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H^∞ FUNCTIONAL CALCULUS AND SQUARE FUNCTIONS ON NONCOMMUTATIVE L^p-SPACES

Marius Junge, Christian Le Merdy, Quanhua Xu

Abstract. — We investigate sectorial operators and semigroups acting on noncommutative L^p-spaces. We introduce new square functions in this context and study their connection with H^∞ functional calculus, extending some famous work by Cowling, Doust, McIntoch and Yagi concerning *commutative* L^p-spaces. This requires natural variants of Rademacher sectoriality and the use of the matricial structure of noncommutative L^p-spaces. We mainly focus on noncommutative diffusion semigroups, that is, semigroups $(T_t)_{t\geq 0}$ of normal selfadjoint operators on a semifinite von Neumann algebra $(\mathcal{M},\tau)$ such that $T_t\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is a contraction for any $p \geq 1$ and any $t \geq 0$. We discuss several examples of such semigroups for which we establish bounded H^∞ functional calculus and square function estimates. This includes semigroups generated by certain Hamiltonians or Schur multipliers, q-Ornstein-Uhlenbeck semigroups acting on the q-deformed von Neumann algebras of Bozejko-Speicher, and the noncommutative Poisson semigroup acting on the group von Neumann algebra of a free group.

***Résumé* (Calcul fonctionnel H^∞ et fonctions carrées dans les espaces L^p non commutatifs)**

Nous étudions les opérateurs sectoriels et les semigroupes opérant sur un espace L^p non commutatif. Nous introduisons de nouvelles fonctions carrées adaptées à ce contexte et étudions leurs interactions avec le calcul fonctionnel H^∞. Nous obtenons des extensions de travaux fameux de Cowling, Doust, McIntoch et Yagi qui concernaient le cas commutatif. Cette étude nécessite l'introduction de variantes de la Rademacher sectorialité et l'usage des structures matricielles sur les espaces L^p non commutatifs. Nous traitons de façon approfondie les semigroupes de diffusion non commutatifs. Il s'agit des semigroupes $(T_t)_{t\geq 0}$ d'opérateurs normaux et auto-adjoints opérant sur une algèbre de von Neumann semifinie $(\mathcal{M},\tau)$, tels que $T_t\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ est une contraction pour tout $p \geq 1$ et pour tout $t \geq 0$. Nous présentons et étudions plusieurs exemples de tels semigroupes, pour lesquels nous sommes en mesure d'établir une propriété de calcul H^∞ borné, ainsi que des estimations quadratiques. Cette étude inclut certains semigroupes engendrés par des opérateurs Hamiltoniens ou des multiplicateurs de Schur, des semigroupes d'Ornstein-Uhlenbeck opérant sur les algèbres de von Neumann de q-déformation de Bozejko-Speicher, et le semigroupe de Poisson non commutatif défini sur l'algèbre de von Neumann d'un groupe libre.

CONTENTS

CHAPTER 1

INTRODUCTION

In the recent past, noncommutative analysis (in a wide sense) has developed rapidly because of its interesting and fruitful interactions with classical theories such as C^*-algebras, Banach spaces, probability, or harmonic analysis. The theory of operator spaces has played a prominent role in these developments, leading to new fields of research in either operator theory, operator algebras or quantum probability. The recent theory of martingale inequalities in noncommutative L^p-spaces is a good example for this development. Indeed, square functions associated to martingales and most of the classical martingale inequalities have been successfully transferred to the noncommutative setting. See in particular [**64, 33, 68, 38**], and also the recent survey [**80**] and the references therein. The noncommutative maximal ergodic theorem in [**36**] is our starting point for the study of noncommutative diffusion semigroups. On this line we investigate noncommutative analogs of classical square function inequalities.

It is remarkable that operator space techniques have led to new results on classical analysis. We mention in particular completely bounded Fourier multipliers and Schur multipliers on Schatten classes [**31**]. In our treatment of semigroups no prior knowledge on operator space theory is required. However, operator space concepts underlie our understanding of the subject.

Our objectives are to introduce natural square functions associated with a sectorial operator or a semigroup on some noncommutative L^p-space, to investigate their connections with H^∞ functional calculus, and to give various concrete examples and applications. H^∞ functional calculus was introduced by McIntosh [**53**], and then developed by him and his coauthors in a series of remarkable papers [**54, 21, 3**]. Nowadays this is a classical and powerful subject which plays an important role in spectral theory for unbounded operators, abstract maximal L^p-regularity, or multiplier theory. See e.g. [**43**] for more information and references.

Square functions for generators of semigroups appeared earlier in Stein's classical book [**70**] on the Littlewood-Paley theory for semigroups acting on usual (=commutative) L^p-spaces. Stein gave several remarkable applications of these square functions to functional calculus and multiplier theorems for diffusion semigroups. Later on, Cowling [**20**] obtained several extensions of these results and used them to prove maximal theorems.

The fundamental paper [**21**] established tight connections between McIntosh's H^∞ functional calculus and Stein's approach. Assume that A is a sectorial operator on $L^p(\Sigma)$, with $1 < p < \infty$, and let F be a non zero bounded analytic function on a sector $\{|\mathrm{Arg}(z)| < \theta\}$ containing the spectrum of A, and such that F tends to 0 with an appropriate estimate as $|z| \to \infty$ and as $|z| \to 0$ (see Chapter 3 for details). The associated square function is defined by

$$\|x\|_F \,=\, \Bigl\| \Bigl(\int_0^\infty \bigl|F(tA)x\bigr|^2 \,\frac{dt}{t}\Bigr)^{\frac{1}{2}} \Bigr\|_p, \qquad x \in L^p(\Sigma).$$

For example if $-A$ is the generator of a bounded analytic semigroup $(T_t)_{t\geq 0}$ on $L^p(\Sigma)$, then we can apply the above with the function $F(z) = ze^{-z}$ and in this case, we obtain the familiar square function

$$\|x\|_F = \Bigl\| \Bigl(\int_0^\infty t\,\Bigl|\frac{\partial}{\partial t}\bigl(T_t(x)\bigr)\Bigr|^2 \,dt\Bigr)^{\frac{1}{2}} \Bigr\|_p$$

from [**70**, Chapters III-IV]. One of the most remarkable connections between H^∞ functional calculus and square functions on L^p-spaces is as follows. If A admits a bounded H^∞ functional calculus, then we have an equivalence $K_1\|x\| \leq \|x\|_F \leq K_2\|x\|$ for any F as above. Indeed this follows from [**21**] (see also [**49**]).

In this paper we consider a sectorial operator A acting on a noncommutative L^p-space $L^p(\mathcal{M})$ associated with a semifinite von Neumann algebra $(\mathcal{M},\tau)$. For an appropriate bounded analytic function F as before, we introduce two square functions which are *approximately* defined as

$$\|x\|_{F,c} \,=\, \Bigl\| \Bigl(\int_0^\infty \bigl(F(tA)x\bigr)^* \bigl(F(tA)x\bigr)\,\frac{dt}{t}\Bigr)^{\frac{1}{2}} \Bigr\|_p$$

and

$$\|x\|_{F,r} \,=\, \Bigl\| \Bigl(\int_0^\infty \bigl(F(tA)x\bigr) \bigl(F(tA)x\bigr)^*\,\frac{dt}{t}\Bigr)^{\frac{1}{2}} \Bigr\|_p$$

(see Chapter 6 for details). The functions $\|\ \|_{F,c}$ and $\|\ \|_{F,r}$ are called column and row square functions respectively. Using them we define a symmetric square function $\|x\|_F$. As with the noncommutative Khintchine inequalities (see [**51, 52**]), this definition depends upon whether $p \geq 2$ or $p < 2$. If $p \geq 2$, we set $\|x\|_F = \max\{\|x\|_{F,c}\,;\|x\|_{F,r}\}$. See Chapter 6 for the more complicated case $p < 2$. Then one of our main results is that if A admits a bounded H^∞ functional calculus on $L^p(\mathcal{M})$, with $1 < p < \infty$, we have an equivalence

$$K_1\|x\| \leq \|x\|_F \leq K_2\|x\| \tag{1.1}$$

for these square functions.

After a short introduction to noncommutative L^p-spaces, Chapter 2 is devoted to preliminary results on noncommutative Hilbert space valued L^p-spaces, which are central for the definition of square functions. These spaces and related ideas first appeared in [**51**] (see also [**52, 62**]). In fact operator valued matrices and operator space techniques (see e.g. [**58, 62, 63**]) play a natural role in our context. However we tried to make the paper accessible to readers not familiar with operator space theory and completely bounded maps.

In Chapter 3 we give the necessary background on sectorial operators, semigroups, and H^∞ functional calculus. Then we introduce a completely bounded H^∞ functional calculus for an operator A acting on a noncommutative $L^p(\mathcal{M})$. Again this is quite natural in our context and indeed it turns out to be important in our study of square functions (see in particular Corollary 7.9).

Rademacher boundedness and Rademacher sectoriality now play a prominent role in H^∞ functional calculus. We refer the reader e.g. to [**41**], [**79**], [**78**], [**47**], [**49**] or [**43**] for developments and applications. On noncommutative L^p-spaces, it is natural to introduce two related concepts, namely the column boundedness and the row boundedness. If $\mathcal{F}$ is a set of bounded operators on $L^p(\mathcal{M})$, we will say that $\mathcal{F}$ is Col-bounded if we have an estimate

$$\Big\|\Big(\sum_k T_k(x_k)^*T_k(x_k)\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathcal{M})} \leq C\Big\|\Big(\sum_k x_k^*x_k\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathcal{M})}$$

for any finite families $T_1,\ldots,T_n$ in $\mathcal{F}$, and $x_1,\ldots,x_n$ in $L^p(\mathcal{M})$. Row boundedness is defined similarly. We develop these concepts in Chapter 4, along with the related notions of column and row sectoriality.

Chapters 6 and 7 are devoted to square functions and their interplay with H^∞ functional calculus. As a consequence of the main result of Chapter 4, we prove that if A is Col-sectorial (resp. Rad-sectorial), then we have an equivalence

$$K_1\|x\|_{G,c} \leq \|x\|_{F,c} \leq K_2\|x\|_{G,c} \qquad (\text{resp. } K_1\|x\|_G \leq \|x\|_F \leq K_2\|x\|_G)$$

for any pair of non zero functions F, G defining square functions. This is a noncommutative generalization of the main result of [**49**]. Then we prove the aforementioned result that (1.1) holds true if A has a bounded H^∞ functional calculus. We also show that conversely, appropriate square function estimates for an operator A on $L^p(\mathcal{M})$ imply that A has a bounded H^∞ functional calculus.

Chapter 5 (which is independent of Chapters 6 and 7) is devoted to a noncommutative generalization of Stein's diffusion semigroups considered in [**70**]. We define a noncommutative diffusion semigroup to be a point w^*-continuous semigroup $(T_t)_{t\geq 0}$ of normal contractions on $(\mathcal{M}, \tau)$, such that each T_t is selfadjoint with respect to τ. In this case, $(T_t)_{t\geq 0}$ extends to a c_0-semigroup of contractions on $L^p(\mathcal{M})$ for any $1 \leq p < \infty$. Let $-A_p$ denote the negative generator of the L^p-realization of $(T_t)_{t\geq 0}$. Our main result in this chapter is that if further each $T_t \colon \mathcal{M} \to \mathcal{M}$ is positive (resp. completely positive), then A_p is Rad-sectorial (resp. Col-sectorial and Row-sectorial). The proof is based on a noncommutative maximal theorem from [**37, 36**], where such diffusion semigroups were considered for the first time.

If $(T_t)_{t\geq 0}$ is a noncommutative diffusion semigroup as above, the most interesting general question is whether A_p admits a bounded H^∞ functional calculus on $L^p(\mathcal{M})$ for $1 < p < \infty$. This question has an affirmative answer in the commutative case [**20**] but it is open in the noncommutative setting. The last three chapters are devoted to examples of natural diffusion semigroups, for which we are able to show that A_p admits a bounded H^∞ functional calculus. Here is a brief description.

In Chapter 8, we consider left and right multiplication operators, Hamiltonians, and Schur multipliers on Schatten space S^p. Let H be a real Hilbert space, and let $(\alpha_k)_{k\geq 1}$ and $(\beta_k)_{k\geq 1}$ be two sequences of H. We consider the semigroup $(T_t)_{t\geq 0}$ of Schur multipliers which are determined by $T_t(E_{ij}) = e^{-t(\|\alpha_i - \beta_j\|)}E_{ij}$, where the E_{ij}'s are the standard matrix units. This is a diffusion semigroup on $B(\ell^2)$ and we show that the associated negative generators A_p have a bounded H^∞ functional calculus for any $1 < p < \infty$.

Let H be a real Hilbert space. In Chapter 9, we consider the q-deformed von Neumann algebras $\Gamma_q(H)$ of Bozejko and Speicher [**14, 15**], equipped with its canonical trace. To any contraction $a \colon H \to H$ we may associate a second quantization operator $\Gamma_q(a) \colon \Gamma_q(H) \to \Gamma_q(H)$, which is a normal unital completely positive map. We consider semigroups defined by $T_t = \Gamma_q(a_t)$, where $(a_t)_{t\geq 0}$ is a selfadjoint contraction semigroup on H. This includes the q-Ornstein-Uhlenbeck semigroup [**9, 11**]. These semigroups $(T_t)_{t\geq 0}$ are completely positive noncommutative diffusion semigroups and we show that the associated A_p's have a bounded H^∞ functional calculus for any $1 < p < \infty$.

In Chapter 10 we consider the noncommutative Poisson semigroup of a free group. Let $G = \mathbb{F}_n$ be the free group with n generators and let $|\cdot|$ be the usual length function on G. Let $VN(G)$ be the group von Neumann algebra and let $\lambda(g) \in VN(G)$ be the left translation operator for any $g \in G$. For any $t \geq 0$, T_t is defined by $T_t(\lambda(g)) = e^{-t|g|}\lambda(g)$. This semigroup was introduced by Haagerup [**30**]. Again this is a completely positive noncommutative diffusion semigroup and we prove that the associated A_p's have a bounded H^∞ functional calculus for any $1 < p < \infty$. The proof uses noncommutative martingales in the sense of [**64**], and we establish new square function estimates of independent interest for these martingales.

Chapter 11 is a brief account on the non tracial case. We consider noncommutative L^p-spaces $L^p(\mathcal{M}, \varphi)$ associated with a (possibly non tracial) normal faithful state φ on $\mathcal{M}$, and we give several generalizations or variants of the results obtained so far in the semifinite setting.

We end this introduction with a few notations. If X is a Banach space, the algebra of all bounded operators on X is denoted by $B(X)$. Further we let I_X denote the identity operator on X.

We usually let $(e_k)_{k\geq 1}$ denote the canonical basis of ℓ^2, or any orthonormal family on Hilbert space. Further we let $E_{ij} = e_i \otimes \overline{e_j} \in B(\ell^2)$ denote the standard matrix units.

We will use the symbol " $\asymp$ " to indicate that two functions are equivalent up to positive constants. For example, (1.1) will be abbreviated by $\|x\|_F \asymp \|x\|$. Next we will write $X \approx Y$ to indicate that two Banach spaces X and Y are isomorphic.

We refer the reader to e.g. [**69**] and [**40**] for the necessary background on C^*-algebras and von Neumann algebras. We will make use of UMD Banach spaces, for which we refer to [**17**].

The main results of the present work were announced in [**34**]. We refer to related work of Mei's [**55**] in the semicommutative case.

CHAPTER 2

NONCOMMUTATIVE HILBERT SPACE VALUED L^p-SPACES

2.A. Noncommutative L^p-spaces

We start with a brief presentation of noncommutative L^p-spaces associated with a trace. We mainly refer the reader to [**73**, Chapter I] and [**26**] for details, as well as to [**65**] and the references therein for further information on these spaces.

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. We let $\mathcal{M}_+$ denote the positive part of $\mathcal{M}$. Let $\mathcal{S}_+$ be the set of all $x \in \mathcal{M}_+$ whose support projection have a finite trace. Then any $x \in \mathcal{S}_+$ has a finite trace. Let $\mathcal{S} \subset \mathcal{M}$ be the linear span of $\mathcal{S}_+$, then $\mathcal{S}$ is a w^*-dense $*$-subalgebra of $\mathcal{M}$.

Let $0 < p < \infty$. For any $x \in \mathcal{S}$, the operator $|x|^p$ belongs to $\mathcal{S}_+$ and we set

$$\|x\|_p \,=\, \bigl(\tau(|x|^p)\bigr)^{\frac{1}{p}}, \qquad x \in \mathcal{S}.$$

Here $|x| = (x^*x)^{\frac{1}{2}}$ denotes the modulus of x. It turns out that $\|\ \|_p$ is a norm on $\mathcal{S}$ if $p \geq 1$, and a p-norm if $p < 1$. By definition, the noncommutative L^p-space associated with $(\mathcal{M}, \tau)$ is the completion of $(\mathcal{S}, \|\ \|_p)$. It is denoted by $L^p(\mathcal{M})$. For convenience, we also set $L^\infty(\mathcal{M}) = \mathcal{M}$ equipped with its operator norm. Note that by definition, $L^p(\mathcal{M}) \cap \mathcal{M}$ is dense in $L^p(\mathcal{M})$ for any $1 < p \leq \infty$.

Assume that $\mathcal{M} \subset B(\mathcal{H})$ acts on some Hilbert space $\mathcal{H}$. It will be fruitful to also have a description of the elements of $L^p(\mathcal{M})$ as (possibly unbounded) operators on $\mathcal{H}$. Let $\mathcal{M}' \subset B(\mathcal{H})$ denote the commmutant of $\mathcal{M}$. We say that a closed and densely defined operator x on $\mathcal{H}$ is affiliated with $\mathcal{M}$ if x commutes with any unitary of $\mathcal{M}'$. Then we say that an affiliated operator x is measurable (with respect to the trace τ) provided that there is a positive integer $n \geq 1$ such that $\tau(1 - p_n) < \infty$, where $p_n = \chi_{[0,n]}(|x|)$ is the projection associated to the indicator function of $[0, n]$ in the Borel functional calculus of $|x|$. It turns out that the set $L^0(\mathcal{M})$ of all measurable operators is a $*$-algebra (see e.g. [**73**] for a precise definition of the sum and product on $L^0(\mathcal{M})$). Indeed, this $*$-algebra has a lot of remarkable stability properties. First for any x in $L^0(\mathcal{M})$ and any $0 < p < \infty$, the operator $|x|^p = (x^*x)^{\frac{p}{2}}$ belongs to

$L^0(\mathcal{M})$. Second, let $L^0(\mathcal{M})_+$ be the positive part of $L^0(\mathcal{M})$, that is, the set of all selfadjoint positive operators in $L^0(\mathcal{M})$. Then the trace τ extends to a positive tracial functional on $L^0(\mathcal{M})_+$, still denoted by τ, in such a way that for any $0 < p < \infty$, we have

$$L^p(\mathcal{M}) = \big\{x \in L^0(\mathcal{M}) \, : \, \tau(|x|^p) < \infty\big\},$$

equipped with $\|x\|_p = (\tau(|x|^p))^{\frac{1}{p}}$. Furthermore, τ uniquely extends to a bounded linear functional on $L^1(\mathcal{M})$, still denoted by τ. Indeed we have $|\tau(x)| \leq \tau(|x|) = \|x\|_1$ for any $x \in L^1(\mathcal{M})$.

