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THE TWISTED WEIGHTED FUNDAMENTAL LEMMA FOR THE TRANSFER OF AUTOMORPHIC FORMS FROM GSp(4) TO GL(4)

by

David Whitehouse

Abstract. — We prove the twisted weighted fundamental lemma for the group $\mathrm{GL}(4) \times \mathrm{GL}(1)$ relative to a certain outer automorphism α, which yields $\mathrm{GSp}(4)$ as a twisted endoscopic group. This version of the fundamental lemma is needed to stabilize the twisted trace formula for the pair $(\mathrm{GL}(4) \times \mathrm{GL}(1), \alpha)$. This stabilized twisted trace formula is required for Arthur's classification of the discrete spectrum of $\mathrm{GSp}(4)$ in terms of automorphic representations of $\mathrm{GL}(4)$.

Résumé **(Le lemme fondamental tordu pondéré pour le transfert des formes automorphes de** $\mathrm{GSp}(4)$ **à** $\mathrm{GL}(4)$**)**

Nous démontrons le lemme fondamental tordu pondéré pour le groupe $\mathrm{GL}(4) \times \mathrm{GL}(1)$ relativement à un certain automorphisme extérieur α qui permet de décrire $\mathrm{GSp}(4)$ comme groupe endoscopique tordu. Cette version du lemme fondamental est nécessaire pour stabiliser la formule des traces tordue pour le couple $(\mathrm{GL}(4) \times \mathrm{GL}(1), \alpha)$. Cette formule des traces tordues est requise pour la classification d'Arthur du spectre discret de $\mathrm{GSp}(4)$ en termes des représentations automorphes de $\mathrm{GL}(4)$.

1. Introduction

Langlands' functoriality conjecture predicts, in a very precise way, relationships between automorphic representations on different groups. The trace formula is an important tool in proving such relationships. For a reductive group G the trace formula (see [**Art88a**]) gives two expressions for a certain linear form $I(f)$; here f is a suitable function on the adelic points of G. One expression, the geometric side of the trace formula, is given as a sum over conjugacy classes of terms involving orbital integrals, while the other, the spectral side of the trace formula, expresses $I(f)$ in terms associated to the automorphic representations of G. Therefore as the group G is allowed to vary identities between geometric sides produce identities between spectral

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sides; out of these one can hope to deduce the relationships between automorphic representations as suggested by Langlands' functoriality conjecture.

Suppose now we are given two groups G_1 and G_2 defined over a number field F. In order to compare the trace formulas for these groups one needs to be able to compare the conjugacy classes in $G_1(F)$ and $G_2(F)$ and to be able to transfer functions from one group to the other given by identities between orbital integrals. In practice, however, for example in the case that G_1 and G_2 are inner forms of each other, one is not quite able to do this. Instead one can only carry out these comparisons over the algebraic closure of F. One is therefore lead to the need for a refinement of this trace formula to a stable trace formula; one in which the geometric side is indexed by stable conjugacy classes and given in terms of stable orbital integrals.

The stabilization of the trace formula was initiated by Langlands in [**Lan83**]. The first problem one encounters is that the distribution $I(f)$ is not stable. In [**Lan83**] Langlands suggested a stabilization of the form

$$I(f) = \sum_H \iota(G,H) S^H(f^H).$$

This sum is over a family of groups $\{H\}$, called elliptic endoscopic groups, attached to G. The distributions S^H are themselves stable and Langlands conjectured a transfer of functions $f \mapsto f^H$ from G to H. One can then hope to compare these stable distributions for different groups. The first case considered was that of SL(2) by Labesse and Langlands in [**LL79**]. For general G, the stabilization of the regular elliptic part of the trace formula was carried out by Langlands in [**Lan83**] under the assumption of a transfer of functions $f \mapsto f^H$. The stabilization of the elliptic singular terms was carried out by Kottwitz in [**Kot86**]. Building on the work of Kottwitz and Langlands, Arthur has now stabilized the full trace formula in a series of papers [**Art02**], [**Art01**] and [**Art03**] under the assumption of certain local conjectures, known as fundamental lemmas, for orbital integrals and weighted orbital integrals; see [**Art02**, Section 5].

The fundamental lemma for orbital integrals has been established in certain cases. It is known for a few groups of low rank, namely for SL(2) by [**LL79**], U(3) by [**Rog90**] and for Sp(4) and GSp(4) by [**Hal97**]. The fundamental lemma has also been established for certain families of groups, for SL(n) by [**Wal91**] and for unitary groups by [**LN04**]. As mentioned in [**Art02**, Section 5] much less is known about the generalization of the fundamental lemma to weighted orbital integrals.

In this paper we are interested in the stabilization of a twisted trace formula. Such a trace formula applies to a group together with an automorphism. The stabilization of the twisted trace formula was begun by Kottwitz and Shelstad in [**KS99**]. For the stabilization of the full twisted trace formula one needs to prove fundamental lemmas for twisted weighted orbital integrals. The statement of the twisted weighted fundamental lemmas is given in the appendix to this paper.

We now turn to the functorial transfer we are concerned with in this paper. We take the group GSp(4) over a number field F. The dual group of GSp(4) is GSp(4, $\mathbf{C}$) which has a natural inclusion into GL(4, $\mathbf{C}$). Associated to this map of dual groups functoriality suggests a transfer of automorphic representations from GSp(4)/F to GL(4)/F.

There has been much interest in this transfer. Unpublished work of Jacquet, Piatetski-Shapiro and Shalika produced this transfer for generic automorphic representations of GSp(4); this result is proven in [**AS**] using different methods. Results on the transfer from PGSp(4) to PGL(4) have been announced in [**Fli04**]. Flicker uses a special form of the trace formula valid only for certain test functions and so, as mentioned in his paper, the transfer is achieved only for automorphic representations satisfying certain local conditions. The transfer of all automorphic representations from GSp(4) to GL(4) is announced by Arthur in [**Art04**]. In this paper Arthur describes the results of his monograph [**Art**] in the case of GSp(4). The main theorem in [**Art04**] is phrased as a classification theorem for representations of GSp(4). This classification includes a parameterization of the representations of the local groups GSp(4, F_v) into packets together with a decomposition of the discrete spectrum of GSp(4) in terms of automorphic representations of GL(4).

The results of [**Art04**] are achieved by a comparison of the stable trace formula for GSp(4) with a stable twisted trace formula for GL(4) $\times$ GL(1) and a certain automorphism α given in Section 2.4 below. The stabilization of these trace formulas, and hence Arthur's result, is conditional on cases of the fundamental lemma. We now describe which fundamental lemmas are required.

For GSp(4) the fundamental lemma for invariant orbital integrals is proven in [**Hal97**]; see also [**Wei94**]. The weighted fundamental lemma in [**Art02**, Section 5] required for the stabilization of the full trace formula does not apply to GSp(4) since its proper Levi subgroups are products of general linear groups, and therefore do not possess proper elliptic endoscopic groups. Therefore, all the local conjectures required for the stabilization of the trace formula for GSp(4) have been established.

For the stabilization of the twisted trace formula for GL(4) $\times$ GL(1) and the automorphism α, the twisted fundamental lemma for invariant orbital integrals is proven in [**Fli99**]. Flicker's proof is for fields of odd residual characteristic, however, this is sufficient for global applications. A weighted variant of the twisted fundamental lemma, stated in the appendix, is also needed. This is because there are Levi subgroups of GL(4) $\times$ GL(1) that have elliptic twisted endoscopic groups. It is this fundamental lemma which we prove in this paper, we again restrict ourselves to local fields of odd residual characteristic.

The outline of this paper is as follows. We begin in Section 2 by giving some definitions and notations used throughout this paper.

The conjectured twisted weighted fundamental lemma is given by the identity

$$\sum_{k\in\Gamma_{G-\mathrm{reg}}(M(F))} \Delta_{M,K}(\ell',k)r_M^G(k) = \sum_{G'\in\mathcal{E}_{M'}(G)} \iota_{M'}(G,G')s_{M'}^{G'}(\ell').$$

The left hand side consists of a finite linear combination of twisted weighted orbital integrals on the group G^0 with respect to the Levi subset $M = M^0 \rtimes \alpha$. We take M' to be an elliptic twisted endoscopic group for M^0; the right hand side is then a finite linear combination of stable weighted orbital integrals on certain groups G' that contain M' as a Levi subgroup.

From Section 3 onwards we specialize to the twisted weighted fundamental lemma for G^0 equal to $\mathrm{GL}(4)\times\mathrm{GL}(1)$. We begin in Section 3 by determining all endoscopic groups that appear in the statement of the twisted weighted fundamental lemma, and in Section 4 we compute the necessary weight functions, which appear in our weighted orbital integrals.

As above, the twisted weighted fundamental lemma applies to a pair (M, M') of a Levi subset $M = M^0 \rtimes \alpha$ of $G = G^0 \rtimes \alpha$ and an unramified elliptic twisted endoscopic group M' for M^0. When $M^0 = G^0$ we recover the statement of the fundamental lemma proven in [**Fli99**], hence we only consider proper Levi subgroups M^0. There are four pairs (M, M') given in the table below, where E denotes the unramified quadratic extension of the local nonarchimedean field F.

M^0	M'
$(\mathrm{GL}(2)\times\mathrm{GL}(2))\times\mathrm{GL}(1)$	$\mathrm{GL}(2)\times\mathrm{GL}(1)$
$(\mathrm{GL}(1)\times\mathrm{GL}(2)\times\mathrm{GL}(1))\times\mathrm{GL}(1)$	$\mathrm{GL}(2)\times\mathrm{GL}(1)$
$(\mathrm{GL}(1)\times\mathrm{GL}(2)\times\mathrm{GL}(1))\times\mathrm{GL}(1)$	$\mathrm{Res}_{E/F}(\mathrm{GL}(1))\times\mathrm{GL}(1)$
$\mathrm{GL}(1)^4\times\mathrm{GL}(1)$	$\mathrm{GL}(1)^3$

The theorem we prove in this paper is:

Theorem 1.1. — *For each pair (M, M') as above the twisted weighted fundamental lemma is true over local fields of characteristic zero and odd residual characteristic.*

The proof of this theorem is given in Sections 5 through 8. We now outline the proof for each pair. We take F to be a local field of characteristic zero. We let R denote the ring of integers in F. We denote by q the cardinality of the residue field of F that for now we assume is odd and greater than three.

In Section 5 we prove the fundamental lemma for the first pair. We begin by writing both sides of the fundamental lemma in this case as untwisted orbital integrals on $\mathrm{GL}(2,F)$. The identity to be proven then takes the form

$$FL(A) : L(A) = R(A)$$

indexed by elements $A \in \mathrm{GL}(2,F)$. Moreover, since both sides vanish if the conjugacy class of A in $\mathrm{GL}(2,F)$ does not intersect $\mathrm{GL}(2,R)$ we may assume that $A \in \mathrm{GL}(2,R)$.

We split the proof into two cases depending on whether A lies in a split or elliptic torus. In the former case we may assume that

$$A = \begin{pmatrix} a & \\ & d \end{pmatrix}$$

is diagonal. We find that both $L(A)$ and $R(A)$ depend only on $|a-1|, |d-1|, |a-d|$ and $|ad-1|$. Since we are assuming that F has odd residual characteristic we have the following three cases

Case 1: $q^{-M} = |ad-1| = |a-d| = |d-1| \geqslant |a-1| = q^{-N}$
Case 2: $q^{-M} = |a-1| = |d-1| = |ad-1| \geqslant |a-d| = q^{-N}$
Case 3: $q^{-M} = |a-1| = |d-1| = |a-d| \geqslant |ad-1| = q^{-N}$.

In each case we denote $L(A)$ (resp. $R(A)$) by $L(M,N)$ (resp. $R(M,N)$). In cases 1 and 3 we prove that

$$qL(M,N+1) - L(M,N) = qR(M,N+1) - R(M,N)$$

and in case 2 we prove that

$$L(M,N+1) - L(M,N) = R(M,N+1) - R(M,N).$$

In each case we exploit cancellations between the integrals on either side of $FL(A)$ allowing us to readily compute the differences. Thus the proof of the identity $FL(A)$, when A lies in a split torus, is reduced to proving the identity under the assumption

$$|ad-1| = |a-d| = |d-1| = |a-1|.$$

We then compute both sides of $FL(A)$ under this assumption and show that they are equal. In the case that A lies in an elliptic torus we again reduce the proof to certain cases, which we then prove, by following a similar strategy.

The proofs of the fundamental lemma for the Levi subgroup

$$M^0 = (\mathrm{GL}(1) \times \mathrm{GL}(2) \times \mathrm{GL}(1)) \times \mathrm{GL}(1)$$

and both its unramified elliptic twisted endoscopic groups are given in Sections 6 and 7. The proof uses the twisted topological Jordan decomposition which is described in Section 4.5. We can write any element $\gamma\alpha$ with $\gamma \in M(R)$ uniquely as

$$\gamma\alpha = us\alpha = s\alpha u,$$

where $s\alpha$ has finite order prime to q and u is topologically unipotent, i.e., $u^{q^n} \to I$ as $n \to \infty$. Using this decomposition allows us to write the twisted weighted orbital integral at $\gamma\alpha$ as an (untwisted) weighted orbital integral at u on the group $G_{s\alpha}$, the centralizer of $s\alpha$ in G. The main part of the proof of the fundamental lemma is when s is the identity. In this case the twisted weighted orbital integrals become untwisted weighted orbital integrals on $\mathrm{Sp}(4)$. These integrals are of a type that appear on the right hand side of the fundamental lemma treated in Section 5. We are then able to use the calculations from there to prove the fundamental lemma for both pairs

(M, M'). When s is not the identity, the groups $G_{s\alpha}$ have dimension strictly smaller than Sp(4) and the fundamental lemma can be readily verified in these cases.

In Section 8 we prove the fundamental lemma for the diagonal torus in GL(4) $\times$ GL(1). We again use the twisted topological Jordan decomposition. The main part of the proof comes down to proving an identity between weighted orbital integrals on Sp(4) with respect to the diagonal torus. We establish this identity by exploiting cancellations between the relevant integrals on Sp(4).

We delay to Section 9 the computation of certain p-adic integrals that are needed in the proof of the fundamental lemma.

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2. Preliminaries

In this section we give some definitions and notations that will be used throughout this paper. Further to the notations introduced below we also adopt the notations introduced in the appendix throughout this paper.

2.1. Twisted conjugacy. — For the moment we take F to be a field of characteristic zero and G^0 to be a connected reductive algebraic group defined over F. We let α be a quasi-semisimple automorphism of G^0 by which we mean, as in [**KS99**, Section 1.1], an automorphism which preserves a pair (B, T) of a Borel subgroup B and maximal torus $T \subset B$.

An element $\gamma \in G^0$ is α-semisimple if the element $\gamma\alpha \in G = G^0 \rtimes \alpha$ is semisimple, i.e., the automorphism of G^0 given by $\mathrm{Int}(\gamma) \circ \alpha$ is quasi-semisimple in the sense above. The twisted conjugacy class of $\gamma \in G^0$ is

$$\left\{g^{-1}\gamma\alpha(g) : g \in G^0\right\}.$$

We note that for $g \in G^0$ we have

$$g^{-1}\gamma\alpha g = g^{-1}\gamma\alpha(g)\alpha,$$

and so the notion of twisted conjugacy of γ is equivalent to conjugacy of $\gamma\alpha$ by elements of G^0; these notions are used interchangeably. The twisted centralizer of $\gamma \in G^0$ is

$$Z_{G^0}(\gamma\alpha) = \left\{g \in G^0 : g^{-1}\gamma\alpha(g) = \gamma\right\}.$$

The element $\gamma\alpha$ is strongly regular if $Z_{G^0}(\gamma\alpha)$ is abelian. The connected component of $Z_{G^0}(\gamma\alpha)$ is denoted by $G_{\gamma\alpha}$.

If M^0 is a Levi subgroup of G^0 which is stable under the automorphism α then we say an element $\gamma \in M^0$ is strongly G^0-regular if it is strongly regular as an element of G^0.

2.2. GSp(4) **and** Sp(4). — We let J denote the matrix

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix},$$

and we set

$$\mathrm{Sp}(4) = \{g \in \mathrm{GL}(4) : J{}^t g^{-1} J^{-1} = g\},$$

and

$$\mathrm{GSp}(4) = \{g \in \mathrm{GL}(4) : J{}^t g^{-1} J^{-1} = \lambda g, \lambda \in \mathrm{GL}(1)\}.$$

The intersection with GSp(4) of the upper triangular Borel subgroup of GL(4) is a Borel subgroup of GSp(4). The proper parabolic subgroups of GSp(4) that contain this Borel subgroup are the Siegel parabolic, which has Levi decomposition

$$\left\{ \begin{pmatrix} g & \\ & aw{}^t g^{-1} w \end{pmatrix} \begin{pmatrix} 1 & & x & r \\ & 1 & r & s \\ & & 1 & \\ & & & 1 \end{pmatrix} : g \in \mathrm{GL}(2) \right\},$$

where

$$w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

and the Klingen parabolic which has Levi decomposition

$$\left\{ \begin{pmatrix} a & & \\ & g & \\ & & a^{-1}\det g \end{pmatrix} \begin{pmatrix} 1 & x & r & s \\ & 1 & & r \\ & & 1 & -x \\ & & & 1 \end{pmatrix} : g \in \mathrm{GL}(2) \right\}.$$

The intersection of each of these parabolic subgroups with Sp(4) is a parabolic subgroup of Sp(4), we refer to their intersection with Sp(4) by the same name.

The dual group of GSp(4) is $\mathrm{GSp}(4, \mathbf{C})$ and under the bijection between parabolic subgroups of G and $\widehat{G}$ the Siegel and Klingen parabolics are interchanged.

2.3. Notation. — From now on F will denote a local nonarchimedean field of characteristic zero. We let R denote the ring of integers in F and we let π denote a uniformizer in R. We fix the Haar measure on F that gives R volume one. We let q denote the order of the residue field of F, which we take to have characteristic p. In Sections 5 through 8 we assume p is odd. We use v and $|\ |$ to denote the additive and multiplicative valuations on F, which we normalized so that $v(\pi) = 1$ and $|\pi| = q^{-1}$.

We let U_F denote the group of units in R and U_F^m denotes the subgroups of U_F defined by

$$U_F^m = \begin{cases} U_F, & \text{if } m = 0; \\ 1 + \pi^m R, & \text{if } m > 0. \end{cases}$$

We fix an algebraic closure $\overline{F}$ of F and denote again by $|\ |$ the extension of $|\ |$ to $\overline{F}$. The letter Γ is used to denote the Galois group $\mathrm{Gal}(\overline{F}/F)$. For a group G_1 with an action of $\mathrm{Gal}(\overline{F}/F)$ we let G_1^Γ denote the elements of G_1 that are fixed by Γ.

We use ln to denote the natural logarithm and log to denote the logarithm to the base q.

For an algebraic group H we let $X(H)$ denote the group of characters of H and H^0 denotes the connected component of the identity in H. For a finite extension of fields E/F and an algebraic group H defined over E we let $\mathrm{Res}_{E/F} H$ denote the restriction of scalars of H to F.

For a compact open subgroup K of a p-adic group we use 1_K or char_K to denote the characteristic function of K.

For ease of notation we frequently use blank entries in matrices to denote zeros. Given $A_i \in \mathrm{GL}(n_i)$, $1 \leqslant i \leqslant k$ we let $\mathrm{diag}(A_1, \ldots, A_k)$ denote the block diagonal matrix in $\mathrm{GL}(n_1 + \ldots + n_k)$ with block diagonal entries $A_1, \ldots, A_k$.

2.4. Our case. — We now describe the situation we are considering in this paper. We take $G^0 = \mathrm{GL}(4) \times \mathrm{GL}(1)$ over the local field F and we take α to be the automorphism of G^0 given by

$$\alpha : (g, e) \longmapsto (J^t g^{-1} J^{-1}, e \det g),$$

where J is as above. We set $G^+ = G^0 \rtimes \langle\alpha\rangle$, $G = G^0 \rtimes \alpha$ and $K = G^0(R)$.

The dual group of G^0 is $\widehat{G}^0 = \mathrm{GL}(4, \mathbf{C}) \times \mathrm{GL}(1, \mathbf{C})$. Since the automorphism α is quasi-semisimple it induces an automorphism $\widehat{\alpha}$ on $\widehat{G}^0$ as in [**KS99**, Section 1.2]. This automorphism is given by

$$\widehat{\alpha} : (h, t) \longmapsto (t J^t h^{-1} J^{-1}, t).$$

The proper standard parabolics P^0 of G^0, which are stable under α are those whose projection onto GL(4) are of the form

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}, \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix}, \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}.$$

We take the Levi component M^0 in each of these parabolic subgroups that contains the diagonal torus in G^0. We refer to these Levi subgroups as the (2,2) Levi, the (1,2,1) Levi and the diagonal Levi.

The transfer factor Δ which appears in the statement of the twisted weighted fundamental lemma is the product of the terms Δ_{I}, Δ_{II} and Δ_{III} from [**KS99**, Chapters 4 & 5].

Let M' be a twisted endoscopic group for M^0. The stable twisted conjugacy class of a strongly regular element $\delta \in M^0(F)$ is the intersection of $M^0(F)$ with the twisted conjugacy class of δ in $M^0(\overline{F})$. The stable conjugacy class of a strongly regular element of $M'(F)$ is defined similarly. By [**KS99**, Theorem 3.3.A] we have a map

$$\mathcal{A}_{M'/M} : \mathrm{Cl}_{\mathrm{ss}}(M') \longrightarrow \mathrm{Cl}_{\mathrm{ss}}(M^0, \alpha)$$

between semisimple stable conjugacy classes of M' and M. The semisimple element $\gamma \in M'(F)$ is called strongly G^0-regular if the image of the conjugacy class of γ under this map is strongly G^0-regular is the sense above.

The integrals $r_M^G(\gamma\alpha)$ depend on the choice of a measure on $G_{\gamma\alpha}$, there is a similar such dependence in the definition of $s_{M'}^{G'}(\ell')$. Within a stable conjugacy class these measures are chosen so that stable conjugacy is measure preserving. Having done this, if we are now given $\gamma\alpha \in M(F)$ and $\gamma' \in M'(F)$, such that $\Delta(\gamma', \gamma) \neq 0$, we normalize the measures on $M_{\gamma\alpha}$ and $M'_{\gamma'}$ such that under this normalization the (unweighted) twisted fundamental lemma holds for the pair (M, M').

3. Endoscopic groups

We now determine the unramified elliptic twisted endoscopic groups M' for each of the Levi subgroups M^0 of G^0 given in Section 2.4. We refer to [**KS99**, Section 2.1] for the definition of twisted endoscopic groups. For each such endoscopic group M' we also compute the set of elliptic twisted endoscopic groups for G^0 in $\mathcal{E}_{M'}(G)$, which contain M' as a Levi subgroup; and for each group G' in $\mathcal{E}_{M'}(G)$ we compute the coefficient $\iota_{M'}(G, G')$. We note that for non-elliptic endoscopic groups in $\mathcal{E}_{M'}(G)$ the coefficient $\iota_{M'}(G, G')$ is zero.

The elliptic twisted endoscopic groups for G^0 itself are computed in [**Fli99**, Section I.F]; these results are recalled in Section 3.1. We use these results below in computing the sets $\mathcal{E}_{M'}(G)$ and the norm maps from M to M'.

3.1. Twisted endoscopic groups for $\mathrm{GL}(4) \times \mathrm{GL}(1)$. — In this section we recall results from [**Fli99**, Section I.F] on the twisted endoscopic groups for G^0. First we note that given $s\widehat{\alpha} \in \widehat{G}$, assumed semisimple, the twisted centralizer $Z_{\widehat{G}^0}(s\widehat{\alpha})$ depends only on the component of s lying in $\mathrm{GL}(4, \mathbf{C})$. Moreover, after twisted conjugation, we can assume that we have

$$s = (\mathrm{diag}(1, 1, c, d), 1)\,.$$

Furthermore the $\widehat{\alpha}$-conjugacy class of s does not change if c is replaced by c^{-1}, d by d^{-1} and (c, d) by (d, c). We recall that a twisted endoscopic group H is called elliptic if $(Z(\widehat{H})^\Gamma)^0$ is contained in $Z(\widehat{G}^0)$. The elliptic twisted endoscopic groups of G^0 are given below.

(1) $c = d = 1$: The twisted centralizer of s is isomorphic to $\mathrm{GSp}(4, \mathbf{C})$ and we get GSp(4) as a twisted endoscopic group.

(2) $c = d = -1$: The connected component of the twisted centralizer of s is isomorphic to $\mathrm{GL}(2, \mathbf{C})^2/\mathbf{C}^\times$ with $\mathbf{C}^\times$ embedded via $z \mapsto (z, z^{-1})$. If we have a trivial Galois action then we obtain $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$, where the prime denotes the subgroup of pairs (A, B) with $\det A = \det B$, as a twisted endoscopic group. We can also have a non-trivial Galois action with Γ acting through a quadratic extension E/F in which case we obtain $\mathrm{Res}_{E/F}\,\mathrm{GL}(2)'$, with the prime here denoting determinant in $F^\times$, as a twisted endoscopic group.

(3) $c = 1$, $d = -1$: The connected component of the twisted centralizer of s is isomorphic to $(\mathrm{GL}(2, \mathbf{C}) \times \mathrm{GL}(1, \mathbf{C})^2)'$ with the prime denoting the subgroup of triples (A, a, b) with $\det A = ab$. In this case we only obtain elliptic endoscopic datum if Γ acts through a quadratic extension E/F; in which case we obtain $(\mathrm{GL}(2) \times \mathrm{Res}_{E/F}\,\mathrm{GL}(1))/\,\mathrm{GL}(1)$, with GL(1) embedded as (z, z^{-1}), as a twisted endoscopic group.

Let H be a twisted endoscopic group for G^0 and let T_H denote the diagonal torus in H. Let T denote the diagonal torus in G^0. For each such group H and $\gamma = (\mathrm{diag}(x, y, z, t), w) \in T$ the image of $\gamma\alpha$ under the norm map $N : T \to T_H$ is given below.

(1) GSp(4): $N(\gamma\alpha) = \mathrm{diag}(xyw, xzw, tyw, ztw)$
(2) $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$: $N(\gamma\alpha) = (\mathrm{diag}(xyw, ztw), \mathrm{diag}(xzw, ytw))$
(3) $\mathrm{Res}_{E/F}\,\mathrm{GL}(2)'$: $N(\gamma\alpha) = (\mathrm{diag}(xyw, ztw), \mathrm{diag}(xzw, ytw))$
(4) $(\mathrm{GL}(2) \times \mathrm{Res}_{E/F}\,\mathrm{GL}(1))/\,\mathrm{GL}(1)$: $N(\gamma\alpha) = (\mathrm{diag}(xw, tw), y, z)$.

3.2. Twisted endoscopic groups for the (2,2) Levi. — In this section we take M^0 to be the (2,2) Levi in G^0. We have $\widehat{M}^0 = \mathrm{GL}(2, \mathbf{C}) \times \mathrm{GL}(2, \mathbf{C}) \times \mathrm{GL}(1, \mathbf{C})$, which sits inside $\widehat{G}^0$ as the (2,2) Levi. The restriction of $\widehat{\alpha}$ to $\widehat{M}^0$ is given by

$$(A, B, t) \longmapsto (tw{}^tB^{-1}w, tw{}^tA^{-1}w, t),$$

where

$$w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

Lemma 3.1. — *The only elliptic twisted endoscopic group for* M^0 *is* $\mathrm{GL}(2) \times \mathrm{GL}(1)$.

Proof. — Let $s \in \widehat{M}^0$ be such that $s\widehat{\alpha}$ is semisimple. We may assume that s is diagonal and, after twisted conjugacy in $\widehat{M}^0$, we can assume that it is of the form

$$s = (\mathrm{diag}(1, 1, \lambda_1, \lambda_2), s_2).$$

We now compute $Z_{\widehat{M}^0}(s\widehat{\alpha})$. We see that $(A, B, t) \in Z_{\widehat{M}^0}(s\widehat{\alpha})$ if and only if we have

$$Aw^tBw = \begin{pmatrix} t & \\ & t \end{pmatrix}$$

and

$$B\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} w^tAw = \begin{pmatrix} t\lambda_1 & \\ & t\lambda_2 \end{pmatrix}.$$

This is the case if and only if we have $A = tw^tB^{-1}w$ and

$$B\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} B^{-1} = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}.$$

So if $\lambda_1 = \lambda_2$, then we have

$$Z_{\widehat{M}^0}(s\widehat{\alpha}) = \left\{ \left(A, tw^tA^{-1}w, t\right) \in \widehat{M}^0 : A \in \mathrm{GL}(2, \mathbf{C}), t \in \mathbf{C}^\times \right\},$$

while if $\lambda_1 \neq \lambda_2$, then we have

$$Z_{\widehat{M}^0}(s\widehat{\alpha}) = \left\{ \left(\begin{pmatrix} x & \\ & y \end{pmatrix}, \begin{pmatrix} ty^{-1} & \\ & tx^{-1} \end{pmatrix}, t \right) \in \widehat{M}^0 : x, y, t \in \mathbf{C}^\times \right\}.$$

Both of these centralizers are connected; hence we can only have a trivial Galois action. Therefore only when we have $\lambda_1 = \lambda_2$ do we get elliptic twisted endoscopic data for M^0. In this case we have $Z_{\widehat{M}^0}(s\widehat{\alpha}) \cong \mathrm{GL}(2, \mathbf{C}) \times \mathrm{GL}(1, \mathbf{C})$ and hence we get $\mathrm{GL}(2) \times \mathrm{GL}(1)$ as a twisted endoscopic group for M^0. □

We now compute $\mathcal{E}_{M'}(G)$.

Lemma 3.2. — *Let M' represent the elliptic twisted endoscopic datum for M^0. Then the elliptic twisted endoscopic groups for G^0 in $\mathcal{E}_{M'}(G)$ are* GSp(4) *and* (GL(2) × GL(2))′*, the prime denoting the subgroup of pairs (A, B) with* $\det A = \det B$*. Each group occurs with multiplicity one.*

Proof. — We may as well take $s = (I, I, 1) \in \widehat{M}^0$ which gives rise to $M' = \mathrm{GL}(2) \times \mathrm{GL}(1)$. We need to look at the translations of $s\widehat{\alpha}$ by elements in $Z(\widehat{M})$ taken modulo $Z(\widehat{G})$. We have

$$Z(\widehat{M}) = \{(\mathrm{diag}(a, a, b, b), ab)\}$$

and

$$Z(\widehat{G}) = \left\{ \left(\mathrm{diag}(a, a, a, a), a^2\right) \right\}.$$

Thus we need to look for elliptic endoscopic datum for G^0 arising from elements of the form $(\mathrm{diag}(1, 1, \lambda, \lambda), \lambda)\widehat{\alpha} \in \widehat{G}$. So we get endoscopic datum only when we have $\lambda = \pm 1$ and we must have a trivial Galois action in both cases. □

We note that $\widehat{M}'$ sits inside $\mathrm{GSp}(4, \mathbf{C})$ as the Siegel Levi, hence we have $M' = \mathrm{GL}(2) \times \mathrm{GL}(1)$ sitting inside GSp(4) as the Klingen Levi. We have M' sitting inside $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$ as $(T \times \mathrm{GL}(2))'$ where T is the diagonal torus in GL(2) and the prime again denotes the subgroup of pairs with equal determinant. The coefficients $\iota_{M'}(G, G')$ are equal to 1 for G' equal to GSp(4) and $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$.

3.3. Twisted endoscopic groups for the (1,2,1) Levi. — In this section we take M^0 to be the (1,2,1) Levi in G^0. We have

$$\widehat{M}^0 = \mathrm{GL}(1, \mathbf{C}) \times \mathrm{GL}(2, \mathbf{C}) \times \mathrm{GL}(1, \mathbf{C}) \times \mathrm{GL}(1, \mathbf{C}),$$

which sits inside $\widehat{G}^0$ as the (1,2,1) Levi. The restriction of $\widehat{\alpha}$ to $\widehat{M}^0$ is given by

$$(a, g, b, t) \longmapsto (tb^{-1}, t(\det g)^{-1}g, ta^{-1}, t).$$

Lemma 3.3. — *The unramified elliptic twisted endoscopic groups for M^0 are* $\mathrm{GL}(2) \times \mathrm{GL}(1)$ *and* $\mathrm{GL}(1) \times \mathrm{Res}_{E/F}\,\mathrm{GL}(1)$*, where E/F is the unramified quadratic extension.*

Proof. — After twisted conjugacy in $\widehat{M}^0$ we can assume that we have

$$s = \left(1, \begin{pmatrix} 1 & \\ & \lambda_1 \end{pmatrix}, \lambda_2, s_2\right).$$

Then $(a, g, b, t) \in Z_{\widehat{M}^0}(s\widehat{\alpha})$ if and only if

$$\left(ab, g\begin{pmatrix} 1 & \\ & \lambda_1 \end{pmatrix}(\det g)g^{-1}, ab\lambda_2\right) = \left(t, \begin{pmatrix} t & \\ & t\lambda_1 \end{pmatrix}, t\lambda_2\right).$$

Hence we need $ab = t$ and

$$g\begin{pmatrix} 1 & \\ & \lambda_1 \end{pmatrix}g^{-1} = \begin{pmatrix} \det g^{-1}t & \\ & \det g^{-1}t\lambda_1 \end{pmatrix}.$$

Therefore if $\lambda_1 = 1$, then we have g is any element of $\mathrm{GL}(2, \mathbf{C})$, while if $\lambda_1 \neq 1$, then

$$g \in \left\{\begin{pmatrix} x & \\ & y \end{pmatrix}, \begin{pmatrix} & z \\ w & \end{pmatrix}\right\}.$$

Thus we see that if $\lambda_1 = 1$, then

$$Z_{\widehat{M}^0}(s\widehat{\alpha}) = \left\{(a, g, a^{-1}\det g, \det g) \in \widehat{M}^0 : g \in \mathrm{GL}(2, \mathbf{C}), a \in \mathbf{C}^\times\right\},$$

while if $\lambda_1 = -1$, then

$$Z_{\widehat{M}^0}(s\widehat{\alpha}) = \left\{\left(a, \begin{pmatrix} x & \\ & y \end{pmatrix}, a^{-1}xy, xy\right)\right\} \bigcup \left\{\left(a, \begin{pmatrix} & x \\ y & \end{pmatrix}, a^{-1}xy, xy\right)\right\},$$

and if $\lambda_1 \neq \pm 1$, then

$$Z_{\widehat{M}^0}(s\widehat{\alpha}) = \left\{\left(a, \begin{pmatrix} x & \\ & y \end{pmatrix}, a^{-1}xy, xy\right)\right\}.$$

When $\lambda_1 = 1$ we have a connected centralizer and hence we have a trivial Galois action. In this case we have elliptic endoscopic data and we get $\mathrm{GL}(2) \times \mathrm{GL}(1)$ as a twisted endoscopic group for M^0.

When $\lambda_1 = -1$ to get elliptic endoscopic datum we need to have a non-trivial Galois action acting through a quadratic extension by

$$\left(a, \begin{pmatrix} x & \\ & y \end{pmatrix}, a^{-1}xy, xy, xy\right) \longmapsto \left(a, \begin{pmatrix} y & \\ & x \end{pmatrix}, a^{-1}xy, xy\right).$$

In order for our endoscopic data to be unramified we need this quadratic extension to be unramified. In this case we get $\mathrm{GL}(1) \times \mathrm{Res}_{E/F}\,\mathrm{GL}(1)$ as a twisted endoscopic group for M^0.

Finally, when $\lambda_1 \neq \pm 1$ the data is never elliptic. □

We now compute $\mathcal{E}_{M'}(G)$ for $M' = \mathrm{GL}(2) \times \mathrm{GL}(1)$.

Lemma 3.4. — *Let $M' = \mathrm{GL}(2) \times \mathrm{GL}(1)$. Then the only elliptic twisted endoscopic group for G^0 in $\mathcal{E}_{M'}(G)$ is $\mathrm{GSp}(4)$ with multiplicity two.*

Proof. — Recall that M' is given by the element $s\widehat{\alpha} = (\mathrm{diag}(1,1,1,\lambda_2), s_2)\widehat{\alpha} \in \widehat{M}$. We have

$$Z(\widehat{M}) = \left\{\left(\mathrm{diag}(a, c, c, a^{-1}c^2), c^2\right)\right\}$$

and so we need to look for elliptic twisted endoscopic groups for G^0 given by translating $s\widehat{\alpha}$ by elements of the form $\left(\mathrm{diag}(1, \lambda, \lambda, \lambda^2), \lambda^2\right) \in \widehat{G}^0$. Thus we need to look at elements of the form $\left(\mathrm{diag}(1, \lambda, \lambda, \lambda^2\lambda_2), \lambda^2 s_2\right)\widehat{\alpha} \in \widehat{G}$. After twisted conjugacy we can look at the elements of the form $\left(\mathrm{diag}(1, 1, 1, \lambda^2\lambda_2), s_2\right)\widehat{\alpha}$. Since we must have a trivial Galois action we get elliptic endoscopic data if and only if $\lambda^2 = \lambda_2^{-1}$; in which case we get $\mathrm{GSp}(4)$. □

We have $\widehat{M'}$ sitting inside $\mathrm{GSp}(4, \mathbf{C})$ as the Klingen Levi and so we get $M' = \mathrm{GL}(2) \times \mathrm{GL}(1)$ sitting inside $\mathrm{GSp}(4)$ as the Siegel Levi. We also have $\iota_{M'}(G, \mathrm{GSp}(4)) = 1$.

Lemma 3.5. — *Let $M' = \mathrm{GL}(1) \times \mathrm{Res}_{E/F}\,\mathrm{GL}(1)$. Then the elliptic twisted endoscopic groups for G^0 in $\mathcal{E}_{M'}(G)$ consists of $(\mathrm{GL}(2) \times \mathrm{Res}_{E/F}\,\mathrm{GL}(1))/\,\mathrm{GL}(1)$ and $\mathrm{Res}_{E/F}\,\mathrm{GL}(2)'$. Each group appears with multiplicity two.*

Proof. — Recall that M' is given by the element $(\mathrm{diag}(1, 1, -1, \lambda_2), s_2)\,\widehat{\alpha} \in \widehat{M}$. We need to look for elliptic twisted endoscopic groups for G^0 given by translating $s\widehat{\alpha}$ by elements of the form

$$\left(\mathrm{diag}(1, \lambda, \lambda, \lambda^2), \lambda^2\right).$$

Thus we need to look at elements of the form $\left(\mathrm{diag}(1, \lambda, -\lambda, \lambda^2\lambda_2), \lambda^2 s_2\right)\widehat{\alpha} \in \widehat{G}$. After conjugacy we can look at the elements $\left(\mathrm{diag}(1, 1, -1, \lambda^2\lambda_2), s_2\right)\widehat{\alpha} \in \widehat{G}$. Thus we get elliptic data if $\lambda^2 = \pm\lambda_2^{-1}$. When $\lambda^2 = \lambda_2^{-1}$ we get $(\mathrm{GL}(2) \times \mathrm{Res}_{E/F}\,\mathrm{GL}(1))/\,\mathrm{GL}(1)$, while if $\lambda^2 = -\lambda_2^{-1}$ we get $\mathrm{Res}_{E/F}\,\mathrm{GL}(2)'$. □

In this case we have M' sitting inside each group in $\mathcal{E}_{M'}(G)$ as the diagonal torus. And we have $\iota_{M'}(G,G') = 1$ for both $G' = (\mathrm{GL}(2) \times \mathrm{Res}_{E/F}\mathrm{GL}(1))/\mathrm{GL}(1)$ and $G' = \mathrm{Res}_{E/F}\mathrm{GL}(2)'$.

3.4. Twisted endoscopic groups for the diagonal Levi. — We now take M^0 to be the diagonal torus in G^0. We have $\widehat{M}^0 = \mathrm{GL}(1,\mathbf{C})^5$, which sits inside $\widehat{G}^0$ as the diagonal torus.

Lemma 3.6. — *The unramified elliptic twisted endoscopic group for M^0 is $\mathrm{GL}(1)^3$.*

Proof. — Since $\widehat{M}^0$ is abelian we see that for any $s \in \widehat{M}^0$ we have

$$Z_{\widehat{M}^0}(s\widehat{\alpha}) = \{(x,y,z,w,t) \in \widehat{M}^0 : xw = yz = t\}.$$

Hence we have $Z_{\widehat{M}^0}(s\widehat{\alpha}) \cong (\mathbf{C}^\times)^3$ and we get $\mathrm{GL}(1)^3$ as the only twisted endoscopic group for M^0. Furthermore, it is both elliptic and unramified. □

We now compute $\mathcal{E}_{M'}(G)$.

Lemma 3.7. — *Let $M' = \mathrm{GL}(1)^3$. Then the elliptic twisted endoscopic groups for G^0 in $\mathcal{E}_{M'}(G)$ are $\mathrm{GSp}(4)$ and $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$; each group appears with multiplicity two.*

Proof. — We have

$$Z(\widehat{M}) = \left\{\left(\mathrm{diag}(x,y,ty^{-1},tx^{-1}),t\right)\right\}.$$

Thus we need to look for the elliptic twisted endoscopic groups for G^0 given by elements of the form

$$\left(\mathrm{diag}(1,y^{-1},yw,w),w\right)\widehat{\alpha} \in \widehat{G}.$$

We can conjugate such an element to $\left(\mathrm{diag}(1,1,y^2w,w),w\right)\widehat{\alpha}$. Since we must have a trivial Galois action we get elliptic data when we have $w = 1$ and $y^2 = 1$, in which case we get $\mathrm{GSp}(4)$, or when we have $w = -1$ and $y^2 = 1$, in which case we get $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$. □

For G' equal to both $\mathrm{GSp}(4)$ and $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$ we have M' sitting inside as the diagonal torus and we have $\iota_{M'}(G,G') = 1$

3.5. Endoscopic groups for $\mathrm{GSp}(4)$. — We will also need to know the endoscopic groups for $\mathrm{GSp}(4)$. There is only one proper elliptic endoscopic group for $\mathrm{GSp}(4)$ namely $(\mathrm{GL}(2) \times \mathrm{GL}(2))/\mathrm{GL}(1)$ with $\mathrm{GL}(1)$ embedded as $a \mapsto (a,a^{-1})$, see [**Fli99**, Section 1.F]. It is given by the element $\mathrm{diag}(1,-1,-1,1) \in \mathrm{GSp}(4,\mathbf{C})$. The norm map is given by

$$\mathrm{diag}(a,b,cb^{-1},ca^{-1}) \longmapsto \left(\mathrm{diag}(1,(ab)^{-1}c),\mathrm{diag}(a,b)\right).$$

For each proper Levi subgroup M of $\mathrm{GSp}(4)$ we also need to compute the elliptic endoscopic groups for $\mathrm{GSp}(4)$ in $\mathcal{E}_M(\mathrm{GSp}(4))$. Since we are taking M as an endoscopic group for itself the elements of $\mathcal{E}_M(\mathrm{GSp}(4))$ are given by elements $s \in Z(\widehat{M})$ taken modulo translation by $Z(\mathrm{GSp}(4,\mathbf{C}))$, which equals $\{\mathrm{diag}(x,x,x,x)\}$.

Lemma 3.8. — *Let M be the Siegel Levi in* $\mathrm{GSp}(4)$. *Then the elliptic endoscopic groups in* $\mathcal{E}_M(\mathrm{GSp}(4))$ *are* $\mathrm{GSp}(4)$ *and* $(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)$ *each with multiplicity one.*

Proof. — We have $\widehat{M}$ sitting inside $\mathrm{GSp}(4,\mathbf{C})$ as the Klingen Levi. So we have

$$Z(\widehat{M}) = \left\{\mathrm{diag}(x,y,y,x^{-1}y^2)\right\}.$$

And we get that the elliptic endoscopic groups in $\mathcal{E}_M(\mathrm{GSp}(4))$ are $\mathrm{GSp}(4)$ and $(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)$ each with multiplicity one. □

We have M sitting inside $(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)$ as $(T\times\mathrm{GL}(2))/\mathrm{GL}(1)$ where T is the diagonal torus in $\mathrm{GL}(2)$. And we have $\iota_M(\mathrm{GSp}(4),(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)) = \frac{1}{2}$.

Lemma 3.9. — *Let M be the Klingen Levi in* $\mathrm{GSp}(4)$. *Then the only elliptic endoscopic group in* $\mathcal{E}_M(\mathrm{GSp}(4))$ *is* $\mathrm{GSp}(4)$ *with multiplicity one.*

Proof. — We have $\widehat{M}$ sitting inside $\mathrm{GSp}(4,\mathbf{C})$ as the Siegel Levi. So we have

$$Z(\widehat{M}) = \{\mathrm{diag}(x,x,y,y)\}.$$

The only elliptic endoscopic group given by such an element is $\mathrm{GSp}(4)$ itself which we obtain when $x=y=1$. □

Lemma 3.10. — *Let M be the diagonal Levi in* $\mathrm{GSp}(4)$. *Then the elliptic endoscopic groups in* $\mathcal{E}_M(\mathrm{GSp}(4))$ *are* $\mathrm{GSp}(4)$ *and* $(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)$, *each with multiplicity one.*

Proof. — We have $\widehat{M}$ sitting inside $\mathrm{GSp}(4,\mathbf{C})$ as the diagonal torus. So we have

$$Z(\widehat{M}) = \left\{\mathrm{diag}(x,y,y^{-1}z,x^{-1}z)\right\},$$

and we get that the elliptic endoscopic groups in $\mathcal{E}_M(\mathrm{GSp}(4))$ are $\mathrm{GSp}(4)$ and $(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)$, each with multiplicity one. □

We have M sitting inside $(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)$ as the diagonal torus and we have

$$\iota_M(\mathrm{GSp}(4),(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)) = \frac{1}{2}.$$

4. Weight functions

In this section we compute all the weight functions needed in the proof of the fundamental lemma. For a Levi subset M of $G = G^0 \rtimes \alpha$ the weight function v_M is defined in [**Art88b**, Section 1]; throughout this section we adopt the notations and definitions given there.

4.1. Twisted weight functions. — In this section we adopt the notation of Section 2.4 and compute the weight functions for the relevant Levi subgroups of $G^0 = \mathrm{GL}(4) \times \mathrm{GL}(1)$. We will use the following basic fact in computing the weight functions below.

Lemma 4.1. — *For $v = (v_1, \ldots, v_n) \in F^n$ define $|v| = \max\{|v_1|, \ldots, |v_n|\}$. Then for all $k \in \mathrm{GL}(n, R)$ and $v \in F^n$ we have $|vk| = |v|$.*

Proof. — We clearly have $|vk| \leqslant |v|$ and replacing v by vk^{-1} yields the result. □

4.1.1. The (2,2) Levi. — In this section we take M^0 to be the (2,2) Levi in G^0. We have $M = M^0 \rtimes \alpha$. Let P^0 (resp. Q^0) be the upper (resp. lower) block triangular parabolic in G^0 with M^0 as its Levi component. We have $M = M^0 \rtimes \alpha$ and if we set $P = P^0 \rtimes \alpha$ and $Q = Q^0 \rtimes \alpha$ then we have $\mathcal{P}(M) = \{P, Q\}$. We let N_P (resp. N_Q) denote the unipotent radical of P^0 (resp. Q^0). Let $x \in G^0(F)$ and write

$$x = n_P m_P k_P = n_Q m_Q k_Q$$

with obvious notation. We write $m_P = (A_P, B_P, c_P) \in \mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(1)$ and similarly we write $m_Q = (A_Q, B_Q, c_Q)$.

Lemma 4.2. — *With notation as above we have*

$$v_M(x) = \mathrm{vol}(\mathfrak{a}_P^G / \mathbf{Z}(\Delta_P^\vee)) \left(\ln|\det A_Q| - \ln|\det A_P|\right).$$

Proof. — For $(A, B, c) \in M^0$ we have

$$\alpha : (A, B, c) \longmapsto (w^t B^{-1} w, w^t A^{-1} w, c \det AB),$$

and hence

$$A_M = \left\{a = \left(\mathrm{diag}(a_1, a_1), \mathrm{diag}(a_1^{-1}, a_1^{-1})\right), a_2\right)\right\}.$$

We fix the basis $\{\chi_1, \chi_2\}$ of $X(A_M)$ given by $\chi_i : a \mapsto a_i$. We have

$$A_{M^0} = \{b = (\mathrm{diag}(b_1, b_1), \mathrm{diag}(b_2, b_2)), b_3)\}$$

and we fix the basis $\{\varphi_1, \varphi_2, \varphi_3\}$ of $X(A_{M^0})$ given by $\varphi_i : b \mapsto b_i$. We have $\Delta_{P^0} = \{\varphi_1 - \varphi_2\}$. We now compute $(\varphi_1 - \varphi_2)^\vee$. Let $\delta_{\varphi_1}, \delta_{\varphi_2}, \delta_{\varphi_3}$ denote the basis of $\mathfrak{a}_{M^0}^*$ given by $\delta_{\varphi_i}(\varphi_j) = \delta_{ij}$, the Kronecker delta symbol.

To determine $(\varphi_1 - \varphi_2)^\vee$ we may as well work inside GL(4). We set P_0 equal to the upper triangular Borel subgroup of GL(4) and we take M_0 to be the diagonal torus in P_0. We have

$$A_{M_0} = M_0 = \{c = \mathrm{diag}(c_1, c_2, c_3, c_4)\}$$

and we fix the basis $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ of $X(M_0)$ given by $\beta_i : c \mapsto c_i$. We define $\delta_{\beta_i} \in \mathfrak{a}_{M_0}$ similarly.

