ASTÉRISQUE 297

# ANALYSE COMPLEXE, SYSTÈMES DYNAMIQUES, SOMMABILITÉ DES SÉRIES DIVERGENTES ET THÉORIES GALOISIENNES (II)

## VOLUME EN L'HONNEUR DE JEAN-PIERRE RAMIS

édité par

Michèle Loday-Richaud

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***Classification mathématique par sujets* (2000). —** 14F10, 14H70, 32D20, 32Gxx, 32S65, 33E17, 34C08, 34C10, 34D20, 34D23, 34Exx, 34Mxx, 35Q53, 37D30, 37Jxx, 37K10, 37K20.

***Mots clefs.* —** Feuilletages analytiques singuliers réels ou complexes, séparabilité, enlacement, formes normales d'un nœud-col ; tores invariants associés à des EDP ; connexions et équations différentielles dans le champ complexe, monodromie, asymptotique, analyse WKB, géométrie de Stokes, équations de Painlevé ou de type Painlevé, espaces de modules ; applications symplectiques, stabilité de Lyapounov, diffusion d'Arnold.

# ANALYSE COMPLEXE, SYSTÈMES DYNAMIQUES, SOMMABILITÉ DES SÉRIES DIVERGENTES ET THÉORIES GALOISIENNES (II)

## VOLUME EN L'HONNEUR DE JEAN-PIERRE RAMIS

### édité par Michèle Loday-Richaud

***Résumé.*** — Cet ouvrage en deux volumes rassemble les actes du colloque *Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes* organisé à Toulouse du 22 au 26 septembre 2003 à l'occasion du soixantième anniversaire de Jean-Pierre Ramis.

En introduction, le premier volume propose deux textes de souvenirs et trois textes de synthèse des travaux de J.-P. Ramis en analyse complexe, en théorie des équations différentielles linéaires et en théorie des équations différentielles non-linéaires. Suivent des textes essentiellement consacrés aux théories galoisiennes, à l'arithmétique et à l'intégrabilité : analogies entre théories différentielles et théories arithmétiques, équations aux $q$-différences classiques ou $p$-adiques, problème de Riemann-Hilbert et renormalisation, $b$-fonctions, problèmes de descente, modules de Krichever, lieu d'intégrabilité, théorie de Drach et équation de Painlevé VI.

Le deuxième volume rassemble des textes plutôt liés à des questions d'analyse et de géométrie : stabilité de Lyapounov, analyse asymptotique et dynamique pour des pinceaux de trajectoires, analyse WKB et géométrie de Stokes, équations de Painlevé I et II, formes normales des singularités de type nœud-col, tores invariants d'équations aux dérivées partielles.

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***Abstract* (Complex analysis, dynamical systems, summability of divergent series and Galois theories (II), Volume in honnor of Jean-Pierre Ramis)**

These two bound volumes present the proceedings of the conference *Complex Analysis, Dynamical Systems, Summability of Divergent Series and Galois Theories* held in Toulouse from September $22^{\text{nd}}$ to September $26^{\text{th}}$ 2003, on the occasion of J.-P. Ramis' $60^{\text{th}}$ birthday.

The first volume opens with two texts composed of recollections and three texts on J.-P. Ramis' works on Complex Analysis and Ordinary Differential Equations Theory, both linear and non-linear. This introduction is followed by papers concerned with Galois Theories, Arithmetic or Integrability: analogies between differential and arithmetical theories, $q$-difference equations, classical or $p$-adic, the Riemann-Hilbert problem and renormalisation, $b$-functions, descent problems, Krichever modules, the set of integrability, Drach theory and the $\mathrm{VI}^{\text{th}}$ Painlevé equation.

The second volume contains papers dealing with analytical or geometrical aspects: Lyapunov stability, asymptotic and dynamical analysis for pencils of trajectories, monodromy in moduli spaces, WKB analysis and Stokes geometry, first and second Painlevé equations, normal forms for saddle-node type singularities, invariant tori for PDEs.

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Nous proposons d'appliquer la méthode des développements de Stokes à la construction perturbative de tores invariants associés à des solutions d'EDP quasi-périodiques en les variables d'espace et de temps. Pour les EDP intégrables, nous nous intéressons à la compensation de presque tous les petits diviseurs apparaissant dans l'analyse pertubative, *i.e.*, la compensation de tous sauf un nombre fini. Nous traitons de cette compensation en détail sur l'exemple de l'équation KP et nous montrons que dans ce cas, sous des hypothèses faibles portant sur la décroissance de l'amplitude des modes de Fourier, toutes les familles analytiques à tores invariants de dimension finie sont données par la construction de Krichever en termes de fonctions thêta de surfaces de Riemann. Nous donnons une construction explicite de fonctions thêta réelles de dimension infinie et des solutions de KP quasi-périodiques correspondantes comme somme d'une infinité d'ondes planes en interaction.

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On s'intéresse à la formule déterminant de Hankel pour les solutions génériques de l'équation de Painlevé II. On établit une relation reliant les fonctions génératrices des coefficients des déterminants de Hankel aux solutions asymptotiques à l'infini du problème linéaire dont les déformations isomonodromiques sont décrites par cette équation de Painlevé II.



Un théorème célèbre de Moser établit la stabilité au sens de Lyapounov des points fixes elliptiques génériques des applications qui conservent l'aire. On étudie la stabilité de Lyapounov des points fixes totalement elliptiques résonnants d'applications symplectiques en dimension 4. On montre que, génériquement, un point totalement elliptique résonnant convexe d'une application symplectique est instable au sens de Lyapounov. La démonstration s'appuie de façon essentielle sur celle donnée par J. Mather pour l'existence d'une diffusion d'Arnold pour les hamiltoniens convexes à 2,5 degrés de liberté. Celle-ci, annoncée dans [**Ma5**], n'est pas encore publiée.



Nous exhibons plusieurs propriétés fondamentales liant, d'une part, la géométrie de Stokes (*i.e.*, la configuration des courbes de Stokes et des points tournants) d'une équation de Painlevé d'ordre supérieur à grand paramètre et, d'autre part, la géométrie de Stokes de l'une des paires de Lax sous-jacentes. L'équation de Painlevé d'ordre supérieur à grand paramètre considérée est l'une des équations de la hiérarchie $P_J$ pour $J = \mathrm{I}$, II-1 ou II-2 que nous détaillons dans le paragraphe 1. Les équations étant d'ordre supérieur leurs lignes de Stokes peuvent se croiser et l'anomalie connue sous le nom de « phénomène de Nishikawa » peut se produire aux points de croisement. Nous analysons le mécanisme par lequel ce phénomène de Nishikawa apparaît. Plusieurs exemples de géométrie de Stokes sont donnés dans le paragraphe 5 en vue d'une visualisation de la partie centrale de nos résultats.



Dans la continuité de [**10**], nous construisons une forme normale simple pour un feuilletage analytique au voisinage d'une singularité de type nœud-col dans le plan réel ou complexe. Nous obtenons une telle forme en recollant des variétés complexes feuilletées. Nous en déduisons une déformation analytique miniverselle dans un cas simple. Nous donnons une forme unique pour un nœud-col possédant une variété centrale analytique. Nous retrouvons ainsi géométriquement et nous généralisons des résultats obtenus par J. Écalle à l'aide de la théorie des moules. Ce travail répond partiellement à des questions ouvertes posées par J. Martinet et J.-P. Ramis à la fin de [**11**].

Pour une courbe standard allant vers un point général à l'infini dans l'espace des modules $M_{\mathrm{DR}}$ des connexions sur une surface de Riemann compacte, nous montrons que le transformé de Laplace de la famille des matrices de monodromie admet un prolongement analytique avec ramification localement finie. En particulier, l'ensemble convexe qui représente la croissance exponentielle est un polygone dont les sommets sont dans un ensemble qu'on peut expliciter en termes de la courbe spectrale. Malheureusement, nous n'obtenons pas d'information sur la taille des singularités du transformé de Laplace et donc pas de développement asymptotique pour la monodromie.

# ABSTRACTS

*Pinceaux de courbes intégrales d'un champ de vecteurs analytique*
FELIPE CANO, ROBERT MOUSSU & FERNANDO SANZ .......................... 1

Let $\gamma_0$ be an integral curve of an analytic vector field $X$ in a real three dimensional manifold. Suppose that $\gamma_0$ has a single limit point and that it has all iterated tangents. The integral pencil $\mathrm{PI}(\gamma_0)$ is the set of all integral curves of $X$ having the same (oriented) iterated tangents as $\gamma_0$. We prove that two arbitrary curves in $\mathrm{PI}(\gamma_0)$ are either subanalytically separated or asymptotically linked. In this last case, $\mathrm{PI}(\gamma_0)$ has a formal axis which is divergent if and only if the curves of $\mathrm{PI}(\gamma_0)$ are not oscillatory.

*On analytic families of invariant tori for PDEs*
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We propose to apply a version of the classical Stokes expansion method to the perturbative construction of invariant tori for PDEs corresponding to solutions quasiperiodic in space and time variables. We argue that, for integrable PDEs all but finite number of the small divisors arising in the perturbative analysis cancel. As an illustrative example we establish such cancellations for the case of KP equation. It is proved that, under mild assumptions about decay of the magnitude of the Fourier modes all analytic families of finite-dimensional invariant tori for KP are given by the Krichever construction in terms of theta-functions of Riemann surfaces. We also present an explicit construction of infinite dimensional real theta-functions and of the corresponding quasiperiodic solutions to KP as sums of an infinite number of interacting plane waves.

In this paper we consider a Hankel determinant formula for generic solutions of the Painlevé II equation. We show that the generating functions for the entries of the Hankel determinants are related to the asymptotic solution at infinity of the linear problem of which the Painlevé II equation describes the isomonodromic deformations.



A well known Moser stability theorem states that a generic elliptic fixed point of an area-preserving mapping is Lyapunov stable. We investigate the question of Lyapunov stability for 4-dimensional resonant totally elliptic fixed points of symplectic maps. We show that generically a convex, resonant, totally elliptic point of a symplectic map is Lyapunov unstable. The proof heavily relies on a proof of J. Mather of existence of Arnold diffusion for convex Hamiltonians in 2.5 degrees of freedom. The latter proof is announced in [**Ma5**], but still unpublished.



We show several basic properties concerning the relation between the Stokes geometry (*i.e.*, configuration of Stokes curves and turning points) of a higher order Painlevé equation with a large parameter and the Stokes geometry of (one of) the underlying Lax pair. The higher-order Painlevé equation with a large parameter to be considered in this paper is one of the members of $P_J$-hierarchy with $J = \mathrm{I}$, II-1 or II-2, which are concretely given in Section 1. Since we deal with higher order equations, the Stokes curves may cross; some anomaly called the Nishikawa phenomenon may occur at the crossing point, and in this paper we analyze the mechanism why and how the Nishikawa phenomenon occurs. Several examples of Stokes geometry are given in Section 5 to visualize the core part of our results.



In the continuation of [**10**], we derive simple forms for saddle-node singular points of analytic foliations in the real or complex plane just by gluing foliated complex manifolds. We give a miniversal analytic deformation of the simplest model. We also derive a unique analytic form for those saddle-node having an analytic central manifold. By this way, we recover and generalize results earlier proved by J. Écalle by using mould theory and partially answer to some questions asked by J. Martinet and J.-P. Ramis at the end of [**11**].

For a standard path of connections going to a generic point at infinity in the moduli space $M_{\mathrm{DR}}$ of connections on a compact Riemann surface, we show that the Laplace transform of the family of monodromy matrices has an analytic continuation with locally finite branching. In particular, the convex subset representing the exponential growth rate of the monodromy is a polygon whose vertices are in a subset of points described explicitly in terms of the spectral curve. Unfortunately, we don't get any information about the size of the singularities of the Laplace transform, which is why we can't get asymptotic expansions for the monodromy.

# AVANT-PROPOS

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## Jean-Pierre Ramis

Jean-Pierre Ramis, né le 26 mars 1943 à Montpellier, a été élève de l'École Normale Supérieure d'octobre 1962 à septembre 1966. Assistant à la Faculté des Sciences de Paris puis attaché de recherches au CNRS il a soutenu, en mars 1969, un doctorat d'état préparé sous la direction de Henri Cartan (*Sous-ensembles analytiques d'une variété analytique banachique*). Maître de conférences à la Faculté des Sciences de Paris en 1968-69 puis à Tunis d'octobre 1969 à septembre 1971, il est nommé ensuite à l'Université Louis Pasteur à Strasbourg où il reste jusqu'en 1994, d'abord, comme maître de conférences puis, rapidement, comme professeur. Il est, depuis septembre 1994, professeur à l'Université Paul Sabatier à Toulouse et, depuis 1996, membre de l'Institut Universitaire de France. Il a reçu le prix Doisteau-Blutel de l'Académie des Sciences en 1982 et le prix A. Joannidès de l'Académie des Sciences en 2002. Il a été nommé chevalier des palmes académiques en 1999.

Strasbourg, novembre 2004

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# PINCEAUX DE COURBES INTÉGRALES D'UN CHAMP DE VECTEURS ANALYTIQUE

*par*

Felipe Cano, Robert Moussu & Fernando Sanz

***Résumé.*** — Soit $\gamma_0$ une courbe intégrale d'un champ de vecteurs analytique $X$ dans une variété réelle de dimension trois. Supposons que $\gamma_0$ ait un seul point limite et qu'elle possède des tangentes itérées. Le pinceau intégral $\mathrm{PI}(\gamma_0)$ est l'ensemble des courbes intégrales de $X$ qui ont les mêmes tangentes itérées (orientées) que $\gamma_0$. Nous montrons que les courbes de $\mathrm{PI}(\gamma_0)$ sont, soit deux à deux sous-analytiquement séparables, soit deux à deux asymptotiquement enlacées. Dans ce dernier cas, $\mathrm{PI}(\gamma_0)$ possède un axe formel qui est divergent si et seulement si les courbes de $\mathrm{PI}(\gamma_0)$ sont non oscillantes.

***Abstract* (Integral pencils of trajectories of an analytic vector field). —** Let $\gamma_0$ be an integral curve of an analytic vector field $X$ in a real three dimensional manifold. Suppose that $\gamma_0$ has a single limit point and that it has all iterated tangents. The integral pencil $\mathrm{PI}(\gamma_0)$ is the set of all integral curves of $X$ having the same (oriented) iterated tangents as $\gamma_0$. We prove that two arbitrary curves in $\mathrm{PI}(\gamma_0)$ are either subanalytically separated or asymptotically linked. In this last case, $\mathrm{PI}(\gamma_0)$ has a formal axis which is divergent if and only if the curves of $\mathrm{PI}(\gamma_0)$ are not oscillatory.

## 0. Introduction

Soit $X$ un champ de vecteurs analytique sur une variété $M$ de dimension trois et soit $\gamma_0$ une courbe intégrale de $X$ dont l'ensemble $\omega$-limite, $\omega(\gamma_0)$, est un point singulier $p$ de $X$. Nous nous intéresserons à la question suivante. Comment, d'un point de vue analytique, $\gamma_0$ peut-elle tendre vers $p$ ? Cette question n'est pertinente que si $\gamma_0$ possède une tangente en $p$. En effet, soit $\pi_1 : M_1 \to M_0$ l'éclatement de centre $p_0$ et soit $\gamma_1$ le relevé de $\gamma$ par $\pi_1$. Son ensemble $\omega$-limite, $\omega(\gamma_1)$, est contenu dans le diviseur exceptionnel de $\pi_1$ qui est identifié à $\mathbb{RP}(2)$. La courbe $\gamma_0$ a une tangente en $p_0$ de direction $p_1$ si et seulement si $\omega(\gamma_1) = p_1$. Si ce n'est pas le cas, l'étude de $\gamma_1$ au voisinage de $p_1$ est un problème de dynamique globale. Ce n'est plus

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***Mots clefs.* —** Champ de vecteurs, EDO, éclatement, oscillation, variété invariante.

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un problème local de géométrie analytique. Pour cette raison nous nous intéressons seulement aux courbes $\gamma_0$ qui possèdent des *tangentes itérées* $\mathrm{TI}(\gamma_0) = \{p_n\}$, c'est-à-dire, à celles pour lesquelles il existe une suite infinie d'éclatements ponctuels

$$M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} M_2 \cdots \longleftarrow M_{n-1} \xleftarrow{\pi_n} M_n \cdots$$

de centres les points $p_0, p_1, \ldots, p_{n-1}, \ldots$, telle que $p_n = \omega(\gamma_n)$ où $\gamma_n$ est le relevé de $\gamma_{n-1}$ par $\pi_n$. La droite tangente à $\gamma_{n-1}$ en $p_{n-1}$ est naturellement orientée par $\gamma_{n-1}$. Nous notons $p_n^+$ le point correspondant de $\mathbb{S}^2$ et $\mathrm{TI}^+(\gamma_0) = \{p_n^+\}$ la suite des *tangentes itérées orientées* de $\gamma_0$. L'ensemble $\mathrm{PI}(\gamma_0)$ des courbes $\gamma$ telles que $\mathrm{TI}^+(\gamma) = \mathrm{TI}^+(\gamma_0)$ est le *pinceau intégral* de $\gamma_0$ pour $X$. Si $\widehat{\Gamma}$ est une courbe formelle en $p_0$, nous notons encore $\mathrm{TI}(\widehat{\Gamma})$ sa suite de points infiniment proches au sens de [**2**]. Si $\mathrm{TI}(\widehat{\Gamma}) = \mathrm{TI}(\gamma_0)$ nous dirons que $\widehat{\Gamma}$ est *l'axe de* $\mathrm{PI}(\gamma_0)$ ou que $\gamma_0$ a un *contact plat* avec $\widehat{\Gamma}$.

S'il existe une surface analytique qui ne contient pas $\gamma_0$ et qui coupe $\gamma_0$ selon une infinité de points, on dit que $\gamma_0$ est *oscillante*. Dans ce cas, le théorème suivant décrit les propriétés du pinceau $\mathrm{PI}(\gamma_0)$.

***Théorème du spiralement axial*** **([7])**. — *Si $\gamma_0$ est oscillante et possède des tangentes itérées,* $\mathrm{PI}(\gamma_0)$ *possède un axe $\Gamma$ convergent $X$-invariant et $\gamma_0$ « spirale » autour de $\Gamma$. De plus, si $\Gamma$ n'est pas contenu dans* $\operatorname{Sing} X$ *le lieu singulier de $X$, toutes les courbes de* $\mathrm{PI}(\gamma_0) \smallsetminus \Gamma$ *sont oscillantes et spiralent autour de $\Gamma$.*

Si la courbe $\gamma_0$ n'est pas oscillante, elle possède des tangentes itérées. Le but de ce travail est l'étude des pinceaux $\mathrm{PI}(\gamma_0)$ constitués de courbes non oscillantes. Ces objets ne sont pas rares. D'après le théorème précédent, c'est le cas si $\gamma_0$ est non oscillante et n'a pas un contact plat avec $\operatorname{Sing} X$.

***Théorème I***. — *Si les courbes de* $\mathrm{PI}(\gamma_0)$ *sont non oscillantes on a l'une des propriétés suivantes :*

s) *Deux courbes distinctes, quelconques, de* $\mathrm{PI}(\gamma_0)$ *sont sous-analytiquement séparables.*

e) *Deux courbes distinctes, quelconques, de* $\mathrm{PI}(\gamma_0)$ *sont asymptotiquement enlacées.*

*De plus, ces propriétés ne peuvent pas être satisfaites simultanément.*

Dans le cas s) nous dirons que $\mathrm{PI}(\gamma_0)$ est un *pinceau intégral séparé.* Dans le cas e) nous dirons que $\mathrm{PI}(\gamma_0)$ est un *pinceau intégral enlacé.* La structure de tels pinceaux est décrite dans le théorème II ci-dessous. Avant de l'énoncer, définissons brièvement les concepts qui apparaissent dans le théorème précédent. Soient $\gamma$, $\gamma'$ deux courbes intégrales de $\mathrm{PI}(\gamma_0)$ et soient $|\gamma|$, $|\gamma'|$ leurs images. On dit que $\gamma$, $\gamma'$ sont « distinctes » si $|\gamma|$, $|\gamma'|$ ne sont pas contenues dans une même orbite du flot de $X$, que $\gamma$, $\gamma'$ sont *sous-analytiquement séparables* s'il existe une application $f$ bornée, non constante, sous-analytique sur un voisinage de $|\gamma| \cup |\gamma'|$ dans $\mathbb{R}^2$ telle que le nombre de points de $f(|\gamma|) \cap f(|\gamma'|)$ est fini. Des coordonnées $w = (x, y, z)$ centrées en $p$ sont dites $z$-positives pour $\gamma$ si $|\gamma| \subset \{z > 0\}$ et si $\gamma$ coupe transversalement les plans $z =$

constante. On peut alors reparamétrer $|\gamma|$ par $z$. Ce que nous écrivons $z \mapsto \gamma(z) = (x(z), y(z), z)$, $z > 0$. Deux courbes $\gamma$, $\gamma'$ sont *asymptotiquement enlacées* s'il existe des coordonnées $w$, $z$-positives pour $\gamma$, $\gamma'$, telles que la courbe $z \mapsto (\gamma(z) - \gamma'(z))$ dans $\mathbb{R}^2 \times \{0\} \equiv \mathbb{R}^2$ spirale autour de 0. Ce concept est indépendant des coordonnées choisies pour le définir lorsque $\gamma$, $\gamma'$ sont non oscillantes.

***Théorème II***. — *Soit* $\mathrm{PI}(\gamma_0)$ *un pinceau intégral enlacé de courbes non oscillantes.*

(1) $\mathrm{PI}(\gamma_0)$ *possède un axe formel* $\widehat{\Gamma}$ *non convergent transcendant ; c'est-à-dire qu'il n'existe pas de surface analytique qui contienne* $\widehat{\Gamma}$.

(2) *Si* $V$ *est un voisinage de* $p$, *il existe un ouvert sous-analytique connexe* $U \subset V$ *positivement invariant par le flot de* $X$ *tel qu'une courbe intégrale* $\gamma$ *de* $X$ *appartient à* $\mathrm{PI}(\gamma_0)$ *si et seulement si* $|\gamma| \cap U \neq \varnothing$.

Un pinceau enlacé de courbes non oscillantes est $X$-irréductible au sens suivant. Un ouvert $U$ comme dans le théorème II n'est pas la réunion de deux ensembles sous-analytiques non vides, disjoints, positivement invariants par le flot de $X$. En effet, sinon, $U$ contiendrait un sous-ensemble sous-analytique $A$ positivement invariant par le flot de $X$ de dimension inférieure ou égale à deux. L'axe formel $\widehat{\Gamma}$ de $\mathrm{PI}(\gamma_0)$ serait contenu dans $A$, ce qui contredirait l'assertion 1 du théorème II.

Supposons que $\gamma_0$ soit une courbe oscillante qui possède des tangentes itérées et que $\gamma_0$ n'a pas un contact plat avec une courbe contenue dans $\mathrm{Sing}\, X$. D'après le théorème du spiralement axial, toutes les courbes de $\mathrm{PI}(\gamma_0) \setminus \Gamma$ sont oscillantes et spiralent autour d'un axe $\Gamma$. Une des composantes connexes de $\Gamma \setminus \{p\}$, notée $\Gamma^+$ est une courbe intégrale de $\mathrm{PI}(\gamma_0)$. Deux courbes quelconques, distinctes de $\mathrm{PI}(\gamma_0)$ sont asymptotiquement enlacées et il existe encore $U$, un ouvert $X$-invariant comme dans le théorème II [**7**]. Mais dans ce cas, $\mathrm{PI}(\gamma_0)$ n'est pas $X$-irréductible puisque $U = (U \setminus \Gamma^+) \cup \Gamma^+$.

***Exemple* (L'équation d'Euler)**. — Dans $\mathbb{C}^2$ muni des coordonnées $(u, v)$ l'équation différentielle

$$(E_\varepsilon) \qquad \frac{du}{dt} = -u + \varepsilon v, \quad \frac{dv}{dt} = -v^2 \quad \text{avec } \varepsilon = 0, 1$$

définit un feuilletage holomorphe $\mathcal{F}_\varepsilon$ dont $v = 0$ est une séparatrice. Le plongement $j_\omega$ de $\mathbb{C} \times \mathbb{R}$ dans $\mathbb{C}^2$ défini par $j_\omega(u, z) = (u, \omega z)$ avec $\omega = \exp(-i\alpha)$ où $\alpha$ est réel, $|\alpha| < \pi/2$ est transverse à $\mathcal{F}_\varepsilon$. L'image inverse $\mathcal{F}_{\varepsilon,\omega}$ de $\mathcal{F}_\varepsilon$ par $j_\omega$ est un feuilletage en courbes réelles. Sa description permet de mieux comprendre la géométrie de $\mathcal{F}_\varepsilon$ du « côté noeud-col » [**23**]. En identifiant $\mathbb{C}$ à $\mathbb{R}^2$ via l'écriture $u = x + iy = (x, y)$, les feuilles de $\mathcal{F}_{\varepsilon,\omega}$ sont les courbes intégrales de l'équation différentielle

$$(E_{\varepsilon,\omega}) \qquad \frac{dx}{dt} = -\cos\alpha x + \sin\alpha y + \varepsilon z, \quad \frac{dy}{dt} = -\sin\alpha x - \cos\alpha y, \quad \frac{dz}{dt} = -z^2.$$

Le plan $z = 0$ est la variété stable de $E_{\varepsilon,\omega}$. Les courbes intégrales $\gamma$ de $E_{\varepsilon,\omega}$ contenues dans $z > 0$ sont transverses aux plans $z =$constante. Ce sont des graphes $\gamma(z) =$

$(u(z), z)$, $z > 0$ de fonctions $u(z)$ solutions de

$$z^2 \frac{du}{dz} = \frac{u}{\omega} - \varepsilon z, \quad \text{avec } u(z) = x(z) + iy(z),\ z > 0.$$

Si $\varepsilon = 0$, on a $u(z) = c \exp(-1/\omega z)$, $c \in \mathbb{C}$ L'ensemble des $|\gamma| \subset \{z > 0\}$ est un pinceau intégral $P_0$ d'axe $\Gamma = \{u = 0\}$. Si $\omega = 1$, elles sont non oscillantes et $P_0$ est séparé. Si $\omega \neq 1$, les courbes de $P_0 \setminus \Gamma$ sont oscillantes, elles spiralent autour de $\Gamma$. Si $\varepsilon = 1$, l'ensemble des $|\gamma| \subset \{z > 0\}$ est un pinceau intégral $P_1$ d'axe formel

$$\widehat{\Gamma}_\omega(z) = (\widehat{x}(z), \widehat{y}(x), z) \equiv (\widehat{u}(z), z), \quad \text{avec } \widehat{u}(z) = \sum_{n \geqslant 1} (n-1)!\, \omega^{n-1} z^n.$$

Soient $\gamma$, $\gamma'$, distinctes, appartenant à $P_1$, $\gamma(z) = (u(z), z)$, $\gamma'(z) = (u'(z), z)$. Alors on a $(u(z) - u'(z)) = c \exp(-1/\omega z)$, $c \in \mathbb{C}$. Si $\omega = 1$, $P_1$ est un pinceau séparé de courbes non oscillantes. Si $\omega \neq 1$, $P_1$ est un pinceau enlacé de courbes non oscillantes.

Les concepts oscillation, tangentes itérées, enlacement asymptotique, séparation sont définis de façon précise dans le chapitre suivant. Par des arguments classiques de géométrie analytique réelle nous montrons qu'ils sont stables par des morphismes permis. Ce sont des composés d'éclatements de points, de courbes lisses et de ramifications au-dessus de surfaces lisses. Ainsi, d'après le « théorème d'uniformisation polarisée » de [**6**], il suffit de démontrer les théorèmes I et II lorsque la partie linéaire $DX(p)$ n'est pas nilpotente. Il est alors nécessaire de distinguer différents cas selon la nature du spectre de $DX(p)$. Cette démarche nécessite encore quelques définitions : pinceau final hyperbolique, pinceau final de type I, pinceau final de type II.

Les chapitres 2, 3, 4 sont consacrés aux démonstrations des théorèmes I (sans l'alternative) et II pour les pinceaux hyperboliques, finaux de type I, finaux de type II, respectivement. Enfin, dans le chapitre 5, nous montrons l'alternative du théorème I et nous prouvons que l'étude des pinceaux intégraux se ramène à celui des pinceaux finaux.

Ce travail doit beaucoup à une question de F. Dumortier et à des conversations avec J.-M. Lion. Nous les remercions vivement.

## 1. Enlacement asymptotique et pinceau intégral

Dans toute cette partie, $X$ désigne un champ de vecteurs analytique sur une variété $M$ de dimension trois et $\gamma : t \mapsto \gamma(t)$, $t \geqslant 0$ une courbe intégrale non constante de $X$ dont l'ensemble $\omega$-limite noté $p = \omega(\gamma)$ est un point singulier de $X$. Dans un premier paragraphe nous rappelons quelques concepts définis dans [**7**] : tangentes itérées, courbes oscillantes, spiralement axial. Dans le paragraphe suivant nous étudions les propriétés « individuelles » d'une courbe $\gamma$ non oscillante. Nous montrons essentiellement que la non oscillation est une propriété stable par des *morphismes admissibles*. Ce sont des composés d'éclatements de points, de courbes et des ramifications. Le concept d'enlacement asymptotique de deux courbes intégrales est défini dans le paragraphe trois. Après avoir constaté qu'il est indépendant des coordonnées choisies pour le définir, pour des courbes non oscillantes, nous montrons qu'il est stable par morphismes admissibles. Dans le dernier paragraphe, nous donnons la définition de pinceau intégral et précisons quelques propriétés d'un tel objet. Nous étudions la stabilité de ce concept par transformation admissible.

**1.1. Tangentes itérées et oscillation.** — Nous reprenons les notations de l'introduction, $\gamma : t \mapsto \gamma(t)$, $t \geqslant 0$ est une courbe intégrale de $X$ sur $M$ telle que $\omega(\gamma) = p$. Soit $\pi_1 : M_1 \to M$ l'éclatement de centre $p$. Il existe une unique courbe $\gamma_1$ dans $M_1$ telle que $\pi_1 \circ \gamma_1 = \gamma$. C'est une courbe intégrale du champ de vecteurs $X_1$ sur $M_1$ défini par $\pi_{1*}(X_1) = X$. On dit que $\gamma_1$, $X_1$ sont les relevés de $\gamma$, $X$ par $\pi_1$. L'ensemble $\omega$-limite, $\omega(\gamma_1)$, de $\gamma_1$ est contenu dans le diviseur exceptionnel $\pi_1^{-1}(p)$ qui est identifié au projectif réel $\mathbb{RP}(2)$. Dire que $\omega(\gamma_1)$ est un point $p_1$ de $\mathbb{RP}(2)$ signifie que $\gamma$ possède une tangente en $p$ dans la direction $p_1$. Si c'est le cas, à la courbe $\gamma$ correspond une des demies droites $p_1^+ \in \mathbb{S}^2$ associées à $p_1 \in \mathbb{RP}(2)$. Le point $p_1^+$ est la *tangente orientée* en $p$ à $\gamma$. Nous dirons que $\gamma$ possède des *tangentes itérées* si l'on peut construire une suite infinie

$$M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} M_2 \cdots \longleftarrow M_{n-1} \xleftarrow{\pi_n} M_n \cdots, \quad \text{avec } M_0 = M,\ p_0 = p,$$

d'éclatements ponctuels $\pi_n$, $n \geqslant 1$, de centre le point $p_{n-1}$ de $M_{n-1}$, telle que $p_n$ soit l'ensemble $\omega$-limite du relevé $\gamma_n = \pi_n^{-1} \circ \gamma_{n-1}$ de $\gamma_{n-1}$ par $\pi_n$. Si c'est le cas nous dirons que $\mathrm{TI}(\gamma) = \{p_n\}$ est la *suite des tangentes itérées* de $\gamma$ et que $\mathrm{TI}^+(\gamma) = \{p_n^+\}$ est la suite des *tangentes itérées orientées* de $\gamma$.

Soit $\widehat{\Gamma}$ une courbe formelle au point $p$ et $\pi_1 : M_1 \to M$ l'éclatement de centre $p$. Il existe un unique point $p_1 \in \pi_1^{-1}(p)$ et une courbe formelle $\widehat{\Gamma}_1$ en $p_1$, le *transformé strict* de $\widehat{\Gamma}$, telle que $\pi_1 \circ \widehat{\Gamma}_1 = \widehat{\Gamma}$. Par une induction évidente, on construit une suite infinie de points $\mathrm{TI}(\widehat{\Gamma}) = \{p_n\}$ associés à $\widehat{\Gamma}$. C'est *la suite de points infiniment proches* de $\widehat{\Gamma}$ au sens de [**2**]. Si $\mathrm{TI}(\widehat{\Gamma}) = \mathrm{TI}(\gamma)$ nous dirons que $\gamma$ a un *contact plat* avec $\widehat{\Gamma}$. Lorsque $\gamma$ a un contact avec une courbe $\Gamma$ analytique, la courbe $\Gamma$ est $X$-invariante ([**7**]). Plus généralement supposons que $\gamma$ ait un contact plat avec une courbe formelle $\widehat{\Gamma}$ et soit $\widehat{\Gamma}(z) = (\widehat{x}(z), \widehat{y}(z), z)$ le développement de Puiseux de $\widehat{\Gamma}$ dans des coordonnées $w = (x, y, z)$ centrées en $p$. Si $|\gamma|$ possède un paramétrage $z \mapsto \gamma(z) = (x(z), y(z), z)$ on peut montrer que $\widehat{\Gamma}(z)$ est un développement asymptotique de $\gamma(z)$. Cette assertion est prouvée dans [**6**] lorsque $\gamma$ est non oscillante. Nous ne l'utiliserons que dans ce cadre.

Soit $S$ une surface analytique dans $M$. On dit que $\gamma$ est *$S$-oscillante* si $|\gamma| \not\subset S$ et $\gamma$ coupe $S$ selon un nombre infini de points. La courbe $\gamma$ est *non oscillante* si, pour toute surface analytique $S$ dans $M$, $\gamma$ n'est pas $S$-oscillante. En dimension deux, une courbe qui possède des tangentes itérées est toujours non oscillante. Cette dichotomie oscillant-tangentes itérées n'est plus vraie en dimension trois. On a alors le théorème suivant du spiralement axial [**7**] :

***Théorème 1.1***. — *La courbe $\gamma$ est oscillante et possède des tangentes itérées si et seulement si $\gamma$ spirale autour d'une demie branche analytique $\Gamma^+$.*

Le concept de spiralement axial est assez intuitif. Au lieu de rappeler sa définition rappelons sa propriété caractéristique : $\gamma$ *spirale autour de* $\Gamma^+$ *si et seulement si* $\gamma$ *a un contact plat avec* $\Gamma^+$ *et* $\gamma$ *est* $S$*-oscillante pour toute surface analytique* $S$ *contenant* $\Gamma^+$. On en déduit que le spiralement axial est stable par éclatement (et effondrement) ponctuel.

**1.2. Courbes non oscillantes et transformations admissibles.** — Nous conservons les notations $M$, $X$, $\gamma$, $\omega(\gamma) = p$ et nous supposons que $\gamma$ est non oscillante. Dans [**7**] nous avons montré que si $\gamma$ est non oscillante, alors $\gamma$ possède des tangentes itérées $\mathrm{TI}(\gamma)$. Nous allons déduire du théorème de spiralement axial la stabilité de la non-oscillation par éclatement ponctuel :

***Lemme 1.2***. — *Soit $\gamma$ non oscillante. Alors son relevé $\gamma_1$ par l'éclatement de centre $\omega(\gamma) = p$ est une courbe non oscillante.*

*Démonstration*. — Supposons que $\gamma_1$ soit oscillante. C'est une courbe intégrale du champ de vecteurs $X_1$ relevé de $X$ par $\pi_1$ qui est oscillante et qui possède des tangentes itérées (comme $\gamma$). Ainsi $\gamma_1$ spirale autour d'un axe $\Gamma_1^+$. Le spiralement axial étant stable par éclatement de point, la courbe $\gamma$ est $S$-oscillante pour toute surface analytique $S$ contenant $\pi_1(\Gamma_1^+)$. $\square$

Soit $Y$ un germe de courbe analytique lisse dans $M$ passant par $p$ qui ne rencontre pas $|\gamma|$. Quitte à localiser en $p$, on définit l'éclatement $\pi_1 : M_1 \to M$ de centre $Y$. Il existe une unique courbe $\gamma_1$, le relevé de $\gamma$ par $\pi_1$, telle que $\pi_1 \circ \gamma_1 = \gamma$. Nous dirons que $\pi_1$ est un éclatement de *courbe lisse $\gamma$-admissible.* Si $Y$ n'est pas $X$-invariant, on ne peut pas en général « relever » $X$ en un champ de vecteurs analytique sur $M_1$. Cependant, il existe une fonction analytique $g$, strictement positive sur $|\gamma|$ et un champ de vecteurs $X_1$ analytique sur $M_1$ avec $\pi_{1*}(X_1) = gX$ tels que $|\gamma_1|$ soit l'image d'une courbe intégrale de $X_1$. Nous dirons encore que cette courbe est un relevé de $\gamma$ par $\pi_1$ et nous la noterons toujours $\gamma_1$. Cette ambiguïté est sans importance, l'objet de ce travail étant l'étude des propriétés des images $|\gamma|$ des courbes intégrales $\gamma$ de $X$.

***Lemme 1.3***. — *Si $\gamma$ est non oscillante, le relevé $\gamma_1$ de $\gamma$ par l'éclatement $\pi_1 : M_1 \to M$ d'une courbe $Y$ lisse $\gamma$-admissible est aussi une courbe non oscillante.*

*Démonstration*. — Montrons tout d'abord que $\gamma_1$ possède des tangentes itérées. Supposons que ce ne soit pas le cas. Il existe une suite finie d'éclatements ponctuels (vide si $\#\omega(\gamma_1) > 1$) $\pi_{k+1} : M_{k+1} \to M_k$, pour $k = 1, 2, \ldots, n-1$, de centres des points $p_k$, telle que $\omega(\gamma_k) = p_k$ avec $\pi_k \circ \gamma_k = \gamma_{k-1}$ pour $k = 2, 3, \ldots, n$ et telle que $\#\omega(\gamma_n) > 1$.

On peut choisir des coordonnées en $p$, $p_n$ telles que le composé $\pi = \pi_n \circ \pi_{n-1} \circ \cdots \circ \pi_1$ ait une écriture polynomiale. Un argument élémentaire de connexité (voir [**7**] page 288) permet de construire une surface analytique $S_n$ dans $M_n$, d'équation polynomiale, telle que $\gamma_n$ soit $S_n$-oscillante. D'après le théorème de Tarski ([**34**]), $\pi(S_n)$ est contenu dans une surface algébrique $S$ pour les coordonnées choisies et la courbe $\gamma$ est $S$-oscillante. Montrons que $\gamma_1$ est non oscillante. Si ce n'est pas le cas, $\gamma_1$ spirale autour d'un axe $\Gamma_1^+$. Son image $\gamma$ spirale autour de $\pi(\Gamma_1^+)$. C'est une courbe oscillante. $\square$

Un morphisme analytique $\pi_1 : M_1 \to M$ est une *$q$-ramification* en $p_1$ s'il existe des coordonnées $w_1 = (x_1, y_1, z_1)$ centrées en $p_1$, des coordonnées $w = (x, y, z)$ centrées en $p = \pi_1(p_1)$ telles que $\pi_1(x_1, y_1, z_1) = (x_1, y_1, z_1^q) = (x, y, z)$. Nous dirons que $\pi_1$ est *une ramification $\gamma$-admissible* si $|\gamma| \subset \{z > 0\}$. Si c'est le cas, il existe une unique courbe $\gamma_1$ telle que $\pi_1 \circ \gamma_1 = \gamma$ et $|\gamma_1| \subset \{z_1 > 0\}$. C'est une courbe intégrale du champ de vecteurs $X_1$ sur $M_1$ telle que $\pi_{1*}(X_1) = X$. Nous dirons encore que $\gamma_1$, $X_1$ sont les relevés de $\gamma$, $X$ par $\pi_1$. Un argument (comme dans le lemme 2) montre que $\gamma_1$ est non oscillante si et seulement si $\gamma$ est non oscillante.

L'éclatement ponctuel de centre $p$, les éclatements de courbes lisses $\gamma$-admissibles, les ramifications $\gamma$-admissibles sont appelés des *morphismes $\gamma$-admissibles élémentaires.* Pour les définir, nous sommes amenés à localiser. Nous les écrirons, avec la notation germifiée, $\pi_1 : (M_1, \gamma_1, p_1) \to (M, \gamma, p)$ où $\omega(\gamma_1) = p_1$, $\pi_1 \circ \gamma_1 = \gamma$. Un *morphisme $\gamma$-admissible* $\pi$ est un composé de morphismes $\gamma$-admissibles élémentaires. Nous l'écrirons encore $\pi : (\widetilde{M}, \widetilde{\gamma}, \widetilde{p}) \to (M, \gamma, p)$, où $\pi \circ \widetilde{\gamma} = \gamma$ et $\omega(\widetilde{\gamma}) = \widetilde{p}$. On peut résumer les définitions et les résultats de ce paragraphe avec la proposition suivante.

***Proposition 1.4***. — *Soient $\gamma$ une courbe intégrale non-oscillante de $X$ et $\pi : (\widetilde{M}, \widetilde{\gamma}, \widetilde{p}) \to (M, \gamma, p)$ un morphisme $\gamma$-admissible. Alors, le relevé $\widetilde{\gamma}$ de $\gamma$ est non oscillant et c'est une courbe intégrale d'un champ vecteurs analytique $\widetilde{X}$ sur $\widetilde{M}$.*

Le corollaire suivant est une conséquence du théorème de [**14**] de rectilineairization des ensembles sous-analytiques, du théorème du spiralement axial et de la proposition précédente.

***Corollaire 1.5* (Non-oscillation sous-analytique)**. — *Si $\gamma$ est une courbe intégrale non oscillante de $X$ et $Z$ un ensemble sous-analytique de $M$ alors $|\gamma| \cap Z$ est un ensemble fini si $|\gamma| \not\subset Z$.*

**1.3. Enlacement asymptotique.** — Ce concept repose sur la notion intuitive de courbe plane qui spirale autour d'un point que nous allons préciser. Soit $v : ]0, z_0] \to \mathbb{R}^2 \setminus \{0\}$ une courbe analytique telle que $0 = \lim_{z \to 0} v(z)$. Dans des coordonnées $u = (x, y)$ on écrit :

$$v(z) = (\alpha(z), \quad \beta(z)) = (\rho(z) \cos \theta(z), \quad \rho(z) \sin \theta(z))$$

où $\rho = \|v\|$ et $\theta(z)$ est une fonction analytique sur $]0, z_0]$. On dit que $v$ *spirale autour de* $0$ si $\lim_{z\to 0} |\theta(z)| = \infty$. La fonction $\beta(z)$ étant analytique, la courbe $v$ coupe le demi axe $\Delta = \mathbb{R}_{>0} \times \{0\}$ selon une suite de points isolés $v(z_n)$ avec $z_n < z_{n-1}$. L'indice d'intersection $i(n)$ de $v$ et $\Delta$ en $z_n$ prend ses valeurs dans $\{1, -1, 0\}$ selon la règle : $i(n) = 1$ s'il existe $\varepsilon > 0$ tel que $\beta(z) < 0$ pour $z_n - \varepsilon < z < z_n$ et $\beta(z) > 0$ pour $z_n < z < z_n + \varepsilon$, $i(n) = -1$ si $\beta(z) > 0$ pour $z_n - \varepsilon < z < z_n$ et $\beta(z) < 0$ pour $z_n < z < z_n + \varepsilon$, $i(n) = 0$ dans les autres cas. On a clairement l'équivalence : $v$ spirale autour de 0 si et seulement si $\lim_{n\to\infty} |I(n)| = \infty$ où $I(n) = \sum_{k=1}^{n} i(n)$. Avec cette caractérisation du spiralement, la démonstration de l'assertion suivante est un exercice élémentaire.

***Assertion***. — *Le concept de spiralement de $v$ autour de $0$ est indépendant des coordonnées (analytiques) choisies pour le définir.*

Nous reprenons les notations des paragraphes précédents : $X$ est un champ de vecteurs sur $M$, $\gamma$ une courbe intégrale de $X$ avec $\omega(\gamma) = p$. Nous dirons que des coordonnées $w = (x, y, z) = (u, z)$ centrées en $p$ sont *$z$-positives* pour $\gamma$ si $|\gamma| \subset \{z > 0\}$ et si $\gamma$ coupe transversalement les plans $z$=constante. Si c'est le cas, on peut paramétrer $\gamma$ par $z$. Ce que nous écrirons $\gamma(z) = (x(z), y(z), z) = (u(z), z)$ avec $z > 0$. Fixons des coordonnées $w = (x, y, z) = (u, z)$ centrées en $p$.

***Définition 1.6***. — Soient $\gamma$, $\gamma'$ deux courbes intégrales distinctes de $X$ avec $\omega(\gamma) = \omega(\gamma') = p$. Fixons des coordonnées $w = (x, y, z) = (u, z)$ centrées en $p$. Nous dirons que $\gamma$, $\gamma'$ sont *$w$-asymptotiquement enlacées* si les coordonnées $w$ sont $z$-positives pour $\gamma$, $\gamma'$ et si la courbe $z \mapsto v(z) = u(z) - u'(z)$, avec $\gamma(z) = (u(z), z)$, $\gamma'(z) = (u'(z), z)$, spirale autour de 0.

D'après l'assertion précédente on a :

***Lemme 1.7***. — *Soient $w = (x, y, z)$, $w' = (x', y', z)$ des coordonnées centrées en $p$. Les courbes $\gamma$, $\gamma'$ sont $w$-asymptotiquement enlacées si et seulement si $\gamma$, $\gamma'$ sont $w'$-asymptotiquement enlacées.*

Soit $S_w(\gamma)$ la surface réglée, à bord $|\gamma|$, image dans la carte $w$ de l'application

$$(s, z) \longmapsto (x(z) + s, y(z), z) \quad \text{avec } z > 0,\ s \geqslant 0.$$

Les points d'intersection de $\gamma'$ avec $S_w(\gamma)$ correspondent au points d'intersection de la courbe $z \mapsto v(z)$ avec $\Delta = \mathbb{R}_{>0} \times \{0\}$. Ce sont les points $\gamma'(z_n) = (x'(z_n), 0, z_n)$ avec $x'(z_n) > 0, z_n < z_{n-1}$, où $\{z_n\}$ est la suite définie plus haut pour la courbe $v$. L'indice d'intersection de $\gamma'$ avec $S_w(\gamma)$ en $\gamma'(z_n)$ est encore $i(n)$ et $I(n) = I^w_{\gamma,\gamma'}(n)$ est l'indice d'intersection de $\gamma'|_{]z_0, z_{n+1}[}$ avec $S_w(\gamma)$. Il est clair que $\gamma, \gamma'$ sont $w$-asymptotiquement enlacées si et seulement si $\lim_{n\to\infty} |I^w_{\gamma,\gamma'}(n)| = \infty$.

**Lemme 1.8**. — *Soient $w = (x, y, z)$ des coordonnées centrées en $p$, $z$-positives et $y$-positives pour $\gamma$, $\gamma'$. Alors les courbes $\gamma$, $\gamma'$ sont $w = (x, y, z)$-asymptotiquement enlacées si et seulement si $\gamma$, $\gamma'$ sont $w' = (x, z, y)$-asymptotiquement enlacées.*

*Démonstration.* — La première partie du lemme est une conséquence de la définition de $I^w_{\gamma,\gamma'}(n)$. La seconde partie résulte de l'égalité $S_w(\gamma) = S_{w'}(\gamma)$. □

**Lemme 1.9**. — *Soit $\gamma$ une courbe intégrale non oscillante de $X$ et soient $w = (x, y, z)$ des coordonnées centrées en $p = w(\gamma)$ telles que $|\gamma| \not\subset \{z = 0\}$. Quitte à restreindre le domaine de définition de $\gamma$ et à changer $z$ en $-z$ ces coordonnées sont $z$-positives pour $\gamma$.*

*Démonstration.* — Soient $X = a\,\partial/\partial x + b\,\partial/\partial y + c\,\partial/\partial z$ et $\gamma(t) = (x(t), y(t), z(t))$ les écritures de $X$ et $\gamma$ dans la carte $w$. Puisque $\gamma$ est non oscillante les fonctions $z(t) = z(\gamma(t))$ et $c(\gamma(t)) = dz(t)/dt$ ne s'annulent qu'un nombre fini de fois. □

**Proposition 1.10**. — *Soient $\gamma$, $\gamma'$ deux courbes intégrales non oscillantes de $X$ et soient $w = (x, y, z)$, $w' = (x', y', z')$ des coordonnées centrées en $p = \omega(\gamma) = \omega(\gamma')$ qui sont $z$-positives, $z'$-positives pour $\gamma$, $\gamma'$. Les courbes $\gamma$, $\gamma'$ sont $w$-asymptotiquement enlacées si et seulement si $\gamma$, $\gamma'$ sont $w'$-asymptotiquement enlacées.*

*Démonstration.* — D'après le lemme 1.9 on peut supposer que les coordonnées $w' = (x', y', z')$ sont $x'$, $y'$, $z'$-positives pour $\gamma$, $\gamma'$. La matrice jacobienne $D_w\, w'(0)$ étant inversible, le vecteur $\partial w'/\partial z(0)$ n'est pas nul. Quitte à permuter les positions de $x', y', z'$ dans $w'$ on peut supposer que $\partial z'/\partial z(0) \neq 0$. Cette permutation est encore légitime d'après le lemme 1.8. Cette dernière propriété du couple $w$, $w'$ implique que $w'' = (x', y', z)$ sont des coordonnées centrées en $p$. D'après le lemme 1.7, $\gamma$, $\gamma'$ sont $w$-asymptotiquement enlacées si et seulement si $\gamma$, $\gamma'$ sont $w''$-asymptotiquement enlacées. D'après le lemme 1.8, $\gamma$, $\gamma'$ sont $w'' = (x', y', z)$-asymptotiquement enlacées si et seulement si $\gamma$, $\gamma'$ sont $(x', z, y')$-asymptotiquement enlacées. D'après le lemme 1.7, $\gamma$, $\gamma'$ sont $(x', z, y')$-asymptotiquement enlacées si et seulement si $\gamma$, $\gamma'$ sont $(x', z', y')$-asymptotiquement enlacées. On conclut en appliquant le lemme 1.8 au couple $(x', z', y')$, $(x', y', z') = w'$ □

La proposition précédente et l'abondance de coordonnées $z$-positives pour $\gamma$, $\gamma'$, donnée par le lemme 1.9, justifient la définition suivante :

***Définition 1.11***. — Soient $\gamma$, $\gamma'$ deux courbes intégrales non oscillantes de $X$ telles que $w(\gamma) = w(\gamma') = p$. Nous dirons que $\gamma$, $\gamma'$ sont *asymptotiquement enlacées* s'il existe des coordonnées $w = (x, y, z)$ telles que $\gamma$, $\gamma'$ soient $w$-asymptotiquement enlacées.

On peut aussi montrer que l'enlacement asymptotique de deux courbes intégrales oscillantes qui possèdent des tangentes itérées est indépendant des coordonnées $z$-positives choisies pour le définir. Mais dans ce cas, on n'a plus le lemme 1.9 « d'abondance » de coordonnées $z$-positives et la démonstration est plus difficile. Nous n'aurons pas à utiliser ce résultat dans ce travail qui porte, essentiellement, sur les courbes non oscillantes.

***Proposition 1.12***. — *Soient $\gamma$, $\gamma'$ deux courbes intégrales de $X$ non oscillantes asymptotiquement enlaceés telles que $p = \omega(\gamma) = \omega(\gamma')$ et soit $H$ un germe de surface analytique lisse en $p$. Alors, $|\gamma|$, $|\gamma'|$ sont contenues dans la même composante connexe de $M \setminus H$.*

*Démonstration*. — Il suffit de choisir des coordonnées $w = (x, y, z)$ telles que $H = \{y = 0\}$ et qui sont $z$-positives pour $\gamma$, $\gamma'$. $\square$

En particulier, dans les conditions de la proposition 1.12, $\gamma$ n'est pas contenue dans $H$ et toute courbe analytique lisse ou surface analytique lisse est un centre $\gamma$ ou $\gamma'$-admissible d'éclatement ou de ramification. La proposition suivante montre la stabilité de l'enlacement asymptotique par des morphismes admissibles :

***Proposition 1.13***. — *Soient $\gamma$, $\gamma'$ deux courbes intégrales de $X$ non oscillantes telles que $p = \omega(\gamma) = \omega(\gamma')$. Supposons que $\gamma$, $\gamma'$ sont asymptotiquement enlacées. Alors un morphisme de germes d'espaces analytiques $\pi : (\widetilde{M}, \widetilde{p}) \to (M, p)$ est $\gamma$-admissible si et seulement s'il est $\gamma'$-admissible. En particulier* $\mathrm{TI}(\gamma) = \mathrm{TI}(\gamma')$. *Les relevés $\widetilde{\gamma}$, $\widetilde{\gamma}'$ de $\gamma$, $\gamma'$ par un morphisme admissible pour $\gamma, \gamma'$ sont asymptotiquement enlacés. Réciproquement, si $\pi : (\widetilde{M}, \widetilde{p}) \to (M, p)$ est admissible pour $\gamma, \gamma'$ et les relevés $\widetilde{\gamma}$, $\widetilde{\gamma}'$ de $\gamma$, $\gamma'$ par $\pi$ sont asymptotiquement enlacés, alors $\gamma$, $\gamma'$ sont asymptotiquement enlacées.*

*Démonstration*. — D'après la proposition 1.12, si $\gamma, \gamma'$ sont asymptotiquement enlacées, toute courbe analytique lisse ou surface analytique lisse est un centre admissible d'éclatement ou de ramification pour $\gamma, \gamma'$. Par récurrence, il suffit de considérer le cas où $\pi : \widetilde{M} \to M$ est un morphisme admissible élémentaire : soit l'éclatement du point $p$, soit l'éclatement d'une courbe analytique lisse $Y$ passant par $p$, soit une $q$-ramification. Montrons tout d'abord que si $\widetilde{p} = \omega(\widetilde{\gamma})$, $\widetilde{p}' = \omega(\widetilde{\gamma}')$ alors $\widetilde{p} = \widetilde{p}'$. Supposons que ce ne soit pas le cas. Soient $w = (x, y, z)$ des coordonnées $z$-positives pour $\gamma, \gamma'$ centrées en $p$ choisies de telle manière qu'il existe des coordonnées $\widetilde{w} = (\widetilde{x}, \widetilde{y}, \widetilde{z}) : \widetilde{U} \to \mathbb{R}^3$ centrées en $\widetilde{p}$ avec $\widetilde{p}' \in \widetilde{U}$ telles que

$$\begin{aligned}
&\pi(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (\widetilde{z}\widetilde{x}, \widetilde{z}\widetilde{y}, \widetilde{z}) \text{ si } \pi \text{ est l'éclatement de centre } p\,;\\
&\pi(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (\widetilde{x}\widetilde{y}, \widetilde{y}, \widetilde{z}) \quad \text{si } \pi \text{ est l'éclatement de centre } Y = \{x = y = 0\}\,;\\
&\pi(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (\widetilde{x}, \widetilde{y}, \widetilde{z}^q) \quad \text{si } \pi \text{ est une ramification.}
\end{aligned}$$

Dans tous les cas, il existe un plan affine $\widetilde{H}$ dans la carte $\widetilde{w}$ tel que $\widetilde{p}, \widetilde{p}'$ appartiennent à deux composantes connexes distinctes de $\widetilde{U} \setminus \widetilde{H}$ et tel que $H = \pi(\widetilde{H})$ soit contenue dans une surface lisse. Les courbes $|\gamma|, |\gamma'|$ appartiennent à deux composantes distinctes de $\pi(\widetilde{U}) \setminus H$. D'après la proposition précédente, elles ne sont pas asymptotiquement enlacées. On peut supposer que $\widetilde{p} = \widetilde{p}'$. Les coordonnées $\widetilde{w}$ sont $\widetilde{z}$-positives pour $\widetilde{\gamma}, \widetilde{\gamma}'$ et on a

$$\pi(S_{\widetilde{w}}(\widetilde{\gamma})) = S_w(\gamma) \quad \text{et} \quad \pi(S_{\widetilde{w}}(\widetilde{\gamma}) \cap |\widetilde{\gamma}'|) = S_w(\gamma) \cap |\gamma'|.$$

D'après le paragraphe précédent, les courbes $\gamma, \gamma'$ sont asymptotiquement enlacées si et seulement si $\widetilde{\gamma}, \widetilde{\gamma}'$ le sont. □

**1.4. Pinceau intégral.** — Soit $\gamma$ une courbe intégrale de $X$ qui possède des tangentes itérées avec $\omega(\gamma) = p$.

***Définition 1.14***. — Le *pinceau intégral* $\mathrm{PI}(\gamma)$ est l'ensemble des courbes intégrales $\gamma'$ de $X$ telles que $\mathrm{TI}^+(\gamma) = \mathrm{TI}^+(\gamma')$. Nous dirons que :

(1) le pinceau $\mathrm{PI}(\gamma)$ est *un pinceau intégral asymptotiquement enlacé* s'il existe des coordonnées $w$ centrées en $p$ telles que deux courbes quelconques distinctes de $\mathrm{PI}(\gamma)$ sont $w$-asymptotiquement enlacées ;

(2) le pinceau $\mathrm{PI}(\gamma)$ *est un pinceau intégral séparé* si tout couple $\gamma$, $\gamma'$ de $\mathrm{PI}(\gamma)$ est *séparable*, c'est-à-dire, s'il existe une application sous-analytique bornée non constante $f$ d'un voisinage de $|\gamma| \cup |\gamma'|$ dans $\mathbb{R}^2$ telle que $\mathrm{card}(f(|\gamma|) \cap f(|\gamma'|)) < \infty$.

Lorsqu'il existe une courbe formelle $\widehat{\Gamma}$ telle que $\mathrm{TI}(\widehat{\Gamma}) = \mathrm{TI}(\gamma)$ nous dirons que $\widehat{\Gamma}$ est *l'axe* du pinceau intégral $\mathrm{PI}(\gamma)$. Par exemple, si $\gamma$ est oscillante, $\mathrm{PI}(\gamma)$ est un pinceau d'axe $\Gamma$ convergent d'après le théorème du spiralement axial. De plus, si $\Gamma \not\subset \mathrm{Sing}\, X$, l'une des demi-courbes $\Gamma^+ \subset \Gamma$ appartient à $\mathrm{PI}(\gamma)$ et, en précisant le théorème 2 de [**7**], on peut montrer que $\mathrm{PI}(\gamma)$ est un pinceau asymptotiquement enlacé et qu'il n'est pas séparé. D'autre part, il existe des pinceaux intégraux qui ne possèdent pas d'axe. C'est le cas de $\mathrm{PI}(\gamma)$ où $\gamma$ est une courbe intégrale de $X = -x\,\partial/\partial x - \lambda y\,\partial/\partial y - \mu z\,\partial/\partial z$, où $\lambda, \mu \in \mathbb{R}_{>0}$ et $(1, \lambda, \mu)$ sont rationellement indépendants et $|\gamma| \not\subset \{xyz = 0\}$.

Dans la suite de ce paragraphe nous supposons que toutes les courbes de $\mathrm{PI}(\gamma)$ sont non oscillantes. C'est par exemple le cas si $\mathrm{PI}(\gamma)$ ne possède pas un axe convergent. D'après la proposition 1.10, la définition de pinceau enlacé composé de courbes non oscillantes est intrinséque, elle est indépendante des coordonnées $w$ choisies.

Le concept de pinceau intégral n'est pas en général stable par morphisme admissible. Illustrons ceci par un exemple. Considérons $X = -x\,\partial/\partial x - y\,\partial/\partial y - z^2\,\partial/\partial z$. Le plan $z = 0$ est $X$-invariant et les courbes intégrales de $X$ situées dans $z > 0$ s'écrivent $\gamma_{a,b}(z) = (a\exp(-1/z),\, b\exp(-1/z),\, z)$. Ce sont les courbes d'un pinceau intégral $\mathrm{PI}(\gamma_{a_0,b_0})$ d'axe $\Gamma = \{x = y = 0\}$. L'éclatement $\pi_1$ de centre $x = y = 0$ est $\gamma$-admissible pour toute $\gamma_{a,b}$ avec $a^2 + b^2 \neq 0$. Soit $\gamma_{a,b;1}$ le relevé de $\gamma_{a,b}$ par $\pi_1$. Alors $\mathrm{PI}(\gamma_{a,b;1}) = \{\gamma_{a',b',1} \mid [a:b] = [a':b'] \in \mathbb{RP}(1)\}$. L'image inverse de $\mathrm{PI}(\gamma_{a_0,b_0})$ est une union infinie de pinceaux intégraux.

Soit $\widehat{\Gamma}$ une courbe formelle en $p \in M$. On dit que $\widehat{\Gamma}$ est (*sous-analytiquement*) *transcendante* si $\widehat{\Gamma}$ n'est pas contenue dans une surface sous-analytique ; c'est-à-dire, $\mathrm{TI}(\widehat{\Gamma})$ n'est pas un suite de points infiniment proches d'une surface sous-analytique. Soit $\widehat{\Gamma}(z) = (\widehat{x}(z), \widehat{y}(z), z)$ une paramétrisation de Puiseux de $\widehat{\Gamma}$. La courbe $\widehat{\Gamma}$ est transcendante si et seulement si pour tout germe non nul de fonction analytique $f$ en $p \in M$ la série $f(\widehat{x}(z), \widehat{y}(z), z)$ est non nulle et cette propriété est stable par éclatements de centre lisse. La proposition suivante montre que les pinceaux d'axe formel transcendant sont stables par des morphismes admissibles :

***Proposition 1.15***. — *Soit* $\mathrm{PI}(\gamma_0)$ *un pinceau intégral de* $X$ *d'axe formel transcendant* $\widehat{\Gamma}$ *et soit* $\pi : (M', p') \to (M, p)$ *un morphisme local composé d'éclatements de centre lisse ou de ramifications tel que* $\widetilde{p}$ *appartient au transformé strict* $\widehat{\Gamma}'$ *de* $\widehat{\Gamma}$ *par* $\pi$. *Alors* $\pi$ *est admissible pour tout courbe* $\gamma \in \mathrm{PI}(\gamma_0)$ *et l'ensemble* $\pi^{-1}(\mathrm{PI}(\gamma_0))$ *des relevés des courbes de* $\mathrm{PI}(\gamma_0)$ *par* $\pi$ *est un pinceau intégral d'axe formel* $\widehat{\Gamma}'$. *Réciproquement, si* $\pi : (M', \gamma_0', p') \to (M, \gamma_0, p)$ *est un morphisme admissible tel que le pinceau intégral* $\mathrm{PI}(\gamma_0')$ *a un axe formel transcendant* $\widehat{\Gamma}'$, *alors,* $\widehat{\Gamma} = \pi(\widehat{\Gamma}')$ *est un axe transcendant du pinceau* $\mathrm{PI}(\gamma_0)$.

*Démonstration*. — C'est une conséquence de la caractérisation suivante du contact plat entre $\gamma_0$ et $\widehat{\Gamma}$ ([**6**]) : si $\widehat{\Gamma}(z) = (\widehat{x}(z), \widehat{y}(z), z)$ est une paramétrisation de Puiseux de $\widehat{\Gamma}$ et $\gamma_0(z) = (x(z), y(z), z)$ est la paramétrization de la courbe $\gamma_0$ par $z$, alors $\mathrm{TI}(\widehat{\Gamma}) = \mathrm{TI}(\gamma_0)$ si et seulement si $\widehat{\Gamma}(z)$ est le développement asymptotique de $\gamma_0(z)$. Notons que toutes les courbes du pinceau $\mathrm{PI}(\gamma_0)$ sont non oscillantes d'après le théorème de spiralement axial puisque $\widehat{\Gamma}$ est non convergente. $\square$

Le lemme suivant nous sera utile pour définir certains types de pinceaux intégraux finaux.

***Lemme 1.16***. — *Soit* $\gamma$ *une courbe intégrale de* $X$ *qui possède une tangente en* $\omega(\gamma) = p$. *Cette tangente est une direction propre de* $DX(p)$ *de valeur propre* $\lambda(\gamma)$.

*Démonstration*. — Ce lemme est déjà prouvé dans [**7**] lorsque $\gamma$ a un contact plat avec une courbe analytique. L'argument utilisé se généralise de la façon suivante. Soit $\pi_1 : M_1 \mapsto M$ l'éclatement de centre $p$ et $\gamma_1$, $X_1$ les relevés de $\gamma$, $X$ par $\pi_1$. Un point $p$ de $\pi_1^{-1}(p) \equiv \mathbb{RP}(2)$ est un point singulier de $X_1$ si et seulement si $p$ est une direction propre de $DX(p)$. $\square$

***Définition 1.17***. — Soit $\gamma$ une courbe intégrale de $X$ qui possède des tangentes itérées. On dit que $\mathrm{PI}(\gamma)$ est *hyperbolique* si $\lambda(\gamma) \neq 0$ et que $\mathrm{PI}(\gamma)$ est *final de type I* si $\lambda(\gamma) = 0$, $DX(p)$ est diagonalisable et de rang 1.

Dans les deux parties suivantes nous montrerons que les pinceaux hyperboliques et finaux de type I sont des pinceaux séparés de courbes non oscillantes. Dans la partie 4, nous définirons et étudierons les pinceaux finaux de type II. Ils ont un axe formel, ils peuvent être séparés, enlacés, ou posséder des courbes oscillantes. Enfin dans une dernière partie, nous montrerons que ces résultats impliquent les théorèmes I et II de l'introduction via des morphismes admissibles.

## 2. Pinceau intégral hyperbolique

L'objet de cette partie est de démontrer que les pinceaux hyperboliques sont séparés. Dans la suite, $X$ désigne un champ de vecteurs analytique sur un voisinage de 0 dans $\mathbb{R}^3$ et $\gamma_0 : t \mapsto \gamma_0(t)$ une courbe fixée de $X$ avec $\lambda(\gamma_0) \neq 0$. Pour des raisons évidentes de dynamique, $\lambda(\gamma_0) < 0$.

**Théorème 2.1**. — *Un pinceau intégral hyperbolique de $X$ est séparé. Plus précisément, supposons que $\gamma_0$ possède des tangentes itérées et que la valeur propre $\lambda(\gamma_0)$ de $DX(0)$ correspondant à la tangente en 0 à $\gamma_0$ soit strictement négative. Alors $\gamma_0$ est non oscillante ainsi que toutes les courbes $\gamma$ de $\mathrm{PI}(\gamma_0)$. Deux courbes distinctes de $\mathrm{PI}(\gamma_0)$ peuvent être séparées par une submersion analytique d'un voisinage de 0 dans $\mathbb{R}^3$ sur $\mathbb{R}^2$.*

En fait, nous allons démontrer sous les mêmes hypothèses et avec les mêmes notations le résultat suivant :

**Proposition 2.2**. — *Pour tout $\gamma \in \mathrm{PI}(\gamma_0)$, le germe en 0 de $|\gamma|$ est un germe de courbe pfaffienne.*

Avant de rappeler la définition d'un germe de courbe pfaffienne, notons que cette proposition implique le théorème. Tout d'abord « les propriétés de finitude » des sous-ensembles pfaffiens [**18, 25**] impliquent que toutes les courbes de $\mathrm{PI}(\gamma_0)$ sont non oscillantes. D'autre part, d'après [**21**] il existe une structure o-minimale, la famille des $T^\infty$-pfaffiens, qui contient les sous-ensembles pfaffiens de $\mathbb{R}^n$ pour $n \in \mathbb{N}$. Il en résulte que deux courbes distinctes de $\mathrm{PI}(\gamma_0)$ peuvent être séparées par une projection linéaire dans des coordonnées fixées.

Soit $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^3$ avec $\omega(\gamma) = 0$ une immersion analytique, injective et soient $\omega_1$, $\omega_2$ deux 1-formes analytiques sur un voisinage de 0 dans $\mathbb{R}^3$ avec $\omega_1$ intégrable (c'est-à-dire $\omega_1 \wedge d\omega_1 \equiv 0$). Le germe de $|\gamma|$ en 0 est un *germe de courbe* $\{\omega_1, \omega_2\}$-*pfaffien* si l'équation de pfaff $\omega_1 = 0$ possède une *variété intégrale de Rolle* $R_1$ et si la restriction à $R_1$ de $\omega_2$ possède une courbe intégrale de Rolle $R_2$ telle que pour $t$ assez grand $\gamma(t)$ appartient à $R_2$. Rappelons que si $\omega = 0$ est une équation de Pfaff intégrable, analytique sur une variété $M$ et $R$ une variété intégrale (lisse) de $\omega = 0$, on dit que $R$ est de Rolle si toute courbe analytique transverse au feuilletage défini par $\omega = 0$ coupe $R$ en un point au plus.

La démonstration de la proposition 2.2 repose sur le concept de variété stable. Rappelons brièvement sa définition et ses propriétés classiques ([**28, 9, 12**]). Soit $Y$ un champ de vecteurs analytique sur un voisinage $V$ de 0 dans $\mathbb{R}^m$ dont 0 est un point singulier. Notons $\Lambda^s = \{\lambda_1, \lambda_2, \dots, \lambda_\ell\}$ le sous-ensemble du spectre de $DY(0)$ constitué des valeurs propres de partie réelle strictement négative. Soit $E^s$ le sous-espace de $T_0\,\mathbb{R}^m \equiv \mathbb{R}^m$ invariant par $DY(0)$ qui lui correspond.

***Théorème 2.3* (Variété stable).** — *Il existe une unique sous-variété analytique lisse $W^s$ de dimension $\ell$ qui possède les trois propriétés suivantes :*

(i) *La variété $W^s$ contient $0$ et $E^s$ est son espace tangent en $0$.*

(ii) *La variété $W^s$ est positivement invariante par le flot du champ $Y$.*

(iii) *Soit $\gamma : t \mapsto \gamma(t)$, $t \geqslant 0$ une courbe intégrale de $X$, avec $\omega(\gamma) = 0$, qui possède une tangente en $0$ appartenant à $E^s$. Alors le germe de $|\gamma|$ est contenue dans $W^s$.*

La propriété (iii) est moins classique que les deux précédentes (voir [**31**]).

*Démonstration de la proposition 2.2.* — Nous reprenons les notations, hypothèses du théorème : $X$ est un champ de vecteurs analytique sur un voisinage de $0 \in \mathbb{R}^3$, $\gamma_0$ une courbe intégrale de $X$ avec $\omega(\gamma_0) = 0$ qui possède des tangentes itérées, la valeur propre $\lambda(\gamma_0)$ est strictement négative et $\{\lambda(\gamma_0), \lambda_0', \lambda_0''\}$ est le spectre de $DX(0)$. Par définition de $W^s$, les courbes $\gamma$ de $\mathrm{PI}(\gamma_0)$ sont contenues dans $W^s$. Distinguons les cas suivants :

Si $\dim W^s = 1$, la variété $W^s$ est une courbe analytique lisse. Le pinceau $\mathrm{PI}(\gamma_0)$ ne contient que $\gamma_0$. Son image $|\gamma_0|$ est une composante connexe de $W^s \setminus \{0\}$. C'est une demie branche analytique.

Si $\dim W^s = 2$, puisque nous nous intéressons aux germes en $0$ des courbes $\gamma$ de $\mathrm{PI}(\gamma_0)$, nous pouvons supposer que $X$ est analytique sur $\mathbb{R}^3$ et que $W^s$ est le plan $\mathbb{R}^2 \times \{0\}$. Les courbes $\gamma$ de $\mathrm{PI}(\gamma_0)$ sont des courbes intégrales de $X' = X|_{W^s}$ et elles ont une tangente en $0$. D'après [**18**] ou [**7**] page 298, leur germe en $0$ est pfaffien.

Si $\dim W^s = 3$ et $\lambda_0'$, $\lambda_0'' < 0$, nous reprenons un argument de [**19**]. Le point $0$ est une singularité dans le domaine de Poincaré de $X$. D'après le théorème de Poincaré-Dulac ([**9**] ou [**1**], page 179), il existe des coordonnées analytiques $w = (x, y, z)$ centrées en $0$ telles que $X$ s'écrive, modulo une unité multiplicative,

$$X = -x\,\frac{\partial}{\partial x} + f(x,y)\,\frac{\partial}{\partial y} + g(x,y,z)\,\frac{\partial}{\partial z}$$

où $f \in \mathbb{R}[x, y]$, $g \in \mathbb{R}[x, y, z]$. Les courbes intégrales de $X$ sont des courbes intégrales du système : $\omega_1 = xdy - f(x,y,)dx = 0$, $\omega_2 = f(x,y)dz - g(x,y,z)dy = 0$. L'équation $\omega_1$ est intégrable. Soit $R_1(\gamma)$ la variété intégrale de $\omega_1 = 0$ qui contient une courbe intégrale fixée $\gamma \in \mathrm{PI}(\gamma_0)$. C'est un cylindre $|\gamma_1| \times \mathbb{R}$ où $\gamma_1$ est une courbe intégrale du champ de vecteurs $X' = X|_{\mathbb{R}^2}$. Compte tenu de l'écriture de $X'$, la courbe $\gamma_1$ possède une tangente en $0$ et $|\gamma_1|$ est une courbe pfaffienne. Le cylindre $|\gamma_1| \times \mathbb{R} = R_1(\gamma)$ est une variété de Rolle. La courbe $|\gamma|$ est une courbe intégrale de $\omega_2|_{R_1(\gamma)} = 0$. Elle possède une tangente en $0$. C'est une courbe de Rolle, elle est pfaffienne ([**18**]).

Si $\dim W^s = 3$ et $\lambda_0' = \overline{\lambda_0''} = \alpha_0 + i\beta_0$, $\alpha_0 < 0$, $\beta_0 \neq 0$, nous allons nous ramener au cas $\dim W^s = 1$. Soit $n$ le plus petit entier tel que $n > \alpha_0/\lambda(\gamma_0)$ et soit

$$M_0 = \mathbb{R}^3 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} M_2 \cdots \longleftarrow M_{n-1} \xleftarrow{\pi_n} M_n$$

la suite de $n$ éclatements ponctuels de centres respectifs $p_0 = 0$, $p_1$, $p_2, \dots, p_{n-1}$ où $\mathrm{TI}(\gamma_0) = \{p_k\}$. Pour $k = 1, 2, \dots, n$ soient $\gamma_k$, $X_k$ les relevés par $\pi_k$ de $\gamma_{k-1}$, $X_k$, avec $X_0 = X$. D'après le lemme 1.16, on sait que la tangente en $p_k$ à $\gamma_k$ est une direction propre de $DX_k(p_k)$ de valeur propre réelle $\lambda(\gamma_k)$. Soit $\Lambda_k = \{\lambda(\gamma_k), \lambda'_k, \lambda''_k\}$ le spectre de $DX_k(p_k)$. On a $\Lambda_k = \{\lambda(\gamma_{k-1}), \lambda'_{k-1} - \lambda(\gamma_{k-1}), \lambda''_{k-1} - \lambda(\gamma_{k-1})\}$, pour $k \geqslant 2$. On en déduit que $\lambda'_n = \lambda_0 - n\lambda(\gamma_0) = (\alpha_0 - n\lambda(\gamma_0)) + i\beta_0$. La partie réelle de $\lambda'_n$ est strictement positive. La variété stable de $X_n$ en $p_n$ est de dimension 1. Ainsi $\mathrm{PI}(\gamma_0)$ est constitué de $\gamma_0$ qui est une demie branche analytique. $\square$

## 3. Pinceau intégral central de type I

Soit $X$ un champ de vecteurs analytique sur un voisinage de 0 dans $\mathbb{R}^3$ et soit $\gamma_0$ une courbe intégrale de $X$, possèdant des tangentes itérées et telle que $\mathrm{PI}(\gamma_0)$ est un pinceau intégral final de type I. C'est à dire que l'endomorphisme $DX(0)$ est diagonalisable de rang 1, de spectre $\{0, 0, \lambda\}$ avec $\lambda \neq 0$ et $\lambda(\gamma_0) = 0$. Soient $E_0$ le noyau de $DX(p)$ et $E_1$ le sous-espace propre de $DX(0)$ correspondant à $\lambda$. D'après [**12**], il existe une unique courbe analytique lisse $W^h$ tangente en 0 à $E_1$ qui est invariante par le flot de $X$. Dans toute la suite nous fixons des coordonnées analytiques sur $\mathbb{R}^3$

$$w = (x, y, z) = (u, z) \quad \text{avec } W^h = \{u = 0\},\ E_0 = \{z = 0\}.$$

***Théorème 3.1***. — *Les courbes de* $\mathrm{PI}(\gamma_0)$ *sont non oscillantes et*

(i) *Si* $\lambda > 0$, *les courbes de* $\mathrm{PI}(\gamma_0)$ *sont contenues dans une surface* $X$*-invariante de classe* $C^1$ *qui est un graphe au-dessus de* $\{z = 0\}$. *Elles sont séparées par la projection* $(x, y, z) \mapsto (x, y)$.

(ii) *Si* $\lambda < 0$, *toute courbe* $\gamma$ *de* $\mathrm{PI}(\gamma_0)$ *est contenue dans une surface* $L(\gamma)$ *de classe* $C^1$ *et* $X-$*invariante (appelée lamelle) qui possède la propriété suivante. Deux courbes distinctes* $\gamma$, $\gamma'$ *de* $\mathrm{PI}(\gamma_0)$ *sont séparées par la projection* $(x, y, z) \mapsto (x, y)$ *si* $L(\gamma) \neq L(\gamma')$ *et elles sont séparées par la projection linéaire de* $L(\gamma)$ *sur le plan contenant l'axe* $u = 0$ *et la tangente en* 0 *à* $\gamma_0$ *si* $L(\gamma) = L(\gamma')$.

Désignons par $A^c$ (comme attracteur central) l'ensemble des courbes intégrales $\gamma$ de $X$ dont 0 est l'ensemble $\omega$-limite et telles que $|\gamma| \not\subset \{u = 0\}$. D'après [**29**], une courbe $\gamma \in A^c$ possède des tangentes itérées si elle possède une tangente en 0. Ainsi nous pouvons remplacer l'hypothèse « $\gamma_0$ possède des tangentes itérées » par « $\gamma_0$ possède une tangente en 0 ». De plus, on peut montrer, en précisant des arguments et des résultats de [**10, 29**], que toutes les courbes de $A^c$ possèdent une tangente en 0 dès que cette propriété est vraie pour l'une d'entre elles. Dans ce cas, $A^c$ est une union disjointe de pinceaux intégraux. Ceci sort du cadre de notre étude et sera l'objet d'une publication ultérieure.

Dans le paragraphe 1, nous démontrons que les courbes de $\mathrm{PI}(\gamma_0)$ sont non oscillantes et le point (i) du théorème 3.1. La démonstration du point (ii) est essentiellement une conséquence d'un « *théorème de la variété centrale* » relativement bien connu des spécialistes [**15, 8, 26**]. Nous l'énonçons dans le deuxième paragraphe sous la forme que nous a enseignée F. Takens. Nous en déduisons le concept de lamelle et prouvons (ii).

**3.1. Variété centrale.** — Nous reprenons les hypothèses et notations de l'introduction pour $X$, $w = (x, y, z)$. Un calcul classique montre qu'il existe une unique série formelle $\widehat{\psi}$ de $\mathbb{R}[[x, y]]$, telle que

$$\widehat{\psi}(0) = 0, \quad D\widehat{\psi}(0) = 0, \quad L_X(z - \widehat{\psi}(x, y)) \in (z - \widehat{\psi}(x, y))\mathbb{R}[[x, y]].$$

La surface formelle $\widehat{W^c} = \{z - \widehat{\psi}(x, y) = 0\}$ est $X$-invariante et nous l'appelerons *variété centrale formelle* de $X$. En général, elle n'est pas convergente. Cependant, pour tout $k \geqslant 1$, elle peut s'incarner dans « une » surface de classe $C^k$, $W^c_k$, invariante par le flot de $X$, tangente en 0 au plan $z = 0$. C'est une *variété centrale* de $X$ en 0. Plus précisément, $k \geqslant 1$ étant fixé, il existe une fonction $\psi_k$ de classe $C^k$ sur un voisinage $V_k$ de $0 \in \mathbb{R}^2$ telle que $\psi_k$ et $\widehat{\psi}$ ont les mêmes jets d'ordre $k$ en 0 et la surface $z - \psi_k(x, y) = 0$ est $X-$invariante. En général, il n'y a pas unicité des variétés centrales pour un $k$ fixé ni de variété centrale de classe $C^\infty$ [**32**].

Montrons que les courbes de $\mathrm{PI}(\gamma_0)$ sont non oscillantes. Cette assertion étant déjà prouvée dans [**29**], nous allons en donner l'esquisse d'une démonstration qui s'appuie sur d'autres arguments. Supposons qu'une courbe $\gamma$ de $\mathrm{PI}(\gamma_0)$ soit oscillante. D'après le thèoréme 1.1, la courbe $\gamma$ spirale autour d'une demie branche analytique $\Gamma^+$ en 0 qui est $X$-invariante. Distinguons deux cas. Si $\Gamma^+$ n'est pas contenue dans le lieu singulier de $X$, d'après [**7**], c'est l'axe d'un tourbillon de $X$ et toutes les courbes de $\mathrm{PI}(\gamma_0) \smallsetminus \Gamma^+$ sont oscillantes. Quitte à effectuer un nombre fini d'éclatements ponctuels, on peut supposer ([**7**]) que $DX(0)$ est de rang 2. Ce qui contredit l'hypothèse initiale. Supposons que $\Gamma^+$ soit contenue dans le lieu singulier de $X$. Quitte à changer $X$ en $-X$, on peut supposer que la valeur propre $\lambda = \lambda_0$ de $DX(0)$ est strictement négative. En tout point $p$ de $\Gamma^+$ voisin de 0, le spectre de $DX(p)$ est du type $\{0, \lambda'_p, \lambda_p\}$ avec $\lambda_p \neq 0$ et $|\lambda'_p|$ petit par rapport à $|\lambda_p|$. D'après une version à paramètre du théorème d'Hadamard [**12**] (voir par exemple [**24**]), en tout $p \in \Gamma^+$ assez voisin de 0, il existe une courbe lisse, analytique, invariante par $X$ et passant par $p$ tangente à la direction propre correspondante à $\lambda_p$. L'union $S$ de ces courbes est une surface analytique qui contient $\Gamma^+$. Elle est invariante par le flot de $X$ par construction. D'après la propriété caractéristique du spiralement axial décrite dans 1.1, $\gamma$ coupe $S$ une infinité de fois et n'est pas contenue dans $S$, ce qui est incompatible avec « $S$ est $X$-invariante ».

Prouvons maintenant la partie (i) du théorème 3.1. Soit $W^c_1$ une variété centrale de classe $C^1$ de $X$ au point 0. Quitte à restreindre l'ouvert $U$ de définition de $X$, on peut supposer que $W^c_1$ est le graphe d'une fonction de classe $C^1$, $\psi_1 : u \mapsto \psi_1(u)$ où

$u$ appartient à un voisinage $V$ de 0 dans $\mathbb{R}^2$. Dans la carte (de classe $C^1$) $w' = (u, z')$ avec $z' = z - \psi_1(x, y)$ le champ $X$ s'écrit

$$X = a(w')\,\frac{\partial}{\partial x} + b(w')\,\frac{\partial}{\partial y} + \lambda z'(1 + c(w'))\,\frac{\partial}{\partial z},$$

où $a(0) = b(0) = c(0) = 0$. Quitte à restreindre $U$, on peut supposer que $|c(w')| < 1$. Soit $\gamma : t \mapsto (u(t),\, z'(t))$ une courbe intégrale quelconque de $X$ telle que $z'(0) \neq 0$. La fonction $t \mapsto |z'(t)|$ est strictement croissante et ainsi $\omega(\gamma) \neq 0$. Les courbes de $\mathrm{PI}(\gamma_0)$ sont contenues dans $W_1^c$. La restriction à $W_1^c$ de la projection $(x, y, z) \mapsto (x, y)$ est injective. Elle sépare les courbes de $\mathrm{PI}(\gamma_0)$.

**3.2. Partition en lamelles.** — Dans ce paragraphe nous prouvons la partie (ii) du théorème 3.1. Nous reprenons les hypothèses et notations précédentes : $X$ est un champ de vecteurs analytique sur un voisinage $U$ de 0 et l'endomorphisme $DX(0)$ est diagonalisable de spectre $\{0, 0, \lambda\}$, avec $\lambda < 0$. Quitte à effectuer un changement de temps, on peut supposer $\lambda = -1$. Les coordonnées $w = (x, y, z) = (u, z)$ sont fixées telles que $\operatorname{Ker} DX(0) = \{z = 0\}$ et $\{u = 0\}$ est la variété stable $W^s$ de $X$ en 0. Quitte à restreindre $U$ et à changer $z$ en $\mu z$ avec $\mu > 0$, on a :

***Théorème 3.2*** **([8, 15, 26])**. — *Il existe un voisinage $V$ de 0 dans $\mathbb{R}^2$ et un $C^2$-difféomorphisme $H : U \to V \times\, ]-1, 1[$, $H : w = (u, z) \mapsto \overline{w} = H(w) = (\overline{u}, \overline{z}) = (\overline{x}, \overline{y}, \overline{z})$, tangent à l'identité en 0, qui applique l'axe $u = 0$ sur $\overline{u} = 0$ tel que*

$$\overline{X} = H_*(X) = \overline{X}_0(\overline{u}) - \overline{z}(1 + d(\overline{w}))\,\frac{\partial}{\partial \overline{z}},$$

*où $\overline{X}_0$ est un champ de vecteurs de classe $C^1$ sur $V$ vérifiant $\overline{X}_0(0) = 0$, $D\overline{X}_0(0) = 0$ et $d$ est une fonction continue sur $U$ nulle en 0.*

L'espace $\mathbb{R}^3$ est muni des deux normes euclidiennes canoniques associées aux coordonnées $w = (x, y, z)$, $\overline{w} = (\overline{x}, \overline{y}, \overline{z})$ $\|w\|^2 = x^2 + y^2 + z^2$, $\|\overline{w}\|^2 = \overline{x}^2 + \overline{y}^2 + \overline{z}^2$. Le fait de les noter de la même façon simplifie l'écriture et n'apporte pas d'ambiguïté. Quitte à restreindre $U$, on peut supposer en outre que

$$\|DH(w)\| < 3/2 \quad \text{si } w \in U, \quad \|DH^{-1}(\overline{w})\| < 3/2, \quad |d(\overline{w})| < 1/2 \quad \text{si } \overline{w} \in H(U).$$

Dans la suite $\gamma : t \mapsto (u(t), z(t))$, $t \geqslant 0$, $u(t) \neq 0$ est une courbe intégrale de $X$ contenue dans $\mathrm{PI}(\gamma_0)$. Son image $\overline{\gamma} = H \circ \gamma : t \mapsto (\overline{u}(t), \overline{z}(t))$ est une courbe intégrale maximale de $\overline{X}$ telle que $\omega(\overline{\gamma}) = 0$. La composante horizontale $\overline{X}_0(\overline{u})$ de $\overline{X}(\overline{u}, \overline{z})$ étant indépendante de $\overline{z}$, la courbe $t \mapsto \overline{u}(t)$ est une courbe intégrale du champ de vecteurs $\overline{X}_0$. On désigne par $\overline{\gamma}^c$ ($c$ comme central) la courbe intégrale de $\overline{X}$ définie par

$$\overline{\gamma}^c : t \longmapsto \overline{\gamma}^c(t) = (\overline{u}(t), 0), \quad \frac{d}{dt}\,(\overline{u}(t)) = \overline{X}_0(\overline{u}(t)) \neq 0.$$

L'image inverse $L(\gamma)$ de $L(\overline{\gamma}) = |\overline{\gamma}^c| \times\, ]-1, 1[$ par $H$ est appelée *lamelle* de $\gamma$. L'image réciproque par $H$ de $V \times \{0\}$ est une variété centrale $W_1^c$ et $\gamma^c = H^{-1} \circ \overline{\gamma}^c$ est une courbe « accompagnatrice » de $\gamma$ au sens de [**29**]. A priori, $L(\gamma)$ dépend de $H$.

Mais, nous montrerons à la fin de ce chapitre que le germe de $L(\gamma)$ en 0 a un sens intrinsèque. Il est indépendant de $H$ et des coordonnées choisies pour définir $L(\gamma)$. La lamelle $L(\gamma)$ hérite, via $H^{-1}$, des propriétés différentiables des $L(\overline{\gamma})$. Plus précisément, on a l'énoncé suivant :

***Proposition 3.3* (Structure des lamelles).** — *La lamelle $L(\gamma)$ est une surface de classe $C^1$ invariante par le flot de $X$. Plus précisément $L(\gamma)$ est un ouvert d'une surface $S_\gamma$ de classe $C^1$ qui contient $U \cap \{u = 0\}$, tangente en 0 au plan contenant l'axe $u = 0$ et la tangente en 0 à $\gamma$. Le bord de $L(\gamma)$ dans $S_\gamma$ est l'intervalle $U \cap \{u = 0\}$.*

*Démonstration*. — La courbe $\gamma$ étant non oscillante, on vérifie que le germe de $|\gamma|$ en 0 est un graphe de classe $C^1$ sur sa tangente. Quitte à effectuer une rotation dans le plan $z = 0$, on peut supposer que la tangente orientée de $\gamma$ est le demi axe $\{y = z = 0, x > 0\}$. Nous notons encore $x \mapsto \gamma(x) = (x, y(x), z(x))$, $x > 0$ la paramétrisation de $|\gamma|$ par $x$. Il existe une immersion $\sigma$ de classe $C^1$ de $]-\alpha, \alpha[$ dans $\{y = z = 0, x \leqslant 0\} \cup |\gamma|$ qui prolonge $\gamma$. Son image $H \circ \sigma : x \mapsto (\overline{\sigma}^c(x), \overline{z}(x))$ est une courbe de classe $C^1$. L'ensemble $|\overline{\sigma}^c| \times ]-1, 1[$ est une surface de classe $C^1$ dès que $\alpha$ est assez petit. Son image inverse par $H$ est une surface $S_\gamma$ de classe $C^1$ qui possède les propriétés requises. $\square$

***Corollaire 3.4* (Séparation de $\gamma, \gamma'$ si $L(\gamma) = L(\gamma')$).** — *Soit $\gamma'$ une courbe de $\mathrm{PI}(\gamma_0)$ contenue dans $L(\gamma)$ distincte de $\gamma$. La projection $(x, y, z) \mapsto (x, z)$ sépare $\gamma, \gamma'$.*

*Démonstration*. — D'après la proposition précédente, il existe un voisinage $U_\gamma$ de 0 tel que $S_\gamma \cap U_\gamma$ soit un graphe de classe $C^1$ sur son plan tangent en 0, le plan $y = 0$. En particulier $L(\gamma) \cap U_\gamma$ est un graphe sur un ouvert du demi plan $\{y = 0, x > 0\}$. La restriction à $L(\gamma) \cap U_\gamma$ de $(x, y, z) \mapsto (x, z)$ est injective. Elle sépare les courbes $|\gamma| \cap U_\gamma$, $|\gamma'| \cap U_\gamma$. $\square$

Le lemme suivant est un des arguments de la caractérisation intrinsèque des lamelles. Il nous permettra aussi de minorer la distance entre deux courbes de $\mathrm{PI}(\gamma_0)$ qui n'appartiennent pas à la même lamelle.

***Lemme 3.5*.** — *Soit $\gamma'$ une courbe intégrale de $X$ contenue dans la lamelle $L(\gamma)$. Il existe $t_0$ tel que pour $t$ assez grand $\gamma'(t + t_0) = \gamma(t) + \exp(-t/2)\,\varphi(t)$ où $\|\varphi\|$ est une fonction bornée.*

*Démonstration*. — Par définition de $L(\gamma)$, les images $|\overline{\gamma}|$, $|\overline{\gamma}'|$ de $\overline{\gamma} = H \circ \gamma$, $\overline{\gamma}' = H \circ \gamma'$ correspondent à la même orbite de $\overline{X}_0$. Il existe $t_0$ tel que pour $t \gg t_0$,

$$\overline{\gamma}(t) = (\overline{u}(t), \overline{z}(t)), \quad \overline{\gamma}'(t) = (\overline{u}(t - t_0), \overline{z}'(t))$$

avec $\frac{d\overline{u}}{dt}(t) = \overline{X}_0(u(t))$. Les fonctions $\overline{z}(t)$, $\overline{z}'(t)$ sont des solutions de

$$\frac{d\overline{z}}{dt} = -\overline{z}(1 + d(\overline{w})) < -\overline{z}/2 \quad \text{avec } \overline{z} > 0.$$

On en déduit que $\overline{z}(t) = \exp(-t/2)\overline{\varphi}(t)$, $\overline{z}'(t) = \exp(-t/2)\overline{\varphi}'(t)$ où $\|\overline{\varphi}\|$, $\|\overline{\varphi}'\|$ sont des fonctions bornées. Puisque $\|DH^{-1}(\overline{w})\| < 3/2$, on a la majoration

$$\|\gamma'(t+t_0) - \gamma(t)\| = \|H^{-1}(\overline{u}(t), \overline{z}'(t+t_0)) - H^{-1}(\overline{u}(t), \overline{z}(t))\| < \frac{3}{2}|\overline{z}'(t+t_0) - \overline{z}(t)|.$$

ceci démontre le lemme compte tenu de l'écriture de $\overline{z}(t)$, $\overline{z}'(t)$. $\square$

***Proposition 3.6***. — *Deux courbes intégrales $\gamma$, $\gamma'$ distinctes qui n'appartiennent pas à la même lamelle sont séparées par la projection $(x,y,z) \mapsto (x,y)$. Plus précisément, si $\gamma(t) = (u(t), z(t))$, $\gamma'(t) = (u'(t), z'(t))$, alors quel que soit $\varepsilon > 0$, il existe $T$ et $K > 0$ tels que*

$$\|u'(t_1) - u(t)\| > K \exp(-\varepsilon t), \quad \textit{si } t_1 \geqslant t > T.$$

La démonstration de cette proposition repose sur le lemme suivant (certainement classique) qui minore la vitesse à laquelle se rapprochent deux courbes intégrales qui tendent vers le même point singulier dégénéré d'un champ de vecteurs.