For any $0 < p \leq \infty$ and any $x \in L^p(\mathcal{M})$, the adjoint operator x^* belongs to $L^p(\mathcal{M})$ as well, with $\|x^*\|_p = \|x\|_p$. Clearly, we also have that $x^*x \in L^{\frac{p}{2}}(\mathcal{M})$ and $|x| \in L^p(\mathcal{M})$, with $\|\,|x|\,\|_p = \|x\|_p$. We let $L^p(\mathcal{M})_+ = L^0(\mathcal{M})_+ \cap L^p(\mathcal{M})$ denote the positive part of $L^p(\mathcal{M})$. The space $L^p(\mathcal{M})$ is spanned by $L^p(\mathcal{M})_+$.

We recall the noncommutative Hölder inequality. If $0 < p, q, r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then

$$\|xy\|_r \leq \|x\|_p \|y\|_q, \qquad x \in L^p(\mathcal{M}),\ y \in L^q(\mathcal{M}). \tag{2.1}$$

Conversely for any $z \in L^r(\mathcal{M})$, there exist $x \in L^p(\mathcal{M})$ and $y \in L^q(\mathcal{M})$ such that $z = xy$, and $\|z\|_r = \|x\|_p \|y\|_q$.

For any $1 \leq p < \infty$, let $p' = p/(p-1)$ be the conjugate number of p. Applying (2.1) with $q = p'$ and $r = 1$, we may define a duality pairing between $L^p(\mathcal{M})$ and $L^{p'}(\mathcal{M})$ by

$$\langle x, y\rangle = \tau(xy), \qquad x \in L^p(\mathcal{M}),\ y \in L^{p'}(\mathcal{M}). \tag{2.2}$$

This induces an isometric isomorphism

$$L^p(\mathcal{M})^* = L^{p'}(\mathcal{M}), \qquad 1 \leq p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.3}$$

In particular, we may identify $L^1(\mathcal{M})$ with the (unique) predual $\mathcal{M}_*$ of $\mathcal{M}$.

Another remarkable property of noncommutative L^p-spaces which will play a crucial role is that they form an interpolation scale. By means of the natural embeddings of $L^\infty(\mathcal{M}) = \mathcal{M}$ and $L^1(\mathcal{M}) = \mathcal{M}_*$ into $L^0(\mathcal{M})$, one may regard $(L^\infty(\mathcal{M}), L^1(\mathcal{M}))$ as a compatible couple of Banach spaces. Then we have

$$[L^\infty(\mathcal{M}), L^1(\mathcal{M})]_{\frac{1}{p}} = L^p(\mathcal{M}), \qquad 1 \leq p \leq \infty, \tag{2.4}$$

where $[\ ,\ ]_\theta$ stands for the interpolation space obtained by the complex interpolation method (see e.g. [**6**]).

The space $L^2(\mathcal{M})$ is a Hilbert space, with inner product given by $(x, y) \mapsto \langle x, y^*\rangle = \tau(xy^*)$. We will need to pay attention to the fact that the identity (2.3) provided by (2.2) for $p = 2$ differs from the canonical (antilinear) identification of a Hilbert space with its dual space. This leads to two different notions of adjoints and we will use different notations for them. Let $T \colon L^2(\mathcal{M}) \to L^2(\mathcal{M})$ be any bounded operator. On the one hand, we will denote by T^* the Banach space adjoint of T provided by (2.3) and (2.2), so that

$$\tau\big(T(x)y\big) = \tau\big(xT^*(y)\big), \qquad x,\, y \in L^2(\mathcal{M}).$$

On the other hand, we will denote by $T^\dagger$ the adjoint of T in the usual sense of Hilbertian operator theory. That is,

$$\tau\big(T(x)y^*\big) = \tau\big(x(T^\dagger(y))^*\big), \qquad x,\, y \in L^2(\mathcal{M}).$$

For any $1 \leq p \leq \infty$ and any $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$, let $T^\circ\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ be defined by

$$T^\circ(x) = T(x^*)^*, \qquad x \in L^p(\mathcal{M}). \tag{2.5}$$

If $p = 2$, we see from above that

$$T^\dagger = T^{*\circ}. \tag{2.6}$$

In particular $T\colon L^2(\mathcal{M}) \to L^2(\mathcal{M})$ being selfadjoint means that $T^* = T^\circ$.

The above notations will be used as well when T is an unbounded operator.

We finally mention for further use that for any $1 < p < \infty$, $L^p(\mathcal{M})$ is a UMD Banach space (see [**8**] or [**65**, Section 7]).

Throughout the rest of this chapter, $(\mathcal{M}, \tau)$ will be an arbitrary semifinite von Neumann algebra.

2.B. Tensor products

Let H be a Hilbert space. If the von Neumann algebra $B(H)$ is equipped with its usual trace tr, the associated noncommutative L^p-spaces are the Schatten spaces $S^p(H)$ for any $0 < p < \infty$. We will simply write S^p for $S^p(\ell^2)$. If $n \geq 1$ is any integer, then $B(\ell^2_n) \simeq M_n$, the algebra of $n \times n$ matrices with complex entries, and we will write S^p_n for the corresponding matrix space $S^p(\ell^2_n)$.

We equip the von Neumann algebra $\mathcal{M}\overline{\otimes}B(H)$ with the trace $\tau \otimes tr$. Then for any $0 < p < \infty$, we let

$$S^p[H; L^p(\mathcal{M})] \;=\; L^p(\mathcal{M}\overline{\otimes}B(H)). \tag{2.7}$$

Again in the case when $H = \ell^2$ (resp. $H = \ell^2_n$), we simply write $S^p[L^p(\mathcal{M})]$ (resp. $S^p_n[L^p(\mathcal{M})] = L^p(M_n(\mathcal{M}))$) for these spaces. These definitions are a special case of Pisier's notion of noncommutative vector valued L^p-spaces [**62**]. Further comments on these spaces and their connection with operator space theory will be given in the paragraph 2.D below.

Lemma 2.1. — *For any $0 < p < \infty$, $S^p(H) \otimes L^p(\mathcal{M})$ is dense in $S^p[H; L^p(\mathcal{M})]$.*

Proof. — Let $(p_t)_t$ be a nondecreasing net of finite rank projections on H converging to I_H in the w^*-topology. Then $1 \otimes p_t$ converges to $1 \otimes I_H$ in the w^*-topology of $\mathcal{M}\overline{\otimes}B(H)$. As is well-known, this implies that $\|(1 \otimes p_t)x(1 \otimes p_t) - x\|_p \to 0$ for any $x \in L^p(\mathcal{M}\overline{\otimes}B(H))$. Each $H_t = p_t(H)$ is finite dimensional, hence we have

$$(1 \otimes p_t)x(1 \otimes p_t) \in L^p(\mathcal{M} \otimes B(H_t)) = L^p(\mathcal{M}) \otimes S^p(H_t) \subset L^p(\mathcal{M}) \otimes S^p(H)$$

for any $x \in L^p(\mathcal{M})$. This shows the density of $S^p(H) \otimes L^p(\mathcal{M})$. □

We shall now define various H-valued noncommutative L^p-spaces. For any $a, b \in H$, we let $a \otimes \bar{b} \in B(H)$ denote the rank one operator taking any $\xi \in H$ to $\langle \xi, b\rangle a$. We fix some $e \in H$ with $\|e\| = 1$, and we let $p_e = e \otimes \bar{e}$ be the rank one projection onto $\mathrm{Span}\{e\}$. Then for any $0 < p \leq \infty$, we let

$$L^p(\mathcal{M}; H_c) \ = \ L^p(\mathcal{M}\overline{\otimes}B(H))(1 \otimes p_e).$$

We will give momentarily further descriptions of that space showing that its definition is essentially independent of the choice of e. For any $0 < p \leq \infty$, let us regard

$$L^p(\mathcal{M}) \subset L^p(\mathcal{M}) \otimes S^p(H) \subset L^p(\mathcal{M}\overline{\otimes}B(H))$$

as a subspace of $L^p(\mathcal{M}\overline{\otimes}B(H))$ by identifying any $c \in L^p(\mathcal{M})$ with $c \otimes p_e$. Clearly this is an isometric embedding. This identification is equivalent to writing that

$$L^p(\mathcal{M}) \ = \ (1 \otimes p_e)L^p(\mathcal{M}\overline{\otimes}B(H))(1 \otimes p_e).$$

For any element $u \in L^p(\mathcal{M}; H_c) \subset L^p(\mathcal{M}\overline{\otimes}B(H))$, the product u^*u belongs to the subspace $(1 \otimes p_e)L^{\frac{p}{2}}(\mathcal{M}\overline{\otimes}B(H))(1 \otimes p_e)$ of $L^{\frac{p}{2}}(\mathcal{M}\overline{\otimes}B(H))$. Applying the above identifications for $\frac{p}{2}$, we may therefore regard u^*u as an element of $L^{\frac{p}{2}}(\mathcal{M})$. Hence $(u^*u)^{\frac{1}{2}} \in L^p(\mathcal{M})$, and we have

$$\|u\|_{L^p(\mathcal{M};H_c)} = \big\|(u^*u)^{\frac{1}{2}}\big\|_{L^p(\mathcal{M})}, \qquad u \in L^p(\mathcal{M}; H_c). \tag{2.8}$$

Let $u \in L^p(\mathcal{M}) \otimes H$ and let $(x_k)_k$ and $(a_k)_k$ be finite families in $L^p(\mathcal{M})$ and H such that $u = \sum_k x_k \otimes a_k$. Let $\tilde{u} \in L^p(\mathcal{M}) \otimes S^p(H)$ be defined by $\tilde{u} = \sum_k x_k \otimes (a_k \otimes \bar{e})$. Then the mapping $u \mapsto \tilde{u}$ induces a linear embedding

$$L^p(\mathcal{M}) \otimes H \subset L^p(\mathcal{M}; H_c).$$

Moreover the argument for Lemma 2.1 shows the following.

Lemma 2.2. — *For any $0 < p < \infty$, $L^p(\mathcal{M}) \otimes H$ is dense in $L^p(\mathcal{M}; H_c)$.*

We shall now compute the norm on $L^p(\mathcal{M}) \otimes H$ induced by $L^p(\mathcal{M}; H_c)$. Let us consider $u = \sum_k x_k \otimes a_k$ as above. Then we have

$$\tilde{u} = \sum_k x_k \otimes a_k \otimes \bar{e} \qquad \text{and} \qquad \tilde{u}^* = \sum_k x_k^* \otimes e \otimes \bar{a_k}.$$

Hence
$$\tilde{u}^*\tilde{u} = \sum_{i,j} \langle a_j, a_i\rangle\, x_i^* x_j \otimes p_e\,.$$
According to (2.8), this shows that

$$\Big\|\sum_k x_k \otimes a_k\Big\|_{L^p(\mathcal{M};H_c)} \ =\ \Big\|\Big(\sum_{i,j} \langle a_j, a_i\rangle\, x_i^* x_j\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathcal{M})}. \tag{2.9}$$

In the above definitions, the index 'c' stands for 'column'. Indeed, if $(e_1, \ldots, e_n)$ is an orthonormal family of H and if $x_1, \ldots, x_n$ belong to $L^p(\mathcal{M})$, it follows from (2.9) that

(2.10)
$$\Big\|\sum_k x_k \otimes e_k\Big\|_{L^p(\mathcal{M};H_c)} = \Big\|\Big(\sum_k x_k^* x_k\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathcal{M})} = \left\|\begin{bmatrix} x_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_n & 0 & \cdots & 0 \end{bmatrix}\right\|_{L^p(M_n(\mathcal{M}))}.$$

Note that according to Lemma 2.2, we can now regard $L^p(\mathcal{M}; H_c)$ as the completion of $L^p(\mathcal{M}) \otimes H$ for the tensor norm given by (2.9), if p is finite. See Remark 2.3 (2) for the case $p = \infty$.

We now turn to analogous definitions with columns replaced by rows. Let $e \in H$ with $\|e\| = 1$ as above, and let $p_{\bar{e}} = \bar{e} \otimes e \in B(\overline{H})$. For any $0 < p \leq \infty$, we let
$$L^p(\mathcal{M}; H_r) \ =\ (1 \otimes p_{\bar{e}}) L^p(\mathcal{M} \overline{\otimes} B(\overline{H})).$$
Then any of the above results for $L^p(\mathcal{M}; H_c)$ has a version for $L^p(\mathcal{M}; H_r)$. In particular, let $u = \sum_k x_k \otimes a_k$ in $L^p(\mathcal{M}) \otimes H$, with $x_k \in L^p(\mathcal{M})$ and $a_k \in H$. Then identifying u with the element $\sum_k x_k \otimes \bar{e} \otimes a_k$ in $L^p(\mathcal{M} \overline{\otimes} B(\overline{H}))$ yields a linear embedding
$$L^p(\mathcal{M}) \otimes H \subset L^p(\mathcal{M}; H_r),$$
and we have

$$\Big\|\sum_k x_k \otimes a_k\Big\|_{L^p(\mathcal{M};H_r)} \ =\ \Big\|\Big(\sum_{i,j} \langle a_i, a_j\rangle\, x_i x_j^*\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathcal{M})}. \tag{2.11}$$

Thus if $(e_1, \ldots, e_n)$ is an orthonormal family of H and if $x_1, \ldots, x_n$ belong to $L^p(\mathcal{M})$, then we have

(2.12)
$$\Big\|\sum_k x_k \otimes e_k\Big\|_{L^p(\mathcal{M};H_r)} = \Big\|\Big(\sum_k x_k x_k^*\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathcal{M})} = \left\|\begin{bmatrix} x_1 & \ldots & x_n \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\right\|_{L^p(M_n(\mathcal{M}))}.$$

Moreover for any $0 < p < \infty$, $L^p(\mathcal{M}) \otimes H$ is a dense subspace of $L^p(\mathcal{M}; H_r)$.

Throughout this work, we will have to deal both with column spaces $L^p(\mathcal{M}; H_c)$ and row spaces $L^p(\mathcal{M}; H_r)$. In most cases, they will play symmetric roles. Thus we will often state some results for $L^p(\mathcal{M}; H_c)$ only and then take for granted that they also have a row version, that will be used without any further comment.

Remark 2.3

(1) Applying (2.10) and (2.12), we see that

$$\Big\|\sum_k x_k \otimes e_k\Big\|_{L^2(\mathcal{M};H_c)} = \Big\|\sum_k x_k \otimes e_k\Big\|_{L^2(\mathcal{M};H_r)} = \Big(\sum_k \|x_k\|_2^2\Big)^{\frac{1}{2}}$$

for any $x_1, \ldots, x_n$ in $L^2(\mathcal{M})$. Thus $L^2(\mathcal{M}; H_c)$ and $L^2(\mathcal{M}; H_r)$ both coincide with the Hilbertian tensor product of $L^2(\mathcal{M})$ and H.

(2) The space $L^\infty(\mathcal{M}; H_c) \subset \mathcal{M}\overline{\otimes}B(H)$ is w^*-closed, and arguing as in the proof of Lemma 2.1, it is clear that $\mathcal{M} \otimes H \subset L^\infty(\mathcal{M}; H_c)$ is w^*-dense. Indeed if $(e_i)_{i\in I}$ is a basis of H for some set I, then $L^\infty(\mathcal{M}; H_c)$ coincides with the well-known space of all families $(x_i)_{i\in I}$ in $\mathcal{M}$ such that

$$\big\|(x_i)_{i\in I}\big\|_{L^\infty(\mathcal{M};H_c)} = \sup\Big\{\Big\|\Big(\sum_{i\in J} x_i^* x_i\Big)^{\frac{1}{2}}\Big\|_{\mathcal{M}} \ : J \subset I \text{ finite} \Big\} < \infty.$$

(3) Let $\{E_{ij} \,:\, i,j \geq 1\}$ be the standard matrix units on $B(\ell^2)$, and let $(e_k)_{k\geq 1}$ be the canonical basis of ℓ^2. It follows either from the definition of $L^p(\mathcal{M}; \ell^2_c)$, or from (2.10), that for any finite family $(x_k)_k$ in $L^p(\mathcal{M})$, we have

$$\Big\|\sum_k x_k \otimes e_k\Big\|_{L^p(\mathcal{M};\ell^2_c)} = \Big\|\sum_k E_{k1} \otimes x_k\Big\|_{S^p[L^p(\mathcal{M})]}.$$

A similar result holds true for row norms.

For any $1 \leq p \leq \infty$, the linear mapping

$$Q_p \colon L^p(\mathcal{M}\overline{\otimes}B(H)) \longrightarrow L^p(\mathcal{M}\overline{\otimes}B(H))$$

taking any $x \in L^p(\mathcal{M}\overline{\otimes}B(H))$ to $x(1 \otimes p_e)$ is a contractive projection whose range is equal to $L^p(\mathcal{M}; H_c)$. Moreover these projections are compatible. Thus applying (2.4) for $\mathcal{M}\overline{\otimes}B(H)$, we obtain that

$$[L^\infty(\mathcal{M}; H_c), L^1(\mathcal{M}; H_c)]_{\frac{1}{p}} = L^p(\mathcal{M}; H_c), \qquad 1 \leq p \leq \infty, \tag{2.13}$$

A similar result holds for row spaces.