We now describe the splittings $\mathfrak{a}_{P_0}^* = \mathfrak{a}_{P^0}^* \oplus (\mathfrak{a}_{P_0}^{P^0})^*$ and $\mathfrak{a}_{P_0} = \mathfrak{a}_{P^0} \oplus \mathfrak{a}_{P_0}^{P^0}$. The map $X(A_{P^0}) \twoheadrightarrow X(M_0)$ is given by

$$\beta_1 \longmapsto \varphi_1 \quad \beta_2 \longmapsto \varphi_1 \quad \beta_3 \longmapsto \varphi_2 \quad \beta_4 \longmapsto \varphi_2$$

and the map $\mathfrak{a}_{P^0} \hookrightarrow \mathfrak{a}_{P_0}$ is given by

$$\varphi_1 \longmapsto \tfrac{1}{2}(\beta_1 + \beta_2) \quad \varphi_2 \longmapsto \tfrac{1}{2}(\beta_3 + \beta_4).$$

Thus we have

$$\mathfrak{a}_{P_0} = \mathfrak{a}_{P^0} \oplus \mathfrak{a}_{P_0}^{P^0} = \mathrm{Span}\{\delta_{\beta_1} + \delta_{\beta_2}, \delta_{\beta_3} + \delta_{\beta_4}\} \oplus \mathrm{Span}\{\delta_{\beta_1} - \delta_{\beta_2}, \delta_{\beta_3} - \delta_{\beta_4}\}$$

and

$$\mathfrak{a}_{P_0}^* = \mathfrak{a}_{P^0}^* \oplus (\mathfrak{a}_{P_0}^{P^0})^* = \mathrm{Span}\{\beta_1 + \beta_2, \beta_3 + \beta_4\} \oplus \mathrm{Span}\{\beta_1 - \beta_2, \beta_3 - \beta_4\}.$$

Therefore we have

$$\varphi_1 - \varphi_2 = \frac{1}{2}(\beta_1 + \beta_2) - \frac{1}{2}(\beta_3 + \beta_4) = \beta_2 - \beta_3 + \frac{1}{2}(\beta_1 - \beta_2) + \frac{1}{2}(\beta_3 - \beta_4)$$

equal to the projection of $\beta_2 - \beta_3$ onto $\mathfrak{a}_{P^0}^*$. Now $(\beta_2 - \beta_3)^\vee = \delta_{\beta_2} - \delta_{\beta_3}$ whose projection onto $\mathfrak{a}_{P^0}$ is

$$\frac{1}{2}(\delta_{\beta_1} + \delta_{\beta_2}) - \frac{1}{2}(\delta_{\beta_3} + \delta_{\beta_4}).$$

Hence we have $(\varphi_1 - \varphi_2)^\vee = \frac{1}{2}(\delta_{\varphi_1} - \delta_{\varphi_2})$.

The map $X(A_{M^0}) \to X(A_M)$ is given by

$$\varphi_1 \longmapsto \chi_1 \quad \varphi_2 \longmapsto -\chi_1 \quad \varphi_3 \longmapsto \chi_2.$$

We have $\Delta_P = \{2\chi_1\}$, $\Delta_Q = \{-2\chi_1\}$ and

$$(2\chi_1)^\vee : \ \chi_1 \longmapsto \tfrac{1}{2} \quad \chi_2 \longmapsto 0.$$

Hence for $\lambda = a_1\chi_1 + a_2\chi_2 \in \mathfrak{a}_{M,\mathbf{C}}^*$ we have

$$\theta_P(\lambda) = \frac{a_1}{2\,\mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))}$$

and

$$\theta_Q(\lambda) = -\frac{a_1}{2\,\mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))}.$$

We now make explicit the isomorphism between $X(A_M) \otimes_{\mathbf{Z}} \mathbf{R}$ and $X(M)_F \otimes_{\mathbf{Z}} \mathbf{R}$. We have a basis for $X(M)_F$ given by the characters

$$\psi_1 : ((A,B),c) \longmapsto \det A \det B^{-1} \quad \psi_2 : ((A,B),c) \longmapsto c$$

of M^0. The restriction map $X(A_M) \to X(M)_F$ is given by $\psi_1 \mapsto 4\chi_1$ and $\psi_2 \mapsto \chi_2$. Now we have

$$H_M(m_P) : \ \chi_1 \longmapsto \tfrac{1}{4}\ln|\det A_P B_P^{-1}| \quad \chi_2 \longmapsto \ln|c_P|.$$

Therefore,

$$v_P(\lambda, x) = \exp\Big(-\frac{a_1}{4}\ln|\det A_P B_P^{-1}| - a_2 \ln|c_P|\Big)$$

and similarly for $v_Q(\lambda, x)$. Hence $v_M(\lambda, x)$ equals

$$\frac{2\operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))}{a_1}$$
$$\left[\exp\left(-\frac{a_1}{4}\ln|\det A_P B_P^{-1}| - a_2 \ln|e_P|\right) - \exp\left(-\frac{a_1}{4}\ln|\det A_Q B_Q^{-1}| - a_2\ln|e_Q|\right)\right].$$

Taking the limit as $\lambda = a_1\chi_1 + a_2\chi_2 \to 0$ we get

$$v_M(x) = -\frac{\operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))}{2}\left(\ln|\det A_P B_P^{-1}| - \ln|\det A_Q B_Q^{-1}|\right).$$

But we have $|\det A_P B_P| = |\det A_Q B_Q|$ and hence

$$v_M(x) = \operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))\left(\ln|\det A_Q| - \ln|\det A_P|\right),$$

which completes our computation. □

We now compute v_M on the unipotent radical of P^0.

Lemma 4.3. — *We have*

$$v_M\left(\begin{pmatrix}1 & & x_1 & x_2\\ & 1 & x_3 & x_4\\ & & 1 & \\ & & & 1\end{pmatrix}, 1\right) = \operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))\ln\max\{1, |x_1|, |x_2|, |x_3|, |x_4|, |x_1x_4 - x_2x_3|\}.$$

Proof. — We write

$$\begin{pmatrix}1 & & x_1 & x_2\\ & 1 & x_3 & x_4\\ & & 1 & \\ & & & 1\end{pmatrix} = \begin{pmatrix}1 & & & \\ & 1 & & \\ y_1 & y_2 & 1 & \\ y_3 & y_4 & & 1\end{pmatrix}\begin{pmatrix}A_Q & \\ & B_Q\end{pmatrix}k_Q.$$

Applying the vector $(1,0,0,0)\wedge(0,1,0,0)$ and using Lemma 4.1 allows us to deduce that

$$\ln|\det A_Q| = \ln\max\{1, |x_1|, |x_2|, |x_3|, |x_4|, |x_1x_4 - x_2x_3|\}$$

and the result follows. □

4.1.2. The (1,2,1) Levi. — In this section we take M^0 to be the (1,2,1) Levi in G. Let P^0 (resp. Q^0) be the upper (resp. lower) triangular parabolic in G^0 with M^0 as its Levi component. We have $M = M^0 \rtimes \alpha$ and if we set $P = P^0 \rtimes \alpha$ and $Q = Q^0 \rtimes \alpha$ then we have $\mathcal{P}(M) = \{P, Q\}$. We let N_P (resp. N_Q) denote the unipotent radical of P^0 (resp. Q^0). Let $x \in G^0(F)$ and write

$$x = n_P m_P k_P = n_Q m_Q k_Q$$

with obvious notation. We write $m_P = (a_P, B_P, c_P, d_P) \in \mathrm{GL}(1)\times\mathrm{GL}(2)\times\mathrm{GL}(1)\times\mathrm{GL}(1)$ and similarly we write $m_Q = (a_Q, B_Q, c_Q, d_Q)$.

Lemma 4.4. — *With notation as above we have*

$$v_M(x) = \frac{\mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))}{2}\left(\ln|a_Q c_Q^{-1}| - \ln|a_P c_P^{-1}|\right).$$

Proof. — We have

$$A_M = \{a = (a_1, I, a_1^{-1}, a_2)\}.$$

We fix the basis $\{\chi_1,\chi_2\}$ of $X(A_M)$ given by $\chi_i : a \mapsto a_i$. We have

$$A_{M^0} = \{b = (b_1, \mathrm{diag}(b_2,b_2), b_3, b_4)\}$$

and we fix the basis $\{\varphi_1,\varphi_2,\varphi_3,\varphi_4\}$ of $X(A_{M^0})$ given by $\varphi_i : b \mapsto b_i$. We have

$$\Delta_{P^0} = \{\varphi_1 - \varphi_2, \varphi_2 - \varphi_3\}.$$

We now compute $(\varphi_1-\varphi_2)^\vee$ and $(\varphi_2-\varphi_3)^\vee$. Let $\delta_{\varphi_1},\delta_{\varphi_2},\delta_{\varphi_3},\delta_{\varphi_4}$ denote the basis of $\mathfrak{a}_{M^0}^*$ given by $\delta_{\varphi_i}(\varphi_j) = \delta_{ij}$.

To determine $(\varphi_1-\varphi_2)^\vee$ and $(\varphi_2-\varphi_3)^\vee$ we may as well work inside GL(4). We set P_0 equal to the upper triangular Borel subgroup of GL(4) and we take M_0 to be the diagonal torus in P_0. We have

$$A_{M_0} = M_0 = \{c = \mathrm{diag}(c_1,c_2,c_3,c_4)\}$$

and we fix the basis $\{\beta_1,\beta_2,\beta_3,\beta_4\}$ of $X(M_0)$ given by $\beta_i : c \mapsto c_i$. We define $\delta_{\beta_i} \in \mathfrak{a}_{M_0}$ similarly.

We now need to describe the splittings $\mathfrak{a}_{P_0}^* = \mathfrak{a}_{P^0}^* \oplus (\mathfrak{a}_{P_0}^{P^0})^*$ and $\mathfrak{a}_{P_0} = \mathfrak{a}_{P^0} \oplus \mathfrak{a}_{P_0}^{P^0}$. The map $X(A_{P^0}) \twoheadrightarrow X(M_0)$ is given by

$$\beta_1 \longmapsto \varphi_1 \quad \beta_2 \longmapsto \varphi_2 \quad \beta_3 \longmapsto \varphi_2 \quad \beta_4 \longmapsto \varphi_3,$$

and the map $\mathfrak{a}_{P^0} \hookrightarrow \mathfrak{a}_{P_0}$ is given by

$$\varphi_1 \longmapsto \beta_1 \quad \varphi_2 \longmapsto \tfrac{1}{2}(\beta_2+\beta_3) \quad \varphi_3 \longmapsto \beta_4.$$

Thus we have

$$\mathfrak{a}_{P_0} = \mathfrak{a}_{P^0} \oplus \mathfrak{a}_{P_0}^{P^0} = \mathrm{Span}\{\delta_{\beta_1}, \delta_{\beta_2}+\delta_{\beta_3}, \delta_{\beta_4}\} \oplus \mathrm{Span}\{\delta_{\beta_2}-\delta_{\beta_3}\},$$

and

$$\mathfrak{a}_{P_0}^* = \mathfrak{a}_{P^0}^* \oplus (\mathfrak{a}_{P_0}^{P^0})^* = \mathrm{Span}\{\beta_1, \beta_2+\beta_3, \beta_4\} \oplus \mathrm{Span}\{\beta_2-\beta_3\}.$$

Therefore we have

$$\varphi_1 - \varphi_2 = \beta_1 - \frac{1}{2}(\beta_2+\beta_3) = \beta_1 - \beta_2 + \frac{1}{2}(\beta_2-\beta_3)$$

equal to the projection of $\beta_1-\beta_2$ onto $\mathfrak{a}_{P^0}^*$. Now we have $(\beta_1-\beta_2)^\vee = \delta_{\beta_1}-\delta_{\beta_2}$ whose projection onto $\mathfrak{a}_{P^0}$ is

$$\delta_{\beta_1} - \frac{1}{2}(\delta_{\beta_2}+\delta_{\beta_3}).$$

Hence we have $(\varphi_1-\varphi_2)^\vee=\delta_{\varphi_1}-\frac{1}{2}\delta_{\varphi_2}$. Now

$$\varphi_2-\varphi_3=\frac{1}{2}(\beta_2+\beta_3)-\beta_4=\beta_3-\beta_4+\frac{1}{2}(\beta_2-\beta_3)$$

equals the projection of $\beta_3-\beta_4$ onto $\mathfrak{a}^*_{P^0}$. Now we have $(\beta_3-\beta_4)^\vee=\delta_{\beta_3}-\delta_{\beta_4}$ whose projection onto $\mathfrak{a}_{P^0}$ is

$$\frac{1}{2}(\delta_{\beta_2}+\delta_{\beta_3})-\delta_{\beta_4}.$$

Hence we have $(\varphi_2-\varphi_3)^\vee=\frac{1}{2}\delta_{\varphi_2}-\delta_{\varphi_3}$.

The map $X(A_{M^0})\to X(A_M)$ is given by

$$\varphi_1\longmapsto\chi_1\quad\varphi_2\longmapsto 0\quad\varphi_3\longmapsto-\chi_1\quad\varphi_4\longmapsto\chi_2.$$

We have $\Delta_P=\{\chi_1\}$, $\Delta_Q=\{-\chi_1\}$ and

$$(\chi_1)^\vee:\ \chi_1\longmapsto 1\quad\chi_2\longmapsto 0.$$

Hence for $\lambda=a_1\chi_1+a_2\chi_2\in\mathfrak{a}^*_{M,\mathbf{C}}$ we have

$$\theta_P(\lambda)=\frac{a_1}{\operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))},$$

and

$$\theta_Q(\lambda)=-\frac{a_1}{\operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))}.$$

We now make explicit the isomorphism between $X(A_M)\otimes_{\mathbf{Z}}\mathbf{R}$ and $X(M)_F\otimes_{\mathbf{Z}}\mathbf{R}$. We have a basis for $X(M)_F$ given by the characters

$$\psi_1:(a,B,c,d)\longmapsto ac^{-1},\quad\psi_2:(a,B,c,d)\longmapsto acd^2\det B$$

on M^0. The restriction map $X(A_M)\to X(M)_F$ is given by $\psi_1\mapsto 2\chi_1$ and $\psi_2\mapsto 2\chi_2$. Now we have

$$H_M(m_P):\ \chi_1\longmapsto\tfrac{1}{2}\ln|a_Pc_P^{-1}|\quad\chi_2\longmapsto\tfrac{1}{2}\ln|a_Pc_Pd_P^2\det B_P|.$$

Therefore,

$$v_P(\lambda,x)=\exp\Big(-\frac{a_1}{2}\ln|a_Pb_P^{-1}|-\frac{a_2}{2}\ln|a_Pc_Pd_P^2\det B_P|\Big)$$

and similarly for $v_Q(\lambda,x)$. We can set $a_2=0$ and take the limit as $a_1\to 0$ to give

$$v_M(x)=\frac{\operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))}{2}\Big(\ln|a_Qc_Q^{-1}|-\ln|a_Pc_P^{-1}|\Big)$$

as wished. □

We now compute v_M on the unipotent radical of P^0.

Lemma 4.5. — *We have*

$$v_M\left(\begin{pmatrix}1&x_1&x_2&x_3\\&1&&x_4\\&&1&x_5\\&&&1\end{pmatrix},1\right)$$

equal to

$$\frac{\mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))}{2}\left(\ln\max\{1,|x_1|,|x_2|,|x_3|\}+\ln\max\{1,|x_4|,|x_5|,|x_1x_4+x_2x_5-x_3|\}\right).$$

Proof. — We write

$$\begin{pmatrix}1 & x_1 & x_2 & x_3\\ & 1 & & x_4\\ & & 1 & x_5\\ & & & 1\end{pmatrix}=\begin{pmatrix}1 & & & \\ y_1 & 1 & & \\ y_2 & & 1 & \\ y_3 & y_4 & y_4 & 1\end{pmatrix}\begin{pmatrix}a_Q & & \\ & B_Q & \\ & & c_Q\end{pmatrix}k_Q.$$

Applying the vector $(1,0,0,0)$ allows us to deduce that

$$\ln|a_Q|=\ln\max\{1,|x_1|,|x_2|,|x_3|\}.$$

Taking the transpose inverse of the above matrix equation and applying the vector $(0,0,0,1)$ allows us to deduce that

$$\ln|c_Q^{-1}|=\ln\max\{1,|x_4|,|x_5|,|x_1x_4+x_2x_5-x_3|\}$$

and the result follows. □

4.1.3. The diagonal Levi. — Let M^0 be the diagonal Levi subgroup in G^0. For the proof of the fundamental lemma it is (essentially) sufficient to compute v_M on elements of G^0 fixed by α, i.e., elements of the form $g=(g_1,g_2)\in\mathrm{Sp}(4)\times\mathrm{GL}(1)$. For now we show that for such a g $v_M(g)$ is, up to a scalar, equal to $v_{M_1}(g_1)$ where M_1 is the diagonal Levi in $\mathrm{Sp}(4)$. We will then compute v_{M_1} on the unipotent radical of the upper triangular Borel subgroup in $\mathrm{Sp}(4)$.

Let B^0 (resp. B_1) denote the upper triangular Borel subgroup of G^0 (resp. $\mathrm{Sp}(4)$).

Lemma 4.6. — *For $g\in(g_1,g_2)\in G^0(F)$ with $g_1\in\mathrm{Sp}(4)$ we have*

$$v_M(g)=\frac{\mathrm{vol}(\mathfrak{a}_B^G/\mathbf{Z}(\Delta_B^\vee))}{\mathrm{vol}(\mathfrak{a}_{B_1}^{\mathrm{Sp}(4)}/\mathbf{Z}(\Delta_{B_1}^\vee))}\,v_{M_1}(g_1).$$

Proof. — We have

$$A_M=\left\{a=(\mathrm{diag}(a_1,a_2,a_2^{-1},a_1^{-1}),a_3)\right\},$$

and we fix the basis $\{\chi_1,\chi_2,\chi_3\}$ of $X(A_M)$ given by $\chi_i:a\mapsto a_i$. We have

$$A_{M^0}=\{b=(\mathrm{diag}(b_1,b_2,b_3,b_4),b_5)\},$$

and we fix the basis $\{\varphi_1,\ldots,\varphi_5\}$ of $X(A_{M^0})$ given by $\varphi_i:b\mapsto b_i$. The map $X(A_{M^0})\to X(A_M)$ given by restriction is given by

$$\varphi_1\longmapsto\chi_1\quad\varphi_2\longmapsto\chi_2\quad\varphi_3\longmapsto-\chi_2\quad\varphi_4\longmapsto-\chi_1\quad\varphi_5\longmapsto\chi_3.$$

We have

$$A_{M_1}=\left\{a=\mathrm{diag}(a_1,a_2,a_2^{-1},a_1^{-1})\right\}.$$

We identify $\mathfrak{a}_{M_1}$ with the subspace of $\mathfrak{a}_M$ of elements which are zero on χ_3 and we identify $\mathfrak{a}^*_{M_1}$ with the subspace $\{a_1\chi_1 + a_2\chi_2\}$ of $\mathfrak{a}^*_M$.

We now compute $\theta_B(\lambda)$ for $\lambda = a_1\chi_1 + a_2\chi_2 + a_3\chi_3 \in \mathfrak{a}^*_{M,\mathbf{C}}$. We have

$$\Delta_{B^0} = \{\varphi_1 - \varphi_2, \varphi_2 - \varphi_3, \varphi_3 - \varphi_4\},$$

and

$$\Delta_B = \{\chi_1 - \chi_2, 2\chi_2\}.$$

We have

$$(\chi_1 - \chi_2)^\vee : \ \chi_1 \longmapsto 2 \ \ \chi_2 \longmapsto -2 \ \ \chi_3 \longmapsto 0,$$

and

$$(2\chi_2)^\vee : \ \chi_1 \longmapsto 0 \ \ \chi_2 \longmapsto 1 \ \ \chi_3 \longmapsto 0.$$

On the other hand

$$\Delta_{B_1} = \{\chi_1 - \chi_2, 2\chi_2\},$$

and we have

$$(\chi_1 - \chi_2)^\vee : \ \chi_1 \longmapsto 1 \ \ \chi_2 \longmapsto -1,$$

and

$$(2\chi_2)^\vee : \ \chi_1 \longmapsto 0 \ \ \chi_2 \longmapsto 1.$$

Hence we see that for $\lambda = \lambda_1 + a_3\chi_3 \in \mathfrak{a}^*_{M,\mathbf{C}}$ with $\lambda_1 \in \mathfrak{a}^*_{M_1,\mathbf{C}}$ we have $\theta_B(\lambda) = \theta_{B_1}(\lambda_1)$. Now each Borel subgroup of G^0, which is α stable and which contains M^0 is of the form $w^{-1}B^0w$ with $w = (w_1, 1)$ where w_1 is an element of the Weyl group of Sp(4). Hence we deduce that for each Borel subgroup P^0 of G^0 which contains M^0 we have

$$\mathrm{vol}(\mathfrak{a}^G_P/\mathbf{Z}(\Delta^\vee_P))\theta_P(\lambda) = \mathrm{vol}(\mathfrak{a}^{\mathrm{Sp}(4)}_{P_1}/\mathbf{Z}(\Delta^\vee_{P_1}))\theta_{P_1}(\lambda_1),$$

where P_1 denotes the Borel subgroup of Sp(4) which is contained in P^0.

Next we compute $v_P(\lambda, g)$ and $v_{P_1}(\lambda_1, g_1)$. In order to compute $v_P(\lambda, g)$ we need to write $g = n_Pm_Pk_P$ with $n_P \in N_P(F)$, $m_P \in M^0(F)$ and $k_P \in K$. But if we write $g_1 = n_{P_1}m_{P_1}k_{P_1}$ with obvious notation then we have

$$g = (g_1, g_2) = (n_{P_1}, 1)(m_{P_1}, g_2)(k_{P_1}, 1).$$

Hence we have for $\lambda = \lambda_1 + a_3\chi_3$ that

$$v_P(\lambda, g) = v_{P_1}(\lambda_1, g_1)|g_2|^{-a_3}.$$

Thus we get

$$\begin{aligned} v_M(\lambda, g) &= \sum_{P\in\mathcal{P}(M)} v_P(\lambda, g)\theta_P(\lambda)^{-1} \\ &= \frac{\mathrm{vol}(\mathfrak{a}^G_B/\mathbf{Z}(\Delta^\vee_B))}{\mathrm{vol}(\mathfrak{a}^{\mathrm{Sp}(4)}_{B_1}/\mathbf{Z}(\Delta^\vee_{B_1}))}|g_2|^{-a_3} \sum_{P_1\in\mathcal{P}(M_1)} v_{P_1}(\lambda_1, g_1)\theta_{P_1}(\lambda_1)^{-1}. \end{aligned}$$

And now taking the limit as $\lambda \to 0$ gives the result. □

Multiplying n on the left by such an element we can put n in the form

$$\begin{pmatrix} 1 & x_1 & & \\ & 1 & & \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix}.$$

For $m > 0$ and $u \in U_F$ we have

$$\begin{pmatrix} 1 & u\pi^{-m} \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ u^{-1}\pi^m & 1 \end{pmatrix} \begin{pmatrix} \pi^{-m} & \\ & \pi^m \end{pmatrix} \begin{pmatrix} \pi^m & u \\ -u^{-1} & \end{pmatrix}$$

and hence we deduce that

$$|a| = \max\{1, |x_1|\}$$

and

$$|b|^{-1} = \max\{1, |x_1|\}.$$

- $w = (23)$. In this case we have

$$w^{-1} N_{B_1} w = \left\{ \begin{pmatrix} 1 & -y_2 & y_1 & y_3 \\ & 1 & & y_1 \\ & -y_4 & 1 & y_2 - y_1 y_4 \\ & & & 1 \end{pmatrix} \right\}.$$

Multiplying n on the left by such an element we can put n in the form

$$\begin{pmatrix} 1 & & & \\ & 1 & x_4 & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

And as above we deduce that

$$|a| = 1$$

and

$$|b| = \max\{1, |x_4|\}.$$

- $w = (14)$. In this case we have

$$w^{-1} N_{B_1} w = \left\{ \begin{pmatrix} 1 & & & \\ y_4 y_1 - y_2 & 1 & y_4 & \\ y_1 & & 1 & \\ -y_3 & y_1 & y_2 & 1 \end{pmatrix} \right\}.$$

Using the vector $(1, 0, 0, 0)$ we deduce that

$$|a| = \max\{1, |x_1|, |x_2 + x_1 x_4|, |x_3|\}.$$

and using $(1, 0, 0, 0) \wedge (0, 0, 1, 0)$ we deduce that

$$|ab^{-1}| = \max\{1, |x_1|^2, |x_3 + x_1 x_2 + x_1^2 x_4|\}.$$

We now compute v_{M_1} on the unipotent radical of B_1. We set

$$n = \begin{pmatrix} 1 & x_1 & x_2 + x_1x_4 & x_3 \\ & 1 & x_4 & x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix} \in N_{B_1}(F).$$

In order to do this we need to write $n = n_1m_1k_1$ for each Borel subgroup of Sp(4) containing M_1 and then if we write

$$m_1 = \mathrm{diag}(a, b, b^{-1}, a^{-1})$$

we need to compute $|a|$ and $|b|$.

The Weyl group of Sp(4) is isomorphic to D_8 with generators

$$w_1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, w_2 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix}.$$

Explicitly the Weyl group is given by

$$\{e, (12)(34), (23), (14), (1243), (1342), (13)(24), (14)(23)\},$$

where we have

$$\begin{aligned} e &= I \\ (12)(34) &= w_1 \\ (23) &= w_2 \\ (14) &= w_1w_2w_1 \\ (1243) &= w_2w_1 \\ (1342) &= w_1w_2 \\ (13)(24) &= w_2w_1w_2 \\ (14)(23) &= w_1w_2w_1w_2. \end{aligned}$$

- $w = e$. In this case we have $|a| = |b| = 1$.
- $w = (12)(34)$. In this case we have

$$w^{-1}N_{B_1}w = \left\{ \begin{pmatrix} 1 & & y_2 - y_1y_4 & y_4 \\ y_1 & 1 & y_3 & y_2 \\ & & 1 & \\ & & -y_1 & 1 \end{pmatrix} \right\}.$$

- $w = (1243)$. In this case we have

$$w^{-1}N_{B_1}w = \left\{\begin{pmatrix} 1 & & y_1 & \\ -y_2 & 1 & y_3 & y_1 \\ & & 1 & \\ -y_4 & & y_2 - y_1y_4 & 1 \end{pmatrix}\right\}.$$

Using the vector $(0,0,1,0)$ we deduce that

$$|b| = \max\{1, |x_1|\}^{-1},$$

and using $(1,0,0,0) \wedge (0,0,1,0)$ we deduce that

$$|ab^{-1}| = \max\{1, |x_1|^2, |x_3 + x_1x_2 + x_1^2x_4|\}.$$

- $w = (1342)$. In this case we have

$$w^{-1}N_{B_1}w = \left\{\begin{pmatrix} 1 & y_1y_4 - y_2 & & y_4 \\ & 1 & & \\ y_1 & -y_3 & 1 & y_2 \\ & y_1 & & 1 \end{pmatrix}\right\}.$$

Using the vector $(0,1,0,0)$ we deduce that

$$|b| = \max\{1, |x_4|, |x_4|\},$$

and using $(0,1,0,0) \wedge (0,0,0,1)$ we deduce that

$$|a^{-1}b| = \max\{1, |x_4|\}.$$

- $w = (13)(24)$. In this case we have

$$w^{-1}N_{B_1}w = \left\{\begin{pmatrix} 1 & -y_1 & & \\ & 1 & & \\ -y_2 & -y_3 & 1 & y_1 \\ -y_4 & y_1y_4 - y_2 & & 1 \end{pmatrix}\right\}.$$

Using the vector $(0,1,0,0)$ we deduce that

$$|b| = \max\{1, |x_2|, |x_4|\},$$

and using $(1,0,0,0) \wedge (0,1,0,0)$ we deduce that

$$|ab| = \max\{1, |x_2|, |x_4|, |x_3 - x_1x_2|, |x_2^2 - x_3x_4 + x_1x_2x_4|\}.$$

- $w = (14)(23)$. In this case we have

$$w^{-1}N_{B_1}w = \left\{\begin{pmatrix} 1 & & & \\ -y_1 & 1 & & \\ y_1y_4 - y_2 & -y_4 & 1 & \\ -y_3 & -y_2 & y_1 & 1 \end{pmatrix}\right\}.$$

Using the vector $(1,0,0,0)$ we deduce that

$$|a| = \max\{1, |x_1|, |x_2 + x_1x_4|, |x_3|\},$$

and using $(1,0,0,0)\wedge(0,1,0,0)$ we deduce that

$$|ab| = \max\{1, |x_2|, |x_4|, |x_3 - x_1x_2|, |x_2^2 - x_3x_4 + x_1x_2x_4|\}.$$

Let's set $\lambda = a_1\chi_1 + a_2\chi_2 \in \mathfrak{a}^*_{M_1,\mathbf{C}}$. Where χ_i is the character of M_1 mapping $\mathrm{diag}(a_1, a_2, a_1^{-1}, a_1^{-1})$ to a_i. Let $x \in \mathrm{Sp}(4, F)$ and let P_1 be a Borel subgroup containing M_1. We write $x = n_{P_1}m_{P_1}k_{P_1}$ with the usual notation where $m_{P_1} = \mathrm{diag}(a_{P_1}, b_{P_1}, b_{P_1}^{-1}, a_{P_1}^{-1})$. Then we have

$$H_{P_1}(x):\ \chi_1 \longmapsto \ln|a_{P_1}|\ \ \chi_2 \longmapsto \ln|b_{P_1}|.$$

Hence we have $v_{P_1}(\lambda, x) = |a_{P_1}|^{a_1}|b_{P_1}|^{a_2}$ and therefore for $\lambda = \beta a_2\chi_1 + a_2\chi_2 \in \mathfrak{a}^*_{M_1,\mathbf{C}}$ we have

$$v_{P_1}(\lambda, x) = (|a_{P_1}|^{\beta}|b_{P_1}|)^{a_2}.$$

Next we compute θ_{P_1} for each of these Borel subgroups $P_1 = w^{-1}B_1w$ and $\lambda = \beta a_2\chi_1 + a_2\chi_2 \in \mathfrak{a}^*_{M,\mathbf{C}}$. These functions are given in the table below.

w	Δ_{P_1}	$\theta_{P_1}(\lambda)/a_2^2$
e	$2\chi_2, \chi_1 - \chi_2$	$\beta - 1$
$(12)(34)$	$2\chi_1, \chi_2 - \chi_1$	$\beta(1-\beta)$
(23)	$-2\chi_2, \chi_1 + \chi_2$	$-(\beta+1)$
(14)	$2\chi_2, -\chi_1 - \chi_2$	$-(\beta+1)$
(1243)	$-2\chi_1, \chi_1 + \chi_2$	$-\beta(\beta+1)$
(1342)	$2\chi_1, -\chi_1 - \chi_2$	$-\beta(\beta+1)$
$(13)(24)$	$-2\chi_1, \chi_1 - \chi_2$	$\beta(1-\beta)$
$(14)(23)$	$-2\chi_2, -\chi_1 + \chi_2$	$\beta - 1$

For $\beta \in \mathbf{C}$ we set $\theta_{P_1}(\beta) = \theta_P(\lambda)/a_2^2$. We have

$$v_{M_1}(x, \varphi) = \sum_{P_1} \frac{\mathrm{vol}(\mathfrak{a}_{B_1}^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_{B_1}^{\vee}))}{a_2^2\theta_{P_1}(\beta)}(|a_{P_1}|^{\beta}|b_{P_1}|)^{a_2}.$$

The value at $a_2 = 0$ of this expression is equal to

$$v_{M_1}(x) = \frac{\mathrm{vol}(\mathfrak{a}_{B_1}^{\mathrm{Sp}(4)}/\mathbf{Z}(\Delta_{B_1}^{\vee}))}{2}\sum_{P_1}\frac{1}{\theta_{P_1}(\beta)}(\beta\ln|a_{P_1}| + \ln|b_{P_1}|)^2$$

for any value of β. The calculations above give the following.

Lemma 4.7. — *We have $v_{M_1}(n)$ equal to $\frac{\mathrm{vol}(\mathfrak{a}_{B_1}^{\mathrm{Sp}(4)}/\mathbf{Z}(\Delta_{B_1}^{\vee}))}{2}$ times*

$$-(A^2 + 2B^2 + 2C^2 + D^2 + 2E^2 + F^2) + 2(AB + AE + BD + CD + EF)$$

where

$$\begin{aligned}
A &= \ln\max\{1, |x_2|, |x_4|, |x_3 - x_1x_2|, |x_2^2 - x_3x_4 + x_1x_2x_4|\}\\
B &= \ln\max\{1, |x_1|, |x_2 + x_1x_4|, |x_3|\}\\
C &= \ln\max\{1, |x_1|\}\\
D &= \ln\max\{1, |x_1|^2, |x_3 + x_1x_2 + x_1^2x_4|\}\\
E &= \ln\max\{1, |x_2|, |x_4|\}\\
F &= \ln\max\{1, |x_4|\}.
\end{aligned}$$

Combining Lemmas 4.6 and 4.7 we get the following.

Corollary 4.8. — *For*

$$n = \left(\begin{pmatrix} 1 & x_1 & x_2 + x_1x_4 & x_3\\ & 1 & x_4 & x_2\\ & & 1 & -x_1\\ & & & 1\end{pmatrix}, 1\right) \in N_B(F)$$

we have $v_M(n)$ *equal to* $\frac{\mathrm{vol}(\mathfrak{a}_B^G/\mathbf{Z}(\Delta_B^\vee))}{2}$ *times*

$$-(A^2 + 2B^2 + 2C^2 + D^2 + 2E^2 + F^2) + 2(AB + AE + BD + CD + EF).$$

where $A, \dots, F$ *are as in Lemma 4.7.*

4.2. Weight functions for GSp(4). — In this section we compute the weight functions for the Levi subgroups of GSp(4).

4.2.1. The Siegel Levi. — In this section we take M to be the Siegel Levi in GSp(4). Let P (resp. Q) be the upper (resp. lower) triangular parabolic in GSp(4) with M as its Levi component. Then we have $\mathcal{P}(M) = \{P, Q\}$. We let N_P (resp. N_Q) denote the unipotent radical in P (resp. Q). Let $x \in \mathrm{GSp}(4, F)$ and write

$$x = n_P m_P k_P = n_Q m_Q k_Q$$

with obvious notation. We write

$$m_P = \begin{pmatrix} A_P & \\ & b_P w\, {}^tA_P^{-1} w\end{pmatrix} \in M(F),$$

and similarly for m_Q.

Lemma 4.9. — *With notation as above we have*

$$v_M(x) = \mathrm{vol}(\mathfrak{a}_P^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_P^\vee))\,(\ln|\det A_Q| - \ln|\det A_P|)\,.$$

Proof. — We have

$$A_M = \{a = \mathrm{diag}(a_1, a_1, a_1^{-1}a_2, a_1^{-1}a_2)\}.$$

We fix the basis $\{\chi_1, \chi_2\}$ of $X(A_M)$ given by $\chi_i : a \mapsto a_i$. We have $\Delta_P = \{2\chi_1 - \chi_2\}$ and $\Delta_Q = \{-2\chi_1 + \chi_2\}$. We now compute $(2\chi_1 - \chi_2)^\vee$.

Let $\delta_{\chi_1}, \delta_{\chi_2}$ denote the basis of $\mathfrak{a}_M^*$ given by $\delta_{\chi_i}(\chi_j) = \delta_{ij}$. We set P_0 equal to the upper triangular Borel subgroup of $\mathrm{GSp}(4)$ and we take M_0 to be the diagonal torus in P_0. We have

$$A_{M_0} = M_0 = \{c = \mathrm{diag}(c_1, c_2, c_2^{-1}c_3, c_1^{-1}c_3)\}$$

and we fix the basis $\{\beta_1, \beta_2, \beta_3\}$ of $X(M_0)$ given by $\beta_i : c \mapsto c_i$. We define $\delta_{\beta_i} \in \mathfrak{a}_{M_0}$ similarly.

We now need to describe the splittings $\mathfrak{a}_{P_0}^* = \mathfrak{a}_P^* \oplus (\mathfrak{a}_{P_0}^P)^*$ and $\mathfrak{a}_{P_0} = \mathfrak{a}_P \oplus \mathfrak{a}_{P_0}^P$. The map $X(A_P) \twoheadrightarrow X(M_0)$ is given by

$$\beta_1 \longmapsto \chi_1 \quad \beta_2 \longmapsto \chi_1 \quad \beta_3 \longmapsto \chi_2$$

and the map $\mathfrak{a}_P \hookrightarrow \mathfrak{a}_{P_0}$ is given by

$$\chi_1 \longmapsto \tfrac{1}{2}\beta_1 + \beta_2 \quad \chi_2 \longmapsto \beta_3.$$

Thus we have

$$\mathfrak{a}_{P_0} = \mathfrak{a}_P \oplus \mathfrak{a}_{P_0}^P = \mathrm{Span}\{\delta_{\beta_1} + \delta_{\beta_2}, \delta_{\beta_3}\} \oplus \mathrm{Span}\{\delta_{\beta_1} - \delta_{\beta_2}\},$$

and

$$\mathfrak{a}_{P_0}^* = \mathfrak{a}_P^* \oplus (\mathfrak{a}_{P_0}^P)^* = \mathrm{Span}\{\beta_1 + \beta_2, \beta_3\} \oplus \mathrm{Span}\{\beta_1 - \beta_2\}.$$

Therefore we have

$$2\chi_1 - \chi_2 = \beta_1 + \beta_2 - \beta_3 = 2\beta_2 - \beta_3 + (\beta_1 - \beta_2)$$

equal to the projection of $2\beta_2 - \beta_3$ onto $\mathfrak{a}_P$. Now we have $(2\beta_2 - \beta_3)^\vee = \delta_{\beta_2}$ whose projection onto $\mathfrak{a}_P^*$ is $\frac{1}{2}(\delta_{\beta_1} + \delta_{\beta_2})$. Hence we have $(2\chi_1 - \chi_2)^\vee = \frac{1}{2}\delta_{\chi_1}$.

Hence for $\lambda = a_1\chi_1 + a_2\chi_2 \in \mathfrak{a}_{M,\mathbf{C}}^*$ we have

$$\theta_P(\lambda) = \frac{a_1}{2\,\mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))},$$

and

$$\theta_Q(\lambda) = -\frac{a_1}{2\,\mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))}.$$

We now make explicit the isomorphism between $X(A_M) \otimes_{\mathbf{Z}} \mathbf{R}$ and $X(M)_F \otimes_{\mathbf{Z}} \mathbf{R}$. We have a basis for $X(M)_F$ given by the characters

$$\psi_1 : \begin{pmatrix} A & \\ & bw^tA^{-1}w \end{pmatrix} \longmapsto \det A$$

and

$$\psi_2 : \begin{pmatrix} A & \\ & bw^tA^{-1}w \end{pmatrix} \longmapsto b.$$

The restriction map $X(M)_F \to X(A_M)$ is given by $\psi_1 \mapsto 2\chi_1$ and $\psi_2 \mapsto \chi_2$. Therefore,

$$H_M(m_P): \ \chi_1 \longmapsto \tfrac{1}{2}\ln|\det A_P| \ \ \chi_2 \longmapsto \ln|b_P|,$$

and so

$$v_P(\lambda, x) = \exp\left(-\frac{a_1}{2}\ln|\det A_P| - a_2\ln|b_P|\right).$$

We have a similar expression for $v_Q(\lambda, x)$. Taking $a_2 = 0$ and letting $a_1 \to 0$ gives

$$v_M(x) = \mathrm{vol}(\mathfrak{a}_P^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_P^\vee))\,(\ln|\det A_Q| - \ln|\det A_P|)$$

as desired. □

The computation of v_M on the unipotent radical of P follows directly from the proof of Lemma 4.3.

Lemma 4.10. — *We have*

$$v_M\begin{pmatrix}1 & & x & r\\ & 1 & r & s\\ & & 1 & \\ & & & 1\end{pmatrix} = \mathrm{vol}(\mathfrak{a}_P^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_P^\vee)))\ln\max\{1,|x|,|r|,|s|,|xs-r^2|\}.$$

4.2.2. The Klingen Levi. — In this section we take M to be the Klingen Levi in GSp(4). Let P (resp. Q) be the upper (resp. lower) triangular parabolic in GSp(4) with M as its Levi component, then we have $\mathcal{P}(M) = \{P, Q\}$. We let N_P (resp. N_Q) denote the unipotent radical in P (resp. Q). Let $x \in \mathrm{GSp}(4, F)$ and write

$$x = n_P m_P k_P = n_Q m_Q k_Q$$

with obvious notation. We write

$$m_P = \begin{pmatrix}a_P & & \\ & B_P & \\ & & a_P^{-1}\det B_P\end{pmatrix} \in M(F)$$

and similarly for m_Q.

Lemma 4.11. — *With notation as above we have*

$$v_M(x) = \mathrm{vol}(\mathfrak{a}_P^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_P^\vee))\,(\ln|a_Q| - \ln|a_P|).$$

Proof. — We have

$$A_M = \left\{a = \mathrm{diag}(a_1, a_2, a_2, a_1^{-1}a_2^2)\right\}.$$

We fix the basis $\{\chi_1, \chi_2\}$ of $X(A_M)$ given by $\chi_i : a \mapsto a_i$. We have $\Delta_P = \{\chi_1 - \chi_2\}$ and $\Delta_P = \{\chi_2 - \chi_1\}$. We now compute $(\chi_1 - \chi_2)^\vee$.

Let $\delta_{\chi_1}, \delta_{\chi_2}$ denote the basis of $\mathfrak{a}_M^*$ given by $\delta_{\chi_i}(\chi_j) = \delta_{ij}$. We set P_0 equal to the upper triangular Borel subgroup of GSp(4) and we take M_0 to be the diagonal torus in P_0. We have

$$A_{M_0} = M_0 = \left\{c = \operatorname{diag}(c_1, c_2, c_2^{-1}c_3, c_1^{-1}c_3)\right\},$$

and we fix the basis $\{\beta_1, \beta_2, \beta_3\}$ of $X(M_0)$ given by $\beta_i : c \mapsto c_i$. We define $\delta_{\beta_i} \in \mathfrak{a}_{M_0}$ similarly.

We now describe the splittings $\mathfrak{a}_{P_0}^* = \mathfrak{a}_P^* \oplus (\mathfrak{a}_{P_0}^P)^*$ and $\mathfrak{a}_{P_0} = \mathfrak{a}_P \oplus \mathfrak{a}_{P_0}^P$. The map $X(A_P) \twoheadrightarrow X(M_0)$ is given by

$$\beta_1 \longmapsto \chi_1 \quad \beta_2 \longmapsto \chi_2 \quad \beta_3 \longmapsto 2\chi_2,$$

and the map $\mathfrak{a}_P \hookrightarrow \mathfrak{a}_{P_0}$ is given by

$$\chi_1 \longmapsto \beta_1 \quad \chi_2 \longmapsto \tfrac{1}{2}\beta_3.$$

Thus we have

$$\mathfrak{a}_{P_0} = \mathfrak{a}_P \oplus \mathfrak{a}_{P_0}^P = \operatorname{Span}\{\delta_{\beta_1}, \delta_{\beta_2} + 2\delta_{\beta_3}\} \oplus \operatorname{Span}\{\delta_{\beta_2}\},$$

and

$$\mathfrak{a}_{P_0}^* = \mathfrak{a}_P^* \oplus (\mathfrak{a}_{P_0}^P)^* = \operatorname{Span}\{\beta_1, \beta_3\} \oplus \operatorname{Span}\{2\beta_2 - \beta_3\}.$$

Therefore we have

$$\chi_1 - \chi_2 = \beta_1 - \frac{1}{2}\beta_3 = \beta_1 - \beta_2 + \left(\beta_2 - \frac{1}{2}\beta_3\right)$$

equal to the projection of $\beta_1 - \beta_2$ onto $\mathfrak{a}_P^*$. Now we have $(\beta_1 - \beta_2)^\vee = \delta_{\beta_1} - \delta_{\beta_2}$ whose projection onto $\mathfrak{a}_P$ is δ_{β_1}. Hence we have $(\chi_1 - \chi_2)^\vee = \delta_{\chi_1}$.

Therefore for $\lambda = a_1\chi_1 + a_2\chi_2 \in \mathfrak{a}_{M,\mathbf{C}}^*$ we have

$$\theta_P(\lambda) = \frac{a_1}{\operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))}$$

and

$$\theta_Q(\lambda) = -\frac{a_1}{\operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))}.$$

We now make explicit the isomorphism between $X(A_M) \otimes_{\mathbf{Z}} \mathbf{R}$ and $X(M)_F \otimes_{\mathbf{Z}} \mathbf{R}$. We have a basis for $X(M)_F$ given by the characters

$$\psi_1 : \begin{pmatrix} a & & \\ & B & \\ & & a^{-1}\det B \end{pmatrix} \longmapsto a$$

and

$$\psi_2 : \begin{pmatrix} a & & \\ & B & \\ & & a^{-1}\det B \end{pmatrix} \longmapsto \det B.$$

The restriction map $X(M)_F \to X(A_M)$ is given by $\psi_1 \mapsto \chi_1$ and $\psi_2 \mapsto 2\chi_2$. So we have

$$H_M(m_P) : \chi_1 \longmapsto \ln|\det a_P| \quad \chi_2 \longmapsto \tfrac{1}{2}\ln|\det B_P|,$$

and therefore,

$$v_P(\lambda, x) = \exp\Big(-a_1 \ln|a_P| - \frac{a_2}{2}\ln|\det B_P|\Big).$$

We have a similar expression for $v_Q(\lambda, x)$. Setting $a_2 = 0$ and taking the limit as $a_1 \to 0$ gives

$$v_M(x) = \mathrm{vol}(\mathfrak{a}_P^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_P^\vee))\,(\ln|a_Q| - \ln|a_P|)$$

as desired. □

The computation of v_M on the unipotent radical of P follows directly from the proof of Lemma 4.5.

Lemma 4.12. — *We have*

$$v_M\begin{pmatrix}1 & x & r & s\\ & 1 & & r\\ & & 1 & -x\\ & & & 1\end{pmatrix} = \mathrm{vol}(\mathfrak{a}_P^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_P^\vee)))\ln\max\{1, |x|, |r|, |s|\}.$$

4.2.3. The diagonal Levi. — In this section we take M to be the diagonal Levi in GSp(4). We will compute v_M on the unipotent radical of the upper triangular Borel subgroup of GSp(4). We follow the strategy in the twisted case; we first relate the function v_M to v_{M_1}, where M_1 is the diagonal torus in Sp(4) and then use Lemma 4.7.

Let B denote the upper triangular Borel subgroup of GSp(4) and let B_1 denote its intersection with Sp(4).

Lemma 4.13. — *For $g \in \mathrm{Sp}(4, F)$ we have*

$$v_M(g) = \frac{\mathrm{vol}(\mathfrak{a}_B^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_B^\vee))}{\mathrm{vol}(\mathfrak{a}_{B_1}^{\mathrm{Sp}(4)}/\mathbf{Z}(\Delta_{B_1}^\vee))} v_{M_1}(g).$$

Proof. — We have

$$A_M = \{a = (\mathrm{diag}(a_1, a_2, a_2^{-1}a_3, a_1^{-1}a_3)\}$$

and we fix the basis $\{\chi_1, \chi_2, \chi_3\}$ of $X(A_M)$ given by $\chi_i : a \mapsto a_i$. We have

$$A_{M_1} = \{a = \mathrm{diag}(a_1, a_2, a_2^{-1}, a_1^{-1})\}.$$

We identify $\mathfrak{a}_{M_1}$ with the subspace of $\mathfrak{a}_M$ given by those elements which are zero on χ_3 and we identify $\mathfrak{a}_{M_1}^*$ with the subspace $\{a_1\chi_1 + a_2\chi_2\}$ of $\mathfrak{a}_M^*$.

We now compute $\theta_B(\lambda)$ for $\lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$. We have

$$\Delta_B = \{\chi_1 - \chi_2, 2\chi_2 - \chi_3\}$$

and

$$(\chi_1 - \chi_2)^\vee : \ \chi_1 \longmapsto 1 \ \ \chi_2 \longmapsto -1 \ \ \chi_3 \longmapsto 0$$

and

$$(2\chi_2 - \chi_3)^\vee : \ \chi_1 \longmapsto 0 \ \ \chi_2 \longmapsto 1 \ \ \chi_3 \longmapsto 0.$$

We have

$$\Delta_{B_1} = \{\chi_1 - \chi_2, 2\chi_2\},$$

and

$$(\chi_1 - \chi_2)^\vee : \ \chi_1 \longmapsto 1 \ \ \chi_2 \longmapsto -1,$$

and

$$(2\chi_2)^\vee : \ \chi_1 \longmapsto 0 \ \ \chi_2 \longmapsto 1.$$

Hence we see that for $\lambda = \lambda_1 + a_3\chi_3 \in \mathfrak{a}^*_{M,\mathbf{C}}$, with $\lambda_1 \in \mathfrak{a}^*_{M_1,\mathbf{C}}$, we have $\theta_B(\lambda) = \theta_{B_1}(\lambda_1)$. Now each Borel subgroup of GSp(4) is of the form $w^{-1}Bw$ with w an element of the Weyl group of Sp(4). Hence we deduce that for each Borel subgroup P of GSp(4) that contains M we have

$$\mathrm{vol}(\mathfrak{a}_P^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_P^\vee))\theta_P(\lambda) = \mathrm{vol}(\mathfrak{a}_{P_1}^{\mathrm{Sp}(4)}/\mathbf{Z}(\Delta_{P_1}^\vee))\theta_{P_1}(\lambda_1),$$

where $P_1 = P \cap \mathrm{Sp}(4)$.

Next we compute $v_P(\lambda, g)$ and $v_{P_1}(\lambda_1, g)$. In order to compute $v_P(\lambda, g)$ we need to write $g = n_P m_P k_P$ with $n_P \in N_P(F)$, $m_P \in M^0(F)$ and $k_P \in K$. Since we are assuming that $g \in \mathrm{Sp}(4)$ we can do this inside Sp(4) and assume that $m_P \in M_1$ for each P. Hence we have for $\lambda = \lambda_1 + a_3\chi_3$ that

$$v_P(\lambda, g) = v_{P_1}(\lambda_1, g).$$

And we get

$$\begin{aligned} v_M(\lambda, g) &= \sum_{P \in \mathcal{P}(M)} v_P(\lambda, g)\theta_P(\lambda)^{-1} \\ &= \frac{\mathrm{vol}(\mathfrak{a}_B^G/\mathbf{Z}(\Delta_B^\vee))}{\mathrm{vol}(\mathfrak{a}_{B_1}^{\mathrm{Sp}(4)}/\mathbf{Z}(\Delta_{B_1}^\vee))} \sum_{P_1 \in \mathcal{P}(M_1)} v_{P_1}(\lambda_1, g_1)\theta_{P_1}(\lambda_1)^{-1}. \end{aligned}$$

Taking the limit as $\lambda \to 0$ gives the result. □

Since the unipotent radical of B lies inside Sp(4) we conclude the following Corollary of Lemmas 4.13 and 4.7.