***Lemme 3.7***. — *Soit $F$ une application de classe $C^1$ d'un voisinage $V$ de $0 \in \mathbb{R}^n$ dans $\mathbb{R}^n$ telle que $F(0) = 0$, $DF(0) = 0$ et soient $t \mapsto v(t)$, $t \mapsto v'(t)$ deux courbes intégrales de $dv/dt = F(v)$ dont $0$ est l'ensemble $\omega$-limite. Pour tout $\varepsilon > 0$, il existe $T$ tel que*

$$\|v'(t_1) - v(t)\| > \exp(-\varepsilon t)\, \|v'(T + t_1 - t) - v(T)\| \quad \textit{si } t_1 \geqslant t > T.$$

*Démonstration*. — Puisque $F$ est $C^1$ et $F(0) = 0$ on a

$$F(v') - F(v) = DF(v)(v' - v) + \|v' - v\|\varphi(v', v)$$

où $DF$, $\varphi$ sont continues et $DF(0) = 0$, $\varphi(0,0) = 0$. Soit $\eta > 0$ tel que $\|DF(v)\| < \varepsilon/2$ et $\|\varphi(v, v')\| < \varepsilon/2$ si $\|v\|$, $\|v'\| < \eta$. Il existe $T > 0$ tel que si $t > T$ alors $\|v(t)\|$, $\|v'(t)\| < \eta$. Fixons $t_0 \geqslant 0$. La fonction $\rho(t) = \|v'(t+t_0) - v(t)\|$ satisfait l'équation

$$\frac{1}{2}\frac{d}{dt}(\rho(t)^2) = \big\langle v'(t+t_0) - v(t), F(v'(t+t_0)) - F(v(t))\big\rangle.$$

Pour $t > T$ on a $d\rho(t)^2/dt > -2\varepsilon\rho(t)^2$. Ainsi $\rho(t) > \exp(-\varepsilon t)\,\rho(T)$ si $t > T$, indépendemment de $t_0$. L'inégalité requise s'en déduit en prenant $t_0 = t_1 - t$. $\square$

*Démonstration de la proposition 3.6*. — En reprenant les notations précédentes on pose

$$\overline{\gamma}(t) = H \circ \gamma(t) = (\overline{u}(t), \overline{z}(t)), \overline{\gamma}^c(t) = (\overline{u}(t), 0), \gamma^c(t) = H^{-1} \circ \overline{\gamma}^c(t) = (u^c(t), z^c(t)).$$

On définit de la même façon $\overline{\gamma}'(t)$, $\overline{\gamma}'^c(t)$, $\gamma'^c(t)$ et on écrit leurs composantes dans les coordonnées $w$, $\overline{w}$. Les courbes $\gamma^c$, $\gamma'^c$ appartiennent à la variété centrale $W_1^c$ qui est tangente en 0 au plan $z = 0$. Puisque $\omega(\gamma^c) = \omega(\gamma'^c) = 0$, il existe $T > 0$ tel que

$$\frac{1}{2}\|\gamma'^c(t_1) - \gamma^c(t)\| < \|u'^c(t_1) - u^c(t))\| \quad \text{si } t_1 \geqslant t > T.$$

Puisque $\gamma^c$ appartient à la lamelle $L(\gamma)$, d'après le lemme 3.5, on peut choisir $T$ tel que

$$\|u^c(t) - u(t)\| \leqslant \|\gamma(t) - \gamma^c(t)\| < 1/2\, K_1 \exp(-t/2) \quad \text{si } t > T,$$

où $K_1$ est une constante positive. On a évidemment la même majoration (avec les mêmes $T$, $K_1$) pour le couple $u'^c$, $u'$. On en déduit, par l'inégalité triangulaire entre $u$, $u^c$, etc. que

$$1/2\, \|\gamma'^c(t_1) - \gamma^c(t)\| < \|u'(t_1) - u(t)\| + K_1 \exp(-t/2) \quad \text{si } t_1 \geqslant t > T.$$

Puisque $\|DH(w)\| < 3/2$, on a $\|\overline{u}' - \overline{u}\| \leqslant \|\overline{w} - \overline{w}'\| \leqslant 3/2\, \|w - w'\|$. En appliquant cette majoration à $w = \gamma^c(t)$, $w' = \gamma'^c(t_1)$, on obtient pour $T$ assez grand :

$$(*) \qquad \|\overline{u}'(t_1) - \overline{u}(t)\| \leqslant 3\|u'(t_1) - u(t)\| + 3K_1 \exp(-t/2) \quad \text{si } t_1 \geqslant t > T.$$

Fixons $\varepsilon < 1/2$ et appliquons le lemme 3.7 précédent aux solutions $\overline{u}(t)$, $\overline{u}'(t)$ de $d\overline{u}/dt = X_0(\overline{u})$. Puisque $\gamma$, $\gamma'$ n'appartiennent pas à la même lamelle, $\overline{u}(t) \neq \overline{u}'(t_1)$ pour $t, t_1 \geqslant 0$ et, en choisissant $T$ assez grand, il existe $K_2 > 0$ tel que

$$\|\overline{u}'(t_1) - \overline{u}(t)\| > K_2 \exp(-\varepsilon t) \quad \text{si } t_1 \geqslant t > T.$$

Cette inégalité avec $(*)$ prouve la minoration annoncée. □

***Corollaire 3.8***. — *La lamelle $L(\gamma)$ à un sens intrinsèque. Plus précisément $L(\gamma)$ est la réunion des courbes accompagnatrices $\gamma'$ de $\gamma$ au sens de* [**29**] *: il existe $T$, $K > 0$ et $t_0 > 0$ tels que $\|\gamma'(t + t_0) - \gamma(t)\| \leqslant K \exp(-t/2)$ si $t > T$.*

*Démonstration.* — D'après le lemme 3.5, si $\gamma'$ est contenue dans $L(\gamma)$ cette majoration est vérifiée. D'après la proposition 3.6, elle ne peut pas l'être si $\gamma'$ n'est pas contenue dans $L(\gamma)$ puisque l'on aurait, pour $\varepsilon = 1/2$, la minoration

$$\exp(-t/2) \leqslant \|u'(t + t_0) - u(t)\| \leqslant \|\gamma'(t + t_0) - \gamma(t)\|$$

dès que $t$ serait assez grand pour tout $t_0 \geqslant 0$. □

## 4. Pinceau intégral final de type II

L'objet de cette section est l'étude d'un pinceau intégral $\mathrm{PI}(\gamma_0)$ non hyperbolique de $X$ lorsque $DX(p)$ est de rang 2 et le germe de $X$ en $p$ vérifie des conditions algébriques « génériques ». Un tel pinceau sera appelé un pinceau final de type II. Le premier paragraphe est consacré à leur définition et à l'énoncé des résultats, les suivants à la démonstration de ces résultats. C'est le cas le plus riche, le pinceau $\mathrm{PI}(\gamma_0)$ peut alors être séparé ou enlacé. Les courbes qui le constituent peuvent être oscillantes ou non oscillantes.

**4.1. Définition d'un pinceau final de type II.** — Dans toute cette partie $X$ désigne un champ de vecteurs analytique sur $\mathbb{R}^3$ qui s'écrit dans des coordonnées $w = (x, y, z) = (u, z)$

$$(*) \qquad X = L(u,z) - z^{q+1}\Big(\frac{\partial}{\partial z} + Y\Big), \quad L(u,z) = \sum_{i=0}^{q} z^i\, L_i(u)$$

où $q \geqslant 1$, les $L_i(u)$ sont des champs de vecteurs linéaires sur $\mathbb{R}^2 \equiv \mathbb{R}^2 \times \{0\}$ avec rang $DL_0(0) = 2$ et $Y$ est un champ de vecteurs sur $\mathbb{R}^3$ tel que $Y(0)=0$, $dz(Y)=0$. Nous dirons que 0 est une singularité *préfinale (de type II) pour* $X$ et que les coordonnées $w = (u, z)$ sont *de bonnes coordonnées* pour $X$. Ces singularités ont été étudiées par P. Bonckaert, F. Dumortier dans le cadre $C^\infty$ ([**3**]). Le plan $z = 0$ est l'unique plan invariant par $X$ tangent à $z = 0$. Soit $\gamma : t \mapsto \gamma(t)$, $t \geqslant 0$ une courbe intégrale de $X$ telle que $z(0) \neq 0$. Quitte à changer $z$ en $-z$, on peut supposer $z(0) > 0$. Pour $t \geqslant 0$, on a $z(t) > 0$ et $z(t)$ est strictement décroissante. La courbe $\gamma$ est transverse aux plans $z$ = constante. Elle peut être reparamétrée par $z$, ce que nous écrirons $\gamma(z) = (x(z), y(z), z) = (u(z), z)$, $z > 0$. L'application $u(z)$ est solution de l'équation différentielle

$$(*_u) \qquad z^{q+1}\,\frac{du}{dz} = -\sum_{i=0}^{q} z^i L_i(u) + z^{q+1} Y(u,z).$$

Nous avons identifié les champs de vecteurs $L_i(u)$, $Y(u, z)$ et leur écriture dans la base $\partial/\partial x$, $\partial/\partial y$. La première partie du résultat suivant est classique ([**9**]) et sa démonstration est élémentaire. La deuxième partie est démontrée dans [**3**].

***Proposition 4.1***. — *Supposons que* 0 *soit une singularité préfinale de* $X$ *et que* $w = (x, y, z) = (u, z)$ *soient de bonnes coordonnées pour* $X$. *Il existe une unique courbe formelle lisse* $\widehat{\Gamma}$ *invariante par* $X$ *et tangente à l'axe* $u = 0$. *De plus, il existe une courbe intégrale* $\gamma_0$ *de* $X$ *avec* $\omega(\gamma_0) = 0$ *qui a un contact plat avec* $\widehat{\Gamma}$.

L'entier $q + 1$ qui apparaît dans $(*)$ est la multiplicité de la restriction de $X$ à $\widehat{\Gamma}$. C'est un invariant du germe de $X$ en 0. Un entier $r \geqslant q + 1$, étant fixé, on peut toujours choisir de bonnes coordonnées en 0 telles que $\widehat{\Gamma}$ soit tangente à l'ordre $r$ à $u = 0$.

*Ordre* $\ell(X)$ *de la Trace de* $X$. — Soient $w = (u, z)$, $w' = (u', z')$ de bonnes coordonnées pour $X$. Les plans $z = 0$, $z' = 0$ sont les mêmes. Un calcul élémentaire montre que $z = z'$ et que $u' = A_z(u) + z^{q+1}\,\varphi(u, z)$ avec $D_u\varphi(0, 0) \equiv 0$, $A_z \in GL(2, \mathbb{C}\{z\})$. Comparons les écritures de l'équation $(*_u)$ dans les coordonnées $w$, $w'$ :

$$z^{q+1}\,\frac{du}{dz} = -L_z(u) + z^{q+1}\,Y(u,z), \quad z^{q+1}\,\frac{du'}{dz'} = -L'_z(u') + z^{q+1}\,Y'(u',z).$$

En considérant $L(u, z) = L_z(u)$, $L'(u', z) = {L'}_z(u')$ comme des endomorphismes de $\mathbb{R}^2$ à paramètre $z$, on montre que $L'_z = A_z \circ L_z \circ A_z^{-1}$(modulo $z^{q+1}$). La trace $T_X(z) = T(z)$ de l'endomorphisme $L_z(u)$ est un polynôme en $z$ de degré $\leqslant q$ qui est indépendant des bonnes coordonnées choisies. Si $T(z) \not\equiv 0$, nous l'écrivons $T(z) = z^\ell \operatorname{Trace} L_\ell + \cdots$ avec $\operatorname{Trace} L_\ell \neq 0$. Par définition, *l'ordre de la trace* $T_X(z)$ *de* $X$ est l'entier $\ell(X)$ défini par $\ell(X) = \ell$ si $T(z) \not\equiv 0$, $\ell(X) = q$ si $T(z) \equiv 0$.

*Indice de radialité* $k(X)$ *de* $X$. — Notons $\wedge$ le produit extérieur de deux vecteurs de $\mathbb{R}^2$ et $R(u) = x\,\partial/\partial x + y\,\partial/\partial y$ le champ radial de $\mathbb{R}^2$. Si $R(u) \wedge L_z(u) \not\equiv 0$ on a

$$R(u) \wedge L_z(u) = z^k(Q_k(u) + \cdots), \quad \text{avec } k \leqslant q,$$

où $Q_k(u)$ est une forme quadratique non nulle. Par définition *l'indice de radialité de* $X$ est l'entier $k(X)$ défini par $k(X) = k$ si $R(u) \wedge L_z(u) \not\equiv 0$ et $k(X) = q+1$ si $R(u) \wedge L_z(u) \equiv 0$. C'est un invariant analytique de $X$. En effet, on a $k(X) = k$ (avec éventuellement $k = q+1$) si et seulement si $R(u) \wedge L_i(u) = 0$ pour $i = 0, 1, \ldots, k-1$ et $R(u) \wedge L_k(u) \neq 0$. Si $w' = (u', z)$ sont aussi de bonnes coordonnées, en reprenant les notations et la formule $L'_z = A_z \circ L_z \circ A_z^{-1}$ (modulo $z^{q+1}$) du paragraphe précédent, on vérifie que si $R(u') \wedge L'_z(u') = z^{k'}(Q'_{k'}(u') + \cdots)$ on a $k = k'$ et $Q'_k = Q_k \circ A_0$. Si $k \leqslant q$ le *discriminant de la forme quadratique* $Q_k$ est un invariant de $X$, on le note $\Delta_k(X) = \Delta_k$.

***Définition 4.2***. — Supposons que 0 soit une singularité préfinale de $X$. Nous dirons que 0 est une singularité *finale de type II* si $k(X) = q+1$ ou si $k(X) \leqslant q$ et $\Delta_k(X) \neq 0$. De plus, si $\gamma_0$ est une courbe intégrale de $X$ comme dans la proposition 4.1, nous dirons que $\mathrm{PI}(\gamma_0)$ est un pinceau *final de type II.*

Dans le chapitre 5, nous démontrerons que, modulo un morphisme admissible, l'étude des singularités préfinales se ramène à l'étude des singularités finales.

**4.2. Propriétés des pinceaux finaux de type II.** — Soit $\mathrm{PI}(\gamma_0)$ un pinceau intégral final de type II de $X$ et soient $w = (x, y, z)$ des bonnes coordonnées pour $X$. Nous dirons que $\mathrm{PI}(\gamma_0)$ est *$w$-enlacé* si deux courbes distinctes quelconques de $\mathrm{PI}(\gamma_0)$ sont $w$-asymptotiquement enlacées.

***Théorème 4.3***. — *Soit* $\mathrm{PI}(\gamma_0)$ *un pinceau final de type II.*

(1) $\mathrm{PI}(\gamma_0)$ *est un pinceau $w$-enlacé si les conditions suivantes sont satisfaites :*

$$(e) \qquad \ell = \ell(X) < q, \quad \operatorname{Trace} L_\ell < 0, \quad k = k(X) \leqslant q, \quad \Delta_k(X) < 0.$$

(2) $\mathrm{PI}(\gamma_0)$ *est un pinceau séparé de courbes non oscillantes si une des quatre conditions de* $(e)$ *n'est pas satisfaite.*

D'après le paragraphe précédent, les conditions $(e)$ sont complètement déterminées par $L(u, z)$, la partie linéaire de $X$ en $u$. Ainsi, le corollaire suivant est une conséquence immédiate du théorème.

***Corollaire 4.4***. — *La nature (séparé-enlacé) d'un pinceau intégral final* $\mathrm{PI}(\gamma_0)$ *est déterminée par le jet d'ordre* $q+1$ *du champ de vecteurs* $X$.

Il est possible, dans le cas (ii), de décrire plus précisément la « taille » et la « forme » de $\mathrm{PI}(\gamma_0)$. Mais ceci sort un peu des objectifs de ce travail. D'autre part, certains travaux de J. Ecalle ([**11**], v. II) laissent à penser que les courbes intégrales d'un pinceau final possèdent des *développements transasymptotiques* et que ces développements les caractérisent. On peut alors se poser les questions : Ces développements sont-ils resommables-réels ?. Si oui, comment se traduit sur les développements transasymptotiques l'alternative pinceau enlacé - pinceau séparé ?.

***Théorème 4.5***. — *Supposons que* $0$ *soit une singularité finale de type II de* $X$ *dans des coordonnées* $w = (u, z)$ *et que* $\mathrm{PI}(\gamma_0)$ *soit* $w$*-enlacé d'axe formel* $\widehat{\Gamma}$. *Il existe un cylindre* $U = \{(u, z)/|u| < \eta,\ 0 < z < \varepsilon\}$ *positivement invariant par le flot de* $X$ *tel que* $\gamma$ *appartient à* $\mathrm{PI}(\gamma_0)$ *si et seulement si* $|\gamma| \cap U \neq \varnothing$. *De plus on a l'alternative suivante :*

(1) *Si* $\mathrm{PI}(\gamma_0)$ *contient une courbe oscillante, alors son axe* $\widehat{\Gamma} = \Gamma$ *est convergent et les courbes intégrales de* $\mathrm{PI}(\gamma_0)$ *distinctes de* $\Gamma$ *sont oscillantes.*

(2) *Les courbes de* $\mathrm{PI}(\gamma_0)$ *sont non oscillantes et alors son axe* $\widehat{\Gamma}$ *est divergent.*

Les démonstrations du théorème 4.3 et de la premier partie du théorème 4.5 sont l'objet des paragraphes 4.4 et 4.5. Quant à la démonstration de l'alternative du théorème 4.5, c'est une conséquence immédiate du théorème du spiralement axial.

***Proposition 4.6***. — *L'axe* $\widehat{\Gamma}$ *d'un pinceau intégral final enlacé de courbes non oscillantes n'est pas contenu dans une surface analytique.*

*Démonstration.* — Supposons que ce ne soit pas le cas. Il existe un élément irréductible $h$ de $\mathbb{R}\{x, y, z\}$ qui appartient à l'idéal de définition de $\widehat{\Gamma}$,

$$I(\widehat{\Gamma}) = \{\widehat{h} \in \mathbb{R}[[x, y, z]] \mid \widehat{h} \circ \widehat{\Gamma}\,(z) = \widehat{h}(\widehat{x}(z), \widehat{y}(z), z) \equiv 0\},$$

tel que $H = \{h = 0\}$ soit une surface analytique. La courbe $\widehat{\Gamma}$ étant (formellement) $X$-invariante, la dérivée de Lie $g = L_X h = dh(X)$ est aussi un élément de $I(\widehat{\Gamma}) \cap \mathbb{R}\{x, y, z\}$. Il est divisible par $h$ dans $\mathbb{R}\{x, y, z\}$. En effet, si ce n'était pas le cas, $\{h = g = 0\}$ serait un ensemble analytique de dimension un avec $h, g \in I(\widehat{\Gamma})$ et $\widehat{\Gamma}$ serait une courbe convergente. Ainsi, la restriction de $g$ à $H = \{h = 0\}$ est identiquement nulle et $H$ est une surface $X$-invariante. Soit $U$ l'ouvert $X$-invariant du théorème 4.5. Quitte à le restreindre, on peut supposer que $h$ est analytique sur $U$. Montrons que $H \cap U$ n'est pas vide. Supposons le contraire. On a par example $h(w) > 0$ pour $w = (x, y, z) \in U$. D'après l'inégalité de Łojasiewicz [**22**], il existe $C, \alpha > 0$ tels que

$$h(w) > C\,\mathrm{dist}(w, H)^\alpha > C z^\alpha, \quad \text{si } w \in U.$$

D'autre part, si $\gamma : z \mapsto (x(z), y(z), z)$, $z > 0$, est une courbe de $\mathrm{PI}(\gamma_0)$, son développement asymptotique en 0 est $\widehat{\Gamma}(z)$. Toutes les dérivées de $h \circ \gamma(z)$ en 0 sont nulles, ce qui est incompatible avec l'inégalité ci-dessus. L'ensemble semi-analytique $H \cap U$ est de dimension 2. En effet, si ce n'est pas le cas, c'est une union finie de courbes analytiques lisses $|\gamma|$ avec $\gamma \in \mathrm{PI}(\gamma_0)$ et ces courbes ne sont pas enlacées. D'après le théorème de reduction de singularités (voir par example [**13**]), il existe un morphisme $\pi : M_1 \to M$ composé d'éclatements de points et de courbes tel que le transformé strict $H_1$ de $H$ soit à croisements normaux. Il existe une composante lisse $H_1'$ de $H_1$ telle que $\pi(H_1') \cap U$ soit une surface invariante par $X$. Soient $\gamma_1, \gamma_1'$ les relevés par $\pi$ de deux courbes $\gamma, \gamma' \in \mathrm{PI}(\gamma_0)$ qui vérifient $|\gamma|, |\gamma'| \subset \pi(H_1') \cap U$. D'après la proposition 1.13, $\gamma_1, \gamma_1'$ sont (comme $\gamma, \gamma'$) asymptotiquement enlacées. Mais ceci est impossible puisque $\gamma_1, \gamma_1'$ sont contenues dans la surface lisse $H_1'$. $\square$

**4.3. Equations de la dynamique « relative ».** — Dans toute la suite 0 est une singularité finale de type II d'un champ vecteurs $X$. Il s'écrit dans de bonnes coordonnées $w = (u, z) = (x, y, z)$

$$(*) \qquad X = \sum_{i=0}^{q} z^i L_i(u) - z^{q+1}\Big(\frac{\partial}{\partial z} + Y(u,z)\Big), \quad Y(0) = 0, \quad dz(Y) = 0.$$

On fixe une courbe intégrale $\gamma_0$ de $X$, $|\gamma_0| \subset \{z > 0\}$ qui a un contact plat avec la courbe formelle $\widehat{\Gamma}$ donnée par la proposition 4.1. On désigne par $\gamma$, $\gamma'$ deux courbes intégrales de $X$ contenues dans $\{z > 0\}$ avec $\omega(\gamma) = \omega(\gamma') = 0$. On les reparamétrise par $\gamma : z \mapsto (u(z), z)$, $\gamma' : z \mapsto (u'(z), z)$, où $u(z) = (x(z), y(z))$, $u'(z) = (x'(z), y'(z))$ sont des solutions de l'équation différentielle

$$(*_u) \qquad z^{q+1}\,\frac{du}{dz} = -\sum_{i=0}^{q} z^i\, L_k(u) + z^{q+1}\, Y(u, z).$$

Le plan $\mathbb{R}^2$ est muni de sa structure euclidienne canonique $\langle\,,\,\rangle$ et du « produit vectoriel » $\wedge$. Pour décrire le comportement analytique relatif de $\gamma$, $\gamma'$ au voisinage de 0 nous étudierons les fonctions

$$v : z \longmapsto v(z) = u(z) - u'(z), \quad \rho : z \longmapsto \rho(z) = \|v(z)\|,$$
$$e : z \longmapsto \frac{v(z)}{\rho(z)}, \quad \theta : z \longmapsto \theta(z) \quad \text{avec } \exp 2i\pi\theta(z) = e(z).$$

Elles sont reliées par les relations différentielles suivantes :

$$(*_v) \qquad z^{q+1}\,\frac{d}{dz}\,v(z) = -\sum_{i=0}^{q} z^i\, L_i(v(z)) + z^{q+1}(Y(u(z),z) - Y(u'(z),z))$$

$$(*_\rho) \qquad z^{q+1}\,\frac{1}{\rho(z)}\,\frac{d}{dz}\rho(z) = -\sum_{i=0}^{q} z^i\big\langle L_i(e(z)),\ e(z)\big\rangle + z^q\, O(z).$$

$$(*_\theta) \qquad z^{q+1}\,\frac{d}{dz}\,\theta(z) = -\sum_{i=0}^{q} z^i(L_i(e(z)) \wedge e(z)) + z^q\, O(z).$$

L'indice de radialité $k = k(X)$ est déterminé par la formule

$$R(u) \wedge \sum_{i=0}^{q} z^i \, L_i(u) = z^k (Q_k(u) + \|u\|^2 \, O(z))$$

où $Q_k(u)$ est une forme quadratique dont le discriminant $\Delta_k$ est non nul si $k \leqslant q$. Ainsi $(*_\theta)$ s'écrit encore

$$(*_\theta) \qquad z^{q+1} \frac{d}{dz} \theta(z) = -z^k (Q_k(e(z)) + O(z)).$$

Rappelons que l'indice $\ell(X) = \ell$ de la trace de $X$ est défini par $T_X(z) = \operatorname{Trace} L_z = z^\ell (\operatorname{Trace} L_\ell + O(z))$ si $\operatorname{Trace} L_\ell \neq 0$ et $\ell < q$.

***Lemme 4.7***. — *Si $k > 0$ ou si $k = 0$ et $\Delta_0 < 0$ on peut écrire $(*_\rho)$ sous la forme*

$$(*_\rho) \qquad \frac{1}{\rho(z)} \frac{d}{dz} \rho(z) = -z^{\ell-q-1} (\operatorname{Trace} L_\ell + O(z)).$$

*Démonstration*. — Si $k > 0$, alors $\ell = 0$, $L_0(u) = (\operatorname{Trace} L_0) u$ et la formule $(*_\rho)$ est vraie. Si $k = 0$ et $\Delta_0 < 0$ les valeurs propres de $L_0$ ne sont pas réelles. Un argument de F. Takens [**33**] montre que l'on peut choisir une base de $\mathbb{R}^2$ telle que les matrices $A_i$ des $L_i$ s'écrivent dans cette base comme

$$A_i = \begin{pmatrix} 0 & -\beta_i \\ \beta_i & 0 \end{pmatrix} \quad \text{pour } 0 \leqslant i < \ell, \qquad A_\ell = \begin{pmatrix} \alpha_\ell & -\beta_\ell \\ \beta_\ell & \alpha_\ell \end{pmatrix},$$

avec $\alpha_\ell \neq 0$. La formule $(*_\rho)$ s'en déduit immédiatement. □

**4.4. Pinceau final séparé.** — Il s'agit de démontrer que si l'une des quatre conditions $(e)$ ($\ell = \ell(X) < q$, $\operatorname{Trace} L_\ell < 0$, $k = k(X) \leqslant q$, $\Delta_k < 0$) n'est pas satisfaite, alors $\mathrm{PI}(\gamma_0)$ est séparé. Pour le démontrer, nous distinguons deux cas.

***Assertion 1***. — *Si $\ell = q$ ou si $\ell < q$ avec* $\operatorname{Trace} L_\ell > 0$ *et $\Delta_0 < 0$ si $k = 0$ alors* $\mathrm{PI}(\gamma_0) = \{\gamma_0\}$ *et $\gamma_0$ est non oscillante.*

Remarquons tout d'abord que la condition $l = q$ implique $k \geqslant q$. Supposons que $\mathrm{PI}(\gamma_0)$ possède une courbe $\gamma'$ distincte de $\gamma_0$. Avec les notations de 4.3, étudions le couple $\gamma = \gamma_0$, $\gamma'$. Posons $v(z) = u(z) - u'(z)$. La fonction $\rho(z) = \|v(z)\|$ vérifie $(*_\rho)$. Dans le cas $\ell < q$ et $\operatorname{Trace} L_\ell > 0$, la fonction $\rho(z)$ ne peut pas tendre vers 0 avec $z$, c'est-à-dire que $\omega(\gamma') \neq 0$. Supposons $\ell = q$, on a alors

$$\rho(z)^{-1} \frac{d}{dz} \rho(z) = -z^{-1} (\operatorname{Trace} L_\ell + O(z))$$

Si $\operatorname{Trace} L_\ell \geqslant 0$, on vérifie que $\rho(z)$ ne tends pas vers 0 avec $z$. Si $\operatorname{Trace} L_\ell < 0$, il existe $\alpha > 0$ tel que $d\rho(z)/dz > \alpha z^{-1} \rho(z)$. Par intégration on obtient, pour $z$ assez petit, $\rho(z) > c z^\alpha$ avec $c > 0$. Ceci est incompatible avec $\gamma_0$, $\gamma'$ ont un contact plat avec $\widehat{\Gamma}$. Enfin, d'après le théorème du spiralement axial, $\gamma_0$ est non oscillante.

***Assertion 2***. — *Si $k = q + 1$ ou si $k \leqslant q$ avec $\Delta_k > 0$, deux courbes distinctes $\gamma$, $\gamma'$ de* $\mathrm{PI}(\gamma_0)$ *peuvent être séparées par une projection linéaire. De plus, les courbes de* $\mathrm{PI}(\gamma_0)$ *sont non oscillantes.*

Signalons que la première partie de cette assertion est vraie si $\theta(z)$ possède une limite $\theta_0$ lorsque $z$ tend vers 0. En effet, on peut supposer $\theta_0 = 0$, c'est-à-dire $\lim_{z\to 0} e(z) = (1, 0)$. Dans les coordonnées $u = (x, y)$, on a aussi $\lim_{z\to 0}(x(z) - x'(z))/|x(z) - x'(z)| = 1$. La projection $(x, y, z) \mapsto (x, z)$ sépare $\gamma$, $\gamma'$. Pour démontrer que $\theta$ tend vers une limite finie nous allons travailler avec $(*_\theta)$. Distinguons les deux cas de l'énoncé. Si $k \geqslant q + 1$, c'est-à-dire $0 = e(z) \wedge L_i(e(z))$ pour $0 \leqslant i \leqslant q$, on déduit de $(*_\theta)$ que la dérivée $d\theta(z)/dz$ est bornée et, par intégration, que $\theta(z)$ tend vers une limite si $z$ tend vers 0. Supposons maintenant que $k \leqslant q$ et $\Delta_k > 0$. L'endomorphsime $L_k$ possède deux valeurs propres réelles distinctes $\lambda, \mu$. Choisissons des coordonnées $u = (x, y)$ telles que les axes $(y = 0)$, $(x = 0)$ soient les directions propres correspondantes. Dans ces coordonnées on a

$$e(z) = (\cos\theta(z), \sin\theta(z)), \quad Q_k(e(z)) = (\lambda - \mu)\sin\theta(z)\cos\theta(z).$$

La relation $(*_\theta)$ s'écrit, avec $r = q + 1 - k$, $a = 1/2(\mu - \lambda)$,

$$az^r \frac{d}{dz}\theta(z) = \sin 2\theta(z) + O(z).$$

On peut toujours supposer $a > 0$ quitte à changer l'orientation. En reprenant un joli argument élémentaire de Hu ([**16**]) montrons que $\theta(z)$ tend vers une limite $\theta_0 = 0$ (mod $\pi/2$). La dérivée $d\theta(z)/dz$ étant du signe de $\sin 2\theta(z) + O(z)$, un réel $\varepsilon > 0$ étant fixé, il existe $z_0(\varepsilon) = z_0$, $b(\varepsilon) = b > 0$ tels que

$$b < (-1)^n \frac{d}{dz}\theta(z) \quad \text{si } z \leqslant z_0 \quad \text{et} \quad n\,\frac{\pi}{2} + \varepsilon \leqslant \theta(z) \leqslant (n+1)\,\frac{\pi}{2} - \varepsilon.$$

La fonction $\theta(z)$ est strictement croissante sur $\theta^{-1}(]n\pi/2 + \varepsilon, (n+1)\pi/2 - \varepsilon[)$. On peut supposer que $z_0$ appartient à un de ces intervalles, par exemple avec $n = 0$. Il existe $z_1(\varepsilon) = z_1 < z_0$ tel que $\theta(z_1) = \pi/2 - \varepsilon$. Puisque la valeur $\pi/2 + \varepsilon$ ne peut être franchie par $\theta(z)$ qu'en décroissant on a $\pi/2 - \varepsilon \leqslant \theta(z) \leqslant \pi/2 + \varepsilon$ si $0 < z < z_1(\varepsilon)$.

Il reste à démontrer que les courbes de $\mathrm{PI}(\gamma_0)$ sont non oscillantes. Supposons qu'il existe $\gamma \in \mathrm{PI}(\gamma_0)$ oscillante. D'après le théorème du spiralement axial, l'axe du pinceau est une courbe lisse analytique $\Gamma$. Soient $w = (x, y, z) = (u, z)$ de bonnes coordonnées telles que $\Gamma = \{u = 0\}$. Les courbes $\Gamma^+ = \Gamma \cap \{z > 0\}$ et $\gamma$ ne sont pas séparées par une projection linéaire sur un plan contenant $\Gamma$. Ce qui contredit la première partie de l'assertion.

**4.5. Pinceau final enlacé.** — Nous supposons que les quatre conditions $(e)$ sont réalisées. Elles s'écrivent encore

$(e_\rho)$ $\quad \mathrm{Trace}(L_z) = z^\ell(\mathrm{Trace}(L_\ell) + O(z)) \quad$ avec $\ell < q \quad$ et $\quad \mathrm{Trace}(L_\ell) < 0$,

$(e_\theta)$ $\quad L_z(e(z)) \wedge e(z) = z^k(Q_k(e(z)) + O(z)) \quad$ avec $k \leqslant q$ et le discriminant $\Delta_k$ de $Q_k(u)$ est strictement négatif.

Pour achever la démonstration des théorèmes 4.3 et 4.5 nous démontrons les trois assertions suivantes.

***Assertion 1***. — *Toute courbe intégrale $\gamma'$ de $X$ avec $\omega(\gamma')=0$, $|\gamma'|\subset\{z>0\}$ appartient à* $\mathrm{PI}(\gamma_0)$.

Appliquons $(*_\rho)$ au couple $\gamma=\gamma_0$, $\gamma'$. Si $\rho(z)=\|v(z)\|=\|u(z)-u'(z)\|$, pour $z$ assez petit on a, compte tenu de l'hypothèse $(e_\rho)$,

$$z^{q+1}\,\frac{1}{\rho(z)}\,\frac{d}{dz}\,\rho(z)=-z^{\ell}(\operatorname{Trace}L_\ell+O(z)).$$

Fixons $\beta$ avec $0<\beta<-\operatorname{Trace}L_\ell$. Il existe $z_0>0$ tel que pour $0<z<z_0$,

$$\frac{1}{\rho(z)}\,\frac{d}{dz}\,\rho(z)>\frac{\beta}{z^r}\quad\text{avec } r=q+1-\ell\geqslant 2.$$

On en déduit, par intégration, qu'il existe des constantes $c$, $c'>0$ telles que

$$0<\rho(z)<c\,\exp\Bigl(-\frac{c'}{z^{r-1}}\Bigr)\quad\text{si } 0<z<z_0.$$

Les courbes $z\mapsto\gamma'(z)$, $z\mapsto\gamma_0(z)$ ont un contact plat en 0. Cette propriété est conservée par éclatements de points et donc $\mathrm{TI}(\gamma')=\mathrm{TI}(\gamma_0)$.

***Assertion 2***. — *Deux courbes distinctes de* $\mathrm{PI}(\gamma_0)$ *sont asymptotiquement enlacées.*

Appliquons $(*_\theta)$ à une couple $\gamma$, $\gamma'$ de $\mathrm{PI}(\gamma_0)$. On a

$$z^{q+1}\,\frac{d}{dz}\theta(z)=-z^k(Q_k(e(z))+O(z))$$

Compte tenu de l'hypothèse $(e_\theta)$, nous pouvons supposer que $Q_k(u)$ est définie positive, quitte à changer l'orientation de $\mathbb{R}^2$. Il existe $\alpha>0$ et $z_0>0$ tels que pour $0<z<z_0$ on a

$$\frac{d}{dz}\,\theta(z)<-\frac{\alpha}{z^r}\quad\text{avec } r=q+1-k>0.$$

Il existe des constantes $c$, $c'>0$ telles que

$$\theta(z)>\frac{c}{z^{r-1}}+c'\text{ si } r>1\quad\text{ou}\quad\theta(z)>c\operatorname{Log}1/z+c'\text{ si } r=1.$$

***Assertion 3***. — *Il existe $\varepsilon$, $\eta>0$ tels que $C=C_{\varepsilon,\eta}=\{(u,z)\mid\|u\|\leqslant\eta,\ 0<z\leqslant\eta\}$ soit $X$-positivement invariant et toute courbe $\gamma$ qui coupe $C$ appartient à* $\mathrm{PI}(\gamma_0)$.

Soit $\gamma: z\mapsto(u(z),z)$ une courbe intégrale de $X$ contenue dans $z>0$. Pour établir une relation différentielle sur $r(z)=\|u(z)\|$, prenons le produit $<u,(*_u)>$ avec

$$(*_u)\qquad z^{q+1}\,\frac{du}{dz}=-\sum_{i=0}^{q}z^i\,L_i(u)+z^{q+1}\,Y(u,z).$$

On obtient, compte tenu de l'hypothèse $(e_\rho)$,

$$(*_r)\qquad z^{q+1}r\,\frac{dr(z)}{dz}=-z^{\ell}r^2(\operatorname{Trace}L_\ell+O(z))+z^{q+1}\langle u,Y(u,z)\rangle.$$

Puisque $Y(0) = 0$, on peut écrire $Y(u,z) = z\,Y_0(z) + \|u\|\,Y_1(u,z)$ où $Y_0$, $Y_1$ sont des applications bornées au voisinage de 0. La relation $(*_r)$ s'écrit, avec $s = q+1-\ell \geqslant 2$,

$$\frac{dr(z)}{dz} = -\frac{r}{z^s}(\operatorname{Trace} L_\ell + O(z)) + zO(z,u).$$

Puisque $\operatorname{Trace} L_\ell < 0$, il existe $\varepsilon, \eta$ tels que la dérivée $dr(z)/dz$ est strictement positive pour $0 < z < \varepsilon$ et $r = \|u\| = \eta$. Ainsi $C_{\varepsilon,\eta}$ est $X$-positivement invariant. De plus, la fonction $r(z)$ est strictement croissante sur une région de la forme $r > cz^{s+1}$, avec $c > 0$. Ainsi, $\lim_{z\to 0} r(z) = 0$ s'il existe $z_0$ tel que $(u(z_0), z_0)$ appartienne à $C_{\varepsilon,\eta}$. D'autre part, en écrivant $\gamma_0(z) = (u_0(z), z)$, $\rho(z) = \|u(z) - u_0(z)\|$, on a

$$(*_\rho) \qquad \frac{d\rho}{dz} = \frac{\rho(z)}{z^s}(-\operatorname{Trace} L_\ell + O(z)).$$

Ainsi $\rho(z)$ est une fonction plate en $z = 0$ puisque $s \geqslant 2$. La courbe $\gamma$ a un contact plat avec $\widehat{\Gamma}$. Elle appartient à $\mathrm{PI}(\gamma_0)$.

## 5. Le cas général

Soient $X$ un champ de vecteurs analytique sur $M$, $\gamma_0$ une courbe intégrale non oscillante de $X$ avec $\omega(\gamma_0) = p$ et $\mathrm{PI}(\gamma_0)$ son pinceau intégral. Nous allons démontrer les deux théorèmes énoncés dans l'introduction :

***Théorème I***. — *Si les courbes de* $\mathrm{PI}(\gamma_0)$ *sont non oscillantes on a l'une des propriétés suivantes :*

s) *Deux courbes distinctes, quelconques, de* $\mathrm{PI}(\gamma_0)$ *sont sous-analytiquement séparées.*

e) *Deux courbes distinctes, quelconques, de* $\mathrm{PI}(\gamma_0)$ *sont asymptotiquement enlacées.*

*De plus, ces propriétés ne peuvent pas être satisfaites simultanément.*

***Théorème II***. — *Soit* $\mathrm{PI}(\gamma_0)$ *est un pinceau intégral enlacé de courbes non oscillantes.*

(1) $\mathrm{PI}(\gamma_0)$ *possède un axe formel* $\widehat{\Gamma}$ *non convergent, transcendant ; c'est-à-dire, il n'existe pas de surface analytique qui contienne* $\widehat{\Gamma}$.

(2) *Si* $V$ *est un voisinage de* $p_0$, *il existe un ouvert sous-analytique* $U \subset V$ *positivement invariant par le flot de* $X$ *tel qu'une courbe intégrale* $\gamma$ *de* $X$ *appartient à* $\mathrm{PI}(\gamma_0)$ *si et seulement si* $|\gamma| \cap U \neq \varnothing$.

Pour démontrer ces théorèmes nous allons tout d'abord montrer que si $\mathrm{PI}(\gamma_0)$ n'est pas séparé (par exemple il existe $\gamma_1 \in \mathrm{PI}(\gamma_0)$ distincte de $\gamma_0$ non séparable de $\gamma_0$), alors les trois assertions suivantes sont vraies :

(i) Il existe une courbe formelle $\widehat{\Gamma}$ transcendante telle que $\mathrm{TI}(\gamma_0) = \mathrm{TI}(\widehat{\Gamma})$.

(ii) Deux courbes distinctes de $\mathrm{PI}(\gamma_0)$ sont asymptotiquement enlacées.

(iii) Pour tout voisinage $V$ de $p \in M$, il existe un ouvert sous-analytique connexe $U$ positivement invariant par $X$ tel que $\gamma \in \mathrm{PI}(\gamma_0)$ si et seulement si $|\gamma| \cap U \neq \varnothing$.

Leurs démonstrations reposent sur la proposition suivante :

***Proposition 5.1***. — *Supposons que $\gamma_0, \gamma_1$ soient non séparables. Alors il existe un morphisme admissible $\pi : (M', \gamma'_0, p') \to (M, \gamma_0, p)$ tel que $\mathrm{PI}(\gamma'_0)$ soit un pinceau final de type II.*

Enfin, l'alternative de théorème I est une conséquence de la proposition suivante :

***Proposition 5.2***. — *Soient $\gamma_0$, $\gamma_1$ deux courbes non oscillantes asymptotiquement enlacées. Si $f$ est une application sous-analytique bornée d'un voisinage $U$ de $|\gamma_0| \cup |\gamma_1|$ dans $\mathbb{R}^2$, alors $\mathrm{card}(f|\gamma_0| \cap f|\gamma_1|) = \infty$.*

**5.1. La proposition 5.1 implique (i), (ii), (iii).** — Supposons que la proposition 5.1 soit vraie et que $\gamma_0, \gamma_1$ ne soient pas séparables. Le relevé $\gamma'_1 = \pi^{-1}(\gamma_1)$ appartient à $\mathrm{PI}(\gamma'_0)$ et de plus $\gamma'_0$ et $\gamma'_1$ ne sont pas séparables. Les conditions (e) du théorème 4.3 sont satisfaites en $p'$ pour $\mathrm{PI}(\gamma'_0)$. En particulier, il existe une courbe formelle $\widehat{\Gamma}'$ de $(M', p')$ analytiquement transcendante et telle que $\mathrm{TI}(\gamma'_0) = \mathrm{TI}(\widehat{\Gamma}')$ d'après la proposition 4.6. Soit $\widehat{\Gamma}$ la projection de $\widehat{\Gamma}'$ par $\pi$. En utilisant les mêmes arguments que dans la démonstration de la proposition 1.15 on montre que $\mathrm{TI}(\gamma_1) = \mathrm{TI}(\gamma_0) = \mathrm{TI}(\widehat{\Gamma})$. Il est clair que $\widehat{\Gamma}$ est analytiquement transcendante. Pour montrer que $\widehat{\Gamma}$ est sous-analytiquement transcendante, il suffit de montrer que son relevé $\sigma^{-1}(\widehat{\Gamma})$ par un morphisme $\sigma : (\widetilde{M}, \widetilde{p}) \to (M, p)$ est aussi analytiquement transcendant. Les relevés $\widetilde{\gamma}_0$, $\widetilde{\gamma}_1$ de $\gamma_0$, $\gamma_1$ par $\sigma$ sont toujours non séparables. L'argument précédent montre l'existence d'un axe $\widetilde{\Gamma}$ analytiquement transcendant pour $\mathrm{PI}(\widetilde{\gamma}_1)$. Par projection, $\mathrm{TI}(\gamma_1) = \mathrm{TI}(\sigma(\widetilde{\Gamma}))$ et ainsi $\widetilde{\Gamma} = \sigma^{-1}(\widehat{\Gamma})$. Ceci prouve l'assertion (i).

Montrons l'assertion (ii). Soient $\gamma_2, \gamma_3$ deux courbes distinctes de $\mathrm{PI}(\gamma_0)$. D'après la proposition 1.15, le morphisme $\pi$ est admissible pour $\gamma_2$ et $\gamma_3$. Leurs relevés $\gamma'_2$, $\gamma'_3$ appartiennent au pinceau $\mathrm{PI}(\gamma'_0)$. Puisque les courbes $\gamma'_0$, $\gamma'_1$ ne sont pas séparables, d'après le théorème 4.3, le pinceau $\mathrm{PI}(\gamma'_0)$ est un pinceau final de type II enlacé. Ceci prouve que $\gamma_2$ et $\gamma_3$ sont asymptotiquement enlacées, d'après la proposition 1.13. Ceci prouve (ii).

Montrons l'assertion (iii). Si $V$ est un voisinage de $p \in M$, $V' = \pi^{-1}(V)$ est un voisinage de $p' \in M'$. D'après le théorème 4.5, il existe un ouvert sous-analytique $U' \subset V'$ tel que $\gamma' \in \mathrm{PI}(\gamma'_0)$ si et seulement si $|\gamma'| \cap U' \neq \varnothing$. La projection $U = \pi(U')$ possède la même propriété relativement au pinceau $\mathrm{PI}(\gamma_0)$, d'après la proposition 1.15. □

**5.2. Démonstration de la proposition 5.1.** — Le résultat suivant de [**6**] nous permet de supposer que $p$ est une singularité élémentaire du champ de vecteurs $X$, c'est-à-dire, que $DX(p)$ n'est pas nilpotente.

***Théorème 5.3*** **(Uniformisation $\gamma_0$-polarisé)**. — *Soit $\gamma_0$ une courbe intégrale non oscillante d'un champ de vecteurs $X$ analytique sur un voisinage de $p \in M$. Il existe un morphisme admissible $\pi : (M', \gamma'_0, p') \to (M, \gamma_0, p)$ et un champ de vecteurs $X'$ analytique sur un voisinage de $p'$ tels que $p'$ soit une singularité élémentaire de $X'$ et $\gamma'_0$ une courbe intégrale de $X'$.*

D'après le théorème 2.1, nous pouvons supposer que $\lambda(\gamma_0) = 0$. Sinon $\gamma_0$ et $\gamma_1$ seraient séparables. On a alors deux possibilités pour le spectre de $DX(p)$. Dans le premier cas, le spectre de $DX(p)$ s'écrit $\{0,0,\lambda\}$ avec $\lambda \neq 0$. Quitte à effectuer un éclatement local de centre $p$, on peut supposer que $DX(p)$ est diagonalisable de rang 1. Alors, $\gamma_0$ appartient à un pinceau final de type I. D'après le théorème 3.1, $\gamma_0$ et $\gamma_1$ sont séparables, ce qui est impossible. Dans le deuxième cas, le spectre de $DX(p)$ s'écrit $\{0,\lambda,\lambda'\}$ avec $\lambda, \lambda' \neq 0$. Montrons d'abord qu'on peut se ramener au cas d'une singularité préfinale avec le lemme suivant :

***Lemme 5.4***. — *Il existe un morphisme $\gamma_0$-admissible $\pi : (M', \gamma_0', p') \to (M, \gamma_0, p)$ tel que $\gamma_0'$ soit une courbe intégrale d'un champ vecteurs $X'$ tel que $p' = \omega(\gamma_0')$ est une singularité préfinale de type II.*

*Démonstration*. — Soient $w = (x,y,z) = (u,z)$ des coordonnées centrées en $p$ telles que $\operatorname{Ker} DX(p) = \{u = 0\}$. Il existe (comme dans la proposition 4.1, voir [**3**]) une unique courbe formelle (ou analytique) $\widehat{\Gamma}$ lisse, tangente à $\{u = 0\}$, invariante par $X$ qui vérifie aussi que $\mathrm{TI}(\widehat{\Gamma}) = \mathrm{TI}(\gamma_0)$. De plus, quitte à éclater le point $p$, on peut supposer que le plan $\{z = 0\}$ est $X$-invariant. Montrons tout d'abord que $\widehat{\Gamma}$ n'est pas contenue dans le lieu singulier $\operatorname{Sing} X$ de $X$. En effet, si $\widehat{\Gamma} \subset \operatorname{Sing} X$, on peut choisir les coordonnées $w$ telles que $\{u = 0\} = \widehat{\Gamma}$. Si les valeurs propres $\lambda$, $\lambda'$ de $DX(p)$ ne sont pas imaginaires, la courbe $\widehat{\Gamma}$ est une variété centrale de $X$ en $p$. D'après le théorème de la variété centrale (voir [**15, 26**]), la dynamique du flot de $X$ est topologiquement conjuguée à la dynamique du champ $X|_{\{z=0\}}$. Les courbes intégrales de $X$ dont $p$ est le point $\omega$-limite sont contenues en $\{z = 0\}$. En particulier, la courbe $\gamma_0$, ce qui n'est pas possible puisque $\lambda(\gamma_0) = 0$. Si $\lambda = \overline{\lambda'} = ia$, $a \in \mathbb{R}^*$, considérons $\sigma : \widetilde{M} \to M$ l'éclatement de l'axe $\{u = 0\}$. Sa fibre $D_0 = \sigma^{-1}(p)$ est un cercle invariant par le relevé $\widetilde{X}$ de $X$. Il possède une application premier retour de Poincaré non triviale. Ainsi, toute courbe $\gamma$ telle que $\omega(\gamma) = p$ est une courbe oscillante. Ceci est impossible puisque $\gamma_0$ est non oscillante.

La multiplicité de la restriction de $X$ à la courbe invariante $\widehat{\Gamma} \not\subset \operatorname{Sing} X_0$ est un entier $q+1$ avec $q \geqslant 1$, puisque $\widehat{\Gamma}$ est tangente au noyau de $DX(p)$. Choississons des coordonnées $w = (x,y,z) = (u,z)$ avec $|\gamma_0| \subset \{z > 0\}$ telles que $\widehat{\Gamma}$ soit tangente à l'ordre au moins $2q+1$ avec $\{u = 0\}$. Un calcul élémentaire montre que, modulo une unité multiplicative, $X$ s'écrit dans ces coordonnées sous la forme

$$(*_0) \qquad X = \sum_{i=0}^{\infty} z^i L_i(u) + Y_0(u,z) - z^{q+1}\,\frac{\partial}{\partial z}.$$

où les $L_i(u)$ sont des champs de vecteurs linéaires sur $\mathbb{R}^2 \equiv \mathbb{R}^2 \times \{0\}$ avec $DL_0(0)$ de rang 2 et $Y_0$ est un champ de vecteurs qui vérifie

$$Y_0(0) = 0, \quad dz(Y_0) = 0, \quad Y_0(0,z) = O(z^{2q+1}), \quad D_u\, Y_0(0,z) \equiv 0.$$

Soit $\pi : (M', p') \to (M, p)$ le morphisme composé des éclatements des $q+1$ premiers points infiniment proches de $\widehat{\Gamma}$ et soient $(u', z)$ des coordonnées en $p'$ telles que $\pi(u', z) = (z^{q+1}u', z)$. On vérifie aisément que le transformé $X' = \pi^*(X)$ s'écrit sous la forme

$$(*) \qquad X' = L_z(u') - z^{q+1}\Big(\frac{\partial}{\partial z} + Y\Big), \ \ L_z(u') = \sum_{i=0}^{q} z^i\, L_i(u')$$

avec $DL_0(0)$ de rang 2, $dz(Y) = 0$, $Y(0) = 0$. Ceci montre que $p'$ est une singularité préfinale de type II. □

Le lemme suivant achève la démonstration de la proposition 5.1 (voir aussi [**30**]) :

**Lemme 5.5**. — *Supposons que $p$ est une singularité préfinale du champ de vecteurs $X$ avec $\lambda(\gamma_0) = 0$. Il existe un morphisme admissible $\pi : (M', \gamma_0', p') \to (M, \gamma_0, p)$ tel que* $\mathrm{PI}(\gamma_0')$ *soit un pinceau final de type II.*

*Démonstration.* — D'après le lemme précédent, $\gamma_0$ a un contact plat avec une courbe $X$-invariante $\widehat{\Gamma}$ qui n'est pas contenue dans le lieu singulier de $X$. Soient $w = (x, y, z) = (u, z)$ de bonnes coordonnées pour $X$ telles que $|\gamma_0| \subset \{z > 0\}$. Écrivons $X$ sous la forme $(*)$ dans ces coordonnées. Notons $q+1$ la multiplicité de la restriction de $X$ à $\widehat{\Gamma}$ et $k = k(X)$ l'indice de radialité de $X$. Par définition, $k \leqslant q+1$. Si $k = q+1$, $\mathrm{PI}(\gamma_0)$ est un pinceau final de type II. Supposons que $k \leqslant q$. Si $R(u)$ est le champ radial de $\mathbb{R}^2$, l'entier $k$ est défini par

$$R(u) \wedge L_z(u) = z^k(Q_k(u) + O(z)), \quad k \leqslant q,$$

où $Q_k(u)$ est une forme quadratique non nulle. Notons $\Delta_k$ le discriminant de $Q_k$. Nous allons montrer qu'il existe un morphisme $\gamma_0$-admissible $\pi : (M', p') \to (M, p)$ tel que $p'$ soit une singularité préfinale du relevé $X'$ de $X$ et tel que le discriminant $\Delta_k'$ correspondant à $X'$ soit non nul. Supposons que $\Delta_k = 0$. Posons

$$L_i(x, y) = (a_i x + b_i y)\,\frac{\partial}{\partial x} + (c_i x + d_i y)\,\frac{\partial}{\partial y}, \quad i = 0, 1, 2, \ldots, q.$$

Les conditions précédentes sur $L_0, L_1, \ldots, L_{k-1}$ se traduisent par les égalités $a_i = d_i$, $b_i = c_i = 0$ pour $i = 0, 1, 2, \ldots k-1$ avec $a_0 \neq 0$. De plus, la condition $\Delta_k = 0$ signifie que $DL_k(0)$ a une valeur propre double et n'est pas diagonalisable. On peut choisir $u = (x, y)$ tel que $a_k = d_k$, $b_k = 1$, $c_k = 0$. La droite $\{y = z = 0\}$ est invariante par $X$ et $L_k(u)$. L'éclatement de cette droite est un morphisme admissible $\sigma : (\widetilde{M}, \widetilde{\gamma}_1, \widetilde{p}) \to (M, \gamma_1, p)$. Soit $\widetilde{X}$ le relevé de $X$ par $\pi$ et soient $\widetilde{w} = (\widetilde{x}, \widetilde{y}, \widetilde{z}) = (\widetilde{u}, \widetilde{z})$ des coordonnées centrées en $\widetilde{p}$ telles que $\sigma(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (\widetilde{x}, \widetilde{x}\widetilde{y}, \widetilde{z})$. Le champ $\widetilde{X}$ s'écrit sous la forme

$$\widetilde{X} = \sum_i^q \widetilde{z}^i \widetilde{L}_i(\widetilde{u}) - \widetilde{z}^{q+1}\Big(\frac{\partial}{\partial \widetilde{z}} + \widetilde{Y}(\widetilde{u}, \widetilde{z})\Big)$$

où $\widetilde{a}_i = \widetilde{d}_i = a_i$, $\widetilde{b}_i = b_{i-1}$, $\widetilde{c}_i = c_{i+1}$, $i = 0, 1, \ldots, q-1$, $\widetilde{a}_q = a_q$, $\widetilde{d}_q = d_q + 1$, $\widetilde{c}_q = c_{q-1}$. Si $q - k = 0$, $R(\widetilde{u}) \wedge \widetilde{L}_z(\widetilde{z}) = \widetilde{z}^q \widetilde{Q}_q(\widetilde{u})$. Le discriminant $\widetilde{\Delta}_q$ de $\widetilde{Q}_q(\widetilde{u})$ est égal à 1. Si $q - k > 0$ et $c_{k+1} = 0$ on a $\widetilde{L}_i(\widetilde{u}) = L_i(u)$ pour $i < k-1$ et $\widetilde{L}_k(u') = a_k R(u')$. Dans cette situation, l'indice de radialité $\widetilde{k}$ de $\widetilde{X}$ satisfait $\widetilde{k} > k$ et on se ramène au cas précédent par une induction élémentaire sur $q - k$. Si $q - k > 0$ et $c_{k+1} \neq 0$, effectuons tout d'abord une 2-ramification $T$ au-dessus du plan $z = 0$. C'est une transformation admissible $T : (\underline{M}, \underline{p}) \mapsto (M, p)$. La droite $\{\underline{y} = \underline{z} = 0\}$ est invariante par $\underline{X}$. Soit $\sigma$ l'éclatement de centre cette droite. La composition $\pi = \sigma \circ T : (M', \gamma_0', p') \to (M, \gamma_0, p)$ est un morphisme admissible. Soient $w' = (u', z') = (x', y', z')$ des coordonnées centrées en $p' = \omega(\gamma_0')$ telles que $\pi(x', y', z') = (x', y'z', z'^2)$. Dans ces coordonnées le relevé $X'$ de $X$ s'écrit

$$X' = \sum_{i=0}^{q} z'^i L_i'(u') - z'^{q'+1}\Big(1/2 \, \frac{\partial}{\partial z} + Y'(u', z')\Big)$$

avec $q' = 2q - 1$, $Y'$ possédant les propriétés habituelles. Un petit calcul montre que, pour $i < 2k - 1$, $L_i' \equiv 0$ si $i$ est impair et $L_i' = L_j$ si $i = 2j$, que $L_{2k}' = a_k I_2$ et $L_{2k+1}' = y \, \partial/\partial x + c_{k+1} x \, \partial/\partial y$. L'indice de radialité de $X'$ est égal à $k' = 2k + 1$. Le discriminant $\Delta_k'$, correspondant est $4c_{k+1} \neq 0$. $\square$

**5.3. Démonstration de la proposition 5.2.** — Soient $\gamma_0, \gamma_1$ et $f = (f_1, f_2) \to \mathbb{R}^2$ comme dans l'enoncé de la proposition 5.2. Les courbes $\gamma_0, \gamma_1$ étant non oscillantes et enlacées, on peut supposer que $f_1, f_2$ sont analytiques sur un voisinage $U$ de $p$. Cette affirmation est une conséquence d'un résultat de [**27**]. Elle peut être aussi montrée en utilisant le théorème de préparation de [**20**].

D'après le théorème II, les courbes $\gamma_0, \gamma_1$ ont un contact plat avec une courbe formelle $\widehat{\Gamma}$ transcendante. Quitte à effectuer une suite d'éclatements de points ou de courbes lisses on peut choisir des coordonnées $w = (x, y, z)$ centrées en $p$ telles que (voir [**2, 13**]) :

(1) Les courbes $\gamma_0, \gamma_1$ sont $z$-positives.
(2) La courbe $\widehat{\Gamma}$ est lisse transverse à $\{z = 0\}$.
(3) Pour $i = 1, 2$, la fonction $f_i$ s'écrit dans $w$ sous la forme

$$f_i(x, y, z) = x^{p_i} y^{q_i} z^{r_i} U_i(x, y, z), \quad \text{avec } U_i(0) \neq 0.$$

Puisque les séries formelles $f_i \circ \widehat{\Gamma}$ ne sont pas nulles, on peut supposer que $p_i = q_i = 0$ pour $i = 1, 2$. En remplaçant la coordonnée $z$ par $z(U_1)^{1/r_1}$, on peut supposer que $f_1(x, y, z) = z^{r_1}$. Les applications $f$ et $(x, y, z) \mapsto (z, f_2(x, y, z))$ ayant les mêmes fibres dans $\{z > 0\}$, on peut supposer que

$$f_1(x, y, z) = z \quad \text{et} \quad f_2(x, y, z) = z^{r_2} U_2(x, y, z).$$

Si le rang générique de $f$ est égal à 1, les images de $|\gamma_0|$, $|\gamma_1|$ sont confondues. La proposition est alors vraie. Supposons que $f$ est de rang générique 2. La courbe $\widehat{\Gamma}'$ n'est pas contenue dans l'ensemble analytique $\{\partial f_2/\partial x = \partial f_2/\partial y = 0\}$. Quitte à effectuer des éclatements de points on peut supposer que $\frac{\partial f_2}{\partial y} \neq 0$ sur $\{z > 0\}$. Soient $\gamma_j(z) = (x_j(x), y_j(z), z)$, $j = 0, 1$ les paramétrisations de $|\gamma_0|$, $|\gamma_1|$ par $z > 0$. Pour chaque $z_0 > 0$, la courbe

$$C_{z_0} = \{(x, y, z_0) \mid f_2(x, y, z) = f_2(\gamma_0(z_0))\}$$

partage le plan $\{z = z_0\}$ en deux composantes connexes $U^+_{z_0}$, $U^-_{z_0}$ telles que

$$\{(x, y, z_0) \mid x = x_0(z_0), y > y_0(z_0)\} \subset U^+_{z_0},$$
$$\{(x, y, z_0) \mid x = x_0(z_0), y < y_0(z_0)\} \subset U^-_{z_0}.$$

Puisque $\gamma_0, \gamma_1$ sont enlacées, il existe deux suites $(z_n^+)$, $(z_n^-)$ qui tendent vers 0 telles que $y_0(z_n^+) > y_1(z_n^+)$ et $x_0(z_n^+) = x_1(z_n^+)$, $y_0(z_n^-) < y_1(z_n^-)$ et $x_0(z_n^-) = x_1(z_n^-)$. Par continuité, il existe une suite $(z_n)$ qui tend vers 0 telle que $\gamma_1(z_n)$ appartient à $C_{z_n}$. Ainsi, les points $\gamma_0(z_n)$ et $\gamma_1(z_n)$ sont dans une même fibre de $f$. □

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# ON ANALYTIC FAMILIES OF INVARIANT TORI FOR PDES

*by*

Boris Dubrovin

*Dedicated to J.-P. Ramis on the occasion of his 60th birthday*

***Abstract.*** **—** We propose to apply a version of the classical Stokes expansion method to the perturbative construction of invariant tori for PDEs corresponding to solutions quasiperiodic in space and time variables. We argue that, for integrable PDEs all but finite number of the small divisors arising in the perturbative analysis cancel. As an illustrative example we establish such cancellations for the case of KP equation. It is proved that, under mild assumptions about decay of the magnitude of the Fourier modes all analytic families of finite-dimensional invariant tori for KP are given by the Krichever construction in terms of theta-functions of Riemann surfaces. We also present an explicit construction of infinite dimensional real theta-functions and of the corresponding quasiperiodic solutions to KP as sums of an infinite number of interacting plane waves.

***Résumé*** **(Tores invariants pour certaines EDP). —** Nous proposons d'appliquer la méthode des développements de Stokes à la construction perturbative de tores invariants associés à des solutions d'EDP quasi-périodiques en les variables d'espace et de temps. Pour les EDP intégrables, nous nous intéressons à la compensation de presque tous les petits diviseurs apparaissant dans l'analyse pertubative, *i.e.*, la compensation de tous sauf un nombre fini. Nous traitons de cette compensation en détail sur l'exemple de l'équation KP et nous montrons que dans ce cas, sous des hypothèses faibles portant sur la décroissance de l'amplitude des modes de Fourier, toutes les familles analytiques à tores invariants de dimension finie sont données par la construction de Krichever en termes de fonctions théta de surfaces de Riemann. Nous donnons une construction explicite de fonctions théta réelles de dimension infinie et des solutions de KP quasi-périodiques correspondantes comme somme d'une infinité d'ondes planes en interaction.

## 1. Introduction

Quasiperiodic solutions of the equations of motion

$$\dot{u} = f(u)$$

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in the form

$$u(t) = U(\phi_1, \dots, \phi_n), \quad \phi_j = \omega_j t + \phi_j^0,\ j = 1, \dots, n$$

for a $2\pi$-periodic in each $\phi_1, \dots, \phi_n$ function $U$ has been studied in the classical mechanics since 19th century. The associated geometric image of linear motion on an $n$-dimensional torus became widely accepted after creation of KAM theory and of the Arnold-Liouville theory of completely integrable Hamiltonian systems [**2**], although it was already familiar in the physics literature after the A. Einstein's treatment of the Bohr-Sommerfeld quantization rules for integrable systems with many degrees of freedom [**14**]. In particular, the Arnold-Liouville theory applied to a completely integrable Hamiltonian system on a $2n$-dimensional symplectic manifold $u \in M^{2n}$ establishes existence of families of $n$-dimensional invariant tori depending on $n$ parameters $\boldsymbol{I} = (I_1, \dots, I_n)$

$$u(t\,|\,\boldsymbol{I}) = U(\phi_1, \dots, \phi_n\,|\,\boldsymbol{I}), \quad \phi_j = \omega_j(\boldsymbol{I})t + \phi_j^0,\ j = 1, \dots, n. \tag{1.1}$$

Changing the values of the action variables $I_1, \dots, I_n$ one represents a $2n$-dimensional domain in the symplectic manifold as a torus fibration. Under the nondegeneracy assumption [**2**] the frequencies $\omega_1(\boldsymbol{I}), \dots, \omega_n(\boldsymbol{I})$ run through all possible directions in an open set. In particular, for generic values of the parameters $\boldsymbol{I}$ the solution (1.1) is a quasiperiodic function in time.

Systems of evolutionary PDEs

$$u_t^a = f^a(u, u_{\boldsymbol{x}}, u_{\boldsymbol{xx}}, \dots), \quad \boldsymbol{x} = (x_1, x_2, \dots, x_d), \quad a = 1, \dots, r \tag{1.2}$$

can be considered as an infinite-dimensional analogue of dynamical systems define on a suitable space of functions of $d$ spatial variables $x_1, \dots, x_d$. Although in certain cases it is possible to develop an infinite-dimensional analogue of the Arnold-Liouville theory of completely integrable Hamiltonian systems and to construct families of infinite-dimensional invariant tori for certain nontrivial examples of nonlinear evolutionary PDEs and, moreover, to develop an infinite-dimensional analogue of KAM theory (see [**28, 7, 24, 35**]) and the related theory of Birkhoff normal forms (see [**15, 21, 22**]), in the most physical applications families of low dimensional invariant tori for PDEs play a prominent role.

For linear PDEs families of one-dimensional invariant tori can be readily found in the form of plane waves

$$u(\boldsymbol{x}, t) = A\cos(k_1 x_1 + \cdots + k_d x_d - \omega t + \phi_0). \tag{1.3}$$

The *wave numbers* $k_1, \dots, k_d$ take arbitrary values within some domain of the $d$-dimensional space, the *frequency*

$$\omega = \omega(k_1, \dots, k_d) \tag{1.4}$$

is determined from the so-called *dispersion relation* substituting the ansatz (1.3) into the equation (1.2). It will be assumed that all branches of the dispersion relation (1.3) are real-valued functions. For any such branch $A$ is a $r$-component vector determined, in the generic situation, up to a scalar factor called the *amplitude*. The phase shift $\phi_0$ can also take an arbitrary value. The solution (1.3) in general is quasiperiodic both in space and time variables. Multidimensional invariant tori for linear PDEs are obtained as linear superpositions of plane waves

$$u(\boldsymbol{x}, t) = \sum_{i=1}^{n} A_i \cos(k_1^i x_1 + \cdots + k_d^i x_d - \omega^i t + \phi_0^i)$$

with arbitrary amplitudes, phases and wave numbers, the frequencies determined as above

$$\omega^i = \omega(k_1^i, \ldots, k_d^i), \quad i = 1, \ldots, n.$$

Note that, in the discussion of invariant tori for PDEs, we will not specify the class of functions[(1)] to be considered.

In many cases families of one-dimensional invariant tori can also be obtained for various nonlinear PDEs as travelling wave solutions

$$u(\boldsymbol{x}, t) = U(\phi \,|\, \boldsymbol{A}), \quad \phi = k_1 x_1 + \cdots + k_d x_d - \omega t + \phi_0. \tag{1.5}$$

Here $U(\phi \,|\, \boldsymbol{A})$ is a $2\pi$-periodic function in $\phi$ depending on some number of parameters $\boldsymbol{A} = (A_1, A_2, \ldots)$ that determine the shape of the wave. The wave numbers and phases take arbitrary values. The shape of the wave does not depend on the phase shifts but it may depend on the wave numbers. It is convenient to subdivide the parameters $\boldsymbol{A}$ in two parts

$$\boldsymbol{A} = (k_1, \ldots, k_d; a) \tag{1.6}$$

where the parameter $a$ is a nonlinear analogue of the amplitude. The frequency is to be determined from a nonlinear analogue of the dispersion relation. The latter involves also the amplitude parameters $a$,

$$\omega = \omega(k_1, \ldots, k_d; a). \tag{1.7}$$

For fixed $t$ the solution (1.5) takes constant values along the hyperplanes

$$k_1 x_1 + \cdots + k_d x_d = \text{const}.$$

The points on the hyperplanes move in the orthogonal directions with the constant phase velocity

$$v = \frac{\omega}{|k|}, \quad |k| = \sqrt{k_1^2 + \cdots + k_d^2}.$$

***Example 1.1***. — The periodic travelling wave for the Kadomtsev-Petviashvili (KP) equation

$$u_{xt} + \frac{1}{4}(3u^2 + u_{xx})_{xx} + \frac{3}{4} u_{yy} = 0 \tag{1.8}$$

(1) The dynamic on a suitable class of almost periodic functions would probably be the appropriate framework for considering the families of finite-dimensional invariant tori with arbitrary wave numbers.

(here $d = 2$, $x = x_1$, $y = x_2$) can easily be obtained in terms of elliptic functions

$$
\begin{aligned}
u(x, y, t) &= U(\phi), \quad \phi = kx + ly - \omega t + \phi_0 \\
U(\phi) &= \frac{2k^2}{\pi^2} K^2 \Big( \kappa^2 \operatorname{cn}^2 \Big[ \frac{K}{\pi} \phi; \kappa \Big] - \gamma \Big) + \frac{c}{6} \\
\omega &= -\frac{c}{4} k + \frac{3}{4} \frac{l^2}{k} - k^3 \frac{K^2}{\pi^2} \Big( 3 \frac{E}{K} + \kappa^2 - 2 \Big) \\
\gamma &= \frac{E}{K} - 1 + \kappa^2.
\end{aligned} \tag{1.9}
$$

Here $\operatorname{cn}[z; \kappa]$ is the Jacobi elliptic function with the modulus $0 \leqslant \kappa \leqslant 1$, $K = K(\kappa)$, $E = E(\kappa)$ are complete elliptic integrals of the first and second kind resp., $c$ is an arbitrary constant.

The functions (1.9) are periodic travelling waves propagating with constant speed in the $(x, t)$-plane. For $l = 0$ the above formulae reduce to the so-called cnoidal waves for the Korteweg-de Vries (KdV) equation

$$
u_t + \frac{1}{4}(3u^2 + u_{xx})_x = 0. \tag{1.10}
$$

The KdV equation is known to arise in a fairly general setting of one-dimensional weakly nonlinear waves with small dispersion (see, *e.g.*, [**33**]). In particular it describes one-dimensional shallow water waves of small amplitude. The $y$-dependence of solutions to the KP equation (1.8) describes(2) slow transversal perturbations of the KdV waves [**23**], [**33**].

The elliptic modulus $\kappa$ plays the role of the amplitude parameter. At the limiting value $\kappa = 0$ one obtains trivial solution $u = 0$; the frequency takes the value $\omega = -\frac{1}{4}(c\, k + k^3)$. For small positive values of the parameter

$$
\varepsilon^2 = k \left[ \omega + \frac{1}{4} \left( c\, k + k^3 - 3 \frac{l^2}{k} \right) \right] > 0
$$

one obtains approximately the plane wave solution

$$
u \simeq \frac{c}{6} + A \cos(kx + ly - \omega t + \phi_0), \quad \omega \simeq \frac{1}{4} \left( 3 \frac{l^2}{k} - c\, k - k^3 \right)
$$

with the small amplitude

$$
A \simeq 2 \sqrt{\frac{2}{3}} \varepsilon.
$$

More accurate idea about the shape of the solution (1.9) for small amplitudes can be obtained by using *Stokes expansion* method [**37**]; see also Chapter 13 of the Whitham's book [**39**]. We will represent this classical method of the theory of water waves in a

(2) The equation (1.8) is often called KPII to distinguish it from the KPI case. The latter equation differs from (1.8) by the sign in front of the second derivative in $y$. It also has physical applications but not within the theory of water waves [**23**].

slightly modified version. Let us look for the solution to the KP equation in the form of Fourier series

$$u(x,y,t) = \frac{c}{6} + A_1 \cos\phi + A_2 \cos 2\phi + A_3 \cos 3\phi + \dots, \quad \phi = kx + ly - \omega t + \phi_0 \tag{1.11}$$

depending on a small parameter $\varepsilon$ assuming that

$$A_k = O(\varepsilon^k), \quad k = 1, 2, \dots. \tag{1.12}$$

Also the dispersion law must be expanded in a series with respect to the small parameter

$$\omega = \frac{1}{4}\left(3\frac{l^2}{k} - c\,k - k^3\right) + \omega_1 + \omega_2 + \dots, \quad \omega_k = O(\varepsilon^k). \tag{1.13}$$

The KP equation must hold for an arbitrary $\phi_0$ as an identity for formal series in $\varepsilon$. Without loss of generality one can use the small amplitude $A = A_1$ of the plane wave as the expansion parameter. Substituting the ansatz (1.11)–(1.13) into (1.8) yields, after simple calculation

$$\begin{aligned} u(x,y,t) = \frac{c}{6} + A\cos\phi + \frac{A^2}{2\,k^2}\left(1 - \frac{A^2}{8\,k^4} + O(A^4)\right)\cos 2\phi \\ + \left(\frac{3\,A^3}{16\,k^4} + O(A^5)\right)\cos 3\phi + \left(\frac{A^4}{16\,k^6} + O(A^6)\right)\cos 4\phi + \dots \end{aligned} \tag{1.14}$$

$$\omega = \frac{1}{4}\left(3\frac{l^2}{k} - c\,k - k^3\right) + \frac{3\,A^2}{8\,k} + \frac{3\,A^4}{128\,k^5} + O(A^6). \tag{1.15}$$

For small amplitudes (1.14)–(1.15) gives a reasonably good uniform approximation to the cnoidal wave (1.9).

Multidimensional invariant tori for PDEs is still a not completely understood phenomenon, although there are quite a few nontrivial examples of PDEs where a families of finite-dimensional invariant tori have been constructed, mainly by applying the methods of algebraic geometry (see, *e.g.*, [**12, 26, 10**]). One can think of them as of the result of nonlinear interaction of travelling waves solutions, although this operation in general has to be defined. We suggest the following approach to the definition of the nonlinear interaction.

Let the PDE (or a system of PDEs) possess a family of travelling wave solutions of the form (1.5) depending on some vector parameter

$$\boldsymbol{A} = (k_1, \dots, k_d; \boldsymbol{a}).$$

It is assumed that the wave vector $k_1, \dots, k_d$ assumes arbitrary values in some domain of $\mathbb{R}^d$,

$$(k_1, \dots, k_d) \in \mathcal{K} \subset \mathbb{R}^d.$$

The amplitude parameter $\boldsymbol{a}$ belongs to a $m$-dimensional domain

$$\boldsymbol{a} \in \mathcal{D} \subset \mathbb{R}^m.$$

Denote

$$\mathcal{A} := \mathcal{K} \times \mathcal{D} \subset \mathbb{R}^{d+m}.$$

The solution (1.5) must satisfy the PDE identically in $\phi^0$. Let us assume that, on a certain submanifold of codimension 1,

$$\boldsymbol{a} \in \mathcal{C} \subset \mathcal{D}, \quad \dim \mathcal{C} = m-1$$

the solution (1.5) becomes constant. We will only consider the local situation where $\mathcal{D}$ belongs to a small neighborhood of the manifold of constant solutions. Denote $\varepsilon$ the distance of a point on $\mathcal{D}$ from $\mathcal{C}$. So, the amplitude parameter is subdivided into

$$\boldsymbol{a} = (\varepsilon, \boldsymbol{c}), \quad \boldsymbol{c} \in \mathcal{C}.$$

For small $\varepsilon$ the solution (1.5) must become close to the plane wave

$$\begin{aligned} &u \simeq u_0(\boldsymbol{c}) + A(\varepsilon, \boldsymbol{c}) \cos\phi, \\ &\phi = k_1 x_1 + \cdots + k_d x_d - \omega t + \phi_0, \quad \omega \simeq \omega_0(k_1, \ldots, k_d, \boldsymbol{c})) \end{aligned} \tag{1.16}$$

where $\omega_0(k_1, \ldots, k_d, \boldsymbol{c})$ is the dispersion law of the linearized PDE near the manifold of constant solutions $\boldsymbol{c} \in \mathcal{C}$, $A(0, \boldsymbol{c}) = 0$.

***Definition 1.2***. — We say that the family of $n$-dimensional invariant tori of the form

$$\begin{aligned} &u = U(\phi_1, \ldots, \phi_n | \boldsymbol{A}^{(1)}, \ldots, \boldsymbol{A}^{(n)}), \\ &\phi_i = k_1^i x_1 + k_2^i x_2 + \cdots + k_d^i x_d - \omega^i t + \phi_i^0, \quad i = 1, \ldots, n \end{aligned} \tag{1.17}$$

is obtained as the result of (nonlinear) interaction of $n$ plane waves if the following conditions are fulfilled.

(i) The functions (1.17) are $2\pi$-periodic in $\phi_1, \ldots, \phi_n$.

(ii) As functions of $(\boldsymbol{A}^{(1)}, \ldots, \boldsymbol{A}^{(n)})$ they are analytic on a complement in $\mathcal{A} \times \cdots \times \mathcal{A}$ ($n$ factors) to a collection of finite number of algebraic subvarieties $R_1, \ldots, R_N$

$$(\boldsymbol{A}^{(1)}, \ldots, \boldsymbol{A}^{(n)}) \in \mathcal{A} \times \cdots \times \mathcal{A} \setminus \cup_{k=1}^N R_k \subset \mathbb{R}^{n\,(d+m)}. \tag{1.18}$$

(iii) Near the manifold of constant solutions the Fourier expansion of the functions (1.17) has the form

$$\begin{aligned} U(\phi_1, \ldots, \phi_n | \boldsymbol{A}^{(1)}, \ldots, \boldsymbol{A}^{(n)}) &= u_0(\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n; \varepsilon_1, \ldots, \varepsilon_n) \\ &\quad + A(\varepsilon_1, \boldsymbol{c}_1) \cos\phi_1 + \cdots + A(\varepsilon_n, \boldsymbol{c}_n) \cos\phi_n \\ &\quad + \sum_{\boldsymbol{m} \in \mathbb{Z}^n, |\boldsymbol{m}| > 1} A_{\boldsymbol{m}} e^{i\,(m_1\phi_1 + \cdots + m_n\phi_n)} \\ \phi_j &= k_1^j x_1 + \cdots + k_d^j x_d - \omega^j + \phi_j^0, \quad j = 1, \ldots, n \\ \omega^j &= \omega_0(k_1^j, \ldots, k_d^j, \boldsymbol{c}_j) + \sum_{k \geqslant 1} \omega_k^j \\ u_0(\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n; \varepsilon_1, \ldots, \varepsilon_n) &= u_0(\boldsymbol{c}_1) + \cdots + u_0(\boldsymbol{c}_n) + O(\varepsilon). \end{aligned} \tag{1.19}$$

The Fourier coefficients

$$A_{\boldsymbol{m}} = A_{\boldsymbol{m}}(k_1^1, \dots, k_d^n, \varepsilon_1, \dots, \varepsilon_n, \boldsymbol{c}_1, \dots, \boldsymbol{c}_n)$$

must be analytic functions on (1.18). Their Taylor expansions in $\varepsilon_1, \dots \varepsilon_n$ near $\mathcal{C} \times \cdots \times \mathcal{C}$ must begin with the terms of the order $|\boldsymbol{m}|$,

$$A_{\boldsymbol{m}} = O\big(\varepsilon^{|\boldsymbol{m}|}\big), \quad |\boldsymbol{m}| = |m_1| + \cdots + |m_n|. \tag{1.20}$$

Also in the expansion of the dispersion law the $k$-th term

$$\omega_k^j = \omega_k^j(k_1^1, \dots, k_d^n, \varepsilon_1, \dots, \varepsilon_n, \boldsymbol{c}_1, \dots, \boldsymbol{c}_n) \tag{1.21}$$

must be of the order $k$ in $\varepsilon$. The coefficients of the leading Fourier modes must coincide with the leading coefficients of the plane wave expansions (1.16).

We believe that existence, for any $n \geqslant 1$, of the analytic families of $n$-dimensional invariant tori satisfying the assumptions of the Definition 1.2 implies integrability of the PDE. It would be interesting to prove precise mathematical theorems in this direction.

In this paper we pursue a more modest goal. For the example of KP equation we want to prove that, indeed, the analytic families of invariant tori satisfying the conditions of the Definition 1.2 exist for any $n$. Actually, we will prove that the families of invariant tori obtained by the I.M. Krichever's construction [**25**]) satisfy the assumptions of the Definition. Moreover, we will prove that *all* such analytic families of invariant tori must be given by the Krichever's construction. Our main motivation was the mathematical understanding of the remarkable physical experiments of J. Hammack *et al.* [**20, 19**]. In these experiments the propagation of small amplitude shallow water waves was studied. In a water tank of the size approximately $13 \times 27$ m and depth 20 cm the waves were generated by a wavemaker programmed to create a superposition of two cnoidal waves with different directions of propagation and different amplitudes. The measurements of the resulting wave profile proved to be in a remarkable agreement with the two-dimensional invariant tori for KP given in terms of theta-functions (see below). Also some oceanic observation were analyzed in [**19**]; again the agreement with the theta-functional invariant tori looked encouraging.

To our opinion the experimental results suggest the following main question to be addressed: why the multidimensional invariant tori for KP created by Krichever [**25**] with sophisticated algebro-geometric technique are observable in the physical experiments? Putting this in a different way, the mathematical questions to be answered are

– does the Krichever's construction cover all finite-dimensional invariant tori for KP?

– are these tori stable?

One of the difficulties in proving exact statements in this direction is quasiperiodicity of the solutions with respect to the spatial directions. The extension of the finite-dimensional Arnold-Liouville and KAM theory to the infinite-dimensional situation developed in [**28, 7, 24, 35**] mainly refers to the space of spatially periodic functions.

In Section 2 we prove a simple uniqueness statement (see Theorem 2.2 below): all finite-dimensional invariant tori for KP obtained as a result of nonlinear interaction of plane waves in the sense explained above are expressed in terms of theta-functions of Riemann surfaces via the Krichever construction.

In the last Section we extend the technique developed in the proof of Theorem 2.2 to the explicit construction of the moduli space of the KP theta-functions of infinite genus. They are obtained as infinite superpositions of plane waves satisfying certain requirements to ensure convergence of the infinite sums. The KP solutions given in terms of these theta-functions will be quasiperiodic in both space and time variables. For the case of hyperelliptic Riemann surfaces the theory of theta-functions of infinite genus and associated KdV periodic and quasiperiodic solutions was initiated by H. McKean and E. Trubowitz [**31**]. For the KP case, where arbitrary Riemann surfaces can appear in the finite genus case, the infinite genus theory for the *doubly periodic* in $(x, y)$ KP solutions was created by I. Krichever [**27**] (see also [**6**]). The state-of-the-art of the theory of the associated infinite genus theta-functions can be found in the monograph of J. Feldman, H. Knörrer and E. Trubowitz [**17**]. Observe that our approach does not require any assumption about spatial periodicity.

## 2. Can one see the shape of a Riemann surface looking at the water waves?

The question in the title of this Section clearly alludes the famous problem, due to M. Kac, regarding the possibility of hearing the shape of a drum. In the situation of the theory of water waves, however, one does not know *a priori* whether a "drum", *i.e.*, a Riemann surface determining the shape of the wave profile, is hidden behind the sufficiently rich class of the water waves. In this Section we suggest an analytic approach to this problem based on a uniqueness theorem for analytic families of invariant tori for the KP equation given by the Krichever construction.

Let us begin with some preliminaries of the theory of KP equation. Although (1.8) is strictly speaking not an evolutionary PDE, our definition of nonlinear interaction of plane waves makes sense also for the KP case. Observe that the mean value

$$\int u(x, y, t)\, dx$$

is a first integral. We will always consider the solutions with zero mean value. This is not a serious constraint. Indeed, the KP equation is invariant with respect to the action of the group of scaling/Galilean transformations

$$\begin{aligned} x &= cx' + ac^2y' - \frac{1}{2}bc^3t' \\ y &= c^2y' - \frac{3}{2}ac^3t' \\ t &= c^3t' \\ u &= c^{-2}\Big[u' + \frac{1}{2}a^2 - \frac{1}{3}b\Big] \end{aligned} \tag{2.1}$$

depending on three arbitrary parameters $c \neq 0$, $a$, $b$. Using these transformations one can always kill the mean value.

Technically it is more convenient to work with the so-called bilinear form of KP. The substitution

$$u = 2\partial_x^2 \log \tau(x, y, t) \tag{2.2}$$

reduces (1.8) to

$$3\tau_{xx}^2 - 4\tau_x\tau_{xxx} + \tau\,\tau_{xxxx} + 3(\tau_{yy}\tau - \tau_y^2) + 4(\tau_{xt}\tau - \tau_x\tau_t) + 2b\tau^2 = 0. \tag{2.3}$$

Here $b$ is an integration constant. Actually what will be studied is the invariant tori for (2.3) of the form

$$\begin{aligned} \tau(x, y, t) &= A_0 + \sum_{\boldsymbol{m}\neq 0} A_{\boldsymbol{m}} e^{i\,(m_1\phi_1 + \cdots + m_n\phi_n)}, \\ \phi_j &= k_jx + l_jy - \omega_jt + \phi_j^0, \quad j = 1, \dots, n \end{aligned} \tag{2.4}$$

Without loss of generality one can assume

$$A_0 = 1.$$

Moreover, doing if necessary suitable shifts along $\phi_1^0, \dots, \phi_n^0$ one can normalize the leading coefficients in such a way that

$$A_{(-1,0,\dots,0)} = A_{(1,0,\dots,0)}, \;\dots,\; A_{(0,\dots,0,-1)} = A_{(0,\dots,0,1)}. \tag{2.5}$$

Let us first recall the construction of the algebro-geometric invariant tori for KP. They are parametrized by quadruples $(\Sigma_n, \infty, \zeta, \sigma)$ where $\Sigma_n$ is a Riemann surface of genus $n$ with a marked point $\infty \in \Sigma_n$, $\zeta$ is a 3-jet of a local parameter on $\Sigma_n$ near $\infty$, $\zeta(\infty) = 0$. Finally, $\sigma$ must be an anticomplex involution

$$\sigma : \Sigma_n \longrightarrow \Sigma_n, \quad \sigma(\infty) = \infty,\; \sigma^*\zeta = \overline{\zeta} \tag{2.6}$$

such that the fixed-point set of the involution $\sigma$ consists of $n+1$ components. Call $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ the (homology classes of) the suitably oriented components not containing the point $\infty$. These will be the basic a-cycles on the Riemann surface $\Sigma_n$. The conjugated b-cycles can be choosen arbitrarily provided that

$$\sigma_*(\boldsymbol{a}_j) = \boldsymbol{a}_j,\; \sigma_*(\boldsymbol{b}_j) = -\boldsymbol{b}_j, \quad j = 1, \dots, n. \tag{2.7}$$

The Fourier coefficients of the algebro-geometric solutions have the form

$$A_{\boldsymbol{m}} = e^{-\pi\langle \boldsymbol{m},\beta\, \boldsymbol{m}\rangle} \tag{2.8}$$

where $\beta = (\beta_{ij})$ is the real symmetric positive definite $n \times n$ matrix given by the periods of holomorphic differentials

$$\beta_{ij} = -i \oint_{\boldsymbol{b}_j} w_i, \quad \oint_{\boldsymbol{a}_j} w_i = \delta_{ij}. \tag{2.9}$$

The wave numbers and frequencies are given in terms of the coefficients of expansions of the basic holomorphic differentials near $\infty \in \Sigma_n$,

$$w_i(P) = \frac{1}{2\,\pi}\left(k_i + l_i\zeta + \omega_i\zeta^2 + O(\zeta^3)\right)d\zeta, \quad P \longrightarrow \infty. \tag{2.10}$$

The phase shifts $\phi_j^0$ can be arbitrary real numbers.

The formulae (2.8)–(2.7) is nothing but the Krichever's construction [**25**] of the algebro-geometric solutions to KP (see also [**10, 13**] regarding the reality conditions). We will refer to the class of quadruples $(\Sigma_n, \infty, \zeta, \sigma)$ described above as the *KP Riemann surfaces*, and their theta-functions as to the *KP theta-functions*. Recall that, besides the reality conditions no other constraints are to be imposed on the triple $(\Sigma, \infty, \zeta)$. More precisely, the following statement holds true [**13**].

***Theorem 2.1***. — *For any KP Riemann surface $(\Sigma_n, \infty, \zeta, \sigma)$ the formulae (2.2), (2.4), (2.8)–(2.10) with arbitrary real phase shifts $\phi_1^0, \dots, \phi_n^0$ define a real smooth solution to the KP equation (1.8). Conversely, if the real smooth KP solution $u(x, y, t)$ of the form (2.2), (2.4), (2.8)–(2.10) for some triple $(\Sigma_n, \infty, \zeta)$ remains smooth under evolution along all flows of the KP hierarchy then the Riemann surface $\Sigma_n$ must admit an involution $\sigma$ with the above properties. Moreover, all the phase shifts must be real.*

Let us call the wave numbers $k_1, \dots, k_n$, $l_1, \dots, l_n$ *resonant* if, for some $i \neq j$

$$k_i = \pm k_j \quad \text{and} \quad l_i k_j = l_j k_i. \tag{2.11}$$

If this is not the case and $k_1 \neq 0, \dots, k_n \neq 0$ the wave numbers will be called *non-resonant*. From the definition of plane waves it follows that one can assume all wave numbers $k_j$ to be positive.

***Theorem 2.2***. — *Let (2.4) be a family of solutions to (2.3), for arbitrary phase shifts $\phi_1^0, \dots, \phi_n^0$, depending analytically on the small parameter $\varepsilon$ and on the "amplitudes"*

$$a_1 = A_{(1,0,\dots,0)} > 0,\ a_2 = A_{(0,1,0,\dots,0)} > 0, \dots, a_n = A_{(0,0,\dots,1)} > 0 \tag{2.12}$$

*and on arbitrary nonresonant wave numbers $k_1 \neq 0, \dots, k_n \neq 0$, $l_1, \dots, l_n$ such that*

$$A_{\boldsymbol{m}} = O\big(\varepsilon^{|m_1|+\dots+|m_n|}\big). \tag{2.13}$$

*Then this family is given by (2.8)–(2.7) for some KP Riemann surface $(\Sigma_n, \infty, \zeta, \sigma)$ of the above form.*

*Proof.* — Let us begin with algebraic preliminaries. Denote

$$\mathcal{R} = \mathbb{C}\left[z_1^{\pm 1}, \dots, z_n^{\pm 1}\right]$$

the ring of Laurent polynomials of $n$ variables. The *degree* of a monomial in $\mathcal{R}$ is defined by

$$\deg z_1^{i_1} \dots z_n^{i_n} = |i_1| + \dots + |i_n|.$$

Denote $\mathcal{R}_m$ the subspace of Laurent polynomials of the degree $m$. The product of Laurent polynomials satisfies

$$\mathcal{R}_i \mathcal{R}_j \subset \oplus_{k=0}^{i+j} \mathcal{R}_k. \tag{2.14}$$

The ring of trigonometric polynomials in $\phi_1, \dots, \phi_n$ is naturally identified with $\mathcal{R}$ by putting

$$z_j = e^{i\,\phi_j}, \quad j = 1, \dots, n.$$

So, the above definition and properties of the degree holds true also for trigonometric polynomials.

We can now reformulate the assumptions of the Theorem in the following way. We are looking for a solution to the equation (2.3) in the form

$$\tau = 1 + \varepsilon \sum_{j=1}^{n} a_j (z_j + z_j^{-1}) + \sum_{m \geqslant 2} \varepsilon^m \tau^{[m]} \tag{2.15}$$

where

$$\tau^{[m]} = \sum_{k=2}^{m} \tau_k^{[m]}, \quad \tau_k^{[m]} \in \mathcal{R}_k. \tag{2.16}$$

In these formulae we use the superscript $[m]$ for labelling the terms of the order $m$ with respect to $\varepsilon$. The coefficients of these trigonometric polynomials along with the coefficients of the expansions

$$\omega_j = \frac{1}{4}\left(3k_j\lambda_j^2 - k_j^3\right) + \sum_{m \geqslant 1} \varepsilon^m \omega_j^{[m]}, \quad j = 1, \dots, n \tag{2.17}$$

$$b = \sum_{m \geqslant 1} \varepsilon^m b^{[m]} \tag{2.18}$$

are to be determined from the KP equation (2.3). Here we introduce the notation

$$\lambda_j := \frac{l_j}{k_j}, \quad j = 1, \dots, n.$$

Let us now describe more precisely the result of substitution of the ansatz (2.15) to the KP equation (2.3). We need to introduce the following notations. Put

$$\partial_x = \sum_{j=1}^{n} k_j \frac{\partial}{\partial \phi_j}, \quad \partial_y = \sum_{j=1}^{n} k_j \lambda_j \frac{\partial}{\partial \phi_j}, \quad \partial_t = \sum_{j=1}^{n} \frac{1}{4}(k_j^3 - 3k_j\lambda_j^2) \frac{\partial}{\partial \phi_j}.$$

We also introduce operators

$$\partial_t^{[m]} = \sum_{j=1}^n \omega_j^{[m]} \frac{\partial}{\partial \phi_j}, \quad m \geqslant 1.$$

Finally, the fourth order linear differential operator $L$ will be defined by

$$L = \partial_x^4 + 3\partial_y^2 + 4\partial_x \partial_t. \tag{2.19}$$

Using these notations one can rewrite the result of substitution of the ansatz (2.15) to the KP equation at the order $m \geqslant 2$ approximation as the following equation:

$$\begin{aligned} L\,\tau^{[m]} + b^{[m]} + {\sum_{k+l=m}}' \Big[4\partial_t^{[k]}\partial_x \tau^{[l]} + 2b^{[k]}\tau^{[l]}\Big] \\ + {\sum_{i+j=m}}' \Big[3\partial_x^2\tau^{[i]}\partial_x^2\tau^{[j]} - 4\partial_x\tau^{[i]}\partial_x^3\tau^{[j]} + \tau^{[i]}\partial_x^4\tau^{[j]} + 4\tau^{[i]}\partial_x\partial_t\tau^{[j]} - 4\partial_x\tau^{[i]}\partial_t\tau^{[j]}\Big] \\ + {\sum_{i+j+k=m}}' \Big[4\partial_t^{[k]}\partial_x\tau^{[i]}\tau^{[j]} - 4\partial_x\tau^{[i]}\partial_t^{[k]}\tau^{[j]} + b^{[k]}\tau^{[i]}\tau^{[j]}\Big] = 0. \end{aligned} \tag{2.20}$$

In this formula it is understood that, in the sums $\sum'$ all the summation indices are distinct from zero.