Likewise, applying (2.3) to $\mathcal{M}\overline{\otimes}B(H)$, we obtain that

$$L^p(\mathcal{M}; H_c)^* = L^{p'}(\mathcal{M}; \overline{H}_r), \qquad 1 \leq p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{2.14}$$

for the duality pairing defined by taking $(x \otimes a, y \otimes \bar{b})$ to $\langle a, b\rangle\tau(xy)$ for any $x \in L^p(\mathcal{M})$, $y \in L^{p'}(\mathcal{M})$, and $a, b \in H$. By (2.10) and (2.12), an essentially equivalent reformulation of this duality result is that for any $1 \le p < \infty$ and for any $x_1, \ldots, x_n \in L^p(\mathcal{M})$, we have

(2.15)
$$\Big\|\Big(\sum_{k=1}^n x_k^* x_k\Big)^{\frac{1}{2}}\Big\|_p = \sup\Big\{\Big|\sum_{k=1}^n \langle y_k, x_k\rangle\Big| \,:\, y_k \in L^{p'}(\mathcal{M}),\ \Big\|\Big(\sum_{k=1}^n y_k y_k^*\Big)^{\frac{1}{2}}\Big\|_{p'} \le 1\Big\}$$

We need to introduce two more H-valued noncommutative L^p-spaces, namely the intersection and the sum of row and column spaces. These spaces naturally appear in the so-called noncommutative Khintchine inequalities (see below). Let $1 \le p < \infty$. We will regard $(L^p(\mathcal{M}; H_c), L^p(\mathcal{M}; H_r))$ as a compatible couple of Banach spaces, in the sense of interpolation theory (see e.g. [**6**]). Indeed if we let W be the injective tensor product of $L^p(\mathcal{M})$ and H, say, Lemma 2.2 and its row counterpart yield natural one-one linear mappings $L^p(\mathcal{M}; H_c) \to W$ and $L^p(\mathcal{M}; H_r) \to W$. According to this convention, we define the intersection

$$L^p(\mathcal{M}; H_{r\cap c}) = L^p(\mathcal{M}; H_c) \cap L^p(\mathcal{M}; H_r), \tag{2.16}$$

equipped with the norm

$$\|u\|_{L^p(\mathcal{M}; H_{r\cap c})} = \max\big\{\|u\|_{L^p(\mathcal{M}; H_c)}, \|u\|_{L^p(\mathcal{M}; H_r)}\big\}. \tag{2.17}$$

Then we define the sum

$$L^p(\mathcal{M}; H_{r+c}) = L^p(\mathcal{M}; H_c) + L^p(\mathcal{M}; H_r), \tag{2.18}$$

equipped with the norm

$$\|u\|_{L^p(\mathcal{M}; H_{r+c})} = \inf\big\{\|u_1\|_{L^p(\mathcal{M}; H_c)} + \|u_2\|_{L^p(\mathcal{M}; H_r)} \,:\, u = u_1 + u_2\big\}. \tag{2.19}$$

We now introduce Rademacher averages. Let $(\varepsilon_k)_{k\ge 1}$ be a Rademacher sequence, that is, a sequence of independent random variables on a probability space $(\Sigma, \mathbb{P})$ such that $\mathbb{P}\big(\varepsilon_k = 1\big) = \mathbb{P}\big(\varepsilon_k = -1\big) = \frac{1}{2}$ for any $k \ge 1$. Then for any finite family $x_1, \ldots, x_n$ in an arbitrary Banach space X, we let

$$\Big\|\sum_{k=1}^n \varepsilon_k\, x_k\Big\|_{\mathrm{Rad}(X)} = \int_\Sigma \Big\|\sum_{k=1}^n \varepsilon_k(\lambda) x_k\Big\|_X \, d\mathbb{P}(\lambda)\,. \tag{2.20}$$

If $X = L^p(\mathcal{M})$ is a noncommutative L^p-space for some $1 \le p < \infty$, the above norms satisfy the following remarkable estimates (called the noncommutative Khintchine inequalities). Let H be a Hilbert space and let $(e_k)_{k\ge 1}$ be an orthonormal sequence in H.

(i) If $2 \le p < \infty$, there is a constant $C_p > 0$ (only depending on p) such that for any $x_1, \ldots, x_n$ in $L^p(\mathcal{M})$, we have

(2.21)
$$\frac{1}{\sqrt{2}}\Big\|\sum_{k=1}^n x_k \otimes e_k\Big\|_{L^p(\mathcal{M}; H_{r\cap c})} \le \Big\|\sum_{k=1}^n \varepsilon_k\, x_k\Big\|_{\mathrm{Rad}(L^p(\mathcal{M}))} \le C_p \Big\|\sum_{k=1}^n x_k \otimes e_k\Big\|_{L^p(\mathcal{M}; H_{r\cap c})}.$$

(ii) There is a constant C_1 such that for any $1 \leq p \leq 2$ and any $x_1, \ldots, x_n$ in $L^p(\mathcal{M})$, we have

$$C_1 \Big\| \sum_{k=1}^n x_k \otimes e_k \Big\|_{L^p(\mathcal{M};H_{r+c})} \leq \Big\| \sum_{k=1}^n \varepsilon_k\, x_k \Big\|_{\mathrm{Rad}(L^p(\mathcal{M}))} \leq \Big\| \sum_{k=1}^n x_k \otimes e_k \Big\|_{L^p(\mathcal{M};H_{r+c})}. \tag{2.22}$$

These fundamental inequalities were proved by Lust-Piquard [**51**] for the Schatten spaces when $p > 1$ and then extended to the general case by Lust-Piquard and Pisier [**52**]. In accordance with (2.21) and (2.22), we let for any Hilbert space H

$$L^p(\mathcal{M}; H_{rad}) = L^p(\mathcal{M}; H_{r+c}) \qquad \text{if } \ 1 \leq p \leq 2; \tag{2.23}$$

$$L^p(\mathcal{M}; H_{rad}) = L^p(\mathcal{M}; H_{r\cap c}) \qquad \text{if } \ 2 \leq p < \infty. \tag{2.24}$$

Then it easily follows from (2.14) and its row counterpart that we have an isometric identification

$$L^p(\mathcal{M}; H_{rad})^* = L^{p'}(\mathcal{M}; \overline{H}_{rad}), \qquad 1 < p, p' < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.25}$$

We conclude this paragraph by a simple lemma concerning tensor extensions.

Lemma 2.4. — *Let H, K be two Hilbert spaces and let $L^p(\mathcal{M})$ be a noncommutative L^p-space, for some $1 \leq p < \infty$. Then for any bounded operator $T\colon H \to K$, the mapping $I_{L^p} \otimes T$ (uniquely) extends to a bounded operator from $L^p(\mathcal{M}; H_c)$ into $L^p(\mathcal{M}; K_c)$, with*

$$\big\| I_{L^p} \otimes T\colon L^p(\mathcal{M}; H_c) \longrightarrow L^p(\mathcal{M}; K_c) \big\| = \|T\|.$$

Likewise $I_{L^p} \otimes T$ extends to bounded operators of norm $\|T\|$ from $L^p(\mathcal{M}; H_r)$ into $L^p(\mathcal{M}; K_r)$, from $L^p(\mathcal{M}; H_{r\cap c})$ into $L^p(\mathcal{M}; K_{r\cap c})$, and from $L^p(\mathcal{M}; H_{r+c})$ into $L^p(\mathcal{M}; K_{r+c})$.

All these extensions will be usually denoted by $\widehat{T}$.

Proof. — Let $T\colon H \to K$ be a bounded operator, and let $1 \leq p < \infty$. Let $(e_1, \ldots, e_n)$ be a finite orthonormal family in H, and let $x_1, \ldots, x_n$ be arbitrary elements in $L^p(\mathcal{M})$. We consider $u = \sum_k x_k \otimes e_k$ and $\widehat{T}(u) = \sum_k x_k \otimes T(e_k)$. Then its norm in $L^p(\mathcal{M}; K_c)$ is equal to

$$\|\widehat{T}(u)\| \ = \ \Big\| \Big(\sum_{i,j} \langle T(e_j), T(e_i)\rangle\, x_i^* x_j \Big)^{\frac{1}{2}} \Big\|_p,$$

by (2.9). The $n \times n$ matrix $[\langle T(e_j), T(e_i)\rangle]$ is nonnegative and its norm is less than or equal to $\|T\|^2$. Hence we may find a matrix $\Delta = [d_{ij}] \in M_n$ such that

$$\Delta^*\Delta = \big[\langle T(e_j), T(e_i)\rangle\big] \qquad \text{and} \qquad \|\Delta\| \leq \|T\|.$$

Then we have

$$\begin{aligned}\sum_{i,j}\langle T(e_j), T(e_i)\rangle\, x_i^*x_j &= \sum_{i,j,k}\overline{d_{ki}}d_{kj}\, x_i^*x_j\\ &= \sum_k\Big(\sum_i d_{ki}\,x_i\Big)^*\Big(\sum_j d_{kj}\,x_j\Big).\end{aligned}$$

Hence

$$\|\widehat{T}(u)\| = \left\|\begin{bmatrix} & \cdot & \\ \cdot & d_{ij} & \cdot \\ & \cdot & \end{bmatrix}\begin{bmatrix} x_1 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ x_n & 0 & \cdots & 0\end{bmatrix}\right\|_p \leq \|\Delta\|\,\left\|\begin{bmatrix} x_1 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ x_n & 0 & \cdots & 0\end{bmatrix}\right\|_p.$$

According to (2.10), this implies that $\|\widehat{T}(u)\| \leq \|\Delta\|\|u\| \leq \|T\|\|u\|$ and proves the column version of our lemma.

The proof of the row version is similar and the other two results are straightforward consequences. □

2.C. Vector-valued functions

In this paragraph, we give preliminary results in the case when the Hilbert space H is a concrete (commutative) L^2-space. We let (Ω,μ) denote an arbitrary σ-finite measure space, and we shall consider Banach space valued L^2-spaces $L^2(\Omega;X)$. For any Banach space X, this space consists of all (strongly) measurable functions $u\colon\Omega\to X$ such that $\int_\Omega \|u(t)\|_X^2\, d\mu(t)$ is finite. The norm on this space is given by

$$\|u\|_{L^2(\Omega;X)} = \Big(\int_\Omega \|u(t)\|_X^2\, d\mu(t)\Big)^{\frac{1}{2}},\qquad u\in L^2(\Omega;X).$$

The main reference for these spaces is [**23**], to which we refer the reader for more information and background. We merely recall a few facts.

First, the tensor product $L^2(\Omega)\otimes X$ is dense in $L^2(\Omega;X)$.

Second, for any $u\in L^2(\Omega;X)$, and for any $v\in L^2(\Omega;X^*)$, the function $t\mapsto \langle v(t),u(t)\rangle$ is integrable and we may define a duality pairing

$$\langle v,u\rangle = \int_\Omega \langle v(t),u(t)\rangle\, d\mu(t)\,. \tag{2.26}$$

This pairing induces an isometric inclusion

$$L^2(\Omega;X^*) \hookrightarrow L^2(\Omega;X)^*. \tag{2.27}$$

If further X is reflexive, then this isometric inclusion is onto, and we obtain an isometric isomorphism $L^2(\Omega;X)^* = L^2(\Omega;X^*)$ (see e.g. [**23**, IV;1]).

Third, as a consequence of (2.4), we have

$$\big[L^2(\Omega;L^\infty(\mathcal{M})), L^2(\Omega,L^1(\mathcal{M}))\big]_{\frac{1}{p}} = L^2(\Omega;L^p(\mathcal{M})),\qquad 1\leq p\leq\infty, \tag{2.28}$$

whenever $(\mathcal{M},\tau)$ is a semifinite von Neumann algebra.

Proposition 2.5. — *Let* (Ω, μ) *be a measure space.*

(1) *For any* $1 \leq p \leq 2$, *we have contractive inclusions*

$$L^p(\mathcal{M}; L^2(\Omega)_c) \subset L^2(\Omega; L^p(\mathcal{M}))\,, \quad L^p(\mathcal{M}; L^2(\Omega)_r) \subset L^2(\Omega; L^p(\mathcal{M}))\,,$$
$$\textit{and} \quad L^p(\mathcal{M}; L^2(\Omega)_{rad}) \subset L^2(\Omega; L^p(\mathcal{M})).$$

(2) *For any* $2 \leq p \leq \infty$, *we have contractive inclusions*

$$L^2(\Omega; L^p(\mathcal{M})) \subset L^p(\mathcal{M}; L^2(\Omega)_c) \quad \textit{and} \quad L^2(\Omega; L^p(\mathcal{M})) \subset L^p(\mathcal{M}; L^2(\Omega)_r).$$

For $p \neq \infty$, *we also have a contractive inclusion*

$$L^2(\Omega; L^p(\mathcal{M})) \subset L^p(\mathcal{M}; L^2(\Omega)_{rad}).$$

Proof. — Given a measurable subset $I \subset \Omega$, we let χ_I denote the indicator function of I. Let $x_1, \ldots, x_n$ be in $L^1(\mathcal{M})$. Then for any $y_1, \ldots, y_n$ in $\mathcal{M} = L^1(\mathcal{M})^*$, we have

$$\begin{aligned}\Big|\sum_k \langle y_k, x_k\rangle\Big| &\leq \Big\|\Big(\sum_k y_k y_k^*\Big)^{\frac12}\Big\|_{\mathcal{M}} \Big\|\Big(\sum_k x_k^* x_k\Big)^{\frac12}\Big\|_{L^1(\mathcal{M})} \\ &\leq \Big(\sum_k \|y_k\|_\infty^2\Big)^{\frac12} \Big\|\Big(\sum_k x_k^* x_k\Big)^{\frac12}\Big\|_1.\end{aligned}$$

Taking the supremum over all $y_1, \ldots, y_n$ with $\sum_k \|y_k\|_\infty^2 \leq 1$ yields

$$\Big(\sum_k \|x_k\|_1^2\Big)^{\frac12} \leq \Big\|\Big(\sum_k x_k^* x_k\Big)^{\frac12}\Big\|_1. \tag{2.29}$$

Now changing x_k into $\mu(I_k)^{\frac12} x_k$ for a sequence $I_1, \ldots, I_n$ of pairwise disjoint measurable subsets of finite measure in Ω, and using (2.9), we derive that

$$\Big\|\sum_{k=1}^n x_k \otimes \chi_{I_k}\Big\|_{L^2(\Omega; L^1(\mathcal{M}))} \leq \Big\|\sum_{k=1}^n x_k \otimes \chi_{I_k}\Big\|_{L^1(\mathcal{M}; L^2(\Omega)_c)}.$$

By density this shows that $L^1(\mathcal{M}; L^2(\Omega)_c) \subset L^2(\Omega; L^1(\mathcal{M}))$ contractively. On the other hand, we have an isometric isomorphism $L^2(\mathcal{M}; L^2(\Omega)_c) = L^2(\Omega; L^2(\mathcal{M}))$ by Remark 2.3 (1). Thus in the column case, the result for $1 \leq p \leq 2$ follows by interpolation, using (2.13) and (2.28). The row case can be treated similarly and the Rademacher case follows from the previous two cases. Once (1) is proved, (2) follows by duality. □

Remark 2.6. — Let $1 \leq p < \infty$ and let $p' = p/(p-1)$ be its conjugate number. If we identify $H = L^2(\Omega)$ with its complex conjugate in the usual way, and if we set $X = L^p(\mathcal{M})$, then the duality pairing given by (2.26) is consistent with the one in (2.14). Namely if $1 \leq p \leq 2$, if $u \in L^p(\mathcal{M}, L^2(\Omega)_c)$ and if $v \in L^2(\Omega; L^{p'}(\mathcal{M}))$, then the action of v on u induced by (2.14) is given by (2.26). Indeed, this is clear when $u \in L^p(\mathcal{M}) \otimes L^2(\Omega)$ and $v \in L^{p'}(\mathcal{M}) \otimes L^2(\Omega)$, and the general case follows by a density argument. This property will be extended in Lemma 2.8 below.

Definition 2.7. — *Let $1 \leq p < \infty$.*

(1) *Let $u\colon \Omega \to L^p(\mathcal{M})$ be a measurable function. We say that u belongs to $L^p(\mathcal{M}; L^2(\Omega)_c)$ if $\langle y, u(\cdot)\rangle$ belongs to $L^2(\Omega)$ for any $y \in L^{p'}(\mathcal{M})$ and if there exists $\theta \in L^p(\mathcal{M}; L^2(\Omega)_c)$ such that*

$$\langle y \otimes b, \theta\rangle = \int_\Omega \langle y, u(t)\rangle\, b(t)\, d\mu(t)\,, \qquad y \in L^{p'}(\mathcal{M}),\ b \in L^2(\Omega). \tag{2.30}$$

(2) *Let $\theta \in L^p(\mathcal{M}; L^2(\Omega)_c)$. We say that θ is representable by a measurable function is there exists a measurable $u\colon \Omega \to L^p(\mathcal{M})$ such that $\langle y, u(\cdot)\rangle$ belongs to $L^2(\Omega)$ for any $y \in L^{p'}(\mathcal{M})$ and (2.30) holds true.*

If (1) (resp. (2)) holds, then θ (resp. u) is necessarily unique. Therefore we will make no notational difference between θ and u in this case.

A similar terminology will be used for row spaces $L^p(\mathcal{M}; L^2(\Omega)_r)$ or Rademacher spaces $L^p(\mathcal{M}; L^2(\Omega)_{rad})$.

It is clear from Remark 2.6 that any $u \in L^p(\mathcal{M}; L^2(\Omega)_c) \cap L^2(\Omega; L^p(\mathcal{M}))$ is representable by a measurable function. Hence if $1 \leq p \leq 2$, any element of $L^p(\mathcal{M}; L^2(\Omega)_c)$ is representable by a measurable function. However we will see in Appendix B that this is no longer the case if $p > 2$.