Corollary 4.14. — *Let*

$$n = \begin{pmatrix} 1 & x_1 & x_2 + x_1x_4 & x_3 \\ & 1 & x_4 & x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix} \in N_B(F)$$

Then $v_M(n)$ *is equal to* $\frac{\mathrm{vol}(\mathfrak{a}_B^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_B^\vee))}{2}$ *times*

$$-(A^2+2B^2+2C^2+D^2+2E^2+F^2)+2(AB+AE+BD+CD+EF),$$

where

$$\begin{aligned}
A &= \ln\max\{1,|x_2|,|x_4|,|x_3-x_1x_2|,|x_2^2-x_3x_4+x_1x_2x_4|\}\\
B &= \ln\max\{1,|x_1|,|x_2+x_1x_4|,|x_3|\}\\
C &= \ln\max\{1,|x_1|\}\\
D &= \ln\max\{1,|x_1|^2,|x_3+x_1x_2+x_1^2x_4|\}\\
E &= \ln\max\{1,|x_2|,|x_4|\}\\
F &= \ln\max\{1,|x_4|\}.
\end{aligned}$$

4.3. Other groups. — We will also need to compute weighted orbital integrals on groups closely related to GL(2). We now compute v_M for M the diagonal torus in GL(2).

Lemma 4.15. — *Let* M *be the diagonal torus in* GL(2) *and* B *the upper triangular Borel subgroup containing* M*. Then we have*

$$v_M\begin{pmatrix}1 & x\\ & 1\end{pmatrix} = \mathrm{vol}(\mathfrak{a}_B^{\mathrm{GL}(2)}/\mathbf{Z}(\Delta_B^\vee))\ln\max\{1,|x|\}.$$

Proof. — Let Q denote the lower triangular Borel subgroup of GL(2). Then we have $\mathcal{P}(M)=\{P,Q\}$. We have

$$A_M=\{a=(a_1,a_2)\}$$

and we let $\chi_i\in X(M)$ be given by $\chi_i : a\mapsto a_i$. We have $\Delta_P=\{\chi_1-\chi_2\}$, $\Delta_Q=\{\chi_2-\chi_1\}$ and

$$(\chi_1-\chi_2)^\vee:\ \chi_1\longmapsto 1\quad \chi_2\longmapsto -1.$$

Let $\lambda=a_1\chi_1+a_2\chi_2\in\mathfrak{a}_{M,\mathbf{C}}^*$ then

$$\theta_Q(\lambda)=\frac{a_2-a_1}{\mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))}.$$

We set

$$n=\begin{pmatrix}1 & x\\ & 1\end{pmatrix}.$$

If $x\in R$ then we have $n\in\mathrm{GL}(2,R)$ and $v_M(n)=0$. Next we note that for $m>0$ and $u\in U_F$ we have

$$\begin{pmatrix}1 & u\pi^{-m}\\ & 1\end{pmatrix}=\begin{pmatrix}1 & \\ u^{-1}\pi^m & 1\end{pmatrix}\begin{pmatrix}\pi^{-m} & \\ & \pi^m\end{pmatrix}\begin{pmatrix}\pi^m & u\\ -u^{-1} & \end{pmatrix}\in N_Q(F)M(F)\,\mathrm{GL}(2,R)$$

Therefore, if $x \notin R$ then

$$v_M(a_1\chi_1 + a_2\chi_2, n) = \frac{\mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))}{a_2 - a_1} \exp\left(-a_1 \ln|x| + a_2 \ln|x|\right)$$

and taking the limit as $\lambda \to 0$ gives $v_M(n) = \mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee)) \ln|x|$ as required. □

4.4. Normalization of volumes. — Let M^0 be one of our Levi subgroups of G^0 and let M' be a twisted endoscopic group for M^0. We need to normalize $\mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee))$ for P a parabolic subset of G with Levi component M with $\mathrm{vol}(\mathfrak{a}_{P'}^{G'}/\mathbf{Z}(\Delta_{P'}^\vee))$ where $G' \in \mathcal{E}_{M'}(G)$ and P' is a parabolic subgroup of G' with levi component M'.

The norm map gives an isomorphism between $\mathfrak{a}_P$ and $\mathfrak{a}_{P'}$; and restricts to give an isomorphism between $\mathfrak{a}_P^G$ and $\mathfrak{a}_{P'}^{G'}$. We choose measures on these spaces, which are preserved by this isomorphism.

First we take M^0 to be the (2,2) Levi in G^0 and P^0 the upper triangular parabolic in G^0 with M^0 as a Levi component. Then we have

$$A_M = \left\{a = ((\mathrm{diag}(a_1, a_1), \mathrm{diag}(a_1^{-1}, a_1^{-1})), a_2)\right\},$$

and

$$N(a\alpha) = \mathrm{diag}(a_1^2 a_2, a_2, a_2, a_1^{-2} a_2) \in \mathrm{GSp}(4),$$

and

$$N(a\alpha) = \big(\mathrm{diag}(a_1^2 a_2, a_1^{-2} a_2), \mathrm{diag}(a_2, a_2)\big) \in (\mathrm{GL}(2) \times \mathrm{GL}(2))'.$$

Using this we see that we have

$$\mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee)) = \mathrm{vol}(\mathfrak{a}_{P_1}^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_P^\vee)) = \mathrm{vol}(\mathfrak{a}_{P_2}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))'}/\mathbf{Z}(\Delta_{P_2}^\vee)).$$

Next we take M^0 to be the (1,2,1) Levi in G^0 and P^0 the upper triangular parabolic in G^0 with M^0 as a Levi component. First we take $M' = \mathrm{GL}(2) \times \mathrm{GL}(1)$. Then we have

$$A_M = \left\{a = (\mathrm{diag}(a_1, 1, 1, a_1^{-1}), a_2)\right\},$$

and

$$N(a\alpha) = \mathrm{diag}(a_1 a_2, a_1 a_2, a_1^{-1} a_2, a_1^{-1} a_2) \in \mathrm{GSp}(4).$$

Using this we see that we have

$$\mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee)) = 2\,\mathrm{vol}(\mathfrak{a}_{P_1}^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_P^\vee)).$$

Next we take M^0 to be the (1,2,1) Levi in G^0 and P^0 the upper triangular parabolic in G^0 with M^0 as a Levi component. We take $M' = \mathrm{GL}(1) \times \mathrm{Res}_{E/F}\,\mathrm{GL}(1)$. Then we have

$$A_M = \left\{a = (\mathrm{diag}(a_1, 1, 1, a_1^{-1}), a_2)\right\},$$

and

$$N(a\alpha) = (\mathrm{diag}(a_1 a_2, a_1^{-1} a_2), \mathrm{diag}(a_1 a_2, a_1^{-1} a_2)) \in \mathrm{Res}_{E/F}\,\mathrm{GL}(2)',$$

and

$$N(a\alpha) = (\operatorname{diag}(a_1a_2, a_1^{-1}a_2), 1, 1) \in (\mathrm{GL}(2) \times \operatorname{Res}_{E/F} \mathrm{GL}(1))/\,\mathrm{GL}(1).$$

Using this we see that we have

$$\operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee)) = \operatorname{vol}(\mathfrak{a}_{P'}^{G'}/\mathbf{Z}(\Delta_{P'}^\vee))$$

for each elliptic endoscopic group $G' \in \mathcal{E}_{M'}(G)$.

Next we take M^0 equal to the diagonal Levi in G^0 and P^0 the upper triangular parabolic in G^0 with M^0 as a Levi component. We have

$$A_M = \left\{a = (\operatorname{diag}(a_1, a_2, a_2^{-1}, a_1^{-1}), a_3)\right\},$$

and

$$N(a\alpha) = \operatorname{diag}(a_1a_2a_3, a_1a_2^{-1}a_3, a_1^{-1}a_2a_3, a_1^{-1}a_2^{-1}a_3) \in \mathrm{GSp}(4),$$

and

$$N(a\alpha) = (\operatorname{diag}(a_1a_2a_3, a_1^{-1}a_2^{-1}a_3), \operatorname{diag}(a_1a_2^{-1}a_3, a_1^{-1}a_2a_3)) \in (\mathrm{GL}(2) \times \mathrm{GL}(2))'.$$

Using this we see that we have

$$\operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee)) = 2\operatorname{vol}(\mathfrak{a}_{P_1}^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_{P_1}^\vee)),$$

and

$$\operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee)) = 2\operatorname{vol}(\mathfrak{a}_{P_2}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))'}/\mathbf{Z}(\Delta_{P_2}^\vee)).$$

We also need to do the same for $\mathrm{GSp}(4)$ and its elliptic endoscopic group $(\mathrm{GL}(2) \times \mathrm{GL}(2))/\,\mathrm{GL}(1)$. First we take M equal to the Siegel Levi in $\mathrm{GSp}(4)$. Then we have

$$A_M = \left\{a = \operatorname{diag}(a_1, a_1, a_1^{-1}a_2, a_1^{-1}a_2)\right\},$$

and

$$N(a\alpha) = (\operatorname{diag}(1, a_1^{-2}a_2), \operatorname{diag}(a_1, a_1)) \in (\mathrm{GL}(2) \times \mathrm{GL}(2))/\,\mathrm{GL}(1).$$

Using this we see that we have

$$\operatorname{vol}(\mathfrak{a}_P^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_P^\vee)) = \frac{1}{2}\operatorname{vol}(\mathfrak{a}_{P'}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))/\,\mathrm{GL}(1)}/\mathbf{Z}(\Delta_{P'}^\vee)).$$

Next we take M equal to the diagonal Levi in $\mathrm{GSp}(4)$. We have

$$A_M = \left\{\operatorname{diag}(a_1, a_2, a_2^{-1}a_3, a_1^{-1}a_3)\right\},$$

and

$$N(a) = (\operatorname{diag}(1, a_1^{-1}a_2^{-1}a_3), \operatorname{diag}(a_1, a_2)) \in (\mathrm{GL}(2) \times \mathrm{GL}(2))/\,\mathrm{GL}(1).$$

Therefore we have

$$\operatorname{vol}(\mathfrak{a}_P^{\mathrm{GSp}(4)}/\mathbf{Z}(\Delta_P^\vee)) = \frac{1}{2}\operatorname{vol}(\mathfrak{a}_{P'}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))/\,\mathrm{GL}(1)}/\mathbf{Z}(\Delta_{P'}^\vee)).$$

4.5. Weighted orbital integrals. — In this section we prove a couple of lemmas that will be useful in the computation of our weighted orbital integrals. We begin with the following lemma, which allows us to write our weighted orbital integrals as integrals over the Levi subgroup itself.

For this section we take G^0 to be a connected reductive group over F. We let α be a quasi-semisimple automorphism of G^0 which we assume to be of finite order and defined over F. We take M^0 to be a Levi component of a parabolic subgroup P^0 with unipotent radical N; we assume all these groups are defined over F. We let K be a hyperspecial maximal compact subgroup of G^0 which is in good position relative to M^0. We assume that M^0, P^0, N and K are all stable under α.

Lemma 4.16. — *Let $K_M = M^0(F) \cap K$. For $a \in M^0(F)$ for which $a\alpha$ is strongly G^0-regular let $\varphi_a : N \to N$ denote the inverse of the bijection $N \to N : n \mapsto a^{-1}na\alpha(n)$ and define*

$$\sigma_P(a) = \int_{N(F)\cap K} v_M(\varphi_a(n))\, dn,$$

where the Haar measure on $N(F)$ is normalized to give $N(F) \cap K$ volume one. Let $\gamma\alpha \in M(F)$ be strongly G^0-regular then

$$r_M^G(\gamma\alpha) = |D_M(\gamma\alpha)|^{1/2} \int_{M_{\gamma\alpha}(F)\backslash M^0(F)} 1_{K_M}(m^{-1}\gamma\alpha(m))\sigma_P(m^{-1}\gamma\alpha(m))\, dm,$$

where the Haar measure on $M^0(F)$ gives K_M volume one.

Proof. — By the Iwasawa decomposition we have $G^0(F) = M^0(F)N(F)K$ and we can write the Haar measure on $G^0(F)$ as $dg = dm\, dn\, dk$. By definition we have

$$\begin{aligned} r_M^G(\gamma\alpha) &= |D_G(\gamma\alpha)|^{1/2} \int_{G_{\gamma\alpha}(F)\backslash G^0(F)} 1_K(g^{-1}\gamma\alpha(g))v_M(g)\, dg \\ &= |D_G(\gamma\alpha)|^{1/2} \int_K \int_{N(F)} \int_{M_{\gamma\alpha}(F)\backslash M^0(F)} 1_K(k^{-1}n^{-1}m^{-1}\gamma\alpha(m)\alpha(n)\alpha(k)) \\ &\qquad\qquad \cdot v_M(mnk)\, dm\, dn\, dk \\ &= |D_G(\gamma\alpha)|^{1/2} \int_{N(F)} \int_{M_{\gamma\alpha}(F)\backslash M^0(F)} 1_K(n^{-1}m^{-1}\gamma\alpha(m)\alpha(n))v_M(n)\, dm\, dn. \end{aligned}$$

If we set $a = m^{-1}\gamma\alpha(m) \in M^0(F)$ then we have

$$n^{-1}m^{-1}\gamma\alpha(m)\alpha(n) = a(a^{-1}n^{-1}a\alpha(n)),$$

which lies in K if and only if $a \in K_M$ and $a^{-1}n^{-1}a\alpha(n) \in N(F) \cap K$. Hence we have $r_M^G(\gamma\alpha)$ equal to

$$|D_G(\gamma\alpha)|^{1/2} \int_{M_{\gamma\alpha}(F)\backslash M^0(F)} 1_{K_M}(m^{-1}\gamma\alpha(m)) \cdot \int_{N(F)} 1_{N(F)\cap K}(a^{-1}n^{-1}a\alpha(n))v_M(n)\, dn\, dm.$$

Let $n' = a^{-1}n^{-1}a\alpha(n)$ so that $n = \varphi_a(n')$ then we have

$$\int_{N(F)} 1_{N(F)\cap K}(a^{-1}n^{-1}a\alpha(n))v_M(n)\,dn = \int_{N(F)\cap K} v_M(\varphi_a(n'))\left|\frac{\partial n}{\partial n'}\right|\,dn'.$$

But we have

$$\left|\frac{\partial n}{\partial n'}\right| = \left|\frac{D_M(\gamma\alpha)}{D_G(\gamma\alpha)}\right|^{1/2}$$

and hence

$$r_M^G(\gamma\alpha) = |D_M(\gamma\alpha)|^{1/2}\int_{M_{\gamma\alpha}(F)\backslash M^0(F)} 1_{K_M}(m^{-1}\gamma\alpha(m))\sigma_P(m^{-1}\gamma\alpha(m))\,dm$$

as wished. □

We now give a reduction for weighted orbital integrals using the topological Jordan decomposition; see [**BWW02**, Section 3].

We now make the assumptions that all our groups are defined over R and we take $K = G^0(R)$. We further assume that α is defined over R and is of finite order prime to p, the residual characteristic of F.

We continue with the notation above and assume that G^0 is defined over R and let $K = G^0(R)$. Assume further that the automorphism α has order prime to the residual characteristic of F and that K is stable under α. For $\gamma \in G^0(R)$ we can write $\gamma\alpha \in G$ uniquely as

$$\gamma\alpha = us\alpha = s\alpha u$$

with $s\alpha$ absolutely semisimple (i.e., $s\alpha$ has finite order prime to the residual characteristic of F) and u topologically unipotent (i.e., $u^{q^n} \to 1$, the identity in G^0, as $n \to \infty$).

We now make the assumption of [**BWW02**, Lemma 5.5]. That is, we assume that if $s_1\alpha$ and $s_2\alpha$ for $s_1, s_2 \in K$ are residually semisimple and conjugate by an element of $G^0(F)$ then they are also conjugate by an element of K. This is automatic in the case that α is trivial. In the case that $G^0 = \mathrm{GL}(4) \times \mathrm{GL}(1)$ and α is as in Section 2.4 this is verified in [**BWW02**]; see also [**Fli99**, Section I.H]. Under this assumption we have for $g \in G^0(F)$ that if $g^{-1}\gamma\alpha(g) \in G^0(R)$ then $g \in Z_{G^0}(s\alpha)(F)K$. For $g \in Z_{G^0}(s\alpha)$ we have

$$g^{-1}us\alpha(g) = g^{-1}ugs.$$

Hence $g^{-1}us\alpha(g) \in K$ if and only if $g^{-1}ug \in K$. Furthermore, if we fix $s\alpha$ and set $G_1 = Z_{G^0}(s\alpha)$ then we have

$$Z_{G^0}(us\alpha) = Z_{G_1}(u).$$

Assume now that $\gamma \in M^0(R)$. Then we have $u, s \in M^0(R)$ and, as in Lemma [**BWW02**, Lemma 5.5],

$$r_M^G(us\alpha) = |D_G(us\alpha)|^{1/2}\int_{G_{1,u}(F)\backslash G_1(F)} 1_{K_1}(g^{-1}ug)v_M(g)\,dg,$$

where $G_{1,u}$ denotes the connected component of the centralizer of u in G_1 and the measure on $G_1(F)$ is taken to give $K_1 = G_1(F) \cap K$ volume one.

We now assume further that G_1 is connected. We note that this is the case if $G^0 = \mathrm{GSp}(4)$ and α is trivial or if $G^0 = \mathrm{GL}(4) \times \mathrm{GL}(1)$ and α is as in Section 2.4. Then K_1 is a hyperspecial maximal compact subgroup of $G_1(F)$ and $P_1 = Z_{P^0}(s\alpha)$ is a parabolic subgroup of G_1. Hence by the Iwasawa decomposition we again have

$$G_1(F) = P_1(F)K_1.$$

Moreover, P_1 has Levi decomposition M_1N_1 where $M_1 = Z_{M^0}(s\alpha)$ and $N_1 = Z_{N_P}(s\alpha)$. We normalize the Haar measures on $M_1(F)$ and $N_1(F)$ to give $M_1 \cap K_1$ and $N_1 \cap K_1$ volume one. We can now mimic the proof of Lemma 4.16 to deduce the following.

Lemma 4.17. — *For $a \in M_1(F)$ strongly G_1-regular let $\varphi_a : N_1 \to N_1$ denote the inverse of the bijection $N_1 \to N_1 : n \mapsto a^{-1}n^{-1}an$ and define*

$$\sigma_{P_1}(a) = \int_{N_1(F)\cap K_1} v_M(\varphi_a(n))\, dn.$$

With the notations above we have

$$r_M^G(us\alpha) = |D_{M_1}(u)|^{1/2} \int_{M_{1,u}(F)\backslash M_1(F)} 1_{K_{M_1}}(m^{-1}um)\sigma_{P_1}(m^{-1}um)\, dm.$$

5. The fundamental lemma for the (2,2) Levi

In this section we take M^0 to be the (2,2) Levi in G^0. We have

$$M^0 = \left\{ \left(\begin{pmatrix} A & \\ & B \end{pmatrix}, c \right) : A, B \in \mathrm{GL}(2), c \in \mathrm{GL}(1) \right\}$$

and we write such an element as a triple (A, B, c). The restriction of α to M^0 is given by

$$\alpha : (A, B, c) \longmapsto (w\,{}^tB^{-1}w, w\,{}^tA^{-1}w, c \det AB),$$

where

$$w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

We set $M' = \mathrm{GL}(2) \times \mathrm{GL}(1)$ the unramified elliptic twisted endoscopic group for M. In this Section we prove the fundamental lemma for the pair (M, M').

5.1. Twisted integrals. — In this section we concentrate on the calculation of the twisted integrals. Note that we have

$$(I,B,1)^{-1}(A,B,c)\alpha(I,B,1)=(Aw^tB^{-1}w,I,c\det B),$$

and hence every twisted conjugacy class in M^0 contains a representative of the form (A,I,c). We now determine the stable twisted conjugacy class of such an element.

Lemma 5.1. — *Assume that $\gamma\alpha\in M(F)$ be semisimple. Let $m\in M(\overline{F})$ such that $m^{-1}\gamma\alpha(m)\in M(F)$. Then there exists $m_1\in M(F)$ such that*

$$m\gamma\alpha(m^{-1})=m_1\gamma\alpha(m_1^{-1}).$$

Proof. — We may assume that $\gamma=(A,I,c)$. We take $m=(D,E,f)\in M(\overline{F})$ and assume that $m^{-1}(A,I,c)\alpha(m)\in M(F)$. We have

$$m^{-1}(A,I,c)\alpha(m)=(D^{-1}Aw^tE^{-1}w,E^{-1}w^tD^{-1}w,c\det DE).$$

Hence we have $E_1=E^{-1}w^tD^{-1}w\in \mathrm{GL}(2,F)$ and therefore,

$$\mathrm{GL}(2,F)\ni D^{-1}Aw^tE^{-1}w=D^{-1}ADw^tE_1w$$

from which it follows that $D^{-1}AD\in GL(2,F)$. Now there exists $D_1\in\mathrm{GL}(2,F)$ such that $D_1^{-1}AD_1=D^{-1}AD$. Then we can take $m_1=(D_1,w^tD_1^{-1}wE_1^{-1},1)$. □

Thus the stable twisted conjugacy class of a strongly regular element γ is equal to the twisted conjugacy class of γ. We now show that the twisted orbital integrals on G^0 can be written as untwisted orbital integrals on GL(2).

Lemma 5.2. — *Let $\gamma\alpha=(A,I,c)\alpha\in M(F)$ be semisimple and strongly G^0-regular. Then if $c\notin U_F$ we have $r_M^G(\gamma\alpha)=0$. Otherwise, let T_1 denote the centralizer of A in* GL(2) *then we have*

$$r_M^G(\gamma\alpha)=|D_M(\gamma\alpha)|^{1/2}\int_{T_1(F)\backslash\,\mathrm{GL}(2,F)}1_{\mathrm{GL}(2,R)}(C^{-1}AC)\sigma_P(C^{-1}AC,I,1)\;dC.$$

Proof. — By Lemma 4.16 we have

$$r_M^G(\gamma\alpha)=|D_M(\gamma\alpha)|^{1/2}\int_{T(F)\backslash M^0(F)}1_{K_M}(m^{-1}\gamma\alpha(m))\sigma_P(m^{-1}\gamma\alpha(m))\;dm.$$

But now let $m=(C,D,e)\in M^0(F)$ then we have

$$m^{-1}\gamma\alpha(m)=(C^{-1}Aw^tD^{-1}w,D^{-1}w^tC^{-1}w,c\det CD).$$

Thus we see that if $m^{-1}\gamma\alpha(m)\in K_M$ then we have $D^{-1}w^tC^{-1}w\in\mathrm{GL}(2,R)$ from which it follows that we must have $\det CD\in U_F$. But this then forces $c\in U_F$ and hence if $c\notin U_F$ then $r_M^G(\gamma\alpha)$ vanishes.

Now assume that $c \in U_F$. Then we have that $m^{-1}\gamma\alpha(m) \in K_M$ if and only if $D^{-1}w^tC^{-1}w = C_1 \in \mathrm{GL}(2,R)$ and

$$C^{-1}Aw^tD^{-1}w = C^{-1}ACw^tC_1w \in \mathrm{GL}(2,R).$$

Which is if, and only if, $C^{-1}AC \in \mathrm{GL}(2,R)$ and $D = w^tC^{-1}wC_1$ with $C_1 \in \mathrm{GL}(2,R)$. So we have $m^{-1}\gamma\alpha(m) \in K_M$ if, and only if,

$$m = (C, w^tC^{-1}w, e)(I, C_1, 1)$$

with $C^{-1}AC, C_1 \in \mathrm{GL}(2,R)$.

Now we note that for $k \in K_M$ and $n \in N(F)$ we have

$$\varphi_{k^{-1}\gamma\alpha(k)}(n) = k^{-1}\varphi_\gamma(\alpha(k)n\alpha(k)^{-1})k,$$

and hence

$$\begin{aligned}\sigma_P(k^{-1}\gamma\alpha(k)) &= \int_{N(F)\cap K} v_M(\varphi_{k^{-1}\gamma\alpha(k)}(n))\,dn\\ &= \int_{N(F)\cap K} v_M(k^{-1}\varphi_\gamma(\alpha(k)n\alpha(k)^{-1})k)\,dn\\ &= \int_{N(F)\cap K} v_M(\varphi_\gamma(\alpha(k)n\alpha(k)^{-1}))\,dn,\end{aligned}$$

which equals $\sigma_P(\gamma)$ after a suitable change of variables.

Therefore the integrand in $r_M^G(\gamma\alpha)$ is invariant under right multiplication of m by an element of K_M. Thus if we set T_1 equal to the centralizer of A in $\mathrm{GL}(2)$ then we have

$$r_M^G(\gamma\alpha) = |D_M(\gamma\alpha)|^{1/2}\int_{T_1(F)\backslash \mathrm{GL}(2,F)} 1_{\mathrm{GL}(2,R)}(C^{-1}AC)\sigma_P(C^{-1}AC, I, 1)\,dC$$

as wished. □

5.2. Explicit statement of the fundamental lemma. — We now give an explicit statement of the fundamental lemma for the pair (M, M'). Let $\gamma\alpha = (A, I, c)\alpha \in M(F)$ be semisimple. Under the norm maps we have

$$N(\gamma\alpha) = \begin{pmatrix} c\det A & & \\ & cA & \\ & & c\end{pmatrix} \in M'(F) \subset \mathrm{GSp}(4,F),$$

and to

$$N(\gamma\alpha) = \left(\begin{pmatrix} c\det A & \\ & c\end{pmatrix}, cA\right) \in M'(F) \subset (\mathrm{GL}(2,F)\times\mathrm{GL}(2,F))'.$$

By Lemma 5.1 the fundamental lemma for the pair (M, M') is the assertion that for all $A \in \mathrm{GL}(2,F)$ and $c \in F^\times$ for which $(A, I, c)\alpha \in M$ is strongly G^0-regular we have

$$r_M^G((A,I,c)\alpha) = r_{M'}^{\mathrm{GSp}(4)}(\mathrm{diag}(c\det A, cA, c)) + r_{M'}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))'}(\mathrm{diag}(c\det A, c), cA).$$

From Lemma 5.2 we know that the twisted integral vanishes if $c \notin U_F$. It is also clear from Lemma 4.16 that the integrals on GSp(4) and $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$ vanish if $c \notin U_F$. Thus the fundamental lemma is proven in this case. Moreover, if $c \in U_F$ then all integrals that appear in the statement of the fundamental lemma are independent of c and so we may assume that $c = 1$. Furthermore, we may as well assume that $A \in K_1 = \mathrm{GL}(2, R)$. Having fixed A we let T_1 denote the centralizer of A in GL(2). Then we can write

$$\mathrm{GL}(2, F) = \coprod_{m \geqslant 0} T_1(F) z_m K_1$$

for an explicit set of representatives z_m to be given below.

Let P_1 (resp. P_2) denote the upper triangular parabolics in GSp(4) (resp. $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$) of which M' is a Levi component. By abuse of notation we write

$$\sigma_P(B) = \sigma_P\left(\begin{pmatrix} B & \\ & I \end{pmatrix}, 1\right)$$

$$\sigma_{P_1}(B) = \sigma_{P_1}\begin{pmatrix} \det B & & \\ & B & \\ & & 1 \end{pmatrix}$$

$$\sigma_{P_2}(B) = \sigma_{P_2}\left(\begin{pmatrix} \det B & \\ & 1 \end{pmatrix}, B\right)$$

for $B \in \mathrm{GL}(2, F)$.

Therefore the fundamental lemma we wish to prove is given by the following.

Proposition 5.3. — *Let $A \in \mathrm{GL}(2, R)$ be such that $\gamma\alpha = (A, I, 1)\alpha$ is strongly G^0-regular. Assume that we have $z_m^{-1} A z_m \in \mathrm{GL}(2, R)$ if and only if $m \leqslant N(A)$. Then*

$$|D_M(\gamma\alpha)|^{1/2} \sum_{m=0}^{N(A)} \mathrm{vol}(K_1 \cap z_m^{-1} T_1(F) z_m \backslash K_1) \sigma_P(z_m^{-1} A z_m)$$

is equal to

$$|D_{M'}(N(\gamma\alpha))|^{1/2} \sum_{m=0}^{N(A)} \mathrm{vol}(K_1 \cap z_m^{-1} T_1(F) z_m \backslash K_1) \left(\sigma_{P_1}(z_m^{-1} A z_m) + \sigma_{P_2}(z_m^{-1} A z_m)\right).$$

We label the identity of this Proposition by $FL(A)$. We now proceed to prove $FL(A)$. We split the proof into two cases, in the first we assume that A lies in a split torus, while in the second we assume that A lies in an elliptic torus.

5.3. Computation of σ_P, σ_{P_1} and σ_{P_2}. — In this section we give the expressions for σ_P, σ_{P_1} and σ_{P_2}. We set $\mathrm{vol}(\mathfrak{a}_P^G / \mathbf{Z}(\Delta_P^\vee))$ equal to $1/\ln q$ and normalize the other volumes as in Section 4.4.

5.3.1. Calculation of σ_P. — We have

$$N_P = \left\{ \left(\begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & x_3 & x_4 \\ & & 1 & \\ & & & 1 \end{pmatrix}, 1 \right) \right\}.$$

If we identify $N_P(F)$ with F^4 using $x_1, \ldots, x_4$ as our coordinates then for $x = (A, I, 1) \in M^0(F)$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the map $n \mapsto x^{-1}n^{-1}x\alpha(n)$ is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \det A^{-1} \begin{pmatrix} -d & 0 & b & \det A \\ 0 & \det A - d & 0 & b \\ c & 0 & \det A - a & 0 \\ \det A & c & 0 & -a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Let B denote this matrix then we have

$$\det B = -\det A^{-2}(\det A - 1)(\det A - \operatorname{tr} A + 1);$$

and after a change of variables we have, for $A \in \mathrm{GL}(2, R)$, $\sigma_P(A)$ equal to the product of $|(\det A - 1)(\det A - \operatorname{tr} A + 1)|$ with

$$\int_{F^4} \operatorname{char}_{R^4}(B^t(x_1, x_2, x_3, x_4)) \log \max\{1, |x_1|, |x_2|, |x_3|, |x_4|, |x_1x_4 - x_2x_3|\}.$$

5.3.2. Calculation of σ_{P_1}. — We have

$$N_{P_1} = \left\{ \begin{pmatrix} 1 & x & r & s \\ & 1 & & r \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \right\}.$$

We identify N_{P_1} with F^3 using x, r and s as our coordinates. For

$$y = \begin{pmatrix} \det A & & \\ & A & \\ & & 1 \end{pmatrix}$$

with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the map $n \mapsto y^{-1}n^{-1}yn$ is given by

$$f : \begin{pmatrix} x \\ r \\ s \end{pmatrix} \longmapsto \det A^{-1} \begin{pmatrix} (\det A - a)x - cr \\ -bx + (\det A - d)r \\ (\det A - 1)s + bx^2 + (d - a)xr - cr^2 \end{pmatrix}.$$

Therefore after a change of variables we have

$$\sigma_{P_1}(A) = |(\det A - 1)(\det A - \operatorname{tr} A + 1)| \int_{F^3} \operatorname{char}_{R^3}(f(x, r, s)) \log \max\{1, |x|, |r|, |s|\}.$$

5.3.3. Calculation of σ_{P_2}. — In this case we have

$$\sigma_{P_2}(A) = |\det A - 1| \int_{|x| \leqslant |\det A - 1|^{-1}} \log \max\{1, |x|\}.$$

5.4. Proof of the fundamental lemma for split tori. — In this section we prove Proposition 5.3 when A lies in a split torus. After conjugation we may assume that A lies in the diagonal torus T_1. We begin by giving a double coset decomposition for $\mathrm{GL}(2, F)$.

Lemma 5.4. — *For each $m \geqslant 0$ let $x_m \in F$ be an element of valuation of $-m$. Then we have*

$$\mathrm{GL}(2, F) = \coprod_{m \geqslant 0} T_1(F) \begin{pmatrix} 1 & x_m \\ & 1 \end{pmatrix} K_1.$$

Proof. — By the Iwasawa decomposition we have $\mathrm{GL}(2, F) = T_1(F)U(F)K_1$, where U denotes the subgroup of $\mathrm{GL}(2)$ of upper triangular unipotent matrices. But for $u \in U_F$ and $x \in F$ we have

$$\begin{pmatrix} 1 & ux \\ & 1 \end{pmatrix} = \begin{pmatrix} u & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & \\ & 1 \end{pmatrix}.$$

To check that the union of double cosets is disjoint we note that

$$\begin{pmatrix} 1 & -x_m \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & x_n \\ & 1 \end{pmatrix} = \begin{pmatrix} a & ax_n - bx_m \\ & b \end{pmatrix},$$

and for this matrix to lie in K_1 we would need $a, b \in U_F$ and $m = n$. □

We now fix a sequence of elements (x_m) as in Lemma 5.4 and we set

$$z_m = \begin{pmatrix} 1 & x_m \\ & 1 \end{pmatrix}.$$

Note that we have

$$z_m^{-1} \begin{pmatrix} a & \\ & d \end{pmatrix} z_m = \begin{pmatrix} a & (a - d)x_m \\ & d \end{pmatrix},$$

and therefore,

$$\operatorname{vol}(K_1 \cap z_m^{-1} T_1(F) z_m \backslash K_1) = \begin{cases} 1, & \text{if } m = 0; \\ (q - 1)q^{m-1}, & \text{if } m > 0. \end{cases}$$

We now set

$$A = \begin{pmatrix} a & \\ & d \end{pmatrix}$$

then in the notation of Proposition 5.3 we have $N(A) = v(a-d)$. Using the action of the Weyl group in GL(2) we can assume that we have $|a-1| \leqslant |d-1|$. We recall that we are assuming that F has odd residual characteristic, so we can split the proof of Proposition 5.3 into the following three cases

Case 1: $|ad-1| = |a-d| = |d-1| \geqslant |a-1|$
Case 2: $|a-1| = |d-1| = |ad-1| \geqslant |a-d|$
Case 3: $|a-1| = |d-1| = |a-d| \geqslant |ad-1|$.

Our strategy will be to show that each case follows from proving the identity $FL(A)$ when $|ad-1| = |a-d| = |d-1| = |a-1|$. We then prove that the identity $FL(A)$ holds in this case. In order to guarantee that, for any $M \geqslant 0$, there exists $a, d \in U_F$ such that

$$|ad-1| = |a-d| = |d-1| = |a-1| = q^{-M}$$

we need to make the additional assumption that $q > 3$. See Remark 5.9 below for the case that $q = 3$.

We will need to compute σ_P, σ_{P_1} and σ_{P_2} at elements of the form

$$\begin{pmatrix} a & b \\ & d \end{pmatrix}$$

with $a, d \in U_F$ and $0 < |a-d| \leqslant |b| \leqslant 1$. For σ_P the matrix B of Section 5.3.1 equals

$$\begin{pmatrix} -d & 0 & b & ad \\ 0 & d(a-1) & 0 & b \\ 0 & 0 & a(d-1) & 0 \\ ad & 0 & 0 & -a \end{pmatrix}.$$

After suitable row operations, invertible over R, we can put B in the form

$$\begin{pmatrix} 0 & 0 & b & ad-1 \\ 0 & (a-1)d & 0 & b \\ 0 & 0 & d-1 & 0 \\ d & 0 & 0 & -1 \end{pmatrix}.$$

Since the function v_M is invariant under right multiplication by K we may assume that $x_1 = d^{-1}x_4$. After multiplying x_2 by d^{-1} we get that $\sigma_P(A)$ is given by $|a-1||d-1||ad-1|$ times the integral of

$$\log \max\{1, |x_2|, |x_3|, |x_4|, |x_4^2 - x_2x_3|\}$$

over the region in F^3 given by

- $|x_3| \leqslant |d-1|^{-1}$
- $(ad-1)x_4 + bx_3 \in R$
- $(a-1)x_2 + bx_4 \in R$.

We have σ_{P_1} at the element

$$\begin{pmatrix} a & b \\ & d \end{pmatrix}$$

equal to $|a-1||d-1||ad-1|$ times the integral of

$$\log\max\{1,|x|,|r|,|s|\}$$

over the region in F^3 given by

- $|x| \leqslant |d-1|^{-1}$
- $-bx + d(a-1)r \in R$
- $(ad-1)s + x(bx-(a-d)r) \in R$.

5.4.1. Reduction in case 1. — We assume that we have $N \geqslant M$ and

$$q^{-M} = |ad-1| = |a-d| = |d-1| \geqslant |a-1| = q^{-N}.$$

We let $L(M,N)$ (resp. $R(M,N)$) denote the left (resp. right) hand side of the identity $FL(A)$ in this case. We will see that $L(M,N)$ and $R(M,N)$ are well defined. In this section we prove the following Proposition.

Proposition 5.5. — *For all $N \geqslant M$ we have*

$$qL(M,N+1)-L(M,N) = 3q^{-M}-3+(3M+N+1)(q-1) = qR(M,N+1)-R(M,N).$$

Proof. — We begin by considering the twisted integrals $\sigma_P(z_m^{-1}Az_m)$. We need to integrate

$$\log\max\{1,|x_2|,|x_3|,|x_4|,|x_4^2-x_2x_3|\}$$

over the region given by

- $|x_3| \leqslant |d-1|^{-1}$
- $bx_3 + (ad-1)x_4 \in R$
- $(a-1)x_2 + bx_4 \in R$

where $b \in R$ with $|a-d| \leqslant |b| \leqslant 1$.

We first consider when $|b|^{-1} < |x_3| \leqslant |d-1|^{-1}$. Then we have

$$x_4 = -(ad-1)^{-1}bx_3u_1$$

with $u_1 \in U_F^{-v(bx_3)}$ and $|x_4| = |ad-1|^{-1}|bx_3| > |b|^{-1}$. Therefore,

$$x_2 = -(a-1)^{-1}bx_4u_2 = (a-1)^{-1}(ad-1)^{-1}b^2x_3u_1u_2$$

with $u_2 \in U_F^{-v(bx_4)}$ and $|x_2| = |a-1|^{-1}|ad-1|^{-1}|b^2x_3|$. Therefore,

$$x_4^2 - x_2x_3 = b^2x_3^2u_1(ad-1)^{-2}(a-1)^{-1}((a-1)u_1-(ad-1)u_2)$$

Since

$$|(a-1)u_1-(ad-1)u_2| = |d-1|$$

for all such u_1 and u_2 we have

$$|x_4^2-x_2x_3| = |(ad-1)^{-1}(a-1)^{-1}||bx_3|^2.$$

The contribution to the integral is

$$|a-1|^{-1}|ad-1|^{-1}\int_{|b|^{-1}<|x_3|\leqslant|d-1|^{-1}} \log|(ad-1)^{-1}(a-1)^{-1}||bx_3|^2.$$

We are now left with the region given by

- $|x_3| \leqslant |b|^{-1}$
- $|x_4| \leqslant |ad-1|^{-1}$
- $(a-1)x_2 + bx_4 \in R$.

We now consider the case that $|x_4| > |b|^{-1}$. Then we have

$$x_2 = (a-1)^{-1} b x_4 u$$

with $u \in U_F^{-v(bx_4)}$ and $|x_2| = |a-1|^{-1}|bx_4|$. Now

$$|x_4^2 - x_2 x_3| = |x_2||x_4^2 x_2^{-1} - x_3|,$$

and

$$|x_4^2 x_2^{-1}| = |a-1||b|^{-1}|x_4| \leqslant |b|^{-1}.$$

Therefore making the change of variables $x_3 \mapsto x_3 - x_4^2 x_2^{-1}$ gives the contribution to the integral as

$$|a-1|^{-1} \int_{|x_3| \leqslant |b|^{-1}} \int_{|b|^{-1} < |x_4| \leqslant |ad-1|^{-1}} \log \max\{|a-1|^{-1}|bx_4|, |a-1|^{-1}|bx_4||x_3|\},$$

which we can write as the sum of

$$|a-1|^{-1}|b|^{-1} \int_{|b|^{-1} < |x_4| \leqslant |ad-1|^{-1}} \log |a-1|^{-1}|bx_4|,$$

and

$$|a-1|^{-1}(|ad-1|^{-1} - |b|^{-1}) \int_{1 < |x_3| \leqslant |b|^{-1}} \log |x_3|.$$

Finally we are left with the remaining contribution, which is

$$\int_{|x_3| \leqslant |b|^{-1}} \int_{|x_4| \leqslant |b|^{-1}} \int_{|x_2| \leqslant |a-1|^{-1}} \log \max\{1, |x_2|, |x_3|, |x_4|, |x_4^2 - x_2 x_3|\}.$$

We note that the integrals above depend only on M, N and $|b|$. We now compute the difference $qL(M, N+1) - L(M, N)$. For b with $|b| = q^{-k}$ where $0 \leqslant k \leqslant M$ we set

$$\sigma_P(M, N, k) = \sigma_P \begin{pmatrix} a & b \\ & d \end{pmatrix}.$$

We need to compute $q\sigma_P(M, N+1, k) - \sigma_P(M, N, k)$. From the first contribution to the integral the difference is given by

$$q^{-M+1} \int_{q^k < |x_3| \leqslant q^M} (M + N + 1 - 2k + 2\log|x_3|)$$

minus

$$q^{-M} \int_{q^k < |x_3| \leqslant q^M} (M + N - 2k + 2\log|x_3|).$$

The difference between the second contributions is given by

$$q^{-2M+k+1} \int_{q^k < |x_4| \leqslant q^M} (N + 1 - k + \log|x_4|) + q^{-2M+1}(q^M - q^k) \int_{1 < |x_3| \leqslant q^k} \log|x_3|$$

minus

$$q^{-2M+k}\int_{q^k<|x_4|\leqslant q^M}(N-k+\log|x_4|)+q^{-2M}(q^M-q^k)\int_{1<|x_3|\leqslant q^k}\log|x_3|.$$

And the difference between the third contributions is

$$q^{-2M-N}\int_{|x_3|\leqslant q^k}\int_{|x_4|\leqslant q^k}\int_{|x_2|=q^{N+1}}\log\max\{1,|x_2|,|x_4^2-x_2x_3|\}.$$

We note that $|x_2^{-1}x_4^2|\leqslant q^{2k-N-1}<q^k$ and so making the change of variables $x_3\mapsto x_3+x_2^{-1}x_4^2$ in this last integral gives

$$q^{-2M-N}\int_{|x_3|\leqslant q^k}\int_{|x_4|\leqslant q^k}\int_{|x_2|=q^{N+1}}N+1+\log\max\{1,|x_3|\}.$$

Using Lemma 9.1 we get

$$q\sigma_P(M,N+1,k)-\sigma_P(M,N,k)=(3M+N-2k+1)(q-1)-1+q^{-M}.$$

Now we have $qL(M,N+1)-L(M,N)$ equal to q^{-M} times

$$(q\sigma_P(M,N{+}1,M)-\sigma_P(M,N,M))+(q-1)\sum_{k=0}^{M-1}(q\sigma_P(M,N{+}1,k)-\sigma_P(M,N,k))q^{M-k-1}.$$

Using the fact that

$$(1-q^{-1})\sum_{i=0}^{m}iq^i=mq^m-\frac{q^m-1}{q-1}$$

for all $m\geqslant -1$ we get

$$qL(M,N+1)-L(M,N)=3q^{-M}+(3M+N+1)(q-1)-3.$$

We now consider the right hand side of the identity $FL(A)$. First we consider the relevant integrals on GSp(4). Here we need to integrate

$$\log\max\{1,|x|,|r|,|s|\}$$

over the region in F^3 given by

- $|x|\leqslant|d-1|^{-1}$
- $-bx+d(a-1)r\in R$
- $(ad-1)s+x(bx-(a-d)r)\in R$.

First we suppose that $|b|^{-1}<|x|$. Then $r=d^{-1}(a-1)^{-1}bxu$ with $u\in U_F^{-v(bx)}$. We have

$$bx-(a-d)r=bx-(a-d)d^{-1}(a-1)^{-1}bxu=bx(a-1)^{-1}d^{-1}(d(a-1)-(a-d)u)$$

and we note that

$$|d(a-1)-(a-d)u|=|d-1|$$

for all $u \in U_F^{-v(bx)}$. Hence we must have $|s| = |a-1|^{-1}|bx^2|$. Thus the contribution to the integral is

$$|a-1|^{-1}|ad-1|^{-1}\int_{|b|^{-1}<|x|\leqslant|d-1|^{-1}} \log|a-1|^{-1}|bx^2|.$$

We are now left with the region

- $|x| \leqslant |b|^{-1}$
- $|r| \leqslant |a-1|^{-1}$
- $(ad-1)s + x(bx-(a-d)r) \in R$

to integrate over. Making the change of variables $s \mapsto s-(ad-1)^{-1}x(bx-(a-d)r)$ we see that the contribution to the integral is

$$\int_{|x|\leqslant|b|^{-1}}\int_{|r|\leqslant|a-1|^{-1}}\int_{|s|\leqslant|ad-1|^{-1}} \log\max\{1,|x|,|r|,|s-(ad-1)^{-1}x(bx-(a-d)r)|\}.$$

Multiplying x, r and s by suitable units this integral equals

$$\int_{|x|\leqslant|b|^{-1}}\int_{|r|\leqslant|a-1|^{-1}}\int_{|s|\leqslant|ad-1|^{-1}} \log\max\{1,|x|,|r|,|s-\pi^{-M}x(bx-\pi^M r)|\}.$$

The integral on $(\mathrm{GL}(2)\times\mathrm{GL}(2))'$ is given by

$$\sigma_{P_2}(z_m^{-1}Az_m) = |ad-1|\int_{1<|x|\leqslant|ad-1|^{-1}} \log|x|.$$

We note that the integrals above depend only on M, N and $|b|$. For $|b| = q^{-k}$, $0 \leqslant k \leqslant M$, we define $\sigma_{P_1}(M,N,k)$ and $\sigma_{P_2}(M,N,k)$ as we did for $\sigma_P(M,N,k)$. We now compute

$$(q\sigma_{P_1}(M,N+1,k)-\sigma_{P_1}(M,N,k)) + (q\sigma_{P_2}(M,N+1,k)-\sigma_{P_2}(M,N,k)).$$

First we compute $q\sigma_{P_1}(M,N+1,k)-\sigma_{P_1}(M,N,k)$. The first part of the integral contributes

$$q^{-M+1}\int_{q^k<|x|\leqslant q^M} N-k+1+2\log|x|$$

minus

$$q^{-M}\int_{q^k<|x|\leqslant q^M} N-k+2\log|x|.$$

While the second part of the integral contributes

$$q^{-N-2M}\int_{|x|\leqslant q^k}\int_{|r|=q^{N+1}}\int_{|s|\leqslant q^M} \log\max\{1,|x||r|,|s-\pi^{-M}x(bx-\pi^M r)|\},$$

which equals

$$q^{-N-2M}\int_{|x|\leqslant q^k}\int_{|r|=q^{N+1}}\int_{|s|\leqslant q^M} \log\max\{|r|,|s-xr|\},$$

which equals

$$q^{-N-M}\int_{|x|\leqslant q^k}\int_{|r|=q^{N+1}} N+1+\log\max\{1,|x|\},$$

since $k \leqslant M \leqslant N$.

Putting this together and using Lemma 9.1 gives

$$(q\sigma_{P_1}(M,N+1,k)-\sigma_{P_1}(M,N,k))+(q\sigma_{P_2}(M,N+1,k)-\sigma_{P_2}(M,N,k))$$

equal to

$$(3M+N-k+1)(q-1)-2+2q^{-M}.$$

And we get $qR(M,N+1)-R(M,N)$ equal to

$$3q^{-M}-3+(3M+N+1)(q-1)$$

as required. □

5.4.2. Reduction in case 2. — We assume that $N\geqslant M$ and

$$q^{-M}=|a-1|=|d-1|=|ad-1|\geqslant|a-d|=q^{-N}.$$

We let $L(M,N)$ (resp. $R(M,N)$) denote the left (resp. right) hand side of the identity $FL(A)$ in this case. We will see that $L(M,N)$ and $R(M,N)$ are well defined. In this section we prove the following Proposition.

Proposition 5.6. — *For all $N\geqslant M$ we have*

$$L(M,N+1)-L(M,N)=0=R(M,N+1)-R(M,N).$$

Proof. — We begin by analyzing the twisted integrals σ_P. For b with $|b|=q^{-k}$ we write

$$\sigma_P(M,N,k)=\sigma_P\begin{pmatrix}a&b\\&d\end{pmatrix}$$

and we define, for $0\leqslant k\leqslant N$,

$$e(M,N,k)=\sigma_P(M,N+1,k)-\sigma_P(M,N,k).$$

Now we have

$$q^{N+1}L(M,N+1)=\sigma_P(M,N+1,N+1)+(q-1)\sum_{k=0}^{N}\sigma_P(M,N+1,k)q^{N-k}$$

$$=\sigma_P(M,N+1,N+1)+(q-1)\sum_{k=0}^{N}\sigma_P(M,N,k)q^{N-k}+(q-1)\sum_{k=0}^{N}e(M,N,k)q^{N-k}$$

$$=\sigma_P(M,N+1,N+1)-\sigma_P(M,N,N)+q^{N+1}L(M,N)+(q-1)\sum_{k=0}^{N}e(M,N,k)q^{N-k}.$$

Therefore, $q^{N+1}\left(L(M,N+1)-L(M,N)\right)$ is equal to

$$\sigma_P(M,N+1,N+1)-\sigma_P(M,N,N)+(q-1)\sum_{k=0}^{N}e(M,N,k)q^{N-k}.$$

Thus we will be done with the left hand side if we can show that $\sigma_P(M,N+1,N+1)=\sigma_P(M,N,N)$ and $e(M,N,k)=0$ for all k.