The left hand side of this equation is a trigonometric polynomial in $\phi_1^0, \ldots, \phi_n^0$. Because of the property (2.14), the degree of this differential polynomial is less or equal to $m$. Since $\phi_1^0, \ldots, \phi_n^0$ are arbitrary variables, we can determine the unknown coefficients just equating the coefficients of the trigonometric polynomials. More specifically, in order to determine the coefficient $a_{i_1\ldots i_n}^{[m]}$ of the trigonometric polynomial

$$\tau^{[m]} = \sum_{2 \leqslant |i_1| + \cdots + |i_n| \leqslant m} a_{i_1\ldots i_n}^{[m]} e^{i(i_1\phi_1^0 + \cdots + i_n\phi_n^0)}$$

one is to collect the coefficients of $z^{i_1}\ldots z^{i_n}$ in (2.20). Clearly the resulting expression will depend linearly on $a_{i_1\ldots i_n}^{[m]}$. It will also depend on the lower order coefficients $a_{j_1\ldots j_n}^{[m']}$, $b^{[m']}$ with $m' < m$, and on $\omega_j^{[m']}$ with $m' < m-1$. Here we use the assumption $|i_1| + \cdots + |i_n| \geqslant 2$. Similarly, in order to compute the coefficient $\omega_j^{[m-1]}$ of the expansion (2.17) one is to collect the terms containing the monomial $z_j$. Again, it is easy to see that all the coefficients of this monomial depend at most linearly on $\omega_j^{[m-1]}$ and also on $a_{j_1\ldots j_n}^{[m']}$, $b^{[m']}$ with $m' < m$, and on $\omega_j^{[m']}$ with $m' < m-1$. Finally, to determine $b^{[m]}$ it suffices to collect the constant term of the trigonometric polynomial (2.20).

We obtain a recursive procedure for computing the coefficients of the expansions (2.15)–(2.18). This procedure is an analogue of the classical Stokes expansion method explained in the introduction; it also resembles the Lindstedt series method of the classical mechanics (see Chapter XIII of the Poincaré book [**34**]). Let us prove that this procedure works to produce a unique solution for any $m$.

It is easy that the equations for $b^{[m]}$ and $\omega_j^{[m-1]}$ have unique solutions. Indeed, from the first line of (2.20) it follows that the coefficients of these unknowns are equal to 1. Let us prove that the coefficient of $a^{[m]}_{i_1\dots i_n}$ is not identically equal to zero.

Let us introduce the polynomial in $2n$ variables $k_1, \dots, k_n$, $\lambda_1, \dots, \lambda_n$ depending on $n$ integer indices $i_1, \dots, i_n$,

$$D(i_1, \dots, i_n) := \left(\sum_{s=1}^{n} k_s i_s\right)^4 - 3\left(\sum_{s=1}^{n} k_s \lambda_s i_s\right)^2 - \sum_{s=1}^{n} k_s i_s \sum_{s=1}^{n} (k_s^3 - 3k_s\lambda_s^2) i_s. \tag{2.21}$$

Clearly, the following identity holds true

$$L\, e^{i\,(m_1\phi_1+\cdots+m_n\phi_n)} = D(m_1, \dots, m_n) e^{i\,(m_1\phi_1+\cdots+m_n\phi_n)} \tag{2.22}$$

if

$$\phi_j = k_j x + k_j \lambda_j y + \frac{1}{4}(k_j^3 - 3k_j\lambda_j^2)t + \phi_j^0, \quad j = 1, \dots, n.$$

For example,

$$D(\pm 1, 0, \dots, 0) = \cdots = D(0, \dots, 0, \pm 1) = 0$$
$$D(1, 1, 0, \dots, 0) = 3k_1k_2\left[(k_1+k_2)^2 + (\lambda_1 - \lambda_2)^2\right]$$
$$D(1, -1, 0, \dots, 0) = -3k_1k_2\left[(k_1-k_2)^2 + (\lambda_1 - \lambda_2)^2\right]$$

etc. Let us prove that, for arbitrary integers $i_1, \dots, i_n$ satisfying

$$|i_1| + \cdots + |i_n| \geqslant 2 \tag{2.23}$$

the polynomial $D(i_1, \dots, i_n)$ is not an identical zero. Indeed, collecting the terms of the polynomial that contain the third and fourth powers of the variables $k_1, \dots, k_n$ yields

$$D(i_1, \dots, i_n) = \sum_{s=1}^{n} i_s^2(i_s^2 - 1)\, k_s^4 + \sum_{s\neq t} i_s i_t (4i_s^2 - 1) k_s^3 k_t + \dots$$

where the periods stand for the terms of lower degree in $k_j$. If at least one of the indices $i_1, \dots i_n$ is not equal to zero or to $\pm 1$, then the sum of the fourth powers of $k_j$ does not identically vanish. If this is not the case, at least two indices, say $i_s$ and $i_t$, $s \neq t$ do not vanish, due to the assumption (2.23). In this case the coefficient of $k_s^3 k_t$ is not equal to zero.

From the above arguments it follows that, all the coefficients $a^{[m]}_{i_1\dots i_n}$, $\omega_j^{[m-1]}$, $b^{[m]}$ for $m \geqslant 2$ are uniquely determined from the equation (2.20) in the form of polynomials in $a_1, \dots, a_n$ with the coefficients being rational functions in $k_1, \dots, k_n$.

We are now to prove existence of the analytic families of invariant tori of the described form. This will imply, last but not least, the proof of *cancellation of all the divisors* $D(i_1, \dots, i_n)$ with $|i_1| + \cdots + |i_n| > 2$.

To prove existence of the families of invariant tori with needed analytic properties we will use the Krichever construction [**25**] of algebro-geometric solutions of KP. According to this construction an arbitrary Riemann surface $\Sigma_n$ of genus $n$ with an arbitrary marked point $\infty \in \Sigma_n$ and a 3-jet of a local parameter $\zeta$ near $\infty$, $\zeta(\infty) = 0$, gives rise to a family of solutions of KP of the form

$$u(x, y, t) = \partial_x^2 \log \theta + \frac{c}{6}$$
$$\theta = \sum_{\boldsymbol{m} \in \mathbb{Z}^n} e^{-\pi \langle \boldsymbol{m}, \beta \boldsymbol{m} \rangle} e^{i(m_1 \phi_1 + \cdots + m_n \phi_n)}$$
$$\phi_j = k_j x + l_j y - \omega_j t + \phi_j^0, \quad j = 1, \dots, n. \tag{2.24}$$

In this formulae $\beta = (\beta_{ij})$ is the period matrix (2.9) of holomorphic differentials on $\Sigma_n$ with respect to a basis of cycles $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n, \boldsymbol{b}_1, \dots, \boldsymbol{b}_n \in H_1(\Sigma_n, \mathbb{Z})$ normalized by the standard form of the intersection pairing matrix

$$\boldsymbol{a}_i \circ \boldsymbol{a}_j = \boldsymbol{b}_i \circ \boldsymbol{b}_j = 0, \quad \boldsymbol{a}_i \circ \boldsymbol{b}_j = \delta_{ij}, \tag{2.25}$$

the wave numbers $k_j$, $l_j$ and frequencies $\omega_j$ are given by the expansions (2.10) of the normalized holomorphic differentials $w_j$ near $\infty$, $\phi_1^0, \dots, \phi_n^0$ are arbitrary phase shifts, $c$ is a certain constant. The constant $c$ can be killed by the Galilean transformation

$$u \longmapsto u - \frac{c}{6}, \quad x \longmapsto x + \frac{c}{4} t$$

corresponding to a suitable change of the 3-jet of the local parameter $\zeta$

$$\zeta \longmapsto \zeta - \frac{c}{12} \zeta^3.$$

We will always assume $c = 0$.

The solution (2.24) in general is a complex valued meromorphic function of the variables $x$, $y$, $t$, $\phi_1^0, \dots, \phi_n^0$. If the triple $(\Sigma_n, \infty, \zeta)$ admits an antiholomorphic involution $\sigma$ satisfying (2.6) such that the fixed-point set of the involution consists of $n+1$ ovals, then the period matrix $\beta_{ij}$ and the wave numbers and frequencies are all real provided the basis of cycles is chosen in the way described in the Theorem. Moreover [**16**], the theta-function in (2.24) takes positive values for all real phase shifts $\phi_1^0, \dots, \phi_n^0$, and the solution $u(x, y, t)$ is real-valued and smooth. Therefore, in this case, the Krichever formulae (2.24) define a $n$-dimensional invariant torus for KP. It will also be invariant for all the flows of the KP hierarchy. Conversely, from reality and smoothness of the solution (2.24) on the torus generated by the flows of the KP hierarchy it follows that $(\Sigma_n, \infty, \zeta)$ must admit the antiholomorphic involution with the above properties (see the Theorem 2.1 above).

We will now produce the needed analytic family of $n$-dimensional invariant tori for KP considering the families of solutions (2.24) with "small" a-cycles.

Let us consider the family of Riemann surfaces of the above form depending on $n$ sufficiently small parameters $s_1, \dots, s_n$ such that, in the limit $s_j \to 0$ the $j$-th cycle $\boldsymbol{a}_j$ is squeezed to zero such that

$$\Sigma_n(s)|_{s_j = 0} \tag{2.26}$$

is a genus $n-1$ curve with an ordinary double point. Construction of such a deformation can be found in the Chapter III of the Fay's book [**16**]. The following statements proved in [**16**] will be essential for us.

First, denote $\widehat{\Sigma}_n^j$ the normalization of (2.26) and $P_j^{\mp}$ the two points of the normalization to be identified on the nodal curve. The basic normalized holomorphic differential $w_j(s)$ on $\Sigma_n(s)$ in the limit $s_j \to 0$ goes to the normalized third kind differential on $\widehat{\Sigma}_n^j$ with simple points with the residues $\pm 1/2\pi i$ in the points $P_j^{\mp}$ resp. Other normalized holomorphic differentials $w_k$ on $\Sigma_n(s)$ go to the normalized holomorphic differentials on $\widehat{\Sigma}_n^j$. The same claim holds true for limits of normalized 2nd and 3d kind differentials on $\Sigma_n(s)$ with pole away from the pinched cycle. The diagonal entry $\beta_{jj}$ has logarythmic behaviour as $s_j \to 0$,

$$\beta_{jj} = -\log s_j + O(1),$$

other matrix entries have regular expansions in $s_j$.

Iterating this procedure, in the limit $s_1 \to 0, \dots, s_n \to 0$ the Riemann surface $\Sigma_n$ goes to the rational nodal curve with $n$ pairs of identified points $z_1^{\mp}, \dots, z_n^{\mp}$. The basic holomorphic differentials take the limiting values

$$w_j = \frac{1}{2\pi i}\Big(\frac{1}{z - z_j^-} - \frac{1}{z - z_j^+}\Big)\, dz, \quad j = 1, \dots, n. \tag{2.27}$$

We will assume that the marked point $\infty \in \Sigma_n(s)$ corresponds to the point $z = \infty$ of the limiting Riemann sphere and that the local parameter $\zeta$ on $\Sigma_n(s)$ goes to

$$\zeta = \frac{1}{z}$$

on the Riemann sphere near infinity. Comparing the expansions

$$w_j = -\frac{1}{2\pi i}\left[(z_j^- - z_j^+) + ((z_j^-)^2 - (z_j^+)^2)\zeta + ((z_j^-)^3 - (z_j^+)^3)\zeta^2 + O(\zeta^3)\right]\, d\zeta$$

with the formulae (2.10) expressing the wave numbers and frequencies in terms of expansion near $\infty$ of the basic normalized holomorphic differentials we conclude that the identified points must have the form

$$z_j^{\pm} = \frac{1}{2}(\lambda_j \pm i k_j), \quad j = 1, \dots, n \tag{2.28}$$

Observe that the nonresonance condition (2.11) means that all $2n$ points (2.28) are pairwise distinct.

The crucial point in proving cancellation of all small divisors but those corresponding to the resonances (2.11) is in proving that arbitrary configuration of the pairwise distinct double points (2.28) on the Riemann sphere can be obtained by the above $n$-parametric degeneration procedure within the family of KP Riemann surfaces.

Let $\beta_{ij}(s)$ be the period matrix (2.9) of the family of Riemann surfaces with respect to the basis of cycles that will be assumed to be continuosly depending on the parameter $s$. Denote

$$a_j(s) = e^{-\pi\beta_{jj}(s)}, \quad j = 1, \dots, n. \tag{2.29}$$

At $s = 0$ one has

$$a_1(0) = \cdots = a_n(0) = 0.$$

The off-diagonal entries of the matrix $\beta_{ij}(s)$ admit finite limits at $s \to 0$ and

$$e^{-2\pi\beta_{ij}(0)} = \frac{(k_i - k_j)^2 + (\lambda_i - \lambda_j)^2}{(k_i + k_j)^2 + (\lambda_i - \lambda_j)^2}, \quad i \neq j. \tag{2.30}$$

The wave numbers $k_j(s), l_j(s)$ and the frequencies $\omega_j(s)$ defined from the expansions (2.10) also admit the limits as $s \to 0$ of the form

$$k_j(0) = k_j, \quad l_j(0) = k_j\lambda_j, \quad \omega_j(0) = \frac{1}{4}(3k_j\lambda_j^2 - k_j^3), \quad j = 1, \dots, n. \tag{2.31}$$

We are now to prove that, for arbitrary nonresonant real numbers $k_1, \dots, k_n$ and arbitrary real numbers $l_1, \dots, l_n$ and for arbitrary sufficiently small positive numbers $a_1, \dots, a_n$ there exists a family of triples $(\Sigma_n, \infty, \zeta)$ of the above form depending analytically on the parameters $a_1, \dots, a_n, k_1, \dots, k_n, l_1, \dots, l_n$. To this end we are to introduce theta-functions of the second order.

Let $\nu = (\nu_1, \dots, \nu_n)$ be a vector with all components $\nu_j = 0$ or 1. Such a vector will be called *characteristic*. Define second order theta-function $\widetilde{\theta}[\nu](\phi|\beta)$ with the characteristic $\nu$ by

$$\begin{aligned}\widetilde{\theta}[\nu](\phi|\beta) \\ = \sum_{\boldsymbol{m}\in\mathbb{Z}^n} \prod_{i=1}^n a_i^{2(m_i^2+m_i\nu_i)} \prod_{i<j} Z_{ij}^{2m_im_j+m_i\nu_j+m_j\nu_i} e^{i((2m_1+\nu_1)\phi_1+\cdots+(2m_n+\nu_n)\phi_n)}.\end{aligned} \tag{2.32}$$

Here

$$a_j = e^{-\pi\beta_{jj}}, \quad Z_{ij} = e^{-2\pi\beta_{ij}}, \quad i \neq j. \tag{2.33}$$

Our definition of the second order theta-functions differs from the standard one (see, *e.g.*, [**16**]) by the factor

$$\frac{1}{2}\prod_{i=1}^n a_i^{-\nu_i^2/2} \prod_{i<j} Z_{ij}^{-\nu_i\nu_j/2}.$$

The advantage of our normalization is that, the functions (2.32) are real analytic in the variables $Z_{ij} > 0$, $a_j \geqslant 0$, $\phi_k \in \mathbb{R}$ provided that the lowest eigenvalue $\rho$ of the symmetric off-diagonal matrix

$$\log Z_{ij}$$

satisfies

$$\rho < 2\pi \log a_j^{-2}, \quad j = 1, \dots, n. \tag{2.34}$$

Actually, (2.32) are even functions in $a_1, \dots, a_n$. In particular,

$$\widetilde{\theta}[0] = 1 + 2\sum_{i=1}^{n} a_i^2 \cos\phi_i + O(a^4) \tag{2.35}$$

$$\widetilde{\theta}[\boldsymbol{n}_i] = \cos\phi_i + \sum_{j\neq i} a_j^2 \left[Z_{ij}\cos(2\phi_j + \phi_i) + Z_{ij}^{-1}\cos(2\phi_j - \phi_i)\right] + O(a^4) \tag{2.36}$$

$$\begin{aligned}\widetilde{\theta}[\boldsymbol{n}_{ij}] = {}& \cos(\phi_i + \phi_j) + Z_{ij}^{-1}\cos(\phi_i - \phi_j)\\ &+ \sum_{k\neq i,j} a_k^2 \big[Z_{ik}Z_{jk}\cos(2\phi_k + \phi_i + \phi_j) + Z_{ik}^{-1}Z_{jk}^{-1}\cos(2\phi_k - \phi_i - \phi_j)\\ &+ Z_{ij}^{-1}\big(Z_{ik}Z_{jk}^{-1}\cos(\phi_i - \phi_j + 2\phi_k) + Z_{ik}^{-1}Z_{jk}\cos(\phi_i - \phi_j - 2\phi_k)\big)\big]\\ &+ O(a^4)\end{aligned} \tag{2.37}$$

In these formulae $\boldsymbol{n}_i$ stands for the characteristic with the $i$-th component 1 and all others 0,

$$\boldsymbol{n}_{ij} = \boldsymbol{n}_i + \boldsymbol{n}_j, \quad i \neq j.$$

The following statement was proven in [**8**] (*cf.* also [**32**], [**10**]).

***Lemma 2.3***. — *The function*

$$\tau(x,y,t) = \theta(\phi|\beta) = \sum_{\boldsymbol{m}\in\mathbb{Z}^n} e^{-\pi\langle\boldsymbol{m},\beta\boldsymbol{m}\rangle} e^{i(m_1\phi_1+\cdots+m_n\phi_n)},$$
$$\phi_j = k_j x + l_j y - \omega_j t + \phi_j^0, \quad j = 1,\dots,n$$

*satisfies (2.3) for arbitrary phase shifts $\phi_1^0,\dots,\phi_n^0$* iff *the vectors $k = (k_1,\dots,k_n)$, $l = (l_1,\dots,l_n)$, $\omega = (\omega_1,\dots,\omega_n)$ and the matrix $\beta = (\beta_{ij})$ satisfy the following system of equations*

$$f[\nu](k,l,\omega,\beta) := \left(\partial_k^4 + 3\partial_l^2 - 4\partial_k\partial_\omega + b\right)\widetilde{\theta}[\nu](\phi|\beta)|_{\phi=0} = 0 \tag{2.38}$$

*for some constant $b = b(k,l,\omega,\beta)$ and for arbitrary characteristic $\nu \in (\mathbb{Z}/2\mathbb{Z})^n$. Here*

$$\partial_k := \sum k_j \frac{\partial}{\partial\phi_j}, \quad \partial_l := \sum l_j \frac{\partial}{\partial\phi_j}, \quad \partial_\omega := \sum \omega_j \frac{\partial}{\partial\phi_j}.$$

In particular, the equations (2.38) remain valid for the values

$$k = k(s), \quad l = l(s), \quad \omega = \omega(s), \quad \beta = \beta(s)$$

of our family of Riemann surfaces for a suitable constant $b = b(s)$. Indeed, it can be readily checked that, at the limit $s = 0$ the equations (2.38) hold true by substituting $a_1^2 = \cdots = a_n^2 = 0$ and the values $Z_{ij}$, $k_j$, $l_j$, $\omega_j$ from (2.30), (2.31) and $b = 0$.

We will now prove that the system (2.38) has unique solution of the form

$$
\begin{aligned}
&Z_{ij} = Z_{ij}(a_1^2, \dots, a_n^2, k_1, \dots, k_n, l_1, \dots, l_n),\\
&\omega_j = \omega_j(a_1^2, \dots, a_n^2, k_1, \dots, k_n, l_1, \dots, l_n),\\
&b = b(a_1^2, \dots, a_n^2, k_1, \dots, k_n, l_1, \dots, l_n)\\
&Z_{ij}(0, \dots, 0, k_1, \dots, k_n, l_1, \dots, l_n) = \frac{(k_i - k_j)^2 + (\lambda_i - \lambda_j)^2}{(k_i + k_j)^2 + (\lambda_i - \lambda_j)^2}\\
&\omega_j(0, \dots, 0, k_1, \dots, k_n, l_1, \dots, l_n) = \frac{1}{4}\left(3k_j\lambda_j^2 - k_j^3\right)\\
&b(0, \dots, 0, k_1, \dots, k_n, l_1, \dots, l_n) = 0
\end{aligned}
\tag{2.39}
$$

analytic for sufficiently small $a_1^2, \dots, a_n^2$ and for arbitrary nonresonant vectors $k$ and $l$. Let us first construct such analytic solution for the subsystem

$$
f[0] = 0, \quad f[\boldsymbol{n}_i] = 0,\ i = 1, \dots, n, \quad f[\boldsymbol{n}_{ij}] = 0,\ 1 \leqslant i < j \leqslant n. \tag{2.40}
$$

To this end let us fix the nonresonant vectors $k^0$ and $\lambda^0$ and choose a real positive number $A$ such that the symmetric matrix $\beta_{ij}^0$ with

$$
\beta_{jj}^0 = -\frac{1}{\pi}\log a_j, \quad \beta_{ij}^0 = -\frac{1}{2\pi}\log\frac{(k_i^0 - k_j^0)^2 + (\lambda_i^0 - \lambda_j^0)^2}{(k_i^0 + k_j^0)^2 + (\lambda_i^0 - \lambda_j^0)^2},\ i \neq j
$$

is positive definite for

$$
0 < a_j < A, \quad j = 1, \dots, n.
$$

Then the functions $f[0]$, $f[\boldsymbol{n}_i]$, $f[\boldsymbol{n}_{ij}]$ will be real analytic in $a$, $Z$, $k$, $\lambda$, $\omega$, $b$ for

$$
0 \leqslant a_j < A', \quad j = 1, \dots, n
$$

for some $A' < A$ and for $Z$, $k$, $l$, $\omega$, $b$ sufficiently close to

$$
Z_{ij}^0 = \frac{(k_i^0 - k_j^0)^2 + (\lambda_i^0 - \lambda_j^0)^2}{(k_i^0 + k_j^0)^2 + (\lambda_i^0 - \lambda_j^0)^2}, \quad k^0, \quad \lambda^0, \quad \omega_j^0 = \frac{k_j^0}{4}\left(3(\lambda_j^0)^2 - (k_j^0)^2\right), b^0 = 0
$$

respectively. For $a_1 = \cdots = a_n = 0$ the system (2.40) has unique solution given by (2.39). We derive existence of such solution to (2.40) for positive small $a$ by applying the implicit function theorem (*cf.* [**9**]). Indeed, from the formulae (2.35)–(2.37) it readily follows that, at $a_1^2 = 0, \dots, a_n^2 = 0$

$$
\begin{aligned}
&\frac{\partial f[0]}{\partial b} = 1,\\
&\frac{\partial f[0]}{\partial \omega_j} = 0, \quad \frac{\partial f[\boldsymbol{n}_i]}{\partial \omega_j} = k_i\delta_{ij},\\
&\frac{\partial f[0]}{\partial Z_{pq}} = 0, \quad \frac{\partial f[\boldsymbol{n}_i]}{\partial Z_{pq}} = 0,\\
&\frac{\partial f[\boldsymbol{n}_{ij}]}{\partial Z_{pq}} = 3Z_{ij}^{-2}k_ik_j\left[(k_i - k_j)^2 + (\lambda_i - \lambda_j)^2\right]\delta_{ip}\delta_{jq}, \quad i < j,\ p < q.
\end{aligned}
\tag{2.41}
$$

We obtain a triangular Jacobi matrix with the nonvanishing diagonal. This proves existence of the needed analytic solution.

Explicitly, the expansion of the needed solution reads

$$Z_{ij} = \frac{\rho_{ij}^-}{\rho_{ij}^+}\left\{1 + 32\frac{k_i k_j}{\left[\rho_{ij}^+\rho_{ij}^-\right]^2}\left[a_i^2k_i^2p_{ij} + a_j^2k_j^2p_{ji}\right] + 256\frac{k_ik_j}{\rho_{ij}^+\rho_{ij}^-}\sum_{k\neq i,j}\frac{a_k^2k_k^4q_{ijk}}{\rho_{ik}^+\rho_{ik}^-\rho_{jk}^+\rho_{jk}^-}\right\} + O(a^4) \tag{2.42}$$

where

$$\rho_{ij}^{\pm} = (k_i \pm k_j)^2 + (\lambda_i - \lambda_j)^2, \quad i \neq j \tag{2.43}$$

$$p_{ij} = \left(k_i^2 - k_j^2\right)^2 + 2\left(3k_i^2 - k_j^2\right)(\lambda_i - \lambda_j)^2 - 3(\lambda_i - \lambda_j)^4 \tag{2.44}$$

$$\begin{aligned} q_{ijk} = {} & \left[(\lambda_k - \lambda_j)(k_i^2 - 3\lambda_i^2) + (\lambda_j - \lambda_i)(k_k^2 - 3\lambda_k^2) + (\lambda_i - \lambda_k)(k_j^2 - 3\lambda_j^2)\right] \\ & \times \left[(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_i)\right. \\ & \left. + \lambda_i(k_j^2 - k_k^2) + \lambda_j(k_k^2 - k_i^2) + \lambda_k(k_i^2 - k_j^2)\right] \end{aligned} \tag{2.45}$$

$$\omega_i = \frac{1}{4}\left(3k_i\lambda_i^2 - k_i^3\right) + 6k_i\Big[a_i^2k_i^2 + 8\sum_{j\neq i}\frac{a_j^2k_j^4(\lambda_i - \lambda_j)^2}{\rho_{ij}^+\rho_{ij}^-}\Big] + O(a^4) \tag{2.46}$$

$$b = -6\sum a_i^2k_i^4 + O(a^4). \tag{2.47}$$

Let us now prove that the solution (2.42)–(2.47) to the subsystem (2.40) also satisfies the whole system (2.38).

***Lemma 2.4***. — *Let $Z_{ij}^0$ be the value of the functions (2.42) at a point $a^0$, $k^0$, $l^0$ (nonresonance of $k_j^0$, $l_j^0$ is assumed). Then the system of equations*

$$Z_{ij}(a_1^2, \dots, a_n^2, k_1, \dots, k_n, l_1, \dots, l_n) = Z_{ij}^0, \quad 1 \leqslant i < j \leqslant n \tag{2.48}$$

*for sufficiently small*

$$\sum\left(a_j^2 - (a_j^0)^2\right)^2 + \sum(k_j - k_j^0)^2 + \sum(\lambda_j - \lambda_j^0)^2$$

*has three-dimensional variety of solutions.*

*Proof*. — Let us first establish validity of the claim of the Lemma for $a_1^0 = \cdots = a_n^0$. Let us rewrite the formula (2.30) in the form of the cross-ratio

$$\frac{(k_i - k_j)^2 + (\lambda_i - \lambda_j)^2}{(k_i + k_j)^2 + (\lambda_i - \lambda_j)^2} = (z_i^+, z_i^-, z_j^+, z_j^-) \equiv \frac{z_i^+ - z_j^+}{z_j^+ - z_i^-}\cdot\frac{z_j^- - z_i^-}{z_i^+ - z_j^-} \tag{2.49}$$

where the complex numbers $z_i^{\pm}$ are defined in (2.28). Because of invariance of the cross-ratio with respect to the Möbius group

$$z \longmapsto \frac{az + b}{cz + d}, \quad ad - bc \neq 0,$$

the space of complex solutions to the system

$$\frac{(k_i - k_j)^2 + (\lambda_i - \lambda_j)^2}{(k_i + k_j)^2 + (\lambda_i - \lambda_j)^2} = \frac{(k_i^0 - k_j^0)^2 + (\lambda_i^0 - \lambda_j^0)^2}{(k_i^0 + k_j^0)^2 + (\lambda_i^0 - \lambda_j^0)^2}, \quad 1 \leqslant i < j \leqslant n$$

is at least three-dimensional. The subgroup $PSL_2(\mathbb{R})$ of the Möbius group preserves reality of the numbers $k_j$, $\lambda_j$. So the dimension of the space of real solutions is also greater or equal to three. It is easy to see that this dimension cannot be greater than 3. This proves the Lemma in the limiting case $a^0 = 0$.

Let us now extend the $PSL_2(\mathbb{R})$-symmetry onto the whole space of solutions to the equations (2.40). We first rewrite the symmetry in the infinitesimal form with the generator

$$X_0 = \sum \left[\frac{1}{4}p(\lambda_j^2 - k_j^2) + q\lambda_j + r\right]\frac{\partial}{\partial \lambda_j} + \sum\left[\frac{1}{2}p\lambda_j k_j + qk_j\right]\frac{\partial}{\partial k_j}. \tag{2.50}$$

Here $p$, $q$, $r$ are arbitrary real parameters. The one-parameter subgroups corresponding to $q$ and $r$ have a clear meaning: these are the groups of scaling transformations of $k$ and $\lambda$ and diagonal shifts of $\lambda$ respectively,

$$\begin{aligned} &k_j \longmapsto c\,k_j,\ \lambda_j \longmapsto c\,\lambda_j,\ j = 1, \dots, n, \quad c \neq 0 \\ &\lambda_j \longmapsto \lambda_j + a,\ j = 1, \dots, n. \end{aligned} \tag{2.51}$$

They are clearly also symmetries of the full system inducing the transformation

$$l_j \longmapsto l_j + ak_j,\ \omega_j \longmapsto \omega_j + \frac{3}{2}al_j + \frac{3}{4}a^2 k_j,\ j = 1, \dots, n.$$

The generator of the one-parameter subgroup corresponding to $p$ can be recast into the form

$$X_0^{(p)} = \sum \frac{1}{2} l_j \frac{\partial}{\partial k_j} + \omega_j^0 \frac{\partial}{\partial l_j}, \quad \omega_j^0 = \frac{1}{4}(3k_j\lambda_j^2 - k_j^3). \tag{2.52}$$

Remarkably, in this form the transformations (2.52) yield symmetries of the full system (2.38) when $\omega_j^0$ is replaced by the exact solution $\omega_j$ of the system. This deep result is one of the important steps in the proof of the Shiota theorem [**36**]. It follows from the following claim [**36**]: compatibility of the system (2.38) implies compatibility of the system

$$\left(2\partial_k^3\partial_l + 4\partial_l\partial_\omega - 4\partial_k\partial_{\dot\omega} + \dot b\right)\widetilde{\theta}[\nu](\phi|\beta)|_{\phi=0} = 0 \tag{2.53}$$

for some vector $\dot\omega$ and some constant $\dot b$. From uniqueness of such a vector it follows that $\dot\omega$ coincides with the derivative of $\omega$ along the vector field

$$X^{(p)} = \sum \frac{1}{2} l_j \frac{\partial}{\partial k_j} + \omega_j \frac{\partial}{\partial l_j}. \tag{2.54}$$

The lemma is proved. □

We are now ready to complete the proof of the Theorem. According to Lemma 2.4 combined with Torelli theorem [**18**], the dimension of the space of solutions to the system (2.38) is equal to the dimension of the moduli space of (real) Riemann surfaces of genus $n$ plus 3, *i.e.*, it is equal to $3n$ for $n \geqslant 2$. We have described the $3n$-dimensional manifold of solutions (2.42), (2.46) to the subsystem (2.40) that by construction contains the solutions of the form (2.9)–(2.7) for Riemann surfaces with sufficiently small real ovals $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$. The dimension counting proves coincidence of these two families. In particular this implies that all the remaining equations of the system (2.38) hold true on the space of solutions (2.42)–(2.47). Therefore the unique solution to KP defined in (2.15)–(2.17) starting from a given nonresonant wave numbers $k_1, \dots, k_n$, $l_1, \dots, l_n$ and arbitrary sufficiently small amplitudes $a_1, \dots, a_n$ must have the form (2.8)–(2.7). Uniform convergence of the series (2.24) for theta-functions together with cancellation of all the divisors but $D(\boldsymbol{n}_i \pm \boldsymbol{n}_j)$, $i \neq j$ implies analyticity of the family of invariant tori. The Theorem is proved. □

***Remark 2.5***. — Explicitly, the extension of the symmetry (2.52) onto the full space of solutions to (2.38) reads

$$X^{(p)} = \sum \frac{1}{2}\lambda_j k_j \frac{\partial}{\partial k_j} + \frac{1}{4}(\lambda_j^2 - k_j^2)\frac{\partial}{\partial \lambda_j} + 6\sum \left[a_i^2 k_i^2 + 8\sum_{j\neq i} \frac{a_j^2 k_j^4 (\lambda_i - \lambda_j)^2}{\rho_{ij}^+ \rho_{ij}^-} + O(a^4)\right]\frac{\partial}{\partial \lambda_j} \tag{2.55}$$

Together with the fields

$$X^{(q)} = \sum k_j \frac{\partial}{\partial k_j} + \lambda_j \frac{\partial}{\partial \lambda_j}$$

and

$$X^{(r)} = \sum \frac{\partial}{\partial \lambda_j}$$

it generates the action of $PSL_2(\mathbb{R})$ on the space of solutions of the system (2.38):

$$\left[X^{(q)}, X^{(p)}\right] = X^{(p)}, \quad \left[X^{(r)}, X^{(p)}\right] = \frac{1}{2}X^{(q)}, \quad \left[X^{(r)}, X^{(q)}\right] = X^{(r)}.$$

The vector field $X^{(p)}$ generates infinitesimal changes of the marked point $\infty \in \Sigma_n$. In other words, integrating the vector field (2.55) one obtains, for $n > 1$, the Riemann surface with the parameters $a_1^2, \dots, a_n^2$, $k_1, \dots, k_n$, $\lambda_1, \dots, \lambda_n$. This construction gives an answer to the question put in the title of the Section. It would be important however to elaborate more practical tools in the analysis of the experimental water wave data in order to measure the moduli of the Riemann surface "hidden" behind the water wave profile. For the case of two interacting plane waves such tools has been developed in [**20, 19**].

## 3. Infinite genus theta-functions of Riemann surfaces without Riemann surfaces

The invariant tori for KP identified in the previous Section as the result of nonlinear interaction of $n$ plane waves with small amplitudes $2a_1, \ldots, 2a_n$ can be represented as infinite sums of homogeneous polynomials in $a_1, \ldots, a_n$ of various degrees with coefficients depending on the phases $\phi_1, \ldots, \phi_n$ and on the wave numbers $k_1, \ldots, k_n$, $l_1 = k_1\lambda_1, \ldots, l_n = k_n\lambda_n$. Let us recast this sum in the following way. For any subset

$$I = \{i_1, \ldots, i_g\} \subset \{1, 2, \ldots, n\}, \quad g > 0$$

denote

$$\theta_I = a_{i_1} \ldots a_{i_g} \Delta\theta_I \tag{3.1}$$

the sum of all monomials that contain only $a_i$ for $i \in I$. We put

$$\theta_\varnothing = 1.$$

Denote also

$$\phi_I = \{\phi_{i_1}, \ldots, \phi_{i_g}\}, \quad a_I = \{a_{i_1}, \ldots, a_{i_g}\},$$
$$k_I = \{k_{i_1}, \ldots, k_{i_g}\}, \quad \lambda_I = \{\lambda_{i_1}, \ldots, \lambda_{i_g}\}.$$

***Lemma 3.1***. — *The genus $n$ KP theta-function described in the Theorem 2.2 can be represented in the form*

$$\theta(\phi_1, \ldots, \phi_n|\beta) = \sum_I \theta_{|I|}(\phi_I|a_I, k_I, \lambda_I) \tag{3.2}$$

*where the summation takes place over all subsets $I \subset \{1, 2, \ldots, n\}$. The functions $\theta_{|I|}(\phi_I|a_I, k_I, \lambda_I)$ are real analytic for all real nonresonant vectors $k_I$, $\lambda_I$ and for sufficiently small nonnegative amplitudes $a_I$. The terms of this expansion can be uniquely determined from the system of the form (2.38) with $n \mapsto |I|$ by requiring that the sum*

$$\sum_{J \subset I} \theta_{|J|}(\phi_J|a_J, k_J, \lambda_J) \tag{3.3}$$

*with*

$$\phi_i = k_i(x + \lambda_i y) - \omega_i^I t + \phi_i^0, \quad i \in I$$

*with some vector $\omega^I$ satisfies KP.*

Here $|I|$ is the cardinality of the set $I$. It should be emphasized that the radii of convergence

$$a_{i_1} < r_{i_1}, \quad a_{i_g} < r_{i_g}$$

of the series depend on $k_I$, $\lambda_I$.

*Proof.* — This statement is almost obvious since, supressing all the amplitudes $a_j = 0$ for $j \in \{1, 2, \ldots, n\} \setminus I$ one reduces a theta-function of the genus $n$ to another one of the genus $|I|$. □

We will also redenote the functions $\Delta\theta_I$ by $\Delta\theta_g$ with $g = |I|$. Explicitly, from (2.42)–(2.45) it follows that

$$\Delta\theta_1(\phi|a) = 2\sum_{n>0} a^{n^2-1}\cos n\phi \tag{3.4}$$

$$\begin{aligned}\Delta\theta_2(\phi_1,\phi_2|a_1,a_2,k_1,k_2,l_1,l_2) &= 2\left[\frac{\rho_{12}^-}{\rho_{12}^+}\cos(\phi_1+\phi_2)+\frac{\rho_{12}^+}{\rho_{12}^-}\cos(\phi_1-\phi_2)\right]\\ &+64\frac{k_1k_2}{[\rho_{12}^+\rho_{12}^-]^2}\left(a_1^2k_1^2p_{12}+a_2^2k_2^2p_{21}\right)\left[\frac{\rho_{12}^-}{\rho_{12}^+}\cos(\phi_1+\phi_2)-\frac{\rho_{12}^+}{\rho_{12}^-}\cos(\phi_1-\phi_2)\right]\\ &+O(a^6)\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\Delta\theta_3(\phi_1,\phi_2,\phi_3|a_1,a_2,a_3,k_1,k_2,k_3,l_1,l_2,l_3)\\ &= 2\Big[\frac{\rho_{12}^-\rho_{23}^-\rho_{31}^-}{\rho_{12}^+\rho_{23}^+\rho_{31}^+}\cos(\phi_1+\phi_2+\phi_3)+\frac{\rho_{12}^-\rho_{23}^+\rho_{31}^+}{\rho_{12}^+\rho_{23}^-\rho_{31}^-}\cos(\phi_1+\phi_2-\phi_3)\\ &\quad+\frac{\rho_{12}^+\rho_{23}^+\rho_{31}^-}{\rho_{12}^-\rho_{23}^-\rho_{31}^+}\cos(\phi_1-\phi_2+\phi_3)+\frac{\rho_{12}^+\rho_{23}^-\rho_{31}^+}{\rho_{12}^-\rho_{23}^+\rho_{31}^-}\cos(-\phi_1+\phi_2+\phi_3)\Big]\\ &\quad+512\frac{k_1k_2k_3q_{123}}{\rho_{12}^+\rho_{12}^-\rho_{23}^+\rho_{23}^-\rho_{31}^+\rho_{31}^-}\Big\{a_1k_1^3\Big[\frac{\rho_{23}^-}{\rho_{23}^+}\cos(\phi_2+\phi_3)-\frac{\rho_{23}^+}{\rho_{23}^-}\cos(\phi_2-\phi_3)\Big]\\ &\qquad+a_2k_2^3\Big[\frac{\rho_{31}^-}{\rho_{31}^+}\cos(\phi_3+\phi_1)-\frac{\rho_{31}^+}{\rho_{31}^-}\cos(\phi_3-\phi_1)\Big]\\ &\qquad+a_3k_3^3\Big[\frac{\rho_{12}^-}{\rho_{12}^+}\cos(\phi_1+\phi_2)-\frac{\rho_{12}^+}{\rho_{12}^-}\cos(\phi_1-\phi_2)\Big]\Big\}+O(a^5)\end{aligned} \tag{3.6}$$

In these formulae, we use the same notations as in the previous Section, *i.e.*, the polynomials $\rho_{ij}^{\pm}$, $p_{ij}$, $q_{ijk}$ in the variables $k_1,\dots,k_n$, $\lambda_1,\dots,\lambda_n$ with

$$\lambda_j = l_j/k_j$$

are defined in (2.43)–(2.45). Recall that, in order to obtain a solution $\tau(x,y,t)$ to the KP equation (2.3) one has to substitute in (3.2)

$$\phi_j = k_jx + l_jy - \omega_jt + \phi_j^0$$

with arbitrary phase shifts $\phi_j^0$ and the frequencies represented by a decomposition similar to (3.2)

$$\omega_j = \omega_j^0(k_j,\lambda_j) + \Delta\omega_j^1(a_j,k_j) + \sum_{i\neq j}\Delta\omega_j^2(a_j,a_i,k_i,k_j,\lambda_i,\lambda_j)+\dots \tag{3.7}$$

In this expansion,

$$\omega_j^0(k_j,\lambda_j) = \frac{1}{4}(k_j\lambda_j^2 - k_j^3)$$

is the dispersion law of the linearized KP,

$$\omega_j^n(a_1,\dots,a_n,k_1,\dots,k_n,\lambda_1,\dots,\lambda_n)$$

is the "pure genus $n$" contribution into the nonlinear dispersion law (3.7) to be found from the system (2.40) of the genus $n$ and then subtracting the lower genera contributions. Explicitly,

$$\omega_i^1 = 6k_i^3\left(a_i^2 + 3a_i^4 + O(a_i^6)\right) \tag{3.8}$$

$$\omega_i^2 = 48k_i \sum_{j\neq i} \frac{a_j^2 k_j^4(\lambda_i - \lambda_j)^2}{\rho_{ij}^+\rho_{ij}^-} + O(a^4) \tag{3.9}$$

etc. The genus $g$ term

$$\begin{aligned} & a_{i_1}\dots a_{i_g}\Delta\theta_g(\phi_{i_1},\dots,\phi_{i_g}|a_{i_1},\dots,a_{i_g},k_{i_1},\dots,k_{i_g},\lambda_{i_1},\dots,\lambda_{i_g})\\ & \phi_j = k_j x + k_j\lambda_j y - \omega_j t + \phi_j^0\\ & \omega_j = \omega_j^0(k_j,\lambda_j)\\ & \qquad + \Delta\omega_j^1(a_j,k_j) + \dots + \Delta\omega_i^g(a_{i_1},\dots,a_{i_g},k_{i_1},\dots,k_{i_g},\lambda_{i_1},\dots,\lambda_{i_g}) \end{aligned} \tag{3.10}$$

is created as the result of interaction of $g$ plane waves

$$\begin{aligned} & 2a_{i_1}\cos[k_{i_1}(x+\lambda_{i_1}y) - \omega_{i_1}^0(k_{i_1},\lambda_{i_1})t + \phi_{i_1}^0] + \cdots\\ & \qquad + 2a_{i_g}\cos[k_{i_g}(x+\lambda_{i_g}y) - \omega_{i_g}^0(k_{i_g},\lambda_{i_g})t + \phi_{i_g}^0] \end{aligned} \tag{3.11}$$

and their harmonics. If the amplitudes of the plane waves are of order $\varepsilon$ then their $g$-tuple interaction is of the order $\varepsilon^g$. In other words, to compute the solution $\tau(x,y,t)$ of (2.3) of genus $N >> 1$ with the accuracy $\varepsilon^n$ for $n < N$ it suffices to sum the expansions of the form (3.10) with $g \leqslant n$ truncating them at the order $n$. The result of the truncation will give uniform in the whole plane $(x,y)\in\mathbb{R}^2$ approximation of the genus $N$ solution for the times $|t| < O(\varepsilon^{-n})$. Observe that the representation (3.2) resembles the virial expansion wellknown in the statistical mechanics (see, *e.g.*, [**29**], § 72).

We want to generalize the expansion (3.2) to the case of interaction of infinite number of plane waves. Given infinite sequences of real numbers

$$a = (a_1, a_2, \dots),\ a_j > 0, \quad k = (k_1, k_2, \dots),\ k_j > 0, \quad \lambda = (\lambda_1, \lambda_2, \dots) \tag{3.12}$$

we can construct a formal Fourier series of infinite number of variables $\phi = (\phi_1, \phi_2, \dots)$ representing it as the following power series in $a$

$$\theta(\phi|a,k,\lambda) = \sum_{g=0}^{\infty}\sum_{|I|=g} a_I \Delta\theta_g(\phi_I|a_I,k_I,\lambda_I). \tag{3.13}$$

The summation takes place over all finite subsets $I\subset\mathbb{N}$. This formal expression makes sense for finite sequences of amplitudes $a$, *i.e.*, assuming that $a_j = 0$ for $j \geqslant N$ for some big $N$. In that case it reduces, for sufficiently small $a_1,\dots,a_N$, to the KP theta function of genus $N$.

If all the amplitudes $a_1$, $a_2$ do not vanish, then, at each order in $a$ one is to summate infinite series. E.g., at the order one (3.13) gives

$$2\sum_{i=1}^{\infty} a_i \cos\phi_i,$$

at the order two

$$2\sum_{i<j} a_i a_j \Big[\frac{\rho_{ij}^-}{\rho_{ij}^+}\cos(\phi_i+\phi_j) + \frac{\rho_{ij}^+}{\rho_{ij}^-}\cos(\phi_i-\phi_j)\Big]$$

etc. We will now give simple sufficient conditions for convergence of the series (3.13) for infinite sequencies of the data (3.12). To this end we are to recall some important points of the theory of infinite dimensional theta-functions, following the book [**17**].

Let $\beta = \beta_{ij}$ be an infinite symmetric matric with real values, $i$, $j = 1, 2, \ldots$. We say that the matrix $\beta$ satisfies the FKT condition if there exists a sequence of positive numbers $\sigma = (\sigma_1, \sigma_2, \ldots)$ and a number $\kappa$ satisfying

$$0 < \kappa < \pi$$

such that

(i) the following series converges

$$\sum_{j=1}^{\infty} e^{-\kappa\,\sigma_j} < \infty; \tag{3.14}$$

(ii) for all finite sequencies of integers $\boldsymbol{m} = (m_1, m_2, \ldots)$, $|\boldsymbol{m}| = |m_1| + |m_2| + \cdots < \infty$ the following inequality holds true

$$\langle \boldsymbol{m}, \beta\boldsymbol{m}\rangle \equiv \sum_{ij} \beta_{ij} m_i m_j \geqslant \sum_j \sigma_j m_j^2. \tag{3.15}$$

For a given sequence $\sigma$ introduce the Banach space $B_\sigma$ given by

$$B_\sigma = \Big\{\boldsymbol{z} = (z_1, z_2, \ldots,) \in \mathbb{C}^\infty \;\Big|\; \lim_{j\to\infty} \frac{|z_j|}{\sigma_j} = 0.\Big\} \tag{3.16}$$

with the norm

$$\|\boldsymbol{z}\| = \sup_j \frac{|z_j|}{\sigma_j}. \tag{3.17}$$

According to the Theorem 4.6 of [**17**] for a symmetric matrix $\beta$ satisfying the FKT condition for some $\sigma$ the theta-series

$$\theta(\phi|\beta) = \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^\infty,\\ |\boldsymbol{m}|<\infty}} e^{-\pi\langle \boldsymbol{m},\beta\boldsymbol{m}\rangle} e^{i\langle \boldsymbol{m},\phi\rangle} \tag{3.18}$$

converges absolutely and uniformely on a sufficiently small ball around any point $\phi \in B_\sigma$ to a holomorphic function.

It is clear that, for a given symmetric matrix $\beta_{ij}$ satisfying the FKT condition, another symmetric matrix $\beta'_{ij}$ with the same off-diagonal terms $\beta'_{ij} = \beta_{ij}$ for $i \neq j$

and with arbitrary diagonal terms satisfying $\beta'_{jj} \geqslant \beta_{jj}$ for all $j = 1, 2, \ldots$ will also satisfy the FKT condition with the same $\sigma$.

Let us first give a simple sufficient condition for an *off-diagonal* symmetric matrix $\beta_{ij}$ to ensure a possibility to choose positive numbers $\beta_{11}$, $\beta_{22}$, ... in such a way that the whole symmetric matrix $\beta_{ij}$ satisfies the FKT condition for some sequence $\sigma$.

***Lemma 3.2***. — *Let the real symmetric off-diagonal matrix $\beta_{ij}$ satisfies the condition*

$$\mu_i^2 := \sum_{j>i} \beta_{ij}^2 < \infty, \quad i = 1, 2, \ldots. \tag{3.19}$$

*Let $\sigma$ be any sequence of positive numbers satisfying the convergence condition (3.14) with some positive $\kappa < \pi$. Let $\beta_{jj}^0$ be another sequence of positive numbers defined by*

$$\beta_{jj}^0 = \sigma_j + 2\sum_{k=1}^{j} \mu_k, \quad j \geqslant 1$$

*(it is assumed that all numbers $\mu_j$ are nonnegative). Then, for any choice of the diagonal entries satisfying*

$$\beta_{jj} > \beta_{jj}^0, \quad j \geqslant 1 \tag{3.20}$$

*the matrix $\beta$ satisfies the FKT condition.*

*Proof*. — Because of the obvious inequality

$$\sum_{i,j} \beta_{ij} m_i m_j \geqslant \sum_j \beta_{jj} m_j^2 - 2\sum_i \Big| m_i \sum_{j>i} \beta_{ij} m_j \Big|$$

it suffices to obtain upper estimate for the second term. Let us consider the Hilbert space of square summable sequencies

$$L_2^{(i)} = \{(x_i, x_{i+1}, \ldots) \mid \textstyle\sum_{j\geqslant i} x_j^2 < \infty\}.$$

Applying the standard inequality

$$|(x, A\,x)| \leqslant \|A\|_{L_2}(x, x), \quad x \in L_2^{(i)}$$

valid for an arbitrary Hilbert-Schmidt operator $A$ to the rank one operator

$$(x_i, x_{i+1}, \ldots) \longmapsto \Big(\sum_{j>i} \beta_{ij} x_j, 0, \ldots\Big)$$

we obtain

$$\Big| x_i \sum_{j>i} \beta_{ij} x_j \Big| \leqslant \mu_i \sum_{j\geqslant i} x_j^2.$$

Finite sequencies of integers give vectors in $L_2^{(i)}$. Applying to these vectors the last inequality yields

$$\sum_{i,j} \beta_{ij} m_i m_j \geqslant \sum_j \beta_{jj} m_j^2 - 2\sum_{j=1}^{\infty} \mu_j \sum_{k\geqslant j} m_k^2.$$

This proves the Lemma. □

Using the Lemma, we will give a simple sufficient condition for an infinite sequence of plane waves to generate, via the formula (3.13), an infinite genus KP theta-function for arbitrary sufficiently small amplitudes $a_j$ and given wave numbers $k_j$, $l_j$.

***Lemma 3.3***. — *Let*

$$z_j = \frac{1}{2}(\lambda_j + ik_j), \quad k_j > 0 \quad j \geqslant 1$$

*be a sequence of complex numbers satisfying the following conditions.*

(i) *There exists a small positive number $r > 0$ such that*

$$|z_i - z_j| > r, \quad i \neq j, \quad |z_i - \overline{z}_j| > r, \quad i,j = 1,2,\ldots. \tag{3.21}$$

(ii) *The series*

$$\sum_{j=1}^{\infty} |z_j|^{-2} < \infty \tag{3.22}$$

*converges.*

*Then there exists a sequence of positive numbers $\beta^0_{jj}$ such that the matrix $\beta$ with the off-diagonal entries*

$$\beta^0_{ij} = -\frac{1}{2\pi} \log \frac{(k_i - k_j)^2 + (\lambda_i - \lambda_j)^2}{(k_i + k_j)^2 + (\lambda_i - \lambda_j)^2}, \quad i \neq j \tag{3.23}$$

*satisfies the FKT condition for arbitrary diagonal entries such that*

$$\beta_{jj} > \beta^0_{jj}, \quad j = 1,2,\ldots.$$

*Proof.* — The formula for $\beta^0_{ij}$ can be rewritten in the form

$$\beta^0_{ij} = -\frac{1}{2\pi} \log \left| \frac{z_j - z_i}{z_j - \overline{z}_i} \right|^2.$$

Using the elementary inequality

$$\left| \log \left| \frac{z - w}{z - \overline{w}} \right|^2 \right| < \frac{4}{|z|} |\operatorname{Im} w| \quad \text{for} \quad \left| \frac{w}{z} \right| < \frac{1}{2}$$

we derive that

$$|\beta^0_{ij}| < \frac{2}{\pi} \frac{k_i}{\sqrt{k_j^2 + \lambda_j^2}}$$

for a fixed $i$ and any sufficiently large $j >> i$. Applying Lemma 3.2 we complete the proof of the Lemma. □

We are now ready to prove convergence of the series (3.13) for a suitable class of parameters $a$, $k$, $\lambda$. Let the vectors $k$, $\lambda$ satisfy the conditions of the Lemma 3.3. Choose positive numbers $\sigma_j$ in such a way that the series (3.14) converges for some positive $\kappa < \pi$. Choose numbers $\beta^0_{jj}$ in such a way that

$$\beta^0_{jj} > \sigma_j + 2 \sum_{k=1}^{j} \mu^0_k, \quad j = 1,2,\ldots \tag{3.24}$$

where

$$\mu_i^0 := \Big( \sum_{j>i} \big(\beta_{ij}^0\big)^2 \Big)^{1/2} \tag{3.25}$$

and the off-diagonal matrix $\beta_{ij}^0 = \beta_{ij}^0(k, \lambda)$ is defined in (3.23).

***Theorem 3.4***. — *Let the numbers $k_j$, $\lambda_j$, $j \geqslant 1$, satisfy the assumptions of the Lemma 3.3 and the numbers $\beta_{jj}^0$ satisfy the conditions (3.23)–(3.25). Then for arbitrary positive numbers $a = (a_1, a_2, \dots)$ satisfying*

$$a_j < e^{-\pi\, \beta_{jj}^0}, \quad j = 1, 2, \dots \tag{3.26}$$

*the series (3.13) converges absolutely and uniformely on a sufficiently small ball around any point $\phi \in B_\sigma$ to a holomorphic function. The series expansion (3.7) also converges to a sequence of frequencies $(\omega_1, \omega_2, \dots)$. The theta-function (3.13), after the substitution*

$$\phi_j = k_j(x + \lambda_j y) - \omega_j t + \phi_j^0, \quad j = 1, 2, \dots$$

*for arbitrary real phase shifts, yields a quasiperiodic solution to the KP equation (2.3) for some constant $b = b(a, k, \lambda)$.*

*Proof.* — Let us consider the space of off-diagonal matrices $\beta_{ij}$ satisfying the following inequalities

$$\sum_{j=1}^{k} \Big( \sum_{j>i} \beta_{ij} r \Big)^{1/2} < \frac{1}{2} \left(\beta_{kk}^0 - \sigma_k\right), \quad k = 1, 2, \dots. \tag{3.27}$$

For any $a$ satisfying (3.26) and any off-diagonal $\beta_{ij}$ satisfying (3.27) the theta-series (3.18) converges to an analytic function on $B_\sigma$. It will also depend analytically on the period matrix, moreover, it satisfies the heat equations

$$a_k \frac{\partial \theta}{\partial a_k} = -\frac{\partial^2 \theta}{\partial \phi_k^2}, \quad Z_{ij} \frac{\partial \theta}{\partial Z_{ij}} = -\frac{\partial^2 \theta}{\partial \phi_i \partial \phi_j}.$$

One can also prove analyticity of the theta-functions of the second order (2.32). Like in the proof of the Theorem 2.2, we consider the system of equations (2.40). The functions $f[\nu]$ vanish at $a = 0$ for

$$\beta_{ij} = \beta_{ij}^0, \ i < j, \quad \omega_j = \frac{1}{4}(3k_j\lambda_j^2 - k_j^3), \quad b = 0.$$

The inverse to the Jacobi matrix (2.41) is a bounded operator due to our assumptions about the wave numbers. Applying the implicit function theorem we obtain convergence of the series (3.13), (3.7). The Theorem is proved. □

***Example 3.5***. — Let $\lambda_j = 0$ for all $j \geqslant 1$ and $k_j$ be arbitrary positive numbers satisfying

$$|k_i - k_j| > r, \quad i \neq j$$

for some positive $r$. Then the assumptions of the Theorem 3.4 are fulfilled. In this way one obtains the theta-functions of the hyperelliptic Riemann surfaces of infinite genus (*cf.* [**31**], [**17**]). In particular, if $k_j$ grows linearly with $j$, then the series (3.13) will converge for all $a$ with exponential decay

$$a_j < e^{-cj}$$

for some positive constant $c$. The formulae (3.13), (3.7) define quasiperiodic solutions to the KdV equations.

More generally, our approach describes some neighborhood of the manifold of hyperelliptic Riemann surfaces of infinite genus. In particular, assuming that the points $z_j$ satisfying (3.21) belong to a strip of a finite width along the imaginary axis, one obtain slow transversal perturbations of the KdV quasiperiodic solutions. The condition (3.22) in this case holds automatically true. It would be interesting to prove that the intersection with this neighborhood of the so-called heat curves of [**27**], [**6**], [**17**] associated with doubly periodic in $x$, $y$ solutions $u(x,u,t)$ of KP form a dense subset. For the case of finite genus density was proved in [**5**].

Some of our assumptions about behaviour of the sequence of wave numbers can in fact be relaxed. We will consider more general situation in a subsequent publication. The assumption (3.21) that prevents the interacting waves to be close to resonant seems however to be essential. For example, as it was shown by S. Venakides [**38**], the limits of hyperelliptic theta-functions with the parameters $k_j$ accumulating in the interval $[0,1]$ are weird functions described by a minimization principle of the Lax-Levermore type [**30**]. It would be also interesting to prove that our infinite genus theta-functions (3.13) come from a parabolic Riemann surfaces in the sense of Ahlfors and Sario [**1**].

We also plan to study in subsequent publications the relationship of our approach to the approach of V. Zakharov and E. Schulman to the problem of classification of integrable PDEs.

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# GENERATING FUNCTION ASSOCIATED WITH THE DETERMINANT FORMULA FOR THE SOLUTIONS OF THE PAINLEVÉ II EQUATION

*by*

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***Abstract.*** — In this paper we consider a Hankel determinant formula for generic solutions of the Painlevé II equation. We show that the generating functions for the entries of the Hankel determinants are related to the asymptotic solution at infinity of the linear problem of which the Painlevé II equation describes the isomonodromic deformations.

***Résumé* (Fonction génératrice associée à la formule déterminant pour les solutions de l'équation de Painlevé II)**

On s'intéresse à la formule déterminant de Hankel pour les solutions génériques de l'équation de Painlevé II. On établit une relation reliant les fonctions génératrices des coefficients des déterminants de Hankel aux solutions asymptotiques à l'infini du problème linéaire dont les déformations isomonodromiques sont décrites par cette équation de Painlevé II.

## 1. Introduction

The Painlevé II equation ($P_{II}$),

$$\frac{d^2u}{dx^2} = 2u^3 - 4xu + 4\Big(\alpha + \frac{1}{2}\Big), \tag{1}$$

where $\alpha$ is a parameter, is one of the most important equations in the theory of nonlinear integrable systems. It is well-known that $P_{II}$ admits unique rational solution when $\alpha$ is a half-integer, and one-parameter family of solutions expressible in terms of the solutions of the Airy equation when $\alpha$ is an integer. Otherwise the solution is non-classical [**13, 14, 17**].

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The rational solutions for $\mathrm{P_{II}}(1)$ are expressed as logarithmic derivative of the ratio of certain special polynomials, which are called the "*Yablonski-Vorob'ev polynomials*", [**18, 20**]. Yablonski-Vorob'ev polynomials admit two determinant formulas, namely, Jacobi-Trudi type and Hankel type. The latter is described as follows: For each positive integer $N$, the unique rational solution for $\alpha = N + 1/2$ is given by

$$u = \frac{d}{dx} \log \frac{\sigma_{N+1}}{\sigma_N},$$

where $\sigma_N$ is the Hankel determinant

$$\sigma_N = \begin{vmatrix} a_0 & a_1 & \cdots & a_{N-1} \\ a_1 & a_2 & \cdots & a_N \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_N & \cdots & a_{2N-2} \end{vmatrix},$$

with $a_n = a_n(x)$ being polynomials defined by the recurrence relation

$$\begin{gathered} a_0 = x, \qquad a_1 = 1, \\ a_{n+1} = \frac{da_n}{dx} + \sum_{k=0}^{n-1} a_k\, a_{n-1-k}. \end{gathered} \tag{2}$$

The Jacobi-Trudi type formula implies that the Yablonski-Vorob'ev polynomials are nothing but the specialization of the Schur functions [**12**]. Then, what does the Hankel determinant formula mean? In order to answer this question, a generating function for $a_n$ is constructed in [**6**]:

***Theorem 1.1*** **([6])**. — *Let $\theta(x,t)$ be an entire function of two variables defined by*

$$\theta(x,t) = \exp\left(2t^3/3\right) \mathrm{Ai}(t^2 - x), \tag{3}$$

*where* $\mathrm{Ai}(z)$ *is the Airy function. Then there exists an asymptotic expansion*

$$\frac{\partial}{\partial t} \log \theta(x,t) \sim \sum_{n=0}^{\infty} a_n(x)\,(-2t)^{-n}, \tag{4}$$

*as $t \to \infty$ in any proper subsector of the sector $|\arg t| < \pi/2$.*

This result is quite suggestive, because it shows that the Airy functions enter twice in the theory of classical solutions of the $\mathrm{P_{II}}$:

(1) in the formula [**3**]

$$u = \frac{d}{dx} \log \mathrm{Ai}\big(2^{1/3} x\big), \qquad \alpha = 0.$$

the one parameter family of classical solutions of $\mathrm{P_{II}}$ for integer values of $\alpha$ is expressed by Airy functions,

(2) in formulae (3), (4) the Airy functions generate the entries of determinant formula for the rational solutions.

In this paper we clarify the nature of this phenomenon. First, we reformulate the Hankel determinant formula for generic, namely non-classical, solutions of $\mathrm{P_{II}}$ already found in [**10, 11**]. We next construct generating functions for the entries of our Hankel determinant formula. We then show that the generating functions are related to the asymptotic solution at infinity of the isomonodromic problem introduced by Jimbo and Miwa [**9**]. More explicitly, the generating functions we construct are represented formally by series in powers of a variable $t$ that does not appear in the second Painlevé equation. We show that they satisfy two Riccati equations, one in the $x$ variable of $\mathrm{P_{II}}$, the other in the auxiliary variable $t$. These Riccati equations simultaneously linearise to the two linear systems whose compatibility is given by $\mathrm{P_{II}}$. This is the first time in the literature, to our knowledge, that the construction of the isomonodromic deformation problem has been carried out by starting directly from the Painlevé equation of interest.

This result explains the appearance of the Airy functions in Theorem 1.1. In fact, for rational solutions of $\mathrm{P_{II}}$, the asymptotic solution at infinity of the isomonodromic problem is indeed constructed in terms of Airy functions [**7, 8, 15**].

We expect that the generic solutions of the so-called Painlevé II hierarchy [**1, 2, 4**] should be expressed by some Hankel determinant formula. Of course the generating functions for the entries of Hankel determinant should be related to the asymptotic solution at infinity of the isomonodromic problem for the Painlevé II hierarchy. We also expect that the similar phenomena can be seen for other Painlevé equations. We shall discuss these generalizations in future publications.

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## 2. Hankel Determinant Formula and Isomonodromic Problem

**2.1. Hankel Determinant Formula.** — We first prepare the Hankel determinant formula for generic solutions for $\mathrm{P_{II}}$ (1). To show the parameter dependence explicitly, we denote equation (1) as $\mathrm{P_{II}}[\alpha]$. The formula is based on the fact that the $\tau$ functions for $\mathrm{P_{II}}$ satisfy the Toda equation,

$$\sigma_n''\sigma_n - (\sigma_n')^2 = \sigma_{n+1}\sigma_{n-1}, \quad n \in \mathbb{Z}, \quad ' = d/dx. \tag{5}$$

Putting $\tau_n = \sigma_n/\sigma_0$ so that the $\tau$ function is normalized as $\tau_0 = 1$, equation (5) is rewritten as

$$\tau_n''\tau_n - (\tau_n')^2 = \tau_{n+1}\tau_{n-1} - \varphi\psi\tau_n^2, \quad \tau_{-1} = \psi, \quad \tau_0 = 1, \quad \tau_1 = \varphi, \quad n \in \mathbb{Z}. \tag{6}$$

Then it is known that $\tau_n$ can be written in terms of Hankel determinant as follows [**11**]:

***Proposition 2.1***. — *Let $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ be the sequences defined recursively as*

$$a_n = a'_{n-1} + \psi \sum_{\substack{i+j=n-2\\ i,j\geqslant 0}} a_i a_j, \quad b_n = b'_{n-1} + \varphi \sum_{\substack{i+j=n-2\\ i,j\geqslant 0}} b_i b_j, \quad a_0=\varphi, \quad b_0=\psi. \tag{7}$$

*For any $N\in\mathbb{Z}$, we define Hankel determinant $\tau_N$ by*

$$\tau_N = \begin{cases} \det(a_{i+j-2})_{i,j\leqslant N} & N>0,\\ 1, & N=0,\\ \det(b_{i+j-2})_{i,j\leqslant |N|} & N<0. \end{cases} \tag{8}$$

*Then $\tau_N$ satisfies equation (6).*

Since the above formula involves two arbitrary functions $\varphi$ and $\psi$, it can be regarded as the determinant formula for general solution of the Toda equation. Imposing appropriate conditions on $\varphi$ and $\psi$, we obtain determinant formula for the solutions of $\mathrm{P}_{\mathrm{II}}$:

***Proposition 2.2***. — *Let $\psi$ and $\varphi$ be functions in $x$ satisfying*

$$\frac{\psi''}{\psi} = \frac{\varphi''}{\varphi} = -2\psi\varphi + 2x, \tag{9}$$

$$\varphi'\psi - \varphi\psi' = 2\alpha, \tag{10}$$

*Then we have the following:*

(1) $u_0 = (\log\varphi)'$ *satisfies* $P_{\mathrm{II}}[\alpha]$.
(2) $u_{-1} = -(\log\psi)'$ *satisfies* $P_{\mathrm{II}}[\alpha-1]$.
(3) $u_N = \left(\log\frac{\tau_{N+1}}{\tau_N}\right)'$, *where $\tau_N$ is defined by equation (8), satisfies* $P_{\mathrm{II}}[\alpha+N]$.

*Proof*. — (i) and (ii) can be directly checked by using the relations (9) and (10). Then (iii) is the reformulation of Theorem 4.2 in [**10**]. □

**2.2. Riccati Equations for Generating Functions.** — Consider the generating functions for the entries as the following formal series

$$F_\infty(x,t) = \sum_{n=0}^{\infty} a_n(x)\ t^{-n}, \quad G_\infty(x,t) = \sum_{n=0}^{\infty} b_n(x)\ t^{-n}. \tag{11}$$

It follows from the recursion relations (7) that the generating functions formally satisfy the Riccati equations. In fact, multiplying the recursion relations (7) by $t^{-n}$ and take the summation from $n=0$ to $\infty$, we have:

***Proposition 2.3***. — *The generating functions $F_\infty(x,t)$ and $G_\infty(x,t)$ formally satisfy the Riccati equations*

$$t\frac{\partial F}{\partial x} = -\psi F^2 + t^2 F - t^2\varphi, \tag{12}$$

$$t\frac{\partial G}{\partial x} = -\varphi G^2 + t^2 G - t^2\psi, \tag{13}$$

*respectively.*

Since $F_\infty$ and $G_\infty$ are defined as the formal power series around $t \sim \infty$, it is convenient to derive the differential equations with respect to $t$. In order to do this, the following auxiliary recursion relations are useful.

***Lemma 2.4***. — *Under the condition (9) and (10), $a_n$ and $b_n$ ($n \geqslant 0$) satisfy the following recursion relations,*

$$(\psi a_{n+2} - \psi' a_{n+1})' = 2(n+1)\psi a_n, \tag{14}$$

$$(\varphi b_{n+2} - \varphi' b_{n+1})' = 2(n+1)\varphi b_n, \tag{15}$$

*respectively.*

We omit the details of the proof of Lemma 2.4, because it is proved by straight but tedious induction. Multiplying equations (14) and (15) by $t^{-n}$ and taking summation over $n = 0$ to $\infty$, we have the following differential equations for $F_\infty$ and $G_\infty$:

***Lemma 2.5***. — *The generating functions $F_\infty$ and $G_\infty$ formally satisfy the following differential equations,*

$$2\psi t\frac{\partial F}{\partial t} = t(\psi' - t\psi)\frac{\partial F}{\partial x} + (\psi'' t - \psi' t^2 + 2\psi)F + t^2(\psi\varphi' + \psi'\varphi), \tag{16}$$

$$2\varphi t\frac{\partial G}{\partial t} = t(\varphi' - t\varphi)\frac{\partial G}{\partial x} + (\varphi'' t - \varphi' t^2 + 2\varphi)G + t^2(\psi\varphi' + \psi'\varphi), \tag{17}$$

*respectively.*

Eliminating $x$-derivatives from equations (12), (16), and equations (13), (17), respectively, we obtain the Riccati equations with respect to $t$:

***Proposition 2.6***. — *The generating functions $F_\infty$ and $G_\infty$ formally satisfy the following Riccati equations,*

$$2t\frac{\partial F}{\partial t} = -(\psi' - t\psi)F^2 + \Big(\frac{\psi''}{\psi}t + 2 - t^3\Big)F + t^2(\varphi' + t\varphi), \tag{18}$$

$$2t\frac{\partial G}{\partial t} = -(\varphi' - t\varphi)G^2 + \Big(\frac{\varphi''}{\varphi}t + 2 - t^3\Big)G + t^2(\psi' + t\psi), \tag{19}$$

*respectively.*

**2.3. Isomonodromic Problem.** — The Riccati equations for $F_\infty$ equations (12) and (18) are linearized by standard technique, which yield isomonodromic problem for $\mathrm{P_{II}}$. It is easy to derive the following theorem from the Proposition 2.3 and 2.6:

***Theorem 2.7***

(1) *It is possible to introduce the functions $Y_1(x,t)$, $Y_2(x,t)$ consistently as*

$$F_\infty(x,t) = \frac{t}{\psi}\Big(\frac{1}{Y_1}\frac{\partial Y_1}{\partial x} + \frac{t}{2}\Big) = \frac{2t}{\psi' - t\psi}\left[\frac{1}{Y_1}\frac{\partial Y_1}{\partial t} + \frac{1}{4}\Big(\frac{\psi''}{\psi} - t^2\Big)\right], \tag{20}$$

$$Y_2 = \frac{1}{\psi}\left(\frac{\partial Y_1}{\partial x} + \frac{tY_1}{2}\right). \tag{21}$$

*Then $Y_1$ and $Y_2$ satisfy the following linear system for $Y = \binom{Y_1}{Y_2}$:*

$$\frac{\partial}{\partial t}Y = AY, \quad A = \begin{pmatrix} \frac{t^2}{4} - \frac{z}{2} - \frac{x}{2} & -\frac{\psi}{2}(t+u_{-1}) \\ \frac{1}{\psi}\left\{(u_{-1}-t)\frac{z}{2}+\alpha\right\} & -\frac{t^2}{4}+\frac{z}{2}+\frac{x}{2} \end{pmatrix}, \tag{22}$$

$$\frac{\partial}{\partial x}Y = BY, \quad B = \begin{pmatrix} -\frac{t}{2} & \psi \\ \frac{z}{\psi} & \frac{t}{2} \end{pmatrix}, \tag{23}$$

*where $z = -\psi\varphi$.*

(2) *Similarly, it is possible to introduce the functions $Z_1(x,t)$, $Z_2(x,t)$ consistently as*

$$G_\infty(x,t) = \frac{t}{\varphi}\Big(\frac{1}{Z_1}\frac{\partial Z_1}{\partial x} + \frac{t}{2}\Big) = \frac{2t}{\varphi' - t\varphi}\left[\frac{1}{Z_1}\frac{\partial Z_1}{\partial t} + \frac{1}{4}\Big(\frac{\varphi''}{\varphi} - t^2\Big)\right], \tag{24}$$

$$Z_2 = \frac{1}{\varphi}\Big(\frac{\partial Y_1}{\partial x} + \frac{tY_1}{2}\Big). \tag{25}$$

*Then $Z_1$ and $Z_2$ satisfy the following linear system for $Z = \binom{Z_1}{Z_2}$:*

$$\frac{\partial}{\partial t}Z = CZ, \quad C = \begin{pmatrix} \frac{t^2}{4} - \frac{z}{2} - \frac{x}{2} & -\frac{\varphi}{2}(t-u_0) \\ -\frac{1}{\varphi}\left\{(u_0+t)\frac{z}{2}+\alpha\right\} & -\frac{t^2}{4}+\frac{z}{2}+\frac{x}{2} \end{pmatrix}, \tag{26}$$

$$\frac{\partial}{\partial x}Z = DY, \quad D = \begin{pmatrix} -\frac{t}{2} & \varphi \\ \frac{z}{\varphi} & \frac{t}{2} \end{pmatrix}. \tag{27}$$

***Remark 2.8***. — The linear systems (22), (23) and (26), (27) are the isomonodoromic problems for $\mathrm{P_{II}}[\alpha-1]$ and $\mathrm{P_{II}}[\alpha]$, respectively [**9**]. For example, compatibility condition for equations (22) and (23), namely,

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial t} + [A,B] = 0,$$

gives

$$\begin{cases} \dfrac{dz}{dx} = -2u_{-1}z - 2\alpha, \\ \dfrac{du_{-1}}{dx} = u_{-1}^2 - 2z - 2x, \\ u_{-1} = -\dfrac{1}{\psi}\dfrac{d\psi}{dx}, \end{cases} \tag{28}$$

which yields $\mathrm{P_{II}}[\alpha-1]$ for $u_{-1}$. This fact also guarantees the consistency of two expressions for $F_\infty$ in terms of $Y_1$ in equation (20). Similar remark holds true for $G_\infty$ and $Z_1$.

***Remark 2.9***. — $F_\infty$ and $G_\infty$ are also expressed as,

$$F_\infty = t\,\frac{Y_2}{Y_1},\quad G_\infty = t\,\frac{Z_2}{Z_1}, \tag{29}$$

respectively. Conversely, it is obvious that for any solution $Y_1$ and $Y_2$ for the linear system (22) and (23), $F = tY_2/Y_1$ satisfies the Riccati equations (12) and (18) (Similar for $G$).

***Remark 2.10***. — Theorem 1.1 is recovered by putting $\psi = 1$, $\varphi = x$.

***Remark 2.11***. — $Y_1$ can be formally expressed in terms of $a_n$ by using equation (20) as

$$Y_1 = \text{const.} \times \exp\left(\frac{1}{12}t^3 - \frac{x}{2}t\right)\ t^{-\alpha}\ \exp\left[\frac{1}{2}\sum_{n=1}^{\infty}\frac{\psi a_{n+1} - \psi' a_n}{n}\ t^{-n}\right]. \tag{30}$$

which coincides with known asymptotic behavior of $Y_1$ around $t \sim \infty$ [**9**].

## 3. Solutions of Isomonodromic Problems and Determinant Formula

In the previous section we have shown that the generating functions $F_\infty$ and $G_\infty$ formally satisfy the Riccati equations (12,18) and (13,19), and that their linearization yield isomonodromic problems (22, 23) and (26,27) for $\mathrm{P_{II}}$. Now let us consider the converse. We start from the linear system (22) and (23). We have two linearly independent solutions around $t \sim \infty$, one of which is related with $F_\infty(x,t)$. Then, what is another solution? In fact, it is well-known that linear system (22) and (23) admit the formal solutions around $t \sim \infty$ of the form [**9**],

$$Y^{(1)} = \begin{pmatrix} Y_1^{(1)} \\ Y_2^{(1)} \end{pmatrix} = \exp\left(\frac{t^3}{12} - \frac{xt}{2}\right) t^{-\alpha}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} y_{11}^{(1)} \\ y_{21}^{(1)} \end{pmatrix} t^{-1} + \begin{pmatrix} y_{12}^{(1)} \\ y_{22}^{(1)} \end{pmatrix} t^{-2} + \cdots\right], \tag{31}$$

$$Y^{(2)} = \begin{pmatrix} Y_1^{(2)} \\ Y_2^{(2)} \end{pmatrix} = \exp\left(-\frac{t^3}{12} + \frac{xt}{2}\right) t^{\alpha}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} y_{11}^{(2)} \\ y_{21}^{(2)} \end{pmatrix} t^{-1} + \begin{pmatrix} y_{12}^{(2)} \\ y_{22}^{(2)} \end{pmatrix} t^{-2} + \cdots\right]. \tag{32}$$

From Remark 2.9 we see that there are two possible power-series solutions for the Riccati equation (18) of the form,

$$Y^{(1)} \to F^{(1)} = t\,\frac{Y_2^{(1)}}{Y_1^{(1)}} = t\,\frac{y_{21}^{(1)}t^{-1} + \cdots}{1 + y_{11}^{(1)}t^{-1} + \cdots} = c_0 + c_1 t^{-1} + \cdots\,, \tag{33}$$

$$Y^{(2)} \to F^{(2)} = t\,\frac{Y_2^{(2)}}{Y_1^{(2)}} = t\,\frac{1 + y_{21}^{(2)}t^{-1} + \cdots}{y_{11}^{(2)}t^{-1} + \cdots} = t^2(d_0 + d_1 t^{-1} + \cdots). \tag{34}$$

The above two possibilities of power-series solutions for the Riccati equations are verified directly as follows:

***Proposition 3.1***. — *The Riccati equation (18) admits only the following two kinds of power-series solutions around $t \sim \infty$:*

$$F^{(1)} = \sum_{n=0}^{\infty} c_n\ t^{-n}, \quad F^{(2)} = t^2 \sum_{n=0}^{\infty} d_n\ t^{-n}. \tag{35}$$

*Proof*. — We substitute the expression,

$$F = t^{\rho} \sum_{n=0}^{\infty} c_n\ t^{-n}, \tag{36}$$

for some integer $\rho$ to be determined, into the Riccati equation (18). We have

$$\sum_{n=0}^{\infty} 2(\rho - n) c_n t^{\rho+1-n} = \sum_{n=0}^{\infty} \psi' \Big( \sum_{k=0}^{n} c_k c_{n-k} \Big) t^{2\rho-2n} - \sum_{n=0}^{\infty} \psi \Big( \sum_{k=0}^{n} c_k c_{n-k} \Big) t^{2\rho+1-2n}$$
$$+ \sum_{n=0}^{\infty} \Big( \frac{\psi''}{\psi} + 2 \Big) c_n t^{\rho-n} - \sum_{n=0}^{\infty} c_n t^{\rho+3-n} + t^2(\varphi' + t\varphi)$$

The leading order should be one of $t^{2\rho+1}$, $t^{\rho+3}$ and $t^3$. Investigating the balance of these terms, we have $\rho = 0$ or $\rho = 2$. □

We also have the similar result for the solution of the Riccati equation (19):

***Proposition 3.2***. — *The Riccati equation (19) admits only the following two kinds of power-series solutions around $t \sim \infty$:*

$$G^{(1)} = \sum_{n=0}^{\infty} e_n\ t^{-n}, \quad G^{(2)} = t^2 \sum_{n=0}^{\infty} f_n\ t^{-n}. \tag{37}$$

It is obvious that $F^{(1)}$ and $G^{(1)}$ are nothing but our generating functions $F_\infty$ and $G_\infty$, respectively. Then, what are $F^{(2)}$ and $G^{(2)}$? In the following, we present two observations regarding this point. First, there are unexpectedly simple relations among those functions:

***Proposition 3.3***. — *The following relations holds.*

$$F^{(2)}(x,t) = \frac{t^2}{G^{(1)}(x,-t)}, \quad G^{(2)}(x,t) = \frac{t^2}{F^{(1)}(x,-t)}. \tag{38}$$

*Proof*. — Substitute $F(x,t) = t^2/g(x,t)$ into equation (18). This gives equation (19) for $G(x,t) = g(x,-t)$ by using the relation (9). Choosing $g(x,t) = G^{(1)}(x,t)$, $F(x,t)$ must be $F^{(2)}(x,t)$, since its leading order is $t^2$. We obtain the second equation by the similar argument. □

Second, $F^{(2)}(x,t)$ and $G^{(2)}(x,t)$ are also interpreted as generating functions of entries of Hankel determinant formula for $\mathrm{P_{II}}$. Recall that the determinant formula in Proposition 2.1 is for the $\tau$ sequence $\tau_n = \sigma_n/\sigma_0$ so that it is normalized as $\tau_0 = 1$. We show that $F^{(2)}(x,t)$ and $G^{(2)}(x,t)$ correspond to different normalizations of $\tau$ sequence:

***Proposition 3.4***. — *Let*

$$F^{(2)}(x,t) = -\frac{t^2}{\psi^2}\sum_{n=0}^{\infty} d_n\ (-t)^{-n}, \tag{39}$$

$$G^{(2)}(x,t) = -\frac{t^2}{\varphi^2}\sum_{n=0}^{\infty} f_n\ (-t)^{-n}, \tag{40}$$

*be formal solutions of the Riccati equations (12),(18) and (13), (19), respectively. We put*

$$\kappa_{-n-1} = \det(d_{i+j})_{i,j=1,\dots,n} \quad (n>0), \quad \kappa_{-1} = 1, \tag{41}$$

$$\theta_{n+1} = \det(f_{i+j})_{i,j=1,\dots,n} \quad (n>0), \quad \theta_1 = 1. \tag{42}$$

*Then $\kappa_n$ and $\theta_n$ are related to $\tau_n$ as*

$$\kappa_n = \frac{\tau_n}{\psi} = \frac{\tau_n}{\tau_{-1}} \quad (n<0), \tag{43}$$

$$\theta_n = \frac{\tau_n}{\varphi} = \frac{\tau_n}{\tau_1} \quad (n>0). \tag{44}$$

To prove Proposition 3.4, we first derive recurrence relations that characterize $d_n$ and $f_n$. By substituting equations (39) and (40) into the Riccati equations (12) and (13), respectively, we easily obtain the following lemma:

***Lemma 3.5***

(1) *$d_0$ and $d_1$ are given by $d_0 = -\psi$ and $d_1 = \psi'$, respectively. For $n \geqslant 2$, $d_n$ are characterized by the recursion relation,*

$$d_n = d'_{n-1} + \frac{1}{\psi}\sum_{k=2}^{n-2} d_k d_{n-k}, \quad d_2 = \frac{\psi''\psi - (\psi')^2 + \varphi\psi^3}{\psi}. \tag{45}$$

(2) *$f_0$ and $f_1$ are given by $d_0 = -\varphi$ and $d_1 = \varphi'$, respectively. For $n \geqslant 2$, $f_n$ are characterized by the recursion relation,*

$$f_n = f'_{n-1} + \frac{1}{\varphi}\sum_{k=2}^{n-2} f_k f_{n-k}, \quad f_2 = \frac{\varphi''\varphi - (\varphi')^2 + \varphi^3\psi}{\varphi}. \tag{46}$$

*Proof of Proposition 3.4*. — Consider the Toda equations (5) and (6). Let us put

$$\widetilde{\tau}_n = \frac{\sigma_n}{\sigma_{-1}} = \frac{\tau_n}{\tau_{-1}} \tag{47}$$

s o that $\widetilde{\tau}_{-1} = 1$. Then it is easy to derive the Toda equation for $\widetilde{\tau}_n$:

$$\widetilde{\tau}''_n\widetilde{\tau}_n - (\widetilde{\tau}'_n)^2 = \widetilde{\tau}_{n+1}\widetilde{\tau}_{n-1} - \frac{\psi''\psi - (\psi')^2 + \varphi\psi^3}{\psi^2}\widetilde{\tau}_n^2, \tag{48}$$

$$\widetilde{\tau}_{-2} = \frac{\psi''\psi - (\psi')^2 + \varphi\psi^3}{\psi}, \quad \widetilde{\tau}_{-1} = 1, \quad \widetilde{\tau}_0 = \frac{1}{\psi}. \tag{49}$$

We have the determinant formula for $\widetilde{\tau}_n$ as,

$$\widetilde{\tau}_n = \begin{cases} \det(\widetilde{a}_{i+j-2})_{i,j\leq qn+1} & n>0, \\ 1, & n=0, \\ \det(\widetilde{b}_{i+j-2})_{i,j\leqslant |n|-1} & n<0, \end{cases} \tag{50}$$

$$\widetilde{a}_n = \widetilde{a}'_{n-1} + \frac{\psi''\psi - (\psi')^2 + \varphi\psi^3}{\psi} \sum_{\substack{i+j=n-2\\ i,j\geqslant 0}} \widetilde{a}_i\widetilde{a}_j, \quad \widetilde{a}_0 = \frac{1}{\psi}, \tag{51}$$

$$\widetilde{b}_n = \widetilde{b}'_{n-1} + \frac{1}{\psi} \sum_{\substack{i+j=n-2\\ i,j\geqslant 0}} \widetilde{b}_i\widetilde{b}_j, \quad \widetilde{b}_0 = \frac{\psi''\psi - (\psi')^2 + \varphi\psi^3}{\psi}. \tag{52}$$

Now it is obvious from Lemma 3.5 that

$$d_j = \widetilde{b}_{j-2} \quad (j \geqslant 2), \quad \kappa_n = \widetilde{\tau}_n \quad (n<0), \tag{53}$$

which proves equation (41). Equation (42) can be proved in similar manner. □

We remark that the mysterious relations among the $\tau$ functions and the solutions of isomonodromic problem in Proposition 3.3 and 3.4 should eventually originate from the symmetry of $\mathrm{P_{II}}$, but their meaning is not sufficiently understood yet.

## 4. Summability of the generating function

To study the growth as $n\to\infty$ of the coefficients $a_n(x)$ (or $b_n(x)$) in (11) we use a theorem proved in [**5**], based on a result by Ramis [**16**].

***Theorem 4.1***. — *Consider the following nonlinear differential equation in the variable $s$*

$$s^{k+1}\frac{\mathrm{d}H}{\mathrm{d}s} = c(s)H + s\,b(s,H), \tag{54}$$

*where $k$ is a positive integer, $c(s)$ is holomorphic in the neighbourhood of $s=0$ and $c(0)\neq 0$, and $b(s,H)$ is holomorphic in the neighbourhood of $(s,H)=(0,0)$. Then equation (54) admits one and only one formal solution $H_f(s)$ of the form $H_f(s) = \sum_{n=1}^{\infty} a_n s^n$. Moreover $H_f$ is $k$-summable in any direction $\arg(s)=\vartheta$ except a finite number of values $\vartheta$. Furthermore the sum of $H_f(s)$ in the direction $\arg(s)=\vartheta$ is a solution of equation (54).*

Equation (18) can be put into the form (54) by changing the variable $t=1/s$ and taking $H = F - a_0$. We obtain equation (54) with $k=3$ and

$$\begin{aligned} c(s) &= \frac{1}{2}\Big(1 - \frac{\psi''}{\psi}s^2 - 2s^3\Big), \\ b(s,H) &= -\frac{1}{2}\Big(\varphi\,(\psi''/\psi\,s + 2s^2) \\ &\qquad + \varphi' + s(\psi - \psi'\,s)\varphi^2 + 2s\,\varphi(\psi-\psi'\,s)H + s(\psi-\psi'\,s)H^2.\Big) \end{aligned}$$

Applying theorem 4.1, we obtain that equation (18) admits one and only one formal solution $F_\infty(t)$ of the form $F_\infty(t) = \sum_{n=0}^{\infty} a_n t^{-n}$. This formal solution is 3-summable in any direction $\arg(t) = \vartheta$ except a finite number of values $\vartheta$ and its sum in the direction $\arg(s) = \vartheta$ is a solution of equation (18). The definition of $k$-summability implies that $F_\infty(t)$ is Gevrey of order 3, namely, for each $x$ there exist positive numbers $C(x)$ and $K(x)$ such that

$$|a_n(x)| < C(x)(n!)^{1/3} K(x)^n, \qquad \text{for all } n \geqslant 1.$$

Clearly, one can prove a similar result for the coefficients $d_n$ of the second formal solution $F^{(2)}$ of equation (18). One has to apply theorem 4.1 to a new series $H$ defined as $H(s) = s^2 F^{(2)} - d_0$.

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# INSTABILITY OF RESONANT TOTALLY ELLIPTIC POINTS OF SYMPLECTIC MAPS IN DIMENSION 4

*by*

Vadim Kaloshin, John N. Mather & Enrico Valdinoci

***Abstract.* —** A well known Moser stability theorem states that a generic elliptic fixed point of an area-preserving mapping is Lyapunov stable. We investigate the question of Lyapunov stability for 4-dimensional resonant totally elliptic fixed points of symplectic maps. We show that generically a convex, resonant, totally elliptic point of a symplectic map is Lyapunov unstable. The proof heavily relies on a proof of J. Mather of existence of Arnold diffusion for convex Hamiltonians in 2.5 degrees of freedom. The latter proof is announced in [**Ma5**], but still unpublished.

***Résumé* (Instabilité des points totalement elliptiques résonnants d'applications symplectiques en dimension 4)**

Un théorème célèbre de Moser établit la stabilité au sens de Lyapounov des points fixes elliptiques génériques des applications qui conservent l'aire. On étudie la stabilité de Lyapounov des points fixes totalement elliptiques résonnants d'applications symplectiques en dimension 4. On montre que, génériquement, un point totalement elliptique résonnant convexe d'une application symplectique est instable au sens de Lyapounov. La démonstration s'appuie de façon essentielle sur celle donnée par J. Mather pour l'existence d'une diffusion d'Arnold pour les hamiltoniens convexes à 2,5 degrés de liberté. Celle-ci, annoncée dans [**Ma5**], n'est pas encore publiée.

## 1. Introduction

J. Moser investigated the smooth area-preserving diffeomorphisms $f$ of the plane with elliptic fixed points. He showed [**Mo**] (see also [**LM**] for a simple proof) that, if the linearization $df(p_0)$ of $f$ at a fixed point $p_0$ has eigenvalues $\exp(\pm 2\pi i\omega)$, which is

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not a small root of unity, then generically $p_0$ is Lyapunov stable(1). An application of such result is the stability of the planar restricted three body problem (see *e.g.* [**MH**]).

Let $\mathbb{R}^{2n}$ be the Euclidean space $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$ and $\Omega$ be the standard bilinear skew-symmetric 2-form $\Omega = \sum_{i=1}^n dx_i \wedge dy_i$. A $C^s$ smooth map $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called *symplectic* if it preserves $\omega$, *i.e.* $f^*\Omega = \Omega$. Let $f(0) = 0$ be a fixed point. We say that a fixed point is *totally elliptic* if all the eigenvalues of the linearization $df(0)$ are pairwise complex conjugate, non-real, and of absolute value one, *i.e.* $\exp(\pm 2\pi i \omega_j)$, $2\omega_j \notin \mathbb{Z}$, $j = 1, \ldots, n$. A fixed point 0 is called *Lyapunov stable* if for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that if $|p - 0| < \delta$, then $|f^n p - 0| < \varepsilon$ for all $n$.

In the multidimensional case (*i.e.* $n > 1$), totally elliptic periodic points are the only possible stable periodic points. Indeed, since $df(0)$ preserves the skew-symmetric form $\omega$ if one of eigenvalues $\lambda$ of $df(0)$ is not 1 in absolute value, then $\lambda^{-1}$ is also an eigenvalue. So either $\lambda$ or $\lambda^{-1}$ in absolute value exceeds 1, say $\lambda$. The approximation of the dynamics by linearization shows that $p_0$ is unstable along the eigenspace corresponding to $\lambda$.

R. Douady [**Dou**] proved that the stability or instability property of a totally elliptic point is *a flat phenomenon* for $C^\infty$ mappings. Namely, if a $C^\infty$ symplectic mapping $f_0$ satisfies certain non–degeneracy hypotheses, then there are two mapping $f$ and $g$ such that

– $f_0 - f$ and $f_0 - g$ are flat mappings at the origin and
– the origin is Lyapunov unstable for $f$ and Lyapunov stable for $g$.

In the present paper we begin an investigation of totally elliptic fixed points in dimension 4. Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be a $C^s$ smooth symplectic mapping with a fixed totally elliptic point at the origin $10 \leqslant s \leqslant \infty$. Denote the eigenvalues of $df(0)$ by $\exp(\pm 2\pi i \omega_j^0)$, $j = 1, 2$. We assume that:

(H1, *resonance*) Let $\omega^0 = (\omega_1^0, \omega_2^0)$ have a resonance of order at least 10, *i.e.* for any $k = (k_0, k_1, k_2) \in \mathbb{Z}^3$, $(k_1, k_2) \neq 0$ such that $k_0 + k_1\omega_1^0 + k_2\omega_2^0 = 0$ we have $|k_1| + |k_2| > 9$ and there is at least one $k$ with this property. Denote

$$k_{\omega^0} = \min\{|k_1| + |k_2| : \ k_0 + k_1\omega_1^0 + k_2\omega_2^0 = 0\} \quad \text{and} \quad d_{\omega^0} = \frac{1}{2}\min\{k_{\omega^0}, s\}.$$

In particular, (H1) does not exclude possibility of rational $\omega^0 = (p_1/q, p_2/q)$ with $q > p_1$, $p_2$ and $|p_1| + |p_2| > 9$. We shall not consider low order resonances here.

Denote $\Lambda_k \subset \mathbb{R}^2$ the line of $\omega$'s in the frequency space satisfying this equation. Notice that such line passes through $\omega^0$. As a matter of fact, we shall construct orbits diffusing "along" $\Lambda_k$.

(1)We remark that earlier a weaker result was obtained by V. Arnold [**Ar1**]. Lyapunov stability will be defined below.