Lemma 2.8. — *Let $1 < p, p' < \infty$ be conjugate numbers and let $u \in L^p(\mathcal{M}; L^2(\Omega)_c)$ and $v \in L^{p'}(\mathcal{M}; L^2(\Omega)_r)$ be (representable by) measurable functions in the sense of Definition 2.7. Then the function $t \mapsto \langle v(t), u(t)\rangle$ is integrable on Ω and*

$$\int_\Omega \bigl|\langle v(t), u(t)\rangle\bigr|\, d\mu(t) \ \leq \|u\|_{L^p(\mathcal{M}; L^2(\Omega)_c)} \|v\|_{L^{p'}(\mathcal{M}; L^2(\Omega)_r)}. \tag{2.31}$$

Moreover the action of v on u given by (2.14) for $H = L^2(\Omega)$ is equal to

$$\langle v, u\rangle = \int_\Omega \langle v(t), u(t)\rangle\, d\mu(t)\,. \tag{2.32}$$

Proof. — We may assume that $p > 2$. We fix some measurable u in $L^p(\mathcal{M}, L^2(\Omega)_c)$. By assumption, (2.30) holds true for any $y \in L^{p'}(\mathcal{M})$ and any $b \in L^2(\Omega)$. Hence $t \mapsto \langle v(t), u(t)\rangle$ is integrable and (2.32) holds true for any v in the tensor product $L^{p'}(\mathcal{M}) \otimes L^2(\Omega)$. Let $c \in L^\infty(\Omega)$ and let v in $L^{p'}(\mathcal{M}) \otimes L^2(\Omega)$. Then

$$\int_\Omega \langle v(t), u(t)\rangle\, c(t)\, d\mu(t) \ = \ \langle cv, u\rangle.$$

Hence by the above observation, we have

$$\left|\int_\Omega \langle v(t), u(t)\rangle\, c(t)\, d\mu(t)\right| \ \leq \ \|u\|_{L^p(\mathcal{M}; L^2(\Omega)_c)} \|cv\|_{L^{p'}(\mathcal{M}; L^2(\Omega)_r)}.$$

Applying Lemma 2.4 to the multiplication operator $L^2(\Omega) \to L^2(\Omega)$ taking any $b \in L^2(\Omega)$ to cb, we obtain that the right hand side of the above inequality is less than or equal to

$$\|c\|_\infty \|u\|_{L^p(\mathcal{M};L^2(\Omega)_c)} \|v\|_{L^{p'}(\mathcal{M};L^2(\Omega)_r)}.$$

Taking the supremum over all $c \in L^\infty(\Omega)$ with norm less than 1, we obtain (2.31) for $v \in L^{p'}(\mathcal{M}) \otimes L^2(\Omega)$.

Next we consider an arbitrary $v \in L^{p'}(\mathcal{M}; L^2(\Omega)_r)$. By Proposition 2.5, we can find a sequence $(v_n)_{n\geq 1}$ in $L^{p'}(\mathcal{M}) \otimes L^2(\Omega)$ such that

$$\|v - v_n\|_{L^2(\Omega;L^{p'}(\mathcal{M}))} \leq \|v - v_n\|_{L^{p'}(\mathcal{M};L^2(\Omega)_r)} \longrightarrow 0.$$

Passing to a subsequence, we may assume that $v_n \to v$ a.e. Then $\langle u, v_n\rangle \to \langle u, v\rangle$ a.e., and we deduce (2.31) by Fatou's Lemma.

Finally applying (2.31) with $(v - v_n)$ instead of v, we deduce that since each v_n satisfies (2.32), then v satisfies it as well. $\square$

Remark 2.9. — The previous lemma clearly has variants (with identical proofs) involving the Rademacher spaces. Namely, if $u \in L^p(\mathcal{M}; L^2(\Omega)_{rad})$ and $v \in L^{p'}(\mathcal{M}; L^2(\Omega)_{rad})$ are measurable functions, then the function $t \mapsto \langle v(t), u(t)\rangle$ is integrable on Ω, the identity (2.32) holds true, and

$$\int_\Omega \bigl|\langle v(t), u(t)\rangle\bigr|\, d\mu(t) \leq \|u\|_{L^p(\mathcal{M};L^2(\Omega)_{rad})} \|v\|_{L^{p'}(\mathcal{M};L^2(\Omega)_{rad})}.$$

We conclude our discussion on measurable functions with the following useful converse to Lemma 2.8.

Lemma 2.10. — *Let $1 \leq p < \infty$, and let p' be its conjugate number. Let $u\colon \Omega \to L^p(\mathcal{M})$ be a measurable function. Then $u \in L^p(\mathcal{M}; L^2(\Omega)_c)$ if and only if $t \mapsto \langle y, u(t)\rangle$ belongs to $L^2(\Omega)$ for any $y \in L^{p'}(\mathcal{M})$ and there is a constant $K > 0$ such that for any $v \in L^{p'}(\mathcal{M}) \otimes L^2(\Omega)$, we have*

$$\left| \int_\Omega \langle v(t), u(t)\rangle\, d\mu(t) \right| \leq K \|v\|_{L^{p'}(\mathcal{M};L^2(\Omega)_r)}.$$

In this case, the norm of u in $L^p(\mathcal{M}; L^2(\Omega)_c)$ is equal to the smallest possible K.

Proof. — The 'only if' part follows from Lemma 2.8. If $p > 1$, the 'if' part is a direct consequence of (2.14) and of the density of $L^{p'}(\mathcal{M}) \otimes L^2(\Omega)$ in $L^{p'}(\mathcal{M}; L^2(\Omega)_r)$. Thus it suffices to consider the case when $p = 1$. In this case, the result can be deduced from operator space arguments which will be outlined in the paragraph 2.D, and from [**62**, Lemma 1.12]. However we give a self-contained proof for the convenience of the reader.

We assume for simplicity that $H = L^2(\Omega)$ is infinite dimensional and separable (otherwise, replace sequences by nets in the argument below). Let $u\colon \Omega \to L^1(\mathcal{M})$ be a measurable function, and assume that

$$K = \sup\Big\{\Big|\int_\Omega \langle v(t), u(t)\rangle \, d\mu(t)\Big| \ : \ v \in \mathcal{M} \otimes L^2(\Omega),\ \|v\|_{L^\infty(\mathcal{M}; L^2(\Omega)_r)} \leq 1\Big\} < \infty.$$

By Proposition 2.5 (2), the norm on $\mathcal{M} \otimes L^2(\Omega)$ induced by $L^2(\Omega;\mathcal{M})$ is greater than the one induced by $L^\infty(\mathcal{M}; L^2(\Omega)_r)$. Thus $v \mapsto \int_\Omega \langle v, u\rangle \, d\mu$ extends to an element of $L^2(\Omega;\mathcal{M})^*$. Since u is measurable and valued in $L^1(\mathcal{M})$, we deduce from (2.27) that $u \in L^2(\Omega, L^1(\mathcal{M}))$.

Let $(e_k)_{k\geq 1}$ be an orthonormal basis of $H = L^2(\Omega)$. Since $u \in L^2(\Omega, L^1(\mathcal{M}))$, we can define $x_k \in L^1(\mathcal{M})$ by

$$x_k = \int_\Omega e_k(t) u(t) \, d\mu(t)\,, \qquad k \geq 1.$$

For any $n \geq 1$, we consider

$$u_n = \sum_{k=1}^n x_k \otimes e_k \in L^1(\mathcal{M}) \otimes L^2(\Omega).$$

For convenience we let $Z = L^1(\mathcal{M}, L^2(\Omega)_c)$ in the rest of the proof. Our objective is now to show that $(u_n)_{n\geq 1}$ is a Cauchy sequence in Z. For any $m \geq 1$, let $P_m\colon H \to H$ be the orthogonal projection onto $\mathrm{Span}\{e_1, \ldots, e_m\}$. If $m \leq n$, then we have $(I_{L^1} \otimes P_m)(u_n) = u_m$. Hence $\|u_m\|_Z \leq \|u_n\|_Z$ by Lemma 2.4. Thus the sequence $(\|u_n\|_Z)_n$ is nondecreasing.

Next we note that for any $n \geq 1$, we have

$$\|u_n\|_Z = \sup\Big\{\Big|\sum_{k=1}^n \langle y_k, x_k\rangle\Big| \ : \ y_k \in \mathcal{M},\ \Big\|\sum_{k=1}^n y_k \otimes e_k\Big\|_{L^\infty(\mathcal{M}; L^2(\Omega)_r)} \leq 1\Big\}.$$

However if we write $v = \sum_{k=1}^n y_k \otimes e_k$, we have

$$\sum_{k=1}^n \langle y_k, x_k\rangle = \int_\Omega \langle v(t), u(t)\rangle \, d\mu(t)\,.$$

Hence $(\|u_n\|_Z)_n$ is bounded, with $\sup_n \|u_n\|_Z = K$.

Let $\varepsilon > 0$, and let $N \geq 1$ be chosen such that $\|u_N\|_Z^2 \geq K^2 - \varepsilon^2$. Let $n \geq m \geq N$ be two integers. According to (2.29), we have

$$\|u_m\|_Z^2 + \|u_n - u_m\|_Z^2 \leq \Big\|\big(u_m^* u_m + (u_n - u_m)^*(u_n - u_m)\big)^{\frac{1}{2}}\Big\|_1^2.$$

Since $n \geq m$, we have $u_m^* u_m = u_n^* u_m = u_m^* u_n$, hence

$$u_m^* u_m + (u_n - u_m)^*(u_n - u_m) = u_n^* u_n.$$

Thus we have

$$\|u_m\|_Z^2 + \|u_n - u_m\|_Z^2 \leq \|u_n\|_Z^2,$$

and hence

$$\|u_n - u_m\|_Z^2 \leq K^2 - \|u_m\|_Z^2 \leq \varepsilon^2.$$

This shows that $(u_n)_n$ is a Cauchy sequence in $L^1(\mathcal{M}, L^2(\Omega)_c)$. It has therefore a limit in that space and by construction, this limit is necessarily u. This shows that $u \in L^1(\mathcal{M}, L^2(\Omega)_c)$, with $\|u\|_Z \leq K$. $\square$

Remark 2.11. — Again the previous lemma also has variants involving $L^2(\Omega)_r$, $L^2(\Omega)_{r\cap c}$, or $L^2(\Omega)_{r+c}$. For instance, a measurable function $u\colon \Omega \to L^p(\mathcal{M})$ belongs to $L^p(\mathcal{M}; L^2(\Omega)_{r+c})$ with $\|u\|_{L^p(\mathcal{M};L^2(\Omega)_{r+c})} \leq K$ if and only if

$$\left| \int_\Omega \langle v(t), u(t)\rangle \, d\mu(t) \right| \leq K\|v\|_{L^{p'}(\mathcal{M};L^2(\Omega)_{r\cap c})}$$

for any $v \in L^{p'}(\mathcal{M}) \otimes L^2(\Omega)$.

We also observe that if $\mathcal{V} \subset L^2(\Omega)$ is a dense subspace, then the same result holds true with $L^{p'}(\mathcal{M}) \otimes L^2(\Omega)$ replaced by $L^{p'}(\mathcal{M}) \otimes \mathcal{V}$.

We will now interpret the above results in the case when $H = L^2(\Omega) = \ell^2$, and regard $L^p(\mathcal{M}, \ell^2_c)$, $L^p(\mathcal{M}, \ell^2_r)$, and $L^p(\mathcal{M}, \ell^2_{rad})$ as sequence spaces. Let $(e_k)_{k\geq 1}$ denote the canonical basis of ℓ^2. For any $k \geq 1$, let $\varphi_k = \langle \cdot\,, e_k\rangle$ be the functional on ℓ^2 associated with e_k, and let $\widehat{\varphi_k}\colon L^p(\mathcal{M}; \ell^2_c) \to L^p(\mathcal{M})$ denote the continuous extension of $I_{L^p} \otimes \varphi_k$. We say that a sequence $(x_k)_{k\geq 1}$ of $L^p(\mathcal{M})$ belongs to $L^p(\mathcal{M}, \ell^2_c)$ if there exists some (necessarily unique) u in $L^p(\mathcal{M}, \ell^2_c)$ such that $x_k = \widehat{\varphi_k}(u)$ for any $k \geq 1$. We adopt a similar convention for $L^p(\mathcal{M}, \ell^2_r)$ and $L^p(\mathcal{M}, \ell^2_{rad})$.

Corollary 2.12. — *Let $1 \leq p < \infty$ and let $(x_k)_{k\geq 1}$ be a sequence of $L^p(\mathcal{M})$. Then $(x_k)_{k\geq 1}$ belongs to $L^p(\mathcal{M}, \ell^2_c)$ if and only if there is a constant $K > 0$ such that*

$$\Big\| \sum_{k=1}^n x_k \otimes e_k \Big\|_{L^p(\mathcal{M};\ell^2_c)} \leq K, \qquad n \geq 1.$$

In this case the norm of $(x_k)_{k\geq 1}$ in $L^p(\mathcal{M}, \ell^2_c)$ is equal to the smallest possible K.

Moreover the same result holds with ℓ^2_c replaced by either ℓ^2_r or ℓ^2_{rad}.

Proof. — This clearly follows from Lemma 2.10 and Remark 2.11. $\square$

We end this paragraph with a few observations to be used later on concerning Rademacher norms and vector-valued L^2-spaces.

Let $\mathrm{Rad} \subset L^1(\Sigma)$ be the closed subspace spanned by the ε_k's. For any $x_1, \ldots, x_n$ in some Banach space X, the norm $\big\|\sum_k \varepsilon_k x_k\big\|_{\mathrm{Rad}(X)}$ defined by (2.20) is the norm of the sum $\sum_k \varepsilon_k \otimes x_k$ in the vector valued L^1-space $L^1(\Sigma; X)$. Accordingly, we let $\mathrm{Rad}(X) \subset L^1(\Sigma; X)$ be the closure of $\mathrm{Rad} \otimes X$ in $L^1(\Sigma; X)$. Likewise, we let $\mathrm{Rad}_2 \subset L^2(\Sigma)$ be the closed linear span of the ε_k's in $L^2(\Sigma)$, and we let $\mathrm{Rad}_2(X) \subset L^2(\Sigma; X)$ be the closure of $\mathrm{Rad}_2 \otimes X$. By Kahane's inequality (see e.g. [**50**, Theorem 1.e.13]), the spaces $\mathrm{Rad}(X)$ and $\mathrm{Rad}_2(X)$ are isomorphic.

We let $P\colon L^2(\Sigma) \to L^2(\Sigma)$ denote the orthogonal projection onto Rad_2. Let (E_0, E_1) be an interpolation couple of Banach spaces, and assume that E_0 and E_1 are both K-convex. Following [**60**, p. 43] or [**59**], this means that for $i \in \{0,1\}$, the tensor extension $P \otimes I_{E_i}\colon L^2 \otimes E_i \to L^2 \otimes E_i$ extends to a bounded projection on $L^2(\Sigma; E_i)$, whose range is equal to $\mathrm{Rad}_2(E_i)$. For any $\alpha \in (0,1)$, let $E_\alpha = [E_0, E_1]_\alpha$. Then

$$L^2(\Sigma; E_\alpha) \,=\, [L^2(\Sigma; E_0), L^2(\Sigma; E_1)]_\alpha,$$

where $[\ ,\ ]_\alpha$ is obtained by the complex interpolation method [**6**]. Owing to the projections onto $\mathrm{Rad}_2(E_0)$ and $\mathrm{Rad}_2(E_1)$ given by the K-convexity, this implies that $\mathrm{Rad}_2(E_\alpha)$ is isomorphic to the interpolation space $[\mathrm{Rad}_2(E_0)\,,\,\mathrm{Rad}_2(E_1)]_\alpha$. Applying Kahane's inequality, we finally obtain the isomorphism

$$\mathrm{Rad}(E_\alpha) \,\approx\, \big[\mathrm{Rad}(E_0)\,,\,\mathrm{Rad}(E_1)\big]_\alpha.$$

Now note that for any $1 < p < \infty$, the Banach space $L^p(\mathcal{M})$ is K-convex. Indeed, this follows from [**25**] and [**59**]. (More generally, any UMD Banach space is K-convex.) Thus we deduce from above and from (2.4) that if $1 < r < q < \infty$, we have

$$\mathrm{Rad}(L^p(\mathcal{M})) \,\approx\, \big[\mathrm{Rad}(L^q(\mathcal{M}))\,,\,\mathrm{Rad}(L^r(\mathcal{M}))\big]_\alpha \qquad \text{if} \quad \frac{1}{p} \,=\, \frac{1-\alpha}{q} \,+\, \frac{\alpha}{r}\,. \tag{2.33}$$

Note also that by our definitions in paragraph 2.B, we have

$$\mathrm{Rad}(L^p(\mathcal{M})) \,\approx\, L^p(\mathcal{M}; \ell^2_{rad}), \qquad 1 \le p < \infty\,. \tag{2.34}$$

2.D. Completely positive maps and completely bounded maps

Let $1 \le p \le \infty$. We say that a linear map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is positive if it maps the positive cone $L^p(\mathcal{M})_+$ into itself. Then for an integer $n \ge 2$, we say that T is n-positive if

$$I_{S^p_n} \otimes T\colon S^p_n[L^p(\mathcal{M})] \longrightarrow S^p_n[L^p(\mathcal{M})]$$

is positive. Recall here that $S^p_n[L^p(\mathcal{M})] \,=\, L^p(M_n(\mathcal{M}))$ is a noncommutative L^p-space. Finally we say that T is completely positive if it is n-positive for all n. We refer the reader e.g. to [**58**] for a large information on completely positive maps on C^*-algebras.

Likewise, we say that $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is completely bounded if

$$\|T\|_{cb} \,=\, \sup_n \big\| I_{S^p_n} \otimes T\colon S^p_n[L^p(\mathcal{M})] \longrightarrow S^p_n[L^p(\mathcal{M})]\big\|$$

is finite. In this case $\|T\|_{cb}$ is called the completely bounded norm of T. If p is finite, it is easy to see that T is completely bounded if and only if $I_{S^p} \otimes T$ extends to a bounded operator from $S^p[L^p(\mathcal{M})]$ into itself. In that case, the extension is unique and

$$\|T\|_{cb} = \big\| I_{S^p} \otimes T\colon S^p[L^p(\mathcal{M})] \longrightarrow S^p[L^p(\mathcal{M})]\big\|. \tag{2.35}$$

More generally if $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is completely bounded and H is any Hilbert space, then $I_{S^p(H)} \otimes T$ extends to a bounded operator from $S^p[H; L^p(\mathcal{M})]$ into itself, whose norm is less than or equal to $\|T\|_{cb}$. Consequently, $T \otimes I_H$ both extends to bounded operators on $L^p(\mathcal{M}; H_c)$ and on $L^p(\mathcal{M}; H_r)$, with

$$\big\|T \otimes I_H \colon L^p(\mathcal{M}; H_c) \longrightarrow L^p(\mathcal{M}; H_c)\big\| \,\leq\, \|T\|_{cb} \tag{2.36}$$

and

$$\big\|T \otimes I_H \colon L^p(\mathcal{M}; H_r) \longrightarrow L^p(\mathcal{M}; H_r)\big\| \,\leq\, \|T\|_{cb}. \tag{2.37}$$

If $\|T\|_{cb} \leq 1$, we say that T is completely contractive. Next we say that the operator $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is a complete isometry if $I_{S^p_n} \otimes T$ is an isometry for any $n \geq 1$. In this case, $I_{S^p} \otimes T\colon S^p[L^p(\mathcal{M})] \to S^p[L^p(\mathcal{M})]$ is also an isometry.