Now recall that $\sigma_P(M,N,k)$ is given by q^{-3M} times the integral of

$$\log\max\{1,|x_2|,|x_3|,|x_4|,|x_4^2-x_2x_3\}$$

over the region given by

- $|x_3|\leqslant q^M$
- $bx_3+(ad-1)x_4\in R$
- $(a-1)x_2+bx_4\in R$.

We now consider the integral over this region when $|b|=q^{-k}$. First suppose that $q^k<|x_3|\leqslant q^M$. Then we have

$$x_4=-(ad-1)^{-1}bx_3u_1$$

with $u_1\in U_F^{-v(bx_3)}$ and

$$x_2=-(a-1)^{-1}bx_4u_2=(a-1)^{-1}(ad-1)^{-1}b^2x_3u_2u_1$$

with $u_2\in U_F^{-v(bx_4)}$. Therefore,

$$\begin{aligned}x_4^2-x_2x_3&=(ad-1)^{-2}b^2x_3^2u_1^2-(a-1)^{-1}(ad-1)^{-1}b^2x_3^2u_1u_2\\&=(ad-1)^{-2}(a-1)^{-1}b^2x_3^2u_1((a-1)u_1-(ad-1)u_2).\end{aligned}$$

We have

$$|(a-1)u_1-(ad-1)u_2|=|d-1|$$

for all u_1 and u_2 and hence in the range $q^k<|x_3|\leqslant q^M$ we have

$$\log\max\{1,|x_2|,|x_3|,|x_4|,|x_4^2-x_2x_3\}=\log|x_4^2-x_2x_3|=2M-2k+2\log|x_3|.$$

We are now left to integrate over the region

- $|x_3|\leqslant\min\{q^k,q^M\}$
- $|x_4|\leqslant q^M$
- $(a-1)x_2+bx_4\in R$.

Next we suppose that $q^k<|x_4|\leqslant q^M$. Then we have

$$x_2=-(a-1)^{-1}bx_4u$$

with $u\in U_F^{-v(bx_4)}$. Hence,

$$x_4^2-x_2x_3=x_4^2+(a-1)^{-1}bx_4ux_3=(a-1)^{-1}bx_4u(u^{-1}(a-1)b^{-1}x_4+x_3).$$

Now $|u^{-1}(a-1)b^{-1}x_4|\leqslant q^{-M+k}q^M=q^k$. Hence making the change of variables

$$x_3\longmapsto x_3-u^{-1}(a-1)b^{-1}x_4$$

gives the integral over this region as

$$q^M\int_{|x_3|\leqslant q^k}\int_{q^k<|x_4|\leqslant q^M}(M-k+\log\max\{|x_4|,|x_3x_4|\}).$$

And finally we are left with the integral

$$\int_{|x_3|\leqslant\min\{q^k,q^M\}}\int_{|x_4|\leqslant\min\{q^k,q^M\}}\int_{|x_2|\leqslant q^M}\log\max\{1,|x_2|,|x_3|,|x_4|,|x_4^2-x_2x_3|\}.$$

It's clear from above that $\sigma_P(M,N,k)$ does not depend on N and hence we have $e(M,N,k)=0$ for all k. Moreover, we see that

$$\sigma_P(M,N,N)=q^{-3M}\int_{|x_2|,|x_3|,|x_4|\leqslant q^M}\log\max\{1,|x_2|,|x_3|,|x_4|,|x_4^2-x_2x_3|\}$$

and hence we have $\sigma_P(M,N+1,N+1)=\sigma_P(M,N,N)$.

Now we turn to the right hand side of the identity $FL(A)$. Let $R_1(M,N)$ (resp. $R_2(M,N)$) denote the contribution to $R(M,N)$ from the sum over the σ_{P_1} (resp. σ_{P_2}).

First we consider the integral on $(\mathrm{GL}(2)\times\mathrm{GL}(2))'$. We have for $0\leqslant m\leqslant N$

$$\sigma_{P_2}(z_m^{-1}Az_m)=q^M\int_{|x|\leqslant q^M}\log\max\{1,|x|\}$$

and it's clear from this that we have $R_2(M,N)=R_2(M,N+1)$.

Now we consider the integral on GSp(4). For $|b|=q^{-k}$, $0\leqslant k\leqslant N$, we set

$$\sigma_{P_1}(M,N,k)=\sigma_{P_1}\begin{pmatrix}a&b\\&d\end{pmatrix},$$

and define

$$e_1(M,N,k)=\sigma_{P_1}(M,N+1,k)-\sigma_{P_1}(M,N,k).$$

As above we have $q^{N+1}\left(R(M,N+1)-R(M,N)\right)$ equal to

$$\sigma_{P_1}(M,N+1,N+1)-\sigma_{P_1}(M,N,N)+(q-1)\sum_{k=0}^{N}e_1(M,N,k)q^{N-k}.$$

We now show that this expression is equal to zero.

Having fixed M we set, for $m\in\mathbf{Z}$,

$$I(m)=q^{-3M}\int_{|r|\leqslant q^M}\int_{|s|\leqslant q^M}|r|\log\max\{1,|r|,|s-\pi^m r^2|\}.$$

We note that $I(m)$ is constant for $m\geqslant 2M$. We will express $\sigma_{P_1}(M,N+1,N+1)-\sigma_{P_1}(M,N,N)$ and $e_1(M,N,k)q^{N-k}$ in terms of $I(m)$.

We begin by computing $e_1(M,N,k)$. Recall that $\sigma_{P_1}(M,N,k)$ is equal to q^{-3M} times the integral of

$$\log\max\{1,|x|,|r|,|s|\}$$

over the region

- $|x|\leqslant q^M$
- $-bx+d(a-1)r\in R$
- $(ad-1)s+x(bx-(a-d)r)\in R$.

First we suppose that $q^k<|x|\leqslant q^M$. Then we have

$$r=d^{-1}(a-1)^{-1}bxu$$

with $u \in U_F^{-v(bx)}$. Therefore,

$$x(bx - (a-d)r) = bx^2 d^{-1}(a-1)^{-1}(d(a-1) - (a-d)u)$$

and we have

$$|d(a-1) - (a-d)u| = |d-1|$$

for all such u. Hence over this region the integrand is equal to $\log |ad-1|^{-1}|bx^2|$ and therefore the contribution to $e_1(M,N,k)$ is zero.

We are now left with the region

- $|x| \leqslant \min\{q^k, q^M\}$
- $|r| \leqslant q^M$
- $(ad-1)s + x(bx - (a-d)r) \in R$.

So after scaling our variables by suitable units we can take this region to be

- $|x| \leqslant \min\{q^k, q^M\}$
- $|r| \leqslant q^M$
- $\pi^M s + x(\pi^k x - \pi^N r) \in R$.

Making the change of variables $x \mapsto x + \frac{1}{2}\pi^{N-k}r$ and $r \mapsto 2r$, which doesn't change the integrand, this region becomes

- $|x| \leqslant \min\{q^k, q^M\}$
- $|r| \leqslant q^M$
- $\pi^M s + \pi^k(x + \pi^{N-k}r)(x - \pi^{N-k}r) \in R$.

Thus we see that if $|x| > |\pi^{N-k}r|$ then we have

$$|\pi^k(x+\pi^{N-k}r)(x-\pi^{N-k}r)| = |\pi^k x^2| = |\pi^k(x+\pi^{N+1-k}r)(x-\pi^{N+1-k}r)|$$

and the contribution to $e_1(M,N,k)$ is zero. Therefore $e_1(M,N,k)$ is equal to the difference between the integral of

$$q^{-3M} \log \max\{1, |r|, |s|\}$$

over the regions

- $|r| \leqslant q^M$
- $|x| \leqslant q^{k-N}|r|$
- $\pi^M s + \pi^k(x + \pi^{N+1-k}r)(x - \pi^{N+1-k}r) \in R$,

and

- $|r| \leqslant q^M$
- $|x| \leqslant q^{k-N}|r|$
- $\pi^M s + \pi^k(x + \pi^{N-k}r)(x - \pi^{N-k}r) \in R$.

Over the first region the integral is equal the sum of

$$q^{-3M} q^{k-N}(1-q^{-1}) I(2N-k),$$

the contribution when $|x| = q^{k-N}|r|$,

$$q^{-3M} q^{k-N-2} I(2N-k+2)$$

the contribution when $|x| \leqslant q^{k-N-2}|r|$, and

$$q^{-3M}q^{k-N-1}(1-3q^{-1})I(2N-k+2)+q^{-3M}q^{k-N-1}\sum_{a=1}^{\infty}2q^{-a}(1-q^{-1})I(2N-k+2+a)$$

the contribution when $|x| = q^{k-N-1}|r|$.

Over the second region the integral is equal to the sum of

$$q^{-3M}q^{k-N-1}I(2N-k)$$

the contribution when $|x| \leqslant q^{k-N-1}|r|$, and

$$q^{-3M}q^{k-N}(1-3q^{-1})I(2N-k)+q^{-3M}q^{k-N}\sum_{a=1}^{\infty}2q^{-a}(1-q^{-1})I(2N-k+a)$$

the contribution when $|x| = q^{k-N}|r|$.

Hence we have $e_1(M,N,k)q^{N-k}$ equal to q^{-3M} times

$$q^{-1}I(2N-k)+q^{-1}(1-2q^{-1})I(2N-k+2) + q^{-1}\sum_{a=1}^{\infty}2q^{-a}(1-q^{-1})\left(I(2N-k+2+a)-I(2N-k+a)\right),$$

which equals q^{-3M} times the sum of

$$q^{-1}I(2N-k)-q^{-1}I(2N-k+2),$$

and

$$2q^{-1}(1-q^{-1})\sum_{a=0}^{\infty}q^{-a}I(2N-k+2+a)-2q^{-1}(1-q^{-1})\sum_{a=0}^{\infty}q^{-a}I(2N-k+1+a).$$

We now sum from $k=0$ to N. By telescoping we have

$$\sum_{k=0}^{N}q^{-1}I(2N-k)-q^{-1}I(2N-k+2)=q^{-1}I(N)+q^{-1}I(N+1)-2q^{-1}I(2M).$$

While we have

$$\sum_{k=0}^{N}\sum_{a=0}^{\infty}q^{-a}I(2N-k+2+a)-\sum_{k=0}^{N}\sum_{a=0}^{\infty}q^{-a}I(2N-k+1+a)$$

equal to

$$\sum_{k=1}^{N+1}\sum_{a=0}^{\infty}q^{-a}I(N+k+1+a)-\sum_{k=0}^{N}\sum_{a=0}^{\infty}q^{-a}I(N+k+1+a),$$

which equals

$$\sum_{a=0}^{\infty}q^{-a}I(2N+2+a)-\sum_{a=0}^{\infty}q^{-a}I(N+1+a),$$

which equals

$$\frac{1}{1-q^{-1}}I(2M) - \sum_{a=0}^{\infty} q^{-a} I(N+1+a),$$

using the fact that $I(m)$ is constant for $m \geqslant 2M$. Putting this altogether we get

$$q^{3M}(q-1)\sum_{k=0}^{N} e_1(M,N,k)q^{N-k}$$

equal to

$$(1-q^{-1})I(N) + (1-q^{-1})I(N+1) - 2(1-q^{-1})^2 \sum_{a=0}^{\infty} q^{-a} I(N+1+a).$$

Next we compute $\sigma_{P_1}(M, N+1, N+1) - \sigma_{P_1}(M,N,N)$ in terms of $I(m)$. We have $\sigma_{P_1}(M,N,N)$ equal to q^{-3M} times the integral of

$$\log\max\{1, |x|, |r|, |s|\}$$

over the region

- $|x|, |r| \leqslant q^M$
- $\pi^M s + \pi^N x(x-r) \in R$,

which becomes, after the change of variables $r \mapsto x - r$ that doesn't affect the integrand,

- $|x|, |r| \leqslant q^M$
- $\pi^M s + \pi^N xr \in R$.

Since the region and integrand are symmetric in x and r we can compute this integral as twice the integral when $|x| \leqslant |r|$ minus the integral when $|x| = |r|$. The contribution from when $|x| \leqslant |r|$ is

$$\sum_{a=0}^{\infty} \int_{|r|,|s| \leqslant q^M} (1-q^{-1})|\pi^a r| \log\max\{1, |r|, |s - \pi^{N+a} r^2|\},$$

which equals

$$\sum_{a=0}^{\infty} q^{-a}(1-q^{-1})I(N+a).$$

While the contribution when $|x| = |r|$ is equal to $(1-q^{-1})I(N)$. Hence we have

$$\sigma_{P_1}(M, N+1, N+1) - \sigma_{P_1}(M,N,N)$$

equal to q^{-3M} times

$$2\sum_{a=0}^{\infty} \big(q^{-a}(1-q^{-1})I(N+1+a)\big) - (1-q^{-1})I(N+1)$$

minus

$$2\sum_{a=0}^{\infty} \big(q^{-a}(1-q^{-1})I(N+a)\big) - (1-q^{-1})I(N).$$

But we have

$$2\sum_{a=0}^{\infty} q^{-a}(1-q^{-1})I(N+1+a) - 2\sum_{a=0}^{\infty} q^{-a}(1-q^{-1})I(N+a)$$

equal to

$$2(1-q^{-1})^2 \sum_{a=0}^{\infty} \big(q^{-a} I(N+1+a)\big) - 2(1-q^{-1})I(N),$$

and hence we have $q^{3M}(\sigma_{P_1}(M, N+1, N+1) - \sigma_{P_1}(M, N, N))$ equal to

$$2(1-q^{-1})^2 \sum_{a=0}^{\infty} q^{-a} I(N+1+a) - (1-q^{-1})I(N+1) - (1-q^{-1})I(N).$$

Thus $R_1(M, N+1) - R_1(M, N) = 0$ as required. □

5.4.3. Reduction in case 3. — We assume that $N \geqslant M$ and

$$q^{-M} = |a-1| = |d-1| = |a-d| \geqslant |ad-1| = q^{-N}.$$

We let $L(M, N)$ (resp. $R(M, N)$) denote the left (resp. right) hand side of the identity $FL(A)$ in this case. We will see that $L(M, N)$ and $R(M, N)$ are well defined. In this section we prove the following Proposition.

Proposition 5.7. — *For all $N \geqslant M$ we have*

$$qL(M, N+1) - L(M, N) = 2q^{-M} - 2 + 2(M+N+1)(q-1) = qR(M, N+1) - R(M, N).$$

Proof. — We begin by considering the twisted integrals $\sigma_P(z_m^{-1} A z_m)$. Again we need to integrate

$$\log \max\{1, |x_2|, |x_3|, |x_4|, |x_4^2 - x_2 x_3|\}$$

over the region in F^3 given by

- $|x_3| \leqslant |d-1|^{-1}$
- $bx_3 + (ad-1)x_4 \in R$
- $(a-1)x_2 + bx_4 \in R$.

We first consider the contribution when $|b|^{-1} < |x_3|$. Then we have

$$x_4 = -(ad-1)^{-1} b x_3 u_1$$

with $u_1 \in U_F^{-v(bx_3)}$. Therefore $|x_4| = |ad-1|^{-1}|bx_3| > |b|^{-1}$ and hence

$$x_2 = -(a-1)^{-1} b x_4 u_2 = (a-1)^{-1}(ad-1)^{-1} b^2 x_3 u_1 u_2$$

with $u_2 \in U_F^{-v(bx_4)}$. Thus,

$$x_4^2 - x_2 x_3 = (ad-1)^{-2}(a-1)^{-1} b^2 x_3^2 u_1 (u_1(a-1) - (ad-1)u_2).$$

Since

$$|u_1(a-1) - (ad-1)u_2| = |d-1|$$

for all u_1 and u_2 we have

$$|x_4^2 - x_2x_3| = |(ad-1)^{-2}b^2x_3^2|.$$

So the contribution when $|b|^{-1} < |x_3|$ is

$$|ad-1|^{-1}|a-1|^{-1}\int_{|b|^{-1}<|x_3|\leqslant|d-1|^{-1}} \log|(ad-1)^{-2}b^2x_3^2|.$$

We are now left to integrate over

- $|x_3| \leqslant |b|^{-1}$
- $|x_4| \leqslant |ad-1|^{-1}$
- $(a-1)x_2 + bx_4 \in R$.

Suppose that $|x_4| > |b|^{-1}$. Then we have

$$x_2 = -(a-1)^{-1}bx_4u$$

with $u \in U_F^{-v(bx_4)}$, and

$$x_4^2 - x_2x_3 = x_4^2 + (a-1)^{-1}bx_4ux_3 = x_4(x_4 + (a-1)^{-1}bux_3).$$

So after multiplying x_3 by a suitable unit the contribution to the integral is

$$|a-1|^{-1}\int_{|x_3|\leqslant|b|^{-1}}\int_{|b|^{-1}<|x_4|\leqslant|ad-1|^{-1}} \log\max\{|\pi^{-M}bx_4|, |x_4(x_4+\pi^{-M}bx_3)|\}.$$

Finally, when $|x_4| \leqslant |b|^{-1}$ the contribution is

$$\int_{|x_3|\leqslant|b|^{-1}}\int_{|x_4|\leqslant|b|^{-1}}\int_{|x_2|\leqslant|a-1|^{-1}} \log\max\{1, |x_2|, |x_3|, |x_4|, |x_4^2 - x_2x_3|\}.$$

We define $\sigma_P(M,N,k)$ as before and now compute $q\sigma_P(N+1,M,k) - \sigma_P(N,M,k)$. From the first contribution to the integral the difference is given by

$$q^{-M+1}\int_{q^k<|x_3|\leqslant q^M}(2N - 2k + 2 + 2\log|x_3|)$$

minus

$$q^{-M}\int_{q^k<|x_3|\leqslant q^M}(2N - 2k + 2\log|x_3|).$$

The difference between the second contributions is

$$2q^{-M-N+k}\int_{|x_4|=q^{N+1}} \log|x_4|,$$

and the difference between the third contributions is zero. Using Lemma 9.1 we get

$$q\sigma_P(M,N+1,k) - \sigma_P(M,N,k) = 2(M+N-k+1)(q-1),$$

and we compute

$$qL(M,N+1) - L(M,N) = 2q^{-M} - 2 + 2(M+N+1)(q-1).$$

We now turn our attention to the right hand side of the identity $FL(A)$. First we look at computing the integrals $\sigma_{P_1}(z_m^{-1}Az_m)$. We are integrating the function

$$\log\max\{1,|x|,|r|,|s|\}$$

over the region

- $|x|\leqslant|d-1|^{-1}$
- $-bx+d(a-1)r\in R$
- $(ad-1)s+x(bx-(a-d)r)\in R.$

If $|b|^{-1}<|x|$ then we have

$$r=d^{-1}(a-1)^{-1}bxu$$

with $u\in U_F^{-v(bx)}$. Then

$$bx-(a-d)r=bxd^{-1}(a-1)^{-1}(d(a-1)-(a-d)u),$$

and we have

$$|d(a-1)-(a-d)u|=|d-1|$$

for all such u. Hence we have

$$|s|=|ad-1|^{-1}|bx^2|.$$

Therefore, the contribution to the integral is

$$|ad-1|^{-1}|a-1|^{-1}\int_{|b|^{-1}<|x|\leqslant|d-1|^{-1}}\log|ad-1|^{-1}|bx^2|.$$

The region that's left is given by

- $|x|\leqslant|b|^{-1}$
- $|r|\leqslant|a-1|^{-1}$
- $(ad-1)s+x(bx-(a-d)r)\in R.$

Making the change of variables $s\mapsto s-(ad-1)^{-1}x(bx-(a-d)r)$ gives the remaining integral as

$$\int_{|x|\leqslant|b|^{-1}}\int_{|r|\leqslant|a-1|^{-1}}\int_{|s|\leqslant|ad-1|^{-1}}\log\max\{1,|x|,|r|,|s-(ad-1)^{-1}x(bx-(a-d)r)|\}.$$

And making the change of variables $r\mapsto r+(a-d)^{-1}bx$ gives this integral as

$$\int_{|x|\leqslant|b|^{-1}}\int_{|r|\leqslant|a-1|^{-1}}\int_{|s|\leqslant|ad-1|^{-1}}\log\max\{1,|x|,|r+(a-d)^{-1}bx|,|s-(ad-1)^{-1}(a-d)xr|\}.$$

We see that if $|xr|>|a-d|^{-1}$ then the integrand equals

$$\log|ad-1|^{-1}|a-d||xr|,$$

and so the contribution to the integral from this region is

$$|ad-1|^{-1}\int_{1<|x|\leqslant|b|^{-1}}\int_{|a-1|^{-1}|x|^{-1}<|r|\leqslant|a-1|^{-1}}\log|ad-1|^{-1}|a-d||xr|.$$

Now we look at the contribution when $|xr| \leqslant |a-d|^{-1}$. This is given, after suitable change of variables in x and s, by

$$\int_{|x|\leqslant|b|^{-1}} \int_{|r|\leqslant|a-1|^{-1}, |xr|\leqslant|a-1|^{-1}} \int_{|s|\leqslant|ad-1|^{-1}} \log\max\{1, |x|, |r+\pi^M bx|, |s|\}.$$

We define $\sigma_{P_1}(M,N,k)$ as before and we now compute $q\sigma_{P_1}(M,N+1,k) - \sigma_{P_1}(M,N,k)$. The difference between the first contributions to the integrals gives

$$q^{-M+1}\int_{q^k<|x|\leqslant q^M} (N+1-k+2\log|x|) - q^{-M}\int_{q^k<|x|\leqslant q^M} (N-k+2\log|x|).$$

The difference between the second contributions is

$$\begin{aligned} q^{-2M}(q-1)&\int_{q^M<|y|\leqslant q^{M+k}} (N-M+\log|y|)\int_{q^{-M}|y|\leqslant|x|\leqslant q^k} |x|^{-1} \\ &= q^{-M-1}(q-1)^2\int_{1<|y|\leqslant q^k} (N+\log|y|)(k+1-\log|y|) \end{aligned}$$

plus

$$q^{-2M+1}\int_{q^M<|y|\leqslant q^{M+k}}\int_{q^{-M}|y|\leqslant|x|\leqslant q^k} |x|^{-1} = q^{-M}(q-1)\int_{1<|y|\leqslant q^k} k+1-\log|y|.$$

And the difference between the third contributions is

$$q^{-N-2M}\int_{|x|\leqslant q^k}\int_{|r|\leqslant q^M, |xr|\leqslant q^M}\int_{|s|=q^{N+1}} \log|s|.$$

Putting these altogether gives

$$q\sigma_{P_1}(M,N+1,k) - \sigma_{P_1}(M,N,k) = (2M+N-k+1)(q-1) - 1 + q^{-M}.$$

We note that we have

$$q\sigma_{P_2}(M,N+1,k) - \sigma_{P_2}(M,N,k) = (N+1)(q-1)$$

and hence

$$q(\sigma_{P_1}(M,N+1,k) + \sigma_{P_2}(M,N+1,k)) - (\sigma_{P_1}(M,N+1,k) + \sigma_{P_2}(M,N+1,k))$$

equals

$$(2M+2N-k+2)(q-1) - 1 + q^{-M}.$$

We now compute

$$qR(M,N+1) - R(M,N) = 2q^{-M} - 2 + 2(M+N+1)(q-1)$$

as desired. □

5.4.4. Proof when $M = N$. — We assume that we have

$$|a-1| = |d-1| = |ad-1| = |d-1| = q^{-M}.$$

We let $L(M)$ (resp. $R(M)$) denote the left (resp. right) hand side of the identity $FL(A)$. We now prove the following Proposition which completes the proof of Proposition 5.3 in the case that A lies in a split torus.

Proposition 5.8. — *For all $M \geqslant 0$ we have*

$$L(M) = 4M - 4\frac{1-q^{-M}}{q-1} = R(M).$$

Proof. — We begin by computing the left hand side of $FL(A)$. For b with $|b| = q^{-k}$ we set

$$\sigma_P(M,k) = \sigma_P\begin{pmatrix} a & b \\ & d \end{pmatrix}.$$

As we have seen $\sigma_P(M,k)$ is equal to the sum of

$$q^{-M}\int_{q^k<|x_3|\leqslant q^M} \log(q^{2M-2k}|x_3|^2),$$

and

$$q^{-2M+k}\int_{q^k<|x_4|\leqslant q^M} \log(q^{M-k}|x_4|),$$

and

$$(q^{-M}-q^{-2M+k})\int_{1<|x_3|\leqslant q^k} \log|x_3|,$$

and

$$q^{-3M}\int_{|x_3|\leqslant q^k}\int_{|x_4|\leqslant q^k}\int_{|x_2|\leqslant q^M} \log\max\{1,|x_2|,|x_3|,|x_4|,|x_4^2-x_2x_3|\}.$$

Putting this altogether gives

$$\sigma_P(M,k) = (4M-2k) + \frac{1}{q-1}(-2+q^{-M}+q^{3k-3M}) - \frac{q^{3k-3M}-q^{-3M}}{q^3-1}.$$

And we get

$$L(M) = \sigma_P(M,M) + (q-1)\sum_{k=0}^{M-1}\sigma_P(M,k)q^{M-k-1} = 4M - 4\frac{1-q^{-M}}{q-1}.$$

We now compute $R(M)$. We define $\sigma_{P_1}(M,k)$ and $\sigma_{P_2}(M,k)$ similarly. First we note that

$$\sigma_{P_2}(M,k) = M - \frac{1-q^{-M}}{q-1}.$$

We now compute $\sigma_{P_1}(M,k)$. As we have seen this is equal to the sum of

$$q^{-M}\int_{q^k<|x|\leqslant q^M} \log(q^{M-k}|x|^2),$$

Now we consider the contribution when $|x| \leqslant q^l$. In this case we need to have that $(ad-1)xr \in R$. When $|x| \leqslant 1$ the contribution is

$$F(1) + \int_{1<|r|\leqslant q^M} F(|r|).$$

Finally we are left with the region $1 < |x| \leqslant q^l$ and $|r| \leqslant q^M |x|^{-1}$. Let's set $|x| = q^i$ with $1 \leqslant i \leqslant l$. Then $|r| \leqslant q^{M-i}$. Note that for all such i we have $q^i \leqslant q^{M-i}$. If we split up the cases that $|r| \leqslant q^i$ and $q^i < |r| \leqslant q^M$ then the contribution to the integral is

$$\sum_{i=1}^{l} \operatorname{vol}(|x| = q^i) \left(q^i F(q^i) + \int_{q^i<|r|\leqslant q^{M-i}} F(|r|) \right).$$

Putting this altogether gives $\sigma_{P_1}(M,k) + \sigma_{P_2}(M,k)$ equal to

$$(4M-k) + \frac{-3 + 4q^{-M} - q^{-M-k+l}}{q-1} - \frac{q^{-M-l} - q^{-3M}}{q^2-1} + \frac{q^{-3M+3l+2} - q^{-3M+2}}{(q+1)(q^3-1)}.$$

And we compute the right hand side of $FL(A)$ to be

$$4Mq^M - 4\frac{q^M - 1}{q-1}$$

as required. □

Remark 5.9. — We made the assumption that $q > 3$ in order to ensure that we could reduce to this $M = N$ case. However, in the case that $q > 3$ the reductions made are still valid. The identity proven in the Proposition above is again valid, it's just that it doesn't actually represent a case of the fundamental lemma since there are no elements a and d satisfying the necessary conditions. Hence the fundamental lemma for the (2,2) Levi is proven in the case that $q = 3$ as well.

5.5. Proof of the fundamental lemma for elliptic tori. — In this section we prove Proposition 5.3 in the case that A lies in an elliptic torus. In this case we may assume that

$$A = \begin{pmatrix} a & bD \\ b & a \end{pmatrix} \in \mathrm{GL}(2,R)$$

with $v(D) = 0$ or 1 and $E_D = F(\sqrt{D})$ a quadratic extension of F. We note that for $\gamma = (A, I, 1) \in M^0(F)$ we have

$$|D_M(\gamma\alpha)|^{1/2} = |b\sqrt{D}| = |D_{M'}(N(\gamma\alpha))|^{1/2}.$$

We take the following from [**Fli99**, Section I.I]. Let T_1 denote the torus in GL(2) with

$$T_1(F) = \left\{ \begin{pmatrix} x & yD \\ y & x \end{pmatrix} \in \mathrm{GL}(2,F) : x + y\sqrt{D} \in E_D^\times \right\}.$$

and

$$q^{-3M}\int_{|x|\leqslant q^k}\int_{|r|\leqslant q^M}\int_{|s|\leqslant q^M}\log\max\{1,|x|,|r|,|s-(ad-1)^{-1}x(bx-(ad-1)r)|\}.$$

We turn our attention to computing this latter integral. It's clear that if $|x(bx-(ad-1)r)|>1$ then the final term dominates. We begin by computing the contribution to the integral in this case. We need to compute the volume of x and r such that $|x(bx-(a-d)r)|=q^m$ for $m>0$.

Making the change of variables

$$x\longmapsto x+\frac{1}{2}(ad-1)b^{-1}r,\ r\longmapsto 2r$$

turns this into

$$|b|^{-1}|bx-(a-d)r||bx-(a-d)r|.$$

We now make the change of variables $u=bx-(ad-1)r$ and $v=bx+(ad-1)r$, which multiplies the integral by $|b|^{-1}|ad-1|^{-1}$. Given m with $0\leqslant m<k$ the volume of u and v such that $|uv|=q^{-m}$ is

$$\sum_{n=0}^{m}\operatorname{vol}(|u|=q^{-n})\operatorname{vol}(|v|=q^{-m+n})=(m+1)q^{-m}(1-q^{-1})^2.$$

Thus the contribution to $\sigma_{P_1}(M,k)$ when $|x(bx-(ad-1)r)|>1$ is

$$q^{k-M}\sum_{m=0}^{k-1}(m+1)(M+k-m)q^{-m}(1-q^{-1})^2.$$

We are now left the range of integration

- $|x|\leqslant q^k$, $|r|\leqslant q^M$, $x(bx-(ad-1)r)\in R$
- $|s|\leqslant q^M$

and after making of change of variables in s we can take our integrand to be

$$\log\max\{1,|x|,|r|,|s|\}.$$

We set $l=\lfloor k/2\rfloor$, so that $|bx^2|>1$ if and only if $|x|>q^l$. We define, for $a\geqslant 0$,

$$F(q^a)=\int_{|s|\leqslant q^M}\log\max\{q^a,|s|\}=Mq^M-\frac{q^M-q^a}{q-1}.$$

Let us first consider the case that $q^l<|x|\leqslant q^k$. Then in order that $x(bx-(ad-1)r)\in R$ we need $r=(ad-1)^{-1}bxu$ with $u\in U_F^{-v(bx^2)}$. The volume of such r equals $|(ad-1)^{-1}x^{-1}|$ and the contribution to the integral is

$$\int_{q^l<|x|\leqslant q^k}|(ad-1)^{-1}x^{-1}|F(q^M|bx|).$$

Similarly we reduce the proof of $FL(A)$ when $|\det A - 1| \leqslant |b^2 D|$ to the case that $|\det A - 1| = |b^2 D|$; we then prove $FL(A)$ in the case that $|b^2 D| \leqslant |\det A - 1| < |b|$.

We again need to make the assumption that $q > 3$. However the same argument as in Remark 5.9 allows us to deduce the fundamental lemma in the case that $q = 3$ as well.

5.5.1. Proof when b is a unit. — We begin by proving Proposition 5.3 under the assumption that $b \in U_F$.

Proposition 5.10. — *Let A be as above with $b \in U_F$. If we have $|T(A)| = 1$ then both sides of $FL(A)$ are equal to*

$$2|D|^{1/2}|\det A - 1| \int_{|x| \leqslant |\det A - 1|^{-1}} \log \max\{1, |x|\}.$$

Otherwise we must have $v(D) = 1$ and $a \in U_F^1$, then if we set $|\det A - 1| = q^{-k}$ we have both sides of $FL(A)$ equal to

$$|D|^{1/2} \left(2k + 1 + q^{-k-1} - 2\frac{1 - q^{-k-1}}{q - 1} \right).$$

Proof. — We first compute the twisted integral. In this case after applying row operations invertible over R we get B in the form

$$\begin{pmatrix} 0 & 0 & a - 1 & b \\ 0 & 0 & (\det A - 1)T(A) & 0 \\ b & 0 & \det A - a & 0 \\ 0 & b^2 & -\det A(\det A - a) & -ab \end{pmatrix}.$$

Hence we have

$$|x_3| \leqslant |(\det A - 1)T(A)|^{-1}$$

and we can take $bx_4 = -(a - 1)x_3$, $bx_1 = -(\det A - a)x_3$ and

$$b^2 x_2 = (\det A^2 - a \det A - a^2 + a)x_3.$$

Then

$$b^2(x_1 x_4 - x_2 x_3) = -\det A b^2 T(A) x_3^2$$

and hence $|x_1 x_4 - x_2 x_3| = |T(A) x_3^2|$. So we have

$$\sigma_P(A) = |\det A - 1||T(A)| \int_{|x_3| \leqslant |\det A - 1|^{-1} |T(A)|^{-1}} \log \max\{1, |x_3|, |T(A) x_3^2|\}.$$

The integral on $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$ is

$$|\det A - 1| \int_{|x| \leqslant |\det A - 1|^{-1}} \log \max\{1, |x|\}.$$

In order to compute the integral on $\mathrm{GSp}(4)$ we need to integrate

$$\log \max\{1, |x|, |r|, |s|\}$$

Let $z_m = \mathrm{diag}(1, \pi^m)$ then we have the double coset decomposition

$$\mathrm{GL}(2, F) = \coprod_{m \geqslant 0} T_1(F) z_m K_1,$$

where $K_1 = \mathrm{GL}(2, R)$. We have

$$z_m^{-1} A z_m = \begin{pmatrix} a & \pi^m bD \\ \pi^{-m} b & a \end{pmatrix}.$$

and so $z_m^{-1} A z_m \in K_1$ if and only if $m \leqslant v(b)$. We have

$$K_1 \cap z_m^{-1} T_1(F) z_m = \left\{ \begin{pmatrix} x & \pi^m yD \\ \pi^{-m} y & x \end{pmatrix} \in K_1 \right\}.$$

So if we set $\mathrm{vol}(D, m) = \mathrm{vol}(K_1 \cap z_m^{-1} T_1(F) z_m \backslash K_1)$ then we have

$$\mathrm{vol}(D, m) = \begin{cases} 1, & \text{if } E_D/F \text{ unramified and } m = 0; \\ (q+1)q^{m-1}, & \text{if } E_D/F \text{ unramified and } m > 0; \\ q^m, & \text{if } E_D/F \text{ ramified.} \end{cases}$$

We set $T(A) = \det A - \mathrm{tr}\, A + 1$. Then we have $\sigma_P(z_m^{-1} A z_m)$ equal to the product of $|\det A - 1||T(A)|$ with

$$\int_{F^4} \mathrm{char}_{R^4}(B^t(x_1, x_2, x_3, x_4)) \log \max\{1, |x_1|, |x_2|, |x_3|, |x_4|, |x_1 x_4 - x_2 x_3|\},$$

where B is the matrix

$$\begin{pmatrix} -a & 0 & \pi^m bD & \det A \\ 0 & \det A - a & 0 & \pi^m bD \\ \pi^{-m} b & 0 & \det A - a & 0 \\ \det A & \pi^{-m} b & 0 & -a \end{pmatrix}.$$

We have $\sigma_{P_1}(z_m^{-1} A z_m)$ equal to $|\det A - 1||T(A)|$ times the integral of

$$\log \max\{1, |x|, |r|, |s|\}$$

over the region in F^3 given by

- $(\det A - a)x - \pi^{-m} br \in R$
- $-\pi^m bDx + (\det A - a)r \in R$
- $(\det A - 1)s - \pi^{-m} b(r^2 - \pi^{2m} Dx^2) \in R.$

And we have

$$\sigma_{P_2}(z_m^{-1} A z_m) = |\det A - 1| \int_{1 < |x| \leqslant |\det A - 1|^{-1}} \log |x|.$$

As in the case that A lies in a split torus we will reduce the proof of $FL(A)$ to certain cases. We find, in the course of the proof, that the integrals in the identity $FL(A)$ depend only on $|b|$ and $|\det A - 1|$. We first prove the equality in the case that b is a unit. Using similar reductions as above we reduce the proof of $FL(A)$ when $|b| \leqslant |\det A - 1|$ to the case that $|b| = |\det A - 1|$; we then prove $FL(A)$ in this case.

over $(x, r, s) \in F^3$ such that

$$\begin{pmatrix} \det A - a & -b \\ -bD & \det A - a \end{pmatrix} \begin{pmatrix} x \\ r \end{pmatrix} \in R^2$$

and

$$(\det A - 1)s + b(Dx^2 - r^2) \in R.$$

Doing the row operation $R2 \mapsto bR2 + (\det A - a)R1$ in the matrix above gives

$$\begin{pmatrix} \det A - a & -b \\ T(A) \det A & 0 \end{pmatrix}.$$

Hence we need $|x| \leqslant |T(A)|^{-1}$ and $(\det A - a)x - br \in R$.

Therefore if $|T(A)| = 1$ we have

$$\sigma_{P_1}(A) = |\det A - 1| \int_{|s| \leqslant |\det A - 1|^{-1}} \log \max\{1, |s|\}$$

and the result follows.

Let $\alpha_1 = a + b\sqrt{D}$ and $\alpha_2 = a - b\sqrt{D}$ be the eigenvalues of A in E_D. We have $T(A) = (\alpha_1 - 1)(\alpha_2 - 1)$ and hence if $|T(A)| < 1$ we must have $v(D) = 1$ and $a \in U_F^1$. It follows that $|T(A)| = q^{-1}$. We now assume that this is the case and set $|\det A - 1| = q^{-k}$. The twisted integral is

$$|D|^{1/2} q^{-k-1} \int_{|x_3| \leqslant q^{k+1}} \log \max\{1, q^{-1}|x_3|^2\}$$

and the integral on $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$ is

$$|D|^{1/2} q^{-k} \int_{|x| \leqslant q^k} \log(\max\{1, |x|\}).$$

For the integral on $\mathrm{GSp}(4)$ we first note that $b(r^2 - Dx^2) \in R$ if and only if x and r are in R, and hence if and only if $x \in R$. The integral on $\mathrm{GSp}(4)$ is therefore the sum of

$$|D|^{1/2} q^{-k-1} \int_{|s| \leqslant q^k} \log \max\{1, |s|\},$$

the term contributing when $|x| \leqslant 1$, and

$$|D|^{1/2} q^{-1} \int_{|x| = q} k + 1$$

the term contributing when $|x| = q$.

We compute the twisted integral to be

$$|D|^{1/2} \left(2q^{-k-1} \left((k+1)q^{k+1} - \frac{q^{k+1} - 1}{q - 1} \right) - q^{-k-1}(q^{k+1} - 1) \right).$$

The integral on $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$ equals

$$|D|^{1/2} q^{-k} \left(kq^k - \frac{q^k - 1}{q - 1} \right)$$

and the integral on GSp(4) equals

$$|D|^{1/2}\left(q^{-k-1}\left(kq^k-\frac{q^k-1}{q-1}\right)+(k+1)(1-q^{-1})\right).$$

Hence we get both the left and right hand sides of the identity $FL(A)$ equal to

$$|D|^{1/2}\left(2k+1+q^{-k-1}-2\frac{1-q^{-k-1}}{q-1}\right)$$

and we are done. □

For the rest of this Section we assume that $|b|<1$.

5.5.2. Reduction when $|b|\leqslant|\det A-1|$. — In this section we reduce the proof of Proposition 5.3 in the case that $|b|\leqslant|\det A-1|$ to the case that $|b|=|\det A-1|$. We note that if we have $|b|<1$ and $|\det A-1|=1$ then we have $|T(A)|=1$ and $|\det A-1|=1$. It follows that both sides of $FL(A)$ vanish in this case. Thus we may as well assume that we also have $|\det A-1|<1$.

Under the assumption $|b|\leqslant|\det A-1|<1$ we have

$$|\det A-a|=|a-1|=|\det A-1|=q^{-M}$$

and hence $|T(A)|=|a-1|^2=q^{-2M}$. We set $n=\det A$ then

$$n-a(a-1)(n-a)^{-1}=(n-a)^{-1}(a(a-1)^2(a+1)-b^2D(n+a(a-1))).$$

Hence if $|b|<|a-1|$ we have

$$|n-a(a-1)(n-a)^{-1}|=|n-1|.$$

On the other hand if $|b|=|a-1|$ then, provided $q>3$, given b we can choose a such that $|a-1|=|b|$ and

$$|n-a(a-1)(n-a)^{-1}|=|n-1|,$$

we make this further assumption in the case that $|b|=|a-1|$.

We now assume that $N\geqslant M$ and

$$q^{-N}=|b|\leqslant|\det A-1|=q^{-M}.$$

We let $L(M,N)$ (resp. $R(M,N)$) denote the left (resp. right) hand side of the identity $FL(A)$ in this case. We now prove the following Proposition.

Proposition 5.11. — *With the notations and assumptions above we have, for all $N\geqslant M\geqslant 1$, $L(M,N+1)-L(M,N)$ and $R(M,N+1)-R(M,N)$ equal to*

$$fq^{-N-1}|D|^{1/2}\left(2M-\frac{1-q^{-M}}{q-1}-\frac{1-q^{-3M}}{q^3-1}\right),$$

where $f=f(E_D/F)$ is the degree of the residue field extension.

Proof. — We begin by seeing how to compute $\sigma_P(z_m^{-1}Az_m)$. Recall we have

$$B = \begin{pmatrix} -a & 0 & \pi^m bD & \det A \\ 0 & \det A - a & 0 & \pi^m bD \\ \pi^{-m}b & 0 & \det A - a & 0 \\ \det A & \pi^{-m}b & 0 & -a \end{pmatrix}.$$

We now do a series of row operations invertible over R to get B in a suitable form. The row operation $R1 \mapsto n^{-1}(R1 - (\pi^m bD)(n-a)^{-1}R3)$ gives

$$\begin{pmatrix} -(a-1)(n-a)^{-1} & 0 & 0 & 1 \\ 0 & n-a & 0 & \pi^m bD \\ \pi^{-m}b & 0 & n-a & 0 \\ n & \pi^{-m}b & 0 & -a \end{pmatrix}.$$

Now we do $R2 \mapsto R2 - (\pi^m bD)R1$ and $R1 \mapsto aR1 + R4$ to give

$$\begin{pmatrix} n - a(a-1)(n-a)^{-1} & \pi^{-m}b & 0 & 0 \\ (a-1)(n-a)^{-1}\pi^m bD & n-a & 0 & 0 \\ \pi^{-m}b & 0 & n-a & 0 \\ n & \pi^{-m}b & 0 & -a \end{pmatrix}.$$

Now

$$n - a(a-1)(n-a)^{-1} = -(n-a)^{-1}(-a(a-1)^2(a+1) + b^2D(n + a(a-1)))$$

and therefore provided $a - 1 \notin U_F$ we have

$$|n - a(a-1)(n-a)^{-1}| = |n-a|^{-1}|a-1|^2 = |n-a| > |\pi^m bD|.$$

Next we do $R2 \mapsto R2 - (a-1)\pi^m bD(a - a^2 + n^2 - an)^{-1}R1$ to give

$$\begin{pmatrix} (n-a)^{-1}(a - a^2 + n^2 - an) & \pi^{-m}b & 0 & 0 \\ 0 & n - a - (a-1)(a - a^2 + n^2 - an)^{-1}b^2D & 0 & 0 \\ \pi^{-m}b & 0 & n-a & 0 \\ n & \pi^{-m}b & 0 & -a \end{pmatrix}.$$

But now

$$|(a-1)(a - a^2 + n^2 - an)^{-1}b^2D| = |a-1|^{-1}|b^2D|.$$

After multiplying row 2 by a suitable unit and adding row 1 to row 4 and multiplying it by a^{-1} we get

$$\begin{pmatrix} (n-a)^{-1}(a - a^2 + n^2 - an) & \pi^{-m}b & 0 & 0 \\ 0 & n-a & 0 & 0 \\ \pi^{-m}b & 0 & n-a & 0 \\ (a-1)(n-a)^{-1} & 0 & 0 & -1 \end{pmatrix}.$$

Therefore in order to compute the twisted integral we need to integrate the function

$$\log\max\{1, |x_1|, |x_2|, |x_3|, |x_4|, |x_1x_4 - x_2x_3|\}$$

over the region

- $|x_2| \leqslant |n-1|^{-1}$
- $(n-a)^{-1}(a-a^2+n^2-an)x_1+\pi^{-m}bx_2 \in R$
- $\pi^{-m}bx_1+(n-a)x_3 \in R$
- $(a-1)(n-a)^{-1}x_1-x_4 \in R.$

Note that we can set $x_4 = (a-1)(n-a)^{-1}x_1$ and make the change of variables $x_3 \mapsto (a-1)(n-a)^{-1}x_3$ to give our integral as the integral of

$$\log\max\{1,|x_1|,|x_2|,|x_3|,|x_1^2-x_2x_3|\}$$

over the region

- $|x_2| \leqslant |n-1|^{-1}$
- $(n-a)^{-1}(a-a^2+n^2-an)x_1+\pi^{-m}bx_2 \in R$
- $\pi^{-m}bx_1+(a-1)x_3 \in R.$

First we note that for m such that $|\pi^{-m}b| \leqslant |n-1|$ this region becomes

- $|x_2| \leqslant |n-1|^{-1}$
- $|x_1| \leqslant |n-1|^{-1}$
- $|x_3| \leqslant |n-1|^{-1}.$

Now assume that $|\pi^{-m}b| > |n-1|$. First suppose that $|\pi^{-m}b|^{-1} < |x_2| \leqslant |n-1|^{-1}$. Then we have

$$x_1 = -(n-a)(a-a^2+n^2-an)^{-1}\pi^{-m}bx_2u_1$$

with $u_1 \in U_F^{-v(\pi^{-m}bx_2)}$ and

$$|x_1| = |n-a|^{-1}|\pi^{-m}bx_2| > |\pi^{-m}b|^{-1}$$

hence

$$\begin{aligned} x_3 &= -(a-1)^{-1}\pi^{-m}bx_1u_2 \\ &= (a-1)^{-1}(n-a)(a-a^2+n^2-an)^{-1}\pi^{-2m}b^2x_2u_1u_2 \end{aligned}$$

with $u_2 \in U_F^{v(\pi^{-m}bx_1)}$; and therefore $|x_3| = |(n-1)^{-2}||\pi^{-m}b|^2|x_2|$. Now we have $x_1^2-x_2x_3$ equal to

$$x_2^2\pi^{-2m}b^2u_1(a-1)^{-1}(n-a)(a-a^2+n^2-an)^{-2}((n-a)(a-1)u_1-(a-a^2+n^2-an)u_2).$$

And since

$$(n-a)(a-1)-(a-a^2+n^2-an) = -nT(A)$$

so

$$|(n-a)(a-1)u_1-(a-a^2+n^2-an)u_2| = |n-1|^2$$

for all u_1 and u_2. Hence we deduce that

$$|x_1^2-x_2x_3| = |\pi^{-m}b(n-a)^{-1}x_2|^2.$$

Thus the contribution to the integral is

$$|n-1|^{-2}\int_{|\pi^{-m}b|^{-1}<|x_2|\leqslant|n-1|^{-1}} 2\log|\pi^{-m}b(n-1)^{-1}x_2|.$$

So we are now left with the region

- $|x_2| \leqslant |\pi^{-m}b|^{-1}$
- $|x_1| \leqslant |n-1|^{-1}$
- $\pi^{-m}bx_1 + (a-1)x_3 \in R$.

We first consider the case that $|\pi^{-m}b|^{-1} < |x_1| \leqslant |n-1|^{-1}$. Then we have

$$x_3 = -(a-1)^{-1}\pi^{-m}bx_1u$$

with $u \in U_F^{-v(bx_1)}$. Hence $|x_3| = |(n-1)^{-1}\pi^{-m}b||x_1|$. Then

$$\begin{aligned} x_1^2 - x_2x_3 &= x_1^2 + (a-1)^{-1}\pi^{-m}bx_2x_1u \\ &= (a-1)^{-1}\pi^{-m}bux_1((a-1)\pi^m b^{-1}u^{-1}x_1 + x_2). \end{aligned}$$

Now $|(a-1)\pi^m b^{-1}u^{-1}x_1| \leqslant |\pi^{-m}b|^{-1}$ and so making the change of variables

$$x_2 \longmapsto x_2 - (a-1)\pi^m b^{-1}u^{-1}x_1$$

gives the integral as

$$|n-1|^{-1}\int_{|\pi^{-m}b|^{-1}<|x_1|\leqslant|n-1|^{-1}}\int_{|x_2|\leqslant|\pi^{-m}b|^{-1}} \log\left(|(n-1)^{-1}\pi^{-m}bx_1|\max\{1,|x_2|\}\right).$$

Finally we are left with the region

- $|x_2| \leqslant |\pi^{-m}b|^{-1}$
- $|x_1| \leqslant |\pi^{-m}b|^{-1}$
- $|x_3| \leqslant |n-1|^{-1}$.