Let $(x_1, x_2, y_1, y_2) \in \mathbb{R}^4$ be Euclidean coordinates. Let us introduce "canonical polar coordinates":

$$x_j = \sqrt{2r_j} \cos 2\pi\theta_j, \quad y_j = \sqrt{2r_j} \sin 2\pi\theta_j, \quad j = 1, 2,$$

where $\theta_j$ is determined modulo 1 or simply $\theta_j \in \mathbb{T}$ and $r_j \geqslant 0$. Denote $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}$ the positive semi-axis. To avoid degeneracy of transformation to polar coordinates, it is convenient to introduce cones. For any $0 < \alpha < 1$, $0 < \rho$ denote

$$K_\rho^\alpha = \{(r_1, r_2) : \ 0 < \alpha r_1 < r_2 < \rho, \ 0 < \alpha r_2 < r_1 < \rho\}.$$

In the interior of $K_\rho^\alpha$, the transformation from Euclidean to polar coordinates is non-degenerate. Denote by

$$\mathcal{K}_\rho^\alpha = \{(\theta_1, \theta_2, r_1, r_2) \in \mathbb{T}^2 \times \mathbb{R}_+^2 : (r_1, r_2) \in K_\rho^\alpha\}$$

the cone part of the $\rho$-neighborhood of the origin. Its complement contains neighborhood of the planes $\{r_j = 0\}_{j=1,2}$, where polar coordinates are degenerate.

Suppose we have a totally elliptic point at $r = 0$ satisfying (H1). Birkhoff Normal Form (BNF), *e.g.* [**Ar2**], App. 7A or [**Dou**], states that for small $r$ the map $f(\theta_1, r_1, \theta_2, r_2) = (\Theta_1, R_1, \Theta_2, R_2)$ can be written in the form:

$$\begin{aligned} &\begin{pmatrix} \Theta_j \\ R_j \end{pmatrix} \longmapsto \begin{pmatrix} \theta_j + \omega_j^0 + B_j r + \dfrac{\partial P(r)}{\partial r_j} & \pmod 1 \\ r_j & \end{pmatrix} + \mathrm{Rem}(\theta, r), \\ &B = \{B_j\}_{j=1,2} = \{\beta_{ij}\}_{i,j \leqslant 2}, \end{aligned} \tag{1}$$

where $B : \mathbb{R}^2 \to \mathbb{R}^2$ is a symmetric matrix, $P(r)$ is a polynomial in $(r_1, r_2)$ having zero of order at least 3 at the origin $(r_1, r_2) = 0$. The remainder term $\mathrm{Rem} : U \to \mathbb{R}^2$ is naturally defined near the origin $0 \in U \subset \mathbb{R}^2$ and is $C^s$ smooth away from $\{r_j = 0\}_{j=1,2}$. Since condition (H1) rules out resonances of order up to $k_{\omega^0} - 1$, the smallest term in Taylor expansion of $\mathrm{Rem}(\theta, r)$ at the origin is of order at least $2d_{\omega^0} \geqslant 10$ in $(x_1, y_1, x_2, y_2)$. It implies that inside $\mathcal{K}_\rho^\alpha$ all partial derivative with respect to $(r_1, r_2)$ of $\mathrm{Rem}(\theta, r)$ of order $d_{\omega^0} - 1 \geqslant 4$ (resp. $d_{\omega^0} \geqslant 5$) tend to 0 (resp. stay bounded) as $r \to 0$.

We also make the following assumption:

(H2, *positive torsion*) Let $B$ be symmetric non-degenerate positive definite and let it map degenerate planes $\{r_j = 0\}_{j=1,2}$ transversally to the resonant line $\Lambda_k$, *i.e.* for $j = 1, 2$ the intersection $B\{r_j = 0\} \cap \Lambda_k$ is exactly one point.

Generically $B$ is symmetric non-degenerate and satisfies image condition. However, $B$ is not necessarily positive definite. M. Herman [**Her**] gave an example of Hamiltonian systems and symplectic maps arbitrarily close to integrable, which have elliptic fixed points with $B$ not positive definite. The positive definiteness assumption on $B$ is needed to recover fiber-convexity hypothesis required to apply Mather theory.

Let $\alpha > 0$ be small enough so that the image cone $\omega^0 + B(K_1^{4\alpha})$ contains an nonempty interval of $\Lambda_k$ around $\omega^0$:

$$\big(\Lambda_k \cap \big\{\omega^0 + B(K_1^{4\alpha})\big\}\big) \smallsetminus \omega^0 \neq \varnothing. \tag{2}$$

We shall restrict consideration of the remainder terms of BNF to the cone $\mathcal{K}_\rho^\alpha$ for small $\rho > 0$.

**1.1. Genericity of totally elliptic points and the main result.** — Here we shall formalize the notion of a generic totally elliptic point. Let $\widehat{K}_\rho^\alpha = K_\rho^\alpha \cup \{0\}$ and $\widehat{\mathcal{K}}_\rho^\alpha = \mathbb{T}^2 \times \widehat{K}_\rho^\alpha$. We denote $C^{s,d}(\widehat{\mathcal{K}}_\rho^\alpha)$— the space of $C^s$ functions with the natural $C^s$-topology having all partial derivative of order $d$ bounded and of order $(d-1)$ tending to 0 as $r$ tends to 0 inside $\mathcal{K}_\rho^\alpha$ and $(\theta, r) \in \mathbb{T}^2 \times \widehat{K}_\rho^\alpha = \widehat{\mathcal{K}}_\rho^\alpha$ coordinates.

Fix $\omega^0$ satisfying (H1) and let $d = \frac{1}{2}\{k_{\omega^0}, s\}$. Consider the canonical polar coordinates. Denote

$$\begin{gathered}\left(\Big(\frac{\partial P(r)}{\partial r_1}, \frac{\partial P(r)}{\partial r_1}\Big) \pmod 1, r_1, r_2\right) + \mathrm{Rem}(\theta, r) = \mathcal{R}(\theta, R) \in \mathbb{T}^2 \times \mathbb{R}_+^2 \\ \mathcal{R}(\theta, r) = (\Theta_1(\theta, r), \Theta_2(\theta, r), R_1(\theta, r), R_1(\theta, r)) \in \mathbb{T}^2 \times \mathbb{R}_+^2.\end{gathered} \tag{3}$$

Denote the space of remainder terms $\mathrm{Rem}(\theta, r)$ in BNF (1) defined on $\mathcal{K}_\rho^\alpha$ for some small $\rho > 0$ by $\mathcal{R}_{\alpha,\rho}^s$. In a view of discussion after BNF (1) we have that $\mathcal{R}_{\alpha,\rho}^s \subset C^{s,d}(\widehat{K}_\rho^\alpha)$. With the above notations BNF (1) becomes

$$\begin{pmatrix}\Theta_j \\ R_j\end{pmatrix} \longmapsto \begin{pmatrix}\theta_j + \omega_j^0 + B_j r \\ r_j\end{pmatrix} + \mathcal{R}(\theta, r). \tag{4}$$

Let $s$ be a positive integer. Let $M$ be one of $U$, $\mathbb{T}^2 \times U$, $\mathbb{T}^2 \times U \times \mathbb{T}$, or $\mathcal{K}_\rho^\alpha$. If $f$ is a $C^s$ real valued function on $M$, the $C^s$*–norm* $\|f\|_s$ of $f$

$$\|f\|_s = \sup_{m \in M, |\alpha| \leqslant s} \|\partial^\alpha f(m)\|,$$

where the supremum is taken over the absolute values of all partial derivatives $\partial^\alpha$ of $f$ order $\leqslant s$. The Banach space of $C^s$ real valued functions on $M$ with the $C^s$–norm is denoted $C^s(M)$. The topology associated to the $C^s$–norm is called the $C^s$*–topology.*

Consider the space of remainders $\mathcal{R}_{\alpha,\rho}^s$. We endow it with the strong $C^s$–topology on the space of functions on a non-compact manifold or the $C^s$ Whitney topology. A base for this topology consists of sets of the following type. Let $\Phi = \{\varphi_i, U_i\}_{i \in \Lambda}$ be a locally finite set of charts on $\mathbb{T}^2 \times K_\rho^\alpha$, where $K_\rho^\alpha$ is the open cone. Let $K = \{K_i\}_{i \in \Lambda}$ be a family of compact subsets of $\mathbb{T}^2 \times K_\rho^\alpha$, $K_i \subset U_i$. Let also $\varepsilon = \{\varepsilon_i\}_{i \in \Lambda}$ be a family of positive numbers. A strong basic neighborhood $\mathcal{N}^s(f, \Phi, K, \varepsilon)$ is given by

$$\|(f\varphi_i)(x) - (g\varphi_i)(x)\|_s \leqslant \varepsilon_i,$$

The strong topology has all possible sets of this form.

The set of $C^\infty$ (*i.e.* infinitely differentiable) real valued functions on $M$ is denoted $C^\infty(M)$. The $C^\infty$*–topology* on $C^\infty(M)(= \cap_s C^s(M))$ is the topology generated by the union of $C^s$–topologies, and it may be also described as the projective (or inverse) limit of the $C^s$–topologies.

***Definition 1.1***. — We say that a totally elliptic point satisfying (H1-H2) is of *generic type* if the remainder $\mathcal{R}(\theta, r)$ belongs to a set $C^s$ Whitney open dense set in $\mathcal{R}^s_{\alpha,\rho}$.

The main result, announced in this paper, is the following:

***Theorem 1.2***. — *Suppose hypotheses (H1-H2) hold true, $\alpha > 0$ and satisfies (2), $\rho > 0$ is small, $10 \leqslant s \leqslant \infty$, $\mathcal{R}$ is a remainder term of $f$ given by (3-4). Then, for $\mathcal{R}(\theta, r) \in \mathcal{R}^s_{\alpha,\rho}$ of generic type, the elliptic fixed point $0$ is Lyapunov unstable. Moreover, there is $0 < 4\delta = 4\delta\,(\alpha, \{P_j, Q_j\}_{j=1,2}) < \rho$ such that there is a pair of points $(\theta^{\pm}, r^{\pm})$ with $|r^{\pm}| > \delta$ and $f^n(\theta^{\pm}, r^{\pm}) \to 0$ as $n \to \pm\infty$, respectively, and trajectories $\{f^n(\theta^{\pm}, r^{\pm})\}_{n\in\mathbb{Z}_{\pm}}$ belong to $\mathcal{K}^{2\alpha}_{2\delta}$.*

***Remark 1.3***. — As a matter of fact, in Theorem 8.1 below we shall give further details about diffusing trajectories $\{f^n(\theta^{\pm}, r^{\pm})\}_{n\in\mathbb{Z}_{\pm}}$. An important point is that these trajectories diffuse along the resonant segment $\Lambda_k$ (see (H1)) and, therefore, belong to $\mathcal{K}^{2\alpha}_{2\delta}$ avoiding degenerate planes $\{r_j = 0\}_{j=1,2}$.

***Remark 1.4***. — The above result can be viewed as a counterexample to a 4-dimensional counterpart to Moser stability theorem under hypotheses (H1-H2) of a resonance between eigenvalues.

***Remark 1.5***. — As the reader will see, the proof essentially relies on Mather's proof of existence of Arnold diffusion for a cusp residual set of nearly integrable convex Hamiltonian systems in 2.5 degrees of freedom [**Ma5, Ma4**]. The latter proof is highly involved, long, and extremely complicated. Since it is still unpublished, we do not find it possible to describe it here in full details. This is the main reason why this paper is an announcement of Theorem 1.2. Below we just extract necessary intermediate results from Mather's proof. The application to our result is carried out in Section 9.

***Remark 1.6***. — We hope to get rid of resonant hypothesis (H1) in future work. However, positive torsion (H2) is crucial to apply variational methods and Mather theory.

Assumptions of high differentiability $s \geqslant 10$ and absence of low order resonances $k_{\omega^0} \geqslant 10$ are required to extract sufficient differentiability of the remainder term $\mathcal{R}(\theta, r)$ with respect to $r$ at $r = 0$ in "canonical polar coordinates" inside a cone $\mathcal{K}^{\alpha}_{\rho}$. More presicely, $\mathcal{R} \in C^{s,d}(\mathcal{K}^{\alpha}_{\rho})$ for $d \geqslant 5$. See representation of the remainder in the form (11).

The proof is organized as follows. In Section 2, we suspend a symplectic map $f$ satisfying hypothesis (H1-H2) in the small cone $\mathcal{K}^{\alpha}_{\rho}$ near a totally elliptic point 0 to a time periodic fiber-convex Hamiltonian $H_f(\theta, r, t)$ , *i.e.* we construct a Hamiltonian whose time 1 map equals $f$ in $\mathcal{K}^{\alpha}_{2\delta}$. In Section 2.1, we recall how to switch from Hamiltonian equations to Euler-Lagrange equations. Section 3 is devoted to an outline of the proof of Theorem 1.2, *i.e.* the proof of existence of "diffusing" trajectories. In Section 4, we state Mather Diffusion Theorem [**Ma5**] in terms of Lagrangians. An important part of this result is an explicit list of non-degeneracy hypotheses which guarantee existence of diffusion. In Sections 5 – 7 we state these non-degeneracy hypotheses. In Section 8 we restate Mather Diffusion Theorem in terms of existence of a minima for a certain Variational Principle due to Mather [**Ma5**]. Existence of such a minimum corresponds to existence of a "diffusing" trajectory. In Section 9, we verify that for small positive $\delta_0$ and $\{\delta_j = 2^{-j}\delta_0\}_{j\in\mathbb{Z}_+}$ in each annulus $0 < \delta_{j+1} \leqslant |r| \leqslant \delta_j \leqslant \rho \ll 1$ intersected with $\mathcal{K}^{\alpha}_{2\delta}$ the symplectic map (1) (resp. the suspending Hamiltonian $H_f$) is a small perturbation of an integrable map (resp. an integrable Hamiltonian). Therefore, we manage to apply the above mentioned Variational Principle in each of these annuli. In the final Section, we derive the main result (Theorem 1.2) by "gluing" the annuli. Namely, show existence of a minima to the aforementioned Variational Principle and conclude that it corresponds to one of "diffusing" trajectories from Theorem 1.2. Existence of the other trajectory can be proven in the same way. This would complete the proof. For the reader's convenience, this paper is provided with two appendices: in Appendix A we introduce necessary notions and objects of Mather theory, while Appendix B contains proofs of auxiliary lemmas.

Sections 2, 3, 9, and Appendices A & B are written by the first and the third authors. Sections 4-8 are written by the first author based on the graduate class of the second author [**Ma4**].

## 2. Suspension of a symplectic map near totally elliptic points of a time periodic fiber-convex Hamiltonian

Moser [**Mo2**] showed how to suspend a twist map of a cylinder to a time 1 map of a time periodic *fiber-convexity* Hamiltonian, *i.e.* Hessian $\partial^2_{rr}H$ in $r$ is positive definite everywhere. To the best of our knowledge, there is no general extension of this result to higher dimensional case, even locally. We apply the standard method of generating functions to construct a required suspension. Even though the fact we need seems quite standard we could not find an appropriate reference.

The following suspension results are known to the authors. Bialy and Polterovich (see [**Go**], sect. 41, A) proved existence of smooth suspension theorem with fiber-convexity. However, this result makes use of the restrictive assumption that a generating function $S(\theta,\Theta)$ corresponding to $f(\theta,r) = (\Theta,R)$ has to have a symmetric matrix $\partial^2_{\theta,\Theta}S(\theta,\Theta)$. Since such condition is not satisfied in general, we can not apply this result. Kuksin-Poschel [**KP**] proved existence of global analytic suspension, which does not possess fiber-convexity.

We propose here a way to construct a local suspension *keeping fiber-convexity.* Our proof, given in Appendix B, is a modification of the one by Golé [**Go**] (see sect. 41, B).

It is based on the construction of a suitable family of generating functions and on a local analysis of it.

***Lemma 2.1.*** — *Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be a $C^s$ smooth symplectic map with a totally elliptic point $f(0) = 0$ at the origin satisfying hypothesis (H1-H2) of positive definite torsion and $10 \leqslant s \leqslant \infty$. Then for any $0 < \alpha < 1/2$ and a small positive $\rho$ there is a $C^{s+1,d+1}$ smooth Hamiltonian, written in BNF $(\theta, r)$-polar coordinates (1) as*

$$\begin{aligned} H_f(\theta, r, t) &= \omega_1 r_1 + \omega_2 r_2 + \frac{1}{2}\langle Br, r\rangle + P(r) + r_f(\theta, r, t) \\ &= \omega_1 r_1 + \omega_2 r_2 + \frac{1}{2}\langle Br, r\rangle + h_f(\theta, r, t), \end{aligned} \tag{5}$$

*where $P(r)$ is a polynomial in $r$ having zero of order at least $3$ at $r = 0$ and $r_f(\theta, r, t)$ belongs $C^{s+1,d+1}(\mathcal{K}^{\alpha}_{2\rho} \times \mathbb{T})$, periodic in $t$ of period $1$ and such that the time $1$ map of Hamiltonian flow of $H$ equals $f$ in the cone $\mathcal{K}^{2\alpha}_{\rho}$.*

***Remark 2.2.*** — By definition of $H_f$, we see that $\partial^2_{rr} H_f$ is positive definite for small $r \in K^{\alpha}_{\rho}$ and all $(\theta, t) \in \mathbb{T}^2 \times \mathbb{T}$. Notice that one can not deduce that $H_f$ is positive definite in a full $\rho$-neighborhood of zero, since polar coordinates are degenerate along the planes $\{r_j = 0\}_{j=1,2}$ and the origin. This hides the degeneracy of positive definiteness. This is the reason why we restrict our suspension to the cone $\mathcal{K}^{2\alpha}_{\rho}$, which does not contain those planes.

The proof of this lemma is in Appendix B.

**2.1. Hamiltonian and Euler-Lagrange flows are conjugate.** — In this Section, which may be skipped by an expert, we exhibit the standard duality between Hamiltonian and Lagrangian systems given by the *Legendre* transform. More explicitly, we state that if a Hamiltonian $H$ satisfies certain conditions, then there is a Lagrangian $L$ such that the Hamilton flow of $H$ corresponds to the Euler-Lagrange flow of $L$ after a coordinate change (see *e.g.* [**Ar2**]). Because of this construction, after this Section, we may consider only Euler-Lagrangian flows.

We shall denote $(\theta, v) \in \mathbf{T}\mathbb{T}^n \simeq \mathbb{T}^n \times \mathbb{R}^n$. The Legendre transform associates to a Hamiltonian $H(\theta, r, t)$, $H : \mathbf{T}^*\mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$, which is assumed to be positive-definite in $r$, a Lagrangian $L(\theta, v, t)$, $L : \mathbf{T}\mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$, which is positive-definite in $v$, according to the following scheme:

$$\begin{gathered} L(\theta, v, t) = \sup_{r \in \mathbf{T}^*_{\theta}\mathbb{T}^n} \{\langle r, v\rangle - H(\theta, r, t)\}, \\ \text{where } \langle\ ,\ \rangle : \mathbf{T}\mathbb{T}^n \times \mathbf{T}^*\mathbb{T}^n \longrightarrow \mathbb{R} \quad \text{is pairing between dual spaces.} \end{gathered} \tag{6}$$

When $L$ is related to $H$ as above, we say that $L$ is the *dual* of $H$. Let us consider the *Euler-Lagrange* flow associated to $L$. The latter is defined as a flow on the extended phase space $\mathbf{T}\mathbb{T}^n \times \mathbb{T}$ such that its trajectories $(\theta(t), \dot\theta(t), t) = (d\theta(t), t)$ are solutions of the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \tag{7}$$

The trajectories of the Euler-Lagrange flow can be also characterized as those which minimize locally the action $\int L(d\gamma(t), t)dt$ among absolutely continuous curves with the same boundary conditions. The standard (but crucial for our purposes) observation is that when we subtract a *closed* 1-form $\eta$, defined on $\mathbb{T}^n \times \mathbb{T}$, from the Lagrangian $L$, then both $L - \eta$ and $L$ have *the same* Euler-Lagrange equations (see *e.g.* [**Fa**]).

Let us suppose the Hamiltonian $H(\theta, r, t)$ satisfies the following properties:

(1) *Positive definiteness in $r$:* For each $(\theta, r) \in \mathbf{T}^*\mathbb{T}^n$ and $t \in \mathbb{T}$ the restriction of $H$ to $\mathbf{T}^*_\theta\mathbb{T}^n \times \{t\}$ is positive definite;

(2) *Super-linear Growth in $r$:* For each $(\theta, r) \in \mathbf{T}^*\mathbb{T}^n$ and $t \in \mathbb{T}$

$$\frac{H(\theta, r, t)}{\|r\|} \longrightarrow +\infty \quad \text{as} \quad \|r\| \longrightarrow +\infty$$

(3) *Completeness:* All the solutions of the Hamiltonian equations can be extended for all $t \in \mathbb{R}$.

We need these conditions to be satisfied in order to apply Mather theory (see Appendix A and Section 9). Notice that the Hamiltonian $H$ of the form (5) satisfies all these properties near $r = 0$. The standard result says:

***Lemma 2.3* (see *e.g.* [Ar2], § 15)**. — *If a $C^{s+1}$ Hamiltonian $H(\theta, r, t)$ satisfies the above conditions (1–3) with $s \geqslant 1$ and $L(\theta, v, t)$ is the dual of $H$, then the map $\mathcal{L} : (\theta, r, t) \to (\theta, v, t)$, given by*

$$\mathcal{L}(\theta, r, t) = (\theta, \partial_r H(\theta, r, t), t), \tag{8}$$

*is $C^s$-smooth and invertible, and it conjugates the Hamiltonian flow of $H$ to the Euler-Lagrange flow of $L$, i.e. it provides a one-to-one correspondence between trajectories of both flows. Moreover, the Lagrangian $L$ satisfies properties (1–3) above with $r$ replaced by $v$ and $H$ by $L$.*

Let $H_f$ be given by formula (5). Let $H_f^*$ be the integrable part $H_f^*(r) = \omega_1 r_1 + \omega_2 r_2 + \frac{1}{2}\langle Br, r\rangle + P(r)$. Namely, $P(r)$ is the polynomial in $r$ part of $h_f(\theta, r)$. Notice that Legendre transform of $H_f$ has the form

$$L_f(\theta, v, t) = \ell_0(v - \omega^0) + P_f(\theta, v - \omega^0, t), \tag{9}$$

where $\ell_0(v - \omega^0)$ is the Legendre transform of $H_f^*(r)$ and $P_f$ is a $C^s$ smooth remainder defined on $\mathbb{T}^2 \times \{\partial_r H_f(\theta, K^{2\alpha}_{2\rho}, t) - \omega^0\} \times \mathbb{T}$. The form (5) of $H_f$ shows that $\partial_r H_f(\theta, r, t) = \omega^0 + Br + \partial_r h_f(\theta, r, t)$. Therefore, for small $\rho$ we have that $B(K^\alpha_\rho) \subset \{\partial_r H_f(\theta, K^{2\alpha}_{2\rho}, t) - \omega^0\}$ so we could assume that $L_f$ is well-defined on $B(K^\alpha_\rho)$. Moreover, $P_f$ has a zero in $(v - \omega^0)$ of at least 6-th order, *i.e.* $P_f \in C^{s,d+1}(\mathbb{T}^2 \times B(K^\alpha_\rho) \times \mathbb{T})$

for $d \geqslant 5$. We shall apply Mather's technology to the Lagrangian $L_f$ and its Euler-Lagrange flow.

## 3. Scheme of construction of diffusing trajectories using Mather action functional

In this Section, we outline a variational approach due to Mather [**Ma5, Ma4, Ma3**] to construct diffusing trajectories toward and outward from zero from Theorem 1.2. We concentrate on the one going toward zero first. Construction of the other one is very similar.

**3.1. Rough sketch of the proof of Theorem 1.2.** — Application of lemma 2 for $n = 2$ to the symplectic map $f$, given by (1) provides the suspension Hamiltonian $H_f(\theta, r, t)$ given in $\mathcal{K}^\alpha_{2\delta}$ near $r = 0$. We have that locally, *i.e.*, in $\mathcal{K}^\alpha_{2\delta}$, $H_f$ satisfies fiber-convexity from hypothesis (1, Sect. 2.1) above. To meet hypotheses (2–3, Sect. 2.1), one may smoothly extend $H_f$ for large $r$ keeping convexity in $r$ so that it is an integrable Hamiltonian, *e.g.* given by (36). Thus, Mather theory is applicable (see Section 9).

Let $L_f(\theta, v, t)$ be the dual of $H_f(\theta, r, t)$, given by (9). The Legendre coordinate change (8) in our case has the form $\mathcal{L}(\theta, r, t) = (\theta, \omega^0 + Br + \partial_r h_f(\theta, r, t), t)$. Let us approximate it by its linearization $T_{\omega^0,B} : r \to v = \omega^0 + Br$. Denote $K^\alpha_\rho(\omega^0, B) = T_{\omega^0,B}(K^\alpha_\rho)$ the image cone, whose complement we need to avoid. By (H2), for small $\alpha > 0$ the image cone $K^{2\alpha}_\rho(\omega^0, B)$ contains a segment in $\Lambda_k$ of length $2\rho/\|B^{-1}\|$ centered at $\omega^0$. Fix $\alpha = \alpha(k, B) > 0$ with the above property. We shall "diffuse" inside $K^\alpha_\rho(\omega^0, B)$. Denote by $e_k$ the unit vector parallel to $\Lambda_k$ and fix a small $0 < \delta < \rho/\|B^{-1}\|$ (to be determined later).

Put $2\delta = \delta_0$ and $\delta_j = 2^{-j}\delta_0$ for each $j \in \mathbb{Z}_+$. Fix the sequence of annuli

$$A_j(\omega^0) = \left\{2^{-2}\delta_j < |v - \omega^0| < 2^2\delta_j\right\} \subset \mathbb{R}^2_+. \tag{10}$$

Denote $K^\alpha_{\delta,j}(\omega^0, B) = K^\alpha_{2\delta}(\omega^0, B) \cap A_j(\omega^0)$. By definition for each $j \in \mathbb{Z}_+$ we have $\omega^0 + \delta_{j+1}e_k$, $\omega^0 + \delta_j e_k$, $\omega^0 + \delta_{j-1}e_k \in K^\alpha_{\delta,j}(\omega^0, B)$ and adjacent annuli $A_j(\omega^0)$ and $A_{j+1}(\omega^0)$ overlap, *i.e.* $A_j(\omega^0) \cap A_{j+1}(\omega^0) \neq \varnothing$. We now point out the aim of our constructions and the steps needed to reach it:

*The goal.* — Construct a diffusing trajectory $\{(\theta, \dot\theta)(t)\}_{t\geqslant 0}$ such that at time 0 its velocity $\dot\theta(t)$ is approximately $\omega^0 + \delta_0 e_k$;

*Stage 1.* — At a time $\mathcal{T}_1 > 0$, its velocity $\dot\theta(\mathcal{T}_1)$ is approximately $\omega^0 + \delta_1 e_k$ and, in between 0 and $\mathcal{T}_1$, we have $\dot\theta(t) \in K^\alpha_{\delta,1}(\omega^0, B)$;

*Stage 2.* — At a time $\mathcal{T}_2 > \mathcal{T}_1$, its velocity $\dot\theta(\mathcal{T}_1)$ is approximately $\omega^0 + \delta_2 e_k$ and, in between $\mathcal{T}_1$ and $\mathcal{T}_2$, we have $\dot\theta(t) \in K^\alpha_{\delta,2}(\omega^0, B)$ and so on;

*Stage $j$.* — At a time $\mathcal{T}_j > \mathcal{T}_{j-1}$, its velocity $\dot\theta(\mathcal{T}_1)$ is approximately $\omega^0 + \delta_j e_k$ and, in between $\mathcal{T}_{j-1}$ and $\mathcal{T}_j$, we have $\dot\theta(t) \in K^\alpha_{\delta,j}(\omega^0, B)$, and so on for possibly infinite number of stages.

If we could construct a trajectory with these properties, we would obtain a trajectory for the symplectic map $f$ which goes toward the origin. To construct a trajectory going out from the origin, the arguments involved are analogous. This would indeed prove Theorem 1.2. Formalization of this scheme requires some notions of Mather theory.

### 3.2. A naive idea of Mather's mechanism of diffusion. — We start with

*A Model Example.* — Suppose $f : M \to M$ be a smooth diffeomorphism of an $2n$-dimensional manifold possibly with a boundary. Let $m$ be any positive integer and $p_1, \dots, p_m$ be a collection of hyperbolic periodic points of the same index, *i.e.* the dimensions of stable and unstable manifolds are the same. Suppose that for each $i = 1, \dots, m-1$ the unstable manifold $W^u(p_i)$ intersects the stable manifold $W^s(p_{i+1})$ transversally and both belong to $M$. Then, it is easy to show that $W^s(p_1)$ intersects $W^u(p_m)$.

If $M = \mathbb{T}^2 \times K^\alpha_\rho \ni (\theta, r)$ and $r$-coordinates of $p_i$ are close to $\delta_i e_k$, then there exists a trajectory whose $r$-coordinate change from nearby $\delta_1 e_k$ to nearby $\delta_m e_k$.

As a matter of fact, in Mather's mechanism of diffusion we use the following objects:

– the hyperbolic periodic points $p_i$'s are replaced by Mather sets $\mathcal{M}_i$, whose projection onto $r$-component is localized near $\delta_i e_k$;

– the stable and unstable manifolds $W^s(p_i)$ and $W^u(p_i)$ are replaced by the stable and unstable sets $W^s(\mathcal{M}_i)$ and $W^u(\mathcal{M}_i)$ respectively. $W^s(\mathcal{M}_i)$ and $W^u(\mathcal{M}_i)$ are not necessarily manifolds and not even continuous;

– to verify the intersection of unstable and stable sets $W^s(\mathcal{M}_i)$ and $W^u(\mathcal{M}_{i+1})$, respectively, we shall use the barrier function defined in Section 8 (see formulas (31-32));

– to show that the intersection of $W^s(\mathcal{M}_i)$ and $W^u(\mathcal{M}_{i+1})$ for each $i = 1, \dots, m-1$ implies existence of a connecting trajectory between $\mathcal{M}_1$ and $\mathcal{M}_m$, we define a suitable action functional (24). As it was shown by Mather [**Ma4**], under certain hypotheses, the minimum of such an action functional is achieved on a trajectory of the Euler-Lagrange equation connecting $\mathcal{M}_1$ and $\mathcal{M}_m$ (see Section 8).

In Appendix A, we define Mather sets, barrier functions, and related objects. The reader who is familiar the basics of Mather theory may go directly to next Section. Others may read Appendix A first.

### 3.3. Detailed scheme of the proof of Theorem 1.2. — We now continue our discussion, assuming that the reader familiar with basic notions of Mather theory (see Section 9). The diffusing trajectories we shall construct move along the resonant

segment $\Lambda_k$ from (H1). Consider a sequence of subsegments of $\Lambda_k$ given by $\Gamma_j = [\delta_{j+1}e_k, \delta_j e_k] \subset \Lambda_k$, $j \in \mathbb{Z}_+$. Denote

$$\Gamma_\delta = \cup_{j\geqslant 1}\Gamma_j.$$

On each of the segments $\Gamma_j$'s we mark a sufficiently dense finite set of points $\{\delta_{j,p}e_k\}_{p=1}^{m_j} \subset \Gamma_j$ (we determine later how dense this set has to be). Each stage of diffusion described in Section 3.1 consists of $m_j$ sub-stages. First, we enumerate marked points in $\Gamma_\delta$. We set $M_p = \sum_{i=1}^{p} m_i$ and for $M_p < i \leqslant M_{p+1}$ we set $\omega_i = \omega^0 + \delta_{p,i-M_p}e_k$. Loosely speaking, $\mathcal{M}_i$ as introduced in the previous Section, is an invariant set of trajectories with approximate rotation vector $\omega_i$. We shall formalize this idea in Section 8. To give a precise definition of $\mathcal{M}_i$'s we need further discussion.

Let $H_f(\theta, r, t)$ be the suspension of the symplectic map $f$ under consideration given in $\mathcal{K}_\rho^\alpha$ near $r = 0$ by lemma 2. We have that locally (*i.e.*, near $r = 0$) $H_f$ satisfies convexity condition (1, Sect. 2.1). To meet hypotheses (2–3, Sect. 2.1), one extends $H_f$ for large $r$ by an integrable Hamiltonian, *e.g.* given by (36) keeping convexity in $r$. Thus, Mather theory is applicable to our case (see Section 9).

Consider the Legendre transform of $H_f$, which would lead to a Lagrangian of the form (9). The first term of $L_f$ corresponds to an integrable Lagrangian. Moreover, we show in Section 9 that for small $\delta_0 > 0$ the second term $P_f$ can be considered as a small perturbation. Therefore, we shall be able to apply Mather Diffusion Theorem (see Section 4). We shall write the remainder in the form (34) with $m = 3$ and $r$ replaced by $(v - \omega^0)$

$$P_f(\theta, v-\omega^0, t) = \sum_{p=0}^{3}(v-\omega^0)_1^p(v-\omega^0)_2^{3-p}P_p(\theta, v-\omega^0, t), \tag{11}$$

where $(v - \omega^0)_i$ is $i$-th coordinate of $(v - \omega^0)$, $i = 1, 2$. Let us denote $\mathbf{P}_f = (P_0, P_1, P_2, P_3)$ and define the unit sphere for perturbations $\mathbf{P}_f$

$$S^{s,3} = \Big\{\mathbf{P}_f : \sum_{p=0}^{3}\|P_p\|^2_{C^3(\mathbb{T}^2\times B(K_\rho^{3\alpha})\times\mathbb{T})} = 1\Big\}.$$

Since $P_f \in C^{s,d+1}(\mathbb{T}^2 \times B(K_\rho^{2\alpha}) \times \mathbb{T})$ with $d \geqslant 5$, by lemma B.1 we have that $C^3$ norm of $\mathbf{P}_f$ is well-defined.

We denote also by $\mathcal{L}_{\beta,f}$ the Fenchel-Legendre transform associated with $L_f$ by (29). In the images of each marked frequency, we choose a cohomology class $c_i \in \mathcal{L}_{\beta,f}(\omega_i)$ for each $i = 1, \dots, M_p, \dots$, so that adjacent $c_i$'s are sufficiently close. We are now in position to define the sets $\{\mathcal{M}_i\}_{i=1}^m$ from the previous Section to be Mather sets $\mathcal{M}^{c_i}$. We shall slightly modify the choice of $c_i$'s in Section 8.

***Definition 3.1***. — We say that $\mathcal{L}_{\beta,f}$ has *channel property* with respect to a resonant segment $\Gamma_\delta$ if there is a smooth connected curve $\sigma_\Gamma \subset \mathcal{L}_{\beta,f}(\Gamma_\delta)$ such that for each $\omega' \in \Gamma_\delta$ the curve $\sigma_\Gamma$ intersects $\mathcal{L}_{\beta,f}(\omega')$.

***Lemma 3.2* ([Ma4])**. — *Let $10 \leqslant s \leqslant \infty$. Then, for a $C^s$ Whitney open dense set of $\mathbf{P}_f \in S^{s,3}$, there is $\delta = \delta(\mathbf{P}_f) > 0$ such that the Fenchel-Legendre transform $\mathcal{L}_{\beta,f}$ has channel property with respect to $\Gamma_\delta$. In particular, for any pair of positive integers $i < i'$ the sets $\mathcal{L}_{\beta,f}(\omega_i)$ and $\mathcal{L}_{\beta,f}(\omega_{i'})$ are connected by $\sigma_\Gamma$.*

***Remark 3.3***. — In Section 6.2 we introduce certain non–degeneracy hypothesis (C1)–(C3) and $(\mathrm{C4})_\omega$–$(\mathrm{C8})_\omega$ for perturbations of integrable Lagrangian systems and in Section 9 show how to adapt these hypothesis for remainder terms in BNF (4) of totally elliptic points. The $C^s$ Whitney open dense set of remainders $\mathbf{P}_f \in S^{s,3}$ that satisfy adapted non–degeneracy hypotheses (C1)–(C3) and $(\mathrm{C4})_\omega$–$(\mathrm{C8})_\omega$ fulfills channel property of the lemma.

We construct trajectories that *diffuse along $\sigma_\Gamma$ inside the channel $\mathcal{L}_{\beta,f}(\Gamma_\delta)$*. To accomplish this, roughly speaking, we vary the cohomology $c$ in order to vary the velocity $\dot\theta$.

We shall apply the Mather method of changing Lagrangians [**Ma5**]. Mather applied this method in [**Ma3**] to show the existence of unbounded trajectories for generic time periodic mechanical systems on $\mathbb{T}^2$. We outline some of the key ideas of the method. For simplicity let $L(\theta, v, t) = \frac{1}{2}\langle v, v\rangle + \varepsilon P(\theta, v, t)$ be sufficiently smooth nearly integrable Lagrangian and $\eta^c = c\, d\theta$ be the standard closed one form on $\mathbb{T}^2 \times \mathbb{T}$ for a vector $c \in \mathbb{R}^2 \simeq T_\theta \mathbb{T}^2$, $\theta \in \mathbb{T}^2$. Then the following scheme can be exploited:

(1) Euler-Lagrange flows of $L$ and $(L - \eta^c)$ are the same (see *e.g.* [**Fa**]).

(2) Minimization of $c$-action $\int_a^b (L - \widehat{\eta}^c)(d\gamma(t), t)dt$ with $\varepsilon$-error leads to minimization of

$$\frac{1}{2}\langle \dot\theta, \dot\theta\rangle - \langle \dot\theta, c\rangle = \frac{1}{2}\Big(\langle \dot\theta - c, \dot\theta - c\rangle - \langle c, c\rangle\Big). \tag{12}$$

Therefore, trajectories minimizing $c$-action have approximate velocity $c$. As a matter of fact, even if $\eta^c$ is a closed one form with $[\eta^c]_{\mathbb{T}^2} = c$, $L$ is close to integrable and $b-a$ is large enough, trajectories minimizing $c$-action still have approximate velocity $c$ (see [**Ma4**]). From now on we consider $\eta^c$ as a closed one form.

(3) Suppose we can find an action functional

$$\sum_{i=1}^{j} \int_{t_i}^{t_{i+1}} (L - \widehat{\eta}_i)(d\gamma(t), t)dt \tag{13}$$

for a sequence of closed one forms $\{\eta_i\}_{i=1}^j$ such that $[\eta_i]_{\mathbb{T}^2} = c_i$ and $[\eta_{i+1}]_{\mathbb{T}^2} = c_{i+1}$, where $c_i$ and $c_{i+1}$ are close for each $i = 1, \ldots, j-1$ and the minimum of such integral is achieved on a trajectory $\{(d\gamma(t), t) : t \in [t_1, t_m]\}$ of the Euler-Lagrange flow of $L^{(2)}$. Standard properties of action minimization give that this is indeed true for time $t \neq t_1, \dots, t_m$, but it is a delicate problem to show that this does not happen at connection times $t = t_1, \dots, t_m$. The corresponding minimizing trajectory $\gamma(t)$ might have *corners* $\dot\gamma(t_i^-) \neq \dot\gamma(t_i^+)$. Notice now that at time $t$ in $[t_1, t_2]$ velocity is approximately $c_1$ and at time $t$ in $[t_{m-1}, t_m]$ velocity is approximately $c_m$. Thus, the key to the method is to find an action functional with the above property and justify absence of corners. In (13) we made only a rough attempt. This functional is defined in Section 8. Usually, this construction is quite involved and highly nontrivial [**Ma5, Ma4, Ma3**].

## 4. Mather diffusion theorem

In this Section, we state Mather result about existence of Arnold diffusion in a generality we use for our application. See [**Ma5**] for the most general version. As a matter of fact, to prove our main result (Theorem 1.2) in Section 8) we shall reformulate Mather Diffusion Theorem in terms of a certain variational principle and in Section 9 apply this principle to prove Theorem 1.2.

In the subject of Arnold diffusion, one studies a time–periodic or autonomous Hamiltonians/Lagrangians that are perturbations of integrable Hamiltonians/Lagrangians (see *e.g.* [**AKN**]).

In the time periodic case, the Lagrangian takes the form

$$L(\theta, \dot\theta, t) = \ell_0(\dot\theta) + \varepsilon P(\theta, \dot\theta, t),$$

where $\ell_0$ is a $C^s$ smooth function on a convex closed set with smooth boundary $U \subset \mathbb{R}^2$, $\varepsilon$ is a small positive number, $P$ is a $C^s$ smooth function on $\mathbb{T}^2 \times U \times \mathbb{T}$, and $s \geqslant 3$. In other words, $P$ is periodic of period 1 in $\theta_1, \theta_2$, and $t$. The function $\ell_0$ is called the *unperturbed integrable Lagrangian* and the function $P$ is called the *perturbation term.*

Denote $d^2_{\dot\theta}\ell_0 = \partial^2_{\dot\theta_i\dot\theta_j}\ell_0$ the Hessian matrix of second partial derivatives of $\ell_0$, *i.e.* $d^2_{\dot\theta}\ell_0 = \big(\partial^2_{\dot\theta_i\dot\theta_j}\ell_0\big)_{i,j=1,2}$. We shall assume that $d^2_{\dot\theta}\ell_0$ is everywhere positive definite on $U$, *i.e.* we have $\sum_{i,j=1}^2 \partial^2_{\dot\theta_i\dot\theta_j}\ell_0(\dot\theta)\dot\varphi_i\dot\varphi_j > 0$, for all $\dot\theta \in U$ and all $(\dot\varphi_1, \dot\varphi_2) \in \mathbb{R}^2 \smallsetminus 0$. In the Hamiltonian case, the analogous assumption is that the unperturbed integrable Hamiltonian convex.

Now, we briefly discuss the problem of Arnold diffusion. For the unperturbed integrable Lagrangian $L = \ell_0$, the Euler-Lagrange (E.–L. for short in the sequel) equations reduce to $d^2\theta/dt^2 = 0$. Every solution $\theta$ lies on a torus $\{\dot\theta = \omega\}$, where $\omega = (\omega_1, \omega_2) \in U$. The $\omega_i$'s are called the *frequencies* of the solution.

By a *trajectory* of $L$, we mean a solution of the E.–L. equations associated to $L$. Along a trajectory of $L$, $\dot\theta$ is constant in the case of the unperturbed integrable

(2) Actually, summation could be over infinite number of terms, as formula (25) shows.

system and varies slowly in the case of a small perturbation of the integrable system. The problem of Arnold diffusion is whether $\dot{\theta}$ can vary a lot over long periods of time. Recently a great progress has been achieved in proving Arnold diffusion in so-called *a priori unstable* case by many different groups (see papers [**Be**], [**CY**], [**DLS**], [**T1, T2, T3**], announcements [**X1, X2**], and work in preparation [**KM**]). The result below is for the *a priori stable* case.

Recall that for a positive integer $C^s(M)$ denotes the Banach space of $C^s$ real valued functions on $M$ (see Section 1.1 for notations and definitions). Now let $s$ be $\infty$, or an integer $\geqslant 3$. We let $\mathcal{L}^s$ denote the topological space of $C^s$ functions $\ell_0 : U \to \mathbb{R}$, such that $\ell$ is positive definite in $\dot{\theta}$, *i.e.* the Hessian $d^2_{\dot{\theta}}\ell_0$ is positive definite. Endow $\mathcal{L}^s$ with the $C^s$ topology. We let $\mathcal{P}^s$ denote the topological space of $C^s$ functions $P : \mathbb{T}^2 \times U \times \mathbb{T} \to \mathbb{R}$ endowed with the $C^s$–topology. Denote

$$\mathcal{S}^s_L = \{P \in \mathcal{P}^s : \ \|P\|_{C^3(\mathbb{T}^2\times U\times\mathbb{T})} = 1\}$$

the unit sphere in the space of perturbations. The topology in $\mathcal{S}^s$ is induced from the ambient space $\mathcal{P}^s$.

***Definition 4.1***. — A set $W^s_\delta \subset \mathcal{P}^s$ is called *$\delta$-cusp residual* if

A) there is a non-negative continuous function $\delta$ on $\mathcal{S}^s_L$ such that the set $U^s_\delta = \{P \in \mathcal{S}^s_L : \ \delta(P) > 0\}$ is open and dense in $\mathcal{S}^s_L$;

B) there is a cusp set $V^s_\delta = \{\varepsilon P \in \mathcal{P}^s : \ P \in U^s_\delta,\ 0 < \varepsilon < \delta(P)\}$, which is a subset of homogeneous extension of $U^s_\delta$ which is defined by $\mathbb{R}U^s = \{\lambda P \in \mathcal{P}^s : P \in U^s,\ \lambda > 0\}$;

C) there is an open and dense set $W^s_\delta$ in $V^s_\delta$.

***Definition 4.2***. — If $\Gamma \subset \mathbb{R}^2$ is a line segment, we shall say that it is *rational or resonant* if there is a resonance[(3)] $k = (k_0, k_1, k_2) \in \mathbb{Z}^3$ such that $\Gamma$ is contained in the line $\Lambda_k$.

We say that a curve $\Gamma \subset \mathbb{R}^2$ is a *resonant piecewise linear curve* if it is a finite union of resonant segments $\Gamma = \cup_{s=1}^m \Gamma_s$ so that end points of $\Gamma_s$ belong to end points of $\Gamma_{s-1}$ and $\Gamma_{s+1}$, for all $s = 2, \ldots, m-1$.

The following result is a modified version of the result announced by Mather [**Ma5**] for the time-periodic case:

***Mather Diffusion Theorem***. — *Let $\Gamma$ be a resonant piecewise linear curve in $U$ and let $3 \leqslant s \leqslant \infty$. There exists a non-negative continuous function $\delta(\ell_0, \Gamma) : \mathcal{P}^s \to \mathbb{R}_+$, such that, for any perturbation $\varepsilon P$ in a $\delta(\ell_0, \Gamma)$-cusp residual set $W^s_{\delta(\ell_0,\Gamma)} \subset \mathcal{P}^s$, there is a trajectory $(\theta, \dot{\theta})(t)$ of $L_\varepsilon = \ell_0 + \varepsilon P$, whose velocity moves along $\Gamma$. More precisely, there is a constant $C = C(\ell_0, P, \Gamma) > 0$ and $T = T(\ell_0, P, \Gamma) > 0$ such that* $\operatorname{dist}\left(\cup_{0<t<T} \dot{\theta}(t), \Gamma\right) \leqslant C\sqrt{\varepsilon}$, *where* dist *is the standard Hausdorff distance between sets in $U$.*

(3) Recall that saying that $k$ is a *resonance*, we mean that $k \in \mathbb{Z}^3$ and $(k_1, k_2) \neq 0$.

***Remark 4.3***. — The function $\delta(\ell_0, \Gamma)$ (and consequently the $\delta(\ell_0, \Gamma)$-cusp residual set $W^s_{\delta(\ell_0,\Gamma)}$) depends on the choice of resonant lines $\Gamma_1, \dots, \Gamma_m$. However, they are independent of behavior of the diffusing trajectory $(\theta, \dot\theta)(t)$.

In particular, this implies the following result. Consider a finite collection of non–void open subsets $\Omega_1, \dots, \Omega_{m+1}$ of $U$, then there is a resonant piecewise linear curve $\Gamma$, consisting of $m$ resonant segments $\Gamma = \cup_{s=1}^m \Gamma_s$ connecting distinct $\Omega_k$'s in any pre–assigned order. Then, by Mather Diffusion Theorem there is a trajectory which visits the sets $\Omega_1, \dots, \Omega_{m+1}$ in the pre–assigned order.

We point out that existence of a "diffusing" trajectory $(\theta, \dot\theta)(t)$ moving along any prescribed resonant piecewise linear curve is a strong form of Arnold diffusion. However, existence of such a trajectory is proved only for a $\delta$-cusp residual set of perturbations [**Ma4**].

The purpose of the next four sections is to define qualitatively the function $\delta(\ell_0, \Gamma)$ and the sets $U^s_{\delta(\ell_0,\Gamma)}$ and $W^s_{\delta(\ell_0,\Gamma)}$ mentioned in definition 4.1 and Mather Diffusion Theorem. We start by defining two averaged mechanical systems $L_{\omega,\Lambda}$ and $L_\omega$. For sake of brevity, we shall not say precisely in what sense the trajectories of $L_{\omega,\Lambda}$ and $L_\omega$ approximate certain trajectories of $L$. We also give an heuristic motivation of the notion of these averaged systems $L_{\omega,\Lambda}$ and $L_\omega$. These averaged mechanical systems are used to define a $C^s$ open and dense set $U^s_{\delta(\ell_0,\Gamma)}$ of "good directions" of perturbations on the unit sphere $\mathcal{S}^s$. In Section 7 we define $\delta(\ell_0, \Gamma)$-cusp residual set $W^s_{\delta(\ell_0,\Gamma)}$ using barrier functions. In Section 8 we restate Mather Diffusion Theorem in terms of a certain variational principle. Finally, in Section 9 we apply this variational principle to prove our main result (Theorem 1.2).

## 5. Averaged mechanical systems corresponding to single and double resonances

**5.1. A Single Resonance Averaged System or a First $(\omega, \Lambda)$-Averaged System.** — Let $L(\theta, \dot\theta, t) = \ell_0(\dot\theta) + \varepsilon P(\theta, \dot\theta, t)$ be a $C^s$ small perturbation of an integrable Lagrangian $\ell_0$ on $\mathbb{T}^2 \times U \times \mathbb{T}$, $s \geqslant 3$. Let us assume $d^2\ell_0 > 0$ on $U$. Consider a resonant frequency vector $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$ and its resonance $k = (k_0, k_1, k_2)$. This means that $k \in \mathbb{Z}^3$, $(k_1, k_2) \neq 0$ and $k_0 + k_1\omega_1 + k_2\omega_2 = 0$. If $\omega \in \mathbb{Q}^2$, it admits two linearly independent resonances; otherwise, it admits at most one resonance up to multiplication by scalar.

We denote by $\Lambda = \Lambda_k$ the resonant line from (H1). Thus, $\Lambda$ is the set of all $\omega \in \mathbb{R}^2$ for which $k$ is a resonance. We set

$$\mathbb{T}^2_\Lambda = \left\{(\theta_1, \theta_2, t) \in \mathbb{T}^2 \times \mathbb{T}^1 : \ k_1\theta_1 + k_2\theta_2 + k_0 t = 0 \ (\mathrm{mod}\ 1)\right\}. \tag{14}$$

If $\dot\theta(0) \in \Lambda$, the trajectory of the unperturbed Euler-Lagrange of $\ell_0(\dot\theta)$ either belongs to $\mathbb{T}^2_\Lambda$ or to its parallel translation. Thus, the 2-torus $\mathbb{T}^2_\Lambda$ can be viewed as a subgroup of $\mathbb{T}^2 \times \mathbb{T}^1$. We set $\mathbb{T}^1_\Lambda = \mathbb{T}^2 \times \mathbb{T}^1 / \mathbb{T}^2_\Lambda$ (and we refer to it as the factor space). Since the unperturbed Euler-Lagrange flow is parallel to $\mathbb{T}^2_\Lambda$, we call $\mathbb{T}^2_\Lambda$ — torus of *fast* motion and $\mathbb{T}^1_\Lambda$ — torus of *slow* motion.

Let $(\theta_1, \theta_2, t) = (\varphi^s_\Lambda, \varphi^f_\Lambda) \in \mathbb{T}^1_\Lambda \times \mathbb{T}^2_\Lambda$ denote slow and fast coordinates on $\mathbb{T}^1_\Lambda$ and $\mathbb{T}^2_\Lambda$, respectively. The product decomposition depends on an arbitrary choice. We shall specify our choice later (see lemma 6.3). Denote by $d\mathcal{H}_\Lambda$ the normalized Haar (Lebesgue) measure on the fast torus $\mathbb{T}^2_\Lambda$. Let $\widetilde{P}(\theta, t, \omega) = P(\theta, \omega, t)$. Define *the first* $(\omega, \Lambda)$*-averaged potential*

$$P_{\omega,\Lambda}(\varphi^s_\Lambda) = \int_{\varphi^f_\Lambda} \widetilde{P}(\varphi^s_\Lambda, \varphi^f_\Lambda, \omega) \; d\mathcal{H}_\Lambda(\varphi^f_\Lambda). \tag{15}$$

So $P_{\omega,\Lambda} : \mathbb{T}^1_\Lambda \to \mathbb{R}$ is a real valued function on $\mathbb{T}^1_\Lambda$.

To define the first $(\omega, \Lambda)$-averaged kinetic energy one needs some linear algebra. Actually, the precise form of this kinetic energy is not important for us. What really matters is that the kinetic energy is given by a constant quadratic form on $T(\mathbb{T}^1_\Lambda)$.

Consider the natural projection $\pi_\Lambda : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^1_\Lambda$ along the fast torus $\mathbb{T}^2_\Lambda$. The definition of both slow and fast tori $\mathbb{T}^1_\Lambda$ and $\mathbb{T}^2_\Lambda$ depends only on the resonance $k$ determining $\Gamma \subset \Lambda_k$. The projection $\pi_\Lambda$ induces a linear map $d\pi_\Lambda : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}_\Lambda$. The restriction to $\mathbb{R}^2 \times 0$ has a null space, denoted by $N_\Lambda \subset \mathbb{R}^2$. Denote by $N^\perp_\Lambda$ the orthogonal complement of $N_\Lambda$ with respect to $d^2\ell_0(\omega)$. Define

$$K_{\omega,\Lambda} = (d^2\ell_0(\omega)/2)|_{N^\perp_\Lambda}. \tag{16}$$

$K_{\omega,\Lambda}$ may be regarded as a constant quadratic form on $T\ (\mathbb{T}^1_\Lambda)$ in a view of identification of $N_\Lambda$ and $T\ (\mathbb{T}^1_\Lambda)$ given by $d\pi_\Lambda$.

Let

$$L_{\omega,\Lambda} = K_{\omega,\Lambda} + P_{\omega,\Lambda} \circ \mathrm{pr}_{\omega,\Lambda} : T\ (T^1_\Lambda) \longrightarrow \mathbb{R},$$

where $pr_{\omega,\Lambda} : T\ (T^1_\Lambda) \to \mathbb{T}^1_\Lambda$. We call $L_{\omega,\Lambda}$ *the first* $(\omega, \Lambda)$*-averaged system associated to* $\omega \in \Lambda$, which is an autonomous mechanical system whose kinetic energy is $K_{\omega,\Lambda}$ and whose potential energy is $-P_{\omega,\Lambda} \circ \mathrm{pr}_{\omega,\Lambda}$.

A classical idea of averaging consists in the fact that the trajectories of $L$ with approximate frequency vector $\omega$ can be approximately described in terms of fast and slow variables. The fast variables correspond to the motion parallel to $\mathbb{T}^2_\Lambda$ and the slow variables correspond to the motion normal (in a suitable sense) to $\mathbb{T}^2_\Lambda$. If we average with respect to the fast variables, we obtain a new Lagrangian system $L_{\omega,\Lambda} = K_{\omega,\Lambda} + P_{\omega,\Lambda}$ whose trajectories approximate the trajectories of $L$ with approximate rotation vector $\omega$.

**5.2. A Double Resonance Averaged System or a Second $\omega$-Averaged System Associated to a Rational Frequency.** — Following the notation introduced here above, we let $L(\theta, \dot\theta, t) = \ell_0(\dot\theta) + \varepsilon P(\theta, \dot\theta, t)$ be a $C^s$ small perturbation of an integrable Lagrangian $\ell_0$ on $\mathbb{T}^2 \times U \times \mathbb{T}$, and we assume that $d^2\ell_0 > 0$ on $U$. Consider a rational frequency vector $\omega = (\omega_1, \omega_2) = (p_1/q, p_2/q) \in \mathbb{Q}^2$ and assume that $(p_1/q, p_2/q)$ is the reduced form, *i.e.* the greatest common divisor of integer $p_1, p_2$, and $q$ is 1. For the unperturbed integrable system $\ell_0$, every trajectory with rotation vector $\dot\theta \equiv \omega$ is closed and parallel to the 1-torus

$$\mathbb{T}^1_\omega = \{(\lambda p_1, \lambda p_2, \lambda q) \in \mathbb{T}^2 \times \mathbb{T}^1 : \ \lambda \in \mathbb{R}\}. \tag{17}$$

Since $\mathbb{T}^2 \times \mathbb{T}^1$ is an Abelian group, $\mathbb{T}^1_\omega$ may be considered as a subgroup. Let $\mathbb{T}^2_\omega = \mathbb{T}^2 \times \mathbb{T}^1/\mathbb{T}^1_\omega$ be the 2-torus obtained as a coset of $\mathbb{T}^1_\omega$. Similarly to the previous section, we call $\mathbb{T}^1_\omega$ — *fast* and $\mathbb{T}^2_\omega$ — *slow torus* respectively. Let $(\varphi^s_\omega, \varphi^f_\omega) \in \mathbb{T}^2_\omega \times \mathbb{T}^1_\omega$ denote slow and fast coordinates in $\mathbb{T}^2_\omega$ and $\mathbb{T}^1_\omega$, respectively. Let $\widetilde{P}(\theta, t, \omega) = P(\theta, \omega, t)$. Denote by $d\mathcal{H}_\omega$ the normalized Haar (Lebesgue) measure on the 1-torus $\mathbb{T}^1_\omega$. Recall that $\widetilde{P}(\theta, t, \omega) = P(\theta, \omega, t)$. Define *the second $\omega$-averaged potential*

$$P_\omega(\varphi^s_\omega) = \int_{\varphi^f_\omega} \widetilde{P}(\varphi^s_\omega, \varphi^f_\omega, \omega)\, d\mathcal{H}_\omega(\varphi^f_\omega). \tag{18}$$

Note that $P_\omega : \mathbb{T}^2_\omega \to \mathbb{R}$ is a real valued function. We need now some linear algebra in order to define the second $\omega$-averaged kinetic energy. Consider the natural projection $\pi_\omega : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2_\omega$ along fast torus $\mathbb{T}^1_\omega$. The definition of both $\mathbb{T}^1_\omega$ and $\pi_\omega$ depends only on $(q, p_1, p_2) \in \mathbb{Z}^3$, where $\omega = (p_1/q, p_2/q)$. The projection $\pi_\omega$ induces a linear map $d\pi_\omega : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2_\omega$. The restriction to $\mathbb{R}^2 \times 0$ becomes an isomorphism. Since $\ell_0$ is a $C^2$ smooth function on $\mathbb{R}^2$, its Hessian $d^2\ell_0(\omega)$ can be regarded as a quadratic form on $\mathbb{R}^2$. We define

$$K_\omega = d^2\ell_0(\omega)/2 \tag{19}$$

and we shall identify $\mathbb{R}^2$ with $T\,(\mathbb{T}^2_\omega)$ via $d\pi_\omega$. Let also define

$$L_\omega = K_\omega + P_\omega \circ \mathrm{pr}_\omega : T\,(\mathbb{T}^2_\omega) \longrightarrow \mathbb{R}, \tag{20}$$

where $\mathrm{pr}_\omega : T\,(\mathbb{T}^2_\omega) \to \mathbb{T}^2_\omega$ is the natural projection. We call $L_\omega$ *the second $\omega$-averaged system associated with the rational frequency* $\omega = (\omega_1, \omega_2) = (p_1/q, p_2/q) \in \mathbb{Q}^2$. This is an autonomous mechanical system whose kinetic energy is $K_\omega$ and whose potential energy is $-P_\omega \circ \mathrm{pr}_\omega$.

## 6. Definition of $U^s_{\delta(\ell_0,\Gamma)}$

**6.1. Part I: Building blocks.** — In this Section, we begin the definition of the set of admissible directions $U^s_{\delta(\ell_0,\Gamma)}$ on the unit sphere of perturbations $S^s_L$ or, equivalently, qualitative definition of $\delta(\ell_0,\Gamma)$. Later, we use this to define a $\delta(\ell_0,\Gamma)$-cusp residual set $W^s_{\delta(\ell_0,\Gamma)}$, where Mather Diffusion Theorem holds. We need it for the application of Mather Diffusion Theorem to our main result Theorem 1.2. The set $U^s_{\delta(\ell_0,\Gamma)}$ implicitly appears in Mather Diffusion Theorem and it is defined as a set where a non–negative functional $\delta(\ell_0,\Gamma)$ is positive. We shall not give here a complete definition $\delta(\ell_0,\Gamma)$, since this would need quite a long discussion. We shall only sketch some qualitative aspects of its definition. For the discussion of the size of $\delta(\ell_0,\Gamma)$ we refer to [**Ma4**].

*Step 1.* — Consider a resonant piecewise linear curve $\Gamma=\cup_{s=1}^{m}\Gamma_s\subset B^2$ consisting of $m$ resonant line segments $\Gamma_s\subset\Lambda_s=\{(\omega_1,\omega_2)\in B^2:\ k_0^s+k_1^s\omega_1+k_2^s\omega_2=0\}$, $k^s=(k_0^s,k_1^s,k_2^s)\in\mathbb{Z}^3$ as in definition 4.2.

*Step 2.* — For each resonant segment $\Gamma_s$ we associate a non-negative function $\delta(\ell_0,\Gamma_s): S_L^s\to\mathbb{R}_+$, defined in the next Section. Then, $\delta(\ell_0,\Gamma)=\min_{s=1}^m\delta(\ell_0,\Gamma_s)$.

Now we discuss the qualitative part of the definition of $\delta(\ell_0,\Gamma_s)$ for one segment. For the sake of simplicity, we omit the subindex $s$ in the sequel, hence, in what follows, $\Gamma$ will denote a single resonant segment. For one segment, we state a finite collection of non–degeneracy hypotheses of two types. Each hypothesis turns out to be fulfilled generically [**Ma4**].

*Type 1.* — Non–degeneracy of the 1-parameter family of the first $(\omega,\Gamma)$-averaged mechanical system $\{L_{\omega,\Gamma}\}_{\omega\in\Gamma}$ on $T\ (\mathbb{T}^1_\Gamma)$.

*Type 2.* — $\Gamma$–non–degeneracy of the second $\omega$-averaged mechanical system $L_\omega$ on $T\,(\mathbb{T}^2_\omega)$ associated to a rational frequency $\omega\in\Gamma\cap\mathbb{Q}^2$.

There are countably many rationals $\omega$'s in any resonant segment $\Gamma$. However, we need type 2 non–degeneracy only for finitely many rational $\omega$'s. At the end of Section 6.3, we define a marginal denominator $q_0=q_0(\ell_0,P,\Gamma_s)$ with the following meaning. Let $\omega=(p_1/q,p_2/q)$ be in the reduced form, then we need to impose non–degeneracy hypotheses of type 2 on $\omega$ only if $q<q_0$. In the next two Sections, we define the non-degeneracy hypotheses of type 1 on the family $\{L_{\omega,\Lambda}\}_{\omega\in\Lambda}$ of the first $(\omega,\Lambda)$-averaged system and of type 2 on the second $\omega$-averaged system $L_\omega$, $\omega\in\Lambda\cap\mathbb{Q}^2$ along with $q_0(\ell_0,P,\Gamma_s)$ respectively.

**6.2. Part II: Non-degeneracy of averaged systems associated to a single segment $\Gamma$.** — By means of Step 2 of the last section, we see that it suffices to define $U^s_{\delta(\ell_0,\Gamma)}$ for one segment. Since $\ell_0$ is fixed, we shall omit it from the notation and denote this set $U^s_{\delta(\Gamma)}$.

Let $\Lambda=\Lambda_k$ be the line that contains a bounded segment $\Gamma$. For $\omega\in\Gamma$, we write $P_{\omega,\Gamma}$ for the averaged function $P_{\omega,\Lambda}$ defined in section, and $\mathbb{T}^i_\Gamma$ for $\mathbb{T}^i_\Lambda$ $(i=1,2)$. Thus, $\{P_{\omega,\Gamma}:\omega\in\Gamma\}$ is a $C^s$ smooth 1-parameter family of functions defined on the circle $\mathbb{T}^1_\Gamma$. For $\varepsilon P$ to be in $U^s_{\delta(\Gamma)}$, we require that the global minima of $\{P_{\omega,\Gamma}:\omega\in\Gamma\}$ are of generic type. More precisely, we require the following three hypotheses to be fulfilled:

(C1) For each value $\omega\in\Gamma$, each global minimum of $m_\omega$ of $P_{\omega,\Gamma}$ is non–degenerate, *i.e.* $P''_{\omega,\Gamma}(m_\omega)>0$.

(C2) For each $\omega\in\Gamma$, there are at most two global minima of $P_{\omega,\Gamma}$.

Let $\omega_0\in\Gamma$ and suppose that $P_{\omega_0,\Gamma}$ has two global minima $m_{\omega_0}$ and $m'_{\omega_0}$. We may continue these to local minima $m_\omega$ and $m'_\omega$ of $P_{\omega,\Gamma}$, for $\omega\in\Gamma$ near $\omega_0$, in view of (C1). Thus, $m_\omega$ and $m'_\omega$ depend continuously on $\omega$ and they are the given global minima for $\omega = \omega_0$. In addition to (C1) and (C2), we require that the following *first transversality condition* be fulfilled:

(C3) $$\frac{dP_{\omega,\Gamma}(m_\omega)}{d\omega}\Big|_{\omega=\omega_0} \neq \frac{dP_{\omega,\Gamma}\ (m'_\omega)}{d\omega}\Big|_{\omega=\omega_0}.$$

Next, we require $\ell_0$ and $P$ to fulfill some conditions on the second $\omega$-averaged systems $L_\omega$ associated to $\omega \in \Gamma \cap \mathbb{Q}^2$, defined in Section 5.2. Such an $\omega$ has the form $\omega = p/q = (p_1/q, p_2/q)$, where $p = (p_1, p_2) \in \mathbb{Z}^2$, and $q \in \mathbb{Z}$, $q > 0$. If $p/q$ is in the reduced form, *i.e.* 1 is the greatest common *denominator* of $p_1, p_2$ and $q$, then we say that $q$ is the denominator of $\omega$. We shall require the remaining hypotheses only in the case $\omega$ has *small denominator*, *i.e.* $q \leqslant q_0$, where $q_0 = q_0(\ell_0, P, \Gamma)$ is a positive integer depending on $\ell_0, P$, and $\Gamma$. The definition of $q_0$ is the quantitative aspect of the definition of $U^s_{\delta(\Gamma)}$ that we shall postpone to Section 6.3.

The first condition we require $L_\omega$ to fulfill is a condition on $P_\omega$ alone:

(C4)$_\omega$ The function $P_\omega$ on $\mathbb{T}^2_\omega$ has only one global minimum $m_\omega$ and it is non–degenerate in the sense of Morse, *i.e.* the quadratic form $d^2P_\omega(m_\omega)$ is non–singular.

To state the remaining hypotheses, we need to define a special homology element $h_{\omega,\Gamma}$ of $H_1(\mathbb{T}^2_\omega; \mathbb{R})$:

Since $\omega \in \Gamma \cap \mathbb{Q}^2$, we have $\mathbb{T}^1_\omega \subset \mathbb{T}^2_\Gamma$, so $\mathbb{T}^2_\Gamma/\mathbb{T}^1_\omega$ is a circle in $\mathbb{T}^2_\omega$, and

$$\mathbb{Z} \approx H_1(\mathbb{T}^2_\Gamma/\mathbb{T}^1_\omega; \mathbb{Z}) \subset H_1(\mathbb{T}^2_\omega; \mathbb{Z}) \subset H_1(\mathbb{T}^2_\omega; \mathbb{R}).$$

We let $h_{\omega,\Gamma}$ be a generator of $H_1(\mathbb{T}^2_\Gamma/\mathbb{T}^1_\omega; \mathbb{Z})$. In view of the inclusions above, this is an element of $H_1(\mathbb{T}^2_\omega; \mathbb{R})$. Geometrically, the above situation has the following meaning. Consider a circle $l_{\omega,\Gamma} \subset \mathbb{T}^2_\omega$ in the homology class $h_{\omega,\Gamma}$ and take $\pi_\omega^{-1}(l_{\omega,\Gamma}) \subset \mathbb{T}^2 \times \mathbb{T}$, where the projection $\pi_\omega : \mathbb{T}^2 \times \mathbb{T} \to \mathbb{T}^2_\omega$ is defined in Section 5.2. Therefore, $h_{\omega,\Gamma}$ is such that $\pi_\omega^{-1}(l_{\omega,\Gamma})$ is parallel to $\mathbb{T}^2_\Lambda \subset \mathbb{T}^2 \times \mathbb{T}$.

The Lagrangian $L_\omega$ describes a conservative mechanical system, *i.e.* it has the form *kinetic energy – potential energy*. Here, the kinetic energy $K_\omega$ is associated to the constant Riemannian metric $g_\omega = d^2\ell_0(\omega)$ on $\mathbb{T}^2_\omega$. The potential energy is $-P_\omega \circ pr_\omega$. By a slight abuse of terminology, we shall shorten this to $-P_\omega$.

Next conditions that we require on $L_\omega$ are easily described in terms of the Maupertuis principle:

We let $E_\omega = -P_\omega(m_\omega)$, where $m_\omega \in \mathbb{T}^2_\omega$ is the unique minimum of $P_\omega$, as above. For any $E \geqslant E_\omega$, we let

$$g_E = 2(P_\omega + E)K_\omega. \tag{21}$$

For $E > E_\omega$, the function $P_\omega + E$ is everywhere positive on $\mathbb{T}^2_\omega$. Hence, $g_E$ is a $C^s$ Riemannian metric on $\mathbb{T}^2_\omega$. For $E = E_\omega$, the function $P_\omega + E$ is positive except at $m_\omega$, where it vanishes and has a non–degenerate minimum.

The Maupertuis principle states that trajectories of $L_\omega$ having energy $E$ are the same as geodesics of $g_E$, except for a time–reparametrization. Carneiro [**DC**] has extended the Maupertuis principle and shown that absolute minimizers of $L_\omega$ having energy $E$ correspond to class A geodesics of $g_E$ (in the sense of Morse [**Mor**] and Hedlund [**Hed**]).

Pick a large energy constant $E^* = E^*(\ell_0, P, \Gamma) > 0$. The next condition that we impose on $L_\omega$ concerns the shortest closed geodesics of $g_E$ in the homology class $h_{\omega,\Gamma}$, for $E_\omega + E^* \geqslant E \geqslant E_\omega$. Briefly, we require that these are of generic type. More explicitly, we ask that the following four hypotheses are fulfilled:

$(\mathrm{C5})_\omega$ For $E_\omega + E^* \geqslant E \geqslant E_\omega$, each shortest closed geodesic of $g_E$ in the homology class $h_{\omega,\Gamma}$ is non–degenerate in the sense of Morse.

$(\mathrm{C6})_\omega$ For $E_\omega + E^* \geqslant E \geqslant E_\omega$, there are at most two shortest closed geodesics of $g_E$ in the homology class $h_{\omega,\Gamma}$.

Let $E_1 > E_\omega$ and suppose that there are two shortest geodesics $\gamma$ and $\gamma'$ of $g_{E_1}$ in the homology class $h_{\omega,\Gamma}$. We may continue these to locally shortest geodesics $\gamma_E$ and $\gamma'_E$ of $g_E$ for $E$ near $E_1$, in view of $(\mathrm{C5})_\omega$. If $\mu$ is a closed curve on $\mathbb{T}^2_\omega$, we let $\ell_E(\mu)$ denote its length with respect to $g_E$. We require that the following *second transversality condition* be fulfilled:

$$(\mathrm{C7})_\omega \qquad \frac{d(\ell_E(\gamma_E))}{dE}\Big|_{E=E_1} \neq \frac{d(\ell_E(\gamma'_E))}{dE}\Big|_{E=E_1}.$$

These are the hypotheses that we require $g_E$ to fulfill when $E_\omega + E^* \geqslant E \geqslant E_\omega$. Note that the case $E = E_\omega$ is somehow special, because $g_{E_\omega}$ is not a Riemannian metric, since it vanishes at $m_\omega$[(4)]. Nevertheless, we may define the length of a curve with respect to $g_{E_\omega}$ just as one normally defines the length of a curve with respect to a Riemannian metric. We define a geodesic to be a curve that is the shortest distance between any two sufficiently nearby points. It is easy to see that there exists a shortest geodesic of $g_{E_\omega}$ in the homology class $h_{\omega,\Gamma}$. We require $L_\omega$ to fulfill the following condition:

$(\mathrm{C8})_\omega$ There is only one shortest geodesic $\gamma$ of $g_{E_\omega}$ in the homology class $h_{\omega,\Gamma}$, and $\gamma$ is non–degenerate in the sense of Morse.

In saying that a $g_E$–shortest geodesic $\gamma$ is non–degenerate in the sense of Morse, we mean the following:

Let $\mu$ be a transversal to $\gamma$, intersecting $\gamma$ in one point, not $m_\omega$, in the case that $E = E_\omega$. For each point $P \in \mu$, let $\gamma_P$ be the $g_E$–shortest curve through $P$ in the homology class $h_{\omega,\Gamma}$ and let $\ell_E(\gamma_P)$ denote its $g_E$–length. The function $P \to \ell_E(\gamma_P)$ is $C^s$ near $\mu \cap \gamma$ and the condition that $\gamma$ be non–degenerate means that its second derivative is positive.

In the case that $E > E_\omega$, this is the usual notion of non–degeneracy in the sense of Morse.

(4)This corresponds to a periodic trajectory for the Euler-Lagrangian flow of $L = \ell_0 + \varepsilon P$.

***Definition 6.1***. — $U^s_{\delta(\Gamma)}\big(=U^s_{\delta(\ell_0,\Gamma)}\big)=\{\varepsilon P:\varepsilon>0, P\in\mathcal{P}^s$, and $P$ satisfies hypotheses (C1)–(C3) as well as hypotheses $(\mathrm{C4})_\omega$–$(\mathrm{C8})_\omega$ for $\omega\in\Gamma\cap\mathbb{Q}^2$ with small denominator, *i.e.* such that $q\leqslant q_0(\ell_0,p,\Gamma)$, where $q$ denotes the denominator of $\omega$.$\}$

***Remark 6.2***. — This definition can be considered as an implicit definition of $\delta(\ell_0,\Gamma)$.

**6.3. What denominators are small?**— In this Section, we define the marginal denominator $q_0=q_0(\ell_0,P,\Gamma)$ from the previous definition. This would answer the question for which rational $\omega$'s we need to verify the non-degeneracy hypotheses $(\mathrm{C4})_\omega$–$(\mathrm{C8})_\omega$. Recalling (14) and (17), we associate to a rational frequency $\omega$ and a resonant segment $\Gamma\ni\omega$ two decompositions of $\mathbb{T}^2\times\mathbb{T}^1$ into (the standard) direct product, and we denote the result of this operation by $\mathbb{T}^2_\Gamma\times\mathbb{T}^1_\Gamma$ and $\mathbb{T}^1_\omega\times\mathbb{T}^2_\omega$. These decompositions can be defined by changing the basis on $\mathbb{T}^2\times\mathbb{T}^1$. Based on the lemma below we can define the following decomposition $\mathbb{T}^1_\omega\times\mathbb{T}^1_{\omega,\Gamma}\times\mathbb{T}^1_\Gamma=\mathbb{T}^2\times\mathbb{T}^1$ into a direct sum.

***Lemma 6.3***. — *There is a choice of these decompositions so that $\mathbb{T}^1_\Gamma\subset\mathbb{T}^2_\omega$, $\mathbb{T}^1_\omega\subset\mathbb{T}^2_\Gamma$.*

*Proof*. — It seems easiest to discuss this in terms of a short sequence of topological abelian groups.

$$0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0.$$

Thus, $A$ is a topological subgroup of $B$, and $C$ is a quotient group of $B$. Denote the inclusion of $A$ into $B$ by $i$ bad the projection of $B$ onto $C$ by $j$. To say that the sequence is short exact means that the kernel of $j$ is $i(A)$.

A splitting of such a sequence is given by a continuous homomorphism $k$ of $C$ into $B$ such that $j\circ k$ is the identity. Equally well, it can be given by a continuous homomorphism $l$ of $B$ into $A$. such that $l\circ i$ is the identity. The relation between $k$ and $l$ is that the kernel of $l$ is $k(C)$.

Given $k$ (resp. $l$) there is a unique $l$ (resp. $k$) such that this relation holds. Given such a splitting, $B$ is the direct sum of $i(A)$ and $k(C)$.

There is a splitting, in fact many, for both of the short exact sequences in the case we consider (it suffices $A$ to be a torus). Indeed, notice that when we consider $\mathbb{T}^1_\Gamma$ (resp. $\mathbb{T}^2_\omega$) as a subgroup $\mathbb{T}^2\times\mathbb{T}$ we choose a splitting of the appropriate one of the two exact sequences in question(5). This proves the lemma. □

Below we present a test to determine $q_0$. The idea of the test is to check how dense the unperturbed closed trajectory $\dot\theta=\omega$ in the 2-torus $\mathbb{T}^2_\Gamma$. A precise definition is in terms of averaged systems $L_{\omega,\Gamma}=K_{\omega,\Gamma}+P_{\omega,\Gamma}$ and $L_\omega=K_\omega+P_\omega$ on slow tori $T$ $(\mathbb{T}^1_\Gamma)$ and $T$ $(\mathbb{T}^2_\omega)=T$ $(\mathbb{T}^1_{\omega,\Gamma}\times\mathbb{T}^1_\Gamma)$, respectively. In the notation of the previous section, let $m_\omega$ and $m'_\omega$ be global (or local continuation of global) minima of $P_\omega$ on $\mathbb{T}^1_\Gamma$ and let

(5)Note that the inclusion of $\mathbb{T}^1_\Gamma$ in $\mathbb{T}^2_\omega$ depends on the choice of splitting. One has inclusion for some splittings not for others.

$\pi_{\omega,\Gamma} : \mathbb{T}^1_{\omega,\Gamma} \times \mathbb{T}^1_\Gamma \to \mathbb{T}^1_\Gamma$ be the natural projection. For each energy $E_\omega + E^* \geqslant E \geqslant E_\omega$, we need that the shortest geodesics $\gamma_E$ and $\gamma'_E$ (if it exists) of $g_E$ are contained in a small neighborhood of $\pi^{-1}_{\omega,\Gamma}(m_\omega \cup m'_\omega)$ (or $\pi^{-1}_{\omega,\Gamma}(m_\omega)$ if $m'_\omega$ does not exist). The precise definition is as follows.

On the slow torus $\mathbb{T}^1_\Gamma$, we define the 1-parameter family of first $(\omega, \Gamma)$-averaged potentials $\{P_{\omega,\Gamma} : \mathbb{T}^1_\Gamma \to \mathbb{R}\}_{\omega\in\Gamma}$. Suppose hypotheses (C1)–(C3) are fulfilled. By the first transversality condition (C3), there are finitely many $\omega_0 \in \Gamma$ with $P_{\omega_0,\Gamma}$ having two global minima $m_{\omega_0}$ and $m'_{\omega_0}$. Mark these $\omega_0$'s. By (C1) and (C2), for $\omega$'s nearby the marked $\omega_0$'s, there is a smooth continuation $m_\omega$ and $m'_\omega$ of $m_{\omega_0}$ and $m'_{\omega_0}$, respectively, to local minima nearby. Pick a small $\eta > 0$, so that $\eta$-neighborhoods of the marked $\omega_0$'s, denoted by $\Upsilon^\eta_{\omega_0}$, are disjoint. Moreover, in each neighborhood $\Upsilon^\eta_{\omega_0}$, there is a well defined continuation $m_\omega$ and $m'_\omega$. Such $\eta$ will be called $(\ell_0, P, \Gamma)$-*admissible.*

Consider now a small $\tau > 0$ with the following properties. For each $\omega \in \Gamma$, consider two cases. In the first case $\omega$ is in one of $\Upsilon^\eta_{\omega_0}$. Then, we define a 2-tuple of $\tau$-neighborhoods $\widetilde{D}^\tau_\omega$ and $\widehat{D}^\tau_\omega$ in $\mathbb{T}^1_\Gamma$ are centered at $m_\omega$ and $m'_\omega$ respectively and disjoint. Denote $D^\tau_\omega = \widetilde{D}^\tau_\omega \cup \widehat{D}^\tau_\omega$. In the other case, $\omega$ is outside of neighborhoods of marked frequencies $\Upsilon^\eta_{\omega_0}$'s put $D^\tau_\omega$ to be a $\tau$-neighborhood centered at the global minimum $m_\omega$.

***Definition 6.4***. — A rational frequency $\omega \in \Gamma \cap \mathbb{Q}^2$ is $(\ell_0, P, \Gamma, \eta, \tau)$-admissible with a small $\tau > 0$ if the family of first $(\omega, \Gamma)$-averaged systems $\{L_{\omega,\Gamma} = K_{\omega,\Gamma} + P_{\omega,\Gamma}\}_{\omega\in\Gamma}$ satisfy hypotheses (C1)–(C3) and for an $(\ell_0, P, \Gamma)$-admissible $\eta > 0$ and any $E_\omega + E^* \geqslant E \geqslant E_\omega$ each shortest geodesic (resp. local continuation of a shortest geodesic) $\gamma_E$ (resp. $\gamma'_E$ if it exists) of the Maupertuis metric $g_E$ in the homotopy class $h_{\omega,\Gamma}$ belongs to $\pi^{-1}_{\omega,\Gamma}(D^\tau_\omega)$.[(6)]

Recall that, for each double resonance of a rational frequency $\omega = (p_1/q, p_2/q) \in \Gamma \cap \mathbb{Q}^2$ in (20), we may define the double resonant mechanical system $L_\omega$ on the slow 2-dimensional torus $\mathbb{T}^2_\omega$ and the natural projection $\pi_{\omega,\Gamma} : \mathbb{T}^2_\omega \to \mathbb{T}^1_\Lambda$ onto the slow 1-dimensional torus $\mathbb{T}^1_\Gamma \subset \mathbb{T}^2_\omega$. Then, we have the following result:

***Lemma 6.5* ([Ma4])**. — *Suppose the perturbation term $P(\theta, \dot\theta, t)$ satisfies hypotheses (C1)–(C3). Then, for any $\tau > 0$, there is an integer $q_0 = q_0(\ell_0, P, \Gamma, \tau)$, such that, for any rational frequency $\omega$ with denominator $q > q_0$, we have that $\omega$ is $(\ell_0, P, \Gamma, \eta, \tau)$-admissible. Namely, a corresponding shortest geodesic (resp. local continuation of a shortest geodesic) $\gamma_E$ (resp. $\gamma'_E$ if exists) of the Maupertuis metric $g_E$, defined in (21), belongs to the strip $\pi^{-1}_{\omega,\Gamma}(D^\tau_\omega) \subset \mathbb{T}^2_\omega$.*

(6) As a matter of fact the proof in [**Ma4**] requires a stronger form of admissibility which still fits into the proof of our main result (Theorem 1.2).

For further reference, we need to give a definition of $\eta$ and $\tau$-neighborhoods for double resonances. Let $L_\omega$ and $\mathbb{T}^2_\omega$ be the mechanical Lagrangian on the 2-torus corresponding to a rational frequency $\omega = (p_1/q, p_2/q) \in \Gamma \cap \mathbb{Q}^2$ as above. Let $\{g_E\}_{E\in[E_\omega+E^*,E_\omega]}$ be the 1-parameter family of Maupertuis metrics defined by (21). Suppose hypotheses $(\mathrm{C4})_\omega$–$(\mathrm{C8})_\omega$ are fulfilled. Mark parameters $E_0$ where $g_E$ has two shortest geodesics in the homology class $h_{\omega,\Gamma}$. By the second transversality condition $(\mathrm{C7})_\omega$, there are finitely many $E_0 \in [E_\omega + E^*, E_\omega]$ with metrics $g_{E_0}$ having two shortest geodesics $\gamma_{E_0}$ and $\gamma'_{E_0}$. By $(\mathrm{C8})_\omega$, there is a smooth continuation $\gamma_{E_1}$ and $\gamma'_{E_1}$ to locally shortest geodesics. Pick a small $\eta_\omega > 0$ so that $\eta_\omega$-neighborhoods of marked $E_0$'s (denoted by $\Upsilon^{\eta_\omega}_{E_0}$) are disjoint. Moreover, in each neighborhood $\Upsilon^{\eta_\omega}_{E_0}$ there is a well defined continuation $\gamma_E$ and $\gamma'_E$. Such $\eta_\omega$ is called $(\ell_0, P, \Gamma, \omega)$-*admissible*.

Pick a small $\tau_\omega > 0$ with the following properties. For each $E\in[E_\omega, E_\omega + E^*]$ consider two cases. Either $\omega$ is in one of $\Upsilon^{\eta_\omega}_{E_0}$. Then, we define a 2-tuple of $\tau$-neighborhoods $\widetilde{D}^\tau_E$ and $\widehat{D}^\tau_E$ in $\mathbb{T}^2_\omega$ of the locally shortest geodesics $\gamma_{E_0}$ and $\gamma'_{E_0}$ respectively so that these neighborhoods and disjoint. Denote $D^\tau_E = \widetilde{D}^\tau_E \cup \widehat{D}^\tau_E$. In the other case $E$ is outside of these neighborhoods of marked energies, then $D^\tau_E$ is a $\tau$-neighborhood of the shortest geodesic $\gamma_E$.

## 7. Definition of $W^s_{\delta(\ell_0,\Gamma)}$ using type 2 non-degeneracy (of Barrier functions)

In this Section, we define the non-degeneracy hypotheses of the second type. They are formulated in terms of minima of certain barrier functions, restricted to what we call *Poincaré screens*. First, we explain the meaning of Poincaré screens and we define them. Later on, we define required barrier functions and state the non–degeneracy hypotheses that we need to define $\delta(\ell_0, \Gamma)$-cusp residual set $W^s_{\delta(\ell_0,\Gamma)}$.

As mentioned in Section 3.2 diffusing trajectories stay most of the time close to the corresponding Mather sets $\mathcal{M}_i$ and from time to time make almost heteroclinic excursions along stable and unstable sets $W^s(\mathcal{M}_i)$ and $W^u(\mathcal{M}_{i+1})$ from one set $\mathcal{M}_i$ to the next one $\mathcal{M}_{i+1}$. In order to keep track of those excursions, we pose a smooth hypersurface (Poincaré screen) "in between" $\mathcal{M}_i$ and $\mathcal{M}_{i+1}$. To give a precise definition we need further discussion.

Recall that $\widetilde{\mathcal{M}}_i = \pi\mathcal{M}_i \subset \mathbb{T}^2 \times \mathbb{T}$ is the projected Mather sets. Suppose hypotheses (C1)–(C3) and $(\mathrm{C4})_\omega$–$(\mathrm{C8})_\omega$ for rational $\omega$'s with small denominator are fulfilled. Consider two different cases:

(1) $\omega$ is $C\sqrt{\varepsilon}$-close to a rational $(p_1/q, p_2/q)$ with small denominator $q < q_0$, where $C$ is some positive constant depending only on $\ell_0, P, \Gamma, \tau$ and is closely related to the energy constant $E^*$.

(2) the opposite case.

Recall that, for any frequency $\omega \in \Gamma$, we associate homology class in $H_1(\mathbb{T}^2, \mathbb{R}) \simeq \mathbb{R}^2$ equal to $\omega$. Each Lagrangian satisfying conditions (1–3) of section 2.1 has Fenchel-Legendre transform $\mathcal{L}_\beta$ associated to it by (29). Using $\mathcal{L}_\beta$, we associate to each homology class $\omega$ any cohomology class $c_\omega$ inside $\mathcal{L}_\beta(\omega)$.

It turns out that, in the first case, for a sufficiently small $\varepsilon \neq 0$ and a cohomology $c \in \mathcal{L}_\beta(\omega)$, there is rescaling which relates $c$ and $E \in [E_\omega, E_\omega + E^*]$, such that the projected Mather set $\widetilde{\mathcal{M}}^c$ belongs to $\pi_\omega^{-1}(D_E^{\tau_\omega})$. In the second case, the projected Mather set $\widetilde{\mathcal{M}}^c$ belongs to $\pi_\Gamma^{-1}(D_\omega^\tau)$. In both cases, the projected Mather sets are localized in a $\tau$-neighborhood of one or two hypersurfaces on the base $\mathbb{T}^2 \times \mathbb{T}$. We shall distinguish these cases.

***Definition 7.1***. — Let $\omega \in \Gamma$ and $c \in \mathcal{L}_\beta(\omega)$ be a cohomology class. Distinguish two cases: $D_E^{\tau_\omega}$ (resp. $D_\omega^\tau$) has one or two components.

In the one component case, let us define $S^c \subset \mathbb{T}^2 \times \mathbb{T}$ to be a smooth hypersurface (*i.e.*, a codimension one closed smooth submanifold) topologically parallel $\pi_\omega^{-1}(\gamma_E)$, which is transversal to class A geodesics with respect to $\gamma_E$, and disjoint from its $\tau$-neighborhood $\pi_\omega^{-1}(D_E^{\tau_\omega})$ in the first case and topologically parallel to $\pi_\Gamma^{-1}(m_\omega)$, transversal to class A geodesics with respect to $\gamma_E$ and $\gamma'_E$, and disjoint from its $\tau$-neighborhood $\pi_\Gamma^{-1}(D_\omega^\tau)$ in the second case.

In the two component case: let us define $S_-^c, S_+^c \subset \mathbb{T}^2 \times \mathbb{T}$ to be a pair of smooth hypersurfaces parallel and separating $\pi_\omega^{-1}(\gamma_E)$ and $\pi_\omega^{-1}(\gamma'_E)$ in the double resonance case. "Separating" means that $S_-^c$ and $S_+^c$ cut $\mathbb{T}^2 \times \mathbb{T}$ into two disjoint parts each containing either $\pi_\omega^{-1}(\gamma_E)$ or $\pi_\omega^{-1}(\gamma'_E)$. In the single resonance case define $S_-^c$, $S_+^c \subset \mathbb{T}^2 \times \mathbb{T}$ to be of smooth hypersurfaces parallel and separating $\pi_\Gamma^{-1}(m_\omega)$ and $\pi_\Gamma^{-1}(m'_\omega)$. Call $S^c$ (resp. $S_-^c$ and $S_+^c$) *Poincaré screen (resp. screens)* associated with cohomology class $c \in \mathcal{L}_\beta(\omega)$.

Hypotheses (C1-C8) we impose do not imply that the geodesic $\gamma_E$ and the minimum $m_\omega$ vary continuously with $E$ and $\omega \in \Gamma$. Points of discontinuity are usually call *bifurcations*. However, (C1) and (C4$_\omega$) imply that $\gamma_E$ and $m_\omega$ vary piecewise continuously. Therefore, we can choose Poincaré screens so that they are piecewise constant with respect to $c$. In other words, one could divide $\Gamma$ into a finite number of subintervals, so that for all $\omega$ in a subinterval the Poincaré screen is the same.

By construction, all $S^c$ are topologically parallel. This property essentially relies on the fact that we have only *one* resonant segment $\Gamma$ under consideration. Since $S^c$ is piecewise constant in $c$, we shall treat the case of one Poincaré screen $S^c$ for each $c$. The other case is analogous.

Denote by $S_i = S^{c_i}$ Poincaré screens corresponding to the frequencies $\omega_i$, $i = 1, 2, \ldots$ related to $c_i$, $i = 1, 2, \ldots$ by Fenchel-Legendre transform respectively. We marked these frequencies $\omega_i$'s in Section 3.3. Consider a cyclic cover $\mathbb{T}_\Gamma^2 \times \mathbb{R}$ over $\mathbb{T}_\Gamma^2 \times \mathbb{T}_\Gamma^1$ obtained by cutting along a Poincaré screen $S$ and unrolling. Fix one representative of $\mathbb{T}^2_\Gamma \times \mathbb{T}^1_\Gamma$ in $\mathbb{T}^2_\Gamma \times \mathbb{R}$ and denote it by $\mathbb{T}^2_\Gamma \times \mathbb{T}^1_\Gamma(0)$. For each integer $k$, we denote by $\iota_k : \mathbb{T}^2_\Gamma \times \mathbb{R} \to \mathbb{T}^2_\Gamma \times \mathbb{R}$ the deck transformation along $\mathbb{T}^1_\Gamma$-direction, and by $\mathbb{T}^2_\Gamma \times \mathbb{T}^1_\Gamma(k) = \iota_k(\mathbb{T}^2_\Gamma \times \mathbb{T}^1_\Gamma(0))$ its $k$-th shift. Denote by $S_i^0$ an image of $S_i$ in $\mathbb{T}^2_\Gamma \times \mathbb{R}$ under the natural embedding so that $S_i^0 \cap (\mathbb{T}^2_\Gamma \times \mathbb{T}^1_\Gamma(0)) \neq \varnothing$. By construction, for each $i = 1, 2, \dots$ we have that the corresponding $S_i$ is topologically parallel to $\mathbb{T}^2_\Gamma$ and that $\iota_k(S_i) \cap \iota_{k'}(S_i)$ are disjoint for any $k \neq k'$. Denote $S_i^k = \iota_k(S_i)$ for $k,\, i \in \mathbb{Z}$ and $\widehat{S}_i = S_i^i$. Now we define the $\delta(\ell_0, \Gamma)$-cusp residual set $W^s_{\delta(\ell_0,\Gamma)}$.

Consider a closed one form $\eta$, with $[\eta]_{\mathbb{T}^2} = c$ and $c \in \mathcal{L}_\beta(\Gamma)$. Define the barrier function on $S_i$

$$H_{\eta,T}((\theta,t),(\theta',t')) = \inf\left\{ \int_a^b (L - \widehat{\eta})(d\gamma(t),t)dt \right\}, \tag{22}$$

where the infimum is taken over all absolutely continuous curves $\gamma : [a,b] \to \mathbb{T}^2_\Gamma \times \mathbb{R}$ such that $\gamma(a) = \theta$, $\gamma(b) = \theta'$, $a \equiv t' \pmod 1$, $b \equiv t \pmod 1$, $b - a \geqslant T$, $(\theta,t) \in S_i^0$, $(\theta',t') \in S_i^1$.

For next definition, we need to introduce suitable curves $\iota_1(\theta,t) = (\theta',t')$, which correspond to closed curves on $\mathbb{T}^2_\Gamma \times \mathbb{T}^1_\Gamma$ connecting a point on $S_i$ with itself and making only one turn in $\mathbb{T}^1_\Gamma$-direction. Notice that, in this case, $H_{\eta,T}$ is independent of the choice of $\eta$ in $[\eta]_{\mathbb{T}^2} = c$ and $[\eta]_{\mathbb{T}}$, because such curves are closed. For a Ma$\widetilde{n}$é critical or subcritical form, the barrier function $H_{\eta,T}$ is finite and continuous [**Ma5**]. Let us consider

$$H_c(\theta,t) = \liminf_{T\to+\infty} H_{\eta,T}((\theta,t),(\theta,t)). \tag{23}$$

This definition is a particular case of the definition of barrier function (32). In [**Ma2**], Mather proved that the limit exists.