Assume that $1 \leq p < \infty$, and let p' be its conjugate number. Applying (2.3) with $M_n(\mathcal{M})$, we have an isometric identification $S^p_n[L^p(\mathcal{M})]^* = S^{p'}_n[L^{p'}(\mathcal{M})]$. It clearly follows from this identity that $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is completely bounded if and only its adjoint $T^*\colon L^{p'}(\mathcal{M}) \to L^{p'}(\mathcal{M})$ is completely bounded, with

$$\|T\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M})\|_{cb} \,=\, \|T^*\colon L^{p'}(\mathcal{M}) \longrightarrow L^{p'}(\mathcal{M})\|_{cb}. \tag{2.38}$$

Although we will not use it explicitly, we briefly mention that several notions considered so far have a natural description in the framework of operator space theory.

We need complex interpolation of operator spaces, for which we refer to [**63**, Section 2.7]. Let E_1 be $L^1(\mathcal{M})$ equipped with the predual operator space structure of $\mathcal{M}^{\mathrm{op}}$. Then for any $1 < p < \infty$, equip $L^p(\mathcal{M})$ with the operator space structure obtained by interpolating between $\mathcal{M} = E_\infty$ and E_1 (see [**63**, p.139]). Let E_p be this operator space, so that $E_p = [E_\infty, E_1]_{\frac{1}{p}}$ completely isometrically. Then for any Hilbert space H, and any $1 \leq p < \infty$, the definition (2.7) coincides with Pisier's operator space valued Schatten space $S^p[H; E_p]$ (see [**62**, pp. 24-25]). Thus according to [**62**, Lemma 1.7], a linear map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is completely bounded in the sense of (2.35) if and only if it is completely bounded from E_p into itself in the usual sense of operator space theory.

Let H be a Hilbert space, and let H_c (resp. H_r) be the space H equipped with its column (resp. row) operator space structure (see e.g. [**63**, p.22]). Then for any $\theta \in [0,1]$, let $H_c(\theta) = [H_c, H_r]_\theta$ in the sense of the interpolation of operator spaces. Then

$$L^p(\mathcal{M}; H_c) = H_c(\tfrac{1}{p}) \otimes_h E_p, \qquad 1 \leq p < \infty,$$

where $\otimes_h$ denotes the Haagerup tensor product (see e.g. [**63**, Chapter 5]). Indeed, this identity follows from [**62**, Theorem 1.1]. Likewise, we have

$$L^p(\mathcal{M}; H_r) = E_p \otimes_h H_r(\tfrac{1}{p}), \qquad 1 \leq p < \infty,$$

where we have defined $H_r(\theta) = [H_r, H_c]_\theta$ for any $\theta \in [0,1]$.

Remark 2.13. — Let $\mathcal{M}$ be a commutative von Neumann algebra, and let Σ be a measure space such that $\mathcal{M} \simeq L^\infty(\Sigma)$ as von Neumann algebras (see e.g. [**69**, 1.18]). Then $L^p(\mathcal{M})$ coincides with the usual commutative space $L^p(\Sigma)$, and $S^p[H; L^p(\Sigma)] = L^p(\Sigma; S^p(H))$ for any H and any $1 \leq p < \infty$. Thus a completely bounded map $T\colon L^p(\Sigma) \to L^p(\Sigma)$ on some commutative L^p-space is a bounded mapping whose tensor extension $T \otimes I_{S^p}$ extends to a bounded operator on the vector valued L^p-space $L^p(\Sigma; S^p)$.

Likewise, for any Hilbert space H and any $1 \leq p < \infty$, we have
$$L^p(\mathcal{M}; H_c) = L^p(\mathcal{M}; H_r) = L^p(\mathcal{M}; H_{rad}) = L^p(\Sigma; H)$$
isometrically.

CHAPTER 3

BOUNDED AND COMPLETELY BOUNDED H^∞ FUNCTIONAL CALCULUS

3.A. H^∞ functional calculus

In this paragraph, we give a brief review of H^∞ functional calculus on general Banach spaces, and preliminary results. We mainly follow the fundamental papers [**53**] and [**21**]. See also [**3**] or [**46**] for further details. We refer the reader e.g. to [**29**] or to [**22**] for the necessary background on semigroup theory.

Let X be a Banach space, and let A be a (possibly unbounded) linear operator A on X. We let $D(A)$, $N(A)$ and $R(A)$ denote the domain, kernel and range of A respectively. Next we denote by $\sigma(A)$ and $\rho(A)$ the spectrum and the resolvent set of A respectively. Then for any $z \in \rho(A)$, we let $R(z, A) = (z - A)^{-1}$ denote the corresponding resolvent operator.

For any $\omega \in (0, \pi)$, we let

$$\Sigma_\omega = \{z \in \mathbb{C}^* \, : \, |\mathrm{Arg}(z)| < \omega\}$$

be the open sector of angle 2ω around the half-line $(0, +\infty)$. By definition, A is a sectorial operator of type ω if A is closed and densely defined, $\sigma(A) \subset \overline{\Sigma_\omega}$, and for any $\theta \in (\omega, \pi)$ there is a constant $K_\theta > 0$ such that

$$\|zR(z, A)\| \leq K_\theta, \qquad z \in \mathbb{C} \setminus \overline{\Sigma_\theta}. \tag{3.1}$$

We say that A is sectorial of type 0 if it is of type ω for any $\omega > 0$.

Let $(T_t)_{t\geq 0}$ be a bounded c_0-semigroup on X and let $-A$ denote its infinitesimal generator. Then A is closed and densely defined. Moreover $\sigma(A) \subset \overline{\Sigma_{\frac{\pi}{2}}}$ and for any $z \in \mathbb{C} \setminus \overline{\Sigma_{\frac{\pi}{2}}}$, we have

$$R(z, A) = -\int_0^\infty e^{tz}\, T_t \, dt \tag{3.2}$$

in the strong operator topology (this is the Laplace formula). It is easy to deduce that A is a sectorial operator of type $\frac{\pi}{2}$.

The following lemma is well-known. A semigroup $(T_t)_{t>0}$ which satisfies (i) and/or (ii) below for some $\omega \in (0, \frac{\pi}{2})$ is called a bounded analytic semigroup, see e.g. [**29**, I.5].

Lemma 3.1. — *Let $(T_t)_{t\geq 0}$ be a bounded c_0-semigroup on X with infinitesimal generator $-A$, and let $\omega \in (0, \frac{\pi}{2})$. The following are equivalent.*

(i) *A is sectorial of type ω.*

(ii) *For any $0 < \alpha < \frac{\pi}{2} - \omega$, $(T_t)_{t>0}$ admits a bounded analytic extension $(T_z)_{z\in\Sigma_\alpha}$ in $B(X)$.*

For any $\theta \in (0, \pi)$, let $H^\infty(\Sigma_\theta)$ be the space of all bounded analytic functions $f\colon \Sigma_\theta \to \mathbb{C}$. This is a Banach algebra for the norm

$$\|f\|_{\infty,\theta} = \sup\bigl\{|f(z)| \,:\, z \in \Sigma_\theta\bigr\}.$$

Then we let $H_0^\infty(\Sigma_\theta)$ be the subalgebra of all $f \in H^\infty(\Sigma_\theta)$ for which there exist two positive numbers $s, c > 0$ such that

$$|f(z)| \leq c\,\frac{|z|^s}{(1+|z|)^{2s}}, \qquad z \in \Sigma_\theta. \tag{3.3}$$

Let A be a sectorial operator of type $\omega \in (0, \pi)$ on X. Let $\omega < \gamma < \theta < \pi$, and let Γ_γ be the oriented contour defined by :

$$\Gamma_\gamma(t) = \begin{cases} -te^{i\gamma}, & t \in \mathbb{R}_-; \\ te^{-i\gamma}, & t \in \mathbb{R}_+. \end{cases} \tag{3.4}$$

In other words, Γ_γ is the boundary of Σ_γ oriented counterclockwise. For any $f \in H_0^\infty(\Sigma_\theta)$, we set

$$f(A) \;=\; \frac{1}{2\pi i}\int_{\Gamma_\gamma} f(z)R(z,A)\,dz\,. \tag{3.5}$$

It follows from (3.1) and (3.3) that this integral is absolutely convergent. Indeed (3.3) implies that for any $0 < \gamma < \theta$, we have

$$\int_{\Gamma_\gamma} |f(z)|\,\Bigl|\frac{dz}{z}\Bigr| < \infty\,. \tag{3.6}$$

Thus $f(A)$ is a well defined element of $B(X)$. Using Cauchy's Theorem, it is not hard to check that its definition does not depend on the choice of $\gamma \in (\omega, \theta)$. Furthermore, the mapping $f \mapsto f(A)$ is an algebra homomorphism from $H_0^\infty(\Sigma_\theta)$ into $B(X)$ which is consistent with the functional calculus of rational functions. We say that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus if the latter homomorphism is continuous, that is, there is a constant $K \geq 0$ such that

$$\|f(A)\| \leq K\|f\|_{\infty,\theta}\,, \qquad f \in H_0^\infty(\Sigma_\theta). \tag{3.7}$$

Sectorial operators and H^∞ functional calculus behave nicely with respect to duality. Assume that X is reflexive and that A is a sectorial operator of type ω on X.

Then A^* is a sectorial operator of type ω on X^* as well. Moreover for any $\theta > \omega$ and any $B \in H^\infty(\Sigma_\theta)$, we have

$$f(A^*) = f(A)^*$$

Consequently, A^* admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus if A does.

We now turn to special features of sectorial operators with dense range. For any integer $n \geq 1$, let g_n be the rational function defined by

$$g_n(z) = \frac{n^2 z}{(n+z)(1+nz)}\,. \tag{3.8}$$

If A is a sectorial operator on X, the sequences $\big(n(n+A)^{-1}\big)_n$ and $\big(nA(1+nA)^{-1}\big)_n$ are bounded. Further it is not hard to check that $n(n+A)^{-1}x \to x$ for any $x \in X$ and that $nA(1+nA)^{-1}x \to x$ for any $x \in \overline{R(A)}$ (see e.g. [**21**, Theorem 3.8]). This yields the following.

Lemma 3.2. — *Let A be a sectorial operator on X, and assume that A has dense range. Let $(g_n)_{n\geq 1}$ be defined by (3.8). Then*

$$\sup_n \|g_n(A)\| < \infty \qquad \text{and} \qquad \lim_n g_n(A)x = x \quad \text{for any } x \in X.$$

Consequently, A is one-one.

Let A be a sectorial operator of type $\omega \in (0,\pi)$ and assume that A has dense range. Our next goal is to define an operator $f(A)$ for any $f \in H^\infty(\Sigma_\theta)$, whenever $\theta > \omega$. For any $n \geq 1$, the operator $g_n(A)$ is one-one and we have

$$R(g_n(A)) \;=\; D(A) \cap R(A).$$

The latter space is therefore dense in X. We let $g = g_1$, that is

$$g(z) = \frac{z}{(1+z)^2}\,. \tag{3.9}$$

Then for any $\theta \in (\omega,\pi)$ and any $f \in H^\infty(\Sigma_\theta)$, the product function fg belongs to $H_0^\infty(\Sigma_\theta)$ hence we may define $(fg)(A) \in B(X)$ by means of (3.5). Then using the injectivity of $g(A)$, we set

$$f(A) = g(A)^{-1}(fg)(A),$$

with domain given by

$$D(f(A)) \;=\; \big\{x \in X \,:\, [(fg)(A)](x) \in D(A) \cap R(A)\big\}.$$

It turns out that $f(A)$ is a closed operator and that $D(A) \cap R(A) \subset D(f(A))$, so that $f(A)$ is densely defined. Moreover this definition is consistent with (3.5) in the case when $f \in H_0^\infty(\Sigma_\theta)$. Note however that $f(A)$ may be unbounded in general.

Theorem 3.3. — *([**53**], [**21**]) Let $0 < \omega < \theta < \pi$ and let A be a sectorial operator of type ω on X with dense range. Then $f(A)$ is bounded for any $f \in H^\infty(\Sigma_\theta)$ if and only if A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus. In that case, we have*

$$\|f(A)\| \leq K\|f\|_{\infty,\theta}\,, \qquad f \in H^\infty(\Sigma_\theta),$$

where K is the constant from (3.7).

We also recall that the above construction comprises imaginary powers of sectorial operators. Namely for any $s \in \mathbb{R}$, let f_s be the analytic function on $\mathbb{C} \setminus \mathbb{R}_-$ defined by $f_s(z) = z^{is}$. Then f_s belongs to $H^\infty(\Sigma_\theta)$ for any $\theta \in (0, \pi)$, with

$$\|f_s\|_{\infty,\theta} = e^{\theta|s|}. \tag{3.10}$$

The imaginary powers of a sectorial operator A with dense range may be defined by letting $A^{is} = f_s(A)$ for any $s \in \mathbb{R}$. In particular, A admits bounded imaginay powers if it has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some θ (see [**21**, Section 5]).

Remark 3.4. — It follows e.g. from [**21**, Theorem 3.8] that if A is a sectorial operator on a reflexive Banach space X, then X has a direct sum decomposition

$$X = N(A) \oplus \overline{R(A)}.$$

Hence A has dense range if and only if it is one-one. Moreover the restriction of A to $\overline{R(A)}$ is a sectorial operator which obviously has dense range. Thus changing X into $\overline{R(A)}$, or changing A into the sum $A + P$ where P is the projection onto $N(A)$ with kernel equal to $\overline{R(A)}$, it is fairly easy in concrete situations to reduce to the case when a sectorial operator has dense range.

Another way to reduce to operators with dense range is to replace an operator A by $A + \varepsilon$ for $\varepsilon > 0$ and then let ε tend to 0. Indeed, let A be a sectorial operator of type ω on X and observe that for any $\varepsilon > 0$, $A + \varepsilon$ is an invertible sectorial operator of type ω. In fact it is easy to deduce from the identity

$$zR(z, A + \varepsilon) = \Big[\frac{z}{z - \varepsilon}\Big]\,\big[(z - \varepsilon)R(z - \varepsilon, A)\big]$$

that the operators $A + \varepsilon$ are uniformly sectorial of type ω, that is, for any $\theta \in (\omega, \pi)$ there is a constant $K_\theta > 0$ not depending on $\varepsilon > 0$ such that

$$\big\|zR(z, A + \varepsilon)\big\| \leq K_\theta, \qquad z \in \mathbb{C} \setminus \overline{\Sigma_\theta},\ \varepsilon > 0. \tag{3.11}$$

The following well-known approximation lemma will be used later on. We include a proof for the convenience of the reader.

Lemma 3.5. — *Let A be a sectorial operator of type ω on a Banach space X and let $\theta \in (\omega, \pi)$ be an angle. Then A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus if and only if the operators $A + \varepsilon$ uniformly admit a bounded $H^\infty(\Sigma_\theta)$ functional calculus, that is, there is a constant K such that $\|f(A + \varepsilon)\| \leq K\|f\|_{\infty,\theta}$ for any $f \in H_0^\infty(\Sigma_\theta)$ and any $\varepsilon > 0$.*

Proof. — To prove the 'only if' part, assume that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus and let $U_\theta : H_0^\infty(\Sigma_\theta) \to B(X)$ be the resulting bounded homomorphism. Let $\varepsilon > 0$ and let f be an arbitrary element of $H_0^\infty(\Sigma_\theta)$. We define a function h on Σ_θ by letting

$$h(z) = f(z+\varepsilon) - \frac{f(\varepsilon)}{1+z}, \qquad z \in \Sigma_\theta.$$

It is easy to check that h belongs to $H_0^\infty(\Sigma_\theta)$, and that $h(A) = f(A+\varepsilon) - f(\varepsilon)(1+A)^{-1}$. Moreover

$$\|h\|_{\infty,\theta} \leq C_\theta \|f\|_{\infty,\theta},$$

for some constant C_θ only depending on θ. Then we have

$$\begin{aligned} \|f(A+\varepsilon)\| &\leq \|h(A)\| + |f(\varepsilon)|\big\|(1+A)^{-1}\big\| \\ &\leq \|U_\theta\|\|h\|_{\infty,\theta} + \|f\|_{\infty,\theta}\big\|(1+A)^{-1}\big\| \\ &\leq \big(\|u_\theta\| C_\theta + \|(1+A)^{-1}\|\big)\|f\|_{\infty,\theta}. \end{aligned}$$

This shows the desired uniform estimate.

To prove the 'if' part, first observe that for any $z \notin \overline{\Sigma_\omega}$, $R(z, A+\varepsilon)$ converges to $R(z, A)$ when $\varepsilon \to 0$. Thus given any $f \in H_0^\infty(\Sigma_\theta)$, we have

$$\lim_{\varepsilon \to 0} \|f(A+\varepsilon) - f(A)\| = 0$$

by (3.5), (3.11), and Lebesgue's Theorem. This concludes the proof. □

3.B. Completely bounded H^∞ functional calculus

We will introduce 'completely bounded versions' of sectoriality and H^∞ functional calculus for operators acting on noncommutative L^p-spaces. Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra, let $1 \leq p < \infty$, and let $X = L^p(\mathcal{M})$. We will use the space

$$Y = S^p[L^p(\mathcal{M})]$$

introduced in paragraph 2.B, and we recall from Lemma 2.1 that $S^p \otimes X$ is a dense subspace of Y. Throughout we will use the following two simple facts. First, for any $\xi \in (S^p)^*$, $\xi \otimes I_X$ (uniquely) extends to a bounded operator

$$\xi \overline{\otimes} I_X : S^p[L^p(\mathcal{M})] \longrightarrow L^p(\mathcal{M}).$$

Second, if $y \in S^p[L^p(\mathcal{M})]$ is such that $(\xi \overline{\otimes} I_X)y = 0$ for any $\xi \in (S^p)^*$, then $y = 0$.