We see that the integrals above depend only on $|b|$, $|n-1|$ and m. For $|a-1| = q^{-M}$ and $|b| = q^{-N}$ we set

$$\sigma_P(M,N,m) = \sigma_P(z_m^{-1}Az_m)$$

then it's clear from above that we have

$$\sigma_P(M,N+1,m+1) = \sigma_P(M,N,m)$$

for all m with $0 \leqslant m \leqslant N$. So we have $|D|^{-1/2}q^N\left(qL(M,N+1) - L(M,N)\right)$ equal to

$$\sum_{m=0}^{N+1}\mathrm{vol}(D,m)\sigma_P(M,N+1,m) - \sum_{m=0}^{N}\mathrm{vol}(D,m)\sigma_P(M,N,m),$$

which equals

$$\mathrm{vol}(D,0)\sigma_P(M,N+1,0) + \sum_{m=0}^{N}(\mathrm{vol}(D,m+1) - \mathrm{vol}(D,m))\sigma_P(M,N,m)).$$

In the case that $|D| = q^{-1}$ we have $\mathrm{vol}(D,m) = q^m$ for all m and hence we see that

$$q^{N+1}|D|^{-1/2}\left(L(M,N+1) - L(M,N)\right) = \sigma_P(M,N+1,0).$$

In the case that $|D| = 1$ we have $\mathrm{vol}(D,0) = 1$ and $\mathrm{vol}(D,m) = (q+1)q^{m-1}$ if $m > 0$. Hence

$$\mathrm{vol}(D,1) - \mathrm{vol}(D,0) = q$$

and if $m > 0$ then

$$\mathrm{vol}(D,m+1) - \mathrm{vol}(D,m) = (q+1)q^m - (q+1)q^{m-1} = (q-1)(q+1)q^{m-1}.$$

So we see that if $|D| = 1$ then

$$|D|^{-1/2}q^{N+1}\left(L(M,N+1) - L(M,N)\right) = \sigma_P(M,N+1,0) + \sigma_P(M,N,0).$$

Now for $N \geqslant M$ we have

$$\sigma_P(M,N,0) = q^{-3M}\int_{|x_1|\leqslant q^M}\int_{|x_2|\leqslant q^M}\int_{|x_3|\leqslant q^M} \log\max\{1,|x_1|,|x_2|,|x_3|,|x_1^2 - x_2x_3|\}.$$

Hence we get from Lemma 9.7 that

$$q^{N+1}|D|^{-1/2}\left(L(M,N+1) - L(M,N)\right) = f\left(2M - \frac{1-q^{-M}}{q-1} - \frac{1-q^{-3M}}{q^3-1}\right).$$

We now turn to computing the right hand side of $FL(A)$. First we consider the integral on $\mathrm{GSp}(4)$. Recall we need to integrate

$$\log\max\{1,|x|,|r|,|s|\}$$

over the region in F^3 given by

- $(n-a)x - \pi^{-m}br \in R$
- $-\pi^m bDx + (n-a)r \in R$
- $(n-1)s - \pi^{-m}b(r^2 - \pi^{2m}Dx^2) \in R.$

Now consider

$$\begin{pmatrix} n-a & -\pi^{-m}b \\ -\pi^m bD & n-a \end{pmatrix}.$$

Doing the row operation $R2 \mapsto R2 + \pi^m bD(n-a)^{-1}R1$ gives

$$\begin{pmatrix} n-a & -\pi^{-m}b \\ 0 & (n-a)^{-1}nT(A) \end{pmatrix}.$$

Note that $|T(A)| = |n-1|^2$ and hence we need to integrate

$$\log\max\{1,|x|,|r|,|s|\}$$

over the region in F^3 given by

- $|r| \leqslant |n-1|^{-1}$
- $(n-a)x - \pi^{-m}br \in R$
- $(n-1)s - \pi^{-m}b(r^2 - \pi^{2m}Dx^2) \in R.$

and over the region

- $|r| \leqslant \min\{q^M, q^{N-m}\}$
- $|x| \leqslant q^M$
- $\pi^M s - \pi^{N-m}(r^2 - D(x\pi^m)^2) \in R$.

When $|r| \geqslant |\pi^m x|$ we have

$$|r^2 - D(x\pi^m)^2| = |r|^2 = |r^2 - D(x\pi^{m+1})^2|$$

and the integrals cancel. Hence $e(M,N,m)$ is given by the difference between integrating

$$q^{-3M} \log \max\{1, |x|, |s|\}$$

over the regions

- $|x| \leqslant q^M$
- $|r| \leqslant q^{-m-1}|x|$
- $\pi^M s - \pi^{N-m}(r^2 - D(x\pi^{m+1})^2) \in R$,

and

- $|x| \leqslant q^M$
- $|r| \leqslant q^{-m-1}|x|$
- $\pi^M s - \pi^{N-m}(r^2 - D(x\pi^m)^2) \in R$.

Now note that when $|r| \leqslant q^{-m-2}|x|$ we have

$$|r^2 - D(x\pi^{m+1})^2| = |D(x\pi^{m+1})^2|$$

and

$$|r^2 - D(x\pi^m)^2| = |D(x\pi^m)^2|.$$

Hence $e(M,N,m)$ is given as the difference between integrating

$$q^{-3M} q^{-m-2}|x| \log \max\{1, |x|, |s|\}$$

over the region

- $|x| \leqslant q^M$
- $\pi^M s - \pi^{N+m+2} D x^2 \in R$

and

- $|x| \leqslant q^M$
- $\pi^M s - \pi^{N+m} D x^2 \in R$

plus the difference between integrating

$$q^{-3M}(1 - q^{-1}) q^{-m-1}|x| \log \max\{1, |x|, |s|\}$$

over the region

- $|x| \leqslant q^M$
- $\pi^M s - \pi^{N+m+2} x^2 \in R$

and

- $|x| \leqslant q^M$
- $\pi^M s - \pi^{N+m} D x^2 \in R$.

First suppose that $|\pi^{-m}b|^{-1} < |r| \leqslant |n-1|^{-1}$. Then we have

$$x = (n-a)^{-1}\pi^{-m}bru$$

with $u \in U_F^{-v(\pi^{-m}br)}$. So

$$\begin{aligned}(n-1)s - \pi^{-m}b(r^2 - \pi^{2m}Dx^2) &= (n-1)s - \pi^{-m}b(r^2 - \pi^{2m}D((n-a)^{-1}\pi^{-m}bru)^2)\\ &= (n-1)s - \pi^{-m}br^2(1 - Db^2(n-a)^{-2}u^2).\end{aligned}$$

Hence the contribution to the integral is

$$|n-1|^{-2}\int_{|\pi^{-m}b|^{-1}<|r|\leqslant|n-1|^{-1}} \log|n-1|^{-1}|\pi^{-m}br^2|.$$

We are then left with the region

- $|r| \leqslant \min\{|n-1|^{-1}, |\pi^{-m}b|^{-1}\}$
- $|x| \leqslant |n-1|^{-1}$
- $(n-1)s - \pi^{-m}b(r^2 - \pi^{2m}Dx^2) \in R$.

The integrals above depend only on $|b| = q^{-N}$, $|n-1| = q^{-M}$ and m. We set

$$\sigma_{P_1}(M,N,m) = \sigma_{P_1}(z_m^{-1}Az_m)$$

and write

$$\sigma_{P_1}(M,N+1,m+1) = \sigma_{P_1}(M,N,m) + e(M,N,m).$$

Let $R_1(M,N)$ denote the contribution of the GSp(4) integral to the right hand side of the identity $FL(A)$. Then we have $|D|^{-1/2}\left(q^{N+1}R_1(M,N+1) - q^N R_1(M,N)\right)$ equal to

$$\sigma_{P_1}(M,N+1,0) + \sum_{m=0}^{N}(\mathrm{vol}(D,m+1) - \mathrm{vol}(D,m))\sigma_{P_1}(M,N,m) + \sum_{m=0}^{N}\mathrm{vol}(D,m+1)e(M,N,m).$$

Thus when $|D| = q^{-1}$ we have $q^{N+1}|D|^{-1/2}(R_1(M,N+1) - R_1(M,N))$ equal to

$$\sigma_{P_1}(M,N+1,0) + \sum_{m=0}^{N}\mathrm{vol}(D,m+1)e(M,N,m)$$

and when $|D| = 1$ we have $q^{N+1}|D|^{-1/2}(R_1(M,N+1) - R_1(M,N))$ equal to

$$\sigma_{P_1}(M,N+1,0) + \sigma_{P_1}(M,N,0) + \sum_{m=0}^{N}\mathrm{vol}(D,m+1)e(M,N,m).$$

We now set about computing $e(M,N,m)$, which is given by the difference between integrating

$$q^{-3M}\log\max\{1,|x|,|r|,|s|\}$$

over the region

- $|r| \leqslant \min\{q^M, q^{N-m}\}$
- $|x| \leqslant q^M$
- $\pi^M s - \pi^{N-m}(r^2 - D(x\pi^{m+1})^2) \in R$,

But adding all this together gives $e(M,N,m)$ as the difference between integrating

$$q^{-3M}q^{-m-2}|x|\log\max\{1,|x|,|s|\}$$

over

- $|x|\leqslant q^M$
- $\pi^M s-\pi^{N+m+2}Dx^2\in R$

and

- $|x|\leqslant q^M$
- $\pi^M s-\pi^{N+m+2}x^2\in R$

plus the difference between integrating

$$q^{-3M}q^{-m-1}|x|\log\max\{1,|x|,|s|\}$$

over the region

- $|x|\leqslant q^M$
- $\pi^M s-\pi^{N+m+2}x^2\in R$

and

- $|x|\leqslant q^M$
- $\pi^M s-\pi^{N+m}Dx^2\in R.$

Having fixed M, N and D we set $I(k)$ equal to the integral of

$$|x|\log\max\{1,|x|,|s|\}$$

over the region

- $|x|\leqslant q^M$
- $\pi^M s-\pi^{N+k}x^2\in R.$

Then if $|D|=1$ we have

$$e(M,N,m)=q^{-3M-m-1}(I(m+2)-I(m))$$

and if $|D|=q^{-1}$ we have

$$e(M,N,m)=q^{-3M-m-2}(I(m+3)-I(m+2))+q^{-3M-m-1}(I(m+2)-I(m+1)).$$

We need to compute

$$\sum_{m=0}^{N}\operatorname{vol}(D,m+1)e(M,N,m).$$

When $|D|=1$ this sum is equal to

$$q^{-3M}\sum_{m=0}^{N}(q+1)q^{-1}(I(m+2)-I(m))$$
$$=(1+q^{-1})(I(N+2)+I(N+1)-I(1)-I(0))$$

while if $|D| = q^{-1}$ this sum is equal to

$$q^{-3M}\sum_{m=0}^{N}(q^{-1}(I(m+3)-I(m+2))+I(m+2)-I(m+1))$$
$$= q^{-1}(I(N+3)-I(2))+I(N+2)-I(1).$$

Finally we also need to compute $\sigma_{P_1}(M,N,0)$, which equals q^{-3M} times the integral of

$$\log\max\{1,|x|,|r|,|s|\}$$

over the region

- $|r| \leqslant q^M$
- $|x| \leqslant q^M$
- $\pi^M s - \pi^N(r^2 - Dx^2) \in R$.

This equals

$$\int_{|x|\leqslant|r|\leqslant q^M}\int_{\pi^M s-\pi^N(r^2-Dx^2)\in R}\log\max\{1,|x|,|r|,|s|\}$$

plus

$$\int_{|r|<|x|\leqslant q^M}\int_{\pi^M s-\pi^N(r^2-Dx^2)\in R}\log\max\{1,|x|,|r|,|s|\},$$

which equals the sum of

$$\int_{|r|\leqslant q^M}\int_{\pi^M s-\pi^N r^2\in R}|r|\log\max\{1,|r|,|s|\}$$

and

$$q^{-1}\int_{|x|\leqslant q^M}\int_{\pi^M s-\pi^N Dx^2\in R}|x|\log\max\{1,|x|,|s|\}.$$

Hence we get $\sigma_{P_1}(M,N,0) = q^{-3M}(1+q^{-1})I(0)$ and $\sigma_{P_1}(M,N+1,0) = q^{-3M}(1+q^{-1})I(1)$ if $|D| = 1$. While if $|D| = q^{-1}$ we get $\sigma_{P_1}(M,N+1,0) = I(1)+q^{-1}I(2)$.

We note that when $m \geqslant N$ we have

$$I(m) = \int_{|x|\leqslant q^M}\int_{|s|\leqslant q^M}|x|\log\max\{1,|x|,|s|\}$$

which by Lemma 9.4 is equal to

$$\frac{q}{q+1}\left(Mq^{3M}-\frac{q^{3M}-1}{q^3-1}\right).$$

Therefore we have

$$|D|^{-1/2}q^{N+1}\left(R_1(M,N+1)-R_1(M,N)\right) = f\left(M-\frac{1-q^{-3M}}{q^3-1}\right).$$

Next we do $R2 \mapsto R2-(\pi^{-m}b)^{-1}(n-a)R1$ and multiply the second row by a suitable unit to get

$$\begin{pmatrix} n-a(a-1)(n-a)^{-1} & \pi^{-m}b & 0 & 0 \\ \pi^m b^{-1}T(A) & 0 & 0 & 0 \\ \pi^{-m}b & 0 & n-a & 0 \\ n & \pi^{-m}b & 0 & -a \end{pmatrix}.$$

Next we do $R4 \mapsto a^{-1}(R4-R1)$ to give

$$\begin{pmatrix} n-a(a-1)(n-a)^{-1} & \pi^{-m}b & 0 & 0 \\ \pi^m b^{-1}T(A) & 0 & 0 & 0 \\ \pi^{-m}b & 0 & n-a & 0 \\ (a-1)(n-a)^{-1} & 0 & 0 & -1 \end{pmatrix}.$$

So we wish to integrate

$$\log\max\{1,|x_1|,|x_2|,|x_3|,|x_4|,|x_1x_4-x_2x_3|\}$$

over the region given by

- $|x_1| \leqslant |\pi^m b^{-1}T(A)|^{-1} = |\pi^m b|^{-1}$
- $\pi^{-m}bx_2 + (n-a(a-1)(n-a)^{-1})x_1 \in R$
- $(n-a)x_3 + \pi^{-m}bx_1 \in R$
- $-x_4 + (a-1)(n-a)^{-1}x_1 \in R.$

Thus we can take $x_4 = (a-1)(n-a)^{-1}x_1$ and make the change of variables $x_3 \mapsto (a-1)(n-a)^{-1}x_3$ to give it as the integral of

$$\log\max\{1,|x_1|,|x_2|,|x_3|,|x_1^2-x_2x_3|\}$$

over the region

- $|x_1| \leqslant |\pi^m b|^{-1}$
- $\pi^{-m}bx_2 + (n-a(a-1)(n-a)^{-1})x_1 \in R$
- $(a-1)x_3 + \pi^{-m}bx_1 \in R.$

Let's see how to compute this integral. Recall that $|n-a(a-1)(n-a)^{-1}| \leqslant |b|$. First suppose that $|x_1| > |n-a(a-1)(n-a)^{-1}|^{-1}$. Then we have

$$x_2 = -\pi^m b^{-1}(n-a(a-1)(n-a)^{-1})x_1u_1$$

with $u_1 \in U_F^{-v((n-a(a-1)(n-a)^{-1})x_1)}$ and we note that $|x_2| \leqslant |x_1|$. Moreover since

$$|n-a(a-1)(n-a)^{-1}|^{-1} \geqslant |b|^{-1}$$

we also have $|x_1| > |\pi^{-m}b|^{-1}$ and hence

$$x_3 = -(a-1)^{-1}\pi^{-m}bx_1u_2$$

with $u_2 \in U_F^{-v(\pi^{-m}bx_1)}$ and we have $|x_3| = |\pi^{-m}x_1| \geqslant |x_1|$. Now

$$\begin{aligned} x_1^2 - x_2x_3 &= x_1^2 - (n-a(a-1)(n-a)^{-1})(a-1)^{-1}x_1^2u_1u_2 \\ &= x_1^2(1-(n-a(a-1)(n-a)^{-1})(a-1)^{-1}u_1u_2) \end{aligned}$$

We now compute the integrals on $(\mathrm{GL}(2)\times\mathrm{GL}(2))'$. We have

$$\begin{aligned}\sigma_{P_2}(z_m^{-1}Az_m) &= q^{-M}\int_{|x|\leqslant q^M}\log\max\{1,|x|\}\\ &= M-\frac{1-q^{-M}}{q-1}.\end{aligned}$$

Thus if we set $\sigma_{P_2}(M,N,m)=\sigma_{P_2}(z_m^{-1}Az_m)$ then we have

$$\sigma_{P_2}(M,N+1,m+1)=\sigma_{P_2}(M,N,m).$$

Hence if we let $R_2(M,N)$ equal the contribution to the right hand side of $FL(A)$ from the integral on $(\mathrm{GL}(2)\times\mathrm{GL}(2))'$ then we have

$$q^{N+1}|D|^{-1/2}\left(R_2(M,N+1)-R_2(M,N)\right)=f\left(M-\frac{1-q^{-M}}{q-1}\right).$$

Putting these together gives

$$R(M,N+1)-R(M,N)=fq^{-N-1}|D|^{1/2}\left(2M-\frac{1-q^{-M}}{q-1}-\frac{1-q^{-3M}}{q^3-1}\right)$$

as required. □

5.5.3. Proof when $|b|=|\det A-1|$. — In this section we prove Proposition 5.3 under the assumption that $|b|=|\det A-1|$. It follows that we have $|a-1|=|\det A-a|=|b|$ and $|T(A)|=|b|^2$. Let $N\geqslant 1$ and assume that we have $|b|=|\det A-1|=q^{-N}$. We let $L(N)$ (resp. $R(N)$) denote the left (resp. right) hand side of the identity $FL(A)$. We now prove the following Proposition.

Proposition 5.12. — *With the notations and assumptions above for all $N\geqslant 1$ we have $L(N)$ and $R(N)$ equal to*

$$|D|^{1/2}\left(\frac{4Nq-2Nq^{-N}}{q-1}+\frac{-4q+3q^{-N+1}+2q^{-N}-q^{-2N}}{(q-1)^2}+\frac{q^{-N+3}-q^{-4N}}{(q-1)(q^3-1)}\right)$$

if $|D|=q^{-1}$ and equal to

$$\begin{aligned}(q+1)\frac{4N-2Nq^{-N-1}}{q-1}+&\frac{-4(q+1)+q^{-N-1}(3q^2+6q+1)-2q^{-2N}}{(q-1)^2}\\ &+2\frac{q^{-N+3}-q^{-4N}}{(q-1)(q^3-1)}-(2N+1)q^{-N-1}\end{aligned}$$

if $|D|=1$.

Proof. — We begin by computing the integrals $\sigma_P(z_m^{-1}Az_m)$. As we saw in the proof of Proposition 5.11 we can make row operations to put the matrix B in the form

$$\begin{pmatrix} n-a(a-1)(n-a)^{-1} & \pi^{-m}b & 0 & 0\\ (a-1)(n-a)^{-1}\pi^m bD & n-a & 0 & 0\\ \pi^{-m}b & 0 & n-a & 0\\ n & \pi^{-m}b & 0 & -a\end{pmatrix}.$$

and since

$$1 - (n - a(a-1)(n-a)^{-1})(a-1)^{-1} = -n(a-1)^{-1}(n-a)^{-1}T(A)$$

we have

$$|1 - (n - a(a-1)(n-a)^{-1})(a-1)^{-1}u_1u_2| = 1$$

for all u_1 and u_2. Hence when $|x_1| > |n - a(a-1)(n-a)^{-1}|^{-1}$ the integrand is equal to $2\log|x_1|$.

Now suppose we have $|b|^{-1} < |x_1| \leqslant |n - a(a-1)(n-a)^{-1}|^{-1}$. Then we have $|x_2| \leqslant |\pi^m b^{-1}|$ and

$$x_3 = -(a-1)^{-1}\pi^{-m}bx_1u$$

with $u \in U_F^{-v(\pi^{-m}bx_1)}$. Therefore $|x_3| = |\pi^{-m}x_1| \geqslant |x_1|$. Now

$$\begin{aligned} x_1^2 - x_2x_3 &= x_1^2 + (a-1)^{-1}\pi^{-m}bx_1x_2u \\ &= x_1(x_1 + (a-1)^{-1}\pi^{-m}bx_2u) \end{aligned}$$

but

$$|(a-1)^{-1}\pi^{-m}bx_2u| = |\pi^{-m}x_2| \leqslant |b|^{-1} < |x_1|$$

and hence when $|b|^{-1} < |x_1|$ the integrand is equal to $2\log|x_1|$.

So the contribution to the integral when $|b|^{-1} < |x_1| \leqslant |\pi^m b|^{-1}$ is

$$2|\pi^{-m}b^2|^{-1}\int_{|b|^{-1}<|x_1|\leqslant|\pi^m b|^{-1}} \log|x_1|.$$

We are now left with the region

- $|x_1| \leqslant |b|^{-1}$
- $|x_2| \leqslant |\pi^{-m}b|^{-1}$
- $(a-1)x_3 + \pi^{-m}bx_1 \in R$.

Next we suppose that $|x_1| > |\pi^{-m}b|^{-1}$. Then we have

$$x_3 = -(a-1)^{-1}\pi^{-m}bx_1u$$

with $u \in U_F^{-v(\pi^{-m}bx_1)}$ and so $|x_3| \geqslant |x_1| > |x_2|$. Now

$$\begin{aligned} x_1^2 - x_2x_3 &= x_1^2 + (a-1)^{-1}\pi^{-m}bux_1x_2 \\ &= u\pi^{-m}(a-1)^{-1}bx_1(u^{-1}\pi^m(a-1)b^{-1}x_1 + x_2) \end{aligned}$$

and

$$|u^{-1}\pi^m(a-1)b^{-1}x_1| = |\pi^m x_1| \leqslant |\pi^{-m}b|^{-1}$$

so making the change of variables $x_2 \mapsto x_2 - u^{-1}\pi^m(a-1)b^{-1}x_1$ gives the contribution when $|\pi^{-m}b|^{-1} < |x_1| \leqslant |b|^{-1}$ as

$$|b|^{-1}\int_{|\pi^{-m}b|^{-1}<|x_1|\leqslant|b|^{-1}}\int_{|x_2|\leqslant|\pi^{-m}b|^{-1}} \log\max\{|\pi^{-m}x_1|, |\pi^{-m}x_1||x_2|\},$$

which equals the sum of

$$|b|^{-1}|\pi^{-m}b|^{-1}\int_{|\pi^{-m}b|^{-1}<|x_1|\leqslant|b|^{-1}}\log|\pi^{-m}x_1|$$

and

$$|b|^{-1}(|b|^{-1}-|\pi^{-m}b|^{-1})\int_{1<|x_2|\leqslant|\pi^{-m}b|^{-1}}\log|x_2|.$$

Finally we are left with the region

- $|x_1|\leqslant|\pi^{-m}b|^{-1}$
- $|x_2|\leqslant|\pi^{-m}b|^{-1}$
- $|x_3|\leqslant|b|^{-1}$.

With $|b|=q^{-N}$ we have $\sigma_P(z_m^{-1}Az_m)$ equal to the sum of

$$2q^{-N-m}\int_{q^N<|x_1|\leqslant q^{N+m}}\log|x_1|$$

and

$$q^{-N-m}\int_{q^{N-m}<|x_1|\leqslant q^N}(m+\log|x_1|)$$

and

$$(q^{-N}-q^{-N-m})\int_{1<|x_2|\leqslant q^{N-m}}\log|x_2|$$

and

$$q^{-3N}\int_{|x_1|\leqslant q^{N-m}}\int_{|x_2|\leqslant q^{N-m}}\int_{|x_3|\leqslant q^N}\log\max\{1,|x_1|,|x_2|,|x_3|,|x_1^2-x_2x_3|\}.$$

Putting these together we get

$$\sigma_P(z_m^{-1}Az_m)=(2N+2m)+\frac{q^{-N}+q^{-3m}-2}{q-1}-\frac{q^{-3m}-q^{-3N}}{q^3-1}.$$

Now we compute the left hand side of $FL(A)$. When $|D|=q^{-1}$ we get

$$L(N)=|D|^{1/2}\Big(\frac{4Nq-2Nq^{-N}}{q-1}+\frac{-4q+3q^{-N+1}+2q^{-N}-q^{-2N}}{(q-1)^2}+\frac{q^{-N+3}-q^{-4N}}{(q-1)(q^3-1)}\Big),$$

and when $|D|=1$ we get $L(N)$ equal to

$$(q+1)\frac{4N-2Nq^{-N-1}}{q-1}+\frac{(-4(q+1))+q^{-N-1}(3q^2+6q+1)-2q^{-2N}}{(q-1)^2}\\ +2\frac{q^{-N+3}-q^{-4N}}{(q-1)(q^3-1)}-(2N+1)q^{-N-1}.$$

We now look to compute the right hand side of $FL(A)$. We have $\sigma_{P_1}(z_m^{-1}Az_m)$ equal to q^{-3N} times the integral of

$$\log\max\{1,|x|,|r|,|s|\}$$

over the region in F^3 given by

- $|x|\leqslant|\pi^m b|^{-1}$

- $\pi^{-m}br - (n-a)x \in R$
- $(n-1)s - \pi^{-m}b(r^2 - \pi^{2m}Dx^2) \in R$.

First suppose that $|b|^{-1} < |x| \leqslant |\pi^m b|^{-1}$. Then we have $|r| = |\pi^m x|$ and

$$|\pi^{-m}b(r^2 - \pi^{2m}Dx^2)| = |b\pi^m x^2| > 1.$$

Therefore $|s| = |\pi^m x^2|$ and the contribution to the integral is

$$|\pi^{-m}b^2|^{-1} \int_{|b|^{-1}<|x|\leqslant|\pi^m b|^{-1}} \log|\pi^m x^2|,$$

which equals

$$q^{2N-m} \int_{q^N<|x|\leqslant q^{N+m}} (2\log|x| - m)\,.$$

We are then left to compute, after multiplying s by a suitable unit, the integral of

$$\log\max\{1, |x|, |r|, |s|\}$$

over the region in F^3 given by

- $|x| \leqslant |b|^{-1}$
- $|r| \leqslant |\pi^{-m}b|^{-1}$
- $\pi^N s - \pi^{N-m}(r^2 - D(\pi^m x)^2) \in R$.

The contribution when $|r^2 - D(\pi^m x)^2| > q^{N-m}$ is

$$q^{N+m} \int_{x\in E_D, q^{N-m}<|x|_{E_D}\leqslant q^{2(N-m)}} (\log|x|_{E_D} + m)\,.$$

Having fixed N and m we set $l = \lfloor \frac{N-m}{2} \rfloor$. If $|D| = 1$ then $\pi^{-m}b(r^2 - D(\pi^m x)^2) \in R$ if and only if $|r| \leqslant q^l$ and $|x| \leqslant q^{l+m}$ and the contribution to the integral is

$$\int_{|x|\leqslant q^{l+m}} \int_{|r|\leqslant q^l} \int_{|s|\leqslant q^N} \log\max\{1, |x|, |r|, |s|\}.$$

If $|D| = q^{-1}$ then we have $\pi^{-m}b(r^2 - D(\pi^m x)^2) \in R$ if and only if $|r| \leqslant q^l$ and $|x| \leqslant q^{l_1+m}$ where $l_1 = \lfloor \frac{N-m+1}{2} \rfloor$ and the contribution to the integral is

$$\int_{|x|\leqslant q^{l_1+m}} \int_{|r|\leqslant q^l} \int_{|s|\leqslant q^N} \log\max\{1, |x|, |r|, |s|\}.$$

When $|D| = q^{-1}$, if $N - m = 2l$ we get $\sigma_{P_1}(z_m^{-1}Az_m)$ equal to q^{-3N} times

$$(2N+m)q^{3N} - \frac{2q^{3N} - q^{3N-m}}{q-1} + \frac{q^{2m+3l+1}}{q^2-1} + \frac{1}{q^3-1} + \frac{q^{3l+2}}{(q+1)(q^3-1)},$$

while if $N - m = 2l + 1$ we have $\sigma_{P_1}(z_m^{-1}Az_m)$ equal to q^{-3N} times

$$(2N+m)q^{3N} - \frac{2q^{3N} - q^{3N-m}}{q-1} + \frac{q^{2m+3l+3}}{q^2-1} + \frac{1}{q^3-1} + \frac{q^{3l+2}}{(q+1)(q^3-1)}.$$

We compute the contribution of the integral on GSp(4) to $R(N)$ to be

$$|D|^{1/2}\left(\frac{3Nq - (N-2)q^{-N}}{q-1} + \frac{-3q + 3q^{-N}}{(q-1)^2} + \frac{q^{-N+3} - q^{-4N}}{(q-1)(q^3-1)}\right).$$

Now suppose that $|D| = 1$. Then we have $\sigma_{P_1}(z_m^{-1}Az_m)$ equal to q^{-3N} times the sum of

$$(2N+m)q^{3N} + (N-m-2l-2)q^{N+m+2l} - \frac{2q^{3N} - 2q^{3N-m}}{q-1} - \frac{q^{N+m+2l}}{q-1}$$

and

$$-2\frac{q^{3N-m} - q^{N+m+2l+2}}{q^2-1} + \frac{q^{3l+2m+1}}{q^2-1} + \frac{1}{q^3-1} + \frac{q^{3l+2}}{(q+1)(q^3-1)}.$$

We compute that the contribution of the integral on GSp(4) to $R(N)$ is equal to

$$-2Nq^{-N-1} - 2 + \frac{3N(q+1) - 2Nq^{-N-1}}{q-1} + \frac{-3q-3+(2q^2-2q+6)q^{-N}}{(q-1)^2} + \frac{q^{-N+3} + q^{-N} - 2q^{-4N}}{(q-1)(q^3-1)}.$$

Now we compute the contribution of the integral on $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$ to $R(N)$. We have

$$\sigma_{P_2}(z_m^{-1}Az_m) = q^{-N}\int_{1<|x|\leqslant q^N} \log|x| = N - \frac{1-q^{-N}}{q-1}.$$

And we compute that the contribution when $|D| = q^{-1}$ is

$$|D|^{1/2}\frac{q-q^{-N}}{q-1}\left(N - \frac{1-q^{-N}}{q-1}\right)$$

while when $|D| = 1$ it is

$$\left(q^{-N} + \frac{(q+1)(1-q^{-N})}{q-1}\right)\left(N - \frac{1-q^{-N}}{q-1}\right).$$

Putting these calculations together gives the computation of $R(N)$ and finishes the proof. □

5.5.4. Reduction when $|\det A - 1| \leqslant |b^2D|$. — We now assume that we have $|\det A - 1| \leqslant |b^2D|$. In this section we reduce the proof of Proposition 5.3 in the case that $|\det A - 1| < |b^2D|$ to the case that $|\det A - 1| = |b^2D|$.

So we assume that we have $N \geqslant M$ and

$$q^{-N} = |\det A - 1| \leqslant |b^2D| = q^{-2M}|D|.$$

We let $L(M,N)$ (resp. $R(M,N)$) denote the left (resp. right) hand side of the identity $FL(A)$.

We note that under the assumption that $|\det A - 1| \leqslant |b^2D|$ we have $|a-1|$, $|\det A - a| \leqslant |b^2D|$ and so $|T(A)| = |b^2D|$. For ease of notation we set $n = \det A$. We now prove the following Proposition.

Thus we can do $R2 \mapsto R2 - (a^2 - a - n^2 + an)\pi^{2m}b^{-2}R3$ to give

$$\begin{pmatrix} -a & 0 & \pi^m bD & n \\ 0 & 0 & a(n-1)T(A)\pi^{2m}b^{-2} & 0 \\ 0 & 0 & a-1 & \pi^{-m}b \\ n & \pi^{-m}b & 0 & -a \end{pmatrix}.$$

Thus we need to integrate

$$\log \max\{1, |x_1|, |x_2|, |x_3|, |x_4|, |x_1x_4 - x_2x_3|\}$$

over the region

- $|x_3| \leqslant |(n-1)D\pi^{2m}|^{-1}$
- $\pi^{-m}bx_4 + (a-1)x_3 \in R$
- $-ax_1 + \pi^m bDx_3 + nx_4 \in R$
- $nx_1 + \pi^{-m}bx_2 - ax_4 \in R.$

Therefore we can take $x_1 = a^{-1}\pi^m bDx_3 + a^{-1}nx_4$ and then we need to integrate

$$\log \max\{1, |x_2|, |x_3|, |x_4|, |a^{-1}nx_4^2 + (a^{-1}\pi^m bDx_4 - x_2)x_3|\}$$

over the region

- $|x_3| \leqslant |(n-1)D\pi^{2m}|^{-1}$
- $\pi^{-m}bx_4 + (a-1)x_3 \in R$
- $a^{-1}n\pi^m bDx_3 + a^{-1}n^2x_4 + \pi^{-m}bx_2 - ax_4 \in R.$

Now we make the change of variables $x_2 \mapsto a^{-1}nx_2 + a^{-1}\pi^m bDx_4$ to give our integral as the integral of

$$\log \max\{1, |x_2|, |x_3|, |x_4|, |x_4^2 - x_2x_3|\}$$

over the region

- $|x_3| \leqslant |(n-1)D\pi^{2m}|^{-1}$
- $\pi^{-m}bx_4 + (a-1)x_3 \in R$
- $\pi^{-m}bx_2 + \pi^m bDx_3 + (n-1)x_4 \in R$

and we have $\sigma_P(z_m^{-1}Az_m)$ equal to $q^{-N-2M}|D|$ times this integral.

First suppose that $|x_3| > |a-1|^{-1}$. Then we have

$$|x_4| = |\pi^m b^{-1}(a-1)x_3| < |x_3|.$$

Now

$$|(n-1)x_4| \leqslant |bD\pi^m(a-1)x_3| < |\pi^m bDx_3|,$$

hence

$$|\pi^m bDx_3 + (n-1)x_4| = |\pi^m bDx_3| > 1,$$

and so $|x_2| = |\pi^{2m}Dx_3|$. Therefore we have

$$|x_4|^2 = |\pi^{2m}b^{-2}(a-1)^2x_3^2| < |x_2x_3| = |\pi^{2m}Dx_3^2|$$

and the integrand equals $\log|\pi^{2m}Dx_3^2|$.

We are now left with the region

Proposition 5.13. — *With the notations and assumptions above we have, for all $M \geqslant 1$ and $N \geqslant 2M + v(D)$, $qL(M, N+1) - L(M, N)$ and $qR(M, N+1) - R(M, N)$ equal to*

$$|D|^{1/2}\left((2N+2M+3)q - (2N+1)q^{-M} - 2\frac{q-q^{-M}}{q-1}\right)$$

when $|D| = q^{-1}$, and equal to

$$(2N+2M+2)(q+1) - (4N+4)q^{-M} - 2(q+1)\frac{1-q^{-M}}{q-1}$$

when $|D| = 1$.

Proof. — We begin by computing the twisted integrals $\sigma_P(z_m^{-1}Az_m)$. As above we have

$$B = \begin{pmatrix} -a & 0 & \pi^m bD & n \\ 0 & n-a & 0 & \pi^m bD \\ \pi^{-m}b & 0 & n-a & 0 \\ n & \pi^{-m}b & 0 & -a \end{pmatrix}.$$

We now do a series of row operations invertible over R to get B in a suitable form. First we do $R2 \mapsto R2 - (n-a)(\pi^{-m}b)^{-1}R4$ and then divide by n to give

$$\begin{pmatrix} -a & 0 & \pi^m bD & n \\ -(n-a)\pi^m b^{-1} & 0 & 0 & (a-1)\pi^m b^{-1} \\ \pi^{-m}b & 0 & n-a & 0 \\ n & \pi^{-m}b & 0 & -a \end{pmatrix}.$$

Next we do $R3 \mapsto aR3 + \pi^{-m}bR1$ and then divide by n to give

$$\begin{pmatrix} -a & 0 & \pi^m bD & n \\ -(n-a)\pi^m b^{-1} & 0 & 0 & (a-1)\pi^m b^{-1} \\ 0 & 0 & a-1 & \pi^{-m}b \\ n & \pi^{-m}b & 0 & -a \end{pmatrix}.$$

Next we do $R2 \mapsto aR2 - (n-a)\pi^m b^{-1}R1$ to give

$$\begin{pmatrix} -a & 0 & \pi^m bD & n \\ 0 & 0 & -(n-a)\pi^{2m}D & (a^2-a-n^2+an)\pi^m b^{-1} \\ 0 & 0 & a-1 & \pi^{-m}b \\ n & \pi^{-m}b & 0 & -a \end{pmatrix}.$$

Now we note that

$$a^2 - a - n^2 + an = -a(a-1)^2(a+1) + b^2D(n + a(a-1))$$

and since $|a-1| \leqslant |b| < 1$ so

$$|a^2 - a - n^2 + an| \leqslant \max\{|a-1|^2, |b^2D|\} \leqslant |b^2|.$$

- $|x_3| \leqslant \min\{|b^2D^2\pi^{2m}|^{-1}, |a-1|^{-1}\}$
- $|x_4| \leqslant |\pi^{-m}b|^{-1}$
- $\pi^{-m}bx_2 + \pi^m bDx_3 \in R$.

If $|x_3| > |\pi^m bD|^{-1}$ then we have

$$|x_2| = |\pi^{2m}Dx_3|$$

and so

$$|x_2x_3| = |\pi^{2m}Dx_3^2| > |b^2D|^{-1} \geqslant |\pi^{-m}b|^{-2} \geqslant |x_4|^2$$

therefore the integrand is equal to $\log|\pi^{2m}Dx_3^2|$ in this case as well. So the contribution to the integral when $|\pi^m bD|^{-1} < |x_3| \leqslant |(n-1)D\pi^{2m}|^{-1}$ is

$$|\pi^{-2m}b^2|^{-1}\int_{|\pi^m bD|^{-1}<|x_3|\leqslant|(n-1)D\pi^{2m}|^{-1}} \log|\pi^{2m}Dx_3^2|,$$

which equals

$$q^{2M-2m}\int_{q^{M+m}|D|^{-1}<|x_3|\leqslant q^{N+2m}|D|^{-1}} (2\log|x_3| - 2m + \log|D|)\,.$$

We are then left with the region

- $|x_3| \leqslant |\pi^m bD|^{-1} = q^{M+m}|D|^{-1}$
- $|x_4| \leqslant |\pi^{-m}b|^{-1} = q^{M-m}$
- $|x_2| \leqslant |\pi^{-m}b|^{-1} = q^{M-m}$

to integrate over.

Thus we see that $\sigma_P(z_m^{-1}Az_m)$ depends only on m, $|b|$ and $|n-1|$. We define $\sigma_P(M,N,m) = \sigma_P(z_m^{-1}Az_m)$ and we see that

$$q\sigma_P(M,N+1,m) - \sigma_P(M,N,m) = (q-1)(2N+2m+2-\log|D|).$$

So we have

$$qL(M,N+1) - L(M,N) = q^{-M}|D|^{1/2}\sum_{m=0}^{M}\mathrm{vol}(D,m)(q-1)(2N+2m+2-\log|D|),$$

which equals

$$|D|^{1/2}\left((2N+2M+3)q - (2N+1)q^{-M} - 2\frac{q-q^{-M}}{q-1}\right)$$

if $|D| = q^{-1}$ and

$$(2N+2M+2)(q+1) - (4N+4)q^{-M} - 2(q+1)\frac{1-q^{-M}}{q-1}$$

if $|D| = 1$.

We now turn to the computation of the right hand side of the identity $FL(A)$. First we consider the integrals $\sigma_{P_1}(z_m^{-1}Az_m)$, which are equal to $|n-1||b^2D|$ times the integral of

$$\log\max\{1,|x|,|r|,|s|\}$$

over the region in F^3 given by

- $(n-a)x - \pi^{-m}br \in R$
- $-\pi^m bDx + (n-a)r \in R$
- $(n-1)s - \pi^{-m}b(r^2 - \pi^{2m}Dx^2) \in R.$

We consider

$$\begin{pmatrix} n-a & -\pi^{-m}b \\ -\pi^m bD & n-a \end{pmatrix}$$

then doing $R2 \mapsto R2 + (n-a)\pi^m b^{-1}R1$ gives

$$\begin{pmatrix} n-a & -\pi^{-m}b \\ -\pi^m bD + (n-a)\pi^m b^{-1}(n-a) & 0 \end{pmatrix}.$$

Now

$$\begin{aligned} -\pi^m bD + (n-a)\pi^m b^{-1}(n-a) &= \pi^m b^{-1}(-b^2D + (n-a)^2) \\ &= \pi^m b^{-1} nT(A), \end{aligned}$$

which has absolute value $|\pi^m bD|$. So after more row operations we get the matrix

$$\begin{pmatrix} 0 & -\pi^{-m}b \\ \pi^m bD & 0 \end{pmatrix}.$$

Therefore $\sigma_{P_1}(z_m^{-1}Az_m)$ is equal to $|(n-1)b^2D|$ times the integral of

$$\log\max\{1, |x|, |r|, |s|\}$$

over the region in F^3 given by

- $|x| \leqslant |\pi^m bD|^{-1}$
- $|r| \leqslant |\pi^{-m}b|^{-1}$
- $(n-1)s - \pi^{-m}b(r^2 - D\pi^{2m}x^2) \in R.$

We set $\sigma_{P_1}(M,N,m) = \sigma_{P_1}(z_m^{-1}Az_m)$. Then we have

$$q^{2M+N}|D|^{-1}(q\sigma_{P_1}(M,N+1,m) - \sigma_{P_1}(M,N,m))$$

equal to the sum of

$$(N+1)q^N(q-1)\operatorname{vol}(\{x,r : |\pi^{-m}b(r^2 - D(\pi^m x)^2)| \leqslant 1\}),$$

the contribution when $|\pi^{-m}b(r^2 - D(\pi^m x)^2)| \leqslant 1$, and the sum of

$$(q^{N+1} - q^N)\int_{x,r:|\pi^{-m}b(r^2-D(\pi^m x)^2)|>1} N + \log|\pi^{-m}b(r^2 - D(\pi^m x)^2)|$$

and

$$q^{N+1}\operatorname{vol}(\{x,r : |\pi^{-m}b(r^2 - D(\pi^m x)^2)| \leqslant 1\}),$$

which is the contribution when $|\pi^{-m}b(r^2 - D(\pi^m x)^2)| > 1$. Putting these contributions together gives $q\sigma_{P_1}(M,N+1,m) - \sigma_{P_1}(M,N,m)$ as the sum of

$$(N+1)(q-1)$$

and

$$q^{-2M+m}|D|(q-1)\int_{|x|\leqslant q^M|D|^{-1},|r|\leqslant q^{M-m},|r^2-Dx^2|>q^{M-m}}\log|r^2-Dx^2|-M+m$$

and

$$q^{-2M+m}|D|\operatorname{vol}\{|x|\leqslant q^M|D|^{-1},|r|\leqslant q^{M-m}:|r^2-Dx^2|>q^{M-m}\}.$$

The integral above can be written as the sum of

$$q^{-M}|D|(q-1)\int_{q^{M-m}<|x|\leqslant q^M|D|^{-1}}\log|Dx^2|-M+m$$

and

$$q^{-2M+m}|D|(q-1)\int_{x\in E_D,q^{M-m}<|x|_{E_D}\leqslant q^{2(M-m)}}\log|x|_{E_D}-M+m$$

And we have $\operatorname{vol}\{|x|\leqslant q^M|D|^{-1},|r|\leqslant q^{M-m}:|r^2-Dx^2|>q^{M-m}\}$ equal to

$$q^{M-m}(q^M|D|^{-1}-q^{M-m})+\operatorname{vol}\{x\in E_D:q^{M-m}<|x|_{E_D}\leqslant q^{2(M-m)}\}.$$

Now we compute $q\sigma_{P_1}(M,N+1,m)-\sigma_{P_1}(M,N,m)$ equal to

$$(N+M+m+2)q-(N+M+m+3)+q^{-m}$$

if $|D|=q^{-1}$. And when $|D|=1$ we have $q\sigma_{P_1}(M,N+1,m)-\sigma_{P_1}(M,N,m)$ equal to

$$(N+M+m+1)q-(N+M+m+2)+2q^{-m}-2q^{-M+1}+q^{-M}-2\frac{q^{-m}-q^{-M+2}}{q+1}$$

when $M-m$ is even and equal to

$$(N+M+m+1)q-(N+M+m+2)+2q^{-m}-q^{-M}-2\frac{q^{-m}-q^{-M+1}}{q+1}$$

when $M-m$ is odd.

With similar notation we have

$$q\sigma_{P_2}(M,N+1,m)-\sigma_{P_2}(M,N,m)=(N+1)(q-1).$$

Using these computations we get

$$qR(N+1,M)-R(N,M)=|D|^{1/2}\left((2N+2M+3)q-(2N+1)q^{-M}-2\frac{q-q^{-M}}{q-1}\right)$$

when $|D|=q^{-1}$ and

$$qR(N+1,M)-R(N,M)=(2N+2M+2)(q+1)-(4N+4)q^{-M}-2(q+1)\frac{1-q^{-M}}{q-1}$$

when $|D|=1$. □

5.5.5. Proof when $|b^2D| \leqslant |\det A-1| < |b|$. — In this section we assume that we have $|b^2D| \leqslant |\det A - 1| < |b|$ and we prove Proposition 5.3 in this case. We set $|b| = q^{-M}$ and $|\det A - 1| = q^{-N}$. We then have $|T(A)| = |b^2D| = q^{-2M}|D|$. We let $L(M,N)$ (resp. $R(M,N)$) denote the left (resp. right) hand side of the identity $FL(A)$. Again for ease of notation we set $n = \det A$. We now prove the following Proposition.

Proposition 5.14. — *Let M and N be such that $M < N \leqslant 2M + v(D)$. Then $L(M,N)$ and $R(M,N)$ are equal to*

$$|D|^{1/2}\Big(\frac{(2N+2M+1)q-(2N+1)q^{-M}}{q-1} - \frac{4q-2(q+1)q^{-M}-q^{-N+1}+q^{-N-M}}{(q-1)^2} + \frac{q^{-N+2}-q^{-N-3M-1}}{(q-1)(q^3-1)}\Big)$$

if $|D| = q^{-1}$ and are equal to

$$\frac{2(N+M)(q+1)}{q-1} - \frac{(4N+2)q^{-M}}{q-1} - \frac{4(q+1)-4q^{-M}(q+1)-2q^{-N+1}+2q^{-N-M}}{(q-1)^2} + 2\frac{q^{-N}-q^{-N-3M}}{(q-1)(q^3-1)}$$

if $|D| = 1$.

Proof. — We begin by computing $\sigma_P(z_m^{-1}Az_m)$. As we saw in the proof of Proposition 5.13, we have $\sigma_P(z_m^{-1}Az_m)$ equal to $|(n-1)b^2D|$ times the integral of

$$\log\max\{1, |x_2|, |x_3|, |x_4|, |x_4^2 - x_2x_3|\}$$

over the region

- $|x_3| \leqslant |(n-1)D\pi^{2m}|^{-1}$
- $\pi^{-m}bx_4 + (a-1)x_3 \in R$
- $\pi^{-m}bx_2 + \pi^m bDx_3 + (n-1)x_4 \in R$.

As we saw above the contribution to this integral when $|x_3| > |\pi^m bD|^{-1}$ is

$$|\pi^{-2m}b^2|^{-1}\int_{|\pi^m bD|^{-1}<|x_3|\leqslant|(n-1)\pi^{2m}D|^{-1}} \log|\pi^{2m}Dx_3^2|.$$

We are then left to integrate over the region

- $|x_3| \leqslant |\pi^m bD|^{-1}$
- $\pi^{-m}bx_4 + (a-1)x_3 \in R$
- $\pi^{-m}bx_2 + (n-1)x_4 \in R$.

We note that if $|\pi^m bD|^{-1} \leqslant |n-1|^{-1}$ then this region becomes

- $|x_3| \leqslant |\pi^m bD|^{-1}$
- $|x_4| \leqslant |\pi^{-m}b|^{-1}$
- $|x_2| \leqslant |\pi^{-m}b|^{-1}$.

over the region

- $|x| \leqslant |\pi^m bD|^{-1}$
- $\pi^{-m}br - (n-a)x \in R$
- $(n-1)s - \pi^{-m}b(r^2 - D\pi^{2m}x^2) \in R.$

For $|n-1|^{-1} < |x| \leqslant |\pi^m bD|^{-1}$ we have

$$|r| = |\pi^m b^{-1}(n-1)x|$$

and

$$|s| = |(n-1)^{-1}bD\pi^m x^2|.$$

Hence the contribution to the integral is

$$|n-1|^{-1}|\pi^{-m}b|^{-1}\int_{|n-1|^{-1}<|x|\leqslant|\pi^m bD|^{-1}} \log(|(n-1)^{-1}bD\pi^m x^2|).$$

We are then left with the region

- $|x| \leqslant \min\{|n-1|^{-1}, |\pi^m bD|^{-1}\}$
- $|r| \leqslant |\pi^{-m}b|^{-1}$
- $(n-1)s - \pi^{-m}b(r^2 - D\pi^{2m}x^2) \in R$

and we can compute this integral as in the proof of Proposition 5.12 when $|n-1| = |b|$.

Having fixed M, N and m we set $l = \lfloor \frac{M-m}{2} \rfloor$. When $|D| = q^{-1}$ we compute $\sigma_{P_1}(z_m^{-1}Az_m)$ to be equal to

$$N + M + m + 1 + q^{-m-1} - \frac{2-q^{-m-1}}{q-1} + \frac{q^{-N-2M-1}}{q^3-1} + \frac{q^{-N-l}}{q^2-1} + \frac{q^{-N-2M+3l+1}}{(q+1)(q^3-1)},$$

and when $|D| = 1$ we have $\sigma_{P_1}(z_m^{-1}Az_m)$ equal to the sum of

$$N + M + m - \frac{2}{q-1} + 2\frac{q^{-m+1}}{q^2-1} + \frac{q^{-N-2M}}{q^3-1}$$

and

$$(M-2l-m-2)q^{-2M+2l+m} + q^{-2M+2l+m}\frac{2q+1}{q+1} + \frac{q^{-N-2M+3l+2m+1}}{q^2-1} + \frac{q^{-N-2M+3l+2}}{(q+1)(q^3-1)}.$$

We now assume that $|D| = q^{-1}$. We compute the contribution of the integral on GSp(4) to the right hand side of $FL(A)$ to be equal to $|D|^{1/2}$ times

$$\frac{(N+2M+1)q - Nq^{-M}}{q-1} - \frac{3q - q^{-M+1} - 2q^{-M}}{(q-1)^2} + \frac{q^{-N+2} - q^{-N-3M-1}}{(q-1)(q^3-1)}$$

when $|D| = q^{-1}$. And when $|D| = q^{-1}$ the integral on $(\mathrm{GL}(2) \times \mathrm{GL}(2))'$ contributes $|D|^{1/2}$ times

$$\frac{q-q^{-M}}{q-1}\left(N - \frac{1-q^{-N}}{q-1}\right).$$

The sum of these expressions equals $L(M, N)$.

