtient par la construction d'algèbres simples centrales comme produits croisés d'un corps par un groupe d'automorphismes, le pas suivant consiste à trouver des exemples naturels de construction de facteurs comme produits croisés d'un corps K, extension transcendante de $\mathbb{C}$ par un groupe d'automorphismes. Dans nos recherches sur les variétés sphériques noncommutatives [**8**], avec M. Dubois-Violette, l'algèbre de Sklyanin ([**36**]) est apparue comme solution en dimension 3 du problème de classification formulé dans [**15**]. La représentation « régulière » de cette algèbre engendre une algèbre de von Neumann intégrale directe de facteurs approximativement finis de type II_1, tous isomorphes au facteur hyperfini R. Les homomorphismes correspondants de l'algèbre de Sklyanin ([**36**]) vers le facteur R se factorisent miraculeusement ([**9**]) à travers le produit croisé du corps K_q des fonctions elliptiques, où le module $q = e^{2\pi i\tau}$ est réel, par l'automorphisme de translation par un nombre réel (mais, en général, irrationnel). On obtient ainsi le facteur R comme produit croisé de K_q par un sous-groupe du groupe de Galois, en parfaite analogie avec la construction des algèbres simples centrales sur un corps local. Il reste à obtenir une construction semblable et naturelle du facteur R_∞ de type III_1. La théorie des algèbres de Hecke modulaires ébauchée dans ([**18**]) devrait y jouer un rôle important.

7.3. Les « constantes » en théorie des champs. — Il est sans doute prématuré d'essayer d'identifier le corps K correspondant, qui devrait jouer le rôle de l'extension non ramifiée maximale $\mathbb{C}_{\mathrm{nr}}$ de $\mathbb{C}$ et être doté d'une action naturelle du groupe multiplicatif $\mathbb{R}_+^*$.

Le rôle du corps K en physique des hautes énergies devrait être relié à l'observation suivante concernant les « constantes » qui interviennent en théorie des champs. En fait, les calculs des physiciens regorgent d'exemples de « constantes » telles les constantes de couplage g des interactions (électromagnétiques, faibles et fortes) qui n'ont de « constantes » que le nom. Elles dépendent en réalité du niveau d'énergie μ auquel les expériences sont réalisées et sont donc des fonctions $g(\mu)$. Ainsi les physiciens des hautes énergies étendent implicitement le « corps des constantes » avec lequel ils

travaillent, passant du corps $\mathbb{C}$ des scalaires à un corps de fonctions $g(\mu)$. Le générateur du groupe de renormalisation est simplement $\mu\partial/\partial\mu$ comme dans l'équation (70).

L'on peut mettre l'exemple plus simple du corps K_q des fonctions elliptiques sous la même forme en passant aux fonctions loxodromiques, c'est-à-dire, en posant $\mu = e^{2\pi iz}$ de sorte que la première périodicité (en $z \to z+1$) est automatique alors que la deuxième s'écrit $g(q\,\mu) = g(\mu)$. Le groupe des automorphismes de la courbe elliptique est alors lui aussi engendré par $\mu\partial/\partial\mu$.

Les points fixes du groupe de renormalisation sont les scalaires ordinaires, mais il se pourrait que la physique quantique conspire pour nous empêcher d'espérer une théorie qui englobe toute la physique des particules et soit construite comme point fixe du groupe de renormalisation. Les interactions fortes sont asymptotiquement libres et l'on peut les analyser à très hautes énergies en utilisant les points fixes du groupe de renormalisation, mais la présence du secteur électrodynamique montre qu'il est vain de vouloir s'en tenir à de tels points fixes pour décrire une théorie qui incorpore l'ensemble des forces observées. Le problème est le même dans le domaine infrarouge où les rôles des interactions fortes et électrodynamiques sont inversés.

Il est bien connu des physiciens que le groupe de renormalisation joue le rôle d'un groupe d'ambiguïté ; l'on ne peut distinguer entre elles deux théories physiques qui appartiennent à la même orbite de ce groupe, ce qui nous ramène à Galois dont la « théorie de l'ambiguïté » allait bien au delà des équations algébriques.

7.4. Renormalisation et groupes de Galois motiviques. — Je terminerai cet article par un bref exposé de résultats récents obtenus en collaboration avec M. Marcolli ([**17**]) qui établissent enfin un lien précis entre renormalisation et théorie de Galois.

Nous montrons que les divergences de la théorie des champs codent, en fait, exactement l'action d'un *groupe de Galois motivique* explicite U^* sur l'ensemble des théories physiques. Le groupe de renormalisation apparaît comme un sous-groupe à un paramètre $\mathbb{G}_a \subset U^*$ du groupe de Galois U^*.

Notre point de départ est le Théorème 5.4 que nous améliorons en donnant la classification des familles de lacets $\gamma_\mu(d)$ vérifiant les conditions (65), (68) à valeurs dans un groupe pro-unipotent positivement gradué arbitraire G :

***Théorème 7.1* ([17])**. — *Soit $\gamma_\mu(z)$ une famille de lacets à valeurs dans G vérifiant les conditions* (65), (68). *Il existe alors un unique élément $\beta \in \mathfrak{g}$ de l'algèbre de Lie de G et un lacet $\gamma_{\rm reg}(z)$ régulier en $z=0$ tels que*

$$\gamma_\mu(z) = \mathrm{T}\mathbf{e}^{-\frac{\mathbf{1}}{\mathbf{z}}\int_\infty^{-\mathbf{z}\log\mu}\theta_{-\mathbf{t}}(\beta)\,\mathbf{dt}}\;\theta_{z\log\mu}(\gamma_{\rm reg}(z)).$$

Pour tout $\beta \in \mathfrak{g}$ et tout lacet régulier $\gamma_{\rm reg}(z)$ la décomposition de Birkhoff du lacet $\gamma_\mu(z)$ est donnée par

$$\gamma_\mu^+(z) = \mathrm{T}\mathbf{e}^{-\frac{\mathbf{1}}{\mathbf{z}}\int_0^{-\mathbf{z}\log\mu}\theta_{-\mathbf{t}}(\beta)\,\mathbf{dt}}\;\theta_{z\log\mu}(\gamma_{\rm reg}(z)),$$
$$\gamma_\mu^-(z) = \mathrm{T}\mathbf{e}^{-\frac{\mathbf{1}}{\mathbf{z}}\int_0^{\infty}\theta_{-\mathbf{t}}(\beta)\,\mathbf{dt}}.$$

La notation

$$\mathrm{T}\mathbf{e}^{\int_{\mathbf{a}}^{\mathbf{b}} \alpha(\mathbf{t})\,\mathbf{dt}} = 1 + \sum_{1}^{\infty} \int_{a \leqslant s_1 \leqslant \cdots \leqslant s_n \leqslant b} \alpha(s_1) \cdots \alpha(s_n) \prod ds_j,$$

désigne l'exponentielle ordonnée des physiciens qui est une notation commode pour la valeur en b de la solution A, $A(a) = 1$ de l'équation différentielle

$$dA(t) = A(t)\,\alpha(t)\,dt$$

à valeurs dans le groupe G et qui est, bien entendu, intimement reliée aux intégrales itérées.

Le pas suivant consiste à exhiber la correspondance de Riemann-Hilbert secrètement présente dans le théorème 7.1. Comme toute correspondance de Riemann-Hilbert elle relie des objets géométriques, qui dans notre cas sont les connexions plates équisingulières, aux représentations d'un groupe de « monodromie généralisée ». Notre modèle était le tore exponentiel de J.-P. Ramis dans la théorie locale des points singuliers irréguliers des équations différentielles.

Le procédé de régularisation dimensionnelle discuté en détails plus haut fournit, en fait, du fait de l'arbitraire dans la normalisation de l'intégrale en dimension complexe $d = D - z$, la donnée géométrique suivante :

– Un fibré principal B de groupe $\mathbb{G}_m = \mathbb{C}^*$, de base un disque infinitésimal Δ centré en D.

– Une connexion plate ω à valeurs dans l'algèbre de Lie $\mathfrak{g}$ de G, obtenue en différentiant la théorie des champs vue comme section du fibré $B \times G$ de base B.

La fibre $\pi^{-1}(d)$ du fibré B au dessus de $d \in \Delta$ est l'ensemble des normalisations possibles de l'intégration en dimension d (*cf.* (23)). La fibre spéciale $V = \pi^{-1}(D)$ joue un rôle particulier à cause des divergences de sorte que la connexion ω est singulière sur $V \subset B$.

En général, définissons l'*équivalence* de deux connexions singulières ω_j par l'existence d'une conjugaison régulière h entre elles,

$$\omega_2 = h^{-1}\,dh + h^{-1}\omega_1\,h.$$

La notion géométrique qui abstrait les propriétés (65), (68) est la suivante :

Definition 7.2. — Une connexion plate ω à valeurs dans l'algèbre de Lie $\mathfrak{g}$ définie sur $B^* = B \backslash V$ est *équisingulière* si elle est invariante par $\mathbb{G}_m$ et si la classe d'équivalence de sa restriction à une section $\sigma : \Delta \mapsto B$ ne dépend que de $\sigma(0)$.

Notre résultat principal est la classification complète des connexions équisingulières à équivalence près. En fait, nous construisons une catégorie tannakienne ayant pour objets les *fibrés plats équisinguliers* lesquels sont des couples (E, ∇) où E est un espace vectoriel $\mathbb{Z}$-gradué et ∇ une (classe de) connexion équisingulière sur le fibré équivariant $B \times E$. Notre résultat s'énonce alors ainsi :

***Théorème 7.3* ([17])**. — *La catégorie des fibrés plats équisinguliers est équivalente à la catégorie des représentations de dimension finie d'un (unique) groupe algébrique affine U^*. Ce groupe est le produit semi-direct par $\mathbb{G}_m$ (agissant par la graduation) du groupe pro-unipotent U dont l'algèbre de Lie*

$$\mathrm{Lie}(U) = \mathcal{F}(1, 2, 3, \dots).$$

est librement engendrée par un générateur e_{-n} de degré n pour tout entier $n \geqslant 1$.

Le groupe de renormalisation est un sous-groupe canonique $\mathbf{rg} : \mathbb{G}_a \mapsto U$ et nous construisons un repère singulier universel sur un fibré principal de base B et de groupe U dans lequel les divergences de la théorie des champs disparaissent.

Nous développons de plus l'analogie entre la catégorie des fibrés plats équisinguliers et celle des motifs de Tate mixtes. On sait, en particulier, que le groupe de Galois motivique $G_{\mathcal{M}_T}(\mathcal{O})$ ([**20**], [**26**]) du schéma $S_4 = \mathrm{Spec}(\mathcal{O})$ associé aux racines quatrièmes de l'unité (de sorte que $\mathcal{O}$ est l'anneau $\mathbb{Z}[i][\frac{1}{2}]$) est (non-canoniquement) isomorphe au groupe U^*.

L'ensemble de ces résultats montre que les divergences de la théorie des champs indiquent, en fait, la présence de symétries de nature « galoisienne » et, bien loin d'être des pathologies à éliminer, elles révèlent sans doute la subtilité de la géométrie qui gouverne l'espace-temps, une fois prise en compte la régularisation dimensionnelle.

Références

[1] A. Beauville – « Monodromie des systèmes différentiels linéaires à pôles simples sur la sphère de Riemann », in *Sém. Bourbaki, Vol. 1992/1993*, Astérisque, vol. 216, Société Mathématique de France, Paris, 1993, Exp. n° 765, p. 103–119.

[2] N.N. Bogoliubov & O. Parasiuk – *Acta Math.* **97** (1957), p. 227–266, K. Hepp, *Comm. Math. Phys.* **2** (1966), p. 301–326.

[3] A. Bolibruch – « Fuchsian systems with reducible monodromy and the Riemann-Hilbert problem », in *Global analysis—studies and applications, V*, Lect. Notes in Math., vol. 1520, Springer, Berlin, 1992, p. 139–155.

[4] P. Cartier – « A mad day's work : from Grothendieck to Connes and Kontsevich. The evolution of concepts of space and symmetry », *Bull. Amer. Math. Soc. (N.S.)* **38** (2001), no. 4, p. 389–408.

[5] J. Collins – *Renormalization*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1984.

[6] A. Connes – « Symétries ; de Galois au monde quantique », volume en l'honneur de Louis Michel (à paraître).

[7] ———, « Noncommutative Geometry and the Riemann Zeta Function », in *Mathematics : Frontiers and perspectives*, Amer. Math. Soc., Providence, RI, 2000, p. 35–54.

[8] A. Connes & M. Dubois-Violette – « Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples », *Comm. Math. Phys.* **230** (2002), no. 3, p. 539–579, arXiv : `math.QA/0107070`.

[9] ———, « Moduli Space and Structure of NonCommutative 3-Spheres », *Lett. Math. Phys.* **66** (2003), no. 1–2, p. 91–121, arXiv : `math.QA/0308275`.

[10] A. Connes & D. Kreimer – « Hopf algebras, renormalization and Noncommutative geometry », *Comm. Math. Phys.* **199** (1998), p. 203–242.

[11] ———, « Lessons from quantum field theory : Hopf algebras and spacetime geometries », *Lett. Math. Phys.* **48** (1999), no. 85, arXiv : `hep-th/9904044`.

[12] ———, « Renormalization in Quantum Field Theory and the Riemann-Hilbert problem », *J. High Energy Phys.* **9** (1999), no. 24.

[13] ———, « Renormalization in Quantum Field Theory and the Riemann-Hilbert problem. I », *Comm. Math. Phys.* **210** (2000).

[14] ———, « Renormalization in quantum field theory and the Riemann-Hilbert problem II, the β function, diffeomorphisms and the renormalization group », *Comm. Math. Phys.* (2001), arXiv : `hep-th/0003188`.

[15] A. Connes & G. Landi – « Noncommutative manifolds, the instanton algebra and isospectral deformations », arXiv : `math.QA/0011194`.

[16] A. Connes & M. Marcolli – « From Physics to Number Theory through Noncommutative Geometry », à paraître dans les actes « Les Houches 2003 ».

[17] ———, « Renormalization and Motivic Galois Theory », à paraître dans IMRN, arXiv : `math.NT/0409306`.

[18] A. Connes & H. Moscovici – « Modular Hecke Algebras and their Hopf Symmetry », arXiv : `math.QA/0301089`.

[19] ———, « Hopf algebras, cyclic cohomology and the transverse index theorem », *Comm. Math. Phys.* **198** (1998), p. 199–246.

[20] P. Deligne & A.B. Goncharov – « Groupes fondamentaux motiviques de Tate mixte », arXiv : `math.NT/0302267`, 2003.

[21] M. Dresden – « Renormalization in historical perspective – The first stage », in *Renormalization* (L. Brown, éd.), Springer-Verlag, New York, Berlin, Heidelberg, 1994.

[22] J. Feldman, J. Magnen, V. Rivasseau & R. Seneor – « Massive Gross-Neveu model : a rigorous perturbative construction », *Phys. Rev. Lett.* **54** (1985).

[23] E. Galois – *Écrits et mémoires mathématiques d'Évariste Galois*, Gauthier-Villars, Paris, London, 1962.

[24] K. Gawedzki & A. Kupiainen – « Exact renormalization of the Gross-Neveu model of quantum fields », *Phys. Rev. Lett.* **54** (1985).

[25] J. Glimm & A. Jaffe – *Quantum Physics*, Springer Verlag, New York, Berlin, Heidelberg, 1987.

[26] A.B. Goncharov – « Multiple polylogarithms and mixed Tate motives », arXiv : `math.AG/0103059`, 2001.

[27] G. Green – « Researches on the vibration of Pendulums in Fluid Media », *Royal Society of Edinburgh Transactions* (1836), p. 315–324.

[28] G.'t. Hooft – *Nuclear Phys. B* **61** (1973), no. 455.

[29] K. Iwasawa – « On the rings of valuation vectors », *Ann. of Math. (2)* **57** (1953), p. 331–356.

[30] T. Krajewski & R. Wulkenhaar – *European J. Phys.* **C7** (1999), p. 697–708, arXiv : `hep-th/9805098`.

[31] D. Kreimer – *Adv. Theor. Math. Phys.* **2.2** (1998), no. 303, arXiv : `q-alg/9707029`.

[32] I. Lappo-Danilevskii – *Mémoires sur la théorie des systèmes des équations différentielles linéaires*, Chelsea, 1953.

[33] J. Martinet & J.-P. Ramis – « Elementary acceleration and multisummability », *Ann. Inst. H. Poincaré* **54** (1991), p. 331–401.

[34] F. Menous – « The Birkhoff decomposition in the group of formal diffeomorphisms », *C. R. Acad. Sci. Paris Sér. I Math.* (2004).

[35] J.-P. Ramis – *Séries divergentes et théories asymptotiques*, Panoramas & Synthèses, Société Mathématique de France, Paris, 1993.

[36] E.K. Sklyanin – « Some algebraic structures connected with the Yang-Baxter equation », *Functional Anal. Appl.* **16** (1982), p. 263–270.

[37] A. Weil – « Sur la théorie du corps de classe », *J. Math. Soc. Japan* **3** (1951).

A. Connes, Collège de France, 11 place M. Berthelot, 75005 Paris et IHES, 35 route de Chartres, 91440 Bures-sur-Yvette (France) • *E-mail :* alain@connes.org

Astérisque
296, 2004, p. 145–165

b-FUNCTIONS AND INTEGRABLE SOLUTIONS OF HOLONOMIC $\mathcal{D}$-MODULE

by

Yves Laurent

À Jean-Pierre Ramis, à l'occasion de son 60^e anniversaire.

***Abstract.* —** A famous theorem of Harish-Chandra shows that all invariant eigendistributions on a semi-simple Lie group are locally integrable functions. We give here an algebraic version of this theorem in terms of polynomials associated with a holonomic $\mathcal{D}$-module.

***Résumé* (b-fonctions et solutions intégrables des modules holonomes). —** Un célèbre théorème de Harish-Chandra montre que les distributions invariantes propres sur un groupe de Lie semi-simple sont des fonctions localement intégrables. Nous donnons ici une version algébrique de ce théorème en termes de polynômes associés à un $\mathcal{D}$-module holonome.

Introduction

Let $G_{\mathbb{R}}$ be a real semisimple Lie group and $\mathfrak{g}_{\mathbb{R}}$ be its Lie algebra. An *invariant eigendistribution* T on $G_{\mathbb{R}}$ is a distribution which is invariant under conjugation by elements of $G_{\mathbb{R}}$ and is an eigenvector of every bi-invariant differential operator on $G_{\mathbb{R}}$. The main examples of such distributions are the characters of irreducible representations of $G_{\mathbb{R}}$. A famous theorem of Harish-Chandra sets that all invariant eigendistributions are L^1_{loc}-functions on $G_{\mathbb{R}}$ [**4**]. After transfer to the Lie algebra by the exponential map, such a distribution satisfies a system of partial differential equations.

In the language of $\mathcal{D}$-modules, these equations define a holonomic $\mathcal{D}$-module on the complexified Lie algebra $\mathfrak{g}$. We call this module the Hotta-Kashiwara module as it has been defined and studied first in [**6**]. In [**20**], J. Sekiguchi extended these results to symmetric pairs. He proved in particular that a condition on the symmetric pair is needed to extend Harish-Chandra theorem. In several papers, Levasseur and Stafford [**15, 16, 17**] gave an algebraic proof of the main part of Harish-Chandra theorem.

2000 *Mathematics Subject Classification.* — 35A27, 35D10, 17B15.
***Key words and phrases.* —** Vanishing cycles, D-modules, symmetric pair, b-function.

© Astérisque 296, SMF 2004

In [**3**], we defined a class of holonomic $\mathcal{D}$-modules, which we called *tame $\mathcal{D}$-modules*. These $\mathcal{D}$-modules have no quotients supported by a hypersurface and their distribution solution are locally integrable. We proved in particular that the Hotta-Kashiwara module is tame, recovering Harish-Chandra theorem. The definition of tame is a condition on the roots of the b-functions which are polynomials attached to the $\mathcal{D}$-module and a stratification of the base space. However, the proof of the fact that the Hotta-Kashiwara module is tame involved some non algebraic vector fields.

The first aim of this paper is to give a completely algebraic version of Harish-Chandra theorem. We give a slightly different definition of tame and an algebraic proof of the fact that the Hotta-Kashiwara module is tame. This proof is different from the proof of [**3**] and gives more precise results on the roots of the b-functions. However our first proof was still valid in the case of symmetric pairs while the present proof uses a morphism of Harish-Chandra which does not exist in that case.

Our second aim is to answer to a remark made by Varadarajan during the Ramis congress. He pointed the fact that an invariant eigendistribution, considered as a distribution on the Lie algebra by the exponential map, is not a solution of the Hotta-Kashiwara module. A key point in the original proof of Harish-Chandra is precisely the proof that after multiplication by a function, the eigendistribution is solution of the Hotta-Kashiwara module (see [**23**]). The study of the Hotta-Kashiwara module did not bypass this difficult step. Here we consider a family of holonomic $\mathcal{D}$-module, which we call (H-C)-modules; this family includes the Hotta-Kashiwara modules but also the module satisfied directly by an eigendistribution. We prove that these modules are tame and get a direct proof of Harish-Chandra theorem.

1. V-filtration and b-functions

We first recall the definition and a few properties of the classical V-filtration, then we give a new definition of quasi-homogeneous b-functions and of tame $\mathcal{D}$-modules. We end this section with a result on the inverse image of $\mathcal{D}$-modules which will be a key point of the proof in the next section.

1.1. Standard V-filtrations. — In this paper, $(X, \mathcal{O}_X)$ is a smooth algebraic variety defined over k, an algebraically closed field of characteristic 0. The sheaf of differential operators with coefficients in $\mathcal{O}_X$ is denoted by $\mathcal{D}_X$. Results and proofs are still valid if $k = \mathbb{C}$, X is a complex analytic manifold and $\mathcal{D}_X$ is the sheaf of differential operators with holomorphic coefficients.

Let Y be a smooth subvariety of X and $\mathcal{I}_Y$ the ideal of definition of Y. The V-filtration along Y is given by [**10**]:

$$V_k\mathcal{D}_X = \{\, P \in \mathcal{D}_X|_Y \mid \forall\, l \in \mathbb{Z}, P\mathcal{I}_Y^l \subset \mathcal{I}_Y^{l+k} \,\}$$

(with $\mathcal{I}_Y^l = \mathcal{O}_X$ if $l \leqslant 0$).

This filtration has been widely used in the theory of $\mathcal{D}$-modules, let us recall some of its properties (for the details, we refer to [**19**], [**12**], [**18**], [**14**]). The associated graded ring $\mathrm{gr}_V\mathcal{D}_X$ is the direct image by $p : T_YX \to X$ of the sheaf $\mathcal{D}_{T_YX}$ of differential operators on the normal bundle T_YX. If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, a $V\mathcal{D}_X$-filtration on $\mathcal{M}$ is a good filtration if it is locally finite, *i.e.* if, locally, there are sections $(u_1, \dots, u_N)$ of $\mathcal{M}$ and integers $(k_1, \dots, k_N)$ such that $V_k\mathcal{M} = \sum V_{k-k_i}\mathcal{D}_X u_i$.

If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module provided with a good V-filtration, the associated graded module is a coherent $\mathrm{gr}_V\mathcal{D}_X$-module and if $\mathcal{N}$ is a coherent submodule of $\mathcal{M}$ the induced filtration is a good filtration (see [**19**, Chapter III, Proposition 1.4.3] or [**18**]).

Let θ_Y be the Euler vector field of the fiber bundle T_YX, that is the vector field verifying $\theta_Y(f) = kf$ when f is a function on T_YX homogeneous of degree k in the fibers of p. A *b-function* along Y for a coherent $\mathcal{D}_X$-module with a good V-filtration is a polynomial b such that

$$\forall\, k \in \mathbb{Z}, \qquad b(\theta_Y + k)\mathrm{gr}_V^k\mathcal{M} = 0$$

If the good V-filtration is replaced by another, the roots of b are translated by integers. Here, we always fix the filtration, in particular, if the $\mathcal{D}_X$-module is of the type $\mathcal{D}_X/\mathcal{I}$, the good filtration will be induced by the canonical filtration of $\mathcal{D}_X$.

1.2. Quasi-homogeneous V-filtrations and quasi-b-functions. — Let $\varphi = (\varphi_1, \dots, \varphi_d)$ be a polynomial map from X to the vector space $W = k^d$ and $m_1, \dots, m_d$ be strictly positive and relatively prime integers. We define a filtration on $\mathcal{O}_X$ by:

$$V_k^\varphi\mathcal{O}_X = \sum_{\langle m,\alpha\rangle = -k} \mathcal{O}_X\varphi^\alpha$$

with $\alpha \in \mathbb{N}^d$, $\langle m, \alpha\rangle = \sum m_i\alpha_i$ and $\varphi^\alpha = \varphi_1^{\alpha_1} \cdots \varphi_d^{\alpha_d}$. If $k \geqslant 0$ we set $V_k^\varphi\mathcal{O}_X = \mathcal{O}_X$.

This filtration extends to $\mathcal{D}_X$ by:

$$V_k^\varphi\mathcal{D}_X = \{\, P \in \mathcal{D}_X \mid \forall\, l \in \mathbb{Z}, PV_l^\varphi\mathcal{O}_X \subset V_{l+k}^\varphi\mathcal{O}_X \,\} \tag{1}$$

Definition 1.2.1. — A (φ, m)-weighted Euler vector field is a vector field η in $\sum_i \varphi_i\mathcal{V}_X$ such that $\eta(\varphi_i) = m_i\varphi_i$ for $i = 1, \dots, d$. ($\mathcal{V}_X$ is the sheaf of vector fields on X.)

Lemma 1.2.2. — *Any (φ, m)-weighted Euler vector field is in $V_0^\varphi\mathcal{D}_X$ and if η_1 and η_2 are two (φ, m)-weighted Euler vector fields, $\eta_1 - \eta_2$ is in $V_{-1}^\varphi\mathcal{D}_X$.*

The map φ may be not defined on X but on an étale covering of X. More precisely, let us consider an étale morphism $\nu : X' \to X$ and a morphism $\varphi : X' \to W = k^d$. If $m_1, \dots, m_d$ are strictly positive and relatively prime integers, we define $V_k^\varphi\mathcal{O}_X$ as the sheaf of functions on X such that $f_\circ\nu$ is in $V_k^\varphi\mathcal{O}_{X'}$. This defines a V-filtration on $\mathcal{O}_X$ and on $\mathcal{D}_X$ by the formula (1). The map $TX' \to TX \times_X X'$ is an isomorphism and a vector field η on X defines a unique vector field $\nu^*(\eta)$ on X'. By definition, a vector

field η on X is a (φ, m)-weighted Euler vector field if $\nu^*(\eta)$ is a (φ, m)-weighted Euler vector field on X'.

Definition 1.2.3. — Let u be a section of a coherent $\mathcal{D}_X$-module $\mathcal{M}$. A polynomial b is a quasi-b-function of type (φ, m) for u if there exist a (φ, m)-weighted Euler vector field η and a differential operator Q in $V^{\varphi}_{-1}\mathcal{D}_X$ such that $(b(\eta) + Q)u = 0$.

The quasi-b-function is said *regular* if the order of Q as a differential operator is less or equal to the order of the polynomial b and *monodromic* if $Q = 0$.

The quasi-b-function is said *tame* if the roots of b are strictly greater than $-\sum m_i$.

These definitions are valid for any map φ but here we always assume that φ is smooth. Then if $Y = \varphi^{-1}(0)$, we say for short that b is a quasi-b-function of total weight $|m| = \sum m_i$ along Y. Remark that lemma 1.2.2 shows that the definition is independent of the (φ, m)-weighted Euler vector field η.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. A $V^{\varphi}\mathcal{D}_X$-filtration on $\mathcal{M}$ is a good filtration if it is locally finite.

Definition 1.2.4. — Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and $V^{\varphi}\mathcal{M}$ a good $V^{\varphi}\mathcal{D}_X$-filtration. A polynomial b is a quasi-b-function of type (φ, m) for $V^{\varphi}\mathcal{M}$ if, for any $k \in \mathbb{Z}$, $b(\eta + k)V^{\varphi}_k\mathcal{M} \subset V^{\varphi}_{k-1}\mathcal{M}$ where η is a (φ, m)-weighted Euler vector field.

The quasi-b-function is *monodromic* if $b(\eta + k)V^{\varphi}_k\mathcal{M} = 0$.

Definition 1.2.3 is a special case of definition 1.2.4 if $\mathcal{D}_X u$ is provided with the filtration induced by the canonical filtration of $\mathcal{D}_X$.

Recall that if $\mathcal{M}$ is a $\mathcal{D}_X$-module its inverse image by ν is its inverse image as an $\mathcal{O}_X$-module, that is:

$$\nu^+\mathcal{M} = \mathcal{O}_{X'} \otimes_{\nu^{-1}\mathcal{O}_X} \nu^{-1}\mathcal{M} = \mathcal{D}_{X'\to X} \otimes_{\nu^{-1}\mathcal{D}_X} \nu^{-1}\mathcal{M}$$

where $\mathcal{D}_{X'\to X}$ is the $(\mathcal{D}_{X'}, \nu^{-1}\mathcal{D}_X)$-bimodule $\mathcal{O}_{X'} \otimes_{\nu^{-1}\mathcal{O}_X} \nu^{-1}\mathcal{D}_X$.

Lemma 1.2.5. — *Let $\nu : X' \to X$ be an étale morphism and let φ be a morphism $X' \to W = k^d$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module.*

The polynomial b is a quasi-b-function of type (φ, m) for a section u of $\mathcal{M}$ if and only if it is a quasi-b-function of type (φ, m) for the section $1 \otimes u$ of $\nu^+\mathcal{M}$.

Proof. — If $\nu : X' \to X$ is étale, the canonical morphism $\mathcal{D}_{X'} \to \mathcal{D}_{X'\to X}$ given by $P \mapsto P(1 \otimes 1)$ is an isomorphism and defines an injective morphism $\nu^* : \nu^{-1}\mathcal{D}_X \to \mathcal{D}_{X'}$.

Conversely, the morphism $\widetilde{\nu} : \nu_*\mathcal{O}_{X'} \to \mathcal{O}_X$ given by $\widetilde{\nu}(f)(x) = \sum_{y\in\nu^{-1}(x)} f(y)$ extends to a morphism $\nu_*\mathcal{D}_{X'} \to \mathcal{D}_X$.

These two morphism are compatible with the V-filtration defined by φ and, by definition, a vector field η on X is a (φ, m)-weighted Euler vector field if and only if $\nu^*(\eta)$ is a (φ, m)-weighted Euler vector field on X'. If $(b(\eta) + R)u = 0$ we

have $(b(\nu^*\eta)+\nu^*R)(1\otimes u)=0$ and conversely, if $(b(\nu^*\eta)+R_1)(1\otimes u)=0$ then $(b(\eta)+\nu_*R_1)u=0$. □

Remark 1.2.6. — In [**3**] we gave an other definition of the V^*-filtration and quasi-b-function. The two definitions are essentially equivalent in the analytic framework but may differ in the algebraic case. More precisely, the filtration in [**3**] is given by a vector field η which we called positive definite. For a given V^φ-filtration, we may find a defining vector field with coefficients in formal power series (or in convergent series if $k=\mathbb{C}$) but in general not in rational functions. The definition of [**3**] is more intrinsic in the analytic case but not suitable here.

1.3. Tame $\mathcal{D}$-modules. — Let us recall that a stratification of the manifold X is a union $X=\bigcup_\alpha X_\alpha$ such that

- For each α, $\overline{X}_\alpha$ is an algebraic subset of X and X_α is its regular part.
- $\{X_\alpha\}_\alpha$ is locally finite.
- $X_\alpha\cap X_\beta=\varnothing$ for $\alpha\neq\beta$.
- If $\overline{X}_\alpha\cap X_\beta\neq\varnothing$ then $\overline{X}_\alpha\supset X_\beta$.

If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, its characteristic variety $Ch(\mathcal{M})$ is a homogeneous lagrangian subvariety of T^*X hence there exists a stratification $X=\bigcup X_\alpha$ such that $Ch(\mathcal{M})\subset\bigcup_\alpha\overline{T^*_{X_\alpha}X}$ [**9**, Ch. 5]. The set of points of X where $Ch(\mathcal{M})$ is contained in the zero section of T^*X is a non empty Zarisky open subset of X, its complementary is the *singular support* of $\mathcal{M}$.

For the next definition, we consider a cyclic $\mathcal{D}_X$-module with a canonical generator $\mathcal{M}=\mathcal{D}_Xu=\mathcal{D}_X/\mathcal{I}$ where $\mathcal{I}$ is a coherent ideal of $\mathcal{D}_X$.

Definition 1.3.1. — The cyclic holonomic $\mathcal{D}_X$-module $\mathcal{M}=\mathcal{D}_Xu$ is *tame* if there is a stratification $X=\bigcup X_\alpha$ of X such that $Ch(\mathcal{M})\subset\bigcup_\alpha\overline{T^*_{X_\alpha}X}$ and, for each α, a tame quasi-b-function associated with X_α.

With definition 1.2.3, this means that for each α, there is a smooth map φ_α from a Zarisky open set of X to a vector space V such that $X_\alpha=\varphi_\alpha^{-1}(0)$, positive integers $m_1,\dots,m_d$, a (φ,m)-weighted Euler vector field η and a quasi-b-function b_α for u with roots $>-\sum m_i$. A subvariety of X is conic for η_α if it is invariant under the flow of η_α. The module $\mathcal{M}$ is *conic tame* if it satisfy definition 1.3.1 and if moreover the singular support of $\mathcal{M}$ is conic for each η_α.

The following property of a tame $\mathcal{D}_X$-module has been proved in [**3**]:

Theorem 1.3.2. — *If the $\mathcal{D}_X$-module $\mathcal{M}$ is tame then it has no quotient with support in a hypersurface of X.*

If M is a real analytic manifold and X its complexification, we also proved:

Theorem 1.3.3. — *Let $\mathcal{M}$ be a holonomic and tame $\mathcal{D}_X$-module, assume that its singular support is the complexification of a real subvariety of M, then $\mathcal{M}$ has no distribution solution on M with support in a hypersurface. If $\mathcal{M}$ is conic-tame, its distribution solutions are in L^1_{loc}.*

Remark 1.3.4. — It is important to note that the definition of *tame* and the conclusions of theorem 1.3.3 depend of the choice of a generator for $\mathcal{M}$.

1.4. Inverse image. — Let $\varphi : X \to W$ and $\varphi' : X' \to W'$ be two morphisms from smooth algebraic varieties X and X' to the vector spaces $W = k^d$ and $W' = k^{d'}$, let $m_1, \dots, m_d$ and $m'_1, \dots, m'_{d'}$ be strictly positive integers. Let $f : X' \to X$ and $F : W' \to W$ be two morphisms such that $\varphi \circ f = F \circ \varphi'$. We assume that F is *quasi-homogeneous*, that is $F = (F_1, \dots, F_d)$ with $F_i(\lambda^{m'_1} x_1, \dots, \lambda^{m'_{d'}} x_{d'}) = \lambda^{m_i} F(x_1, \dots, x_{d'})$.

If $\mathcal{N}$ is a $\mathcal{D}_X$-module its inverse image by f is:

$$f^+\mathcal{N} = \mathcal{O}_{X'} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{N} = \mathcal{D}_{X' \to X} \otimes_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{N}$$

where $\mathcal{D}_{X' \to X}$ is the $(\mathcal{D}_{X'}, f^{-1}\mathcal{D}_X)$-bimodule $\mathcal{O}_{X'} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X$.

We define a filtration on $\mathcal{D}_{X' \to X}$ by

$$V_k^{\varphi'}\mathcal{D}_{X' \to X} = \sum_{i+j=k} V_i^{\varphi'}\mathcal{O}_{X'} \otimes f^{-1}V_j^{\varphi}\mathcal{D}_X$$

By the hypothesis, $g \circ f$ is a section of $V_k^{\varphi'}\mathcal{O}_{X'}$ for any g section of $V_k^{\varphi}\mathcal{O}_X$, hence the filtration on $\mathcal{D}_{X' \to X}$ is compatible with the corresponding filtrations on $\mathcal{D}_{X'}$ and $\mathcal{D}_X$.

If a $\mathcal{D}_X$-module $\mathcal{N}$ is provided with a V^{φ}-filtration, this defines a $V^{\varphi'}\mathcal{D}_{X'}$-filtration on $f^+\mathcal{N}$ by

$$V_k^{\varphi'} f^+\mathcal{N} = \sum_{i+j=k} V_i^{\varphi'}\mathcal{O}_{X'} \otimes f^{-1}V_j^{\varphi}\mathcal{N} = \sum_{i+j=k} V_i^{\varphi'}\mathcal{D}_{X' \to X} \otimes f^{-1}V_j^{\varphi}\mathcal{N} \tag{2}$$

The V-filtration has not all the good properties of the usual filtration, in particular non invertible elements may have an invertible principal symbol. In the proof of theorem 1.4.1 we introduce its formal completion given by:

$$\widehat{\mathcal{D}}_{X|Y} = \varinjlim_k V_k\widehat{\mathcal{D}}_{X|Y} \qquad \text{with} \qquad V_k\widehat{\mathcal{D}}_{X|Y} = \varprojlim_l V_k\mathcal{D}_X / V_{k-l}\mathcal{D}_X$$

By definition the graded ring of $\widehat{\mathcal{D}}_{X|Y}$ is the same than the graded ring of $\mathcal{D}_X$. If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module provided with a good V-filtration, its completion $\widehat{V}\mathcal{M}$ is defined in the same way and has the same associated graded module than $V\mathcal{M}$. The following properties may be found in [**19**] and [**14**].

The sheaf $\widehat{\mathcal{D}}_{X|Y}$ is a coherent and noetherian, flat over $\mathcal{D}_X$. We remind that a coherent sheaf of rings $\mathcal{A}$ is noetherian if any increasing sequence of coherent $\mathcal{A}$-submodules of a coherent $\mathcal{A}$-module is stationary. The sheaf of rings $V_0\widehat{\mathcal{D}}_{X|Y}$ is also coherent and noetherian.

If $\mathcal{M}$ is a $\mathcal{D}_X$-module provided with a good V-filtration, the associated graded module is a coherent $\mathrm{gr}_V\mathcal{D}_X$-module and if $\mathcal{N}$ is a coherent submodule of $\mathcal{M}$ the induced filtration is a good filtration. If $\kappa : (\widehat{\mathcal{D}}_{X|Y})^N \to \mathcal{M}$ is a filtered morphism which defines a surjective graded morphism $\mathrm{gr}_V(\widehat{\mathcal{D}}_{X|Y})^N \to \mathrm{gr}_V\mathcal{M} \to 0$ then κ is surjective.

As $\widehat{\mathcal{D}}_{X|Y}$ is flat over $\mathcal{D}_X$, if $\mathcal{M}$ is coherent we have $\widehat{V}\mathcal{M} = \widehat{\mathcal{D}}_{X|Y} \otimes_{\mathcal{D}_X} \mathcal{M}$. Remark also that $\widehat{V}\mathcal{O}_X$, the completion of $\mathcal{O}_X$ for the V-filtration, is the formal completion of $\mathcal{O}_X$ along Y usually denoted by $\mathcal{O}_{\widehat{X|Y}}$ and $\widehat{\mathcal{D}}_{X|Y}$ is a $\mathcal{O}_{\widehat{X|Y}}$-module.

After completion by the V-filtration, we get a similar formula:

$$\widehat{V}_k^{\varphi'} f^+\mathcal{N} = \sum_{i+j=k} \widehat{V}_i^{\varphi'}\mathcal{O}_{X'} \otimes f^{-1}\widehat{V}_j^{\varphi}\mathcal{N} \tag{3}$$

Let $Y = \varphi^{-1}(0)$ and $Y' = \varphi'^{-1}(0)$, let $p : T_Y X \to X$ and $p' : T_{Y'}X' \to X'$ be the normal bundles, $\widetilde{f} : T_{Y'}X' \to T_Y X$ be the map induced by f,

Theorem 1.4.1. — *We assume that φ' is smooth on X'. If $\mathcal{N}$ is a holonomic $\mathcal{D}_X$-module provided with a good $V^{\varphi}\mathcal{D}_X$-filtration, then $f^+\mathcal{N}$ is holonomic, $p'^{-1}\mathrm{gr}_{V^{\varphi'}} f^+\mathcal{N}$ is equal to $\widetilde{f}^+p^{-1}\mathrm{gr}_{V^{\varphi}}\mathcal{N}$ and isomorphic to the graded module associated with a good $V^{\varphi'}\mathcal{D}_{X'}$-filtration of $f^+\mathcal{N}$.*

Proof. — We recall that if $\mathcal{N}$ is coherent, then $f^+\mathcal{N}$ is not coherent in general but if $\mathcal{N}$ is holonomic, then $f^+\mathcal{N}$ is holonomic [**8**].

The filtration on $\mathcal{N}$ is a good $V^{\varphi}\mathcal{D}_X$-filtration hence we may assume that there are sections $(u_1, \dots, u_q)$ of $\mathcal{N}$ and integers $(k_1, \dots, k_q)$ such that $V_k^{\varphi}\mathcal{N} = \sum V_{k-k_i}^{\varphi}\mathcal{D}_X u_i$. Let $\mathcal{D}_{X'\to X}[N]$ be the sub-$\mathcal{D}_{X'}$-module of $\mathcal{D}_{X'\to X}$ generated by the sections of $\mathcal{D}_X$ of order less or equal to N. This submodule is finitely generated hence coherent. For each N, $(u_1, \dots, u_q)$ defines a canonical morphism $(\mathcal{D}_{X'\to X}[N])^q \to f^+\mathcal{N}$ and the family of the images of these morphisms is an increasing sequence of coherent submodules of the coherent $\mathcal{D}_{X'}$-module $f^+\mathcal{N}$. As $\mathcal{D}_{X'}$ is a noetherian sheaf of rings, this sequence is stationary, hence there is some N_0 such that for each $N > N_0$, the morphism $(\mathcal{D}_{X'\to X}[N])^q \to f^+\mathcal{N}$ is onto. The filtration $V^{\varphi'}\mathcal{D}_{X'\to X}$ induces a good filtration on $\mathcal{D}_{X'\to X}[N]$ hence, for $N > N_0$ a good filtration on $f^+\mathcal{N}$ which is denoted by $V_k^{\varphi'}[N]f^+\mathcal{N}$. To prove the theorem, we will prove that if N is large enough, $\mathrm{gr}_V f^+\mathcal{N}$ is equal to the graded module $\mathrm{gr}_{V[N]}f^+\mathcal{N}$ associated with the good filtration $V_k^{\varphi'}[N]f^+\mathcal{N}$.

We assume first that the integers m_i' are equal to 1, that is that the $V^{\varphi'}$-filtration is the usual V-filtration on the non singular variety $Y' = \varphi'^{-1}(0)$. For $N > N_0$,

$p'^{-1}\mathrm{gr}_{V[N]}f^+\mathcal{N}$ is a coherent $\mathcal{D}_{T_{Y'}X'}$-module. A direct calculation shows that $p'^{-1}\mathrm{gr}_V f^+\mathcal{N} = \widetilde{f}^+ p^{-1}\mathrm{gr}_V^\varphi\mathcal{N}$. If $\mathcal{N}$ is holonomic then $\mathrm{gr}_V^\varphi\mathcal{N}$ is also holonomic [**12**, Cor 4.1.2.] hence $p'^{-1}\mathrm{gr}_V f^+\mathcal{N}$ is holonomic hence coherent.

Consider the completion $\widehat{V}f^+\mathcal{N}$ of $f^+\mathcal{N}$ for the V-filtration and $\widehat{V}[N]f^+\mathcal{N}$ of $f^+\mathcal{N}$ for the $V[N]$-filtration. The graded module of $\widehat{V}f^+\mathcal{N}$ is equal to the graded module of $Vf^+\mathcal{N}$ which is coherent. Let $u_1,\dots,u_M$ be local sections of $\widehat{V}f^+\mathcal{N}$ whose classes generate the graded module, then $u_1,\dots,u_M$ generate $\widehat{V}f^+\mathcal{N}$ as a filtered $V\widehat{\mathcal{D}}_{X'|Y'}$-module and applying the same result to the kernel of $(V\widehat{\mathcal{D}}_{X'|Y'})^M \to \widehat{V}f^+\mathcal{N}$ we get that $\widehat{V}f^+\mathcal{N}$ admits a filtered presentation

$$(V\widehat{\mathcal{D}}_{X'|Y'})^L \longrightarrow (V\widehat{\mathcal{D}}_{X'|Y'})^M \longrightarrow \widehat{V}f^+\mathcal{N} \longrightarrow 0.$$

This shows in particular that each $\widehat{V}_k f^+\mathcal{N}$ is a coherent $V_0\widehat{\mathcal{D}}_{X'|Y'}$-module. We know that, for any N, $\mathrm{gr}_{V[N]}f^+\mathcal{N}$ is coherent hence for the same reason, each $\widehat{V}_k[N]f^+\mathcal{N}$ is a coherent $V_0\widehat{\mathcal{D}}_{X'|Y'}$-module.

Consider the family of the images of $\widehat{V}_k[N]f^+\mathcal{N}$ in $\widehat{V}_k f^+\mathcal{N}$, it is an increasing sequence of coherent sub-modules of the coherent $V_0\widehat{\mathcal{D}}_{X'|Y'}$-module $\widehat{V}_k f^+\mathcal{N}$ hence it is stationary because the sheaf of rings $V_0\widehat{\mathcal{D}}_{X'|Y'}$ is noetherian. Moreover, the filtration $\widehat{V}f^+\mathcal{N}$ is separated hence the maps $\widehat{V}_k[N]f^+\mathcal{N} \to \widehat{V}_k f^+\mathcal{N}$ are injective and the union of the images is all $\widehat{V}_k f^+\mathcal{N}$, so there is some N_0 such that for any $N > N_0$, $\widehat{V}_k[N]f^+\mathcal{N} = \widehat{V}_k f^+\mathcal{N}$. This implies that $\mathrm{gr}_V f^+\mathcal{N} = \mathrm{gr}_{V[N]}f^+\mathcal{N}$ is the graded module associated with a good V-filtration of $f^+\mathcal{N}$.

Assume now that the numbers m_i' are positive integers. Let $W'' = W'$, we define the ramification map $F_m : W'' \to W'$ by $F(s_1,\dots,s_d) = (s_1^{m_1'},\dots,s_d^{m_d'})$ and the corresponding map $f_m : X'' = X' \times_{W'} W'' \to X$. Applying the first part of the proof, we get $\widehat{V}[N]f_m^+ f^+\mathcal{N} = \widehat{V}f_m^+ f^+\mathcal{N}$ if N is large. The formula (3) shows that

$$\widehat{V}f_m^+ f^+\mathcal{N} = \widehat{V}\mathcal{O}_{X''} \otimes_{f^{-1}\widehat{V}\mathcal{O}_{X'}} f^{-1}\widehat{V}_j^\varphi\mathcal{N} = \mathcal{O}_{\widehat{W}} \otimes_{\mathcal{O}_{\widehat{V'}}} f^{-1}\widehat{V}_j^\varphi\mathcal{N}.$$

Here $\mathcal{O}_{\widehat{W}}$ is the set of formal power series in $(s_1,\dots,s_d)$ while $\mathcal{O}_{\widehat{V'}}$ is the set of formal power series in $(s_1^{m_1'},\dots,s_d^{m_d'})$ hence $\mathcal{O}_{\widehat{W}}$ is a finite free $\mathcal{O}_{\widehat{V'}}$-module. So, if $M' = \sum m_i'$, $\widehat{V}f_m^+ f^+\mathcal{N}$ is isomorphic to $(\widehat{V}f^+\mathcal{N})^{M'}$ as a $\widehat{V}\mathcal{O}_{X'}$-module.

In the same way, $\widehat{V}[N]f_m^+ f^+\mathcal{N}$ is isomorphic to $(\widehat{V}[N]f^+\mathcal{N})^{M'}$, hence $\widehat{V}[N]f^+\mathcal{N} = \widehat{V}[N]f^+\mathcal{N}$. This shows that $\mathrm{gr}_{V^{\varphi'}}f^+\mathcal{N}$ is the graded module associated with $V^{\varphi'}[N]f^+\mathcal{N}$ which is a good filtration of $f^+\mathcal{N}$. □

Remark 1.4.2. — The result was known when f is a submersion, $Y' = f^{-1}(Y)$ and the V-filtrations being the usual V-filtrations along Y and Y' [**14**]. The introduction of the weights m_i and m_i' allows f to be non submersive and Y' to be a proper subvariety of $f^{-1}(Y)$; the relation between the weights is given by the quasi-homogeneity of F.

Corollary 1.4.3. — *Under the hypothesis of theorem 1.4.1, if $\mathcal{N}$ is a holonomic $\mathcal{D}_X$-module provided with a good $V^{\varphi}\mathcal{D}_X$-filtration, $f^+\mathcal{N}$ is provided with a good $V^{\varphi'}\mathcal{D}_{X'}$-filtration such that a polynomial b is a quasi-b-function of type (φ, m) for the filtration of $\mathcal{N}$ if and only if b is a quasi-b-function of type (φ', m') for the filtration of $f^+\mathcal{N}$.*

Proof. — Let η' be a (φ', m')-weighted Euler vector field, then $\eta = f_*\eta'$ is a (φ, m)-weighted Euler vector field. As definition 1.2.4 is independent of the (φ, m)-weighted Euler vector field, we may assume that the quasi-b-function for $\mathcal{N}$ is relative to η.

By definition, for any Q in $\mathcal{D}_{X'\to X}$, we have $\eta' Q = Q\eta$ hence for any polynomial $b(\eta')Q = Qb(\eta)$ which shows the corollary. □

Corollary 1.4.4. — *Under the hypothesis of theorem 1.4.1, if $\mathcal{N}$ is a holonomic $\mathcal{D}_X$-module and u a section of $\mathcal{N}$ with a quasi-b-function of type (φ, m), then the section $1_{X'\to X} \otimes u$ of $f^+\mathcal{N}$ has the same polynomial b as a quasi-b-function of type (φ', m').*

Proof. — Recall that $1_{X'\to X}$ is the canonical section $1 \otimes 1$ in $\mathcal{D}_{X'\to X} = \mathcal{O}_{X'} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X$. If u is a section of $\mathcal{N}$, we set on $\mathcal{D}_X u$ the filtration image of the filtration of $\mathcal{D}_X$. Then, by definition of the filtration $V^{\varphi'}[N]f^+\mathcal{N}$ used in the proof of theorem 1.4.1, $1_{X'\to X} \otimes u$ is of order 0 for this filtration. Then corollary 1.4.4 is a special case of corollary 1.4.3. □

2. Reductive Lie algebras

2.1. Statement of the main theorem. — Let G be a connected reductive algebraic group with Lie algebra $\mathfrak{g}$, let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. The group G acts on $\mathfrak{g}$ by the adjoint action hence on the symmetric algebra $S(\mathfrak{g})$ identified to the space $\mathcal{O}(\mathfrak{g}^*)$ of polynomial functions on $\mathfrak{g}^*$. By Chevalley's theorem, the space $\mathcal{O}(\mathfrak{g}^*)^G \simeq S(\mathfrak{g})^G$ of invariant polynomials on $\mathfrak{g}^*$ is equal to a polynomial algebra $k[Q_1, \ldots, Q_l]$ where $Q_1, \ldots, Q_l$ are algebraically independent. These spaces are graded and we denote $S_+(\mathfrak{g})^G = \oplus_{k>0} S_k(\mathfrak{g})^G$. It is also the set $\mathcal{O}_+(\mathfrak{g}^*)^G$ of invariant polynomials vanishing at $\{0\}$, their common roots are the nilpotent elements of $\mathfrak{g}^*$.

The differential of the adjoint action induces a Lie algebra morphism $\tau : \mathfrak{g} \to \mathrm{Der}S(\mathfrak{g}^*)$ by:

$$(\tau(A)f)(x) = \frac{d}{dt} f\left(\exp(-tA)\cdot x\right)|_{t=0} \quad \text{for } A \in \mathfrak{g},\ f \in S(\mathfrak{g}^*) = \mathcal{O}(\mathfrak{g}),\ x \in \mathfrak{g}$$

i.e. $\tau(A)$ is the vector field on $\mathfrak{g}$ whose value at $x \in \mathfrak{g}$ is $[x, A]$. We denote by $\tau(\mathfrak{g})$ the set of all vector fields $\tau(A)$ for $A \in \mathfrak{g}$. It generates the set of vector fields on $\mathfrak{g}$ tangent to the orbits of G.

Let $\mathcal{D}_{\mathfrak{g}}^G$ be the sheaf of differential operators on $\mathfrak{g}$ invariant under the adjoint action of G. The principal symbol $\sigma(P)$ of such an operator P is a function on $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^*$ invariant under the action of G. Examples of such invariant functions are the elements of $S(\mathfrak{g})^G$ identified to functions on $\mathfrak{g} \times \mathfrak{g}^*$ constant in the variables of $\mathfrak{g}$. If F is a

subsheaf of $\mathcal{D}_{\mathfrak{g}}^G$, we denote by $\sigma(F)$ the sheaf of the principal symbols of all elements of F.

Definition 2.1.1. — A subsheaf F of $\mathcal{D}_{\mathfrak{g}}^G$ is of (H-C)-type if $\sigma(F)$ contains a power of $S_+(\mathfrak{g})^G$. An (H-C)-type $\mathcal{D}_{\mathfrak{g}}$-module is the quotient $\mathcal{M}_F$ of $\mathcal{D}_{\mathfrak{g}}$ by the ideal $\mathcal{I}_F$ generated by $\tau(\mathfrak{g})$ and by F.

The main result of this paper is

Theorem 2.1.2. — *Any $\mathcal{D}_{\mathfrak{g}}$-module of (H-C)-type is holonomic and conic-tame.*

Here (H-C) stands for Harish-Chandra. There are two main examples of such $\mathcal{D}_{\mathfrak{g}}$-modules which we describe now.

Example 2.1.3. — An element A of $\mathfrak{g}$ defines a vector field with constant coefficients on $\mathfrak{g}$ by:

$$(A(D_x)f)(x) = \frac{d}{dt} f(x + tA)|_{t=0} \quad \text{for } f \in S(\mathfrak{g}^*),\ x \in \mathfrak{g}$$

By multiplication, this extends to an injective morphism from $S(\mathfrak{g})$ to the algebra of differential operators with constant coefficients on $\mathfrak{g}$; we identify $S(\mathfrak{g})$ with its image and denote by $P(D_x)$ the image of $P \in S(\mathfrak{g})$. If F is a finite codimensional ideal of $S(\mathfrak{g})^G$, its graded ideal contains a power of $S_+(\mathfrak{g})^G$ hence when it is identified to a set of differential operators with constant coefficients, F is a subsheaf of $\mathcal{D}_{\mathfrak{g}}$ of (H-C)-type and $\mathcal{M}_F$ is a $\mathcal{D}_{\mathfrak{g}}$-module of (H-C)-type. If $\lambda \in \mathfrak{g}^*$, the module $\mathcal{M}_\lambda^F$ defined by Hotta and Kashiwara [**6**] is the special case where F is the set of polynomials $Q - Q(\lambda)$ for $Q \in S(\mathfrak{g})^G$.

Example 2.1.4. — The enveloping algebra $U(\mathfrak{g})$ is the algebra of left invariant differential operators on G. It is filtered by the order of operators and the associated graded algebra is isomorphic by the symbol map to $S(\mathfrak{g})$. This map is a G-map and defines a morphism from the space of bi-invariant operators on G to the space $S(\mathfrak{g})^G$. This map is a linear isomorphism, its inverse is given by a symmetrization morphism [**22**, Theorem 3.3.4.]. We assume that $k = \mathbb{C}$. Then, through the exponentional map a bi-invariant operator P defines a differential operator $\widetilde{P}$ on the Lie algebra $\mathfrak{g}$ which is invariant under the adjoint action of G (because the exponential intertwines the adjoint action on the group and on the algebra) and the principal symbol $\sigma(\widetilde{P})$ is equal to $\sigma(P)$.

Let U be an open subset of $\mathfrak{g}$ where the exponential is injective and $U_G = \exp(U)$. Let T be an invariant eigendistribution on U_G and $\widetilde{T}$ the distribution on U given by $\langle T, \varphi\rangle = \langle \widetilde{T}, \varphi_o \exp\rangle$. As T is invariant and eigenvalue of all bi-invariant operators, $\widetilde{T}$ is solution of an (H-C)-type $\mathcal{D}_{\mathfrak{g}}$-module. As this module is conic-tame by theorem 2.1.2, the results of theorems 1.3.2 and 1.3.3 are true for it, hence $\widetilde{T}$ and T are a L^1_{loc}-function.

As $\mathfrak{g}$ is reductive, it is the direct sum $\mathfrak{c} \oplus [\mathfrak{g}, \mathfrak{g}]$ of its center and of the semi-simple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. We choose a non-degenerate G-invariant symmetric bilinear form κ on $\mathfrak{g}$ which extend the Killing form of $[\mathfrak{g}, \mathfrak{g}]$. This defines an isomorphism from $\mathfrak{g}^*$ to $\mathfrak{g}$ and the cotangent bundle $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^*$ is identified with $\mathfrak{g} \times \mathfrak{g}$. Then if $\mathcal{N}(\mathfrak{g})$ is the nilpotent cone of $\mathfrak{g}$, the characteristic variety of an (H-C)-type $\mathcal{D}_{\mathfrak{g}}$-module is a subset of:

$$\{ (x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0, y \in \mathcal{N}(\mathfrak{g}) \}$$

so it is a holonomic $\mathcal{D}_{\mathfrak{g}}$-module [**6**].

2.2. Stratification of a reductive Lie algebra. — In this section, we define the stratification which will be used to prove that an (H-C)-type module is tame. This stratification is classical (see [**1**] for example).

The stratification of a reductive Lie algebra is the direct sum of the center by the stratification of the semi-simple part, so we may assume that $\mathfrak{g}$ is semi-simple. An element X of $\mathfrak{g}$ is said to be *semisimple* (resp. *nilpotent*) if $\mathrm{ad}(X)$ is semisimple (resp $\mathrm{ad}(X)$ is nilpotent). Any $X \in \mathfrak{g}$ may be decomposed in a unique way as $X = S + N$ where S is semisimple, N is nilpotent and $[S, N] = 0$ (Jordan decomposition). An element X is said to be *regular* if the dimension of its centralizer $\mathfrak{g}^X = \{ Z \in \mathfrak{g} \mid [X, Z] = 0 \}$ is minimal, that is equal to the rank of $\mathfrak{g}$. The set $\mathfrak{g}_{rs}$ of semisimple regular elements of $\mathfrak{g}$ is Zarisky dense and its complementary $\mathfrak{g}'$ is defined by a G-invariant polynomial equation $\Delta(X) = 0$. The function Δ may be defined from the characteristic polynomial of $\mathrm{ad}(X)$:

$$\det(T \cdot \mathrm{Id} - \mathrm{ad}(X)) = T^n + \sum \lambda_i(X) T^i$$

Here n is the dimension of $\mathfrak{g}$. Then $\lambda_0 \equiv 0$, the rank l of $\mathfrak{g}$ is the lowest i such that $\lambda_i \not\equiv 0$ and $\Delta(X) = \lambda_l(X)$. This function is homogeneous of degree $n - l$.