We simply write I for the identity operator on S^p. Let A be a closed and densely defined operator on $X = L^p(\mathcal{M})$. We claim that the operator

$$I \otimes A \colon S^p \otimes D(A) \longrightarrow S^p[L^p(\mathcal{M})]$$

is closable. Indeed let $(y_n)_{n\geq 0}$ be a sequence of $S^p \otimes D(A)$ converging to 0 and assume that $(I \otimes A)y_n$ converges to some $y \in Y$. Then for any $\xi \in (S^p)^*$, $(\xi \otimes I_X)y_n$ belongs to $D(A)$ and we have

$$A(\xi \otimes I_X)y_n = (\xi \otimes I_X)(I \otimes A)y_n \longrightarrow (\xi \overline{\otimes} I_X)y.$$

On the other hand, we have $(\xi \otimes I_X)y_n \to 0$. Since A is closed, this implies that $(\xi\overline{\otimes}I_X)y = 0$. Since ξ was arbitrary, we deduce that $y = 0$. This proves the claim.

The closure of $I \otimes A$ on $S^p[L^p(\mathcal{M})]$ will be denoted by

$$I\overline{\otimes}A.$$

Note that if $A = T\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is a bounded operator, then $I\overline{\otimes}T$ is bounded if and only if T is completely bounded, with $\|T\|_{cb} = \|I\overline{\otimes}T\|$ (see paragraph 2.D).

Lemma 3.6. — *Let A be a closed and densely defined operator on X, and let $\mathcal{A} = I\overline{\otimes}A$ on Y.*

(1) *For any $\xi \in (S^p)^*$ and any $y \in D(\mathcal{A})$, $(\xi\overline{\otimes}I_X)y$ belongs to $D(A)$ and*

$$A(\xi\overline{\otimes}I_X)y = (\xi\overline{\otimes}I_X)\mathcal{A}y. \tag{3.12}$$

(2) *We have*

$$\rho(\mathcal{A}) \;=\; \big\{z \in \rho(A) \,:\, R(z, A) \;\; \textit{is completely bounded}\big\}.$$

Moreover, $R(z,\mathcal{A}) = I\overline{\otimes}R(z, A)$ for any $z \in \rho(\mathcal{A})$.

Proof. — Part (1) is proved by repeating the argument showing that $I\otimes A$ is closable.

To prove (2), let $z \in \rho(\mathcal{A})$ and let $\xi \in (S^p)^*$. By part (1), $(z - A)(\xi\overline{\otimes}I_X)$ and $(\xi\overline{\otimes}I_X)(z - \mathcal{A})$ coincide on $D(\mathcal{A})$, hence

$$\xi\overline{\otimes}I_X = (z - A)(\xi\overline{\otimes}I_X)R(z,\mathcal{A}).$$

We deduce that for any $e \in S^p$ and any $x \in X$, we have

$$\langle\xi, e\rangle x = (z - A)(\xi\overline{\otimes}I_X)R(z,\mathcal{A})(e \otimes x). \tag{3.13}$$

Consider a pair (e, ξ) verifying $\langle\xi, e\rangle = 1$, and define $R_z\colon X \to X$ by

$$R_z(x) = (\xi\overline{\otimes}I_X)R(z,\mathcal{A})(e \otimes x), \qquad x \in X.$$

It follows from above that R_z is valued in $D(A)$ and that $(z - A)R_z = I_X$. Further it is clear that $R_z(z - A) = I_{D(A)}$. This shows that $z \in \rho(A)$, with $R(z, A) = R_z$. Now (3.13) can be rewritten as

$$\langle\xi, e\rangle R(z, A)x = (\xi\overline{\otimes}I_X)R(z,\mathcal{A})(e \otimes x), \qquad e \in S^p,\ \xi \in (S^p)^*,\ x \in X.$$

This shows that $e \otimes R(z, A)x = R(z,\mathcal{A})(e \otimes x)$ for any $e \in S^p$ and any $x \in X$. Hence $R(z, A)$ is completely bounded and $I\overline{\otimes}R(z, A) = R(z,\mathcal{A})$.

Conversely, let $z \in \rho(A)$ such that $R(z, A)\colon X \to X$ is completely bounded, and consider $\mathcal{R}_z = I\overline{\otimes}R(z, A)\colon Y \to Y$. Let $y \in D(\mathcal{A})$. By definition of this domain, there is a sequence $(y_n)_{n\geq 1}$ in $S^p \otimes D(A)$ such that $y_n \to y$ and $(I \otimes A)y_n \to \mathcal{A}y$. It is clear that $\mathcal{R}_z(z - \mathcal{A})y_n = y_n$ for any $n \geq 1$ and passing to the limit we deduce that $\mathcal{R}_z(z - \mathcal{A})y = y$.

On the other hand, let $y \in Y$ and let $u = \mathcal{R}_z y$. Let $(y_n)_{n\geq 1}$ be a sequence in $S^p \otimes X$ converging to y, and let $u_n = (I \otimes R(z,A))y_n$ for any $n \geq 1$. Then u_n belongs to $S^p \otimes D(A)$ and $u_n \to u$. Moreover

$$(I \otimes A)u_n = (I \otimes AR(z,A))y_n \longrightarrow (I\overline{\otimes} AR(z,A))y.$$

Hence $u = \mathcal{R}_z y \in D(\mathcal{A})$, with $\mathcal{A}\mathcal{R}_z y = (I\overline{\otimes} AR(z,A))y$. This shows that $\mathcal{R}_z$ is valued in $D(\mathcal{A})$ and that $(z - \mathcal{A})R_z y = y$ for any $y \in Y$. These results show that $z \in \rho(\mathcal{A})$. □

Definition 3.7. — *Let A be a sectorial operator of type $\omega \in (0,\pi)$ on $X = L^p(\mathcal{M})$.*

(1) *We say that A is cb-sectorial of type ω if $I\overline{\otimes} A$ is sectorial of type ω on $S^p[L^p(\mathcal{M})]$.*
(2) *Assume that (1) is fulfilled, and let $\theta \in (\omega,\pi)$ be an angle. We say that A admits a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus if $I\overline{\otimes} A$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

Proposition 3.8. — *Let A be a sectorial operator of type $\omega \in (0,\pi)$ on $X = L^p(\mathcal{M})$.*

(1) *A is cb-sectorial of type ω if and only if $R(z,A)$ is completely bounded for any $z \in \mathbb{C} \setminus \overline{\Sigma_\omega}$ and for any $\theta \in (\omega,\pi)$ there is a constant $K_\theta > 0$ such that*
$$\|zR(z,A)\|_{cb} \leq K_\theta, \qquad z \in \mathbb{C} \setminus \overline{\Sigma_\theta}.$$
(2) *Assume that A is cb-sectorial of type ω, and let $\theta > \omega$. For any $f \in H_0^\infty(\Sigma_\theta)$, the operator $f(A)$ is completely bounded and $I\overline{\otimes} f(A) = f(I\overline{\otimes} A)$. Further A admits a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus if and only if there is a constant $K \geq 0$ such that*
$$\|f(A)\|_{cb} \leq K\|f\|_{\infty,\theta}\,, \qquad f \in H_0^\infty(\Sigma_\theta).$$
(3) *Assume that A has dense range and is cb-sectorial of type ω. Then $I\overline{\otimes} A$ has dense range and for any $\theta > \omega$, we have*
$$I\overline{\otimes} f(A) = f(I\overline{\otimes} A), \qquad f \in H^\infty(\Sigma_\theta).$$

Proof. — Parts (1) and (2) are straightforward consequences of Lemma 3.6 and (3.5).

Assume that A has dense range and is cb-sectorial of type ω, and let $\mathcal{A} = I\overline{\otimes} A$. Its range contains $S^p \otimes R(A)$, hence it is a dense subspace of Y. Let $f \in H^\infty(\Sigma_\theta)$ for some $\theta > \omega$. It is clear that the two operators $f(\mathcal{A})$ and $I\overline{\otimes} f(A)$ coincide on $S^p \otimes R(g(A))$. To prove that they are equal, it suffices to check that this space is a core for each of them. Since $R(g(A))$ is a core for $f(A)$ and $I\overline{\otimes} f(A)$ is the closure of $I \otimes f(A)\colon S^p \otimes D(f(A)) \to Y$, we obtain that $S^p \otimes R(g(A))$ is a core of $I\overline{\otimes} f(A)$.

Next, let $y \in D(f(\mathcal{A}))$, and let $(g_n)_{n\geq 1}$ be the sequence defined by (3.8). By Lemma 3.2, $g_n(\mathcal{A})y$ converges to y when $n \to \infty$, and we also have

$$f(\mathcal{A})g_n(\mathcal{A})y = g_n(\mathcal{A})f(\mathcal{A})y \longrightarrow f(\mathcal{A})y \quad \text{when} \quad n \to \infty.$$

Now let $(y_k)_k$ be a sequence of $S^p \otimes X$ converging to y. For any fixed $n \geq 1$, $g_n(\mathcal{A})y_k$ belongs to $S^p \otimes R(g(A))$, and we both have

$$g_n(\mathcal{A})y_k \longrightarrow g_n(\mathcal{A})y \quad \text{and} \quad f(\mathcal{A})g_n(\mathcal{A})y_k \longrightarrow f(\mathcal{A})g_n(\mathcal{A})y$$

when $k \to \infty$. This proves that $S^p \otimes R(g(A))$ is a core of $f(\mathcal{A})$ and completes the proof. □

We now turn to the special case of sectorial operators defined as negative generators of semigroups. Let $(T_t)_{t\geq 0}$ be a bounded c_0-semigroup on $X = L^p(\mathcal{M})$. We say that $(T_t)_{t\geq 0}$ is a completely bounded semigroup if each T_t is completely bounded and $\sup_{t\geq 0} \|T_t\|_{cb} < \infty$. In this case, each $I \otimes T_t$ extends to a bounded operator $I\overline{\otimes}T_t \colon S^p[L^p(\mathcal{M})] \to S^p[L^p(\mathcal{M})]$ and a standard equicontinuity argument shows that $(I\overline{\otimes}T_t)_{t\geq 0}$ is a bounded c_0-semigroup on $Y = S^p[L^p(\mathcal{M})]$.

Lemma 3.9. — *Let $(T_t)_{t\geq 0}$ be a completely bounded c_0-semigroup on $L^p(\mathcal{M})$ and let A denote its negative generator. Then $I\overline{\otimes}A$ is the negative generator of $(I\overline{\otimes}T_t)_{t\geq 0}$, hence A is cb-sectorial of type $\frac{\pi}{2}$.*

Proof. — We let $\mathcal{A} = I\overline{\otimes}A$, and we let $\mathcal{B}$ denote the negative generator of $(I\overline{\otimes}T_t)_{t\geq 0}$ on Y. Applying the Laplace formula (3.2) to $(T_t)_{t\geq 0}$ and to $(I\overline{\otimes}T_t)_{t\geq 0}$, we see that $I \otimes (1+A)^{-1}$ and $(1+\mathcal{B})^{-1}$ coincide on $S^p \otimes X$. According to Lemma 3.6, this implies that $-1 \in \rho(\mathcal{A})$ and that $(1+\mathcal{A})^{-1} = (1+\mathcal{B})^{-1}$. Thus $\mathcal{A} = \mathcal{B}$. □

Example 3.10. — Let $1 \leq p < \infty$, and let $(T_t)_{t\geq 0}$ denote the translation semigroup on $L^p(\mathbb{R})$, that is, $(T_t f)(s) = f(s-t)$ for $s \in \mathbb{R}, t \geq 0$. Its negative generator is the derivation operator $A = \frac{d}{dt}$, with domain equal to the Sobolev space $W^{1,p}(\mathbb{R})$. More generally for any Banach space Z, we can define the translation semigroup $(T_t^Z)_{t\geq 0}$ on $L^p(\mathbb{R}; Z)$ by the same formula, and its negative generator is the derivation $\mathcal{A}^Z$ with domain $W^{1,p}(\mathbb{R}; Z)$. It is clear that $\mathcal{A}^Z$ coincides with $A \otimes I_Z$ on $L^p(\mathbb{R}) \otimes Z$. We noticed in Remark 2.13 that we have a canonical identification $L^p(\mathbb{R}; S^p) = S^p[L^p(\mathbb{R})]$. Hence it follows from Lemma 3.9 that the operator $I\overline{\otimes}\frac{d}{dt}$ coincides with the derivation operator on $L^p(\mathbb{R}; S^p)$.

It turns out that for any $\theta > \frac{\pi}{2}$, the operator $\mathcal{A}^Z$ has a bounded $H^\infty(\Sigma_\theta)$ functional calculus if and only if Z is a UMD Banach space (see [**20, 32, 67**]). Thus if $1 < p < \infty$, the operator $\frac{d}{dt}$ has a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$, because S^p is a UMD Banach space.

3.C. Dilations

We will need the following result due to Hieber and Prüss [**32**].

Proposition 3.11. — *([**32**]) Let Z be a UMD Banach space. Let $(U_t)_t$ be a c_0-group of isometries on Z, and let $-B$ denote its infinitesimal generator. Then B has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$. More precisely there exists for any $\theta > \frac{\pi}{2}$*

a constant $C_{Z,\theta}$ only depending on θ and Z such that $\|f(B)\| \leq C_{Z,\theta}\|f\|_{\infty,\theta}$ for any $f \in H_0^\infty(\Sigma_\theta)$.

Indeed using a transference technique, it is shown in [**32**] that for any B as above and any $f \in H_0^\infty(\Sigma_\theta)$, one has

$$\|f(B)\| \leq \|f(\mathcal{A}^Z)\|,$$

where $\mathcal{A}^Z$ is the derivation operator on $L^2(\mathbb{R}; Z)$ discussed in Example 3.10. Since $\mathcal{A}^Z$ has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$, this yields the result.

Extending previous terminology, we say that a c_0-group $(U_t)_t$ on some noncommutative L^p-space X is a completely isometric c_0-group if each $U_t \colon X \to X$ is a complete isometry. In this case, $(I\overline{\otimes}U_t)_t$ is a c_0-group of isometries on $S^p[X]$.

Proposition 3.12. — *Let $1 < p < \infty$, and let $\mathcal{M}$ be a semifinite von Neumann algebra. Let $(T_t)_{t\geq 0}$ be a contractive c_0-semigroup on $L^p(\mathcal{M})$ and let $-A$ denote its infinitesimal generator. Assume that there exist another semifinite von Neumann algebra $\mathcal{M}'$, a c_0-group $(U_t)_t$ of isometries on $L^p(\mathcal{M}')$, and contrative maps $J\colon L^p(\mathcal{M}) \to L^p(\mathcal{M}')$ and $Q\colon L^p(\mathcal{M}') \to L^p(\mathcal{M})$ such that*

$$T_t = QU_tJ, \qquad t \geq 0. \tag{3.14}$$

Then A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$. If further, $(U_t)_t$ is a completely isometric c_0-group and J and Q are completely contractive, then $(T_t)_{t\geq 0}$ is completely bounded and A admits a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$.

Proof. — Let $-B$ denote the infinitesimal generator of $(U_t)_t$ on $L^p(\mathcal{M}')$. Let z be a complex number with $\mathrm{Re}(z) < 0$. According to the Laplace formula (3.2), we have

$$R(z, A) = -\int_0^\infty e^{tz}T_t\, dt \qquad \text{and} \qquad R(z, B) = -\int_0^\infty e^{tz}U_t\, dt\,.$$

Hence our dilation assumption (3.14) yields

$$R(z, A) = QR(z, B)J.$$

Then for any $\theta > \frac{\pi}{2}$ and any $f \in H_0^\infty(\Sigma_\theta)$, we have $f(A) = Qf(B)J$, by (3.5). Therefore we have

$$\|f(A)\| \leq \|Q\|\|J\|\|f(B)\|.$$

The Banach space $L^p(\mathcal{M}')$ is UMD, hence B has a bounded $H^\infty(\Sigma_\theta)$ functional calculus by Proposition 3.11. Thus A also has a bounded $H^\infty(\Sigma_\theta)$ functional calculus.

If J and Q are completely contractive, $I \otimes J$ and $I \otimes Q$ extend to contractions

$$I\overline{\otimes}J\colon S^p[L^p(\mathcal{M})] \longrightarrow S^p[L^p(\mathcal{M}')] \qquad \text{and} \qquad I\overline{\otimes}Q\colon S^p[L^p(\mathcal{M}')] \longrightarrow S^p[L^p(\mathcal{M})].$$

If we assume that $(U_t)_t$ is a completely isometric group, we obtain that $(T_t)_{t\geq 0}$ is a completely contractive c_0-semigroup and we have

$$I\overline{\otimes}T_t = (I\overline{\otimes}Q)(I\overline{\otimes}U_t)(I\overline{\otimes}J), \qquad t \geq 0.$$

Since $S^p[L^p(\mathcal{M})]$ and $S^p[L^p(\mathcal{M}')]$ are noncommutative L^p-spaces, it follows from the first part of the proof and Lemma 3.9 that $\mathcal{A} = I\overline{\otimes}A$ has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$. $\square$

Let $\mathcal{M} \simeq L^\infty(\Sigma)$ be a commutative von Neumann algebra and let $(T_t)_{t\geq 0}$ be a c_0-semigroup of positive contractions on $L^p(\Sigma)$. Fendler showed in [**27**] that there exist a commutative L^p-space $L^p(\Sigma')$, a c_0-group $(U_t)_t$ of isometries on $L^p(\Sigma')$, and contractive maps $J\colon L^p(\Sigma) \to L^p(\Sigma')$ and $Q\colon L^p(\Sigma') \to L^p(\Sigma)$ such that $T_t = QU_tJ$ for any $t \geq 0$. (This is a continuous version of Akcoglu's dilation Theorem [**1, 2**].) Applying Proposition 3.12, we deduce that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$ provided that $-A$ generates a positive contraction c_0-semigroup on $L^p(\Sigma)$, for $1 < p < \infty$. This result is due to Duong [**24**] (see also [**20**]). However it is still unknown whether an analog of Fendler's Theorem holds on noncommutative L^p-spaces, and this is a significant although interesting drawback for the study of completely positive contractive semigroups on noncommutative L^p-spaces. See Remark 5.9 for more on this.

CHAPTER 4

RADEMACHER BOUNDEDNESS AND RELATED NOTIONS

4.A. Column boundedness and row boundedness

Rademacher boundedness [**7, 19**] has played a prominent role in recent developments of H^∞ functional calculus, see in particular [**41**], [**79**], [**78**], [**47**]. On noncommutative L^p-spaces it will be convenient to consider two natural variants of this notion that we introduce below under the names of column boundedness and row boundedness.