***Lemma 7.2*** **([Ma4])**. — *Let $P \in U^s_{\delta(\ell_0,\Gamma)}$, $\varepsilon$ be sufficiently small and positive, and $\Gamma$ be a resonant line segment in $U$. Then, the Fenchel-Legendre transform $\mathcal{L}_\beta$ associated with $L = \ell_0 + \varepsilon P$ by (29) has the channel property with respect to $\Gamma$.*

By lemma 7.2, there is a closed connected curve $\sigma_\Gamma \subset \mathcal{L}_\beta(\Gamma)$ with the channel property. We could parameterize this curve by a smooth parameter, say $\tau$, *i.e.* $\widehat{\sigma}_\Gamma : [0,1] \to \sigma_\Gamma \subset \mathcal{L}_\beta(\Gamma)$. Thus, we can define a family of barrier functions

$$\{H_\tau : S_\tau \to \mathbb{R}\}_{\tau\in[0,1]}$$

by $c_\tau = \widehat{\sigma}_\Gamma(\tau)$, $S_\tau = S^{c_\tau}$ and $H_\tau(\theta,t) = H_{c_\tau} : S_\tau \to \mathbb{R}$. It turns out that, under our hypotheses, $H_\tau$ is continuous in $\tau$ and even satisfies certain modulus of continuity (see [**Ma4**]). As we pointed out above, hypersurfaces $S_\tau$ can be chosen to be smooth in $\tau$. Recall that a closed subset $D$ of a torus $\mathbb{T}^d$ is called *acyclic in* $\mathbb{T}^d$ if there is a neighborhood $V$ of $D$ in $\mathbb{T}^d$ such that the inclusion map $H_1(V,\mathbb{R}) \subset H_1(\mathbb{T}^d,\mathbb{R})$ is the zero map. Since the ambient manifold $\mathbb{T}^d$ is a torus, the above inclusion map is the zero map if and only if any closed curve in $V$ is contractible. Let

$$D_\tau = \left\{ (\theta, t) \in S_\tau : \; H_\tau(\theta, t) = \min_{(\theta', t') \in S_\tau} H_\tau(\theta', t') \right\}$$

the set where minimum of the barrier $H_\tau$ on $S_\tau$ is achieved. Recall that $S_\tau$ is diffeomorphic to the 2-dimensional torus. The last non–degeneracy hypothesis we require is the following:

(C9) For each $\tau \in [0, 1]$ the set $D_\tau \subset S_\tau$ is acyclic.

Suppose that there is a curve $\sigma_\Gamma \in \mathcal{L}_\beta(\Gamma)$ with channel property such that the family of barrier functions $\{H_\tau\}_{\tau \in [0,1]}$ satisfies hypothesis (C9); then, we denote the set of perturbation terms $\varepsilon P$ with this property by $W^s_{\delta(\ell_0,\Gamma)} \subset \mathbb{R}U^s_{\delta(\ell_0,\Gamma)}$. The following result is not trivial to prove:

***Lemma 7.3*** **([Ma4])**. — *The set $W^s_{\delta(\ell_0,\Gamma)}$ is $C^s$ open and dense in $V^\delta_{\delta(\ell_0,\Gamma)}$.*

The application of Mather theory to the instability of elliptic points requires the following lemmas about the localization of the velocity of the minimizers. Recall that $L(\theta, v, t) = \ell_0(v) + \varepsilon P(\theta, v, t)$ is $C^s$ smooth nearly integrable Lagrangian, defined on $\mathbb{T}^2 \times U \times \mathbb{T}$. Let $\mathcal{L}_\beta$ be Fenchel-Legendre transform associated with $L$. Denote by $\pi_v : \mathbb{T}^2 \times U \times \mathbb{T} \to U$ the natural projection.

***Localization Lemma I***. — *There is $C = C(\ell_0, P) > 0$ such that for any frequency $\omega \in U$ and any cohomology class $c \in \mathcal{L}_\beta(\omega)$ the Mañé set $\mathcal{N}^c(\supset \mathcal{M}^c)$ is contained in $\pi_v^{-1}(B^2_{C\sqrt{\varepsilon}}(\omega))$.*

In other words, velocity of minimizers with approximate velocity $\omega$ may differ from $\omega$ at most by $C\sqrt{\varepsilon}$.

Let the perturbation direction $P \in U^s_{\delta(\ell_0,\Gamma)}$, then, for any frequency $\omega \in \Gamma$ and any cohomology class $c \in \mathcal{L}_\beta(\omega)$, the Poincaré screen $S^c$, the barrier function $H_c$, and its minimum set $D^c \subset S^c$ are well defined.

***Localization Lemma II***. — *The property $D^c$ being acyclic depends only on the values of $L$ inside $\pi_v^1(B^2_{C\sqrt{\varepsilon}}(\omega))$ with the same $C$ as in the Localization Lemma I.*

These lemmas are a restatement of Lemma 3 in [**Ma5**] and they follow from it. Their proof is based on a careful perturbation analysis. First, one proves that, with the standard identification of $H_1(\mathbb{T}^2, \mathbb{R}) \simeq \mathbb{R}^2$, $H^1(\mathbb{T}^2, \mathbb{R}) \simeq \mathbb{R}^2$, Fenchel-Legendre transform $\mathcal{L}_\beta$ is $C\sqrt{\varepsilon}$-close to the map $\nabla_v \ell_0 : U \to \mathbb{R}^2$. Then, using a generalization of (12) to the case of arbitrary $C^s$ smooth convex unperturbed integrable Lagrangian $\ell_0(\dot\theta)$ and the remark that $\langle \dot\theta + c, \dot\theta + c \rangle$ is non-negative, one shows that, if $C$ is too large, $c$-minimality would be contradicted. See also [**BK**] for similar results.

## 8. Variational principle and restatement of Mather diffusion theorem

In this Section, we introduce a variational principle of Mather [**Ma4**]. We shall use the notation of the previous section. By lemma 7.2, we have that $\mathcal{L}_\beta(\Gamma)$ has a smooth connected curve $\sigma_\Gamma \subset \mathcal{L}_\beta(\Gamma)$ having channel property. Fix an orientation on $\sigma_\Gamma$ toward $\mathcal{L}_\beta(\omega^0)$ and a sufficiently dense ordered set of cohomology classes $\mathfrak{C} = \{c_i\}_{i\in\mathbb{Z}_+} \subset \sigma_\Gamma$ so that they are monotonically oriented along $\sigma_\Gamma$ and in between any two $c_{i-1}$ and $c_{i+1}$ on $\sigma_\Gamma$ there is only $c_i$ from $\mathfrak{C}$. How dense this set needs to be will depend on how close the family of barriers $\{H_\tau\}_{\tau\in[0,1]}$ defined above to fail hypothesis (C9). This collection of $c_i$'s plays the role of the collection of $\omega_i$'s from Section 3.3. For each positive integer $i$, denote Poincaré screens by $S_i = S^{c_i}$ on $\mathbb{T}^2_\Gamma \times \mathbb{T}^1_\Gamma$ and by $\widehat{S}_i = \widehat{S}^{c_i}$ on $\mathbb{T}^2_\Gamma \times \mathbb{R}$, Mather sets by $\mathcal{M}_i = \mathcal{M}^{c_i} \subset \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{T}$. Fix a sequence of closed one forms $\eta_i$ such that $[\eta]_{\mathbb{T}^2} = c_i$ and positive numbers $T_i$. For $(\theta,t) \in \widehat{S}_i$, $(\theta',t') \in \widehat{S}_{i+1}$ and $T>0$ define

$$H_{i,T_i}\left((\theta,t),(\theta',t')\right) = \inf\left\{\int_a^b (L-\widehat{\eta}_i)(d\gamma(t),t)dt\right\}, \tag{24}$$

where the infimum is taken over all absolutely continuous curves $\gamma : [a,b] \to \mathbb{T}^2_\Gamma \times \mathbb{R}$ such that $\gamma(a) = \theta$, $\gamma(b) = \theta'$, $a \equiv t \pmod 1$, $b \equiv t' \pmod 1$, $b - a \geqslant T_i$. This leads to a variational principle

$$\sum_{i\in J'} H_{i,T_i}\left((\theta_i,t_i),(\theta_{i+1},t_{i+1})\right). \tag{25}$$

Here above, we understood the following notation: if $J$ is a set of consecutive integers, we denote by $J'$ the index set $J$ without its largest element (provided it exists).

If all one forms $\eta_i$ are critical or subcritical, then each $H_i$ is finite and continuous. Therefore, we can define a *minimizer* of the variational principle to be a sequence $\{(\theta_i,t_i): \ i \in J\}$ such that if $a<b$ and $\{(\theta'_i,t'_i): \ i\in J\}$ is any sequence satisfying $(\theta'_i,t'_i) = (\theta_i,t_i)$ for $i \leqslant a$ and $i \geqslant b$, then

$$\sum_{i\in J'} H_{i,T_i}\left((\theta_i,t_i),(\theta_{i+1},t_{i+1})\right) \leqslant \sum_{i\in J'} H_{i,T_i}\left((\theta'_i,t'_i),(\theta'_{i+1},t'_{i+1})\right).$$

Since each $H_i$ is finite and continuous, an elementary compactness argument shows the existence of a minimizer.

***Theorem 8.1***. — *Let $\varepsilon P \in W^s_{\delta(\ell_0,\Gamma)}$. Then, for any index set $J$, there are sequences of positive numbers $\{\varepsilon_i\}_{i\in J}$ and subcritical closed one forms $\{\eta_i\}_{i\in J}$, such that $[\eta_i]_{\mathbb{T}} - \alpha_L([\eta_i]_{\mathbb{T}^s}) < \varepsilon_i$, and large positive numbers $\{T_i\}_{i\in J}$ satisfying the following property: there exists a minimizer $\{d\gamma(t) : t \in \mathbb{R}\}$ of the variational principle (25), which provides a smooth solution of the Euler-Lagrange equation (7). In other words, the minimizer $\{d\gamma(t) : t\in\mathbb{R}\}$ has no corners.*

## 9. Application

In this Section, we describe how to apply Theorem 8.1 in order to prove Theorem 1.2. It does not seem possible to simplify arguments, because of degeneracy of polar coordinates at the planes $r_j = 0$, $j = 1, 2$.

Consider the rough sketch of the proof from Section 3.1 and Lagrangian $L_f$ defined in (9). We shall modify it by restricting $L_f$ to the annuli $\{A_j(\omega^0)\}_{j\in\mathbb{Z}_+}$ and applying Theorem 8.1 to each of these restrictions. This will allow us to construct a modification of the variational principle (25) appropriate for our problem.

Write the remainder $P_f$ of $L_f$ in the form (11). If $\Gamma \subset \Lambda_k$ is contained in one of the axes, some terms might vanish, but not all of them. For a unit vector $e_k = (e_k^1, e_k^2)$ parallel to $\Gamma$, denote

$$P_{f,\Gamma}(\theta, v-\omega^0, t) = \sum_{p=0}^{3} (e_k^1)^p (e_k^2)^{3-p} P_p(\theta, v-\omega^0, t).$$

To apply Theorem 8.1 to $L_f$ in each of the annuli, we need to verify the following hypotheses

– (C1)–(C3) for all $\omega$'s in $\Gamma$,

– $(\mathrm{C4})_\omega$–$(\mathrm{C8})_\omega$ for $\omega$'s with small denominators in $\Gamma$, and

– (C9) for all $c$'s in $\mathcal{L}_\beta(\sigma_\Gamma)$.

First we shall verify all hypotheses except (C9). For this purpose, we define the following Lagrangian

$$L_{f,\Gamma}(\theta, v, t) = \frac{1}{2}\langle B^{-1}(v-\omega^0), (v-\omega^0)\rangle + \varepsilon P_{f,\Gamma}(\theta, v-\omega^0, t), \tag{26}$$

where $\varepsilon$ is nonzero and small. For this perturbation term $P_{f,\Gamma}$ using (15) (resp. (18)) define the first $(\omega,\Gamma)$-averaged potential, denoted by $P_{f,\omega,\Gamma} : \mathbb{T}^1_\Gamma \to \mathbb{R}$ (resp. the second $\omega$-averaged one, denoted by $P_{f,\omega} : \mathbb{T}^2_\omega \to \mathbb{R}$).

Rescale the annulus $A_j(\omega^0)$ to the unit size. Denote $T^\lambda_{\omega^0} : v \to \lambda(v-\omega^0) + \omega^0$ rescaling centered at $\omega^0$. We have that $T^{2^j/\delta_0}(A_j(\omega^0)) = A(\omega^0) = \{1/4 < |r| < 4\}$. Notice that $\Lambda_k$ is invariant under rescaling $T^\lambda_{\omega^0}$ for any $\lambda > 0$.

Restrict the Lagrangian $L_f$ to $\mathbb{T}^2 \times A_j(\omega^0) \times \mathbb{T}$. Consider the rescaling $T^{2^j/\delta_0}_{\omega^0}$ in $(v-\omega^0)$ of $\mathbb{T}^2 \times A_j(\omega^0) \times \mathbb{T}$ to $\mathbb{T}^2 \times A(\omega^0) \times \mathbb{T}$. it gives the new "rescaled" Lagrangian

$$\begin{aligned} L^j_f(\theta, v, t) = \frac{1}{2}&\langle B^{-1}(v-\omega^0), (v-\omega^0)\rangle \\ &+ \delta_0 2^{-j} \sum_{p=0}^{3} (v-\omega^0)_1^p (v-\omega^0)_2^{3-p} P^j_p(\theta, v-\omega^0, t), \end{aligned} \tag{27}$$

where $\{P^j_p(\theta, v-\omega^0, t) = P_p(\theta, T^{2^j/\delta_0}_{\omega^0} v - \omega^0, t)\}_{p=0}^3$ is $C^{s,d-3}$ smooth and defined in $\mathbb{T}^2 \times K^\alpha_(\omega^0, B) \times \mathbb{T}$. Denote the remainder

$$\textstyle\sum_{p=0}^{3} (v-\omega^0)_1^p (v-\omega^0)_2^{3-p} P^j_p(\theta, T^{2^j/\delta_0}_{\omega^0} v - \omega^0, t)$$

by $P^j_f(\theta, v-\omega^0, t)$.

The definitions of all hypotheses except (C9) involves averaged kinetic and potential energies. Fix any positive integer $j$. Notice that the unperturbed integrable parts are the same for both $L_f$ and $L^j_{f,\Gamma}$. These parts define the averaged kinetic energies (see (16) and (19)). Thus, the averaged kinetic energies of $L_f$ and $L^j_{f,\Gamma}$ coincide. Now consider the perturbation terms. A direct calculation based on (15) and (18) shows that up to a constant the first $\omega$-averaged (resp. the second $(\omega,\Gamma)$-averaged) potentials of $L_f$ and $L_{f,\Gamma}$ are coincide respectively. Therefore, for each $j \in Z_+$ we have that up to a constant

$\Gamma$-*averaged mechanical systems associated to* $L_f$ *and* $L^j_{f,\Gamma}$ *coincide.*

Denote the first $\omega$-averaged and the second $(\omega,\Gamma)$-averaged mechanical systems by $L_{\omega,f}$ and $L_{\omega,f,\Gamma}$ respectively. The definition of a small denominator $q_0$ involves only averaged mechanical systems. After $q_0$ is determined, notice that choosing $\delta_0$ small enough we need to verify $(\mathrm{C4})_\omega$–$(\mathrm{C8})_\omega$ only for at most one $\omega^0$ in the case $\omega^0$ is a rational with small denominator. Suppose $L_{f,\Gamma}$ in (26) satisfies hypotheses (C1)–(C3) and $(\mathrm{C4})_\omega$–$(\mathrm{C8})_\omega$ (if the latter is necessary). Then, there exists $\delta = \delta(\ell_0, \Gamma, P_{f,\Gamma}) > 0$ such that the variational principle (25) is well-defined for the Lagrangian $L_{f,\Gamma}$ and each $0 < |\varepsilon| < \delta$. In notations of Section 3.1, let $\delta$ be given by $2\delta = \delta_0$. Consider also the rescaling of the original Lagrangian $L_f$ in $(v-\omega^0)$, in order to see that the above arguments is applicable to $L_f$ too.

The definition of all hypotheses except (C9) involves averaged kinetic and potential energies. The above verification shows that if $L^1_f$ satisfies hypotheses (C1)–(C3) and $(\mathrm{C4})_\omega$–$(\mathrm{C8})_\omega$ (if the latter is necessary) on $\Gamma_\delta$, then $L^j_f$ satisfies these hypotheses on $\Gamma_\delta$ too. The only difference is that the constant in front of $P^j_f$ decreases as $j$ increases. This implies that *if* $P^1_f \in U^s_{\delta(\ell_0,\Gamma)}$, *then* $P^j_f \in U^s_{\delta(\ell_0,\Gamma)}$.

In notations of Section 3.1 we now verify that, for a $C^s$ Whitney open and dense set of remainders (11), the restriction of $L_f$ to any of $\mathbb{T}^2 \times K^\alpha_{\delta,j}(\omega^0, B) \times \mathbb{T}$ satisfies hypotheses (C1)–(C3) on $\Gamma_\delta$ and $(\mathrm{C4})_\omega$–$(\mathrm{C8})_\omega$ (if the latter is necessary). This implies that the variational principle (25) is well-defined. What is left to verify is hypothesis (C9), and the fact that velocity of minimizers of the variational principle (25) belong to the corresponding cones $K^\alpha_{\delta,j}(\omega^0, B)$.

Suppose the first potential satisfies hypotheses (C1)–(C3) on $\Gamma_\rho = [\omega^0, \omega^0 + \rho e_k]$, where $\rho$ is the radius of the ball such that (9) is defined on $\mathbb{T}^2 \times K^\alpha_\rho(\omega^0, B) \times \mathbb{T}$.

Consider now the rescaling $L^j_f$ of the restriction of $L_f$ on the annulus $A_j(\omega)$. By lemma 3.2, there is a smooth curve $\sigma_\Gamma$ with channel property. Denote by $\sigma^j_\gamma$ a part of this curve which connects $\mathcal{L}_\beta(\omega^0 + \delta_j e_k)$ and $\mathcal{L}_\beta(\omega^0 + \delta_{j+1} e_k)$. We can apply Theorem 8.1 with the curve $\sigma^j_\gamma$ as the curve with channel property. According to the variational principle (25) its minimizers velocity moves along $\Gamma$ with a certain error.

The Localization Lemma I shows that minimizers of the variational principle (25) for $L = L^j_f$ have velocity $C2^{-j/2}$-close to $\Gamma$ for some constant $C$. Therefore, after the backward rescaling, the velocity has to be $C2^{-3j/2}$-close to $\Gamma_j$. It implies that minimizing trajectory of (25) does not leave the cone $\mathbb{T}^2 \times K^{\alpha}_{\delta,j}(\omega^0, B) \times \mathbb{T}$.

The Localization Lemma II shows that non-degeneracy hypothesis (C9) holds for $L^j_f$ taking into account that the velocity value is $C2^{-j/2}$-close to $\Gamma$. By lemma 7.3, the set of restrictions of $L^j_f$ onto $\mathbb{T}^2 \times K^{\alpha}_{\delta,j}(\omega^0, B) \times \mathbb{T}$, where hypothesis (C9) holds, is $C^s$ Whitney open and dense. Therefore, there is a $C^s$ Whitney open and dense set of remainders $P_f$ in (9) such that for all positive integers $j$ the corresponding $P^j_f$ fulfills hypothesis (C9). This completes the proof of Theorem 1.2. $\square$

## Appendix A. Mather minimal sets

In this Appendix, we discuss basic objects of Mather's theory of minimal or action-minimizing measures [**Ma**]. This theory can be considered as an extension of KAM theory. Namely, it provides a large class of invariant sets for a Hamiltonian (or dual Euler-Lagrange) flow. KAM invariant tori and Aubry-Mather sets are examples of these sets. We need to define these notions to give the detailed scheme of the proof of Theorem 1.2 (see Section 3.1).

We start with a positive integer $n$, a smooth $n$-dimensional torus $\mathbb{T}^n$, and a $C^s$-smooth time periodic Lagrangian $L : \mathbf{T}\mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$, $(\theta, v) \in \mathbf{T}\mathbb{T}^n$, $s \geqslant 2$ which *satisfies hypotheses* (1–3) of section 2.1. Note that all definitions and results of this section can be given for any smooth compact manifold instead of $\mathbb{T}^n$. Later we apply it for $n = 2$ and the Lagrangian $L$ given by (9) near the zero section and extended outside to keep fiber-convexity.

We say that $\mu$ is a *probability measure*, if it is a Borel measure of total mass one. Let $\mathcal{P}_L$ be the space of probability measures on $\mathbf{T}\mathbb{T}^n \times \mathbb{T}$ invariant with respect to Euler-Lagrange flow (7). We shall consider probability measures only from $\mathcal{P}_L$. If $\eta$ is a closed one-form on $\mathbb{T}^n \times \mathbb{T}$, we may associate to it a real valued function $\widehat{\eta}$ on $\mathbf{T}\mathbb{T}^n \times \mathbb{T}$ as follows: express $\eta$ in the form

$$\eta = \eta_{\mathbb{T}^n} d\theta + \eta_\tau d\tau,$$

where $\eta_{\mathbb{T}^n}$ is the restriction of $\eta$ to $\mathbb{T}^n$ and $\eta_\tau :\ \mathbb{T} \to \mathbb{R}$ and set

$$\widehat{\eta} = \eta_{\mathbb{T}^n} + \eta_\tau \circ \pi,$$

where $\pi : \mathbf{T}\mathbb{T}^n \times \mathbb{T} \to \mathbb{T}^n \times \mathbb{T}$ denotes the natural projection. This function has the property

$$\int_a^b \widehat{\eta}(d\gamma(t), t) dt = \int_{(\gamma,\tau)} \eta,$$

for every absolutely continuous curve $\gamma : [a, b] \to \mathbb{T}^n$ with the right hand being usual integral over the curve $(\gamma, \tau) : [a, b] \to \mathbb{T}^n \times \mathbb{T}$ defined by $(\gamma, \tau)(t) = (\gamma(t), t \bmod 1)$.

If $\mu$ is an invariant probability measure on $\mathbf{T}\mathbb{T}^n \times \mathbb{T}$, its *average action* is defined as

$$A(\mu) = \int L(\theta, v, t)\ d\mu(\theta, v, t).$$

Since $L$ is bounded below, this integral is well defined, although it may be equal to $+\infty$. Next step is to define an appropriate notion which generalizes the rotation vector of a periodic trajectory. If $A(\mu) < \infty$, one can define *a rotation vector* $\rho(\mu) \in H_1(\mathbb{T}^n, \mathbb{R})$ *of a probability measure* $\mu$ by

$$\langle \rho(\mu), [\eta]_{\mathbb{T}^n} \rangle + [\eta]_{\mathbb{T}} = \int \widehat{\eta}(\theta, v, t)\ d\mu(\theta, v, t) \tag{28}$$

for every $C^1$ closed one form $\eta$ on $\mathbb{T}^n \times \mathbb{T}$, where

$$[\eta] = ([\eta]_{\mathbb{T}^n}, [\eta]_{\mathbb{T}}) \in H^1(\mathbb{T}^n \times \mathbb{T}, \mathbb{R}) = H^1(\mathbb{T}^n, R) \times \mathbb{R}$$

denotes the de Rham cohomology class and $\langle\ ,\ \rangle$ denotes the dual pairing $H_1(\mathbb{T}^n, \mathbb{R}) \times H^1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$. The idea of a rotation vector is classical and goes back to Schwartzman's asymptotic cycles (see [**Ma**] in the time independent case), but in the time dependent case definitions and arguments are the same. In [**Ma**], by using a Krylov-Bogoliuboff type argument, Mather proved the following result:

***Lemma A.1***. — *For every homology class $h \in H_1(\mathbb{T}^n, \mathbb{R})$ there exists a probability measure $\mu \in \mathcal{P}_L$ such that $A(\mu) < \infty$ and $\rho(\mu) = h$.*

Such a probability measure $\mu \in \mathcal{P}_L$ is called *minimal* or *action-minimizing* if

$$A(\mu) = \min\{A(\nu)\ :\ \rho(\nu) = \rho(\mu)\},$$

where $\nu$ ranges in $\mathcal{P}_L$ and $A(\nu) < \infty$. If $\rho(\mu) = h$, we also say that $\mu$ is *$h$-minimal*. Denote by $\mathcal{M}_h$ closure of the union of supports of all $h$-minimal measures from $\mathcal{P}_L$. This set $\mathcal{M}_h \subset T\mathbb{T}^n \times \mathbb{T}$ is called *Mather set*. By the above lemma $\mathcal{M}_h$ is always nonempty.

A probability measure $\mu \in \mathcal{P}_L$ is *$c$-minimal* for $c \in H^1(\mathbb{T}^n, \mathbb{R})$, if it minimizes

$$A_c(\mu) = A(\mu) - \langle \rho(\mu), c \rangle$$

over all invariant probability measures. $A_c(\mu)$ as above is called $c$-action of a measure. Mather [**Ma**] also proved the following result:

***Lemma A.2***. — *For every cohomology class $c \in H^1(\mathbb{T}^n, \mathbb{R})$ there exists a $c$-*minimal *probability measure $\mu \in \mathcal{P}_L$ such that $A(\mu) < \infty$.*

Denote by $\mathcal{M}^c$ closure of supports of the union of all $c$-minimal measures from $\mathcal{P}_L$. $\mathcal{M}^c \subset T\mathbb{T}^n \times \mathbb{T}$ is also called *Mather set*. By the above lemma $\mathcal{M}^c$ is always nonempty. Mather [**Ma**] proved that

$$\cup_{h \in H_1(\mathbb{T}^n, R)} \mathcal{M}_h = \cup_{c \in H^1(\mathbb{T}^n, R)} \mathcal{M}^c.$$

It turns out that $\mathcal{M}^c$ can be "nicely" projected onto the base $\mathbb{T}^n \times \mathbb{T}$.

***Graph Theorem***. — *Let $\pi : \mathbf{T}\mathbb{T}^n \times \mathbb{T} \to \mathbb{T}^n \times \mathbb{T}$ be the natural projection onto the base. Then, for any $c \in H^1(\mathbb{T}^n, \mathbb{R})$, the corresponding Mather set $\mathcal{M}^c$ is a Lipschitz graph over the base $\mathbb{T}^n \times \mathbb{T}$, i.e. $\pi^{-1}|_{\pi\mathcal{M}^c} : \pi\mathcal{M}^c \to \mathcal{M}^c$.*

Call $\pi\mathcal{M}^c$ *projected Mather set* and denote $\widetilde{\mathcal{M}}^c = \pi\mathcal{M}^c$.

***Definition A.3***. — The function

$$\beta_L : \ H_1(\mathbb{T}^n, \mathbb{R}) \longrightarrow \mathbb{R}, \quad \beta_L(h) = A(\mu),$$

where $\mu$ is an $h-$minimal probability measure, is called *Mather's $\beta$-function*. The function

$$\alpha_L \ : \ H^1(\mathbb{T}^n, \mathbb{R}) \longrightarrow \mathbb{R}, \quad \alpha_L(c) = \sup_{h \in H_1(\mathbb{T}^n, R)} \{\langle h, c\rangle - \beta_L(h)\}$$

is called *Mather's $\alpha$-function*.

***Lemma A.4* ([Ma])**. — *Both $\alpha$-function and $\beta$-function are convex and conjugate by the Legendre transform.*

By definition,

$$\beta_L(h) + \alpha_L(c) \geqslant \langle h, c\rangle, \quad h \in H_1(\mathbb{T}^n, \mathbb{R}),\ c \in H^1(\mathbb{T}^n, \mathbb{R}).$$

To distinguish from the standard Legendre transform (6) the map

$$\mathcal{L}_\beta : H_1(\mathbb{T}^n, \mathbb{R}) \longrightarrow \{\text{compact, convex, non-empty subsets of } H^1(\mathbb{T}^n, \mathbb{R})\}, \tag{29}$$

defined by letting $\mathcal{L}_\beta(h)$ be the set of $c \in H^1(\mathbb{T}^n, \mathbb{R})$ for which the inequality in (9) becomes equality, is called *Fenchel-Legendre transform*. In what follows, we shall identify each $h-$minimal invariant probability measure with a $c-$minimal invariant probability measure, provided that $c \in \mathcal{L}_\beta(h)$.

For an absolutely continuous curve $\gamma : [a, b] \to \mathbb{T}^n$, let us denote $d\gamma(t) = (\gamma(t), \dot\gamma(t))$. The above is well defined for a.e. $t$. For such $\gamma$ and a closed one form $\eta$ with $[\eta]_{\mathbb{T}^n} = c$, we call *$c$-action*

$$A_c(\gamma) = \int_a^b (L - \widehat{\eta})(d\gamma(t), t)dt, \tag{30}$$

where $\widehat{\eta}(\theta, \dot\theta, t) = \eta_{T^n}(\theta, t)\dot\theta + \eta_T(\theta, t)$ if $\eta = {\eta_T}^n d\theta + \eta_T dt$. Notice that $c$-action does not depend on a choice of $\eta$ in the cohomology class $c$. A closed one form $\eta$ on $\mathbb{T}^n \times \mathbb{T}$ is called *Mañé critical* if and only if

$$\min_{\mu \in \mathcal{P}_L} \left\{ \int (L - \widehat{\eta}) d\mu \right\} = 0.$$

Since each closed one form can be written as $[\eta] = ([\eta]_{\mathbb{T}^n}, [\eta]_{\mathbb{T}})$, by the definition of $\alpha$-function for Mañé critical one form we have $[\eta]_{\mathbb{T}} = -\alpha([\eta]_{\mathbb{T}^n})$. We also say that $\eta$ is *Mañé supercritical* if $[\eta]_{\mathbb{T}} > -\alpha_L([\eta]_{\mathbb{T}^n})$ and *Mañé subcritical* if $[\eta]_{\mathbb{T}} < -\alpha_L([\eta]_{\mathbb{T}^n})$. We shall explain geometric meaning of sub and super criticality in the next section.

We say that an absolutely continuous curve $\gamma : \mathbb{R} \to \mathbb{T}^n$ is an absolute *c-minimizer* if, for any interval $[a, b]$ and any absolutely continuous curve $\gamma_1 : [d, e] \to \mathbb{T}^n$ such that $d \equiv a \pmod 1$ and $e \equiv b \pmod 1$, we have

$$\int_a^b (L - \widehat{\eta})(d\gamma(t), t)dt \leqslant \int_d^e (L - \widehat{\eta})(d\gamma_1(t), t)dt,$$

where $\eta$ is a Mañé critical closed one form on $\mathbb{T}^n \times \mathbb{T}$ such that $[\eta]_{\mathbb{T}^n} = c$. Notice that the time intervals $b - a$ and $e - d$ are not necessarily the same. Completeness of the Euler-Lagrange flow (see property 3 of Lagrangian) implies that every $c$-minimal curve is $C^1$-smooth and, therefore, as smooth as $L$ is. So it is $C^{s-1}$-smooth. Denote the union of all sets of $c$-minimizers $\{(d\gamma(t), t) : \ t \in \mathbb{R}\} \subset T\mathbb{T}^n \times \mathbb{T}$ by $\mathcal{N}^c$ and call it *Mañé set*. This set is certainly a closed set.

We now introduce the notion of barrier function and we deal with another set of trajectories associated to a cohomology class $c \in H^1(\mathbb{T}^n, \mathbb{R})$. The barrier function is introduced in [**Ma2**] and is a generalization of Peierl's barrier. Let $\theta_1, \theta_2 \in \mathbb{T}^n$, $\tau_1, \tau_2 \in \mathbb{T} \geqslant 0$, and $\eta$ is a Mañé critical one closed one form on $\mathbb{T}^n \times \mathbb{T}$ such that $[\eta]_{\mathbb{T}^n} = c$. Define

$$h_{\eta,T}((\theta_1, \tau_1), (\theta_2, \tau_2)) = \inf \int (L - \widehat{\eta})(d\gamma(t), t)dt, \tag{31}$$

where the infinum is taken over all absolutely continuous curves $\theta : [a, b] \to \mathbb{T}^n$ such that $a \equiv \tau_1 \pmod 1$, $b \equiv \tau_2 \pmod 1$, $\theta(a) = \theta_1$, $\theta(b) = \theta_2$, and $b - a \geqslant T$. Define the barrier function

$$h_\eta((\theta_1, \tau_1), (\theta_2, \tau_2)) = \liminf_{T \to +\infty} h_{\eta,T}((\theta_1, \tau_1), (\theta_2, \tau_2)). \tag{32}$$

In [**Ma2**], Mather has proved that the limit exists. He also introduced a pseudo-metric

$$\rho_c((\theta_1, \tau_1), (\theta_2, \tau_2)) = h_\eta((\theta_1, \tau_1), (\theta_2, \tau_2)) + h_\eta((\theta_2, \tau_2), (\theta_1, \tau_1)). \tag{33}$$

It turns out that this construction is independent of $\eta$, provided $\eta$ is Mañé critical and $[\eta]_{\mathbb{T}^n} = c$. One can show that $\rho_c \geqslant 0$, satisfies the triangle inequality, and is independent of the choice of a Mañé critical closed one form $\eta$. The set

$$\mathcal{A}^c = \{(\theta, \tau) : \ \rho_c((\theta, \tau), (\theta, \tau)) = 0\}$$

is called *Aubry set*. One can show [**Ma2**] that

$$\mathcal{M}^c \subset \mathcal{A}^c \subset \mathcal{N}^c \subset T\mathbb{T}^n \times \mathbb{T}$$

for all $c \in H^1(\mathbb{T}^n, \mathbb{R})$ and $\mathcal{A}^c$ also satisfies the Graph Theorem stated above.

## Appendix B. Proofs of auxiliary lemmas

***Lemma B.1***. — *For positive $\rho$ and $0 < \alpha < 1$ consider the space of $C^{s,d}$ smooth function on $\widehat{\mathcal{K}}^\alpha_\rho \ni (\theta, r)$ with the natural $C^s$ Whitney topology in $\mathcal{K}^\alpha_\rho$, where $m \leqslant d \leqslant$*

*$s \leqslant \infty$, $d, m \in \mathbb{Z}_+$. Then for any $C^s$ Whitney open dense set $\mathcal{D}^s_{d-m}$ of $(m+1)$-tuples $(g_0, \dots, g_m)$ of $C^{s,d-m}$ smooth functions on $\mathcal{K}^\alpha_\rho$, the set of functions of the form*

$$g(\theta, r) = \sum_{p=0}^{m} r_1^p r_2^{m-p} g_p(\theta, r) \tag{34}$$

*with $(m+1)$-tuples $(g_0, \dots, g_m) \in \mathcal{D}^s_{d-m}$ intersected with $C^s(\mathcal{K}^{2\alpha}_\rho)$ is $C^s$ Whitney open dense.*

*Proof of Lemma B.1.* — Pick $r = (r_1, r_2) \in K^\alpha_\rho$, *i.e.* $0 < \alpha r_1 < r_2 < \rho$ and $0 < \alpha r_2 < r_1 < \rho$. We find two functions $f_{11}(\theta, r)$ and $f_{21}(\theta, r)$ defined on $\mathcal{K}^\alpha_\rho$ satisfying

$$r_1 f_{11}(\theta, r) - r_2 f_{21}(\theta, r) = f(\theta, r). \tag{35}$$

Consider two functions equalities:

$$f(\theta, r) - f(\theta, r_1, \frac{\alpha}{2} r_1) = r_2 g_2(\theta, r), \quad f(\theta, r_1, \frac{\alpha}{2} r_2) = r_1 g_1(\theta, r).$$

To define $f_1$ and $f_2$ by explicit formulas inside $K^\alpha_\rho$ consider the coordinate change: $\widetilde{r}_1 = (r_1, \alpha r_1)$, $r_2 = (0, r_2 - \alpha r_1)$, and $\widetilde{f}(\theta, \widetilde{r}_1, \widetilde{r}_2) = f(\theta, r_1, r_2)$. By Hadamard-Torricelli's lemma

$$\begin{aligned}
f_1(\theta, \widetilde{r}_1, 0) &= \int_0^1 \frac{\partial \widetilde{f}}{\partial \widetilde{r}_1}(\theta, t\widetilde{r}_1, 0) dt = \frac{\widetilde{f}(\theta, \widetilde{r}_1, 0)}{r_1} \\
r_1 g_1(\theta, \widetilde{r}_1, 0) &= \widetilde{f}(\theta, \widetilde{r}_1, 0) - \widetilde{f}(\theta, 0) = f(\theta, r_1, \frac{\alpha}{2} r_1) \\
g_2(\theta, \widetilde{r}_1, \widetilde{r}_2) &= \int_0^1 \frac{\partial \widetilde{f}}{\partial \widetilde{r}_2}(\theta, \widetilde{r}_1, t\widetilde{r}_2) dt = \frac{\widetilde{f}(\theta, \widetilde{r}_1, \widetilde{r}_2) - \widetilde{f}(\theta, \widetilde{r}_1, 0)}{r_2 - \frac{\alpha}{2} r_1}.
\end{aligned}$$

This implies that for $f_{11} = g_1 + \frac{\alpha}{2} g_2$ and $f_{21} = g_2$ (35) holds true.

Notice that $f_{11}$ and $f_{21}$ have zero of order $(m-1)$ in $r$ in the sense that they are $C^{s,d-1}$ smooth. Application of Hadamard-Torricelli's lemma to $f_{11}$ and $f_{21}$ gives explicit formulas for functions $f_{02}, f_{12}$, and $f_{22}$, which have zero of order $(m-2)$ in $r$, namely, functions belong to $C^{s,d-2}(\mathcal{K}^\alpha_\rho)$. After $m$ steps we get explicit formulas for functions $f_{0m}, \dots, f_{mm}$. Denote $f_{pm} = f_p$ for $p = 0, \dots, m$. These functions satisfy (34) which completes the proof. $\square$

*Proof of Lemma 2.1.* — We start with an integrable truncation of $f$. Let $\widehat{f}$ be defined by (1) with $P_j, Q_j \equiv 0$. Then, the time 1 map of the Hamiltonian

$$H_0(\theta, r) = \omega_1 r_1 + \omega_2 r_2 + \frac{1}{2}\langle Br, r\rangle \tag{36}$$

coincides with $f_0$, as an easy calculation shows. Construct a time periodic deformation $\{H(\cdot, t)\}_{t \in \mathbb{T}}$ of $H_0$ so that the time 1 map of $H(\cdot, t)$ coincides with $f$ and so that $H(\cdot, t)$ is $C^{s+1}$ close to $H_0(\cdot)$ near $r = 0$ for all $t \in \mathbb{T}$. Since $H_0$ is convex in $r$, $s \geqslant 2$, and we are interested in small $r$, it implies the desired convexity of $H(\cdot, t)$ in $r$ for each $t \in \mathbb{T}$.

The construction of $\{H(\cdot, t)\}_{t \in \mathbb{T}}$ is done using generating functions. We recall a standard fact from Hamiltonian system (see *e.g.* Arnold [**Ar2**] sect. 48) for a $C^s$

smooth symplectic map $g(\theta,r)=(\Theta,R)$, $\theta,\Theta\in\mathbb{T}^n$, $r,R\in\mathbb{R}^n_+$: one can define a $C^{s+1}$ smooth generating function $S_g(\theta,\Theta)$ so that

$$\begin{cases} r=-\partial_\theta S_g(\theta,\Theta),\\ R=\partial_\Theta S_g(\theta,\Theta).\end{cases} \tag{37}$$

The function $S_g(\theta,\Theta)$ above is defined up to a constant. Direct calculation for $\widehat{f}$ and $f$ in a small $r$-neighborhood of zero show that

$$\begin{aligned} &S_{\widehat{f}}(\Theta,\theta)=\frac{1}{2}\langle B^{-1}(\Theta-\theta-\omega),(\Theta-\theta-\omega)\rangle \text{ and}\\ &S_f(\Theta,\theta)=S_{\widehat{f}}(\Theta,\theta)+0(|(\Theta-\theta-\omega)|^3).\end{aligned} \tag{38}$$

Consider a smooth family of generating functions $\{\widetilde{S}_t\}_{t\in[0,1]}$ given by

$$\widetilde{S}_t(\theta,\Theta)=\begin{cases}\dfrac{h(t)}{2}\big\langle B^{-1}(\Theta-\theta-\omega/h(t)),(\Theta-\theta-\omega/h(t))\big\rangle & \text{for } t\in(0,\frac12]\\ h(t)\widetilde{S}_{1/2}(\theta,\Theta)+(1-h(t))S_f(\theta,\Theta) & \text{for } t\in[\frac12,1],\end{cases}$$

where $h$ is a smooth positive function away from zero, $h(1)=h'(1/2)=0$, $h(1/2)=1$, and $1/h(t)=t$ near $t=0+$. The choice of $h$ is designed so that $\widetilde{S}_t$ is sufficiently smooth with respect to $t$ for $t\in(0,1]$. By construction, $\widetilde{S}_t$ generates a smooth family $\{\widetilde{f}_t\}_{t\in(0,1]}$ of exact symplectic twist maps (see [**Go**] sect 26 or [**McS**] sect 9.3). More precisely, $\widetilde{f}_t(\theta,r)=\big(\theta+(h(t))^{-1}(\omega+Br),r\big)$ for $0\leqslant t\leqslant 1/2$ and $\lim_{t\to 0+}\widetilde{f}_t=\mathrm{Id}$. Define $s_t(\theta,r)=\widetilde{S}_t(\theta,\theta+(h(t))^{-1}(\omega+Br))$ for $0\leqslant t\leqslant 1/2$. It becomes

$$s_t(\theta,r)=(2h(t))^{-1}\langle Br,r\rangle.$$

By assumption, $1/h(t)=t$ near zero $s_t$ can be smoothly continued for all $t\in[0,1]$ with $s_0\equiv 0$ and $s_1(\theta,r)=S_f(\theta,\Theta(\theta,r))$, where $\Theta(\theta,r)$ is given by $f(\theta,r)=(\Theta,R)$ for some $R$. Now we can write:

$$\widetilde{f}_t^*(rd\theta)-rd\theta=ds_t,\quad t\in[0,1]. \tag{39}$$

It shows that $\{\widetilde{f}_t\}_{t\in[0,1]}$ is a Hamiltonian isotopy. By standard results from symplectic geometry, obtained by combining homotopy formula and (39) (see *e.g.* Prop. 9.18 in [**McS**] or Thm. 58.9 in [**Go**]) this family generates Hamiltonian functions $\{H(\cdot,t)\}_{t\in[0,1]}$ as follows. Denote by $\widetilde{X}_t$ vector fields generated by isotopy $\widetilde{f}_t$, *i.e.* given by $\widetilde{X}_t(\theta,r)=(d\widetilde{f}_t/dt)\big((\widetilde{f}_t)^{-1}(\theta,r)\big)$, by $i_{\widetilde{X}_t}\alpha$ the interior derivative of a 1-form $\alpha$ by $\widetilde{X}_t$, and by $s_t(\theta,r)$ a form of generating function, given by (39). Then

$$\widetilde{H}_f(\cdot,t)=i_{\widetilde{X}_t}rd\theta-(\widetilde{f}_t^{-1})^*\left(\frac{d}{dt}s_t\right).$$

One can check that $d\widetilde{H}_f(\cdot,t)=-i_{\widetilde{X}_t}dr\wedge d\theta$.

By the construction, the time map of $\widetilde{H}_f(\cdot,t)$ from time $t=0$ to $t=1$ equals $f$. However, $\{H(\cdot,t)\}_{t\in[0,1]}$ is not necessarily periodic in $t$. To attain periodicity we slightly modify the above construction. For small $t$, say $t\in[0,\delta]$, we have $f_t(\theta,r)=$

$(\theta + tBr, r)$ and, therefore, $\widetilde{H}_f(\cdot, t) = H_0(\theta, r)$. Let us define $f_{1-\tau} = f_\tau^{-1} \circ f$ for $\tau \in [0, \delta]$. Let $\widehat{S}_{1-\tau}(\theta, \Theta)$ be the generating function of $f_{1-\tau}$ with $\widehat{S}_{1-\tau}(\theta, \theta + \omega) = 0$. Consider the following family of generating functions

$$S_t(\theta, \Theta) = \begin{cases} \widetilde{S}_t(\theta, \Theta) & \text{for } t \in (0, 1-\delta] \\ (1 - g(t))\widetilde{S}_t(\theta, \Theta) + g(t)\widehat{S}_{1-t}(\theta, \Theta)) & \text{for } t \in [1-\delta, 1], \end{cases}$$

where $g(t)$ is a $C^{s+1}$ smooth function on $[1-\delta, 1]$, with $g(1-\delta) = 0$, $g(1) = 1$, and $g^{(p)}(1) = g^{(p)}(1-\delta) = 0$ for $p = 0, 1, \dots, s+1$. By construction, $S_t$ defines a Hamiltonian isotopy $\{f_t\}_{t \in (0,1]}$ with $f_1 = \widetilde{f}_1 = f$. By the same token as above $\{f_t\}_{t \in (0,1]}$ defines a Hamiltonian function $H_f(\cdot, t)$ which is $C^{s+1}$ smooth and periodic in time $t$.

In order to verify positive definiteness, we consider two cases: $t \in [0, 1/2]$ and $t \in [1/2, 1]$. In the first case, near $t = 0+$, we have $f_t(\theta, r) = (\theta + tBr, r)$ and, therefore, $H_f(\theta, r, t) = H_0(\theta, r)$. Similarly, for $0 \leqslant t \leqslant 1/2$, but not near zero, we have that $H_f(\theta, r, t) = m(t)H_0(\theta, r)$, where $m(t)$ is a smooth strictly positive function (explicitly computable from $h(t)$). Definition (36) of $H_0$ and hypothesis (H2) of positive definiteness of $B$ implies positive definiteness of $\partial^2_{rr} H_f$ for $0 \leqslant t \leqslant 1-\delta$.

In the case $t \in [1-\delta, 1]$, by definition, $S_t(\theta, \Theta) = S_{\widehat{f}}(\theta, \Theta) + 0(|(\Theta - \theta - \omega)|^3)$. Explicit calculation gives that the underlying Hamiltonian has the form

$$H(\theta, r, t) = \omega_1 r_1 + \omega_2 r_2 + \frac{1}{2}\langle Br, r\rangle + 0(|r|^3). \tag{40}$$

It implies the Hessian $\partial^2_{rr} H(\cdot, t)$ is close to $B$ and, therefore, positive definite. This proves the lemma. $\square$

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# ON THE STOKES GEOMETRY OF HIGHER ORDER PAINLEVÉ EQUATIONS

*by*

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***Abstract.* —** We show several basic properties concerning the relation between the Stokes geometry (*i.e.*, configuration of Stokes curves and turning points) of a higher order Painlevé equation with a large parameter and the Stokes geometry of (one of) the underlying Lax pair. The higher-order Painlevé equation with a large parameter to be considered in this paper is one of the members of $P_J$-hierarchy with $J = \mathrm{I}$, II-1 or II-2, which are concretely given in Section 1. Since we deal with higher order equations, the Stokes curves may cross; some anomaly called the Nishikawa phenomenon may occur at the crossing point, and in this paper we analyze the mechanism why and how the Nishikawa phenomenon occurs. Several examples of Stokes geometry are given in Section 5 to visualize the core part of our results.

***Résumé* (Sur la géométrie de Stokes des équations de Painlevé d'ordre supérieur)**
Nous exhibons plusieurs propriétés fondamentales liant, d'une part, la géométrie de Stokes (*i.e.*, la configuration des courbes de Stokes et des points tournants) d'une équation de Painlevé d'ordre supérieur à grand paramètre et, d'autre part, la géométrie de Stokes de l'une des paires de Lax sous-jacentes. L'équation de Painlevé d'ordre supérieur à grand paramètre considérée est l'une des équations de la hiérarchie $P_J$ pour $J = \mathrm{I}$, II-1 ou II-2 que nous détaillons dans le paragraphe 1. Les équations étant d'ordre supérieur leurs lignes de Stokes peuvent se croiser et l'anomalie connue sous le nom de « phénomène de Nishikawa » peut se produire aux points de croisement. Nous analysons le mécanisme par lequel ce phénomène de Nishikawa apparaît. Plusieurs exemples de géométrie de Stokes sont donnés dans le paragraphe 5 en vue d'une visualisation de la partie centrale de nos résultats.

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## 0. Introduction

This paper is the first of a series of our papers on the exact WKB analysis of higher order Painlevé equations. For the sake of the clarity and the uniformity of the description we restrict our consideration in this paper to the $P_{\mathrm{I}}, P_{\mathrm{II\text{-}1}}$ and $P_{\mathrm{II\text{-}2}}$ hierarchies with a large parameter $\eta$, which are described explicitly in Section 1. Although these hierarchies are basically the same as those discussed by Shimomura ([**S2**]), Gordoa-Pickering ([**GP**]) and Gordoa-Joshi-Pickering ([**GJP**]), we need to appropriately introduce a large parameter $\eta$ in their coefficients together with the underlying systems of linear differential equations (the so-called Lax pairs) so that we may develop the WKB analysis of the hierarchies in question. As is evident in the series of papers ([**KT1, AKT2, KT2, T1**]; see [**KT3**] for their résumé), the relations between the Stokes geometry for (one of) the Lax pair and the appropriately defined Stokes geometry for the Painlevé equation play the key role in the WKB analysis of the traditional Painlevé equations, *i.e.*, the second order differential equations first studied by Painlevé and Gambier. One of our main purposes of this paper is to show that the relations observed for the traditional Painlevé equations remain to hold for each member in the Painlevé hierarchies considered in this paper (Section 2). Another main purpose of this paper is to analyze why the novel and interesting phenomena numerically discovered by one of us (Y.N.) should occur in our context (Section 3). To analytically detect where the phenomena (the so-called Nishikawa phenomena) are observed, we introduce the notion of new Stokes curves in Section 4. In Section 5 we present several illuminating examples of Stokes geometry for higher order Painlevé equations and the Stokes geometry of their underlying Lax pair. Appendix A gives a proof of some properties of auxiliary functions $\mathcal{K}_j$ and $K_j$ used in Sections 1 and 2 to write down the $P_{\mathrm{II\text{-}1}}$-hierarchy with a large parameter. In Appendix B we note that the $P_{\mathrm{I}}$-hierarchy with a large parameter is equivalent to a hierarchy discussed by Gordoa and Pickering ([**GP**]) if a large parameter is appropriately introduced.

As the discussion of [**KT1**] etc. uses a Lax pair of single differential equations, the results there may look pretty different from the results in this paper, where a Lax pair of $2\times 2$ systems is used, that is, the framework of Flaschka-Newell ([**FN**]) and Jimbo-Miwa ([**JM**]) is used instead of the framework of Okamoto ([**O**]); in particular, the apparent singularities which played an important role in [**KT1**] etc. do not appear in this paper. Hence we end this introduction with briefly recalling the geometric results in [**KT1**] which are reformulated for a Lax pair of matrix equations. For the sake of simplicity we consider only the first Painlevé equation. Thus, following [**JM**], we start with the following Lax pair:

$$(0.1)\qquad \begin{cases} \left(\dfrac{\partial}{\partial x} - \eta A\right)\psi = 0, & (0.1.\mathrm{a}) \\ \left(\dfrac{\partial}{\partial t} - \eta B\right)\psi = 0, & (0.1.\mathrm{b}) \end{cases}$$

where

$$A = \begin{pmatrix} v(t,\eta) & 4(x-u(t,\eta)) \\ x^2 + u(t,\eta)x + u(t,\eta)^2 + t/2 & -v(t,\eta) \end{pmatrix} \tag{0.2}$$

and

$$B = \begin{pmatrix} 0 & 2 \\ x/2 + u(t,\eta) & 0 \end{pmatrix}. \tag{0.3}$$

That is, we consider an isomonodromic deformation (with respect to the variable $t$) of the first matrix equation (0.1.a); the second equation (0.1.b) explicitly describes this deformation. To obtain (0.1) we have introduced a large parameter $\eta$ to the equation (C.2) of [**JM**, p. 437] so that the resulting compatibility condition may become the first Painlevé equation with a large parameter $\eta$ in [**KT1**] etc. We have also interchanged the first component and the second component of the unknown vector $\psi$ for the sake of uniformity of presentation in this paper. The compatibility condition of the equations (0.1.a) and (0.1.b), *i.e.*,

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + \eta(AB - BA) = 0 \tag{0.4}$$

can be readily seen to be equivalent to the following system $(H_{\mathrm{I}})$:

$$(H_{\mathrm{I}}) : \begin{cases} \dfrac{du}{dt} = \eta v \\ \dfrac{dv}{dt} = \eta(6u^2 + t) \end{cases}. \tag{0.5}$$

We next construct the so-called 0-parameter solution $(\widehat{u}, \widehat{v})$ of $(H_{\mathrm{I}})$ which has the following form:

$$\widehat{u}(t,\eta) = \widehat{u}_0(t) + \eta^{-1}\widehat{u}_1(t) + \cdots, \tag{0.6}$$

$$\widehat{v}(t,\eta) = \widehat{v}_0(t) + \eta^{-1}\widehat{v}_1(t) + \cdots. \tag{0.7}$$

It is known that, although $(\widehat{u}, \widehat{v})$ is a divergent series, it is Borel summable. Note that

$$6\widehat{u}_0^2 + t = 0 \quad \text{and} \quad \widehat{v}_0 = 0 \tag{0.8}$$

hold and that $\widehat{u}_j$ and $\widehat{v}_j$ $(j \geq 1)$ are recursively determined. Substituting $(\widehat{u}, \widehat{v})$ into the coefficients of $A$ and $B$, we let $A_0$ and $B_0$ denote their top degree part in $\eta$, that is,

$$A_0 = \begin{pmatrix} 0 & 4(x - \widehat{u}_0(t)) \\ x^2 + \widehat{u}_0(t)x + \widehat{u}_0(t)^2 + t/2 & 0 \end{pmatrix}, \tag{0.9}$$

$$B_0 = \begin{pmatrix} 0 & 2 \\ x/2 + \widehat{u}_0(t) & 0 \end{pmatrix}. \tag{0.10}$$

To consider the linearization of $(H_{\mathrm{I}})$ at $(\widehat{u}, \widehat{v})$, we set $u = \widehat{u} + \Delta u$ and $v = \widehat{v} + \Delta v$ in (0.5) and consider the part linear in $(\Delta u, \Delta v)$. (Although the terminology "linearization" used here has a completely different meaning from that used in [**JM**], we hope there is no fear of confusions; in [**JM**] etc., the linearization of $(H_{\rm I})$ means the system (0.1) of linear differential equations.) Then we obtain

$$\frac{d}{dt}\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix} = \eta\begin{pmatrix}0 & 1\\ 12\widehat{u} & 0\end{pmatrix}\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix}. \tag{0.11}$$

Let $C$ and $C_0$ respectively denote

$$\begin{pmatrix}0 & 1\\ 12\widehat{u} & 0\end{pmatrix} \tag{0.12}$$

and

$$\begin{pmatrix}0 & 1\\ 12\widehat{u}_0 & 0\end{pmatrix}. \tag{0.13}$$

Concerning the matrices $A_0, B_0$ and $C_0$ we find the following several relations. First of all, (0.8) immediately entails

$$A_0 = 2(x-\widehat{u}_0)B_0. \tag{0.14}$$

This relation leads to the following

***Fact A***

(i) *The equation* (0.1.a) *has one double turning point* $x=\widehat{u}_0(t)$ *if* $\widehat{u}_0 \neq 0$.

(ii) *It has one simple turning point* $x=-2\widehat{u}_0(t)$ *if* $\widehat{u}_0 \neq 0$, *and this point is a turning point of the equation* (0.1.b).

Here and in what follows we use the terminology "a turning point" for a matrix equation like (0.1.a) to mean, as usual, a point where eigenvalues of its highest degree part in $\eta$ (*i.e.*, the matrix $A_0$ in the case of (0.1.a)) merge. In other words, a turning point is a zero of the discriminant of the characteristic equation of the highest degree part, and it is said to be simple (resp. double) if it is a simple (resp. double) zero of the discriminant. We next obtain

$$12\widehat{u}_0(t)\widehat{u}_0(t)' + 1 = 0 \tag{0.15}$$

by differentiating (0.8). Then this relation proves the following

***Fact B***. — *The eigenvalues* $\lambda_\pm$ *of* $A_0$ (*i.e.*, $\pm 2(x-\widehat{u}_0)\sqrt{x+2\widehat{u}_0}$) *and the eigenvalues* $\mu_\pm$ *of* $B_0$ (*i.e.*, $\pm\sqrt{x+2\widehat{u}_0}$) *satisfy the following relation:*

$$\frac{\partial}{\partial t}\lambda_\pm = \frac{\partial}{\partial x}\mu_\pm. \tag{0.16}$$

The following Fact C might look too trivial to note, but for the sake of later references we note it here.

***Fact C***. — *We find*

$$\det(\nu - C_0) = 4\det(\mu - B_0)\Big|_{x=\widehat{u}_0,\mu=\nu/2}. \tag{0.17}$$

In what follows a point is called a turning point of a non-linear equation when it is a turning point of the linearization of the non-linear equation at a 0-parameter solution. (Hence, logically speaking, we have to specify the 0-parameter solution to define the notion of a turning point. However, the situation is usually obvious and we omit the explicit reference to the 0-parameter solution unless it is confusing.)

The following Fact D (actually together with Facts A, B and C) is observed for all traditional Painlevé equations with due modifications and it plays a crucially important role in reducing each Painlevé transcendent to Painlevé I near its simple turning point. (*Cf.* [**KT1, KT2**] and [**KT3**].)

***Fact D***

(i) *At the turning point $t = 0$ of the equation* (0.11)*, the double turning point $x = \widehat{u}_0(t)$ merges with the simple turning point $x = -2\widehat{u}_0(t)$ in the Stokes geometry of* (0.1.a)*.*

(ii) *We find*

$$\frac{1}{2}\int_0^t (\nu_+ - \nu_-)dt = \int_{-2\widehat{u}_0(t)}^{\widehat{u}_0(t)} (\lambda_+ - \lambda_-)dx, \tag{0.18}$$

*where $\nu_\pm$ are the eigenvalues of the matrix $C_0$.*

Since a Stokes curve of (0.1.a) that emanates from a turning point $a$ is, by definition, a curve defined by

$$\operatorname{Im}\int_a^x (\lambda_+ - \lambda_-)dx = 0, \tag{0.19}$$

and since a Stokes curve of (0.11) that emanates from its turning point $\tau$ (actually $\tau = 0$) is given by

$$\operatorname{Im}\int_\tau^t (\nu_+ - \nu_-)dt = 0, \tag{0.20}$$

the relation (0.18) entails the following important

***Fact E***. — *If $t$ ($\neq 0$) lies on a Stokes curve of* (0.11)*, the Stokes geometry of* (0.1.a) *becomes degenerate in the sense that its two turning points are connected by a Stokes curve.*

In this manner the Stokes geometry of (0.11), *i.e.*, the Stokes geometry of $(H_{\mathrm{I}})$ is closely related with that of (0.1.a), one of the underlying Lax pair whose monodromy data (including Stokes multipliers) are preserved.

***Remark 0.1***. — As is common in the literature (*e.g.*, [**V**]) in the exact WKB analysis (*i.e.*, WKB analysis based on the Borel resummation), we employ the above definition of a Stokes curve, that is, the definition making use of the imaginary part of the quantity in question; considering the imaginary part, not the real part, is most appropriate in view of the definition of the Borel resummation.

***Remark 0.2***. — Because of the simple character of the Stokes geometry of (0.1.a) its degeneracy occurs only when the parameter $t$ lies on a Stokes curve of (0.11). As we will see in Section 3, this is not always the case for the higher order Painlevé equations. However, Fact E, together with Facts A, B, C and D, will be confirmed with due modifications in Section 2 for each member in the $P_J$-hierarchy with $J=\mathrm{I}$, II-1 or II-2.

## 1. $P_J$-hierarchy with a large parameter ($J=$ I, II-1 or II-2)

The purpose of this section is to explicitly write down the $P_J$-hierarchy with a large parameter ($J=$ I, II-1 or II-2) together with the underlying Lax pair.

**1.1. $P_{\mathrm{I}}$-hierarchy with a large parameter.** — The $P_{\mathrm{I}}$-hierarchy with a large parameter $\eta$ is, by definition, the following family of systems of non-linear equations which are labeled by a positive integer $m$. As one can readily see, the first member of the family, *i.e.*, $(P_{\mathrm{I}})_1$ is reduced to $(P_{\mathrm{I}})$, the Painlevé I equation with a large parameter $\eta$ (in the notation of [**KT3**] etc.). This fact justifies the name "$P_{\mathrm{I}}$-hierarchy". It was introduced (in a form somewhat different from the expression below) by Shimomura ([**S1, S2**]) in studying the most degenerate Garnier system. It is essentially the same as the $P_{\mathrm{I}}$-hierarchy proposed earlier by Gordoa and Pickering ([**GP**]) through a particular reduction of KdV-hierarchy in a similar way as in the case of $P_{\mathrm{II}\text{-}1}$-hierarchy discussed in the next subsection (*cf.* Appendix B). See also [**KS**].

***Definition 1.1.1* ($P_{\mathrm{I}}$-hierarchy with a large parameter $\eta$)**

$$(1.1.1)\qquad (P_{\mathrm{I}})_m:\begin{cases}\dfrac{du_j}{dt}=2\eta v_j \quad (j=1,\ldots,m), & (1.1.1.a)\\[2mm] \dfrac{dv_j}{dt}=2\eta(u_{j+1}+u_1u_j+w_j)\quad (j=1,\ldots,m), & (1.1.1.b)\\[2mm] u_{m+1}=0,\end{cases}$$

where $w_j$ is a polynomial of $u_l$ and $v_l$ $(1\leqslant l\leqslant j)$ that is determined by the following recursive relation:

$$(1.1.2)\quad w_j=\frac{1}{2}\left(\sum_{k=1}^{j}u_ku_{j+1-k}\right)+\sum_{k=1}^{j-1}u_kw_{j-k} -\frac{1}{2}\left(\sum_{k=1}^{j-1}v_kv_{j-k}\right)+c_j+\delta_{jm}t\quad (j=1,\ldots,m).$$

Here $c_j$ is a constant and $\delta_{jm}$ stands for Kronecker's delta.

***Remark 1.1.1***

(i) $(P_{\mathrm{I}})_1$ is equivalent to

$$u_1'' = \eta^2(6u_1^2 + 4c_1 + 4t). \tag{1.1.3}$$

(ii) $(P_{\mathrm{I}})_2$ is equivalent to

$$u_1'''' = \eta^2(20u_1u_1'' + 10(u_1')^2) + \eta^4(-40u_1^3 - 16c_1u_1 + 16c_2 + 16t). \tag{1.1.4}$$

(iii) $(P_{\mathrm{I}})_3$ is equivalent to

$$\begin{aligned} u_1^{(6)} = \eta^2(28u_1u_1^{(4)} + 56u_1'u_1^{(3)} + 42(u_1'')^2) - \eta^4(280u_1^2u_1'' + 280u_1(u_1')^2 \\ + 16c_1u_1'') + \eta^6(280u_1^4 + 96c_1u_1^2 - 64c_2u_1 - 32c_1^2 + 64c_3 + 64t). \end{aligned} \tag{1.1.5}$$

To present the underlying Lax pair we first introduce the following polynomials in $x$ with coefficients $u_j$, etc.

$$U(x) = x^m - \sum_{j=1}^{m} u_j x^{m-j}, \tag{1.1.6}$$

$$V(x) = \sum_{j=1}^{m} v_j x^{m-j}, \tag{1.1.7}$$

$$W(x) = \sum_{j=1}^{m} w_j x^{m-j}. \tag{1.1.8}$$

We then let $A$ and $B$ denote the following matrices:

$$A = \begin{pmatrix} V(x)/2 & U(x) \\ (2x^{m+1} - xU(x) + 2W(x))/4 & -V(x)/2 \end{pmatrix} \tag{1.1.9}$$

$$B = \begin{pmatrix} 0 & 2 \\ u_1 + x/2 & 0 \end{pmatrix}. \tag{1.1.10}$$

Now the required Lax pair is given by

$$(L_{\mathrm{I}})_m : \begin{cases} \Big(\dfrac{\partial}{\partial x} - \eta A\Big)\psi = 0, & \text{(1.1.11.a)} \\ \Big(\dfrac{\partial}{\partial t} - \eta B\Big)\psi = 0. & \text{(1.1.11.b)} \end{cases} \tag{1.1.11}$$

In order to prove that $(P_{\mathrm{I}})_m$ is the condition for the compatibility of (1.1.11.a) and (1.1.11.b), we first show the following

**Lemma 1.1.1**. — *The system of equations $(P_{\mathrm{I}})_m$ together with the relation* (1.1.2) *entails*

$$\frac{dw_j}{dt} = 2\eta u_1 v_j + \delta_{jm} \quad (j = 1, \dots, m). \tag{1.1.12}$$

*Proof.* — When $m = 1$ the conclusion is obvious. Hence, we suppose $m > 1$. It, then, follows from (1.1.2) that

$$w_1 = \frac{1}{2}u_1^2 + c_1. \tag{1.1.13}$$

Thus we find by (1.1.1.a)

$$w_1' = 2\eta u_1 v_1. \tag{1.1.14}$$

We, now, use the induction on $j$. Suppose that (1.1.12) holds for $j = 1, \ldots, j_0 < m$. Then, by differentiating $w_{j_0+1}$ determined by (1.1.2), we find

$$\begin{aligned} w'_{j_0+1} = \frac{1}{2}\Big(\sum_{k=1}^{j_0+1}(u'_k u_{j_0+2-k} + u_k u'_{j_0+2-k})\Big) \\ + \sum_{k=1}^{j_0}(u'_k w_{j_0+1-k} + u_k w'_{j_0+1-k}) \\ - \frac{1}{2}\Big(\sum_{k=1}^{j_0}(v'_k v_{j_0+1-k} + v_k v'_{j_0+1-k})\Big) + \delta_{j_0+1,m}. \end{aligned} \tag{1.1.15}$$

Then, the induction hypothesis together with $(P_{\mathrm{I}})_m$ entails

$$\begin{aligned} w'_{j_0+1} &= 2\eta\Big(\sum_{l=1}^{j_0+1} v_{j_0+2-l}u_l + \sum_{k=1}^{j_0} v_k w_{j_0+1-k} + \sum_{k=1}^{j_0} u_k u_1 v_{j_0+1-k} \\ &\qquad - \sum_{k=1}^{j_0}(u_{k+1} + u_1 u_k + w_k)v_{j_0+1-k}\Big) + \delta_{j_0+1,m} \\ &= 2\eta\Big(v_{j_0+1}u_1 + \sum_{p=1}^{j_0} v_{j_0+1-p}u_{p+1} + \sum_{k=1}^{j_0} v_k w_{j_0+1-k} \\ &\qquad + \sum_{k=1}^{j_0} u_k u_1 v_{j_0+1-k} - \sum_{k=1}^{j_0} u_{k+1}v_{j_0+1-k} - \sum_{l=1}^{j_0} w_{j_0+1-l}v_l \\ &\qquad - \sum_{k=1}^{j_0} u_1 u_k v_{j_0+1-k}\Big) + \delta_{j_0+1,m} \\ &= 2\eta v_{j_0+1}u_1 + \delta_{j_0+1,m}. \end{aligned} \tag{1.1.16}$$

Thus, the induction proceeds, completing the proof of (1.1.12). □

We, now, prove the following

***Proposition 1.1.1***. — $(P_{\mathrm{I}})_m$ *is the compatibility condition for* (1.1.11.a) *and* (1.1.11.b).

*Proof.* — The compatibility condition for (1.1.11.a) and (1.1.11.b) is given by

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + \eta[A, B] = 0. \tag{1.1.17}$$

It follows from the definition of matrices $A$ and $B$ that

$$[A,B]=\begin{pmatrix} u_1U+xU-x^{m+1}-W & 2V \\ -u_1V-(xV)/2 & x^{m+1}-xU+W-u_1U \end{pmatrix}. \tag{1.1.18}$$

Writing down (1.1.17) componentwise, we find the following three relations.

$$\eta^{-1}\frac{\partial V}{\partial t}+2(u_1U+xU-x^{m+1}-W)=0, \tag{1.1.19}$$

$$\eta^{-1}\frac{\partial U}{\partial t}+2V=0, \tag{1.1.20}$$

$$\eta^{-1}(-x\frac{\partial U}{\partial t}+2\frac{\partial W}{\partial t}-2)-4u_1V-2xV=0. \tag{1.1.21}$$

Clearly, (1.1.20) is the same as (1.1.1.a). As the part of (1.1.19) with degree $m+1$ or $m$ in $x$ trivially vanishes, the relation (1.1.19) is reduced to

$$\eta^{-1}\frac{\partial v_j}{\partial t}+2(-u_1u_j-u_{j+1}-w_j)=0 \quad (j=1,\ldots,m). \tag{1.1.22}$$

This is nothing but (1.1.1.b). Note that $u_{m+1}=0$ by the definition. Let us, next, write down the coefficients of like powers in $x$ in (1.1.21). The coefficient of $x^m$ is

$$\eta^{-1}\frac{\partial u_1}{\partial t}-2v_1=0, \tag{1.1.23}$$

that of $x^{m-j}$ $(1\leqslant j\leqslant m-1)$ is

$$\eta^{-1}\left(\frac{\partial u_{j+1}}{\partial t}+2\frac{\partial w_j}{\partial t}\right)-4u_1v_j-2v_{j+1}=0, \tag{1.1.24}$$

and that of $x^0$ is

$$\eta^{-1}\left(2\frac{\partial w_m}{\partial t}-2\right)-4u_1v_m=0. \tag{1.1.25}$$

Then, Lemma 1.1.1 proves that (1.1.24) is reduced to

$$\eta^{-1}\frac{\partial u_{j+1}}{\partial t}=2v_{j+1} \quad (j=1,\ldots,m-1). \tag{1.1.26}$$

The same lemma entails that (1.1.25) is a trivial relation. The combination of (1.1.23) and (1.1.26) is again the same as (1.1.1.a). Thus we have confirmed that $(P_{\mathrm{I}})_m$ is the compatibility condition of (1.1.11.a) and (1.1.11.b.). □

**1.2. $P_{\text{II-1}}$-hierarchy with a large parameter.** — The $P_{\text{II-1}}$-hierarchy (with a large parameter) is a hierarchy obtained by a similarity reduction of the KdV hierarchy. As is shown by Gordoa and Pickering in [**GP**], this hierarchy together with its underlying Lax pair can be reproduced also by their scheme called “nonisospectral scattering problems”. Here, following the formulation of [**GP**], we define the $P_{\text{II-1}}$-hierarchy with a large parameter in the following manner:

***Definition 1.2.1*** **($P_{\text{II-1}}$-hierarchy with a large parameter $\eta$)**

$$(P_{\text{II-1}})_m : \Big(\eta^{-1}\frac{\partial}{\partial t} + 2v\Big)K_m + g(2tv + \eta^{-1}) + c = 0. \tag{1.2.1}$$

Here, $m$ is a positive integer that labels a member of the hierarchy, $v = v(t)$ is an unknown function, $g$ ($\neq 0$) and $c$ are constants, and $K_j$ is a polynomial of $v$ and its derivatives defined by the following recursive relation

$$\eta^{-1}\partial_t K_{j+1} = (\eta^{-3}\partial_t^3 - 4\eta^{-1}(v^2 - \eta^{-1}v')\partial_t - 2(2vv' - \eta^{-1}v''))K_j \tag{1.2.2}$$

for $j \geqslant 0$ with $K_0 = 1/2$ and $\partial_t = \partial/\partial t$.

***Remark 1.2.1***. — Although the differentiation $\partial_t$ appears in the left-hand side of the recursive relation (1.2.2), we can define each $K_j$ so that it becomes a polynomial only of $v$ and its derivatives and independent of any integrated terms like $\partial_t^{-1}v$. For the proof see Appendix A. For example, first few members of $K_j$ are given as follows:

$$K_0 = 1/2, \tag{1.2.3}$$

$$K_1 = -v^2 + \eta^{-1}v', \tag{1.2.4}$$

$$K_2 = 3v^4 - 6\eta^{-1}v^2v' + \eta^{-2}\big((v')^2 - 2vv''\big) + \eta^{-3}v^{(3)}, \tag{1.2.5}$$

$$\begin{aligned} K_3 = {}& -10v^6 + 30\eta^{-1}v^4v' + \eta^{-2}\big(10v^2(v')^2 + 20v^3v''\big) \\ & +\eta^{-3}\big(-10(v')^3 - 40vv'v'' - 10v^2v^{(3)}\big) \\ & +\eta^{-4}\big(-(v'')^2 + 2v'v^{(3)} - 2vv^{(4)}\big) + \eta^{-5}v^{(5)}. \end{aligned} \tag{1.2.6}$$

***Remark 1.2.2***. — By an induction we can also show that

$$K_j = \frac{(-1)^j 2^{j-1}(2j-1)!!}{j!}v^{2j} + O(\eta^{-1}), \tag{1.2.7}$$

where $(2j-1)!! = (2j-1)\cdot(2j-3)\cdot\cdots 3\cdot 1$.

***Remark 1.2.3***

(i) $(P_{\text{II-1}})_1$ is

$$\eta^{-2}v'' = v^3 - g(2tv + \eta^{-1}) - c. \tag{1.2.8}$$

This is equivalent to $(P_{\text{II}})$, the Painlevé II equation with a large parameter $\eta$.

(ii) $(P_{\text{II-1}})_2$ is

$$\eta^{-4}v^{(4)} = \eta^{-2}(10v^2v'' + 10v(v')^2) - 6v^5 - g(2tv + \eta^{-1}) - c. \tag{1.2.9}$$

The underlying Lax pair of (1.2.1) is

$$(L_{\text{II-1}})_m : \begin{cases} \Big(\dfrac{\partial}{\partial x} - \eta A\Big)\psi = 0, & \text{(1.2.10.a)} \\ \Big(\dfrac{\partial}{\partial t} - \eta B\Big)\psi = 0. & \text{(1.2.10.b)} \end{cases} \tag{1.2.10}$$

where

$$A=\frac{1}{4xg}\begin{pmatrix}-\eta^{-1}\partial_t T_m & 2T_m\\ 2qT_m-\eta^{-2}\partial_t^2 T_m & \eta^{-1}\partial_t T_m\end{pmatrix},\quad B=\begin{pmatrix}0 & 1\\ q & 0\end{pmatrix}. \tag{1.2.11}$$

Here, $T_m$ and $q$ respectively denote the following functions:

$$T_m = gt+\sum_{k=0}^{m}(4x)^k K_{m-k}, \tag{1.2.12}$$

$$q = x+v^2-\eta^{-1}v'. \tag{1.2.13}$$

Our $P_{\text{II-1}}$-hierarchy (1.2.1) is obtained from the hierarchy

$$(\partial_t+2v)\mathcal{K}_m+g(2tv+1)+c=0 \tag{1.2.14}$$

discussed by Gordoa and Pickering through the scaling

$$v\longmapsto \eta^{1/(2m+1)}v,\quad t\longmapsto \eta^{2m/(2m+1)}t,\quad g\longmapsto g,\quad c\longmapsto \eta c. \tag{1.2.15}$$

Here, $\mathcal{K}_j$ is a polynomial of $v$ and its derivatives, satisfying the recursive relation

$$\partial_t\mathcal{K}_{j+1}=(\partial_t^3+4(v'-v^2)\partial_t+2(v'-v^2)')\mathcal{K}_j. \tag{1.2.16}$$

Note that by the scaling (1.2.15) $\mathcal{K}_j$ is transformed to $\eta^{2j/(2m+1)}K_j$ and each $K_j$ can be written as

$$K_j=K_j[v,\eta]=K_{j,0}[v]+\eta^{-1}K_{j,1}[v]+\cdots+\eta^{-2j+1}K_{j,2j-1}[v] \tag{1.2.17}$$

with $K_{j,l}$ being a polynomial of $v$ and its derivatives independent of $\eta$. As is explained also in [**GP**, III, pp. 5751–5755], (1.2.14) is the compatibility condition for the following system of linear ordinary differential equations:

$$\begin{cases} 4xg\dfrac{\partial}{\partial x}\psi=(-\partial_t T_m+2T_m\dfrac{\partial}{\partial x})\psi,\\ \left(\dfrac{\partial^2}{\partial t^2}+v'-v^2-x\right)\psi=0, \end{cases} \tag{1.2.18}$$

or for the system equivalent to it:

$$\frac{\partial}{\partial x}\psi=\widetilde{A}\psi,\quad \frac{\partial}{\partial t}\psi=\widetilde{B}\psi, \tag{1.2.19}$$

where

$$\widetilde{A}=\frac{1}{4xg}\begin{pmatrix}-\partial_t T_m & 2T_m\\ 2qT_m-\partial_t^2 T_m & \partial_t T_m\end{pmatrix},\quad \widetilde{B}=\begin{pmatrix}0 & 1\\ q & 0\end{pmatrix}. \tag{1.2.20}$$

Here,

$$T_m=gt+\sum_{k=0}^{m}(4x)^k\mathcal{K}_{m-k}\quad\text{and}\quad q=x+v^2-v'. \tag{1.2.21}$$

As a matter of fact, by straightforward computations we readily find that

$$\frac{\partial \widetilde{A}}{\partial t} - \frac{\partial \widetilde{B}}{\partial x} + [\widetilde{A}, \widetilde{B}] = \begin{pmatrix} 0 & 0 \\ \Delta & 0 \end{pmatrix} \tag{1.2.22}$$

with

$$\Delta = -\frac{1}{4xg}(\partial_t - 2v)\partial_t\big\{(\partial_t + 2v)\mathcal{K}_m + g(2tv+1)\big\}. \tag{1.2.23}$$

Thus (1.2.14) is the compatibility condition for the Lax pair (1.2.19) with (1.2.20). Our Lax pair (1.2.10) and (1.2.11) are obtained from (1.2.19) and (1.2.20) through the scaling (1.2.15) and $x \mapsto \eta^{2/(2m+1)}x$.

**1.3. $P_{\text{II-2}}$-hierarchy with a large parameter.** — The $P_{\text{II-2}}$-hierarchy with a large parameter is obtained from the hierarchy introduced by Gordoa-Joshi-Pickering in [**GJP**, p. 337] through an appropriate scaling of the variables and constants. Here, we content ourselves with explicitly listing up the final results and we refer the reader to [**N1**] and [**N2**] for the details of the discussion.

***Definition 1.3.1*** **($P_{\text{II-2}}$-hierarchy with a large parameter $\eta$)**

$$(P_{\text{II-2}})_m : \begin{cases} K_{m+1} + \displaystyle\sum_{j=1}^{m-1} c_j K_j + gt = 0, \\ L_{m+1} + \displaystyle\sum_{j=1}^{m-1} c_j L_j \qquad\quad = \delta. \end{cases} \tag{1.3.1}$$

Here, $g$ ($\neq 0$), $c_j$ and $\delta$ are constants, and $K_j$ and $L_j$ are polynomials of unknown functions $u$, $v$ and their derivatives defined by the following recursive relation

$$\eta^{-1}\partial_t \begin{pmatrix} K_{j+1} \\ L_{j+1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \eta^{-1}u' + u\eta^{-1}\partial_t - \eta^{-2}\partial_t^2 & 2\eta^{-1}\partial_t \\ 2\eta^{-1}v\partial_t + \eta^{-1}v_t & u\eta^{-1}\partial_t + \eta^{-2}\partial_t^2 \end{pmatrix}\begin{pmatrix} K_j \\ L_j \end{pmatrix} \tag{1.3.2}$$

($j \geqslant 0$) with $K_0 = 2$ and $L_0 = 0$.

***Remark 1.3.1***. — As in the case of $P_{\text{II-1}}$-hierarchy, we can show that $K_j$ and $L_j$ become polynomials of $u$, $v$ and their derivatives. For the proof see [**N1**] and [**N2**]. First few members of $K_j$ and $L_j$ are given as follows:

$$\begin{pmatrix} K_1 \\ L_1 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.3.3}$$

$$\begin{pmatrix} K_2 \\ L_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} u^2 + 2v - \eta^{-1}u' \\ 2uv + \eta^{-1}v' \end{pmatrix}, \tag{1.3.4}$$

$$\begin{pmatrix} K_3 \\ L_3 \end{pmatrix} = \left(\frac{1}{2}\right)^2 \begin{pmatrix} u^3 + 6uv - 3\eta^{-1}uu' + \eta^{-2}u'' \\ 3u^2v + 3v^2 + 3\eta^{-1}uv' + \eta^{-2}v'' \end{pmatrix}. \tag{1.3.5}$$

***Remark 1.3.2***

(i) $(P_{\mathrm{II}-2})_1$ is reduced to

$$\eta^{-2}u'' = 2u^3 + 2g\left(2tu + \eta^{-1}\right) + 4\delta. \tag{1.3.6}$$

(ii) $(P_{\mathrm{II}-2})_2$ is reduced to

$$\begin{aligned}\eta^{-4}u^{(4)} = \frac{1}{2u^2}\Big[&\eta^{-4}\left(-4(u')^2u'' + 3u(u'')^2 + 4uu'u^{(3)}\right) + 16\eta^{-3}guu' \\ &+ \eta^{-2}\big(-16gt(u')^2 + 5u^3(u')^2 + 16gtuu'' + 10u^4u''\big) - 24\eta^{-1}gu^3 \\ &+ \left(16g^2t^2u - 16{c_1}^2u^3 - 48\delta u^3 - 16gtu^4 - 24c_1u^5 - 5u^7\right)\Big].\end{aligned} \tag{1.3.7}$$

The underlying Lax pair of $(P_{\text{II-2}})_m$ is

$$(L_{\text{II-2}})_m : \begin{cases} \left(\dfrac{\partial}{\partial x} - \eta A\right)\psi = 0, & \text{(1.3.8.a)} \\ \left(\dfrac{\partial}{\partial t} - \eta B\right)\psi = 0, & \text{(1.3.8.b)} \end{cases} \tag{1.3.8}$$

where

$$A = A^{(m)} + c_{m-1}A^{(m-2)} + c_{m-2}A^{(m-3)} + \cdots + c_1A^{(0)} \tag{1.3.9}$$

$$B = \begin{pmatrix} -x + u/2 & 1 \\ -v & x - u/2 \end{pmatrix}. \tag{1.3.10}$$

Here, $A^{(j)}$ denotes

$$A^{(j)} = \frac{1}{g}\begin{pmatrix} -(2x-u)T_j - \eta^{-1}\partial_t T_j & 2T_j \\ \begin{array}{r}-2vT_j - \eta^{-1}\partial_t\left\{(2x-u)T_j\right. \\ \left.+\partial_t T_j + K_{j+1}\right\}\end{array} & (2x-u)T_j + \eta^{-1}\partial_t T_j \end{pmatrix}, \tag{1.3.11}$$

where

$$T_m = \frac{1}{2}\sum_{j=0}^{m} x^{m-j}K_j. \tag{1.3.12}$$

## 2. Relations between the Stokes geometry of the $(P_J)$-hierarchies and that of their underlying Lax pairs

In this section we prove that the relations, being similar to the Facts A $\sim$ E for the traditional Painlevé equations explained in Introduction, also hold between the Stokes geometry of a member in the $(P_J)$-hierarchies ($J =$ I, II-1 and II-2) and that of its underlying Lax pair.

**2.1. Case of the $(P_{\mathrm{I}})$-hierarchy.** — As in the case of the traditional Painlevé equations, we first construct what we call the 0-parameter solution $(\widehat{u}_j, \widehat{v}_j)$ of $(P_{\mathrm{I}})_m$ of the following form:

$$\widehat{u}_j(t,\eta) = \widehat{u}_{j,0}(t) + \eta^{-1}\widehat{u}_{j,1}(t) + \cdots , \tag{2.1.1}$$

$$\widehat{v}_j(t,\eta) = \widehat{v}_{j,0}(t) + \eta^{-1}\widehat{v}_{j,1}(t) + \cdots . \tag{2.1.2}$$

Substituting these expansions into (1.1.1.a) and (1.1.1.b), we readily find that $\widehat{v}_{j,0}$ $(j = 1, \dots, m)$ identically vanishes and $\widehat{u}_{j,0}$ should satisfy

$$\widehat{u}_{j+1,0} + \widehat{u}_{1,0}\widehat{u}_{j,0} + \widehat{w}_{j,0} = 0 \quad (j = 1, \dots, m). \tag{2.1.3}$$

We can also observe that $\widehat{u}_{j,k}$ and $\widehat{v}_{j,k}$ $(k \geq 1)$ are recursively determined once $\widehat{v}_{j,0}$ is taken to be zero and $\widehat{u}_{j,0}$ is chosen so that it satisfies the algebraic equation (2.1.3). Note that the top order part $\widehat{w}_{j,0}$ of $w_j$ satisfies a recursive relation

$$\widehat{w}_{j,0} = \frac{1}{2}\Big(\sum_{k=1}^{j} \widehat{u}_{k,0}\widehat{u}_{j+1-k,0}\Big) + \sum_{k=1}^{j-1} \widehat{u}_{k,0}\widehat{w}_{j-k,0} + c_j + \delta_{jm}t \quad (j = 1, \dots, m) \tag{2.1.4}$$

corresponding to (1.1.2), and that (2.1.3) together with (2.1.4) recursively determines each $\widehat{u}_{j,0}$ $(j = 1, \dots, m)$ as a polynomial of $\widehat{u}_{1,0}$. In particular, as $\widehat{u}_{m+1,0} = 0$ by the definition, (2.1.3) for $j = m$ provides an algebraic equation for $\widehat{u}_{1,0}$. Hence all $\widehat{u}_{j,0}$ and $\widehat{v}_{j,0}$ are determined algebraically and the 0-parameter solution $(\widehat{u}_j, \widehat{v}_j)$ of $(P_{\mathrm{I}})_m$ is thus constructed.

***Remark 2.1.1***. — By using an induction on $j$ we can verify that $\widehat{u}_{j,0}$ is a polynomial of $\widehat{u}_{1,0}$ with degree at most $j$. Furthermore, letting $(-1)^{j-1}\alpha_j\widehat{u}_{1,0}^{j}$ denote the top degree part of $\widehat{u}_{j,0}$, we obtain the following recursive relation for $\{\alpha_j\}$ as a consequence of (2.1.3) and (2.1.4):

$$\alpha_{j+1} = \alpha_j + \frac{1}{2}\Big(\sum_{k=1}^{j} \alpha_k\alpha_{j+1-k}\Big) - \sum_{k=1}^{j-1} \alpha_k(\alpha_{j+1-k} - \alpha_{j-k}) \quad (j = 1, \dots, m) \tag{2.1.5}$$

and $\alpha_1 = 1$. Since

$$\widetilde{\alpha}_j = (-2)^j \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-j+1)}{j!} = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2j-1)}{j!} \tag{2.1.6}$$

satisfies the same recursive relation (2.1.5), we can conclude that $\alpha_j = \widetilde{\alpha}_j \neq 0$. Thus, $\widehat{u}_{1,0}$ is a solution of an algebraic equation with degree exactly equal to $m+1$ and, roughly speaking, there exist $m+1$ 0-parameter solutions of $(P_{\mathrm{I}})_m$.

We, next, substitute the 0-parameter solution $(\widehat{u}_j, \widehat{v}_j)$ of $(P_{\mathrm{I}})_m$ into the coefficients $A$ and $B$ respectively given by (1.1.9) and (1.1.10), *i.e.*, the coefficients of its underlying Lax pair. Then, their top order parts $A_0$ and $B_0$ become

$$A_0 = \begin{pmatrix} V_0(x)/2 & U_0(x) \\ (2x^{m+1} - xU_0(x) + 2W_0(x))/4 & -V_0(x)/2 \end{pmatrix}, \tag{2.1.7}$$

$$B_0 = \begin{pmatrix} 0 & 2 \\ \widehat{u}_{1,0} + x/2 & 0 \end{pmatrix}, \tag{2.1.8}$$

where $U_0(x)$, $V_0(x)$ and $W_0(x)$ respectively denote the top order parts (in $\eta$) of $U(x)$, $V(x)$ and $W(x)$, that is,

$$U_0(x) = x^m - \sum_{j=1}^{m} \widehat{u}_{j,0} x^{m-j}, \tag{2.1.9}$$

$$V_0(x) = \sum_{j=1}^{m} \widehat{v}_{j,0} x^{m-j}, \tag{2.1.10}$$

$$W_0(x) = \sum_{j=1}^{m} \widehat{w}_{j,0} x^{m-j}. \tag{2.1.11}$$

Here, it follows from (2.1.3) that

$$\begin{aligned} 2x^{m+1} &- xU_0(x) + 2W_0(x) \\ &= x^{m+1} + \sum_{j=1}^{m} \widehat{u}_{j,0} x^{m+1-j} + 2\sum_{j=1}^{m} \widehat{w}_{j,0} x^{m-j} \\ &= x^{m+1} + \sum_{j=1}^{m} \widehat{u}_{j,0} x^{m+1-j} - 2\sum_{j=1}^{m} (\widehat{u}_{j+1,0} + \widehat{u}_{1,0}\widehat{u}_{j,0}) x^{m-j} \\ &= x^{m+1} + 2\widehat{u}_{1,0} x^m - \sum_{j=1}^{m} \widehat{u}_{j,0} x^{m+1-j} - 2\widehat{u}_{1,0} \sum_{j=1}^{m} \widehat{u}_{j,0} x^{m-j} \\ &= (x + 2\widehat{u}_{1,0}) U_0(x) \end{aligned} \tag{2.1.12}$$

holds. This immediately entails

$$A_0 = \frac{U_0(x)}{2} B_0, \tag{2.1.13}$$

and hence, as a generalization of Fact A for the traditional Painlevé equations, we obtain the following

***Proposition 2.1.1***

(i) *The equation* (1.1.11.a) *has* $m$ (*generically*) *double turning points* (*which will be denoted by* $x = b_1(t)$, $\ldots$, $x = b_m(t)$ *in what follows*), *and each double turning point is a root of* $U_0(x) = 0$.

(ii) *It has one (generically) simple turning point* $x = -2\widehat{u}_{1,0}(t)$, *(which will be denoted by* $x = a(t)$ *for short in what follows), and this point is simultaneously a turning point of the equation* (1.1.11.b).

We can also prove Fact B in a quite general context, that is, even for $(P_{\mathrm{I}})_m$ we have

***Proposition 2.1.2***. — *The eigenvalues* $\lambda_\pm$ *of* $A_0$ *and the eigenvalues* $\mu_\pm$ *of* $B_0$ *satisfy the following relation:*

$$\frac{\partial}{\partial t}\lambda_\pm = \frac{\partial}{\partial x}\mu_\pm. \tag{2.1.14}$$

For the proof of Proposition 2.1.2 see [**T2**], where the method of diagonalization for the Lax pair $(L_{\mathrm{I}})_m$ is used to prove the proposition in question.

Now, to define the Stokes geometry of $(P_{\mathrm{I}})_m$, we consider the linearization of $(P_{\mathrm{I}})_m$ at the 0-parameter solution $(\widehat{u}_j, \widehat{v}_j)$, that is, we take the part linear in $(\Delta u_j, \Delta v_j)$ after the substitution $u_j = \widehat{u}_j + \Delta u_j$ and $v_j = \widehat{v}_j + \Delta v_j$ in $(P_{\mathrm{I}})_m$. We then obtain

$$\begin{cases} \dfrac{d}{dt}\Delta u_j = 2\eta\Delta v_j \quad (j = 1, \ldots, m), \\ \dfrac{d}{dt}\Delta v_j = 2\eta(\Delta u_{j+1} + \widehat{u}_1\Delta u_j + \widehat{u}_j\Delta u_1 + \Delta w_j) \quad (j = 1, \ldots, m). \end{cases} \tag{2.1.15}$$

This defines a system of first order linear ordinary differential equations for $(\Delta u_j, \Delta v_j)$. We write this system as

$$\frac{d}{dt}\begin{pmatrix} \Delta u_1 \\ \Delta v_1 \\ \Delta u_2 \\ \Delta v_2 \\ \vdots \\ \Delta v_m \end{pmatrix} = \eta C(t,\eta)\begin{pmatrix} \Delta u_1 \\ \Delta v_1 \\ \Delta u_2 \\ \Delta v_2 \\ \vdots \\ \Delta v_m \end{pmatrix}. \tag{2.1.16}$$

As in the case of the traditional Painlevé equations, we then call a turning point (resp. Stokes curve) of (2.1.16) a turning point (resp. Stokes curve) of our non-linear equation $(P_{\mathrm{I}})_m$. That is, if we let $C_0$ denote the top order part (*i.e.*, the part of order 0 in $\eta$) of the coefficient matrix $C(t,\eta)$ of the right-hand side of (2.1.16), a turning point $\tau$ of $(P_{\mathrm{I}})_m$ is a point where two eigenvalues $\nu_j(t)$ $(j = 1, 2)$ of $C_0$ merge and a Stokes curve of $(P_{\mathrm{I}})_m$ emanating from $\tau$ is given by $\operatorname{Im}\int_\tau^t (\nu_1 - \nu_2)dt = 0$. To write down $C_0$ in an explicit manner, we note the following

***Lemma 2.1.1***

$$\Delta w_j = \widehat{u}_{1,0}\Delta u_j + O(\eta^{-1}) \qquad (j = 1, \ldots, m). \tag{2.1.17}$$

*Proof*. — In parallel with the proof of Lemma 1.1.1, we use the induction on $j$ to prove (2.1.17). In the case of $j = 1$ (1.1.13) immediately entails

$$\Delta w_1 = \widehat{u}_1 \Delta u_1. \tag{2.1.18}$$

We now suppose that (2.1.17) holds for $j = 1, \ldots, j_0 (< m)$. It follows from (1.1.2) that

$$\begin{aligned}\Delta w_{j_0+1} = &\sum_{k=1}^{j_0+1} \widehat{u}_{j_0+2-k} \Delta u_k \\ &+ \sum_{k=1}^{j_0} (\widehat{u}_{j_0+1-k} \Delta w_k + \widehat{w}_{j_0+1-k} \Delta u_k) - \sum_{k=1}^{j_0} \widehat{v}_{j_0+1-k} \Delta v_k.\end{aligned} \tag{2.1.19}$$

Then by the induction hypothesis together with the fact $\widehat{v}_{j,0} = 0$ we find

$$\Delta w_{j_0+1} = \sum_{k=1}^{j_0+1} \widehat{u}_{j_0+2-k} \Delta u_k + \sum_{k=1}^{j_0} (\widehat{u}_{j_0+1-k} \widehat{u}_{1,0} + \widehat{w}_{j_0+1-k}) \Delta u_k + O(\eta^{-1}). \tag{2.1.20}$$

Since we know by (2.1.3) that $\widehat{u}_{j+1,0} + \widehat{u}_{1,0} \widehat{u}_{j,0} + \widehat{w}_{j,0} = 0$ holds for $j = 1, \ldots, m$, we obtain from (2.1.20) the following:

$$\Delta w_{j_0+1} = \widehat{u}_{1,0} \Delta u_{j_0+1} + O(\eta^{-1}). \tag{2.1.21}$$

This completes the proof of (2.1.17). □

In view of (2.1.15) and Lemma 2.1.1 we find that the explicit form of $C_0$ is given by

$$C_0 = \left(\begin{array}{cc|cc|cc|c} 0 & 2 & & & & & \\ 6\widehat{u}_{1,0} & 0 & 2 & & & & \\ \hline 0 & & 0 & 2 & & & \\ 2\widehat{u}_{2,0} & & 4\widehat{u}_{1,0} & 0 & 2 & & \\ \hline 0 & & & & 0 & 2 & \\ 2\widehat{u}_{3,0} & & & & 4\widehat{u}_{1,0} & 0 & \\ \hline \vdots & & & & & & \ddots \end{array}\right). \tag{2.1.22}$$

This leads to the following

***Proposition 2.1.3***. — *We have the relation*

$$\begin{aligned}\det(\nu - C_0) &= 4^m \prod_{j=1}^{m} \det(\mu - B_0)\Big|_{x=b_j(t), \mu=\nu/2} \\ &= \prod_{j=1}^{m} \big(\nu^2 - 4(2\widehat{u}_{1,0}(t) + b_j(t))\big),\end{aligned} \tag{2.1.23}$$

*where $b_j(t)$ denotes a double turning point of* (1.1.11.a), *i.e., a root of $U_0(x) = 0$* (*cf. Proposition 2.1.1*).