The set $\mathfrak{N}(\mathfrak{g})$ of nilpotent elements of $\mathfrak{g}$ is a cone equal to:

$$\mathfrak{N}(\mathfrak{g}) = \{ X \in \mathfrak{g} \mid \forall P \in \mathcal{O}(\mathfrak{g})^G \ P(X) = P(0) \}$$

and the set of nilpotent orbits is finite and define a stratification of $\mathfrak{N}$ [**11**, Cor 3.7.].

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote by $\mathfrak{W}$ the Weyl group $\mathfrak{W}(\mathfrak{g}, \mathfrak{h})$. Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ be the root system associated with $\mathfrak{h}$. For each $\alpha \in \Phi$ we denote by $\mathfrak{g}_\alpha$ the root subspace corresponding to α and by $\mathfrak{h}_\alpha$ the subset $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ of $\mathfrak{h}$ (they are all 1-dimensional). Let $\mathcal{F}$ be the set of the subsets P of Φ which are closed and symmetric that is such that $(P + P) \cap \Phi \subset P$ and $P = -P$. For each $P \in \mathcal{F}$ we define $\mathfrak{h}_P = \sum_{\alpha \in P} \mathfrak{h}_\alpha$, $\mathfrak{g}_P = \sum_{\alpha \in P} \mathfrak{g}_\alpha$, $\mathfrak{h}_P^\perp = \{ H \in \mathfrak{h} \mid \alpha(H) = 0 \text{ if } \alpha \in P \}$ and $(\mathfrak{h}_P^\perp)' = \{ H \in \mathfrak{h} \mid \alpha(H) = 0 \text{ if } \alpha \in P, \alpha(H) \neq 0 \text{ if } \alpha \notin P \}$.

The following results are well-known (see [**2**, Ch. VIII, § 3]):

a) $\mathfrak{q}_P = \mathfrak{h}_P + \mathfrak{g}_P$ is a semisimple Lie subalgebra of $\mathfrak{g}$ stable under $\mathrm{ad}\,\mathfrak{h}$ and $\mathfrak{h}_P^\perp$ is an orthocomplement of $\mathfrak{h}_P$ for the Killing form, $\mathfrak{h}_P$ is a Cartan subalgebra of $\mathfrak{q}_P$. The

Weyl group $\mathfrak{W}_P$ of $(\mathfrak{q}_P, \mathfrak{h}_P)$ is identified to the subgroup $\mathfrak{W}'$ of $\mathfrak{W}$ of elements whose restriction to $\mathfrak{h}_P^\perp$ is the identity.

b) $\mathfrak{h} + \mathfrak{g}_P$ is a reductive Lie subalgebra of $\mathfrak{g}$ stable under $\operatorname{ad}\mathfrak{h}$. For any $S \in \mathfrak{h}_P^\perp$, $\mathfrak{h} + \mathfrak{g}_P \subset \mathfrak{g}^S$ and $(\mathfrak{h}_P^\perp)' = \{\, S \in \mathfrak{h}_P^\perp \mid \mathfrak{g}^S = \mathfrak{h} + \mathfrak{g}_P \,\}$.

c) Conversely, if $S \in \mathfrak{h}$, there exists a subset P of Φ which is closed and symmetric such that $\mathfrak{g}^S = \mathfrak{h} + \mathfrak{g}_P$. P is unique up to a conjugation by $\mathfrak{W}$.

To each P of $\mathcal{F}$ and each nilpotent orbit $\mathfrak{O}$ of $\mathfrak{q}_P$ we associate a conic subset of $\mathfrak{g}$

$$S_{(P,\mathfrak{O})} = \bigcup_{X \in (\mathfrak{h}_P^\perp)'} G \cdot (X + \mathfrak{O}) \tag{4}$$

where $G \cdot (X + \mathfrak{O})$ is the union of orbits of points $X + \mathfrak{O}$.

If $X = S + N$ is the Jordan decomposition of $X \in \mathfrak{g}$, the semisimple part S belongs to a Cartan subalgebra which we may assume to be $\mathfrak{h}$ because they are all conjugate. Hence there is some P in $\mathcal{F}$ such that $\mathfrak{g}^S = \mathfrak{h} + \mathfrak{g}_P$. Then, if the orbit of N in $\mathfrak{q}_P$ is $\mathfrak{O}$, $X \in S_{(P,\mathfrak{O})}$. For a detailed proof of the fact that it is a stratification, see [**3**].

2.3. Polynomials and differentials. — Let us begin with some elementary calculations. If $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index of $\mathbb{N}^n$ we denote $|\beta| = \sum \beta_i$ and $\beta! = \beta_1! \cdots \beta_n!$, if α is another element of $\mathbb{N}^n$, we denote by $\alpha \leqslant \beta$ the relation $\alpha_1 \leqslant \beta_1, \dots, \alpha_n \leqslant \beta_n$.

Lemma 2.3.1. — *Let $\beta \in \mathbb{N}^n$ and $M = |\beta|$, let $N \in \mathbb{N}$ such that $N \leqslant M$, then*

$$\sum_{\substack{|\alpha|=N \\ \alpha \leqslant \beta}} \frac{\beta!}{\alpha!(\beta-\alpha)!} = \frac{M!}{N!(M-N)!}$$

Proof

$$\sum_{\alpha \leqslant \beta} \frac{\beta!}{\alpha!(\beta-\alpha)!} x^\alpha = \prod_{i=1}^{n} \sum_{\alpha_i=0}^{\beta_i} \frac{\beta_i!}{\alpha_i!(\beta_i-\alpha_i)!} x_i^{\alpha_i} = (1+x_1)^{\beta_1} \cdots (1+x_n)^{\beta_n}$$

hence if $t = x_1 = \cdots = x_n$ we get:

$$\sum_{\alpha \leqslant \beta} \frac{\beta!}{\alpha!(\beta-\alpha)!} t^{|\alpha|} = (1+t)^M$$

and the coefficient of t^N in both side of the equality gives the lemma. □

Lemma 2.3.2. — *Let us denote $x = (x_1, \dots, x_n)$, $D_x = (D_{x_1}, \dots, D_{x_n})$, $x^\alpha = (x_1^{\alpha_1}, \dots, x_n^{\alpha_n})$ and $D_x^\alpha = (D_{x_1}^{\alpha_1}, \dots, D_{x_n}^{\alpha_n})$, let $\theta = \sum x_i D_{x_i}$, then:*

$$\sum_{|\alpha|=N} \frac{N!}{\alpha!} x^\alpha D_x^\alpha = \theta(\theta-1)\cdots(\theta-N+1)$$

Proof. — To prove the equality of the two differential operators we have to show that they give the same result when acting on a monomial x^β, so lemma 2.3.1 gives:

$$\sum_{|\alpha|=N} \frac{N!}{\alpha!} x^\alpha D_x^\alpha x^\beta = \sum_{\substack{|\alpha|=N \\ \alpha \leqslant \beta}} \frac{N!}{\alpha!} \frac{\beta!}{(\beta-\alpha)!} x^\beta = \frac{|\beta|!}{(|\beta|-N)!} x^\beta = \theta(\theta-1)\cdots(\theta-N+1)x^\beta$$

□

Proposition 2.3.3. — *Let $p_1, \ldots, p_n(\xi)$ be homogeneous polynomial on $X = \mathbb{C}^n$ and assume that:*

$$\bigcap_{i=1}^{n} \{p_i(\xi) = 0\} = \{0\}$$

Let $\mathcal{I}$ be the ideal of $\mathcal{D}_X$ generated by $p_1(D_x), \ldots, p_n(D_x)$ and $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$. The $\mathcal{D}_X$ module $\mathcal{M}$ is holonomic and the b-function of $\mathcal{M}$ relative to $\{0\}$ is equal to

$$b(\theta) = \theta(\theta-1)\cdots(\theta+n-M)$$

where M is the sum of the degrees of the polynomials $p_1, \ldots, p_n$ and θ the Euler vector field of X. This b-function is monodromic in the canonical coordinates of $\mathbb{C}^n$.

Proof. — The Nullstellensatz shows that there is some integer M_1 such that the monomial ξ^α are in the ideal generated by $p_1, \ldots, p_n$ if $|\alpha| > M_1$. In fact it is known that the lowest M_1 is $M - n$ (the proof uses the Hilbert polynomial). Then lemma 2.3.2 shows that the b-function of $\mathcal{M}$ divides $\theta\cdots(\theta+n-M)$. It has been proved by T. Torrelli [**21**] that all integers $0, \ldots, M-n$ appear effectively as roots of b. □

Proposition 2.3.4. — *Let $p_1, \ldots, p_n$ be the same polynomials as in the previous proposition and let $P_1, \ldots P_n$ be differential operators such that $\sigma(P_i) = p_i$. Let $\mathcal{I}$ be the ideal of $\mathcal{D}_X$ generated by the operators $P_1, \ldots, P_n$ and $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$. The $\mathcal{D}_X$-module $\mathcal{M}$ is holonomic and the b-function of $\mathcal{M}$ relative to $\{0\}$ is equal to*

$$b(\theta) = \theta(\theta-1)\cdots(\theta+n-M)$$

The b-function of $\mathcal{M}$ along a vector subspace L of $\mathbb{C}^n$ divides the same polynomial b.

Proof. — Each function ξ^α for $|\alpha| = N = M - n + 1$ is written as $\xi^\alpha = \sum q_i^\alpha(\xi) p_i(\xi)$ and

$$\begin{aligned} b(\theta) &= \sum_{|\alpha|=N} \frac{N!}{\alpha!} x^\alpha D_x^\alpha = \sum_{\substack{|\alpha|=N \\ i=1,\ldots,n}} \frac{N!}{\alpha!} x^\alpha q_i^\alpha(D_x) p_i(D_x) \\ &= \sum_{\substack{|\alpha|=N \\ i=1,\ldots,n}} \frac{N!}{\alpha!} x^\alpha q_i^\alpha(D_x) P_i(x, D_x) + \sum_{\substack{|\alpha|=N \\ i=1,\ldots,n}} \frac{N!}{\alpha!} x^\alpha q_i^\alpha(D_x) \left(p_i(D_x) - P_i(x, D_x)\right) \end{aligned}$$

By definition, the order for the V-filtration along $\{0\}$ is always less then the usual order with equality if the operator has constant coefficients. So the order for the V-filtration of $\sum \frac{N!}{\alpha!} x^\alpha q_i^\alpha(D_x)(p_i(D_x) - P_i(x, D_x))$ is strictly less than the order

of $\sum \frac{N!}{\alpha!} x^\alpha q_i^\alpha(D_x) p_i(D_x)$ which is the order of $b(\theta)$, that is 0. On the other hand $\sum \frac{N!}{\alpha!} x^\alpha q_i^\alpha(D_x) P_i(x, D_x)$ is in the ideal $\mathcal{I}$ hence $b(\theta)$ is a b-function.

For the second part, we choose linear coordinates of $\mathbb{C}^n$ such that $L = \{ (x, t_1, \ldots, t_d) \in \mathbb{C}^n \mid t = 0 \}$ and we write

$$b(\langle t, D_t \rangle) = \sum_{|\beta|=N} \frac{N!}{\beta!} t^\beta D_t^\beta$$

As all D_t^β for $|\beta| = N$ are in the ideal generated by $p_1(D_x, Dt), \ldots, p_n(D_x, Dt)$ the proof is the same then before. □

2.4. Proof of the main theorem. — Let $\varphi : Y \to X$ be an algebraic map. A vector field u on Y is said to be *tangent to the fibers of* φ if $u(f \circ \varphi) = 0$ for all f in $\mathcal{O}_X$. A differential operator P is said to be *invariant under* φ if there exists a k-endomorphism A of $\mathcal{O}_X$ such that $P(f \circ \varphi) = A(f) \circ \varphi$ for all f in $\mathcal{O}_X$. If we assume from now that φ is dominant, A is uniquely determined by P and is a differential operator on X. We denote by $A = \varphi_*(P)$ the image of P in $\mathcal{D}_X$ under this ring homomorphism.

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote by $\mathfrak{W}$ the Weyl group $\mathfrak{W}(\mathfrak{g}, \mathfrak{h})$. The Chevalley theorem shows that $\mathcal{O}(\mathfrak{g})^G$ is equal to $k[P_1, \ldots, P_l]$ where $(P_1, \ldots, P_l)$ are algebraically independent invariant polynomials and l is the rank of $\mathfrak{g}$, that the set of polynomials on $\mathfrak{h}$ invariant under $\mathfrak{W}$ is $\mathcal{O}(\mathfrak{h})^{\mathfrak{W}} = k[p_1, \ldots, p_l]$ where p_j is the restriction to $\mathfrak{h}$ of P_j and that the restriction map $P \mapsto P|_{\mathfrak{h}}$ defines an isomorphism of $\mathcal{O}(\mathfrak{g})^G$ onto $\mathcal{O}(\mathfrak{h})^{\mathfrak{W}}$ [**22**, § 4.9.]. The space $W = \mathfrak{h}/\mathfrak{W}$ is thus isomorphic to k^l and the functions $P_1, \ldots, P_l$ define two morphisms $\psi : \mathfrak{g} \to W$ and $\varphi : \mathfrak{h} \to W$ by $\psi(x) = (P_1(x), \ldots, P_l(x))$ and $\varphi(z) = (p_1(z), \ldots, p_l(z))$.

An operator Q of $\mathcal{D}_{\mathfrak{g}}^G$ transforms invariant functions into invariant functions hence is invariant under ψ and $\psi_*(Q)$ is a differential operator on W. A vector field of $\tau(\mathfrak{g})$ annihilates the functions $P_1, \ldots, P_l$ hence is tangent to the fibers of ψ. In the same way, let $\mathcal{D}_{\mathfrak{h}}^{\mathfrak{W}}$ be the space of differential operators on $\mathfrak{h}$ which are invariant under the action of the Weyl group $\mathfrak{W}$, they are invariant under φ and define operators on W through φ_*.

Let $\mathcal{D}(\mathfrak{g})^G$ (resp. $\mathcal{D}(\mathfrak{h})^{\mathfrak{W}}$) be the set of global sections of $\mathcal{D}_{\mathfrak{g}}^G$ (resp. of $\mathcal{D}_{\mathfrak{h}}^{\mathfrak{W}}$). The morphism of Harish-Chandra [**5**] is a morphism of sheaves of rings $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^{\mathfrak{W}}$ which satisfies the following properties:

(1) If $f \in \mathcal{O}(\mathfrak{g})^G \simeq \mathcal{O}(\mathfrak{h})^{\mathfrak{W}}$ then $\delta(P)(f|_{\mathfrak{h}}) = \Delta^{1/2} P(f) \Delta^{-1/2}|_{\mathfrak{h}}$.

(2) If $f \in \mathcal{O}(\mathfrak{g})^G$, $\delta(f)$ is the restriction of f to $\mathfrak{h}$

(3) If $f \in S(\mathfrak{g})^G$ and f is considered as a constant coefficients operator, then $\delta(f)$ is the restriction of f to $\mathfrak{h}^*$.

(4) The morphism δ is surjective onto $\mathcal{D}(\mathfrak{h})^{\mathfrak{W}}$.

(5) The kernel of δ is $\mathcal{D}(\mathfrak{g})^G \cap \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$.

The last two results have been proved algebraically by Levasseur and Stafford in [**15**] and [**16**]. Let E be the Euler vector field of $\mathfrak{g}$ and ϑ the Euler vector field of $\mathfrak{h}$. The function Δ is homogeneous of degree $n-l$ (2.2) hence $\delta(E)$ is equal to $\vartheta-(n-l)/2$.

Let $\mathcal{D}_W[d^{-1}]$ be the sheaf of differential operators on W with poles on $\{d=0\}$ and $\mathcal{D}(W)[d^{-1}]$ be the ring of its global sections. The function Δ is invariant hence of the form $d(P_1,\dots,P_l)$ and the formula $Q \mapsto d^{1/2}Qd^{-1/2}$ defines an isomorphism γ of $\mathcal{D}_W[d^{-1}]$. We get a diagram:

$$\begin{array}{ccc} \mathcal{D}(\mathfrak{g})^G & \xrightarrow{\delta} & \mathcal{D}(\mathfrak{h})^{\mathfrak{W}} \\ \psi_* \downarrow & & \downarrow \varphi_* \\ \mathcal{D}(W)[d^{-1}] & \xrightarrow{\gamma} & \mathcal{D}(W)[d^{-1}] \end{array} \tag{5}$$

If f is a polynomial on W and Q an operator of $\mathcal{D}_{\mathfrak{g}}^G$ we have $\varphi_*(\delta(P))(f) = \gamma(\psi_*(P))(f)$ from the definitions hence the diagram is commutative. We can avoid the denominators $[d^{-1}]$ in the diagram because of the following lemma:

Lemma 2.4.1. — *The morphism γ sends the image of ψ_* into $\mathcal{D}(W)$ while its inverse γ^{-1} sends the image of φ_* into $\mathcal{D}(W)$.*

Proof. — This commutativity of the diagram shows that if an operator of $\mathcal{D}(W)$ is in the range of ψ_* then its image under γ is in $\mathcal{D}(W)$.

Conversely let us choose a positive system of roots for $(\mathfrak{g},\mathfrak{h})$ and define a function by $\pi = \Pi_{\alpha>0}\alpha$. Then π is a product of distinct linear forms, its square π^2 is equal to the restriction of Δ to $\mathfrak{h}$ and it is changed to $-\pi$ under a reflection of the Weyl group.

Let $P \in \mathcal{D}(\mathfrak{h})^{\mathfrak{W}}$ and $f \in \mathcal{O}_{\mathfrak{h}}^{\mathfrak{W}}$, by definition the function Pf is invariant under $\mathfrak{W}$ while the function $\pi^{-1}P(\pi f)$ is in $\mathcal{O}_{\mathfrak{h}}[\pi^{-1}]$ and is invariant under $\mathfrak{W}$. Hence the function $\tau = P(\pi f)$ is in $\mathcal{O}_{\mathfrak{h}}$ and changes its sign under the action of reflections.

Let z be a point of $\{\pi=0\}$, there exists a root α such that $\alpha(z)=0$. Let s be the reflection which let the hyperplane $\{\alpha=0\}$ invariant. We have $\tau(z)=\tau(z^s)=-\tau(z)$ hence $\tau(z)=0$. As τ vanishes on $\{\pi=0\}$ and π has multiplicity 1, τ is divisible by π and $\pi^{-1}P(\pi f)$ has no denominator.

So the operator $\pi^{-1}P\pi$ is in $\mathcal{D}(\mathfrak{h})^{\mathfrak{W}}[\pi^{-1}]$ but applied to an invariant polynomial it gives a polynomial. Its image under φ_* is thus a differential operator of $\mathcal{D}(W)[d^{-1}]$ which sends any polynomial to a polynomial hence an operator of $\mathcal{D}(W)$. □

Let F be an (H-C)-type subsheaf of $\mathcal{D}_{\mathfrak{g}}^G$, we define four $\mathcal{D}$-modules:

– $\mathcal{M}_F$ is the (H-C)-type $\mathcal{D}_{\mathfrak{g}}$-module. It is equal to the quotient of $\mathcal{D}_{\mathfrak{g}}$ by the ideal $\mathcal{I}_F$ generated by $\tau(\mathfrak{g})$ and F.

– $\mathcal{N}_F$ is the quotient of $\mathcal{D}_W$ by the ideal generated by $\psi_*(F)$.

– $\mathcal{M}_F^{\mathfrak{h}}$ is the quotient of $\mathcal{D}_{\mathfrak{h}}$ by the ideal generated by $\delta(F)$.

– $\mathcal{N}_F^{\mathfrak{h}}$ is the quotient of $\mathcal{D}_W$ by the ideal generated by $\varphi_*(\delta(F))$.

Let $1_{\mathfrak{g}\to W}$ be the canonical generator of $\mathcal{D}_{\mathfrak{g}\to V}$ as defined in the proof of corollary 1.4.4 and $u_{\mathfrak{g}\to W}$ its class in $\psi^+\mathcal{N}_F$. We denote by $\mathcal{M}_F^0$ the $\mathcal{D}_{\mathfrak{g}}$-submodule of $\psi^+\mathcal{N}_F$ generated by $u_{\mathfrak{g}\to W}$.

Theorem 2.4.2. — *The module $\mathcal{M}_F^0$ is conic-tame.*

In this section we prove this theorem and in the next section we prove that $\mathcal{M}_F$ is isomorphic to $\mathcal{M}_F^0$.

Proposition 2.4.3. — *Let n be the dimension of $\mathfrak{g}$, l its rank. Then there exit some positive integer N such that*

$$b(T) = (T-N)\cdots T(T+1)\cdots\Big(T+\frac{n-l}{2}\Big)$$

is a quasi-b-function of total weight $(n+l)/2$ for $\mathcal{N}_F$ along $\{0\}$. Moreover, $N=0$ if $\sigma(F) = S_+(\mathfrak{g})$.

Proof. — We recall that the rank l of the algebra $\mathfrak{g}$ is the dimension of a Cartan subalgebra and that the degrees $n_1,\dots,n_l$ of the generators $P_1,\dots,P_l$ of $\mathcal{O}(\mathfrak{g})^G$ are called the primitive degrees of $\mathfrak{g}$ and that their sum is $(n+l)/2$ [**22**]. The map $\psi:\mathfrak{g}\to W$ is defined by $(P_1,\dots,P_l)$, hence if $E=\sum x_iD_{x_i}$ is the Euler vector field of $\mathfrak{g}$, $\eta=\psi_*(E)$ is equal to $\sum n_it_iD_{t_i}$.

The morphism δ is graded and its restriction to $S(\mathfrak{g})^G$ is the map $Q\mapsto q=Q|_{\mathfrak{h}}$ hence $\sigma(\delta(F))$ the set of principal symbols contains a power of $S_+(\mathfrak{h})^{\mathfrak{W}}$ (and is equal to $S_+(\mathfrak{h})^{\mathfrak{W}}$ if $\sigma(F)=S_+(\mathfrak{g})$). We may then apply proposition 2.3.4 to the module $\mathcal{M}_F^{\mathfrak{h}}$ and we find that its b-function is equal to $b_0(\vartheta)=\vartheta(\vartheta-1)\cdots(\vartheta-M)$ where ϑ is the Euler vector field of $\mathfrak{h}$ and M is a positive integer equal to $(n-l)/2$ if $\sigma(\delta(F))=S_+(\mathfrak{h})^{\mathfrak{W}}$. This means that there exist differential operators $R,A_1,\dots,A_l$ on $\mathfrak{h}$ such that R is of order -1 for the V-filtration in $\{0\}$ and

$$b_0(\vartheta)+R(z,D_z)=A_1(z,D_z)q_1(z,D_z)+\cdots+A_l(z,D_z)q_l(z,D_z)$$

The action of $\mathfrak{W}$ on $\mathcal{D}_{\mathfrak{h}}$ does not affect the V-filtration and $b_0(\vartheta)$ and all $q_i(z,D_z)$ are invariant under the Weyl group, so if we take the mean value (that is $\frac{1}{\#\mathfrak{W}}\sum_{w\in\mathfrak{W}}P^w$) we find the same relation with R and all A_i invariant under $\mathfrak{W}$.

Applying φ_* and γ^{-1} we find

$$b_0(\gamma^{-1}(\varphi_*(\vartheta)))+\gamma^{-1}(\varphi_*(R))=B_1\psi_*(Q_1)+\cdots+B_l\psi_*(Q_l) \tag{6}$$

with $B_1,\dots,B_l$ in $\mathcal{D}(W)$ (lemma 2.4.1) and $\gamma^{-1}(\varphi_*(q_i))=\psi_*(Q_i)$ (the diagram 5 is commutative).

As $\varphi=(p_1,\dots,p_l)$ and p_i has degree n_i, $\varphi_*(\vartheta)$ is equal to $\eta=\sum n_it_iD_{t_i}$. We have $\gamma^{-1}(\eta)=d^{-1/2}\varphi_*(\vartheta)d^{1/2}=\varphi_*(\Delta^{-1/2}\vartheta\Delta^{1/2})$ and the function Δ is homogeneous of degree $n-l$ hence $\Delta^{-1/2}\vartheta\Delta^{1/2}=\vartheta+(n-l)/2$ and $\gamma^{-1}(\eta)=\eta+(n-l)/2$. □

Proposition 2.4.4. — *For each nilpotent orbit S of codimension r, $\mathcal{M}_F^0$ has a b-function of total weight $(n+r)/2$ along S equal to*

$$b(T) = (T-N)\cdots T(T+1)\cdots\Big(T+\frac{n-l}{2}\Big)$$

with $N=0$ if $\sigma(F) = S_+(\mathfrak{g})$. Here n is the dimension of $\mathfrak{g}$ and l its rank.

All roots of b are strictly greater than $-(n+r)/2$ hence this b-function is tame.

Proof. — Let us consider first the null orbit $S=\{0\}$. We apply corollary 1.4.3 to $X'=\mathfrak{g}$, $X=W$, $f=\psi$, $W'=\mathfrak{g}$, φ' is the identity map of $\mathfrak{g}$, φ the identity map of W and $F:\mathfrak{g}^Y\to W$ is the map ψ. The weights $(m'_1,\dots,m'_n)$ on $\mathfrak{g}$ are $(1,\dots,1)$, that is the V-filtration on $\mathfrak{g}$ is the usual v-filtration relative to $\{0\}$, and the weights $(m_1,\dots,m_l)$ on W are the primitive degrees considered in the proof of Proposition 2.4.3. Then F is quasi-homogeneous and we get directly the result for $S=\{0\}$.

Consider now the nilpotent orbit S of maximal dimension, then S is the smooth part of the nilpotent cone and $\psi:\mathfrak{g}\to W$ is smooth on S. We apply corollary 1.4.3 to $X'=\mathfrak{g}$, $X=W$, $f=\psi$, $W'=W$, $\varphi'=\psi$, φ and F are both the identity map of W. The weights on $\mathfrak{g}$ and on W are the weights $(m_1,\dots,m_l)$ considered on W in the case of the null orbit.

We consider now a non null nilpotent orbit S. Let $X\in S$, by the Jacobson-Morozov theorem, we can find H and Y in $\mathfrak{g}$ such that (H,X,Y) is a $\mathfrak{sl}_2$-triple. They generate a Lie algebra isomorphic to $\mathfrak{sl}_2$ which acts on $\mathfrak{g}$ by the adjoint representation. The theory of $\mathfrak{sl}_2$-representations shows $\mathfrak{g}$ splits into a direct sum $\bigoplus_{i=1}^r E(\lambda_i)$ of irreducible submodules. The dimension of $E(\lambda_i)$ is λ_i+1 hence $n=\sum(\lambda_i+1)$. Moreover $\mathfrak{g}=[X,\mathfrak{g}]\oplus\mathfrak{g}^Y$, $\dim\mathfrak{g}^Y=r$ and we can select a basis $(Y_1,\dots,Y_r)$ of $\mathfrak{g}^Y$ such that $[H,Y_i]=-\lambda_i Y_i$. The tangent space to S at X is $[X,\mathfrak{g}]$ hence r is the codimension of S.

The map $\nu:G\times\mathfrak{g}^Y\to\mathfrak{g}$ given by $\nu(g,Z)=g\cdot(X+Z)$ is a submersion because its tangent map is the map $\mathfrak{g}\times\mathfrak{g}^Y\to\mathfrak{g}$ given by $(Z',Z)\mapsto[Z',X]+Z$. Let $\mathfrak{g}_1$ be a linear subspace of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{g}^X\oplus\mathfrak{g}_1$, we have $[\mathfrak{g},X]=[\mathfrak{g}_1,X]$. We choose functions $(\alpha_1,\dots,\alpha_r)$ on G whose differentials at the unit e of G are the equations of $\mathfrak{g}_1$ in $\mathfrak{g}$ and define $A=\{\,g\in G\mid\alpha_1(g)=\cdots=\alpha_r(g)\,\}$. Then there is a Zarisky open subset U of $A\times\mathfrak{g}^Y$ containing $(e,0)$ which is smooth and such that the map $\nu:U\to\mathfrak{g}$ is étale.

Let $(s_1,\dots,s_r)$ be the coordinates of $\mathfrak{g}^Y$ associated with the basis $(Y_1,\dots,Y_r)$, they define functions $(s_1,\dots,s_r)$ on U and $\nu(s^{-1}(0))$ is equal to S. Let $\eta_0=\nu^*E$ on U (E be the Euler vector field of $\mathfrak{g}$). A standard calculation [**23**, Part I, § 5.6], shows that $\eta_0(s_i)=(\lambda_i/2+1)s_i$ hence the map $F_0:\mathfrak{g}^Y\to\mathfrak{g}$ defined by $F_0(Z)=X+Z$ is quasi-homogeneous if the weights on $\mathfrak{g}^Y$ are $m'_i=(\lambda_i/2+1)$ for $i=1,\dots,r$ and the weights on $\mathfrak{g}$ are $(1,\dots,1)$. The map $F:\mathfrak{g}^Y\to W$ defined by $F(Z)=\psi(X+Z)$ is thus quasi-homogeneous with the weights $(m'_1,\dots,m'_r)$ on $\mathfrak{g}^Y$ and $(m_1,\dots,m_l)$ on W.

Now, we apply corollary 1.4.3 to $X' = U$, $X = W$, $f = \psi_\circ\nu$, $W' = \mathfrak{g}^Y$, φ' the projection $U \to \mathfrak{g}^Y$ and $F : \mathfrak{g}^Y \to W = k^l$ given by $F(Z) = \psi(X + Z)$. This gives the b-function for $\nu^+\mathcal{M}_F^0$ and thus for $\mathcal{M}_F^0$ by lemma 1.2.5. □

Let us now consider the non-nilpotent strata of the stratification of $\mathfrak{g}$ (§ 2.2):

Proposition 2.4.5. — *The module $\mathcal{M}_F^0$ admits a tame quasi-b-function b_S along each stratum S.*

More precisely, if the stratum is $S_{(P,\mathfrak{O})}$ according to definition (4) *and $\mathfrak{q}_P$ the associated semi-simple Lie subalgebra of $\mathfrak{g}$, then*

a) *b_S depends only on P and its roots are integers greater or equal to $-(m-k)/2$ where m is the dimension of $\mathfrak{q}_P$ and k its rank.*

b) *The total weight of b_S is equal to $(m+r)/2$ where r is the codimension of $\mathfrak{O}$ in $\mathfrak{q}_P$.*

In particular, on the stratum of codimension 1 *in $\mathfrak{g}$, the roots of the usual b-function of $\mathcal{M}_F^0$ are half integers greater or equal to $-1/2$.*

Proof. — We fix a Cartan subalgebra of $\mathfrak{g}$ and a subset P of roots with the notations of § 2.2. This define a semi-simple algebra $\mathfrak{q}_P$ to which are associated the maps $\psi_P : \mathfrak{q}_P \to W_P$ and $\varphi_P : \mathfrak{h}_P \to W_P$. Here W_P is a vector space of dimension the rank of $\mathfrak{q}_P$. The Cartan subalgebra $\mathfrak{h}$ splits into the direct sum $\mathfrak{h} = \mathfrak{h}_P \oplus \mathfrak{h}_P^\perp$ and this define a map $\varphi'_P = \varphi_P \otimes 1 : \mathfrak{h} \to W = k^l \oplus \mathfrak{h}_P^\perp$.

Let $S \in \mathfrak{h}_P^\perp$, we know from section 2.2 that $\mathfrak{g}^S = \mathfrak{h}_P^\perp \oplus \mathfrak{q}_P$ and as S is semisimple we have $\mathfrak{g} = [\mathfrak{g}, S] \oplus \mathfrak{g}^S$. The map $\nu : G \times \mathfrak{g}^S \to \mathfrak{g}$ defined by $\nu(g, Z) = g \cdot (Z + S)$ is thus a submersion. Let $(\alpha_1, \dots, \alpha_r)$ be functions on G whose differentials at e are the equations of $[\mathfrak{g}, S]$ in $\mathfrak{g}$ and define $A = \{\, g \in G \mid \alpha_1(g) = \cdots = \alpha_r(g) \,\}$. Then there is a Zarisky open subset U of $A \times \mathfrak{g}^S$ containing $(e, 0)$ which is smooth and such that the map $\nu : U \to \mathfrak{g}$ is étale. Let $\psi' : U \to W$ be defined as the composition of the canonical projection $A \times \mathfrak{g}^S \to \mathfrak{g}^S$ and of ψ_P.

Now we follow the proof of proposition 2.4.3 with the same notations. Applying the second part of proposition 2.3.4 to $L = \{0\} \times (\mathfrak{h}_P^\perp)'$, we find that $\mathcal{N}_F$ admits a monodromic b-function along L which is equal to $b_0(\vartheta_P) = \vartheta_P(\vartheta_P - 1) \cdots (\vartheta_P - N' + 1)$ $(\vartheta_P - N')$ where ϑ_P is the Euler vector field of $\mathfrak{h}_P$ and N' is less or equal to $N = (n-l)/2$ with n the dimension of $\mathfrak{g}$. This means that there exists l differential operators $R, A_1, \dots, A_l$ on $\mathfrak{h}$ that we may assume invariant under $\mathfrak{W}_P$, with R of order -1 for the Vfiltration associated with L such that $b_0(\vartheta_P) + R = A_1(z, D_z)q_1(D_z) + \cdots + A_l(z, D_z)q_l(D_z)$. If $\lambda = 0$ we have $R = 0$.

As these operators are invariant under $\mathfrak{W}_P$ hence under φ' we may apply φ'_* and γ^{-1} and find an equation $b_0(\gamma^{-1}(\eta)) + \gamma^{-1}\varphi_*(R) = B_1\psi_*(Q_1) + \cdots + B_l\psi_*(Q_l)$ with $B_1, \dots, B_l$ in $\mathcal{D}(W)$ and $\eta = \varphi_*(\vartheta)$. In the coordinates of W defined by the isomorphism $\varphi' : k^l \oplus (\mathfrak{h}_P^\perp)' \to W$, the vector η is equal to $\sum n_i t_i D_{t_i}$ where the n_i are the primitive degrees of $\mathfrak{h}_P$, it is associated with the manifold $L' = \varphi'(\{0\} \oplus (\mathfrak{h}_P^\perp)')$.

As Δ is the product of Δ_P by a function which does not vanish on a neighborhood of L, the function d which defines the morphism γ is the product of the corresponding function d_P associated with $\mathfrak{h}_P$ by a function ϱ which does not vanish in a neighborhood of L'. So we have

$$\gamma^{-1}(\eta) = \varrho^{-1/2} d_P^{-1/2} \eta d_P^{1/2} \varrho^{1/2} = \varrho^{-1/2} (\eta + N_P) \varrho^{1/2} = (\eta + N_P) + a$$

where N_P is $(m-k)/2$ (m is the dimension of $\mathfrak{q}_P$, k its rank) and a is a function which vanishes on L' hence of order at most -1 for the V^η-filtration. The operator $\gamma^{-1}\varphi_*(R)$ is also of order -1 for the V^η-filtration.

We have proved that $\mathcal{N}_F$ admits a $b(\eta)$-function along L' which is equal to $b(T) = (T - N_P) \cdots (T - N_P + N)$. The end of the proof is the same than to the proof of proposition 2.4.4. □

Proposition 2.4.5 shows that $\mathcal{M}_F^0$ is tame. To prove theorem 2.4.2 we have still to prove that it is conic. This come from the fact that the singular support of $\mathcal{M}_F^0$ is conic for the Euler vector field of $\mathfrak{g}$ and the vector fields associated with the strata are equal to this Euler vector field modulo vector fields tangent to the orbits.

2.5. Isomorphism with the inverse image. — We recall that $\mathcal{M}_F^0$ is the submodule of $\psi^+\mathcal{N}_F$ generated by $u_{\mathfrak{g}\to W}$, it is the image of the morphism $\mathcal{M}_F \to \psi^+\mathcal{N}_F$.

Theorem 2.5.1. — *The canonical morphism $\mathcal{M}_F \to \mathcal{M}_F^0$ is an isomorphism.*

Proof

1st step: From semi-simple Lie algebras to reductive algebras. — Assume that the result has been proved for semi-simple Lie algebras and let $\mathfrak{g}$ be a reductive algebra, direct sum of its center and a semisimple Lie algebra. By induction, we may assume that $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}'$ with $\mathfrak{c}$ subspace of the center of dimension 1 and $\mathfrak{g}'$ reductive Lie algebra for which the result has been proved.

Let t a coordinate of $\mathfrak{c}$ and τ the corresponding coordinate of the dual space $\mathfrak{c}^*$. By the hypothesis, there is a differential operator in F whose principal symbol is equal to some power τ^q. This means that $\mathfrak{g}' = \{t = 0\}$ is non characteristic for $\mathcal{M}_F$. Let $\mathcal{K}$ be the kernel of $\mathcal{M}_F \to \mathcal{M}_F^0$. We have an exact sequence $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{M}_F \longrightarrow \mathcal{M}_F^0 \longrightarrow 0$ of non characteristic $\mathcal{D}_{\mathfrak{g}}$-modules. As the inverse image is an exact functor in the non characteristic case, this gives an exact sequence $0 \longrightarrow \mathcal{K}/t\mathcal{K} \longrightarrow \mathcal{M}_F/t\mathcal{M}_F \longrightarrow \mathcal{M}_F^0/t\mathcal{M}_F^0 \longrightarrow 0$. If we prove that $\mathcal{K}/t\mathcal{K} = 0$, we will have $\mathcal{K} = 0$ (as $\mathcal{K}$ is non characteristic).

So, we have to prove that $\mathcal{M}_F/t\mathcal{M}_F \to \mathcal{M}_F^0/t\mathcal{M}_F^0$ is injective. Here we use the same proof than in [**13**, Lemma 2.2.3.]. In fact, as $\mathcal{D}_{\mathfrak{g}'}$-module $\mathcal{M}_F/t\mathcal{M}_F$ is generated by the classes of $1, D_t, \ldots, D_t^{q-1}$ and the submodule generated by D_t^{q-1} is a module on $\mathfrak{g}'$ of the same type than $\mathcal{M}_F$ for which the theorem is true. Then we consider the quotient of $\mathcal{M}_F/t\mathcal{M}_F$ by the module generated by D_t^{q-1} and argue by induction.

2nd step: The result is true at points $X \in \mathfrak{g}$ whose semi-simple part is non zero

By the first step, we may assume that $\mathfrak{g}$ is semisimple. Let S be a non zero semisimple element of $\mathfrak{g}$, $\mathfrak{g}^S$ its centralizer and G^S the corresponding group. The spaces $\mathcal{O}(\mathfrak{g})^G$ and $\mathcal{O}(\mathfrak{g}^S)^{G^S}$ are isomorphic hence the space W_S associated with $\mathfrak{g}^S$ is equal to W and the map $\psi_S : \mathfrak{g}^S \to W$ is the restriction of $\psi : \mathfrak{g} \to W$. Thus the sheaf of differential operators on $\mathfrak{g}^S$ invariant under the action of G^S is isomorphic to $\mathcal{D}_{\mathfrak{g}}^G$.

By induction on the dimension of $\mathfrak{g}$, we may assume that the theorem is true for $\mathfrak{g}^S$ hence that the morphism $\mathcal{M}_F^S \to \psi_S^+ \mathcal{N}_F$ is injective. Here $\mathcal{M}_F^S$ is the $\mathcal{D}_{\mathfrak{g}^S}$ module associated with F and $\mathcal{N}_F$ the quotient of $\mathcal{D}_W$ by the ideal generated by F. By definition, the germ at S of $\psi^+\mathcal{N}_F$ is $(\psi^+\mathcal{N}_F)_S = \mathcal{O}_{\mathfrak{g},S} \otimes_{\mathcal{O}_{\mathfrak{g}^S,S}} (\psi_S^+\mathcal{N}_F)_S$. On the other hand, we have $(\mathcal{D}_{\mathfrak{g}}/\mathcal{D}_{\mathfrak{g}}\tau(\mathfrak{g}))_S = \mathcal{O}_{\mathfrak{g},S} \otimes_{\mathcal{O}_{\mathfrak{g}^S,S}} (\mathcal{D}_{\mathfrak{g}^S}/\mathcal{D}_{\mathfrak{g}^S}\tau(\mathfrak{g}^S))_S$ hence $\mathcal{M}_{F,S} = \mathcal{O}_{\mathfrak{g},S} \otimes_{\mathcal{O}_{\mathfrak{g}^S,S}} \mathcal{M}_{F,S}^S$. The morphism $\mathcal{M}_F \to \psi^+\mathcal{N}_F$ is thus injective at the point S hence at all the orbits whose closure contains S that is in particular at all points X whose semisimple part in the Jordan decomposition is S.

3nd step: The case of nilpotent orbits. — Let $\mathcal{K}$ be the kernel of $\mathcal{M}_F \to \mathcal{M}_F^0$. By the second step, we may assume that the theorem is true at all non nilpotent points of $\mathfrak{g}$ that is that $\mathcal{K}$ is supported by the nilpotent cone. Let $\mathcal{K}(\mathfrak{g})^G$ be the set of global sections of $\mathcal{K}$ invariant under G, we get an exact sequence

$$0 \longrightarrow \mathcal{K}(\mathfrak{g})^G \longrightarrow \mathcal{M}(\mathfrak{g})^G \longrightarrow \mathcal{M}_0(\mathfrak{g})^G \longrightarrow 0$$

and by [**7**, lemma 3.2.] we have $\mathcal{M}(\mathfrak{g})^G = \mathcal{M}_0(\mathfrak{g})^G = \mathcal{N}_F$ hence $\mathcal{K}(\mathfrak{g})^G = 0$. Then $\mathcal{K} = 0$ by [**17**, lemma 3.2.]. □

Remark that the third step is also a consequence of the property (5) of the Harish-Chandra morphism which has been proved by Levasseur-Stafford [**16**].

References

[1] M.F. Atiyah – "Characters of semi-simple Lie groups", Preprint. Oxford, 1976.

[2] N. Bourbaki – *Groupes et algèbres de Lie*, Hermann.

[3] E. Galina & Y. Laurent – "$\mathcal{D}$-modules and characters of semi-simple Lie groups", *Duke Math. J.* **123** (2004), no. 2, p. 265–309.

[4] Harish-Chandra – "Invariant distributions on semi-simple Lie groups", *Bull. Amer. Math. Soc.* **69** (1963), p. 117–123.

[5] ———, "Invariant differential operators and distributions on a semi-simple Lie algebra", *Amer. J. Math.* **86** (1964), p. 534–564.

[6] R. Hotta & M. Kashiwara – "The invariant holonomic system on a semi-simple Lie algebra", *Invent. Math.* **75** (1984), p. 327–358.

[7] M. Hunziker & G.W. Schwarz – "A homomorphism of Harish-Chandra and direct images of D-modules", *Proc. Amer. Math. Soc.* **129** (2001), no. 12, p. 3485–3493.

[8] M. Kashiwara – "On holonomic systems of differential equations II", *Invent. Math.* **49** (1978), p. 121–135.

[9] ———, *Systems of microdifferential equations*, Progress in Math., vol. 34, Birkhäuser, 1983.

[10] ———, "Vanishing cycles and holonomic systems of differential equations", in *Algebraic geometry (Tokyo/Kyoto, 1982)*, Lect. Notes in Math., vol. 1016, Springer, 1983, p. 134–142.

[11] B. Kostant – "The principal three-dimensional subgroups...", *Amer. J. Math.* **81** (1959), p. 973–1032.

[12] Y. Laurent – "Polygone de Newton et b-fonctions pour les modules microdifférentiels", *Ann. scient. Éc. Norm. Sup.* 4^{e} *série* **20** (1987), p. 391–441.

[13] ———, "Regularity of $\mathcal{D}$-module associated to a symmetric pair", in *Autour de l'analyse microlocale*, Astérisque, vol. 284, Société Mathématique de France, Paris, 2003.

[14] Y. Laurent & B. Malgrange – "Cycles proches, spécialisation et $\mathcal{D}$-modules", *Ann. Inst. Fourier (Grenoble)* **45** (1995), p. 1353–1405.

[15] T. Levasseur & J.T. Stafford – "Invariant differential operators and a homomorphism of Harish-Chandra", *J. Amer. Math. Soc.* **8** (1995), no. 2, p. 365–372.

[16] ———, "The kernel of an homomorphism of Harish-Chandra", *Ann. scient. Éc. Norm. Sup.* 4^{e} *série* **29** (1996), p. 385–397.

[17] ———, "Invariant differential operators on the tangent space of some symmetric spaces", *Ann. Inst. Fourier (Grenoble)* **49** (1999), no. 6, p. 1711–1741.

[18] C. Sabbah – "$\mathcal{D}$-modules et cycles évanescents", in *Conférence de La Rábida 1984*, Travaux en cours, vol. 24, Hermann, 1987, p. 53–98.

[19] P. Schapira – *Microdifferential systems in the complex domain*, Grundlehren der math. Wiss., vol. 269, Springer, 1985.

[20] J. Sekiguchi – "Invariant spherical hyperfunctions on the tangent space of a symmetric space", Advanced Studies in pure mathematics, vol. 6, 1985, p. 83–126.

[21] T. Torrelli – "Closed formulae for a b-function associated with a weighted homogeneous isolated complete intersection singularity", to appear in *J. Algebra*.

[22] V.S. Varadarajan – *Lie groups, Lie algebras and their representations*, Prentice-Hall, 1974.

[23] ———, *Harmonic analysis on real reductive groups*, Lect. Notes in Math., vol. 576, Springer, 1977.

Y. Laurent, Institut Fourier Mathématiques, UMR 5582 CNRS/UJF, BP 74, 38402 St Martin d'Hères Cedex • *E-mail :* `Yves.Laurent@ujf-grenoble.fr` • *Url :* `http://www-fourier.ujf-grenoble.fr/~laurenty`

Astérisque
296, 2004, p. 167–190

CURVATURE OF PENCILS OF FOLIATIONS

by

Alcides Lins Neto

Dedicated to J.-P. Ramis in his 60^{th} birthday

Abstract. — Let $\mathcal{F}$ and $\mathcal{G}$ be two distinct singular holomorphic foliations on a compact complex surface M, in the same class, that is $N_{\mathcal{F}} = N_{\mathcal{G}}$. In this case, we can define the *pencil* $\mathcal{P} = \mathcal{P}(\mathcal{F}, \mathcal{G})$ *of foliations generated by* $\mathcal{F}$ *and* $\mathcal{G}$. We can associate to a pencil $\mathcal{P}$ a meromorphic 2-form $\Theta = \Theta(\mathcal{P})$, the form of curvature of the pencil, which is in fact the Chern curvature (*cf.* [**Ch**]). When $\Theta(\mathcal{P}) \equiv 0$ we will say that the pencil is *flat*. In this paper we give some sufficient conditions for a pencil to be flat. (Theorem 2). We will see also how the flatness reflects in the pseudo-group of holonomy of the foliations of $\mathcal{P}$. In particular, we will study the set $\{\mathcal{H} \in \mathcal{P} \mid \mathcal{H}$ has a first integral$\}$ in some cases (Theorem 1).

***Résumé* (Courbure de pinceaux de feuilletages).** — Nous nous intéressons au *pinceau de feuilletages* $\mathcal{P} = \mathcal{P}(\mathcal{F}, \mathcal{G})$ engendré par deux feuilletages $\mathcal{F}$ et $\mathcal{G}$ holomorphes singuliers distincts sur une surface complexe compacte M et appartenant à la même classe, *i.e.*, $N_{\mathcal{F}} = N_{\mathcal{G}}$. La forme de courbure du pinceau $\mathcal{P}$ est une 2-forme $\Theta = \Theta(\mathcal{P})$ qui coïncide avec la courbure de Chern (*cf.* [**Ch**]) ; lorsque $\Theta(\mathcal{P}) \equiv 0$ on dit que le pinceau est *plat*. Dans cet article, nous donnons des conditions suffisantes de platitude d'un pinceau (Théorème 2). Nous regardons comment se traduit la platitude dans le pseudo-groupe d'holonomie des feuilletages de $\mathcal{P}$ et, en particulier, nous étudions dans certains cas l'ensemble $\{\mathcal{H} \in \mathcal{P} \mid \mathcal{H}$ admet une intégrale première$\}$ (Théorème 1).

1. Introduction

Let $\mathcal{F}$ and $\mathcal{G}$ be two distinct singular holomorphic foliations on a compact complex surface M, with isolated singularities, in the same class, that is $N_{\mathcal{F}} = N_{\mathcal{G}}$. This means that there exists a Leray covering $(U_\alpha)_{\alpha \in A}$ of M by open sets, and collections $(\omega_\alpha)_{\alpha \in A}$, $(\eta_\alpha)_{\alpha \in A}$ and $(g_{\alpha\beta})_{U_{\alpha\beta} \neq \varnothing}$, $U_{\alpha\beta} = U_\alpha \cap U_\beta$, such that

2000 ***Mathematics Subject Classification.*** — Primary 37F75; Secondary 32S65.
Key words and phrases. — Holomorphic foliations, curvature, flat pencils.

This research was partially supported by Pronex.

© Astérisque 296, SMF 2004

(I) ω_α and η_α are holomorphic 1-forms on U_α which represent the foliations $\mathcal{F}$ and $\mathcal{G}$, respectively. This means that $\mathcal{F}|_{U_\alpha}$ and $\mathcal{G}|_{U_\alpha}$ are defined by the differential equations $\omega_\alpha = 0$ and $\eta_\alpha = 0$, respectively. Since the singularities of $\mathcal{F}$ and $\mathcal{G}$ are isolated, we have $\mathrm{cod}_{\mathbb{C}}(\omega_\alpha = 0) \geqslant 2$ and $\mathrm{cod}_{\mathbb{C}}(\eta_\alpha = 0) \geqslant 2$ for every $\alpha \in A$.

(II) If $U_{\alpha\beta} \neq \varnothing$ then $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$, $\omega_\alpha = g_{\alpha\beta} \cdot \omega_\beta$ and $\eta_\alpha = g_{\alpha\beta} \cdot \eta_\beta$ on $U_{\alpha\beta}$.

The class of the multiplicative cocycle $(g_{\alpha\beta})_{U_{\alpha\beta}\neq\varnothing}$ in $\mathrm{Pic}(M)$ defines $N_{\mathcal{F}}$ and $N_{\mathcal{G}}$, so that $N_{\mathcal{F}} = N_{\mathcal{G}}$. The *pencil generated by $\mathcal{F}$ and $\mathcal{G}$* is the family $\mathcal{P} = (\mathcal{F}_T)_{T\in\overline{\mathbb{C}}}$, where

(III) $\mathcal{F}_\infty = \mathcal{G}$ and if $T \in \mathbb{C}$, then $\mathcal{F}_T$ is represented on U_α by the form $\omega_\alpha^T := \omega_\alpha + T \cdot \eta_\alpha$.

The singular set of $\mathcal{F}_T$ is defined by $\mathrm{sing}(\mathcal{F}_T) \cap U_\alpha = \{\omega_\alpha^T = 0\}$. The tangency divisor of $\mathcal{F}$ and $\mathcal{G}$ is defined by $\mathrm{Tang}(\mathcal{F},\mathcal{G}) \cap U_\alpha = \{\omega_\alpha \wedge \eta_\alpha = 0\}$. Note that $\mathrm{sing}(\mathcal{F}_T)$ and $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ are analytic subsets of M and that $\mathrm{sing}(\mathcal{F}_T) \subset |\mathrm{Tang}(\mathcal{F},\mathcal{G})|$ for all $T \in \overline{\mathbb{C}}$. Since $\mathcal{F} \neq \mathcal{G}$, $|\mathrm{Tang}(\mathcal{F},\mathcal{G})|$ is a proper analytic subset of pure dimension one. Let $W = M \setminus |\mathrm{Tang}(\mathcal{F},\mathcal{G})|$ and $W_\alpha = W \cap U_\alpha$. Since $\omega_\alpha \wedge \eta_\alpha(p) \neq 0$ for all $p \in W_\alpha$, there exists an unique holomorphic 1-form θ_α on W_α such that

$$(*) \qquad d\omega_\alpha = \theta_\alpha \wedge \omega_\alpha \quad \text{and} \quad d\eta_\alpha = \theta_\alpha \wedge \eta_\alpha$$

for all $\alpha \in A$. It follows from $(*)$, (II) and $\omega_\alpha \wedge \eta_\alpha \not\equiv 0$ that, if $W_{\alpha\beta} := W_\alpha \cap W_\beta \neq \varnothing$ then, $\theta_\alpha = \theta_\beta + dg_{\alpha\beta}/g_{\alpha\beta}$ on $W_{\alpha\beta}$. Hence $d\theta_\alpha = d\theta_\beta$ on $W_{\alpha\beta}$ and we can define a holomorphic 2-form Θ on W by

$$(**) \qquad \Theta|_{U_\alpha} := d\theta_\alpha$$

It can be proved that the form Θ can be extended meromorphically to $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ (see § 2). This extension will be called the *curvature of the pencil $\mathcal{P}(\mathcal{F},\mathcal{G})$*. We will say that the pencil is *flat* if $\Theta = 0$. Let us see some examples of flat pencils.

Example 1. — Let ω and η be two meromorphic closed 1-forms on some compact complex surface M, such that $\omega \wedge \eta \not\equiv 0$ and the divisors of poles and zeroes of ω and η coincide. Let $\mathcal{F}$ and $\mathcal{G}$ be the foliations generated by ω and η, respectively. It is known that $N_{\mathcal{F}} = N_{\mathcal{G}}$ in this case (*cf.* [**Br**]). Moreover, the pencil generated by $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{P}(\mathcal{F},\mathcal{G})$, is defined by the pencil of forms $\omega_T = \eta + T \cdot \omega$. Therefore, it is flat. We will call a pencil like this *a pencil of closed forms*.

A particular case is given by some families of logarithmic forms in $\mathbb{C}P(2)$. Let $f_1, \ldots, f_k$, $k \geqslant 3$, be irreducible homogeneous polynomials of three variables such that $df_i \wedge df_j \neq 0$ if $i \neq j$. Given $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$, such that $\sum_{j=1}^k \lambda_j \cdot dg(f_j) = 0$, set $\omega_\lambda = \sum_{j=1}^k \lambda_j \cdot df_j/f_j$. The closed form ω_λ can be considered as meromorphic form on $\mathbb{C}P(2)$, so that the family $(\omega_\lambda)_\lambda$ generates a family of foliations $(\mathcal{F}_\lambda)_\lambda$ on $\mathbb{C}P(2)$. It can be checked that any pencil contained in this family is flat.

Another particular case, is the following: let M be the complex two torus $\mathbb{C}^2/\Gamma$, where $\Gamma = \mathbb{Z} \cdot v_1 \oplus \mathbb{Z} \cdot v_2 \oplus \mathbb{Z} \cdot v_3 \oplus \mathbb{Z} \cdot v_4$ is some lattice in $\mathbb{C}^2$, and $\pi\colon \mathbb{C}^2 \to \mathbb{C}^2/\Gamma$ be

the canonical projection. Consider an affine coordinate system (z,w) on $\mathbb{C}^2$ and let $\mathcal{F}$ and $\mathcal{G}$ be the foliations generated by the closed forms ω and η such that $\pi^*(\omega)=dz$ and $\pi^*(\eta)=dw$, respectively.

Example 2. — The pull-back of a flat pencil is a flat pencil. More precisely, let M and N be complex surfaces and $f\colon M- \to N$ be a meromorphic map. If $\mathcal{P}:=\mathcal{P}(\mathcal{F},\mathcal{G})$ is a pencil of foliations on N, then we can define the pencil $f^*(\mathcal{P})=\mathcal{P}(f^*(\mathcal{F}),f^*(\mathcal{G}))$ on M. It is not difficult to prove that, if $\mathcal{P}$ is flat then $f^*(\mathcal{P})$ is also flat.

Example 3. — Suppose that the pencil $\mathcal{P}(\mathcal{F},\mathcal{G})$ is defined by $\omega+T\cdot\eta$, where ω and η are meromorphic 1-forms, and there exists a closed meromorphic 1-form θ such that $d\omega=\theta\wedge\omega$ and $d\eta=\theta\wedge\eta$. Then the pencil $\mathcal{P}(\mathcal{F},\mathcal{G})$ is flat. Of course, the pencils of Example 1 are of this kind, because the forms ω and η are closed. However, the reader can find some examples in [**LN**] or [**LN-1**] which are not generated by closed forms. One example of this kind is the pencil $\mathcal{P}_1$ of foliations of degree two on $\mathbb{C}P(2)$ defined in some affine coordinate system $(x,y)\in\mathbb{C}^2\subset\mathbb{C}P(2)$ by the the forms (see § 2.4 of [**LN**]):

$$\begin{cases}\omega_1=(4x-9x^2+y^2)dy-6y(1-2x)dx\\ \eta_1=2y(1-2x)dy-3(x^2-y^2)dx.\end{cases}\tag{1}$$

A straightforward computation gives $d\omega_1=\frac{5}{6}\frac{dP}{P}\wedge\omega_1$ and $d\eta_1=\frac{5}{6}\frac{dP}{P}\wedge\eta_1$, where $P(x,y)=-4y^2+4x^3+12xy^2-9x^4-6x^2y^2-y^4$. The other examples of [**LN**] can be obtained from the above one by pulling-back $\mathcal{P}_1$ by a meromorphic map $f\colon\mathbb{C}P(2)- \to \mathbb{C}P(2)$.

Another example is the pencil $\mathcal{P}_2$ of degree three generated by

$$\begin{cases}\omega_2=y(x^2-y^2)dy-2x(y^2-1)dx\\ \eta_2=(4x-x^3-x^2y-3xy^2+y^3)dy+2(x+y)(y^2-1)dx.\end{cases}\tag{2}$$

In this case, we have $d\omega_2=\frac{3}{4}\frac{dQ}{Q}\wedge\omega_2$ and $d\eta_2=\frac{3}{4}\frac{dQ}{Q}\wedge\eta_2$, where $Q(x,y)=(y^2-1)(x^2+y^2-2x)(x^2+y^2+2x)$.

We would like to observe that both pencils $\mathcal{P}_1$ and $\mathcal{P}_2$ are exceptional families of foliations in the sense of [**LN-1**]. This means the folowing: Let $\mathcal{F}_T^j$, $T\in\overline{\mathbb{C}}$, be the foliation defined in $\mathbb{C}^2\subset\mathbb{C}P(2)$ by the form $\omega_j+T\cdot\eta_j$ ($\mathcal{F}_\infty^j$ defined by η_j), where ω_j and η_j are as in (j), $j=1,2$, of example 3. Then, for $j=1,2$, we have:

(a) The singularities of $\mathcal{F}_T^j$ are of constant analytic type. In other words, there is a finite subset $F_j\subset\overline{\mathbb{C}}$ such that if $T_1,T_2\in\overline{\mathbb{C}}\smallsetminus F_j$ then every singularity of $\mathcal{F}_{T_1}^j$ is locally analytically equivalent to some singularity of $\mathcal{F}_{T_2}^j$.