Let X be a Banach space and let $\mathcal{F} \subset B(X)$ be a set of bounded operators on X. We say that $\mathcal{F}$ is Rad-bounded if there is a constant $C > 0$ such that for any finite families $T_1, \ldots, T_n$ in $\mathcal{F}$, and $x_1, \ldots, x_n$ in X, we have

$$\Bigl\|\sum_{k=1}^{n} \varepsilon_k\, T_k(x_k)\Bigr\|_{\mathrm{Rad}(X)} \leq C\Bigl\|\sum_{k=1}^{n} \varepsilon_k\, x_k\Bigr\|_{\mathrm{Rad}(X)}. \tag{4.1}$$

In this definition, the norms $\| \ \|_{\mathrm{Rad}(X)}$ are given by (2.20).

Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra, let $1 \leq p < \infty$, and assume that $X = L^p(\mathcal{M})$. We say that a set $\mathcal{F} \subset B(L^p(\mathcal{M}))$ is Col-bounded (resp. Row-bounded) if there there is a constant $C > 0$ such that for any finite families $T_1, \ldots, T_n$ in $\mathcal{F}$, and $x_1, \ldots, x_n$ in $L^p(\mathcal{M})$, we have

$$\Bigl\|\Bigl(\sum_k T_k(x_k)^* T_k(x_k)\Bigr)^{\frac{1}{2}}\Bigr\|_{L^p(\mathcal{M})} \leq C\,\Bigl\|\Bigl(\sum_k x_k^* x_k\Bigr)^{\frac{1}{2}}\Bigr\|_{L^p(\mathcal{M})} \tag{4.2}$$

$$\left(\text{resp.}\quad \Bigl\|\Bigl(\sum_k T_k(x_k) T_k(x_k)^*\Bigr)^{\frac{1}{2}}\Bigr\|_{L^p(\mathcal{M})} \leq C\,\Bigl\|\Bigl(\sum_k x_k x_k^*\Bigr)^{\frac{1}{2}}\Bigr\|_{L^p(\mathcal{M})}.\right) \tag{4.3}$$

The least constant C satisfying (4.1) (resp. (4.2), resp. (4.3)) will be denoted by $\mathrm{Rad}(\mathcal{F})$ (resp. $\mathrm{Col}(\mathcal{F})$, resp. $\mathrm{Row}(\mathcal{F})$). Obviously any Rad-bounded (resp. Col-bounded, resp. Row-bounded) set is bounded but the converse does not hold true except on Hilbert space.

It follows from the noncommutative Khintchine inequalities (2.21) and (2.22) that if a set $\mathcal{F} \subset B(L^p(\mathcal{M}))$ is both Col-bounded and Row-bounded, then it is Rad-bounded. Moreover these three notions coincide on commutative L^p-spaces (see Remark 2.13). However this is no longer the case in the general noncommutative setting. Indeed let $\mathcal{F} = \{T\} \subset B(L^p(\mathcal{M}))$ be a singleton, and let H be an infinite dimensional Hilbert space. Then $\mathcal{F}$ is Rad-bounded with $\mathrm{Rad}(\mathcal{F}) = \|T\|$ whereas $\mathcal{F}$ is Col-bounded if and only if $T \otimes I_H$ extends to a bounded operator on $L^p(\mathcal{M}; H_c)$. Indeed this follows from (2.10). Likewise $\mathcal{F}$ is Row-bounded if and only if $T \otimes I_H$ extends to a bounded operator on $L^p(\mathcal{M}; H_r)$. Thus applying (2.36) and (2.37), the set $\{T\}$ is both Col-bounded and Row-bounded if T is completely bounded.

It turns out that if $p \neq 2$, one may find $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ such that $T \otimes I_H$ is bounded on the column space $L^p(\mathcal{M}; H_c)$, but $T \otimes I_H$ is not bounded on the row space $L^p(\mathcal{M}; H_r)$, see Example 4.1 below. Thus there are sets $\mathcal{F}$ which are Rad-bounded and Col-bounded without being Row-bounded. Similarly, one may find subsets of $B(L^p(\mathcal{M}))$ which are Rad-bounded and Row-bounded without being Col-bounded, or which are Rad-bounded without being either Row-bounded or Col-bounded.

Example 4.1. — Let H be an infinite dimensional Hilbert space and let $1 \leq p \neq 2 < \infty$ be any number. For simplicity we write $S^p[H_c]$ and $S^p[H_r]$ for $L^p(B(\ell^2); H_c)$ and $L^p(B(\ell^2); H_r)$ respectively. It is well-known that there exists an operator $T\colon S^p \to S^p$ whose tensor extension $T \otimes I_H$ extends to a bounded operator on $S^p[H_c]$ but $T \otimes I_H\colon S^p[H_r] \to S^p[H_r]$ is unbounded. We provide an example for the convenience of the reader not familiar with matricial and operator space techniques.

We assume that $p < 2$, the other case being similar. We regard elements of S^p as infinite matrices in the usual way and we let E_{ij} denote the standard matrix units. Let $T\colon S^p \to S^p$ be defined by $T(E_{1j}) = E_{j1}$ for any $j \geq 1$ and $T(E_{ij}) = 0$ for any $i \geq 2$ and any $j \geq 1$. Thus $T = U \circ P$, where $U\colon S^p \to S^p$ is the transpose map, and $P\colon S^p \to S^p$ is the canonical projection onto the space of matrices which have zero entries except on the first row. It is easy to check that $\|P\|_{cb} = 1$ and that $\|U\| = 1$. Hence $\|T\| = 1$. We will show that

$$\big\|T \otimes I_H \colon S^p[H_c] \longrightarrow S^p[H_c]\big\| = 1. \tag{4.4}$$

We may assume that $H = \ell^2$, and we let $(e_k)_{k\geq 1}$ denote its canonical basis. Since P is completely contractive, the operator $P \otimes I_H\colon S^p[H_c] \to S^p[H_c]$ is contractive, by (2.36). Hence it suffices to show that $U \otimes I_H$ is contractive on $\mathrm{Span}\{E_{1j} \otimes e_k \,:\, j, k \geq 1\} \subset S^p \otimes H$. Let $(\alpha_{jk})_{j,k\geq 1}$ be a finite family of complex numbers and let

$$u = \sum_{j,k} \alpha_{jk} E_{1j} \otimes e_k.$$

Applying (2.10), we find that

$$\|u\|_{S^p[H_c]} = \Big\|\Big(\sum_{j,k,m} \overline{\alpha_{jk}}\,\alpha_{mk}E_{jm}\Big)^{\frac{1}{2}}\Big\|_{S^p}.$$

Since $\sum_{j,k,m} \overline{\alpha_{jk}}\,\alpha_{mk}E_{jm} = \Big(\sum_{j,k} \alpha_{jk}\,E_{kj}\Big)^*\Big(\sum_{j,k} \alpha_{jk}\,E_{kj}\Big)$, we deduce that

$$\|u\|_{S^p[H_c]} = \Big\|\sum_{j,k} \alpha_{jk}\,E_{kj}\Big\|_{S^p}.$$

Applying the transpose map U, we have

$$(U \otimes I_H)u = \sum_{j,k} \alpha_{jk}E_{j1} \otimes e_k.$$

Then using (2.10) again we deduce that

$$\|(U \otimes I_H)u\|_{S^p[H_c]} = \Big(\sum_{j,k} |\alpha_{jk}|^2\Big)^{\frac{1}{2}} = \Big\|\sum_{j,k} \alpha_{jk}\,E_{kj}\Big\|_{S^2}.$$

Since $p < 2$, we deduce that $\|(U \otimes I_H)u\|_{S^p[H_c]} \leq \|u\|_{S^p[H_c]}$, which proves (4.4).

Now essentially reversing the above arguments, we see that if $T \otimes I_H$ extends to a bounded operator on $S^p[H_r]$ with norm $\leq K$, then for any finite family $(\alpha_{jk})_{j,k\geq 1}$ of complex numbers, we have

$$\Big\|\sum_{j,k} \alpha_{jk}\,E_{kj}\Big\|_{S^p} \leq K\Big\|\sum_{j,k} \alpha_{jk}\,E_{kj}\Big\|_{S^2},$$

which is wrong.

Throughout the rest of this chapter, $\mathcal{M}$ is a semifinite von Neumann algebra and we fix some $1 \leq p < \infty$. We will require the following lemma which extends [**19**, Lemma 3.2].

Lemma 4.2. — *Let $\mathcal{F} \subset B(L^p(\mathcal{M}))$ be a set of bounded operators, let I be an interval of $\mathbb{R}$, let $C > 0$ be a constant, and let*

$$\mathcal{T} = \Big\{\int_I f(t)R(t)\,dt \;\Big|\; R\colon I \to \mathcal{F} \text{ is continuous, } f \in L^1(I;dt), \text{ and } \int_I |f(t)|\,dt \leq C\Big\}.$$

(1) *If $\mathcal{F}$ is Rad-bounded then $\mathcal{T}$ is Rad-bounded with $Rad(\mathcal{T}) \leq 2C Rad(\mathcal{F})$.*
(2) *If $\mathcal{F}$ is Col-bounded (resp. Row-bounded), then $\mathcal{T}$ is Col-bounded (resp. Row-bounded) with $Col(\mathcal{T}) \leq C Col(\mathcal{F})$ (resp. $Row(\mathcal{T}) \leq C Row(\mathcal{F})$).*

Proof. — For the first assertion, recall that by [**19**, Lemma 3.2], the closed absolute convex hull $\overline{aco}(\mathcal{F})$ of $\mathcal{F}$ is Rad-bounded with $\mathrm{Rad}(\overline{aco}(\mathcal{F})) \leq 2\mathrm{Rad}(\mathcal{F})$. A standard approximation argument shows that $\frac{1}{C}\mathcal{T} \subset \overline{aco}(\mathcal{F})$, which proves the result. The same proof yields the second assertion, except that the factor 2 does not appear. □

It was observed in [**78**, 4.a] that given a measure space Σ, an interval $I \subset \mathbb{R}$, and a strongly continuous function $\Phi\colon I \to B(L^p(\Sigma))$, then the set $\{\Phi(t) : t \in I\}$ is Rad-bounded if and only if there is a constant $C > 0$ such that

$$\Big\| \Big(\int_I \big|\Phi(t)u(t)\big|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_p \leq C \, \Big\| \Big(\int_I \big|u(t)\big|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_p$$

for any measurable function $u\colon I \to L^p(\Sigma)$ belonging to $L^p(\Sigma; L^2(I))$. The aim of Proposition 4.4 below is to extend this result to our noncommutative setting. We will need a standard approximation procedure that we briefly recall (see e.g. [**23**, III.2 Lemma 1] for details).

Let (Ω, μ) be a σ-finite measure space. By a subpartition of Ω, we mean a finite set $\pi = \{I_1, \ldots, I_m\}$ of pairwise disjoint measurable subsets of Ω such that $0 < \mu(I_i) < \infty$ for any $1 \leq i \leq m$. Let Z be a Banach space and let π be a subpartition of Ω. We may define a linear mapping E_π on $L^p(\Omega; Z)$ by letting

$$E_\pi(u) = \sum_{i=1}^{m} \frac{1}{\mu(I_i)} \Big(\int_{I_i} u(t) \, d\mu(t) \Big) \chi_{I_i}, \qquad u \in L^p(\Omega; Z). \tag{4.5}$$

Here χ_I denotes the indicator function of I. Then the mapping $E_\pi\colon L^p(\Omega; Z) \to L^p(\Omega; Z)$ is a contraction. Further if subpartitions are directed by refinement, then we have

$$\lim_{\pi} \|E_\pi u - u\|_p = 0, \qquad u \in L^p(\Omega; Z). \tag{4.6}$$

The use of the same notation E_π for all Z and all p should not create any confusion. The following elementary lemma is easy to deduce from (4.6) and its proof is left to the reader.

Lemma 4.3. — *Let (Ω, μ) be a σ-finite measure space. Then for any $a, b \in L^2(\Omega)$ and for any $c \in L^\infty(\Omega)$, we have*

$$\int_\Omega cab \, d\mu \;=\; \lim_\pi \int_\Omega E_\pi(c) E_\pi(a) E_\pi(b) \, d\mu \, .$$

Let (Ω, μ) be a σ-finite measure space. If $\Phi\colon \Omega \to B(L^p(\mathcal{M}))$ is any bounded measurable function, we may define a multiplication operator $T_\Phi\colon L^2(\Omega; L^p(\mathcal{M})) \to L^2(\Omega; L^p(\mathcal{M}))$ by letting

$$\big(T_\Phi(u)\big)(t) = \Phi(t)u(t), \qquad u \in L^2(\Omega; L^p(\mathcal{M})).$$

Proposition 4.4. — *Let $\Phi\colon (\Omega, \mu) \to B(L^p(\mathcal{M}))$ be a bounded measurable function and consider the bounded set*

$$\mathcal{F} = \Big\{ \frac{1}{\mu(I)} \int_I \Phi(t) \, d\mu(t) \; : I \subset \Omega, \; 0 < \mu(I) < \infty \Big\} \; \subset B(L^p(\mathcal{M})).$$

(1) *If the set $\mathcal{F}$ is Col-bounded, then*

$$T_\Phi\colon L^p(\mathcal{M};L^2(\Omega)_c) \longrightarrow L^p(\mathcal{M};L^2(\Omega)_c) \qquad \textit{boundedly.}$$

(2) *If the set $\mathcal{F}$ is Row-bounded, then*

$$T_\Phi\colon L^p(\mathcal{M};L^2(\Omega)_r) \longrightarrow L^p(\mathcal{M};L^2(\Omega)_r) \qquad \textit{boundedly.}$$

(3) *If the set $\mathcal{F}$ is Rad-bounded, then*

$$T_\Phi\colon L^p(\mathcal{M};L^2(\Omega)_{rad}) \longrightarrow L^p(\mathcal{M};L^2(\Omega)_{rad}) \qquad \textit{boundedly.}$$

Proof. — We first assume that $\mathcal{F}$ is Col-bounded and we shall prove (1) by using duality. We let $p' = p/(p-1)$ be the conjugate number of p. Then we let $u \in L^p(\mathcal{M}) \otimes L^2(\Omega)$ and $v \in L^{p'}(\mathcal{M}) \otimes L^2(\Omega)$. They may be written as

$$u = \sum_k x_k \otimes a_k \qquad \text{and} \qquad v = \sum_j y_j \otimes b_j,$$

for some finite families $(a_k)_k \subset L^2(\Omega)$, $(x_k)_k \subset L^p(\mathcal{M})$, $(b_j)_j \subset L^2(\Omega)$, and $(y_j)_j \subset L^{p'}(\mathcal{M})$. We claim that there is a constant $K > 0$ not depending on u and v such that whenever π is a subpartition of Ω, we have

(4.7)
$$\Big|\sum_{k,j}\int_\Omega E_\pi\big(\langle\Phi(\cdot\,)x_k, y_j\rangle\big)E_\pi(a_k)E_\pi(b_j)\Big| \leq K\|u\|_{L^p(\mathcal{M};L^2(\Omega)_c)}\,\|v\|_{L^{p'}(\mathcal{M};L^2(\Omega)_r)}.$$

Taking this for granted for the moment, we deduce that

$$\begin{aligned}\langle T_\Phi(u), v\rangle &= \int_\Omega \langle\Phi(t)u(t), v(t)\rangle\,d\mu(t)\\ &= \sum_{k,j}\int_\Omega \langle\Phi(t)x_k, y_j\rangle a_k(t)b_j(t)\,d\mu(t)\\ &= \lim_\pi \sum_{k,j}\int_\Omega E_\pi\big(\langle\Phi(\cdot\,)x_k, y_j\rangle\big)\,E_\pi(a_k)E_\pi(b_j)\,d\mu\end{aligned}$$

by Lemma 4.3. It therefore follows from (4.7) that

$$\big|\langle T_\Phi(u), v\rangle\big| \leq K\|u\|_{L^p(\mathcal{M};L^2(\Omega)_c)}\,\|v\|_{L^{p'}(\mathcal{M};L^2(\Omega)_r)}.$$

By Lemma 2.10, we deduce that T_Φ maps $L^2(\Omega)\otimes L^p(\mathcal{M})$ into $L^p(\mathcal{M};L^2(\Omega)_c)$ and that

$$\big\|T_\Phi\colon L^p(\mathcal{M};L^2(\Omega)_c) \longrightarrow L^p(\mathcal{M};L^2(\Omega)_c)\big\| \leq K.$$

To complete the proof of (1), it therefore remains to prove (4.7). We let $E = E_\pi$ along the proof of this estimate and we assume that E is defined by (4.5). Then we have

$$\begin{aligned}\sum_{k,j}\int_\Omega E\big(\langle\Phi(\cdot\,)x_k,y_j\rangle\big)E(a_k)E(b_j)\\ &= \sum_{k,j}\sum_{i=1}^m \frac{1}{\mu(I_i)^2}\Big(\int_{I_i}\langle\Phi(\cdot\,)x_k,y_j\rangle\Big)\Big(\int_{I_i}a_k\Big)\Big(\int_{I_i}b_j\Big)\\ &= \sum_{i=1}^m \frac{1}{\mu(I_i)^2}\Big\langle\Big(\int_{I_i}\Phi\Big)\Big(\int_{I_i}u\Big),\Big(\int_{I_i}v\Big)\Big\rangle.\end{aligned}$$

Let $(e_k)_{k\geq 1}$ be an orthornormal family in some Hilbert space H. Owing to (2.14), we deduce that

$$\begin{aligned}\Big|\sum_{k,j}\int_\Omega E\big(\langle\Phi(\cdot\,)x_k,y_j\rangle\big)E(a_k)E(b_j)\Big|\\ &\leq \Big\|\sum_{i=1}^m \frac{1}{\mu(I_i)^{\frac32}}\Big(\int_{I_i}\Phi\Big)\Big(\int_{I_i}u\Big)\otimes e_i\Big\|_{L^p(\mathcal{M};H_c)}\\ &\qquad\times\Big\|\sum_{i=1}^m \frac{1}{\mu(I_i)^{\frac12}}\Big(\int_{I_i}v\Big)\otimes e_i\Big\|_{L^{p'}(\mathcal{M};H_r)}.\end{aligned}$$

Thus if we let $K = \mathrm{Col}(\mathcal{F})$ denote the column boundedness constant of $\mathcal{F}$, we obtain that

$$\begin{aligned}\Big|\sum_{k,j}\int_\Omega E\big(\langle\Phi(\cdot\,)x_k,y_j\rangle\big)E(a_k)E(b_j)\Big|\\ &\leq K\Big\|\sum_{i=1}^m \frac{1}{\mu(I_i)^{\frac12}}\Big(\int_{I_i}u\Big)\otimes e_i\Big\|_{L^p(\mathcal{M};H_c)}\\ &\qquad\times\Big\|\sum_{i=1}^m \frac{1}{\mu(I_i)^{\frac12}}\Big(\int_{I_i}v\Big)\otimes e_i\Big\|_{L^{p'}(\mathcal{M};H_r)}.\end{aligned}$$

Now recall that $E = E_\pi\colon L^2(\Omega)\to L^2(\Omega)$ is a contraction. Equivalently, the linear mapping $\sigma\colon L^2(\Omega)\to H$ defined by letting

$$\sigma(a) = \sum_{i=1}^m \frac{1}{\mu(I_i)^{\frac12}}\Big(\int_{I_i}a\Big)e_i$$

for any $a\in L^2(\Omega)$ is a contraction. Since

$$\sum_{i=1}^m \frac{1}{\mu(I_i)^{\frac12}}\Big(\int_{I_i}u\Big)\otimes e_i = (I_{L^p}\otimes\sigma)(u),$$

it therefore follows from Lemma 2.4 that

$$\Big\|\sum_{i=1}^m \frac{1}{\mu(I_i)^{\frac12}}\Big(\int_{I_i}u\Big)\otimes e_i\Big\|_{L^p(\mathcal{M};H_c)} \leq \|u\|_{L^p(\mathcal{M};H_c)}. \tag{4.8}$$

Similarly,

$$\Big\|\sum_{i=1}^{m}\frac{1}{\mu(I_i)^{\frac{1}{2}}}\Big(\int_{I_i}v\Big)\otimes e_i\Big\|_{L^{p'}(\mathcal{M};H_r)}\leq\|v\|_{L^{p'}(\mathcal{M};H_r)},\tag{4.9}$$

whence (4.7).