*Proof.* — Expanding

$$\det(2\mu - C_0) = 4^m \begin{vmatrix} \mu & -1 & & & & & \\ -3\widehat{u}_{1,0} & \mu & -1 & & & & \\ 0 & & \mu & -1 & & & \\ -\widehat{u}_{2,0} & & -2\widehat{u}_{1,0} & \mu & -1 & & \\ 0 & & & & \mu & -1 & \\ -\widehat{u}_{3,0} & & & & -2\widehat{u}_{1,0} & \mu & \\ \vdots & & & & & & \ddots \end{vmatrix} \tag{2.1.24}$$

with respect to the first column, we find

$$\begin{aligned} \det(2\mu - C_0) &= 4^m[\mu^2(\mu^2 - 2\widehat{u}_{1,0})^{m-1} - 3\widehat{u}_{1,0}(\mu^2 - 2\widehat{u}_{1,0})^{m-1} \\ &\qquad - \widehat{u}_{2,0}(\mu^2 - 2\widehat{u}_{1,0})^{m-2} - \cdots - \widehat{u}_{m,0}] \\ &= 4^m[(\mu^2 - 2\widehat{u}_{1,0})^m - \widehat{u}_{1,0}(\mu^2 - 2\widehat{u}_{1,0})^{m-1} - \cdots - \widehat{u}_{m,0}] \\ &= 4^m U_0(\mu^2 - 2\widehat{u}_{1,0}). \end{aligned} \tag{2.1.25}$$

This immediately entails (2.1.23). □

Proposition 2.1.3 claims that $\pm 2\sqrt{2\widehat{u}_{1,0}(t) + b_j(t)}$ is an eigenvalue of $C_0$ for $j = 1, \ldots, m$. We can thus label each eigenvalue of $C_0$ by a combination of the index $j$ and the sign; we let $\nu_{j,\pm}$ denote $\pm 2\sqrt{2\widehat{u}_{1,0}(t) + b_j(t)}$ in what follows. Note that $\nu_{j,+} + \nu_{j,-} = 0$ holds for every $j$.

It also follows from Proposition 2.1.3 that $\det(\nu - C_0) = 0$ has the form $f(\nu^2, t)$ with some polynomial $f$ of degree $m$. This implies that there are two kinds of turning points for $(P_{\mathrm{I}})_m$: (i) A turning point where the degree 0 part of $f$ vanishes ("a turning point of the first kind"), and (ii) a turning point where the discriminant of $f$ vanishes ("a turning point of the second kind"). Then, as in the case of the traditional Painlevé equations, we can obtain the following relations between the Stokes geometry of $(P_{\mathrm{I}})_m$ and that of its underlying Lax pair $(L_{\mathrm{I}})_m$.

***Proposition 2.1.4***

(i) *Let $t = \tau^{\mathrm{I}}$ be a turning point of the first kind of $(P_{\mathrm{I}})_m$. Then at $t = \tau^{\mathrm{I}}$ a double turning point $x = b_j(t)$ merges with the simple turning point $x = a(t) = -2\widehat{u}_{1,0}(t)$ in the Stokes geometry of* (1.1.11.a). *Consequently the two eigenvalues $\nu_{j,\pm}$ of $C_0$ merge and vanish at $t = \tau^{\mathrm{I}}$. Furthermore the following relation holds:*

$$\frac{1}{2}\int_{\tau^{\mathrm{I}}}^{t} (\nu_{j,+} - \nu_{j,-})dt = \int_{a(t)}^{b_j(t)} (\lambda_+ - \lambda_-)dx. \tag{2.1.26}$$

(ii) *Let $t = \tau^{\mathrm{II}}$ be a turning point of the second kind of $(P_{\mathrm{I}})_m$. Then at $t = \tau^{\mathrm{II}}$ a double turning point $x = b_j(t)$ merges with another double turning point $x = b_{j'}(t)$. Consequently two eigenvalues $\nu_{j,+}$ and $\nu_{j',+}$ of $C_0$ merge at $t = \tau^{\mathrm{II}}$, and so do $\nu_{j,-}$*

*and $\nu_{j',-}$. Furthermore the following relation holds:*

$$\int_{\tau^{\mathrm{II}}}^{t}(\nu_{j,+}-\nu_{j',+})dt = -\int_{\tau^{\mathrm{II}}}^{t}(\nu_{j,-}-\nu_{j',-})dt = \int_{b_{j'}(t)}^{b_j(t)}(\lambda_+-\lambda_-)dx. \tag{2.1.27}$$

*Proof.* — We first consider the case of a turning point $t=\tau^{\mathrm{I}}$ of the first kind. Proposition 2.1.3 implies that $2\widehat{u}_{1,0}(t)+b_j(t)$ vanishes at $t=\tau^{\mathrm{I}}$ for some $j$. This immediately entails that $x=b_j(t)$ merges with $x=-2\widehat{u}_{1,0}(t)$ at $t=\tau^{\mathrm{I}}$ and that $\nu_{j,\pm}$ merge and vanish there. Note that Proposition 2.1.3 also implies

$$\nu_{j,+}(t)-\nu_{j,-}(t) = 2(\mu_+(x,t)-\mu_-(x,t))\Big|_{x=b_j(t)}. \tag{2.1.28}$$

Hence it follows from Proposition 2.1.2 that

$$\begin{aligned} \frac{d}{dt}\int_{a(t)}^{b_j(t)}(\lambda_+-\lambda_-)dx &= \int_{a(t)}^{b_j(t)}\frac{\partial}{\partial t}(\lambda_+-\lambda_-)dx \\ &= \int_{a(t)}^{b_j(t)}\frac{\partial}{\partial x}(\mu_+-\mu_-)dx \\ &= (\mu_+-\mu_-)\Big|_{x=b_j(t)} \\ &= \frac{1}{2}(\nu_{j,+}-\nu_{j,-}). \end{aligned} \tag{2.1.29}$$

Integrating (2.1.29) from $\tau^{\mathrm{I}}$ to $t$, we then obtain (2.1.26).

We next consider the case of a turning point $t=\tau^{\mathrm{II}}$ of the second kind. Proposition 2.1.3 again implies that $2\widehat{u}_{1,0}(t)+b_j(t)$ coincides with $2\widehat{u}_{1,0}(t)+b_{j'}(t)$ at $t=\tau^{\mathrm{II}}$ for some $j$ and $j'$. This entails that $x=b_j(t)$ merges with $x=b_{j'}(t)$ at $t=\tau^{\mathrm{II}}$ and that $\nu_{j,+}$ and $\nu_{j',+}$ merge there. The proof of the relation (2.1.27) is similar to that of (2.1.26). □

As an immediate consequence of the relations (2.1.26) and (2.1.27) we also observe the following important

***Proposition 2.1.5***. — *If $t$ lies on a Stokes curve of $(P_{\mathrm{I}})_m$ emanating from a turning point $t=\tau^{\mathrm{I}}$ (resp. $t=\tau^{\mathrm{II}}$) of the first (resp. second) kind, the Stokes geometry of* (1.1.11.a) *becomes degenerate in the sense that its two turning points $x=b_j(t)$ and $x=a(t)$ (resp. $x=b_j(t)$ and $x=b_{j'}(t)$) are connected by a Stokes curve.*

Propositions 2.1.4 and 2.1.5 are natural generalizations to $(P_{\mathrm{I}})_m$ of Facts D and E for the traditional Painlevé equations explained in Introduction.

**2.2. Case of the $P_{\mathrm{II}\text{-}1}$-hierarchy.** — As in the case of the $P_{\mathrm{I}}$-hierarchy, by substituting

$$v=\widehat{v}(t,\eta)=\widehat{v}_0(t)+\eta^{-1}\widehat{v}_1(t)+\cdots \tag{2.2.1}$$

into (1.2.1) and comparing like powers of $\eta$, we can construct the 0-parameter solution $\widehat{v}(t,\eta)$ of $(P_{\text{II-1}})_m$. In this case the top order part $\widehat{v}_0$ satisfies

$$2\widehat{v}_0 K_{m,0}(\widehat{v}_0) + 2gt\widehat{v}_0 + c = 0, \tag{2.2.2}$$

or more explicitly

$$\frac{(-1)^m 2^m (2m-1)!!}{m!}\,\widehat{v}_0^{2m+1} + 2gt\widehat{v}_0 + c = 0 \tag{2.2.3}$$

(*cf.* Remark 1.2.2).

We then substitute the 0-parameter solution $\widehat{v}(t,\eta)$ of $(P_{\text{II-1}})_m$ into the coefficients $A$ and $B$ of the underlying Lax pair (1.2.10). Their top order parts $A_0$ and $B_0$ are given by

$$A_0 = \frac{1}{2xg}\begin{pmatrix} 0 & T_{m,0} \\ q_0 T_{m,0} & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 1 \\ q_0 & 0 \end{pmatrix}, \tag{2.2.4}$$

where

$$T_{m,0} = gt + \sum_{k=0}^{m} (4x)^k K_{m-k,0}\Big|_{v=\widehat{v}_0}, \tag{2.2.5}$$

$$q_0 = x + \widehat{v}_0^2. \tag{2.2.6}$$

Thus

$$A_0 = \frac{T_{m,0}}{2xg} B_0 \tag{2.2.7}$$

holds and hence we obtain

***Proposition 2.2.1***

(i) *The equation* (1.2.10.a) *has $m$* (*generically*) *double turning points* (*which will be denoted by $x = b_1(t), \ldots, x = b_m(t)$ in what follows*), *and each double turning point $x = b_j(t)$ is a root of $T_{m,0} = 0$, that is,*

$$T_{m,0} = 2^{2m-1} \prod_{j=1}^{m} (x - b_j(t)). \tag{2.2.8}$$

(ii) *It has one* (*generically*) *simple turning point $x = -(\widehat{v}_0(t))^2$,* (*which will be denoted by $x = a(t)$ for short in what follows*), *and this point is simultaneously a turning point of the equation* (1.2.10.b).

The following proposition corresponding to Fact B also holds for $(P_{\text{II-1}})_m$.

***Proposition 2.2.2***. — *The eigenvalues $\lambda_\pm$ of $A_0$ and the eigenvalues $\mu_\pm$ of $B_0$ satisfy the following relation:*

$$\frac{\partial}{\partial t}\lambda_\pm = \frac{\partial}{\partial x}\mu_\pm. \tag{2.2.9}$$

(For the proof of Proposition 2.2.2 see [**T2**].)

Now we consider the linearization of $(P_{\text{II-1}})_m$ at the 0-parameter solution $\widehat{v}$. Let $\Delta K_j$ denote the linear (in $\Delta v$) part of $K_j$ after the substitution $v = \widehat{v} + \Delta v$, then the linearization of $(P_{\text{II-1}})_m$ is

$$(\eta^{-1}\partial_t + 2\widehat{v})\Delta K_m + 2K_m\Big|_{v=\widehat{v}}\Delta v + 2gt\Delta v = 0. \tag{2.2.10}$$

Since $K_j$ is a polynomial of $v$ and its derivatives, there exists a (formal) differential operator

$$p_j(t,\eta^{-1}\partial_t;\eta^{-1}) = p_{j,0}(t,\eta^{-1}\partial_t) + \eta^{-1}p_{j,1}(t,\eta^{-1}\partial_t) + \cdots \tag{2.2.11}$$

for which the following relation holds:

$$\Delta K_j = p_j(t,\eta^{-1}\partial_t;\eta^{-1})\Delta v. \tag{2.2.12}$$

In terms of this operator $p_j(t,\eta^{-1}\partial_t;\eta^{-1})$, the characteristic equation (*i.e.*, the top order part (with respect to $\eta$) of the symbol obtained by replacing $\eta^{-1}\partial_t$ by $\nu$) of (2.2.10) is expressed as

$$C(t,\nu) = (\nu + 2\widehat{v}_0)p_{m,0}(t,\nu) + 2K_{m,0}\Big|_{v=\widehat{v}_0} + 2gt. \tag{2.2.13}$$

This $C(t,\nu)$ corresponds to the characteristic equation of $C_0$ in the case of $P_{\text{I}}$-hierarchy.

***Proposition 2.2.3***. — *We find*

$$C(t,\nu) = 4^m \prod_{j=1}^{m} \det(\mu - B_0)\Big|_{x=b_j(t),\mu=\nu/2}. \tag{2.2.14}$$

*Proof*. — We first note that the right-hand side of (2.2.14) becomes

$$\begin{aligned} 4^m \prod_{j=1}^{m} \det(\mu - B_0)\Big|_{x=b_j,\mu=\nu/2} &= 4^m \prod_{j=1}^{m} (\mu^2 - \widehat{v}_0^2 - x)\Big|_{x=b_j,\mu=\nu/2} \\ &= 4^m \prod_{j=1}^{m} \Big(\frac{\nu^2}{4} - \widehat{v}_0^2 - b_j\Big) \\ &= 2T_{m,0}\Big|_{x=(\nu^2-4v_0{}^2)/4}. \end{aligned} \tag{2.2.15}$$

To calculate the left-hand side of (2.2.14), we use the recursive relation (1.2.2). Considering the linear (in $\Delta v$) part of both sides of (1.2.2), we find that $\{p_{j,0}\}$ should satisfy the following recursive relation:

$$\nu p_{j+1,0}(t,\nu) = (\nu^3 - 4\widehat{v}_0^2\nu)p_{j,0}(t,\nu) + 2\nu(\nu - 2\widehat{v}_0)K_{j,0}\Big|_{v=\widehat{v}_0}, \tag{2.2.16}$$

that is,

$$p_{j+1,0}(t,\nu) = (\nu^2 - 4\widehat{v}_0^2)p_{j,0}(t,\nu) + 2(\nu - 2\widehat{v}_0)K_{j,0}\Big|_{v=\widehat{v}_0}. \tag{2.2.17}$$

By solving this recursive relation with the initial condition

$$p_{1,0}(t,\nu) = \nu - 2\widehat{v}_0, \tag{2.2.18}$$

we obtain

$$p_{m,0}(t,\nu) = 2(\nu - 2\widehat{v}_0)\sum_{k=0}^{m-1}(\nu^2 - 4\widehat{v}_0^{m-j-1})K_{k,0}\Big|_{v=\widehat{v}_0}. \tag{2.2.19}$$

It then follows from (2.2.13) that the left-hand side of (2.2.14) becomes

$$2\sum_{k=0}^{m}(\nu^2 - 4\widehat{v}_0^2)^{m-k}K_{k,0}\Big|_{v=\widehat{v}_0} + 2gt, \tag{2.2.20}$$

which coincides with $2T_{n,0}\big|_{x=(\nu^2-4\widehat{v}_0^2)/4}$. This completes the proof of Proposition 2.2.3. □

Thus the same propositions as Propositions 2.1.1 $\sim$ 2.1.3 in the case of the $P_{\mathrm{I}}$-hierarchy hold for the $P_{\text{II-1}}$-hierarchy also. In particular, since it follows from Proposition 2.2.3 that $C(t,\nu)$ is of the form $f(\nu^2,t)$ with some polynomial $f$ of degree $m$, we can define a turning point of the first kind and that of the second kind also for the $P_{\text{II-1}}$-hierarchy in a manner similar to the case of the $P_{\mathrm{I}}$-hierarchy. For both kinds of the turning points we can verify the following relations, similar to those for the $P_{\mathrm{I}}$-hierarchy, between the Stokes geometry of $(P_{\text{II-1}})_m$ and that of its underlying Lax pair $(L_{\text{II-1}})_m$.

***Proposition 2.2.4***

(i) *Let $t=\tau^{\mathrm{I}}$ be a turning point of the first kind of $(P_{\text{II-1}})_m$. Then at $t=\tau^{\mathrm{I}}$ a double turning point $x=b_j(t)$ merges with the simple turning point $x=a(t)=-(\widehat{v}_0(t))^2$ in the Stokes geometry of* (1.2.10.a)*. Consequently the two roots $\nu_{j,\pm}$ of $C(t,\nu)$ merge and vanish at $t=\tau^{\mathrm{I}}$. Furthermore the following relation holds:*

$$\frac{1}{2}\int_{\tau^{\mathrm{I}}}^{t}(\nu_{j,+}-\nu_{j,-})dt = \int_{a(t)}^{b_j(t)}(\lambda_+ - \lambda_-)dx. \tag{2.2.21}$$

(ii) *Let $t=\tau^{\mathrm{II}}$ be a turning point of the second kind of $(P_{\text{II-1}})_m$. Then at $t=\tau^{\mathrm{II}}$ a double turning point $x=b_j(t)$ merges with another double turning point $x=b_{j'}(t)$. Consequently two roots $\nu_{j,+}$ and $\nu_{j',+}$ of $C(t,\nu)$ merge at $t=\tau^{\mathrm{II}}$, and so do $\nu_{j,-}$ and $\nu_{j',-}$. Furthermore the following relation holds:*

$$\int_{\tau^{\mathrm{II}}}^{t}(\nu_{j,+}-\nu_{j',+})dt = -\int_{\tau^{\mathrm{II}}}^{t}(\nu_{j,-}-\nu_{j',-})dt = \int_{b_{j'}(t)}^{b_j(t)}(\lambda_+ - \lambda_-)dx. \tag{2.2.22}$$

***Proposition 2.2.5***. — *If $t$ lies on a Stokes curve of $(P_{\text{II-1}})_m$ emanating from a turning point $t=\tau^{\mathrm{I}}$ (resp. $t=\tau^{\mathrm{II}}$) of the first (resp. second) kind, the Stokes geometry of* (1.2.10.a) *becomes degenerate in the sense that its two turning points $x=b_j(t)$ and $x=a(t)$ (resp. $x=b_j(t)$ and $x=b_{j'}(t)$) are connected by a Stokes curve.*

We omit the proof of Propositions 2.2.4 and 2.2.5 as it is the same as that of Propositions 2.1.4 and 2.1.5.

**2.3. Case of the $P_{\text{II-2}}$-hierarchy.** — As is discussed in [**N1**] and [**N2**], the relations between the Stokes geometry of each member of the hierarchy and that of its underlying Lax pair, similar to those for the $P_{\text{I}}$-hierarchy and $P_{\text{II-1}}$-hierarchy, can be confirmed also for the $P_{\text{II-2}}$-hierarchy. We refer the reader to [**N1**] and [**N2**] for their precise formulation and the details of the proofs. Here, we only explain the core part of the discussion. For the sake of simplicity of the notations, we restrict our consideration to the case where $c_0 = c_1 = \cdots = c_{m-1} = 0$.

Substituting the 0-parameter solution

$$\widehat{u}(t,\eta) = \widehat{u}_0(t) + \eta^{-1}\widehat{u}_1(t) + \cdots, \tag{2.3.1}$$

$$\widehat{v}(t,\eta) = \widehat{v}_0(t) + \eta^{-1}\widehat{v}_1(t) + \cdots \tag{2.3.2}$$

of $(P_{\text{II-2}})_m$ into the coefficients $A$ and $B$ of the underlying Lax pair (1.3.8), we find that their top order parts $A_0$ and $B_0$ become

$$A_0 = \frac{1}{g}\begin{pmatrix} -(2x-\widehat{u}_0)T_{m,0} & 2T_{m,0} \\ -2\widehat{v}_0 T_{m,0} & (2x-\widehat{u}_0)T_{m,0} \end{pmatrix}, \tag{2.3.3}$$

$$B_0 = \begin{pmatrix} -x+\widehat{u}_0/2 & 1 \\ -\widehat{v}_0 & x-\widehat{u}_0/2 \end{pmatrix}, \tag{2.3.4}$$

where

$$T_{m,0} = \frac{1}{2}\sum_{j=0}^{m} x^{m-j} K_{j,0}\Big|_{u=\widehat{u}_0, v=\widehat{v}_0}. \tag{2.3.5}$$

This immediately entails that

$$A_0 = \frac{2T_{m,0}}{g} B_0. \tag{2.3.6}$$

Hence, if we let $x = b_j(t)$ $(1 \leqslant j \leqslant m)$ denote a root of $T_{m,0} = 0$, each $b_j(t)$ becomes a (generically) double turning point of the equation (1.3.8.a). Note that in this case there exist two (generically) simple turning points, denoted by $x = a_1(t)$ and $x = a_2(t)$ in what follows, since the characteristic equation of $B_0$ is a quadratic polynomial in $x$.

We, next, consider the linearization of $(P_{\text{II-2}})_m$ at $(u,v) = (\widehat{u},\widehat{v})$. Letting $\Delta K_j$ and $\Delta L_j$ respectively denote the linear part of $K_j$ and $L_j$ in $(\Delta u, \Delta v)$ after the substitution $(u,v) = (\widehat{u},\widehat{v}) + (\Delta u, \Delta v)$, we find that the linearization of $(P_{\text{II-2}})_m$ is

$$\Delta K_{m+1} = \Delta L_{m+1} = 0. \tag{2.3.7}$$

Let $C(t,\nu)$ denote its characteristic equation, then we obtain

$$C(t,\nu) = (-1)^m \prod_{j=1}^{m} \det(\mu - B_0)\Big|_{\mu=\nu/2, x=b_j}. \tag{2.3.8}$$

As in the preceding two subsections, (2.3.8) enables us to define a turning point of the first kind and that of the second kind also for the $P_{\text{II-2}}$-hierarchy. The key relation (2.3.8) can be proved in a similar manner as in Section 2.2; That is, since $K_j$ and $L_j$ are polynomials of $u$, $v$ and their derivatives, there exists a $2\times 2$ matrix of differential operators

$$D_j(t,\eta^{-1}\partial_t;\eta^{-1}) = D_{j,0}(t,\eta^{-1}\partial_t) + \eta^{-1}D_{j,1}(t,\eta^{-1}\partial_t) + \cdots \tag{2.3.9}$$

satisfying

$$\begin{pmatrix}\Delta K_j\\ \Delta L_j\end{pmatrix} = D_j(t,\eta^{-1}\partial_t;\eta^{-1})\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix}. \tag{2.3.10}$$

Then, in terms of $D_j(t,\eta^{-1}\partial_t;\eta^{-1})$, $C(t,\nu)$ is expressed as

$$C(t,\nu) = \det D_{m+1,0}(t,\nu). \tag{2.3.11}$$

On the other hand, considering the linear (in $(\Delta u,\Delta v)$) part of both sides of (1.3.2) and taking its top order term, we find

$$D_{j+1,0}(t,\nu) = \begin{pmatrix}(\widehat{u}_0-\nu)/2 & 1\\ \widehat{v}_0 & (\widehat{u}_0+\nu)/2\end{pmatrix} D_{j,0}(t,\nu) + \frac{1}{2}K_{j,0}I_2, \tag{2.3.12}$$

where $I_2$ stands for the $2\times 2$ identity matrix. By solving this recursive relation under the condition $D_{1,0}(t,\nu) = I_2$, we obtain

$$D_{m+1,0} = \frac{1}{2}\sum_{j=0}^{m} K_{m-j,0}\begin{pmatrix}(\widehat{u}_0-\nu)/2 & 1\\ \widehat{v}_0 & (\widehat{u}_0+\nu)/2\end{pmatrix}^{j}. \tag{2.3.13}$$

Hence (2.3.5) and (2.3.13) entail that

$$\begin{aligned} D_{m+1,0}(t,\nu) &= \prod_{j=1}^{m}\left(\begin{pmatrix}(\widehat{u}_0-\nu)/2 & 1\\ \widehat{v}_0 & (\widehat{u}_0+\nu)/2\end{pmatrix} - b_j I_2\right) \\ &= \prod_{j=1}^{m}\begin{pmatrix}(\widehat{u}_0-\nu)/2-b_j & 1\\ \widehat{v}_0 & (\widehat{u}_0+\nu)/2-b_j\end{pmatrix}. \end{aligned} \tag{2.3.14}$$

The relation (2.3.8) immediately follows from (2.3.4), (2.3.11) and (2.3.14).

## 3. The inevitability of the Nishikawa phenomenon

In a computer-assisted study of the Stokes geometry for $(P_{\text{II-2}})_2$ Nishikawa ([**N1**]) found the following intriguing phenomenon:

There exist points outside the union of all Stokes curves for $(P_{\text{II-2}})_2$ where the Stokes geometry of (1.3.8.a) degenerates. Furthermore the totality of such points forms a curved ray emanating from the intersection of two Stokes curves for $(P_{\text{II-2}})_2$.

The purpose of this section is to show why and how such a phenomenon, which is now known as the Nishikawa phenomenon, should be observed. To fix the notations we consider the case $(P_{\mathrm{I}})_2$, although the reasoning equally applies to $(P_J)_m$ with $m \geqslant 2$ and $J = \mathrm{I}$, II-1 or II-2. We note that the phenomena studied below are not observed when $m = 1$, *i.e.*, for the traditional Painlevé equations. One important reason for this is the fact that the number of the double turning points of the equation (1.1.11.a) is 1 when $m = 1$; at least two double turning points seem to be needed for the occurrence of a Nishikawa phenomenon.

Let $T$ be a crossing point of two Stokes curves of $(P_{\mathrm{I}})_2$. Suppose that, when $t$ lies on one of the Stokes curves of $(P_{\mathrm{I}})_2$, a (double) turning point $A$ is connected with a (simple) turning point $C$ by a Stokes curve in the Stokes geometry of the linear equation (1.1.11.a) and that another (double) turning point $B$ is similarly connected with $C$ by a Stokes curve of (1.1.11.a) when $t$ lies on the other Stokes curve of $(P_{\mathrm{I}})_2$; the (topological) configuration of the Stokes curves of (1.1.11.a) when $t = T$ is seen in Figure 3.1. (As we study the configuration of Stokes curves both for the Painlevé equations (*i.e.*, in $t$-variable) and for one of the underlying Lax pair (*i.e.*, in $x$-variable), we put throughout this article a sign $t$ or $x$ to each figure for the convenience of the reader.) In what follows, having these geometrical situations in mind, we label the two Stokes curves of $(P_{\mathrm{I}})_2$ crossing at $T$ as $[AC]$ and $[BC]$ respectively.

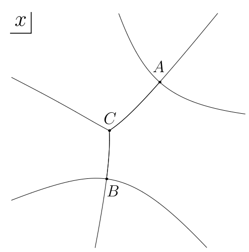


FIGURE 3.1

Let us move around the point $T$ from $t_1$ to $t_4$ as designated by the arrows shown in Figure 3.2.

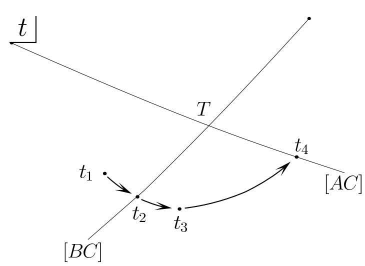


FIGURE 3.2

To fix the notation let us suppose that the configuration of Stokes curves of (1.1.11.a) at $t_j$ $(j = 1, 2, 3)$ is as in Figure 3.3.$j$. The letters $a \sim g$ label the directions into which Stokes curves asymptotically flow. Such configurations are really observed, for example, near the crossing point of Stokes curves of $(P_{\mathrm{I}})_2$ shown in Figure 5.1.2(i) in Section 5.

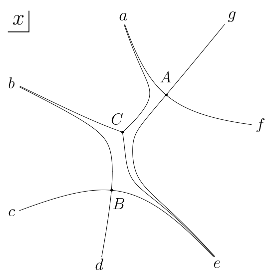


FIGURE 3.3.1

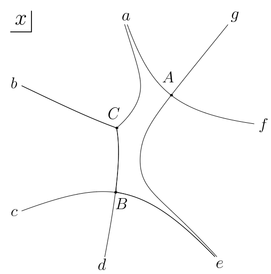


FIGURE 3.3.2

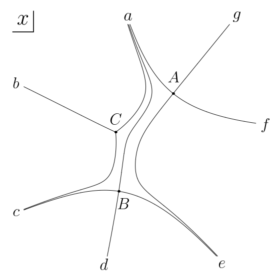


FIGURE 3.3.3

Note that we can detect the configuration in Fig. 3.3.3 by the relation (2.1.26) without resorting to the computer-assisted numerical computations; the Stokes curve emanating from $B$ and flowing to the direction $b$ in Fig. 3.3.1 should now go to some direction looking at $C$ on the left side, but the number of directions to which Stokes curves of (1.1.11.a) flow is 7 and they are exhausted by $a \sim g$. Since (1.1.11.a) is a $2 \times 2$ system, its Stokes curves do not cross. Hence the only direction to which the Stokes curve in question flow is the direction $a$. The same reasoning applies to the Stokes curve emanating from $C$ and flowing to the direction $e$ in Fig. 3.3.1. Thus Fig. 3.3.3 is a logical consequence of Fig. 3.3.1 and Fig. 3.3.2.

Now, is it possible to reach a point $t_4$ in $[AC]$ with keeping the topological configuration designated in Fig. 3.3.3 ? For the convenience of the reader we give the configuration of Stokes curves of (1.1.11.a) when $t = t_4$ in Fig. 3.3.4.

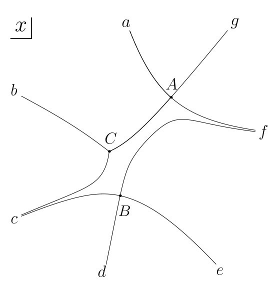


FIGURE 3.3.4

The answer to the above question is clearly "No", because no Stokes curve can connect $A$ and $C$; if such a Stokes curve existed, it should cross the Stokes curve emanating from $B$ and flowing to the direction $a$ or $d$, and it should contradict the requirement that no Stokes curves should cross for $2 \times 2$ systems. Thus the Stokes curve emanating from $B$ and flowing to the direction $a$ should swing further and hit the turning point $A$ as in Fig. 3.3.5 at some point $t = t_5$ during the journey of $t$ from $t_3$ to $t_4$.

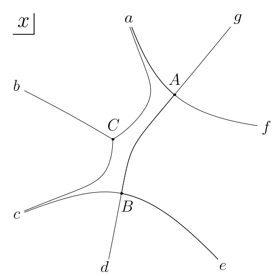


FIGURE 3.3.5

We can then smoothly continue our journey; we find the configuration shown in Fig. 3.3.6 after $t$ passes through $t_5$, as is detected by (2.1.26). Then it is natural to find the configuration shown in Fig. 3.3.4 as we continue our journey to reach $t = t_4$.

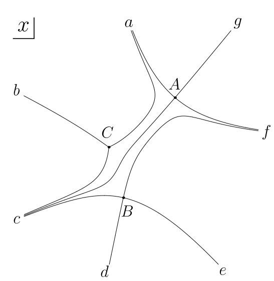


FIGURE 3.3.6

Summing up, during the journey from $t_2$ to $t_4$, unanticipated degeneracy of the Stokes geometry of (1.1.11.a) inevitably occurs at some point, and the totality of such points is a (curved) ray emanating from $T$. This explains why and how the Nishikawa phenomenon should occur.

We note that the above discussion makes essential use of the fact that, although $(P_{\mathrm{I}})_2$ is equivalent to the fourth order equation (and hence its Stokes curves may, and really do, cross), the Lax pair associated with it consists of $2 \times 2$ systems.

## 4. Introduction of a new Stokes curve to explain the Nishikawa phenomenon

The purpose of this section is to introduce a "new" Stokes curve so that the Nishikawa phenomenon may be naturally interpreted as the occurrence of degeneracy of the Stokes geometry of the underlying Lax pair when the parameter $t$ lies on the new Stokes curve. Introduction of a new Stokes curve was first done by Berk-Nevins-Roberts ([**BNR**]) for a linear differential operator with holomorphic coefficients so that the connection formula for WKB solutions may be consistently written down near crossing points of Stokes curves. Because of the complexity of the equation in question, the reasoning of Berk et al. cannot be applied to our case. Instead, in introducing new Stokes curves for the linearization of $(P_J)_m$ such as (2.1.16) we use the graph-theoretical structure of the Stokes curves of the linear equation (1.1.11.a),(1.2.10.a) or (1.3.8.a).

Now, as Nishikawa ([**N1**]) has numerically observed, it is not always the case that we encounter Nishikawa phenomena near a crossing point of Stokes curves for the linearization of Painlevé equations, or for short, Fréchet derivatives. To characterize a crossing point of Stokes curves near which we observe Nishikawa phenomena we make some preparatory discussions.

Let us suppose that two Stokes curves for a Fréchet derivative cross transversally at a point $T$. By the Fact E for $(P_J)_m$ (*cf.* Proposition 2.1.5 and Proposition 2.2.5) each of the Stokes curves corresponds to a pair of turning points of (1.1.11.a) (or (1.2.10.a) or (1.3.8.a)) which are connected by a Stokes curve. Then either one of the following two situations is observed at $t = T$:

Case I: These two pairs share one turning point.

Case II: The four turning points are mutually distinct.

In what follows, we say in Case I that the two Stokes curves of (1.1.11.a) etc. (each of which connects a pair of turning points) are hinged by the shared turning point. We also call the shared turning point a hinging turning point (*cf.* Fig. 4.1). Using these terminologies, we further classify the situations in Case I.

Case Ia: The hinged two Stokes curves of (1.1.11.a) are adjacent at the hinging turning point.

Case Ib: The hinged two Stokes curves of (1.1.11.a) are not adjacent.

Note that, if the hinging turning point $x(T)$ in Fig. 4.1 is simple, then Case Ib is never realized; in fact, only 3 Stokes curves emanate from a simple turning point, and hence two Stokes curves are always adjacent there.

A crossing point $T$ is said to be Lax-adjacent, or for short, LA if the configuration of Stokes curves of (1.1.11.a) etc. at $t = T$ is classified as in Case Ia. Otherwise, it is said to be non-Lax-adjacent or non-LA for short. An important property of two adjacent Stokes curves of (1.1.11.a) etc. is that the dominance relation of each of the Stokes curves is opposite (if the angle formed by the two Stokes curves does not

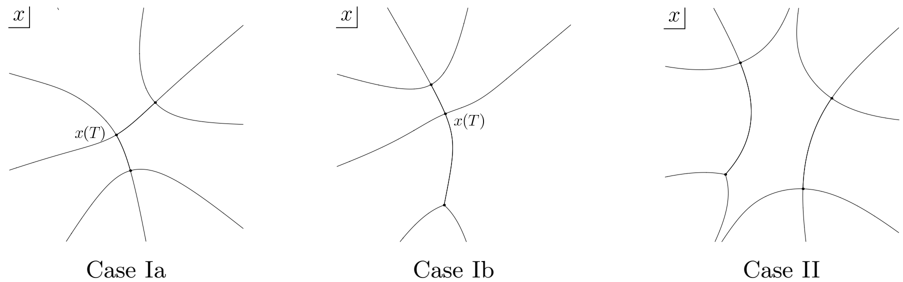


FIGURE 4.1. Example of configurations of relevant Stokes curves of (1.1.11.a) etc. in Cases Ia, Ib and II. In Cases Ia and Ib $x(T)$ designates the hinging turning point, while the pairs are not hinged in Case II.

contain the cut that fixes the branch of the characteristic values of (1.1.11.a) etc.) . In what follows we use this property in a substantial manner.

A new Stokes curve is, by definition, not introduced at a non-LA crossing point. At an LA crossing point $T$ we introduce new Stokes curves that pass through $T$, following the rules given below. Here and in what follows, we attach the symbol "$(j,+) > (k,-)$" etc., to each (ordinary) Stokes curve to mean

$$\mathrm{Re}\int_{\tau}^{t}(\nu_{j,+}-\nu_{k,-})dt > 0 \tag{4.1}$$

holds on the Stokes curve in question. Here, $\nu_{j,+}$ (resp. $\nu_{k,-}$) designates the relevant characteristic root of the Fréchet derivative which is labeled by $(j,+)$ (resp. $(k,-)$), that is, $\nu_{j,+}$ and $\nu_{k,-}$ are solutions of the equation

$$\det(\nu - C_0) = 0. \tag{4.2}$$

(*Cf.* (2.1.23), (2.2.14) and (2.3.8)) We choose the lower end point of the integral in (4.1) to be the turning point from which the Stokes curve emanates. We also note that two symbols like $(j,+) > (k,-)$ and $(k,+) > (j,-)$ are attached to a Stokes curve which emanates from a turning point of the second kind; this means that two Stokes curves determined respectively by $\mathrm{Im}\int_{\tau}^{t}(\nu_{j,+}-\nu_{k,-})dt = 0$ and $\mathrm{Im}\int_{\tau}^{t}(\nu_{k,+}-\nu_{j,-})dt = 0$ sit on one and the same curve.

### Rules for introducing new Stokes curves

***Case A***. — *At a Lax-adjacent crossing point $T$ of two Stokes curves $C_1$ and $C_2$ respectively emanating from turning points $\tau_1 = \tau_1^{\mathrm{I}}$ and $\tau_2 = \tau_2^{\mathrm{I}}$ of the first kind.*

In this case, using the Fact D for $(P_J)_m$ (*cf.* Proposition 2.1.4 and Proposition 2.2.4) and the assumption that $T$ is an LA crossing point, we can find a simple turning point $a(t)$ and two double turning points $b_j(t)$ and $b_k(t)$ for which the configuration of relevant Stokes curves of (1.1.11.a) etc. contains the following portion at $t = T$ (Fig. 4.2).

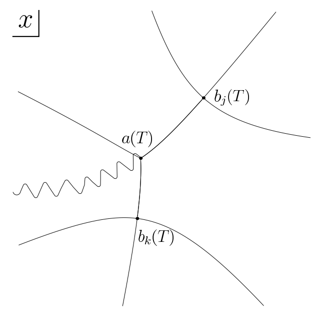


FIGURE 4.2

Here, the wiggly line designates a cut to fix the branch of $\sqrt{-\det A_0}$. Since $\tau_1$ is a turning point of the first kind, we can find characteristic roots $\nu_{j,\pm}$ so that they satisfy

$$\nu_{j,-} = -\nu_{j,+} \tag{4.3}$$

and

$$\nu_{j,+}(\tau_1) = \nu_{j,-}(\tau_1) = 0. \tag{4.4}$$

(*Cf.* the remark after Proposition 2.1.3.) Letting $Q_0$ denote $-\det A_0$, we may assume

$$\frac{1}{2}\int_{\tau_1}^{t}(\nu_{j,+} - \nu_{j,-})dt = 2\int_{a(t)}^{b_j(t)} \sqrt{Q_0}\, dx \tag{4.5}$$

by replacing $\nu_{j,+}$ and $\nu_{j,-}$ if necessary. To fix the notation let us suppose that the Stokes curve $C_1$ is labeled by $(j,+) > (j,-)$. We then find

$$\operatorname{Re}\int_{a(T)}^{b_j(T)} \sqrt{Q_0}\, dx = \frac{1}{4}\int_{\tau_1}^{T}(\nu_{j,+} - \nu_{j,-})dt > 0. \tag{4.6}$$

With a similar reasoning we find characteristic roots $\nu_{k,\pm}$ satisfying

$$\nu_{k,-} = -\nu_{k,+}, \tag{4.7}$$

$$\nu_{k,+}(\tau_2) = \nu_{k,-}(\tau_2) = 0 \tag{4.8}$$

and

$$\int_{\tau_2}^{t}(\nu_{k,+} - \nu_{k,-})dt = 4\varepsilon\int_{a(t)}^{b_k(t)} \sqrt{Q_0}\, dx \tag{4.9}$$

with $\varepsilon = \pm 1$. In view of the location of the cut in Fig. 4.2, we find from (4.6)

$$\operatorname{Re}\int_{a(T)}^{b_k(T)} \sqrt{Q_0}\, dx < 0. \tag{4.10}$$

Hence the Stokes curve $C_2$ is labeled as $(k,+) > (k,-)$ (resp. $(k,-) > (k,+)$) if $\varepsilon = -1$ (resp. $\varepsilon = 1$). Then we introduce a new Stokes curve by the following:

$$\operatorname{Im} \int_T^t (\nu_{j,+} - \nu_{k,-}) dt = \operatorname{Im} \int_T^t (\nu_{k,+} - \nu_{j,-}) dt = 0 \tag{4.11}$$

if $\varepsilon = -1$, and

$$\operatorname{Im} \int_T^t (\nu_{j,+} - \nu_{k,+}) dt = \operatorname{Im} \int_T^t (\nu_{k,-} - \nu_{j,-}) dt = 0 \tag{4.12}$$

if $\varepsilon = 1$. At this moment we label a new Stokes curve by just the pair(s) of indices of the characteristic roots appearing in the definition of the curve, that is, we do not use the inequality symbol. To be concrete, the curve defined by (4.11) (resp. (4.12)) is labeled as $(j,+;k,-),(k,+;j,-)$ (resp. $(j,+;k,+),(k,-;j,-)$). Thus the resulting configuration of (ordinary and new) Stokes curves near $t = T$ is either one of the following two graphs given in Fig. 4.3.

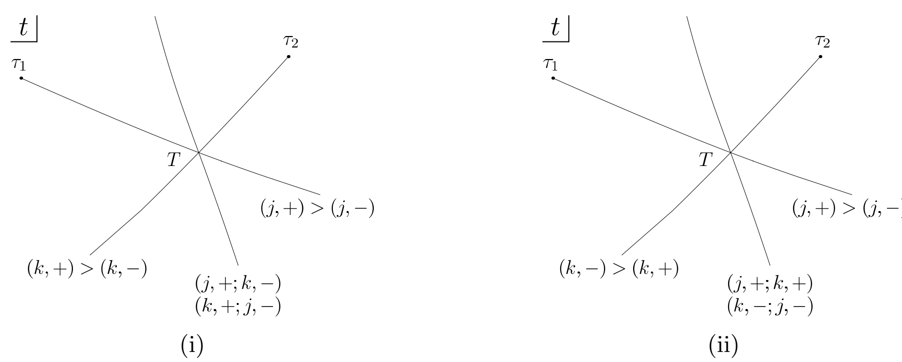


FIGURE 4.3

***Case B***. — *At a Lax-adjacent crossing point $T$ of two Stokes curves $C_1$ and $C_2$ respectively emanating from a turning point $\tau_1 = \tau_1^{\mathrm{I}}$ of the first kind and from a turning point $\tau_2 = \tau_2^{\mathrm{II}}$ of the second kind.*

By the same reasoning as in Case A we find a simple turning point $a(t)$ and two double turning points $b_j(t)$ and $b_k(t)$ for which the configuration of Stokes curves of (1.1.11.a) etc. contains the portion designated in Fig. 4.4 (or its mirror image) at $t = T$.

Let us choose characteristic roots $\nu_{j,\pm}$ so that they satisfy (4.3) $\sim$ (4.6). To fix the situation we assume the Stokes curve $C_1$ is labeled as $(j,+) > (j,-)$. By the Fact D for $(P_J)_m$ (*cf.* Proposition 2.1.4 and Proposition 2.2.4.), we find $\nu_{k,\pm}$ for which the

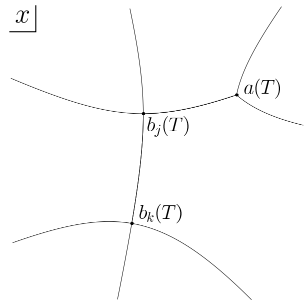


FIGURE 4.4

following relation holds with appropriate $\sigma = \pm$ and $\varepsilon = \pm 1$:

$$\int_{\tau_2}^{t} (\nu_{k,\sigma} - \nu_{j,+}) dt = 2\varepsilon \int_{b_j(t)}^{b_k(t)} \sqrt{Q_0}\, dx. \tag{4.13}$$

Hence we find

$$\nu_{k,\sigma} - \nu_{j,+} = 2\varepsilon \frac{d}{dt}\Big( \int_{b_j(t)}^{a(t)} \sqrt{Q_0}\, dx + \int_{a(t)}^{b_k(t)} \sqrt{Q_0}\, dx \Big). \tag{4.14}$$

On the other hand, (4.3) and (4.5) entail

$$\nu_{j,+} = 2\frac{d}{dt} \int_{a(t)}^{b_j(t)} \sqrt{Q_0}\, dx. \tag{4.15}$$

Thus we conclude $\varepsilon = +1$ in (4.14). Then, as $\sigma$ is rather conventional in our current context, we consider both situations. (If we consider the problem globally, not localizing the problem near $T, \sigma$ should be fixed in concrete problems. See [**NT**] for this point.) Since we have labeled $C_1$ as $(j,+) > (j,-)$, we find

$$\int_{a(T)}^{b_j(T)} \sqrt{Q_0}\, dx > 0. \tag{4.16}$$

Hence the Lax-adjacency assumption implies

$$\int_{b_j(T)}^{b_k(T)} \sqrt{Q_0}\, dx > 0. \tag{4.17}$$

This means that $C_2$ is labeled as

$$(k,+) > (j,+) \text{ and } (j,-) > (k,-) \text{ if } \sigma = + \tag{4.18}$$

$$(k,-) > (j,+) \text{ and } (j,-) > (k,+) \text{ if } \sigma = -. \tag{4.19}$$

The required new Stokes curve is then given by

$$\operatorname{Im} \int_{T}^{t} (\nu_{k,+} - \nu_{k,-}) dt = 0. \tag{4.20}$$

Thus the resulting configuration of (ordinary and new) Stokes curves near $t = T$ is either one of the following two graphs given in Fig. 4.5.

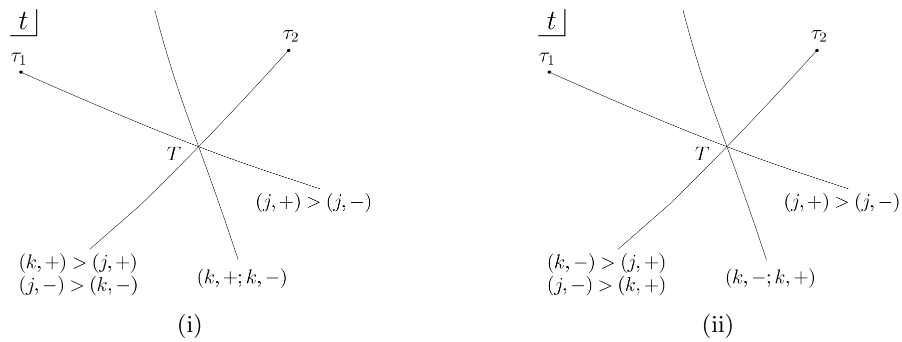


FIGURE 4.5

***Case C***. — *At a Lax-adjacent crossing point $T$ of two Stokes curves $C_1$ and $C_2$ respectively emanating from turning points $\tau_1 = \tau_1^{\mathrm{II}}$ and $\tau_2 = \tau_2^{\mathrm{II}}$ of the second kind.*

In this case, using the Fact D for $(P_J)_m$ (*cf.* Proposition 2.1.4 and Proposition 2.2.4.) we find three double turning points $b_j(t), b_k(t)$ and $b_l(t)$ for which the configuration of Stokes curves of (1.1.11.a) etc. contains the following portion at $t = T$:

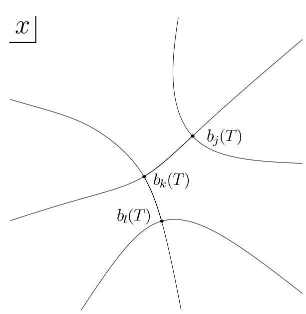


FIGURE 4.6

To fix the situation, let us choose characteristic roots $\nu_{j,\pm}$ and $\nu_{k,\pm}$ so that they satisfy the following:

$$\nu_{j,-} = -\nu_{j,+} \text{ and } \nu_{k,-} = -\nu_{k,+}, \tag{4.21}$$

$$\nu_{j,+}(\tau_1) = \nu_{k,+}(\tau_1) \neq 0, \tag{4.22}$$

$$\begin{aligned}\int_{\tau_1}^{t} (\nu_{j,+} - \nu_{k,+})dt &= -\int_{\tau_1}^{t} (\nu_{j,-} - \nu_{k,-})dt \\ &= 2\int_{b_k(t)}^{b_j(t)} \sqrt{Q_0}\, dx.\end{aligned} \tag{4.23}$$

We also assume

$$\operatorname{Re} \int_{b_k(T)}^{b_j(T)} \sqrt{Q_0}\, dx > 0. \tag{4.24}$$

Otherwise stated, the Stokes curve $C_1$ is labeled as $(j,+) > (k,+)$ and $(k,-) > (j,-)$. In parallel with the argument in Case B, we find characteristic roots $\nu_{l,\pm}$ for which the following relation holds with appropriate $\sigma = \pm$ and $\varepsilon = \pm 1$:

$$\int_{\tau_2}^{t} (\nu_{l,\sigma} - \nu_{k,+})dt = 2\varepsilon \int_{b_k(t)}^{b_l(t)} \sqrt{Q_0}\, dx. \tag{4.25}$$

Then we have

$$\begin{aligned}\nu_{l,\sigma} - \nu_{k,+} &= 2\varepsilon \frac{d}{dt}\Big( \int_{b_k(t)}^{b_j(t)} \sqrt{Q_0}\, dx + \int_{b_j(t)}^{b_l(t)} \sqrt{Q_0}\, dx \Big) \\ &= 2\varepsilon \frac{d}{dt}\Big( \int_{b_j(t)}^{b_l(t)} \sqrt{Q_0}\, dx) + \varepsilon(\nu_{j,+} - \nu_{k,+}) \Big).\end{aligned} \tag{4.26}$$

Hence we conclude $\varepsilon = +1$. Again in parallel with Case B, we do not fix $\sigma$. Since we have assumed (4.24), the Lax-adjacency assumption entails

$$\operatorname{Re} \int_{b_k(T)}^{b_l(T)} \sqrt{Q_0}\, dx < 0. \tag{4.27}$$

As $\varepsilon = +1$ in (4.25), we find that the Stokes curve $C_2$ is labeled as

$$(k,+) > (l,+) \text{ and } (l,-) > (k,-) \text{ if } \sigma = + \tag{4.28}$$

or

$$(k,+) > (l,-) \text{ and } (l,+) > (k,-) \text{ if } \sigma = -. \tag{4.29}$$

Then the required new Stokes curve is given by

$$\operatorname{Im} \int_{T}^{t} (\nu_{j,+} - \nu_{l,\sigma})dt = 0. \tag{4.30}$$

Thus the resulting configuration of Stokes curves near $t = T$ is either one of the following two graphs given in Fig. 4.7.

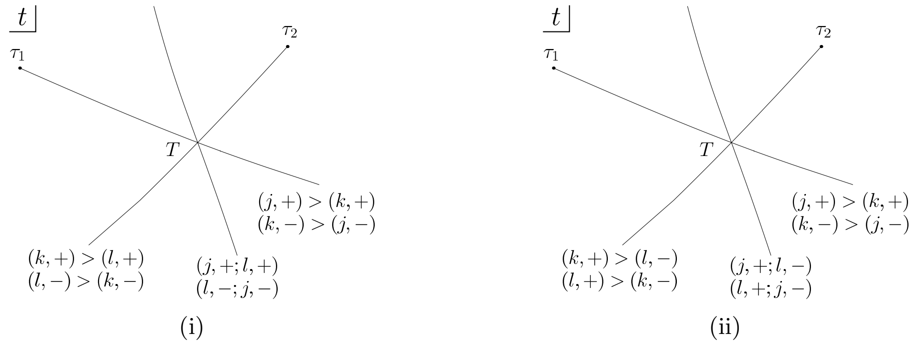


FIGURE 4.7

There exist crossing points of an ordinary Stokes curve and a new Stokes curve introduced above. However, no Nishikawa phenomena have been observed near them, at least in the examples so far studied. (*Cf.* [**N1**]; see also § 5.4). Hence we do not try to define the "secondary" new Stokes curves in this article. At the same time we surmise that we need such new Stokes curves in some more complicated examples.

Now, the importance and the naturality of the notion of new Stokes curves are shown by the following

***Theorem 4.1***. — *If $t$ lies on a new Stokes curve introduced above, then the imaginary part of the integral $\int_{x_1(t)}^{x_2(t)} \sqrt{Q_0}\,dx$ vanishes for appropriately chosen turning points $x_1(t)$ and $x_2(t)$ of the equation* (1.1.11.a), (1.2.10.a) *or* (1.3.8.a). *To be more concrete, we find the following:*

(i) *In Case A, $x_1(t) = b_k(t)$ and $x_2(t) = b_j(t)$.*
(ii) *In Case B, $x_1(t) = a(t)$ and $x_2(t) = b_k(t)$.*
(iii) *In Case C, $x_1(t) = b_l(t)$ and $x_2(t) = b_j(t)$.*

*Proof.* — As the reasoning is the same for all cases, we prove the theorem only in the case (i). In what follows we use the notations in Rules above. Let us consider the case where $\varepsilon = -1$ in (4.9). Then, summing up (4.5) and (4.9), we find

$$\int_{\tau_1}^{t} (\nu_{j,+} - \nu_{j,-})dt + \int_{\tau_2}^{t} (\nu_{k,+} - \nu_{k,-})dt = 4 \int_{b_k(t)}^{b_j(t)} \sqrt{Q_0}\,dx. \tag{4.31}$$

Since $T$ is a crossing point of Stokes curves $C_1$ and $C_2$,

$$\operatorname{Im} \int_{\tau_1}^{T} (\nu_{j,+} - \nu_{j,-})dt = \operatorname{Im} \int_{\tau_2}^{T} (\nu_{k,+} - \nu_{k,-})dt = 0 \tag{4.32}$$

holds. Therefore we obtain

$$\operatorname{Im} \int_{T}^{t} (\nu_{j,+} - \nu_{j,-} + \nu_{k,+} - \nu_{k,-})dt = 4 \operatorname{Im} \int_{b_k(t)}^{b_j(t)} \sqrt{Q_0}\,dx. \tag{4.33}$$

Since the left-hand side of (4.33) vanishes by the definition (4.11) of a new Stokes curve, we find the required fact. □

***Remark 4.1***. — If $x_1(t)$ and $x_2(t)$ are connected by a Stokes curve of (1.1.11.a) etc., then we find

$$\operatorname{Im} \int_{x_1(t)}^{x_2(t)} \sqrt{Q_0}\, dx = 0, \tag{4.34}$$

but not vice versa. The point is that a Stokes curve of (1.1.11.a) etc. is, by definition, an integral curve of the vector field $\operatorname{Im} \sqrt{Q_0}\, dx$ that emanates from a turning point. (*Cf.* [**AKT1**, p. 80])

As a matter of fact, Rules stated above are somewhat loose. A more precise description of a new Stokes curve should be as follows:

If the (real 1-dimensional) curve defined by (4.34) is non-singular,

$$\operatorname{Re} \int_{x_1(t)}^{x_2(t)} \sqrt{Q_0}\, dx \tag{4.35}$$

is monotonically decreasing or increasing along the curve. In particular, we can always find a point $\omega$ in the curve where the integral

$$\int_{x_1(t)}^{x_2(t)} \sqrt{Q_0}\, dx \tag{4.36}$$

vanishes at $t = \omega$. Then, in an analogy with the case of linear differential operators with holomorphic coefficients (*cf.* [**BNR**],[**AKT1**]), the part of the new Stokes curve which contains $\omega$ should be designated by a dotted line (near $t = T$) in the precise definition of a new Stokes curve. As a matter of fact the dotted part of a new Stokes curve is irrelevant to the degeneracy of the Stokes geometry of (1.1.11.a) etc.. This can be confirmed by a similar reasoning as is given in § 3 once concrete description of a new Stokes curve is given. As a typical example we analyze the example we studied in § 3. This time we consider the configuration of the Stokes curves of (1.1.11.a) etc. at $t = t_j$ ($j = 6, 7, 8$) designated in Fig. 4.8.

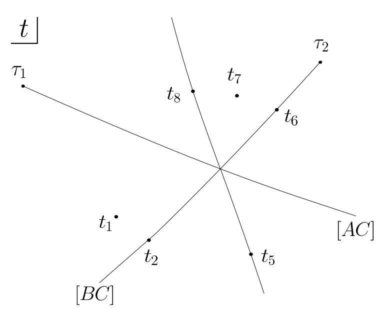


FIGURE 4.8. (*Cf.* Figure 3.2.)

If we move from $t_2$ to $t_6$ along $[BC]$, we cross $[AC]$ at $T$. Hence it follows from (2.1.26) that the Stokes curve emanating from $A$ and flowing to the direction $e$ and the Stokes curve emanating from $C$ and flowing to the direction $a$ in Fig. 3.3.2 should interchange the directions to which they flow when $t$ reaches $t_6$, as shown in Fig. 4.9.6.

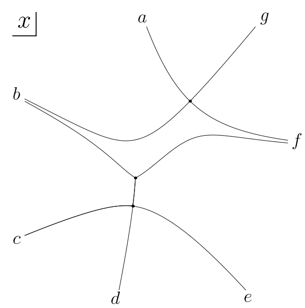


FIGURE 4.9.6

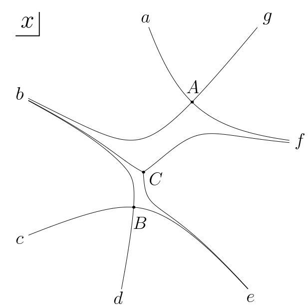


FIGURE 4.9.7

Again by (2.1.26) and the comparison of Fig. 3.3.1 and Fig. 3.3.2, we find from Fig. 4.9.6 that the configuration of Stokes curves at $t = t_7$ is given in Fig. 4.9.7.

Now we know that the unanticipated degeneracy of Stokes curves occurs at $t_5$ (*cf.* § 3), and we can confirm that the point $t_5$ lies on the new Stokes curve described in Fig. 4.3. As the unanticipated degeneracy at $t_5$ means that $A$ and $B$ are connected by a Stokes curve (*cf.* Fig. 3.3.5), we label the curve as $[AB]$. Is it, then, possible to reach a point $t_8$ where $A$ and $B$ are connected by a Stokes curve with keeping the topological configuration designated in Fig. 4.9.7? The answer is clearly "No" by the same reasoning as in § 3, *i.e.*, by the fact that no Stokes curves are allowed to cross each other for a $2 \times 2$ system, like (1.1.11.a). Otherwise stated, if $A$ and $B$ were really connected by a Stokes curve at $t = t_8$, either ($A$ and $C$) or ($B$ and $C$) should be connected by a Stokes curve before $t$ reaches $t_8$. But, neither Stokes curve $[AC]$ nor $[BC]$ exists between $t_7$ and $t_8$. This means that $A$ and $B$ are not connected by a Stokes curve at $t_8$, although

$$\operatorname{Im} \int_A^B \sqrt{Q_0}\, dx = 0 \tag{4.37}$$

holds at $t = t_8$. As a matter of fact, some numerical computation shows that (4.36) vanishes at some point $\omega$ near $t_8$. Thus the precise description of the Stokes curves would be as in Fig. 4.10.

Finally we note that we can actually label a new Stokes curve not by just a pair like $(k, +; k, -)$ but by a more informative label like $(k, +) > (k, -)$; the sign of

$$\operatorname{Re} \int_{x_1(T)}^{x_2(T)} \sqrt{Q_0}\, dx \tag{4.38}$$

can be effectively used for this purpose. Concerning these subtle issues we will report in our forthcoming paper.

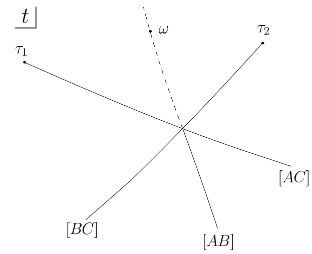


FIGURE 4.10

## 5. Examples of Stokes geometry

**5.1.** As the simplest example of the linearization of a higher order Painlevé equation, we study $(P_{\mathrm{I}})_2$ in this subsection. In this case the configuration of the Stokes curves are shown in Figure 5.1.1. However, if we want to understand the global structure of the configuration, we should take into account the Riemann sheet structure of the coefficients of $(P_{\mathrm{I}})_2$; the coefficients contain a multi-valued function $\widehat{u}_{1,0}$ defined by

$$5\widehat{u}_{1,0}^3 + 2c_1\widehat{u}_{1,0} - 2c_2 - 2t = 0. \tag{5.1}$$

Hence we first prepare three sheets which describes the Riemann sheet structure of $\widehat{u}_{1,0}$, and we then draw the Stokes curves of the linearization of $(P_{\mathrm{I}})_2$ on each sheet. The resulting configurations are described in Figure 5.1.2($j$) ($j = \mathrm{i}, \mathrm{ii}, \mathrm{iii}$) where we have chosen $c_1 = 1 - 1.7i$ and $c_2 = 0$. We note the singular points of $\widehat{u}_{1,0}$ are given by the zeros of discriminant of (5.1), which are coincident with the turning points $\tau_1^{\mathrm{I}}$ and $\tau_2^{\mathrm{I}}$ of $(P_{\mathrm{I}})_2$ of the first kind. The wiggly lines in Figure 5.1.2($j$) designate the cuts to describe the global structure of $\widehat{u}_{1,0}$ with the additional information that the singularity of $\widehat{u}_{1,0}$ is of the square-root type. We note that, if we take into account the sheet structure of $\widehat{u}_{1,0}$, the points $\tau_1^{\mathrm{II}}$ and $\tau_2^{\mathrm{II}}$ on the first sheet (*i.e.*, in Figure 5.1.2(i)), for example, are not the turning points (of the second kind).

We next draw the new Stokes curves in Figure 5.1.2($j$) to find the following Figure 5.1.3($j$) ($j = \mathrm{i}, \mathrm{ii}, \mathrm{iii}$). Here, we employ the precise definition of a new Stokes curve given in Remark 4.1; we will see below that the dotted part is irrelevant to the topological change of the configuration of the Stokes geometry of the linear equation (1.1.11.$a$). In Figure 5.1.5(i).j (resp. Figure 5.1.5(ii).k), we concretely describes the configuration of Stokes curves of (1.1.11.a) when $t$ moves around the crossing point $t = T_{(\mathrm{i})}$ (resp. $T_{(\mathrm{ii})}$) of Stokes curves in Figure 5.1.3(i) (resp. Figure 5.1.3(ii)). The configuration for $t = T_{(\mathrm{i})}$ (resp. $t = T_{(\mathrm{ii})}$) is also given in Figure 5.1.4(i) (resp. Figure 5.1.4(ii)). The specific points to be considered are labeled by $t = t_j$ ($j = 1, \dots 12$) in Figure 5.1.3(i) and by $t = t_k$ ($k = 13, \dots 18$) in Figure 5.1.3(ii). The reader readily finds that the topological changes occur only at $t = t_j$ or $t = t_k$ that lies on an ordinary Stokes curve or on the solid line part of a new Stokes curve.

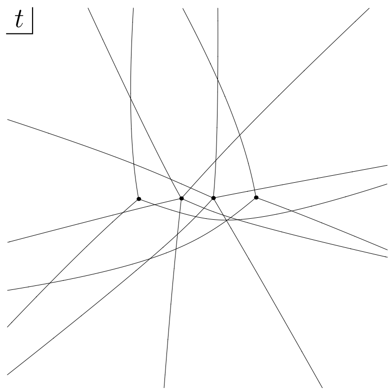


FIGURE 5.1.1

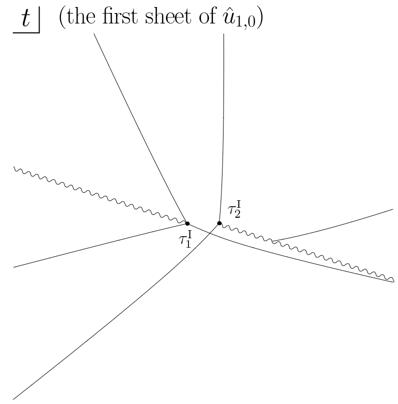


FIGURE 5.1.2(i)

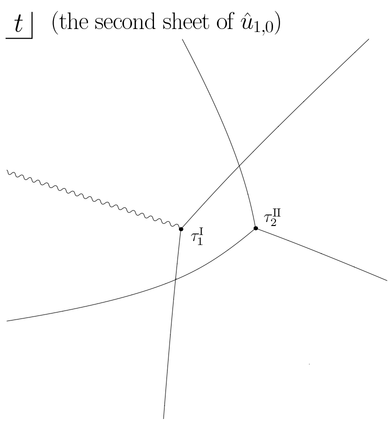


FIGURE 5.1.2(ii)

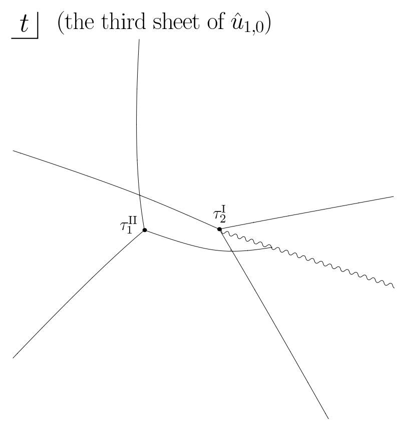


FIGURE 5.1.2(iii)

**5.2.** Since the number of double turning points of (1.1.11.$a$) is 2 for $(P_{\mathrm{I}})_2$, we need to try to study $(P_{\mathrm{I}})_3$, for example, to find a crossing point of two Stokes curves both emanating from a turning point of the second kind. (Case C in Section 4.) Fortunately we can really locate it in the Stokes geometry of $(P_{\mathrm{I}})_3$ (with $c_1 = 1.2{+}0.8i$, $c_2 = -1.7{-}1.5i$ and $c_3 = i$). The Stokes geometry (without the detailed consideration of the sheet structure) is given in Figure 5.2.1. We concentrate our attention to the turning points $\tau_1^{\mathrm{II}}$ and $\tau_2^{\mathrm{II}}$ specified in Figure 5.2.1 and we present in Figure 5.2.3 the configuration of Stokes curves of (1.1.11.$a$) at the crossing point $T$ of the Stokes curve for $(P_{\mathrm{I}})_3$ emanating from $\tau_1^{\mathrm{II}}$ and that from $\tau_2^{\mathrm{II}}$. The configuration of the Stokes curves for $t = t_j$ specified in Figure 5.2.2 is given respectively by Figure 5.2.4.$j$.

**5.3.** In studying $(P_{\mathrm{II}\text{-}1})_m$, one might wonder there would be any effect of the singularity at $x = 0$ in the equation (1.2.10.$a$). As some Stokes curves of (1.2.10.$a$) flow

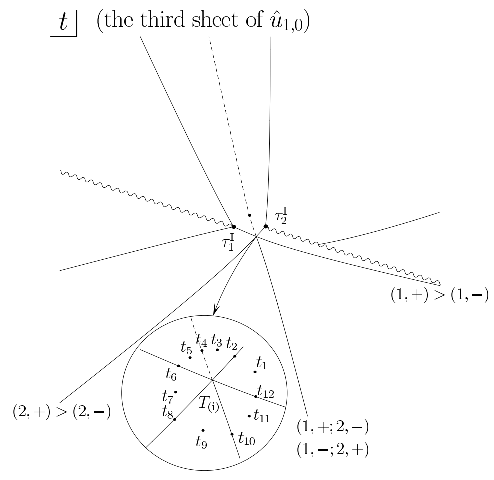


FIGURE 5.1.3(i)

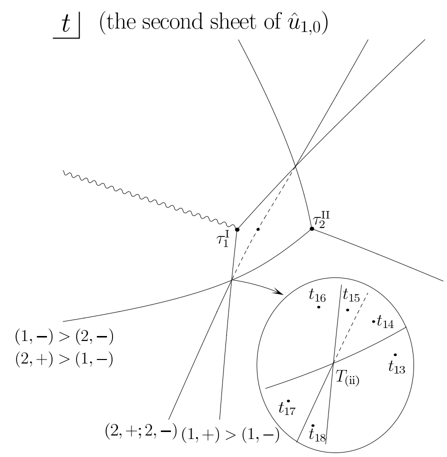


FIGURE 5.1.3(ii)

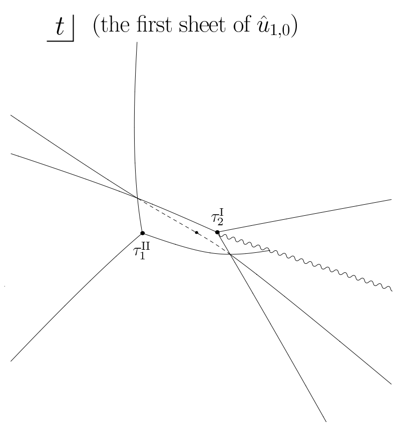


FIGURE 5.1.3(iii)

into the singular point $x = 0$ besides the points at infinity, the appearance of the Stokes geometry of (1.2.10.$a$) is somewhat different from that of the Stokes geometry of (1.1.11.$a$). But, nothing peculiar is observed concerning the relation between the Stokes geometry of the linearization of $(P_{\text{II-1}})_m$ and that of the linear equation (1.2.10.$a$). In order to show this we present the Stokes geometry of $(P_{\text{II-1}})_2$ with $g = -1/2$ and $c = 0.5 - 0.8i$, again ignoring the detailed sheet structure (*cf.* [**NT**]). We concentrate our attention to turning points $\tau^{\mathrm{I}}$ and $\tau^{\mathrm{II}}$ in Figure 5.3.1, and we present the enlarged figure of the Stokes curve emanating from $\tau^{\mathrm{I}}$ and that from $\tau^{\mathrm{II}}$,

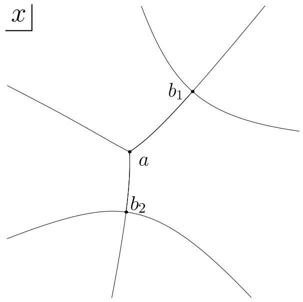


FIGURE 5.1.4(i)

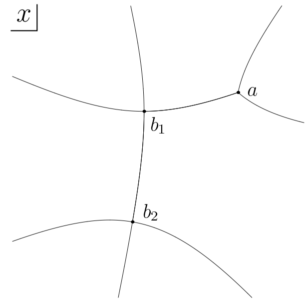


FIGURE 5.1.4(ii)

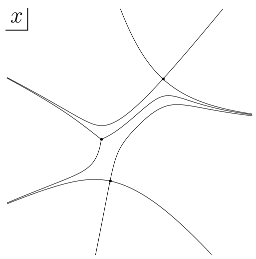


FIGURE 5.1.5(i).1

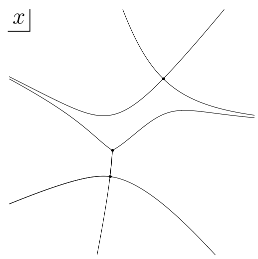


FIGURE 5.1.5(i).2

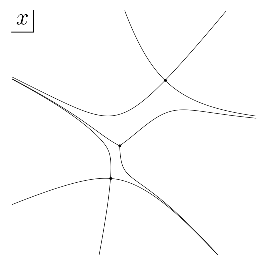


FIGURE 5.1.5(i).3

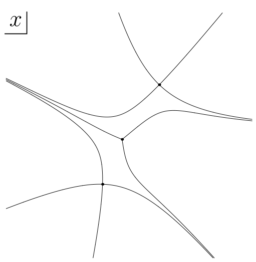


FIGURE 5.1.5(i).4

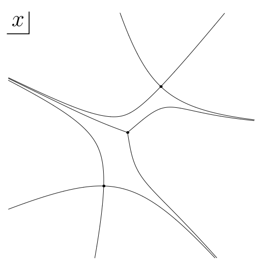


FIGURE 5.1.5(i).5

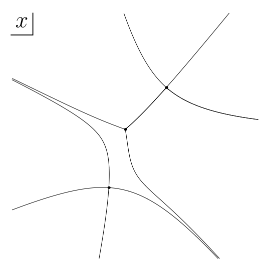


FIGURE 5.1.5(i).6

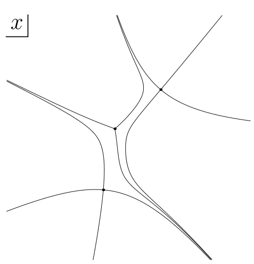


FIGURE 5.1.5(i).7

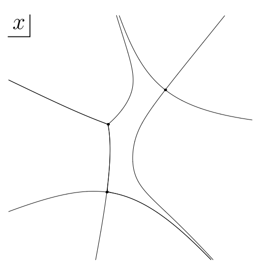


FIGURE 5.1.5(i).8

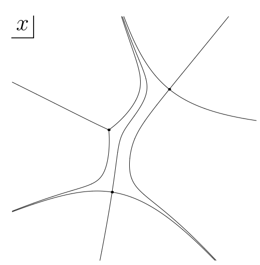


FIGURE 5.1.5(i).9

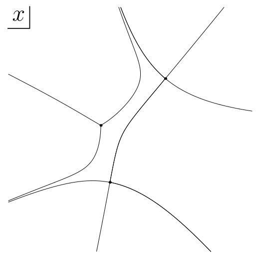


FIGURE 5.1.5(i).10

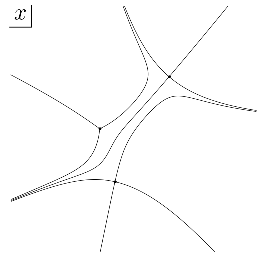


FIGURE 5.1.5(i).11

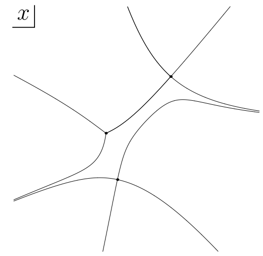


FIGURE 5.1.5(i).12

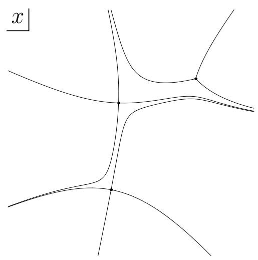


FIGURE 5.1.5(ii).13

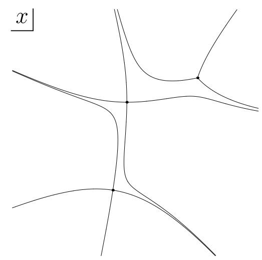


FIGURE 5.1.5(ii).14

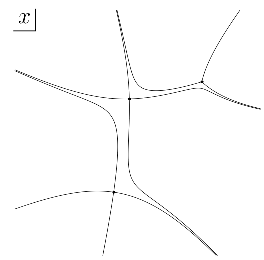


FIGURE 5.1.5(ii).15

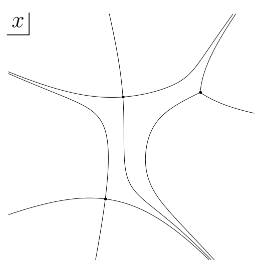


FIGURE 5.1.5(ii).16

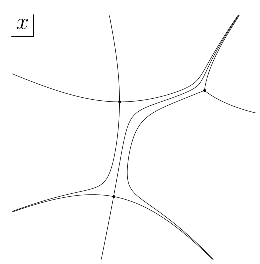


FIGURE 5.1.5(ii).17

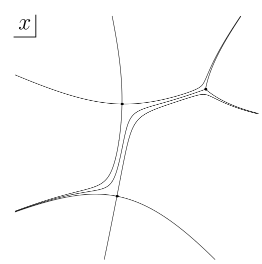


FIGURE 5.1.5(ii).18

together with the required new Stokes curve at the crossing point $T$. The configuration of the Stokes curves of (1.2.10.$a$) for $t = T$ is given by Figure 5.3.3 and that for $t = t_j$ ($j = 1, \ldots, 6$) is given respectively by Figure 5.3.4.$j$.

**5.4.** In connection with a remark before Theorem 4.1, we show an example of a crossing point of a new Stokes curve and an ordinary Stokes curve. The example is observed for $(P_{\text{II-2}})_2$ with $c = 9.8 - 0.1i$, $g = 7,6 + 6.6i$ and $\delta = -6.2 - 5.6i$, as we show below. The Stokes geometry of the linearization of $(P_{\text{II-2}})_2$ is given by Figure 5.4.1, and we concentrate our attention to the portion of Figure 5.4.1 that is enlarged in Figure 5.4.2; we focus our attention to the Stokes curve $C_j$ ($j = 1, 2, 3$) respectively emanating from the turning point $\tau_j$ ($j = 1, 2, 3$), the new Stokes curve $C_4$ emanating from the crossing point $T_0$ of $C_2$ and $C_3$ and the crossing point $T$ of

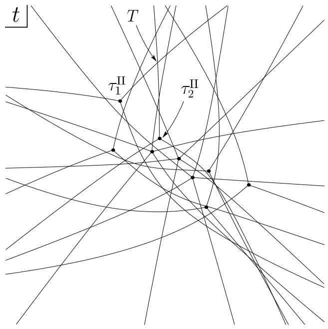


FIGURE 5.2.1

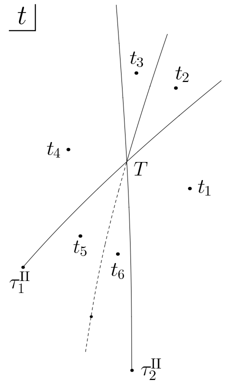


FIGURE 5.2.2

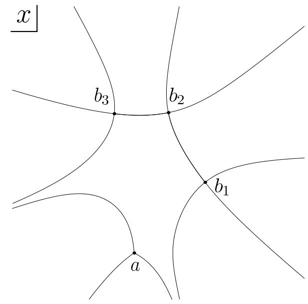


FIGURE 5.2.3

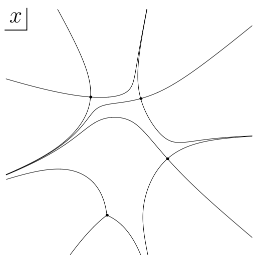


FIGURE 5.2.4.1

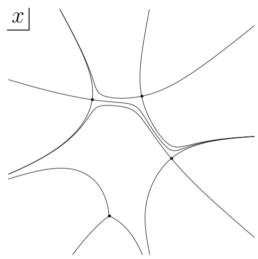


FIGURE 5.2.4.2

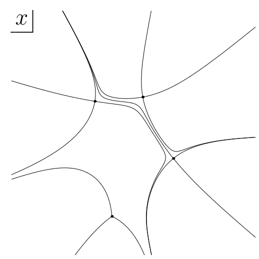


FIGURE 5.2.4.3

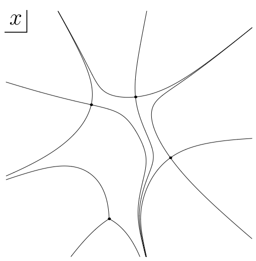


FIGURE 5.2.4.4

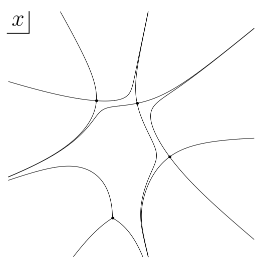


FIGURE 5.2.4.5

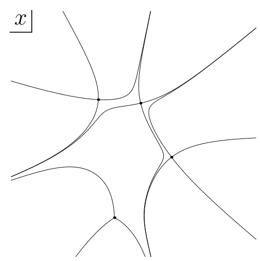


FIGURE 5.2.4.6

the Stokes curve $C_1$ and the new Stokes curve $C_4$; the configuration of Stokes curves of (1.3.8.$a$) at $t = T$ is given by Figure 5.4.3. Although we do not include the figures of the configuration of Stokes curves when the parameter $t$ moves around $T$, we note that the topological change is observed only when $t$ lies on $C_1$ or $C_4$.

*Acknowledgements*. — In making this research the authors, especially T. Koike and Y. Takei, benefited much from their visits to the University of Sydney in 2003 and 2004, supported by JSPS Japan-Australia Research Cooperative Program. They express their sincere gratitude to Prof. N. Joshi for her kind hospitality and to Prof.

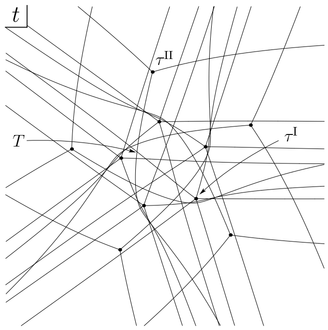


FIGURE 5.3.1

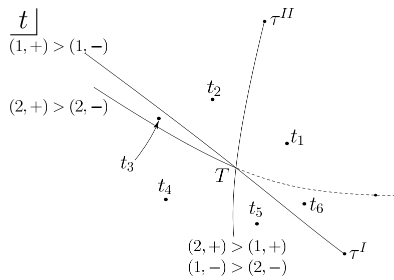


FIGURE 5.3.2

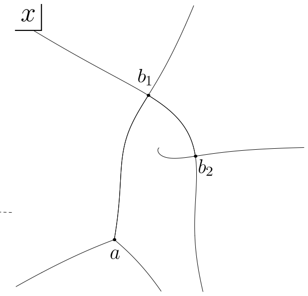


FIGURE 5.3.3

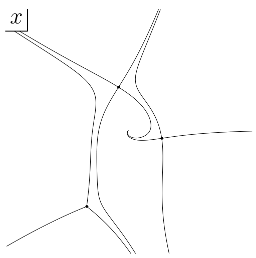


FIGURE 5.3.4.1

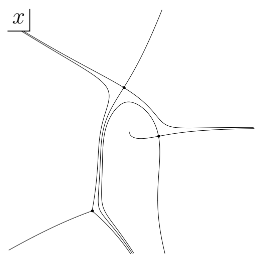


FIGURE 5.3.4.2

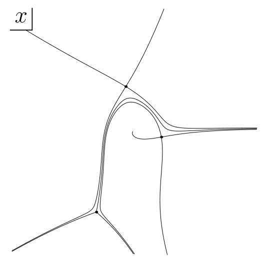


FIGURE 5.3.4.3

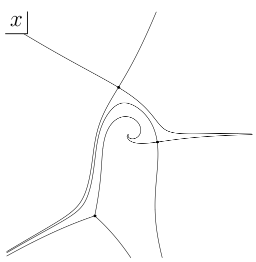


FIGURE 5.3.4.4

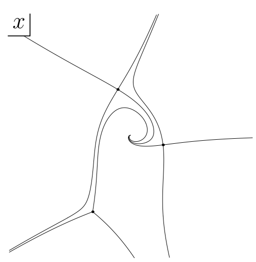


FIGURE 5.3.4.5

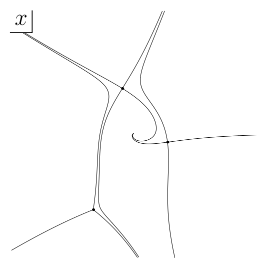


FIGURE 5.3.4.6

T. Aoki, Prof. P.R. Gordoa, Prof. N. Joshi, Prof. A. Pickering and Prof. S. Shimomura for many valuable discussions for hierarchies of higher-order Painlevé equations.

## Appendix A

### Some properties of $\mathcal{K}_j$ and $K_j$

Let us first consider $\{\mathcal{F}_j\}$ defined by the following recursive relation:

(A.1) $$\partial_t \mathcal{F}_{j+1} = (\partial_t^3 + 4u\partial_t + 2u')\mathcal{F}_j$$

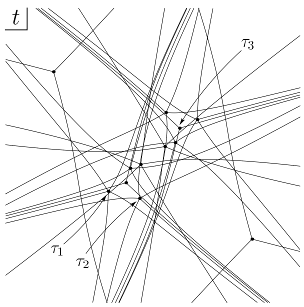


FIGURE 5.4.1

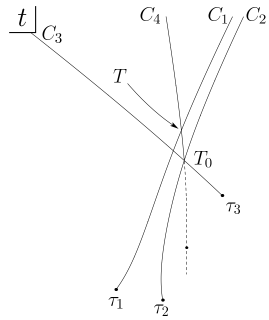


FIGURE 5.4.2

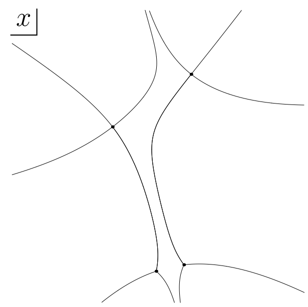


FIGURE 5.4.3

with $\mathcal{F}_0 = 1/2$. Here and in what follows, $'$ denotes the differentiation with respect to the variable $t$. Then, as is proved in [**DT**, Introduction], the following lemma holds for $\{\mathcal{F}_j\}$ thus defined.

***Lemma A.1***. — *If* $\{\mathcal{F}_j\}$ *satisfy* (A.1) *and each* $\mathcal{F}_j$ *does not contain a constant term, then the following relation holds:*

$$\mathcal{F}_{n+1} = -\sum_{j=0}^{n-1} \mathcal{F}_{n-j}\mathcal{F}_{j+1} + 4u\sum_{j=0}^{n} \mathcal{F}_{n-j}\mathcal{F}_j + 2\sum_{j=0}^{n} \mathcal{F}_{n-j}\partial_t^2\mathcal{F}_j - \sum_{j=0}^{n} \partial_t\mathcal{F}_{n-j}\partial_t\mathcal{F}_j. \tag{A.2}$$

Once the relation (A.2) is confirmed, we can readily find that all $\mathcal{F}_j$ are polynomials of $u$ and its derivatives by using an induction. Note that the recursive relation (A.1) itself does not fix integration constants in each step; here, we fix them to be 0.

In what follows we present a proof of Lemma A.1 along the line of [**DT**] for the reader's convenience. (See also [**L**] for another proof different from below.)

*Proof*. — Multiplying both sides of (A.1) by $\mathcal{F}_{n-j}$ and taking the sum from $j = 0$ to $n$, we obtain

$$\sum_{j=0}^{n} \mathcal{F}_{n-j}\partial_t\mathcal{F}_{j+1} = \sum_{j=0}^{n} \mathcal{F}_{n-j}\partial_t^3\mathcal{F}_j + \sum_{j=0}^{n} \mathcal{F}_{n-j}(4u\partial_t + 2u')\mathcal{F}_j. \tag{A.3}$$

The left-hand side of (A.3) can be written as

$$\mathcal{F}_0\partial_t\mathcal{F}_{n+1} + \frac{1}{2}\partial_t\sum_{j=0}^{n-1} \mathcal{F}_{n-j}\mathcal{F}_{j+1} = \frac{1}{2}\partial_t\Bigl(\mathcal{F}_{n+1} + \sum_{j=0}^{n-1} \mathcal{F}_{n-j}\mathcal{F}_{j+1}\Bigr). \tag{A.4}$$

On the other hand, since

$$\sum_{j=0}^{n} \mathcal{F}_{n-j}{\partial_t}^3\mathcal{F}_j = \partial_t\Bigl(\sum_{j=0}^{n} \mathcal{F}_{n-j}{\partial_t}^2\mathcal{F}_j - \frac{1}{2}\sum_{j=0}^{n} \partial_t\mathcal{F}_{n-j}\partial_t\mathcal{F}_j\Bigr) \tag{A.5}$$

and

$$\sum_{j=0}^{n} \mathcal{F}_{n-j}(4u\partial_t + 2u')\mathcal{F}_j = 2\partial_t\Big(u\sum_{j=0}^{n}\mathcal{F}_{n-j}\mathcal{F}_j\Big), \tag{A.6}$$

the right-hand side of (A.3) becomes

$$\partial_t\Big(\sum_{j=0}^{n}\mathcal{F}_{n-j}\partial_t^2\mathcal{F}_j - \frac{1}{2}\sum_{j=0}^{n}\partial_t\mathcal{F}_{n-j}\partial_t\mathcal{F}_j + 2u\sum_{j=0}^{n}\mathcal{F}_{n-j}\mathcal{F}_j\Big). \tag{A.7}$$

This proves the lemma. □

Straightforward computations show that

$$\mathcal{F}_1 = u, \tag{A.8}$$

$$\mathcal{F}_2 = 3u^2 + u'', \tag{A.9}$$

$$\mathcal{F}_3 = 10u^3 + 5(u')^2 + 10uu'' + u^{(4)}, \tag{A.10}$$

$$\begin{aligned}\mathcal{F}_4 = 35u^4 + 70(u(u')^2 + u^2u'') + 21(u'')^2 \\ + 28u'u^{(3)} + 14uu^{(4)} + u^{(6)}.\end{aligned} \tag{A.11}$$

The polynomials $\{\mathcal{F}_j\}$ of $u$ and its derivatives have the following scaling property:

***Lemma A.2***. — *Under the scaling $u \mapsto \lambda^2 u$, $t \mapsto \lambda^{-1}t$, $\{\mathcal{F}_j\}$ is transformed as*

$$\mathcal{F}_j \longmapsto \lambda^{2j}\mathcal{F}_j. \tag{A.12}$$

Employing what is called the Miura map $u = v' - v^2$, we now define a new family $\{\mathcal{K}_j\}$ of polynomials by

$$\mathcal{K}_j = \mathcal{F}_j\Big|_{u=v'-v^2}. \tag{A.13}$$

Then we can readily find that $\{\mathcal{K}_j\}$ satisfies the recursive relation (1.2.16). Hence these polynomials $\{\mathcal{K}_j\}$ coincide with those introduced in Section 1.2 to define the hierarchy (1.2.14) of Gordoa and Pickering. The following scaling property of $\{\mathcal{K}_j\}$ is also an immediate consequence of Lemma A.2:

***Lemma A.3***. — *Under the scaling $v \mapsto \lambda v$, $t \mapsto \lambda^{-1}t$, $\{\mathcal{K}_j\}$ is transformed as*

$$\mathcal{K}_j \longmapsto \lambda^{2j}\mathcal{K}_j. \tag{A.14}$$

Finally, as is explained in Section 1.2, $\{K_j\}$ defined by the recursive relation (1.2.2) is obtained from $\{\mathcal{K}_j\}$ through the scaling $v \mapsto \eta^{1/(2m+1)}v$, $t \mapsto \eta^{2m/(2m+1)}t$ and $\mathcal{K}_j \mapsto \eta^{2j/(2m+1)}K_j$. Hence $K_j$ also becomes a polynomial of $v$ and its derivatives.

## Appendix B
## Another formulation of the $P_{\mathrm{I}}$-hierarchy

In [**GP**] Gordoa and Pickering discuss the following hierarchy of differential equations:

$$\mathcal{G}_{m+1} + gt = 0, \tag{B.1}$$

where $g$ is a non-zero constant and $\{\mathcal{G}_j\}$ is defined by (B.2) below in terms of constants $\{\delta_j\}$ and $\{\mathcal{F}_j\}$ given in Appendix A.

$$\mathcal{G}_j = \mathcal{F}_j + \delta_1 \mathcal{F}_{j-1} + \cdots + \delta_j \mathcal{F}_0 = \sum_{k=0}^{j} \delta_k \mathcal{F}_{j-k} \quad (\delta_0 = 1). \tag{B.2}$$

***Remark B.1***. — We may assume $\delta_1 = 0$ without loss of generality. We also note that $g$ may be changed to be an arbitrary non-zero constant by an appropriate scaling of $u$ and $t$.