On the other hand if $|\pi^m bD|^{-1} > |n-1|^{-1}$ then when $|n-1|^{-1} < |x_3| \leqslant |\pi^m bD|^{-1}$ we have

$$x_4 = -\pi^m b^{-1}(a-1)x_3 u$$

with $u \in U_F^{-v((n-1)x_3)}$ and $|x_2| \leqslant |\pi^{-m}b|^{-1}$. The integrand in this case equals

$$\log\max\{|x_3|, |x_4^2 - x_2x_3|\}.$$

But for x_3 in this range we have $|x_4^2||x_3|^{-1} \leqslant |\pi^{-m}b|^{-1}$ and so after a change of variables in x_2 the integral over this range becomes

$$|\pi^{-m}b|^{-1}\int_{|n-1|^{-1}<|x_3|\leqslant|\pi^m bD|^{-1}}\int_{|x_2|\leqslant|\pi^{-m}b|^{-1}}\log\max\{|x_3|,|x_2x_3|\}.$$

We can write this integral as the sum of

$$|\pi^{-m}b|^{-2}\int_{|n-1|^{-1}<|x_3|\leqslant|\pi^m bD|^{-1}}\log|x_3|$$

and

$$|\pi^{-m}b|^{-1}(|\pi^m bD|^{-1}-|n-1|^{-1})\int_{|x_2|\leqslant|\pi^{-m}b|^{-1}}\log\max\{1,|x_2|\}.$$

And finally we are left to integrate over

- $|x_3| \leqslant |n-1|^{-1}$
- $|x_4| \leqslant |\pi^{-m}b|^{-1}$
- $|x_2| \leqslant |\pi^{-m}b|^{-1}$.

We let $e \in \{0,1\}$ be such that $|D| = q^{-e}$. Using the results of Section 9 we get $\sigma_P(z_m^{-1}Az_m)$ equal to $q^{-N-2M-e}$ times

$$(2N+2m+e)q^{N+2M+e} - \frac{2q^{N+2M+e} - q^{3M-3m} - q^{2M+e}}{q-1} - \frac{q^{3M-3m}-1}{q^3-1}.$$

And we compute $L(M,N)$ to be equal to

$$|D|^{1/2}\Big(\frac{(2N+2M+1)q-(2N+1)q^{-M}}{q-1} - \frac{4q-2(q+1)q^{-M}-q^{-N+1}+q^{-N-M}}{(q-1)^2} + \frac{q^{-N+2}-q^{-N-3M-1}}{(q-1)(q^3-1)}\Big)$$

when $|D| = q^{-1}$ and to be equal to

$$\frac{2(N+M)(q+1)}{q-1} - \frac{(4N+2)q^{-M}}{q-1} - \frac{4(q+1)-4q^{-M}(q+1)-2q^{-N+1}+2q^{-N-M}}{(q-1)^2} + 2\frac{q^{-N}-q^{-N-3M}}{(q-1)(q^3-1)}$$

when $|D| = 1$.

We now turn to the computation of the right hand side of $FL(A)$. We begin with the integrals $\sigma_{P_1}(z_m^{-1}Az_m)$, which equal $|(n-1)b^2D|$ times the integral of

$$\log\max\{1,|x|,|r|,|s|\}$$

We now assume that $|D| = 1$. The contribution of the integral on GSp(4) to $R(M,N)$ is equal to the sum of

$$(N+M)q^{-M} - \frac{2q^{-M}}{q^2-1} + \frac{q^{-N-3M}}{q^3-1}$$

and

$$(q+1)\left(\frac{(N+2M)-(N+M)q^{-M}}{q-1} - 3\frac{1-q^{-M}}{(q-1)^2} + \frac{2Mq^{-M}}{q^2-1} + \frac{q^{-N-2M}-q^{-N-3M}}{(q-1)(q^3-1)}\right)$$

and

$$-\frac{q^{-M}}{q+1} + \frac{q^{-N}(q^3+1) - q^{-N-2M}(q+1)}{(q-1)(q^3-1)}.$$

The integral on $(\mathrm{GL}(2)\times\mathrm{GL}(2))'$ contributes

$$\left(N - \frac{1-q^{-N}}{q-1}\right)\left(q^{-M} + (q+1)\frac{1-q^{-M}}{q-1}\right)$$

to $R(M,N)$. Adding these together we find they are equal to $L(M,N)$. □

6. The fundamental lemma for the (1,2,1) Levi I

In this section we take M^0 to be the (1,2,1) Levi in G^0. We have

$$M^0 = \left\{\left(\begin{pmatrix} a & & \\ & A & \\ & & b\end{pmatrix}, e\right) : A \in \mathrm{GL}(2), a, b, e \in \mathrm{GL}(1)\right\}$$

and we write such an element as a tuple (a, A, b, e). The restriction of α to M^0 is given by

$$\alpha : (a, A, b, e) \longmapsto (b^{-1}, \det A^{-1}A, a^{-1}, abe\det A).$$

We set $M' = \mathrm{GL}(2)\times\mathrm{GL}(1)$ an unramified elliptic twisted endoscopic group for M. In this section we prove the fundamental lemma for the pair (M, M').

6.1. Stable conjugacy. — We begin by determining the stable twisted conjugacy class of an α-semisimple element $\gamma = (a, A, b, e) \in M^0(F)$. For $m = (a_1, A_1, b_1, e_1) \in M^0$ we have

$$m^{-1}\gamma\alpha(m) = ((a_1b_1)^{-1}a, \det A_1^{-1}A_1^{-1}AA_1, (a_1b_1)^{-1}b, a_1b_1\det A_1 e).$$

Now if we assume that $m_1^{-1}m\alpha(m_1) \in M^0(F)$ then it's clear that we must have $a_1b_1 \in F$ and $\det A_1 \in F^\times$. Moreover, after twisted conjugation over F, we can assume that A is either diagonal or else lies in an elliptic torus of the form

$$\left\{\begin{pmatrix} x & Dy \\ y & x\end{pmatrix} : x + y\sqrt{D} \in E_D^\times\right\}$$

with $v(D) \in \{0,1\}$ and $E_D = F(\sqrt{D})$ a quadratic extension of F.

Lemma 6.1. — *Assume that A lies in the diagonal torus. Then the stable twisted conjugacy class of γ is equal to the twisted conjugacy class of γ.*

Proof. — Let T denote the diagonal torus in GL(2). Then the question is, given $A_1 \in \mathrm{GL}(2, \overline{F})$ with $A_1^{-1}AA_1 \in \mathrm{GL}(2,F)$ and $\det A_1 \in F^\times$, does there exist $B \in \mathrm{GL}(2,F)$ such that $B^{-1}AB = A_1^{-1}AA_1$ and $\det B = \det A_1$. We know there exists $C \in \mathrm{GL}(2,F)$ such that $C^{-1}AC = A_1^{-1}AA_1$; and by multiplying C on the left by an element of $T(F)$ we can insist that $\det C = \det A_1$. □

For $\gamma = (a, A, b, e)$ with A diagonal we take the Haar measure on $M_{\gamma\alpha}(F)$, which gives its maximal compact subgroup volume one.

Lemma 6.2. — *Assume that A is non-central and lies in an elliptic torus as above. Then the stable twisted conjugacy class of γ is equal to the disjoint union of the twisted conjugacy classes of $\gamma = (a, A, b, e)$ and $(a, c^{-1}A, b, ce)$ with $c \in F^\times \backslash N_{E_D/F}E_D^\times$.*

Proof. — Let T denote the torus in GL(2) containing A. First it's clear that (a, A, b, e) and $(a, c^{-1}A, b, ce)$ are not twisted conjugate over F. It's also clear that they are stably conjugate, since we can conjugate them by an element of the form $(1, B, 1, 1)$ with $B \in T(\overline{F})$ such that $\det B = c$. Next we show that every element of the stable twisted conjugacy class of γ is conjugate to one of these elements. Let

$$\gamma_1 = m^{-1}\gamma\alpha(m) = ((a_1b_1)^{-1}a, \det A_1^{-1}A_1^{-1}AA_1, (a_1b_1)^{-1}b, a_1b_1 \det A_1 e)$$

lie in the stable twisted conjugacy class of γ. Then we can find $B \in \mathrm{GL}(2,F)$ such that $A_1^{-1}AA_1 = B^{-1}AB$. We can change our choice of B by multiplying B on the left by an element of $T(F)$ and hence change $\det B$ by an element of $N_{E_D/F}(E_D^\times)$. Thus γ_1 is twisted conjugate over F to either (a, A, b, e) or $(a, c^{-1}A, b, ce)$. □

We continue with the assumption that A lies in an elliptic torus as above. First suppose that E_D/F is ramified. Then we may take $c \in U_F$. We note that the weighted orbital integral at the element $(a, c^{-1}A, b, ce)$ is the same as the weighted orbital integral at the element $(ca, A, cb, c^{-1}e)$, having multiplied by the element $(c, \mathrm{diag}(c, c), c, c^{-2})$ which lies in $Z(G^0) \cap K$. But now conjugating this element by $m = (c, I, 1, 1)$ gives (a, A, b, e). Thus the weighted orbital integral along the twisted conjugacy class of $\gamma = (a, A, b, e)$ is equal to the weighted orbital integral along the twisted conjugacy class of (a, cA, b, ce). For such an A we take the measure on $M_{\gamma\alpha}(F)$ that gives its maximal compact subgroup volume two.

Next we assume that E_D/F unramified and we take

$$A = \begin{pmatrix} c & Dd \\ d & c \end{pmatrix}$$

with $v(D) = 0$. In this case $(a, \pi A, b, \pi^{-1}e)$ is stably conjugate but not conjugate to $\gamma = (a, A, b, e)$. Conjugating this element by

$$\left(1, \begin{pmatrix} 1 & \\ & \pi \end{pmatrix}, 1, 1\right)$$

gives

$$\left(a, \begin{pmatrix} c & D\pi d \\ \pi^{-1}d & c \end{pmatrix}, b, e\right).$$

If the stable twisted conjugacy class of $\gamma = (a, A, b, e)$ intersects $M^0(R)$ then we can assume that we have $a, b, e \in U_F$ and $A \in \mathrm{GL}(2, R)$ with A as above. If we assume that $(a, A, b, e) \in M^0(R)$ then we see that the twisted conjugacy class of $(a, \pi A, b, \pi^{-1}e)$ intersects $M^0(R)$ if and only if $v(d) \geqslant 1$; this is clear from the double coset decomposition found in Section 5.5. For such an A we take the measure on $M_{\gamma\alpha}(F)$ that gives its maximal compact subgroup volume one.

6.2. Statement of the fundamental lemma. — In this section we give the statement of the fundamental lemma for the pair (M, M').

We recall that M' sits inside GSp(4) as the Siegel Levi and the only elliptic twisted endoscopic group in $\mathcal{E}_{M'}(G)$ is GSp(4), with multiplicity two. Thus in this case the fundamental lemma states that for ℓ' a strongly G-regular, stable conjugacy class in $M'(F)$ we have

$$\sum_k r_M^G(k\alpha) = 2s_{M'}^{\mathrm{GSp}(4)}(\ell')$$

where the sum on the left is over those twisted conjugacy classes in $M^0(F)$ for which $N(k\alpha) = \ell'$.

We now compute the function $s_{M'}^{\mathrm{GSp}(4)}(\ell')$ whose definition is given in [**Art02**, Section 5]. From Lemma 3.8 we see that for

$$\ell' = \mathrm{diag}(g, aw^tg^{-1}w),$$

a (stable) conjugacy class in $M'(F)$, we have

$$s_{M'}^{G'}(\ell') = r_{M'}^{\mathrm{GSp}(4)}(\mathrm{diag}(g, aw^tg^{-1}w)) - \frac{1}{2}r_{M'}^{G''}(\mathrm{diag}(1, a\det g^{-1}), g)$$

where $G'' = (\mathrm{GL}(2) \times \mathrm{GL}(2))/\mathrm{GL}(1)$. Therefore the fundamental lemma for the pair (M, M') is given by the following Proposition.

Proposition 6.3. — *For $\gamma\alpha = (a, g, b, e)\alpha \in M(F)$ semisimple and strongly G^0-regular we have*

$$\sum_{\gamma'} r_M^G(\gamma'\alpha) = 2r_{M'}^{\mathrm{GSp}(4)}\begin{pmatrix} eag & \\ & eb\det g\; w^tg^{-1}w \end{pmatrix} - r_{M'}^{G''}\left(\begin{pmatrix} 1 & \\ & a^{-1}b \end{pmatrix}, eag\right)$$

where the sum on the left hand side is over representatives for the twisted conjugacy classes within the stable twisted conjugacy class of γ.

For $\gamma' = \mathrm{diag}(eag, eb\det gw^tg^{-1}w) \in M'(F)$ we take the Haar measure on $M'_{\gamma'}(F)$ that gives its maximal compact subgroup volume one.

For P^0 the upper triangular (1,2,1) parabolic in G^0 we set $\mathrm{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee)) = 1/\ln q$ and normalize the other volumes as in Section 4.4.

6.3. Proof of the fundamental lemma. — In this section we prove Proposition 6.3. We begin by noting that for $\gamma = (a, g, b, e) \in M^0(F)$ the stable twisted conjugacy class of γ does not intersect $M^0(R)$ if $|a| \neq |b|$. It's clear that the integrals on GSp(4) and $(\mathrm{GL}(2) \times \mathrm{GL}(2))/\mathrm{GL}(1)$ also vanish in this case.

If $|a| = |b|$ then we may, after twisted conjugation, assume that $a, b \in U_F$. Then the stable twisted conjugacy class of γ intersects $M^0(R)$ if and only if eg is conjugate in GL(2) to an element in $\mathrm{GL}(2, R)$. It's also clear that if eg is not conjugate to an element in $\mathrm{GL}(2, R)$ then the integrals on GSp(4) and $(\mathrm{GL}(2) \times \mathrm{GL}(2))/\mathrm{GL}(1)$ also vanish.

We now assume that we have $\gamma \in M^0(R)$. We use the twisted topological Jordan decomposition to prove the fundamental lemma. We can write $\gamma\alpha \in M(R)$ uniquely as

$$\gamma\alpha = us\alpha = s\alpha u$$

with $u \in M^0(R)$ topologically unipotent and $s\alpha \in M(R)$ absolutely semisimple. The twisted weighted orbital integrals can now be computed using 4.17. We set N equal to the unipotent radical of the upper triangular parabolic of which M^0 is a Levi component, we define N' in GSp(4) similarly.

Given $s = (a_1, g_1, b_1, e_1)$ we have

$$Z_{M^0}(s\alpha) = \{(a, g, a^{-1}, e) \in M^0 : g^{-1}g_1g = g_1, \det g = 1\}.$$

For $u = (a, g, a^{-1}, e) \in Z_{M^0}(s\alpha)$ topologically unipotent we have that the norm of $\gamma\alpha$ in GSp(4) is equal to the product of the absolutely semisimple element

$$a_1e_1\begin{pmatrix} g_1 & \\ & a_1^{-1}b_1\det g_1w^tg_1^{-1}w \end{pmatrix}$$

and topologically unipotent element

$$ae\begin{pmatrix} g & \\ & a^{-2}\det gw^tg^{-1}w \end{pmatrix}.$$

We can then also use Lemma 4.17 to compute the weighted orbital integrals on GSp(4).

We now proceed to prove the fundamental lemma by analyzing the possibilities for s.

6.3.1. s equal to the identity. — We first consider the case that s is the identity. In this case we have $Z_{G^0}(\alpha) = \mathrm{Sp}(4) \times \mathrm{GL}(1)$ and we take $\gamma = (u, e) \in \mathrm{Sp}(4, R) \times U_F$ topologically unipotent.

Lemma 6.4. — *Suppose that s is the identity, then the fundamental lemma holds.*

Proof. — We have

$$u = \begin{pmatrix} a & & \\ & g & \\ & & a^{-1} \end{pmatrix} \in \mathrm{Sp}(4, R)$$

topologically unipotent. By Lemma 4.17 we have

$$r_M^G((u,e)\alpha) = r_{\text{Klingen}}^{\mathrm{Sp}(4)}(u)$$

and hence for $\gamma = (u, e)$ we have

$$\sum_{\gamma'} r_M^G(\gamma'\alpha) = \sum_{u'} r_{\text{Klingen}}^{\mathrm{Sp}(4)}(u')$$

where $\{u'\}$ is a set of representatives for the conjugacy classes within the stable conjugacy class of u. But now using Lemma 4.16 and the double coset decompositions for $\mathrm{SL}(2, F)$ given in [**Fli99**, Lemma I.I.3] we have

$$\sum_{u'} r_{\text{Klingen}}^{\mathrm{Sp}(4)}(u') = r_{\text{Klingen}}^{\mathrm{GSp}(4)}(u).$$

From the fundamental lemma for the (2,2) Levi proven above we have

$$r_{\text{Klingen}}^{\mathrm{GSp}(4)}(u) = r_{(2,2)}^G((\mathrm{diag}(ag,1),1)\alpha) - r_{(T\times \mathrm{GL}(2))'}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))'}(\mathrm{diag}(a^2,1), ag).$$

Therefore to prove Proposition 6.3 we need to show that

$$r_{(2,2)}^G((\mathrm{diag}(ag,1),1)\alpha) - r_{(T\times \mathrm{GL}(2))'}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))'}(\mathrm{diag}(a^2,1), ag)$$

is equal to

$$2r_{M'}^{\mathrm{GSp}(4)}(\mathrm{diag}(ag, w^t(ag)^{-1}w)) - r_{M'}^{G''}(\mathrm{diag}(a,a^{-1}), g).$$

First we note that

$$r_{M'}^{G''}(\mathrm{diag}(a,a^{-1}), g) = r_{(T\times \mathrm{GL}(2))'}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))'}(\mathrm{diag}(a^2,1), ag).$$

Next we note that the element

$$\begin{pmatrix} ag & \\ & w^t(ag)^{-1}w \end{pmatrix} \in \mathrm{GSp}(4)$$

lies in $\mathrm{Sp}(4)$ and by Lemma 4.16 we have

$$2r_{M'}^{\mathrm{GSp}(4)}(\mathrm{diag}(ag, w^t(ag)^{-1}w)) = 2r_{M'}^{\mathrm{Sp}(4)}(\mathrm{diag}(ag, w^t(ag)^{-1}w)).$$

Since this element is topologically unipotent, we can apply Lemma 4.17 to get

$$2r_{M'}^{\mathrm{GSp}(4)}(\mathrm{diag}(ag, w^t(ag)^{-1}w)) = r_{(2,2)}^G((\mathrm{diag}(ag, w^t(ag)^{-1}w),1)\alpha).$$

After twisted conjugation we have

$$r_{(2,2)}^G((\mathrm{diag}(ag, w^t(ag)^{-1}w),1)\alpha) = r_{(2,2)}^G((\mathrm{diag}((ag)^2, I),1)\alpha)$$

and from the calculations of Section 5 we have

$$r_{(2,2)}^G((\mathrm{diag}((ag)^2, I),1)\alpha) = r_{(2,2)}^G((\mathrm{diag}(ag, I),1)\alpha)$$

and we are done. □

6.3.2. s central. — We now assume that $s = (a_1, g_1, b_1, e_1)$ with $g_1 = \mathrm{diag}(c_1, c_1)$ a scalar matrix. Therefore we have $u = (a, g, a^{-1}, e)$ with $a, e \in \mathrm{GL}(1)$ and $g \in \mathrm{SL}(2)$. In this section we prove Proposition 6.3 for $\gamma = us$ either by reducing the proof to Lemma 6.4 or by showing that both sides of the identity in Proposition 6.3 vanish. We begin with the following Lemma.

Lemma 6.5. — *Let $\gamma\alpha = (a, g, b, e)\alpha \in M(F)$ be semisimple and strongly G^0-regular. Then for $\lambda, \mu \in U_F$ we have*

$$r_M^G(\gamma\alpha) = r_M^G((\lambda a, \mu g, \lambda b, e)\alpha).$$

Proof. — Since we are free to scale γ by an element of $Z(G^0) \cap K$ without changing the value of $r_M^G(\gamma\alpha)$ we have

$$r_M^G((\lambda a, \mu g, \lambda b, e)\alpha) = r_M^G((\lambda\mu^{-1}a, g, \lambda\mu^{-1}b, \lambda^{-1}\mu e)\alpha).$$

But now for $m = (\lambda, I, \mu^{-1}, 1)$ we have

$$m^{-1}(\lambda\mu^{-1}a, g, \lambda\mu^{-1}b, \lambda^{-1}\mu e)\alpha(m) = (a, g, b, e)$$

and we are done. □

Now suppose that $a_1 = b_1$. Then by Lemma 6.5 we have $r_M^G(\gamma\alpha) = r_M^G(u\alpha)$ and the fundamental lemma in this case follows from Lemma 6.4. Proposition 6.3 in the case that $a_1 \neq b_1$ follows from the following.

Lemma 6.6. — *With notation as above assume that we have $a_1 \neq b_1$. Then both sides of the fundamental lemma vanish.*

Proof. — We first compute $N \cap Z_{G^0}(s\alpha)$, by abuse of notation we work inside GL(4). For

$$n = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ & 1 & & x_4 \\ & & 1 & x_5 \\ & & & 1 \end{pmatrix} \in N$$

we have

$$\alpha(n) = \begin{pmatrix} 1 & -x_5 & x_4 & x_3 - x_1x_4 - x_2x_5 \\ & 1 & & x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix}$$

and

$$s^{-1}ns = \begin{pmatrix} 1 & a_1^{-1}c_1x_1 & a_1^{-1}c_1x_2 & a_1^{-1}b_1x_3 \\ & 1 & & b_1c_1^{-1}x_4 \\ & & 1 & b_1c_1^{-1}x_5 \\ & & & 1 \end{pmatrix}.$$

Thus we need

$$\begin{aligned} x_1 &= -b_1c_1^{-1}x_5 \\ x_2 &= b_1c_1^{-1}x_4 \\ x_4 &= a_1^{-1}c_1x_2 \\ x_5 &= -a_1^{-1}c_1x_1, \end{aligned}$$

from which it follows that $x_1 = a_1^{-1}b_1x_1$ and $x_2 = a_1^{-1}b_1x_2$. But since we are assuming that $a_1 \neq b_1$, it follows that $x_1 = x_2 = x_4 = x_5 = 0$. But now we need $x_3 = a_1^{-1}b_1x_3$, and hence $x_3 = 0$ in this case as well. Thus when $a_1 \neq b_1$ the twisted integral vanishes by Lemma 4.17.

We now consider the right hand side of the fundamental lemma. First we consider the integral on GSp(4). The absolutely semisimple part of $N(\gamma\alpha)$ is

$$s_1 = a_1e_1\begin{pmatrix} g_1 & \\ & a_1^{-1}b_1 \det g_1 w^t g_1^{-1} w \end{pmatrix}.$$

We now compute $Z_{\mathrm{GSp}(4)}(s_1) \cap N'$. For

$$n = \begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & x_3 & x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix} \in N'$$

we have

$$s_1^{-1}ns_1 = \begin{pmatrix} 1 & & a_1^{-1}b_1x_1 & a_1^{-1}b_1x_2 \\ & 1 & a_1^{-1}b_1x_3 & a_1^{-1}b_1x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

from which it follows that $Z_{\mathrm{GSp}(4)}(s) \cap N' = \{I\}$ if $a_1 \neq b_1$ and hence by Lemma 4.17 the integral on GSp(4) vanishes.

Finally we consider the integral on $(\mathrm{GL}(2) \times \mathrm{GL}(2))/\mathrm{GL}(1)$. The norm of the element $\gamma\alpha$ in $(\mathrm{GL}(2) \times \mathrm{GL}(2))/\mathrm{GL}(1)$ is equal to

$$\left(\begin{pmatrix} 1 & \\ & a^{-2}a_1^{-1}b_1 \end{pmatrix}, e_1a_1eag_1g\right) \in (\mathrm{GL}(2) \times \mathrm{GL}(2))/\mathrm{GL}(1).$$

And therefore if $a_1 \neq b_1$ then $a_1^{-1}b_1 \notin U_F^1$, and since u is topologically unipotent $a^{-2} \in U_F^1$. Hence we have $a^{-2}a_1^{-1}b_1 \notin U_F^1$ and the integral on $(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)$ vanishes. □

6.3.3. s diagonal. — In this section we prove Proposition 6.3 in the case that s is diagonal but not central. So we take

$$s = \left(a_1, \begin{pmatrix} c_1 & \\ & d_1 \end{pmatrix}, b_1, e_1\right)$$

with $c_1 \neq d_1$. After twisted conjugation we may assume that $a_1 = c_1 = 1$. We now compute $N_1 = N \cap Z_G(s\alpha)$; by abuse of notation we consider $N \subset \mathrm{GL}(4)$.

Lemma 6.7. — *Let $s = (1, \mathrm{diag}(1, d_1), b_1, e_1)$. Then we have the following possibilities for N_1.*

(1) *If $b_1 = d_1 = -1$ then*

$$N_1 = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & -x_1x_2 \\ & 1 & & -x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix} \right\}.$$

(2) *If $b_1 = d_1 \neq -1$ then*

$$N_1 = \left\{ \begin{pmatrix} 1 & x_1 & 0 & 0 \\ & 1 & & 0 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix} \right\}.$$

(3) *If $b_1 = d_1^{-1} \neq -1$ then*

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & x_2 & 0 \\ & 1 & & d_1x_2 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\}.$$

(4) *If $b_1 = 1$ and $d_1 \neq 1$ then*

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & x_3 \\ & 1 & & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\}.$$

(5) *In all other cases $N_1 = \{I\}$.*

Proof. — For

$$n = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ & 1 & & x_4 \\ & & 1 & x_5 \\ & & & 1 \end{pmatrix}$$

we have

$$\alpha(n) = \begin{pmatrix} 1 & -x_5 & x_4 & x_3 - x_1x_4 - x_2x_5 \\ & 1 & & x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix}$$

and

$$s^{-1}ns = \begin{pmatrix} 1 & x_1 & d_1x_2 & b_1x_3 \\ & 1 & & b_1x_4 \\ & & 1 & b_1d_1^{-1}x_5 \\ & & & 1 \end{pmatrix}.$$

Hence we need

$$\begin{aligned} x_1 &= -b_1d_1^{-1}x_5 \\ x_2 &= b_1x_4 \\ x_4 &= d_1x_2 \\ x_5 &= -x_1. \end{aligned}$$

Thus unless $b_1 = d_1$ we have $x_1 = x_5 = 0$. And unless $b_1 = d_1^{-1}$ we have $x_2 = x_4 = 0$. And the only way both can happen is if $b_1 = d_1 = -1$ (since we are assuming that $d_1 \neq 1$). We also need to have

$$b_1x_3 = x_3 - x_1x_4 - x_2x_5$$

and hence we need to have

$$(1 - b_1)x_3 = x_1x_4 + x_2x_5 = (d_1 - 1)x_1x_2.$$

Putting this all together completes the proof. □

We now compute the twisted integral in each of the above cases. We have

$$u = \left(a, \begin{pmatrix} c & \\ & c^{-1} \end{pmatrix}, a^{-1}, e\right)$$

and so the stable twisted conjugacy class of $\gamma = us$ is equal to the twisted conjugacy class of γ.

Lemma 6.8. — *With notation as in Lemma 6.7 the twisted integral $r_M^G(\gamma\alpha)$ is given by the following.*

(1) *If $b_1 = d_1 = -1$ then*

$$r_M^G(\gamma\alpha) = |ac - 1||ac^{-1} - 1| \int_{|x_1| \leqslant |ac^{-1}-1|^{-1}} \int_{|x_2| \leqslant |ac-1|} \log\max\{1, |x_1|, |x_2|, |x_1x_2|\}.$$

(2) *If $b_1 = d_1 \neq -1$ then*

$$r_M^G(\gamma\alpha) = |ac^{-1} - 1| \int_{|x_1| \leqslant |ac^{-1}-1|^{-1}} \log\max\{1, |x_1|\}.$$

(3) *If $b_1 = d_1^{-1} \neq -1$ then*

$$r_M^G(\gamma\alpha) = |ac - 1| \int_{|x_2| \leqslant |ac-1|^{-1}} \log\max\{1, |x_2|\}.$$

(4) *If* $b_1 = 1$ *and* $d_1 \neq \pm 1$ *then*

$$r_M^G(\gamma\alpha) = |a-1| \int_{|x_3| \leqslant |a-1|^{-1}} \log \max\{1, |x_3|\}.$$

(5) *In all other cases* $r_M^G(\gamma\alpha) = 0$.

Proof. — In each case we compute $u^{-1}n^{-1}un$ for $n \in N$. In the first case we have

$$u^{-1}n^{-1}un = \begin{pmatrix} 1 & (1-a^{-1}c)x_1 & (1-a^{-1}c^{-1})x_2 & -(1-a^{-1}c)(1-a^{-1}c^{-1})x_1x_2 \\ & 1 & & -(1-a^{-1}c^{-1})x_2 \\ & & 1 & -(1-a^{-1}c)x_1 \\ & & & 1 \end{pmatrix}.$$

In the second case we have

$$u^{-1}n^{-1}un = \begin{pmatrix} 1 & (1-a^{-1}c)x_1 & 0 & 0 \\ & 1 & & 0 \\ & & 1 & -(1-a^{-1}c)x_1 \\ & & & 1 \end{pmatrix}.$$

In the third case we have

$$u^{-1}n^{-1}un = \begin{pmatrix} 1 & 0 & (1-a^{-1}c^{-1})x_2 & 0 \\ & 1 & & (1-a^{-1}c^{-1})d_1x_2 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}.$$

In the fourth case we have

$$u^{-1}n^{-1}un = \begin{pmatrix} 1 & 0 & 0 & (1-a^{-2})x_3 \\ & 1 & & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}.$$

And of course in the fifth case the integral vanishes. □

Now we turn to the corresponding integrals on GSp(4). The absolutely semisimple part of $N(\gamma\alpha)$ is

$$s_1 = e_1 \begin{pmatrix} 1 & & & \\ & d_1 & & \\ & & b_1 & \\ & & & b_1d_1 \end{pmatrix}.$$

Lemma 6.9. — *With notation as above we have the following possibilities for* $N_1' = Z_{\mathrm{GSp}(4)}(s_1) \cap N'$.

(1) *If* $b_1 = 1$ *then*
$$N_1' = \left\{\begin{pmatrix} 1 & & x_1 & 0 \\ & 1 & 0 & x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix}\right\}.$$

(2) *If* $b_1 = d_1 = -1$ *then*
$$N_1' = \left\{\begin{pmatrix} 1 & & 0 & x_2 \\ & 1 & x_3 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix}\right\}.$$

(3) *If* $b_1 = d_1 \notin \{1, -1\}$ *then*
$$N_1' = \left\{\begin{pmatrix} 1 & & 0 & 0 \\ & 1 & x_3 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix}\right\}.$$

(4) *If* $b_1 = d_1^{-1} \notin \{1, -1\}$ *then*
$$N_1' = \left\{\begin{pmatrix} 1 & & 0 & x_2 \\ & 1 & 0 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix}\right\}.$$

(5) *In all other cases* $N_1' = \{I\}$.

Proof. — For
$$n = \begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & x_3 & x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix} \in N'$$
we have
$$s_1^{-1} n s_1 = \begin{pmatrix} 1 & & b_1 x_1 & d_1 b_1 x_2 \\ & 1 & b_1 d_1^{-1} x_3 & b_1 x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix}$$
and the result follows. □

We now need to compute the weighted integral on $Z_{\mathrm{GSp}(4)}(s_1)$ at the element
$$u = \begin{pmatrix} ac & & & \\ & ac^{-1} & & \\ & & a^{-1}c & \\ & & & a^{-1}c^{-1} \end{pmatrix}.$$

These integrals are given in the following Lemma.

Lemma 6.10. — *With notation as above the integral* $2r_{M'}^{\mathrm{GSp}(4)}(N(\gamma\alpha))$ *is given by the following.*

(1) *If* $b_1 = 1$ *then*

$$2r_{M'}^{\mathrm{GSp}(4)}(N(\gamma\alpha)) = 2|a-1|\int_{|x_1|\leqslant|a-1|^{-1}} \log\max\{1,|x_1|\}.$$

(2) *If* $b_1 = d_1 = -1$ *then* $2r_{M'}^{\mathrm{GSp}(4)}(N(\gamma\alpha))$ *is equal to*

$$|ac-1||ac^{-1}-1|\int_{|x_2|\leqslant|ac-1|^{-1}}\int_{|x_3|\leqslant|ac^{-1}-1|^{-1}} \log\max\{1,|x_2|,|x_3|,|x_2x_3|\}.$$

(3) *If* $b_1 = d_1 \neq -1$ *then*

$$2r_{M'}^{\mathrm{GSp}(4)}(N(\gamma\alpha)) = |ac^{-1}-1|\int_{|x_3|\leqslant|ac^{-1}-1|^{-1}} \log\max\{1,|x_3|\}.$$

(4) *If* $b_1 = d_1^{-1} \neq -1$ *then*

$$2r_{M'}^{\mathrm{GSp}(4)}(N(\gamma\alpha)) = |ac-1|\int_{|x_2|\leqslant|ac-1|^{-1}} \log\max\{1,|x_2|\}.$$

(5) *In all other cases* $2r_{M'}^{\mathrm{GSp}(4)}(N(\gamma\alpha)) = 0$.

Proof. — We take $n \in N_1'$. In the first case we have

$$u^{-1}n^{-1}un = \begin{pmatrix} 1 & (1-a^{-2})x_1 & 0 & \\ & 1 & 0 & (1-a^{-2})x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

In the second case we have

$$u^{-1}n^{-1}un = \begin{pmatrix} 1 & 0 & (1-a^{-2}c^{-2})x_2 & \\ & 1 & (1-a^{-2}c^{2})x_3 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

In the third case we have

$$u^{-1}n^{-1}un = \begin{pmatrix} 1 & & 0 & 0 \\ & 1 & (1-a^{-2}c^{2})x_3 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

In the fourth case we have

$$u^{-1}n^{-1}un = \begin{pmatrix} 1 & 0 & (1-a^{-2}c^{-2})x_2 & \\ & 1 & 0 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

And in the fifth case it's clear that the integral vanishes. ☐

For the integral on $(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)$ the norm of $\gamma\alpha$ is

$$\left(\begin{pmatrix}1 & \\ & a^{-2}b_1\end{pmatrix}, e_1ea\,\mathrm{diag}(c, d_1c^{-1})\right)\in(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1).$$

Thus we see that the integral vanishes unless $b_1=1$ in which case it equals

$$|a-1|\int_{|x|\leqslant|a-1|^{-1}}\log\max\{1,|x|\}.$$

Combining the above lemmas proves Proposition 6.3 in this case.

6.3.4. s elliptic. — We now assume that we have $g_1\in\mathrm{GL}(2,F)$ which is non-central and lies in an elliptic torus. After stable twisted conjugation we can assume that we have

$$g_1=\begin{pmatrix}c_1 & Dd_1\\ d_1 & c_1\end{pmatrix}\in\mathrm{GL}(2,R)$$

with $d_1\neq 0$ and $v(D)=\{0,1\}$. We let $E_D=F(\sqrt{D})$. For $s\alpha$ to be absolutely semisimple we need to have

$$g_1^k=\begin{pmatrix}x^k & \\ & x^k\end{pmatrix}$$

for some $x\in F$ and k prime to the residual characteristic of F. But then, as an element of E_D, we have $g_1=\zeta x$ for some k^{th} root of unity ζ. Since we're assuming that g_1 is non-central we must have $\zeta\notin F^\times$. Hence we must have E_D/F unramified and $v(D)=0$. After twisted conjugation we can take

$$s=\left(1,\begin{pmatrix}c_1 & D\\ 1 & c_1\end{pmatrix}, b_1, e_1\right).$$

We now compute $N_1=N\cap Z_{G^0}(s\alpha)$, which by abuse of notation we consider as a subgroup of $\mathrm{GL}(4)$.

Lemma 6.11. — *With notation as above we have the following possibilities for* N_1.

(1) *If* $b_1=-1$ *and* $c_1=0$ *then*

$$N_1=\left\{\begin{pmatrix}1 & x_1 & x_2 & (Dx_1^2-x_2^2)/2\\ & 1 & & Dx_1\\ & & 1 & -x_2\\ & & & 1\end{pmatrix}\right\}.$$

(2) *If* $b_1=1$ *then*

$$N_1=\left\{\begin{pmatrix}1 & 0 & 0 & x_3\\ & 1 & & 0\\ & & 1 & 0\\ & & & 1\end{pmatrix}\right\}.$$

(3) *In all other cases we have* $N_1=\{I\}$.

Proof. — For

$$n = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ & 1 & & x_4 \\ & & 1 & x_5 \\ & & & 1 \end{pmatrix}$$

we have

$$\alpha(n) = \begin{pmatrix} 1 & -x_5 & x_4 & x_3 - x_1x_4 - x_2x_5 \\ & 1 & & x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix}$$

and

$$s^{-1}ns = \begin{pmatrix} 1 & c_1x_1 + x_2 & Dx_1 + c_1x_2 & b_1x_3 \\ & 1 & & (c_1^2 - D)^{-1}(b_1c_1x_4 - b_1Dx_5) \\ & & 1 & (c_1^2 - D)^{-1}(-b_1x_4 + b_1c_1x_5) \\ & & & 1 \end{pmatrix}.$$

Hence we need

$$\begin{aligned} x_1 &= -(c_1^2 - D)^{-1}(-b_1x_4 + b_1c_1x_5) \\ x_2 &= (c_1^2 - D)^{-1}(b_1c_1x_4 - b_1Dx_5) \\ x_4 &= Dx_1 + c_1x_2 \\ x_5 &= -c_1x_1 - x_2. \end{aligned}$$

So we have

$$(c_1^2 - D)x_1 = b_1x_4 - b_1c_1x_5$$

and from the third and fourth equations we get

$$(c_1^2 - D)x_1 = -x_4 - c_1x_5.$$

Hence we have

$$(1 + b_1)x_4 + c_1(1 - b_1)x_5 = 0.$$

We also have

$$(c_1^2 - D)x_2 = b_1c_1x_4 - b_1Dx_5$$

and from the third and fourth equations we get

$$(c_1^2 - D)x_2 = c_1x_4 + Dx_5.$$

So we have

$$\begin{aligned} (1 + b_1)x_4 + c_1(1 - b_1)x_5 &= 0 \\ c_1(b_1 - 1)x_4 - D(1 + b_1)x_5 &= 0. \end{aligned}$$

Hence we deduce that

$$(c_1^2(1 - b_1)^2 - D(1 + b_1)^2)x_4 = 0$$

and

$$(c_1^2(1-b_1)^2 - D(1+b_1)^2)x_5 = 0.$$

Thus unless $b_1 = -1$ and $c_1 = 0$ we have $x_1 = x_2 = x_4 = x_5 = 0$. Now we also need to have

$$x_3 - x_1x_4 - x_2x_5 = b_1x_3.$$

Thus if $b_1 = 1$ then we can take x_3 to be anything we like. On the other hand if $b_1 = -1$ and $c_1 = 0$ we have $x_4 = Dx_1$, $x_5 = -x_2$ and $x_3 = \frac{1}{2}\left(Dx_1^2 - x_2^2\right)$. □

We take

$$u = \left(a, \begin{pmatrix} c & Dd \\ d & c \end{pmatrix}, a^{-1}, e\right) \in Z_{M^0}(s\alpha)$$

to be topologically unipotent; so we have $c \in U_F^1$ and $d \in (\pi)$. We have

$$us = \left(aa_1, \begin{pmatrix} cc_1 + Dd & D(c+dc_1) \\ c + dc_1 & cc_1 + Dd \end{pmatrix}, a^{-1}b_1, ee_1\right).$$

Now $c + dc_1 \in U_F$ and hence we deduce that it is only the twisted conjugacy class of us that intersects $M^0(R)$, i.e., the other twisted conjugacy class within the stable twisted conjugacy class of us does not intersect $M^0(R)$. The twisted integrals at the element us are given by the following lemma.

Lemma 6.12. — *With notation as above the twisted integrals $r_M^G(\gamma\alpha)$ are given by the following.*

(1) *If $b_1 = -1$ and $c_1 = 0$ then*

$$r_M^G(\gamma\alpha) = 2|D_G(\gamma\alpha)|^{1/2} \int \log\max\{1, |x_1|, |x_2|\}$$

over the region

- $(1 - a^{-1}c)x_1 - a^{-1}dx_2 \in R$
- $-a^{-1}dDx_1 + (1 - a^{-1}c)x_2 \in R.$

(2) *If $b_1 = 1$ then*

$$r_M^G(\gamma\alpha) = |a-1| \int_{|x_3| \leqslant |a-1|^{-1}} \log\max\{1, |x_3|\}.$$

(3) *In all other cases $r_M^G(\gamma\alpha) = 0$.*

Proof. — First suppose we have $b_1 = -1$ and $c_1 = 0$ then we have

$$u^{-1}n^{-1}un = \begin{pmatrix} 1 & (1-a^{-1}c)x_1 - a^{-1}dx_2 & -a^{-1}dDx_1 + (1-a^{-1}c)x_2 & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}.$$

If $b_1 = 1$, then we have

$$u^{-1}n^{-1}un = \begin{pmatrix} 1 & 0 & 0 & (1-a^{-2})x_3 \\ & 1 & & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}.$$

And in all other cases it's clear that the integral vanishes. □

Next we look at the integrals on GSp(4). The absolutely semisimple part of $N(\gamma\alpha)$ is

$$s_1 = e_1 \begin{pmatrix} c_1 & D & & \\ 1 & c_1 & & \\ & & b_1c_1 & -b_1D \\ & & -b_1 & b_1c_1 \end{pmatrix}.$$

Lemma 6.13. — *With notation as above we have the following possibilities for $N_1' = Z_{\mathrm{GSp}(4)}(s_1) \cap N'$.*

(1) *If $b_1 = -1$ and $c_1 = 0$ then*

$$N_1' = \left\{ \begin{pmatrix} 1 & & x_1 & Dx_3 \\ & 1 & x_3 & x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}.$$

(2) *If $b_1 = 1$ then*

$$N_1' = \left\{ \begin{pmatrix} 1 & & 0 & -Dx_3 \\ & 1 & x_3 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}.$$

(3) *In all other cases $N_1' = \{I\}$.*

Proof. — For

$$n = \begin{pmatrix} 1 & & x_1 & x_2 \\ & 1 & x_3 & x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

we have $s^{-1}ns$ equal to $(c_1^2 - D)^{-1}$ times

$$\begin{pmatrix} 1 & & b_1(c_1^2+D)x_1 - b_1c_1x_2 - b_1c_1Dx_3 & -2b_1c_1Dx_1 + b_1c_1^2x_2 + b_1D^2x_3 \\ & 1 & -2b_1c_1x_1 + b_1x_2 + c_1^2b_1x_3 & b_1(c_1^2+D)x_1 - b_1c_1x_2 - b_1c_1Dx_3 \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Hence we need

$$\begin{array}{rcccccccc} (c_1^2 - D) & x_1 = & b_1(c_1^2+D) & x_1 & - & b_1c_1 & x_2 & - & b_1c_1D & x_3 \\ (c_1^2 - D) & x_2 = & -2b_1c_1D & x_1 & + & b_1c_1^2 & x_2 & + & b_1D^2 & x_3 \\ (c_1^2 - D) & x_3 = & -2b_1c_1 & x_1 & + & b_1 & x_2 & + & c_1^2b_1 & x_3. \end{array}$$

That is

$$\begin{array}{cccccccc} (b_1c_1^2 + b_1D + D - c_1^2) & x_1 & - & b_1c_1 & x_2 & - & b_1c_1D & x_3 & = 0 \\ -2b_1c_1D & x_1 & + & (D + b_1c_1^2 - c_1^2) & x_2 & + & b_1D^2 & x_3 & = 0 \\ -2b_1c_1 & x_1 & + & b_1 & x_2 & + & (c_1^2b_1 + D - c_1^2) & x_3 & = 0. \end{array}$$

Equation 1 times D plus equation 2 times c_1 gives

$$D(D - c_1^2)(1 + b_1)x_1 + c_1(1 - b_1)(D - c_1^2)x_2 = 0$$

and since $D - c_1^2 \neq 0$ we have

$$D(1 + b_1)x_1 + c_1(1 - b_1)x_2 = 0$$

Next we do equation 2 times $c_1^2b_1 + D - c_1^2$ minus equation 3 times b_1D^2 to give

$$2b_1c_1D(1 - b_1)(c_1^2 - D)x_1 + (b_1 - 1)(c_1^2 - D)(-(c_1^2 - D) + b_1(c_1^2 + D))x_2 = 0$$

and since $D - c_1^2 \neq 0$ we have

$$2b_1c_1D(1 - b_1)x_1 + (b_1 - 1)(-(c_1^2 - D) + b_1(c_1^2 + D))x_2 = 0.$$

Thus we have

$$\begin{aligned} D(1 + b_1)x_1 + c_1(1 - b_1)x_2 &= 0 \\ 2b_1c_1D(1 - b_1)x_1 + (b_1 - 1)(-(c_1^2 - D) + b_1(c_1^2 + D))x_2 &= 0, \end{aligned}$$

which yields

$$(D(b_1 + 1)^2 - c_1^2(b_1 - 1)^2)x_1 = 0$$

and

$$(b_1 - 1)(c_1^2(b_1 - 1)^2 - D(b_1 + 1)^2)x_2 = 0.$$

Therefore if $c_1 = 0$ and $b_1 = -1$ we can take x_1 and x_2 to be whatever we like; and then we have $Dx_3 = x_2$. Now if $b_1 = 1$ then we have $x_1 = 0$ and $x_2 = -Dx_3$. In all other cases we have $x_1 = x_2 = x_3 = 0$. □

Now we compute the integrals on GSp(4). We need to compute the relevant integrals at the element

$$u = e \begin{pmatrix} ac & adD & & \\ ad & ac & & \\ & & a^{-1}c & -a^{-1}dD \\ & & -a^{-1}d & a^{-1}c \end{pmatrix}.$$

Lemma 6.14. — *With notation as above* $2r_{M'}^{\mathrm{GSp}(4)}(N(\gamma\alpha))$ *is given by the following.*

(1) *If* $b_1 = -1$ *and* $c_1 = 0$ *then we have* $2r_{M'}^{\mathrm{GSp}(4)}(N(\gamma\alpha))$ *equal to*

$$2|D_{\mathrm{GSp}(4)}(N(\gamma\alpha))|^{1/2}\int \log\{1, |x_1|, |x_3|\}$$

over the region

- $(a^2 - c^2 - Dd^2)x_1 + 2cdDx_3 \in R$
- $2cdx_1 + (a^2 - c^2 - Dd^2)x_3 \in R.$

(2) *If* $b_1 = 1$ *then we have* $2r_{M'}^{\mathrm{GSp}(4)}(N(\gamma\alpha))$ *equal to*

$$2|a-1|\int_{|x_3|\leqslant|a-1|^{-1}} \log\max\{1, |x_3|\}.$$

(3) *In all other cases we have* $2r_{M'}^{\mathrm{GSp}(4)}(N(\gamma\alpha)) = 0.$

Proof. — Let's consider the first case. We have

$$u^{-1}n^{-1}un = \begin{pmatrix} 1 & & (a^2-c^2-Dd^2)x_1 + 2cdDx_3 & * \\ & 1 & 2cdx_1 + (a^2-c^2-Dd^2)x_3 & * \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

In the second case we have

$$u^{-1}n^{-1}un = \begin{pmatrix} 1 & & 0 & -D(1-a^{-2})x_3 \\ & 1 & (1-a^{-2})x_3 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

And it's clear that in the third case the integral vanishes. □

Again we recall that the integral on $(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)$ vanishes unless $b_1 = 1$ in which case it equals

$$|a-1|\int_{|x|\leqslant|a-1|^{-1}} \log\max\{1, |x|\}.$$

Thus it's clear that the fundamental lemma holds in all cases except perhaps when $b_1 = -1$ and $c_1 = 0$. We have $|D_G(\gamma\alpha)| = |D_{\mathrm{GSp}(4)}(N(\gamma\alpha))|$ and in this case we need to show that the integrals of $\log\max\{1, |x|, |y|\}$ over the regions in F^2 given by

$$\begin{pmatrix} a-c & -d \\ -dD & a-c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$$

and

$$\begin{pmatrix} a^2-c^2-Dd^2 & 2cdD \\ 2cd & a^2-c^2-Dd^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$$

are equal. We readily see that if $|d| \geqslant |a-c|$ then both these matrices lie in

$$\mathrm{GL}(2,R)\begin{pmatrix} d & \\ & d \end{pmatrix}$$

and if $|d| < |a-c|$ then both these matrices lie in

$$\mathrm{GL}(2, R)\begin{pmatrix} a-c & \\ & a-c \end{pmatrix}.$$

Hence the integrals above are equal and the proof of Proposition 6.3 is now complete.

7. The fundamental lemma for the (1,2,1) Levi II

In this section we again take M^0 to be the (1,2,1) Levi in G^0. We have

$$M^0 = \left\{ \left(\begin{pmatrix} a & & \\ & A & \\ & & b \end{pmatrix}, e \right) : A \in \mathrm{GL}(2), a, b, e \in \mathrm{GL}(1) \right\}$$

and we write such an element as a tuple (a, A, b, e). The restriction of α to M^0 is given by

$$\alpha : (a, A, b, e) \longmapsto (b^{-1}, \det A^{-1} A, a^{-1}, abe \det A).$$

We set $M' = \mathrm{GL}(1) \times \mathrm{Res}_{E/F} \mathrm{GL}(1)$ an unramified elliptic twisted endoscopic group for M^0. In this section we prove the fundamental lemma for the pair (M, M').

7.1. Statement of the fundamental lemma. — Let E denote the unramified quadratic extension of F. We fix $D \in F$ with $v(D) = 0$ such that $E = F(\sqrt{D})$. Let R_E denote the ring of integers in E and U_E the group of units. We let $|\ |_E$ denote the multiplicative valuation on E normalized such that $|\pi|_E = q^{-2}$. Given $\beta \in E$ we let $\overline{\beta}$ denote its Galois conjugate. We fix the Haar measure on E that gives R_E volume one.

We recall from Lemma 3.5 that the elliptic twisted endoscopic groups for G^0 in $\mathcal{E}_{M'}(G)$ are $G_1 = \mathrm{Res}_{E/F} \mathrm{GL}(2)'$ and $G_2 = (\mathrm{GL}(2) \times \mathrm{Res}_{E/F} \mathrm{GL}(1)) / \mathrm{GL}(1)$. Moreover each group appears with multiplicity two and we have M' sitting inside both of these groups as the diagonal torus.