(b) If we set

$$E_j=\{T\in\overline{\mathbb{C}}\mid\mathcal{F}_T^j\text{ has a meromorphic first integral}\},$$

then E_j is countable and dense in $\overline{\mathbb{C}}$.

(c) Given $T \in E_j$ denote by $d_j(T)$ the degree of the generic level of the first integral of $\mathcal{F}_T^j$. Then, for any $m \in \mathbb{N}$ the set $\{T \in E_j \mid d_j(T) \leqslant m\}$ is finite. In particular, in both families, there are foliations with first integrals of arbitrarily large degrees.

Concerning the exceptional pencils above, we have the following result:

Theorem 1. — *Let E_j, $j = 1, 2$, be as in* (b). *Then*

$$\begin{cases} E_1 = \mathbb{Q} \cdot \langle 1, e^{2\pi i/3} \rangle \cup \{\infty\} \\ E_2 = \mathbb{Q} \cdot \langle 1, i \rangle \cup \{\infty\}. \end{cases}$$

where $\mathbb{Q} \cdot \langle a, b \rangle = \{q_1 \cdot a + q_2 \cdot b \mid q_1, q_2 \in \mathbb{Q}\}$.

In our last result we will give some sufficient condictions for the flatness of a pencil $\mathcal{P} = \mathcal{P}(\mathcal{F}, \mathcal{G})$ in terms of the singularities of the foliations in $\mathcal{P}$ and the components of the divisor of tangencies. In order to state it, let us consider the singularities of $\mathcal{F}_T$, $T \in \overline{\mathbb{C}}$. Without lost of generality, we will suppose that $\mathcal{F}$ and $\mathcal{G}$ have isolated singularities. This implies that the set $NI := \{T \in \overline{\mathbb{C}} \mid \mathcal{F}_T \text{ has non-isolated singularities}\}$ is finite. Set $IS := \overline{\mathbb{C}} \setminus NI$ and for each $T \in IS$, set $n(T) := \#(\mathrm{sing}(\mathcal{F}_T))$. Note that, if $T \in IS$ then $N_{\mathcal{F}_T} = N_{\mathcal{F}}$. It is well known that the number of singularities of $\mathcal{F}_T$, counted with multiplicities, is given by (*cf.* [**Br**]):

$$m(\mathcal{F}) = m(\mathcal{F}_T) = N_{\mathcal{F}}^2 + N_{\mathcal{F}}.K_M + c_2(M)$$

where K_M is the canonical bundle of M. Hence $n(T) \leqslant m(\mathcal{F})$ for all $T \in IS$. Let $n_0 = \max\{n(T) \mid T \in IS\}$ and $GP = \{T \in IS \mid n(T) = n_0\}$. We need a fact.

Lemma 1. — $\overline{\mathbb{C}} \setminus GP$ *is finite. Moreover, there exist holomorphic maps* $p_j \colon GP \to M$, $j = 1, \ldots, n_0$, *such that* $\mathrm{sing}(\mathcal{F}_T) = \{p_1(T), \ldots, p_{n_0}(T)\}$ *for all* $T \in GP$.

The proof of Lemma 1 is left for the reader.

Definition 1. — We say that the singularity p_j is *fixed* if the map $p_j \colon GP \to M$ is constant. Otherwise, we say that p_j is *movable*. For instance, if p is a singularity of the curve $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ then p is a singularity of all foliations of the pencil and it is a fixed singularity of the pencil.

Note that, for any movable singularity p_j of the pencil, the image $p_j(GP)$ is contained in some irreducible component C of $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$. In this case we will say that p_j *is contained in* C.

Let $C \subset \mathrm{Tang}(\mathcal{F}, \mathcal{G})$ be an irreducible component. We have two possibilities:

(A) C is invariant for both foliations $\mathcal{F}$ and $\mathcal{G}$. In this case, C is invariant for all foliations $\mathcal{F}_T$ in the pencil and we will say that C is *invariant for the pencil*.

(B) C is not invariant for the pencil. In this case, the set $IN(C) = \{T \in \overline{\mathbb{C}} \mid C \text{ is invariant for } \mathcal{F}_T\}$ is finite.

Remark 1. — Given an irreducible component C of $\mathrm{Tang}(\mathcal{F},\mathcal{G})$, we have two possibilities: either C contains a movable singularity, or C does not contain movable singularities. In the second case, we will call C a *NI-component*. The reason is the following: let $(U_\alpha)_{\alpha\in A}$ be a covering of M by open sets and $(\omega_\alpha)_{\alpha\in A}$, $(\eta_\alpha)_{\alpha\in A}$ be collections of holomorphic 1-forms such that the foliations in the pencil are defined on U_α by $\omega_\alpha^T := \omega_\alpha + T\cdot\eta_\alpha$, $T\in\overline{\mathbb{C}}$. Given $p\in U_\alpha\cap C\setminus(\mathrm{sing}(\mathcal{F})\cup\mathrm{sing}(\mathcal{G}))$, there exists an unique T_o such that $\omega_\alpha(p)+T_o\cdot\eta_\alpha(p)=0$, because $\omega_\alpha(p)$ and $\eta_\alpha(p)$ are linearly dependent. However, since C does not contain movable singularities and $p\notin\mathrm{sing}(\mathcal{F})\cup\mathrm{sing}(\mathcal{G})$, the unique possibility is that $\omega_\alpha(q)+T_o\cdot\eta_\alpha(q)=0$ for all $q\in C\cap U_\alpha$. Hence, $T_o\in NI$ and the component C is contained in $\mathrm{sing}(\mathcal{F}_{T_o})$. Note that T_o depends only on C. We will use the notation $T_o=T(C)$. This happens for instance in the case of the Logarithmic forms (see Example 1).

The *divided foliation* associated to $T(C)$ is defined as follows: for each $\alpha\in A$, let $(f_\alpha=0)$ be a reduced equation of $C\cap U_\alpha$. Since $\omega_\alpha^{T(C)}|_{C\cap U_\alpha}\equiv 0$, we can write $\omega_\alpha^{T(C)}=f_\alpha^\ell\cdot\widetilde{\omega}_\alpha$, where $\widetilde{\omega}_\alpha$ has isolated singularities and $\ell\in\mathbb{N}$, does not depend on α. The divided foliation, denoted by $\widetilde{\mathcal{F}}_{T(C)}$, is defined by the collection $(\widetilde{\omega}_\alpha)_{\alpha\in A}$. Note that $N_{\widetilde{\mathcal{F}}_{T(C)}}=N_{\mathcal{F}_{T(C)}}\otimes C^{-\ell}$.

Definition 2. — We say that an irreducible component C of $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ is *nice*, if one of the following condictions hold:

(a) C is invariant for the pencil and contains a movable singularity $p_j(T)$ such that the function $T\in GP\mapsto BB(p_j(T),\mathcal{F}_T)$ is constant, where $BB(p_j(T),\mathcal{F}_T)$ denotes the Baum-Bott index of the singularity (*cf.* [**Br**]).

(b) C is an NI-component, invariant for the pencil.

(c) C is non-invariant for the pencil and C contains a movable singularity, say $p_j(T)$, such that $BB(p_j(T),\mathcal{F}_T)=0$ for all $T\in GP$.

(d) C is an NI-component, non-invariant for the pencil. In this case, we ask that C is invariant for the divided foliation associated to $T(C)$.

The last result, characterizes when the pencil is flat, if we assume that the components of the divisor of tangencies have multiplicity one.

Theorem 2. — *Let $\mathcal{F}$ and $\mathcal{G}$ by two holomorphic foliations on a compact complex surface, such that $N_{\mathcal{F}}=N_{\mathcal{G}}$, and let Θ be the curvature of the pencil generated by them. Suppose that all components of* $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ *have multiplicity one. Then the following condictions are equivalent:*

(a) *The pencil is flat.*

(b) *All components of* $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ *are nice.*

(c) Θ *is holomorphic.*

Let us state one consequence.

Corollary. — *Let $\mathcal{F}$ and $\mathcal{G}$ by two holomorphic foliations on a compact complex surface M. Suppose that $N_{\mathcal{F}} = N_{\mathcal{G}}$ and* $\mathrm{Tang}(\mathcal{F},\mathcal{G}) = \varnothing$. *Then the pencil generated by them is flat. Moreover, M is a complex 2-torus and $\mathcal{F}$, $\mathcal{G}$ are linear foliations.*

We observe that this corollary is a consequence of Theorem 2 and the classification of complex compact surfaces (see [**BPV**]). We would like to pose the following problems:

Problem 1. — Given a flat pencil $\mathcal{P} = \mathcal{P}(\mathcal{F},\mathcal{G})$, describe the set

$$E(\mathcal{P}) = \{\alpha \in \overline{\mathbb{C}} \mid \mathcal{F}_\alpha \text{ has a first integral}\}.$$

Problem 2. — Give necessary and sufficient conditions for a pencil to be flat, like in Theorem 2. Recall that Theorem 2 is true only in the case that all components of $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ have multiplicity one.

Problem 3. — Give necessary and sufficient conditions for a flat pencil to be a pencil of closed 1-forms. We observe that the pencils defined by logarithmic forms satisfy the following properties, when all components of $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ have multiplicity one:

(a) All invariant components of $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ are NI-components.

(b) All non-invariant components of $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ are nice.

We note that the above conditions are necessary in the case that all components of $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ have multiplicity one. It seems that they are also sufficient in some cases.

2. Proofs

2.1. Proof of Theorem 1. — We will use the notation $\mathcal{F}^j_T$ (resp. $\mathcal{F}^j_\infty$) to denote the foliation defined by $\omega_j + T\cdot\eta_j$, $T \in \mathbb{C}$ (resp. η_j), where ω_j and η_j are as in (j) of example 3, $j = 1, 2$. First of all, we observe that, in both cases, it is easy to see that some foliations in the foliations in the pencils have first integrals. Given $\alpha \in E_j$ we will call g^j_α the first integral of $\mathcal{F}^j_\alpha$. For the pencil $\mathcal{P}_1$ we have:

$$\begin{cases} g^1_\infty(x,y) = P(x,y)/(2x-1)^3 \\ g^1_1(x,y) = P(x,y)/(y-x)^3 \\ g^1_{-1}(x,y) = P(x,y)/(y+x)^3 \end{cases} \tag{3}$$

where, $P(x,y) = -4y^2 + 4x^3 + 12xy^2 - 9x^4 - 6x^2y^2 - y^4$.

In particular, $1, -1, \infty \in E_1$. On the other hand, for the pencil $\mathcal{P}_2$ we have

$$\begin{cases} g^2_0(x,y) = C_1(x,y)\cdot C_{-1}(x,y)/4L_1(y)\cdot L_{-1}(y) \\ g^2_\infty(x,y) = L_{-1}(y)\cdot C_1(x,y)/L_1(y)\cdot C_{-1}(x,y) \\ g^2_{1/2}(x,y) = L_1(y)\cdot C_1(x,y)/L_{-1}(y)\cdot C_{-1}(x,y) \end{cases} \tag{4}$$

where

$$\begin{cases} C_1(x,y) = x^2 + y^2 - 2x \\ C_{-1}(x,y) = x^2 + y^2 + 2x \\ L_1(y) = y - 1 \\ L_{-1}(y) = y + 1 \end{cases}$$

In particular, $0, \infty, 1/2 \in E_2$.

Note that, in all above cases, the generic level curves of g_α^j are elliptic curves. There is a difference between the two cases: for $j = 1$ the level curves, after normalization, are of the form $\mathbb{C}/\langle 1, e^{2\pi i/3}\rangle$, whereas for $j = 2$ they are of the form $\mathbb{C}/\langle 1, i\rangle$. In the case $j = 1$, the proof can be found in § 2.4 of [**LN**]. In the case $j = 2$, the fact that the level curves are elliptic can be proved by using the genus formula. For instance, in the case of g_∞^2, the level curve $L_c := (g_\infty^2 = c)$, for generic $c \in \mathbb{C}$, has degree three and no singularities. Hence, $g(L_c) = (3-1)(3-2)/2 = 1$. The proof that the normalization L_c is $\mathbb{C}/\langle 1, i\rangle$ will be sketched next.

Let us give an idea of the proof that the pencil $\mathcal{P}_2$ is exceptional. This proof was done in § 2.2 of [**LN**] for another pencil (of degree four), but the idea is the same. First of all, the divisor of tangency of $\mathcal{F}_0^2$ and $\mathcal{F}_\infty^2$ is

$$Tg := \mathrm{Tang}(\mathcal{F}_0^2, \mathcal{F}_\infty^2) = C_1 + C_{-1} + L_1 + L_{-1} + L_\infty,$$

where L_∞ is the line at infinity of $\mathbb{C}^2 \subset \mathbb{C}P(2)$. The singular set of Tg, which are the fixed singularities of the pencil, is (in homogeneous coordinates)

(I) Fix:=$\{O := (0:0:1), A := (-1:1:1), B := (1:1:1), C := (1:-1:1), D := (-1:-1:1), E := (1:i:0), F := (1:-i:0), G := (1:0:0)\}$. For $T \notin \{1, -1, i, -i, \infty\}$ the points E, F, G are radial singularities for the foliation $\mathcal{F}_T^2$ (of type $1:1$), whereas the points A, B, C, D and O are singularities of type $2:1$. We say that a singularity is of type $p:q$ if the foliation has a local first integral of the form u^p/v^q, in some local coordinate system (u, v).

On the other hand, each component of Tg contains exactly one movable singularity of $\mathcal{F}_\alpha^2$, $\alpha \in \overline{\mathbb{C}}$:

(II) The points $P_{-1}(\alpha) := (\alpha, -1) \in L_{-1}$, $P_1(\alpha) := (-\alpha, 1) \in L_1$, $Q_{-1}(\alpha) := (-2/(1+\alpha^2), 2\alpha/(1+\alpha^2)) \in C_{-1}$ and $Q_1(\alpha) := (2/(1+\alpha^2), -2\alpha/(1+\alpha^2)) \in C_1$. These singularities are of the type $1:-4$ (with local first integral of the type $u \cdot v^4$).

(III) The point $P_\infty(\alpha) := [\alpha : 1 : 0] \in L_\infty$. This singularity is of the type $1:-2$.

The next step is to reduce the fixed singularities (which are dicritical) by blowing-ups. This can be done for all foliations in the pencil simultaneously by doing one blowing-up at each radial singularity and two at each singularity of the type $2:1$. After this procedure, we find a rational surface M and a bimeromorphism $\pi\colon M \to \mathbb{C}P(2)$. We will use the notation $\mathcal{F}_\alpha = \pi^*(\mathcal{F}_\alpha^2)$, $\alpha \in \overline{\mathbb{C}}$, and $\mathcal{P}$ for the pencil in M so obtained. The pencil $\mathcal{P}$ has ten invariant curves (rational): five of them are the strict

transforms of the components of Tg and the other five are the divisors introduced in the first blowing-up at the singularities of the type $2:1$ (A,B,C,D,O). For each $\alpha\in E_2$, the foliation $\mathcal{F}_\alpha$, which corresponds to the first integral g_α^2, has also a first integral $g_\alpha:=g_\alpha^2\circ\pi$. We observe that g_α is holomorphic, because the foliation $\mathcal{F}_\alpha$ has no dicritical singularities. In fact, for any $\alpha\in\overline{\mathbb{C}}$, $\mathcal{F}_\alpha$ has ten singularities, one in each invariant curve, which are the folowing: four of the type $1:-4$, which come from the singularities $P_1(\alpha)$, $P_{-1}(\alpha)$, $Q_1(\alpha)$ and $Q_{-1}(\alpha)$, and six of the type $1:-2$. One of these six singularities come from $P_\infty(\alpha)$ and the other five are contained in the five invariant divisors introduced in the blowing-up procedure. We leave the details of the proof of these facts for the reader.

Let us describe briefly the (singular) fibration g_∞. We will denote by T_c the level curve $g_\infty^{-1}(c)\subset M$. It has three critical levels: T_0,T_1 and T_∞. If we call $U=M\setminus(T_0\cup T_1\cup T_\infty)$, then $f:=g_\infty|_U\colon U\to\overline{\mathbb{C}}\setminus\{0,1,\infty\}:=W$ is a (regular) elliptic fibration. The main fact is the following

Lemma 2.1.1. — *If $\alpha\neq\infty$ then $\mathcal{F}_\alpha$ is tranverse to the fibers of f in all points of the set U.*

Proof. — Since the divisors introduced by π are contained in $T_0\cup T_1\cup T_\infty$, it is sufficient to prove that the foliations $\mathcal{F}_\infty^2$ and $\mathcal{F}_\alpha^2$ are transverse outside Tg, because $\pi|_U\colon U\to\pi(U)=\mathbb{C}P(2)\setminus Tg$ is a biholomorphism. On the other hand, we have:

$$\begin{aligned}(\omega_2+\alpha\cdot\eta_2)\wedge\eta_2&=2(x^2+y^2-2x)(x^2+y^2+2x)(y-1)(y+1)dx\wedge dy\\&=2\,C_1\cdot C_{-1}\cdot L_1\cdot L_{-1}\,dx\wedge dy.\end{aligned}$$

Hence $\mathcal{F}_\alpha^2$ and $\mathcal{F}_\infty^2$ are transverse outside Tg, which implies the lemma. □

Now, we use Ehresmann's theory of foliations tranverse to a fibration (*cf.* [E-R]). According to this theory, if L is a leaf of $\mathcal{F}_\alpha|_U$ then $f|_L\colon L\to W$ is a covering map. Moreover, if we fix a (regular) fiber T_c and a closed curve $\gamma\colon[0,1]\to W=\overline{\mathbb{C}}\setminus\{0,1,\infty\}$ with $\gamma(0)=\gamma(1)=c$, then we can define an automorphism $H_{\gamma,\alpha}\colon T_c\to T_c$, as follows: given $p\in T_c$, let $L_\alpha(p)$ be the leaf of $\mathcal{F}_\alpha$ through p. Since $f|_{L_\alpha(p)}\colon L_\alpha(p)\to W$ is a covering map, there exists an unique curve $\widehat{\gamma}$ on $L_\alpha(p)$ such that $f\circ\widehat{\gamma}=\gamma$ and $\widehat{\gamma}(0)=p$. The automorphism is defined by $H_{\gamma,\alpha}(p)=\widehat{\gamma}(1)$. It is called the *global holonomy transformation* associated to γ. We will use the following facts:

(i) For every $\alpha\in\mathbb{C}$ the automorphism $H_{\gamma,\alpha}$ is holomorphic and depends only of the the class of γ in $\Pi_1(W,c)$. This follows from Ehresmann's theory and the fact that the foliations are holomorphic.

(ii) If $\gamma_1,\gamma_2\in\Pi_1(W,c)$ and $\alpha\in\mathbb{C}$ then $H_{\gamma_1*\gamma_2,\alpha}=H_{\gamma_1,\alpha}\circ H_{\gamma_2,\alpha}$. In particular, for each $\alpha\in\mathbb{C}$, we can define an action $H_\alpha\colon\Pi_1(W,c)\to\mathrm{Aut}(T_c)$ by $H_\alpha(\gamma)=H_{\gamma,\alpha}$, called the *holonomy representation*. The image $H_\alpha(\Pi_1(W,c))=:G(\alpha,c)$ is called the *global holonomy group* of $\mathcal{F}_\alpha$.

(iii) For each fixed $\gamma \in \Pi_1(W,c)$, the map $H_\gamma\colon \mathbb{C}\times T_c \to T_c$ defined by $H_\gamma(\alpha,p) = H_{\gamma,\alpha}(p)$ is holomorphic. This follows from the theorem of holomorphic dependency of the solutions with respect to initial conditions and parameters and the fact that $H_{\gamma,\alpha}$ can be found by integrating the equation $\omega_2 + \alpha\cdot\eta_2 = 0$.

(iv) For any $p \in T_c$, the orbit of p by H_α coincides with the intersection of the leaf $L_\alpha(p)$ with the fiber T_c.

(v) If c_1 is another point of W and γ_1 is a curve in W connecting c_1 to c, then, for each $\alpha\in\mathbb{C}$ it can be defined a biholomorphism $F_\alpha\colon T_{c_1}\to T_c$ (by lifting γ_1 to leaves of $\mathcal{F}_\alpha$) such that

$$H_\alpha(\gamma_1^{-1} * \gamma * \gamma_1) = F^{-1}\circ H_\alpha(\gamma)\circ F.$$

In particular, the holonomy representations are conjugated and the fibration f is isotrivial, that is, all regular fibers are biholomorphic.

Now, consider the two closed curves $\gamma_0,\gamma_1\colon [0,1]\to W$, where $\gamma_k(0)=\gamma_k(1)=c$, $k=0,1$, γ_0 goes around 0 once and γ_1 goes around ∞ once. It is known that γ_0,γ_1 generate $\Pi_1(W,c)$. We will call $f_{1,\alpha}=H_\alpha(\gamma_0)$ and $g_{1,\alpha}=H_\alpha(\gamma_1)$. Fix a holomorphic universal covering $P\colon\mathbb{C}\to T_c$ and let $f_\alpha, g_\alpha\in \mathrm{Aut}(\mathbb{C})$ be coverings of $f_{1,\alpha}$ and $g_{1,\alpha}$, respectively ($P\circ f_\alpha = f_{1,\alpha}\circ P$ and $P\circ g_\alpha = g_{1,\alpha}\circ P$).

Lemma 2.1.2. — *If we choose well the orientation of the curves γ_0 and γ_1, then for any $\alpha\in\mathbb{C}$ we have $f_\alpha(z) = i\cdot z + A(\alpha)$ and $g_\alpha(z) = i\cdot z + B(\alpha)$, where $A,B\colon\mathbb{C}\to\mathbb{C}$ are holomorphic.*

Idea of the proof. — The proof is analogous to the proof of Proposition 4 of § 2.2 of [**LN**], and so we will give only an idea. Let us consider the case of f_α. The critical fiber $T_0 := f^{-1}(0)$ of the fibration f contains the strict transforms, by $\pi\colon M\to\mathbb{C}P(2)$, of the curves C_1 and L_{-1}, which we call C and L, respectively. On the other hand, C_1 and L_{-1} contain the movable singularities $Q_1(\alpha)$ and $P_{-1}(\alpha)$ of $\mathcal{F}^2_\alpha$, which are of the type $1:-4$. These singularities give origin to movable singularities of the pencil $\mathcal{P}$, $Q(\alpha) := \pi^{-1}(Q_1(\alpha))\in C$ and $P(\alpha)=\pi^{-1}(P_{-1}(\alpha))\in L$, which are also of the type $1:-4$. Since $Q(\alpha)$ is the unique singularity of $\mathcal{F}_\alpha$ on C and C is a rational curve, $Q(\alpha)$ is linearizable for the foliation $\mathcal{F}_\alpha$ (because the holonomy of C is trivial, and so linearizable). The same argument applies to $P(\alpha)$, which is the unique singularity of $\mathcal{F}_\alpha$ on L. On the other hand, the foliation $\mathcal{F}_\alpha$ has an unique local smooth separatrix, say $S(\alpha)$, which is transversal to C. Since the quotient of the eigenvalues is $-1/4$, the holonomy of $S(\alpha)$, in a suitable coordinate system u of a transversal Σ, is linear of the form $u\mapsto e^{-2\pi i/4}\cdot u = -i\cdot u$. If we choose c near 0 then the separatrix $S(\alpha)$ cuts the fiber T_c in an unique point, say $p(\alpha)$. It can be checked that $f|_{S(\alpha)}\colon S(\alpha)\to D := f(S(\alpha))$ is a bijection. If we choose the curve γ_0 as a small circle sorrounding 0 contained in D, then when we go around γ_0 in order to evaluate $f_{1,\alpha}$ we see that $p(\alpha)$ is a fixed point of $f_{1,\alpha}$. Moreover, the section Σ can be choosed to be contained in T_c. This implies that $f_{1,\alpha}$ is locally conjugated to $u\mapsto \pm i\cdot u$. The

sign $\pm$ depends on the orientation of γ_0. We choose this orientation in such a way that $f_{1,\alpha}$ is locally conjugated to $u \mapsto i \cdot u$. This implies that $f_{1,\alpha}$ has period four and that $f_\alpha(z) = i \cdot z + A(\alpha)$. Analogously, we can choose the orientation of γ_1 in such a way that $g_\alpha(z) = i \cdot z + B(\alpha)$. The maps $\alpha \in \mathbb{C} \mapsto A(\alpha), B(\alpha)$ are holomorphic by (iii). □

As a consequence of Lemma 2.1.2, we obtain that T_c is biholomorphic to $\mathbb{C}/\langle 1, i\rangle$. This implies that all regular fibers of f are biholomorphic to $\mathbb{C}/\langle 1, i\rangle$, because the fibration is isotrivial. We will fix an universal covering $P\colon \mathbb{C} \to T_c$ such that the associated lattice is $\langle 1, i\rangle$. The crucial result is the following:

Lemma 2.1.3. — *$A(\alpha)$ and $B(\alpha)$ are affine, that is, $A(\alpha) = a_1 \cdot \alpha + a_0$ and $B(\alpha) = b_1 \cdot \alpha + b_0$, where $a_0, a_1, b_0, b_1 \in \mathbb{C}$.*

Proof. — We need another lemma.

Lemma 2.1.4. — *Let $\mathcal{P}(\mathcal{F}, \mathcal{G})$ be a flat pencil on a surface M. Given $p \in M \setminus \mathrm{Tang}(\mathcal{F}, \mathcal{G})$, there exists a local coordinate system $(U, (x, y))$, $p \in U$, $(x, y)\colon U \to \mathbb{C}^2$, such that the foliation $\mathcal{F}_\alpha$ of the pencil, $\alpha \in \overline{\mathbb{C}}$, is defined on U by $dy + \alpha \cdot dx = 0$. Moreover, if $(V, (u, v))$ is another coordinate system such that $U \cap V \neq \varnothing$ is connected and $\mathcal{F}_\alpha|_V$ is defined by $dv + \alpha \cdot du = 0$, $\alpha \in \overline{\mathbb{C}}$, then $du = \lambda \cdot dx$ and $dv = \lambda \cdot dy$ on $U \cap V$, where $\lambda \in \mathbb{C}^*$.*

Proof. — Let $W \subset M \setminus \mathrm{Tang}(\mathcal{F}, \mathcal{G})$ be a small simply connected open neighborhood of p and ω, η be holomorphic 1-forms such that the foliation $\mathcal{F}_\alpha|_W$ is defined by $\omega + \alpha \cdot \eta = 0$. Note that $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_\infty = \mathcal{G}$ are defined on W by $\omega = 0$ and $\eta = 0$, respectively. Since $W \cap \mathrm{Tang}(\mathcal{F}, \mathcal{G}) = \varnothing$, we have $\omega \wedge \eta \neq 0$ on W. Hence, we can write $d\omega = \theta \wedge \omega$ and $d\eta = \theta \wedge \eta$, where θ is holomorphic on W. Since the pencil is flat, θ is closed. Therefore, there exists $h \in \mathcal{O}(W)$ such that $\theta = dh$. If we set $f = \exp(h)$ then we get

$$d\omega = \frac{df}{f} \wedge \omega \quad \text{and} \quad d\eta = \frac{df}{f} \wedge \eta \quad \Longrightarrow \quad d\Big(\frac{\omega}{f}\Big) = d\Big(\frac{\eta}{f}\Big) = 0.$$

Again, since W is simply connected, there exist $x, y \in \mathcal{O}(W)$ such that $dy = \omega/f$ and $dx = \eta/f$. The foliation $\mathcal{F}_\alpha$ is defined on W by $dy + \alpha \cdot dx = \frac{1}{f}(\omega + \alpha \cdot \eta) = 0$. Note that $dx \wedge dy \neq 0$ on W. It follows that $(x, y)\colon W \to \mathbb{C}^2$ is an immersion. This implies that we can take a smaller neighborhood $U \subset W$ of p such that $(x, y)|_U$ is a biholomorphism from U to an open set of $\mathbb{C}^2$.

Let $(V, (u, v))$ be another coordinate system such that $U \cap V \neq \varnothing$ is connected and $\mathcal{F}_\alpha|_V$ is defined by $dv + \alpha \cdot du = 0$. Note that $\mathcal{F}|_V$ and $\mathcal{G}|_V$ are defined by $dv = 0$ and $du = 0$, respectively. Since $\mathcal{F}_\alpha|_{U \cap V}$ is defined by $dy + \alpha \cdot dx$ and $du + \alpha \cdot dv = 0$, we get

$$(*) \qquad dv + \alpha \cdot du = h(x, y, \alpha)(dy + \alpha \cdot dx)$$

where h is holomorphic. Differenciating both members of $(*)$ with respect to α, we get

$$du = \frac{\partial h}{\partial \alpha}(dy + \alpha \cdot dx) + h \cdot dx \implies \frac{\partial h}{\partial \alpha} \equiv 0,$$

because du is a multiple of dx on $U \cap V$. Hence, $h(x, y, \alpha) = h(x, y)$, does not depend on α. Therefore, $du = h \cdot dx$ and $dv = h \cdot dy$ on $U \cap V$. This implies that $dh \wedge dy = dh \wedge dx = 0$ and $h \in \mathbb{C}^*$, is a constant. This finishes the proof of lemma 2.1.4. □

Let us finish the proof of Lemma 2.1.3. Fix $\alpha_0 \in \mathbb{C}$ and $p \in T_c$. Set $q = f_{1,\alpha_0}(p) \in T_c$. Denote by $L_\alpha(p)$ the leaf of $\mathcal{F}_\alpha$ through p. Let $\gamma_p \colon [0,1] \to L_{\alpha_0}(p)$ be the lifting of γ_0 on the leaf $L_{\alpha_0}(p)$ through the fibration f. Note that $\gamma_p(0) = p$ and $\gamma_p(1) = q$. Let $(U_n)_{1 \leqslant n \leqslant m}$ be a covering of $\gamma_p[0,1]$ by open sets as in Lemma 2.1.4. For each $n = 1, \ldots, m$ there exists a coordinate system (x_n, y_n) on U_n such $\mathcal{F}_\alpha|_{U_n}$ is defined by $dy_n + \alpha \cdot dx_n = 0$. We can choose the enumeration in such a way that there is a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of $[0,1]$ such that $\gamma_p[t_{n-1}, t_n] \subset U_n$, for all $n = 1, \ldots, m$. We can suppose that $U_n \cap U_{n+1}$ is connected for every $n = 1, \ldots, m-1$. It follows from Lema 2.1.4 that there exist constants $\lambda_n \in \mathbb{C}^*$ such that $dx_{n+1} = \lambda_n \cdot dx_n$ and $dy_{n+1} = \lambda_n \cdot dy_n$, $n = 1, \ldots, m-1$. Hence,

(i) $y_{n+1} = \lambda_n \cdot y_n + a_n$, where $a_n \in \mathbb{C}$, $n = 1, \ldots, m-1$.

Fix transversal sections to the foliation $\mathcal{F}_0$, $\Sigma_0, \ldots, \Sigma_m$, such that:

(ii) $\gamma_p(t_n) \in \Sigma_n$, $n = 0, 1, \ldots, m$.

(iii) $\Sigma_n \subset (x_n = ct)$, that is Σ_n is contained in a leaf of $\mathcal{F}_\infty$. Note that $\Sigma_0, \Sigma_m \subset T_c$.

Since $\mathcal{F}_\alpha$ is defined by $dy_n + \alpha \cdot dx_n = 0$ on U_n, the holonomy transformation of $\mathcal{F}_\alpha$, α near α_0, from the section $\Sigma_{n-1} \subset (x_n = c_1)$ to the section $\Sigma_n \subset (x_n = c_2)$, in terms of the parameter y_n is of the form $y_n \mapsto H_n(y_n, \alpha) = y_n - \alpha \cdot b_n$, $b_n = c_2 - c_1$. It follows from (i) that, in the section Σ_n, we have $y_{n+1} = \lambda_n \cdot y_n + a_n$, and so the holonomy transformation H_n, can be written in terms of the parameter y_{n+1} (in the immage) as $y_{n+1}(y_n, \alpha) = \lambda_n . H_n(y_n) + a_n = \lambda_n \cdot y_n - \alpha . \lambda_n \cdot b_n + a_n$. As the reader can check, this implies that the holonomy transformation from the section $\Sigma_0 \subset U_1 \cap T_c$ to the section $\Sigma_m \subset U_M \cap T_c$, which is the composition of the intermediate holonomies, is of the form

$$y_m = H(y_1, \alpha) = \mu \cdot y_1 + \alpha \cdot b + c, \quad \text{where } \mu \in \mathbb{C}^*,\ b, c \in \mathbb{C}.$$

Now, let us relate the parameters $y_1 \in \Sigma_1$ and $y_m \in \Sigma_m$ with the parametrization which comes from the universal covering $P \colon \mathbb{C} \to T_c$. Since $\mathcal{F}_0$ is transverse to T_c, there exists a neighborhood V of T_c such that

(iv) $f|_V \colon V \to D := f(V)$ is a trivial fibration. In particular, $V \simeq D \times T_c$, where $f|_V = \pi_1$, the first projection, and the fibers of the second projection $\pi_2 \colon V \to T_c$ are the leaves of $\mathcal{F}_0|_V$.

Let τ be a non-vanishing 1-form on T_c such that $P^*(\tau) = dz$.

Claim. — *There exist constants* $k_1, k_m \in \mathbb{C}^*$ *such that* $dy_1|_{\Sigma_1} = k_1 \cdot \tau|_{\Sigma_1}$ *and* $dy_m|\Sigma_m = k_m \cdot \tau|_{\Sigma_m}$.

Proof. — Set $\omega = \pi_2^*(\tau)$. Note that $\omega(p) \neq 0$, for all $p \in V$, and that the foliation $\mathcal{F}_0|_V$ is defined by $\omega = 0$. We can suppose that $D \subset \mathbb{C}$ and consider $x := f|_V : V \to \mathbb{C}$. This implies that $\mathcal{F}_\infty|_V$ is defined by $dx = 0$. We assert that there exists $g \in \mathcal{O}^*(D)$ such that the foliation $\mathcal{F}_\alpha|_V$ is defined by $\omega + \alpha \cdot g(x) \cdot dx = 0$.

In fact, since ω and dx are linearly independent on V, the foliation $\mathcal{F}_\alpha|_V$ is defined by a 1-form of the type $\omega_\alpha = \omega + g_\alpha \cdot dx$, where $g_\alpha \in \mathcal{O}^*(V)$. Since the fiber $T_x = f^{-1}(x)$ is compact, the function $g_\alpha|_{T_x}$ is constant. Hence, we can write $g_\alpha = g_\alpha(x)$ and $\omega_\alpha = \omega + g_\alpha(x) \cdot dx$. Fix a point $q \in V$ and a coordinate system $(U_q, (x_q, y_q))$ such that $U_q \subset V$ and $\mathcal{F}_\alpha|_{U_q}$ is defined by $dy_q + \alpha \cdot dx_q = 0$. It follows that $dy_q + \alpha \cdot dx_q = h_\alpha(\omega + g_\alpha(x) \cdot dx)$ on U_q, where $h_\alpha \in \mathcal{O}^*(U_q)$. Differentiating twice both members with respect to α and by an argument similar to the proof of Lemma 2.1.4, we get $\partial h_\alpha/\partial\alpha = 0$ and $\partial^2 g_\alpha/\partial\alpha^2 = 0$. This implies that $g_\alpha(x) = \alpha \cdot g(x)$, where $g \in \mathcal{O}^*(V)$.

Since ω and $g(x)dx$ are closed, they are locally exact and we can apply Lemma 2.1.4 to them and the forms dy_1 and dx_1. It follows that $dy_1 = k_1 \cdot \omega|_{U_1}$, $k_1 \in \mathbb{C}^*$. Similarly, $dy_m = k_m \cdot \omega|_{U_m}$, $k_m \in \mathbb{C}^*$. Hence, $dy_j|_{\Sigma_j} = k_j \cdot \tau|_{\Sigma_j}$, $j = 1, m$. □

Now, fix a disk $D_1 \subset \mathbb{C}$ such that $\phi_1 := P|_{D_1} : D_1 \to \Sigma_1$ is a biholomorphism. The claim implies that $\phi_1^*(dy_1) = k_1 \cdot dz$. Therefore, $y_1 \circ \phi_1(z) = k_1 \cdot z + d_1$, $d_1 \in \mathbb{C}$. Similarly, $y_m \circ \phi_m(z) = k_m \cdot z + d_m$, $d_m \in \mathbb{C}$ ($\phi_m = P|_{D_m}$). It follows that the holonomy transformation f_α can be written, in terms of the parameter $z \in \mathbb{C}$, as

$$f_\alpha(z) = k_m^{-1}.H(y_1 \circ \phi_1(z), \alpha) - k_m^{-1} \cdot d_m = i \cdot z + a_1 \cdot \alpha + a_0,$$

where $a_1 = k_m^{-1} \cdot b$ and $a_0 = k_m^{-1}(c - d_m) + \mu \cdot d_1$. Hence, $A(\alpha) = a_1 \cdot \alpha + a_0$, where $a_1, a_0 \in \mathbb{C}$. Similarly, $B(\alpha) = b_1 \cdot \alpha + b_0$. □

Now, the point $z_0 = A(\alpha)/(1 - i)$ is a fixed point of f_α. Let $Q_\alpha(z) = z - z_0$. The global holonomy group $G(\alpha, c)$ (viewed in the universal covering) is conjugated to the group generated by $F_\alpha(z) = Q_\alpha \circ f_\alpha \circ Q_\alpha^{-1}(z) = i \cdot z$ and $G_\alpha(z) = Q_\alpha \circ g_\alpha \circ Q_\alpha^{-1}(z) = i \cdot z + C(\alpha)$, where $C(\alpha) = B(\alpha) - A(\alpha) = a \cdot \alpha + b$, $a = b_1 - a_1$ and $b = b_0 - a_0$. Let us finish the proof of Theorem 1. We need two more results. We will give only an idea of the proof of these results (see Proposition 5 and its corollary in [**LN**]).

Lemma 2.1.5. — *The following assertions are equivalent:*

(a) *The group* $G(\alpha, c)$ *is finite.*
(b) $G(\alpha, c)$ *has a finite orbit in* T_c.
(c) *There exists* $m \in \mathbb{N}$ *such that* $m \cdot C(\alpha) \in \langle 1, i \rangle$.
(d) $\mathcal{F}_\alpha$ *has a first integral. In particular,* $\alpha \in E_2$.

Idea of the proof. — The proof of the equivalences (a) $\iff$ (b) $\iff$ (c) is based in the fact that the group generated by F_α and G_α is

$$G = \{z \mapsto c \cdot z + d \cdot C(\alpha) \mid c \in \{1, -1, i, -i\} \text{ and } d \in \langle 1, i\rangle\}.$$

This is done in Proposition 5 of [**LN**] in another case, but the proof is similar for the above case. On the other hand, if $\mathcal{F}_\alpha$ has a first integral, then all leaves of $\mathcal{F}_a$ are algebraic and cut T_c in a finite number of points. Hence, (d) $\implies$ (b). Finally, if the group $G(\alpha, c)$ is finite, say $\#\, G(\alpha, c) = m$, then each leaf of $\mathcal{F}_\alpha$ cut each fiber $T_x = f^{-1}(x)$ in m points. This implies that all leaves $\mathcal{F}_\alpha$ are algebraic. There is a delicate point here, which involves the fact that the leaves of $\mathcal{F}_\alpha$ cut transversely the components of the critical fibers of f which are not invariant for $\mathcal{F}_\alpha$. We have not proved this fact here, but the proof can be done by studing carefully the blowing-up process π. We leave the details for the reader. Now, we can use Darboux's theorem which asserts that if all leaves of a foliation are algebraic then the foliation has a first integral. Therefore, (a) $\implies$ (d). □

Lemma 2.1.6. — *The map $\alpha \mapsto C(\alpha)$ is non-constant. In particular, $a \neq 0$.*

Idea of the proof. — If $\alpha \mapsto C(\alpha)$ were constant then all holonomy groups $G(\alpha, c)$ would be isomorphic. Therefore, it is sufficient to prove that there are $\alpha_0, \alpha_1 \in E_2$ such that $\#(G(\alpha_0, c)) \neq \#(G(\alpha_1, c))$. In the case of this pencil, we have $0, 1/2 \in E_2$ and the first integrals g_0^2 and $g_{1/2}^2$ given in (4). It can be checked by using Bézout's theorem and the explicit expressions for g_∞^2, g_0^2 and $g_{1/2}^2$ that the generic leaf of $\mathcal{F}_0$ cuts T_c in eight points, whereas the generic leaf of $\mathcal{F}_{1/2}$ cuts T_c in four points. This implies that $\#(G(0, c)) = 8$ and $\#(G(1/2, c)) = 4$. Therefore, $\alpha \mapsto C(\alpha)$ is not constant. □

End of the proof of Theorem 1. — We have seen that $C(\alpha) = a \cdot \alpha + b$, where $a \neq 0$. On the other hand, $0, 1/2 \in E_2$, which implies that there exist $m, n \in \mathbb{N}$ and $m_0, n_0, m_1, n_1 \in \mathbb{N}$ such that

$$m \cdot b = m_0 + n_0 \cdot i \quad \text{and} \quad n\Big(\frac{a}{2} + b\Big) = m_1 + n_1 \cdot i \quad \implies \quad a, b \in \mathbb{Q} \cdot \langle 1, i\rangle.$$

Since $\mathbb{Q} \cdot \langle 1, i\rangle$ is a field, we get

$$m(\alpha \cdot a + b) \in \mathbb{Q} \cdot \langle 1, i\rangle,\ m \in \mathbb{N} \quad \iff \quad \alpha \in \mathbb{Q} \cdot \langle 1, i\rangle.$$

This finishes the proof in the case of the pencil $\mathcal{P}_2$.

In the case of the pencil $\mathcal{P}_1$ the proof is similar. In this case, the non-singular fibers of f are biholomorphic to $\mathbb{C}/\langle 1, k\rangle$ ($k = e^{\pi i/3}$) and the holonomy group of $\mathcal{F}_\alpha$ is isomorphic to the group generated by the transformations $F_\alpha(z) = k \cdot z$ and $G_\alpha(z) = k^2 \cdot z + C(\alpha)$ (in the universal covering), where and $C(\alpha) = a \cdot \alpha + b$, $a \neq 0$. This group is

$$G = \{z \mapsto c \cdot z + d \cdot C(\alpha) \,|\, c \in \{1, k, k^2, k^3, k^4, k^5\} \text{ and } d \in \langle 1, k\rangle\}.$$

By the analogous of Lemma 2.1.5 we have that $\alpha \in E_1$ if, and only if, there exists $m \in \mathbb{N}$ such that $m \cdot C(\alpha) \in \langle 1, k\rangle$. On the other hand, we know that $1, -1 \in E_1$, because we have the explicit first integrals g_1^1 and g_{-1}^1 (see (3)). Therefore, there exist $m, n \in \mathbb{N}$ and $m_0, n_0, m_1, n_1 \in \mathbb{Z}$ such that

$$m(a+b) = m_0 + n_0 \cdot k \text{ and } n(-a+b) = m_1 + n_1 \cdot k \implies a, b \in \mathbb{Q} \cdot \langle 1, k\rangle.$$

Since $\mathbb{Q} \cdot \langle 1, k\rangle$ is a field, we get

$$m(a \cdot \alpha + b) \in \langle 1, k\rangle,\ m \in \mathbb{N} \iff \alpha \in \mathbb{Q} \cdot \langle 1, k\rangle.$$

This finishes the proof of the theorem. □

2.2. Proof of Theorem 2. — Let $\mathcal{P}(\mathcal{F}, \mathcal{G})$ be a pencil of foliations on the compact complex surface M.

Definition 3. — Suppose that $\mathcal{F}$ and $\mathcal{G}$ are defined on an open set $U \subset M$ by $\omega = 0$ and $\eta = 0$, where ω and η are holomorphic 1-forms on U. We will say that (U, ω, η) are *compatible with the pencil* if the foliation $\mathcal{F}_\alpha$ is defined on U by $\omega + \alpha \cdot \eta = 0$, $\alpha \in \mathbb{C}$.

We need a Lemma.

Lemma 2.2.1. — *Let C be an irreducible component of $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ of multiplicity $k \geqslant 1$. There exists a finite set $F \subset |C|$ such that if $p \in |C| \setminus F$ then there is a holomorphic coordinate system $(U, (x, y))$ with $p \in U$, $x(p) = y(p) = 0$, $|C| \cap U = (y = 0)$, and holomorphic 1-forms ω and η, representing $\mathcal{F}|_U$ and $\mathcal{G}|_U$ respectively, such that (U, ω, η) is compatible with the pencil and*

(a) *If C is invariant for the pencil then*

$$\begin{cases} \omega = dy \\ \eta = P(x, y)\, dy - y^k\, dx \end{cases}$$

where $P \in \mathcal{O}(U)$. If θ is such that $d\omega = \theta \wedge \omega$ and $d\eta = \theta \wedge \eta$, then

$$\theta = \Big(\frac{P_x}{y^k} + \frac{k}{y}\Big) dy$$

In particular, $\Theta|_U = y^{-k}\, P_{xx}(x, y)\, dx \wedge dy$ in these coordinates.

(b) *If C is non-invariant for $\mathcal{F}$ (and so for the pencil) then*

$$\begin{cases} \omega = dx \\ \eta = y^k\, dy - Q(x, y)\, dx \end{cases}$$

where $Q \in \mathcal{O}(U)$. If θ is such that $d\omega = \theta \wedge \omega$ and $d\eta = \theta \wedge \eta$, then

$$\theta = \frac{Q_y}{y^k}\, dx$$

In particular $\Theta|_U = -\frac{\partial}{\partial y}(y^{-k}\, Q_y)\, dx \wedge dy$ in these coordinates.

Proof. — Consider a covering $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ of M by open sets and collections $\Omega = (\omega_\alpha)_{\alpha\in A}$, $\Xi = (\eta_\alpha)_{\alpha\in A}$ and $\Lambda = (g_{\alpha\beta})_{U_{\alpha\beta}\neq\varnothing}$, such that $(U_\alpha, \omega_\alpha, \eta_\alpha)$ is compatible with the pencil for every $\alpha \in A$ and, if $U_{\alpha\beta} \neq \varnothing$ then $\omega_\alpha = g_{\alpha\beta}\cdot\omega_\beta$ and $\eta_\alpha = g_{\alpha\beta}\cdot\eta_\beta$ on $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Let $F_1 = |C| \cap \mathrm{sing}(\mathcal{F})$. Given $p \in |C| \setminus F_1$, let $(V, (u, v))$ be a holomorphic coordinate system around p such that $u(p) = v(p) = 0$ and $V \cap |C| = (v = 0)$. We can suppose that $V \subset U_\alpha$, for some $\alpha \in A$. Suppose first that C is invariant for the pencil. Since $p \notin \mathrm{sing}(\mathcal{F})$ and C is invariant for $\mathcal{F}$, by taking a smaller V if necessary, we can suppose that the leaves of $\mathcal{F}|_C$ are the level curves of v, so that $\omega_\alpha|_V = f\cdot dv$, where $f \in \mathcal{O}^*(V)$. Set $\omega = f^{-1}\cdot\omega_\alpha = dv$ and $\eta = f^{-1}\cdot\eta_\alpha$. Note that (V, ω, η) is compatible with the pencil. Let $\eta = A(u, v)dv - B(u, v)du$. Since $\omega \wedge \eta = B(u, v)du \wedge dv$ and the multiplicity of C in $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ is k, then $B(u, v) = v^k \cdot b(u, v)$, where $b \in \mathcal{O}(V)$ and $b(u, 0) \not\equiv 0$. Let $F_V = \{(u, 0) \in |C| \cap V\,;\ b(u, 0) = 0\}$ and $F = \cup_V F_V \cup F_1$. We leave for the reader the proof that F is finite. If $p \in |C| \setminus F$ then, in the above coordinate system we have $b(0, 0) \neq 0$. Therefore, there exists a neighborhood U of p, with $U \subset\subset V$, and a function $x \in \mathcal{O}(U)$ such that $x(p) = 0$, $\partial x/\partial u = b$ and $\Phi(u, v) = (x(u, v), v)$ is biholomorphism onto $\Phi(U) \subset \mathbb{C}^2$. In the coordinate system $(x, y) := (x, v)$, we have $\omega = dy$ and

$$\eta = A\,dv - v^k\,b\,du = A\,dy - y^k\Big(dx - \frac{\partial x}{\partial v}dy\Big) = \Big(A + y^k\,\frac{\partial x}{\partial v}\Big)dy - y^k\,dx := P\,dy - y^k\,dx$$

Let us compute $\Theta|_U$. If θ is such that $d\omega = \theta \wedge \omega$ and $d\eta = \theta \wedge \eta$ then $\theta = \phi \cdot dy$, because $\omega = dy$ and $d\omega = 0$. Since

$$d\eta = (P_x + k\,y^{k-1})dx \wedge dy = \Big(\frac{P_x}{y^k} + \frac{k}{y}\Big)dy \wedge \eta$$

we get that

$$\theta = \Big(\frac{P_x}{y^k} + \frac{k}{y}\Big)dy \quad\Longrightarrow\quad \Theta|_U = d\theta = \frac{P_{xx}}{y^k}dx \wedge dy$$

Now, suppose that C is non-invariant for $\mathcal{F}$. Let

$$F_1 = \{p \in |C|\,;\ \mathcal{F} \text{ is tangent to } |C| \text{ at } p\}.$$

Clearly F_1 is finite and if $p \in |C| \setminus F_1$ then there exists a holomorphic coordinate system $(V, (u, v))$ around p such that $V \subset U_\alpha$ for some $\alpha \in A$, $u(p) = v(p) = 0$, $|C| \cap V = (v = 0)$ and the leaves of $\mathcal{F}|_V$ are the level curves of u. In this case, $\omega_\alpha|_V = f \cdot du$ where $f \in \mathcal{O}^*(V)$. Set $\omega := du = f^{-1}\omega_\alpha|_V$ and $\eta = f^{-1} \cdot \eta_\alpha|_V$. Note that (V, ω, η) is compatible with the pencil. Let $\eta = A\,dv - B\,du$, where $A, B \in \mathcal{O}(U)$. Since $\omega\wedge\eta = A\,du\wedge dv$ and C is a component of multiplicity k, we can write $A = v^k\cdot a$, where $a(u, 0) \not\equiv 0$. Let $F_V = \{(u, 0) \in |C| \cap V\,;\ a(u, 0) = 0\}$ and set $F = \cup_V F_V \cup F_1$. We leave for the reader the proof that F is finite. If $p \in |C| \setminus F$ then in the above coordinate system we have $a(0, 0) \neq 0$. We assert that there exists a coordinate system $(U, (x, y))$ around p such that $U \subset V$, $u = x$, $y = v \cdot \phi(u, v)$ and

$$(*) \qquad \frac{\partial\, y^{k+1}}{\partial v} = (k + 1)\,v^k\,a(u, v)$$

In fact, in a neighborhood of $p = (0,0) \in V$, we can write $(k+1)\,v^k\,a(u,v) = \sum_{j=k}^{\infty} a_j(u)\,v^j$, where $a_k(0) = (k+1)\,a(0,0) \neq 0$. Let

$$\phi(u,v) = \sum_{j=k+1}^{\infty} \frac{1}{j}\,a_{j-1}(u)\,v^j := v^{k+1}\cdot b(u,v).$$

Note that $b(0,0) = a(0,0) \neq 0$ and $\partial\phi/\partial v = (k+1)\,v^k\,a(u,v)$. Let $U_1 \subset V$ be a simply connected open neighborhood of $(0,0)$ such that $b \in \mathcal{O}^*(U_1)$. Let $c \in \mathcal{O}^*(U_1)$ be such that $c^{k+1} = b$ and $y \in \mathcal{O}(U_1)$ be defined by $y(u,v) = v\cdot c(u,v)$. Note that $y^{k+1} = \phi$ and the map $\Phi(u,v) = (u, y(u,v)) = (x,y)$ is a biholomorphism from some open neighborhood $U \subset U_1$ onto an open subset of $\mathbb{C}^2$. Clearly, the coordinate system $(U,(x,y))$ satisfies $(*)$. In these coordinates, we have $\omega = dx$ and

$$\begin{aligned}\eta = v^k\,a(u,v)\,dv - B(u,v)\,du &= \frac{1}{k+1}\frac{\partial y^{k+1}}{\partial v}\,dv - B(u,v)\,du\\ &= y^k\,dy - \Big(\frac{1}{k+1}\frac{\partial y^{k+1}}{\partial u} + B(u,v)\Big)du := y^k\,dy - Q(x,y)\,dx\end{aligned}$$

If θ is such that $d\omega = \theta\wedge\omega$ and $d\eta = \theta\wedge\eta$ then $\theta = \psi\cdot dx$, because $\omega = dx$ and $d\omega = 0$. Since

$$d\eta = Q_y\,dx\wedge dy = \frac{Q_y}{y^k}\,dx\wedge\eta$$

we get that

$$\theta = \frac{Q_y}{y^k}\,dx \quad\Longrightarrow\quad \Theta = d\theta = -\frac{\partial}{\partial y}\Big(\frac{Q_y}{y^k}\Big)\,dx\wedge dy$$

This finishes the proof of Lemma 2.2.1. □

From now on, in this section, we will suppose that all irreducible components of $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ have multiplicity one.

2.2.2. (b) $\Longrightarrow$ (c). — Denote by D_∞ the divisor of poles of Θ. Let C be a component of $\mathrm{Tang}(\mathcal{F},\mathcal{G})$. Suppose first that C is invariant for the pencil. Since the multiplicity of C in $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ is one, by Lemma 2.2.1, we can choose a coordinate system $(U,(x,y))$ such that $U\cap C = (y=0)$, $p = (0,0) \in U$ and

$$\begin{cases}\omega = dy\\ \eta = P(x,y)\,dy - y\,dx\end{cases} \tag{5}$$

Let $P(x,y) = p_0(x) + y\,p(x,y)$, where $p \in \mathcal{O}(U)$ and $p_0(x) = \sum_{j=0}^{\infty} a_j\,x^j$. Since $\Theta = h\cdot dx\wedge dy$, where $h = y^{-1}\,P_{xx} = y^{-1}\,p_0''(x) + p_{xx}(x,y)$, then $C \not\subset D_\infty$ if, and only if, $p_0(x) = a_0 + a_1\,x$. Note that the foliation $\mathcal{F}_T$ associated to $\eta_T = \eta + T\cdot\omega = (T + P(x,y))dy - y\,dx$, is defined on U by the vector field

$$X_T(x,y) = (T + p_0(x) + y\,p(x,y))\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

Hence, the singularities of $\mathcal{F}_T$ on U are given by $y = T + p_0(x) = 0$.

We have two possibilities: either p_0 is a constant ($p_0(x) = a_0$), or p_0 is not a constant. In the first case, we get that $\eta_{-a_0} = y(p\,dy - dx)$. In this case, $-a_0 \in NI$ and there is no movable singularity on C. Moreover $\Theta = p_{xx}\,dx \wedge dy$, which implies that $C \not\subset D_\infty$. In the second case, there is a movable singularity on C: if $x(T)$ is such that $T + p_0(x(T)) = 0$ and $-T$ is a regular value of p_0 then $x(T)$ is a movable singularity of $\mathcal{P}$ and $T \in GP = \{T \in IS \mid n(T) = n_0\}$. Without lost of generality, we can suppose that this singularity satisfies (a) of Definition 2. This singularity is non-degenerate, in the sense that zero is not an eigenvalue of $DX_T(q(T))$, where $q(T) = (x(T), 0)$. In this case, the Baum-Bott index of $\mathcal{F}_T$ at $p(T)$ is given by (*cf.* [**Br**]):

$$B(T) := BB(q(T), \mathcal{F}_T) = \frac{\mathrm{tr}^2(DX_T(q(T))}{\det(DX_T(q(T))} = \frac{(p_0'(x(T)) + 1)^2}{p_0'(x(T))}$$

$$= p_0'(x(T)) + \frac{1}{p_0'(x(T))} + 2 \tag{6}$$

Since C is nice, we have $B'(T) \equiv 0$. As the reader can check, this condiction is equivalent to

$$p_0''(x(T))\big(1 - \frac{1}{(p_0'(x(T)))^2}\big)x'(T) \equiv 0$$

Since $q(T)$ is a movable singularity, we have $x'(T) \not\equiv 0$. Therefore, $p_0''(x(T)) \equiv 0$, which implies that $p_0'' \equiv 0$ and $p_0(x) = a_0 + a_1\,x$ (note that $p_0'(x(T)) = \pm 1$ implies also that $p_0'' = 0$). Therefore, $C \not\subset D_\infty$.

Suppose now that C is non-invariant for $\mathcal{P}$. Consider a coordinate system $(U, (x, y))$ such that $U \cap C = (y = 0)$, $p = (0, 0) \in U$ and

$$\begin{cases} \omega = dx \\ \eta = y\,dy - Q(x, y)\,dx \end{cases} \tag{7}$$

where $Q(x, y) = q_0(x) + q_1(x)\,y + y^2\,q(x, y)$, where q_0, q_1 and q are holomorphic. Since $\Theta = -(y^{-1}\,Q_y)_y\,dx \wedge dy$, then $C \not\subset D_\infty$ if, and only if, $q_1(x) \equiv 0$. Note that the foliation $\mathcal{F}_T$ associated to $\eta_T = \eta + T \cdot \omega = y\,dy + (T - Q(x, y)dx$, is defined on U by the vector field

$$X_T(x, y) = y\frac{\partial}{\partial x} + (q_0(x) + q_1(x)\,y + y^2\,q(x, y) - T)\frac{\partial}{\partial y}$$

Hence, the singularities of $\mathcal{F}_T$ on U are given by $y = q_0(x) - T = 0$.

We have two possilities: either q_0 is a constant, or q_0 is not a constant. In the first case, we get $\eta_{q_0} = y[dy - (q_1(x) + y\,q(x, y))dx]$, and so $q_0 \in NI$ and there is no movable singularity on C. Since C is nice, the curve C is invariant for the divided foliation associated to q_0, which is defined by $\widetilde{\omega} = dy - (q_1(x) + y\,q(x, y))dx$ on U. But, $C \cap U = (y = 0)$ and this curve is invariant for $\widetilde{\omega} = 0$ if, and only if, $q_1 \equiv 0$. Therefore, $C \not\subset D_\infty$. In the second case, there is a movable singularity: $p(T) = (x(T), 0) \in U \cap C$, where $x(T)$ is such that $q_0(x(T)) - T = 0$. Set $q_0(0) = T_0$. If T is a regular value

of q_0 near T_0, then $T \in GP$. Without lost of generality, we can suppose that this singularity satisfies (c) of Definition 2. This singularity is non-degenerate, and so:

$$B(T) := BB(p(T), \mathcal{F}_T) = \frac{\mathrm{tr}^2(DX_T(p(T))}{\det(DX_T(p(T))} = \frac{q_1^2(x(T))}{q_0'(x(T))} \tag{8}$$

Since C is nice, we get $B \equiv 0$, and so $q_1 \equiv 0$, which implies that $C \not\subset D_\infty$.