***Remark B.2***. — $\{\mathcal{G}_j\}$ satisfies

$$\partial_t \mathcal{G}_{j+1} = (\partial_t^3 + 4u\partial_t + 2u')\mathcal{G}_j. \tag{B.3}$$

Note that each $\mathcal{G}_j$ contains the constant term $\delta_j/2$. Hence an argument similar to that employed in the proof of Lemma A.1 entails that

$$\begin{aligned}\mathcal{G}_{n+1} = -\sum_{j=0}^{n-1} \mathcal{G}_{n-j}\mathcal{G}_{j+1} + 4u \sum_{j=0}^{n} \mathcal{G}_{n-j}\mathcal{G}_j \\ + 2\sum_{j=0}^{n} \mathcal{G}_{n-j}\partial_t^2 \mathcal{G}_j - \sum_{j=0}^{n} \partial_t \mathcal{G}_{n-j}\partial_t \mathcal{G}_j \\ + \frac{1}{2}\delta_{n+1} + \frac{1}{4}\sum_{j=0}^{n-1} \delta_{n-j}\delta_{j+1}.\end{aligned} \tag{B.4}$$

We now introduce a large parameter $\eta$ to (B.1) through a scaling

$$u \longmapsto \eta^{2\alpha} u, \quad t \longmapsto \eta^{\beta} t, \quad x \longmapsto \eta^{2\alpha} x, \quad g \longmapsto \eta^{2(m+1)\alpha - \beta} g, \quad \delta_j \longmapsto \eta^{2\alpha j}\delta. \tag{B.5}$$

Here, $\alpha$ and $\beta$ are arbitrary constants satisfying $\alpha + \beta = 1$. Under this scaling $\{\mathcal{G}_j\}$ is transformed as

$$\mathcal{G}_j \longmapsto \eta^{2j\alpha} G_j, \tag{B.6}$$

where

$$G_j = \sum_{k=0}^{j} \delta_k F_{j-k}. \tag{B.7}$$

We thus obtain from (B.1) the following hierarchy of differential equations with a large parameter $\eta$:

$$G_{m+1} + gt = 0. \tag{B.8}$$

We now claim that the hierarchy (B.8) is equivalent to the $P_{\mathrm{I}}$-hierarchy formulated in Section 1.1. That is, we can prove the following

***Proposition B.1***. — *Assume that $g = 2^{2m+1}$ and that $\delta_1 = 0$. Then, for a given solution $u$ of* (B.8)*, if we let $u_j$ and $v_j$ $(1 \leqslant j \leqslant m)$ be respectively given*

$$u_j = -2^{1-2j} G_j, \tag{B.9}$$

$$v_j = -2^{-2j} \eta^{-1} \partial_t G_j, \tag{B.10}$$

*$(u_j, v_j)$ satisfies $(P_{\mathrm{I}})_m$ with*

$$c_j = 2^{-2j-2}\Big(\delta_{j+1} + \frac{1}{2}\sum_{k=0}^{j-1} \delta_{j-k}\delta_{k+1}\Big) \quad (1 \leqslant j \leqslant m). \tag{B.11}$$

*Proof*. — If we define $w_j$ by

$$w_j = \begin{cases} 2^{-2j-1}\left(G_{j+1} - 2G_1 G_j - \eta^{-2}\partial_t^2 G_j\right) & (1 \leqslant j \leqslant m-1), \\ -2^{-2m-1}\left(2G_1 G_m + \eta^{-2}\partial_t^2 G_m\right) & (j = m), \end{cases} \tag{B.12}$$

we readily find that $u_j$, $v_j$ and $w_j$ satisfy the system (1.1.1). Thus what remains to be verified is that $u_j$, $v_j$ and $w_j$ thus defined should satisfy the recursive relation (1.1.2). Note that it follows from (B.4) and (B.6) that $G_j$ satisfies the following relation:

$$\begin{aligned} G_{n+1} = &-\sum_{j=0}^{n-1} G_{n-j}G_{j+1} + 4u\sum_{j=0}^{n} G_{n-j}G_j \\ &+ 2\eta^{-2}\sum_{j=0}^{n} G_{n-j}\partial_t^2 G_j - \eta^{-2}\sum_{j=0}^{n}\partial_t G_{n-j}\partial_t G_j \\ &+ \frac{1}{2}\delta_{n+1} + \frac{1}{4}\sum_{j=0}^{n-1}\delta_{n-j}\delta_{j+1}. \end{aligned} \tag{B.13}$$

Using this relation (B.13), we obtain

(B.14) L.H.S of (1.1.2) – R.H.S of (1.1.2)

$$= \begin{cases} 2^{-2j-2}\Big(\delta_{j+1} + \frac{1}{2}\sum_{k=0}^{j-1}\delta_{j-k}\delta_{k+1}\Big) - c_j & (j \neq m), \\ -2^{-2m-1}(G_{m+1} + 2^{2m+1}t) & \\ \qquad + 2^{-2m-2}\Big(\delta_{m+1} + \frac{1}{2}\sum_{k=0}^{m-1}\delta_{m-k}\delta_{k+1}\Big) - c_m & (j = m). \end{cases}$$

Hence (B.11) entails (1.1.2). This completes the proof of Proposition B.1. □

Each member of the hierarchy (B.8) has the following Lax pair:

$$\frac{\partial}{\partial x}\psi = \eta A\psi, \quad \frac{\partial}{\partial t}\psi = \eta B\psi, \tag{B.15}$$

where

$$A = \frac{1}{g}\begin{pmatrix} -\eta^{-1}\partial_t T_m & 2T_m \\ 2(x-u)T_m - \eta^{-2}\partial_t^2 T_m & \eta^{-1}\partial_t T_m \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ x-u & 0 \end{pmatrix} \tag{B.16}$$

and

$$T_m = \sum_{j=0}^{m} (4x)^j G_{m-j}. \tag{B.17}$$

As the form of this Lax pair is similar to that of $(L_{\text{II-1}})_m$ (*i.e.*, the underlying Lax pair of the $P_{\text{II-1}}$-hierarchy), we can develop a similar argument as in Section 2.2 also for the hierarchy (B.8). This gives us another proof of Propositions 2.1.1 $\sim$ 2.1.5 for the $P_{\text{I}}$-hierarchy.

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# VERSAL DEFORMATION OF THE ANALYTIC SADDLE-NODE

*by*

Frank Loray

*To Jean-Pierre Ramis for his 60th birthday*

***Abstract.*** — In the continuation of [**10**], we derive simple forms for saddle-node singular points of analytic foliations in the real or complex plane just by gluing foliated complex manifolds. We give a miniversal analytic deformation of the simplest model. We also derive a unique analytic form for those saddle-node having an analytic central manifold. By this way, we recover and generalize results earlier proved by J. Écalle by using mould theory and partially answer to some questions asked by J. Martinet and J.-P. Ramis at the end of [**11**].

***Résumé* (Déformation verselle d'un nœud-col analytique).** — Dans la continuité de [**10**], nous construisons une forme normale simple pour un feuilletage analytique au voisinage d'une singularité de type nœud-col dans le plan réel ou complexe. Nous obtenons une telle forme en recollant des variétés complexes feuilletées. Nous en déduisons une déformation analytique miniverselle dans un cas simple. Nous donnons une forme unique pour un nœud-col possédant une variété centrale analytique. Nous retrouvons ainsi géométriquement et nous généralisons des résultats obtenus par J. Écalle à l'aide de la théorie des moules. Ce travail répond partiellement à des questions ouvertes posées par J. Martinet et J.-P. Ramis à la fin de [**11**].

## Introduction and results

Let $X$ be a germ of analytic vector field at the origin of $\mathbb{C}^2$

$$X = f(x,y)\partial_x + g(x,y)\partial_y, \quad f,g \in \mathbb{R}\{x,y\} \text{ or } \mathbb{C}\{x,y\}$$

having a singularity at 0: $f(0) = g(0) = 0$. Consider $\mathcal{F}$ the germ of singular holomorphic foliation induced by the complex integral curves of $X$ near 0. A question going back to H. Poincaré is the following:

***Problem***. — Find local coordinates in which the foliation is defined by a vector field having coefficients as simple as possible.

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In this problem, the vector field is considered up to analytic change of coordinates and up to multiplication by a germ of analytic function. For instance, if the vector field $X$ has a linear part (in the matrix form)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax+by)\partial_x + (cx+dy)\partial_y$$

having non zero eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ with eigenratio $\lambda_2/\lambda_1 \notin \mathbb{R}$, then H. Poincaré proved that the vector field $X$ is actually linear in convenient analytic coordinates. In this situation, the eigenvalues $\{\lambda_1, \lambda_2\}$ (resp. the eigenratio $\lambda_2/\lambda_1$) provide a complete set of invariants for such vector fields (resp. foliations) modulo analytic change of coordinates.

In this paper, we consider *unramified saddle-nodes*, *i.e.* foliations defined by a vector field having (exactly) one zero eigenvalue and multiplicity 2. Following H. Dulac, such a foliation is defined in convenient coordinates by a vector field of the form

$$X = x^2\partial_x + y\partial_y + xf(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\}. \tag{1}$$

and one can further *formally* reduce the vector field $X$ to a unique form

$$X_\mu = x^2\partial_x + y\partial_y + \mu xy\partial_y, \quad \mu \in \mathbb{C} \tag{2}$$

The complete *analytic* classification of those singular points has been given by J. Martinet and J.-P. Ramis in 1982 (see [**11**] or section 1), giving rise to infinitely many invariants additional to the formal one $\mu$ above. The resulting moduli space is huge and we expect that a generic saddle-node cannot be defined by a polynomial vector field in any analytic coordinates (although this is open, as far as I know). A direct application of our recent work [**10**] provides the following

***Theorem 1***. — *Let $\mathcal{F}$ be a germ of saddle-node foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) in the form (1) above. Then, there exist local analytic coordinates in which $\mathcal{F}$ is defined by a vector field of the form*

$$X_f = x^2\partial_x + y\partial_y + xf(y)\partial_y, \quad f \in \mathbb{C}\{y\} \tag{3}$$

*where $f'(0) = \mu$.*

This statement is a particular case of a general simple analytic form independantly announced by A.D. Bruno and P.M. Elizarov for all resonant saddles ($\lambda_2/\lambda_1 \in \mathbb{Q}^-$) and saddle-nodes in 1983 (see [**3, 6**]). So far, only the case of Theorem 1 with $\mu = 0$ has been proved: it is presented by J. Écalle as an application of *resurgent functions and mould theory* at the end of [**5**], p. 535. In 1994, P.M. Elizarov made an important step toward the analytic form announced by solving in [**7**] the associate cohomological equation. One can immediately deduce from his computations that the family $X_f$ of Theorem 1 is *miniversal* at $f \equiv 0$: the coefficients of $f$ play the role of Martinet-Ramis' invariants at the first order. This will be rigorously stated in section 1, once we have recalled the definition (and construction) of Martinet-Ramis' invariants.

It is important to notice that the form (3) is not unique. Of course, we can modify the functional coefficient $f$ by conjugating the vector field with an homothety $y \mapsto c \cdot y$, $c \in \mathbb{C}^*$. But even if we restrict to tangent-to-the-identity conjugacies, the form (3) is perhaps locally unique at $X_0$ ($f \equiv 0$), but not globally for the following reason. By construction (see proof of Theorem 1), the form (3) is obtained with $f(0) \neq 0$, even if the saddle node has a *central manifold* (see below). For instance, the model $X_0$ has also another form (3) with $f(0) \neq 0$.

From preliminary form (1), we see that $\{x = 0\}$ is an invariant curve for the vector field that we will call *strong manifold* throughout the paper. Tangent to the zero eigendirection, there is also a unique "formal invariant curve" $\{y = \varphi(x)\}$, $\varphi \in \mathbb{R}[[x]]$ or $\mathbb{C}[[x]]$, which is generically divergent. When this curve is convergent, we call it *central manifold*. A remarquable result of Martinet-Ramis' classification is that saddle-nodes having a central manifold form an analytic submanifold of codimension one (in the unramified case). For instance, saddle-nodes in the form (3) with $f(0) = 0$ have the central manifold $\{y = 0\}$. Conversely, a natural question is:

***Problem***. — Given a saddle-node like in Theorem 1 having a central manifold, is it possible to put it analytically into the form (3) with $f(0) = 0$ (*i.e.* simultaneously straightening the central manifold onto $\{y = 0\}$) ?

For generic $\mu$, the answer is yes:

***Theorem 2***. — *Let $\mathcal{F}$ be a germ of saddle-node foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) like in Theorem 1 with $\mu \in \mathbb{C} - \mathbb{R}^-$. If $\mathcal{F}$ has a central manifold, then there exist local analytic coordinates in which $\mathcal{F}$ is defined by*

$$X_f = x^2\partial_x + y\partial_y + xf(y)\partial_y, \quad \text{with } f(0) = 0. \tag{4}$$

*Moreover, this form is unique up to homothety $y \mapsto c \cdot y$, $c \in \mathbb{C}^*$.*

In the remaining case $\mu \in \mathbb{R}^-$, we will give necessary and suffisant conditions in section 4 in terms of Martinet-Ramis' invariants (see Theorem 8), thus providing a complete answer to the question above; in the case $\mu = 0$, the condition was already given by J. Écalle in [**5**], p. 539. It turns out that these conditions are very restrictive (infinite codimension). For instance, when $\mu \in -\mathbb{N}^*$, only the saddle-nodes analytically conjugated to the formal model (2) can be normalized to the form (4). In particular, for each $\mu \in -\mathbb{N}^*$, the subfamily of those $X_f$ satisfying $f(0) = 0$ and $f'(0) = \mu$ provides a codimension two analytically trivial deformation of the formal model (2).

Accidentally, our method to prove Theorem 2 provides in turn a simple form for saddles:

***Theorem 3***. — *Let $\mathcal{F}$ be a germ of saddle foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) with eigenratio $-\mu < 0$. Then there exist local analytic coordinates in which $\mathcal{F}$ is* *defined by a vector field of the form*

$$X_f = -x\partial_x + \mu(f(y)+x)y\partial_y, \quad \textit{with } f(0)=1. \tag{5}$$

This latter form is not unique: for generic $\mu$, all $X_f$ are conjugated. For saddle-nodes having a central manifold that cannot be transformed into the form (4), it is possible to give an alternate unique form as follows.

**Theorem 4**. — *Let $\mathcal{F}$ be a germ of saddle-node foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) like in Theorem 1 having a central manifold. Let $n \in \mathbb{N}$ be such that $\mu + n \notin \mathbb{R}^-$. Then, there exist local analytic coordinates in which $\mathcal{F}$ is defined by a vector field of the form*

$$X_f = x^2\partial_x + y\partial_y + xyf(x^n y)\partial_y, \quad \textit{where } f(0)=\mu. \tag{6}$$

*Moreover, this form is unique up to homothety $y \mapsto c\cdot y$, $c \in \mathbb{C}^*$.*

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## 1. Martinet-Ramis' invariants

We recall the construction of [**11**]. Consider a saddle-node in Dulac preliminary form (1)

$$X = x^2\partial_x + y\partial_y + xf(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\}.$$

The Sectorial Normalization Theorem due to Hukuhara, Kimura and Matuda reads as follows. For a sufficiently small $r, \varepsilon > 0$, there exists on each of the two sectorial domains $V^+$ and $V^-$

$$V^\pm := \{|x| < r,\ |y| < r,\ 0-\varepsilon < \arg(\pm x) < \pi + \varepsilon\}$$

a unique holomorphic diffeomorphism $\Phi^\pm : V^\pm \to \Phi^\pm(V^\pm) \subset \mathbb{C}^2$ of the form $\Phi(x,y) = (x, \phi(x,y))$, which is tangent to the identity at $(0,0)$ and conjugating the saddle-node above to its formal normal form (2)

$$X_\mu := x^2\partial_x + y\partial_y + \mu xy\partial_y.$$

The model $X_\mu$ admits the first integral $H_\mu(x,y) := yx^{-\mu}e^{1/x}$. Once we have fixed determinations $H_\mu^\pm$ of $H_\mu$ on the sectors $V^\pm$ coinciding over $\{-\varepsilon < \arg(x) < +\varepsilon\}$, we immediately deduce sectorial first integrals $H^\pm := H_\mu^\pm \circ \Phi^\pm$ for the initial saddle-node.

On the overlapping $V^+ \cap V^-$, the two first integrals $H^+$ and $H^-$ factorize in the following way. Over $V^0 = \{\pi-\varepsilon < \arg(x) < \pi+\varepsilon\}$, the first integrals $H^+$ and $H^-$ both identify the space of leaves with a neighborhood of $0 \in \mathbb{C}$, the size of which depending on the radius $r$: one can write $H^- = \varphi^0 \circ H^+$ for some germ of diffeomorphism $\varphi^0 \in \mathrm{Diff}(\mathbb{C},0)$. Over the other overlapping $V^\infty = \{-\varepsilon < \arg(x) < +\varepsilon\}$, the first integrals $H^+$ and $H^-$ both identify the space of leaves with $\mathbb{C}$: one can write $H^- = \varphi^\infty \circ H^+$ for some affine automorphism $\varphi^\infty$ of $\mathbb{C}$. From the asymptotics of $\Phi^\pm$ and the choice of the determinations $H_\mu^{\pm}$, one easily deduce that the linear parts of $\varphi^0$ and $\varphi^\infty$ are respectively $e^{2i\pi\mu}$ and 1.

We have thus defined the *moduli map*:

$$X \longmapsto \begin{cases} \varphi^0(\zeta) \ = e^{2i\pi\mu}\zeta + \sum_{n\geqslant 2} a_n\zeta^n \ \in \mathrm{Diff}(\mathbb{C},0) \\ \varphi^\infty(\zeta) = \zeta + t \ \in \mathbb{C} \qquad\qquad \text{(a translation)} \end{cases} \tag{7}$$

The main result of [**11**] is

***Theorem* (Martinet-Ramis).** — *Any two saddle-nodes into the form (1) are conjugated by a tangent-to-the-identity diffeomorphism $\Phi : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$, $D\Phi(0) = I$, if, and only if, they have the same image through the moduli map above.*

*Moreover, the moduli map is surjective: any pair $(\varphi^0,\varphi^\infty) \in \mathrm{Diff}(\mathbb{C},0)\times\mathbb{C}$ can be realized by a saddle-node of the form (1).*

Two saddle-nodes $X$ and $\widetilde{X}$ in the form (1) can be conjugated by a diffeomorphism $\Phi : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ with a non trivial linear part, namely an homothety in variable $y$. In this case, the corresponding pairs are conjugated by an homothety:

$$(c\cdot\widetilde{\varphi}^0(\zeta), c\cdot\widetilde{\varphi}^\infty(\zeta)) = (\varphi^0(c\cdot\zeta), \varphi^\infty(c\cdot\zeta)), \quad \text{for some } c \in \mathbb{C}^*.$$

This equivalence relation on $\mathrm{Diff}(\mathbb{C},0)\times\mathbb{C}$ provides a complete set of invariants for saddle-nodes with multiplicity 2 with respect to the analytic conjugacy.

The classification above is a foliated version of Ecalle-Malgrange-Voronin classification of tangent-to-the-identity maps. Let us recall the Martinet-Ramis presentation in the case of multiplicity 2. Any $\varphi(x) = x + 2i\pi x^2 + \cdots \in \mathrm{Diff}(\mathbb{C},0)$ is conjugate by formal change of the coordinate to the 1-time map $\varphi_\mu := \exp(2i\pi\frac{x^2}{1+\mu x}\partial_x)$ for a unique $\mu \in \mathbb{C}$. On sectors $V^\pm$ like the ones above (without variable $y$), Leau's Theorem says that one can conjugate the dynamics of $\varphi$ with that of $\varphi_\mu$ by tangent-to-the-identity sectorial diffeomorphisms $\Phi^\pm$. After composition with convenient determinations of the $\varphi_\mu$-invariant function $H_\mu(x) := x^{-\mu}e^{1/x}$, one deduce sectorial invariant functions $H^\pm$ identifying the quotients of $V^\pm$ by the dynamics with $\mathbb{C}^*$. On $V^0$ (resp. $V^\infty$) defined as before, the functions $H^\pm$ identify the set of $\varphi$-orbits with a punctured neighborhood of 0 (resp. $\infty$) whose size depend on the radius of the sectors $V^\pm$. Therefore, one can write $H^- = \varphi^0\circ H^+$ (resp. $H^- = \varphi^\infty\circ H^+$) for some germ of diffeomorphism $\varphi^0 \in \mathrm{Diff}(\mathbb{C},0)$ (resp. $\varphi^\infty \in \mathrm{Diff}(\mathbb{C},\infty)$). The respective linear parts of those diffeomorphisms are $e^{2i\pi\mu}$ and 1. The Ecalle-Malgrange-Voronin Theorem can be stated like Martinet-Ramis Theorem above except that $\varphi^\infty$ can be any convergent power series $\zeta + \sum_{n<0} a_n\zeta^n$.

***Theorem* (Martinet-Ramis).** — *The analytic invariants $(\varphi^0,\varphi^\infty)$ of a saddle-node into the form (1) coincide with the analytic invariants of the holonomy map $\varphi(x) = x + 2i\pi x^2 + \cdots$ of the strong manifold $\{x=0\}$.*

Therefore, any two saddle-nodes in the form (1) are analytically conjugated if and only if the holonomy maps of the corresponding strong manifolds are analytically conjugated in $\mathrm{Diff}(\mathbb{C},0)$.

Another consequence is that very few tangent-to-the-identity maps $\varphi(x) = x + 2i\pi x^2 + \cdots \in \mathrm{Diff}(\mathbb{C},0)$ are the holonomy map of the strong manifold of a saddle-node into the form (1).

***Theorem* (Martinet-Ramis).** — *A saddle-node into the form (1) admits a central manifold if and only if the translation part $\varphi^\infty$ of the analytic invariants $(\varphi^0,\varphi^\infty)$ is trivial. In this case, the holonomy of the central manifold coincide with $\varphi^0$.*

When there is a central manifold, we note that the analytic class of the saddle-node is given by $\varphi^0$ up to linear conjugacy; the conjugacy class of $\varphi^0$ in $\mathrm{Diff}(\mathbb{C},0)$ does not characterize the saddle-node in general.

We also note that any germ of diffeomorphism $\varphi^0(\zeta) = e^{2i\pi\mu}\zeta + \cdots \in \mathrm{Diff}(\mathbb{C},0)$ is the holonomy map of the central manifold of a saddle-node of the form (1) with formal invariant $\mu$.

There are similar constructions and results for saddle-nodes

$$X = x^{k+1}\partial_x + y\partial_y + x^k f(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\}.$$

with higher multiplicity, $k \in \mathbb{N}^*$, and for tangent-to-the-identity germs $\varphi(x) = x + 2i\pi x^{k+1} + \cdots \in \mathrm{Diff}(\mathbb{C},0)$ giving rise to multiple moduli $(\varphi_l^0,\varphi_l^\infty)_{l=1,\ldots,k}$

$$X \longmapsto \begin{cases} \varphi_l^0(\zeta) \ = e^{2i\pi\mu/k}\zeta + \cdots \ \in \mathrm{Diff}(\mathbb{C},0) \\ \varphi_l^\infty(\zeta) = \zeta + \cdots \ \in \mathrm{Diff}(\mathbb{C},\infty) \end{cases} \qquad l = 1,\ldots,k \tag{8}$$

where, in the saddle-node case, all $\varphi_l^\infty$ are translations. Those $2k$-uple have to be considered up to simultaneous conjugacy by an homothety and up to a cyclic permutation of the indices $\{1,\ldots,k\}$. We omit the precise statements here.

Let us now consider the following family of saddle-nodes ($\varepsilon > 0$)

$$X_\varepsilon = \frac{x^2}{1+\mu x}\partial_x + y\partial_y + \varepsilon f(x,y)y\partial_y, \quad f = \sum_{m\leqslant 0,\ n\leqslant -1} f_{m,n}x^m y^n \in \frac{1}{y}\mathbb{C}\{x,y\}$$

with $f_{0,0} = f_{0,1} = f_{1,1} = 0$, so that multiplicity is 2 and formal invariant $\mu$, and consider its Martinet-Ramis' invariants (depending on $\varepsilon$)

$$\varphi^0(\zeta) = e^{2i\pi\mu}\zeta + \sum_{n>0}\varphi_n\zeta^{n+1} \quad \text{and} \quad \varphi^\infty(\zeta) = \zeta + t$$

Then, the main result of [**7**] reads

***Theorem* (Elizarov)**. — *The derivative (in the sense of Gâteau) of Martinet-Ramis'moduli at $\varepsilon = 0$ is given by*

$$\frac{d\varphi_n}{\partial\varepsilon}|_{\varepsilon=0} = n^{\mu n-1}e^{-2i\pi n\mu}\sum_{m>0}\frac{m}{\Gamma(1+m+\mu n)}f_{m,n}(-n)^m$$

$$\text{and} \quad \frac{dt}{\partial\varepsilon}|_{\varepsilon=0} = (-1)^{-\mu}e^{2i\pi\mu}\sum_{m>0}\frac{m}{\Gamma(1+m-\mu)}f_{m,-1}(-n)^m$$

*where* $\Gamma$ *is the Euler's Gamma Function.*

For instance, if we restrict to the family (3) of Theorem 1, we have

$$f(y) = \sum_{n\geqslant 0} a_n y^n \longmapsto \begin{cases} \varphi^0(\zeta) &= e^{2i\pi a_1}\zeta + \sum_{n\geqslant 2} a_n\zeta^n \\ \varphi^\infty(\zeta) &= \zeta + a_0 \end{cases}$$

In particular, the derivative at $X_0$ is bijective. The theorem above motivates the following analytic form announced in [**3**]

***Conjecture* (Bruno-Elizarov)**. — *Any saddle-node in the form (1) with formal invariant* $\mu$ *can be analytically reduced to the form*

$$x^2\partial_x + y\partial_y + x\Big(f_0 + \mu y + \sum_{(m,n)\in E_s} f_{m,n}x^m y^{n+1}\Big)\partial_y \tag{9}$$

*with support in the strip* $E_s = \{(m,n); n > 0, \frac{n}{s}+1 \leqslant m < \frac{n}{s}+2\}$ *for any slope* $0 < s \leqslant +\infty$ *such that* $E_s$ *does not intersect the set of resonances* $\{(m,n); m+\mu n \in -\mathbb{N}\}$.

For $s = +\infty$, Bruno's form (9) coincides with our (3) without restriction on $\mu$ ($E_{+\infty}$ contains resonances for $\mu \in \mathbb{Q}_*^-$).

## 2. Proof of Theorem 1

We repeat the geometric construction of [**10**]. Consider the germ of foliation $\mathcal{F}_0$ defined by a vector field $X_0$ of the form (1)

$$X_0 = x^2\partial_x + y\partial_y + xf(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\}.$$

Maybe replacing $y$ by $x + y$, the linear part of $X_0$ is given by

$$\begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix} = (cx + y)\partial_y \quad \text{with} \quad c = f(0) \neq 0.$$

Therefore, the vector field $X_0$ is well-defined on the neighborhood of any small horizontal disc $\Delta_0 = \{|x| < \varepsilon\} \times \{0\}$, $\varepsilon > 0$, and transversal to $\Delta_0$ outside the singular point. Consider inside the horizontal line $L = \overline{\mathbb{C}} \times \{0\}$ the covering given by $\Delta_0$ and $\Delta_\infty = \{|x| > \varepsilon/2\} \times \{0\}$, and denote by $C = \Delta_0 \cap \Delta_\infty$ the intersection corona. By the flow-box Theorem, there exists a unique germ of diffeomorphism of the form

$$\Phi : (\mathbb{C}^2, C) \longrightarrow (\mathbb{C}^2, C)\ ;\ (x,y) \longmapsto (\phi(x,y), y), \quad \phi(x,0) = x$$

straightening $\mathcal{F}_0$ onto the vertical foliation $\mathcal{F}_\infty$ (defined by $\partial_y$) at the neighborhood of the corona $C$. Therefore, after gluing the germs of complex surfaces $(\overline{\mathbb{C}} \times \mathbb{C}, \Delta_0)$ and $(\overline{\mathbb{C}} \times \mathbb{C}, \Delta_\infty)$ along the corona by means of $\Phi$, we obtain a germ of smooth complex surface $S$ along a rational curve $L$ equipped with a singular holomorphic foliation $\mathcal{F}$ and a (germ of) *rational fibration* $y : (S, L) \to (\mathbb{C}, 0)$ (an holomorphic fibration whose fibers are biholomorphic to $\overline{\mathbb{C}}$). Following [**8**], there exists a germ of submersion $x : (S, L) \to \overline{\mathbb{C}}$ completing $y$ into a system of trivializing coordinates: $(x, y) : (S, L) \to \overline{\mathbb{C}} \times (\mathbb{C}, 0)$.

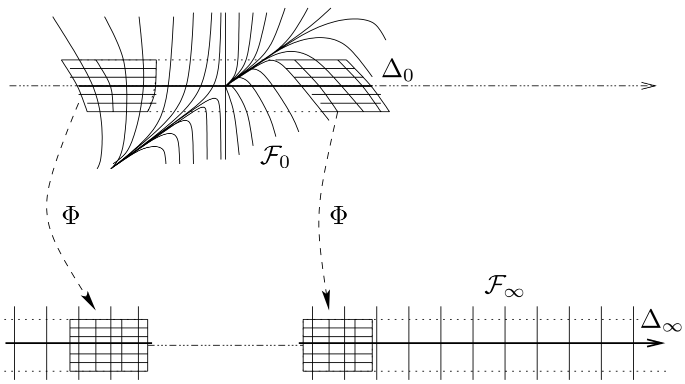


FIGURE 1. Gluing (bi)foliated surfaces

At the neighborhood of any point $p \in L$, the foliation $\mathcal{F}$ is defined by a (non unique) germ of holomorphic vector field, or equivalently by a unique germ of meromorphic vector field of the form

$$X = f(x, y)\partial_x + \partial_y$$

with $f$ meromorphic at $p$. By unicity, this meromorphic vector field is actually globally defined on the neighborhood of $L$ and is therefore rational in $x$, *i.e.* $f$ is the quotient of two Weierstrass polynomials. For $y$ fixed (close to 0), the horizontal component $f(x, y)\partial_x$ defines a meromorphic vector field on the corresponding horizontal line $\overline{\mathbb{C}} \times \{y\}$ whose zeroes and poles coincide with the tangencies between $\mathcal{F}$ and the respective vertical and horizontal fibrations. By construction, we control the number of poles: in the second chart, $\mathcal{F} = \mathcal{F}_\infty$ is transversal to $y$, although in the first chart, $\mathcal{F} = \mathcal{F}_0$ has exactly one simple tangency with any horizontal line. It follows that, for $y$ fixed, the meromorphic vector field $f(x, y)\partial_x$ has exactly 1 simple pole and thus 3 zeroes (counted with multiplicity).

Of course, in restriction to $L$, the pole vanishes together with one zero at the singular point of $\mathcal{F}$. We conclude that the vector field $X$ defining the foliation $\mathcal{F}$ takes the form

$$X = \frac{f_0(y) + f_1(y)x + f_2(y)x^2 + f_3(y)x^3}{g_0(y) + g_1(y)x}\partial_x + \partial_y \tag{10}$$

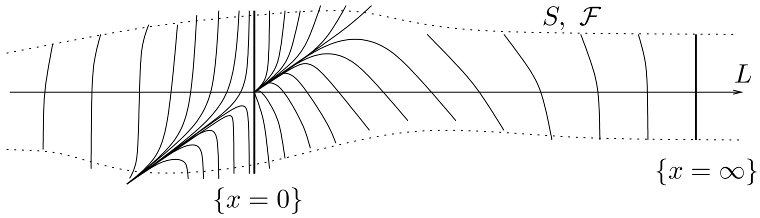


FIGURE 2. Uniformization

with $f_i, g_j \in \mathbb{C}\{y\}$. Up to a change of projective horizontal coordinate $x := \{\frac{a(y)x+b(y)}{c(y)x+d(y)}\}$ on $S$, one can assume that $\{x=\infty\}$ is a vertical leaf of $\mathcal{F}$, that $\{x=0\}$ is the invariant curve of the saddle-node tangent to the non zero eigendirection and that $\mathcal{F}$ has a contact of order 2 with the vertical fibration along $\{x=0\}$ (likely as in the local form (1)). Therefore, $f_0, f_1, f_3 \equiv 0$ and, reminding that $\mathcal{F}_0$ is an unramified saddle-node with 0-eigendirection transversal to $L$, we also have $f_2(0) \neq 0$, $g_1(0) \neq 0$, $g_0(0) = 0$ and $g_0'(0) \neq 0$. After division, $\mathcal{F}$ is actually defined by a vector field of the form

$$\widetilde{X} = x^2\partial_x + (f(y)x + yg(y))\partial_y, \quad f(0), g(0) \neq 0.$$

After change of $y$-coordinate, one may normalize the holomorphic vector field $yg(y)\partial_y$ to $g(0)y\partial_y$; after division by $g(0)$ and linear change of the $x$-coordinate, we finally obtain the form (3).

## 3. Gluing Lemmae

Although Theorems 2, 3 and 4 can be shortly proved by using Savelev Theorem [**15**] like in [**10**], we provide an alternate proof more "down to the earth" where we simultaneously construct the auxiliary fibration during the gluing construction. In order to do this, we need some lemmae allowing us to glue pairs of non transversal foliations.

The *order of contact* between two germs of regular holomorphic vector fields $X_1$ and $X_2$ at $0 \in \mathbb{C}^2$, or between the corresponding foliations, is by definition the order at 0 of the determinant $\det(X_1, X_2)$. For instance, $X_1$ and $X_2$ are transversal if and only if they have a contact of order $k = 0$. Now, if those two foliations share a common leaf, and if moreover there is no contact between them outside this leaf, then the contact order $k \in \mathbb{N}^*$ is constant along this common leaf and classifies locally the pair of foliations:

***Lemma 5***. — *Let $\mathcal{F}$ be a germ of regular analytic foliation at the origin of $\mathbb{C}^2$ (or $\mathbb{R}^2$) having the horizontal axis $L_0 : \{y=0\}$ as a particular leaf and having no other contact with the horizontal fibration $\{y = constant\}$: $\mathcal{F}$ is defined by a unique function (or vector field) of the form*

$$F(x,y) = y + y^k x f(x,y) \quad \textit{with } f(0,0) \neq 0$$
$$(\textit{or } X = g(x,y)\partial_x + y^k\partial_y, \quad \textit{with } g(0,0) \neq 0)$$

*where $k \in \mathbb{N}^*$ denotes the contact order between $\mathcal{F}$ and the horizontal fibration. Then, up to a change of coordinates of the form $\Phi(x,y) = (\phi(x,y), y)$, the foliation $\mathcal{F}$ is defined by the function (or vector field)*

$$F_0(x,y) = y + xy^k \quad (\textit{or } X_0 = \partial_x + y^k\partial_y).$$

*The restriction of $\Phi$ to $L_0$ is the identity if, and only if, $f(0,w) \equiv 1$. Moreover, the normalizing coordinate $\Phi$ is unique once we have decided that it fixes the vertical axis, i.e. $\phi(x,y) = x\widetilde{\phi}(x,y)$.*

*Proof.* — Given $\mathcal{F}$ as in the statement, choose $F(x,y)$ to be the unique function which is constant on the leaves and has restriction $F(0,y) = y$ on the vertical axis: $F(x,y) = y(1 + x\widetilde{F}(x,y))$. The assumption $dF \wedge dy = y^k u(x,y)$, $u(0,0) \neq 0$, yields $\widetilde{F}(x,y) = y^{k-1}f(x,y)$ with $f(0,0) \neq 0$, whence the form $F(x,y) = y + xy^k f(x,y)$. Now, we have

$$F = F_0 \circ \Phi_0 \quad \text{with} \quad \Phi_0(x,y) = (xf(x,y), y).$$

Thus, $\Phi_0$ is the unique change of $x$-coordinate which conjugates the functions $F$ and $F_0$; in particular, it conjugates the induced foliations.

Conversely, assume that $\Phi(x,y) = (\phi(x,y), y)$ is conjugating the foliations respectively induced by $F$ and $F_0$: we have

$$F_0 \circ \Phi(x,y) = \varphi \circ F(x,y) \quad \text{with} \quad \varphi(y) = y + y^k\phi(0,y)$$

(the germ $\varphi$ is determined by the equality restricted to $\{w = 0\}$). If we decompose $f(x,y) = u(x) + yv(x,y)$, we notice that $\varphi \circ F(x,y) = y + xy^k(u(x) + y\widetilde{v}(x,y))$, so that $\phi(x,0) = xu(x) = xf(x,0)$. Finally, if $\phi(x,y) = x\widetilde{\phi}(x,y)$, then $\varphi(y) = y$ and $\Phi$ actually conjugates the functions: we must have $\Phi = \Phi_0$ whence the unicity.

Now, if $\mathcal{F}$ is defined by $X = f(x,y)\partial_x + g(x,y)\partial_y$, assumption gives $dy(X) = g(x,y) = y^k\widetilde{g}(x,y)$ with $f(0,0), \widetilde{g}(0,0) \neq 0$. After dividing $X$ by $g$, we can write $X = f(x,y)\partial x + y^k\partial_y$. We have already proved that any two such foliations (in particular those induced by $X$ and $X_0$) are conjugate by a unique diffeomorphism of the form $\Phi(x,y) = (x\widetilde{\phi}(x,y), y)$. Now, if $\Phi(x,y) = (\phi(x,y), y)$ conjugates the foliations respectively induced by $X$ and $X_0$, it actually conjugates these vector fields. In restriction to the trajectory $L_0$, we see that $\phi(x,0)$ conjugates $\widetilde{X}|_{L_0} = f(x,0)\partial_x$ to the constant vector field $\partial_x$. Therefore, $\phi(x,0) = \int_0^x \frac{1}{f(\zeta,0)}d\zeta$ and $\phi(x,0) \equiv x$ if, and only if, $f(x,0) \equiv 1$. □

For the next statement, denote by $\Omega \subset (\mathbb{C} \times \{0\})$ a connected open domain inside the horizontal axis.

***Lemma 6***. — *Let $\mathcal{F}$ and $\mathcal{F}'$ be regular holomorphic foliations defined at the neighborhood of $\Omega$ in $\mathbb{C}^2$ both having $\Omega$ as a particular leaf. Assume that the contact between each foliation with the horizontal fibration $\{y = \text{constant}\}$ reduces to $\Omega$, with same* *order $k \in \mathbb{N}^*$. In other words, $\mathcal{F}$ and $\mathcal{F}'$ are respectively defined by vector fields*

$$X = f(x,y)\partial_x + y^k\partial_y \quad \text{and} \quad X' = f'(x,y)\partial_x + y^k\partial_y$$

*where $f$ and $f'$ are non vanishing functions in the neighborhood of $\Omega$. Then, $\mathcal{F}$ and $\mathcal{F}'$ are conjugated in a neighborhood of $\Omega$ by a diffeomorphism of the form $\Phi(x,y) = (x + y\phi(x,y), \psi(y))$ (fixing $\Omega$) if, and only if, the two following conditions hold*

(1) *$f(x,0) \equiv f'(x,0)$;*

(2) *the respective holonomies $\varphi$ and $\varphi'$ of $\mathcal{F}$ and $\mathcal{F}'$ along $\Omega$ are analytically conjugated: $\psi \circ \varphi = \varphi' \circ \psi$.*

*Proof.* — Following Lemma 5, condition (1) is the necessary and sufficient condition for the existence of local conjugacies $\Phi = (y\phi(x,y), y)$ between $\mathcal{F}$ and $\mathcal{F}'$ at the neighborhood of any point $w_0 \in \Omega$. Fix one of these points and consider the respective holonomy maps $\varphi$ and $\varphi'$ computed on the transversal $T : \{x = x_0\}$ in the variable $y$. By condition (2), up to conjugate, say $\mathcal{F}'$, by a diffeomorphism of the form $(x, \psi(y))$, we may assume without loss of generality $\varphi(y) = \varphi'(y)$. We start with the local diffeomorphism $\Phi(x,y) = (y\phi(x,y), y)$ given by Lemma 5 conjugating the foliations and fixing $T$. Since $\Phi$ conjugates the corresponding vector fields $X$ and $X'$, it extends analytically along the whole of $\Omega$ by the formula $\Phi(p) := \Phi_{X'}^{-t} \circ \Phi \circ \Phi_X^t(p)$. The condition $\varphi(y) = \varphi'(y)$ implies that $\Phi$ is uniform. □

Here is a last gluing Lemma for pairs of regular foliations $\mathcal{F}$ and $\mathcal{G}$ at the neighborhood of a common leaf $\Omega$. Again, $\Omega$ is a connected open subset of the horizontal axis $\Omega \subset (\mathbb{C} \times \{0\})$. When $\mathcal{G}$ is the horizontal fibration, the following Lemma reduces to the previous one.

***Lemma 7***. — *Let $\mathcal{F}$ and $\mathcal{G}$ (resp. $\mathcal{F}'$ and $\mathcal{G}'$) be regular holomorphic foliations defined at the neighborhood of $\Omega$ in $\mathbb{C}^2$ both having $\Omega$ as a regular leaf. Assume that the contact between $\mathcal{F}$ and $\mathcal{G}$ (resp. $\mathcal{F}'$ and $\mathcal{G}'$) reduces to $\Omega$, with same order $k \in \mathbb{N}^*$. In other words, the foliations above are respectively defined by vector fields*

$$X = \partial_x + yf(x,y)\partial_y \quad \text{and} \quad Y = X + y^k g(x,y)\partial_y,$$

$$(\text{resp.} \quad X' = \partial_x + yf'(x,y)\partial_y \quad \text{and} \quad Y' = X' + y^k g'(x,y)\partial_y)$$

*where $g$ and $g'$ are non vanishing functions in the neighborhood of $\Omega$. Then, $\mathcal{F}$ and $\mathcal{G}$ are simultaneously conjugated to $\mathcal{F}'$ and $\mathcal{G}'$ in a neighborhood of $\Omega$ by a diffeomorphism of the form $\Phi(x,y) = (x + y\phi(x,y), y\psi(x,y))$ (fixing point-wise $\Omega$) if, and only if, the two conditions hold*

(i) *for any (and for all) $x_0 \in \Omega$, we have*

$$\frac{g(x,0)}{\exp(-\int_{x_0}^x f(\zeta,0)d\zeta)} \equiv \frac{g'(x,0)}{\exp(-\int_{x_0}^x f'(\zeta,0)d\zeta)};$$

(ii) *the respective pairs of holonomies $(\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}})$ and $(\varphi_{\mathcal{F}'}, \varphi_{\mathcal{G}'})$ along $\Omega$ are simultaneously analytically conjugated: $\psi \circ \varphi_{\mathcal{F}} = \varphi_{\mathcal{F}'} \circ \psi$ and $\psi \circ \varphi_{\mathcal{G}} = \varphi_{\mathcal{G}'} \circ \psi$.*

*Proof.* — It is similar to that of the previous Lemma. Up to a change of coordinate $y := \psi(y)$ (which does not affect neither $f(x,0)$, nor $g(x,0)$ and hence preserves equality (i)), we may assume that holonomies $(\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}}) \equiv (\varphi_{\mathcal{F}'}, \varphi_{\mathcal{G}'})$ actually coincide on a transversal $T : \{x = x_0\}$. We just detail that condition (i) exactly provides the existence of local conjugacies between the given pairs of foliations fixing point-wise $\Omega$; the unique conjugacy fixing $T$ will extend uniformly along $\Omega$ by (ii).

At the neighborhood of any point $(x_0, 0) \in \Omega$, say $x_0 = 0$ for simplicity, we preliminary conjugate $X$ to $X_0 = \partial_x$ by respective local changes of $y$-coordinate $\Psi(x,y) = (x, y\psi(x,y))$, $\psi(0,0) \neq 0$

$$\Psi_* X = X_0 = \partial_x \quad \text{and} \quad \Psi_* Y = Y_0 = z^k g_0(x,y)\partial_z + \partial_w$$

Doing the same with the pair $X'$ and $Y'$, we see by Lemma 5 that the corresponding pairs of foliations are conjugated by a diffeomorphism fixing point-wise $\{y = 0\}$ if, and only if, the differential form $\omega = g_0(x,0)dx$ along $\Omega$ coincide with the corresponding one $\omega' = g_0'(x,0)dx$ for $X_0' = X_0$ and $Y_0' = \partial_x + y^k g_0(x,y)\partial_y$. This 1-form $\omega$ can be redefined in the following intrinsic way: the holonomy of $\mathcal{G}$ between two transversal cross-sections $T_0$ and $T_1$ computed in any coordinate $y$ which is $\mathcal{F}$-invariant (here $\mathcal{F}$ is defined by $\partial_x$) takes the form

$$\varphi(y) = y + \left(\int_{x_0}^{x_1} \omega\right) y^k + (\text{higher order terms})$$

where $(x_i, 0) := T_i \cap \Omega$, $i = 0, 1$. Since

$$\Psi^* X_0 = \partial_x - \frac{\psi_x}{\psi + y\psi_y} y\partial_y \quad \text{and} \quad \Psi^* Y_0 = \Psi^* X_0 + \frac{g_0}{\psi + y\psi_y} y^k \partial_y,$$

($\psi_x$ and $\psi_y$ are partial derivatives of $\psi$) we derive in restriction to $\Omega$

$$f(x,0) = -\frac{\psi_x(x,0)}{\psi(x,0)} \quad \text{and} \quad g_0(x,0) = \psi(x,0) \cdot g(x,0)$$

yielding the formula for the local invariant of our conjugacy problem

$$\omega = \frac{g(x,0)}{\exp(-\int_{x_0}^{x} f(\zeta,0)d\zeta)} dx. \qquad \square$$

## 4. Proof of Theorem 2

Given a saddle-node foliation $\mathcal{F}$ of the form (4), it is easy to verify that its analytic continuation at the neighborhood of the horizontal line $L = \overline{\mathbb{C}} \times \{0\}$ satisfies

(1) the line $L$ is a global invariant curve for $\mathcal{F}$, the union of a smooth leaf together with 2 singular points;

(2) the point $x = 0$ is a saddle-node singular point with multiplicity 2, formal invariant $\mu$ and invariant curve $\{xy = 0\}$; in particular, the saddle-node has a central manifold which is contained in $L$;

(3) the point $x = \infty$ is a singular point with eigenratio $-\mu$ and invariant curve $\{x = \infty\} \cup \{y = 0\}$;

(4) the foliation $\mathcal{F}$ has a contact of order 2 with the vertical fibration along the invariant curve $\{x = 0\}$ (in the sense of section 3).

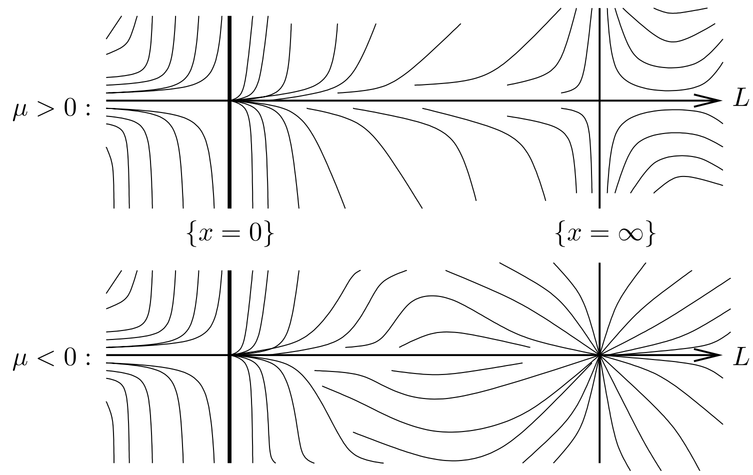


FIGURE 3. Geometry of the second normal form

Conversely, a germ of foliation $\mathcal{F}$ on $\overline{\mathbb{C}} \times (\mathbb{C}, 0)$ satisfying conditions above can easily be transformed into the form (4). Indeed, $\mathcal{F}$ is defined by a unique vector field of the form $F(x,y)\partial_x + \partial_y$ with $F$ meromorphic at the neighborhood of the line $L$. In restriction to the horizontal lines, the vector field $F(x,y)\partial_x$ is rational; its zeroes and poles coincide with the points where the foliation $\mathcal{F}$ is respectively vertical and horizontal. Because we have two singular points of multiplicities 1 and 2 along $L$, we deduce that $F(x,y)\partial_x$ has 3 zeroes (counted with multiplicity) in restriction to each fiber; hence, it it has exactly 1 pole (the divisor of a vector field has degree 2 on $\overline{\mathbb{C}}$). From conditions (2) and (4), we actually see that the zeroes are supported by the vertical invariant curves, that $L$ gives contribution for 1 pole and one can write $F(x,y) = \frac{x^2}{y(f(y)x+g(y))}$ for holomorphic functions $f, g \in \mathbb{C}\{y\}$. It is easy to verify that $f$ and $g$ do not vanish at $y = 0$ otherwise the singular points would be more degenerate. Therefore, the foliation $\mathcal{F}$ is also defined by the holomorphic vector field

$$\widetilde{X} = x^2\partial_x + (f(y)x + g(y))y\partial_y, \quad f(0), g(0) \neq 0.$$

After a change of $y$-coordinate, one may linearize the holomorphic vector field $yg(y)\partial_y$ to $g(0)y\partial_y$; after division by $g(0)$ and linear change of the $x$-coordinate, we finally obtain the form (4).

A *necessary condition* for a saddle-node to admit a form (4) is that the holonomy of the central manifold, which actually coincides with Martinet-Ramis' invariant $\varphi^0$

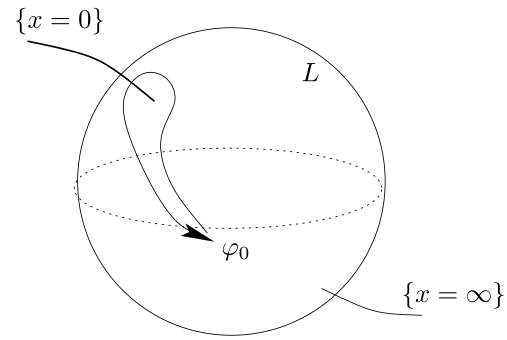


FIGURE 4. Holonomy compatibility

(see section 1), is also the anti-holonomy of the invariant curve $L$ around the singular point $x=\infty$. This gives restriction for $\varphi^0$, and hence for the saddle-node, at least when $\mu \in \mathbb{R}^-$. In the case $\mu<0$, the other singular point is linearizable by Poincaré's Theorem implying the linearizability of the holonomy map $\varphi^0$. Here, we use property (3) above and the fact that, in the resonant (non linearizable) case, the node has only one irreducible germ of invariant curve. In the case $\mu=0$, the holonomy $\varphi^0$ is tangent to the identity and its inverse $(\varphi^0)^{-1}$ must be the holonomy of the *strong manifold* (the invariant curve tangent to the non zero eigendirection) of a saddle-node having a central manifold. Following section 1, this is equivalent to condition (3) of Theorem 8 below.

***Theorem 8***. — *Let $\mathcal{F}$ be a germ of saddle-node with multiplicity* 2 *at the origin of* $\mathbb{R}^2$ *(resp. of* $\mathbb{C}^2$*) having a central manifold. Then, there exist analytic coordinates in which $\mathcal{F}$ is defined by a vector field of the form (4)*

$$X_f = x^2\partial_x + y\partial_y + xf(y)\partial_y, \quad \textit{with } f(0)=0.$$

*(and $\mu = f'(0)$) if, and only if, we are in one of the following cases*

(1) $\mu \in \mathbb{C} - \mathbb{R}^-$,

(2) $\mu < 0$ *and* $\varphi_0$ *is linearizable up to conjugacy in* $\mathrm{Diff}(\mathbb{C},0)$,

(3) $\mu = 0$ *and Martinet-Ramis' invariants* $(\widetilde{\varphi}_0^i, \widetilde{\varphi}_\infty^i)_i$ *of* $\varphi_0$ *satisfy: all* $\widetilde{\varphi}_0^i$ *are linear.*

*When $\mu \notin \mathbb{Q}_*^-$, the form (4) is unique up to homothety $y \mapsto c\cdot y$, $c \in \mathbb{C}^*$.*

Recall that condition (2) is automatic as soon as $\mu$ is a Bruno number:

$$\mu \in \mathcal{B} \quad \Longleftrightarrow \quad \sum_{n\geqslant 0} \frac{\log(q_{n+1})}{q_n} < \infty$$

(where $p_n/q_n$ stand for successive truncatures of the continued fraction of $|\mu|$). The set $\mathcal{B}$ has full Lebesgue measure in $\mathbb{R}$. For all other values $\mu \in \mathbb{R}^- - \mathcal{B}$, condition (2) is very restrictive for $\varphi_0$, and thus for the saddle-node.

Like in Section 2, we start with a germ of saddle-node $\mathcal{F}_0$ defined on the neighborhood of some disc $\Delta_0$ and glue it with a germ of foliation $\mathcal{F}_\infty$ along a complementary disc $\Delta_\infty$ in order to obtain a germ of 2-dimensional neighborhood $S$ along a rational curve $L$ equipped with a singular foliation $\mathcal{F}$. The difference with Section 2 is that we now glue $\mathcal{F}_0$ and $\mathcal{F}_\infty$ along a common invariant curve, in such a way that $L$ becomes a global invariant curve for the foliation $\mathcal{F}$. We do it first respect to the vertical fibration; this is very easy but we need the difficult Savelev's Theorem to recover the triviality of the neighborhood (and the rational fibration). Then, we give an alternate gluing using technical (but elementary) Lemmae of section 3 in which we keep on constructing by hands the rational fibration.

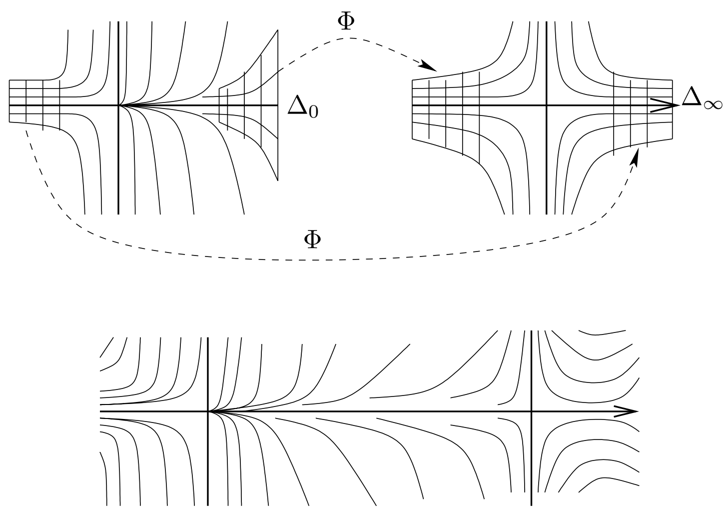


FIGURE 5. Gluing picture

We start with $\mathcal{F}_0$ into Dulac preliminary form

$$X_0 = x^2\partial_x + y\partial_y + xyf(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\},\ f(0) = \mu$$

(when the saddle-node has a central manifold, the form (1) can be achieved with the central manifold contained in $\{y = 0\}$, see [**11**]). Consider, in local coordinates $(\widetilde{x} = 1/x, y)$ at infinity, a germ of singular foliation $\mathcal{F}_\infty$ defined by

$$X_\infty = \widetilde{x}\partial_{\widetilde{x}} - (\mu + g(\widetilde{x},y))y\partial_y, \quad g(0) = 0.$$

Up to a linear change of $\widetilde{x}$-coordinate, one may assume that $\mathcal{F}_\infty$ is actually defined on the neighborhood of $\Delta_\infty$. Obviously, there exists a germ of diffeomorphism of the form

$$\Phi : (\mathbb{C}^2, C) \longrightarrow (\mathbb{C}^2, C)\ ;\ (x,y) \longmapsto (x, \phi(x,y)), \quad \phi(x,0) = 0$$

gluing $\mathcal{F}_0$ with $\mathcal{F}_\infty$ if, and only if, the respective holonomy maps around the corona $C = \Delta_0 \cap \Delta_\infty$ are conjugated in $\mathrm{Diff}(\mathbb{C}, 0)$.

When $\mu \notin \mathbb{R}$, the holonomy map $\varphi^0$ of $\mathcal{F}_0$ around $C$ (or $x = 0$) is hyperbolic and hence linearizable by Kœnigs' Theorem. It is therefore enough to choose $X_\infty$ linear. When $\mu > 0$, then the holonomy map $\varphi^0$ can be realized as the holonomy of a saddle $\mathcal{F}_\infty$ like above following [**12, 14**]. When $\mu < 0$ and $\varphi^0$ is linearizable, we obviously realize it with $X_\infty$ linear. Finally, when $\mu = 0$, condition (3) of Theorem 2 is exactly the one to realize $(\varphi^0)^{-1}$ as the holonomy of the strong manifold of a saddle node $\mathcal{F}_\infty$ having a central manifold. After gluing $\mathcal{F}_0$ and $\mathcal{F}_\infty$ along $C$, we obtain a germ of surface $S$ containing a rational curve $L$ which, by Camacho-Sad's Formula (see [**4**]), has 0 self-intersection in $S$. Following Savelev's Theorem [**15**], there exists a system of trivializing coordinates: $(x, y) : (S, L) \to \overline{\mathbb{C}} \times (\mathbb{C}, 0)$. Up to a change of trivializing coordinates $x := \{\frac{a(y)x+b(y)}{c(y)x+d(y)}\}$ and $y = \varphi(y)$ on $S$, one may assume properties (1), (2), (3) and (4) of the begining of the section all satisfied. Therefore, $\mathcal{F}$ is defined by a vector field of the form (4). The existence part is proved. □

Let us now show how to avoid with Savelev Theorem by using section 3. We first choose germs of foliations $\mathcal{F}_0$ and $\mathcal{F}_\infty$ with compatible holonomy as in the previous proof. Instead of $X_0$, we define the foliation $\mathcal{F}_0$ by the meromorphic vector field

$$\widetilde{X}_0 = \frac{x^2}{1 + xf(x, y)}\partial_x + y\partial_y, \quad f(0) = \mu.$$

After a local change of the $x$-coordinate, we may assume that the restriction $\widetilde{X}_0|_L = \frac{x^2}{1+xf(x,0)}\partial_x$ to $L = \{y = 0\}$ coincides with the global meromorphic vector field $\frac{x^2}{1+\mu x}\partial_x$. By the same way, the alternate meromorphic vector field

$$\widetilde{X}_\infty = \frac{x}{\mu + g(1/x, y)}\partial_x + y\partial_y, \quad g(0) = 0.$$

defines $\mathcal{F}_\infty$ at the neighborhood of $x = \infty$ and its restriction $\frac{x}{\mu+g(1/x,0)}\partial_x$ coincide with $\frac{x^2}{1+\mu x}\partial_x$ after a local change of $x$-coordinate at infinity (they are both conjugated to $\frac{1}{\mu}x\partial_x$ at $x = \infty$).

Assume first that $\widetilde{X}_0$ and $\widetilde{X}_\infty$ are defined at the neighborhood of some horizontal discs $\Delta_0$ and $\Delta_\infty$ covering $L$. Maybe restricting to slightly smaller discs, one may assume that the intersecting corona $C = \Delta_0 \cap \Delta_\infty$ does not contain $-1/\mu$ (the pole of $\frac{1}{\mu}x\partial_x$): therefore, the vector fields $\widetilde{X}_0$ and $\widetilde{X}_\infty$ are both holomorphic on the neighborhood of $C$ and can be glued by means of Lemma 6. By this way, we construct a surface $S$ equipped with a global foliation $\mathcal{F}$ and a rational fibration $y : S \to (\mathbb{C}, 0)$. By Fisher-Grauert [**8**], $S$ is a germ of trivial $\overline{\mathbb{C}}$-bundle and we can end the proof as before.

The problem is that $\widetilde{X}_0$ and $\widetilde{X}_\infty$ are *a priori* defined on small respective neighborhoods $\Omega_0$ and $\Omega_\infty$ of $x = 0$ and $x = \infty$. We would like to apply a change of coordinate in variable $x$ in order to enlarge $\Omega_\infty$, for instance. We cannot do this with an homothety anymore because we need to preserve the restrition of $\widetilde{X}_\infty$ to $L$ in order to apply Lemma 6. We can only use the changes of coordinates which commute with $\frac{x^2}{1+\mu x}\partial_x$, *i.e.* those ones given by an element of the flow $\exp(t\frac{x^2}{1+\mu x}\partial_x)$, $t\in\mathbb{C}$. By this way, we will not be able to cover the complement of $\Omega_0$ with the new domain $\Omega_\infty$, but at least, we may assume that $\Omega_0$ and $\Omega_\infty$ intersect. Then, we can complete a covering of $L$ by adding a third open set $\Omega_1$ in such a way that the intersections $\Omega_0\cap\Omega_1$, $\Omega_0\cap\Omega_\infty$ and $\Omega_1\cap\Omega_\infty$ do not contain neither 0, $-1/\mu$ nor $\infty$. We refer to [**1**] for a complete description of the flow $\exp(t\frac{x^2}{1+\mu x}\partial_x)$ in function of $\mu$. Finally, consider the third foliation defined on the neighborhood of $\Omega_1$ by the rational vector field

$$\widetilde{X}_\mu = \frac{x^2}{1+\mu x}\partial_x + y\partial_y.$$

By means of Lemma 6, we can glue the 3 foliations together on the neighborhood of $L$, simultaneously preserving the $y$-coordinate. This finishes the second proof of the construction of form (4). □

It remains to prove the unicity (up to homothety) of form (4) in case $\mu$ is not rational negative. Assume that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are of the form (4) and are analytically conjugated on a neighborhood of $(x,y)=0$. Following [**11**], they are also conjugated by a germ of diffeomorphism of the form

$$\Phi_0 : (\mathbb{C}^2,0) \longrightarrow (\mathbb{C}^2,0)\ ;\ (x,y) \longmapsto (x,\phi_0(x,y))$$

which must preserve the central manifold: $\phi_0(x,0)=0$. One can extend analytically $\Phi_0$ on a neighborhood of $L-\{x=\infty\}$ in the obvious way, by lifting-path-property. We *claim* that $\Phi_0$ extends until the other singular point $x=\infty$. Before proving this, let us show how to conclude the proof. Therefore, we obtain a global diffeomorphism $\Phi$ along $L$ conjugating $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. By Blanchard's argument, $\Phi$ permutes the horizontal lines: for any line $L'$ close to $L$, the restriction of $y$ along the image $\Phi(L')$ is an holomorphic map from a compact manifold into a bounded domain; therefore, $y|_{\Phi(L')}$ is constant and $\Phi(L')$ is actually a fiber of $y$. Therefore, one can write $\Phi(x,y) = (x,\phi(y))$ and due to the form (4), $\phi$ has to commute with $y\partial_y$ and must be linear. This concludes the proof of Theorem 8. □

We first prove the *claim* in case $\mathcal{F}_\infty$ is in the Poincaré domain ($\mu\in\mathbb{C}-\mathbb{R}^-$). Recall that property (3) implies that $\mathcal{F}_\infty$ is non resonant and hence linearizable by a local change of coordinates of the form $(\widetilde{x},y)\mapsto(\widetilde{x},\phi_\infty(\widetilde{x},y))$. Therefore, we can assume that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are defined by

$$X_\infty = \widetilde{x}\partial_{\widetilde{x}} - \mu y\partial_y, \quad \mu \not\geqslant 0.$$

and that $\Phi_0(\widetilde{x},y) = (\widetilde{x},\phi(\widetilde{x},y))$ is a self-conjugacy of $\mathcal{F}_\infty$ at the neighborhood of the punctured disc $\Delta^* := \Delta_\infty - \{\widetilde{x}=0\}$. The question is, when does $\Phi_0$ coincide with a symetry of $\mathcal{F}_\infty$

$$\Phi_\infty(\widetilde{x},y) = (\widetilde{x}, c\cdot y), \quad c\in\mathbb{C}^*.$$

Of course, this is the case if, and only if, $\phi(\widetilde{x}, y)$ is linear in $y$. In fact, for $x$ fixed, $\phi(\widetilde{x}, y)$ commutes with the holonomy $y \mapsto e^{-2i\pi\mu}y$ of $\mathcal{F}_\infty$ and is therefore linear as soon as $\mu$ is not rational.

Finally, in the remaining case $\mu \geqslant 0$, the fact that $\Phi_0$ extends at the singular point at infinity is due to J.-F. Mattei and R. Moussu ([**13**], p. 484-485 or [**12**], p. 595-596) in the case $\mu > 0$ and to M. Berthier, R. Meziani and P. Sad ([**2**], Theorem 1.1) in the case $\mu = 0$. Actually, in both cases, it is proved that any conjugacy between the holonomy maps of two saddles ($\mu > 0$) or strong manifolds of two saddle-nodes with a central manifold ($\mu = 0$) extends as a conjugacy of the respective foliations of the form $\Phi_\infty(x, y) = (x, \phi_\infty(x, y))$; this $\Phi_\infty$ will automatically coincide with $\Phi_0$ and extend it at the singular point $x = \infty$. The claim is proved. $\square$

***Remark 9***. — In the case $\mu \in \mathbb{Q}_*^-$, it is easy to construct examples of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ like above that are not globally conjugated and giving rise to non unique form (4).

*Proof of Theorem 3*. — It is the same with $\mu > 0$, except that we start with the saddle $\mathcal{F}_\infty$ at $x = \infty$. Following Martinet-Ramis (see section 1), the anti-holonomy $\varphi^0(y) = e^{2i\pi\mu}y + \cdots$ of $\mathcal{F}_\infty$ can be realized as the holonomy of the central manifold of a saddle-node $\mathcal{F}_0$. Like above, we can glue those two foliations and obtain normal form (4). We deduce the normal form (5) for the saddle $\mathcal{F}_\infty$ by setting $\widetilde{x} = 1/x$ in the form (4). $\square$

## 5. Proof of Theorem 4

Let us start by blowing-up a saddle-node of the form (4)

$$X_f = x^2\partial_x + y\partial_y + xyf(y)\partial_y, \quad f(0) = \mu.$$

Along the exceptional divisor, we have one saddle with eigenratio $-1$ and a saddle-node, given in the chart $(x, t)$, $y = tx$, by

$$\widetilde{X}_f = x^2\partial_x + t\partial_t + xt(f(xt) - 1)\partial_t.$$

In particular, $\widetilde{X}_f$ takes the form (6) of Theorem 4 with $n = 1$ and has formal invariant $\widetilde{\mu} = \mu - 1$.

After $n$ successive blow-ups of the saddle-nodes, we obtain an exceptional divisor like in the picture below where the new saddle-node takes the form (6) of Theorem 4 with formal invariant $\widetilde{\mu} = \mu - n$. All other singular points are saddles with $-1$ eigenratio.

The rough idea to put a given saddle-node $\widetilde{\mathcal{F}}$ into the form (6) is to realize it as the $n^{\text{th}}$ blowing-up of a saddle-node $\mathcal{F}$, then apply Theorem 2 to put $\mathcal{F}$ into the form (4). We first detail the case $n = 1$.

Since the holonomy map $\varphi$ of the strong manifold of $\widetilde{\mathcal{F}}$ is tangent-to-the-identity, it can be realized as the holonomy map of a saddle with $-1$ eigenratio following

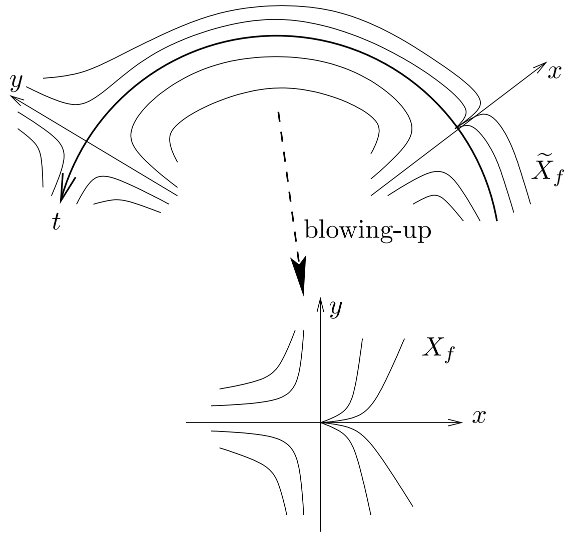


FIGURE 6. Blowing-up a saddle-node

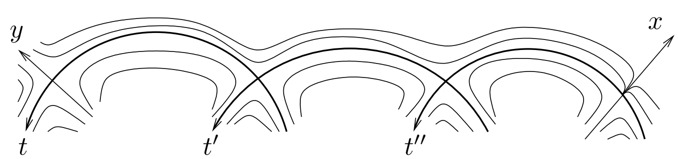


FIGURE 7. After 3 blowing-ups

Martinet-Ramis (see [**12**]). Therefore, one can glue those two foliations along their invariant curve like we did in section 4 to prove Theorem 2 (first gluing construction). By this way, we obtain a germ of surface $S$ around a rational curve $L$ having self-intersection $-1$ by Camacho-Sad index Theorem [**4**]. Following Grauert (see [**9**]), *the neighborhood of a smooth rational curve with negative self-intersection in a surface is rigid*: maybe replacing $S$ by a smaller neighborhood of $L$, $S$ is biholomorphic to the neighborhood of the exceptional divisor after blowing-up the origin of $\mathbb{C}^2$ ($-1$ self-intersection). After making this identification, the global foliation $\widetilde{\mathcal{F}}$ on $S$ becomes the germ of a saddle-node $\mathcal{F}$ at the origin of $\mathbb{C}^2$. The corresponding formal invariants are related by $\mu = \widetilde{\mu} - 1$ so that if $\widetilde{\mathcal{F}}$ satisfies the assumptions of Theorem 4 with $n = 1$, then one can apply Theorem 2 to $\mathcal{F}$. Once $\mathcal{F}$ is in the form (4), we obtain the form (6) for $\widetilde{\mathcal{F}}$. Here, we implicitely use the known fact that one can blow up a diffeomorphism: the conjugacy from $\mathcal{F}$ to its normal form (4) induces after blowing up a conjugacy from $\widetilde{\mathcal{F}}$ to its normal form (6). This proves the existence part. □

The unicity also follows from that of form (4) proved in Section 4. Indeed, if two such foliations $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ are locally conjugated, then the corresponding holonomies along the exceptional divisor $L$ are conjugated. By Mattei-Moussu [**13**], this implies that the $-1$ saddles are conjugated; therefore, the holonomies of the saddles along the other invariant curve $\{t=\infty\}$ are conjugated as well. This latter means that after blowing down, the holonomies of the strong manifold of the corresponding saddle-nodes $\mathcal{F}$ and $\mathcal{F}'$ are conjugated. We can apply unicity of Theorem 2. $\square$

The general case $n \in \mathbb{N}^*$ is proved by the same way. Starting from a saddle-node $\widetilde{\mathcal{F}}$ with formal invariant $\mu > -n$ (or $\mu \notin \mathbb{R}$), we glue it successively with $-1$ saddles in order to construct a $n$-blow-up configuration as in the picture; then, Grauert's Theorem permits to blow down successively all irreducible components of the divisor: at each step, the component which contains the saddle-node has again self-intersection $-1$ by Camacho-Sad. After blowing down the whole divisor, we can apply Theorem 2 to the resulting saddle-node.

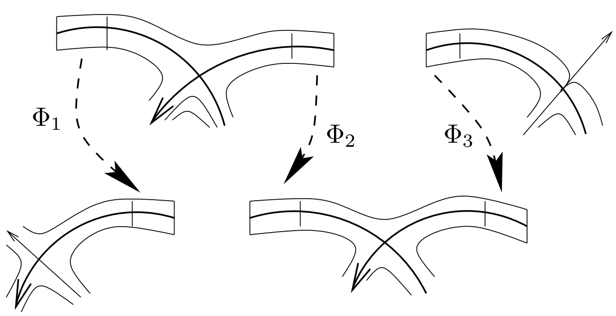


FIGURE 8. Gluing foliations along an exceptional divisor

For the unicity, given a conjugacy between two saddle-nodes $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ like above, we successively deduce by Mattei-Moussu the conjugacy of all respective $-1$ saddles and finally of the resulting saddle-nodes $\mathcal{F}$ and $\mathcal{F}'$ after blowing down. The unicity follows again from that of Theorem 2.

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# ASYMPTOTICS FOR GENERAL CONNECTIONS AT INFINITY

*by*

Carlos Simpson

***Abstract.*** — For a standard path of connections going to a generic point at infinity in the moduli space $M_{\mathrm{DR}}$ of connections on a compact Riemann surface, we show that the Laplace transform of the family of monodromy matrices has an analytic continuation with locally finite branching. In particular, the convex subset representing the exponential growth rate of the monodromy is a polygon whose vertices are in a subset of points described explicitly in terms of the spectral curve. Unfortunately, we don't get any information about the size of the singularities of the Laplace transform, which is why we can't get asymptotic expansions for the monodromy.

***Résumé* (Asymptotique des connexions génériques à l'infini).** — Pour une courbe standard allant vers un point général à l'infini dans l'espace des modules $M_{\mathrm{DR}}$ des connexions sur une surface de Riemann compacte, nous montrons que le transformé de Laplace de la famille des matrices de monodromie admet un prolongement analytique avec ramification localement finie. En particulier, l'ensemble convexe qui représente la croissance exponentielle est un polygone dont les sommets sont dans un ensemble qu'on peut expliciter en termes de la courbe spectrale. Malheureusement, nous n'obtenons pas d'information sur la taille des singularités du transformé de Laplace et donc pas de développement asymptotique pour la monodromie.

## 1. Introduction

We study the asymptotic behavior of the monodromy of connections near a general point at $\infty$ in the space $M_{\mathrm{DR}}$ of connections on a compact Riemann surface $X$. We will consider a path of connections of the form $(E, \nabla + t\theta)$ which approaches the boundary divisor transversally at the point on the boundary of $M_{\mathrm{DR}}$ corresponding to a general Higgs bundle $(E, \theta)$. By some meromorphic gauge transformations in § 5 we reduce to the case of a family of connections of the form $d + B + tA$. This is very

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similar to what was treated in [**36**] except that here our matrix $B$ may have poles. We import the vast majority of our techniques directly from there. The difficulty posed by the poles of $B$ is the new phenomenon which is treated here. We are not able to get results as good as the precise asymptotic expansions of [**36**]. We just show in Theorem 6.3 (p. 205) that if $m(t)$ denotes the family of monodromy or transport matrices for a given path, then the Laplace transform $f(\zeta)$ of $m$ has an analytic continuation with locally finite singularities over the complex plane (see Definition 6.2, p. 205). The singularities are what determine the asymptotic behavior of $m(t)$. The upside of this situation is that since we are aiming for less, we can considerably simplify certain parts of the argument. What we don't know is the behavior of $f(\zeta)$ near the singularities: the main question left open is whether $f$ has polynomial growth at the singularities, and if so, to what extent the generalized Laurent series can be calculated from the individual terms in our integral expression for $f$.

We can get some information about where the singularities are. Fix a general point $(E,\theta)$. Recall from [**26, 27, 19, 30**] that the *spectral curve* $V$ is the subset of points in $T^*(X)$ corresponding to eigenforms of $\theta$. We have a proper mapping $\pi : V \to X$. In the case of a general point, $V$ is smooth and the mapping has only simple ramification points. Also there is a tautological one-form

$$\alpha \in H^0(V, \pi^*\Omega^1_X) \subset H^0(V, \Omega^1_V).$$

Finally there is a line bundle $L$ over $V$ such that $E \cong \pi_*(L)$ and $\theta$ corresponds to the action of $\alpha$ on the direct image bundle. This is all just a geometric version of the diagonalization of $\theta$ considered as a matrix over the function field of $X$.

Let $\mathcal{R} \subset X$ denote the subset of points over which the spectral curve is ramified, that is the image of the set of branch points of $\pi$. It is the set of *turning points* of our singular perturbation problem. Suppose $p$ and $q$ are points in $X$ joined by a path $\gamma$. A *piecewise homotopy lifting* of $\gamma$ to the spectral curve $V$ consists of a collection of paths

$$\widetilde{\gamma} = \{\widetilde{\gamma}_i\}_{i=1,\ldots,k}$$

such that each $\widetilde{\gamma}_i$ is a continuous path in $V$, and such that if we denote by $\gamma_i := \pi\circ\widetilde{\gamma}_i$ the image paths in $V$, then $\gamma_1$ starts at $p$, $\gamma_k$ ends at $q$, and for $i = 1,\ldots,k-1$, the endpoint of $\gamma_i$ is equal to the starting point of $\gamma_{i+1}$ and this is a point in $\mathcal{R}$. Among these there is a much more natural class of paths which are the *continuous homotopy liftings*, namely those where the endpoint of $\widetilde{\gamma}_i$ is equal to the starting point of $\widetilde{\gamma}_{i+1}$ (which is not necessarily the case for a general piecewise lifting).

Denote by $\Sigma(\gamma) \subset \mathbb{C}$ the set of integrals of the tautological form $\alpha$ along piecewise homotopy liftings of $\gamma$, *i.e.* the set of complex numbers of the form

$$\sigma = \int_{\widetilde{\gamma}} \alpha := \sum_{i=1}^{k} \int_{\widetilde{\gamma}_i} \alpha.$$

Let $\Sigma^{\mathrm{cont}}(\gamma)$ be the subset of integrals along the continuous homotopy liftings. The following is the statement of Theorem 6.3 augmented with a little bit of information about where the singularities are.

***Theorem 1.1***. — *Let $p, q$ be two points on $X$, and let $\gamma$ denote a path from $p$ to $q$. Let $\{(E, \nabla + t\theta)\}$ denote a curve of connections cutting the divisor $P_{\mathrm{DR}}$ at a general point $(E, \theta)$ and let $(V, \alpha, L)$ denote the spectral data for this Higgs bundle. Let $m(t)$ be the function (with values in $Hom(E_p, E_q)$) whose value at $t \in \mathbb{C}$ is the transport matrix for the connection $\nabla + t\theta$ from $p$ to $q$ along the path $\gamma$. Let $f(\zeta)$ denote the Laplace transform of $m$. Then, $f$ has an analytic continuation with locally finite singularities over the complex plane. The set of singularities which are ever encountered is a subset of the set $\Sigma(\gamma) \subset \mathbb{C}$ of integrals of the tautological form along piecewise homotopy liftings defined above.*

It would have been much nicer to be able to say that the set of singularities is contained in $\Sigma^{\mathrm{cont}}(\gamma)$, however I don't see that this is necessarily the case. However, it might be that the singularities in $\Sigma^{\mathrm{cont}}(\gamma)$ have a special form different from the others. This is an interesting question for further research.

Ths first singularities which are encountered in the analytic continuation of $f$ determine the growth rate of $m(t)$ in a way which we briefly formalize. Suppose that $m(t)$ is an entire function with exponentially bounded growth. We say that $m(t)$ is *rapidly decreasing in a sector*, if for some (open) sector of complex numbers going to $\infty$, there is $\varepsilon > 0$ giving a bound of the form $|m(t)| \leqslant e^{-\varepsilon|t|}$. Define the *hull* of $m$ by

$$\mathbf{hull}(m) := \{\zeta \in \mathbb{C} \mid e^{-\zeta t} m(t) \text{ not rapidly decreasing in any sector}\}.$$

It is clear from the definition that the set of $\zeta$ such that $e^{-\zeta t} m(t)$ is rapidly decreasing in some sector, is open. Therefore $\mathbf{hull}(m)$ is closed. It is also not too hard to see that it is convex (see § 13). Note that the hull is defined entirely in terms of the growth rate of the function $m$.

***Corollary 1.2***. — *In the situation of Theorem 1.1, the hull of $m$ is a finite convex polygon with at least two vertices, and all of its vertices are contained in $\Sigma(\gamma)$.*

The above results fall into the realm of *singular perturbation theory* for systems of ordinary differential equations, which goes back at least to Liouville. A steady stream of progress in this theory has led to a vast literature which we don't attempt completely to cover here (and which the reader can explore by using internet and database search techniques, starting for example from the authors mentionned in the bibliography).

Recall that following [**4**], Voros and Ecalle looked at these questions from the viewpoint of "resurgent functions" [**43, 44, 42, 21, 20, 22, 7, 9, 15**]. In the terminology of Ecalle's article in [**7**], the singular perturbation problem we are considering here is an example of *co-equational resurgence*. Our approach is very related to this viewpoint, though self-contained. We use a notion of analytic continuation of the Laplace transform 6.2 which is a sort of weak version of resurgence, like that used in [**15**] and [**9**]. The elements of our expansion 6.1 are what Ecalle calls the "elementary monomials" and the trees which appear in § 8 are related to *(co)moulds (co)arborescents*, see [**7**]. Conversion properties related to the trees have been discussed in [**23**] (which is on the subject of KAM theory [**24**]). The relationship with integrals on a spectral curve was explicit in [**14**], [**15**]. The works [**42**], Ecalle's article in [**7**], and [**15**], raise a number of questions about how to prove resurgence for certain classes of singular perturbation problems notably some arising in quantum mechanics. A number of subsequent articles treat these questions; I haven't been able to include everything here but some examples are [**23**], [**16**], [**17**], . . . (and apparently [**46**]). In particular [**17**] discuss extensively the way in which the singularities of the Laplace transform determine the asymptotic behavior of the original function, specially in the case of the kinds of integrals which appear as terms in the decomposition 6.1.

There are a number of other currents of thought about the problem of singular perturbations. It is undoubtedly important to pursue the relationship with all of these. For example, the study initiated in [**6**] and continuing with several articles in [**7**], as well as the more modern [**1**] (also Prof. Kawai's talk at this conference) indicates that there is an intricate and fascinating geometry in the propagation of the Stokes phenomenon. And on the other hand it would be good to understand the relationship with the local study of turning points such as in [**8**], [**41**]. The article [**16**] incorporates some aspects of all of these approaches, and one can see [**5**] for a physical perspective. Also works on Painlevé's equations and isomonodromy such as [**11, 28, 34, 45**] are probably relevant .

Even though he doesn't appear in the references of [**36**], the ideas of J.-P. Ramis indirectly had a profound influence on that work (and hence on the present note). This can be traced to at least two inputs as follows:

(1) I had previously followed G. Laumon's course about $\ell$-adic Fourier transform, which was partly inspired by the corresponding notions in complex function theory, a subject in which Ramis (and Ecalle, Voros, . . . ) had a great influence; and

(2) at the time of writing [**36**] I was following N. Katz's course about exponential sums, where again much of the inspiration came from Ramis' work (which Katz mentionned very often) on irregular singularities.

Thus I would like to take this opportunity to thank Jean-Pierre for inspiring such a rich mathematical context.

I would also like to thank the several participants who made interesting remarks and posed interesting questions. In particular F. Pham pointed out that it would be a good idea to look at what the formula for the location of the singularities actually said, leading to the statement of Theorem 6.3 in its above form. I haven't been able to treat other suggestions (D. Sauzin, . . . ), such as looking at the differential equation satisfied by $f(\zeta)$.

## 2. The compactified moduli space of connections

Let $X$ be a smooth projective curve over the complex numbers $\mathbb{C}$. Fix $r$ and suppose $E$ is a vector bundle of rank $r$ over $X$. A *connection* (by which we mean an algebraic one) on $E$ is a $\mathbb{C}$-linear morphism of sheaves $\nabla : E \to E \otimes_{\mathcal{O}} \Omega^1_X$ satisfying the Leibniz rule $\nabla(ae) = (da)e + a\nabla(e)$. If $p$ and $q$ are points joined by a path $\gamma : [0,1] \to X$, $\gamma(0) = P$, $\gamma(1) = Q$ then local solutions of $\nabla(e) = 0$ continue along $\gamma$, giving a *transport matrix* $m_\gamma(E, \nabla) : E_P \to E_Q$. The transport matrix, our main object of study, is the fundamental solution of a linear system of ODE's. If $E$ is a trivial bundle (which will always be the case at least on a Zariski open subset of $X$ containing $\gamma$) then there is a formula for the transport matrix as a sum of iterated integrals [**10, 25**]. A modified version of this formula is basic to the argument below, although we mostly refer to [**36**] for the details of that part of the argument.

Recall that we have a moduli space $M_{\mathrm{DR}}$ of rank $r$ vector bundles with integrable connection on $X$ [**39**], which has a compactification $M_{\mathrm{DR}} \subset \overline{M}_{\mathrm{DR}}$ constructed as follows. A *Higgs bundle* is a pair $(E, \theta)$ where $\theta : E \to E \otimes_{\mathcal{O}} \Omega^1_X$ is an $\mathcal{O}_X$-linear bundle map (rather than a connection) [**26, 27, 38**], which is *semistable of degree* 0 if $E$ has degree zero and if any sub-Higgs bundle has degree $\leqslant 0$. In fact for any $\lambda \in \mathbb{A}^1_{\mathbb{C}}$ we can look at the notion of *vector bundle with $\lambda$-connection* [**18**]—related in an obvious way to the notion of singular perturbation—which is a pair $(E, \nabla)$ of a bundle plus a connection-like operator satisfying Leibniz' rule with a factor of $\lambda$ in front of the first term. For $\lambda = 0$ this is just a Higgs bundle and for any $\lambda \neq 0$ the operator $\lambda^{-1}\nabla$ is a connection.

With these definitions, there is a moduli space [**37, 40, 39**] $M_{\mathrm{Hod}} \to \mathbf{A}^1$ for vector bundles with $\lambda$-connection, $\lambda \in \mathbf{A}^1$. The fiber over $\lambda = 0$ is the moduli space $M_{\mathrm{Dol}}$ for semistable Higgs bundles of degree zero, whereas for any $\lambda \neq 0$ the fiber is isomorphic to $M_{\mathrm{DR}}$.

The Higgs-bundle moduli space has a subvariety $M^{\mathrm{nil}}_{\mathrm{Dol}}$ parametrizing the Higgs bundles $(E, \theta)$ such that $\theta$ is nilpotent as an $\Omega^1_X$-valued endomorphism of $E$. Let $M^*_{\mathrm{Dol}}$ denote the complement of $M^{\mathrm{nil}}_{\mathrm{Dol}}$ in $M_{\mathrm{Dol}}$ and let $M^*_{\mathrm{Hod}}$ denote the complement of $M^{\mathrm{nil}}_{\mathrm{Dol}}$ in $M_{\mathrm{Hod}}$. Then the algebraic group $\mathbf{G}_m$ acts on $M_{\mathrm{Hod}}$ preserving all of the above subvarieties, and the compactification is obtained as the quotient [**37, 40**]

$$\overline{M}_{\mathrm{DR}} := M^*_{\mathrm{Hod}}/\mathbf{G}_m.$$

The complement of $M_{\mathrm{Dol}}$ in $M_{\mathrm{Hod}}$ (which is also the complement of $M^*_{\mathrm{Dol}}$ in $M^*_{\mathrm{Hod}}$) is isomorphic to $M_{\mathrm{DR}} \times \mathbf{G}_m$ and this gives the embedding $M_{\mathrm{DR}} \hookrightarrow \overline{M}_{\mathrm{DR}}$. The complementary divisor is given by

$$P_{\mathrm{DR}} = M^*_{\mathrm{Dol}}/\mathbf{G}_m.$$

In conclusion, this means that the points at $\infty$ in $\overline{M}_{\mathrm{DR}}$ correspond to equivalence classes of semistable, degree 0, non-nilpotent Higgs bundles $(E, \theta)$ under the equivalence relation

$$(E, \theta) \sim (E, u\theta)$$

for any $u \in \mathbf{G}_m$.

Recall that the moduli space $M_{\mathrm{Dol}}$ is an irreducible algebraic variety [**39**], so $P_{\mathrm{DR}}$ is also irreducible. The general point therefore corresponds to a "general" Higgs bundle $(E, \theta)$ (in what follows we often forget to add the adjectives "semistable, degree 0"). For a general point, the *spectral curve* of $\theta$ (described in more detail in the section after next) is an irreducible curve with ramified map to $X$, such that the ramification points are all of the simplest type.

We should note that Arinkin [**2, 3**] has defined a finer compactification by modifying the notion of $\lambda$-connection, and this is taken up by Inaba, Iwasaki and Saito [**28**].

## 3. Curves going to infinity

The moduli spaces considered above are coarse only. In an etale neighborhood of the generic point, though, they are fine and smooth. At a general point of the divisor $P_{\mathrm{DR}}$, both $\overline{M}_{\mathrm{DR}}$ and $P_{\mathrm{DR}}$ are smooth. Thus we can look for a curve cutting $P_{\mathrm{DR}}$ transversally at a general point. Such a curve may be obtained by taking the projection of a curve in $M_{\mathrm{Hod}}$ cutting $M_{\mathrm{Dol}}$ at a general point. In turn, this amounts to giving a family $(E_c, \nabla_c)$ where $\nabla_c$ is a $\lambda(c)$-connection, parametrized by $c \in C$ for some curve $C$. In an etale neighborhood of the point $\lambda = 0$, the function $\lambda(c)$ should be etale. Note also that $(E_0, \nabla_0)$ should be a general semistable Higgs bundle of degree zero.

The easiest way to obtain such a curve is as follows: let $(E, \theta)$ be a general Higgs bundle, stable of degree zero. The bundle $E$ is stable as a vector bundle (since stability is an open condition and it certainly holds on the subset of Higgs bundles with $\theta = 0$, so it holds at general points). By Narasimhan-Seshadri, $E$ supports a (flat unitary) connection $\nabla$ compatible with the holomorphic structure and we can set

$$\nabla_\lambda := \lambda\nabla + \theta$$

for $\lambda \in \mathbf{A}^1$. Here the parameter is $\lambda$ itself. The subset $\mathbf{G}_m \subset \mathbf{A}^1$ corresponds to points which are mapped into $M_{\mathrm{DR}}$, and indeed the vector bundle with connection corresponding to the above $\lambda$-connection is

$$(E, \nabla + t\theta), \quad t = \lambda^{-1}.$$

The map actually extends to a map from $\mathbf{A}^1$ into $M_{\mathrm{DR}}$ for the other coordinate chart $\mathbf{A}^1$ providing a neighborhood at $\infty$ in $\mathbf{P}^1$. In conclusion, the family of connections $\{(E, \nabla + t\theta)\}$ corresponds to a morphism

$$\mathbf{P}^1 \longrightarrow \overline{M}_{\mathrm{DR}}$$

sending $t \in \mathbf{A}^1$ into $M_{\mathrm{DR}}$, sending the point $t = \infty$ to a general point in the divisor $P_{\mathrm{DR}}$, and the curve is transverse to the divisor at that point. This type of curve was called a *pencil of connections* by Losev and Manin [**31**].

We will look only at curves of the above form. It should be possible to obtain similar results for other curves cutting $P_{\mathrm{DR}}$ transversally at a general point, but that is left as a problem for future study.

We will investigate the asymptotic behavior of the monodromy representations of the connections $(E, \nabla + t\theta)$ as $t \to \infty$. Recall that the *Betti moduli space* $M_B$ is the moduli space for representations of $\pi_1(X)$ up to conjugation, and we have an analytic isomorphism $M_{\mathrm{DR}}^{\mathrm{an}} \cong M_B^{\mathrm{an}}$ sending a connection to its monodromy representation. We will look at the asymptotics of the resulting analytic curve $\mathbf{A}^1 \to M_B$.

In order to set things up it will be useful to fix a basepoint $p \in X$ and a trivialization $\tau : E_p \cong \mathbb{C}^r$. Then for any $\gamma \in \pi_1(X, x)$ we obtain the monodromy matrix

$$\rho(E, \nabla + t\theta, \tau, \gamma) \in GL(r, \mathbb{C}).$$

Of course the monodromy matrices don't directly give functions on the moduli space $M_B$ of representations, because they depend on the choice of trivialization $\tau$. However, one has the Procesi coordinates (see Culler and Shalen [**12**] and Procesi [**33**]) which are certain polynomials in the monodromy matrices (for several $\gamma$ at once) which are invariant under change of trivialization and give an embedding of the Betti moduli space $M_B$ into an affine space. We will look at the asymptotic behavior of the monodromy matrices, but the resulting asymptotic information will also hold for any polynomials (see Corollary 14.2), and in particular for the Procesi coordinates. This will give asymptotic information about the image curve in $M_B$.

Notationally it is easier to start right out considering the transport matrices between points $p$ and $q$. In any case, the functions we shall consider, be they the matrix coefficients of the monodromy $\rho$ or some other polynomials in these or the transport matrices, will be entire functions $m(t)$ on the complex line $t \in \mathbb{C}$. We will be looking to characterize their asymptotic properties.

The method we will use is the same as the method already used in [**36**] to treat exactly this question, for a more special class of curves going to infinity in $M_{\mathrm{DR}}$. In that book was treated the case of families of connections $(E, \nabla + t\theta)$ where

$$E = \mathcal{O}_X^r, \quad \nabla = d + B, \quad \theta = A$$

with $A$ and $B$ being $r \times r$ matrices of one-forms on $X$ such that $A$ is diagonal and $B$ contains only zeros on the diagonal. In [**36**], a fairly precise description of the asymptotic behavior of the monodromy was obtained. It was also indicated how one should be able to reduce to this case in general; we shall explain that below. The only problem is that in the course of this reduction, one obtains the special situation but with $B$ being a matrix of one-forms which has some poles on $X$. In this case the exact method used in [**36**] breaks down.

The purpose of the present paper is to try to remedy this situation as far as possible. We change very slightly the method (essentially by taking the more canonical gradient flows of the functions $\Re g_{ij}$ rather than the flows defined in Chapter 3 of [**36**], and also stopping the flows before arriving at the poles of $B$). However, we don't obtain the full results of [**36**], namely we can show an analytic continuation result for the Laplace transform of $m(t)$ (this Laplace transform is explained in more detail below), however we don't get good bounds or information about the singularities of the Laplace transform other than that they are locally finite sets of points. In particular we obtain information about the growth rate of $m(t)$ but not asymptotic expansions.

Even in order to obtain the analytic continuation, a much more detailed examination of the dynamics generated by the general method of [**36**] is necessary. This is the main body of the present paper (see Theorem 12.5). For the remainder of the technique we mostly refer to [**36**].

Thus while we treat a much more general type of curve going to infinity than was treated in [**36**], we obtain a weaker set of results for these curves. This leaves open the difficult question of what kinds of singularities the Laplace transforms have, and thus what type of asymptotic expansion we can get for $m(t)$.

## 4. Genericity results for the spectral data

Before beginning to look more closely at the monodromy representations, we will consider some properties of general points $(E, \theta)$ on $P_{\mathrm{DR}}$, best expressed in terms of the *spectral curve* [**26, 27, 19, 30, 14, 34**].

Suppose $(E, \theta)$ is a Higgs bundle. Suppose $P \in X$ and $v \in T_P X$; then we obtain the fiber $E_P$ which is a vector space of rank $r$, with an endomorphism $\theta_P(v) \in \mathrm{End}(E_P)$. We say that $P$ is *singular* if $\theta_P(v)$ has an eigenvalue (*i.e.* zero of the characteristic polynomial) of multiplicity $\geqslant 2$. It is more natural to look at the *eigenforms* of $\theta$ obtained by dividing out the vector $v$. The eigenforms are elements of the cotangent space $T_P^* X = (\Omega_X^1)_P$.

We say that a singular point $P$ is *generic* if there is exactly one eigenform of multiplicity $\geqslant 2$; if it has multiplicity exactly 2; and if the two eigenforms $\alpha^{\pm}$ of $\theta$ which come together at $P$, may be expressed in a neighborhood with coordinate $z$ as

$$\alpha^{\pm} = c dz \pm a z^{1/2} dz + \cdots .$$

The condition that all singular points are generic is a Zariski open condition on the moduli space of Higgs bundles.

Suppose $P$ is a generic singular point. The eigenforms give a set of $r-1$ distinct elements of $T_P^* X$, consisting of the values of the multiplicity-one eigenvalues of $\theta$ at $P$, plus the leading term $c dz$ for the pair $\alpha^{\pm}$. Call this set $EF_P$. We say that $P$ is *non-parallel* if $EF_P$, viewed as a subset of the real two-dimensional space $T_P^* X$, doesn't have any colinear triples, nor any quadruples of points defining two parallel lines.

In terms of a coordinate $z$ at $P$ we can write the elements of $EF_P$ as

$$\alpha_i(P) = a_i dz$$

with $a_i$ being distinct complex numbers, and say $a_1 = c$ in the previous formulation. Then $P$ is non-parallel if and only if the set of $a_i \in \mathbb{C} \cong \mathbb{R}^2$ doesn't have any colinear triples or parallel quadruples. In turn this is equivalent to saying that the angular coordinates of the complex numbers $a_i - a_j$ are distinct.