The stable twisted conjugacy classes in $M^0(F)$, which transfer to $M'(F)$, are those with representatives of the form

$$\gamma = \left(\begin{pmatrix} c & & & \\ & a & bD & \\ & b & a & \\ & & & d \end{pmatrix}, e \right).$$

Moreover as we saw in Section 6.1 the stable twisted conjugacy class of γ is the disjoint union of the twisted conjugacy classes of γ and

$$\gamma' = \left(\begin{pmatrix} c & & & \\ & a & b\pi^{-1} D & \\ & b\pi & a & \\ & & & d \end{pmatrix}, e \right).$$

And we have, using [**KS99**, Chapter 4], $\Delta(N(\gamma\alpha),\gamma) = (-1)^{v(b)}$ and $\Delta(N(\gamma\alpha),\gamma') = (-1)^{v(b)+1}$.

We let $\beta = a + b\sqrt{D} \in E^{\times}$. The fundamental lemma is given by the following Proposition.

Proposition 7.1. — *Let γ and γ' be as above, then we have*

$$r_M^G(\gamma\alpha) - r_M^G(\gamma'\alpha) = (-1)^{v(b)}\left(2r_{M'}^{G_1}\begin{pmatrix} ce\beta & \\ & de\overline{\beta}\end{pmatrix} + 2r_{M'}^{G_2}\left(\begin{pmatrix} ce & \\ & de\end{pmatrix}, \beta\right)\right).$$

For P^0 the upper triangular (1,2,1) parabolic in G^0 we set $\operatorname{vol}(\mathfrak{a}_P^G/\mathbf{Z}(\Delta_P^\vee)) = 1/\ln q$ and normalize the other volumes as in Section 4.4.

7.2. Proof of the fundamental lemma. — We note that both sides of the identity in Proposition 7.1 vanish if the stable twisted conjugacy class of γ does not intersect $M^0(R)$. Thus we may assume that we have

$$\gamma = \left(\begin{pmatrix} c & & & \\ & a & bD & \\ & b & a & \\ & & & d\end{pmatrix}, e\right) \in M^0(R).$$

We now compute $2r_{M'}^{G_1}(N(\gamma\alpha))$ and $2r_{M'}^{G_2}(N(\gamma\alpha))$. We have

$$2r_{M'}^{G_1}(N(\gamma\alpha)) = |c\beta - d\overline{\beta}|_E \int_{|x|\leqslant|c\beta-d\overline{\beta}|_E^{-1}} \log\max\{1,|x|_E\}$$

and

$$2r_{M'}^{G_2}(N(\gamma\alpha)) = |c-d| \int_{|x|\leqslant|c-d|^{-1}} \log\max\{1,|x|\}.$$

As in the previous section we use the twisted topological Jordan decomposition of $\gamma\alpha$ to prove the fundamental lemma. So we write $\gamma\alpha = us\alpha = s\alpha u$ as a commuting product of an absolutely semisimple element $s\alpha$ and a topological unipotent element u. We again analyze the possibilities for $s\alpha$ and prove the fundamental lemma for each such $s\alpha$ and every topologically unipotent element u that commutes with it.

7.2.1. s equals the identity. — We now assume that s is equal to the identity. With a slight change in notation we take

$$\gamma = \left(\begin{pmatrix} c & & & \\ & ac^{-1} & bc^{-1}D & \\ & bc^{-1} & ac^{-1} & \\ & & & c^{-1}\end{pmatrix}, e\right) \in \operatorname{Sp}(4,R)\times U_F$$

with $c, e \in U_F^1$, $\beta = a + b\sqrt{D} \in U_E^1$ and $a^2 - Db^2 = c^2$.

In order to use the calculations and reductions of Section 5 we make the further assumption that $q > 3$. However, arguing as in Remark 5.9 will give the fundamental lemma in the case $q = 3$ as well.

We set $|b| = q^{-M}$ and $|c-1| = q^{-N}$. Then we have

$$|D_G(\gamma\alpha)|^{1/2} = \begin{cases} q^{-3N-M}, & \text{if } N \leqslant M, \\ q^{-N-3M}, & \text{if } N \geqslant M, \end{cases}$$
$$|D_M(\gamma\alpha)|^{1/2} = q^{-M},$$
$$|D_{G_1}(N(\gamma\alpha))|^{1/2} = \begin{cases} q^{-2N}, & \text{if } N \leqslant M, \\ q^{-2M}, & \text{if } N \geqslant M, \end{cases}$$
$$|D_{G_2}(N(\gamma\alpha))|^{1/2} = q^{-N}.$$

Using Lemma 4.17 we note that the twisted weighted orbital integrals we need to compute on G^0 are equal to the weighted orbital integrals on GSp(4) with respect to the Klingen Levi. Let M_1 denote the Klingen Levi in GSp(4) and P_1 the upper triangular parabolic of which M_1 is a Levi component. We also set N_1 equal to the unipotent radical in P_1. We let σ_{P_1} denote the function

$$\sigma_{P_1}(a) = \int_{N_1(F)\cap \mathrm{GSp}(4,R)} v_{M_1}(\varphi_a(n))\, dn$$

where $\varphi_a : N_1 \to N_1$ the inverse of the map $N_1 \to N_1 : n \mapsto a^{-1}n^{-1}an$. By abuse of notation we identify γ with it's component lying in Sp(4); then we have $r_M^G(\gamma\alpha) - r_M^G(\gamma'\alpha)$ equal to $|D_M(\gamma\alpha)|^{1/2}$ times

$$\sigma_{P_1}(\gamma) + (q+1)\sum_{m=1}^{M}(-1)^m q^{m-1}\sigma_{P_1}(z_m^{-1}\gamma z_m)$$

where

$$z_m = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \pi^m & \\ & & & 1 \end{pmatrix}.$$

And Proposition 7.1 says that it equals

$$(-1)^M\left(2q^{-2N}\left(Nq^{2N} - \frac{q^{2N}-1}{q^2-1}\right) + q^{-N}\left(Nq^N - \frac{q^N-1}{q-1}\right)\right)$$

if $N \leqslant M$ and equals

$$(-1)^M\left(2q^{-2M}\left(Mq^{2M} - \frac{q^{2M}-1}{q^2-1}\right) + q^{-N}\left(Nq^N - \frac{q^N-1}{q-1}\right)\right)$$

if $N \geqslant M$.

We now set about computing

$$\sigma_{P_1}(\gamma) + (q+1)\sum_{m=1}^{M}(-1)^m q^{m-1}\sigma_{P_1}(z_m^{-1}\gamma z_m).$$

To put us in the same shape as Section 5 we scale our element γ by c to give

$$\gamma = \begin{pmatrix} c^2 & & & \\ & a & bD & \\ & b & a & \\ & & & 1 \end{pmatrix},$$

which of course doesn't change the value of $\sigma_{P_1}(z_m^{-1}\gamma z_m)$. And in the notation of Section 5 we have

$$n = \det \begin{pmatrix} a & bD \\ b & a \end{pmatrix} = c^2.$$

We have $|n-1| = |c-1| = q^{-N}$ and $|b| = q^{-M}$. Note also that we have $|\beta - 1| = \max\{|a-1|, |b|\}$. But since $a^2 - b^2D = c^2$ it follows that $|\beta - 1| = \max\{|c-1|, |b|\}$.

We begin by proving Proposition 7.1 under the assumption that $|b|^2 \leqslant |n-1| \leqslant |b|$. So we have $M \leqslant N$. As we have seen in Sections 5.5.3 and 5.5.5, $\sigma_{P_1}(z_m^{-1}\gamma z_m)$ for $0 \leqslant m \leqslant M$ equals the sum of

$$N + M + m - \frac{2}{q-1} + 2\frac{q^{-m+1}}{q^2-1} + \frac{q^{-N-2M}}{q^3-1}$$

and

$$(M-2l-m-2)q^{-2M+2l+m} + q^{-2M+2l+m}\frac{2q+1}{q+1} + \frac{q^{-N-2M+3l+2m+1}}{q^2-1} + \frac{q^{-N-2M+3l+2}}{(q+1)(q^3-1)}$$

where $l = \lfloor \frac{N-m}{2} \rfloor$. Using this we compute that

$$\sigma_{P_1}(\gamma) + (q+1)\sum_{m=1}^{M}(-1)^m q^{m-1}\sigma_{P_1}(z_m^{-1}\gamma z_m)$$

equals

$$(-1)^M \left(2q^{-M}\left(Mq^{2M} - \frac{q^{2M}-1}{q^2-1} \right) + q^{-N+M}\left(Nq^N - \frac{q^N-1}{q-1} \right) \right).$$

And since $|D_M(\gamma\alpha)| = q^{-M}$ Proposition 7.1 follows in this case.

In proving the fundamental lemma for the (2,2) Levi in the case of an elliptic torus we reduced the proof to this case. We now follow these same reductions for the fundamental lemma here. First we assume that we have $|c-1| \leqslant |b|^2$. We set

$$\sigma_{P_1}(M, N, m) = \sigma_{P_1}(z_m^{-1}\gamma z_m)$$

and

$$L(M,N) = q^{-M}\left(\sigma_{P_1}(N,M,0) + (q+1)\sum_{m=1}^{M}(-1)^m q^{m-1}\sigma_{P_1}(N,M,m) \right).$$

We now compute $qL(M, N+1) - L(M,N)$. As we have seen in the proof of Proposition 5.13 we have $q\sigma_{P_1}(M, N+1, m) - \sigma_{P_1}(M,N,m)$ equal to

$$(N+M+m+1)q - (N+M+m+2) + 2q^{-m} - 2q^{-M+1} + q^{-M} - 2\frac{q^{-m} - q^{-M+2}}{q+1}$$

if $M-m$ is even and equal to

$$(N+M+m+1)q-(N+M+m+2)+2q^{-m}-q^{-M}-2\frac{q^{-m}-q^{-M+1}}{q+1}$$

if $M-m$ is odd.

Using this we compute that $qL(M,N+1)-L(M,N)$ equals $(-1)^M$ times the sum of

$$q^{-N+M}\left((N+1)q^{N+1}-\frac{q^{N+1}-1}{q-1}\right)-q^{-N+M}\left(Nq^N-\frac{q^N-1}{q-1}\right)$$

and

$$2q^{-M+1}\left(Mq^{2M}-\frac{q^{2M}-1}{q^2-1}\right)-2q^{-M}\left(Mq^{2M}-\frac{q^{2M}-1}{q^2-1}\right)$$

as required.

Now assume that we have $q^{-N}=|c-1|=|n-1|\geqslant|b|=q^{-M}$. Again we set

$$\sigma_{P_1}(M,N,m)=\sigma_{P_1}(z_m^{-1}\gamma z_m)$$

and

$$L(M,N)=\sigma_{P_1}(M,N,0)+(q+1)\sum_{m=1}^{M}(-1)^m q^{m-1}\sigma_{P_1}(M,N,m).$$

We denote, as in the case of the (2,2) Levi,

$$e(M,N,m)=\sigma_{P_1}(M+1,N,m+1)-\sigma_{P_1}(M,N,m),$$

and we have $L(M+1,N)+L(M,N)$ equal to

$$\sigma_{P_1}(M+1,N,0)-\sigma_{P_1}(M,N,0)+(q+1)\sum_{m=0}^{M}(-1)^{m+1}q^m e(M,N,m).$$

And as we have seen in the proof of Proposition 5.11

$$e(M,N,m)=q^{-3N-m-1}(I(m+2)-I(m)),$$

where $I(m)$ is equal to the integral of

$$|x|\log\max\{1,|x|,|s|\}$$

over the region

- $|x|\leqslant q^N$
- $\pi^N s-\pi^{m+M}x^2\in R$.

And we have

$$\sigma_{P_1}(M,N,0)=q^{-3N}(1+q^{-1})I(0)\quad\text{and}\quad\sigma_{P_1}(M+1,N,0)=q^{-3N}(1+q^{-1})I(1).$$

Hence we have

$$L(M+1,N)+L(M,N)=(q+1)q^{-3N-1}\left((-1)^{M+1}I(M+2)+(-1)^M I(M+1)\right).$$

But since $I(m)$ is constant for $m\geqslant M$ so we have $L(M+1,N)+L(M,N)=0$ as required.

The proof of Proposition 7.1 is now complete under the assumption that s equals the identity.

7.2.2. s not equal to the identity. — We now analyze the other possibilities for s. Let's take

$$s = (a_1, g_1, b_1, e_1).$$

First we assume that $g_1 \in Z(\mathrm{GL}(2))$. Then we have $u \in M^0(F)$ topologically unipotent and $\alpha(u) = u$. If $a_1 = b_1$ then from Lemma 6.5 we have

$$r_M^G(us\alpha) = r_M^G(u\alpha).$$

It's clear that when $a_1 = b_1$ we also have

$$r_{M'}^{G_1}(N(us\alpha)) = r_{M'}^{G_1}(N(u\alpha))$$

and

$$r_{M'}^{G_2}(N(us\alpha)) = r_{M'}^{G_2}(N(u\alpha)).$$

Hence in this case Proposition 7.1 follows from the case that s is equal to the identity.

Next we assume that g_1 is central and $a_1 \neq b_1$. Then from Lemma 6.6 we see that the left hand side of the identity in Proposition 7.1 vanishes. It's clear that the corresponding integrals on G_1 and G_2 also vanish. Thus we are done with the case that g_1 is the identity.

Now we suppose that $g_1 \notin Z(\mathrm{GL}(2))$. Then we can take

$$s = \left(1, \begin{pmatrix} c_1 & D \\ 1 & c_1 \end{pmatrix}, b_1, e_1\right)$$

and

$$u = \left(a, \begin{pmatrix} c & Dd \\ d & c \end{pmatrix}, a^{-1}, e\right)$$

topologically unipotent with $c^2 - Dd^2 = 1$. In this case, as remarked before Lemma 6.12, the other twisted conjugacy class within the twisted conjugacy class of $us\alpha$ does not intersect $M^0(R)$. The twisted integrals in this case have been computed in Lemma 6.12.

We now compute the integrals on G_1 and G_2.

Lemma 7.2. — *We have* $2r_{M'}^{G_1}(N(us\alpha)) = 0$ *unless* $b_1 = -1$ *and* $c_1 = 0$ *in which case it equals*

$$\max\{|a-c|_E, |d|_E\} \int_{|x|_E \leqslant \max\{|a-c|_E, |d|_E\}^{-1}} \log \max\{1, |x|_E\}.$$

Proof. — We have the norm of s in $\mathrm{GL}(2,E)'$ equal to

$$\begin{pmatrix} e(c_1 + \sqrt{D}) & \\ & eb_1(c_1 - \sqrt{D}) \end{pmatrix}.$$

If we let N' denote the unipotent radical of a Borel subgroup containing M' then we have $N' \cap Z_{G_1}(N(s\alpha)) = \{I\}$ unless

$$e(c_1 + \sqrt{D}) = eb_1(c_1 - \sqrt{D});$$

which is if and only if $c_1 = 0$ and $b_1 = -1$. Let $\beta = c + d\sqrt{D}$ then when $c_1 = 0$ and $b_1 = -1$ we have

$$\begin{aligned}
2r_{M'}^{G_1}(N(us\alpha)) &= |a\beta - a^{-1}\beta^{-1}|_E \int_{|x|_E \leqslant |a\beta - a^{-1}\beta^{-1}|_E^{-1}} \log\max\{1, |x|_E\} \\
&= |1 - a^{-1}\beta^{-1}|_E \int_{|x|_E \leqslant |1 - a^{-1}\beta^{-1}|_E^{-1}} \log\max\{1, |x|_E\} \\
&= \max\{|a - c|_E, |d|_E\} \int_{|x|_E \leqslant \max\{|a-c|_E, |d|_E\}^{-1}} \log\max\{1, |x|_E\}
\end{aligned}$$

as required. □

Lemma 7.3. — *We have* $2r_{M'}^{G_2}(N(us\alpha)) = 0$ *unless* $b_1 = 1$ *in which case it equals*

$$|a - 1| \int_{|x| \leqslant |a-1|^{-1}} \log\max\{1, |x|\}.$$

Proof. — We have the norm of s in $(\mathrm{GL}(2, F) \times E^\times)/F^\times$ equal to

$$\left(\begin{pmatrix} e & \\ & eb_1 \end{pmatrix}, c_1 + \sqrt{D} \right).$$

If we let N' denote the unipotent radical of a Borel subgroup containing M' then $N' \cap Z_{G_2}(N(s\alpha)) = \{I\}$ unless $b_1 = 1$. In this case we see from above that we have

$$2r_{M'}^{G_2}(N(us\alpha)) = |a - 1| \int_{|x| \leqslant |a-1|^{-1}} \log\max\{1, |x|\}$$

and we are done. □

So, unless either $b_1 = 1$ or $b_1 = -1$ and $c_1 = 0$, all integrals vanish and the fundamental lemma holds. If we have $b_1 = 1$ then by Lemma 6.12 the twisted integral is equal to

$$|a - 1| \int_{|x_3| \leqslant |a-1|^{-1}} \log\max\{1, |x_3|\}$$

and we are done in this case.

Now suppose that $b_1 = -1$ and $c_1 = 0$. Then by Lemma 6.12 we need to show that the integral of

$$2 \int \log\max\{1, |x_1|, |x_2|\}$$

over the region

- $(a - c)x_1 - dx_2 \in R$
- $-dDx_1 + (a - c)x_2 \in R$

is equal to

$$\int_{|x|_E \leqslant \max\{|a-c|_E, |d|_E\}^{-1}} \log \max\{1, |x|_E\}.$$

If we let $\max\{|a-c|, |d|\} = q^{-n}$, then this latter integral is equal to

$$2\left(nq^{2n} - \frac{q^{2n}-1}{q^2-1}\right).$$

We now turn to the first integral. As we saw in Section 6.3.4 this integral is equal to

$$2\int_{|x_1| \leqslant \max\{|a-c|, |d|\}^{-1}} \int_{|x_2| \leqslant \max\{|a-c|, |d|\}^{-1}} \log \max\{1, |x_1|, |x_2|\},$$

which equals, by Lemma 9.8,

$$2\left(nq^{2n} - \frac{q^{2n}}{q-1} + \frac{q^{2n+1}}{q^2-1} + \frac{1}{q^3-1} + \frac{q^2}{(q+1)(q^3-1)}\right)$$

which equals

$$2\left(nq^{2n} - \frac{q^{2n}-1}{q^2-1}\right).$$

The proof of Proposition 7.1 is now complete.

8. The fundamental lemma for the diagonal Levi

In this section we prove the fundamental lemma for M^0 equal to the diagonal torus in G^0 and M' equal to $\mathrm{GL}(1)^3$, the unique unramified elliptic twisted endoscopic group for M^0. The restriction of α to M^0 is given by

$$\alpha : (\mathrm{diag}(a,b,c,d), e) \longmapsto (\mathrm{diag}(d^{-1}, c^{-1}, b^{-1}, a^{-1}), abcde).$$

8.1. Statement of the fundamental lemma. — We note that for $\gamma = (\mathrm{diag}(a,b,c,d), e) \in M^0(F)$ and $m = (\mathrm{diag}(a_1,b_1,c_1,d_1), e_1) \in M^0(F)$ we have

$$m^{-1}\gamma\alpha(m) = (\mathrm{diag}((a_1d_1)^{-1}a, (b_1c_1)^{-1}b, (b_1c_1)^{-1}c, (a_1d_1)^{-1}d), a_1b_1c_1d_1e).$$

It's clear from this that the stable twisted conjugacy class of γ is equal to the twisted conjugacy class of γ. Therefore the fundamental lemma for the pair (M, M') is given by the following Proposition.

Proposition 8.1. — *For* $(\mathrm{diag}(a,b,c,d), e)\alpha \in M(F)$ *which is strongly* G^0*-regular we have*

$$r_M^G((\mathrm{diag}(a,b,c,d), e)\alpha) - 2r_{M'}^{\mathrm{GSp}(4)}(\mathrm{diag}(abe, ace, bde, cde))$$

equal to

$$2r_{M'}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))'}(\mathrm{diag}(abe, cde), \mathrm{diag}(ace, bde)) - r_{M'}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)}(\mathrm{diag}(1, a^{-1}d), \mathrm{diag}(abe, ace)).$$

We set $\mathrm{vol}(\mathfrak{a}_B^G/\mathbf{Z}(\Delta_B^\vee)) = 2/\ln q$ and normalize the other volumes as in Section 4.4.

8.2. Proof of the fundamental lemma. — As above for

$$m = (\mathrm{diag}(a_1, b_1, c_1, d_1), e_1) \in M^0(F)$$

we have

$$m^{-1}\gamma\alpha(m) = (\mathrm{diag}((a_1d_1)^{-1}a, (b_1c_1)^{-1}b, (b_1c_1)^{-1}c, (a_1d_1)^{-1}d), a_1b_1c_1d_1e).$$

Hence we see that the twisted conjugacy class of $\gamma = (\mathrm{diag}(a, b, c, d), e) \in M^0(F)$ intersects $M^0(R)$ if and only if we have $|a| = |d|$, $|b| = |c|$ and $|abe| = 1$. It's clear that unless these conditions are met then the same is true of the conjugacy class of $N(\gamma\alpha)$ in $M'(F)$. Thus we may as well assume that we have

$$\gamma = (\mathrm{diag}(a, b, c, d), e) \in M^0(R).$$

Under this assumption we have $2r_{M'}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))'}(\mathrm{diag}(abe, cde), \mathrm{diag}(ace, bde))$ equal to

$$2|ab - cd||ac - bd| \int_{|x|\leqslant|ab-cd|^{-1}} \log\max\{1, |x|\} \int_{|y|\leqslant|ac-bd|^{-1}} \log\max\{1, |y|\}$$

and $r_{M'}^{(\mathrm{GL}(2)\times\mathrm{GL}(2))/\,\mathrm{GL}(1)}(\mathrm{diag}(1, a^{-1}d), \mathrm{diag}(abe, ace))$ equal to

$$2|a - d||b - c| \int_{|x|\leqslant|a-d|^{-1}} \log\max\{1, |x|\} \int_{|y|\leqslant|b-c|^{-1}} \log\max\{1, |y|\}.$$

We now prove Proposition 8.1 using the twisted topological Jordan decomposition. As before we write $\gamma\alpha = us\alpha = s\alpha u$ and analyze the possibilities for s.

8.2.1. s equal to the identity. — We begin by proving Proposition 8.1 in the case that s is the identity. We take $\gamma = (u, e) \in \mathrm{Sp}(4, R) \times \mathrm{GL}(1, R)$ topologically unipotent. We write

$$u = \mathrm{diag}(a, b, b^{-1}, a^{-1}).$$

Then with the normalizations above we have, from Lemma 4.17, $r_M^G(\gamma\alpha) = 2r_{M'}^{\mathrm{GSp}(4)}(u)$. Thus in order to prove Proposition 8.1 in this case we need to prove that

$$2r_{M'}^{\mathrm{GSp}(4)}(\mathrm{diag}(a, b, b^{-1}, a^{-1})) - 2r_{M'}^{\mathrm{GSp}(4)}(\mathrm{diag}(ab, ab^{-1}, a^{-1}b, a^{-1}b^{-1}))$$

is equal to

$$2|ab - 1||a - b| \int_{|x|\leqslant|ab-1|^{-1}} \log\max\{1, |x|\} \int_{|y|\leqslant|a-b|^{-1}} \log\max\{1, |y|\}$$

minus

$$2|a - 1||b - 1| \int_{|x|\leqslant|a-1|^{-1}} \log\max\{1, |x|\} \int_{|y|\leqslant|b-1|^{-1}} \log\max\{1, |y|\}.$$

We have $|a - 1| < 1$ and $|b - 1| < 1$. Since we are in odd residual characteristic we have at least three of $|ab - 1|$, $|a - b|$, $|a - 1|$ and $|b - 1|$ equal. For $N \geqslant M$ we define $I(N, M)$ to be equal to $2r_{M'}^{\mathrm{GSp}(4)}(\mathrm{diag}(a, b, b^{-1}, a^{-1}))$ for a and b such that

$$|a - 1| = q^{-N}, \; |b - 1| = |a - b| = |ab - 1| = q^{-M}$$

and we define $I(M,N)$ to be equal to $2r_{M'}^{\mathrm{GSp}(4)}(\mathrm{diag}(a,b,b^{-1},a^{-1}))$ for a and b such that

$$|ab-1| = q^{-N},\ |a-1| = |b-1| = |a-b| = q^{-M}.$$

Using the action of the Weyl group in Sp(4) we see that in order to prove Proposition 8.1 in the case that s is the identity it suffices to prove the following Lemma.

Lemma 8.2. — *For $N \geqslant M$ we have $I(N,M) - I(M,N)$ equal to*

$$2q^{-2M}\left(Mq^M - \frac{q^M-1}{q-1}\right)\left(Mq^M - \frac{q^M-1}{q-1}\right)$$

minus

$$2q^{-N-M}\left(Nq^N - \frac{q^N-1}{q-1}\right)\left(Mq^M - \frac{q^M-1}{q-1}\right).$$

We now see how to compute $2r_{M'}^{\mathrm{GSp}(4)}(a,b,b^{-1},a^{-1})$. Using the notation of Lemma 4.7 we need to integrate

$$-(A^2+2B^2+2C^2+D^2+2E^2+F^2)+2(AB+AE+BD+CD+EF)$$

over the region

- $|x_1| \leqslant |a-b|^{-1}$
- $|x_4| \leqslant |b-1|^{-1}$
- $(ab-1)x_2 + b(a-b)x_1x_4 \in R$
- $(a^2-1)x_3 + ab^{-1}(1-b^2)x_1x_2 \in R$.

We assume that $|a-b| = |b-1|$. We first note that if $|x_1x_4| > |b-1|^{-1}$ then we must have both $|x_1| > 1$ and $|x_4| > 1$. Now

$$\begin{aligned}|x_1x_4| > |b-1|^{-1} &\Longrightarrow |x_2| = |ab-1|^{-1}|a-b||x_1x_4| \geqslant |x_1x_4| > |b-1|^{-1}\\ &\Longrightarrow |x_1x_2| > |b-1|^{-1}.\end{aligned}$$

So if $|x_1x_4| > |b-1|^{-1}$ then we have

$$x_2 = -(ab-1)^{-1}(a-b)bx_1x_4u$$

with $u \in U_F^{-v((b-1)x_1x_4)}$ and we have

$$\begin{aligned}x_3 &= -(a^2-1)^{-1}(1-b^2)ab^{-1}x_1x_2v\\ &= (a^2-1)^{-1}(ab-1)^{-1}(1-b^2)(a-b)ax_1^2x_4uv\end{aligned}$$

with $v \in U_F^{-v((b-1)x_1x_2)}$.

Now suppose that $|x_1x_4| \leqslant |b-1|^{-1}$, then we have $|x_2| \leqslant |ab-1|^{-1}$. Now if $|x_1x_2| > |b-1|^{-1}$ then we have

$$x_3 = -(a^2-1)^{-1}(1-b^2)ab^{-1}x_1x_2w$$

with $w \in U_F^{-v((b-1)x_1x_2)}$.

And finally if we have $|x_1x_4| \leqslant |b-1|^{-1}$ and $|x_1x_2| \leqslant |b-1|^{-1}$ then we have $|x_3| \leqslant |a-1|^{-1}$. So we have divided our region of integration into three regions. The first is given by

- $|x_1x_4| > |b-1|^{-1}$
- $x_2 = -(ab-1)^{-1}(a-b)bx_1x_4u$, $u \in U_F^{-v((b-1)x_1x_4)}$
- $x_3 = (a^2-1)^{-1}(ab-1)^{-1}(1-b^2)(a-b)ax_1^2x_4uv$, $v \in U_F^{-v((b-1)x_1x_2)}$.

The second is given by

- $|x_1x_4| \leqslant |b-1|^{-1}$
- $|x_2| \leqslant |ab-1|^{-1}$, $|x_1x_2| > |b-1|^{-1}$
- $x_3 = -(a^2-1)^{-1}(1-b^2)ab^{-1}x_1x_2w$, $w \in U_F^{-v((b-1)x_1x_2)}$.

And the third by

- $|x_1x_4| \leqslant |b-1|^{-1}$
- $|x_2| \leqslant |ab-1|^{-1}$, $|x_1x_2| \leqslant |b-1|^{-1}$
- $|x_3| \leqslant |a-1|^{-1}$.

We now compute $I(N,M) - I(M,N)$ over each of these three regions.

Region 1. — Over the first region we clearly have

$$B = \log|x_3|$$
$$C = \log|x_1|$$
$$E = \log|x_2|$$
$$F = \log|x_4|$$

for both $I(N,M)$ and $I(M,N)$.

Next we compute A over region 1 under the assumption that $|a-b| = |b-1| < 1$. We have

$$A = \log\max\{|x_2|, |x_3 - x_1x_2|, |x_2^2 - x_3x_4 + x_1x_2x_4|\}.$$

Now, $x_2^2 - x_3x_4 + x_1x_2x_4$ equals

$$x_1^2x_4^2u(ab-1)^{-2}(a^2-1)^{-1}(a-b)((a-b)(a^2-1)b^2u-(ab-1)(1-b^2)av-(ab-1)(a^2-1)b)$$

and

$$(a-b)(a^2-1)b^2 - (ab-1)(1-b^2)a - (ab-1)(a^2-1)b = (a-b)(1-b^2).$$

Therefore

$$|x_2^2 - x_3x_4 + x_1x_2x_4| = |x_1^2x_4^2||ab-1|^{-2}|a^2-1|^{-1}|b-1|^3$$

and so

$$A = \log\left(|x_1^2x_4^2||ab-1|^{-2}|a^2-1|^{-1}|b-1|^3\right).$$

For D we note that $x_3 + x_1x_2 + x_1^2x_4$ equals

$$x_1^2x_4(a^2-1)^{-1}(ab-1)^{-1}((1-b^2)(a-b)auv - (a^2-1)(a-b)bu + (a^2-1)(ab-1))$$

and

$$(1-b^2)(a-b)a-(a^2-1)(a-b)b+(a^2-1)(ab-1)=(b^2-1)(ab-1).$$

First we look at $I(N,M)$. In this case over region 1 we have

- $A=2\log|x_1|+2\log|x_4|+N-M$
- $B=2\log|x_1|+\log|x_4|+N-M$
- $C=\log|x_1|$
- $D=2\log|x_1|+\log|x_4|+N-M$
- $E=\log|x_1|+\log|x_4|$
- $F=\log|x_4|$

and so

$$-(A^2+2B^2+2C^2+D^2+2E^2+F^2)+2(AB+AE+BD+CD+EF)$$

equals

$$4(N-M)\log|x_1|+2(N-M)\log|x_4|+4(\log|x_1|)^2+8\log|x_1|\log|x_4|+2(\log|x_4|)^2.$$

Next we compute $I(M,N)$ over region 1. In this case we have

- $A=2\log|x_1|+2\log|x_4|+2(N-M)$
- $B=2\log|x_1|+\log|x_4|+N-M$
- $C=\log|x_1|$
- $E=\log|x_1|+\log|x_4|+N-M$
- $F=\log|x_4|$

For D, we have $x_3+x_1x_2+x_1^2x_4$ equal to

$$(a^2-1)^{-1}(ab-1)^{-1}x_1^2x_4u\left((1-b^2)(a-b)av+(a^2-1)(ab-1)u^{-1}-(a^2-1)(a-b)b\right).$$

Now

$$v=1+(b-1)^{-2}(ab-1)x_1^{-2}x_4^{-1}y$$

with $y\in R$, so

$$(1-b^2)(a-b)av=(1-b^2)(a-b)a+(1-b^2)(a-b)a(b-1)^{-2}(ab-1)x_1^{-2}x_4^{-1}y$$

and

$$|(1-b^2)(a-b)a(b-1)^{-2}(ab-1)x_1^{-2}x_4^{-1}y|=|(ab-1)x_1^{-2}x_4^{-1}y|<q^{-M-N}.$$

Since

$$(a-b^2)(a-b)a+(a^2-1)(ab-1)-(a^2-1)(a-b)b=(ab-1)(b^2-1)$$

we get

$$|x_3+x_1x_2+x_1^2x_4|=|x_1^2x_4|$$

and so $D=2\log|x_1|+\log|x_4|$. Therefore for $I(M,N)$ over region 1 we have

$$-(A^2+2B^2+2C^2+D^2+2E^2+F^2)+2(AB+AE+BD+CD+EF)$$

equal to

$4(N-M)\log|x_1| + 4(N-M)\log|x_4| + 4(\log|x_1|)^2 + 8\log|x_1|\log|x_4| + 2(\log|x_4|)^2.$

Hence we see that the contribution from region 1 to $I(N,M) - I(M,N)$ is equal to

$$2(M-N)q^{-M-N}\int_{1\leqslant|x_4|\leqslant q^M}\int_{q^M|x_4|^{-1}<|x_1|\leqslant q^M}\log|x_4|$$

which equals

$$2(M-N)q^{-M}\int_{1\leqslant|x_4|\leqslant q^M}(1-|x_4|^{-1})\log|x_4|.$$

Region 2. — We now compute the contribution from the integrals over region 2 to $I(N,M) - I(M,N)$. We begin by computing the contribution to $I(N,M)$. In this case region 2 is given by

- $|x_1|, |x_2|, |x_4| \leqslant q^M$
- $|x_1x_4| \leqslant q^M < |x_1x_2|$
- $x_3 = -(a^2-1)^{-1}(1-b^2)ab^{-1}x_1x_2w,\ w \in U_F^{-v((b-1)x_1x_2)}$.

We note that we have $|x_1|, |x_2| > 1$, $|x_4| < |x_2|$ and $|x_3| = q^{N-M}|x_1x_2| > q^N$. So we have

$$\begin{aligned}
A &= \log\max\{|x_2|, |x_3 - x_1x_2|, |x_2^2 - x_3x_4 + x_1x_2x_4|\}\\
B &= N - M + \log|x_1| + \log|x_2|\\
C &= \log|x_1|\\
D &= \log\max\{|x_1|^2, |x_3 + x_1x_2 + x_1^2x_4|\}\\
E &= \log|x_2|\\
F &= \log\max\{1, |x_4|\}.
\end{aligned}$$

For A we have

$$x_3 - x_1x_2 = -(a^2-1)^{-1}b^{-1}((1-b^2)aw + (a^2-1)b)x_1x_2$$

and since

$$(1-b^2)a + (a^2-1)b = (a-b)(1+ab)$$

so

$$|x_3 - x_1x_2| = q^{N-M}|x_1x_2|.$$

And we have

$$\begin{aligned}
x_2^2 - x_3x_4 + x_1x_2x_4 &= x_2^2 + (a^2-1)^{-1}(1-b^2)ab^{-1}wx_1x_2x_4 + x_1x_2x_4\\
&= x_2(x_2 + (a^2-1)^{-1}b^{-1}((1-b^2)aw + (a^2-1)b)x_1x_4).
\end{aligned}$$

We note that

$$(1-b^2)a + (a^2-1)b = (a-b)(1+ab)$$

and hence that $|(1-b^2)aw+(a^2-1)b| = q^{-M}$ for all w. Therefore after scaling x_1 and x_2 by suitable units, which doesn't affect B or E, we get

$$A = \log\max\{q^{N-M}|x_1x_2|, |x_2(x_2+\pi^{M-N}x_1x_4)|\}.$$

We now make the change of variables $x_4 \mapsto x_4 - \pi^{N-M}x_1^{-1}x_2$, which again doesn't affect B or E, to give

$$A = N - M + \log|x_1| + \log|x_2| + \log\max\{1, |x_4|\}.$$

For D we have

$$x_3 + x_1x_2 + x_1^2x_4 = x_1(((a^2-1)b - (1-b^2)aw)(a^2-1)^{-1}b^{-1}x_2 + x_1x_4).$$

Since

$$(a^2-1)b - (1-b^2)a = (a+b)(ab-1)$$

we have

$$|((a^2-1)b - (1-b^2)aw)(a^2-1)^{-1}b^{-1}| = q^{N-M}$$

for all w. Thus after scaling x_2 by a suitable unit we have

$$D = \log\max\{|x_1|^2, |x_1(\pi^{M-N}x_2 + x_1x_4)|\}.$$

So we have

$$\begin{aligned}
A &= N - M + \log|x_1| + \log|x_2| + \log\max\{1, |x_4|\}\\
B &= N - M + \log|x_1| + \log|x_2|\\
C &= \log|x_1|\\
D &= \log|x_1| + \log\max\{|x_1|, |\pi^{M-N}x_2 + x_1x_4|\}\\
E &= \log|x_2|\\
F &= \log\max\{1, |x_4|\}.
\end{aligned}$$

If we have $|x_2| > q^{2M-N}$ then

$$D = N - M + \log|x_1| + \log|x_2|$$

on the other hand if $|x_2| \leqslant q^{2M-N}$ then we can do the change of variables $x_4 \mapsto x_4 - \pi^{M-N}x_1^{-1}x_2$, which doesn't change the value of B or C, to give

$$D = 2\log|x_1| + \log\max\{1, |x_4|\}.$$

The difference between the integrand

$$-(A^2 + 2B^2 + 2C^2 + D^2 + 2E^2 + F^2) + 2(AB + AE + BD + CD + EF)$$

when

$$D = N - M + \log|x_1| + \log|x_2|$$

and

$$D = 2\log|x_1| + \log\max\{1, |x_4|\}$$

is

$$(N-M+\log|x_2|-\log\max\{1,|x_4|\})^2-(\log|x_1|)^2.$$

Lemma 8.3. — *The integral of*

$$(N-M+\log|x_2|-\log\max\{1,|x_4|\})^2-(\log|x_1|)^2$$

over the region

- $|x_1|,|x_4|\leqslant q^M$
- $|x_2|\leqslant q^{2M-N}$
- $|x_1x_4|\leqslant q^M<|x_1x_2|$

is zero.

Proof. — We assume that $N<2M$ so that this region is non-empty. We note that we must have $|x_1|>q^{N-M}$. The volume of x_1,x_2,x_4 such that $\log|x_1|=k$ with $N-M<k\leqslant M$ is

$$q^k(1-q^{-1})(q^{2M-N}-q^{M-k})q^{M-k}=(1-q^{-1})(q^{3M-N}-q^{2M-k}).$$

We now compute the volume of x_1,x_2,x_4 such that $\log|x_2|-\log\max\{1,|x_4|\}=M-N+k$, with $N-M<k\leqslant M$, is the sum of

$$q^{M-N+k}(1-q^{-1})(q^M-q^{N-k})=(1-q^{-1})(q^{2M-N+k}-q^M),$$

the contribution when $|x_4|\leqslant 1$, and

$$(1-q^{-1})^2(q^M-q^{N-k})\sum_{i=M-N+k+1}^{2M-N}q^i=(1-q^{-1})(q^M-q^{N-k})(q^{2M-N}-q^{M-N+k}),$$

the contribution when $|x_4|>1$. This sum equals

$$(1-q^{-1})(q^{3M-N}-q^{2M-k})$$

as required. □

Therefore we can assume that $D=N-M+\log|x_1|+\log|x_2|$ in all cases and then we have

$$-(A^2+2B^2+2C^2+D^2+2E^2+F^2)+2(AB+AE+BD+CD+EF)$$

equal to

$$2(N-M)\log|x_1x_2|+4(\log|x_1|+\log\max\{1,|x_4|\})\log|x_2|-2(\log\max\{1,|x_4|\})^2.$$

So $I(N,M)$ is equal to q^{-3M} times the integral of this function over the region

- $|x_1|,|x_2|,|x_4|\leqslant q^M$
- $|x_1x_4|\leqslant q^M<|x_1x_2|$.

We compute the contribution from the $2(N-M)\log|x_1x_2|$ term. If we make the change of variables $z=x_1x_2$ then the integral becomes

$$2(N-M)q^{-3M}\int_{1\leqslant|x_1|\leqslant q^M}\int_{|x_4|\leqslant q^M|x_1|^{-1}}\int_{q^M<|z|\leqslant q^M|x_1|}|x_1|^{-1}\log|z|,$$

which equals the integral of

$$2(N-M)q^{-M}\left((M+\log|x_1|)|x_1|^{-1}-M|x_1|^{-2}-\frac{|x_1|^{-1}-|x_1|^{-2}}{q-1}\right)$$

over $1\leqslant|x_1|\leqslant q^M$. We will compute the remaining terms when we compute $I(M,N)$ over region 2.

We now compute the contribution to $I(M,N)$ over region 2. This region is given by

- $|x_1|,|x_4|\leqslant q^M$, $|x_1x_4|\leqslant q^M$
- $|x_2|\leqslant q^N$, $|x_1x_2|>q^M$
- $x_3=-(a^2-1)^{-1}(1-b^2)ab^{-1}x_1x_2w$, $w\in U_F^{-v((b-1)x_1x_2)}$.

We note that we must have $|x_2|>1$, $|x_3|=|x_1x_2|$ and $|x_4|<|x_2|$. So we have

$$\begin{aligned}A&=\log\max\{|x_2|,|x_3-x_1x_2|,|x_2^2-x_3x_4+x_1x_2x_4|\}\\B&=\log\max\{|x_1x_2|,|x_2+x_1x_4|\}\\C&=\log\max\{1,|x_1|\}\\D&=\log\max\{1,|x_1|^2,|x_3+x_1x_2+x_1^2x_4|\}\\E&=\log|x_2|\\F&=\log\max\{1,|x_4|\}.\end{aligned}$$

We note that

$$B=\log|x_2|+\log\max\{1,|x_1|\}.$$

As we saw above $|x_3-x_1x_2|=|x_1x_2|$ and so

$$A=\log\max\{|x_2|,|x_1x_2|,|x_2^2-x_3x_4+x_1x_2x_4|\}.$$

We have

$$x_2^2-x_3x_4+x_1x_2x_4=x_2(x_2+(a^2-1)^{-1}b^{-1}x_1x_4((1-b^2)aw+(a^2-1)b))$$

and we note that

$$(1-b^2)a+(a^2-1)b=(a-b)(1+ab).$$

Hence

$$|(a^2-1)^{-1}b^{-1}x_1x_4((1-b^2)aw+(a^2-1)b|=|x_1x_4|$$

for all w. So after multiplying x_4 by a suitable unit we can take

$$A=\log|x_2|+\log\max\{1,|x_1|,|x_2-x_1x_4|\}.$$

Making the change of variables $x_4 \mapsto x_4 + x_1^{-1}x_2$ when $|x_2| \leqslant q^M$ gives

$$A = \begin{cases} 2\log|x_2|, & \text{if } |x_2| > q^M; \\ \log|x_1| + \log|x_2| + \log\max\{1,|x_4|\}, & \text{if } |x_2| \leqslant q^M. \end{cases}$$

Now we look at D. We have

$$x_3 + x_1x_2 + x_1^2x_4 = x_1((a^2-1)^{-1}b^{-1}((a^2-1)b - (1-b^2)aw)x_2 + x_1x_4).$$

We write

$$w = 1 + a^{-1}bx_1^{-1}x_2^{-1}x$$

with $|x| \leqslant q^M$. Then

$$\begin{aligned} x_3 + x_1x_2 + x_1^2x_4 &= x_1((a^2-1)^{-1}b^{-1}((a+b)(ab-1) + bx_1^{-1}x_2^{-1}x)x_2 + x_1x_4) \\ &= (a^2-1)^{-1}b^{-1}(a+b)(ab-1)x_1x_2 + (a^2-1)^{-1}(b^2-1)x + x_1^2x_4. \end{aligned}$$

Multiplying x_2 and x by suitable units gives

$$x_3 + x_1x_2 + x_1^2x_4 = \pi^{N-M}x_1x_2 + \pi^{-M}x + x_1^2x_4.$$

Now if $|x_1| > 1$ then this equals

$$x_1^2(x_4 + \pi^{N-M}x_1^{-1}x_2 + xx_1^{-2})$$

and we can make the change of variables $x_4 \to x_4 - \pi^{N-M}x_1^{-1}x_2 + \pi^{-M}xx_1^{-2}$ to get $x_1^2x_4$. On the other hand if $|x_1| \leqslant 1$ then we have

$$x + \pi^{N-M}x_1x_2 + x_1^2x_4$$

and we can make a change of variables $x \mapsto x - \pi^{N-M}x_1x_2 - x_1^2x_4$ to get x. So we have

$$D = \begin{cases} 2\log|x_1| + \log\max\{1,|x_4|\}, & \text{if } |x_1| > 1; \\ \log\max\{1,|x|\}, & \text{if } |x_1| \leqslant 1. \end{cases}$$

Putting this altogether gives

$$\begin{aligned} A &= \begin{cases} 2\log|x_2|, & \text{if } |x_2| > q^M, \\ \log|x_2| + \log|x_1| + \log\max\{1,|x_4|\}, & \text{if } |x_2| \leqslant q^M \end{cases} \\ B &= \log|x_2| + \log\max\{1,|x_1|\} \\ C &= \log\max\{1,|x_1|\} \\ D &= \begin{cases} 2\log|x_1| + \log\max\{1,|x_4|\}, & \text{if } |x_1| > 1, \\ \log\max\{1,|x|\}, & \text{if } |x_1| \leqslant 1 \end{cases} \\ E &= \log|x_2| \\ F &= \log\max\{1,|x_4|\}. \end{aligned}$$

and we need to integrate the function

$$-(A^2 + 2B^2 + 2C^2 + D^2 + 2E^2 + F^2) + 2(AB + AE + BD + CD + EF)$$

over the region

- $|x|, |x_1|, |x_4| \leqslant q^M$, $|x_2| \leqslant q^N$
- $|x_1x_4| \leqslant q^M < |x_1x_2|$.

When $|x_1| \leqslant 1$ the integrand is equal to

$$2(\log\max\{1,|x|\}+\log\max\{1,|x_4|\})\log|x_2|-(\log\max\{1,|x|\})^2-(\log\max\{1,|x_4|\})^2.$$

But if we have $|x_1| \leqslant 1$ then integrating over x is the same as integrating over x_4 and we can replace this function by

$$4\log\max\{1,|x_4|\}\log|x_2|-2(\log\max\{1,|x_4|\})^2.$$

If we now assume we have $|x_2| \leqslant q^M$ so that $|x_1| > 1$ then we are in the situation considered above, when computing $I(N,M)$, and we take our integrand to be

$$4(\log|x_1|+\log\max\{1,|x_4|\})\log|x_2|-2(\log\max\{1,|x_4|\})^2.$$

Finally we have the region $|x_2| > q^M$ and $|x_1| > 1$ then the integrand is equal to

$$4(\log\max\{1,|x_4|\}+\log|x_1|)\log|x_2|-2(\log\max\{1,|x_4|\})^2.$$

Thus we can take our integrand to be

$$4(\log\max\{1,|x_4|\}+\log\max\{1,|x_1|\})\log|x_2|-2(\log\max\{1,|x_4|\})^2$$

in all cases. Therefore the contribution to $I(M,N)$ from region 2 is given by q^{-N-2M} times the integral of this function over the region

- $|x_1|,|x_4| \leqslant q^M$, $|x_1x_4| \leqslant q^M$
- $|x_2| \leqslant q^N$, $|x_1x_2| > q^M$.

So the contribution from region 2 to $I(N,M)-I(M,N)$ is equal to the integral of

$$2(N-M)q^{-M}\left((M+\log|x_1|)|x_1|^{-1}-M|x_1|^{-2}-\frac{|x_1|^{-1}-|x_1|^{-2}}{q-1}\right)$$

over $1 \leqslant |x_1| \leqslant q^M$, plus the integral of

$$q^{-3M}\left(4(\log|x_1|+\log\max\{1,|x_4|\})\log|x_2|-2(\log\max\{1,|x_4|\})^2\right)$$

over the region

- $|x_1|,|x_2|,|x_4| \leqslant q^M$
- $|x_1x_4| \leqslant q^M < |x_1x_2|$

minus the integral of

$$q^{-N-2M}\left(4(\log\max\{1,|x_1|\}+\log\max\{1,|x_4|\})\log|x_2|-2(\log\max\{1,|x_4|\})^2\right)$$

over the region

- $|x_1|,|x_4| \leqslant q^M$, $|x_2| \leqslant q^N$
- $|x_1x_4| \leqslant q^M < |x_1x_2|$.

We now compute the difference of these integrals. We begin with the contribution from the $\log\max\{1,|x_1|\}\log|x_2|$ term. Given $|x_1| \geqslant 1$ the volume of x_4 is $q^M|x_1|^{-1}$. So over the first region we compute

$$q^{-2M}\int_{1<|x_1|\leqslant q^M}|x_1|^{-1}\log|x_1|\int_{q^M|x_1|^{-1}<|x_2|\leqslant q^M}\log|x_2|$$

while over the second we compute

$$q^{-N-M}\int_{1<|x_1|\leqslant q^M}|x_1|^{-1}\log|x_1|\int_{q^M|x_1|^{-1}<|x_2|\leqslant q^N}\log|x_2|.$$

The integral over x_2 over the first region gives

$$Mq^M-(M-\log|x_1|)q^M|x_1|^{-1}-\frac{q^M-q^M|x_1|^{-1}}{q-1}$$

while over the second region we get

$$Nq^N-(M-\log|x_1|)q^M|x_1|^{-1}-\frac{q^N-q^M|x_1|^{-1}}{q-1}.$$

Multiplying the first by q^{-2M} and the second by q^{-N-M} and subtracting gives

$$(M-N)q^{-M}+q^{-N-M}(q^M-q^N)(M-\log|x_1|)|x_1|^{-1}+q^{-N-M}\frac{q^N-q^M}{q-1}|x_1|^{-1},$$

which we then need to multiply by $|x_1|^{-1}\log|x_1|$ and integrate over $1\leqslant|x_1|\leqslant q^M$.