2.2.3. (a) $\Longrightarrow$ (b). — Suppose first that C is invariant for $\mathcal{P}(\mathcal{F}, \mathcal{G})$. Let $(U, (x, y))$ be a coordinate system like in (5), around a point $p = (0, 0) \in U \cap C$. Since $\Theta \equiv 0$, by Lemma 2.2.1, we have $P_{xx} = 0$. This implies that $P(x, y) = p_0(y) + p_1(y)x$, where p_0, p_1 are holomorphic. Hence, the singularities of $\mathcal{F}_T$ on $C \cap U$ are the solutions of $y = T + p_0(0) + p_1(0)x = 0$. We have two possibilities: either $p_1(0) \neq 0$, or $p_1(0) = 0$. If $p_1(0) = 0$, then $T = -p_0(0) \in NI$ and we are in the situation of (b) of Definition 2. Therefore, C is nice. If $p_1(0) \neq 0$, then C contains an unique movable singularity: $q(T) = (x(T), 0)$, where $x(T) = -(T + p_0(0))/p_1(0)$ (clearly $q(T) \in U$ for $|T + p_0(0)|$ small enough). This singularity is non-degenerate, and so by (6) we get:

$$BB(p(T), \mathcal{F}_T) = \frac{\mathrm{tr}^2(DX_T(p(T))}{\det(DX_T(p(T))} = \frac{(p_1(0) + 1)^2}{p_1(0)}$$

Hence, C is nice in this case.

Suppose now that C is non-invariant for the pencil. Consider a coordinate system $(U, (x, y))$ around $p = (0, 0) \in U$ as in (7). Since $\Theta \equiv 0$, Lemma 2.2.1 implies that

$$\frac{\partial}{\partial y}(y^{-1}\, Q_y) = 0 \quad \Longrightarrow \quad Q(x, y) = q_0(x) + q_2(x)y^2$$

This implies that C is nice, as the reader can check by using (8).

2.2.4. (c) $\Longrightarrow$ (a). — Suppose that Θ is holomorphic. The idea is to use the well-known fact that

$$\Theta \equiv 0 \iff \int_M \Theta \wedge \overline{\Theta} = 0 \iff [\Theta] = 0 \text{ in } H^2_{DR}(M)$$

The proof will be based in the following:

Claim 1. — $\int_M \Theta \wedge \overline{\Theta} = -2\pi i \int_M c_1(N_\mathcal{F}) \wedge \overline{\Theta}$, *where* $c_1(N_\mathcal{F})$ *is any representative of the first Chern class of* $N_\mathcal{F}$ *in* $H^2_{DR}(M)$.

Proof. — Let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be a covering of M by open sets, $\Omega = (\omega_\alpha)_{\alpha \in A}$, $\Xi = (\eta_\alpha)_{\alpha \in A}$ and $\Lambda = (g_{\alpha\beta})_{U_{\alpha\beta} \neq \varnothing}$ be as in (I), (II) and (III) of § 1. Let $(\theta_\alpha)_{\alpha \in A}$ be a collection of 1-forms, where θ_α is meromorphic on U_α, $d\omega_\alpha = \theta_\alpha \wedge \omega_\alpha$ and $d\eta_\alpha = \theta_\alpha \wedge \eta_\alpha$. Recall that, if $U_{\alpha\beta} \neq \varnothing$ then $\theta_\alpha - \theta_\beta = dg_{\alpha\beta}/g_{\alpha\beta}$. On the other hand, by taking a C^∞ resolution of the additive cocycle $(dg_{\alpha\beta}/g_{\alpha\beta})_{U_{\alpha\beta} \neq \varnothing}$, we can write $dg_{\alpha\beta}/g_{\alpha\beta} = \mu_\alpha - \mu_\beta$, where the closed 2-form Λ defined by $\Lambda|_{U_\alpha} = \frac{i}{2\pi} d\mu_\alpha$, represents $c_1(N_\mathcal{F})$ on $H^2_{DR}(M)$ (*cf.* [**G-H**], p. 141). If $U_{\alpha\beta} \neq \varnothing$, then $dg_{\alpha\beta}/g_{\alpha\beta} = \theta_\alpha - \theta_\beta = \mu_\alpha - \mu_\beta$. Hence, we can

define a C^∞ 1-form φ on $W := M \setminus \mathrm{Tang}(\mathcal{F},\mathcal{G})$ by $\varphi|_{U_\alpha\cap W} = \frac{i}{2\pi}(\theta_\alpha - \mu_\alpha)$. Note that $d\varphi = \frac{i}{2\pi}\Theta - \Lambda$. This implies that $d\varphi$ extends to a C^∞ form in M. Moreover,

$$\int_M (\tfrac{i}{2\pi}\Theta - \Lambda)\wedge\overline{\Theta} = \int_M d\varphi\wedge\overline{\Theta}. \tag{9}$$

The idea is to prove that $\int_M d\varphi\wedge\overline{\Theta} = 0$. Let us study the behavior of φ near an irreducible component of $\mathrm{Tang}(\mathcal{F},\mathcal{G})$. Set $\mathrm{Tang}(\mathcal{F},\mathcal{G}) = \sum_{j=1}^k C_j + \sum_{i=1}^\ell D_i$, where C_j is invariant for the pencil, $j = 1,\dots,k$, and D_i is non-invariant, $i = 1,\dots,\ell$. Consider first the non-invariant case. Let $p \in |D_i| \cap U_\alpha$ be a point such that we have a normal form like in (b) of Lemma 2.2.1, in a coordinate system $(U,(x,y))$, where $U \subset U_\alpha$. As we have seen, $\omega_\alpha|_U = f\,\omega$ and $\eta_\alpha|_U = f\,\eta$, where $f \in \mathcal{O}^*(U)$, $\omega = dx$ and $\eta = ydy - Q(x,y)dx$, $Q(x,y) = q_0(x) + q_1(x)\,y + y^2\,q(x,y)$. This implies that $\theta_\alpha = \theta + df/f$, where $\theta = \frac{Q_y}{y}dx$. Note that Θ is holomorphic in U if, and only if, Q_y/y is holomorphic, which implies that θ_a is holomorphic in U and φ is C^∞ in U. This implies that φ is C^∞ on $M \setminus |C|$, where $C = \sum_j C_j$ and $|C| = \cup_j |C_j|$.

Consider now a point $p \in |C|$. Let $(f_1\cdots f_k = 0)$ be a (reduced) equation of C in a small Stein neighborhood U of p. We assert that there exist $\lambda_1,\dots,\lambda_k \in \mathbb{C}$ and a C^∞ 1-form ν such that

$$\varphi|_U = \sum_{j=1}^r \lambda_j \frac{df_j}{f_j} + \nu. \tag{10}$$

In fact, suppose first that p belongs to an invariant component C_j and we have a normal form like in (a) of Lemma 2.2.1 on a coordinate system $(U,(x,y))$, where $U \subset U_\alpha$, for some $\alpha \in A$. As before, we have $\omega_\alpha|_U = f\cdot\omega = f\cdot dy$ and $\eta_\alpha|_U = f\cdot\eta$, where $f \in \mathcal{O}^*(U)$ and $\eta = P(x,y)\,dy - y\,dx$. From the first part of the proof and the fact that Θ is holomorphic, we get

$$\theta_\alpha = \theta + \frac{df}{f} = \frac{1+P_x}{y}dy + \frac{df}{f} = \frac{1}{2\pi i}\lambda_U\frac{dy}{y} + \varsigma_U, \tag{*}$$

where $\lambda_U \in \mathbb{C}$ and ς_U is a holomorphic 1-form. This implies that $\varphi|_U = \lambda_U\dfrac{dy}{y} + \nu_U$, where ν_U is a C^∞ 1-form.

Let us prove that λ_U depends only of C_j. It follows from (*) that

$$\frac{1}{2\pi i}\lambda_U = \mathrm{Res}(\theta_\alpha, C_j) = \frac{1}{2\pi i}\int_\gamma \theta_\alpha,$$

where γ is a small cicle surrounding C_j. If $\beta \in A$ is such that $U_\alpha\cap U_\beta\cap C_j \neq \varnothing$ then $\theta_\alpha - \theta_\beta = dg_{\alpha\beta}/g_{\alpha\beta}$. Hence,

$$\frac{1}{2\pi i}\int_\gamma \theta_\alpha = \frac{1}{2\pi i}\int_\gamma \theta_\beta,$$

if $\gamma \subset U_\alpha \cap U_\beta$. This proves that λ_U depends only of C_j. Set $\lambda_U = \lambda_j$.

Note that λ_j satisfies the following property

(A) Let $(f_{j\alpha}=0)$ be a reduced equation of $C_j\cap U_\alpha$. Then $\theta_\alpha-\frac{1}{2\pi i}\lambda_j df_{j\alpha}/f_{j\alpha}$ has no poles along $C_j\cap U_\alpha$.

We leave the proof of (A) for the reader. Let $p\in|C|\cap U_\alpha$ and $(f_{j\alpha}=0)$ be a reduced equation of C_j on U_α. It follows from (A) that $\theta_\alpha-\sum_{j=1}^k\frac{1}{2\pi i}\lambda_j\,df_{j\alpha}/f_{j\alpha}$ is holomorphic on U_α. Hence, $\nu=\varphi|_{U_\alpha}-\sum_{j=1}^k\lambda_j\,df_{j\alpha}/f_{j\alpha}$ is C^∞. This proves (10).

Let us prove that $\int_M d\varphi\wedge\overline{\Theta}=0$. We will consider two cases:

First case. — All the singularities of C are nodes. In this case, we can find a finite open covering $\mathcal{V}=(V_\alpha)_{\alpha\in A}$ of M with the following properties:

(i) For every $\alpha\in A$, V_α is a domain of a coordinate system $\psi_\alpha=(x_\alpha,y_\alpha)\colon U_\alpha\to\mathbb{C}^2$ such that $\psi_\alpha(U_\alpha)=D_2\times D_2$, where $D_r=\{z\in\mathbb{C}\,|\,|z|<r\}$.

(ii) If $U_\alpha=\psi_\alpha^{-1}(D_1\times D_1)$ then $\cup_\alpha U_\alpha=M$.

(iii) If $|C|\cap V_\alpha\neq\varnothing$ is smooth then $(y_\alpha=0)$ is an equation of $C\cap V_\alpha$. In particular, $\varphi|_{V_\alpha}=\lambda\frac{dy_\alpha}{y_\alpha}+\nu$, where $\lambda\in\mathbb{C}$ and ν is C^∞.

(iv) If $|C|\cap V_\alpha$ has a singularity in V_α then $(x_\alpha\cdot y_\alpha=0)$ is an equation of $C\cap V_\alpha$. In particular, $\varphi|_{V_\alpha}=\lambda_a\frac{dx_\alpha}{x_\alpha}+\lambda_b\frac{dy_\alpha}{y_\alpha}+\nu$, where $\lambda_a,\lambda_b\in\mathbb{C}$ and ν is C^∞.

In general, let $(f_\alpha=0)$ be an equation of $C\cap V_\alpha$. Let $(\varphi_\alpha)_{\alpha\in A}$ be a C^∞ partition of the unity such that $\operatorname{supp}(\varphi_\alpha)\subset V_\alpha$ for all $\alpha\in A$ and set $f=\exp(\sum_\alpha\varphi_\alpha\cdot\ln|f_\alpha|)$. If $\beta\in A$ is fixed, then

$$\begin{aligned}f|_{V_\beta}&=\exp\Big(\sum_{\alpha,V_{\alpha,\beta}\neq\varnothing}\varphi_\alpha\cdot\ln|f_\alpha|\Big)\cdot\exp\Big(\sum_{\alpha,V_{\alpha,\beta}=\varnothing}\varphi_\alpha\cdot\ln|f_\alpha|\Big)\\&=\exp\Big(\sum_{\alpha,V_{\alpha,\beta}\neq\varnothing}\varphi_\alpha\cdot\ln|g_{\alpha\beta}f_\beta|\Big)\cdot\exp\Big(\sum_{\alpha,V_{\alpha,\beta}=\varnothing}\varphi_\alpha\cdot\ln|f_\alpha|\Big)=|f_\beta|\cdot g_\beta\end{aligned}$$

where $g_\beta\colon V_\beta\to(0,+\infty)$ is C^∞.

(v) $f|_{V_\alpha}=|f_\alpha|\cdot g_\alpha$, where $g_\alpha\in C^\infty(V_\alpha)$. In particular, f can be extended continually to M as $f|_{|C|}\equiv 0$.

(vi) $f>0$ on $M\smallsetminus|C|$ and $f^{-1}(0)=|C|$.

Set $M_\varepsilon=\{p\in M|\,f(p)\geqslant\varepsilon\}$ and $C_\varepsilon=\{p\in M|\,f(p)\leqslant\varepsilon\}$. For all $\varepsilon>0$ we have

$$\begin{aligned}\int_M d\varphi\wedge\overline{\Theta}&=\int_{M_\varepsilon}d\varphi\wedge\overline{\Theta}+\int_{C_\varepsilon}d\varphi\wedge\overline{\Theta}=\int_{M_\varepsilon}d(\varphi\wedge\overline{\Theta})+\int_{C_\varepsilon}d\varphi\wedge\overline{\Theta}\\&=\int_{\partial M_\varepsilon}\varphi\wedge\overline{\Theta}+\int_{C_\varepsilon}d\varphi\wedge\overline{\Theta}.\end{aligned}$$

Since $\lim_{\varepsilon\to0}\big(\int_{C_\varepsilon}d\varphi\wedge\overline{\Theta}\big)$, we get

(vii) $\int_M d\varphi\wedge\overline{\Theta}=\lim_{\varepsilon\to0}\big(\int_{\partial M_\varepsilon}\varphi\wedge\overline{\Theta}\big)$.

It is enough to prove that $\lim_{\varepsilon\to0}\big(\int_{\partial M_\varepsilon}\varphi\wedge\overline{\Theta}\big)=0$. In order to prove this fact, consider a covering $\{V_1:=V_{\alpha_1},\dots,V_n:=V_{\alpha_n}\}$ of $|C|$ by sets of $\mathcal{V}$, such that $\{U_j:=U_{\alpha_j}\mid 1\leqslant j\leqslant n\}$ is still a covering of $|C|$. If $U=\cup_{j=1}^n U_j$ then there

exists ε_0 such that, if $\varepsilon < \varepsilon_0$ then $\partial M_\varepsilon \subset V$. Hence, if $S_j(\varepsilon) = \partial M_\varepsilon \cap \overline{U}_\varepsilon$ and $I_j(\varepsilon) = \int_{S_j(\varepsilon)} |\varphi \wedge \overline{\Theta}|$, we get that

$$\Big| \int_{\partial M_\varepsilon} \varphi \wedge \overline{\Theta} \Big| \leqslant \sum_{j=1}^{n} I_j(\varepsilon), \quad \text{if } \varepsilon < \varepsilon_0$$

It follows that, it is sufficient to prove that $\lim_{\varepsilon\to 0} I_j(\varepsilon) = 0$ for all $j = 1, \ldots, n$. We will prove this fact in the case where V_j is like in (iv) and leave the other case for the reader.

Consider a coordinate system (x, y) on V_j as in (iv), that is $|C| \cap V_j = (x \cdot y = 0)$. As we have seen before, $\Theta|_{V_j} = g(x, y)\, dx \wedge dy$ and $\varphi|_{V_j} = \lambda_a \frac{dx}{x} + \lambda_b \frac{dy}{y} + \nu$, where $g \in \mathcal{O}(V_j)$, $\lambda_a, \lambda_b \in \mathbb{C}$ and ν is C^∞. Therefore, there exists a constant $c > 0$ such that on $\overline{U}_j$ we have

$$|\varphi \wedge \overline{\Theta}| \leqslant c \left(\Big| \frac{dx}{x} \wedge d\overline{x} \wedge d\overline{y} \Big| + \Big| \frac{dy}{y} \wedge d\overline{x} \wedge d\overline{y} \Big| + \Big| \nu \wedge d\overline{x} \wedge d\overline{y} \Big| \right)$$

If we set

$$A_j(\varepsilon) = \int_{S_j(\varepsilon)} \Big| \frac{dx}{x} \wedge d\overline{x} \wedge d\overline{y} \Big|,$$
$$B_j(\varepsilon) = \int_{S_j(\varepsilon)} \Big| \frac{dy}{y} \wedge d\overline{x} \wedge d\overline{y} \Big|$$
$$C_j(\varepsilon) = \int_{S_j(\varepsilon)} \Big| \nu \wedge d\overline{x} \wedge d\overline{y} \Big|,$$

then $I_j(\varepsilon) \leqslant c.(A_j(\varepsilon) + B_j(\varepsilon) + C_j(\varepsilon))$. Hence, it is sufficient to prove that $\lim_{\varepsilon\to 0} A_j(\varepsilon) = \lim_{\varepsilon\to 0} B_j(\varepsilon) = \lim_{\varepsilon\to 0} C_j(\varepsilon) = 0$. We will prove that $\lim_{\varepsilon\to 0} A_j(\varepsilon) = 0$ and leave the proof that $\lim_{\varepsilon\to 0} B_j(\varepsilon) = \lim_{\varepsilon\to 0} C_j(\varepsilon) = 0$ for the reader (note that $\lim_{\varepsilon\to 0} C_j(\varepsilon) = 0$ because ν is C^∞). Given $0 < a < 1$, define

$$J(a, \varepsilon) = \int_{S_j(\varepsilon)\cap(|x|\geqslant a)} \Big| \frac{dx}{x} \wedge d\overline{x} \wedge d\overline{y} \Big| \quad \text{and} \quad K(a, \varepsilon) = \int_{S_j(\varepsilon)\cap(|x|\leqslant a)} \Big| \frac{dx}{x} \wedge d\overline{x} \wedge d\overline{y} \Big|$$

so that $A_j(\varepsilon) = J(a, \varepsilon) + K(a, \varepsilon)$. Since $|\frac{dx}{x} \wedge d\overline{x} \wedge d\overline{y}|$ is C^∞ on $(|x| \geqslant a)$, we get that $\lim_{\varepsilon\to 0} J(a, \varepsilon) = 0$ for all $a > 0$. Therefore, it is sufficient to prove that there exists $0 < a < 1$ such that $\lim_{\varepsilon\to 0} K(a, \varepsilon) = 0$.

Set $x = r\, e^{i\alpha}$ and $y = s\, e^{i\beta}$, so that $|\frac{dx}{x} \wedge d\overline{x} \wedge d\overline{y}| = 2|dr \wedge d\alpha \wedge d\overline{y}|$. In the coordinate system (r, α, y) we have $f(r, \alpha, y) = r \cdot s \cdot g(r, \alpha, y)$ (by (iv)), where $g \in C^\infty$ and $g > 0$. Since $\partial r \cdot g/\partial r(0, \alpha, y) = g(0, \alpha, y) > 0$, there exists $0 < a < 1$ such that the map $\psi(r, \alpha, y) = (r \cdot g(r, \alpha, y), \alpha, y) = (R, \alpha, y)$ is diffeomorphism from a neighborhood W of $(r = 0) \cap (|y| \leqslant 1)$ onto $W_1 = (R < \delta) \cap (|y| < 1 + \delta)$, where $W \sup(r \leqslant a) \cap (|y| \leqslant 1)$. Note that $\psi^{-1}(R, \alpha, y) = (R \cdot h(R, \alpha, y), \alpha, y)$, where h is C^∞. In the coordinate system (R, α, y) we have

$$S_j(\varepsilon)\cap W_1 = (R\cdot|y| = R\cdot s = \varepsilon)\cap(s \leqslant 1) := T(\varepsilon) \implies K(a, \varepsilon) = \int_{T(\varepsilon)} 2|d(R\cdot h)\wedge d\alpha\wedge d\overline{y}|$$

if $\varepsilon > 0$ is small. We assert that there exists a constant $c > 0$ such that $2|d(R\cdot h)\wedge d\alpha\wedge d\overline{y}| \leqslant c\cdot R|ds\wedge d\alpha\wedge d\beta|$ on $T(\varepsilon)$, if ε is small (the restriction to $T(\varepsilon)$). In fact,

$$\begin{aligned}2|d(R\cdot h)\wedge d\alpha\wedge d\overline{y}| &\leqslant 2R\,|dh\wedge d\alpha\wedge d\overline{y}| + 2|h|\,|dR\wedge d\alpha\wedge d\overline{y}|\\ &\leqslant\ 2R\,|h_R|\,|dR\wedge d\alpha\wedge d\overline{y}| + 2R\,|h_y|\,|d\alpha\wedge dy\wedge d\overline{y}|\\ &\quad + 2|h|\,|dR\wedge d\alpha\wedge d\overline{y}|\end{aligned}$$

Since $K := \psi((r\leqslant a)\cap(|y|\leqslant 1))$ is compact, $2|h|, 2|h_R|, 2|h_y|, R$ are bounded in K, so that there exists a constants $c_1 > 0$ such that

$$\begin{aligned}2|d(R\cdot h)\wedge d\alpha\wedge d\overline{y}| &\leqslant c_1\big(R\,|d\alpha\wedge dy\wedge d\overline{y}| + |dR\wedge d\alpha\wedge d\overline{y}|\big)\\ &\leqslant c_1\big(2\,R\,|ds\wedge d\alpha\wedge d\beta| + |dR\wedge d\alpha\wedge d\overline{y}|\big)\end{aligned}$$

on K, because $|d\alpha\wedge dy\wedge d\overline{y}| = 2|ds\wedge d\alpha\wedge d\beta|$. On the other hand, $\overline{y} = s\cdot e^{-i\beta}$ and $R\cdot s = \varepsilon$ on $T(\varepsilon)$. Hence, if $\varepsilon > 0$ is small, we get

$$\begin{aligned}|dR\wedge d\alpha\wedge d\overline{y}| &= |d(R\,d\overline{y})\wedge d\alpha| = |d(-R\,s\,i\,e^{-i\beta}d\beta)\wedge d\alpha + d(R\,e^{-i\beta}ds)\wedge d\alpha|\\ &= |d(-\varepsilon\,i\,e^{-i\beta}d\beta)\wedge d\alpha + d(R\,e^{-i\beta}ds)\wedge d\alpha|\\ &= |d(R\,e^{-i\beta}ds)\wedge d\alpha| \leqslant\ R\,|ds\wedge d\alpha\wedge d\beta| + |dR\wedge ds\wedge d\alpha|\\ &= R\,|ds\wedge d\alpha\wedge d\beta|\end{aligned}$$

because $dR\wedge ds = 0$ on $T(\varepsilon)$. Therefore, on $T(\varepsilon)$ we have $2|d(R\cdot h)\wedge d\alpha\wedge d\overline{y}| \leqslant c\cdot R|ds\wedge d\alpha\wedge d\beta|$, where $c = 3\,c_1$. From this, we get that

$$K(a,\varepsilon) \leqslant c\int_{T(\varepsilon)} R|ds\wedge d\alpha\wedge d\beta| = c\,\varepsilon\int_{T(\varepsilon)}\Big|\frac{ds}{s}\wedge d\alpha\wedge d\beta\Big|.$$

On the other hand, the region $T(\varepsilon)$ in the real hypersurface $R\cdot s = \varepsilon$, is contained in a region of the form

$$T_1(\varepsilon) := \{(R,s,\alpha,\beta)\mid R\cdot s = \varepsilon,\ \alpha,\beta\in[0,2\pi],\ 1\geqslant s\geqslant \varepsilon/R_0\},$$

where $R_0 = \sup\{R(r,\alpha,y)|\,(r,\alpha,y)\in S_j(\varepsilon)\}$. This implies that

$$K(a,\varepsilon) \leqslant c\,\varepsilon\int_{T_1(\varepsilon)}\Big|\frac{ds}{s}\wedge d\alpha\wedge d\beta\Big| = 4\pi^2\,c\,\varepsilon\cdot|\log(\varepsilon/R_0)| \quad\Longrightarrow\quad \lim_{\varepsilon\to 0}K(a,\varepsilon) = 0$$

This finishes the proof of Claim 1 in the first case.

Second case: general case. — Consider a resolution of the curve C by blowing-ups $\pi\colon \widehat{M}\to M$ and let $C^* = \pi^{-1}(C)$, $\Theta^* = \pi^*(\Theta)$ and $\varphi^* = \pi^*(\varphi)$. Then $\int_M d\varphi\wedge\overline{\Theta} = 0$ if, and only if, $\int_{\widehat{M}} d\varphi^*\wedge\overline{\Theta}^* = 0$. Note that the singularities of C^* are of nodal type. It is sufficient to prove that $|C^*|$ admits an open covering satisfying (i), (ii), (iii) and (iv). Let $p\in\mathrm{sing}(C)$ (which is not a node) and $q\in\pi^{-1}(p)$. Since the singularities of C^* are nodes, we have two possibilities: either q is a smooth point of C^*, or q is in the normal crossing of two local components, say D_1 and D_2 of C^*. Let us consider, for instance, the second case. Let $(W,(x,y))$ be a coordinate system

around p, where $C \cap U$ has a reduced equation $(f_1 \cdots f_k = 0)$. As we have seen, we can write $\varphi|_W = \sum_{j=1}^{k} \lambda_j \frac{df_j}{f_j} + \nu$, where $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and ν is C^∞. Consider a coordinate system $(V, \phi = (u, v))$ around $q = (0, 0)$ such that $\pi(V) \subset W$, $\phi(V) = \{(u, v) \in \mathbb{C}^2 \mid |u|, |v| \leqslant 2\}$, $D_1 \cap V = (u = 0)$ and $D_2 \cap V = (v = 0)$. We have still two possibilities: either $\pi(D_1) = \pi(D_2) = \{p\}$, or $\pi(D_j) = \{p\}$ for just one $j \in \{1, 2\}$. Let us consider, for instance, the first case. In this case, if $\widehat{f}_j$ is the strict transform of f_j, then $F_j := \widehat{f}_j|_V \in \mathcal{O}^*(V)$. On the other hand, $f_j \circ \pi(u, v) = u^{m_j} \cdot v^{n_j} \cdot F_j$. Hence, in the coordinates (u, v) we have, $\pi^*(\varphi) = \lambda_a \frac{du}{u} + \lambda_b \frac{dv}{v} + \nu^*$, where $\lambda_a = \sum_j m_j \cdot \lambda_j$, $\lambda_b = \sum_j n_j \cdot \lambda_j$ and $\nu^* = \pi^*(\nu) + \sum_j \lambda_j dF_j/F_j$. Since $F_j \in \mathcal{O}^*(V)$ for all j, we get that ν^* is C^∞. We leave the proof of the other cases for the reader. This finishes the proof of Claim 1. □

Let us finish the proof of (c) $\Longrightarrow$ (a). Suppose by contradiction that Θ is holomorphic and $\Theta \not\equiv 0$. Let $Z := (\Theta)_0$ be the divisor of zeroes of Θ. Given a divisor D on M we will denote by $[D]$ its class in $\mathrm{Pic}(M)$. Since Θ is a non-vanishing section of $\Omega^2(M)$, we have $K_M = [Z]$. On the other hand, it is known that $\mathrm{Tang}(\mathcal{F}, \mathcal{G}) = K_M + N_{\mathcal{F}} + N_{\mathcal{G}}$ (*cf.* [**Br**]). Since $N_{\mathcal{F}} = N_{\mathcal{G}}$ we get that $2N_{\mathcal{F}} = \mathrm{Tang}(\mathcal{F}, \mathcal{G}) - [Z] = \sum_{j=1}^{m} n_j [D_j]$, where $n_j \in \mathbb{Z}$ and D_j is an irreducible component of $\mathrm{Tang}(\mathcal{F}, \mathcal{G}) \cup Z$, $1 \leqslant j \leqslant m$. It follows from Claim 1 that

$$\int_M \Theta \wedge \overline{\Theta} = \sum_{j=1}^{m} -i\pi\, m_j \int_M c_1(D_j) \wedge \overline{\Theta}$$

On the other hand, it is known that (*cf.* [**G-H**])

$$\int_M c_1(D_j) \wedge \overline{\Theta} = \int_{D_j} \overline{\Theta} = 0$$

because $\overline{\Theta}$ is a $(0, 2)$-form. This finishes the proof of Theorem 2. □

References

[BPV] W. Barth, C. Peters & A. van de Ven – *Complex Compact Surfaces*, Springer Verlag, 1984.

[Br] M. Brunella – *Birational geometry of foliations*, IMPA, 2000, text book for a course in the First Latin American Congress of Mathematics.

[Ch] S.S. Chern – "Web Geometry", *Bull. Amer. Math. Soc.* **6** (1982), p. 1–8.

[E-R] C. Ehresmann – "Les connexions infinitésimales dans un espace fibré différentiable", in *Colloque de topologie (Bruxelles, 1950)*, Georges Thone, Liège, 1951, p. 29–55.

[G-H] P.A. Griffiths & J. Harris – *Principles of Algebraic Geometry*, John-Wiley and Sons, 1994.

[LN-1] A. Lins Neto – "Exceptional families of foliations and the Poincaré problem", Preprint IMPA, 2000.

[LN] ———, "Some examples for the Poincaré and Painlevé problems", *Ann. scient. Éc. Norm. Sup.* 4$^{\mathrm{e}}$ *série* **35** (2002), p. 231–266.

[Sc2] B. SCÁRDUA – "Transversely affine and transversely projective holomorphic foliations", *Ann. scient. Éc. Norm. Sup.* 4^{e} *série* **30** (1997), p. 169–204.

[S] M.F. SINGER – "Liouvillian first integrals of differential equations", *Trans. Amer. Math. Soc.* **333** (1992), no. 2, p. 673–687.

A. LINS NETO, Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina, 110, Horto, Rio de Janeiro, Brasil • *E-mail :* alcides@impa.br

Astérisque
296, 2004, p. 191–205

SKEW DIFFERENTIAL FIELDS, DIFFERENTIAL AND DIFFERENCE EQUATIONS

by

Marius van der Put

Dedicated to Jean-Pierre Ramis on the occasion of his 60th birthday

***Abstract.* —** The central question is: Let a differential or difference equation over a field K be isomorphic to all its Galois twists w.r.t. the group $\mathrm{Gal}(K/k)$. Does the equation descend to k? For a number of categories of equations an answer is given.

***Résumé* (Corps différentiels non commutatifs, équations différentielles et équations aux différences)**

On étudie la descente sur un corps k d'une équation différentielle ou aux différences donnée sur un corps K et qui est isomorphe à toutes ses conjuguées sous l'action du groupe de Galois $\mathrm{Gal}(K/k)$ de K sur k. On traite le cas de plusieurs classes d'équations.

Introduction

Rationality questions for differential modules and differential operators are strongly related to skew differential fields. This theme has been developed in [**H-P**]. An open question in [**H-P**] has found an answer, namely the existence and unicity of the differentiation on a skew field of finite dimension over its center, that is, a differential field in the usual sense. The present paper, *written in honour of Jean-Pierre Ramis*, reviews these descent problems but now in the context of meromorphic differential equations. A remarkable family of examples is the result. Equally surprising is that descent does hold for meromorphic q-difference equations. This is shown using recent work of J.-P. Ramis and J. Sauloy on moduli for these equations. Finally, it is shown that descent does not hold for meromorphic ordinary difference equations.

2000 ***Mathematics Subject Classification.* —** Primary 12H, 34M35, 34M40, 39A11, 39A13; Secondary 12G05, 14F22.
***Key words and phrases.* —** Differential equation, difference equation, descent, Galois cohomology.

© Astérisque 296, SMF 2004

1. The construction of skew differential fields

Let k denote a differential field having characteristic 0. The skew fields, or more generally the algebras F, that we consider here are central, simple, finite dimensional over their center k. A differentiation on F, extending the differentiation of k, is an additive map $\partial : F \to F$ such that $\partial(ab) = \partial(a)b + a\partial(b)$ holds for all $a, b \in F$. Moreover, we require that $\partial(a) = a'$ for every $a \in k$. For special cases, such differentiations are constructed in [**H-P**]. Here we prove a general result on differentiations.

Theorem 1.1. — *Let k be a differential field of characteristic 0. Let F be a central, simple algebra over k of finite dimension. Suppose that F contains a maximal commutative subfield K which is Galois over k. There exists a differentiation ∂_0 on F extending the differentiation of k and having the property $\partial_0 K \subset K$.*

Moreover, for any differentiation ∂ on F extending the differentiation of k, there is an element $c \in F$ (unique up to an element in k) such that $\partial(a) = \partial_0(a) + ac - ca$ for all $a \in F$.

Proof. — The asumptions on F imply that F is a crossed product algebra (see [**Bl**], Chapitre IV). The structure of F is the following:

The elements of F are uniquely given by expressions $\sum_{\sigma \in G} d_\sigma[\sigma]$, where G is the Galois group of K/k and all $d_\sigma \in K$. The multiplication is given by the rules $[\sigma]d = \sigma(d)[\sigma]$ (for $\sigma \in G$, $d \in K$) and $[\sigma][\tau] = c(\sigma, \tau)[\sigma\tau]$. Here $(\sigma, \tau) \mapsto c(\sigma, \tau)$ is a 2-cocycle representing an element of $H^2(G, K^*)$.

Let $'$ denote the unique differentiation on K, extending the one of k. Then $(\sigma, \tau) \mapsto c(\sigma, \tau)'/c(\sigma, \tau)$ is a 2-cocycle for $H^2(G, K)$. Since the latter group is trivial (see [**Se**]), there are elements $\{a(\sigma)\}$ in K such that $a(\sigma) + {}^\sigma a(\tau) - a(\sigma\tau) = c(\sigma, \tau)'/c(\sigma, \tau)$. Now, define ∂_0 by the formula

$$\partial_0\big(\sum d_\sigma[\sigma]\big) = \sum (d'_\sigma + d_\sigma a(\sigma))[\sigma].$$

The verification that ∂_0 has the required properties is straightforward.

Let ∂ be another derivation on F extending the one of k. Then $\partial - \partial_0$ is a k-linear derivation on F. It is known that these derivations are given by $a \mapsto [a, c] := ac - ca$ for $c \in F$. (See [**Ren**], Corollaire 3 on p. 111). □

We note that the differentiation ∂ on F, extending the differentiation of k, is *almost unique* if one prescribes that ∂ is the usual differentiation on the maximal commutative subfield K of F. Indeed, $\partial(a) = \partial_0(a) + [a, c]$ for some $c \in F$. For $a \in K$ one has $[a, c] = 0$. Further, K is a maximal commutative subfield of F and, thus, $c \in K$.

2. Skew differential fields over $\mathbf{R}(\{x\})$

Notations. — $k := \mathbf{R}(\{x\})$, $K := \mathbf{C}(\{x\})$ are the fields of convergent Laurent series over $\mathbf{R}$ and $\mathbf{C}$. The differentiation on these fields is given by $f \mapsto f' := x df/dx$.

Hamilton's quaternion field is denoted by $\mathbf{H} := \mathbf{R}1 + \mathbf{R}i + \mathbf{R}j + \mathbf{R}k$. Then $F = \mathbf{H} \otimes_{\mathbf{R}} k$ is a quaternion field with center k. Let $\| \ \|$ denote the usual norm on $\mathbf{H}$. The differentiation on F is defined by $(h \otimes f)' = h \otimes f'$ for all $h \in \mathbf{H}$ and $f \in k$. The elements of F are represented by convergent Laurent series with coefficients in $\mathbf{H}$. Thus, an element of F has the form $\sum a_n x^n$ with all $a_n \in \mathbf{H}$ and such that only finitely many negative powers of x are present and, moreover, there are positive constants C, R with $\|a_n\| \leqslant CR^n$ for all $n \geqslant 0$. One observes that $(\sum a_n x^n)' = \sum n a_n x^n$.

Consider the 1-dimensional differential module $M = Fe$ over F, defined by the formula $\partial e = de$. After identifying F with M, via $v \mapsto ve$, one has $\partial(v) = v' + vd$. For d we make the choice $d = i + x^{-1}j$. One can consider M as a differential module over K of dimension 2, by the obvious inclusion $K \subset F$. Further, M is also a differential module over k of dimension 4.

Proposition 2.1. — $\mathrm{End}_{k[\partial]}(M)$, *the* $\mathbf{R}$*-algebra of the endomorphisms of the* k*-differential module* M*, is equal to* $\mathbf{H}$.

Proof. — Every k-linear map $L : M \to M$ has uniquely the form $L(v) = va_0 + iva_1 + jva_2 + kva_3$ with $a_0, \ldots, a_3 \in F$. A calculation shows that

$$(\partial L - L\partial)(v) = v(a_0' + [a_0, d]) + iv(a_1' + [a_1, d]) + jv(a_2' + [a_2, d]) + kv(a_3' + [a_3, d]).$$

Hence $L \in \mathrm{End}_{k[\partial]}(M)$ if and only if $a_i' + [a_i, d] = 0$ for $i = 0, \ldots, 3$. Therefore, the proposition follows from the statement:

The only solutions $a \in F$ *of* $a' + [a, d] = 0$ *are* $a \in \mathbf{R}$.

The proof of this statement is as follows. Write $a = \sum a_n x^n$ with all $a_n \in \mathbf{H}$. Then, $a' + [a, d] = 0$ translates into

$$\sum (na_n + [a_n, i] + [a_{n+1}, j])x^n = 0.$$

For $n > 0$ and $t = t_0 + t_1 i + t_2 j + t_3 k \in \mathbf{H}$ one has

$$nt + [t, i] = nt_0 + nt_1 i + (nt_2 + 2t_3)j + (nt_3 - 2t_2)k.$$

It follows that $\|nt + [t, i]\| \geqslant n\|t\|$.

For $s = s_0 + s_1 i + s_2 j + s_3 k \in \mathbf{H}$ one has $[s, j] = 2s_i k - 2s_3 i$ and thus $\|[s, j]\| \leqslant 2\|s\|$. One concludes that for $n > 0$ one has $\|a_{n+1}\| \geqslant \frac{n}{2}\|a_n\|$. If some $a_m \neq 0$ with $m > 0$, then $a_n \neq 0$ for all $n \geqslant m$. Moreover, for a suitable constant $C > 0$ one has $\|a_n\| \geqslant C2^{-n}n!$ for all $n \geqslant m$. This contradicts the assumption that the Laurent series a is convergent. The conclusion is that $a_n = 0$ for all $n \geqslant 1$.

$0 \cdot a_0 + [a_0, i] + [a_1, j] = 0$ implies that $a_0 \in \mathbf{R} + \mathbf{R}i$. After subtracting from a a real number, one has $a_0 \in \mathbf{R}i$. In the sequel we will write $*$ for a non-zero real number. Suppose that $a_0 = *i$. Then $-a_{-1} + [a_{-1}, i] + [a_0, j] = 0$ implies that $a_{-1} = *j + *k$. The equation $-2a_{-2} + [a_{-2}, i] + [a_{-1}, j] = 0$ implies $a_{-2} = *i$. By induction, one finds that $a_{-2m} = *i$ and $a_{-2m-1} = *j + *k$. This contradicts the fact that a is a Laurent series. One concludes that $a_0 = 0$. □

In the proof of the following corollary we will use some ideas and results of [**H-P**]. For any differential fields $k \subset K$, one says that a differential module M over K descends to k, if there exists a differential module N over k such that $M \cong K \otimes_k N$. Suppose that K/k is a Galois extension with group G. For a differential module M over K and for $\sigma \in G$, one defines the twisted differential module ${}^{\sigma}M$ by:

– ${}^{\sigma}M$ is equal to M as an additive group.
– For $f \in K$ and $m \in {}^{\sigma}M$ one puts $fm = \sigma^{-1}(f)m$.
– The ∂ on ${}^{\sigma}M$ coincides with ∂ on M.

If M descends to k, then clearly ${}^{\sigma}M \cong M$ for all $\sigma \in G$. The descent problem of [**H-P**] asks whether the converse is true. In general, there is an obstruction given by the class of a 2-cocycle.

Corollary 2.2. — *We keep the above notations.*

(a) *$M = Fe$ is an irreducible differential module over k.*

(b) *Let σ denote the non-trivial element of the Galois group of K/k. Then the twisted differential module ${}^{\sigma}M$ over K is isomorphic to M.*

(c) *The K-differential module M does not descend to k.*

(d) *Let $\widehat{k} = \mathbf{R}((x))$ and $\widehat{K} = \mathbf{C}((x))$. The $\widehat{K}$-differential module $\widehat{K} \otimes_K M$ descends to $\widehat{k}$.*

Proof

(a) Suppose that M is reducible as a K-differential module. Let $N \subset M$ be a 1-dimensional K-submodule. Then jN is also a 1-dimensional K-submodule and $M = N + jN$. In particular, M is semi-simple as K-differential module. If M is irreducible as K-differential module, then M is semi-simple, too. According to [**H-P**], proposition 2.7, M is also semi-simple as k-differential module. Since $\mathrm{End}_{k[\partial]}(M)$ is a skew field, one has that M is irreducible as k-differential module.

(b) The map $\Phi(\sigma) : M \to M$, given by $fe \mapsto jfe$, is a σ-linear bijection commuting with ∂. This proves the statement.

(c) Since $\Phi(\sigma)\Phi(\sigma) = -1$, the 2-cocycle class in $H^2(\{1, \sigma\}, \mathbf{C}^*)$, associated to M, is not trivial. It follows from [**H-P**], theorem 2.8, that the K-differential module M does not descend to k.

(d) Put $\widehat{M} = \widehat{K} \otimes_K M$. The twisted $\widehat{K}$-module ${}^{\sigma}\widehat{M}$ is isomorphic to $\widehat{M}$. According to [**H-P**], theorem 2.4, $\widehat{M}$ descends to $\widehat{k}$. □

Explicit calculations. — The element e of the K-differential module $M = Fe$ is a cyclic vector. The minimal monic operator $L_2 \in K[\partial]$ with $L_2 e = 0$ can be calculated to be

$$\delta^2 + \delta + (1 + x^{-2} - i).$$

Here, we prefer to write $\delta = xd/dx$ instead of ∂, since the latter may be confusing. Note that $\delta x = x\delta + x$. The corollary translates into the following:

L_2 is equivalent to its conjugate $\sigma(L_2) = \delta^2 + \delta + (1 + x^{-2} + i)$. In fact, one has

$$\sigma(L_2)x(\delta - i) = x(\delta - i - 2)L_2.$$

Furthermore, L_2 is not equivalent over K to an operator in $k[\delta]$.

Finally, L_2 is over $\widehat{K}$, equivalent to the operator $\delta^2 + \delta + x^{-2}$ in $k[\delta]$.

Now, we apply the methods of [**M-R**] (see also [**P-S2**]) in order to obtain the formal classification, the formal differential Galois group G_{formal}, the monodromy group and the differential Galois group G of the 2-dimensional differential module M over $K = \mathbf{C}(\{x\})$.

(i) $\widehat{M} := \widehat{K} \otimes_K M$ has a basis e_1, e_2 such that $\delta e_j = q_j e_j$ for $j = 1, 2$ and with $q_1 = ix^{-1}$, $q_2 = -ix^{-1}$. This is the formal classification. One observes that $\sigma(q_1) = q_2$. This implies what we know already, namely that $\widehat{M}$ descends to $\widehat{k} = \mathbf{R}((x))$.

(ii) The formal differential Galois group, *i.e.*, the differential Galois group of $\widehat{M}$ over $\widehat{K}$, is the group

$$G_{\text{formal}} = \left\{ \left(\begin{smallmatrix} c & 0 \\ 0 & c^{-1} \end{smallmatrix} \right) \mid c \in \mathbf{C}^* \right\} \cong \mathbf{G}_m.$$

The formal monodromy is the identity.

(iii) The topological monodromy group of $\delta^2 + \delta + (1 + x^{-2} - i)$ can be calculated at the point ∞. The local exponents are $-i$, $i - 1$ and the eigenvalues of the topological monodromy are $e^{2\pi}, e^{-2\pi}$.

(iv) The differential Galois group G is contained in SL_2, since the second exterior power of M corresponds to the equation $\delta - 1$. Since M is irreducible, the group G is an irreducible subgroup of SL_2. According to [**M-R**], the group G is generated, as an algebraic group, by G_{formal} and the Stokes matrices. There are two singular directions $\pm\pi/2$ and two Stokes matrices $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix} \right)$. Since G is irreducible both Stokes matrices are different from 1. One concludes that $G = \mathrm{SL}_2$.

(v) The two Stokes matrices correspond to the singular directions for $q_1 - q_2$ and $q_2 - q_1$. From these one sees that the formal solutions involve Gevrey-1 functions. This is explicitly seen in the calculation of proposition 2.1, where the factor $n!$ occurs as measure for the divergence.

(vi) According to [**M-R**] and [**P-S2**], the product of the two Stokes matrices is conjugated to the topological monodromy. In particular, $2 + ab = e^{2\pi} + e^{-2\pi}$. This confirms again that $a \neq 0 \neq b$.

Moreover, one can convince oneself that the Stokes matrices of the twisted differential module ${}^{\sigma}M$ are $\left(\begin{smallmatrix} 1 & -\overline{b} \\ 0 & 1 \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ -\overline{a} & 1 \end{smallmatrix} \right)$ (where $\overline{a}, \overline{b}$ are the complex conjugates of a, b). The statement ${}^{\sigma}M \cong M$ is equivalent to $b = -\overline{a}$.

(vii) We have shown that the above condition $b = -\overline{a}$ on the Stokes matrices does not imply that M descends to $\mathbf{R}(\{x\})$. The 'explanation' is rather deep. We will use differential modules. Let the differential module N over K be given by the matrix

equation

$$\delta + \begin{pmatrix} 0 & -x^{-1} \\ x^{-1} & 0 \end{pmatrix} \quad \text{with } \delta = x\frac{d}{dx}.$$

Babbitt and Varadarajan have considered the moduli set BV consisting of equivalence classes of pairs (M, ϕ) consisting of a differential module M over K and an isomorphism $\phi : \widehat{K} \otimes_K M \to \widehat{K} \otimes_K N$. This moduli set has the structure of an algebraic variety over $\mathbf{C}$. In this special case $BV \cong \mathbf{A}^2_{\mathbf{C}}$. This is a coarse moduli space. An element $(a, b) \in BV$ corresponds to the two Stokes matrices $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right)$ that multisummation associates with (M, ϕ).

There is also a fine moduli space $\mathbf{M}$, again $\cong \mathbf{A}^2_{\mathbf{C}}$, with a universal family on it, namely

$$\delta + \begin{pmatrix} \alpha & -x^{-1} + \beta \\ x^{-1} + \beta & -\alpha \end{pmatrix} \quad \text{with } (\alpha, \beta) \in \mathbf{M}(\mathbf{C}).$$

$\mathbf{M}$ has an obvious natural structure over $\mathbf{R}$. In this special case, the analytic morphism $\mathrm{St} : \mathbf{M} \to BV$ is explicitly known, namely

$$\mathrm{St}(\alpha, \beta) = if(\alpha^2 + \beta^2) \cdot (\alpha + i\beta, \alpha - i\beta), \quad \text{where } f(t) = 2\,\frac{\sin\sqrt{t}}{\sqrt{t}}.$$

Take $\xi \in \mathbf{M}(\mathbf{C})$. Suppose that ξ is isomorphic to $\overline{\xi}$. Is there a $\eta \in \mathbf{M}(\mathbf{R})$ with ξ equivalent to η? This question translates into: Suppose that $\mathrm{St}(\xi) = \mathrm{St}(\overline{\xi})$. Is there an $\eta \in \mathrm{St}^{-1}(\mathrm{St}(\xi))$ with $\eta \in \mathbf{M}(\mathbf{R})$. One sees that the answer is negative in general.

(viii) The element e is also a cyclic element for M considered as differential module over k. The minimal monic operator $L_4 \in k[\delta]$ with $L_4 e = 0$ is calculated to be:

$$L_4 = \delta^4 + 2\delta^3 + (3 + 2x^{-2})\delta^2 + (2 - 2x^{-2})\delta + (2 + 4x^{-2} + x^{-4}).$$

Clearly, L_2 is an irreducible right hand factor of L_4 in the operator ring $K[\delta]$. Also $\sigma(L_2)$ is an irreducible right hand factor of L_4. Since L_2 and $\sigma(L_2)$ commute, one has in fact $L_4 = \sigma(L_2)L_2$. By the above, L_4 is irreducible in $k[\delta]$. However L_4 has a right hand factor in $\widehat{k}[\delta]$.

Remarks. — The above example can be largely extended as follows. Let F be the skew differential field considered above. On the left vector space $M := F^a$ over F one considers a structure ∂ of F-differential module given by $\partial(v) = v' + vD$, where $v' = (v_1, \ldots, v_a)' := (v_1', \ldots, v_a')$ and where $D \in \mathrm{Matr}(a, F)$. As in [**H-P**], proposition 2.14, for a suitable choice of D, one has:

(a) $\mathrm{End}_{K[\partial]}(M) = \mathbf{C}$.
(b) *The twist* ${}^{\sigma}M$ *is isomorphic to the* K*-differential module* M*.*
(c) $\mathrm{End}_{k[\partial]}(M) = \mathbf{H}$ *and* M *is an irreducible* k*-differential module.*
(d) *The* K*-differential module* M *does not descend to* k*.*
(e) *The* $\widehat{K}$*-differential module* $\widehat{K} \otimes_K M$ *descends to* $\widehat{k}$*.*

Other skew fields over $k = \mathbf{R}(\{x\})$. — It is known that $K = \mathbf{C}(\{x\})$ is a C_1-field. It follows that the Brauer group of K is trivial. As a consequence, K is a splitting field for any skew field F, of finite dimension over its center k. Thus the Brauer group of k is equal to the cohomology group $H^2(\{1,\sigma\}, K^*)$. The map $\mathbf{C}^* \times \mathbf{Z} \times x\mathbf{C}\{x\} \to K^*$, given by $(c, n, f) \mapsto cx^n \exp(f)$ is an isomorphism of $\{1,\sigma\}$-modules. One calculates that $H^2(\{1,\sigma\}, \mathbf{C}^*)$ and $H^2(\{1,\sigma\}, \mathbf{Z})$ are both isomorphic to $\mathbf{Z}/2\mathbf{Z}$. The $\{1,\sigma\}$-module $x\mathbf{C}\{x\}$ is an induced module and its H^2 is 0. Therefore the Brauer group of k is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. The three non-zero elements of this Brauer group correspond to the three distinct skew fields of finite dimension over their center k. One can give them explicitly as follows.

A quaternion algebra with center k is given by a basis b_0, b_1, b_2, b_3 over k and rules of multiplication: $b_0 = 1$; $b_i^2 = A_i \in k$ for $i = 1, 2, 3$; $b_1 b_2 = -b_2 b_1 = b_3$ and thus $A_1 A_2 = -A_3$.

For suitable choices of A_1, A_2 one obtains the three skew fields mentioned above, namely:

(1) $A_1 = -1,\ A_2 = -1$. This is the skew field $\mathbf{H} \otimes_{\mathbf{R}} k$ that we have used.
(2) $A_1 = -1,\ A_2 = x$.
(3) $A_1 = -1,\ A_2 = -x$.

The cases (2) and (3) lead to other examples of differential equations over k having interesting descent properties.

Another family of examples. — F denotes the quaternion field over $k = \mathbf{R}(\{x\})$ with basis $b_0, \ldots, b_3$ and multiplication given by $b_0 = 1$, $b_1 b_2 = -b_2 b_1 = b_3$ and $b_1^2 = -1$, $b_2^2 = x$. The differentiation $'$ on F, extending $x d/dx$ on k, is defined by $b_0' = b_1' = 0$ and $b_2' = 1/2 \cdot b_2$, $b_3' = 1/2 \cdot b_3$. The field $K = \mathbf{C}(\{x\})$ is seen as a subfield of F by the identification of i with b_1.

One considers the differential module $M = Fe$ over F, defined by $\partial e = de$. Later on we will choose $d = b_1 + x^{-1} b_2$. Now M is a differential module over K of dimension 2, with basis $e, b_2 e$. The twisted module ${}^{\sigma}M$ is isomorphic to $M \otimes_K L$, where $L = Kr$ with $\partial r = 1/2 \cdot r$. Indeed, the σ-linear bijection $A(\sigma) : M \to M$, defined by $A(\sigma)m = b_2 m$, has the property $\partial A(\sigma) = A(\sigma)(\partial + 1/2)$. The second symmetric power $N := \mathrm{Sym}^2_K M$ is a differential module of dimension 3 with basis $e \otimes e, b_2 e \otimes e, b_2 e \otimes b_2 e$. The σ-linear bijection $B : N \to N$, with formula $B(\sigma) m_1 \otimes m_2 = x^{-1} b_2 m_1 \otimes b_2 m_2$ commutes with ∂. Further $B(\sigma)B(\sigma)$ is the identity on N. Put $N^o = \{n \in N \mid B(\sigma)n = n\}$. Then one calculates that N^o is a vector space over k with basis

$$n_1^o = e \otimes e + x^{-1} b_2 e \otimes b_2 e,\ \ n_2^0 = b_1 e \otimes e - b_1 x^{-1} b_2 e \otimes b_2 e,\ \ n_3^o = b_2 e \otimes e.$$

Clearly N^o is a differential module over k and $K \otimes_k N^o \cong N$. In other words, N descends to k. For the choice $d = b_1 + x^{-1} b_2$, the minimal monic operator L_3 with

$L_3 n_3^o = 0$ can be calculated:

$$L_3 = \delta^3 - 1/2 \cdot \delta^2 + (4 - 4x^{-1})\delta + 4x^{-1} - 2.$$

Here one has written δ instead of ∂ in order to emphasize that δ is the operator xd/dx.

The main interesting properties of the 3-dimensional differential module N^o over k are:

Proposition 2.3 **(Properties of M and N^o)**

(i) *N^o is not the second symmetric power of a differential module over k.*

(ii) *For a finite field extension $L \supset k$, which is a splitting field of F, the module $L \otimes_k N^o$ is a second symmetric power. This holds in particular for $L = K$ and $L = k(\sqrt{x})$.*

(iii) *$\widehat{k} \otimes_k N^o$ is a second symmetric power.*

(iv) *$\widehat{M} := \widehat{K} \otimes_K M$ descends to $\widehat{k}$.*

Again with $d = b_1 + x^{-1}b_2$, one computes that the monic minimal operator L_2 for the cyclic element e of M (over the differential field K);

$$L_2 = \delta^2 + 1/2 \cdot \delta - x^{-1} + 1 - i/2.$$

In this formula we have again written $\delta := xd/dx$, in order to emphasize that the differentiation on K is given by $f \mapsto xdf/dx$. By construction L_3 is equivalent, but not equal (!), to the second symmetric power of L_2.

(i) states that L_3 is not equivalent over k to the second symmetric power of an operator in $k[\delta]$.

(ii) states that L_3 is equivalent to a second symmetric power in $L[\delta]$ if $L \supset k$ is a splitting field for F.

(iii) states that L_3 is equivalent over $\widehat{k}$ to the second symmetric power of an element in $k[\delta]$.

(iv) states that L_2 is equivalent over $\widehat{K}$ to an operator in $k[\delta]$.

Proof. — For the proof of (i) and (ii), we use methods of [**M-R**] and [**P-S2**]. In particular we determine the formal differential Galois group G_{formal}, the topological monodromy group, the differential Galois G, et cetera of L_2.

(a) G and G_{formal} are contained in $\{A \in \mathrm{GL}_2 \mid \det(A)^2 = 1\}$. (because of the term $1/2 \cdot \delta$ in L_2).

(b) The eigenvalues of the operator L_2 are $\pm x^{-1/2}$. (follows from a computation with the Riccati equation $u' + u^2 + 1/2 \cdot u + (-x^{-1} + 1 - i/2) = 0$).

(c) G_{formal} is the dihedral group generated by $\{\left(\begin{smallmatrix} c & 0 \\ 0 & c^{-1} \end{smallmatrix}\right) \mid c \in \mathbf{C}^*\}$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. (follows from the explicit form $\widehat{K}(x^{1/2}, \exp(2x^{-1/2}))$ of the Picard-Vessiot field).

(d) $G = \{A \in \mathrm{GL}_2 \mid \det(A)^2 = 1\}$.

(The topological monodromy can be calculated at ∞. The formal monodromy is easy to calculate. The product of the formal monodromy and the Stokes matrices is conjugated to the topological monodromy. Hence there are non-trivial Stokes matrices. Finally G, as algebraic group, is generated by G_{formal} and the Stokes matrices. Thus G is the above group).

The test for a differential module Z of dimension 3 to be a second symmetric power, is the existence of a 1-dimensional submodule of $\mathrm{Sym}^2 Z$, which corresponds to non-singular quadratic form having a rational point over the differential field. We refer to [**H-P**] for a proof of this and for the statement that the only candidate in case N^o is the quadratic form corresponding to F. Here one uses that $sym^4_K M$ has no 1-dimensional submodules, which is a consequence of the above calculation of G. Now (i) and (ii) follow.

Let $\widehat{F}$ denote the quaternion field over $\widehat{k}$, given by the same formulas as F. Then $\widehat{M} := \widehat{K} \otimes_K M = \widehat{F}e$ is a differential module over $\widehat{F}$ with the same formula $\partial e = (b_1 + x^{-1}b_2)e$. We will show that there exists an element $f = ue$, with $u \in \widehat{F}$, $u \neq 0$, such that $\partial f = x^{-1}b_2 f$. Assuming this, one considers $\widehat{M}$ again as a $\widehat{K}$-differential module with cyclic vector f. The mimimal monic operator $\widetilde{L}_2 \in \widehat{K}[\delta]$ with $\widetilde{L}_2 f = 0$ is easily calculated, namely

$$\widetilde{L}_2 = \delta^2 + 1/2 \cdot \delta - x^{-1}.$$

This proves (iv). Part (iii) follows at once from (iv), since L_3 is equivalent over $\widehat{k}$ to the second symmetric power of $\widetilde{L}_2$.

The element $u \in \widehat{F}$ that we are looking for must be a non-zero solution of the equation $\partial(ue) = x^{-1}b_2 ue$. This translates into

$$u' + [u, x^{-1}b_2] + ub_1 = 0.$$

Consider the operator $L : \widehat{F} \to \widehat{F}$, given by $L(u) := u' + [u, x^{-1}b_2] + ub_1$. One observes that:

$$L(1) = b_1 \qquad L(b_1 x) = 2b_3 + (b_1 - 1)x$$
$$L(b_2) = 1/2 \cdot b_2 + b_3 \qquad L(b_3) = 1/2 \cdot b_3 + 2b_1 - b_2$$

and for $n \geqslant 1$ one has the formulas

$$L(x^n) = (n + b_1)x^n \qquad L(b_1 x^{n+1}) = 2b_3 x^n + ((n+1)b_1 - 1)x^{n+1}$$
$$L(b_2 x^n) = ((n + 1/2)b_2 + b_3)x^n \qquad L(b_3 x^n) = ((n + 1/2)b_3 + 2b_1 - b_2)x^n.$$

Let u_0 be a non-trivial $\mathbf{R}$-linear combination of $1, b_1 x, b_2, b_3$ such that $L(u_0)$ lies in $\mathbf{R}[[x]]$-module W spanned by $x1, xb_1, xb_2, xb_3$. Let V be the $\mathbf{R}[[x]]$-module spanned by $x, x^2 b_1, xb_2, xb_3$. The formulas show that $L : V \to W$ is bijective. Let $u_1 \in V$ satisfy $L(u_1) = -L(u_0)$. Then $u = u_0 + u_1$ is a non-zero element of $\widehat{F}$ and $L(u) = 0$. □

The method of proposition 2.3 extends and produces a large class of differential equations over k, having interesting descent properties.