***Lemma 4.1***. — *The set of Higgs bundles $(E, \theta)$ such that the singularities are generic and satisfy the non-parallel condition, is a dense real Zariski-open subset of the moduli space.*

*Proof*. — The condition of being non-parallel is a real Zariski open condition. In particular, the condition that all singular points be generic and non-colinear, holds in the complement of a closed real algebraic subset of the moduli space. Therefore, if there is one such point then the set of such points is a dense real Zariski open subset.

To show that there is one point $(E, \theta)$ such that all of the singular points are generic and non-parallel, we can restrict to the case where $E = \mathcal{O}^{\oplus r}$ is a trivial bundle. In this case, $\theta$ corresponds to a matrix of holomorphic one-forms on $X$. We will consider a matrix of the form $A + \lambda B$ with $A$ diagonal having entries $\alpha_i$, and $B$ is off-diagonal with $\lambda$ small. The singular points are perturbations of the points where $\alpha_i(P) = \alpha_j(P)$. A simple calculation with a $2 \times 2$ matrix shows that the singularities are generic in this case. In order to obtain the non-colinear condition, it suffices to have that for a point $P$ where $\alpha_i(P) = \alpha_j(P)$, the subset of $r - 1$ values of all the $\alpha_k(P)$ is non-parallel.

For a general choice of the $\alpha_k$, this is the case. Suppose we are at a point $P$ where $\alpha_1(P) = \alpha_2(P)$ for example. Then moving the remaining $\alpha_k$ for $k \geqslant 3$ shows that the remaining points are general with respect to the first one. A set of $r - 1$ points such that the last $r - 2$ are general with respect to the first one (whatever it is), satisfies the non-parallel condition. □

***Lemma 4.2***. — *If $(E, \theta)$ is generic in the sense of the previous lemma, then the spectral curve $V$ is actually an irreducible smooth curve sitting in the cotangent bundle $T^*X$. There is a line bundle $L$ on $V$ such that $E \cong \pi_*(L)$ and $\theta$ is given by multiplication by the tautological one-form over $V$.*

*Proof*. — The genericity condition on the way the eigenforms come together at any point where the multiplicity is $\geqslant 2$, guarantees that at any point where the projection $\pi : V \to X$ is not locally etale, the curve $V$ is a smooth ramified covering of order 2 in the usual standard form. This shows that $V$ is smooth. It is irreducible, because this is so for at least some points (for example the deformations used in the previous proof) and Zariski's connectedness implies that in a connected family of smooth projective curves if one is irreducible then all are. For connectedness of the family we use the irreducibility of the moduli space of Higgs bundles *cf.* [**39**]. The last statement is standard in theory of spectral curves [**26, 27, 19, 30**]. □

***Remark***. — Once $p$ and $q$ are fixed, then for general $\theta$ the endpoints $p, q$ will not be contained in the set $\mathcal{R}$ of turning points.

## 5. Pullback to a ramified covering and gauge transformations

Fix a general Higgs bundle $(E, \theta)$ on $X$. By taking a Galois completion of the spectral curve of $\theta$ and Galois-completing a further two-fold ramified covering if necessary, we can obtain a ramified Galois covering

$$\varphi : Y \longrightarrow X$$

such that the pullback Higgs field $\varphi^*\theta$ has a full set of eigen-one-forms defined on $Y$; and such that the ramification powers over singular points of $\theta$ are divisible by 4.

We have one-forms $\alpha_1, \ldots, \alpha_r$ and line sub-bundles

$$L_1, \ldots, L_r \subset \varphi^* E$$

such that at a general point of $Y$ we have

$$\psi : \varphi^* E \cong L_1 \oplus \cdots \oplus L_r$$

with $\varphi^*\theta$ represented by the diagonal matrix with entries $\alpha_i$. Note that $\varphi^*\theta$ preserves $L_i$ (acting there by multiplication by $\alpha_i$) globally on $Y$. However, the isomorphism $\psi$ will only be meromorphic, and also the $L_i$ are of degree $< 0$. Choose modifications $L_i'$ of $L_i$ (see Lemma 5.1 below, also the modifications are made only over singular points) such that $L_i'$ is of degree zero, and set

$$E' := L_1' \oplus \cdots \oplus L_r'.$$

Let $\theta'$ denote the diagonal Higgs field with entries $\alpha_i$ on $E'$. Let $\nabla'$ be a diagonal flat connection on $E'$. We have a meromorphic map

$$\psi : E \longrightarrow E',$$

and

$$\psi \circ \varphi^*\theta \circ \psi^{-1} = \theta'.$$

Suppose now that $\nabla$ was a connection on $E$, giving a connection $\varphi^*\nabla$ on $\varphi^*E$. We can write

$$\psi \circ \varphi^*\nabla \circ \psi^{-1} = \nabla' + \beta$$

with $\beta$ a meromorphic section of $\mathrm{End}(E') \otimes_{\mathcal{O}} \Omega^1_Y$.

A transport matrix of $(E, \nabla + t\theta)$ may be recovered as a transport matrix for the pullback bundle on $Y$. Indeed if $\gamma$ is a path in $X$ going from $p$ to $q$ then it lifts to a path going from a lift $p'$ of $p$ to a lift $q'$ of $q$. Thus it suffices to look at the problem of the asymptotics for transport matrices for the pullback family

$$\{(\varphi^* E, \varphi^* \nabla + t\varphi^* \theta)\}.$$

We may assume that $p$ and $q$ are not singular points of $\theta$, so $p'$ and $q'$ will not be singular points of $\varphi^*\theta$. Then the transport matrices for this family are conjugate (by a conjugation which is constant in $t$) to the transport matrices for the family

$$\{(E', \nabla' + \beta + t\theta')\}.$$

**Lemma 5.1**. — *In the above situation, the modifications $L_i'$ of $L_i$ may be chosen so that the diagonal entries of $\beta$ are holomorphic. Furthermore the poles of the remaining entries of $\beta$ are restricted to the points lying over singular points in $X$ for the original Higgs field $\theta$ (the "turning points").*

*Proof*. — Note first that, by definition, away from the singular points of $\theta$ the eigen-one-forms are distinct so the eigenvectors form a basis for $E$, in other words the direct sum decomposition $\psi$ is an isomorphism at these points. Thus $\psi$ only has poles over the singular points of $\theta$ (hence the same for $\beta$).

We will describe a choice of $L_i'$ locally at a singular point.

Look now in a neighborhood of a point $P' \in Y$, lying over a singular point $P \in X$. Let $z'$ denote a local coordinate at $P'$ on $Y$, with $z$ a local coordinate at $P$ on $X$ and with

$$z = (z')^m.$$

Our assumption on $Y$ was that $m$ is divisible by 4. In fact we may as well assume that $m = 4$ since raising to a further power doesn't modify the argument. Thus we can write

$$z' = z^{1/4}.$$

There are two eigenforms of $\theta$ which come together at $P$. Suppose that their lifts are $\alpha_1$ and $\alpha_2$. Then near $P'$ we can write

$$\varphi^* E = U \oplus L_3 \oplus \cdots \oplus L_r$$

where $U$ is the rank two subbundle of $\varphi^* E$ corresponding to eigenvalues $\alpha_1$ and $\alpha_2$. The direct sum decomposition is holomorphic at $P'$ because the other eigenvalues of $\theta$ were distinct at $P$ and different from the two singular ones (of course after the pullback all of the eigenforms have a value of zero at $P'$ but the decomposition still holds nonetheless).

Now we use a little bit more detailed information about spectral curves for Higgs bundles: the general $(E, \theta)$ is obtained as the direct image of a line bundle on the spectral curve (Lemma 4.2). This means that locally near $P$ there is a two-fold branched covering with coordinate $u = z^{1/2}$ such that the rank 2 subbundle of $E$ corresponding to the singular values looks like the direct image of the trivial bundle on the covering, and the $2 \times 2$ piece of $\theta$ looks like the action of multiplication by

$udz = 2u^2du$. The direct image, considered as a module over the series in $z$, is just the series in $u$. One can obtain a basis by looking at the odd and even powers of $u$: the basis vectors are $e_1 = 1$ and $e_2 = u$. In these terms we have

$$\theta e_1 = e_2 dz; \quad \theta e_2 = z e_1 dz.$$

Thus the $2 \times 2$ singular part of $\theta$ has matrix

$$\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} dz.$$

Pulling back now to the covering $Y$ which is locally 4-fold, we have a basis for $U$ in which

$$\varphi^*\theta|_U = \begin{pmatrix} 0 & (z')^7 \\ (z')^3 & 0 \end{pmatrix} dz'.$$

On the other hand, since up until now our decomposition is holomorphic, the pullback connection $\varphi^*\nabla$ may be written (in terms of our basis for $U$ plus trivializations of the $L_i$ for $i \geqslant 3$) as $d + B'$ where $B'$ is a holomorphic matrix of one-forms. Since the basis can be pulled back from downstairs, we can even say that $B'$ consists of one-forms pulled back from $X$.

To choose the modifications $L'_i$ (for $i = 1, 2$) locally at $P'$ we have to find a meromorphic change of basis for the bundle $U$, which diagonalizes $\varphi^*\theta|_U$. The eigenforms of the matrix are $\pm(z')^5 dz'$ and we can choose eigenvectors

$$e_\pm := \begin{pmatrix} z' \\ \pm(z')^{-1} \end{pmatrix}.$$

Note by calculation that

$$(\varphi^*\theta|_U)e_\pm = (\pm(z')^5 dz')e_\pm.$$

Choose the line bundles $L'_1$ and $L'_2$ to be spanned by the meromorphic sections $e_+$ and $e_-$ of $U$. These are indeed eigen-subbundles for $\varphi^*\theta$. We just have to calculate the connection $\varphi^*\nabla$ on the bundle $U' = L'_1 \oplus L'_2$. Which is the same as the modification of $U$ given by the meromorphic basis $z'e_1, (z')^{-1}e_2$.

Note first that the matrix $B'$ of one-forms pulled back from $X$ consists of one-forms which have zeros at least like $(z')^3 dz'$. Thus $B'$ transported to $U'$ is still a matrix of holomorphic one-forms so it doesn't affect our lemma. In particular we just have to consider the transport to $U'$ of the connection $d_U$ constant with respect to the basis $(e_1, e_2)$ on the bundle $U$.

Calculate

$$\begin{aligned} d_U(a_+e_+ + a_-e_-) &= d_U \begin{pmatrix} (a_+ + a_-)z' \\ (a_+ - a_-)(z')^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (da_+ + da_-)z' \\ (da_+ - da_-)(z')^{-1} \end{pmatrix} + \begin{pmatrix} (a_+ + a_-)(d\log z')z' \\ -(a_+ - a_-)(d\log z')(z')^{-1} \end{pmatrix} \end{aligned}$$

and with the notation $d_{U'}$ for the constant connection on the bundle $U'$ with respect to its basis $e_{\pm}$, this is equal to

$$= d_{U'}(a_+e_+ + a_-e_-) + a_+(d\log z')e_- + a_-(d\log z')e_+.$$

We conclude that the connection matrix $\beta$ is, up to a holomorphic piece, just the $2\times 2$ matrix

$$\begin{pmatrix} 0 & (z')^{-1} \\ (z')^{-1} & 0 \end{pmatrix} dz'.$$

In particular the diagonal terms of $\beta$ are holomorphic, as desired for the lemma.

These local modifications piece together to give global modifications $L_i'$ of the $L_i$. We have to show that the $L_i'$ are of degree zero.

In general, given a meromorphic connection on a bundle which is a direct sum of line bundles, we can extract its "diagonal" part, which in terms of a local framing compatible with the direct sum is just the connection given by the diagonal entries of the original connection matrix. Denote this operation by $(\ )_{\mathrm{diag}}$. Note that for any diagonal connection $\nabla'$ and meromorphic endomorphism-valued one-form $\beta$, the diagonal connection is given by $(\nabla' + \beta)_{\mathrm{diag}} = \nabla' + \beta_{\mathrm{diag}}$ where $\beta_{\mathrm{diag}}$ is the matrix of diagonal entries of $\beta$.

Setting $E' := \bigoplus L_i'$ we have a meromorphic map $\psi : E \to E'$. We obtain a meromorphic connection $\psi \circ \varphi^*\nabla \circ \psi^{-1}$ on $E'$, and by the above choice of $L_i'$ the associated diagonal connection is holomorphic at the singularities. On the other hand, $\psi\circ\varphi^*\nabla\circ\psi^{-1}$ is holomorphic away from the singularities, so its diagonal part is holomorphic there too. Therefore the global diagonal connection $(\psi\circ\varphi^*\nabla\circ\psi^{-1})_{\mathrm{diag}}$ on $\bigoplus L_i'$ is holomorphic. This proves that the $L_i'$ are of degree zero. In particular, our choice of modification is allowable for the argument given at the start of the present section. This proves the lemma. □

***Remarks***

(i) The above proof gives further information: the only terms with poles in the matrix $\beta$ are the off-diagonal terms corresponding to the two eigenvalues which came together originally downstairs in $X$; and these terms have exactly logarithmic (*i.e.* first-order) poles with residue 1. This information might be useful in trying to improve the current results in order to obtain precise expansions at the singularities of the Laplace transform of the monodromy.

(ii) This gauge transformation is probably not new, but I don't currently have a good reference. It looks related to [**29**], [**45**] and [**34**], and indeed may go back to [**14, 43**].

(iii) The fact that we had to go to a covering whose ramification power is divisible by 4 rather than just 2 (as would be sufficient for diagonalizing $\theta$) is somewhat mysterious; it probably indicates that we (or some of us at least) don't fully understand what is going on here.

Let $\beta^{\text{diag}}$ denote the matrix of diagonal entries of $\beta$. Let $Z = \widetilde{Y}$ be the universal covering. Over $Z$ we can use the diagonal connection $\nabla' + \beta^{\text{diag}}$ to trivialize

$$E'|_Z \cong \mathcal{O}_Z^r.$$

With respect to this trivialization , our family now has the form of a family of connections

$$\{(\mathcal{O}_Z^r, d + B + tA)\}$$

where $A$ (corresponding to the pullback of $\theta'$ to $Z$) is the diagonal matrix whose entries are the pullbacks of the $\alpha_i$; and where $B$ is a matrix whose diagonal entries are zero, and whose off-diagonal entries are meromorphic with poles at the points lying over singular points for $\theta$.

We can now apply the method developped in [**36**] to this family of connections. Note that it is important to know that the diagonal entries of $A$ come from forms on the compact Riemann surface $Y$; on the other hand the fact that $B$ is only defined over the universal covering $Z$ is not a problem. The next two sections will constitute a brief discussion of how the method of [**36**] works; however the reader is refered back there for the full details.

## 6. Laplace transform of the monodromy operators

We now look at a family of connections of the form $d+B+tA$ on the trivial bundle $\mathcal{O}^r$ on the universal covering $Z$ of the ramified cover $Y$, where $A$ is a diagonal matrix with one-forms $\alpha_i$ along the diagonal, and $B$ is a matrix of meromorphic one-forms with zeros on the diagonal. We assume that the poles of $B$ are at points $P \in \mathcal{R}$ coming from the original singular points of the Higgs field $\theta$ on $X$. We make no assumption about the order of poles, in spite of the additional information given by Remark (i) after the proof of Lemma 5.1 above.

Assume that $p$ and $q$ are two points in $Z$, not on the singular points. Choose a path $\gamma$ from $p$ to $q$ not passing through the singular points. We obtain the *transport matrix* $m(t)$ for continuing solutions of the ordinary differential equation $(d + B + tA)f = 0$ from $p$ to $q$ along the path $\gamma$. Note that $m(t)$ is a holomorphic $r \times r$-matrix-valued function defined for all $t \in \mathbb{C}$.

Denote by $Z^*$ (resp. $Z^\varepsilon$) the complement of the inverse image of $\mathcal{R}$ (resp. the complement of the union of open discs of radius $\varepsilon$ around points in the inverse image of $\mathcal{R}$). The poles of $B$ force us to work in $Z^*$ rather than $Z$, and in the course of the argument an $\varepsilon$ will be chosen so that we really work in $Z^\varepsilon$. Actually it turns out that the fact of staying inside these regions will be guaranteed by our choice of vector fields, so we don't need to worry about any modification of the procedure of [**36**] because of this difference.

Recall that after a gauge transformation and an expansion as a sum of iterated integrals, we obtain a formula for the transport matrix. One way of thinking of this formula is to look at the transport for the connection $d + sB + tA$ and expand in a Taylor series in $s$ about the point $s = 0$, then evaluate at $s = 1$. The terms in the expansion are the higher derivatives in $s$, at $s = 0$, which are functions of $t$. A concrete derivation of the formula is given in [**36**]. It says

$$m(t) = \sum_I \int_{\eta_I} b_I e^{t g_I}$$

where:

– the sum is taken over multi-indices of the form $I = (i_0, i_1, \dots, i_k)$ where we note $k = |I|$;

– for a multi-index $I$ we denote by $Z_I^*$ the product of $k = |I|$ factors $Z^* \times \cdots \times Z^*$;

– in $Z_I^*$ we have a cycle

$$\eta_I := \{(\gamma(t_1), \dots, \gamma(t_k))\}$$

for $0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant 1$ where $\gamma$ is viewed as a path parametrized by $t \in [0, 1]$;

– the cycle $\eta_I$ should be thought of as representing a class in a relative homology group of $Z_I^*$ relative to the simplex formed by points where $z_i = z_{i+1}$ or at the ends $z_1 = p$ or $z_k = q$;

– the matrix $B$ leads to a (now meromorphic) matrix-valued $k$-form $b_I$ on $Z_I^*$ defined as follows: if the entries of $B$ are denoted $b_{ij}(z)dz$ then

$$b_I = b_{i_k i_{k-1}}(z_k)dz_k \wedge \cdots \wedge b_{i_1 i_0}(z_1)dz_1 \boldsymbol{e}_{i_k i_0}$$

where $\boldsymbol{e}_{i_k i_0}$ denotes the elementary matrix with zeros everywhere except for a 1 in the $i_k i_0$ place;

– and finally $g_I$ is a holomorphic function $Z_I^* \to \mathbb{C}$ defined by integrating the one-forms $\alpha_i$ as follows:

$$g_I(z_1, \dots, z_k) = \int_p^{z_1} \alpha_{i_0} + \cdots + \int_{z_k}^q \alpha_{i_k}.$$

The terms in the above expression correspond to what Ecalle calls the *elementary monomials* som, see his article in [**7**].

The fact that $b_I$ is meromorphic rather than holomorphic is the only difference between our present situation and the situation of [**36**]. Note that because our path $\gamma$ misses the singular points and thus the poles of $B$, the cycle $\eta_I$ is supported away from the poles of $b_I$. We will be applying essentially the same technique of moving the cycle of integration $\eta$, but we need to do additional work to make sure it stays away from the poles of $b_I$.

It is useful to have the formula

$$g_I(z_1, \dots, z_k) = g_{i_0 i_1}(z_1) + \cdots + g_{i_{k-1} i_k}(z_k) + \int_p^q \alpha_{i_k},$$

where

$$g_{ij}(z) := \int_p^z \alpha_i - \alpha_j.$$

Our formula for $m$ gives a preliminary bound of the form

$$|m(t)| \leqslant Ce^{a|t|}.$$

Indeed, along the path $\gamma$ the one-forms $b_{ij}$ are bounded, so

$$|b_I| \leqslant C^k$$

on $\eta_I$; also we have a bound $|g_I(z)| \leqslant a$ for $z \in \eta_I$, uniform in $I$; and finally the cycle of integration $\eta_I$ has size $(k!)^{-1}$. Putting these together gives the bound for $m(t)$ (and, incidentally, shows why the formula for $m$ converged in the first place).

Recall now that the *Laplace transform* of a function $m(t)$ which satisfies a bound such as the above, is by definition the integral

$$f(\zeta) := \int_0^\infty m(t)e^{-\zeta t}dt$$

where $\zeta \in \mathbb{C}$ with $|\zeta| > a$ and the path of integration is taken in a suitably chosen direction so that the integrand is rapidly decreasing at infinity. In our case since $m(t)$ is a matrix, $f(\zeta)$ is also a matrix. We can recover $m(t)$ by the inverse transform

$$m(t) = \frac{1}{2\pi i}\oint f(\zeta)e^{\zeta t}d\zeta$$

with the integral being taken over a loop going around once counterclockwise in the region $|\zeta| > a$.

The singularities of $f(\zeta)$ are directly related to the asymptotic behavior of $m(t)$. This is a classical subject which we discuss a little bit more in § 14. One can note for example that by the inverse transform, there exist functions $m(t)$ satisfying the preliminary bound $|m(t)| \leqslant Ce^{a|t|}$ but such that the Laplace transforms $f(\zeta)$ have arbitrarily bad singularities in the region $|\zeta| \leqslant a$. Thus getting any nontrivial restrictions on the singularities of $f$ amounts to a restriction on which types of functions $m(t)$ can occur.

In our case, the expansion formula for $m(t)$ leads to a similar formula for the Laplace tranform, which we state as a lemma. Define the image support of a collection $\eta = \{\eta_I\}$ by the collection of functions $g = \{g_I\}$ to be the closure of the union of the images of the component pieces:

$$g(\eta) := \overline{\bigcup_I g_I(|\eta_I|)} \subset \mathbb{C},$$

where $|\eta_I| \subset Z_I$ is the support of the chain $\eta_I$. As in [**36**], p. 42, the support $|\eta_I|$ is defined as the smallest closed subset of $Z_I$ such that the integral of $\eta_I$ against any form vanishing on $|\eta_I|$ is zero.

***Lemma 6.1***. — *With the functions $g_I$, the forms $b_I$, and the chains $\eta_I$ intervening above, for any $\zeta$ in the complement of the region $g(\eta)$ the formula*

$$f(\zeta) = \sum_I \int_{\eta_I} \frac{b_I}{g_I - \zeta}$$

*converges, and gives an analytic continuation of the Laplace transform in the (unique) unbounded connected component of the complement of $g(\eta)$.*

*Proof*. — The convergence comes from the same bounds on $b_I$ and the size of $\eta_I$ which allowed us to bound $m$. The fact that this formula gives the Laplace transform is an exercise in complex path integrals. □

The terms in this expansion correspond to Ecalle's elementary monomials "soc" in [**7**].

A first approach would be to try to move the path $\gamma$ so as to move the union of images $g(\eta)$ and analytically continue $f$ to a larger region. This works quite well for rank 2, where one can get an analytic continuation to a large region meeting the singularities [**14**]. In higher rank, the $3 \times 3$ example at the end of [**36**] shows that this approach cannot be optimal. In fact, we should instead move each cycle of integration $\eta_I$ individually. Unfortunately this has to be done with great care in order to maintain control of the sizes of the individual terms so that the infinite sum over $I$ still converges.

Now we get to the main definition. It is a weak version of resurgence, see [**15, 9**].

***Definition 6.2***. — A function such as $f(\zeta)$ defined on $|\zeta| > a$ is said to have an *analytic continuation with locally finite branching* if for every $M > 0$ there is a finite set of points $S_M \subset \mathbb{C}$ such that if $\sigma$ is any piecewise linear path in $\mathbb{C} - S_M$ starting at a point where $|\zeta| > a$ and such that the length of $\sigma$ is $\leqslant M$, then $f(\zeta)$ can be analytically continued along $\sigma$.

And the statement of the main theorem.

***Theorem 6.3***. — *Suppose $m(t)$ is the transport matrix from $p$ to $q$ for a family of connections on the trivial bundle $\mathcal{O}_Z^r$ of the form $\{d + B + tA\}$. Suppose that $A$ is diagonal with one-forms $\alpha_i$, coming from the pullback of a general Higgs field $\theta$ over the original curve $X$, and suppose that $B$ is a meromorphic matrix of one-forms with poles only at points lying over the singular points of $\theta$. Let $f(\zeta)$ denote the Laplace transform of $m(t)$. Then $f$ has an analytic continuation with locally finite branching.*

Most of the remainder of these notes is devoted to explaining the proof.

## 7. Analytic continuation of the Laplace transform

We now recall the basic method of [**36**] for moving the cycles $\eta_I$ to obtain an analytic continuation of $f(\zeta)$. We refer there for most details and concentrate here just on stating what the end result is. Still we need a minimal amount of notation. Before starting we should refer to [**17**] (and the references therein) for an extensive discussion of this process for each individual integral in the sum, including numerical results on how the singularities of the analytic continuations determine the asymptotics of the pre-transformed integrals.

We work with *pro-chains* which are formal sums of the form $\eta = \sum_I \eta_I$ of chains on the $Z_I^*$. Here, technically speaking *chains* are the "singular-de Rham chains" of [**36**] p. 41, made up of direct images of differential forms under maps from simplices into the $Z_I^*$. These arise because of the use of cutoff functions in the argument.

We have a boundary operator denoted $\partial + A$ where $\partial$ is the usual boundary operator on each $\eta_I$ individually, and $A$ (different from the matrix of one-forms considered above) is a signed sum of face maps corresponding to the inclusions $Z_{I'}^* \to Z_I^*$ obtained when some $z_i = z_{i+1}$. Our original pro-chain of integration in the integral expansion satisfies $(\partial + A)\eta = 0$. We can write the expansion formula of Lemma 6.1 as an integral over the pro-chain $\eta = \sum_I \eta_I$,

$$f(\zeta) = \int_\eta \frac{b}{g - \zeta}$$

where $b$ is the collection of forms $b_I$ on $Z_I^*$ and $g$ is collection of functions $g_I$. Such a formula is of course subject to the condition that the infinite sum of integrals converges.

In a formal way (*i.e.* element-by-element in the infinite sums implicit in the above notation), if we add to $\eta$ a boundary term of the form $(\partial + A)\kappa$ then the integral doesn't change:

$$\int_{\eta + (\partial + A)\kappa} \frac{b}{g - \zeta} = \int_\eta \frac{b}{g - \zeta}.$$

This again is subject to the condition that the infinite sums on both sides converge absolutely and in fact that the individual terms in the rearrangement (*i.e.* separating $\partial$ and $A$) converge absolutely. Whenever we use this, we will be refering (perhaps without mentionning it further) to the work on convergence which was done in [**36**].

Our analytic continuation procedure rests upon consideration of the locations of the images by the function $g$, of the pro-chains of integration. Recall the notation

$$g(\eta) := \overline{\bigcup_I g_I(|\eta_I|)}$$

where $|\eta_I|$ is the support of the chain $\eta_I$.

If $f$ is defined by the right-hand integral over $\eta$ in a neighborhood of a point $\zeta_0$, meaning that the image $g(\eta)$ misses an open neighborhood of $\zeta_0$, and if the image

$g(\eta + (\partial + A)\kappa)$ misses an entire segment going from $\zeta_0$ to $\zeta_1$, then the integral over $\eta + (\partial + A)\kappa$ defines an analytic continuation of $f$ along the segment. The procedure can be repeated with $\eta$ replaced by $\eta + (\partial + A)\kappa$.

At this point we let our notation slide a little bit, and denote by $\eta$ any pro-chain which would be obtained from the original chain of integration by a sequence of modifications of the kind we are presently considering, such that the integral over $\eta$ serves to define an analytic continuation of $f(\zeta)$ to a neighborhood of a point $\zeta_0 \in \mathbb{C}$. The original pro-chain $\eta$ of Lemma 6.1 is the initial case. Our assumption on $\eta$ says among other things that the image $g(\eta)$ doesn't meet a disc around $\zeta_0$. Fix a line segment $S$ going from $\zeta_0$ to another point $\zeta_1$; we would like to continue $f$ in a neighborhood of $S$. By making a rotation in the complex plane (which can be seen as a rotation of the original Higgs field) we may without loss of generality assume that the segment $S$ is parallel to the real axis and the real part of $\zeta_1$ is smaller than the real part of $\zeta_0$. Let $u$ be a cut-off function for a neighborhood of $S$ and write

$$\eta = \eta' + \eta'', \quad \eta' = g^*(u) \cdot \eta.$$

We will apply the method of [**36**] to move the piece $\eta'$ (this piece corresponds to what was called $\eta$ in Chapter 4 of [**36**]).

The first step is to choose flows. This corresponds to Chapter 3 of [**36**]. In our case, we will use flows along vector fields $W_{ij}$ which are $\mathcal{C}^\infty$ multiples of the gradient vector fields of the real parts $\Re g_{ij}$. To link up with the terminology of [**36**], these vector fields determine flowing functions $f_{ij}(z,t)$ (for $z \in Z$ and $t \in \mathbb{R}^+$ taking values in $Z$) by the equations

$$\frac{\partial}{\partial t} f_{ij}(z,t) = W_{ij}(f_{ij}(z,t)), \quad f_{ij}(z,0) = z.$$

Note that this choice is considerably simpler than that of [**36**]. The choice of vector fields will be discussed in detail below, and will in particular be subject to the following constraints.

***Condition 7.1***

(i) the vector fields $W_{ij}$ are lifts to $Z$ of vector fields defined on the compact surface $Y$;

(ii) the differential $d\Re g_{ij}$ applied to $W_{ij}$ at any point, is a real number $\leqslant 0$;

(iii) there exists $\varepsilon$ such that the flows preserve $Z^\varepsilon$ *i.e.* the vector fields $W_{ij}$ are identically zero in the discs of radius $\varepsilon$ around the singular points; and

(iv) the $W_{ii}$ are identically zero.

The flows given by our vector fields lead to a number of operators $F$, $K$ and $H$ defined as in Chapters 4 and 5 of [**36**]. These give pro-chains

$$F\tau = \sum_{r,s} F(-KA)^r H(AK)^s \eta',$$
$$F\psi = \sum_{r} F(-KA)^r K(\partial + A)\eta',$$
$$FK\varphi = \sum_{r} FK(AK)^r \eta'.$$

The reader can get a fairly good idea of these definitions from our discussion of the points on $|F\tau|$ in § 8 below.

***Lemma 7.2***. — *With these notations, and assuming that the vector fields satisfy the constraints marked above, we can write*

$$\eta + (\partial + A)FK\varphi = \eta'' + F\tau - F\psi.$$

*On the right, the images $g(\eta'')$ and $g(F\psi)$ miss a neighborhood of the segment $S$.* Assuming *we can show that the image $g(F\tau)$ also misses a neighborhood of the segment $S$, then*

$$f(\zeta) = \int_{\eta + (\partial + A)FK\varphi} \frac{b}{g - \zeta}$$

*gives an analytic continuation of $f$ from $\zeta_0$ to $\zeta_1$ along the segment $S$.*

*Proof*. — The operator $K$ corresponds to applying the flows defined by $W_{ij}$ in the various coordinates. This has the effect of decreasing the real part $\Re g$. The fact that in our case we use flows along vector fields which are positive real multiples of $-\mathbf{grad}\, \Re g_{ij}$ (this is the second of the constraints on $W_{ij}$) implies that the flows strictly respect the imaginary part of $g$. This differs from the case of [**36**] and means we can avoid discussion of "angular sectors" such as on pages 52–53 there. Thus, in our case, when we apply a flow to a point, the new point has the same value of $\Im g$, and the real part $\Re g$ is decreased.

The operator $F$ is related to the use of buffers; we refer to [**36**] for that discussion and heretofore ignore it. The operator $A$ is the boundary operator discussed above; and the operator $H$ is just the result of doing the flows $K$ after unit time. In particular, $A$ doesn't affect the value of $g$. And $H$ decreases $\Re g$ while fixing $\Im g$ just as $K$ did (this point will perhaps become clearer with the explicit description of points in the supports of $F\tau$ and $FK\varphi$ in the next section).

The proof of the first formula is the same as in [**36**] Lemma 4.4, and we refer there for it.

To show that the supports of $g(\eta'')$ and $g(F\psi)$ miss a neighborhood of $S$, it is useful to be a little bit more precise about the neighborhoods which are involved. Let $N_1$ be the support of $u$, which is a neigborhood of $S$ (we assume it is convex), and let $N_2$ be the support of $du$ which is an oval going around $S$ but not touching it. Let $N_3$ be the neighborhood of $S$ where $u$ is identically 1. Let $D$ be a disc around $\zeta_0$, such that $g(\eta)$ misses $D$, and which we may assume has radius bigger than the width of $N_1$. Then

$$g(\eta') \subset N_1 - (N_1 \cap D),$$
$$g(\eta'') \subset \mathbb{C} - (N_3 \cup D),$$

and $(\partial + A)\eta' = -(\partial + A)\eta''$ with

$$g((\partial + A)\eta') \subset N_2 - (N_2 \cap D).$$

In particular the support of $g(\eta'')$ misses the neighborhood $N_3$ of $S$. Also, given that the boundary term $(\partial + A)\eta'$ is supported in the $U$-shaped region $N_2$, the effect of our operators on $\Re g$ and $\Im g$ described above implies that $g(F\psi)$ is supported away from $N_3$. This completes the proof of the second statement of the lemma.

For the last statement, assume that we have chosen things such that the support of $g(F\tau)$ also misses $S$. This is certainly what we hope, because of the inclusion of the operator $H$ applying all the flows for unit time. The only possible problem would be if we get too close to singular points; that is the technical difficulty which is to be treated in the remainder of the paper. For now, we assume that this is done.

Formally speaking, the first equation of the lemma means that

$$\int_\eta \frac{b}{g-\zeta} = \int_{\eta''+F\tau-F\psi} \frac{b}{g-\zeta}.$$

By our starting assumption $f(\zeta)$ is defined by the integral on the left, in a neighborhood of $\zeta_0$. On the other hand, the integral on the right defines an analytic continuation along the segment $S$.

An important part of justifying the argument of the preceding paragraph (and indeed, of showing that the integral on the right is convergent) is to bound the sizes and numbers of all the chains appearing here. This was done in [**36**].

The only difference in our present case is the poles in the integrand $b$. However, thanks to the third constraint on the vector fields $W_{ij}$, everything takes place in $Z^\varepsilon_I := Z^\varepsilon \times \cdots \times Z^\varepsilon$, and on $Z^\varepsilon$ there is a uniform bound on the size of $b_{ij}$. Also, everything takes place inside a relatively compact subset of $Z$, see § 9. Thus the integrand in the multivariable integral is bounded by

$$\sup_{Z^\varepsilon_I} |b| \leqslant C^k$$

for $k = |I|$. With this information the remainder of the argument of [**36**] works identically the same way (it is too lengthy to recall here). This justifies the formal argument of two paragraphs ago and completes the proof of the lemma. □

***Remark***. — It is clear from the end of the proof that the bounds depend on $\varepsilon$, which in turn will depend on how close we want to get to a singularity. This is the root of why we don't get any good information about the order of growth of the Laplace transform at its singularities.

## 8. Description of cells using trees

As was used in [**36**], the chains defined above can be expressed as sums of cells. We are most interested in the chain $F\tau$ although what we say also applies to the other ones such as $FK\varphi$. These chains are unions of cells which have the form of a family of cubes parametrized by points in one of the original cells $\eta'_I$. We call these things just cubes. In the cubes which occur the points are parametrized by "trees" furnished with lots of additional information[(1)]. We make this precise as follows: a *furnished tree* is:

– a binary planar tree $T$ (not necessarily connected) sandwiched between a top horizontal line and a bottom horizontal line;

– with leftmost and rightmost vertical strands whose edges are called the *side edges*;

– for each top vertex of the tree (*i.e.* where an edge meets the top horizontal line) we should specify a point $z \in Z^*$ (the point corresponding to the left resp. right side edge is $p$ resp. $q$);

– for each region in the complement of the tree between the top and bottom horizontal lines and between the side edges we should specify an index, so that each (non-side) edge of the tree is provided with left and right indices which will be denoted $i_e$ and $j_e$ below; and

– each edge $e$ is assigned a "length" $s(e) \in [0, 1]$.

Suppose $T$ is a furnished tree. By looking at the indices assigned to the regions meeting the top and bottom horizontal lines we obtain multi-indices $I^{\rm top}$ and $I^{\rm bot}$, so the collection of points $(z_1, \ldots, z_k)$ attached to the top vertices gives a point $z^{\rm top} \in Z^*_{I^{\rm top}}$.

We can now explain how a furnished tree leads to a point $z^{\rm bot} \in Z_{I^{\rm bot}(T)}$. This depends on a choice of vector fields $W_{ij}$ for each pair of indices $i, j$, which we now assume as having been made. A *flowing map* $\Phi : T \to Z$ is a map from the topological realization of the tree, into $Z$, satisfying the following properties:

(i) if $v$ is a top vertex which is assigned a point $z$ in the information contained in $T$, then $\Phi(v) = z$;

(ii) the side edges are mapped by constant maps to the points $p$ or $q$ respectively; and

(iii) if $e$ is an edge with left and right indices $i_e$ and $j_e$ and with initial vertex $v$ and terminal vertex $v'$, then $\Phi(e)$ is the flow curve for flowing along the vector field

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(1) The occurence of trees here is certainly related to and probably the same as Ecalle's notions of *(co)mould (co)arborescent cf.* [**22**]. In another direction, John Conway pointed out at the time of [**36**] that cubes parametrized by trees in this way glue together into Stasheff polytopes. I didn't know what those were at the time, but retrospectively this still remains mysterious since we are dealing with representations of the fundamental group and it isn't clear what that has to do with homotopy-associativity. This is certainly a good subject for further thought.

$W_{i_e j_e}$ from $\Phi(v)$ to $\Phi(v')$, where the flow is done for time $s = s(e)$. This determines $\Phi(v')$ as a function of $\Phi(v)$ and the information in the tree. Thus by recursion we determine the $\Phi(v)$ for all vertices, as well as the paths $\Phi(e)$ for the edges $e$ (the map $\Phi$ on the edges is only well determined up to reparametrization because we don't fix a parametrization of the edges; the length $s$ is abstract, since it is convenient to picture even edges assigned $s = 0$ as being actual edges).

For a given choice of vector fields $W_{ij}$ and of information attached to the tree $T$, the flowing map exists and is unique. This determines a point given by the values $z$ at the bottom vertices,

$$z^{\mathrm{bot}}(W, T) \in Z_{I^{\mathrm{bot}}}.$$

Now go back to the situation of the previous section. Starting from a chain $\eta'$ we obtained a chain $F\tau$.

***Lemma 8.1***. — *The points in the support of $F\tau$ are described as the $z^{\mathrm{bot}}(W, T)$, where $W = \{W_{ij}\}$ is the collection of vector fields used to define the flows $K$ and $H$, and where $T$ is a furnished tree such that $z^{\mathrm{top}}(T)$ is in the support of $\eta'$ and satisfying the following auxiliary condition:*
($*$) *there exists (up to reparametrization of the planar embedding) a horizontal line which cuts the tree along a sequence of edges, such that all of these edges are assigned the fixed length value $s = 1$.*

*Proof*. — See [**36**], pages 54–55. The auxiliary condition comes from the term $H$ in the formula for $\tau$. □

***Remark***. — For the chain $FK\varphi$ the same statement holds except that the furnished trees $T$ might not necessarily satisfy the auxiliary condition.

We finish this section by pointing out the relationship between $g(z^{\mathrm{top}})$ and $g(z^{\mathrm{bot}})$. This is the key point in our discussion, because $z^{\mathrm{top}}$ is the input point coming from the chain $\eta'$ and $z^{\mathrm{bot}}$ is the output point which goes into the resulting chain $F\tau$. We want to prove that the real part of $g(z^{\mathrm{bot}})$ can be moved down past the end of the segment $S$.

***Lemma 8.2***. — *If $T$ is a furnished tree and $W$ a choice of vector fields, then*

$$g(z^{\mathrm{bot}}(T)) = g(z^{\mathrm{top}}(T)) + \sum_e \int_{\Phi(e)} dg_{i_e j_e},$$

*In particular if $W$ saitsfies Condition 7.1 then*

$$g(z^{\mathrm{bot}}(T)) - g(z^{\mathrm{top}}(T)) \in \mathbb{R}_{\leqslant 0}.$$

*Proof*. — If $e$ is an edge of $T$ and $s' \in [0, s(e)]$ then we can define the tree $T'$ obtained by *pruning* $T$ at $(e, s')$. This is obtained by cutting off everything below $e$ and sending the bottom vertex of $e$ to the line at the bottom. The indices associated to regions in the complement follow accordingly. Finally we set $s(e) := s'$ in the new tree $T'$.

Suppose for the same edge $e$ we also pick $s'' \in [s', s(e)]$. Then we obtain a different pruning denoted $T''$ (which has almost all the same information except for the length of the edge $e$). Let $v'$ (resp. $v''$) denote the bottom vertices corresponding to $e$ in the trees $T'$ (resp. $T''$). Let $\Phi'$ (resp. $\Phi''$) denote the flowing map for $T'$ (resp. $T''$). These coincide and coincide with $\Phi$ on the parts of the trees that are in common (the unpruned parts). We have

$$g(z^{\mathrm{bot}}(T'')) = g(z^{\mathrm{bot}}(T')) + \int_{\Phi'(v')}^{\Phi''(v'')} dg_{i_e j_e}.$$

Note that the segment of $\Phi(e)$ going from $\Phi'(v')$ to $\Phi''(v'')$ is a flow curve for the vector field $W_{i_e j_e}$, and it flows for time $s'' - s'$.

If we prune at an edge $e$ with $s' = s(e)$ then it amounts to cutting off the tree at the lower vertex of $e$. If furthermore all of the length vectors assigned to edges below $e$ are 0, then $g(z^{\mathrm{bot}}(T')) = g(z^{\mathrm{bot}}(T))$.

By recurrence we obtain the first statement in the lemma.

Recall that one of the constraints was the condition that the vector fields $W_{i_e j_e}$ be negative multiples of the gradient vector fields for the real functions $\Re g_{i_e j_e}$. With this condition we get that the integral of $dg_{i_e j_e}$ along a flow curve for $W_{i_e j_e}$ is a negative real number, so this gives at each stage of the recurrence

$$g(z^{\mathrm{bot}}(T'')) - g(z^{\mathrm{bot}}(T')) \in \mathbb{R}_{\leqslant 0}.$$

Putting these together gives the second statement of the lemma. □

There is also another way to prune a tree: if $e$ is an edge such that $i_e = j_e$ then we can cut off $e$ and all of the edges below it, and consolidate the two edges above and to the side of $e$ into one edge. The only difficulty here is that the consolidated edge might have total length $> 1$ but this doesn't affect the remainder of our argument (since at this point we can ignore questions about the sizes of the cells). Let $T'$ denote the pruned tree obtained in this way. We again have

$$g(z^{\mathrm{bot}}(T)) - g(z^{\mathrm{bot}}(T')) \in \mathbb{R}_{\leqslant 0}.$$

In general we will be trying to show for the trees which arise in $F\tau$, that the real part of $g(z^{\mathrm{bot}}(T))$ is small enough. If we can show it for $T'$ then it follows also for $T$. In this way we can reduce for the remainder of the argument, to the case where $i_e \neq j_e$ for all edges of $T$. This is the content of the following lemma. For its statement, recall the neighborhood $S \subset N_1$ appearing in the proof of Lemma 7.2.

***Lemma 8.3***. — *Let $|S| = \zeta_0 - \zeta_1$ denote the length of the segment along which we want to continue $f$. In order to show that the image $g(F\tau)$ misses a neighborhood, say $N_1$, of the segment $S$ it suffices to choose our vector fields $W$ (satisfying Condition 7.1) so that if $T$ is any furnished tree satisfying:*

(i) *the auxiliary condition $(*)$ of Lemma 8.1;*

(ii) *that* $i_e \neq j_e$ *for all edges* $e$ *of* $T$*; and*

(iii) *that* $z^{\mathrm{top}}(T)$ *is in the support of* $\eta'$*;*

*then* $g(z^{\mathrm{bot}}(T))$ *lies outside of our neighborhood* $N_1$ *of* $S$*.*

*Proof.* — Assume that we have chosen the vector fields to give the reduced condition of this statement. Suppose $z$ is a point on the support of $F\tau$. Then there is a furnished tree $T^1$ as in Lemma 8.1 such that $z = z^{\mathrm{bot}}(T^1)$ and such that $z^{\mathrm{top}}(T^1)$ is on the support of $\eta'$. Let $T := (T^1)'$ be the pruning of $T^1$ described directly above. It still satisfies (i), *i.e.* the condition $(*)$ of Lemma 8.1, and by the pruning process it automatically satisfies (ii). Also $z^{\mathrm{top}}(T) = z^{\mathrm{top}}(T^1)$ is on the support of $\eta'$, so our condition gives that $g(z^{\mathrm{bot}}(T))$ lies outside of $N_1$. On the other hand,

$$g(z^{\mathrm{bot}}(T^1)) - g(z^{\mathrm{bot}}((T^1)')) \in \mathbb{R}_{\leqslant 0}, \quad g(z^{\mathrm{bot}}(T)) - g(z^{\mathrm{top}}(T)) \in \mathbb{R}_{\leqslant 0}.$$

Thus $g(z) = g(z^{\mathrm{bot}}(T^1))$, then $g(z^{\mathrm{bot}}(T)) = g(z^{\mathrm{bot}}((T^1)'))$, and then $g(z^{\mathrm{top}}(T))$ lie in order on a line segment parallel to the real axis. Given that $g(z^{\mathrm{top}}(T)) \in N_1$ but $g(z^{\mathrm{bot}}(T)) \notin N_1$, and that $N_1$ is a convex, we obtain $g(z) \notin N_1$ as desired. □

***Remark***. — The condition of the lemma will not be possible, of course, when the segment $S$ passes through a turning point. Finding out the conditions on $S$ to make it possible will tell us where the turning points are.

## 9. Remoteness of points

One of the important facets of the statements of theorems 6.3 and 1.1 is the local finiteness of the set of singularities. We describe here briefly how this works. It reproduces the discussion of [**36**], but with considerable simplification due to Condition 7.1 (iv) which says that when $i = j$ the flow $f_{ij}(z,t)$ is constant.

It should be noted that the local finiteness notion 6.2 is fairly strong in that one can wind arbitrarily many times around a given singularity for an arbitrarily small cost in terms of length of the path. In our mechanism, this is achieved by analytically continuing along a large number of very small segments.

We can choose a metric $d\sigma$ on $Z^*$ (and which is a singular but finite metric on $Z$) with the property that for any distinct pair of indices $i \neq j$, if $\xi : [0,1] \to Z$ is a path whose derivative is a negative real multiple of $\mathbf{grad}\,\Re g_{ij}$ then

$$\int_{\xi} d\sigma \leqslant \Re(g_{ij}(\xi(0)) - g_{ij}(\xi(1))).$$

Later in § 11 we will choose a smooth metric $h$ on $Z$. We can choose $d\sigma$ to be conformally equivalent to $h$, so that gradients point in the same direction for both metrics. Thus the above condition (using the metric $d\sigma$) will also be true for any path whose derivative is a negative real multiple of the gradient $\mathbf{grad}_h\,\Re g_{ij}$ with respect to $h$.

Now suppose $z = (z_i) \in Z_I$, and suppose $T$ is a binary planar tree embedded in $Z$, with one top vertex at $p$ and whose bottom vertices are the $z_i$. Let

$$\boldsymbol{r}_T(z) := \int_T d\sigma$$

be the total length of the tree with respect to our metric. Define the *remoteness* $\boldsymbol{r}(z)$ to be the infimum of $\boldsymbol{r}_T(z)$ over all such trees.

***Lemma 9.1***. — *Suppose $T$ is a furnished tree, and use flows defined by vector fields satisfying Condition 7.1 to define $z^{\mathrm{bot}}(T)$. Then*

$$\boldsymbol{r}(z^{\mathrm{bot}}(T)) \leqslant \boldsymbol{r}(z^{\mathrm{top}}(T)) + g(z^{\mathrm{top}}(T)) - g(z^{\mathrm{bot}}(T)).$$

*Proof*. — If $T^1$ is any tree as in the definition of remoteness for $z^{\mathrm{top}}(T)$ then we can add $T$ to $T^1$ (the top vertices of $T$ being the same as the bottom vertices of $T^1$) to obtain a tree $T^2$ as in the definition of remoteness for $z^{\mathrm{bot}}(T)$. The formula

$$\boldsymbol{r}_{T^2}(z^{\mathrm{bot}}(T)) \leqslant \boldsymbol{r}_{T^1}(z^{\mathrm{top}}(T)) + g(z^{\mathrm{top}}(T)) - g(z^{\mathrm{bot}}(T))$$

is immediate from Lemma 8.2 and the property of $d\sigma$; use Condition 7.1 (iv) to deal with edges of $T$ having $i_e = j_e$. $\square$

***Lemma 9.2***. — *Let $\gamma$ be a path from $p$ to $q$, which leads to the original pro-chain $\eta$ appearing in Lemma 6.1. Suppose $M_0$ is the length of $\gamma$ in the metric $d\sigma$. Then for any point $z$ on the support of $\eta$ we have $\boldsymbol{r}(z) \leqslant M_0$.*

*Proof*. — For any point $z$ on the support of $\eta$, we have $z_i = \gamma(t_i)$ for $t_1 \leqslant \cdots \leqslant t_k \leqslant 1$. The path $\gamma$ can be considered as a tree (of total length $M_0$) starting at $p$ with one spine and $k$ edges of length 0 coming off at the points $z_i$. $\square$

In our procedure for analytic continuation along a path of length $\leqslant M$, we obtain chains whose support consists only of points with $\boldsymbol{r}(z) \leqslant M_0 + 2M$ (see § 13 below). In particular each $z_i$ is at distance $\leqslant M_0 + 2M$ from $p$ with respect to $d\sigma$. Thus everything we do takes place in a relatively compact subset of $Z$ (and concerns only a finite number of singular points $P \in Z$).

## 10. Calculations of gradient flows

We express the gradient of the real part of a holomorphic function, as a vector field in a usual coordinate and in logarithmic coordinates. This is of course elementary but we do the calculation just to get the formula right. Suppose $z$ is a coordinate in a coordinate patch on $X$. The metric on $X$ may be expressed by the real-valued positive function

$$h(z) := \frac{|dz|^2}{2}.$$

Write $z = x + iy$. Note that $dx$ and $dy$ are perpendicular and have the same length, so

$$h(z) = |dx|^2.$$

The real tangent space has orthogonal basis

$$\Big\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big\}$$

and the formula

$$1 = \Big|\frac{\partial}{\partial x} \cdot dx\Big| = h^{1/2}\Big|\frac{\partial}{\partial x}\Big|$$

yields

$$\Big|\frac{\partial}{\partial x}\Big| = h^{-1/2}.$$

In particular an orthonormal basis for the real tangent space is given by

$$\Big\{h^{1/2}\frac{\partial}{\partial x}, h^{1/2}\frac{\partial}{\partial y}\Big\}.$$

Thus we have the formula, for any function $a$:

$$\mathbf{grad}\, a = h\,\frac{\partial a}{\partial x}\frac{\partial}{\partial x} + h\,\frac{\partial a}{\partial y}\frac{\partial}{\partial y}.$$

Now suppose $g = a + ib$ is a holomorphic function (with $a, b$ real), and pose $f(z) := \partial g/\partial z$ so that $dg = f(z)dz$. Write $f(z) = u + iv$ with $u, v$ real, and expand:

$$(u + iv)(dx + idy) = \frac{\partial a}{\partial x}dx + \frac{\partial a}{\partial y}dy + i\frac{\partial b}{\partial x}dx + i\frac{\partial b}{\partial y}dy.$$

Comparing both sides we get

$$u = \frac{\partial a}{\partial x}, \qquad v = -\frac{\partial a}{\partial y}.$$

Note that $a = \Re g$ is the real part of $g$, so finally we have the formula

$$\mathbf{grad}\, \Re g = h(z)\left((\Re\frac{\partial g}{\partial z})\frac{\partial}{\partial x} - (\Im\frac{\partial g}{\partial z})\frac{\partial}{\partial y}\right).$$

Suppose now that $w$ is a local coordinate at a point $P$, and consider

$$g = a_m w^m.$$

Let $z = -i\log w$ so $w = e^{iz}$, and writing $z = x + iy$ we have $w = e^{ix-y}$. Then

$$g(z) = a_m e^{imz}; \quad \frac{\partial g}{\partial z} = mia_m e^{imz}.$$

If we write $mia_m = e^{r+is}$ then

$$\frac{\partial g}{\partial z} = e^{r-my+i(s+mx)},$$

so

$$\mathbf{grad}\, \Re g = h(z)e^{r-my}\left(\cos(s + mx)\frac{\partial}{\partial x}, \sin(s + mx)\frac{\partial}{\partial y}\right).$$

The *asymptotes* are the values $x = B$ where $\cos(s + mx) = 0$. At these points, the gradient flow vector field is vertical (going either up or down, depending on the sign of $\sin(s + mx)$). If the flow goes up, then it stays on the vertical line until $y = \infty$.

Note that the gradient of $\Re g$ is perpendicular to the level curves of $\Re g$, so it is parallel to the level curves of $\Im g$. Which is to say that the level curves of $\Im g$ are the flow lines. This gives an idea of the dynamics of the flow. We have

$$\Im g = \Im((im)^{-1}e^{(r-my)+i(s+mx)}) = -m^{-1}e^{r-my}\cos(s + mx).$$

Thus a curve $\Im g = C$ is given by

$$e^{-my} = \frac{-mC}{e^r \cos(s + mx)}$$

or (noting that the sign of $C$ must be chosen so that the right hand side is positive)

$$y = m^{-1}r + m^{-1}\log|\cos(s + mx)| - m^{-1}\log|mC|.$$

In particular the level curves are all vertical translates of the same curve; this curve $y = m^{-1}r + m^{-1}\log|\cos(s + mx)|$ has vertical asymptotes at the points where $\cos(s + mx) = 0$. Note however that at the asymptotes, we get $y \to -\infty$; whereas our coordinate patch corresponds to a region $y > y_0$. Thus, every gradient flow except for the inbound (*i.e.* upward) flows directly on the asymptotes, eventually turns around and exits the coordinate patch. This of course corresponds to what the classical picture looks like in terms of the original coordinate $w$.

Also we can calculate the second derivative (which depends only on $x$ and not on which level curve we are on, since they are all vertical translates). Consider for example points where $\cos(s + mx) > 0$. There

$$\frac{dy}{dx} = \frac{\sin(s + mx)}{\cos(s + mx)}$$

and

$$\frac{d^2y}{dx^2} = \frac{-m}{\cos^2(s + mx)}$$

In particular note that we have a uniform bound everywhere:

$$\frac{d^2y}{dx^2} \leqslant -\gamma,$$

here with $\gamma = m$.

Suppose now more generally that $g$ is a holomorphic function with Taylor expansion

$$g = a_m w^m + a_{m+1}w^{m+1} + \dots .$$

Then we will get

$$h(z)^{-1}e^{my-r}\,\mathbf{grad}\,\Re g = \left(\cos(s + mx)\frac{\partial}{\partial x}, \sin(s + mx)\frac{\partial}{\partial y}\right) + O(e^{-y}).$$

In particular, the direction of the gradient flow for $g$ is determined, up to an error term in $O(e^{-y})$, by the vector $(\cos(s + mx), \sin(s + mx))$.

The asymptotes are no longer vertical curves, but they remain in bands $x \in B_{ij,a}$. Also we can choose $A$ in the definition of steepness, so that at non-steep parts of the level curves we still have a bound

$$\frac{d^2y}{dx^2} \leqslant -\gamma.$$

## 11. Choice of the vector fields $W_{ij}$

The only thing left to be determined in order to fix our procedure for moving the cycle of integration is to choose the vector fields. Before going further, fix a smooth metric $h$ on $Z$, for example coming from the pullback of a smooth metric on $Y$. Use this to calculate gradients (note that $h$ is different from the singular metric $d\sigma$ considered in § 9, but they are conformally equivalent so the remoteness estimate of § 9 holds for $h$-gradient paths). Suppose $\varepsilon$ is given. Let $\rho$ denote a cutoff function which is identically 0 in the discs $D_{\varepsilon/2}(P)$ (for all points $P$ in the inverse image of $\mathcal{R}$), and is identically 1 outside the (closed) discs $D_\varepsilon(P)$. Of course $\varepsilon$ will be small enough that the discs don't intersect. Consider also a positive real constant $\mu \in \mathbb{R}_{>0}$. Then we put

$$W_{ij} := \mu\,\mathbf{grad}\,\Re g_{ij},$$

and

$$W'_{ij} := \rho W_{ij}.$$

The vector fields $W'_{ij}$ satisfy Condition 7.1 (with $\varepsilon/2$ in place of $\varepsilon$). We will use these vector fields for our choice of flows, and apply the criterion of Lemma 8.3.

The point we want to make in the present section is that the flow curves for the cut-off gradient vector field $W'_{ij}$ are the same as those of the true gradient flow along $W_{ij}$, up until any point where they enter some $D_\varepsilon(P)$. This will allow the notational simplification of looking at $W_{ij}$ rather than $W'_{ij}$ in the next section.

Let $\nu > 0$ be the radius used to define the oval neighborhood $N_1$, *i.e.* choose $N_1$ equal to the set of points of distance $< \nu$ from $S$. Once $\varepsilon$ is given, choose $\mu$ large enough so that the following property holds:

***Condition 11.1***. — If $z(t) = f_{ij}(z_0, t)$ is a flow curve for $W_{ij}$ (for distinct indices $i \neq j$) which never enters into any $D_\varepsilon(P)$ flowing for $t \in [0, s]$ with $s \geqslant 1$, then

$$g_{ij}(f_{ij}(z_0, s)) - g_{ij}(z_0) < \zeta_1 - \zeta_0 - 2\nu.$$

Recall that $\zeta_0, \zeta_1$ were the endpoints of the segment $S$ with $\zeta_1 - \zeta_0$ a negative real number.

It is possible to choose $\mu$ (we only need to do it over a relatively compact subset of $z_0 \in Z$ by the remark at the end of § 9, but in any case everything involved is pulled back from the compact $Y$ so the choice of $\mu$ is uniform in $z_0$).

The next lemma formalizes the following reduction: the trees which show up in Lemma 8.3 have a horizontal line of edges assigned length 1. If the flow for at least one of these edges stays outside of all the $D_\varepsilon(P)$ then by Condition 11.1 the value of $g$ is decreased sufficiently to get us out of $N_1$. Thus the only case which poses a problem is when every downward branch of the tree ends up flowing into some $D_\varepsilon(P)$. In this case we prune the tree at the points where it enters these discs.

***Lemma 11.2***. — *Suppose $\varepsilon$ is given, and $\mu$ chosen to satisfy Condition 11.1. Use the vector fields $W'_{ij}$ to define the flows. In order to show that the image $g(F\tau)$ misses our neighborhood $N_1$ of the segment $S$, it suffices to show that if $T$ is any furnished tree satisfying the following conditions:*

(i) *that $z^{\rm top}(T)$ lies on the support of $\eta'$;*

(ii) *that $i_e \neq j_e$ for any edge of $T$;*

(iii) *that for each bottom vertex $v$ of $T$ (except for bottom vertices on side edges) there is a singular point $P(v)$ such that $\Phi(v) \in D_\varepsilon(P(v))$; and*

(iv) *that all other points of $\Phi(T)$ are outside the discs $D_\varepsilon(P)$,*

*then $g(z^{\rm bot}(T))$ is not in the neighborhood $N_1$ of $S$.*

*Proof*. — Suppose $T$ is a furnished tree as in the reduction of Lemma 8.3. Prune $T$ at any point where the flowing map $\Phi$ enters into one of the closed discs $D_\varepsilon(P)$. If this prunes all branches of the tree, then by an argument using 8.2 similar to the previous reductions, that puts us in the case described here so we are done.

Thus we may assume that there is at least one branch which is not pruned. By condition 8.3 (i) which is the same as Condition $(*)$ of Lemma 8.1, the branch going to the bottom has at least one edge assigned length 1. This edge has $i_e \neq j_e$. By Condition 11.1 we have for this edge

$$\int_{\Phi(e)} dg_{i_e j_e} < \zeta_1 - \zeta_0 - 2\nu.$$

Therefore, by the formula of Lemma 8.2 we have

$$g(z^{\rm bot}(T)) - g(z^{\rm top}(T)) < \zeta_1 - \zeta_0 - 2\nu.$$

Given that $g(z^{\rm top}(T)) \in g(\eta') \subset N_1$ but $N_1$ is an oval with largest diameter $2\nu+\zeta_0-\zeta_1$, we get $g(z^{\rm bot}(T)) \notin N_1$. ☐

***Corollary 11.3***. — *Define the chain $F\tau$ using the vector fields $W'_{ij}$. Then, in order to show that $g(F\tau)$ misses $N_1$ it suffices to show that for any furnished tree $T$ satisfying the conditions (i)-(iv) of 11.2 with respect to the flowing map $\Phi$ defined by the vector fields $W_{ij}$ (rather than $W'_{ij}$), we have $g(z^{\rm bot}(T)) \notin N_1$.*

*Proof*. — The two flowing maps coincide, in view of condition (iv). ☐

In view of this corollary, we can in the next section ignore the cutoff functions $\rho$ and look directly at the gradient flows $W_{ij}$.

## 12. Results on the dynamics of our flowing maps

We will consider a system of discs centered at our singular points $P$:

$$D_\varepsilon(P) \subset D_\xi(P) \subset D_u(P) \subset D_w(P).$$

We will first fix $u$ and $w$ so that certain things are true in a coordinate system for $D_w(P)$ (and say $u = w/2$). Then once $u$ and $w$ are fixed we will let $\varepsilon \to 0$. Finally $\xi > \varepsilon$ will be a function of $\varepsilon$ with $\xi \to 0$ when $\varepsilon \to 0$.

The innermost discs $D_\varepsilon(P)$ are those which will enter into the reduction of Lemma 11.2. Recall that $\mu$ is chosen after $\varepsilon$. In view of the Corollary 11.3, we henceforth look directly at the gradient flows $W_{ij} = \mu\,\mathbf{grad}\,\Re g_{ij}$.

Our first lemma bounds the number of outgoing subtrees.

***Lemma 12.1***. — *If $T$ is a furnished tree with one top edge $e$, and if $\Phi : T \to X$ is a flowing map such that the images of all bottom vertices are contained in some $D_\varepsilon(P_i)$, and if $\Phi(e)$ exits from $D_u(P)$ then $T$ contains a strand $\sigma$ such that $\Phi(\sigma)$ exits from $D_w(P)$ also.*

Our next lemma gives a normal form for any subtree which stays entirely within $D_u(P)$.

***Lemma 12.2***. — *If $T$ is a furnished tree with one edge $e$ at the top, and if $\Phi$ is a flowing map from $T$ into $D_u(P) \subset X$ such that all of the bottom vertices are mapped into $D_\varepsilon(P)$, then the curve $\Phi(e)$ passes into $D_\xi(P)$, and flows along a vector field $W_{i_e j_e}$ in an ingoing sector near an ingoing curve $G_{i_e j_e}$.*

The last of our preliminary lemmas bounds the number of subtrees having the previous normal form.

***Lemma 12.3***. — *There is a number $K$ (depending on $u, w, A$ but independent of $\varepsilon$, $\xi$ and $\mu$) such that if $T$ is a furnished tree consisting of one edge strand $\kappa$ plus a number of sub-trees coming out of $\kappa$, and if $\Phi$ is a flowing map from $T$ into $D_u(P)$ with the property that all the sub-trees coming out of $\kappa$ are covered by Lemma 12.2, then there are $\leqslant K$ of these sub-trees.*

For the proofs of these lemmas, we will use a logarithmic coordinate system for $D_w(P)$. If $z_D$ denotes the coordinate in the disc then we introduce $z_L = -i \log z_D$ and write $z_L = x + iy$ $z_D = e^{ix-y}$.

The disc $D_w(P)$ is given by $y > y_0$.

The vector fields $W_{ij} = \mu\,\mathbf{grad}\,\Re g_{ij}$ are approximately equal (up to a term smaller by a factor of $O(\mu e^{-y})$) to the standard vector fields $W'_{ij} = \mu\,\mathbf{grad}\,\Re g'_{ij}$ where $g'_{ij}$ is the leading term in the Taylor expansion for $g_{ij}$ at $P$.

Because of this, we obtain the following facts. The asymptotic directions (which are close to vertical lines) occur in bands of the form $x \in B_{ij,a}$ where $B_{ij,a} \subset \mathbb{R}$ are intervals which can be made as small as we like by modifying $y_0$. These intervals are disjoint, except for the asymptotes of the pairs $\{W_{ik}, W_{jk}\}$ or $\{W_{ki}, W_{kj}\}$, where $i, j$

are the two indices attached to $P$, and $k$ is any index different from these two. In those cases the pairs share the same values $B_{ij,a}$ and the same bands. We say that a vector field $W_{ij}$ is *attached* to an interval $B$ if $B = B_{ij,a}$. The only intervals with more than one vector field attached to them are those described above.

It is worth mentionning why we have this disjointness property. It is because of the non-parallel condition on the eigenforms of $\theta$ at the singular points. The non-parallel condition implies that the bands, which are the solutions of $s+mx=0$ modulo $\pi$, are distinct, because the values of $s$ (which are the angular coordinates of the constants attached to the leading terms of $g_{ij}$ as explained in the preceding section) are different exactly because of it. Notice that the exponents $m$ are the same for all of the values $ij$ except the two attached to the singular point; for those which are attached the value $m'$ is bigger. The non-parallel condition gives disjointness for all of the bands except the ones corresponding to the attached indices $ij$ and $ji$. For those, note that if we make a general rotation of everything, the asymptotic solutions of $s+m'x$ move differently than the solutions of $s+mx$, so those bands are disjoint from all the other ones. The general rotation of everything corresponds to a condition that the line segments in the complex plane along which we analytically continue, might be constrained not to be parallel to a certain finite number of directions. This doesn't hurt our ability to analytically-continue the function.

We can fix a number $A>0$ with the following properties: outside of an asymptotic band for $W_{ij}$ or $W_{ji}$, the slope of the vector $W_{ij}$ satisfies

$$\left|\frac{dy}{dx}(W_{ij})\right| \leqslant A.$$

Inside an asymptotic band $B$, only the vector fields $W_{ij}$ which are attached to $B$ can have slope bigger than $A$ or less than $-A$.

Suppose now that $(x(t), y(t))$ is a flow along one of the vector fields $W_{ij}$. We say that the path is *steep* if

$$\left|\frac{dy}{dx}\right| > A,$$

and we say that it is *not steep* otherwise. We say that the path is *ingoing* if $dy/dx>0$ and outgoing otherwise. Note that with our logarithmic coordinate system, outgoing is downward and ingoing is upward. The coordinate patch (*i.e.* choice of $y_0$) and the choice of $A$ can be made so that all of the paths satisfy the following property:

– once the path is steep and outgoing, it remains steep and outgoing for the remainder of the time of definition, and ends up leaving the region $y>y_0$.

This is true even though the vector field is not exactly equal to the standard model but only close to it.

On the other hand, the direction, *i.e.* the sign of $dx/dt$ remains the same throughout the interval where the path is not steep. Call this sign $(-1)^m$. In particular we can think of the path as being parametrized by $x$. Define the *slope* to be the signed derivative $(-1)^m dy/dx$.

We have a bound, in the region where the path is not steep:

$$\frac{d^2y}{dx^2} \leqslant -\gamma$$

with $\gamma > 0$ a positive constant. Note that the second derivative is also the variation of the slope with respect to $x$ when we go in the direction of the path.

In particular, once the path is outgoing it remains outgoing for the remainder of its period of definition. This is because of the second derivative when it is not steep, and the fact that when it becomes steep and outgoing then it stays that way.

We now note the *additive relation* for the vector fields at vertices of a tree.

***Lemma 12.4***. — *Suppose we are in the situation of a flowing map $\Phi : T \to X$ defined by vector fields $W_{ij} = \mu\,\mathbf{grad}\,\Re g_{ij}$. At any vertex $v$ of $T$ with edges noted $e_1, e_2, e_3$ (say $e_1$ ingoing and $e_2, e_3$ outgoing), we have three indices $i, j, k$ such that*

$$i_{e_1} = i_{e_2} = i; \quad j_{e_2} = i_{e_3} = j; \quad j_{e_1} = j_{e_3} = k.$$

*For the three vector fields $W_{ik}, W_{ij}, W_{jk}$ corresponding to the edges $e_1, e_2, e_3$ we have the relation*

$$W_{ik}(\Phi(v)) = W_{ij}(\Phi(v)) + W_{jk}(\Phi(v)).$$

*Proof*. — The vector fields $W_{ij}$ are all the same multiple of the gradients $\mathbf{grad}\,\Re g_{ij}$. The fact that $dg_{ij} = \alpha_i - \alpha_j$ implies that $dg_{ij} + dg_{jk} = dg_{ik}$ giving the relation in question. □

*Proof of Lemma 12.1*. — The disc $D_u(P)$ will be determined by $y > y_1$ for some $y_1$ fixed as a function of $y_0$ (and in fact one could take $y_1 = y_0 + 1$ for example). A consequence of the additive relation is that if $W_{ik}(\Phi(v))$ is outgoing (*i.e.* $dy/dt \leqslant 0$ along this vector) then one of the other two $W_{ij}(\Phi(v))$ or $W_{jk}(\Phi(v))$ will also be outgoing. As we have noted above, if the flow along any edge is outgoing at some point then it is outgoing for all further points. In particular if at any point in the tree the flow is outgoing then we can choose a strand going down to the bottom, along which the flow is always outgoing. If there is an edge which crosses out of $D_u(P)$, at the crossing point it has $dy/dt \leqslant 0$, so we get a strand which maintains $dy/dt \leqslant 0$ as long as it stays inside $D_w(P)$. In particular the strand cannot go back to $D_\varepsilon(P)$ so it must exit from $D_w(P)$ (here using the hypothesis that any strand must end in some $D_\varepsilon(P_i)$). This completes the proof of Lemma 12.1. □

Now we come to the proofs of Lemmas 12.2 and 12.3. Fix notations $L := -\log\varepsilon$ and $L_1 := -\log\xi$. Thus we will let $L \to \infty$ and we have to specify $L_1$ as a function of $L$ such that $L_1 \to \infty$ too. Our discs $D_\varepsilon(P)$ and $D_\xi(P)$ respectively become the regions $y > L$ and $y > L_1$. We will specify $L_1$ as a function of $L$ so as to make the proofs of Lemmas 12.2 and 12.3 work.

In both lemmas, we lift the maps $\Phi$ into maps into the coordinate chart for the logarithmic coordinates.

*Proof of Lemma 12.2.* — At any point where the flow is not steep, the second derivative is bounded above by $-\gamma$. In particular the flow becomes outgoing before it becomes steep again. Furthermore, if $v$ is a vertex with indices $i, j, k$ as above, such that the vector field $W_{ik}(\Phi(v))$ is not steep but is ingoing, then the additive relation insures that one of the other two flows $W_{ij}(\Phi(v))$ or $W_{jk}(\Phi(v))$ has slope less than or equal to the slope of $W_{ik}(\Phi(v))$. For this, draw a line through the first vector, and note that one of the two other vectors has to lie below or on the line. Note that this gives two cases: either the new vector changes direction (*i.e.* the sign $(-1)^m$ changes) and the new vector is in fact outgoing; or else the direction stays the same and the slope decreases. Thus if $t_0$ is any point in $T$ where the flow is ingoing but not steep, then we can choose a strand $\sigma$ below $t_0$ with the property that at the end of the strand the flow becomes outgoing; and along the strand the direction stays the same and the second derivative satisfies

$$\frac{d}{dx}((-1)^m \frac{dy}{dx}) \leqslant -\gamma$$

in a distributional sense. Then (noting by $(x(t), y(t))$ the coordinates of the image point $\Phi(t)$ for $t \in \sigma$) we have

$$y(t) \leqslant y(t_0) + A(-1)^m(x(t) - x(t_0)) - \frac{\gamma}{2}(x(t) - x(t_0))^2$$

for any $t \geqslant t_0$. In particular there is a number $N$ such that

$$y(t) \leqslant y(t_0) + N$$

further along the strand. We will choose $L_1 = L - N$.

Recall now that in the hypotheses of the lemma, we suppose that all strands in the tree remain inside $D_u(P)$ and also finish in $D_\varepsilon(P)$. However, we construct above a strand which eventually becomes outgoing; therefore the strand must enter the region corresponding to $D_\varepsilon(P)$ before it becomes outgoing (and notice also that it could simply stop inside this region before becoming outgoing, a case not mentionned above). In particular, if there is any point $t_0$ corresponding to a non-steep ingoing flow, or of course to any sort of outgoing flow, then we have to have $y(t_0) + N > L$ or $y(t_0) > L_1$.

Now we can complete the proof of the lemma. If $v$ is any vertex, such that the incoming edge is steep and ingoing, then one of the two outgoing edges has to be either non-steep and ingoing, or outgoing. This is verified from the fact that at most two different vector fields can be attached as ingoing asymptotic vector fields for the same band $B$. From what was said above, the bottom vertex of the first edge $e$ of the tree must satisfy $y(\Phi(v)) > L_1$, in other words the first edge continues all the way until $D_\xi(P)$. Also the part of the edge $e$ which is outside of $D_\xi(P)$ must be contained in an ingoing asymptotic band for its vector field $W_{ij}$ and the flow is steep at all points of $\Phi(e)$ which are outside of $D_\xi(P)$. This completes the proof of Lemma 12.2. □

*Proof of Lemma 12.3.* — Consider a vertex $v$ along $\kappa$ where a subtree in the normal form of Lemma 12.2 comes off. Use the same notation as previously for the edjes and indices adjoining $v$. For the sake of simplicity we assume that $\kappa$ corresponds to the two leftmost edges $e_1$ and $e_2$ at $v$. The upper edge of the subtree is thus $e_3$ with indices $jk$.

Note from the proof of 12.2 that $W_{jk}$ is ingoing and steep at $\Phi(v)$.

As a first case, note that if the anterior edge $e_1$ of $\kappa$ has $W_{ik}$ which is outgoing and steep, then the subsequent edge $e_2$ of $\kappa$ is also outgoing and steep. In particular at any point where $\kappa$ becomes outgoing and steep, it remains that way and in fact will leave the region $y > y_1$ before it goes into any other band $B$. By looking at the possible combinatorics of the indices one sees, even in the case of two vector fields sharing the same band, that there can be no further normal-form vertices on $\kappa$.

In view of the previous paragraph we may restrict our attention to the places where $\kappa$ is either not steep, or else steep but ingoing. However, if it is steep but ingoing then again at most one vertex with a normal-form subtree can correspond to the current band; thus at some point $\kappa$ leaves this band and must become non-steep. On the other hand, once $\kappa$ is non-steep, it doesn't change to become steep and ingoing. It doesn't do this in the middle of an edge, because of the second derivative condition. It doesn't do it at a vertex because the edge $e_3$ which comes off is steep and ingoing, and a $W_{ik}$ which is not steep couldn't be the sum of two steep and ingoing vectors.

The two previous paragraphs show that we may (at the price of at most two extra normal-form subtrees) restrict our attention to the region where $\kappa$ is non-steep. Now one sees again from the additive relation that if $W_{ik}$ and $W_{ij}$ are non-steep, whereas $W_{jk}$ is steep and ingoing, then the directions of $W_{ik}$ and $W_{ij}$ must be the same. Indeed, if not then we would have $W_{jk} = W_{ik} + (-W_{ij})$ which would be a sum of two vectors in the same non-steep quadrant, so $W_{jk}$ in a steep quadrant would be impossible.

Since the sign $(-1)^m$ of $dx/dt$ doesn't change, we can use $x$ to parametrize $\kappa$. Furthermore the slope $(-1)^m dy/dx$ is decreasing along $\kappa$ (note that at any vertices where a subtree in normal form comes off, the remaining outgoing edge of $\kappa$ has a smaller slope than the ingoing edge, because of the additive relation).

In other words, the second derivative is distributionally less than the constant $-\gamma$, so at some time $t$ with $|x(t)-x_0| \leqslant 2A/\gamma$ we get to $(-1)^m dy/dx \leqslant -A$, *i.e.* $\kappa$ becomes steep and outgoing. We get that the non-steep part of the path $\kappa$ is parametrized by an interval in the $x$-coordinate, of length $\leqslant 2A/\gamma$. There is a bound $K$ so that such an interval can cross (or go near) at most $K-2$ asymptotic bands. A band is attached to at most two pairs of indices, but only one of these can lead correspond to a normal-form subtree. Thus (counting the two we may have missed above) the number of normal-form subtrees attached to $\gamma$ is $\leqslant K$. This completes the proof of Lemma 12.3. □

We now come to the main result of this section. Fix $u, w$ as above, and let $L_1 := L + N$ be the function determined by the above proofs. For any $\varepsilon$ put $L := \log \varepsilon$ and set $\xi := e^{L_1} = e^N \varepsilon$. Note that $\xi \to 0$ as $\varepsilon \to 0$.

***Theorem 12.5***. — *There is a bound $K$ depending on $u, w$ and a real constant $F$ (which will be $\zeta_0 + 2\nu - \zeta_1$ later on), but with $K$ independent of $\varepsilon$, $\xi$ and $\mu$, with the following properties. Suppose $T$ is a furnished tree and $\Phi : T \to X$ is a flowing map such that the top vertices are outside of any $D_w(P_i)$ and such that the bottom vertices are each mapped into some $D_\varepsilon(P_i)$. Suppose furthermore that $g(z^{\rm bot}) \geqslant g(z^{\rm top}) - F$. Then we can cut $T$ into a tree $T'$ onto which are attached subtrees, such that $\Phi$ maps the bottom vertices of $T'$ into various $D_\xi(P_i)$ and such that the number of bottom vertices of $T'$ is bounded by $K$.*

*Proof*. — Among the subtrees that we strip off are any ones starting with edges $e$ for which $i_e = j_e$. In particular we may assume from the start that $T$ has no such edges.

Next group the bottom vertices into series connected by intervals where the bounding path of the interval (*i.e.* the path of edges in the tree which goes from the bottom vertex at one end of the interval to the bottom vertex at the other end) is mapped into $D_u(P)$. There is a bound $K_1$ for the number of such series, because any path which goes out of $D_u(P)$ and back has to contribute at least a certain fixed amount to $g(z^{\rm top}) - g(z^{\rm bot})$. Next we can look at a specific series. It is the set of bottom vertices of a subtree $T_1$ obtained by taking the union of all of the paths joining the bottom vertices together. Note in particular that $\Phi(T_1) \subset D_u(P)$. Let $\kappa$ denote the boundary path of $T_1$. Note that the subtree $T_1$ doesn't necessarily include all strands emanating from all of its vertices. However, if $v$ is a vertex on $\kappa$ corresponding to an adjoining edge $e$ not in $\kappa$, then either $e$ goes into the interior of the region bounded by $\kappa$, in which case $e$ starts a subtree mapped into $D_u(P)$ and such that all bottom edges go into $D_\varepsilon(P)$; or else it goes out of the region bounded by $\kappa$ in which case $e$ is not a part of the tree $T_1$. In the former case, the normal form of Lemma 12.2 applies to the subtree starting at $e$. In the latter case, the subtree starting at $e$ could be in normal form or not. However, if $e$ is an edge going out of $\kappa$ such that the subtree starting at $e$ is not in the normal form of Lemma 12.2, then this subtree contains at least one strand which goes out of $D_u(P)$. By Lemma 12.1 it also contains a strand which goes out of $D_w(P)$ and there is a global bound $K_2$ on the number of such edges $e$. If we cut $\kappa$ at vertices $v$ where such edges $e$ go out, then it is cut into $\leqslant K_2$ strands $\kappa'$ and each little strand has only vertices corresponding to normal-form subtrees. Finally, by the bound of Lemma 12.3 there are no more than $K_3$ such vertices on each little strand $\kappa'$. Each of these normal-form subtrees can be cut at the point where it goes into $D_\xi(P)$, and there is only one such point for each subtree. Thus if we trim off the tree $T_1$ at all of the points where the strands enter $D_\xi(P)$, there are at most $K_2 K_3$ bottom vertices. Finally, since there were at most $K_1$ subtrees $T_1$ corresponding to series of bottom vertices, we can trim off $T$ to a tree $T'$ where there are at most $K_1 K_2 K_3$ bottom vertices, all going inside some $D_\xi(P_i)$. This proves the theorem. $\square$

## 13. Proofs

By a *multisingular point* we mean a point $y = (y_1, \ldots, y_k) \in Z_I^*$ such that the $y_n$ are singular points of the functions $g_{i_n i_{n+1}}$. Note that in our situation the singular points in $Z$ are the preimages of the turning points $P \in \mathcal{R}$ corresponding to the places where the Higgs field $\theta$ has singular eigenvalues.

If $z$ is a point in $Z_I^*$ with $\boldsymbol{r}(z) \leqslant M$ then in particular each $z_i$ is at distance $\leqslant M$ from $p$ with respect to $d\sigma$ (using the notations of § 9). This defines a relatively compact subset of $Z$, containing a finite number of singular points. It is improved with the lemma below.

Define $S_M$ to be the set of complex values of the form $g(y)$ where $y$ are multisingular points with $\boldsymbol{r}(y) \leqslant M_0 + 2M$. This is the subset which is to enter into the definition of analytic continuation with locally finite branching for $f(\zeta)$.

***Lemma 13.1***. — *For each $M$, the set $S_M$ is finite.*

*Proof*. — There is a positive constant $c$ such that if $P_1$ and $P_2$ are distinct singular points, then the distance from $P_1$ to $P_2$ using the metric $d\sigma$ is at least $c$. Suppose $y = (y_1, \ldots, y_k)$ is a multisingular point, so each $y_i$ is a singular point. If $\boldsymbol{r}(y) \leqslant M_0 + 2M$ then, in view of the definition of $\boldsymbol{r}$ there are at most $(M_0+2M)/c$ indices $i$ such that $y_i \neq y_{i+1}$. Let $y' := (y'_1, \ldots, y'_{k'})$ be the sequence of distinct different points in the sequence $y$. Define a new multi-index $I'$ by setting $i'_a = i_{b(a)}$ where $b(a)$ is the place with $y_{b(a)} = y'_a$ and $y_{b(a)+1} = y'_{a+1}$. Then $y' \in Z_{I'}$ and $g_{I'}(y') = g_I(y)$. Now $k' \leqslant (M_0+2M)/c$ so there are only a finite number of possibilities for $y'$ (the singular points themselves occuring in a fixed relatively compact subset of $Z$ as pointed out above). Thus there are only a finite number of possible values. □

*Proof of Theorem 6.3*. — Suppose we have already analytically continued $f$ along a piecewise linear path of length $\leqslant M_1$. Inductively we may assume that the points of $\eta$ have remoteness $\leqslant M_0 + 2M_1$. If we add a segment $S$ then the total length of the path is $\leqslant M$ where $M = M_1 + |S|$. We assume that $S$ doesn't meet any of the points in $S_M$.

Fix a number $\nu > 0$ so that the segment $S$ stays at a distance $> 2\nu$ away from the points of $S_M$. Choose our neighborhoods $N_i$ with $N_1$ being the oval around $S$ of radius $\nu$, so $N_1$ stays at a distance $> \nu$ away from the points of $S_M$. Let $K$ be the bound of Theorem 12.5. Choose $\varepsilon$ small enough so that if $z \in D_\xi(P)$ then for any $i$

$$\left| \int_P^z \alpha_i \right| < \frac{\nu}{K}.$$

We show that all points of the chain $F\tau$ are sent (by $g$) outside of $N_3$. Suppose on the contrary that we had a point, corresponding to a tree $T$, such that $g(z^{\mathrm{bot}}(T)) \in N_3$.

By Theorem 12.5 there exists a pruning $T'$ of $T$ with $k \leqslant K$ bottom vertices, such that for every bottom vertex $v$ of $T'$ we have $\Phi(v) \in D_\xi(P(v))$ for some singular point

$P(v)$. In particular, the point $z^{\mathrm{bot}}(T')$ which is the vector of these $\Phi(v)$ is near to a point $y = (\ldots, P(v), \ldots)$. More precisely we obtain from $k \leqslant K$ and the bound above,

$$|g(y) - g(z^{\mathrm{bot}}(T'))| < \nu.$$

On the other hand, if $g(z^{\mathrm{bot}}(T'))$ were inside $N_3$ then the singular point $y$ would occur below points of $\eta$ at distance $\leqslant 2|S|$, and hence below points of $p$ at distance $\leqslant M_0 + 2M_1 + 2|S| < M_0 + 2M$, therefore $g(y)$ must be included in $S_M$. On the other hand, the point $g(z^{\mathrm{bot}}(T'))$ occurs on the real segment between $g(z^{\mathrm{bot}}(T))$ and some point of $g(\eta')$. This contradicts the assumption that the neighborhood $N_1$ stays away from $S_M$ by distance at least $\nu$. This shows that all points of $g(F\tau)$ are outside of $N_3$, and completes the proof that we can analytically continue $f(\zeta)$ along the segment $S$.

Finally in order to maintain the inductive hypothesis we note that, cutting everything off fairly close to the segment $S$ we can insure that the points of the new cycle of integration $F\tau$ (and also $F\psi$) are remote from points of $\eta$ at distance $\leqslant 2|S|$, hence they have remoteness $\leqslant M_0 + 2M$ as required. □

*Proof of Theorem 1.1.* — — The statement is essentially contained in that of Theorem 6.3, but we need to show that $S_M \subset \Sigma(\gamma)$. In other words, if $z \in Z_I$ is a multisingular point, we need to show that $g(z)$ is the integral of the tautological form on a piecewise homotopy lifting $\widetilde{\gamma}$. Recall the formula

$$g_I(z) = \int_p^{z_1} \alpha_{i_0} + \cdots + \int_{z_k}^{q} \alpha_{i_k}.$$

Let $\widetilde{\gamma}'_i$ be the path joining $z_i$ to $z_{i+1}$ where by convention $z_0 = p$ and $z_{k+1} = q$. These paths are unique up to homotopy because we are working in the contractible universal cover $Z$. Composing the main projection $Z \to Y$ with Galois automorphisms of $Y$ and then the projection $Y \to V$, gives projections $\tau_i : Z \to V$ which commute with the projection to $X$, such that $\alpha_i$ is the pullback of the tautological form $\alpha$ on $V$, *i.e.* $\alpha_i = \tau_i^*(\alpha)$. We can put $\widetilde{\gamma}_i := \tau_i \circ \widetilde{\gamma}'_i$. The collection $\widetilde{\gamma} = \{\widetilde{\gamma}_i\}$ is a piecewise homotopy lifting of $\gamma$. To see this, note that the projections to $X$ of the $\widetilde{\gamma}_i$ are equal to the projections of the original $\widetilde{\gamma}'_i$, so these join together to give a path homotopic to the projection of the path from $p$ to $q$ in $Z$. Since the lifts $p, q \in Z$ were chosen to correspond to our original path $\gamma$ in $X$, so the composite path in $X$ is homotopic to $\gamma$. Our formula for $g_I(z)$ becomes

$$g_I(z) = \sum_i \int_{\widetilde{\gamma}_i} \alpha = \int_{\widetilde{\gamma}} \alpha.$$

This shows that $S_M$ is a subset of $\Sigma(\gamma)$. □

## 14. Conclusion

We close with a few more general remarks about the consequences of Theorem 6.3. The first is to note that it also applies to any polynomials in the transport matrix coefficients, in particular to the Procesi coordinates for $M_B$.

**Lemma 14.1**. — *Suppose $f_1$ and $f_2$ have analytic continuations with locally finite branching, then the same is true for their convolution $f_1 * f_2$.*

*Proof*. — This was proven in [**9**]. See also the proof of [**36**] Lemma 11.1. There, the proof of locally finite branching for the convolutions uses only locally finite branching for the two functions. □

**Corollary 14.2**. — *If $P(t)$ is a polynomial in the transport matrices for various paths, then then the Laplace transform of $P(t)$ has an analytic continuation with locally finite branching.*

*Proof*. — The Laplace transform of a product of functions $m_1(t)m_2(t)$ is the convolution of their Laplace transforms, so Lemma 14.1 and Theorem 6.3 give the result. □

The next remark is about the growth rate of $m(t)$. This is measured by the hull $\mathbf{hull}(m)$ defined in the introduction.

For reference we indicate first an elementary argument showing that $\mathbf{hull}(m)$ is convex. Indeed, if $\zeta_0$ is a point which is not in $\mathbf{hull}(m)$, then by definition there is an angular sector $\boldsymbol{s}$ in which $m(t)e^{-\zeta_0 t}$ is rapidly decreasing. Suppose $u$ is a complex number such that $\zeta_0 + u$ is in $\mathbf{hull}(m)$. Again by the definition of $\mathbf{hull}(m)$ this implies that $m(t)e^{-\zeta_0 t}e^{-ut}$ is no longer rapidly decreasing in any part of $\boldsymbol{s}$. This means that $\boldsymbol{s}$ is contained in the half-plane $\Re ut \leqslant 0$. In particular, for any vector $u'$ which is a negative real multiple of $u$, we have that $\Re u't \geqslant 0$ so $m(t)e^{-\zeta_0 t}e^{-u't}$ is rapidly decreasing on $\boldsymbol{s}$, therefore $\zeta_0 + u'$ is not in $\mathbf{hull}(m)$. This proves the convexity.

Next we can characterize $\mathbf{hull}(m)$ as the intersection of all closed half-planes $H \subset \mathbb{C}$ such that the Laplace transform $f$ of $m$ admits an analytic continuation over the complementary open half-plane (this would give another proof of convexity). Indeed, if a point $\zeta$ is in the complement of $\mathbf{hull}(m)$ then the sector along which $m(t)e^{-\zeta t}$ is rapidly decreasing provides an open half-plane containing $\zeta$ over which $f$ can be analytically continued. This shows one inclusion. The other inclusion is clear from the inverse Laplace transform.

The hull is related to growth rates as follows. If $\mathbf{hull}(m)$ is a single point, then some multiplicative translate of the form $m(t)e^{\zeta t}$ has sub-exponential growth. If $\mathbf{hull}(m)$ contains at least a line segment, then we say $m$ is *semistrictly exponential*: for subsequences in sectors covering all but two directions we have a lower bound of the form $|m(t)| \geqslant ce^{a|t|}$, and in particular there is a positive lower bound for the possible exponents $a$ which can enter into bounds of the form $|m(t)| \leqslant Ce^{a|t|}$. If $\mathbf{hull}(m)$ contains a nonempty interior (say, containing the origin) then we say $m$ is *strictly* *exponential*: there is a lower bound of the form $|m(t)| \geqslant ce^{a|t|}$ valid on subsequences in all directions.

Unfortunately we are only able to show that some monodromy matrix is semistrictly exponential in the generic case of Corollary 1.2 of the introduction.

*Proof of Corollary 1.2.* — By Theorem 1.1, the Laplace transform has locally finite branching (Definition 6.2). Choose $M$ big enough so that one goes all the way around $\mathbf{hull}(m)$ with a path of length $\leqslant M$. Let $S_M^{\mathrm{real}} \subset S_M$ be the subset of non-removable singularities of the Laplace transform attainable by a path of length $\leqslant M$ (which is finite because $S_M$ is finite). Then $f$ admits an analytic continuation to an open half-plane if and only if this half-plane doesn't meet $S_M^{\mathrm{real}}$. Therefore $\mathbf{hull}(m)$ is a polygon.

We show by specialization that for some fundamental group elements at least, $\mathbf{hull}(m)$ is not reduced to a single point. General considerations using Hartogs' theorem show that if the monodromy is semistrictly exponential for a special curve going to infinity, then the same will be true away from a piecewise holomorphic real codimension 2 divisor.

We choose as special curve the family of connections on the trivial bundle of the form $d+B+tA$ with $A$ diagonal and $B$ off-diagonal, everything being holmorphic on $X$, that was originally considered in [**36**]. In that case, we get asymptotic expansions whose coefficients can be calculated. One route is to note that for generic values of $A$ and $B$, calculation of the coefficients gives nonzero coefficients at more than one singular point. Another route would be to note that if there were only one singularity for the monodromy matrices for this family, then the monodromy representation would actually have polynomial growth. That possibility is ruled out by specializing again to a direct sum of a $2\times 2$ system and trivial systems, and noting that for $2\times 2$ systems we have proven (in the paper [**35**]) that the monodromy representation always has growth at least $e^{t^{1/k}}$ for some integer $k$.

In any case by either of these two routes we can conclude that the Laplace transform for at least one monodromy matrix has at least two singularities. □

It is perhaps more interesting to note that the same thing also works for the Procesi coordinates. This improves, at least for certain generic points at infinity approached from certain sectors, the bound given in [**35**].

**Corollary 14.3**. — *For each family $(E, \nabla + t\theta)$ going to infinity at a generic Higgs bundle $(E,\theta)$, let $\rho_t$ denote the family of monodromy representations, thought of as a point in $M_B$. Let $R_i : M_B \to \mathbb{C}$ denote a set of Procesi coordinates giving an affine embedding. Write by abuse of notation $R_i(t) := R_i(\rho_t)$. Then each $\mathbf{hull}(R_i)$ is a polygon, and for general $(E,\theta)$ (in a dense open set) at least one $R_i$ is semistrictly exponential (i.e. its hull has at least two vertices). If we define $|\rho_t| := \sup_i |R_i(t)|$ then for general $(E,\theta)$ and for a family of subsequences $t \to \infty$ covering all but possibly* *two opposite directions, we have bounds of the form*

$$|\rho_t| \geqslant ce^{a|t|}$$

*with* $a > 0$.

*Proof.* — The same proof as for Corollary 1.2 works here too. □

Lastly it is important to reiterate that, in spite of the above consequences, the result of Theorem 6.3 is highly unsatisfactory in that it doesn't say anything about the behavior of the Laplace transform $f(\zeta)$ near the singularities. It doesn't even seem clear what the answer will be: on the one hand one can imagine that an improvement of the present analysis, potentially based on Remark (i) following the proof of Lemma 5.1, might lead to a polynomial bound for the singularities. On the other hand, a crude look at the present argument yields no such bound, and it is also quite concievable that the poles in the matrix $B$ lead unavoidably to more complicated singularities of $f(\zeta)$. This is undoubtedly true in the general case where $B$ has poles of order $> 1$.

This problem also leads to the unsatisfactory statement of Corollary 14.3: if we could calculate exactly where the singularities were we could probably show that for generic values of $(E, \theta)$ the singularities would span a convex hull with nonempty interior, in other words that the monodromy families $\rho_t$ would be strictly exponential. This would be a more significant improvement of the result of [**35**].

The result of Theorem 6.3 should be thought of as a weak form of "resurgence" for the monodromy function $m(t)$ and its Laplace transform. The problem of getting more precise information about this behaviour is probably most naturally attacked using new ideas and techniques for resummation such as have been developped by the school of J.-P. Ramis.

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