Next we consider the $\log\max\{1,|x_4|\}\log|x_2|$ term. Given $|x_4|\geqslant 1$ and x_2 with $|x_2|>|x_4|$ the volume of x_1 is $q^M(|x_4|^{-1}-|x_2|^{-1})$. So over the first region we need to compute

$$q^{-2M}\int_{1\leqslant|x_4|\leqslant q^M}\int_{|x_4|<|x_2|\leqslant q^M}|x_4|^{-1}\log|x_4|\log|x_2|-|x_2|^{-1}\log|x_4|\log|x_2|$$

while over the second we need to compute

$$q^{-N-M}\int_{1\leqslant|x_4|\leqslant q^M}\int_{|x_4|<|x_2|\leqslant q^N}|x_4|^{-1}\log|x_4|\log|x_2|-|x_2|^{-1}\log|x_4|\log|x_2|.$$

We consider the $|x_4|^{-1}\log|x_4|\log|x_2|$ term. Taking the difference over these two regions means we need to compute the integral of

$$\left((M-N)q^{-M}-(q^{-2M}-q^{-N-M})|x_4|\log|x_4|+\frac{q^{-2M}-q^{-N-M}}{q-1}|x_4|\right)|x_4|^{-1}\log|x_4|$$

over the region $1\leqslant|x_4|\leqslant q^M$. Next we consider the $|x_2|^{-1}\log|x_2|\log|x_4|$ term. Taking the difference over these two regions means we need to compute the integral of $(1-q^{-1})\log|x_4|$ times

$$\frac{N(N+1)}{2}q^{-N-M}-\frac{\log|x_4|(\log|x_4|+1)}{2}q^{-N-M}-\frac{M(M+1)}{2}q^{-2M}+\frac{\log|x_4|(\log|x_4|+1)}{2}q^{-2M}$$

over the region $1\leqslant|x_4|\leqslant q^M$.

Finally we consider the $(\log\max\{1,|x_4|\})^2$ term. Given $|x_4| = q^k \geqslant 1$ the volume of x_1 and x_2 in the first region is

$$\begin{aligned}\sum_{a=1}^{M-k} q^a(1-q^{-1}) \sum_{b=M-a+1}^{M} q^b(1-q^{-1}) &= \sum_{a=1}^{M-k} q^a(1-q^{-1})(q^M - q^{M-a})\\ &= (1-q^{-1})\left(\frac{q^{2M-k+1}-q^{M+1}}{q-1} - (M-k)q^M\right)\\ &= q^{2M-k} - q^M - (M-k)q^M(1-q^{-1}),\end{aligned}$$

while the volume of x_1 and x_4 in the second region is

$$\begin{aligned}\sum_{a=M-N+1}^{M-k} q^a(1-q^{-1}) \sum_{b=M-a+1}^{N} q^b(1-q^{-1}) &= \sum_{a=M-N+1}^{M-k} q^a(1-q^{-1})(q^N - q^{M-a})\\ &= q^{N+M-k} - q^M - (N-k)(1-q^{-1})q^M.\end{aligned}$$

Thus in computing the difference between the two regions we need to integrate

$$\big(q^{-N-M} - q^{-2M} + (N - \log|x_4|)(1-q^{-1})q^{-N-M} - (M-\log|x_4|)q^{-2M}(1-q^{-1})\big) \cdot (\log|x_4|)^2$$

over $1 \leqslant |x_4| \leqslant q^M$. Adding this altogether gives the contribution to $I(N,M) - I(M,N)$ over region 2 as the integral of the sum of

$$6(M-N)q^{-M}|x|^{-1}\log|x|,$$

$$4q^{-N-M}\left(M(q^M-q^N) + \frac{q^N-q^M}{q-1}\right)|x|^{-2}\log|x|,$$

$$2\Big(2\frac{q^{-2M}-q^{-N-M}}{q-1} + N(N+1)(1-q^{-1})q^{-N-M} - M(M+1)(1-q^{-1})q^{-2M}\Big)\log|x|,$$

$$4q^{-N-M}(q^N-q^M)|x|^{-2}(\log|x|)^2,$$

$$2(Mq^{-2M} - (M+1)q^{-2M-1} - Nq^{-N-M} + (N+1)q^{-N-M-1})(\log|x|)^2,$$

and

$$2(N-M)q^{-M}\left(M|x|^{-1} - M|x|^{-2} - \frac{|x|^{-1}-|x|^{-2}}{q-1}\right)$$

over the region $1 \leqslant |x| \leqslant q^M$.

Region 3. — We have $I(N,M)$ given by q^{-N-3M} times the integral of

$$-(A^2+2B^2+2C^2+D^2+2E^2+F^2) + 2(AB+AE+BD+CD+EF)$$

over the region

- $|x_1|, |x_2|, |x_4| \leqslant q^M$
- $|x_1x_4|, |x_1x_2| \leqslant q^M$
- $|x_3| \leqslant q^N$.

And we have $I(M, N)$ given by q^{-N-3M} times the integral of the same function over the region

- $|x_1|, |x_4| \leqslant q^M$, $|x_1 x_4| \leqslant q^M$
- $|x_2| \leqslant q^N$, $|x_1 x_2| \leqslant q^M$
- $|x_3| \leqslant q^M$.

Thus after removing the common region we need to compute the integral of our function over the region

- $|x_1|, |x_2|, |x_4| \leqslant q^M$
- $|x_1 x_4|, |x_1 x_2| \leqslant q^M$
- $q^M < |x_3| \leqslant q^N$

and subtract from it the integral over the region

- $|x_1|, |x_4| \leqslant q^M$, $|x_1 x_4| \leqslant q^M$
- $q^M < |x_2| \leqslant q^N$, $|x_1 x_2| \leqslant q^M$
- $|x_3| \leqslant q^M$.

We first compute the integrand over the first of these subregions. We have

$$\begin{aligned}
A &= \log \max\{|x_3|, |x_2^2 - x_3 x_4 + x_1 x_2 x_4|\} \\
B &= \log |x_3| \\
C &= \log \max\{1, |x_1|\} \\
D &= \log \max\{1, |x_1|^2, |x_3 + x_1 x_2 + x_1^2 x_4|\} \\
E &= \log \max\{1, |x_2|, |x_4|\} \\
F &= \log \max\{1, |x_4|\}.
\end{aligned}$$

After the change of variables $x_3 \mapsto x_3 \pm x_1 x_2$, which doesn't change the region of integration, we have

$$\begin{aligned}
A &= \log \max\{|x_3|, |x_2^2 - x_3 x_4|\} \\
B &= \log |x_3| \\
C &= \log \max\{1, |x_1|\} \\
D &= \log \max\{1, |x_1|^2, |x_3 + x_1^2 x_4|\} \\
E &= \log \max\{1, |x_2|, |x_4|\} \\
F &= \log \max\{1, |x_4|\}.
\end{aligned}$$

We can make the change of variables $x_4 \mapsto x_4 + x_2^2 x_3^{-1}$ which doesn't alter E since $x_2 x_3^{-1} \in R$ to get

$$\begin{aligned}
A &= \log|x_3| + \log\max\{1,|x_4|\} \\
B &= \log|x_3| \\
C &= \log\max\{1,|x_1|\} \\
D &= \log\max\{|x_1|^2, |x_3 + x_1^2 x_4|\} \\
E &= \log\max\{1,|x_2|,|x_4|\} \\
F &= \log\max\{1,|x_4|\}.
\end{aligned}$$

If $|x_3| > q^M|x_1|$ then $D = \log|x_3|$. On the other hand if $|x_3| \leqslant q^M|x_1|$ then we can do a change of variables in x_4 to get

$$D = 2\log|x_1| + \log\max\{1,|x_4|\}.$$

The difference in the integrand between taking

$$D = 2\log|x_1| + \log\max\{1,|x_4|\}$$

and taking $D = \log|x_3|$ is

$$(\log|x_3| - (\log|x_1| + \log\max\{1,|x_4|\}))^2 - (\log|x_1|)^2.$$

Lemma 8.4. — *The integral of*

$$(\log|x_3| - (\log|x_1| + \log\max\{1,|x_4|\}))^2 - (\log|x_1|)^2$$

over the region

- $1 \leqslant |x_1| \leqslant q^M$
- $|x_2|, |x_4| \leqslant q^M|x_1|^{-1}$
- $q^M < |x_3| \leqslant \min\{q^M|x_1|, q^N\}$

is zero.

Proof. — We fix k with $0 \leqslant k \leqslant M$ and set $M_1 = \min\{M+k, N\}$. The volume of x_1, x_2, x_3, x_4 with $|x_1| = q^k$ is

$$q^k(1-q^{-1})q^{M-k}q^{M-k}(q^{M_1} - q^M) = (1-q^{-1})(q^{2M+M_1-k} - q^{3M-k}).$$

Now we compute the volume of x_1, x_2, x_3, x_4 such that $\log|x_3| - (\log|x_1| + \log\max\{1,|x_4|\}) = k$. If $|x_4| \leqslant 1$ then the volume is

$$\sum_{i=M-k+1}^{M_1-k} q^i(1-q^{-1})q^{M-i}q^{k+i}(1-q^{-1}) = (1-q^{-1})(q^{M+M_1} - q^{2M}).$$

Now assume that $|x_4| > 1$. Then the region is given by

- $1 < |x_4| \leqslant q^{M-k}$
- $q^{M-k}|x_4|^{-1} < |x_1| \leqslant q^M|x_4|^{-1}$
- $q^M < |x_3| \leqslant \min\{q^M|x_1|^{-1}, q^N\}$.

So we need
$$q^M < |x_3| = q^k|x_1x_4| \leqslant \min\{q^M|x_1|^{-1}, q^N\}.$$
So the total volume of x_1, x_2, x_3, x_4 with $|x_4| > 1$ and $|x_3| = q^k|x_1x_4|$ is
$$\sum_{i=1}^{M-k} q^i(1-q^{-1}) \sum_{j=M-k-i+1}^{M_1-k-i} q^j(1-q^{-1})q^{M-j}q^{k+i+j}(1-q^{-1}),$$
which equals
$$(1-q^{-1})(q^{2M+M_1-k} - q^{3M-k} - q^{M+M_1} + q^{2M})$$
as required. □

By this lemma we can assume that we have $D = \log|x_3|$. Then over the first subregion we have
$$-(A^2 + 2B^2 + 2C^2 + D^2 + 2E^2 + F^2) + 2(AB + AE + BD + CD + EF)$$
equal to
$$2(\log\max\{1, |x_1|\} + \log\max\{1, |x_2|, |x_4|\})\log|x_3|$$
plus
$$-2(\log\max\{1, |x_1|\})^2 - 2(\log\max\{1, |x_2|, |x_4|\} - \log\max\{1, |x_4|\})^2.$$

The contribution from the $2\log\max\{1, |x_1|\}\log|x_3|$ term is
$$2q^{2M}\left(Nq^N - Mq^M - \frac{q^N - q^M}{q-1}\right)\int_{1\leqslant|x_1|\leqslant q^M} |x_1|^{-2}\log|x_1|.$$
The contribution from the $-2(\log\max\{1, |x_1|\})^2$ term is
$$-2q^{2M}(q^N - q^M)\int_{1\leqslant|x_1|\leqslant q^M} |x_1|^{-2}(\log|x_1|)^2.$$
The contribution from the $2\log\max\{1, |x_2|, |x_4|\}\log|x_3|$ term is
$$4q^M\left(Nq^N - Mq^M - \frac{q^N - q^M}{q-1}\right)\int_{1\leqslant|x_4|\leqslant q^M} \log|x_4|$$
plus
$$-2q^M(1-q^{-1})\left(Nq^N - Mq^M - \frac{q^N - q^M}{q-1}\right)\int_{1\leqslant|x_4|\leqslant q^M} \log|x_4|.$$
The contribution from the $-2(\log\max\{1, |x_2|, |x_4|\} - \log\max\{1, |x_4|\})^2$ term is
$$-2q^M(q^N - q^M)\int_{|x_4|\leqslant|x_2|\leqslant q^M} |x_2|^{-1}(\log\max\{1, |x_2|\} - \log\max\{1, |x_4|\})^2,$$
which equals
$$-2q^{M-1}(q^N - q^M)\int_{1\leqslant|x_2|\leqslant q^M} |x_2|^{-1}(\log|x_2|)^2$$

plus

$$-2q^M(q^N - q^M)\int_{1\leqslant|x_4|\leqslant|x_2|\leqslant q^M} |x_2|^{-1}(\log|x_2x_4^{-1}|)^2.$$

Making the change of variables $y = x_2x_4^{-1}$, this latter term equals

$$-2q^M(q^N - q^M)\int_{1\leqslant|y|\leqslant q^M} |y|^{-1}(q^M|y|^{-1} - q^{-1}(\log|y|)^2.$$

Adding this altogether gives the integral of

$$-(A^2 + 2B^2 + 2C^2 + D^2 + 2E^2 + F^2) + 2(AB + AE + BD + CD + EF)$$

over the first subregion as the integral of

$$2q^{2M}\left(Nq^N - Mq^M - \frac{q^N - q^M}{q-1}\right)|x|^{-2}\log|x| - 4q^{2M}(q^N - q^M)|x|^{-2}(\log|x|)^2$$

plus

$$2q^{M-1}(q+1)\left(Nq^N - Mq^M - \frac{q^N - q^M}{q-1}\right)\log|x|$$

over $1 \leqslant |x| \leqslant q^M$.

We now compute the integral over the second subregion. This subregion is given by

- $|x_3|, |x_4| \leqslant q^M$
- $q^M < |x_2| \leqslant q^N$
- $|x_1| \leqslant q^M|x_2|^{-1}$.

We note that $|x_1| < 1$ and we have, after a change of variables,

$$\begin{aligned}
A &= 2\log|x_2|\\
B &= \log|x_2|\\
C &= 0\\
D &= \log\max\{1, |x_3|\}\\
E &= \log|x_2|\\
F &= \log\max\{1, |x_4|\}.
\end{aligned}$$

And

$$-(A^2 + 2B^2 + 2C^2 + D^2 + 2E^2 + F^2) + 2(AB + AE + BD + CD + EF)$$

equals

$$\begin{aligned}
&-(\log\max\{1, |x_4|\})^2 + 2\log|x_2|\log\max\{1, |x_4|\} + 2\log|x_2|\log\max\{1, |x_3|\}\\
&\qquad - (\log\max\{1, |x_3|\})^2.
\end{aligned}$$

But integrating over x_4 is the same as integrating over x_3. Hence we can replace this function by

$$-2(\log\max\{1, |x_4|\})^2 + 4\log|x_2|\log\max\{1, |x_4|\}.$$

Thus to compute the integral we need to multiply this function by $q^{2M}|x_2|^{-1}$ and integrate over $q^M < |x_2| \leqslant q^N$ and $|x_4| \leqslant q^M$. We have

$$\int_{q^M<|x_2|\leqslant q^N} |x_2|^{-1} = (N-M)(1-q^{-1}),$$

and so the integral of $-2(\log\max\{1,|x_4|\})^2$ yields

$$-2(N-M)q^{2M}(1-q^{-1})\int_{1\leqslant|x_4|\leqslant q^M} (\log|x_4|)^2.$$

We have

$$\int_{q^M<|x_2|\leqslant q^N} |x_2|^{-1}\log|x_2| = \left(\frac{N(N+1)}{2}-\frac{M(M+1)}{2}\right)(1-q^{-1})$$

and so the contribution of $4\log|x_2|\log\max\{1,|x_4|\}$ is

$$2\left(N(N+1)-M(M+1)\right)q^{2M}(1-q^{-1})\int_{1\leqslant|x_4|\leqslant q^M}\log|x_4|.$$

Thus the integral over the second subregion is equal to the integral over $1\leqslant|x|\leqslant q^M$ of

$$2\left(N(N+1)-M(M+1)\right)q^{2M}(1-q^{-1})\log|x| - 2(N-M)q^{2M}(1-q^{-1})(\log|x|)^2.$$

Combining this together we get that the contribution to $I(N,M)-I(M,N)$ over region 3 is equal to q^{-3M-N} times the integral over $1\leqslant|x|\leqslant q^M$ of the sum of

$$\begin{gathered}
2q^{2M}\left(Nq^N - Mq^M - \frac{q^N-q^M}{q-1}\right)|x|^{-2}\log|x|,\\
2q^{M-1}(q+1)\left(Nq^N - Mq^M - \frac{q^N-q^M}{q-1}\right)\log|x|,\\
-2\left(N(N+1)-M(M+1)\right)q^{2M}(1-q^{-1})\log|x|,
\end{gathered}$$

and
$$2q^{2M}\left((N-M)(1-q^{-1})-2(q^N-q^M)|x|^{-2}\right)(\log|x|)^2.$$

Putting it altogether. — Gathering together the computations above we get that $I(N,M)-I(M,N)$ is equal to the integral of the sum of

$$2(M-N)q^{-M}\log|x|,$$
$$2\big((-(M^2+M-N)q+(N+M^2+M+1))q^{-2M-1} +(M^2q-(M+1)^2)q^{-N-M-1}\big)\log|x|,$$
$$4(M-N)q^{-M}|x|^{-1}\log|x|,$$
$$2q^{-N-M}\Big(Mq^M+(N-2M)q^N+\frac{q^N-q^M}{q-1}\Big)|x|^{-2}\log|x|,$$
$$2(M(q^{-2M}-q^{-N-M})-(M+1)(q^{-2M-1}-q^{-N-M-1}))(\log|x|)^2,$$

and
$$2M(N-M)q^{-M}|x|^{-1}-2M(N-M)q^{-M}|x|^{-2}-2(N-M)q^{-M}\frac{|x|^{-1}}{q-1} +2(N-M)q^{-M}\frac{|x|^{-2}}{q-1}$$

over the region $1\leqslant|x|\leqslant q^M$. Using the results of Section 9 we compute this integral to be equal to

$$2q^{-2M}\left(Mq^M-\frac{q^M-1}{q-1}\right)\left(Mq^M-\frac{q^M-1}{q-1}\right)$$

minus

$$2q^{-N-M}\left(Nq^N-\frac{q^N-1}{q-1}\right)\left(Mq^M-\frac{q^M-1}{q-1}\right)$$

and the proof of Lemma 8.2 is now complete.

8.2.2. s not equal to the identity. — We now assume that s is not the identity. After twisted conjugation we may assume that we have

$$s=\left(\begin{pmatrix}1&&&\\&1&&\\&&a_1&\\&&&b_1\end{pmatrix},c_1\right)$$

with $a_1^k=b_1^k=c_1^k=1$ for some k prime to the residual characteristic of F and with a_1 and b_1 not both 1. Since M^0 is abelian $u\in M(F)$ commutes with $s\alpha$ if and only if $\alpha(u)=u$ and hence if and only if u is of the form

$$u=\left(\begin{pmatrix}a&&&\\&b&&\\&&b^{-1}&\\&&&a^{-1}\end{pmatrix},e\right).$$

We take N equal to the unipotent radical of the upper triangular Borel in G^0 and compute the possibilities for $N_1=N\cap Z_{G^0}(s\alpha)$. By abuse of notation we consider $N\subset \mathrm{GL}(4)$.

Lemma 8.5. — *With notation as above we have the following possibilities for N_1.*

(1) *If $a_1 = 1$ then we have*

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & x_4 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\}.$$

(2) *If $a_1 = b_1 = -1$ then we have*

$$N_1 = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & -x_1x_2 \\ & 1 & 0 & -x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix} \right\}.$$

(3) *If $a_1 \neq \pm 1$ and $b_1 = a_1$ then we have*

$$N_1 = \left\{ \begin{pmatrix} 1 & x_1 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix} \right\}.$$

(4) *If $a_1 \neq \pm 1$ and $b_1 = a_1^{-1}$, then we have*

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & x_2 & 0 \\ & 1 & 0 & a_1x_2 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\}.$$

(5) *If $b_1 = 1$ then we have*

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & x_3 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\}.$$

(6) *In all other cases we have $N_1 = \{I\}$.*

Proof. — We take

$$n = \begin{pmatrix} 1 & x_1 & x_2 + x_1x_4 & x_3 \\ & 1 & x_4 & x_5 \\ & & 1 & x_6 \\ & & & 1 \end{pmatrix}.$$

We have

$$\alpha(n) = \begin{pmatrix} 1 & -x_6 & x_5 - x_4x_6 & x_3 - x_2x_6 - x_1x_5 \\ & 1 & x_4 & x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix}$$

and

$$s^{-1}ns = \begin{pmatrix} 1 & x_1 & a_1(x_2+x_1x_4) & b_1x_3 \\ & 1 & a_1x_4 & b_1x_5 \\ & & 1 & b_1a_1^{-1}x_6 \\ & & & 1 \end{pmatrix}.$$

We now find n such that $\alpha(n) = s^{-1}ns$. First we note that

- $x_1 = -x_6$
- $x_1 = x_6 = 0$ unless $a_1 = b_1$
- $x_4 = 0$ unless $a_1 = 1$.

Let's first assume that $a_1 = 1$. Then we have $b_1 \neq 1$ and so $x_1 = x_6 = 0$, $x_2 = x_5 = 0$ and $x_3 = 0$. We now assume that we have $a_1 \neq 1$. Therefore we must have $x_4 = 0$. We have $x_2 = x_5 = 0$ unless $a_1 = b_1^{-1}$ and we also need to have

$$(1-b_1)x_3 = x_2x_6 + x_1x_5 = (b_1^{-1}-1)x_1x_2.$$

The result now follows. □

We now compute the integral $r_M^G(us\alpha)$ in each of these cases.

Lemma 8.6. — *With notation as above we have the following possibilities for $r_M^G(us\alpha)$.*

(1) *If $a_1 = b_1 = -1$ then*

$$r_M^G(us\alpha) = 4|a-b||ab-1| \int_{|x_1|\leqslant|a-b|^{-1}} \log\max\{1,|x_1|\} \int_{|x_2|\leqslant|ab-1|^{-1}} \log\max\{1,|x_2|\}.$$

(2) *In all other cases $r_M^G(us\alpha) = 0$.*

Proof. — We let $n \in N_1(F)$ and compute $v_M(n)$. When $a_1 = 1$ we have $n \in \mathrm{Sp}(4)$ and $v_M(n) = 0$ by Corollary 4.8. Similarly when $a_1 \neq \pm 1$ and $b_1 = a_1$ we have $n \in \mathrm{Sp}(4)$ and $v_M(n) = 0$ by Corollary 4.8. When $a_1 \neq \pm1$ and $b_1 = a_1^{-1}$ we have

$$\begin{pmatrix} 1 & 0 & x_2 & 0 \\ & 1 & 0 & a_1x_2 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & a_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & x_2 & 0 \\ & 1 & 0 & x_2 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & a_1 \end{pmatrix}$$

and

$$v_M \begin{pmatrix} 1 & 0 & x_2 & 0 \\ & 1 & 0 & x_2 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} = 0$$

by Corollary 4.8. Finally when $a_1 = b_1 = -1$ we have

$$n = \begin{pmatrix} 1 & x_1 & x_2 & -x_1x_2 \\ & 1 & 0 & -x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix}$$

and one can compute as in the proof of Lemma 4.7 that

$$v_M(n) = 4\log\max\{1,|x_1|\}\log\max\{1,|x_2|\}.$$

Moreover for $u = \mathrm{diag}(a,b,a^{-1},b^{-1})$ we have

$$u^{-1}n^{-1}un = \begin{pmatrix} 1 & (1-a^{-1}b)x_1 & (1-a^{-1}b^{-1})x_2 & -(1-a^{-1}b)(1-a^{-1}b^{-1})x_1x_2 \\ & 1 & 0 & -(1-a^{-1}b^{-1})x_2 \\ & & 1 & -(1-a^{-1}b)x_1 \\ & & & 1 \end{pmatrix}$$

and the result now follows. □

We now consider the integral on GSp(4). We have $N(us\alpha)$ equal to the product of

$$s_1 = c_1 \begin{pmatrix} 1 & & & \\ & a_1 & & \\ & & b_1 & \\ & & & a_1b_1 \end{pmatrix}$$

and

$$\begin{pmatrix} ab & & & \\ & ab^{-1} & & \\ & & a^{-1}b & \\ & & & a^{-1}b^{-1} \end{pmatrix}.$$

We take N' equal to the unipotent radical of the upper triangular Borel in GSp(4) and we compute the possibilities for $N'_1 = N \cap Z_{\mathrm{GSp}(4)}(s_1)$.

Lemma 8.7. — *With notation as above we have the following possibilities for N'_1.*

(1) *If $a_1 = 1$ then we have*

$$N'_1 = \left\{\begin{pmatrix} 1 & x_1 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix}\right\}.$$

(2) *If $a_1 = b_1 = -1$ then we have*

$$N'_1 = \left\{\begin{pmatrix} 1 & 0 & 0 & x_3 \\ & 1 & x_4 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\right\}.$$

(3) *If $a_1 \neq \pm 1$ and $b_1 = a_1$ then we have*

$$N'_1 = \left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & x_4 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\right\}.$$

(4) *If* $a_1 \neq \pm 1$ *and* $b_1 = a_1^{-1}$, *then we have*

$$N_1' = \left\{ \begin{pmatrix} 1 & 0 & 0 & x_3 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\}.$$

(5) *If* $b_1 = 1$ *then we have*

$$N_1' = \left\{ \begin{pmatrix} 1 & 0 & x_2 & 0 \\ & 1 & 0 & x_2 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\}.$$

(6) *In all other cases we have* $N_1' = \{I\}$.

Proof. — We take

$$n = \begin{pmatrix} 1 & x_1 & x_2 + x_1 x_4 & x_3 \\ & 1 & x_4 & x_2 \\ & & 1 & -x_1 \\ & & & 1 \end{pmatrix}$$

and we have

$$s_1^{-1} n s_1 = \begin{pmatrix} 1 & a_1 x_1 & b_1(x_2 + x_1 x_4) & a_1 b_1 x_3 \\ & 1 & b_1 a_1^{-1} x_4 & b_1 x_2 \\ & & 1 & -a_1 x_1 \\ & & & 1 \end{pmatrix}.$$

So if we have $s_1^{-1} n s_1 = n$ then we have the following implications

- $a_1 \neq 1$: $x_1 = 0$
- $b_1 \neq 1$: $x_2 = 0$
- $a_1 \neq b_1^{-1}$: $x_3 = 0$
- $a_1 \neq b_1$: $x_4 = 0$

and the result now follows. □

We now compute the integral $2r_{M'}^{\mathrm{GSp}(4)}(N(\gamma\alpha))$ in each of these cases.

Lemma 8.8. — *With notation as above we have the following.*

(1) *If* $a_1 = b_1 = -1$ *then* $2r_{M'}^{\mathrm{GSp}(4)}(N(\gamma\alpha))$ *is equal to*

$$2|a-b||ab-1| \int_{|x_4| \leqslant |a-b|^{-1}} \log\max\{1, |x_4|\} \int_{|x_3| \leqslant |ab-1|^{-1}} \log\max\{1, |x_3|\}.$$

(2) *In all other cases we have* $2r_{M'}^{\mathrm{GSp}(4)}(N(\gamma\alpha)) = 0$.

Proof. — We let $n \in N_1(F)$. Suppose first that $a_1 = b_1 = -1$ then

$$v_{M'}\begin{pmatrix}1 & 0 & 0 & x_3\\ & 1 & x_4 & 0\\ & & 1 & 0\\ & & & 1\end{pmatrix} = 2\log\max\{1,|x_3|\}\log\max\{1,|x_4|\}.$$

Now for $u = \operatorname{diag}(ab, ab^{-1}, a^{-1}b, a^{-1}b^{-1})$ and n as above we have

$$u^{-1}n^{-1}un = \begin{pmatrix}1 & 0 & 0 & (1-a^{-2}b^{-2})x_3\\ & 1 & (1-a^{-2}b^{2})x_4 & 0\\ & & 1 & 0\\ & & & 1\end{pmatrix}$$

and the result is clear in this case. In all other cases one can check that for $n \in N_1'$ we have $v_{M'}(n) = 0$. □

Finally we consider the integrals on $(\mathrm{GL}(2)\times\mathrm{GL}(2))'$ and $(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)$. We have $\gamma = us = \operatorname{diag}(a, b, a_1b^{-1}, b_1a^{-1})$ and as we saw above the integral on $(\mathrm{GL}(2)\times\mathrm{GL}(2))'$ is equal to $2|ab - a_1b^{-1}b_1a^{-1}||aa_1b^{-1} - bb_1a^{-1}|$ times

$$\int_{|x|\leqslant|ab-a_1b^{-1}b_1a^{-1}|^{-1}} \log\max\{1,|x|\}\int_{|y|\leqslant|aa_1b^{-1}-bb_1a^{-1}|^{-1}}\log\max\{1,|y|\}.$$

and the integral on $(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)$ is equal to

$$2|a-b_1a^{-1}||b-a_1b^{-1}|\int_{|x|\leqslant|a-b_1a^{-1}|^{-1}}\log\max\{1,|x|\}\int_{|y|\leqslant|b-a_1b^{-1}|^{-1}}\log\max\{1,|y|\}.$$

Now it's clear that the integral on $(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1)$ vanishes unless we have $a_1 = b_1 = 1$ and the integral on $(\mathrm{GL}(2)\times\mathrm{GL}(2))'$ vanishes unless $a_1 = b_1 = \pm 1$ in which case it is equal to

$$2|ab-1||a-b|\int_{|x|\leqslant|ab-1|^{-1}}\log\max\{1,|x|\}\int_{|y|\leqslant|a-b|^{-1}}\log\max\{1,|y|\}.$$

The fundamental lemma is now proven!

9. Some p-adic integrals

In this section we compute certain p-adic integrals that were required in the proof of the fundamental lemma. All these integrals are over open subsets of F^n. In each case we take the measure on F^n that gives R^n volume one; and we suppress it from our notation.

Lemma 9.1. — *For $k \geqslant 0$ we have*

$$\int_{1\leqslant|x|\leqslant q^k}\log|x| = kq^k - \frac{q^k-1}{q-1}.$$

Proof. — We have

$$\int_{1\leqslant|x|\leqslant q^k}\log|x| = \sum_{i=0}^{k} iq^i(1-q^{-1}) = kq^k - \sum_{i=0}^{k-1} q^i = kq^k - \frac{q^k-1}{q-1}$$

as wished. □

As a corollary we have the following.

Lemma 9.2. — *Assume that* $0 \leqslant a \leqslant b$ *then*

$$\int_{q^a<|x|\leqslant q^b}\log|x| = bq^b - aq^a - \frac{q^b-q^a}{q-1}.$$

Lemma 9.3. — *Let* $M \geqslant 0$. *Then we have*

$$\int_{1\leqslant|x|\leqslant q^M}|x|^k\log|x| = (1-q^{-1})\left(M\frac{q^{(M+1)(k+1)}}{q^{k+1}-1} - \frac{q^{(M+1)(k+1)}-q^{k+1}}{(q^{k+1}-1)^2}\right)$$

if $k \neq -1$ *and*

$$\int_{1\leqslant|x|\leqslant q^M}|x|^{-1}\log|x| = \frac{M(M+1)}{2}(1-q^{-1}).$$

Proof. — We have

$$\int_{1<|x|\leqslant q^M}|x|^k\log|x| = (1-q^{-1})\sum_{m=1}^{M} mq^{(k+1)m}.$$

If $k=-1$ then it's clear that this integral is equal to

$$\frac{M(M+1)}{2}(1-q^{-1}).$$

On the other hand if $k \neq -1$ then we have

$$\begin{aligned}\sum_{m=1}^{M} mq^{(k+1)m} &= \sum_{m=1}^{M}\frac{q^{(M+1)(k+1)}-q^{m(k+1)}}{q^{k+1}-1}\\ &= \left(M\frac{q^{(M+1)(k+1)}}{q^{k+1}-1} - \sum_{m=1}^{M}\frac{q^{m(k+1)}}{q^{k+1}-1}\right)\\ &= \left(M\frac{q^{(M+1)(k+1)}}{q^{k+1}-1} - \frac{q^{(M+1)(k+1)}-q^{k+1}}{(q^{k+1}-1)^2}\right)\end{aligned}$$

as wished. □

Lemma 9.4. — *Let* $M \geqslant 0$ *then we have*

$$\int_{|x|\leqslant q^M}\int_{|s|\leqslant q^M}|x|\log\max\{1,|x|,|s|\}$$

equal to

$$\frac{q}{q+1}\left(Mq^{3M}-\frac{q^{3M}-1}{q^3-1}\right).$$

Proof. — We write this integral as the sum of

$$\int_{|s|\leqslant|x|\leqslant q^M}|x|\log\max\{1,|x|\}$$

and

$$\int_{|x|<|s|\leqslant q^M}|x|\log\max\{1,|x|,|s|\}.$$

The first integral equals

$$\int_{|x|\leqslant q^M}|x|^2\log\max\{1,|x|\}.$$

The second equals

$$\int_{|s|\leqslant q^M}\log\max\{1,|s|\}\int_{|x|<|s|}|x|,$$

which equals

$$\frac{q^{-1}}{q+1}\int_{|s|\leqslant q^M}|s|^2\log\max\{1,|s|\}.$$

Thus the sum of the two integrals is

$$\left(1+\frac{q^{-1}}{q+1}\right)\int_{|x|\leqslant q^M}|x|^2\log\max\{1,|x|\},$$

which equals

$$\frac{q}{q+1}\left(Mq^{3M}-\frac{q^{3M}-1}{q^3-1}\right)$$

by Lemma 9.3.

Lemma 9.5. — *Let $M\geqslant 0$ then we have*

$$\int_{1\leqslant|x|\leqslant q^M}(\log|x|)^2=M^2q^M-\frac{(2M-1)q^M}{q-1}+2\frac{q^M-q}{(q-1)^2}+\frac{1}{q-1}.$$

Proof. — We have

$$\begin{aligned}\int_{1\leqslant|x|\leqslant q^M}(\log|x|)^2 &= \sum_{k=0}^{M} k^2q^k(1-q^{-1})\\ &= \sum_{k=0}^{M} k^2q^k - \sum_{k=0}^{M-1}(k+1)^2q^k\\ &= M^2q^M - \sum_{k=0}^{M-1}(2k+1)q^k\\ &= M^2q^M - 2\left(\frac{(M-1)q^M}{q-1} - \frac{q^M-q}{(q-1)^2}\right) - \frac{q^M-1}{q-1}\\ &= M^2q^M - \frac{(2M-1)q^M}{q-1} + 2\frac{q^M-q}{(q-1)^2} + \frac{1}{q-1}\end{aligned}$$

as wished. □

Lemma 9.6. — *Let $M\geqslant 0$ then we have*

$$\int_{1\leqslant|x|\leqslant q^M}|x|^k = (1-q^{-1})\frac{q^{(M+1)(k+1)}-1}{q^{k+1}-1}$$

if $k\neq -1$ and we have

$$\int_{1\leqslant|x|\leqslant q^M}|x|^{-1} = (M+1)(1-q^{-1}).$$

Proof. — Assume that $k\neq -1$ then we have

$$\begin{aligned}\int_{1\leqslant|x|\leqslant q^M}|x|^k &= (1-q^{-1})\sum_{m=0}^{M} q^{(k+1)m}\\ &= (1-q^{-1})\frac{q^{(M+1)(k+1)}-1}{q^{k+1}-1}.\end{aligned}$$

And when $k=-1$ the result is clear. □

Lemma 9.7. — *Assume that $0\leqslant k\leqslant M$. Then*

$$\int_{|x_2|\leqslant q^M}\int_{|x_3|\leqslant q^k}\int_{|x_4|\leqslant q^k}\log\max\{1,|x_2|,|x_3|,|x_4|,|x_4^2-x_2x_3|\}$$

equals

$$(M+k)q^{M+2k} - \frac{2q^{M+2k}-q^{M+k}-q^{3k}}{q-1} - \frac{q^{3k}-1}{q^3-1}.$$

Proof. — For ease of notation we define, for $m \geqslant 0$,

$$I_1(m) = \int_{1\leqslant|x|\leqslant q^m} \log|x|$$

and

$$I_2(m) = \int_{1\leqslant|x|\leqslant q^m} |x|^2\log|x|.$$

We begin by considering the contribution to the integral when $q^k < |x_2| \leqslant q^M$. In this case the integral is

$$\int_{|x_3|\leqslant q^k}\int_{|x_4|\leqslant q^k}\int_{q^k<|x_2|\leqslant q^M} \log\max\{|x_2|, |x_2||x_4^2x_2^{-1} - x_3|\}.$$

Now $|x_4^2x_2^{-1}| < q^k$ and so we can make a change of variables $x_3 \mapsto x_3 + x_4^2x_2^{-1}$ to give

$$\int_{|x_3|\leqslant q^k}\int_{|x_4|\leqslant q^k}\int_{q^k<|x_2|\leqslant q^M} \log\max\{|x_2|, |x_2||x_3|\},$$

which equals

$$q^{2k}(I_1(M) - I_1(k)) + q^k(q^M - q^k)I_2(k). \tag{1}$$

We are now left to integrate over the region $|x_2|, |x_3|, |x_4| \leqslant q^k$. Since the integrand is symmetric in x_2 and x_3 we can take twice the integral with $|x_3| < |x_2|$ plus the integral with $|x_3| = |x_2|$.

We begin by computing the contribution when $|x_3| < |x_2|$. This is equal to

$$\int_{|x_4|\leqslant q^k}\int_{|x_3|<|x_2|}\int_{|x_2|\leqslant q^k} \log\max\{1, |x_2|, |x_4|, |x_4^2 - x_2x_3|\}.$$

If $|x_2| > |x_4|$ then $|x_4^2 - x_2x_3| = |x_2||x_2^{-1}x_4^2 - x_3|$ and $|x_2^{-1}x_4^2| < |x_4| < |x_2|$. Hence we can make the change of variables $x_3 \mapsto x_3 + x_2^{-1}x_4^2$ to get the integral

$$\int_{|x_4|<|x_2|}\int_{|x_3|<|x_2|}\int_{|x_2|\leqslant q^k} \log\max\{1, |x_2|, |x_2||x_3|\},$$

which equals

$$\int_{1\leqslant|x_2|\leqslant q^k} (|\pi x_2|)^2\log|x_2| + \int_{|x_4|<|x_2|}\int_{1\leqslant|x_3|<|x_2|}\int_{1<|x_2|\leqslant q^k} \log|x_3|.$$

The first of these integrals is

$$q^{-2}I_2(k). \tag{2}$$

The second integral can be written as

$$\int_{|x_4|<|x_2|}\int_{|x_3|<|x_2|\leqslant q^k}\int_{1\leqslant|x_3|\leqslant q^k} \log|x_3|,$$

which equals

$$q^{-1}\int_{|x_3|<|x_2|\leqslant q^k} |x_2| \int_{1\leqslant|x_3|\leqslant q^k} \log|x_3|,$$

and by Lemma 9.6 this equals

$$\frac{q^{2k}}{q+1} I_1(k) - \frac{1}{q+1} I_2(k). \tag{3}$$

On the other hand if we have $|x_3| < |x_2| \leqslant |x_4|$ then the integral becomes

$$2 \int_{|x_3|<|x_2|\leqslant|x_4|\leqslant q^k} \log \max\{1, |x_4|\},$$

which equals

$$2q^{-1} \int_{|x_2|\leqslant|x_4|} |x_2| \int_{|x_4|\leqslant q^k} \log \max\{1, |x_4|\}.$$

We have

$$\int_{|x_2|\leqslant|x_4|} |x_2| = \frac{q|x_4|^2}{q+1}$$

and therefore

$$2 \int_{|x_3|<|x_2|\leqslant|x_4|\leqslant q^k} \log \max\{1, |x_4|\} = 2\frac{q}{q+1} I_2(k). \tag{4}$$

Now we look at the contribution to the integral when $|x_2| = |x_3|$. We split it up into three cases

(a) $|x_2| < |x_4|$, integrand equals $\log \max\{1, |x_4|^2\}$
(b) $|x_2| = |x_4|$
(c) $|x_2| > |x_4|$, integrand equals $\log \max\{1, |x_2|^2\}$.

In case (a) the contribution is

$$\int_{|x_2|<|x_4|\leqslant q^k} \int_{|x_3|=|x_2|} \int_{|x_2|\leqslant q^k} \log \max\{1, |x_4|^2\}.$$

This integral is

$$(1-q^{-1}) \int_{|x_2|<|x_4|} |x_2| \int_{|x_4|\leqslant q^k} \log \max\{1, |x_4|^2\} = 2\frac{1-q^{-1}}{q+1} I_2(k). \tag{5}$$

Similarly the contribution in case (c) is

$$2q^{-1}(1-q^{-1}) I_2(k). \tag{6}$$

We now deal with the contribution in case (b). We can write this as

$$\int_{1\leqslant|x_2|=|x_3|=|x_4|\leqslant q^k} \log|x_2| + \log \max\{1, |x_3||1 - x_4^2(x_2x_3)^{-1}|\}.$$

Firstly we have

$$\int_{1\leqslant|x_2|=|x_3|=|x_4|\leqslant q^k} \log|x_2| = (1-q^{-1})^2 I_2(k). \tag{7}$$

Now we deal with

$$\int_{1\leqslant|x_2|=|x_3|=|x_4|\leqslant q^k} \log \max\{1, |x_3||1 - x_4^2(x_2x_3)^{-1}|\}.$$

We have

$$\mathrm{vol}\{u \in U_F : |1-u| = q^{-i}\} = \begin{cases} 1-2q^{-1}, & \text{if } i=0; \\ q^{-i}(1-q^{-1}), & \text{if } i>0. \end{cases}$$

And therefore

$$\int_{1\leqslant |x_2|=|x_3|=|x_4|\leqslant q^k} \log\max\{1, |x_3||1-x_4^2(x_2x_3)^{-1}|\}$$

is equal to

$$(1-q^{-1})\int_{1\leqslant |x_2|\leqslant q^k}\int_{|x|\leqslant 1} |x_2|^2 \log\max\{1,|x_2||x|\}$$

minus

$$q^{-1}(1-q^{-1})\int_{1\leqslant |x_2|\leqslant q^k} |x_2|^2 \log\max\{1,|x_2|\}.$$

Now in the integral

$$(1-q^{-1})\int_{1\leqslant |x_2|\leqslant q^k}\int_{|x|\leqslant 1} |x_2|^2 \log\max\{1,|x_2||x|\}$$

if we make the change of variables $y = xx_2$ then this integral becomes

$$(1-q^{-1})\int_{|y|\leqslant |x_2|\leqslant q^k} |x_2| \int_{|y|\leqslant q^k} \log\max\{1,|y|\},$$

which equals, by Lemma 9.6,

$$(1-q^{-1})^2\int_{|y|\leqslant q^k}\left(\frac{q^{2(k+1)}-|y|^2}{q^2-1}\right)\log\max\{1,|y|\}.$$

Hence we have

$$\int_{1\leqslant |x_2|=|x_3|=|x_4|\leqslant q^k} \log\max\{1, |x_3||1-x_4^2(x_2x_3)^{-1}|\}$$

equal to

$$q^{2k}\frac{q-1}{q+1}I_1(k) - q^{-2}\frac{q-1}{q+1}I_2(k) - q^{-2}(q-1)I_2(k). \tag{8}$$

Putting this altogether, the integral

$$\int_{|x_2|\leqslant q^M}\int_{|x_3|\leqslant q^k}\int_{|x_4|\leqslant q^k} \log\max\{1,|x_2|,|x_3|,|x_4|,|x_4^2-x_2x_3|\}$$

is equal to

$$(1)+2\times(2)+2\times(3)+2\times(4)+(5)+(6)+(7)+(8).$$

Collecting together like terms in this sum gives

$$q^{2k}I_1(M) + (q^{M+k}-q^{2k})I_1(k) + q^{-2}(q^2+q+1)I_2(k).$$

Applying Lemmas 9.1 and 9.3 now gives the result.

Lemma 9.8. — *Assume that* $0 \leqslant a \leqslant b \leqslant c$. *Then*

$$\int_{|x|\leqslant q^a}\int_{|r|\leqslant q^b}\int_{|s|\leqslant q^c} \log\max\{1,|x|,|r|,|s|\}$$

is equal to

$$cq^{a+b+c} - \frac{q^{a+b+c}}{q-1} + \frac{q^{a+2b+1}}{q^2-1} + \frac{1}{q^3-1} + \frac{q^{3a+2}}{(q+1)(q^3-1)}.$$

Proof. — The contribution when $q^b < |s| \leqslant q^c$ is

$$q^{a+b}\int_{q^b<|s|\leqslant q^c} \log|s| = q^{a+b}\left(cq^c - bq^b - \frac{q^c-q^b}{q-1}\right).$$

We are now left with

$$\int_{|x|\leqslant q^a}\int_{|r|\leqslant q^b}\int_{|s|\leqslant q^b} \log\max\{1,|x|,|r|,|s|\},$$

which equals

$$2\int_{|x|\leqslant q^a}\int_{|r|\leqslant|s|\leqslant q^b} \log\max\{1,|x|,|s|\} - \int_{|x|\leqslant q^a}\int_{|r|=|s|\leqslant q^b} \log\max\{1,|x|,|s|\}.$$

This equals

$$2\int_{|x|\leqslant q^a}\int_{|s|\leqslant q^b} |s|\log\max\{1,|x|,|s|\} - \int_{|x|\leqslant q^a}\int_{|s|\leqslant q^b} (1-q^{-1})|s|\log\max\{1,|x|,|s|\},$$

which equals

$$(1+q^{-1})\int_{|x|\leqslant q^a}\int_{|s|\leqslant q^b} |s|\log\max\{1,|x|,|s|\}.$$

Now the contribution when $|s| > q^a$ is

$$(1+q^{-1})q^a\int_{q^a<|s|\leqslant q^b} |s|\log|s| = q^a\left(bq^{2b} - (a+1)q^{2a} - \frac{q^{2b}-q^{2a+2}}{q^2-1}\right)$$

by Lemma 9.3.

We are now left with

$$(1+q^{-1})\int_{|x|\leqslant q^a}\int_{|s|\leqslant q^a} |s|\log\max\{1,|x|,|s|\} = aq^{3a} - \frac{q^{3a}-1}{q^3-1}$$

by Lemma 9.4.

Thus

$$\int_{|x|\leqslant q^a}\int_{|r|\leqslant q^b}\int_{|s|\leqslant q^c} \log\max\{1,|x|,|r|,|s|\}$$

is equal to the sum of

$$q^{a+b}\left(cq^c - bq^b - \frac{q^c-q^b}{q-1}\right),$$

$$q^a\left(bq^{2b} - (a+1)q^{2a} - \frac{q^{2b}-q^{2a+2}}{q^2-1}\right),$$

and

$$aq^{3a} - \frac{q^{3a}-1}{q^3-1}.$$

Adding these terms together gives

$$cq^{a+b+c} - \frac{q^{a+b+c}}{q-1} + \frac{q^{a+2b+1}}{q^2-1} + \frac{1}{q^3-1} + \frac{q^{3a+2}}{(q+1)(q^3-1)}$$

as wished. □

Appendix

The twisted weighted fundamental lemma

In this appendix we give the formulation of the twisted weighted fundamental taken from a letter from James Arthur to Dinakar Ramakrishnan, dated March 11, 2002.

Let me try to convince you that the statement of the conjectural twisted, weighted fundamental lemma is similar to that of the untwisted case. We can in fact state them identically if we take G to be the connected component $G^0 \rtimes \alpha$ of the nonconnected group $G^+ = G^0 \rtimes \langle\alpha\rangle$. We assume that G is unramified over a local p-adic field F (of characteristic 0).

Let $M = M^0 \rtimes \alpha$ be a "Levi subset" of G, in the sense of my paper [**Art88b**, p. 228]. Then $\mathcal{P}(M)$ denotes the set of "parabolic subsets" $P = MN_P$ of G with Levi component M, and A_M the split component of the centralizer of M in M^0. We define the weighted orbital integral

$$J_M(\gamma, f) = |D(\gamma)|^{1/2} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x) v_M(x)\, dx,$$

$f \in C_c^\infty(G(F))$, $\gamma \in M(F)$ strongly G^0-regular, $G_\gamma = \mathrm{Cent}(G^0, \gamma)^0$, as a special case of [**Art88b**, p. 233].

Suppose that M' represents an unramified elliptic, twisted endoscopic datum $(M', \mathcal{M}', s'_M, \xi'_M)$ for M^0. Here, s'_M is a semisimple element in the nonconnected component $\widehat{M} = \widehat{M}^0 \rtimes \widehat{\alpha} \subset \widehat{G} = \widehat{G}^0 \rtimes \widehat{\alpha}$. (I trust that this slightly nonstandard formulation is OK.) We suppose that $\mathcal{M}'$ is an L-subgroup of ${}^LM^0 = \widehat{M}^0 \rtimes W_F$, and that ξ'_M is the identity embedding. We then define $\mathcal{E}_{M'}(G)$ as in the untwisted case, in [**Art99**, § 3]. Thus, if $Z(\widehat{M})^\Gamma$ denotes the group of Γ-invariants in the centralizer of $\widehat{M}$ in $\widehat{M}^0$, $\mathcal{E}_{M'}(G)$ is the set of twisted endoscopic data for G^0 of the form $(G', \mathcal{G}', s', \xi')$, where s' lies in $s'_M Z(\widehat{M})^\Gamma$, $\widehat{G}'$ is the connected centralizer of s' in $\widehat{G}$ and ξ' is the identity embedding of $\mathcal{G}' = \mathcal{M}'\widehat{G}'$ into ${}^LG^0$. The elements in $\mathcal{E}_{M'}(G)$ are taken up to translation of s' by $Z(\widehat{G})^\Gamma$.

We can now proceed as in [**Art02**, § 5]. Set

$$\iota_{M'}(G, G') = |Z(\widehat{M}')^\Gamma / Z(\widehat{M})^\Gamma||Z(\widehat{G}')^\Gamma / Z(\widehat{G})^\Gamma|^{-1},$$

and

$$r_M^G(k) = J_M(k, u)$$

where $k \in M(F)$ is strongly G^0-regular, and $u = u_K$ is the stabilizer in $G(F)$ of the unit in a Hecke algebra of $G^0(F)$.

Conjecture. — *Let ℓ' be a strongly G^0-regular, stable conjugacy class in $M'(F)$. Then*

$$\sum_{k \in \Gamma_{G\text{-}reg}(M(F))} \Delta_{M,K}(\ell', k) r_M^G(k)$$

equals

$$\sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') s_{M'}^{G'}(\ell'),$$

where $s_{M'}^{G'}(\ell')$ is the function defined uniquely for the unramified connected pair (G', M') in [**Art02**, Conjecture 5.1]*, and $\Delta_{M,K}$ is the twisted transfer factor for M^0, normalized relative to the hyperspecial maximal compact $K \cap M^0(F)$.*

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