3. Descent for q-difference equations

In this section we compare q-difference equations over the fields $\mathbf{R}(\{x\})$ and $\mathbf{C}(\{x\})$. For this purpose one supposes that $q \in \mathbf{R}$ and $0 < |q| < 1$. Let σ denote the complex conjugation of $\mathbf{C}$ and its natural extension to an automorphism of $\mathbf{C}(\{x\})$ over $\mathbf{R}(\{x\})$. The automorphism ϕ on both fields $\mathbf{C}(\{x\})$ and $\mathbf{R}(\{x\})$ is defined by $\phi x = qx$. A q-difference module $M = (M, \Phi)$ over $\mathbf{R}(\{x\})$ or $\mathbf{C}(\{x\})$ is a finite dimensional vector space over that field together with a bijective additive map $\Phi : M \to M$ satisfying $\Phi(fm) = \phi(f)\Phi(m)$ for $m \in M$ and f in the field. For a q-difference module M over $\mathbf{C}(\{x\})$ one defines the twist ${}^{\sigma}M = ({}^{\sigma}M, {}^{\sigma}\Phi)$ as follows:

(i) ${}^{\sigma}M$ is equal to M as additive group, ${}^{\sigma}\Phi$ is equal to Φ and

(ii) ${}^{\sigma}M$ has a new structure as $\mathbf{C}(\{x\})$-vector space given by the formula $f * m := \sigma(f)m$.

***Theorem 3.1* (Descent for q-difference modules).** — *Let q-difference module M over $\mathbf{C}(\{x\})$ satisfy ${}^{\sigma}M \cong M$. There exists a q-difference module N over $\mathbf{R}(\{x\})$ such that $M \cong \mathbf{C} \otimes_{\mathbf{R}} N$.*

Proof. — We sketch here the proof which is based on the work of J.-P. Ramis and J. Sauloy ([**D-R-S-Z**] and a preprint in preparation) concerning moduli spaces for convergent q-difference equations (*i.e.*, q-difference modules over $\mathbf{C}(\{x\})$).

We consider first the case of a regular singular q-difference module M over $\mathbf{C}(\{x\})$. The classification of regular singular modules can be formulated as follows. There is a unique $\mathbf{C}$-vector space $W \subset M$ with the following properties:

(a) the natural map $\mathbf{C}(\{x\}) \otimes_{\mathbf{C}} W \to M$ is a bijection,

(b) $\Phi(W) = W$ and all eigenvalues λ of the restriction ψ of Φ to W satisfy $|q| < |\lambda| \leqslant 1$.

We write $W(M)$ and $\psi(M)$ for the W and ψ above. A morphism $f : M \to N$ between regular singular modules induces a linear map $\widetilde{f} : W(M) \to W(N)$ satisfying $\widetilde{f} \circ \psi(M) = \psi(N) \circ \widetilde{f}$. One obtains in this way a $\mathbf{C}$-linear bijection (with obvious notations):

$$\mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}\big((W(M), \psi(M)), (W(N), \psi(N))\big).$$

The pair (W, ψ) associated to ${}^{\sigma}M$ is equal to $({}^{\sigma}W(M), \psi(M))$, where ${}^{\sigma}W(M)$ is the twist of $W(M)$. More precisely, ${}^{\sigma}W(M)$ coincides with $W(M)$ as an additive group. The new scalar multiplication on ${}^{\sigma}W(M)$ is given by $a * w = \overline{a}w$ (with $a \in \mathbf{C}$, $w \in W(M)$ and $\overline{a}$ the complex conjugate of a). If ${}^{\sigma}M \cong M$, then there exists an additive bijective $A : W(M) \to W(M)$, commuting with $\psi(M)$ and such that $A(aw) = \overline{a}A(w)$ for all $a \in \mathbf{C}$ and $w \in W(M)$. It is not dificult to verify that there exists a real vector space V together with an $\mathbf{R}$-linear automorphism τ such that $\mathbf{C} \otimes_{\mathbf{R}} (V, \tau) \cong (W(M), \psi(M))$. Moreover $N := \mathbf{R}(\{x\}) \otimes_{\mathbf{R}} (V, \tau)$, with its obvious structure of a q-difference module over $\mathbf{R}(\{x\})$, satisfies $\mathbf{C} \otimes_{\mathbf{R}} N \cong M$.

One associates, in a canonical way, to a general q-difference module M a Newton polygon and a decreasing slope filtration $\{M^{\geqslant\mu}\}$ of M by submodules. Each non zero quotient $M^{\geqslant\mu}/M^{>\mu}$ is a pure q-difference module. The latter means that the Newton polygon of this quotient has only one slope, namely μ.

A pure q-difference module such that its slope is an integer has the form $N \otimes E(n)$ with N a regular singular module and $E(n) = \mathbf{C}(\{x\})e$ with $\Phi e = x^n e$ (for a suitable integer n).

In the case of non-integer slopes μ one has to make a few modifications to describe the pure difference modules with slope μ. Let $d \geqslant 1$ denote the denominator of μ. First of all, one extends ϕ to an automorphism of $\mathbf{C}(\{x^{1/d}\})$ by $\phi(x^{1/d}) = q^{1/d}x^{1/d}$ for a suitable choice of $q^{1/d}$. A coherent choice can be made as follows. Choose an element τ in the upper half plane with $e^{2\pi i\tau} = q$. Then $q^{1/d} := e^{2\pi i\tau/d}$. One defines the module $E(\mu)^+ := \mathbf{C}(\{x^{1/d}\})e$ over $\mathbf{C}(\{x^{1/d}\})$ by $\Phi e = x^\mu e$. Let $E(\mu)$ denote $E(\mu)^+$, considered as a q-difference module of dimension d over the field $\mathbf{C}(\{x\})$. One can show that $E(\mu)$ does not depend on the choice of $q^{1/d}$. Moreover, one can show that any pure q-difference module with slope μ has uniquely the form $E(\mu)\otimes N$ where N is a regular singular module.

One associates to M the graded module $\mathrm{gr}(M) := \oplus_\mu M^{\geqslant\mu}/M^{>\mu}$. This is a direct sum of pure modules. If one works over the field of formal Laurent series, *i.e.*, the field $\mathbf{C}((x))$, then M is isomorphic to $\mathrm{gr}(M)$. In the convergent situation one can only say that M is a multiple extension of the pure modules ocurring in $\mathrm{gr}(M)$ (and taken in the correct order). The equivalence classes of the multiple extension are the $\mathbf{C}$-valued points of a finite dimensional moduli variety over $\mathbf{C}$, which we will call $\mathrm{Ext}(\mathrm{gr}(M))$. In fact, $\mathrm{Ext}(\mathrm{gr}(M))$ is an affine space over $\mathbf{C}$. In the simplest situation, one considers extensions of two pure modules M_1, M_2. Then $\mathrm{Ext}(\mathrm{gr}(M))$ is just $\mathrm{Ext}^1(M_1, M_2)$ (where M_1, M_2 are interpreted as left modules over $\mathbf{C}(\{x\})[\Phi, \Phi^{-1}]$). This is a vector space of finite dimension over $\mathbf{C}$. In short, the q-difference module M corresponds to a pair $(\mathrm{gr}(M), \xi)$ with ξ a (closed) point of $\mathrm{Ext}(\mathrm{gr}(M))$.

Now we return to the proof of the theorem. Let M be given and suppose that ${}^\sigma M \cong M$. This isomorphism is provided by an additive bijective map $A : M \to M$, which commutes with Φ and is semi-linear, *i.e.*, $A(fm) = \sigma(f)A(m)$ for all $m \in M$ and $f \in \mathbf{C}(\{x\})$. The map A preserves the canonical filtration and induces a semi-linear bijection $\mathrm{gr}(A) : \mathrm{gr}(M) \to \mathrm{gr}(M)$, commuting with Φ. Each $E(\mu)$, as defined above, is obviously of the form $\mathbf{C} \otimes_{\mathbf{R}} E_{\mathbf{R}}(\mu)$, where $E_{\mathbf{R}}(\mu)$ is a q-difference module over $\mathbf{R}(\{x\})$. Using the above results for regular singular q-difference modules one concludes that $\mathrm{gr}(M)$ is equal to $\mathbf{C}\otimes_{\mathbf{R}}\mathrm{gr}(M)_{\mathbf{R}}$, where $\mathrm{gr}(M)_{\mathbf{R}}$ is a direct sum of pure modules over $\mathbf{R}(\{x\})$. The moduli variety $\mathrm{Ext}(\mathrm{gr}(M))$ is the complexification of a real affine variety $\mathrm{Ext}(\mathrm{gr}(M)_{\mathbf{R}})$. The closed point ξ of $\mathrm{Ext}(\mathrm{gr}(M))$ is invariant under complex conjugation and comes therefore from a real point η of $\mathrm{Ext}(\mathrm{gr}(M)_{\mathbf{R}})$. The q-difference module N over $\mathbf{R}(\{x\})$, given by the pair $(\mathrm{gr}(M)_{\mathbf{R}}, \eta)$, has by construction the property $\mathbf{C} \otimes_{\mathbf{R}} N \cong M$. □

Remarks 3.2

(1) Descent of q-difference modules over the field $\mathbf{C}(x)$ (again with $q \in \mathbf{R}$ and $0 < |q| < 1$) does not hold. Indeed, one introduces the skew difference field $F := \mathbf{H} \otimes_{\mathbf{R}} \mathbf{R}(x)$ provided with the operator ϕ given by $\phi(x) = qx$ and ϕ is the identity on $\mathbf{H}$. One considers the one dimensional skew q-difference module Fe given by $\Phi(e) = (i + jx)e$. Let $N = Fe$, viewed as a 4-dimensional q-difference module over $\mathbf{R}(x)$. The essential step is to verify that $\mathrm{End}_{\mathbf{R}(x)[\Phi,\Phi^{-1}]}(N) = \mathbf{H}$. Assuming this, one defines $M = Fe$, viewed as a q-difference module over $\mathbf{C}(x)$ of dimension 2. A variation on the proof of corollary 2.2 will show that M is irreducible, ${}^{\sigma}M \cong M$ and M does not descend to $\mathbf{R}(x)$. The element e is a cyclic vector for M. The scalar q-difference equation corresponding to e can be calculated to be

$$y(q^2x) + i(q-1)y(qx) + (1 + q^2x^2 + q - 1)y(x) = 0 \text{ with } q \in \mathbf{R}, 0 < |q| < 1.$$

This equation is irreducible and equivalent to its complex conjugate. However, the equation is not equivalent to an equation over $\mathbf{R}(x)$.

For the verification of $\mathrm{End}_{\mathbf{R}(x)[\Phi,\Phi^{-1}]}(N) = \mathbf{H}$, one takes an $\mathbf{R}(x)$-linear map $L : N \to N$, which has the form $L(v) = va_0 + iva_1 + jva_2 + kva_3$ (where $v \in Fe = F$ and $a_0, \ldots, a_3 \in F$). The equation $\Phi L = L\Phi$ implies that each a_i satisfies the equation $\phi(a)(i + jx) = (i + jx)a$. The real vector space $V := \{a \in F | \phi(a)(i + jx) = (i + jx)a\}$ has dimension $\leqslant 4$, since it is the set of solutions of a 4-dimensional q-difference equation. Moreover, V is a subalgebra of F. Take a non zero $a \in V$ and let $P \in \mathbf{R}[x]$ be the monic polynomial of minimal degree such that $P \cdot a \in \mathbf{H} \otimes \mathbf{R}[x]$. The equality $\phi(a)d = da$ implies that $(1 + x^2)\phi(P) \cdot a \in \mathbf{H} \otimes \mathbf{R}[x]$. A small computation shows that P must be a power of x. Therefore $a \in \mathbf{H} \otimes \mathbf{R}[x, x^{-1}]$. Write $a = \sum_{i=s}^{t} a_i x^i$ with $s, t \in \mathbf{Z},\ s \leqslant t$ and all $a_i \in \mathbf{H}$. Observing that all powers of a lie in V and that V is finite dimensional, one concludes that $a \in \mathbf{H}$. Now a commutes with $(i + jx)$ and this leads to $a \in \mathbf{R}$. The conclusion is that $L \in \mathbf{H}$, as required.

(2) The reason for the rather striking difference between theorem 3.1 and the examples in section 2 lies in the structure of the moduli space. In the first case, this moduli space has (in the situation that we are interested in) a natural real structure. In the second case, the moduli space is a coarse one.

4. Descent for ordinary difference equations

We consider the difference fields $F(\{x^{-1}\})$ and $F((x^{-1}))$, where F is either $\mathbf{R}$ or $\mathbf{C}$ and with the automorphism ϕ given by $\phi(x) = x + 1$. Let M be a difference module over $\mathbf{C}(\{x^{-1}\})$ such that its twist ${}^{\sigma}M$, where σ is the complex conjugation extended to $\mathbf{C}(\{x^{-1}\})$ in the obvious way, is isomorphic to M. The descent problem is to determine whether M is isomorphic to $\mathbf{C} \otimes_{\mathbf{R}} N$ for some difference module over $\mathbf{R}(\{x^{-1}\})$.

The meromorphic classification of ordinary difference modules is rather complicated. We will restrict ourselves to the class of the *regular difference modules*. We recall from [**P-S1**] some definitions and results. A difference module $M = (M, \Phi)$ is called regular singular if there is a $\mathbf{C}\{x^{-1}\}$-lattice $\Lambda \subset M$ that is invariant under Φ and moreover Φ is the identity on $\Lambda/x^{-1}\Lambda$. Equivalently, M is represented by a matrix difference equation $y(x+1) = Ay(x)$ with A a convergent matrix of the form $1 + A_2x^{-2} + \cdots$. Furthermore, M is regular if and only if $\mathbf{C}((x^{-1})) \otimes M$ is a trivial difference module over $\mathbf{C}((x^{-1}))$. The class of all regular difference modules forms a Tannakian category.

Theorem 4.1. — *Descent does not hold for the category of the regular difference modules over* $\mathbf{C}(\{x^{-1}\})$.

For the construction of an example showing that descent does not hold, we replace the category of the regular difference modules by an equivalent Tannakian category. An object of this Tannakian category is a triple $(V, T_{\text{upper}}, T_{\text{lower}})$ where:

(a) V is a complex vector space of finite dimension,

(b) $T_{\text{upper}} = T_{\text{upper}}(u)$ is a $\mathbf{C}\{u\}$-linear automorphism of $\mathbf{C}\{u\} \otimes_{\mathbf{C}} V$ such that $T_{\text{upper}}(0) = 1$ and,

(c) $T_{\text{lower}} = T_{\text{lower}}(u^{-1})$ is a $\mathbf{C}\{u^{-1}\}$-linear automorphism of $\mathbf{C}\{u^{-1}\} \otimes_{\mathbf{C}} V$ such that $T_{\text{lower}}(0) = 1$.

We note that the symbol u stands for $e^{2\pi ix}$.

A morphism $f : (V, T_{\text{upper}}, T_{\text{lower}}) \to (V', T'_{\text{upper}}, T'_{\text{lower}})$ is a linear map $f : V \to V'$ satisfying $f \circ T_{\text{upper}} = T'_{\text{upper}} \circ f$ and $f \circ T_{\text{lower}} = T'_{\text{lower}} \circ f$.

There is a summation method for regular difference modules. The right summation F_{right} and the left summation F_{left} of a formal fundamental matrix F for M are compared by considering $F^{-1}_{\text{right}} F_{\text{left}}$. This matrix exists on an upper half plane and on a lower half plane and yields convergent matrices in the variables u and u^{-1}. These two matrices are the T_{upper} and T_{lower} above. The term V of the triple is the formal solution space of M (or of the matrix equation $y(x+1) = Ay(x)$). This describes the functor $M \mapsto (V, T_{\text{upper}}, T_{\text{lower}})$ from the category of the regular difference equations to the category of the triples. In subsections 8.5 and 10.1 of [**P-S1**] details and explicit calculations are given.

For a vector space V over $\mathbf{C}$, we write (as before) ${}^{\sigma}V$ for the twisted vector space. This means that ${}^{\sigma}V = V$ as additive group and the new scalar multiplication $*$ on ${}^{\sigma}V$ is given by $\lambda * v = \sigma(\lambda)v$ (with $\lambda \in \mathbf{C}$, $\sigma(\lambda) = \overline{\lambda} =$ the complex conjugate of λ and $v \in V$). A linear map $F : V \to W$ can also be considered as a map ${}^{\sigma}V \to {}^{\sigma}W$. This map is again linear and will be denoted by ${}^{\sigma}F$. A linear map $A : {}^{\sigma}V_1 \to V_2$ can also be considered as a semi-linear map $A : V_1 \to V_2$, *i.e.*, A is additive and $A(\lambda v) = \overline{\lambda} v$ for $\lambda \in \mathbf{C}$ and $v \in V_1$. A *real structure* for a complex vector space V is a real subspace $W \subset V$ such that the natural map $\mathbf{C} \otimes_{\mathbf{R}} W \to V$ is an isomorphism.

Suppose that a real structure for V is given. Then one defines complex conjugation on V by $\overline{w_1+iw_2}=w_1-iw_2$ for all $w_1,w_2\in W$. Let $F:V\to V$ be a linear map and suppose that a real structure for V is given. Then one defines a linear map $\overline{F}$, called the complex conjugate of F, by the formula $\overline{F}(w)=\overline{F(w)}$ for all $w\in W$.

Let M correspond to the triple $(V,T_{\rm upper}(u),T_{\rm lower}(u^{-1}))$. Using the explicit formulation of subsection 10.1 of [**P-S1**], one can see that the triple corresponding to ${}^{\sigma}M$ is

$$({}^{\sigma}V,{}^{\sigma}T_{\rm lower}(u),{}^{\sigma}T_{\rm upper}(u^{-1})).$$

We will write $T_{\rm upper}=1+\sum_{n\geqslant 1}T_{{\rm upper},n}u^n$ and $T_{\rm lower}=1+\sum_{n\geqslant 1}T_{{\rm lower},n}u^{-n}$.

Suppose that M descends to $\mathbf{R}(\{x^{-1}\})$ and so $M=\mathbf{C}\otimes_{\mathbf{R}}N$ for some regular difference module N over $\mathbf{R}(\{x^{-1}\})$. Then the solution space V has a natural real structure, namely $V=\mathbf{C}\otimes_{\mathbf{R}}W$, where W is the real vector space $\ker(\Phi-1,\mathbf{R}((x^{-1}))\otimes N)$. For this real structure one has that $\overline{T_{{\rm upper},n}}=T_{{\rm lower},n}$ for all $n\geqslant 1$. Equivalently, for each $n\geqslant 1$, the maps $T_{{\rm upper},n}+T_{{\rm lower},n}$ and $iT_{{\rm upper},n}-iT_{{\rm lower},n}$ are real (*i.e.*, they are invariant under conjugation).

On the other hand let be given a triple $(V,T_{\rm upper},T_{\rm lower})$ together with a real structure for V such that $T_{{\rm upper},n}+T_{{\rm lower},n}$ and $iT_{{\rm upper},n}-iT_{{\rm lower},n}$ are real for all $n\geqslant 1$, then the corresponding regular difference module M descends to $\mathbf{R}(\{x^{-1}\})$.

By the above considerations, theorem 4.1 will be proven if we can produce an example of a triple $(V,T_{\rm upper},T_{\rm lower})$ and an isomorphism

$$A:({}^{\sigma}V,{}^{\sigma}T_{\rm lower}(u),{}^{\sigma}T_{\rm upper}(u^{-1}))\longrightarrow(V,T_{\rm upper}(u),T_{\rm lower}(u^{-1}))$$

such that V does not have a real structure for which all maps $T_{{\rm upper},n}+T_{{\rm lower},n}$ and $iT_{{\rm upper},n}-iT_{{\rm lower},n}$ are real.

We prefer to see the map A as a semi-linear isomorphism satisfying $AT_{{\rm upper},n}=T_{{\rm lower},n}A$ and $AT_{{\rm lower},n}=T_{{\rm upper},n}A$ for all $n\geqslant 1$. Since $Ai=-iA$, the latter conditions can also be formulated as A commutes with $T_{{\rm upper},n}+T_{{\rm lower},n}$ and $iT_{{\rm upper},n}-iT_{{\rm lower},n}$ for all $n\geqslant 1$. The example is now constructed as follows:

$\mathbf{H}$ denotes the skew field of the Hamilton quaternions over $\mathbf{R}$. Let $V=\mathbf{H}e$ be a 1-dimensional left vector space over $\mathbf{H}$. In particular, V is a 2-dimensional vector space over $\mathbf{C}$. The semi-linear map $A:V\to V$ is given by $A(he)=jhe$. Consider two sequences $\{A_n\}_{n\geqslant 1}$ and $\{B_n\}_{n\geqslant 1}$ of $\mathbf{H}$-linear maps from V to itself, with bounded norms. Define the $\mathbf{C}$-linear maps $T_{{\rm upper},n},T_{{\rm lower},n}$ by the formulas $A_n=T_{{\rm upper},n}+T_{{\rm lower},n}$ and $B_n=iT_{{\rm upper},n}-iT_{{\rm lower},n}$. Put $T_{\rm upper}=1+\sum_{n\geqslant 1}T_{{\rm upper},n}u^n$ and $T_{\rm lower}=1+\sum_{n\geqslant 1}T_{{\rm lower},n}u^{-n}$. Then A is an isomorphism

$$({}^{\sigma}V,{}^{\sigma}T_{\rm lower}(u),{}^{\sigma}T_{\rm upper}(u^{-1}))\longrightarrow(V,T_{\rm upper}(u),T_{\rm lower}(u^{-1})).$$

Suppose that $W\subset V$ is a real structure such that all A_n,B_n are real. We may suppose that W contains e. Then W contains $\mathbf{R}[S]e$ for $S=A_n$ and $S=B_n$. We note that $\mathbf{R}[S]$ is a commutative subfield of $\mathbf{H}$. We may have chosen the A_n,B_n such that the

fields $\mathbf{R}[S]$ with $S = A_n$ or $S = B_n$ are distinct maximal commutative subfields of $\mathbf{H}$. We conclude that W does not exist.

From the construction of this example for theorem 4.1 one can guess the explicit form of the difference module. Let $F := \mathbf{H} \otimes_{\mathbf{R}} \mathbf{R}(\{x^{-1}\})$ denote the skew difference field with ϕ-action defined by $\phi(x) = x + 1$. The left vector space Fe is made into a skew difference module by $\Phi e = de$ for a suitable $d \in F$. One takes a d of the form $d = 1 + d_2x^{-2} + d_3x^{-3} + \cdots$. Then $M := Fe$, viewed as a difference module over $\mathbf{C}(\{x^{-1}\})$ of dimension 2, is regular and moreover ${}^{\sigma}M \cong M$. We follow the method explained in Remarks 3.2 part (1). In order to show that M does not descend to $\mathbf{R}(\{x^{-1}\})$, one has to consider $N := Fe$ as a difference module over $\mathbf{R}(\{x^{-1}\})$ of dimension 4. The essential step is to show that the $\mathbf{R}$-algebra of endomorphism of the difference module N is equal to $\mathbf{H}$. This amounts to showing that the only solutions $a \in F$ of the equation $\phi(a)d = da$ are the $a \in \mathbf{R}$. We note that there are more solutions in $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{R}((x^{-1}))$. For a specific example, say $d = 1 + ix^{-2} + jx^{-3}$, one has to verify that the only *convergent* solutions a of $\phi(a)d = da$ are the $a \in \mathbf{R}$. It seems rather difficult to prove this by direct computation.

References

[Bl] A. Blanchard – *Les corps non commutatifs*, Collection SUP, vol. 9, Presses universitaires de France, 1972.

[D-R-S-Z] L. Di Vizio, J.-P. Ramis, J. Sauloy & C. Zhang – "Équations aux q-différences", *Gaz. Math.* **96** (2003), et laboratoire Émile Picard, U.F.R. M.I.G. Toulouse, (2003).

[H-P] M. van Hoeij & M. van der Put – "Descent for differential modules and skew fields", Preprint, September 2002.

[M-R] J. Martinet & J.-P. Ramis – "Elementary acceleration and multisummability", *Ann. Inst. H. Poincaré. Phys. Théor.* **54** (1991), no. 4, p. 331–401.

[P-S1] M. van der Put & M.F. Singer – *Galois theory of difference equations*, Lect. Notes in Math., vol. 1666, Springer Verlag, 1997.

[P-S2] ———, *Galois theory of linear differential equations*, Ergebnisse der Mathematik, Springer Verlag, 2003.

[Ren] G. Renault – *Algèbre non commutative*, Gauthier-Villars, Paris, Bruxelles, Montreal, 1975.

[Se] J.-P. Serre – *Corps locaux*, Hermann, Paris, 1968.

M. van der Put, Department of mathematics, University of Groningen, P.O. Box 800, 9700 AV Groningen (The Netherlands) • *E-mail :* mvdput@math.rug.nl

Astérisque
296, 2004, p. 207–225

KRICHEVER MODULES FOR DIFFERENCE AND DIFFERENTIAL EQUATIONS

by

Marius van der Put & Marc Reversat

Dedicated to Jean-Pierre Ramis on the occasion of his 60th birthday

***Abstract.* —** In order to understand an analogue of Kronecker-Weber and some abelian Langlands theory for function fields, an explicit comparison between Krichever differential (or difference) modules of rank one and abelian differential (or difference) equations is carried out.

***Résumé* (Modules de Krichever pour les équations aux différences et différentielles)**
On compare explicitement les modules de Krichever différentiels ou aux différences de rang 1 aux extensions abéliennes différentielles ou aux différences. Ceci permet de donner, dans ces situations, un analogue du théorème de Kronecker-Weber.

Introduction

The observation that the "elliptic module" of Drinfel'd has an analogue for difference and differential operators was made around 1977 by Krichever, Drinfel'd, Mumford et al. (see [**Dr, Kr, Mu, L1**]). Geometric Langlands theory (see [**L2, F-G-K-V, F-G-V**]) for differential equations is developed on the basis of this observation. In this paper we investigate some rather basic questions for the analogues of Drinfel'd modules for difference and differential equations. The main question is the relation between abelian differential (resp. difference) equations and Krichever differential (resp. difference) modules of rank one. By *abelian differential* (resp. *difference*) *equation* we mean a differential (resp. difference) module such that its differential (resp. difference) Galois group is an abelian (linear algebraic) group. For this purpose, explicit calculations of the universal Picard-Vessiot ring and the universal differential (resp. difference) Galois group for abelian differential (resp. difference) equations are carried out. In other words, we investigate Langlands theory for function fields where

2000 ***Mathematics Subject Classification.* —** 12H, 13N, 11R, 11F.
***Key words and phrases.* —** Krichever module, Class field theory, Differential, Difference.

© Astérisque 296, SMF 2004

the Frobenius endomorphism is replaced by a differential operator or a difference operator. This is seen as a first step towards GL_1-theory for Krichever modules. The differential field extension defined by the "torsion" of a rank one Krichever differential module has a vague resemblance to the torsion of rank one Drinfel'd modules. In contrast to this, the rank one Krichever difference modules provide abelian extensions in perfect analogy with rank one Drinfel'd modules. For the theory of (q-)difference equations we refer to [**P-S**] and the recent survey [**D-R-S-Z**].

1. Abelian differential equations

Let K be a differential field. We suppose that the field of constants C of K is algebraically closed and has characteristic 0. The Tannakian category of all differential modules over K is called Diff_K. One associates to Diff_K a universal differential Picard-Vessiot ring Univ_K. This is the direct limit of Picard-Vessiot rings for all linear differential equations over K. The universal differential Galois group $G_{\mathrm{univ},K}$ is the group of the differential automorphisms of Univ_K/K. This is an affine group scheme. For a full Tannakian subcategory $\mathcal{C}$ of Diff_K, there is also a universal Picard-Vessiot ring $\mathrm{Univ}_{\mathcal{C}}$ and a universal differential Galois group $G_{\mathcal{C}}$. From the Tannakian formulation of differential Galois theory it is clear that $\mathrm{Univ}_{\mathcal{C}}$ is a subring of Univ_K and that $G_{\mathcal{C}}$ is a quotient of $G_{\mathrm{univ},K}$. In this section, we present explicit calculations of $\mathrm{Univ}_{\mathcal{C}}$ and $G_{\mathcal{C}}$ for the case when $\mathcal{C}$ consists of the differential modules for which the differential Galois group is commutative. We write, in this situation, $G_{\mathrm{univ},K,\mathrm{ab}}$ for $G_{\mathcal{C}}$ and this group scheme is equal to $(G_{\mathrm{univ},K})_{\mathrm{ab}}$, the abelianized group scheme obtained from $G_{\mathrm{univ},K}$.

Lemma 1.1. — *Let M be a differential module over K with a differential Galois group G that is the product of two (algebraic) groups H_1 and H_2. Let R be the Picard-Vessiot ring of M. Then $R = R_1 \otimes_K R_2$ where R_i, $i = 1, 2$ is the Picard-Vessiot ring for a differential module M_i over K with differential Galois group H_i.*

Proof. — Let L be the field of fractions of R and put $L_1 = L^{H_2}$, $L_2 = L^{H_1}$. By Galois correspondence L_i is the Picard-Vessiot field of some differential module M_i over K with differential Galois group H_i. Let $R_i \subset L_i$ denote the K-subalgebra of L_i consisting of the elements $f \in L_i$ such that the K-vector subspace of L_i, generated by f and all its derivatives is finite dimensional. Then R_i is the Picard-Vessiot ring of M_i (compare [**P-S03**], Corollary 1.38). One considers the natural map

$$\psi : R_1 \otimes_K R_2 \longrightarrow L_1 \otimes_K L_2 \xrightarrow{\alpha} L.$$

This is a morphism of differential algebras over K. Moreover, ψ is equivariant for the action of $H_1 \times H_2$. Let $L_3 \subset L$ denote the field of fractions of the image of α. This is the Picard-Vessiot field for the differential module $L_1 \otimes_K M_2$ over L_1 since (compare [**P-S03**], Proposition 1.22):

(a) L_3 is generated over L_1 by the entries of a fundamental matrix for the differential module $L_1 \otimes M_2$ over L_1.

(b) the field of constants of L_3 is C.

Let H_3 denote the group of the differential automorphisms of L_3/L_1. The embedding $L_3 \subset L$ induces a morphism $H_2 \to H_3$. From the equalities $L_3^{H_3} = L_1$, $L^{H_2} = L_1$ and the Galois correspondence of differential Galois theory one deduces that $L_3 = L$ and $H_2 \to H_3$ is a bijection. The Picard-Vessiot ring R_3 of $L_1 \otimes_K M_2$ over L_1 is equal to the image $L_1 \otimes_K R_2/I$ of $L_1 \otimes_K R_2$ in L_3. In particular I is a maximal differential ideal of $L_1 \otimes_K R_2$. The group of differential automorphisms of R_2/K is H_2. If the ideal I is not zero, then the group H_3 of the differential automorphisms of $L_1 \otimes_K R_2/I$ is a proper subgroup of H_2. This contradicts the equality $H_2 = H_3$. We conclude that ψ is injective and that L is the field of fractions of the image of ψ. Any element f in the image of ψ has the property that the K-vector space generated by f and all its derivatives is finite dimensional. Thus, we have obtained an injective $\psi : R_1 \otimes_K R_2 \to R$ which induces a bijection for the corresponding fields of fractions. We will regard ψ as an inclusion $R_1 \otimes_K R_2 \subset R$. Consider an element $f \in R$. Let W be the K-vector space generated by f and all its derivatives. The dimension of W over K is finite. Define the ideal $J \subset R_1 \otimes_K R_2$ to consist of the elements $g \in R_1 \otimes_K R_2$ satisfying $gW \subset R_1 \otimes_K R_2$. Then, J is a non zero differential ideal and hence, equal to $R_1 \otimes_K R_2$. Thus, $R_1 \otimes_K R_2 = R$ □

Corollary 1.2. — *The universal Picard-Vessiot ring* $\mathrm{Univ}_{K,\mathrm{ab}}$ *for the category of the differential modules with abelian differential Galois group has the form* $K[\{s(a)_{a\in K}\}, \{y(b)\}_{b\in K}]$ *where the "symbols"* $s(a), y(b)$ *satisfy:*

$$\begin{aligned} s(0) = 1, \quad & s(a_1)s(a_2) = s(a_1 + a_2), \\ y(0) = 0, \quad & y(b_1) + y(b_2) = y(b_1 + b_2), \\ s(a)' = as(a), \quad & y(b)' = b. \end{aligned}$$

The relations between the symbols depend on the field K*. They are generated by:*

$$\begin{aligned} s(f'/f) = c(f)f \quad & \textit{for } f \in K^* \textit{ and suitable } c(f) \in C^*; \\ y(f') = f + d(f) \quad & \textit{for suitable } d(f) \in C. \end{aligned}$$

Proof. — Any abelian linear algebraic group G (over C) is a finite product of copies of the groups $\mathbf{G}_m$ (the multiplicative group), $\mathbf{G}_a$ (the additive group) and finite cyclic groups. According to the lemma we have only to consider differential modules M over K with differential Galois group in $\{\mathbf{G}_m, \mathbf{Z}/n\mathbf{Z}, \mathbf{G}_a\}$. For the first two classes of groups, the differential module is direct sum of 1-dimensional modules. Indeed, the solution space V of M, is a direct sum of 1-dimensional spaces, invariant under the differential Galois group. Hence, for every equation $y' = ay$ (with $a \in K$) the universal Picard-Vessiot ring $\mathrm{Univ}_{K,\mathrm{ab}}$ should contain a non zero solution which we call $s(a)$. Let M be an indecomposable differential module of dimension m having

differential Galois group $\mathbf{G}_a$. The corresponding solution space V of dimension m over C has an indecomposable $\mathbf{G}_a$-action. This action has the form $t \mapsto \exp(tN)$, where $N : V \to V$ is a nilpotent map such that $N^{m-1} \neq 0$ and $N^m = 0$. Let $V_i = \ker(N^i)$ for $i = 1, 2, \ldots$. There are submodules $M_i \subset M$ corresponding to the V_i. One considers the 2-dimensional M_2 with its trivial 1-dimensional subspace. This produces an inhomogenous equation $y' = b$ for some $b \in K$. Let $L \supset K$ be the Picard-Vessiot field for M. Then there is a differential subfield $L_2 = K(y(b))$ of L, with $y(b)' = b$. This is the Picard-Vessiot field of M_2. Since L_2/K and L/K have the same differential Galois group, namely $\mathbf{G}_a$, one has (by the Galois correspondence for differential fields) that $L_2 = L$. Therefore, $\mathrm{Univ}_{K,\mathrm{ab}}$ must contain an element $y(b)$ with $y(b)' = b$ for every $b \in K$. Moreover, $\mathrm{Univ}_{K,\mathrm{ab}}$ is generated over K by the elements $\{s(a)\}_{a\in K}$ and $\{y(b)\}_{b\in K}$. One can normalize the $y(b)$, such that $y(0) = 0$ and $y(b_1 + b_2) = y(b_1) + y(b_2)$ for all $b_1, b_2 \in K$. This is done by considering a basis $\{B_i\}_{i\in I}$ of K as vector space over $\mathbf{Q}$. One chooses symbols $y(B_i)$ for every $i \in I$. Any b is a finite sum $\sum_i \lambda_i B_i$ (with all $\lambda_i \in \mathbf{Q}$. Define now $y(b) := \sum_i \lambda_i y(B_i)$.

For every integer $n \geqslant 1$ and every $i \in I$ we choose an invertible symbol $s(\frac{1}{n!}B_i)$. This can be done such that $s(\frac{1}{n!}B_i)^n = s(\frac{1}{(n-1)!}B_i)$ for all $i \in I$ and $n \geqslant 2$. An arbitrary element $a \in K$ can be written as a finite sum $\sum_i \frac{t_i}{(n_i)!}B_i$ with $t_i \in \mathbf{Z}$ (almost all equal to zero) and $n_i \geqslant 1$. One defines $s(a) := \prod_i s(\frac{1}{(n_i)!}B_i)^{t_i}$. Clearly $s(0) = 1$ and $s(a_1 + a_2) = s(a_1)s(a_2)$ for all s_1, s_2.

Relations between symbols clearly depend on the field K. Between the symbols $\{s(a)\}$ and the symbols $\{y(b)\}$ there are no relations since the corresponding differential equations $y' = ay$ and $y' = b$ have unrelated differential Galois groups (namely a subgroup of $\mathbf{G}_m$ and $\mathbf{G}_a$). The symbols $\{y(b)\}$ form a group for the addition. Hence it suffices to introduce a relation $y(b) = f$ with $f \in K$ whenever the equation $y' = b$ has a solution in K. Consider the exact sequence $0 \to C \to K \to K' \to 0$, where $K' := \{f' \mid f \in K\}$ and the second map is $f \mapsto f'$. This sequence splits and we fix an additive (or even C-linear) map $\gamma : K' \to K$ satisfying $\gamma(b)' = b$ for every $b \in K'$. Then the relations that we want are $y(b) = \gamma(b)$ for every $b \in K'$.

Consider the exact sequence

$$1 \longrightarrow C^* \longrightarrow K^* \longrightarrow K^*/C^* \longrightarrow 1.$$

This exact sequence splits since the group C^* is divisible. One identifies K^*/C^* with the group $\{f'/f \mid f \in K^*\}$. Let $\delta : \{f'/f \mid f \in K^*\} \to K^*$ be a homomorphism of groups satisfying $\delta(a)'/\delta(a) = a$. Then the relations that we want are $s(a) = \delta(a) \in K^*$ for every $a \in \{f'/f \mid f \in K^*\}$. □

Define the group $\mathrm{Isom}_{K,1}$ and the C-vector space $A(K)$ by the exact sequences

$$1 \longrightarrow K^*/C^* \xrightarrow{\alpha} K \longrightarrow \mathrm{Isom}_{K,1} \longrightarrow 0,$$

$$0 \longrightarrow K/C \xrightarrow{\beta} K \longrightarrow A(K) \longrightarrow 0,$$

where α is the map $f \mapsto f'/f$ and β is given by $f \mapsto f'$. Then $\mathrm{Isom}_{K,1}$ is equal to the group of the isomorphy classes of the 1-dimensional differential modules over K. Indeed, consider the 1-dimensional module $M(a) := Ke$ with $\partial e = ae$ and $a \in K$. Then $M(a) \cong M(b)$ if and only if $a - b = f'/f$ for some $f \in K^*$. Moreover, $M(a) \otimes_K M(b) \cong M(a+b)$.

Let $C[\mathrm{Isom}_{K,1}]$ denote the group algebra of the group $\mathrm{Isom}_{K,1}$ over the field C. We recall that the elements of this algebra are the finite formal expressions $\sum_{a \in \mathrm{Isom}_{K,1}} \lambda_a [a]$ with all $\lambda_a \in C$. The addition is the obvious one. The multiplication is induced by the rule $[a] \cdot [b] = [a+b]$. Take a basis $\{b_i\}_{i \in I}$ of $A(K)$ over C. One forms the C-algebra $C[\mathrm{Isom}_{K,1}, \{t_i\}_{i \in I}]$ where the $\{t_i\}_{i \in I}$ is a family of indeterminates. This algebra is made into a Hopf algebra by the formulas $[a] \mapsto [a] \otimes [a]$ and $t_i \mapsto t_i \otimes 1 + 1 \otimes t_i$.

Corollary 1.3. — *The affine group scheme $G_{\mathrm{univ},K,\mathrm{ab}}$ is equal to*

$$\mathrm{Spec}(C[\mathrm{Isom}_{K,1}, \{t_i\}_{i \in I}]).$$

Proof. — An element g in $G_{\mathrm{univ},K,\mathrm{ab}}(C)$ is a differential automorphism of the differential ring $\mathrm{Univ}_{K,\mathrm{ab}}/K$. Then g is given by elements $c(g,a) \in C^*$ (for $a \in K$) and elements $d(g,b)$ (for $b \in K$) satisfying $g(s(a)) = c(g,a) \cdot s(a)$ and $g(y(b)) = y(b) + d(g,b)$. Now $a \mapsto c(g,a)$ is a homomorphism from K to C^* satisfying $c(g,a) = 1$ if the equation $y' = ay$ has a solution in K^*. In other words $a \mapsto c(g,a)$ is any homomorphism $\mathrm{Isom}_{K,1} \to C^*$. The map $b \mapsto d(g,b)$ is a C-linear map and $d(g,b) = 0$ if there exists a $y \in K$ with $y' = b$. In other words, $b \mapsto d(g,b)$ is any C-linear map $A(K) \to C$. A C-linear map $\ell : A(K) \to C$ is determined by the collection $\{\ell(b_i)\}_{i \in I}$. Hence, g corresponds to a C-algebra homomorphism $C[\mathrm{Isom}_{K,1}, \{t_i\}_{i \in I}] \to C$. The multiplication of elements in $G_{\mathrm{univ},K,\mathrm{ab}}(C)$ is induced by the Hopf algebra structure of $C[\mathrm{Isom}_{K,1}, \{t_i\}_{i \in I}]$. This description remains valid if C is replaced by any C-algebra (commutative and with identity). This proves the statement of the corollary. □

1.1. K is the field of formal Laurent series $C((x))$. — One provides $K = C((x))$ with the usual differentiation $f \mapsto df/dx$. An easy calculation shows that $\mathrm{Isom}_{K,1}$ is equal to $x^{-1}C[x^{-1}]/\mathbf{Z}x^{-1}$. Indeed, the image of the map $f \mapsto f'/f$ consists of the Laurent series $m/x + C[[x]]$ with $m \in \mathbf{Z}$. As an additive group $\mathrm{Isom}_{K,1}$ is the direct sum of $\mathbf{Q}/\mathbf{Z}$ and an infinite dimensional vector space over $\mathbf{Q}$. Furthermore, we need only to introduce the symbols $s(a)$ for $a \in x^{-1}C[x^{-1}]$. The only relation is $s(1/x) = x$.

For the"additive" equations $y' = b$ we need only to consider $b = x^{-1}$. Indeed, the map $f \mapsto f'$ from K to itself has a 1-dimensional cokernel, generated by the image of x^{-1}. We will write $\log x$ for the symbol $y(x^{-1})$. The conclusion is:

$$\mathrm{Univ}_{K,\mathrm{ab}} = K[\{s(a)\}_{a \in x^{-1}C[x^{-1}]}, \log x]$$

and the only relation between the symbols is given by $s(1/x) = x$. The affine group scheme $G_{\mathrm{univ},K,\mathrm{ab}}$ is the Spec of the Hopf algebra $C[\mathrm{Isom}_{K,1}, t]$ over C. We note that the part $C[t]$ of the Hopf algebra comes from the one factor $\mathbf{G}_a$ corresponding to $\log x$. Finally, as it should be, $\mathrm{Spec}(\mathrm{Univ}_{K,\mathrm{ab}})$ is a torsor over $G_{\mathrm{univ},K,\mathrm{ab}}$. This torsor is not trivial, due to the torsion subgroup $\mathbf{Q}/\mathbf{Z}$ of $\mathrm{Isom}_{K,1}$.

1.2. K is the field $C(x)$. — We follow the same procedure as in the last subsection. The group $\mathrm{Isom}_{K,1}$ is equal to K/V, where V consists of the elements $f \in K = C(x)$ which have at most poles of order one and such that all the residues of $f\,dx$ are in $\mathbf{Z}$. The surjective map $K \to K/V$ has no section because the group K/V has torsion elements. One way to make $\mathrm{Isom}_{K,1}$ more explicit is to consider partial fraction decompostion of the elements in K. An element $f \in C(x)$ is written as a finite sum $f_\infty + \sum_{c\in C} f_c$, with $f_\infty \in C[x]$ and $f_c \in (x-c)^{-1}C[(x-c)^{-1}]$ for each $c \in C$. This yields a direct sum decomposition $\oplus_{v\in\mathbf{P}^1(C)} \mathrm{Isom}_{K,1,v}$ of $\mathrm{Isom}_{K,1}$. Furthermore, $\mathrm{Isom}_{K,1,\infty} = C[x]$ and for every $c \in C$ one has $\mathrm{Isom}_{K,1,c} = (x-c)^{-1}C[(x-c)^{-1}]/\mathbf{Z}\frac{1}{x-c}$.

For the "additive" equations $y' = b$ with $b \in K$, we use also partial fractions. The only equations that one has to consider are $y' = 1/(x-c)$ with $c \in C$. We will write $\log(x-c)$ for the symbol $y(1/(x-c))$. The K-algebra $\mathrm{Univ}_{K,\mathrm{ab}}$ can now be written as

$$K[\{s(a)\}_{a\in K}, \{\log(x-c)\}_{c\in C}]$$

with, as before, the relations $s(0) = 1$, $s(a_1 + a_2) = s(a_1)s(a_2)$ and the new relations $s(1/(x-c)) = x - c$ for every $c \in C$. Furthermore $s(a)' = as(a)$ for all $a \in K$ and $\log(x-c)' = 1/(x-c)$ for all $c \in C$. The group scheme $G_{\mathrm{univ},K,\mathrm{ab}}$ is the Spec of the Hopf algebra $C[\mathrm{Isom}_{K,1}, \{t_c\}_{c\in C}]$.

One can give this the following interpretation. The universal Picard-Vessiot ring for the global abelian situation has as symbols the union of all local symbols for the completions $C((x-c))$ (all $c \in C$) and $C((x^{-1}))$ of $C(x)$. We note that the completion $C((x^{-1}))$ of $C(x)$ at ∞ behaves somewhat differently because its differentiation d/dx is not the one associated to the local parameter x^{-1}. A better way to formulate the differential equations that we are dealing with:

(a) the multiplicate equation in the form $dy/y = \omega$ with ω a differential form for K (or for the curve $\mathbf{P}^1_C$).

(b) the additive equation in the form $dy = \omega$ with ω a differential form.

We will make this more explicit for the case of a function field over C.

1.3. K is a function field over C. — K is the function field of a smooth, irreducible, projective curve X over C. Let Ω denote the C-vector space of all differential forms for K. In other words, Ω is the set of meromorphic 1-forms on X. We have to consider two types of equations:

(a) $dy/y = \omega$ with $\omega \in \Omega$ (multiplicative equations) and

(b) $dy = \omega$ with $\omega \in \Omega$ (additive equations).

For any (closed) point $x \in X$ one considers the field K_x, which is the completion of K w.r.t. the discrete valuation attached to x. Let Ω_x denotes the universal finite differential module of K_x over C. This means that after identifying K_x with $C((t))$ one has $\Omega_x = C((t))dt$. For the equations of type (a) one has an exact sequence

$$0 \longrightarrow \mathrm{Jac}(X) \longrightarrow \Omega/\{dy/y \mid y \in K^*\} \xrightarrow{L} \oplus_{x\in X}\Omega_x/\{dy/y \mid y \in K_x^*\} \longrightarrow C/\mathbf{Z} \longrightarrow 0.$$

By $\mathrm{Jac}(X)$ we mean in fact the group of the C-valued points of the Jacobian variety of X. The term $\Omega/\{dy/y \mid y \in K^*\}$ is equal to $\mathrm{Isom}_{K,1}$. The term $\Omega_x/\{dy/y \mid y \in K_x^*\}$ is equal to $\mathrm{Isom}_{K_x,1}$. The last map is induced by the map $\oplus_{x\in X}\Omega_x \to C$, given by the formula $(\omega_x)_{x\in X} \mapsto \sum_{x\in X} \mathrm{Res}_x(\omega_x)$. The map L is induced by the obvious map $\Omega \to \prod_{x\in X} \Omega_x$. Let $\omega \in \Omega$, $\omega \neq 0$ represent an element in the kernel of L. Then ω has at most poles of order 1 and all the residues of ω are in $\mathbf{Z}$. One associates to ω the divisor $D = \sum_{x\in X} \mathrm{Res}_x(\omega)x$. Then $\omega \in \{dy/y \mid y \in K^*\}$ if and only if D is a principal divisor. We will omit the verification that the above sequence is exact. This sequence provides the relation between the global multiplicative symbols (*i.e.*, for $\mathrm{Univ}_{K,\mathrm{ab}}$) with the local multiplicative symbols (*i.e.*, for $\mathrm{Univ}_{K_x,\mathrm{ab}}$). One observes that the collection of the multiplicative local symbols have relations induced by "the sum of the residues is zero". For $X = \mathbf{P}^1_C$ an example of these relations is

$$s_c\Big(\frac{\lambda}{x-c}\Big) \cdot s_\infty\Big(\frac{\lambda x}{1-cx^{-1}}\Big) = 1.$$

Here s_c and s_∞ denote local symbols at the points c and ∞. Further $\lambda \in C^*$. The above relation is obtained by transforming

$$\frac{d}{d(x-c)}\, y = \frac{\lambda}{x-c}\, y$$

into the equation

$$\frac{d}{dx^{-1}}\, y = \frac{-\lambda x}{1-cx^{-1}}\, y.$$

If the curve X has genus $g > 0$, then apart from the local multiplicative symbols one needs also symbols for the group $\mathrm{Jac}(X)$ in order to obtain all global symbols.

For the equations of type (b) one has an exact sequence

$$0 \longrightarrow H^1_{\mathrm{DR}}(X, C) \longrightarrow \Omega/\{dy \mid y \in K\} \xrightarrow{L} \oplus_{x\in X}\Omega_x/\{dy \mid y \in K_x\} \longrightarrow C \longrightarrow 0.$$

The map L is induced by the obvious map $\Omega \to \prod_{x\in X} \Omega_x$. Let $\omega \in \Omega$ represent an element in the kernel of L. Then the residue of ω at every point $x \in X$ is zero. The group $H^1_{\mathrm{DR}}(X, C)$ is the De Rham cohomology group with coefficients in C is equal to $\{\omega \in \Omega \mid \text{all residues } 0\}/\{dy \mid y \in K\}$. This cohomology group has dimension $2g$ over C where g is the genus of X. The group $A(K)$ is equal to $\Omega/\{dy \mid y \in K\}$ and $A(K_x)$ is equal to $\Omega_x/\{dy \mid y \in K_x\}$. Therefore the exact sequence provides the relation between the global additive symbols (*i.e.*, for $\mathrm{Univ}_{K,\mathrm{ab}}$) and the local additive symbols (*i.e.*, for $\mathrm{Univ}_{K_x,\mathrm{ab}}$). For every $x \in X$ there is in fact one local symbol, namely $\log t_x$ where t_x is a local parameter. Between these local symbols

there is just one C-linear relation, induced by " the sum of the residues is zero". We note that for $X = \mathbf{P}^1_C$, this relation is obviously $\log x + \log x^{-1} = 0$. Furthermore, if the genus g of X is strictly positive then one needs $2g$ new additive symbols coming from a basis of $H^1_{\mathrm{DR}}(X, C)$ over C.

1.4. Krichever differential modules of rank one. — Let X be a smooth, irreducible, projective curve over the algebraically closed field C of characteristic 0. Choose a (closed) point $\infty \in X$ and let A be the coordinate ring of $X \setminus \{\infty\}$. Let K denote a differential field (or more generally, differential ring) with field of constants C. A *differential Krichever module over* K is an injective C-algebra homomorphism $\phi : A \to K[\partial]$ such that $\phi(A) \not\subset K$. We denote by Univ_K the universal Picard-Vessiot ring (or field) of K. The following result is sketched in [**L1**]. Here we provide a more complete proof.

Proposition 1.4. — *Let A, K, ϕ be as above.*

(1) *There is a positive integer r, called the* rank of ϕ *such that for every $a \in A$, $a \neq 0$ the degree of the operator $\phi(a)$ is equal to $r \cdot \deg(a)$.*

(2) *For every non zero ideal $I \subset A$ the C-vector space*

$$V(I) := \{v \in \mathrm{Univ}_K \mid \phi(f)v = 0 \text{ for all } f \in I\}$$

has a natural structure of A-module. For this structure $V(I)$ is isomorphic to $(A/I)^r$. Moreover, the torsion A-module

$$W(\phi) := \{v \in \mathrm{Univ}_K \mid \exists\, a \in A,\, a \neq 0 \text{ with } \phi(a)(v) = 0\}$$

is isomorphic to $(\mathrm{Qt}(A)/A)^r$, where $\mathrm{Qt}(A)$ denotes the field of fractions of A.

(3) *The differential Galois group of the set of operators $\{\phi(f) \mid f \in I\}$ is an algebraic subgroup of $\mathrm{GL}_r(A/I)$.*

Proof. — For any non zero ideal $I \subset A$, the set of operators $\{\phi(f) \mid f \in I\}$ generates a left ideal in $K[\partial]L_I$ where L_I is a (monic) operator. Indeed, $K[\partial]$ is a (left and right) Euclidean ring. This observation clarifies the statements (2) and (3).

The map from $a \in A \setminus \{0\}$ to $\mathbf{Z}$, given by $a \mapsto -\deg(\phi(a))$, extends to a discrete valuation on the field of fractions of A. This valuation has negative values on A and is therefore equivalent to the discrete valuation attached to the point ∞. It follows that there is an rational number $r > 0$ such that $\deg(\phi(a)) = r \cdot \deg(a)$ holds for every $a \in A$, $a \neq 0$.

Consider an element $v \in V(I)$ and an element $a \in A$. The operator $\phi(a)$ acts on Univ_K in the obvious way. The element $\phi(a)v$ belongs again to $V(I)$ because for $f \in I$ one has $\phi(f)\phi(a)v = \phi(fa)v$ and $fa \in I$. In this way $V(I)$ is an A-module and also an A/I-module.

Let $\underline{P}$ be a non zero prime ideal of A. Then $V(\underline{P}^n)$ is a torsion module over the valuation ring $B = A_{\underline{P}}$. Choose $\pi \in A$ such that π generates the maximal ideal $\underline{P}B$ of B. The map $\phi(\pi) : V(\underline{P}^{n+1}) \to V(\underline{P}^n)$ is surjective. It follows that

$W := \cup_{n\geqslant 1} V(\underline{P}^n)$ is a divisible torsion module over B. Let $\mathrm{Qt}(B)$ denote the field of fractions of B. Then W is as B-module isomorphic to $(\mathrm{Qt}(B)/B)^{N(\underline{P})}$ for some integer $N(\underline{P}) \geqslant 1$. Hence $V(\underline{P}^n) \cong (A/\underline{P}^n)^{N(\underline{P})}$ for all $n \geqslant 1$. In particular, $N(\underline{P})$ is the dimension of $V(\underline{P})$ as vector space over $A/\underline{P}$. Using that A is a Dedekind ring one finds that $V(IJ) = V(I) \oplus V(J)$, if I and J are relatively prime ideals. For $I = \underline{P}_1^{n_1} \cdots \underline{P}_s^{n_s}$ (with distinct prime ideals $\underline{P}_1, \ldots, \underline{P}_s$) one has that the dimension of $V(I)$ over C is equal to $\sum_{i=1}^s n_i \cdot N(\underline{P}_i) \cdot \dim_C A/\underline{P}_i$. This is also equal to $\sum_{i=1}^s N(\underline{P}_i) \cdot \dim_C A/\underline{P}_i^{n_i}$. For a principal ideal $I = (a)$ one has $\dim_C V((a)) = r \cdot \deg(a) = r \cdot \dim_C A/(a) = \sum_{i=1}^s r \cdot \dim_C A/\underline{P}_i^{n_i}$. It follows that all $N(\underline{P}_i)$ are equal to r. If one varies a then one finds that all $N(\underline{P})$ are equal to r. In particular, r is a positive integer and $V(I) \cong (A/I)^r$ for every non zero ideal I.

Finally, the elements of the differential Galois group G of the set of differential operators $\phi(I)$, act on $V(I)$. This action commutes with the action of $\phi(a)$ for $a \in A$. Indeed, $g \in G$ commutes with ∂ and the multiplication by elements in K. Therefore G is an algebraic subgroup of $\mathrm{GL}_r(A/I)$. □

Remarks 1.5 **(Isogenies and isomorphisms)**

(1) An *isogeny* $u : \phi \to \phi'$ between two Krichever differential modules $\phi, \phi' : A \to K[\partial]$ is a non zero element of $K[\partial]$ satisfying $u \cdot \phi(a) = \phi'(a) \cdot u$ for all $a \in A$. This formula implies that $\deg \phi(a) = \deg \phi'(a)$ for all $a \in A$. Therefore ϕ and ϕ' have the same rank. An isogeny $u : \phi \to \phi'$ is called an *isomorphism* if $u \in K^*$.

(2) We will prove the following result:

Let $u : \phi \to \phi'$ be an isogeny. There exists a $v \in K[\partial]$, $v \neq 0$ and an $a \in A$ such that $v \cdot u = \phi(a)$.

Consider $\ker(u) := \{f \in \mathrm{Univ}_K \mid u(f) = 0\}$. For any $b \in A$ and any $f \in \ker(u)$ one has $\phi(b)(f) \in \ker(u)$. Indeed $u \cdot \phi(b) = \phi'(b) \cdot u$. Hence $\ker(u)$ is an A-module of finite dimension over C. Therefore there is an $a \in A$, $a \neq 0$ with $\phi(a)(\ker(u)) = 0$. In particular, $\ker(u) \subset \ker(\phi(a))$. This implies that $\phi(a) = v \cdot u$ for some $v \in K[\partial]$.

Examples 1.6 **(The rank one Krichever modules $\phi : C[t] \to C(x)[\partial]$)**

Put $\phi(t) = a_0 + a_1\partial$ with $a_0, a_1 \in C(x)$ and $a_1 \neq 0$. For every $f \in C[t] \setminus C$, the differential operator $\phi(f)$ has an abelian differential Galois group. We want to determine the Picard-Vessiot ring $\mathrm{Univ}_{C(x),\phi}$ obtained from this category of differential operators. In $\mathrm{Univ}_{C(x)}$ we consider the following elements:

(a) an invertible f, solution of the operator $\phi(t)$.

(b) invertible solutions $y(c)$ of $y(c)' = \frac{c}{a_1} y(c)$ for all $c \in C$. We impose the rules (as we may) $y(0) = 1$ and $y(c_1 + c_2) = y(c_1)y(c_2)$.

(c) a solution z of $z' = 1/a_1$.

We claim that

$$\mathrm{Univ}_{C(x),\phi} = C(x)[f, f^{-1}, \{y(c)\}_{c\in C}, z].$$

We observe that $fy(c)$ is a solution of the operator $\phi(t-c)$ and fz is a solution of the operator $\phi(t^2)$. This proves one inclusion. The elements $fy(c)z^i$, $i=0,\dots,n-1$ are solutions of the operator $\phi((t-c)^n)$. We will show, by induction on n, that the elements $1,z,\dots,z^n$ are linearly independent over C for every $n\geqslant 1$. For $n=1$ this is obvious. Suppose that $1,\dots,z^{n-1}$ are linearly independent over C and consider an expression $c_0+c_1z+\cdots+c_nz^n=0$ with $c_0,\dots,c_n\in C$. Differentiation yields that $\frac{1}{a_1}(c_1+2c_2z+\cdots+nc_nz^{n-1})=0$ and so $c_1=\cdots=c_n=0$. Hence $c_0=0$, too. This shows the other inclusion and proves the claim.

The structure of the differential ring $C(x)[f,f^{-1},\{y(c)\}_{c\in C},z]$ depends of course on a_0,a_1. For "generic" a_0,a_1, the only relations is this ring are the ones imposed above, namely $y(0)=1$ and $y(c_1+c_2)=y(c_1)y(c_2)$. The differential Galois group is then a product of $\mathbf{G}_m$ (this factor comes from f), $\mathbf{G}_a$ (this factor comes from z) and $\mathrm{Spec}(C[C])$ (this factor comes from $\{y(c)\}_{c\in C}$), where $C[C]$ is the group algebra of the group C over the field C.

In special cases, the factor $\mathbf{G}_m$ can become a finite group if the equation $y'=\frac{-n\cdot a_0}{a_1}y$ has a non zero solution in $C(x)$ for some integer $n\geqslant 1$. If a_0 happens to be an element of C, then the factor $\mathbf{G}_m$ disappears. Furthermore, the factor $\mathbf{G}_a$ disappears if $y'=1/a_1$ happens to have a solution in $C(x)$. Finally, it is possible that $y'=\frac{c}{a_1}y$ has a non zero solution in $C(x)$ for some $c\in C$, $c\neq 0$. In this case the group $\{c\in C\mid y'=\frac{c}{a_1}y \text{ has a solution in } C(x)\}$ is generated by some element $c_0\neq 0$. One finds a relation $y(c_0)=g$ for a suitable $g\in C(x)^*$.

The set $V\subset \mathrm{Univ}_{C(x)}$ of all solutions of all $\phi(f)$ has a $C[t]$-action and according to proposition 1.4, V is isomorphic to $C(t)/C[t]$. The differential Galois group G for the collection of all these differential equations is an algebraic subgroup of the automorphism group of $C(t)/C[t]$. The latter is equal to $\widehat{C[t]}^*$, where $\widehat{C[t]}$ is the projective limit of all $C[t]/(f)$. One observes that G, which is the group of the differential automorphisms of $\mathrm{Univ}_{C(x),\phi}$, is much smaller (even in the generic case) than $\widehat{C[t]}^*$. Indeed, only one additive equation is needed and there is (at most) one factor $\mathbf{G}_a$ in this group.

It is clear that the "torsion" of a single Krichever module ϕ of rank one cannot generate the full $\mathrm{Univ}_{C(x),\mathrm{ab}}$. One reason for this is that the differential operators $\phi(f)$ (with $f\in C[t]$) have their singularities in the union of the set of poles of a_0/a_1, the set of zeros of a_1 and possibly ∞. If one considers the differential operators $\phi(f)$ for all $f\in C[t]$ and all ϕ, then one obtains the Picard-Vessiot ring $\mathrm{Univ}_{C(x),\mathrm{ab}}$. Even in the local situation, *i.e.*, $K=C((x))$, the set of differential operators $\phi(f)$ (all $f\in C[t]\smallsetminus C$) and a single Krichever module ϕ, given by $\phi(t)=a_0+a_1\partial$, will not produce the local universal abelian Picard-Vessiot ring $\mathrm{Univ}_{C((x)),\mathrm{ab}}$. Indeed, the set of equations $y'=\frac{-a_0+c}{a_1}y$ (all $c\in C$) and $y'=1/a_1$, is too small for that.

We compare this with the Drinfel'd situation, where $\phi:\mathbf{F}_q[t]\to\mathbf{F}_q(t)[\tau]$ is a Drinfel'd module of rank one. Then $\phi(t)=t+a\tau$. Let ϕ' be another rank one

Drinfel'd module, given by $\phi'(t) = t + a'\tau$. Then ϕ and ϕ' become isomorphic after taking the cyclic extension $\mathbf{F}_q(t)(b)$ of $\mathbf{F}_q(t)$ of degree dividing $q-1$, given by the equation $ab^{q-1} = a'$. Hence the torsion points of ϕ and ϕ' produce essentially the same infinite abelian extension of $\mathbf{F}_q(t)$. For the case $\phi(t) = t - t\tau$, this extension is characterized as the maximal abelian extension of $\mathbf{F}_q(t)$ which is totally split at the place ∞. □

***Remarks 1.7* (Some interpretations of $\mathrm{Univ}_{C(x),\phi}$).** — The term a_0 in the operator $\phi(t) = a_0 + a_1\partial$ is of minor importance for $\mathrm{Univ}_{C(x),\phi}$. In the following examples we will take $a_0 = 0$.

(1) For the case $a_1 = x$, all $\phi(f)$ are regular singular at the two points $x = 0$ and $x = \infty$ (and regular outside $0, \infty$). The differential ring $C((x)) \otimes_{C(x)} \mathrm{Univ}_{C(x),\phi}$ is the universal extension of $C((x))$ for the collection of all regular differential equations over $C((x))$ (the same holds with x replaced by x^{-1}).

(2) Suppose that $a_1 = (x-p_1)\cdots(x-p_s)$ with distinct $p_1, \ldots, p_s$. For each i the differential ring $C((x-p_i)) \otimes_{C(x)} \mathrm{Univ}_{C(x),\phi}$ is again the universal extension of $C((x-p_i))$ for the collection of all regular singular equations at p_i. However, the various local solutions, like $(x-p_i)^{1/2}$, $\log(x-p_i)$ et cetera, are not independently present in $\mathrm{Univ}_{C(x),\phi}$.

(3) In the general case, the interpretation of the differential ring $\mathrm{Univ}_{C(x),\phi}$ remains unclear. Take for instance $a_1 = x^2$. Then $\mathrm{Univ}_{C(x),\phi} = C(x)[\{y(c)\}_{c\in C}]$ where the $y(c)$ satisfy $y(c)' = \frac{c}{x^2}\, y(c)$. The $y(c)$ are solutions of some irregular equations at $x = 0$.

2. Abelian difference equations

The field C is algebraically closed and has characterstic 0. The two types of difference equations, namely ordinary difference equations and q-difference equations, that we will study correspond to the two automorphisms σ and σ_q of $C(x)$ given by $\sigma x = x+1$ and $\sigma_q x = qx$. It is assumed that $q \in C^*$ is not a root of unity. Difference Galois theory, as developed in [**P-S**], provides an adequate Picard-Vessiot theory and difference Galois groups. We note that (q-)difference equations is a *multiplicative* theory. Let us make this explicit. A (q-)difference module is a vector space M over K of finite dimension (and $K = C(x)$ or $C((x^{-1}))$ or $C((x))$), provided with an additive invertible operator $\Phi : M \to M$ satisfying $\Phi(fm) = \sigma(f)\Phi(m)$ (or $\Phi(fm) = \sigma_q(f)\Phi(m)$) for $m \in M$ and $f \in K$. The tensor product of two (q)-difference modules M and N is the vector space $M \otimes_K N$ provided with the operation Φ given by $\Phi(m \otimes n) = \Phi(m) \otimes \Phi(n)$.

An important difference with differential Galois theory is that Picard-Vessiot rings are reduced but may have zero divisors. The role of the differential Picard-Vessiot field is taken over by the total ring of quotients of the Picard-Vessiot ring. Lemma 1.1

and its proof remain valid for (q-)difference equations. Indeed, the one ingredient for this proof, which is not available in [**P-S**], is the following theorem.

Theorem 2.1. — *Let K be a field of characteristic zero provided with an automorphism σ of infinite order. The field of constants $C = \{a \in K \mid \sigma a = a\}$ is supposed to be algebraically closed. Let M be a difference module over K with Picard-Vessiot ring R and difference Galois group G. Let $\mathrm{Qt}(R)$ denote the total ring of quotients of R. Then the following properties for an element $f \in \mathrm{Qt}(R)$ are equivalent:*

(1) *$f \in R$.*
(2) *The C-vector space generated by the orbit $\{gf \mid g \in G\}$ is finite dimensional.*
(3) *The K-vector space generated by $\{\sigma^n f \mid n \in \mathbf{Z}\}$ is finite dimensional.*

For the differential case, this theorem is proved in [**P-S03**], Corollary 1.38. This proof can be adapted for the difference case.

We will make the universal Picard-Vessiot ring $\mathrm{Univ}_{\mathrm{ab}}$ for the category of the (q-)difference modules with abelian difference Galois group explicit as well as the structure of the universal abelian (q-)difference Galois group $G_{\mathrm{univ,ab}}$.

2.1. Abelian q-difference equations. — As in section 1, we have to consider separately, two types of equations:

(a) (multiplicative equations) $\sigma_q y = ay$ with $a \in K^*$.
(b) (additive equations) $\sigma_q y = y + b$ with $b \in K$.

For K we have the possibilities: $K = C(x)$, $K = C((x))$ and $K = C((x^{-1}))$. For the multiplicative equations we consider the exact sequence

$$1 \longrightarrow K^*/C^* \xrightarrow{\alpha} K^* \longrightarrow \mathrm{Isom}_{K,1} \longrightarrow 1,$$

where α is given by $f \mapsto \sigma_q f/f$. As before, $\mathrm{Isom}_{K,1}$ is the group of the isomorphism classes of the 1-dimensional q-difference modules. For the field $K = C((x))$ one calculates that the image of α is equal to the subgroup $q^{\mathbf{Z}} \times (1 + xC[[x]])$ of K^*. Hence $\mathrm{Isom}_{K,1} \cong x^{\mathbf{Z}} \times C^*/q^{\mathbf{Z}}$. Hence the local multiplicative equations that contribute to the universal abelian Picard-Vessiot ring are $\sigma_q y = xy$ and $\sigma_q y = cy$ with $c \in C^*$.

The same description holds for the field $K = C((x^{-1}))$.

For the field $K = C(x)$ the description of $\mathrm{Isom}_{K,1}$ is more involved. One considers the exact sequence of groups

$$1 \longrightarrow C^* x^{\mathbf{Z}} \longrightarrow C(x)^* \longrightarrow \oplus_{\overline{c} \in C^*/q^{\mathbf{Z}}} \mathrm{Div}(\overline{c}) \longrightarrow 0.$$

Here $\overline{c}$ is seen as a subset $cq^{\mathbf{Z}}$ of C^* and $\mathrm{Div}(\overline{c})$ is the group of the divisors on this subset. An element $f \in C(x)^*$ is mapped to $(\mathrm{div}_{\overline{c}}(f))_{\overline{c}}$, where $\mathrm{div}_{\overline{c}}(f)$ is the restriction of the divisor of f to the set $\overline{c}$. We note that $\oplus_{\overline{c}} \mathrm{Div}(\overline{c})$ is equal to the group of the divisors on C^*. Every divisor $\sum_i n_i[\alpha_i]$ on C^* is the divisor of $\prod_i \left((x - \alpha_i)/x\right)^{n_i} \in C(x)^*$. In particular, the above sequence splits an this leads to an exact sequence

$$1 \longrightarrow C^*/q^{\mathbf{Z}} \times x^{\mathbf{Z}} \longrightarrow \mathrm{Isom}_{K,1} \longrightarrow \oplus_{\overline{c}} \mathbf{Z} \longrightarrow 0.$$

The main calculation takes place for the group $\mathrm{Div}(\overline{c})$. One has to calculate the cokernel of the operator $-1+\sigma_q$. One observes that $\mathrm{Div}(\overline{c})$ is in fact a free module over $\mathbf{Z}[\sigma_q,\sigma_q^{-1}]$, generated by the divisor $[c]$. Hence the cokernel of $-1+\sigma_q$ is isomorphic to $\mathbf{Z}[\sigma_q,\sigma_q^{-1}]/(-1+\sigma_q)\cong\mathbf{Z}$. Since the above exact sequence splits one has $\mathrm{Isom}_{K,1}\cong C^*/q^{\mathbf{Z}}\times x^{\mathbf{Z}}\oplus_{\overline{c}\in C^*/q^{\mathbf{Z}}}\mathbf{Z}$. More explicitly, the multiplicative equations that one has to consider for the construction of $\mathrm{Univ}_{K,\mathrm{ab}}$ are:

(i) $\sigma_q y = xy$,
(ii) $\sigma_q y = \dfrac{x-c}{x}\,y$ for c in a set of representatives of $C^*/q^{\mathbf{Z}}$ and
(iii) $\sigma_q y = cy$ with $c\in C^*$, again in a set of representatives of $C^*/q^{\mathbf{Z}}$.

For later use we introduce names for the invertible solutions in $\mathrm{Univ}_{C(x),\,\mathrm{ab}}$ of these equations, namely f_0 for (i), f_c for (ii) and $y(c)$ for (iii). They only relations between these elements are:

$\sigma f_c = f_{c/q}$ for $c\in C^*$,
$y(q)=x$ and $y(c_1c_2)=y(c_1)\cdot y(c_2)$ for all $c_1,c_2\in C^*$.

Define the C-vector space $A(K)$ by the exactness of the sequence

$$0\longrightarrow K/C\xrightarrow{\beta}K\longrightarrow A(K)\longrightarrow 0,$$

where $\beta(f)=\sigma_q(f)-f$. Take $K=C(x)$. For the investigation of $A(K)$, which is the cokernel of the map form K to itself, given by $f\mapsto\sigma_q(f)-f$, we use partial fractions. Write $C(x)=\oplus_{m\in\mathbf{Z}}Cx^m\oplus\oplus_{\overline{c}\in C^*/q^{\mathbf{Z}},\,n\geqslant 1}C(x)_{\overline{c},n}$. The space $C(x)_{\overline{c},n}$ consists of the partial fractions $\sum_{m\in\mathbf{Z}}c(m)/(q^mx-c)^n$, where $c\in C^*$ is chosen such that $c \mod q^{\mathbf{Z}}=\overline{c}$ and where all $c(m)\in C$. One observes that the spaces Cx^m and $C(x)_{\overline{c},n}$ are invariant under the action of σ_q. Hence the cokernel of the operator $-1+\sigma_q$ is the direct sum of the cokernels for each of these spaces. For the spaces Cx^m with $m\neq 0$, the cokernel is 0. On $C\cdot 1$ the action of $-1+\sigma_q$ is trivial. The action of σ_q on the space $C(x)_{\overline{c},n}$ makes $C(x)_{\overline{c},n}$ into a module over the ring $C[\sigma_q,\sigma_q^{-1}]$. This module is in fact the free module generated by one element, namely $1/(x-c)^n$. Hence the cokernel of $-1+\sigma_q$ on this vector space is isomorphic to $C[\sigma_q,\sigma_q^{-1}]/(-1+\sigma_q)\cong C$. The cokernel of $-1+\sigma_q$ on $C(x)_{\overline{c},n}$ is represented by the element $1/(x-c)^n$. Therefore we have found a basis for $A(K)$ over C, namely the images of the elements 1 and $1/(x-c)^n$ with $n\geqslant 1$ and c in a set of representatives of $C^*/q^{\mathbf{Z}}$. The corresponding additive equations are $\sigma_q y=y+1$ and $\sigma_q y=y+1/(x-c)^n$ (for $n\geqslant 1$ and c in a set of representatives of $C^*/q^{\mathbf{Z}}$).

For the field $K=C((x))$ the cokernel of $-1+\sigma_q$ is 1-dimensional and the corresponding additive equation is $\sigma_q y=y+1$. The same holds for the field $C((x^{-1}))$.

One can combine the above information in a straightforward way to obtain an explicit descriptions of $\mathrm{Univ}_{K,\mathrm{ab}}$ and $G_{\mathrm{univ},K,\mathrm{ab}}$ for the fields $K=C(x)$, $C((x))$, $C((x^{-1}))$.

2.2. Abelian ordinary difference equations. — $K = C(x)$ and $\sigma(x) = x + 1$. The methods used in the last subsection lead to explicit formulas for $\mathrm{Isom}_{K,1}$, $A(K)$, $\mathrm{Univ}_{K,\mathrm{ab}}$ and $G_{\mathrm{univ},K,\mathrm{ab}}$.

$$1 \longrightarrow C^* \longrightarrow C(x)^* \longrightarrow \oplus_{\overline{c}\in C/\mathbf{Z}} \mathrm{Div}(\overline{c}) \longrightarrow 0.$$

Here $\overline{c}$ is seen as the subset $c + \mathbf{Z}$ of C and $\mathrm{Div}(\overline{c})$ is the set of the divisors on $\overline{c}$.

$$1 \longrightarrow C^* \longrightarrow \mathrm{Isom}_{K,1} \longrightarrow \oplus_{\overline{c}\in C/\mathbf{Z}} \mathbf{Z} \longrightarrow 0.$$

The multiplicate equations that contribute to the formation of $\mathrm{Univ}_{K,\mathrm{ab}}$ are $\sigma y = (x-c)y$ with c in a set of representatives of $C/\mathbf{Z}$ and moreover the equations $\sigma y = cy$ with $c \in C^*$. We prefer to split up the first collection of multiplicative equations as $\sigma y = \frac{x-c}{x}\, y$ for $c \in C^*$ and $\sigma y = xy$.

For the calculation of the cokernel of $-1 + \sigma$ on $C(x)$ one considers the decomposition

$$C(x) = C[x] \oplus \oplus_{\overline{c}\in C/\mathbf{Z},\ n\geqslant 1} C(x)_{\overline{c},n},$$

where $C(x)_{\overline{c},n}$ consists of the finite expressions $\sum_{m\in\mathbf{Z}} d(m)/(x+m-c)^n$ with all $d(m) \in C$ and with a fixed choice of $c \in \overline{c}$. It is easily seen that $C(x)_{\overline{c},n}$ is a free $C[\sigma, \sigma^{-1}]$-module generated by $1/(x-c)^n$. Hence the cokernel of $-1+\sigma$ on $C(x)_{\overline{c},n}$ has dimension 1. The corresponding additive equation is $\sigma y = y + 1/(x-c)^n$. Furthermore, $-1+\sigma$ is surjective on $C[x]$ and has kernel $C \cdot 1$. This describes the additive part of $\mathrm{Univ}_{C(x),\mathrm{ab}}$.

Finally, we consider the difference field $C((x^{-1}))$. The cokernel of $f \mapsto \sigma f/f$ on $C((x^{-1}))^*$ is easily seen to be

$$x^{\mathbf{Z}} \times C^* \times (1 + x^{-1}C[[x^{-1}]])/(1+x^{-1})^{\mathbf{Z}}(1 + x^{-2}C[[x^{-1}]]).$$

The third group in this product can be identified with $C/\mathbf{Z}$. The corresponding multiplicative equations are: $\sigma y = xy$, $\sigma y = cy$ with $c \in C^*$ and $\sigma y = (1+cx^{-1})y$ with c in a class of representatives of $C/\mathbf{Z}$. The cokernel of $-1+\sigma$ on $C((x^{-1}))$ has dimension 1. The corresponding additive equation is $\sigma y = y + x^{-1}$.

2.3. Krichever difference modules. — C is an algebraically closed field of characteristic 0. One considers as before a smooth, irreducible, projective curve X over C with a specified point $\infty \in X$. Then A denotes the C-algebra of the regular functions on $X \smallsetminus \{\infty\}$. One considers a C-algebra R (commutative and with a unit element) provided with a specified C-algebra homomorphism $\gamma : A \to R$ and an endomorphism of infinite order σ. The skew polynomial ring $R[\sigma]$ is defined as usual. The elements are finite formal sums $\sum_{i\geqslant 0} r_i \cdot \sigma^i$ with all $r_i \in R$. The definition of the addition is obvious. The multiplication is induced by the formula $\sigma \cdot r = \sigma(r) \cdot \sigma$. A *Krichever difference module of rank r over R* is a C-algebra homomorphism $\phi : A \to R[\sigma]$ satisfying:

(a) For $a \in A$, $a \neq 0$, the element $\phi(a)$ has degree $r \cdot \deg(a)$, and the leading coefficient of $\phi(a)$ is invertible,

(b) $\phi(a) = \gamma(a) + *\sigma + *\sigma^2 + \cdots$

With the above data, one considers $R[\sigma]$ as a left $R[\sigma] \otimes_C A$-module by the formula $(f \otimes a)g = f \cdot g \cdot \phi(a)$ for $f, g \in R[\sigma]$ and $a \in A$. For any non zero ideal $I \subset A$, the subset $R[\sigma]\phi(I)$ of $R[\sigma]$ is also a left $R[\sigma] \otimes_C A$-module. Hence $R[\sigma]/R[\sigma]\phi(I)$ is a again a left $R[\sigma] \otimes_C A$-module.

A *level I-structure for ϕ* is an isomorphism of left $R[\sigma] \otimes_C A$-modules

$$\lambda : R \otimes_C (I^{-1}/A)^r \longrightarrow R[\sigma]/R[\sigma]\phi(I).$$

One regards R as a left $R[\sigma]$-module by the formula $(\sum r_i\sigma^i)r = \sum r_i\sigma^i(r)$. This explains also the left $R[\sigma] \otimes_C A$-module structure on $R \otimes_C (I^{-1}/A)^r$.

The two basic cases are $A = C[x]$ and (R, σ) is an extension of

(a) $(C(x), \sigma)$ with $\sigma(x) = x + 1$ or

(b) $(C(x), \sigma_q)$ with $\sigma_q(x) = qx$ ($q \neq 0$ and q not a root of unity).

In the above cases, $\gamma : A = C[x] \to R$ is injective. We note that more general situations, e.g., $\ker(\gamma) \neq \{0\}$ or $A \neq C[x]$, seem of interest.

As for Drinfel'd modules with a suitable level structure, one may try to show the existence of a universal Krichever difference module of a fixed rank and provided with a suitable level structure. In other words, one may want to construct a fine moduli space.

We consider the basic example: ϕ of rank 1, (R, σ) of type (a) or (b) and with level structure λ for the ideal (x). Put $\phi(x) = x + a\sigma$ with $a \in R^*$. The level structure λ prescribes an element (say) $b \in R^*$ with $xb + a\sigma(b) = 0$. The universal object that one can make for this situation is a ring $U = C(x)[\{b_n\}_{n\geqslant 0}, \{b_n^{-1}\}]$ (*i.e.*, U is a polynomial ring over $C(x)$ in the variables $\{b_n\}_{n\geqslant 0}$, localized at the set of all b_n). The operation σ on U is given by $\sigma(b_n) = b_{n+1}$ for every $n \geqslant 0$. The universal ϕ is given by $\phi(x) = x - x\frac{b_0}{b_1}\sigma$ and b_0 is the prescribed invertible solution of $\phi(x)y = 0$. We observe that U is not finitely generated over the field $C(x)$ and so the "fine moduli space" $\mathrm{Spec}(U)$ is not of finite type. It is however of "finite difference type." The definition of this concept can be given as follows. A *free difference algebra of finite type over $C(x)$* is an the polynomial ring $R = C(x)[\{b(i)_n\}_{i=1,\ldots,s,\ n\geqslant 0}]$ in the variables $\{b(i)_n\}$. Moreover, the endomorphism σ on R extends the given endomorphism of $C(x)$ (*i.e.*, either $\sigma(x) = x + 1$ or $\sigma(x) = qx$) and satisfies $\sigma(b(i)_n) = b(i)_{n+1}$ for all $n \geqslant 0$. A *difference algebra of finite type over $C(x)$* is the quotient of a free difference algebra of finite type over $C(x)$ by an ideal that is invariant under σ.

Theorem 2.2. — *Let the ring A, the ideal $I \subset A$ (with $I \neq A, \{0\}$) and the rank r be given. There exists a universal Krichever difference module of rank r and level I (of*

type (a) or (b)). The corresponding universal ring U is of finite difference type over $C(x)$.

Proof. — The proof is analogous to the one given in [**L1**] for the case of differential Krichever modules. We only sketch the proof for the difference case. One wants to obtain a universal U, σ, ϕ, λ. Write $A = C[f_1, \ldots, f_s] = C[F_1, \ldots, F_s]/(G_1, \ldots, G_t)$. For the definition of the universal ϕ one has to introduce variables for the coefficients of all $\phi(f_i)$. The variables corresponding to the leading coefficients should be invertible. Moreover one has to introduce the σ^i-images of all these variables. For the formulation of the level structure $\lambda : U \otimes_C (I^{-1}/A)^r \to U[\sigma]/U[\sigma]\phi(I)$ one has to introduce again a set of variables and all their σ^i-images. This produces a free difference algebra of finite type over $C(x)$ (suitably localized). In this free object one has to divide out the ideal given by all relations $G_1, \ldots, G_t$ and the relations which make the map λ into an isomorphism of left modules over $U[\sigma] \otimes_C A$. We note that $I \neq A$ garantees that the Krichever difference modules with level I structure have no automorphisms. □

2.4. Krichever q-difference modules of rank one. — We consider the rank one Krichever module $\phi : C[x] \to C(x)[\sigma]$ with $\phi(x) = x + a\sigma$ for the case $\sigma = \sigma_q$. The interesting question is the structure of the subring Univ_ϕ of $\mathrm{Univ}_{C(x),\mathrm{ab}}$, generated by all the solutions of all $\phi(f)$ (with $f \in C[x] \smallsetminus C$). In the sequel we will use the notations of subsection 2.1.

First we consider the multiplicative equations involved in ϕ. They are $\sigma_q(y) = \frac{-x}{a}\, y$ and $\sigma(y) = \frac{c-x}{a}\, y$ for all $c \in C^*$. Let $g \in \mathrm{Univ}_{C(x),\,\mathrm{ab}}$ be an invertible solution of the first equation. Then the solution of the other equations are $g \cdot f_c$ with $c \in C^*$. Hence $\{f_c\}_{c \in C^*} \subset \mathrm{Univ}_\phi$. Using the f_c's one can modify the first equation into an equation of the form $\sigma_q y = dx^n y$ for certain $d \in C^*$ and $n \in \mathbf{Z}$. One concludes that Univ_ϕ contains almost the complete multiplicative part of $\mathrm{Univ}_{C(x),\,\mathrm{ab}}$. The only possible exceptions are the elements f_0 and the $\{y(c)\}$. In case $a = -x$ this is precisely the part of the multiplicative symbols of $\mathrm{Univ}_{C(x),\mathrm{ab}}$ that is missing in Univ_ϕ.

Now we investigate the additive equations involved in ϕ. They are incorporated in the difference operators $\phi((x-c)^n)$. An invertible solution h of the operator $\phi(x-c)$ is gf_c if $c \in C^*$ and g otherwise. Define $z_0 = 1$ and for $n \geqslant 1$ the elements $z_n \in \mathrm{Univ}_{C(x),\,\mathrm{ab}}$ by the formula $\phi(x-c)hz_n = hz_{n-1}$. Then all $z_n \in \mathrm{Univ}_\phi$. One has the following recurrence relation

$$z_0 = 1 \quad \text{and} \quad (x-c)(1-\sigma_q)z_n = z_{n-1} \text{ for } n \geqslant 1.$$

For every $n \geqslant 1$, there are polynomial expression A_n in $z_1, \ldots, z_n$ such that $\sigma_q A_n = A_n + 1/(x-c)^n$. A calcuation yields

$$A_1 = z_1, \quad A_2 = z_1^2 - 2z_2, \quad A_3 = z_1^3 - 3z_1z_2 + 3z_3.$$

For the general case we consider a "generating function" $F := \sum_{n \geqslant 0} z_n T^n$ which belongs to $\mathrm{Univ}_\phi[[T]]$. The action of σ_q on T is supposed to be trivial, *i.e.*, $\sigma_q T = T$.

One observes that $(x-c)(1-\sigma_q)F = T\cdot F$ and thus $\sigma_q F/F = (1-T/(x-c))$. Write $F = \prod_{n\geqslant 1}(1+a_nT^n)$ (with all $a_n \in \mathrm{Univ}_\phi$). Then

$$\prod_{n\geqslant 1}\Big(1-\frac{(1-\sigma_q)a_nT^n}{(1+a_nT^n)}\Big) = 1-\frac{T}{(x-c)}.$$

Taking $-\log$ on both sides of this formula one obtains

$$\sum_{n\geqslant 1}\sum_{m\geqslant 1}\frac{1}{m}\Big(\frac{(1-\sigma_q)a_nT^n}{1+a_nT^n}\Big)^m = \sum_{m\geqslant 1}\frac{T^m}{m(x-c)^m}.$$

Comparing the coefficient of T^s on both sides implies that:
$(1-\sigma_q)a_s+$ a polynomial expression in the $a_1,\dots,a_{s-1}$ and their images under σ_q is equal to $1/(x-c)^s$. We conclude from this that Univ_ϕ contains all the additive symbols of $\mathrm{Univ}_{C(x),\mathrm{ab}}$ with the exception of the symbol corresponding to $\sigma_q y = y+1$.

For the formulation of the theorem below we introduce the following terminology. A ring R with $C(x)\subset R\subset C((x^{-1}))$ will be called an *abelian difference subring* if:

(a) $\sigma_q(R) = R$,

(b) For every finite dimensional $C(x)$-vector space $V\subset R$ the $C(x)$-vector space $\widetilde{V}$ generated by $\{\sigma_q^n r \mid n\in\mathbf{Z},\ r\in V\}$ is finite dimensional, and

(c) the group of the $C(x)$-automorphisms of the field $C(x)(\widetilde{V})$ which commute with σ_q is abelian.

It follows from this definition that $C(x)(\widetilde{V})$ is the field of fractions of the Picard-Vessiot ring of a q-difference equation over $C(x)$ with abelian difference Galois group. Furthermore there is a unique maximal abelian difference subring of $C((x^{-1}))$, namely the one generated by all multiplicative equations $\sigma_q y = ay$ and all additive equations $\sigma_q y = y+b$ over $C(x)$ that have a (non trivial) solution in $C((x^{-1}))$. The next proposition easily follows from the preceding calculations

Theorem 2.3. — *The rank one Krichever q-difference module*

$$\phi : C[x]\longrightarrow C(x)[\sigma_q] \text{ with } \phi(x) = x - x\sigma_q \text{ satisfies:}$$

Univ_ϕ *has an σ_q-equivariant embedding in $C((x^{-1}))$. Its image is the maximal abelian difference subring of $C((x^{-1}))$.*

In the general case $\psi(x) = x+a\sigma_q$ with $a\in C(x)^*$, one observes that Univ_ψ contains an invertible element b with $\sigma_q b = \frac{-x}{a}\,b$. Moreover $b^{-1}\psi(x)b = \phi(x)$. Hence $\mathrm{Univ}_\psi = \mathrm{Univ}_\phi[b,b^{-1}]$. Using the elements in Univ_ϕ one can transform the equation $\sigma_q y = \frac{-x}{a}\,y$ in to an equation $\sigma_q y = cx^n y$ for suitable $c\in C^*$ and $n\in\mathbf{Z}$. The Picard-Vessiot ring of this equation over $C(x)$ is $C(x)[b,b^{-1}]$ and $\mathrm{Univ}_\psi = C(x)[b,b^{-1}]\otimes_{C(x)}\mathrm{Univ}_\phi$. We note that the difference Galois group G of $\sigma_q y = cx^n y$ over $C(x)$ is a subgroup of $\mathbf{G}_m$. If cx^n is not a root of unity, then $G=\mathbf{G}_m$ and b is transcendental over $C(x)$. If $cx^n = d$ where d is an m^{th}root of unity, then G is a cyclic group of order m. Moreover the ring $C(x)[b,b^{-1}] = C(x)[T,T^{-1}]/(T^m-1)$ has zero divisors if $m>1$.

We conclude that Univ_ψ is equal to $C(x)[b,b^{-1}]\otimes_{C(x)} U$ with $C(x)[b,b^{-1}]$ as above and U the maximal abelian difference subring of $C((x^{-1}))$.

***Remarks 2.4* (Comparison with rank one Drinfel'd modules)**. — The rank one Drinfel'd module $\phi : \mathbf{F}_q[t] \to \mathbf{F}_q(t)[\tau]$, given by $\phi(t) = t - t\tau$, has the property that the field extension of $\mathbf{F}_q(t)$ obtained by all the torsion elements for ϕ is the maximal abelian extension of $\mathbf{F}_q(t)$ which is totally split at $t = \infty$ (see [**G-P-R-V**]). Theorem 2.3 is the perfect analogue of this.

2.5. Krichever difference modules of rank 1. — The same questions as in the last subsection are studied but now for σ given by $\sigma(x) = x + 1$. Let $\phi(x) = x + a\sigma$. The multiplicative equations involved in Univ_ϕ are $\sigma y = \frac{c-x}{a}\, y$. This can be separated into the equations $\sigma y = \frac{-x}{a}\, y$ and $\sigma y = \frac{x-c}{x}\, y$ for all $c \in C^*$. The last collection of equations produces all multiplicative equations for $C(x)$ with the exception of $\sigma y = xy$ and $\sigma y = cy$ with $c \in C^*$. For the additive equations, incorporated in the difference operators $\phi((x-c)^n)$, one finds, as in the q-difference case, all the additive equations $\sigma y = y + 1/(x-c)^n$ with $c \in C$ and $n \geqslant 1$. One observes that Univ_ϕ involves equations that have no (non trivial) solution in $C((x^{-1}))$. The equations are $\sigma y = (1 + cx^{-1})y$ for $c \in C^*$ and $\sigma y = y + x^{-1}$. We enlarge $C((x^{-1}))$ to $C((x^{-1}))[\{x^c\}_{c\in C}, \log(x)]$. This algebra is well known from the theory of differential equations. It is the universal Picard-Vessiot extension for the collection of all regular singular differential equations over $C((x^{-1}))$. The formal rules are $x\frac{d}{dx}(x^c) = cx^c$, $x\frac{d}{dx}\log(x) = 1$ and relations $x^{c_1}x^{c_2} = x^{c_1+c_2}$, $x^c = x^n \in C((x^{-1}))$ if c is equal to the integer n. Their behaviour under σ is given by $\sigma(x^c) = (1+x^{-1})^c x^c$ and $\sigma\log(x) = \log(x) + \sum_{n\geqslant 1} \frac{(-1)^{n-1}}{n} x^{-n}$. As for q-difference equations one defines abelian difference subrings of $C((x^{-1}))[\{x^c\}_{c\in C}, \log(x)]$. The maximal abelian difference subring is denoted by U.

Theorem 2.5. — *The rank one Krichever difference module*

$$\phi : C[x] \longrightarrow C(x)[\sigma] \text{ with } \phi(x) = x - x\sigma \text{ satisfies:}$$

Univ_ϕ *has a σ-equivariant embedding in $C((x^{-1}))[\{x^c\}_{c\in C}, \log(x)]$. Its image is the maximal abelian difference subring U.*

For a general rank one Krichever difference module ψ, given by $\psi(x) = x + a\sigma$, one has that $\mathrm{Univ}_\psi = C(x)[b,b^{-1}]\otimes_{C(x)} U$, where $C(x)[b,b^{-1}]$ is the Picard-Vessiot ring for the equation $\sigma y = Ay$ (suitable $A \in C(x)^*$) and U is the maximal abelian difference subring defined above.

References

[D-R-S-Z] L. Di Vizio, J.-P. Ramis, J. Sauloy & C. Zhang – "Équations aux q-différences", *Gaz. Math.* **96** (2003), et laboratoire Émile Picard, U.F.R. M.I.G. Toulouse, (2003).

[Dr] V.G. Drinfel'd – "Commutative subrings of certain non commutative rings", *Functional Anal. Appl.* **10** (1977), p. 9–12.

[F-G-K-V] E. Frenkel, D. Gaitsgory, D. Kazhdan & K. Vilonen – "Geometric realization of Whittaker functions and the Langlands conjecture", *J. Amer. Math. Soc.* **11** (1988), p. 451–484.

[F-G-V] E. Frenkel, D. Gaitsgory & K. Vilonen – "On the geometric Langlands conjecture", *J. Amer. Math. Soc.* **15** (2002), p. 367–417.

[G-P-R-V] U.E. Gekeler, M. van der Put, M. Reversat & J. van Geel (eds.) – *Drinfeld Modules, Modular Schemes and Applications*, World Scientific Publications Co, Inc., River Edge, 1997.

[Kr] I.M. Krichever – "Methods of algebraic geometry in the theory of non-linear equations", *Russian Math. Surveys* **32** (1977), no. 6, p. 185–213.

[L1] G. Laumon – Preprint Orsay, 1986.

[L2] ———, "Correspondance de Langlands géométrique pour les corps de fonctions", *Duke Math. J.* **54** (1987), no. 2, p. 309–351.

[Mu] D. Mumford – "An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg de Vries equation and related non-linear equations", in *Proceedings of the International Symposium on Algebraic Geometry (Kyoto, 1977)*, 1977, p. 115–153.

[P-S] M. van der Put & M.F. Singer – *Galois Theory of Difference Equations*, Lect. Notes in Math., vol. 1666, Springer-Verlag, Berlin, 1997.

[P-S03] ———, *Galois Theory of Linear Differential Equations*, Ergebnisse der Mathematik, Springer Verlag, 2003.

M. van der Put, Department of mathematics, University of Groningen, P.O. Box 800, 9700 AV Groningen (The Netherlands) • *E-mail :* `mvdput@math.rug.nl`

M. Reversat, Laboratoire de Mathématiques Émile Picard, Université Paul Sabatier, U.F.R. M.I.G., 118, route de Narbonne, 31602 Toulouse (France) • *E-mail :* `reversat@picard.ups-tlse.fr`

Astérisque
296, 2004, p. 227–251

ALGEBRAIC CONSTRUCTION OF THE STOKES SHEAF FOR IRREGULAR LINEAR q-DIFFERENCE EQUATIONS

by

Jacques Sauloy

Je laisse aux nombreux avenirs (non à tous)
mon jardin aux sentiers qui bifurquent.
Jorge Luis Borges, *Fictions*

***Abstract.* —** The local analytic classification of irregular linear q-difference equations has recently been obtained by J.-P. Ramis, J. Sauloy and C. Zhang. Their description involves a q-analog of the Stokes sheaf and theorems of Malgrange-Sibuya type and is based on a discrete summation process due to C. Zhang. We show here another road to some of these results by algebraic means and we describe the q-Gevrey devissage of the q-Stokes sheaf by holomorphic vector bundles over an elliptic curve.

***Résumé* (Construction algébrique du faisceau de Stokes pour les équations aux q-différences linéaires irrégulières)**

La classification analytique locale des équations aux q-différences linéaires irrégulières a été récemment réalisée par J.-P. Ramis, J. Sauloy et C. Zhang. Leur description fait intervenir un q-analogue du faisceau de Stokes et des théorèmes de type Malgrange-Sibuya et elle s'appuie sur la sommation discrète de C. Zhang. Nous montrons ici comment retrouver une partie de ces résultats par voie algébrique et nous décrivons le dévissage q-Gevrey du q-faisceau de Stokes par des fibrés vectoriels holomorphes sur une courbe elliptique.

1. Introduction and general conventions

1.1. Introduction. — This paper deals with Birkhoff's program of 1941 ([**3**], see also [**2**]) towards the local analytic classification of q-difference equations and some extensions stated by J.-P. Ramis in 1990 ([**13**]).

A full treatment of the Birkhoff program including the case of irregular q-difference equations is being given in [**16**]. The method used there closely follows the analytic procedure developed in the last decades by B. Malgrange, Y. Sibuya, J.-P. Ramis,... for the "classical" case, *i.e.*, the case of differential equations: adequate asymptotics,

2000 *Mathematics Subject Classification.* — Primary 39A13; Secondary 34M40, 32G34.
***Key words and phrases.* —** q-difference equations, Stokes sheaf.

© Astérisque 296, SMF 2004

q-Stokes phenomenon, q-Stokes sheaves and theorems of Malgrange-Sibuya type; explicit cocycles are built using a *discrete summation process* due to C. Zhang ([**27**]) where the Jackson q-integral and theta functions are introduced in place of the Laplace integral and exponential kernels.

To get an idea of the classical theory for linear differential equations one should look at the survey [**25**] by V.S. Varadarajan, especially section 6, and to get some feeling of how the change of landscape from differential equations to q-difference equations operates, at the survey [**7**] by L. Di Vizio, J.-P. Ramis, J. Sauloy and C. Zhang.

The aim of this paper is to show how the harder analytic tools can, to some extent, be replaced by much simpler algebraic arguments. The problem under consideration being a transcendental one we necessarily keep using analytic arguments but in their most basic, "19th century style", features only. In particular, we avoid here using the discrete summation process.

Again our motivation is strongly pushed ahead by the classical model of which we recall three main steps: the *dévissage Gevrey* introduced by J.-P. Ramis ([**12**]) occured to be the fundamental tool for understanding the Stokes phenomenon. The underlying algebra was clarified by P. Deligne in [**4**], then put at work by D.G. Babbitt and V.S. Varadarajan in [**1**] (see also [**25**]) for moduli theoretic purposes. On the same basis, effective methods, a natural summation and galoisian properties were thoroughly explored by M. Loday-Richaud in [**9**].

Some specificities of our problem are due, on one hand, to the fact that the sheaves to be considered are quite similar to holomorphic vector bundles over an elliptic curve, whence the benefit of GAGA theorems, and, on another hand, to the existence for q-difference operators of an analytic factorisation without equivalent for differential operators. Such a factorisation originates in Birkhoff ([**3**]), where it was rather stated in terms of a triangular form of the system. It has been revived by C. Zhang ([**26**], [**10**]) in terms of factorisation and we will use it in its linear guise, as a filtration of q-difference modules ([**22**]).

In this paper, following the classical theory recalled above, we build a q-Gevrey filtration on the q-Stokes sheaf, thereby providing a q-analog of the Gevrey devissage in the classical case. This q-devissage jointly with a natural summation argument allows us to prove the q-analog of a Malgrange-Sibuya theorem (theorem 3 of [**25**]) in quite a direct and easy way; in particular, we avoid here the Newlander-Nirenberg structural theorem used in [**16**]. Our filtration is, in some way, easier to get than the classical one: indeed, due to the forementionned canonical filtration of q-difference modules, our systems admit a natural triangularisation which is independent of the choice of a Stokes direction and of the domination order of exponentials (here replaced by theta functions). Also, our filtration has a much nicer structure than the classical Gevrey filtration since the so-called elementary sheaves of the classical theory are here

replaced by holomorphic vector bundles endowed with a very simple structure over an elliptic curve (they are tensor products of flat bundles by line bundles).

On the side of what this paper does not contain, there is neither a study of confluency when q goes to 1, nor any application to Galois theory. As for the former, we hope to extend the results in [**20**] to the irregular case, but this seems a difficult matter. Only partial results by C. Zhang are presently available, on significant examples. As for the latter, it is easier to obtain as a consequence of the present results that, under natural restrictions, "canonical Stokes operators are Galoisian" like in [**9**]. However, to give this statement its full meaning, we have to generalize the results of [**21**] and to associate vector bundles to arbitrary equations. This is a quite different mood that we will develop in a forthcoming paper ([**23**]; meanwhile, a survey is given in [**24**]). Here, we give some hints in remarks 3.11 and 4.5.

Also, let us point out that there has been little effort made towards systematisation and generalisation. The intent is to get as efficiently as possible to the striking specific features of q-difference theory. For instance, most of the results about morphisms between q-difference modules can be obtained by seing these morphisms as meromorphic solutions of other modules (internal Hom) and they can therefore be seen as resulting from more general statements. These facts, evenso quite often sorites, deserve to be written. In the same way, the many regularity properties of the homological equation $X(qz)A(z) - B(z)X(z) = Y(z)$ should retain some particular attention and be clarified in the language of functional analysis. They are implicitly or explicitly present in many places in the work of C. Zhang. Last, the q-Gevrey filtration should be translated in terms of factorisation of Stokes operators, like in [**9**].

Let us now describe the organization of the paper.

Notations and conventions are given in subsection 1.2.

Section 2 deals with the recent developments of the theory of q-difference equations and some improvements. In subsection 2.1, we recall the local classification of fuchsian systems by means of flat vector bundles as it can be found in [**21**] and its easy extension to the so-called "tamely irregular" q-difference modules. We then describe the filtration by the slopes ([**22**]). In subsections 2.2, we summarize results from [**16**] about the local analytic classification of irregular q-difference systems, based on the Stokes sheaf. The lemma 2.7 provides a needed improvement about Gevrey decay; proposition 2.8 and corollary 2.10 an improvement about polynomial normal forms.

In chapter 3, we first build our main tool, the algebraic summation process (theorem 3.7). Its application to the local classification is then developed in subsection 3.2. We state there and partially prove the second main result of this paper (theorem 3.18): a q-analog of the Malgrange-Sibuya theorem for the local analytic classification of linear differential equations.

Section 4 is devoted to studying the q-Gevrey filtration of the Stokes sheaf and proving the theorem 3.18. In subsection 4.1, we show how conditions of flatness

(otherwise said, of q-Gevrey decay) of solutions near 0 translate algebraically and how to provide the devissage for the Stokes sheaf of a "tamely irregular" module. In subsection 4.2 we derive some cohomological consequences and we finish the proof of the theorem 3.18. Finally, in subsection 4.3, we sketch the Stokes sheaf of a general module.

The symbol $\square$ indicates the end of a proof or the absence of proof if considered straightforward. Theorems, propositions and lemmas considered as "prerequisites" and coming from the quoted references are not followed by the symbol $\square$.

Acknowledgements. — The present work is directly related with the paper [**16**], written in collaboration with Jean-Pierre Ramis and with Changgui Zhang. It has been a great pleasure to talk with them, confronting very different points of view and sharing a common excitement.

The epigraph at the beginning of this paper is intended to convey the happiness of wandering and daydreaming in Jean-Pierre Ramis' garden; and the overwhelming surprise of all its bifurcations. Like in Borges' story, pathes fork and then unite, the same landscapes are viewed from many points with renewed pleasure. This strong feeling of the unity of mathematics without any uniformity is typical of Jean-Pierre.

1.2. Notations and general conventions. — We fix once for all a complex number $q \in \mathbf{C}$ such that $|q| > 1$. We then define the automorphism σ_q on various rings, fields or spaces of functions by putting $\sigma_q f(z) = f(qz)$. This holds in particular for the field $\mathbf{C}(z)$ of complex rational functions, the ring $\mathbf{C}\{z\}$ of convergent power series and its field of fractions $\mathbf{C}(\{z\})$, the ring $\mathbf{C}[[z]]$ of formal power series and its field of fractions $\mathbf{C}((z))$, the ring $\mathcal{O}(\mathbf{C}^*, 0)$ of holomorphic germs and the field $\mathcal{M}(\mathbf{C}^*, 0)$ of meromorphic germs in the punctured neighborhood of 0, the ring $\mathcal{O}(\mathbf{C}^*)$ of holomorphic functions and the field $\mathcal{M}(\mathbf{C}^*)$ of meromorphic functions on $\mathbf{C}^*$; this also holds for all modules or spaces of vectors or matrices over these rings and fields.

For any such ring (resp. field) R, the σ_q-invariants elements make up the subring (resp. subfield) R^{σ_q} of constants. For instance, the field of constants of $\mathcal{M}(\mathbf{C}^*, 0)$ or that of $\mathcal{M}(\mathbf{C}^*)$ can be identified with a field of elliptic functions, the field $\mathcal{M}(\mathbf{E}_q)$ of meromorphic functions over the complex torus (or elliptic curve) $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. We shall use heavily the theta function of Jacobi defined by the following equality:

$$\theta_q(z) = \sum_{n \in \mathbf{Z}} q^{-n(n+1)/2} z^n.$$

This function is holomorphic in $\mathbf{C}^*$ with simple zeroes, all located on the discrete q-spiral $[-1; q]$, where we write $[a; q] = aq^{\mathbf{Z}}$, $(a \in \mathbf{C}^*)$. It satisfies the functional equation: $\sigma_q \theta_q = z\theta_q$. We shall also use its multiplicative translates $\theta_{q,c}(z) = \theta_q(z/c)$ (for $c \in \mathbf{C}^*$); the function $\theta_{q,c}$ is holomorphic in $\mathbf{C}^*$ with simple zeroes, all located on the discrete q-spiral $[-c; q]$ and satisfies the functional equation: $\sigma_q \theta_{q,c} = \frac{z}{c}\theta_{q,c}$.

As is customary for congruence classes, we shall write $\overline{a} = a \pmod{q^{\mathbf{Z}}}$ for the image of $a \in \mathbf{C}^*$ in the elliptic curve $\mathbf{E}_q$. This notation extends to a subset A of $\mathbf{C}^*$, so that $\overline{A}$ does *not* denote its topological closure. Then, for a divisor $D = \sum n_i[\alpha_i]$ over $\mathbf{E}_q$ (*i.e.*, the $n_i \in \mathbf{Z}$, the $\alpha_i \in \mathbf{E}_q$), we shall write $\mathrm{ev}_{\mathbf{E}_q}(D) = \sum n_i\alpha_i \in \mathbf{E}_q$ for its evaluation, computed with the group law on $\mathbf{E}_q$.

Let K denote any one of the forementioned fields of functions. Then, we write $\mathcal{D}_{q,K} = K\langle\sigma, \sigma^{-1}\rangle$ for the Öre algebra of non commutative Laurent polynomials characterized by the relation $\sigma \cdot f = \sigma_q(f) \cdot \sigma$. We now define the category of q-difference modules in three clearly equivalent ways:

$$\begin{aligned}\mathrm{DiffMod}(K, \sigma_q) &= \{(E, \Phi) \mid E \text{ a } K\text{-vector space of finite rank,} \\ &\qquad \Phi : E \longrightarrow E \text{ a } \sigma_q\text{-linear map}\} \\ &= \{(K^n, \Phi_A) \mid A \in \mathrm{GL}_n(K), \Phi_A(X) = A^{-1}\sigma_q X\} \\ &= \text{ finite length left } \mathcal{D}_{q,K}\text{-modules.}\end{aligned}$$

This is a $\mathbf{C}$-linear abelian rigid tensor category, hence a tannakian category. For basic facts and terminology about these, see [**21**], [**11**], [**6**], [**5**]. Last, we note that all objects in $\mathrm{DiffMod}(K, \sigma_q)$ have the form $\mathcal{D}_{q,K}/\mathcal{D}_{q,K}P$.

2. Local analytic classification

2.1. Devissage of irregular equations ([21],[22])

Fuchsian and tamely irregular modules. — For a q-difference module M over any of the fields $\mathbf{C}(z)$, $\mathbf{C}(\{z\})$, $\mathbf{C}((z))$, it is possible to define its Newton polygon at 0, or, equivalently, the slopes of M, which we write in descending order: $\mu_1 > \cdots > \mu_k \in \mathbf{Q}$, and their multiplicities $r_1, \ldots, r_k \in \mathbf{N}^*$. The module M is said to be pure of slope μ_1 if $k = 1$ and fuchsian if it is pure of slope 0. The latter condition is equivalent to M having the shape $M = (K^n, \Phi_A)$ with $A(0) \in \mathrm{GL}_n(\mathbf{C})$. There are also criteria of growth (or decay) of solutions near 0, see further below, in section 2.2, the subsection about flatness conditions.

Call $\mathcal{E}$ the category $DiffMod(\mathbf{C}(z), \sigma_q)$ of rational equations. Fuchsian modules at 0 and ∞ over $\mathbf{C}(z)$ make up a tannakian subcategory $\mathcal{E}_f$ of $\mathcal{E}$. In order to study them, one "localizes" these categories by extending the class of morphisms, precisely, by allowing morphisms defined over $\mathbf{C}(\{z\})$. This gives "thickened" categories $\mathcal{E}^{(0)}$ and $\mathcal{E}_f^{(0)}$. A classical lemma says that any fuchsian system is locally equivalent to one with constant coefficients. This suggests the introduction of the full subcategory $\mathcal{P}_f^{(0)}$ of $\mathcal{E}_f^{(0)}$ made up of "flat" objects, that is, the $(\mathbf{C}(z)^n, \Phi_A)$ with $A \in \mathrm{GL}_n(\mathbf{C})$. Thus, the inclusion of $\mathcal{P}_f^{(0)}$ into $\mathcal{E}_f^{(0)}$ is actually an isomorphism of tannakian categories.

To any $A \in \mathrm{GL}_n(\mathbf{C})$ one associates the holomorphic vector bundle F_A over $\mathbf{E}_q$ obtained by quotienting $\mathbf{C}^* \times \mathbf{C}^n$ by the equivalence relation $\sim_A$ generated by the relations $(z, X) \sim_A (qz, AX)$. This defines a functor from $\mathcal{P}_f^{(0)}$ to the category $\mathrm{Fib}_p(\mathbf{E}_q)$ of flat holomorphic vector bundles over $\mathbf{E}_q$. This is an equivalence of tannakian categories. Note that the classical lemma alluded above equally holds for any fuchsian q-difference module over $\mathbf{C}(\{z\})$ or over $\mathbf{C}((z))$, which implies that this local classification applies to $\mathrm{DiffMod}(\mathbf{C}(\{z\}), \sigma_q)$ and $\mathrm{DiffMod}(\mathbf{C}((z)), \sigma_q)$ as well. The galoisian aspects of this local correspondence and its global counterpart are detailed in [**21**].

A pure module of *integral* slope μ over $K = \mathbf{C}(\{z\})$ or $\mathbf{C}((z))$ has the shape $(K^n, \Phi_{z^{-\mu}A})$ with $A \in \mathrm{GL}_n(\mathbf{C})$. For such a module, the above construction of a vector bundle extends trivially, yielding the tensor product of a flat bundle by a line bundle. We shall call pure such a bundle.

Direct sums of pure modules play a special role in [**22**], [**16**] and in the present paper. We shall call them tamely irregular, in an intended analogy with tamely ramified extensions in algebraic number theory: for us, they are irregular objects without wild monodromy, as follows from [**16**]. The category of tamely irregular modules with integral slopes over $\mathbf{C}(\{z\})$ can, for the same reasons as above, be seen either as a subcategory of $\mathrm{DiffMod}(\mathbf{C}(\{z\}), \sigma_q)$ or of $\mathcal{E}^{(0)}$. We write it $\mathcal{E}^{(0)}_{\mathrm{mi},1}$ [(1)]. It is generated (as a tannakian category) by the fuchsian modules and by the pure module $(\mathbf{C}(\{z\}), z^{-1}\sigma_q)$ of slope 1. We can thus associate to any such module a direct sum of pure modules, thereby defining a functor from $\mathcal{E}^{(0)}_{\mathrm{mi},1}$ to the category $\mathrm{Fib}(\mathbf{E}_q)$ of holomorphic vector bundles over $\mathbf{E}_q$. This functor is easily seen to be compatible with all linear operations (it is a functor of tannakian categories).

Filtration by the slopes. — The following theorem is proved in [**22**].

Theorem 2.1. — *Let the letter K stand for the field $\mathbf{C}(\{z\})$ (convergent case) or the field $\mathbf{C}((z))$ (formal case). In any case, any object M of* $\mathrm{DiffMod}(K, \sigma_q)$ *admits a unique filtration $(F^{\geqslant \mu}(M))_{\mu \in \mathbf{Q}}$ by subobjects such that each $F^{(\mu)}(M) = F^{\geqslant \mu}(M)/F^{>\mu}(M)$ is pure of slope μ. The $F^{(\mu)}$ are endofunctors of* $\mathrm{DiffMod}(K, \sigma_q)$ *and $\mathrm{gr} = \bigoplus F^{(\mu)}$ is a faithful exact $\mathbf{C}$-linear $\otimes$-compatible functor and a retraction of the inclusion of $\mathcal{E}^{(0)}_{\mathrm{mi}}$ into $\mathcal{E}^{(0)}$. In the formal case, gr is isomorphic to the identity functor.*

From now on, we only consider the full subcategory $\mathcal{E}^{(0)}_1$ of modules with integral slopes.

The notation $\mathcal{E}^{(0)}_1$ will be justified *a posteriori* by the fact that all its objects are locally equivalent to objects of $\mathcal{E}$ (existence of a normal polynomial form). This is an abelian tensor subcategory of $\mathcal{E}^{(0)}$ and the functor *gr* retracts $\mathcal{E}^{(0)}_1$ to $\mathcal{E}^{(0)}_{\mathrm{mi},1}$. We

(1) The subscript "mi" stands for "modérément irrégulier", the subscript 1 for restricting to slopes with denominator 1.

also introduce notational conventions which will be used all along this paper for a module M in $\mathcal{E}_1^{(0)}$ and its associated graded module $M_0 = \mathrm{gr}(M)$, an object of $\mathcal{E}_{\mathrm{mi},1}^{(0)}$.

The module M may be given the shape $M = (\mathbf{C}(\{z\})^n, \Phi_A)$, with:

$$A = A_U \underset{\mathrm{def}}{=} \begin{pmatrix} z^{-\mu_1}A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & z^{-\mu_k}A_k \end{pmatrix}, \tag{2.1.1}$$

where $\mu_1 > \cdots > \mu_k$ are integers, $r_i \in \mathbf{N}^*$, $A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$ $(i = 1, \dots, k)$ and

$$U = (U_{i,j})_{1 \leqslant i < j \leqslant k} \in \prod_{1 \leqslant i < j \leqslant k} \mathrm{Mat}_{r_i, r_j}(\mathbf{C}(\{z\})).$$

The associated graded module is then a direct sum $M_0 = P_1 \oplus \cdots \oplus P_k$, where, for $1 \leqslant i < j \leqslant k$, the module P_i is pure of rank r_i and slope μ_i and can be put into the form $P_i = (\mathbf{C}(\{z\})^{r_i}, \Phi_{z^{-r_i}A_i})$. Therefore, one has $M_0 = (\mathbf{C}(\{z\})^n, \Phi_{A_0})$, where the matrix A_0 is block-diagonal:

$$A_0 = \begin{pmatrix} z^{-\mu_1}A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & z^{-\mu_k}A_k \end{pmatrix}. \tag{2.1.2}$$

The set of analytic isoformal classes. — This section comes from [**16**]. The definitions here should be compared to those in [**25**], p. 29 or [**1**].

In $\mathrm{DiffMod}(\mathbf{C}((z)), \sigma_q)$, the canonical filtration of a module M is split; more precisely, the associated graded module $\mathrm{gr}(M)$ is the unique formal classifyer of M. The isoformal analytic classification is therefore the same as the isograded classification, whence the following definitions.

Definition 2.2. — Let $P_1, \dots, P_k$ be pure modules with ranks $r_1, \dots, r_k$ and with integral slopes $\mu_1 > \cdots > \mu_k$. The module $M_0 = P_1 \oplus \cdots \oplus P_k$ has rank $n = r_1 + \cdots + r_k$. We shall write $\mathcal{F}(M_0)$ for the set of equivalence classes of pairs (M, g) of a module M and an isomorphism $g : \mathrm{gr}(M) \to M_0$, where (M, g) is said to be equivalent to (M', g') if there exists a morphism $u : M \to M'$ such that $g = g' \circ \mathrm{gr}(u)$ (u is automatically an isomorphism).

We write $\mathfrak{G}$ for the algebraic subgroup of GL_n made up of matrices of the form

$$F = \begin{pmatrix} I_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & F_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & I_{r_k} \end{pmatrix}. \tag{2.2.1}$$

Its Lie algebra $\mathfrak{g}$ consists in matrices of the form

$$f = \begin{pmatrix} 0_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & f_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0_{r_k} \end{pmatrix}. \tag{2.2.2}$$

For F in $\mathfrak{G}$, we shall write $F[A] = (\sigma_q F)\, AF^{-1}$ for the result of the gauge transformation F on the matrix A.

We shall identify P_i with $(\mathbf{C}(\{z\})^{r_i}, \Phi_{z^{-\mu_i} A_i})$, where $A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$. The datum of a pair (M, g) then amounts to that of a matrix A in the form 2.1.1. Two such matrices A, A' are equivalent iff there exists a matrix $F \in \mathfrak{G}(\mathbf{C}(\{z\}))$ such that $F[A] = A'$.

Write $\mathfrak{G}^{A_0}(\mathbf{C}((z))) = \{F \in \mathfrak{G}(\mathbf{C}((z))) \mid F[A_0] \in \mathrm{GL}_n(\mathbf{C}(\{z\}))\}$. The subgroup $\mathfrak{G}(\mathbf{C}(\{z\}))$ of $\mathfrak{G}(\mathbf{C}((z)))$ operates at left on the latter (by translation) and $\mathfrak{G}^{A_0}(\mathbf{C}((z)))$ is stable for that operation. The theory in the previous section entails:

$$\forall\, (U_{i,j})_{1 \leqslant i < j \leqslant k} \in \prod_{1 \leqslant i < j \leqslant k} \mathrm{Mat}_{r_i, r_j}(\mathbf{C}(\{z\})),\ \exists !\, \widehat{F} \in \mathfrak{G}(\mathbf{C}((z)))\ :\ \widehat{F}[A_0] = A_U.$$

This $\widehat{F}$ will be written $\widehat{F}(U)$. Its existence can also be proved by direct computation, solving by iteration the fixpoint equation of the z-adically contracting operator: $F \mapsto (A_U)^{-1} \left(\sigma_q \widehat{F}\right) A_0$. It follows that the unique formal gauge transformation of $\mathfrak{G}(\mathbf{C}((z)))$ taking A_U to A_V is $\widehat{F}(U, V) = \widehat{F}(V)\widehat{F}(U)^{-1}$. Besides, A_U is equivalent to A_V in the above sense if and only if $\widehat{F}(U, V) \in \mathfrak{G}(\mathbf{C}(\{z\}))$, or, equivalently, $\widehat{F}(V) \in \mathfrak{G}(\mathbf{C}(\{z\}))\widehat{F}(U)$. This translates into the following proposition.

Proposition 2.3. — *Sending A_U to $\widehat{F}(U)$ induces a one-to-one correspondence between $\mathcal{F}(M_0)$ and the left quotient $\mathfrak{G}(\mathbf{C}(\{z\}))\backslash\mathfrak{G}^{A_0}(\mathbf{C}((z)))$.*

One, thus, recognizes in the isoformal classification a classical problem of summation of divergent power series. In order to illustrate the possible strategies, we shall end this section by examining a specific example. We shall try, as far as possible, to mimic the methods and the terminology of the "classical" theory (Stokes operators for linear differential equations and summation in sectors along directions).

Example 2.4. — The module $M_u = (\mathbf{C}(\{z\})^2, \Phi_{A_u})$ corresponding to the matrix $A_u = \left(\begin{smallmatrix} 1 & u \\ 0 & z \end{smallmatrix}\right)$ is formally isomorphic to its associated graded module M_0. More precisely, there exists a formal gauge transformation F such that $F[A_0] = A_u$, that is, $F(qz)A_0(z) = A_u(z)F(z)$. If one moreover requires F to be compatible with the graduation, that is, to have the form $F = \left(\begin{smallmatrix} 1 & f \\ 0 & 1 \end{smallmatrix}\right)$, then there is unicity of the formal series f, which must satisfy the functional equation

$$f(z) = -u(z) + zf(qz).$$

We call $\widehat{f}_u$ this unique formal solution (it can be computed by iterating the above fixpoint equation) and $\widehat{F}_u$ the corresponding formal gauge transformation. One checks that two such matrices A_u and A_v are analytically equivalent if and only if the formal power series $\widehat{f}_{u-v} = \widehat{f}_u - \widehat{f}_v$ is convergent. In this case (two slopes), the problem is additive.

For $u = 1$, the unique solution is

$$\widehat{f}_1 = -\sum_{n \geqslant 0} q^{n(n-1)/2} z^n,$$

the so-called Tschakaloff series (up to the sign). It is divergent and may be seen as a natural q-analog of the Euler series. Thus, A_1 is not equivalent to A_0.

In general, we apply the formal q-Borel-Ramis transform of level 1, defined by

$$\mathcal{B}_{q,1} \sum_n a_n z^n = \sum_n q^{-n(n-1)/2} a_n \xi^n.$$

It sends convergent series to series with an infinite radius of convergence. Our functional equation is transformed into

$$(1-\xi)\mathcal{B}_{q,1} f(\xi) = -\mathcal{B}_{q,1} u(\xi).$$

The existence of a convergent solution f has only one obstruction, the number $\nu = \mathcal{B}_{q,1} u(1)$. This number can therefore be considered as the unique analytic invariant of M_u within the formal class of M_0. It can also be considered as giving a normal form, since A_ν is the unique matrix in the analytic class of A_u such that $\nu \in \mathbf{C}$. It is a particular case of normal polynomial form (see further below).

The functional equation can also be solved by a variant of the method of "varying constants". We look for the solution in the form $g = \theta_{q,\lambda} f$. For convenience, we also write $v = \theta_{q,\lambda} u$, which is an element of $\mathcal{O}(\mathbf{C}^*, 0)$. We compare their Laurent series coefficientwise and we get

$$\forall\, n \in \mathbf{Z},\ (1 - \lambda q^n) g_n = v_n.$$

If $\lambda \notin [1; q]$ (prohibited direction of summation), there is a unique solution $g \in \mathcal{O}(\mathbf{C}^*, 0)$ (it does converge where it should), thus a unique solution $f \in \mathcal{M}(\mathbf{C}^*, 0)$ such that $\theta_{q,\lambda} f$ has no poles in $\mathbf{C}^*$. We then get a unique solution $f_{\lambda,u}$ with (at most) simple poles over $[-\lambda; q]$: it is the summation of $\widehat{f}_u$ in the direction $\overline{\lambda} \in \mathbf{E}_q$ and its "sector" of validity is (the germ at 0) of $\mathbf{C}^* \smallsetminus [-\lambda; q]$, the preimage by the canonical projection $\mathbf{C}^* \to \mathbf{E}_q$ of the Zariski open set $\mathbf{E}_q \smallsetminus \{\overline{-\lambda}\}$.

There is another way of looking at this summation process, with a deeper analytical meaning. We can consider $\mathcal{B}_{q,1} f(\xi)$ as a meromorphic function ϕ over the ξ-plane and apply to it some q-analog of the Laplace transform. In our case, putting

$$\mathcal{L}^{\lambda}_{q,1} \phi(z) = \sum_{\xi \in [\lambda; q]} \frac{\phi(\xi)}{\theta_q(z/\xi)},$$

gives again $f_{\lambda,u}$. This *discrete summation* process is due to Changgui Zhang see ([**27**], also see [**7**]) and it is heavily used in [**16**]. In this work, we rather use the first more algebraic and more naive method.

2.2. Classification through the Stokes sheaf ([16],[15])

The Stokes sheaf and its Lie algebra. — First, we recall the relevant definitions about asymptotic expansions. The semigroup $\Sigma = q^{-\mathbf{N}}$ operates on $\mathbf{C}^*$ with quotient $\mathbf{E}_q$ (its *horizon*); in the classical setting, one would rather have an operation of the semigroup $\Sigma = e^{]-\infty,0]}$ with horizon the circle S^1 of directions. We consider as sectors the germs at 0 of invariant open subsets of $\mathbf{C}^*$. We introduce two sheaves of differential algebras over $\mathbf{C}^*$ by putting, for any sector U,

$$\mathcal{B}(U) = \{f \in \mathcal{O}(U) \mid f \text{ is bounded on all invariant relatively compact subset of } U\}$$
$$\mathcal{A}'(U) = \{f \in \mathcal{O}(U) \mid \exists\, \widehat{f} \in \mathbf{C}[[z]] \ :\ \forall n \in \mathbf{N},\ z^{-n}\big(f - S_{n-1}\widehat{f}\big) \in \mathcal{B}(U)\},$$

where, as usually, $S_{n-1}\widehat{f}$ stands for the truncation. For any sector U, we write $U_\infty = U/\Sigma$ for its horizon (an open subset of $\mathbf{E}_q$). We now define a sheaf of differential algebras over $\mathbf{E}_q$ by putting

$$\mathcal{A}(V) = \varinjlim \mathcal{A}'(U),$$

the direct limit being taken for the system of those open subsets U such that their horizon is $U_\infty = V$. There is a natural morphism from $\mathcal{A}$ to the constant sheaf with fibre $\mathbf{C}\{z\}$ over $\mathbf{E}_q$ and it is an epimorphism (q-analog of Borel-Ritt lemma). We call $\mathcal{A}_0$ its kernel, the sheaf of infinitely flat functions. For instance, it is easy to see that a solution of a Fuchsian equation divided by a product of theta functions is flat within its domain (more on this in the next subsection).

We, then, write $\Lambda_I = I_n + \mathrm{Mat}_n(\mathcal{A}_0)$ for the subsheaf of groups of $\mathrm{GL}_n(\mathcal{A})$ made up of matrices infinitely tangent to the identity and we put $\Lambda_I^{\mathfrak{G}} = \Lambda_I \cap \mathfrak{G}(\mathcal{A})$. This is a sheaf of matrices of the form 2.2.1 with all the $F_{i,j}$ flat. Last, for a module $M = (\mathbf{C}(\{z\})^n, \Phi_A)$, we consider the subsheaf $\Lambda_I(M)$ of $\Lambda_I^{\mathfrak{G}}$ whose sections F satisfy the equality: $F[A] = A$ (automorphisms of M infinitely tangent to identity). This is the *Stokes sheaf* of the module M.

Note, for further use, that $\Lambda_I(M)$ is a sheaf of unipotent groups so that one can define algebraically the sheaf $\lambda_I(M)$ of their Lie algebras: we put $\lambda_I = \mathrm{Mat}_n(\mathcal{A}_0)$, $\lambda_I^{\mathfrak{g}} = \lambda_I \cap \mathfrak{g}(\mathcal{A})$ (see 2.2.2) and take as sections of $\lambda_I(M)$ those sections of $\lambda_I^{\mathfrak{g}}$ such that $(\sigma_q f)\, A = Af$. Obviously, f is a section of $\lambda_I(M)$ if and only if $I_n + f$ is a section of $\Lambda_I(M)$, or, equivalently, $\exp(f)$ is a section of $\Lambda_I(M)$. Indeed, the triangular form and the functional equations are easily checked, and the flatness properties stem from the well known fact that, for nilpotent matrices, f and $\exp(f) - I_n$ are polynomials in each other, without constant terms.

The q-analogs of Malgrange-Sibuya theorems. — One can find in [**16**] the following q-analogs of classical theorems by Malgrange and Sibuya:

Theorem 2.5. — *There are natural bijective mappings*

$$\mathfrak{G}(\mathbf{C}\{z\})\backslash\mathfrak{G}^{A_0}(\mathbf{C}[[z]]) \longrightarrow \mathfrak{G}(\mathbf{C}(\{z\}))\backslash\mathfrak{G}^{A_0}(\mathbf{C}((z))) \longrightarrow H^1(\mathbf{E}_q, \Lambda_I^{\mathfrak{G}}).$$

Actually, the following more general theorem is proven in *loc. cit.*, dealing with an arbitrary algebraic subgroup G of GL_n. Its proof relies on some heavy analysis (Newlander-Nirenberg theorem).

Theorem 2.6. — *Let M_0 be as above. There are natural bijective mappings*

$$\mathcal{F}(M_0) \longrightarrow G(\mathbf{C}\{z\})\backslash G^{A_0}(\mathbf{C}[[z]]) \longrightarrow G(\mathbf{C}(\{z\}))\backslash G^{A_0}(\mathbf{C}((z))) \longrightarrow H^1(\mathbf{E}_q, \Lambda_I^G),$$

where $\Lambda_I^G = \Lambda_I \cap G(\mathcal{A})$.

The former theorem is deduced from the latter together with the existence of asymptotic solutions. One can explicitly build, by discrete resummation, privileged cocycles associated to a class in $\mathcal{F}(M_0)$ and to "Stokes directions". In the next chapter, I shall exhibit an algebraic variant of this construction. Morally, it is possible because the sheaf $\Lambda_I(M_0)$ is almost a vector bundle over the elliptic curve $\mathbf{E}_q$.

Flatness conditions. — Details about the contents of this section can be found in [**17**] and [**16**]; see also the older references [**14**] and [**13**].

The above notion of flatness can be refined, introducing q-Gevrey levels. These may be characterized either in terms of growth (or decay) of functions near 0, or in terms of growth of coefficients of power series. We shall use here the following simple terminology and facts.

We start from a proper germ of $q^{-\mathbf{N}}$ invariant subset U of $(\mathbf{C}^*, 0)$. Then any solution of a fuchsian system that is holomorphic on U has polynomial growth at 0 (see for instance [**21**]); this is for instance true for a quotient of theta functions. We say that $f \in \mathcal{O}(U)$ has level of flatness $\geqslant t$ (where t is an integer) if, for one (hence any) theta function $\theta = \theta_{q,\lambda}$, the function $f|\theta|^t$ has polynomial growth near 0. We easily get the following implications.

Lemma 2.7

(i) *For $t > 0$, t-flatness implies flatness in the sense of asymptotics.*
(ii) *Solutions of pure systems of slope μ are μ-flat.*
(iii) *If a solution of a pure system of slope μ is t-flat with $t > \mu$, then it is 0.*

Normal polynomial forms. — The computations will follow the same pattern as in [**16**], [**15**]. However, we shall need a slightly more general version afterwards (proposition 2.8).

We start with a computation with two slopes. Take integers $\mu > \mu'$, square invertible matrices $A \in \mathrm{GL}_r(\mathbf{C})$ and $A' \in \mathrm{GL}_{r'}(\mathbf{C})$. Just for this section, call $\mathcal{V}(r, r', \mu, \mu')$

the subspace of $\mathcal{M}_{r,r'}(\mathbf{C}(\{z\}))$ spanned by matrices all of whose coefficients belong to $\sum_{\mu'\leqslant k<\mu}\mathbf{C}z^k$.

For $U\in\mathcal{M}_{r,r'}(\mathbf{C}(\{z\}))$, write $B_U=\left(\begin{smallmatrix}z^{-\mu}A & U\\ 0 & z^{-\mu'}A'\end{smallmatrix}\right)$. Then, for any such U, there exists a unique pair (F,V) with $F\in\mathcal{M}_{r,r'}(\mathbf{C}(\{z\}))$ and $V\in\mathcal{V}(r,r',\mu,\mu')$ such that the matrix $\left(\begin{smallmatrix}I_r & F\\ 0 & I_{r'}\end{smallmatrix}\right)$ defines an isomorphism from B_U to B_V. This amounts to solving:

$$(\sigma_q F)\,(z^{-\mu'}A')-(z^{-\mu}A)F=V-U. \tag{2.7.1}$$

Successive reductions boil the problem down to example 2.4. We shall write $\mathrm{Red}(\mu,A,\mu',A',U)$ for the pair (F,V).

Now, we come back to our usual notations 2.1.1 and 2.1.2. We consider the matrix A_U associated to $U=(U_{i,j})\in\prod_{1\leqslant i<j\leqslant k}\mathcal{M}_{r_i,r_j}(\mathbf{C}(\{z\}))$. Then, there is a unique pair $(\underline{F},V)$ with $\underline{F}=(F_{i,j})\in\prod_{1\leqslant i<j\leqslant k}\mathcal{M}_{r_i,r_j}(\mathbf{C}(\{z\}))$ and $V=(V_{i,j})\in\prod_{1\leqslant i<j\leqslant k}\mathcal{V}(r_i,r_j,\mu_i,\mu_j)$ such that the associated gauge transformation $F\in\mathfrak{G}(\mathbf{C}(\{z\}))$ defines an isomorphism from A_U to A_V. The pair $(\underline{F},V)$ can be computed by solving iteratively a system of equations of the type 2.7.1 for $1\leqslant i<j\leqslant k$. This is done by inductively with the help of the formula:

$$(F_{i,j},V_{i,j})=\mathrm{Red}\left(\mu_i,A_i,\mu_j,A_j,U_{i,j}+\sum_{i<l<j}(\sigma_q F_{i,l})\,U_{l,j}-\sum_{i<l<j}V_{i,l}F_{l,j}\right).$$

What we get is, in essence, the canonical form of Birkhoff and Guenther. Standing alone, this statement confirms our earlier contention in section 2.1, to the effect that all objects of $\mathcal{E}_1^{(0)}$ are locally equivalent to objects of $\mathcal{E}$.

Now, we shall have use for an extension of these results allowing for coefficients in $\mathcal{O}(\mathbf{C}^*,0)$ (instead of $\mathbf{C}(\{z\})$).

Proposition 2.8. — *Let A_U be as above, but with $U=(U_{i,j})\in\prod_{1\leqslant i<j\leqslant k}\mathcal{M}_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*,0))$. Then, there exists a unique pair $(\underline{F},V)$ with $\underline{F}=(F_{i,j})\in\prod_{1\leqslant i<j\leqslant k}\mathcal{M}_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*,0))$ and $V=(V_{i,j})\in\prod_{1\leqslant i<j\leqslant k}\mathcal{V}(r_i,r_j,\mu_i,\mu_j)$ such that the associated gauge transformation $F\in\mathfrak{G}(\mathcal{O}(\mathbf{C}^*,0))$ defines an isomorphism from A_U to A_V.*

Proof. — The same induction as before can be used, and the proof boils down to the following lemma.

Lemma 2.9. — *Let $\mu>\mu'$ in $\mathbf{Z}$, $A\in\mathrm{GL}_r(\mathbf{C})$,$A'\in\mathrm{GL}_{r'}(\mathbf{C})$ and $U\in\mathcal{M}_{r,r'}(\mathcal{O}(\mathbf{C}^*,0))$. There exists a unique pair (F,V) with $F\in\mathcal{M}_{r,r'}(\mathcal{O}(\mathbf{C}^*,0))$ and $V\in\mathcal{V}(r,r',\mu,\mu')$ satisfying 2.7.1.*

Proof. — The same reductions as in *loc. cit.* entail that we may as well assume from the beginning that $\mu=0$ and $\mu'=-1$. The equation as written has unknown F and right hand side $V-U$ in a space of rectangular matrices. Call s the rank of this space and call B the matrix of its automorphism $F\mapsto AFA'^{-1}$ relative to some basis. Multiplying both sides of 2.7.1 by A'^{-1} at right, we get an equivalent equation

of the shape $z\,\sigma_q X - BX = Y - Y^{(0)}$, for which we want to show that, for arbitrary $B \in \mathrm{GL}_s(\mathbf{C})$ and $Y \in \mathcal{O}(\mathbf{C}^*,0)^s$, there is a unique solution $(X, Y^{(0)}) \in \mathcal{O}(\mathbf{C}^*,0)^s \times \mathbf{C}^s$. Note that, replacing $X = \sum_{n\in\mathbf{Z}} X_n z^n$, $Y = \sum_{n\in\mathbf{Z}} Y_n z^n$ and $Y^{(0)}$ respectively by $\sum_{n\in\mathbf{Z}} B^n X_n z^n$, $\sum_{n\in\mathbf{Z}} B^{n-1} Y_n z^n$ and $B^{-1}Y^{(0)}$, we do not change the conditions on X, Y, $Y^{(0)}$, and we are led to study a similar problem with $B = I_s$. The latter problem can be tackled componentwise: we are to show that, for any $u \in \mathcal{O}(\mathbf{C}^*,0)^s$, there is a unique pair $(f,\nu) \in \mathcal{O}(\mathbf{C}^*,0) \times \mathbf{C}$ such that $z\sigma_q f - f = u - \nu$ (compare to example 2.4).

We apply the q-Borel-Ramis transform of level 1. This clearly sends $\mathcal{O}(\mathbf{C}^*,0)$ to $\mathcal{O}(\mathbf{C}^*)$: indeed, for any $A > 0$, $A^n q^{-n(n-1)/2}$ tends to 0 when $n \to \pm\infty$. From the computations in example 2.4, we deduce that we have to take $\nu = \mathcal{B}_{q,1}u(1) = \sum_{n\in\mathbf{Z}} q^{-n(n-1)/2} u_n$; we must then prove the existence and unicity of f. Replacing u by $u - \nu$, we may assume that $\mathcal{B}_{q,1}u(1) = 0$. We write $f'_n = q^{-n(n-1)/2} f_n$ and $u'_n = q^{-n(n-1)/2} u_n$ the coefficients of the q-Borel-Ramis transforms $\mathcal{B}_{q,1}f$ and $\mathcal{B}_{q,1}u$. We know that $\sum_{-\infty}^{+\infty} u'_k = 0$ and we require that $\forall\, n \in \mathbf{Z}, f'_{n-1} - f'_n = u'_n$. The only possibility allowing $f'_n \to 0$ for $n \to \pm\infty$ is given by the two equivalent definitions

$$\begin{aligned} f'_n &\underset{\text{def}}{=} \sum_{k=n+1}^{+\infty} u'_k \\ &\underset{\text{def}}{=} -\sum_{k=-\infty}^{n} u'_k. \end{aligned}$$

For $n \to +\infty$, we thus take (using the first definition of f'_n)

$$f_n = q^{n(n-1)/2} \sum_{k=n+1}^{+\infty} \frac{u_k}{q^{k(k-1)/2}}.$$

By assumption on u, there exists $A > 0$ and $C > 0$ such that, $\forall\, n \geqslant 0$, $|u_n| \leqslant CA^n$. Then,

$$|f_n| \leqslant \frac{CA^{n+1}}{|q|^n}\left(1 + \frac{A}{|q|^{n+1}} + \frac{A^2}{|q|^{(n+1)+(n+2)}} + \cdots\right),$$

whence $|f_n| = \mathrm{O}\left((A/|q|)^n\right)$ when $n \to +\infty$.

On the side of negative powers, putting, for convenience, $g_n = f_{-n}$ and $v_k = u_{-k}$ and, using the second definition for f'_n, we see that

$$g_n = q^{n(n+1)/2} \sum_{k=n}^{+\infty} \frac{v_k}{q^{k(k+1)/2}}.$$

By assumption on u, we have, for *any* $B > 0$, $|v_k| = \mathrm{O}(B^k)$ when $k \to +\infty$ and a similar computation as before then yields that, for *any* $B > 0$, $|g_n| = \mathrm{O}(B^n)$ when $n \to +\infty$, allowing one to conclude that $f \in \mathcal{O}(\mathbf{C}^*,0)$ as desired. □

We shall actually need only the following consequence of the proposition.

Corollary 2.10. — *Let $A = A_U$ in the canonical form 2.1.1, with $U = (U_{i,j}) \in \prod_{1\leqslant i<j\leqslant k} \mathcal{M}_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*,0))$. Then, there exists $F \in \mathfrak{G}(\mathcal{O}(\mathbf{C}^*,0))$ such that $A_V = F[A_U]$ has the same form, but with $V = (V_{i,j}) \in \prod_{1\leqslant i<j\leqslant k} \mathcal{M}_{r_i,r_j}(\mathbf{C}(\{z\}))$.* □

Obviously the same properties hold if one replaces $\mathcal{O}(\mathbf{C}^*,0)$ by $\mathcal{O}(\mathbf{C}^*)$.

3. Algebraic summation

3.1. The algorithm. — We keep the notations M_0, A_0, A_U of 2.1.1 and 2.1.2 and the corresponding conventions from section 2.1. Also, we shall, for $1 \leqslant i < j \leqslant k$, use the abreviation $\mu_{i,j} = \mu_i - \mu_j \in \mathbf{N}^*$.

Definition 3.1

(i) A summation divisor adapted to A_0 is a family $(D_{i,j})_{1\leqslant i<j\leqslant k}$ of effective divisors over the elliptic curve $\mathbf{E}_q$, each $D_{i,j}$ having degree $\mu_{i,j}$, the family satisfying moreover the following compatibility condition:

$$\forall\, i,l,j \text{ such that } 1 \leqslant i < l < j \leqslant k,\ D_{i,j} = D_{i,l} + D_{l,j}.$$

Obviously, it amounts to the same thing to give only the $k-1$ divisors $D_{i,i+1}$, $i = 1,\dots,k-1$.

(ii) We say that the adapted summation divisor $(D_{i,j})_{1\leqslant i<j\leqslant k}$ is allowed if it satisfies the following conditions:

$$\forall\, i,j \text{ such that } 1 \leqslant i < j \leqslant k,\ \mathrm{ev}_{\mathbf{E}_q}(D_{i,j}) \notin \overline{(-1)^{\mu_{i,j}} \frac{\mathrm{Sp}(A_i)}{\mathrm{Sp}(A_j)}}.$$

Here, for $S,T \subset \mathbf{C}^*$, we put $S/T = \{s/t \mid s \in S, t \in T\}$; $\overline{X}$ and $\mathrm{ev}_{\mathbf{E}_q}$ were defined in the introduction.

Note that, for an adapted summation divisor, the condition of being allowed is a generic one.

Example 3.2. — A special case is that of an adapted summation divisor concentrated on a point $\alpha \in \mathbf{E}_q$, that is, each $D_{i,j} = \mu_{i,j}[\alpha]$. Then the condition that D is allowed is equivalent to:

$$\forall\, i,j \text{ such that } 1 \leqslant i < j \leqslant k,\ \mu_{i,j}\,\alpha \notin \overline{\mathrm{Sp}(A_i)} - \overline{\mathrm{Sp}(A_j)}.$$

It is generically (that is, over a non empty Zariski open subset) satisfied by $\alpha \in \mathbf{E}_q$.

Now, let $(D_{i,j})_{1\leqslant i<j\leqslant k}$ be a summation divisor adapted to A_0. We choose points $a_l \in \mathbf{C}^*$ for $\mu_k < l \leqslant \mu_1$ such that, for $1 \leqslant i < j \leqslant k$,

$$D_{i,j} = \sum_{\mu_j < l \leqslant \mu_i} [\overline{a_l}].$$

These certainly exist. We then put:

$$t_i = \theta_q^{\mu_k} \prod_{\mu_k < l \leqslant \mu_i} \theta_{q,-a_l}.$$

Lemma 3.3

(i) *The functions $t_1, \dots, t_k \in \mathcal{M}(\mathbf{C}^*)$ are such that:*

(i1) *For $i = 1, \dots, k$, $\sigma_q t_i = \alpha_i z^{\mu_i} t_i$, where $\alpha_i \in \mathbf{C}^*$.*

(i2) *For $1 \leqslant i < j \leqslant k$, $\mathrm{div}_{\mathbf{E}_q}(t_i) - \mathrm{div}_{\mathbf{E}_q}(t_j) = D_{i,j}$ (the notation is explained in the course of the proof).*

(i3) *For $1 \leqslant i < j \leqslant k$, the function $t_{i,j} = \sigma_q t_i / t_j$ belongs to $\mathcal{O}(\mathbf{C}^*)$.*

(ii) *If the summation divisor $(D_{i,j})_{1 \leqslant i < j \leqslant k}$ is moreover allowed, for $1 \leqslant i < j \leqslant k$, the spectra of $\alpha_i A_i$ and $\alpha_j A_j$ have empty intersection on $\mathbf{E}_q$:*

$$\overline{\mathrm{Sp}(\alpha_i A_i)} \cap \overline{\mathrm{Sp}(\alpha_j A_j)} = \varnothing.$$

Proof. — It is an immediate consequence of the properties recalled in the introduction that these functions t_i indeed satisfy (i1). Moreover, the functional equation implies that the divisor $\mathrm{div}_{\mathbf{C}^*}(t_i)$ of zeroes and poles of t_i on $\mathbf{C}^*$ is invariant under the action of $q^{\mathbf{Z}}$, so that it makes sense to consider it as a divisor $\mathrm{div}_{\mathbf{E}_q}(t_i)$ on $\mathbf{E}_q$ (alternatively, one can consider t_i as a section of a line bundle over $\mathbf{E}_q$ and the notation is then classical). Again, because of the properties of theta functions, one clearly gets (i2). Assertion (i3) comes from the equalities

$$\begin{aligned} t_{i,j} &= \frac{\sigma_q t_i}{t_i} \times \frac{t_i}{t_j} \\ &= \alpha_i z^{\mu_i} \times \text{ a function with positive divisor.} \end{aligned}$$

The function $\theta_q^{\mu_i}\, \theta_q / \theta_{q,\alpha_i}$ satisfies the same functional equation as t_i, which means that their quotient is elliptic so that its divisor on $\mathbf{E}_q$ has trivial evaluation. Therefore,

$$\mathrm{div}_{\mathbf{E}_q}(t_i) = \overline{\frac{(-1)^{\mu_i}}{\alpha_i}}.$$

The conclusion (ii) then follows from the definition of an allowed divisor. □

We now introduce a temporary and slightly ambiguous notation. For an adapted summation divisor $D = (D_{i,j})_{1 \leqslant i < j \leqslant k}$, we write Θ_D for the following block-diagonal matrix:

$$\Theta_D = \begin{pmatrix} t_1 I_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & t_k I_{r_k} \end{pmatrix}.$$

Of course, it does not only depend on D, but on a particular choice of the functions $t_1, \dots, t_k$ whose existence has just been proved. However, the summation process we

are defining will produce a result that only depends on D. For a family of rectangular blocks $U'_{I,j}$, we shall use the following abreviation:

$$A'_{U'} = \begin{pmatrix} A'_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U'_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & A'_k \end{pmatrix}.$$

Lemma 3.4

(i) *The effect of the gauge transformation Θ_D is to "regularize" the diagonal blocks of $A = A_U$: $\Theta_D[A] = A'_{U'}$, where, for $1 \leqslant i < j \leqslant k$, $U'_{i,j} = t_{i,j}U_{i,j} \in M_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*, 0))$ and, for $i = 1, \dots, k$, $A'_i = \alpha_i A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$. If moreover the adapted summation divisor D is allowed, then, for $1 \leqslant i < j \leqslant k$, $\overline{\mathrm{Sp}(A'_i)} \cap \overline{\mathrm{Sp}(A'_j)} = \varnothing$.*

(ii) *Suppose we started with A_U in polynomial normal form. Then, we get $A'_{U'}$ such that $U'_{i,j} \in M_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*))$.*

Proof. — The computations are immediate. □

We shall now take two matrices A_U and A_V in the formal class of A_0, flatten their slopes through the gauge transformation Θ_D, and then link the resulting matrices $A'_{U'}$ and $A'_{V'}$ by an isomorphism defined over $\mathbf{C}^*$. This relies on the following

Proposition 3.5

(i) *Let*

$$A'_{U'} = \begin{pmatrix} A'_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U'_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & A'_k \end{pmatrix} \quad \text{and} \quad A'_{V'} = \begin{pmatrix} A'_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & V'_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & A'_k \end{pmatrix},$$

where, for $i = 1, \dots, k$, $A'_i \in \mathrm{GL}_{r_i}(\mathbf{C})$ are such that, for $1 \leqslant i < j \leqslant k$, $\overline{\mathrm{Sp}(A'_i)} \cap \overline{\mathrm{Sp}(A'_j)} = \varnothing$ and, for $1 \leqslant i < j \leqslant k$, $U'_{i,j}, V'_{i,j} \in M_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*, 0))$.

Then, there exists a unique $F' \in \mathfrak{G}(\mathcal{O}(\mathbf{C}^*, 0))$ such that $F'[A'_{U'}] = A'_{V'}$.

(ii) *If, for $1 \leqslant i < j \leqslant k$, $U'_{i,j}, V'_{i,j} \in M_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*))$, then $F' \in \mathfrak{G}(\mathcal{O}(\mathbf{C}^*))$.*

Proof. — We have to solve inductively the system of equations

$$\left(\sigma_q F'_{i,j}\right) - A'_i F'_{i,j} = V'_{i,j} - U'_{i,j} + \sum_{i<l<j} V'_{i,l} F'_{l,j} - \sum_{i<l<j} \left(\sigma_q F'_{i,l}\right) U'_{l,j}.$$

The induction is the same as the one we met when building normal polynomial forms. The proposition then follows from the following lemma.

Lemma 3.6

(i) *Let $B \in \mathrm{GL}_s(\mathbf{C})$ and $C \in \mathrm{GL}_t(\mathbf{C})$ be invertible complex matrices such that $\overline{\mathrm{Sp}(B)} \cap \overline{\mathrm{Sp}(C)} = \varnothing$. Then, for $Y' \in M_{t,s}(\mathcal{O}(\mathbf{C}^*, 0))$, the equation*

$$(\sigma_q X')\, B - CX' = Y'$$

has a unique solution $X' \in M_{t,s}(\mathcal{O}(\mathbf{C}^*, 0))$.

(ii) *If $Y' \in M_{t,s}(\mathcal{O}(\mathbf{C}^*))$, then $X' \in M_{t,s}(\mathcal{O}(\mathbf{C}^*))$.*

Proof. — We write the Laurent series:

$$X' = \sum_{n \in \mathbf{Z}} X'_n z^n \quad \text{and} \quad Y' = \sum_{n \in \mathbf{Z}} Y'_n z^n.$$

By identification, we obtain $X'_n = \Phi^{-1}_{q^n B, C}(Y'_n)$, where $\Phi_{q^n B, C}$ is the automorphism $M \mapsto M(q^n B) - CM$ of $M_{t,s}(\mathbf{C})$; that it is indeed an automorphism comes from the assumption that $q^n B$ and C have non intersecting spectra. For $n \to +\infty$, $\Phi^{-1}_{q^n B, C} \sim q^{-n} \Phi^{-1}_{B,0}$ and, for $n \to -\infty$, $\Phi^{-1}_{q^n B, C} \to \Phi^{-1}_{0,C}$. Taking $]r, R[\times e^{\imath \mathbf{R}}$ to be the annulus of convergence of Y', we conclude that the annulus of convergence of X' is $]r, |q|R[\times e^{\imath \mathbf{R}}$. Annuli of definition actually grow, again an illustration of the good regularity properties of the homological equation. □

Putting all together, we now get our first fundamental theorem.

Theorem 3.7

(i) *Let A_U, A_V be defined as above, in the formal class of A_0. Then, there exists a unique $F \in \mathfrak{G}(\mathcal{M}(\mathbf{C}^*, 0))$ such that $F[A_U] = A_V$ and, for $1 \leqslant i < j \leqslant k$, $\mathrm{div}_{\mathbf{E}_q}(F_{i,j}) \geqslant -D_{i,j}$ (the notation is explained in the course of the proof).*

(ii) *If A_U, A_V are in polynomial normal form, $F \in \mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$.*

Proof. — We put $A'_{U'} = \Theta_D[A_U]$ and $A'_{V'} = \Theta_D[A_V]$, then $F[A_U] = A_V$ is equivalent to $F'[A'_{U'}] = A'_{V'}$, where $F' = \Theta_D F \Theta_D^{-1}$. The matrices F and F' together are upper-triangular with diagonal blocks $I_{r_1}, \ldots, I_{r_k}$ and their over-diagonal blocks are related by the relations: $F_{i,j} = \frac{t_j}{t_i} F'_{i,j}$. This implies the unicity of $F \in \mathfrak{G}(\mathcal{M}(\mathbf{C}^*, 0))$ (resp. $\mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$) subject to the constraint that the coefficients of $F_{i,j}$ belong to $\frac{t_j}{t_i}\mathcal{O}(\mathbf{C}^*, 0)$ (resp. $\frac{t_j}{t_i}\mathcal{O}(\mathbf{C}^*)$). Since $\mathrm{div}_{\mathbf{E}_q}(t_i) - \mathrm{div}_{\mathbf{E}_q}(t_j) = D_{i,j}$, this proves (and explains) the given condition. □

As a matter of notation, we shall write $F_D(U, V)$ for the F obtained in the theorem: it does indeed depend only on D. We see it as the canonical resummation of $\widehat{F}(U, V)$ along the "direction" D. We shall write, in particular, $F_D(U) = F_D(0, U)$.

Let us call $\mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*, 0))$ (resp. $\mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*))$) the subset of $\mathfrak{G}(\mathcal{M}(\mathbf{C}^*, 0))$ (resp. of $\mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$) defined by the constraints $\mathrm{div}_{\mathbf{E}_q}(F_{i,j}) \geqslant -D_{i,j}$ for $1 \leqslant i < j \leqslant k$.

Corollary 3.8. — $F_D(U, V) = F_D(V) F_D(U)^{-1}$.

Proof. — Actually, $\mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*,0)) = \Theta_D\mathfrak{G}(\mathcal{O}(\mathbf{C}^*,0))\Theta_D^{-1}$ and $\mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*)) = \Theta_D\mathfrak{G}(\mathcal{O}(\mathbf{C}^*))\Theta_D^{-1}$, so that these subsets are subgroups. Then, the statement follows from the unicity property in the theorem. □

Corollary 3.9. — *The conclusion of the theorem still holds if one only assumes that* $U = (U_{i,j}) \in \prod_{1\leqslant i<j\leqslant k} \mathcal{M}_{r_i,r_j}(\mathcal{O}(\mathbf{C}^*,0))$.

Proof. — This immediately follows from corollary 2.10. □

Remark 3.10. — It is not difficult to prove that $\widehat{F}(U,V)$ is the aymptotic expansion of $F_D(U,V)$ in the sense of section 2.2. One first has to extend the definitions so as to allow for a pole at 0. The proof then proceeds in two steps.

(i) First, one proves that, in its domain of definition, $F_D(U,V)$ is a section of the sheaf $z^d\mathcal{B}$ for some $d \in \mathbf{Z}$. This is done using only the functional equation that it satisfies, and studying inductively its upper diagonal blocks $F_{i,j}$.

(ii) Then, one proves that the operator $F \mapsto A_V^{-1}\,(\sigma_q F)\,A_U$, sends $\mathfrak{G}(z^d\mathcal{B})$ to $\mathfrak{G}(z^{d+1}\mathcal{B})$. Starting from $F_D(U,V)$ and iterating yields the conclusion.

Actually, in [**16**], a stronger result is proved. It relies on a refined definition of asymptotics taking in account the position of poles; this is essential to get summation by discrete integral formulas.

Remark 3.11. — To give our theorem its functorial meaning, one should proceed as follows. One generalizes the construction of a vector bundle F_M from a q-difference module M. This defines a fibre functor ω over $\mathbf{E}_q$. Then, for each D, if one restricts to an appropriate subcategory of $\mathcal{E}_1^{(0)}$, $M \rightsquigarrow F_D(M)$ is an isomorphism from the fibre functor $\omega\circ\mathrm{gr}$ to ω. On the other hand, endowing F_M with the filtration coming from that of M, one defines an enriched functor and the underlying principle of all our uses of the homological equation is that this functor is fully faithful. This is exploited in [**23**].

3.2. Applications to classification

One direction of summation. — Let A_U, A_V be defined over $\mathbf{C}(\{z\})$. Suppose A_U and A_V are analytically equivalent. Then the power series $\widehat{F}(U,V)$ is convergent and satisfies the conclusion of theorem 3.7, so that, by unicity, $F_D(U,V) = \widehat{F}(U,V)$ for any allowed summation divisor. Conversely,

Proposition 3.12. — *Suppose* $F_D(U,V) \in \mathfrak{G}(\mathcal{O}(\mathbf{C}^*,0))$. *Then* A_U *and* A_V *are analytically equivalent (and all the above holds).*

Proof. — The gauge transform $F = F_D(U,V)$ is obtained by solving the system of equations

$$z^{-\mu_j}\,(\sigma_q F_{i,j})\,A_j - z^{-\mu_i}A_iF_{i,j} = V_{i,j} - U_{i,j} + \sum_{i<l<j} V_{i,l}F_{l,j} - \sum_{i<l<j} (\sigma_q F_{i,l})\,U_{l,j}.$$

By induction, we are reduced to the following lemma:

Lemma 3.13. — *Let $\mu > \mu'$ in $\mathbf{Z}$, $A \in \mathrm{GL}_r(\mathbf{C})$,$A' \in \mathrm{GL}_{r'}(\mathbf{C})$ and $Y \in \mathcal{M}_{r,r'}(\mathbf{C}(\{z\}))$. Let $X \in \mathcal{M}_{r,r'}(\mathcal{O}(\mathbf{C}^*,0))$ be a solution of the equation:*

$$(\sigma_q X)\,(z^{-\mu'}A') - (z^{-\mu}A)X = Y.$$

Then, one actually has $X \in \mathcal{M}_{r,r'}(\mathbf{C}(\{z\}))$.

Proof. — Going to the Laurent series and identifying coefficients, one finds

$$\forall\, n \in \mathbf{Z},\ q^{n+\mu'} X_{n+\mu'} A' - A X_{n+\mu} = Y_n.$$

Since $Y \in \mathcal{M}_{r,r'}(\mathbf{C}(\{z\}))$, $Y_n = 0$ for $n \ll 0$. Therefore, for $n \ll 0$, writing $d = \mu - \mu' \in \mathbf{N}^*$, one has $X_n = q^{-n} A X_{n+d} {A'}^{-1}$. Since $|q| > 1$, either $X_n = 0$ for $n \ll 0$, or the coefficients of X are rapidly growing for indices near $-\infty$ prohibiting convergence and contradicting the assumption that $X \in \mathcal{M}_{r,r'}(\mathcal{O}(\mathbf{C}^*,0))$. □

In order to make this a statement about classification, we introduce one more notation. We write

$$\mathfrak{G}_D^{A_0}(\mathcal{M}(\mathbf{C}^*,0)) = \{F \in \mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*,0)) \mid F[A_0] \in \mathrm{GL}_n(\mathbf{C}(\{z\}))\}.$$

Clearly, the subset $\mathfrak{G}_D^{A_0}(\mathcal{M}(\mathbf{C}^*,0))$ of the group $\mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*,0))$ is stable under the action by left translations of the subgroup $\mathfrak{G}(\mathcal{O}(\mathbf{C}^*,0))$. Now, the above proposition immediately entails

Proposition 3.14. — *Mapping A_U to $F_D(U)$ yields a bijection*

$$\mathcal{F}(M_0) \longrightarrow \mathfrak{G}(\mathcal{O}(\mathbf{C}^*)) \backslash \mathfrak{G}_D^{A_0}(\mathcal{M}(\mathbf{C}^*,0)).$$ □

This is strikingly similar to the corresponding "formal modulo analytic" description in proposition 2.3.

Varying the direction of summation. — Let $D = (D_{i,j})_{1 \leqslant i < j \leqslant k}$ be an allowed summation divisor for M_0, A_0. We consider as its support and write $\mathrm{Supp}(D)$ the union $\bigcup_{1 \leqslant i < j \leqslant k} \mathrm{Supp}(D_{i,j})$ and define the following Zariski open subset of $\mathbf{E}_q$: $V_D = \mathbf{E}_q \setminus \mathrm{Supp}(D)$. We also write U_D for the preimage of V_D in $\mathbf{C}^*$. Thus, the elements of $\mathfrak{G}_D(\mathcal{M}(\mathbf{C}^*,0))$ are holomorphic germs over $(U_D,0)$.

Now let D' be another allowed summation divisor. Then, for any A_U in the formal class of A_0, the gauge transformation $F_{D,D'}(U) \underset{\text{def}}{=} F_D(U)^{-1} F_{D'}(U)$ sends A_0 to itself. It is holomorphic on the open subset $(U_D \cap U_{D'}, 0)$. We call $U_{D,D'}$ the sector $U_D \cap U_{D'}$, which is the preimage of the open subset $V_{D,D'} = V_D \cap V_{D'}$ of $\mathbf{E}_q$. Note that, if D and D' have non intersecting supports (which is easy to realize), then U_D and $U_{D'}$ cover $\mathbf{C}^*$ and V_D and $V_{D'}$ cover $\mathbf{E}_q$.

Lemma 3.15. — *$F_{D,D'}(U)$ is a section of the sheaf $\Lambda_I(M_0)$ over $V_{D,D'}$.*

Proof. — One only has to prove that the upper-diagonal part of $F = F_{D,D'}(U)$ is flat. But its rectangular blocks satisfy: $(\sigma_q F_{i,j})\,(z^{-\mu_j} A_j) = (z^{-\mu_i} A_i) F_{i,j}$. This is a pure system of slope $\mu_{i,j} > 0$, hence $F_{i,j}$ is indeed flat by lemma 2.7. □

We now call $\mathfrak{U}$ (resp. $\mathfrak{V}$) the covering of $\mathbf{C}^*$ (resp. of $\mathbf{E}_q$) by the open subsets U_D (resp. V_D), where D runs among all the allowed summation divisors for A_0, M_0. The following is immediate.

Corollary 3.16. — *The family $(F_{D,D'}(U))_{D,D'}$ is a Cech cocycle of the sheaf $\Lambda_I(M_0)$ for the covering $\mathfrak{V}$ of $\mathbf{E}_q$.* □

Proposition 3.17. — *Mapping A_U to the cocycle $(F_{D,D'}(U))$ defines a one-to-one mapping:*

$$\mathcal{F}(M_0) \longhookrightarrow Z^1(\mathfrak{V}, \Lambda_I(M_0)).$$

Proof. — If A_U is analytically equivalent to A_V, we have $A_V = F[A_U]$ for a unique $F \in \mathfrak{G}(\mathbf{C}(\{z\}))$ and it is clear (by unicity) that $F_D(V) = F F_D(U)$ for all allowed divisors D, whence $F_{D,D'}(V) = F_{D,D'}(U)$ by immediate computation. This shows that the above mapping is well defined.

Conversely, just assume that $F_{D,D'}(V) = F_{D,D'}(U)$ for two allowed divisors with non intersecting supports. This equality gives $F_D(U,V) = F_{D'}(U,V)$. Hence, both sides are holomorphic over $U_D \cup U_{D'} = \mathbf{C}^*$ (near 0), and we already saw in proposition 3.12 that this implies the analytic equivalence of A_U and A_V. □

We now come to the second fundamental result of this paper.

Theorem 3.18. — *The above mapping yields a bijective correspondence:*

$$\mathcal{F}(M_0) \simeq H^1(\mathbf{E}_q, \Lambda_I(M_0)).$$

Proof. — Let A_U and A_V have the same image in $H^1(\mathbf{E}_q, \Lambda_I(M_0)$, hence in $H^1(\mathfrak{V}, \Lambda_I(M_0)$ (in Cech cohomology, the H^1 of a covering embeds into the direct limit). There is, for each allowed divisor D, a matrix $G(D) \in \mathfrak{G}(\mathcal{O}(U_D)) \cap \mathrm{Aut}(M_0)$ in such a way that:

$$\forall\, D, D' \;:\; F_{D,D'}(U) = (G(D))^{-1}\, F_{D,D'}(V) G(D').$$

One draws that $F_D(V) G(D)\,(F_D(U))^{-1}$ does not depend on D, so that it is holomorphic on $\bigcup U_D = \mathbf{C}^*$; call Φ their common value. As a gauge transformation, it sends A_U to A_V. By proposition 3.12, A_U and A_V are analytically equivalent, which proves the injectivity.

We now prove that our mapping from $\mathcal{F}(M_0)$ to $H^1(\mathfrak{V}, \Lambda_I(M_0))$ is onto. For that, we take $(\Phi_{D,D'})_{D,D'} \in Z^1(\mathfrak{V}, \Lambda_I(M_0))$. By definition of the sheaf $\Lambda_I(M_0)$, each component $\Phi_{D,D'}$ is an element of $\mathfrak{G}(\mathcal{O}(U_{D,D'}))$, so that our cocycle can be considered as describing a vector bundle over $\mathbf{C}^*$, trivialized by the covering $\mathfrak{U}$ and with structural group in $\mathfrak{G}$. By [**19**], theorem 1.0 (see also [**18**]) it is trivial in the following sense: there

is, for each D, a $\Phi_D \in \mathfrak{G}(\mathcal{O}(U_D))$ in such a way that, for all D, D', $\Phi_{D,D'} = \Phi_D^{-1}\Phi_{D'}$. Since the $\Phi_{D,D'}$ are automorphisms of A_0, the $\Phi_D[A_0]$ are all equal to a same matrix $A_{U'}$. Moreover, this is holomorphic over $\mathbf{C}^*$. By corollary 2.10, there is a $\Psi \in \mathfrak{G}(\mathcal{O}(\mathbf{C}^*, 0))$ such that $A_U = \Psi[A_{U'}]$ is meromorphic at 0. Then $\Psi\Phi_D$ is holomorphic on $(U_D, 0)$ and sends A_0 to A_U. Put $G_D = F_D(U)^{-1}\Psi\Phi_D$. This is a section of $\Lambda_I(M_0)$ over V_D. The equalities $F_D(U)G_D = \Psi\Phi_D$ entail $\Phi_D^{-1}\Phi_{D'} = G_D^{-1}F_{D,D'}(U)G_{D'}$, that is, the cocycle $(\Phi_{D,D'})_{D,D'}$ is equivalent to the cocycle $(F_{D,D'}(U))_{D,D'}$ which ends the proof of our statement.

There remains to check that the natural mapping from $H^1(\mathfrak{V}, \Lambda_I(M_0))$ to $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ is onto (we already said it was one-to-one). This is the content of proposition 4.4, to be proved after the discussion on the q-Gevrey filtration of the Stokes sheaf the in next chapter. □

4. The q-Gevrey filtration on the Stokes sheaf

The sources of inspiration for the contents of this chapter are [**12**], [**1**], [**9**], [**25**] and [**4**].

4.1. The filtration for the Stokes sheaf of a tamely irregular module

We stick to the conventions of section 2.1, in particular, the notations of 2.1.1 and 2.1.2.

Conditions of flatness. — Let F be a section of the sheaf $\Lambda_I(M_0)$. Then, for $1 \leqslant i < j \leqslant k$, the block $F_{i,j}$ is solution of the equation

$$\sigma_q F_{i,j}\left(z^{-\mu_j}A_j\right) = \left(z^{-\mu_i}A_i\right)F_{i,j}.$$

From this and lemma 2.7, we deduce that $F_{i,j}$ is $(\mu_i - \mu_j)$-flat and that, if it is t-flat for some $t > \mu_i - \mu_j$, then it vanishes.

We now introduce a filtration of the Stokes sheaf and a filtration of the sheaf of its Lie algebras. For real non-negative t, we call $\lambda_I^t(M_0)$ the subsheaf of $\lambda_I(M_0)$ made of t-flat sections and $\Lambda_I^t(M_0)$ the subsheaf $I_n + \lambda_I^t(M_0)$ of $\Lambda_I(M_0)$. The latter is a sheaf of unipotent subgroups, while the former is the sheaf of its Lie algebras (see the discussion in section 2.2). Both filtrations are decreasing and exhaustive (the 0-term is the total sheaf, the t-term is the trivial sheaf for $t > \mu_1 - \mu_k$).

From the previous argument, we see that $\lambda_I^t(M_0)$ has a very simple concrete description in terms of matrices: its sections have non trivial blocks only over the "curved over-diagonal" consisting of those (i,j)-blocks such that $\mu_i - \mu_j = t$. There is a similar description for $\Lambda_I^t(M_0)$ (taking in account the block-diagonal of identities). We shall however have use for a more intrinsic definition of these filtrations. We first describe the filtration of $\lambda_I(M_0)$.

Proposition 4.1

(i) *The sheaf $\lambda_I(M_0)$ is the sheaf of sections of the vector bundle associated (see section 2.1) to the tamely irregular module* $\mathrm{End}^{>0}(M_0)$.

(ii) *The above filtration on the sheaf $\lambda_I(M_0)$ is the decreasing filtration associated to the graduation inherited from* $\mathrm{End}^{>0}(M_0)$.

Proof. — We have $M_0 = P_1 \oplus \cdots \oplus P_k$, whence:

$$\mathrm{End}(M_0) = \bigoplus_{1 \leqslant i,j \leqslant k} \mathrm{Hom}(P_j, P_i).$$

The internal Hom $\mathrm{Hom}(P_j, P_i)$ is a pure module of slope $\mu_i - \mu_j$. Therefore, $\mathrm{End}^{>0}(M_0)$ is the sum of those $\mathrm{Hom}(P_j, P_i)$ such that $\mu_i > \mu_j$, *i.e.* $i < j$.

On the other hand, the vector bundle associated to $\mathrm{Hom}(P_j, P_i)$ has as sections on an open subset V of $\mathbf{E}_q$ the morphisms from P_j to P_i that are holomorphic on the preimage U of V in $\mathbf{C}^*$. This implies that the sheaf of sections of the vector bundle associated to the module $\mathrm{End}^{>0}(M_0)$ is indeed $\lambda_I(M_0)$; that it is the direct sum of its subsheaves $\lambda_I^{(t)}(M_0)$, where $\lambda_I^{(t)}(M_0)$ is the sheaf of sections of the pure vector bundle associated to the pure module

$$\mathrm{End}^{(t)}(M_0) = \bigoplus_{\mu_i - \mu_j = t} \mathrm{Hom}(P_j, P_i);$$

and that $\lambda_I^t(M_0)$ is the direct sum of the $\lambda_I^{(t')}(M_0)$ for all $t' \geqslant t$. □

Actually the whole structure only depends on the filtrations and the properties of internal Homs, so that it can be extended to an arbitrary tannakian category.

Proposition 4.2. — *Let t be a nonnegative integer.*

(i) *$\Lambda_I^t(M_0)$ is a sheaf of normal subgroups of $\Lambda_I(M_0)$.*

(ii) *The map $f \mapsto 1 + f$ induces an isomorphism:*

$$\lambda_I^{(t)}(M_0) \simeq \frac{\Lambda_I^t(M_0)}{\Lambda_I^{t+1}(M_0)}.$$

Proof. — Actually, these are purely algebraic properties: for a nilpotent two sided ideal I of a non commutative algebra A, the subgroups $1 + I^t$ of the unit group are normal and their successive quotients are isomorphic to the quotient modules I^t/I^{t+1}. □

We now are in position to reconstruct the Stokes sheaf by successive exact sequences:

$$1 \longrightarrow \Lambda_I^{t+1}(M_0) \longrightarrow \Lambda_I^t(M_0) \longrightarrow \lambda_I^{(t)}(M_0) \longrightarrow 0. \tag{4.2.1}$$

Note also that, again from general algebraic considerations, we have a sequence of central extensions:

$$0 \longrightarrow \lambda_I^{(t)}(M_0) \longrightarrow \frac{\Lambda_I(M_0)}{\Lambda_I^{t+1}(M_0)} \longrightarrow \frac{\Lambda_I(M_0)}{\Lambda_I^{t}(M_0)} \longrightarrow 1. \tag{4.2.2}$$

4.2. Cohomological consequences

Lemma 4.3. — *Let V be a proper open subset of $\mathbf{E}_q$. Then $H^1(V, \Lambda_I(M_0))$ is trivial.*

Proof. — We apply theorem I.2 of [**8**] to the exact sequence 4.2.1. This gives an exact sequence of pointed sets:

$$H^1(V, \Lambda_I^{t+1}(M_0)) \longrightarrow H^1(V, \Lambda_I^{t}(M_0)) \longrightarrow H^1(V, \lambda_I^{t}(M_0)).$$

If the extreme terms are trivial, so must be the central one (this, by the very definition of an exact sequence of pointed sets). The rightmost term is the first cohomology group of a vector bundle (after proposition 4.1) over an open Riemann surface. Such a bundle being a trivial bundle, its H^1 is trivial. The leftmost term is trivial for $t > \mu_1 - \mu_k$. By descending induction, the inner term is trivial for all t, hence for $t = 0$. □

Proposition 4.4. — *The covering $\mathfrak{V}$ is good.*

Proof. — This means that the map from $H^1(\mathfrak{V}, \Lambda_I(M_0))$ to $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ is an isomorphism. After [**1**], cor. 1.2.4, p. 113, this follows from the lemma. □

Note that this ends the proof of theorem 3.18.

4.3. The Stokes sheaf of a general module. — We briefly sketch here how the previous results extend to the Stokes sheaf of a module M. We take M in the formal class of M_0 and identify it with $(\mathbf{C}(\{z\})^n, A_U)$, according to the conventions of section 2.1.

The mapping $\Phi \mapsto F_D(U)\Phi F_D(U)^{-1}$ defines an isomorphism from $\Lambda_I(M_0)$ to $\Lambda_I(M)$ over V_D. Therefore, the two sheaves are locally isomorphic. Actually, the latter is obtained from the former by the operation of twisting by the cocycle $(F_{D,D'}(U))_{D,D'}$, described in [**8**], prop. 4.2 (also see [**1**], II.1 or [**25**], pp. 30-31). According to the same references, their H^1 are isomorphic. Moreover, the same operations provide a local isomorphism of the sheaves of Lie algebras, so that $\lambda_I(M)$ is a vector bundle. Last, these isomorphisms preserve the filtrations by levels of flatness.

Remark 4.5. — One should also note that the mapping $X \mapsto F_D(U)X$ defines an isomorphism from the space of solutions of A_0 holomorphic over $(U_D, 0)$ to the same space for A_U. This means that their sheaves of solutions are locally isomorphic. Since the former is a vector bundle, so is the latter. This yields an explicit way of associating

a vector bundle with an arbitrary module with integral slopes (for arbitrary slopes, see [**23**], [**24**]).

References

[1] D.G. Babbitt & V.S. Varadarajan – *Local Moduli for Meromorphic Differential Equations*, Astérisque, vol. 169-170, Société Mathématique de France, 1989.

[2] G.D. Birkhoff – The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations, *Proc. Amer. Acad. Arts and Sci.* **49** (1913), p. 521–568.

[3] G.D. Birkhoff & P.E. Guenther – Note on a Canonical Form for the Linear q-Difference System, *Proc. Nat. Acad. Sci. U.S.A.* **27** (1941), no. 4, p. 218–222.

[4] P. Deligne – Lettres à J.-P. Ramis, janvier & février 1986.

[5] ———, Catégories Tannakiennes, in *Grothendieck Festschrift, Vol. II* (P. Cartier et al., eds.), Progress in Math., Birkhäuser, 1990.

[6] P. Deligne & J. Milne – Tannakian Categories, in *Hodge Cycles, Motives and Shimura Varieties* (P. Deligne et al., eds.), Lect. Notes in Math., vol. 900, Springer Verlag, 1989.

[7] L. Di Vizio, J.-P. Ramis, J. Sauloy & C. Zhang – Equations aux q-différences, *Gaz. Math.* **96** (2003), p. 20–49.

[8] J. Frenkel – Cohomologie non abélienne et espaces fibrés, *Bull. Soc. math. France* **85** (1957), p. 135–220.

[9] M. Loday-Richaud – Stokes phenomenon, multisummability and differential Galois groups, *Ann. Inst. Fourier (Grenoble)* **44** (1994), no. 3, p. 849–906.

[10] F. Marotte & C. Zhang – Multisommabilité des séries entières solutions formelles d'une équation aux q-différences linéaire analytique, *Ann. Inst. Fourier (Grenoble)* **50** (2000), no. 6, p. 1859–1890.

[11] M. van der Put & M.F. Singer – *Galois theory of difference equations*, Lect. Notes in Math., vol. 1666, Springer Verlag, 1997.

[12] J.-P. Ramis – Dévissage Gevrey, in *Journées singulières de Dijon*, Astérisque, vol. 59-60, Société Mathématique de France, 1978, p. 173–204.

[13] ———, Fonctions θ et équations aux q-différences, non publié, Strasbourg, 1990.

[14] ———, About the growth of entire functions solutions to linear algebraic q-difference equations, *Ann. Fac. Sci. Toulouse Math. (6)* **I** (1992), no. 1, p. 53–94.

[15] J.-P. Ramis, J. Sauloy & C. Zhang – La variété des classes analytiques d'équations aux q-différences dans une classe formelle, *C. R. Math.* **338** (2004), no. 4, p. 277–280.

[16] ———, Local analytic classification of irregular q-difference equations, Article in preparation.

[17] J.-P. Ramis & C. Zhang – Développement asymptotique q-Gevrey et fonction thêta de Jacobi, *C. R. Acad. Sci. Paris Sér. I Math.* **335** (2002), p. 899–902.

[18] H. Röhrl – Das Riemann-Hilbertsche Problem der Theorie der linearen Differentialgleichungen, *Math. Ann.* **133** (1957), p. 1–25.

[19] ———, Holomorphic fiber bundles over Riemann surfaces, *Bull. Amer. Math. Soc. (N.S.)* **68** (1962), p. 125–160.

[20] J. Sauloy – Systèmes aux q-différences singuliers réguliers: classification, matrice de connexion et monodromie, *Ann. Inst. Fourier (Grenoble)* **50** (2000), no. 4, p. 1021–1071.

[21] ———, Galois theory of fuchsian q-difference equations, *Ann. scient. Éc. Norm. Sup.* 4^{e} *série* **36** (2003), no. 6, p. 925–968.

[22] ———, La filtration canonique par les pentes d'un module aux q-différences et le gradué associé, *Ann. Inst. Fourier (Grenoble)* **54** (2004), no. 1, p. 181–210.

[23] ———, Local Galois theory of irregular q-difference equations, article in preparation.

[24] ———, Isomonodromy for q-difference equations, article to be submitted.

[25] V.S. Varadarajan – Linear meromorphic differential equations: a modern point of view, *Bull. Amer. Math. Soc. (N.S.)* **33** (1996), no. 1, p. 1–42.

[26] C. Zhang – Développements asymptotiques q-Gevrey et séries Gq-sommables, *Ann. Inst. Fourier (Grenoble)* **49** (1999), no. 1, p. 227–261.

[27] ———, Une sommation discrète pour des équations aux q-différences linéaires et à coefficients analytiques: théorie générale et exemples, in *Differential equations and the Stokes phenomenon* (Braaksma, Immink, van der Put & Top, eds.), World scientific, 2003.

J. Sauloy, Laboratoire Émile Picard, CNRS UMR 5580, U.F.R. M.I.G., 118, route de Narbonne, 31062 Toulouse Cedex 4 (France) • *E-mail :* sauloy@picard.ups-tlse.fr
Url : http://picard.ups-tlse.fr/~sauloy/

Astérisque
296, 2004, p. 253–269

MONODROMY PRESERVING DEFORMATION AND DIFFERENTIAL GALOIS GROUP I

by

Hiroshi Umemura

For J.-P. Ramis on the occasion of his 60th birthday

***Abstract.* —** In 1914, J. Drach interpreted in terms of his infinite dimensional differential Galois theory R. Fuchs' work on the monodromy preserving deformation and the sixth Painlevé equation. This note of Drach contains a quite original idea but it is difficult to understand. We analyze his note by our infinite dimensional differential Galois theory. We get non-trivial examples of which we can calculate our Galois group.

***Résumé* (Déformation isomonodromique et groupe de Galois différentiel). —** En 1914, J. Drach interpréta le travail de R. Fuchs sur les déformations isomonodromiques et la sixième équation de Painlevé en termes de sa théorie de Galois de dimension infinie. La note de Drach contient une idée très originale mais difficile à comprendre. Nous analysons sa note en appliquant notre théorie de Galois différentielle de dimension infinie. Cela nous donne des exemples non triviaux dont nous pouvons calculer notre groupe de Galois.

1. Introduction

Today, there are a variety of ways of defining the Painlevé equations. Most of them are unimaginable from the original definition.

(1) Historically the origin of the Painlevé equations goes back to the pursuit of special functions defined by algebraic differential equations of the second order. Around 1900 Painlevé succeeded in classifying algebraic differential equations $y'' = F(t, y, y')$ without movable singular points, where F is a rational function of t, y and y' and t is the independent variable so that $y' = dy/dt$ and $y'' = d^2y/dt^2$. The property of being free from the movable singularities is nowadays called the Painlevé property. After he classified the equations satisfying the condition, Painlevé then threw away those

2000 *Mathematics Subject Classification.* — 12H05, 13N99, 33E17.
***Key words and phrases.* —** Differential Galois theory of infinite dimension, Painlevé equation, Isomonodromic deformation.

© Astérisque 296, SMF 2004

equations that he could integrate by the so far known functions and thus he arrived at the list of the six Painlevé equations. This is the first definition of the Painlevé equations. It is, however, very lucky that he could discover the Painlevé equations in this manner.

(2) In 1907, R. Fuchs discovered that the sixth Painlevé equation describes a monodromy preserving deformation of a second order ordinary linear equation $y'' = p(x)y$. Later R. Garnier generalized this for the other Painlevé equations.

(3) In our former work [**5**], we showed that we can recover the second Painlevé equation form a rational surface with a rational double point. We can regard this as an algebro-geometric definition of the second Painlevé equation.

(4) Masatoshi Noumi and Yasuhiko Yamada interpreted theory of Painlevé equations form the view point of Kač-Moody Lie algebra. They not only uniformly reviewed the theory of τ function of the Painlevé equations but also generalized the Painlevé equations in a natural frame work.

(5) There is another definition due to J. Drach [**1**] in 1914. He asserts the equivalence of the following two conditions for a function $\lambda(t)$.

(i) $\lambda(t)$ satisfies the sixth Painlevé equation.

(ii) The dimension of the Galois group of a non-linear differential equation

$$\frac{dy}{dt} = \frac{y(y-1)(t-\lambda)}{t(t-1)(y-\lambda)}$$

is finite.

In the second condition, the Galois group of general algebraic differential equation is involved. Namely the second condition depends on his infinite dimensional differential Galois theory, which has been an object of discussion since he proposed it in his thesis in 1898.

In this note, we apply our infinite dimensional Galois theory of differential equations [**7**] to study the result of J. Drach. We prove that the first condition (i) implies the second (ii).

Theorem 1.1. — *Let $\lambda(t)$ be a function of t satisfying the sixth Painlevé equation. Let $K = \mathbf{C}(t, \lambda(t), \lambda'(t))$ which is a differential field with derivation d/dt. Let $L = K(y)$ be a differential field extension of K such that y is transcendental over K and such that y satisfies*

$$\frac{dy}{dt} = \frac{y(y-1)(t-\lambda)}{t(t-1)(y-\lambda)}.$$

Then the Galois group $\mathrm{InfGal}(L/K)$ *is at most of dimension* 3.

Remark 1.2. — We can expect that generically the dimension of $\mathrm{InfGal}(L/K)$ is 3. Yet inequality $\dim \mathrm{InfGal}(L/K) < 3$ may occur. So it is important to determine the solutions λ of the sixth Painlevé equation and the corresponding $\mathrm{InfGal}(L/K)$ such that $\dim \mathrm{InfGal}(L/K) < 3$.

For the first Painlevé equation, we can prove a more precise result. However, this still relies on a statement about constant fields, called Proposition 5.3 below, which will be proven in [**8**].

***Theorem 1.3* (assuming Proposition 5.3 in § 5).** — *Let $\lambda(t)$ be a function of t satisfying the first Painlevé equation $\lambda'' = 6\lambda^2 + t$. Let $K = \mathbf{C}(t, \lambda(t), \lambda'(t))$ which is a differential field with derivation d/dt. Let $L = K(y)$ be a differential field extension of K such that y is transcendental over K and such that y satisfies q*

$$\frac{dy}{dt} = \frac{1}{2}\frac{1}{y - \lambda(t)}$$

Then the Galois group

$$\mathrm{InfGal}(L/K) \simeq \widehat{\mathrm{SL}}_{2L^{\natural}}.$$

Remarks 1.4. — As the proof of the Theorems shows, it is difficult to imagine how to deduce the condition (i) from (ii).

The assertion of Drach should be properly understood otherwise we would have counter examples. In fact, the second condition (ii) is closed under the specialization of the function $\lambda(t)$, whereas the first (i) is not so. Hence the first condition (i) should be replaced by

(i)' The function $\lambda(t)$ satisfies the sixth Painlevé equation P_{VI} or a degeneration of P_{VI}.

Why are the Theorems interesting? Because the Galois group, which is a formal group of infinite dimension in general, is very difficult to calculate. We have only two types of examples where we can calculate the Galois group. (1) If L/K is a strongly normal extension in the sense of Kolchin which is his generalization of classical Galois extension so that the Galois group $G := \mathrm{Gal}(L/K)$ of the extension is an algebraic group, then $\mathrm{InfGal}(L/K) = \widehat{G}$ and (2) for differential field extension $L = K(y)/K$ such that y is a solution of a Riccati equation with coefficients in K, $\mathrm{InfGal}(L/K)$ is a formal subgroup of $\widehat{\mathrm{SL}}_2$ (*cf.* Theorem (5.16), [**7**]).

Since we can prove only one direction of the assertion of Drach, our result is not satisfactory in the sense that it does not give us a new definition of the Painlevé equation. It offers us, however, highly non-trivial examples of differential field extensions of which we can calculate our Galois group.

The author would like to acknowledge his indebtedness to Daniel Bertrand. Without his constant interest in the subject and valuable discussions with him, this work would not have been done. It is a pleasure to thank B. Malgrange who kindly permitted us to add his letter to D. Bertrand as an appendix to this note.

2. Review of R. Fuchs' paper

R. Fuchs studied a monodromy preserving deformation of a linear differential equation $d^2y/dx^2 = p(x)y$. Namely he considered a system of linear equations

$$\begin{cases} \dfrac{\partial^2 y_i}{\partial x^2} = p y_i, \\ \dfrac{\partial y_i}{\partial t} = B y_i - A \dfrac{\partial y_i}{\partial x}, \end{cases} \quad \text{for } i = 1, 2, \tag{1}$$

where

$$p = \frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{(x-t)^2} + \frac{e}{(x-\lambda)^2} + \cdots$$

and we assume that λ is not a function of x but it is a function of t, *i.e.*, $\partial\lambda/\partial x = 0$. y_1 and y_2 are linearly independent solutions. The integrability of the system (1) implies

$$A(x, t) = \frac{x(x-1)(t-\lambda)}{t(t-1)(x-\lambda)} \quad \text{and} \quad B(x, t) = \frac{1}{2}\frac{\partial A}{\partial x}$$

and $\lambda(t)$ satisfies the sixth Painlevé equation P_{VI}.

Where does the non-linear differential equation

$$\frac{dy}{dt} = \frac{y(y-1)(y-\lambda)}{t(t-1)(t-\lambda)}$$

in Theorem 1.1 come from?

Lemma 2.1. — *We may assume that the Wronskian*

$$W_r = \begin{vmatrix} y_1 & y_2 \\ \partial y_1/\partial x & \partial y_2/\partial x \end{vmatrix} = 1.$$

Proof. — It is an exercise to check $\partial W_r/\partial t = \partial W_r/\partial x = 0$ so that W_r is a constant. It is sufficient to replace y_i by $(1/\sqrt{W_r})y_i$ for $i = 1, 2$. □

From now on we write T for t, W for x so that we consider the system

$$\begin{cases} \dfrac{\partial^2 y_i}{\partial W^2} = p y_i, \\ \dfrac{\partial y_i}{\partial T} = B(W, T) y_i - A(W, T)\dfrac{\partial y_i}{\partial W}, \end{cases} \quad \text{for } i = 1, 2, \tag{2}$$

Lemma 2.2. — *If we set $y = y_2/y_1$, then we have*

$$\begin{cases} \dfrac{\partial y}{\partial W} = \dfrac{1}{y_1^2}, \\ \dfrac{\partial y}{\partial T} = -A\dfrac{1}{y_1^2}. \end{cases}$$

Proof. — This is a consequence of Lemma 2.1. □

We are working in the partial differential field

$$\mathbf{C}(W,T)\langle\lambda(T)\rangle = \mathbf{C}(W,T,\lambda(T),\lambda'(T),\dots)(y_1,y_2,\partial y_1/\partial T,\partial y_2/\partial T) \tag{3}$$

with derivations $\{\partial/\partial W,\, \partial/\partial T\}$. The differential field extension

$$\mathbf{C}(W,T)\langle\lambda(T)\rangle(y_1,y_2,\partial y_1/\partial T,\partial y_2/\partial T)/\mathbf{C}(W,T)\langle\lambda(T)\rangle$$

is defined by the adjunction of the solutions $y_i,\ y_2$ of the system (2) of linear equations.

Now we introduce differential operators

$$\begin{cases} D_t = \dfrac{\partial}{\partial T} + A\,\dfrac{\partial}{\partial W}, \\ D_w = \quad y_1^2\,\dfrac{\partial}{\partial W}. \end{cases} \tag{4}$$

Then $[D_t, D_w] = 0$ so that the field (3) is a differential field with derivations $\{D_t,\ D_w\}$. If we regard the the field (3) as a partial differential field with derivations $\{D_t,\ D_w\}$, then it involves non-linear differential equations.

Lemma 2.3. — *$D_tW = A(W,\, T)$.*

Proof. — This follows from the definition of the operator D_t. □

Lemma 2.4. — $\dfrac{\partial y}{\partial T} + A\dfrac{\partial y}{\partial W} = 0$ *so that* $D_t y = 0$.

Proof. — This is a consequence of Lemma 2.2. □

Lemma 2.4 shows that y is a first integral of a non-linear ordinary differential equation $dY/dT = A(Y,\, T)$.

It follows from the definition of the operator(4) $D_w(W) = y_1^2$ and hence y_1 is algebraic of degree at most 2 over $\mathbf{C}(T)\langle\lambda\rangle\langle y\rangle\langle W\rangle$. Here $\langle\,\rangle$ should be interpreted in the partial differential field (3) with derivations $\{D_t,\ D_w\}$. Since $y_2 = yy_1$,

$$(\mathbf{C}(W,\,T)\langle\lambda\rangle(y_1,y_2,\partial y_1/\partial W,\,\partial y_2/\partial W : \mathbf{C}(T)\langle\lambda\rangle\langle y\rangle\langle W\rangle) \leqslant 2. \tag{5}$$

3. Infinite dimensional differential Galois theory

Let us now explain our differential Galois theory of infinite dimension using a particular example. Namely we consider a differential field extension $L = \mathbf{C}(T)\langle\lambda\rangle(W)$ over $K = \mathbf{C}(T)\langle\lambda\rangle$ with derivation D_t such that

(1) $D_t(T) = 1$,
(2) W is transcendental over the field K.
(3) $D_tW = \dfrac{W(W-1)(T-\lambda)}{T(T-1)(W-\lambda)}$ and
(4) $\lambda(T)$ is an arbitrary function of T.

Remark 3.1. — The extension $L = \mathbf{C}(T)\langle\lambda\rangle(W)/K = \mathbf{C}(T)\langle\lambda\rangle$ of the previous section is a special case of the extension introduced above. In the previous section, we assumed moreover that λ is a solution of the sixth Painlevé equation and that the fields L and K are differential subfields of

$$\mathbf{C}(W,\, T)\langle\lambda\rangle(y_1, y_2, \partial y_1/\partial W,\, \partial y_2/\partial W).$$

We start from the differential field extension $L = K(W)/K$ with derivation D_t. We define its Galois group. We consider the universal Taylor morphism $i : L \to L^\natural[[\tau]]$. Namely we set for an element $a \in L^\natural[[\tau]]$

$$i(a) = \sum_{n=0}^{\infty} \frac{1}{n!} D_t^n(a)\tau^n.$$

Here $L^\natural$ is the abstract field structure of the differential field L. Namely we forget in the differential field L the derivation D_t. The map i introduced above is a morphism of rings compatible with the derivations D_t and $\partial/\partial\tau$.

Consider now on $L^\natural$, the derivation $\partial/\partial W$, which we denote by $(\partial/\partial W)^\natural$ to avoid confusions. The differential field endowed with $(\partial/\partial W)^\natural$ will be denoted by $L^\sharp$. So we have in the power series ring $L^\sharp[[\tau]]$ two mutually commutative derivations $\partial/\partial\tau$ and $(\partial/\partial W)^\natural$. The latter operates as a derivation of coefficients of a power series.

The quotient field of $L^\sharp[[\tau]]$ is the field $L^\sharp[[\tau]][\tau^{-1}]$ of Laurent series that is the differential field with derivations $\partial/\partial\tau$ and $(\partial/\partial W)^\natural$. In this partial differential field $L^\sharp[[\tau]][\tau^{-1}]$, let $\mathcal{L}$ be the partial differential subfield generated by $i(L)$ and $L^\natural$ and we define $\mathcal{K}$ as the partial differential subfield generated by $i(K)$ and $L^\natural$.

Remark 3.2. — The $L^\natural$-vector space $\mathrm{Der}(L^\natural/K^\natural)$ of $K^\natural$-derivations of $L^\natural$ is 1-dimensional, and so it is spanned by any non zero element of the $L^\natural$-vector space $\mathrm{Der}(L^\natural/K^\natural)$. Hence we have

$$\mathrm{Der}(L^\natural/K^\natural) = L^\natural(\partial/\partial W)^\natural$$

Therefore the construction of $\mathcal{L}$ and $\mathcal{K}$ is independent of the choice of a generator of the $L^\natural$-vector space $\mathrm{Der}(L^\natural/K^\natural)$.

Now considering again the Taylor expansion of the coefficients of a Laurent series, we have a differential algebra morphism $L^\sharp[[\tau]][\tau^{-1}] \to L^\natural[[\xi]][[\tau]][\tau^{-1}]$, where ξ is the variable appearing when we expand the coefficients of our Laurent series.

$$L^\sharp \longrightarrow L^\natural[[\xi]], \quad a \longmapsto \sum_{n=0}^{\infty} \frac{1}{n!} \left((\partial/\partial W)^\natural\right)^n (a)\xi^n.$$

So now $\mathcal{L}$ and $\mathcal{K}$ are differential subfields of $L^\natural[[\xi]][[\tau]][\tau^{-1}]$ with derivations $\{\partial/\partial\xi, \partial/\partial\tau\}$.

Now we consider the functor of infinitesimal deformations of $\mathcal{L}/\mathcal{K}$ in

$$L^\natural[[\xi]][[\tau]][\tau^{-1}]$$

that is a principal homogeneous space of a formal group $\mathrm{InfGal}(L/K)$ of infinite dimension in general. This is the definition of our Galois group. To be more precise, we consider the category $\mathrm{Alg}(L^\natural)$ of commutative $L^\natural$-algebras. We define the functor $F : \mathrm{Alg}(L^\natural) \to (\mathrm{Sets})$ by setting for $A \in \mathrm{Alg}(L^\natural)$

$$F(A) := \{\varphi \in \mathcal{L} \to A[[\xi]][[\tau]][\tau^{-1}] \mid \varphi \text{ is a differential algebra morphism satisfying the following two conditions below}\}$$

(1) φ induces the identity map on $\mathcal{K}$.

(2) Let $N(A)$ be the ideal of the algebra A consisting of all the nilpotent elements of A. So we have a canonical morphism

$$r : A[[\xi]][[\tau]][\tau^{-1}] \longrightarrow A/N(A)[[\xi]][[\tau]][\tau^{-1}]$$

of reducing the coefficients of Laurent series modulo the ideal $N(A)$. Let $j : \mathcal{L} \to A[[\xi]][[\tau]][\tau^{-1}]$ be the composite of the inclusions

$$\mathcal{L} \subset L^\natural[[\xi]][[\tau]][\tau^{-1}] \subset A[[\xi]][[\tau]][\tau^{-1}].$$

Using this notation, the condition that we require is $r \circ \varphi = r \circ j$.

Intuitively φ is an infinitesimal deformation of the inclusion map j. Let

$$\mathcal{W}(\xi, \tau) \in L^\natural[[\xi]][[\tau]][\tau^{-1}]$$

be the image of $W \in L$ by the canonical map

$$L \longrightarrow L^\natural[[\xi]][[\tau]][\tau^{-1}].$$

Let $\varphi \in F(A)$. Then there exists a power series

$$\psi(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots \in A[[\xi]]$$

such that

$$\varphi(W) = \mathcal{W}(\psi(\xi), \tau)$$

and such that $\psi(\xi)$ is congruent to ξ modulo $N(A)$. More precisely

$$a_0,\ a_1 - 1,\ a_2,\ a_3,\ \cdots \in N(A).$$

The infinitesimal deformation φ is determined by the power series $\psi(\xi)$ because the $\{D_t, (\partial/\partial W)^\natural\}$-differential field $\mathcal{L}$ is generated over $\mathcal{K}$ by $\mathcal{W}(\xi, \tau)$. The set

$$G(A) = \{\psi(\xi) = a_0 + a_1\xi +_2 \xi^2 + \cdots \in A[[\xi]] \mid a_0,\ a_1 - 1,\ a_2,\ a_3,\ \cdots \in N(A).\}$$

of formal power series congruent to the identity ξ modulo $N(A)$ forms a group by the composite of power series. The group functor G plays the role of the Lie pseudo-group of all the coordinate transformations of 1-variable. We can show that

$$H(A) = \{\psi \mid \mathcal{W}(\psi(\xi), \tau) \text{ defines an element of } F(A)\}$$

forms a subgroup of $G(A)$. The subgroup functor H is defined by a set of algebraic differential equations. We baptized such a functor a Lie-Ritt functor in [**7**]. In the

classical language H is a Lie pseudo-group and $H(A)$ operates on $F(A)$ functorially. The group functor H is the Galois group $\mathrm{InfGal}(L/K)$ of L/K.

4. Proof of Theorem 1.1

We prove Theorem 1.1. Namely, we consider a differential field extension $L = \mathbf{C}(T)\langle\lambda(T)\rangle(W)$ over $K = \mathbf{C}(T)\langle\lambda(T)\rangle$ with derivation D_t satisfying the following conditions.

(1) D_t is a derivation over $\mathbf{C}$. Or equivalently $D_t(\mathbf{C}) = 0$.
(2) T is a variable over $\mathbf{C}$ and $D_t(T) = 1$.
(3) $\lambda(T)$ is a function of T satisfying the sixth Painlevé equation.
(4) W is transcendental over K.
(5) $D_t W = \dfrac{W(W-1)(T-\lambda)}{T(T-1)(W-\lambda)}$.

We have to show that the Galois group $\mathrm{InfGal}(L/K)$ is at most of dimension 3. We know that since λ satisfies the sixth Painlevé equation, we are in the framework of monodromy preserving deformation of R. Fuchs treated in § 2 and § 3. The differential fields L and K with derivation D_t are differential subfields of the partial differential field

$$\mathbf{C}(W,\, T)\langle\lambda\rangle(y_1, y_2, \partial y_1/\partial W,\, \partial y_2/\partial W).$$

with derivations D_t, D_w. We will show in [**8**] the following general fact. We prove, however, a part of the assertion that allows us to show Theorem 1.1.

Proposition 4.1. — *Consider in general an ordinary differential field extension L/K of an ordinary differential field K with derivation D_t such that L is finitely generated over K as an abstract field. Then, the dimension of the formal group* $\mathrm{InfGal}(L/K)$ *is equal to the transcendence degree* $\mathrm{tr.\,d.}[\mathcal{L}^\natural : \mathcal{K}^\natural]$ *of the abstract field extension $\mathcal{L}^\natural/\mathcal{K}^\natural$.*

Proof. — We give a sketch of a proof. Let us content ourselves to prove that the dimension of the Lie algebra $\mathrm{Lie}\,(\mathrm{InfGal}(L/K))$ is equal to the transcendence degree $\mathrm{tr.\,d.}[\mathcal{L}^\natural : \mathcal{K}^\natural]$. We construct $\mathcal{L}$ and $\mathcal{K}$ by the general procedure of [**7**]. Namely we consider the universal Taylor morphism

$$i : L \longrightarrow L^\natural[[\tau]].$$

Let $y_1, y_2, \ldots, y_n$ be a transcendence base of the abstract field extension $L^\natural/K^\natural$. Let

$$\partial/\partial y_i : K(y_1,\, y_2,\, \ldots, y_n) \longrightarrow K(y_1,\, y_2,\, \ldots, y_n)$$

be the derivation over $K^\natural$ of the rational function field

$$K(y_1,\, y_2,\, \ldots, y_n)^\natural$$

for $1 \leqslant i \leqslant n$. The derivation $\partial/\partial y_i$ can be extended to a unique derivation over $K^\natural$ of $L^\natural$ that we denote also by $\partial/\partial y_i$ for $1 \leqslant i \leqslant n$. The partial differential field $L^\natural$

endowed with derivations $\partial/\partial y_i$ $(1 \leqslant i \leqslant n)$ will be denoted by $L^\sharp$. $\mathcal{L}$ is the partial differential field generated by the image $i(L)$ and $L^\sharp$ in $L^\sharp[[\tau]][\tau^{-1}]$ with derivations

$$(\partial/\partial y_1)^\natural, (\partial/\partial y_2)^\natural, \dots, (\partial/\partial y_n)^\natural,\ d/d\tau, \tag{6}$$

where

$$(\partial/\partial y_1)^\natural, (\partial/\partial y_2)^\natural, \dots, (\partial/\partial y_n)^\natural$$

are derivations of coefficients of Laurent series so that derivations (6) of $\mathcal{L}$ commute. The partial differential field $\mathcal{K}$ is defined in a similar way. Namely it is the partial differential subfield of $L^\sharp[[\tau]][\tau^{-1}]$ generated by $L^\sharp$ and $i(K)$. So $\mathcal{L}$ and $\mathcal{K}$ are partial differential fields of $L^\sharp[[\tau]][\tau^{-1}]]$ with derivations

$$(\partial/\partial y_1)^\natural, (\partial/\partial y_2)^\natural, \dots, (\partial/\partial y_n)^\natural,\ d/d\tau.$$

The Galois group $\mathrm{InfGal}(L/K)$ is a formal group or a Lie-Ritt functor on the category of $\mathrm{Alg}(L^\natural)$ of $L^\natural$-commutative algebras. We consider the restriction

$$\mathrm{InfGal}(L/K) \otimes_{L^\natural} \mathcal{L}^\natural$$

of the functor

$$\mathrm{InfGal}(L/K)$$

on the subcategory $\mathrm{Alg}(\mathcal{L}^\natural)$ of $\mathcal{L}^\natural$ commutative algebras. Another ingredient is the functor

$$\mathfrak{F}_{L/K} : \mathrm{Alg}(\mathcal{L}^\natural) \longrightarrow (\mathrm{Sets}),$$

where (Sets) is the category of sets. Now we consider the universal Taylor morphism

$$\mathcal{L} \longrightarrow \mathcal{L}^\natural[[\boldsymbol{t},\ \boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n]],$$

where $\mathbf{t}$ and $\boldsymbol{w}_i$ are introduced by the derivations $\partial/\partial\tau$ and $\partial/\partial w_i$, which we identify with $\partial/\partial y_i$, for $1 \leqslant i \leqslant n$. So the universal Taylor morphism

$$\mathcal{L} \longrightarrow \mathcal{L}^\natural[[\boldsymbol{t}, \boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n]]$$

maps an element $a \in \mathcal{L}$ to

$$\sum_{m=(m_0,m_1,\dots,m_n)\in\mathbf{N}^{n+1}} \frac{1}{m!} \frac{\partial^{m_0}}{\partial\tau^{m_0}} \frac{\partial^{m_1}}{\partial w_1^{m_1}} \frac{\partial^{m_2}}{\partial w_1^{m_2}} \cdots \frac{\partial^{m_n}}{\partial w_1^{m_n}} (a)\, \boldsymbol{t}^{m_0} \boldsymbol{w}_1^{m_1} \mathbf{w_2}^{m_2} \cdots \boldsymbol{w}_n^{m_n}.$$

For every $\mathcal{L}^\natural$-algebra A, we have

$$\mathrm{Hom}_{\text{differential algebra}}(\mathcal{L},\ A[[\boldsymbol{t}, \boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n]]) \simeq \mathrm{Hom}_{\text{algebra}}(\mathcal{L}^\natural,\ A). \tag{7}$$

We set for an $\mathcal{L}^\natural$-algebra A

$$\begin{aligned}\mathfrak{F}_{L/K}(A) := \{f : \mathcal{L} \to A[[\boldsymbol{t}, \boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n]] \mid f \text{ is a partial differential algebra}\\ \text{morphism that is an infinitesimal deformation over } \mathcal{K} \text{ of the universal}\\ \text{Taylor morphism}\}\end{aligned}$$

We know that the functor $\mathrm{InfGal}(L/K) \otimes_{L^\natural} \mathcal{L}^\natural$ operates on the functor $\mathfrak{F}_{L/K}$ in such a way that

$$(\mathrm{InfGal}(L/K) \otimes_{L^\natural} \mathcal{L}^\natural,\ \mathfrak{F}_{L/K})$$

is a principal homogeneous space. See Theorem (5.11), [**7**]. So we have to show that the tangent space of the functor $\mathfrak{F}_{L/K}$ at the point $\mathfrak{F}_{L/K}(\mathcal{L}^\natural)$ is equal to $\mathrm{tr.\,d.}[\mathcal{L}^\natural : \mathcal{K}^\natural]$. In fact by (7),

$$\mathfrak{F}_{L/K}(\mathcal{L}^\natural)[\varepsilon]) = \{f : \mathcal{L} \to \mathcal{L}^\natural[\varepsilon][[\boldsymbol{t}, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n]] \mid f \text{ is a partial differential algebra morphism that is an infinitesimal } \mathcal{K}\text{-deformation of the universal Taylor morphism}\}$$

coincides with

$$\mathrm{Hom}_{\mathcal{K}^\natural}(\mathcal{L}^\natural,\ \mathcal{L}^\natural[\varepsilon]).$$

The latter is nothing but

$$\mathrm{Der}_{\mathcal{K}^\natural}(\mathcal{L}^\natural)$$

of which the dimension is $\mathrm{tr.\,d.}[\mathcal{L}^\natural : \mathcal{K}^\natural]$ □

To prove Theorem 1.1, by Proposition 4.1, we have to show $\mathrm{tr.\,d.}[\mathcal{L} : \mathcal{K}] \leqslant 3$. The following fact is well-known. See [**6**], Lemma (1.1) for example.

Lemma 4.2. — *Let F be an ordinary differential field and A a differential F-algebra. Let C_K and C_A be the rings of constants of F and A respectively. Then F and C_A are linearly disjoint over C_F.*

Let us denote

$$\mathbf{C}(W, T)\langle\lambda\rangle(y_1, y_2, \partial y_1/\partial W, \partial y_2/\partial W)$$

by M, which is a partial differential field with derivations $\{D_t, D_w\}$. M is an over field of L/K so that we have

$$L^\natural[[\tau]][\tau^{-1}] \subset M^\natural[[\tau]][\tau^{-1}].$$

Let us consider subfields $M^\natural\mathcal{L}$ and $M^\natural\mathcal{K}$ in $M^\natural[[\tau]][\tau^{-1}]$. They are closed under the derivation $d/d\tau$.

Lemma 4.3. — *In $M^\natural[[\tau]][\tau^{-1}]$, (i)$M^\natural$ and $\mathcal{L}$ are linearly disjoint over $L^\natural$ and (ii)$M^\natural$ and $\mathcal{K}$ are linearly disjoint over $L^\natural$.*

Proof. — We prove the first assertion, the second being proved similarly. Let us consider the composite field $M^\natural\mathcal{L}$ and $\mathcal{L}$ in $M^\natural[[\tau]][\tau^{-1}]$. They are partial differential fields. We consider only the derivation $d/d\tau$. If we notice that the ring of $d/d\tau$-constants of $M^\natural\mathcal{L}$ is $M^\natural$ and that the field of $d/d\tau$-constants of $\mathcal{L}$ is $L^\natural$, then the assertion (i) follows from Lemma 4.2. □

It follows from Lemma 4.3 that

$$\text{tr. d.}[\mathcal{L} : \mathcal{K}] = \text{tr. d.}[M^\natural \otimes_{L^\natural} \mathcal{L} : M^\natural \otimes_{L^\natural} \mathcal{K}] \tag{8}$$

$$= \text{tr. d.}[M^\natural\mathcal{L} : M^\natural\mathcal{K}]. \tag{9}$$

$\mathcal{L}$ is the partial differential field in $L^\natural[[\tau]][\tau^{-1}]$ generated by $i(L)$ and $L^\natural$ with respect to the derivations $(\partial/\partial W)^\natural$ and $d/d\tau$. Hence $M^\natural\mathcal{L}$ is the differential field in $M^\natural[[\tau]][\tau^{-1}]$ generated by $i(L)$ and $M^\natural$ with respect to the derivations $(\partial/\partial W)^\natural$ and $d/d\tau$. Since $\partial/\partial W = y_1^2 D_w$ in M by (4) and since $y_1^2 \in M$, $M^\natural\mathcal{L}$ is the partial differential field in

$$M^\natural[[\tau]][\tau^{-1}]$$

generated by $i(L)$ and $M^\natural$ with respect to the derivations $D_w^\natural$ and $d/d\tau$. Let us denote by abuse of notation by

$$i : M \longrightarrow M^\natural[[\tau]] \subset M^\natural[[\tau]][\tau^{-1}]$$

the universal Taylor morphism of D_t-differential field M. Since D_t commutes with D_w,

$$\begin{aligned} i(D_w(a)) &= \sum_{n=0}^{\infty} \frac{1}{n!} D_t^n(D_w(a))\tau^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} D_w(D_t^n(a))\tau^n \\ &= D_w^\natural(i(a)) \end{aligned}$$

for every element $a \in M$. Hence

$$i(D_w(W)) = D_w^\natural(i(W))$$

is in $M^\natural\mathcal{L}$. Since $D_t y = 0$,

$$i(y) = y^\natural \in M^\natural \subset M^\natural\mathcal{L} \subset M^\natural[[\tau]][\tau^{-1}].$$

So let

$$\mathbf{C}(T)\langle\lambda\rangle\langle y\rangle\langle W\rangle$$

denote as in § 2 the partial differential field generated by λ, y over $\mathbf{C}(T)$ with respect to the derivations $\{D_t, D_w\}$, then

$$i(\mathbf{C}(T)\langle\lambda\rangle\langle y\rangle\langle W\rangle) \subset M^\natural\mathcal{L}.$$

Hence

$$M^\natural i(\mathbf{C}(T)\langle\lambda\rangle\langle y\rangle\langle W\rangle) = M^\natural\mathcal{L}.$$

It follows from (5) and from Lemma 4.2

$$\text{tr. d.}[M^\natural\mathcal{L} : M^\natural\mathcal{K}] = \text{tr. d.}[M^\natural i(M) : M^\natural i(K)] \tag{10}$$

$$= \text{tr. d.}[M^\natural \otimes_{C_M} M : M^\natural \otimes_{C_K} K]. \tag{11}$$

Here C_M, C_K denote respectively the field of D_t-constants of M and K.

Lemma 4.4. — $\text{tr. d.}[M^\natural \otimes_{C_M} M : M^\natural \otimes_{C_K} K] \leqslant 3.$

Proof. — Let us recall

$$M = \mathbf{C}(W, T)\langle\lambda\rangle(y_1, y_2, \partial y_1/\partial W, \partial y_2/\partial W), \qquad K = \mathbf{C}(T)\langle\lambda\rangle,$$

with

$$W_r = y_1\frac{\partial y_2}{\partial W} - y_2\frac{\partial y_1}{\partial W} = 1$$

and $y = y_2/y_1 \in C_M$. Therefore tr. d.$[M : KC_M] \leqslant 3$. □

Now Theorem 1.1 follows from (8), (9), (10), (11) and Lemma 4.4.

5. Framework of proving Theorem 1.3

Now we show a framework of proving Theorem 1.3. We consider the system (12) below according to Garnier [**2**], p. 49.

$$\text{(12)} \qquad \begin{cases} \dfrac{\partial^2 y_i}{\partial W^2} = p y_i, \\ \dfrac{\partial y_i}{\partial T} = -\dfrac{1}{2}\dfrac{\partial B}{\partial W}\, y_i + B\,\dfrac{\partial y_i}{\partial W}, \end{cases} \qquad \text{for } i = 1, 2,$$

where

$$p(W) = c_4[W^4 - \lambda^4 + T(W^2 - \lambda^2)] + 2c_3[2(W^3 - \lambda^3) + T(W - \lambda)] + c_2(W^2 - \lambda^2) + 2c_1(W - \lambda) + \frac{3}{4(W-\lambda)^2} - \frac{\lambda'}{W - \lambda} + \lambda'^2$$

and

$$B(W) = \frac{1}{2}\,\frac{1}{W - \lambda},$$

and y_1, y_2 are linearly independent solutions of the first equation of (12), λ being a function of T, $\lambda' = d\lambda/dT$ and $\lambda'' = d^2\lambda/dT^2$. After Garnier, the integrability condition of (12) implies that $\lambda(T)$ satisfies

$$\text{(13)} \qquad \lambda'' = c_4(2\lambda^3 + \lambda T) + c_3(6\lambda^2 + t) + c_2\lambda + c_1,$$

where c_1, c_2, c_3, c_4 are constants.

Remark 5.1. — The ordinary differential equation of the second order (13) contains both the second Painlevé equation

$$\lambda'' = 2\lambda^3 + \lambda T + c_1,$$

when $c_4 = 1$, $c_3 = 0\, c_2 = 0$, and the first Painlevé equation

$$\lambda'' = 6\lambda^2 + T.$$

when $c_4 = 0$, $c_3 = 1$, $c_2 = c_1 = 0$.

Using the notation of this section, we consider the partial differential fields

$$M = \mathbf{C}(W, T)\langle\lambda\rangle(y_1, y_2, \partial y_1/\partial W, \partial y_2/\partial W) \tag{14}$$

with respect to the derivations $\partial/\partial W$, $\partial/\partial T$. We introduce mutually commutative differential operators on M

$$\begin{cases} D_t = \dfrac{\partial}{\partial T} + \dfrac{1}{2}\dfrac{1}{W - \lambda(T)}\dfrac{\partial}{\partial W}, \\ D_w = y_1^2 \dfrac{\partial}{\partial W}. \end{cases}$$

Remark 5.2. — It follows from the definition of differential operator D_t that

$$D_t T = 1, \qquad D_t^n \lambda(T) = \frac{\partial^n \lambda(T)}{\partial T^n} \quad \text{for } n = 0, 1, 2, \ldots.$$

Hence the first Painlevé equation

$$\lambda'' = 6\lambda^2 + T$$

is equivalent to

$$D_t^2 \lambda = 6\lambda^2 + T$$

in M.

Drach's idea for the sixth Painlevé equation suggests the equivalence of the following two conditions on a function $\lambda(T)$, where T is an independent variable. We use the same notation as above but logically they have nothing to do with the variables W, T and derivations D_t, D_w for a moment.

(1) $\lambda(T)$ satisfies the first Painlevé equation (13) with respect to the derivation D_t with

$$D_t(\mathbf{C}) = 0,\ D_t(T) = 1.$$

Namely $D_t^2\lambda = 6\lambda^2 + T$.

(2) $\mathrm{InfGal}(L/K)$ is finite dimensional, where $L = K(W)$, $K = \mathbf{C}(T)\langle\lambda\rangle$ such that D_t is a differential operator on L satisfying

$$D_t(\mathbf{C}) = 0, \qquad D_t T = 1, \qquad D_t W = \frac{1}{2}\frac{1}{W - \lambda},$$

the element W being transcendental over K.

To prove Theorem 1.3, we argue as in § 3. Let us assume the condition (i) above that λ satisfies the first Painlevé equation

$$D_t^2 \lambda = 6\lambda^2 + T.$$

We start from the differential fields L, K in Condition (ii). Then we know by Garnier by virtue of Condition (i) that we are in the situation of the beginning of this section. Namely $K = \mathbf{C}(T)\langle T\rangle$, $L = K(W)$ are partial differential subfields of

$$M = \mathbf{C}(W, T)\langle\lambda\rangle(y_1, y_2, \partial y_1/\partial W, \partial y_2/\partial W)$$

with derivations D_t, D_w. We construct as in § 3, partial differential field $\mathcal{L}$, $\mathcal{K}$. We have to show that

$$\text{tr. d.}[\mathcal{L} : \mathcal{K}] = 3. \tag{15}$$

This implies that the dimension of the Lie algebra of the formal group $\text{InfGal}(L/K) = 3$. Then Lie's classification of Lie algebras operating on a manifold of dimension 1 shows

$$\text{Lie algebra of } \text{InfGal}(L/K) \simeq sl_2$$

and hence

$$\text{InfGal}(L/K) \simeq \widehat{\text{SL}}_{2L^{\natural}}.$$

To prove (15), we have to show

$$\text{tr. d.}[M : K] = 4 \tag{16}$$

and

$$\text{tr. d.}[C_M : C_K] = 1, \tag{17}$$

where $K = \mathbf{C}(T)\langle\lambda\rangle$ and C_M, C_K are respectively D_t constants of M, K.

We will give a proof of (16) and (17) in [**8**]. The equality (16) is proved in a standard way. The proof of equality (17) is reduced to the following assertion.

Proposition 5.3. — *The field of constants of the differential field*

$$\mathbf{C}(T,\, \lambda,\, \lambda',\, W,\, y_1,\, \partial y_1/\partial W)$$

with derivation D_t coincides with $\mathbf{C}$.

The proof of the Proposition 5.3 is as much involved as that of the irreducibility Theorem of the first Painlevé equation. See also Remark 1.2.

6. Questions

Malgrange [**3**] proposes a differential Galois theory of infinite dimension. The relation between his theory and ours remains to be clarified. The following natural questions arises.

Question 6.1. — *Interprete Drach's paper by Malgrange's theory.*

Looking at the first version of this note, Malgrange proved, in the frame work of his differential Galois theory, an analogue of the inequality in Theorem 1.1 for the first and sixth Painlevé equations. So far as the inequality is concerned, his marvelous proof is geometric and more accessible than our algebraic argument. We notice here a difference. Namely his proof depends on the monodromy preserving deformation of a linear differential equation for a 2×2 matrix introduced by Jimbo and Miwa. On the other hand, Noumi and Yamada [**4**] introduced a more symmetric monodromy preserving deformation associated with $\widehat{\mathfrak{so}}(8)$ to define the sixth Painlevé equation. So it is natural to ask

Question 6.2. — *Can one use the Noumi-Yamada system associated with* $\widehat{\mathfrak{so}}(8)$ *to prove Theorem 1.1?*

We naively believe that our extensions L/K offer us totally new examples of which we can determine the Galois group $\mathrm{InfGal}(L/K)$. You may doubt this belief.

Question 6.3. — *Are the extensions* L/K *studied in this note subextensions of strongly normal extensions? To be more affirmative, prove that they are not so.*

7. Appendix: Extract from a letter of B. Malgrange to D. Bertrand

Grenoble, le 16.10.03

[...] Voici les détails dont je te parlais avant-hier.

7.1. Sorite général. — On considère un système linéaire 2×2,

$$dF = \Omega F, \quad F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \text{ et } \Omega = (\omega_{ij}),\ 1 \leqslant i, j \leqslant 2.$$

Les ω_{ij} sont holomorphes sur un ouvert de $\mathbf{C}^n$ (par exemple ; ou aussi polynomiales en $x_1, \ldots, x_n$, etc. ; dans chaque cas, on a un résultat analogue). Ici, n est $\geqslant 2$ et, à part ça, quelconque. Ω doit vérifier la condition d'intégrabilité $d\Omega = \Omega \wedge \Omega$.

Posons $f_2/f_1 = w$.

w vérifie $dw + \alpha + \beta w + \gamma \frac{w^2}{2}$, avec $\alpha = -\omega_{21}, \beta = \omega_{11} - \omega_{22}, \gamma = 2\omega_{12}$; la condition d'intégrabilité s'écrit ici $d\pi = \pi \wedge \partial\pi/\partial w$, $\pi = dw + \alpha + \beta w + \gamma w^2/2$; explicitement, $d\alpha = \alpha \wedge \beta$, $d\beta = \alpha \wedge \gamma$, $d\gamma = \beta \wedge \gamma$.

On a évidemment une structure projective transverse associée au feuilletage $\pi = 0$ dans l'espace (x, w) ; d'où *par restriction à* $w = 0$ une structure projective transverse associée au feuilletage $\omega_{21}(= -\alpha) = 0$ dans l'espace des x (structure projective à prendre dans le sens de mon article[(1)] sur Galois différentiel = des équations différentielles qui définissent localement à un difféomorphisme près une structure projective au sens usuel).

7.2. Application à Painlevé. — On applique ça aux équations de Painlevé qui proviennent de systèmes linéaires du type précédent ; ici $n = 2$, et on écrit (x, t) au lieu de (x_1, x_2).

[(1)] B. Malgrange, *Le groupoïde de Galois d'un feuilletage*, Monogr. Ens. math., 38, 2001, 465-501.

Cas de P_1 (cf. Jimbo-Miwa[2], Appendix C). — On considère le système

$$\frac{\partial F}{\partial x} = AF \ , \ \frac{\partial F}{\partial t} = BF, \text{ avec}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x^2 + \begin{pmatrix} 0 & y \\ 4 & 0 \end{pmatrix} x + \begin{pmatrix} -z & y^2 + t^2/2 \\ -4y & z \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & y \\ 2 & 0 \end{pmatrix}.$$

Les conditions d'intégrabilité s'écrivent ici $dy/dt = z$, $dz/dt = 6y^2 + t$, *i.e.*, y satisfait à P_1 (y et z sont supposés a priori *dépendre de t seul*). Alors, modulo ces équations, on peut appliquer ce qui précède ; on a $\Omega = Adx + Bdt$, $\omega_{12} = 4(x-y)dx + 2dt$, *i.e.*, modulo que y satisfait à P_1, on a une structure projective transverse[3] (*cf.* Umemura[4], th. 1.3).

Cas de P_6

$$A = \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t} \ , B = -\frac{A_t}{x-t}$$

avec les notations A_0, A_1, A_t données dans Miwa-Jimbo (et reprises, avec un léger changement de notations dans Boalch[5], p. 29-32).

On prend ici $\omega_{12} = 0$; en explicitant, on trouve $\dfrac{y-x}{x(x-1)}dx = \dfrac{y-t}{t(t-1)}dt$, avec y fonction de t vérifiant P_6 (*cf.* Umemura, th. 1.1). Détails ci-dessous.

Bien sûr, cela s'appliquerait à tous les sytèmes 2×2 vérifiant la condition d'intégrabilité.

Tout cela est immédiat. La question difficile est de savoir s'il y a ou non des réductions supplémentaires (probablement pas pour P_1, ni pour P_6 dans le cas générique).

B. Malgrange

Détail du calcul sur P_6 (cf. Jimbo-Miwa ou Boalch, loc. cit.).— On a

$$A_0 = \begin{pmatrix} z_0 + \theta_0 & -uz_0 \\ (z_0 + \theta_0)/u & -z_0 \end{pmatrix}, \ A_1 = \begin{pmatrix} z_1 + \theta_1 & -vz_1 \\ (z_1 + \theta_1)/v & -z_1 \end{pmatrix}, \ A_t = \begin{pmatrix} z_t + \theta_t & -wz_t \\ (z_t + \theta_t)/w & -z_t \end{pmatrix}.$$

(Boalch écrit z_1, z_3, z_2 au lieu de z_0, z_1, z_t.)

Conditions : $A_0 + A_1 + A_t = -\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ (les θ et k sont fixés), *i.e.*,

$$z_0 + z_1 + z_t = k_2 \ , \ uz_0 + vz_1 + wz_t = 0 \ , \ (z_0 + \theta_0)/u + (z_1 + \theta_1)/v + (z_t + \theta_t)/w = 0$$

[2] M. Jimbo, T. Miwa, *Monodromy preserving deformations of LDEs with rational coefficients II*, Physica 2D, 1981, 407-448.

[3] Il faut prendre cette assertion au sens de mon article, avec toutefois un grain de sel (interpréter le fait que y satisfait à P_1 au moyen d'un système de 2 feuilletages emboîtés, en regardant des transversales à l'un dans des transversales de l'autre. Il y a un sorite à écrire que je n'ai pas vérifié en détail).

[4] H. Umemura, *Monodromy preserving deformation and differential Galois group I*, ce volume.

[5] P. Boalch, *The Klein solution to Painlevé's sixth equation*, math.AG/0308221.

(et alors, on a : $k_1 + k_2 + \theta_0 + \theta_1 + \theta_t = 0$).

Ici, $a_{12} = -\dfrac{uz_0}{x} - \dfrac{vz_1}{x-1} - \dfrac{wz_t}{x-t}$, $b_{12} = \dfrac{wz_t}{x-t}$, et l'équation est

$$\Big(\frac{uz_0}{x} + \frac{vz_1}{x-1} + \frac{wz_t}{x-t}\Big)dx = \frac{wz_t}{x-t}\,dt.$$

Éliminons les variables inutiles avec les conditions précédentes, auxquelles on peut rajouter (*cf.* Boalch) $(1+t)uz_0 + tvz_1 + wz_t = 1$. On trouve

$$\frac{-x + tuz_1}{x(x-1)}\,dx = wz_t\,dt.$$

Posant $y = tuz_1$, la dernière condition d'intégrabilité est que y *satisfasse à* P_6 (les autres conditions ont déjà été données implicitement, en écrivant que les classes de conjugaison des A sont fixées, et que leur somme est constante).

On trouve, d'autre part, $wz_t = \dfrac{y-t}{t(t-1)}$ (*loc. cit.*), d'où le résultat final

$$\frac{y-x}{x(x-1)}\,dx = \frac{y-t}{t(t-1)}\,dt,$$

annoncé plus haut, pour $\omega_{12} = 0$ (avec y satisfaisant à P_6).

References

[1] J. Drach – "Sur les équations différentielles du premier ordre et du premier degré", *C. R. Acad. Sci. Paris* (1914), p. 926–929.

[2] M.R. Garnier – "Sur les équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes", *Ann. Sci. École Norm. Sup.* (1912), p. 1–126.

[3] B. Malgrange – "Le groupoïde de Galois d'un feuilletage", in *Essays on geometry and related topics. Vol. 1, 2*, Monogr. Enseign. Math., vol. 38, l'Enseignement Mathématique, Genève, 2001, p. 465–501.

[4] M. Noumi & Y. Yamada – "A new Lax pair for the sixth Painlevé equation associated with $\widehat{\mathfrak{so}(8)}$", arXiv: `math-ph/0203029`, 2002.

[5] M.-H. Saito & H. Umemura – "Painlevé equations and deformations of rational surfaces with rational double points", in *Proc. of Nagoya international workshop on Physics and Combinatorics*, World Scientific, Singapore, 2001, p. 320–365.

[6] H. Umemura – "Galois theory of algebraic and differential equations", *Nagoya Math. J.* **144** (1996), p. 1–58.

[7] ———, "Differential Galois theory of infinite dimension", *Nagoya Math. J.* **144** (1996), p. 59–135.

[8] ———, "Monodromy preserving deformation and differential Galois group II", in preparation.

H. Umemura, Graduate School of Mathematics, Nagoya University, Furocho Chikusaku 464-8602, Nagoya (Japan) • *E-mail :* `umemura@math.nagoya-u.ac.jp`

ASTÉRISQUE

2004

297. Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes (II), M. LODAY-RICHAUD éditeur
296. Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes (I), M. LODAY-RICHAUD éditeur
295. Cohomologies p-adiques et applications arithmétiques (III), P. BERTHELOT, J.-M. FONTAINE, L. ILLUSIE, K. KATO, M. RAPOPORT, éditeurs
294. SÉMINAIRE BOURBAKI, volume 2002/2003, exposés 909-923
293. M. EMERTON, M. KISIN – *The Riemann-Hilbert correspondence for unit F-crystals*
292. F. LE ROUX – *Homéomorphismes de surfaces. Théorèmes de la fleur de Leau-Fatou et de la variété stable*
291. L. FARGUES, E. MANTOVAN – *Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales*

2003

290. SÉMINAIRE BOURBAKI, volume 2001/2002, exposés 894-908
289. P. SCOTT, G.A. SWARUP – *Regular neighbourhoods and canonical decompositions for groups*
288. M. REES – *Views of Parameter Space : Topographer and Resident*
287. Geometric Methods in Dynamics (II), volume in honor of Jacob Palis, W. DE MELO, M. VIANA, J.-C. YOCCOZ, editors
286. Geometric Methods in Dynamics (I), volume in honor of Jacob Palis, W. DE MELO, M. VIANA, J.-C. YOCCOZ, editors
285. PO HU – *Duality for Smooth Families in Equivariant Stable Homotopy Theory*
284. Autour de l'analyse microlocale, volume en l'honneur de Jean-Michel Bony, G. LEBEAU, éditeur

2002

283. L. ROBBIANO, C. ZUILY – *Analytic theory for the quadratic scattering wave front set and application to the Schrödinger equation*
282. SÉMINAIRE BOURBAKI, volume 2000/2001, exposés 880-893
281. T. DUQUESNE, J.-F. LE GALL – *Random Trees, Lévy Processes and Spatial Branching Processes*
280. A. MOKRANE, P. POLO, J. TILOUINE – *Cohomology of Siegel varieties*
279. Cohomologies p-adiques et applications arithmétiques (II), P. BERTHELOT, J.-M. FONTAINE, L. ILLUSIE, K. KATO, M. RAPOPORT, éditeurs
278. Cohomologies p-adiques et applications arithmétiques (I), P. BERTHELOT, J.-M. FONTAINE, L. ILLUSIE, K. KATO, M. RAPOPORT, éditeurs
277. B. RÉMY – *Groupes de Kac-Moody déployés et presque déployés*
276. SÉMINAIRE BOURBAKI, volume 1999/2000, exposés 865-879

2001

275. J.-M. BISMUT, S. GOETTE – *Families torsion and Morse functions*
274. A. BONNET, G. DAVID – *Cracktip is a global Mumford-Shah minimizer*
273. K. NISHIYAMA, H. OCHIAI, K. TANIGUCHI, H. YAMASHITA, S. KATO – *Nilpotent orbits, associated cycles and Whittaker models for highest weight representations*
272. M. BOILEAU, J. PORTI – *Geometrization of 3-orbifolds of cyclic type (avec la collaboration de M. HEUSENER)*
271. M. KASHIWARA, P. SCHAPIRA – *Ind-Sheaves*
270. M. BONK, J. HEINONEN, P. KOSKELA – *Uniformizing Gromov hyperbolic spaces*
269. J.-L. WALDSPURGER – *Intégrales orbitales nilpotentes et endoscopie pour les groupes classiques non ramifiés*

2000

268. J. FRANCHETEAU, G. MÉTIVIER – *Existence de chocs faibles pour des systèmes quasi-linéaires hyperboliques multidimensionnels*
267. R. CERF – *Large Deviations for three Dimensional Supercritical Percolation*
266. SÉMINAIRE BOURBAKI, volume 1998/1999, exposés 850-864
265. O. BIQUARD – *Métriques d'Einstein asymptotiquement symétriques*
264. L. LIPSHITZ, Z. ROBINSON – *Rings of Separated Power Series and Quasi-Affinoid Geometry*
263. C. SABBAH – *Équations différentielles à points singuliers irréguliers et phénomène de Stokes en dimension 2*
262. A. VASY – *Propagation of singularities in three-body scattering*
261. Géométrie complexe et systèmes dynamiques, colloque en l'honneur d'Adrien Douady, Orsay 1995, M. FLEXOR, P. SENTENAC et J.-C. YOCCOZ, éditeurs

1999

260. S. D. CUTKOSKY – *Local monomialization and factorization of morphisms*
259. R. KRIKORIAN – *Réductibilité des systèmes produits-croisés à valeurs dans des groupes compacts*
258. Structure Theory of Set Addition, J.-M. DESHOUILLERS, B. LANDREAU and A.A. YUDIN, editors
257. J.-P. LABESSE – *Cohomologie, stabilisation et changement de base (avec la collaboration de L. BREEN et L. CLOZEL)*
256. F. MOREL – *Théorie homotopique des schémas*
255. R.E. KOTTWITZ, D. SHELSTAD – *Foundations of twisted endoscopy*
254. C.J. BUSHNELL, G. HENNIART – *Local tame lifting for* $\mathrm{GL}(n)$ *II : wildly ramified supercuspidals*
253. B. MAGNERON – *Involutions complexes et vecteurs sphériques associés pour les groupes de Lie nilpotents réels*

1998

252. SÉMINAIRE BOURBAKI, volume 1997/1998, exposés 835-849
251. Nombre et répartition de points de hauteur bornée, E. PEYRE, éditeur
250. C. BONATTI, R. LANGEVIN – *Difféomorphismes de Smale des surfaces (avec la collaboration de E. JEANDENANS)*
249. P. AUSCHER, P. TCHAMITCHIAN – *Square root problem for divergence operators and related topics*
248. P. COLMEZ – *Intégration sur les variétés* p*-adiques*
247. G. PISIER – *Non-commutative vector valued* L_p*-spaces and completely* p*-summing maps*

1997

246. V. TARASOV, A. VARCHENKO – *Geometry of* q*-hypergeometric functions, quantum affine algebras and elliptic quantum groups*
245. SÉMINAIRE BOURBAKI, volume 1996/1997, exposés 820-834
244. J.-M. BISMUT – *Holomorphic families of immersions and higher analytic torsion forms*
243. L. LAFFORGUE – *Chtoucas de Drinfeld et conjecture de Ramanujan-Petersson*
242. N. BURQ – *Pôles de diffusion engendrés par un coin*
241. SÉMINAIRE BOURBAKI, volume 1995/1996, exposés 805-819

1996

240. A. SÀ BARETTO, R. B. MELROSE, M. ZWORSKI – *Semilinear diffraction of conormal waves*
239. J.-L. VERDIER – *Des catégories dérivées des catégories abéliennes*
238. A. BROISE, F. DAL'BO, M. PEIGNÉ – *Méthodes des opérateurs de transfert : transformations dilatantes de l'intervalle et dénombrement de géodésiques fermées*
237. SÉMINAIRE BOURBAKI, volume 1994/1995, exposés 790-804
236. Hommage à P. A. MEYER et J. NEVEU
235. J.-P. OTAL – *Le théorème d'hyperbolisation pour les variétés fibrées de dimension 3*
234. A. GENESTIER – *Espaces symétriques de Drinfeld*