The proof of (2) is identical to that of (1) and may be omitted. To prove (3), assume for instance that $1\leq p\leq 2$, the other case being similar. Let u, v and $E=E_\pi$ be as in the previous computation. Arguing as above, we find that

$$\begin{aligned}\Big|\sum_{k,j}\int_\Omega E\big(\langle\Phi(\cdot\,)x_k,y_j\rangle\big)E(a_k)E(b_j)\Big|&\\ \leq\Big\|\sum_{i=1}^{m}\frac{1}{\mu(I_i)^{\frac{3}{2}}}\Big(\int_{I_i}\Phi\Big)\Big(\int_{I_i}u\Big)\otimes e_i\Big\|_{L^p(\mathcal{M};H_{r+c})}&\\ \times\Big\|\sum_{i=1}^{m}\frac{1}{\mu(I_i)^{\frac{1}{2}}}\Big(\int_{I_i}v\Big)\otimes e_i\Big\|_{L^{p'}(\mathcal{M};H_{r\cap c})}&.\end{aligned}$$

Then it follows from (2.22) that

$$\begin{aligned}\Big\|\sum_{i=1}^{m}\frac{1}{\mu(I_i)^{\frac{3}{2}}}\Big(\int_{I_i}\Phi\Big)\Big(\int_{I_i}u\Big)\otimes e_i\Big\|_{L^p(\mathcal{M};H_{r+c})}&\\ \leq\frac{\mathrm{Rad}(\mathcal{F})}{C_1}\Big\|\sum_{i=1}^{m}\frac{1}{\mu(I_i)^{\frac{1}{2}}}\Big(\int_{I_i}u\Big)\otimes e_i\Big\|_{L^p(\mathcal{M};H_{r+c})}&.\end{aligned}$$

Hence using Lemma 2.4 as in the proof of (1), we deduce the following inequality

$$\begin{aligned}\Big|\sum_{k,j}\int_\Omega E_\pi\big(\langle\Phi(\cdot\,)x_k,y_j\rangle\big)E_\pi(a_k)E_\pi(b_j)\Big|&\\ \leq\frac{\mathrm{Rad}(\mathcal{F})}{C_1}\|u\|_{L^p(\mathcal{M};L^2(\Omega)_{r+c})}\|v\|_{L^{p'}(\mathcal{M};L^2(\Omega)_{r\cap c})}&,\end{aligned}$$

which is the analogue of (4.7). The rest of the proof of (3) is identical to that of (1), appealing to Remark 2.11 in due place. □

Remark 4.5. — Let Φ and $\mathcal{F}$ be as in Proposition 4.4. It follows from the above proof that if $\mathcal{F}$ is Col-bounded, then the norm of $T_\Phi\colon L^p(\mathcal{M};L^2(\Omega)_c)\to L^p(\mathcal{M};L^2(\Omega)_c)$ is less than or equal to $\mathrm{Col}(\mathcal{F})$. Similar comments apply to the row case and to the Rademacher case, up to absolute constants.

Remark 4.6. — Let $\Phi\colon\Omega\to B(L^p(\mathcal{M}))$ be a bounded measurable function, and assume that T_Φ maps $L^p(\mathcal{M};L^2(\Omega)_c)$ into itself boundedly. If $u\in L^p(\mathcal{M};L^2(\Omega)_c)$ is a measurable function (in the sense of Definition 2.7), then $T_\Phi u$ also is a measurable function, namely $[T_\Phi u](t)=\Phi(t)u(t)$. Indeed this is obvious if $p\leq 2$. Then if $p > 2$, let us consider $y \in L^{p'}(\mathcal{M})$ and $b \in L^2(\Omega)$. Applying Lemma 2.8 with $v(t) = \big[T_\Phi^*(y \otimes b)\big](t) = b(t)\Phi(t)^* y$ yields

$$\begin{aligned}\langle y \otimes b, T_\Phi u\rangle = \langle T_\Phi^*(y \otimes b), u\rangle &= \int_\Omega \langle \Phi(t)^* y, u(t)\rangle\, b(t)\, d\mu(t)\\ &= \int_\Omega \langle y, \Phi(t)u(t)\rangle\, b(t)\, d\mu(t),\end{aligned}$$

and this proves the claim.

4.B. Col-sectorial, Row-sectorial, and Rad-sectorial operators

Following [**41**], we say that an operator A on some Banach space X is Rad-sectorial of Rad-type ω if A is sectorial of type ω and for any $\theta \in (\omega, \pi)$, the set

$$\big\{ zR(z, A) \,:\, z \in \mathbb{C} \setminus \overline{\Sigma_\theta} \big\} \tag{4.10}$$

is Rad-bounded. This is a strengthening of (3.1), which says that the latter set merely has to be bounded.

Next if $X = L^p(\mathcal{M})$, we say that A is Col-sectorial (resp. Row-sectorial) of Col-type (resp. Row-type) ω if the set in (4.10) is Col-bounded (resp. Row-bounded) for any $\theta \in (\omega, \pi)$. If A is both Col-sectorial of Col-type ω and Row-sectorial of Row-type ω, then is is Rad-sectorial of Rad-type ω.

In this paragraph, we establish a series of simple results concerning these notions.

Lemma 4.7. — *Let $1 < p, p' < \infty$ be conjugate numbers, and let A be a sectorial operator on $L^p(\mathcal{M})$. Let $\omega \in (0, \pi)$ be an angle. Then A is Col-sectorial of Col-type ω on $L^p(\mathcal{M})$ if and only if A^* is Row-sectorial of Row-type ω on $L^{p'}(\mathcal{M})$. Moreover A is Rad-sectorial of Rad-type ω on $L^p(\mathcal{M})$ if and only if A^* is Rad-sectorial of Rad-type ω on $L^{p'}(\mathcal{M})$.*

Proof. — Let $\mathcal{F} \subset B(L^p(\mathcal{M}))$ be a set of operators, and let $\mathcal{F}^* = \{T^* \,:\, T \in \mathcal{F}\} \subset B(L^{p'}(\mathcal{M}))$ be the set of its adjoints. Using (2.15), it is easy to see that $\mathcal{F}$ is Col-bounded if and only if $\mathcal{F}^*$ is Row-bounded. If A is sectorial of type ω on $L^p(\mathcal{M})$, then A^* is sectorial of type ω on $L^{p'}(\mathcal{M})$, and we have $R(z, A)^* = R(\overline{z}, A^*)$ for any $z \in \mathbb{C} \setminus \overline{\Sigma_\omega}$. We deduce that A is Col-sectorial of Col-type ω if and only if A^* is Row-sectorial of Row-type ω. The proof of the 'Rad-sectorial' result is similar. □

Lemma 4.8. — *Let $\theta \in (0, \pi)$ be an angle, and let $U \colon \overline{\Sigma_\theta} \to B(L^p(\mathcal{M}))$ be a strongly continuous bounded function whose restriction to Σ_θ is analytic. If the set $\{U(z) \,:\, z \in \partial\Sigma_\theta\}$ is Col-bounded (resp. Row-bounded, resp. Rad-bounded), then $\{U(z) \,:\, z \in \Sigma_\theta\}$ also is Col-bounded (resp. Row-bounded, resp. Rad-bounded).*

Proof. — In the Rademacher case, this result is proved in [**79**, Proposition 2.8]. The proofs for the other cases are identical, using Lemma 4.2. □

Lemma 4.9. — *Let $(T_t)_{t\geq 0}$ be a bounded c_0-semigroup on $L^p(\mathcal{M})$ with infinitesimal generator $-A$, and assume that A is sectorial of type $\omega \in (0, \frac{\pi}{2})$. Then A is Col-sectorial of Col-type ω if and only if for any angle $\alpha \in (0, \frac{\pi}{2} - \omega)$, the set $\{T_z : z \in \Sigma_\alpha\} \subset B(L^p(\mathcal{M}))$ is Col-bounded. The same result holds with Col-boundedness replaced by Row-boundedness or Rad-boundedness.*

Proof. — This result is an analog of Lemma 3.1. Again it is proved in [**79**, Theorem 4.2] in the Rademacher case, and the proofs for the other cases are identical. □

Remark 4.10. — Let A be a sectorial operator of type $\omega \in (0, \pi)$ on some Banach space X. For any positive real number $\alpha > 0$, we let A^α denote the corresponding fractional power of A. If $\alpha\omega < \pi$, then A^α is a sectorial operator of type $\alpha\omega$ (see e.g. [**5**, Proposition 5.2]). It is well-known to specialists that with the same proof, one obtains that A^α is Rad-sectorial of Rad-type $\alpha\omega$ if A is Rad-sectorial of Rad-type ω. Moreover if θ and $\alpha\theta$ both belong to $(0, \pi)$, if $f \in H_0^\infty(\Sigma_{\alpha\theta})$ and if $f_\alpha \in H_0^\infty(\Sigma_\theta)$ is defined by $f_\alpha(z) = f(z^\alpha)$, then we have $f_\alpha(A) = f(A^\alpha)$. Thus A^α has a bounded $H^\infty(\Sigma_{\alpha\theta})$ functional calculus provided that A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus.

Now assume that $X = L^p(\mathcal{M})$ is a noncommutative L^p-space. We observe that mimicking again the proof of [**5**, Proposition 5.2], and using Lemma 4.2 (2), we have that A^α is Col-sectorial (resp. Row-sectorial) of Col-type (resp. Row-type) equal to $\alpha\omega$ if A is Col-sectorial (resp. Row-sectorial) of Col-type (resp. Row-type) equal to ω.

In [**41**, Theorem 5.3, (3)], Kalton-Weis showed that an operator with a bounded $H^\infty(\Sigma_\theta)$ functional calculus on a Banach space X is Rad-sectorial of Rad-type θ provided that X satisfies a certain geometric property called (Δ). According to [**41**, Proposition 3.2], any UMD Banach space X satisfies this property. We deduce the following statement.

Theorem 4.11. — *Let $1 < p < \infty$ and let A be an operator on $L^p(\mathcal{M})$ with a bounded $H^\infty(\Sigma_\theta)$ functional calculus. Then A is Rad-sectorial of Rad-type θ.*

In the next statement, we establish a variant of the above result for Col-sectoriality and Row-sectoriality (see also Remark 4.13).

Theorem 4.12. — *Let A be a sectorial operator on $L^p(\mathcal{M})$, with $1 < p < \infty$. Assume that A admits a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \pi)$. Then the operator A is both Col-sectorial of Col-type θ and Row-sectorial of Row-type θ.*

Proof. — We will only show the 'column' result, the proof for the 'row' one being the same. Given a number $\nu > \theta$, we wish to show that the set

$$\mathcal{F}_\nu = \{zR(z, A) : z \in \mathbb{C} \setminus \overline{\Sigma_\nu}\}.$$

is Col-bounded. We consider $\mathcal{A} = I\overline{\otimes}A$ on $Y = S^p[L^p(\mathcal{M})]$ (see paragraph 3.B). This is a noncommutative L^p-space, hence applying Theorem 4.11 we obtain that the set

$$\mathcal{T}_\nu = \big\{zR(z,\mathcal{A}) \,:\, z\in\mathbb{C}\setminus\overline{\Sigma_\nu}\big\}$$

is Rad-bounded. Now consider $x_1,\ldots,x_n$ in $L^p(\mathcal{M})$ and $T_1,\ldots,T_n$ in $\mathcal{F}_\nu$. For any finite sequence $(\varepsilon_k)_{1\leq k\leq n}$ valued in $\{-1,1\}$, we have

$$\begin{aligned}\Big\|\Big(\sum_k x_k^*x_k\Big)^{\frac12}\Big\|_{L^p(\mathcal{M})} &= \Big\|\Big(\sum_k (\varepsilon_k x_k)^*(\varepsilon_k x_k)\Big)^{\frac12}\Big\|_{L^p(\mathcal{M})}\\ &= \Big\|\sum_{k=1}^n \varepsilon_k\, E_{k1}\otimes x_k\Big\|_{S^p[L^p(\mathcal{M})]}\end{aligned}$$

(see Remark 2.3 (3)). Then passing to the average over all possible choices of $\varepsilon_k=\pm1$, we deduce that

$$\Big\|\Big(\sum_k x_k^*x_k\Big)^{\frac12}\Big\|_{L^p(\mathcal{M})} = \Big\|\sum_{k=1}^n \varepsilon_k\,(E_{k1}\otimes x_k)\Big\|_{\mathrm{Rad}(S^p[L^p(\mathcal{M})])}$$

Likewise we have

$$\Big\|\Big(\sum_k T_k(x_k)^*T_k(x_k)\Big)^{\frac12}\Big\|_{L^p(\mathcal{M})} = \Big\|\sum_{k=1}^n \varepsilon_k\,(I_{S^p}\otimes T_k)(E_{k1}\otimes x_k)\Big\|_{\mathrm{Rad}(S^p[L^p(\mathcal{M})])}.$$

It therefore follows from Lemma 3.6 (2), that

$$\Big\|\Big(\sum_k T_k(x_k)^*T_k(x_k)\Big)^{\frac12}\Big\|_{L^p(\mathcal{M})} \leq \mathrm{Rad}(\mathcal{T}_\nu)\Big\|\Big(\sum_k x_k^*x_k\Big)^{\frac12}\Big\|_{L^p(\mathcal{M})}.$$

This concludes the proof, with $\mathrm{Col}(\mathcal{T}_\nu)\leq\mathrm{Rad}(\mathcal{T}_\nu)$. ☐

Remark 4.13. — The complete boundedness assumption in Theorem 4.12 cannot be replaced by a boundedness assumption. Indeed assume that $1\leq p\neq 2<\infty$, let $\omega\in(0,\pi)$ be an angle, and assume that $\mathcal{M}=B(\ell^2)$. According to Example 4.1, we have a bounded operator $T\colon L^p(\mathcal{M})\to L^p(\mathcal{M})$ such that $T\otimes I_H$ does not extend to a bounded operator on $L^p(\mathcal{M};H_c)$. Shifting T if necessary we may clearly assume that $\sigma(T)$ is included in the open set Σ_ω. Then T is invertible and $\sigma(T^{-1})\subset\Sigma_\omega$. Hence there exists a positive number $\varepsilon>0$ such that $\sigma(T^{-1}-\varepsilon)\subset\Sigma_\omega$. We let $A=T^{-1}-\varepsilon$. By construction, A is a bounded and invertible sectorial operator of type ω. Hence it admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta>\omega$. However $R(-\varepsilon,A)=-T$, and $\{T\}$ is not Col-bounded. Hence A cannot be Col-sectorial.

4.C. Some operator valued singular integrals

We wish to prove a criterion for the boundedness of certain operator valued singular integrals which will appear both in Chapter 6 and in Chapter 7 below. We shall work on the measure space $\Omega_0 = (\mathbb{R}_+^*, \frac{dt}{t})$. Let $\kappa\colon \Omega_0 \times \Omega_0 \to B(L^p(\mathcal{M}))$ be a bounded continuous function. We may define an operator $T\colon L^1(\Omega_0; L^p(\mathcal{M})) \to L^\infty(\Omega_0; L^p(\mathcal{M}))$ by

$$Tu(s) = \int_0^\infty \kappa(s,t)u(t)\,\frac{dt}{t}\,, \qquad u \in L^1(\Omega_0; L^p(\mathcal{M})).$$

Then we say that $\kappa(s,t)$ is the kernel of T.

If T maps $(L^1(\Omega_0)\cap L^2(\Omega_0)) \otimes L^p(\mathcal{M})$ into $L^p(\mathcal{M}; L^2(\Omega_0)_c)$ and if there is a constant $C > 0$ such that $\|Tu\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)} \leq C\|u\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)}$ for any u in $(L^1(\Omega_0)\cap L^2(\Omega_0)) \otimes L^p(\mathcal{M})$, then T uniquely extends to a bounded linear mapping, that we still denote by

$$T\colon L^p(\mathcal{M}; L^2(\Omega_0)_c) \longrightarrow L^p(\mathcal{M}; L^2(\Omega_0)_c).$$

Indeed, $(L^1(\Omega_0)\cap L^2(\Omega_0)) \otimes L^p(\mathcal{M})$ is dense in $L^p(\mathcal{M}; L^2(\Omega)_c)$. Moreover a standard approximation argument shows that this extension coincides with the original operator T on $L^1(\Omega_0; L^p(\mathcal{M})) \cap L^p(\mathcal{M}; L^2(\Omega_0)_c)$. In this case we simply say that the operator with associated kernel $\kappa(s,t)$ is bounded on $L^p(\mathcal{M}; L^2(\Omega_0)_c)$. We define similarly the boundedness of T on $L^p(\mathcal{M}; L^2(\Omega_0)_r)$, or on $L^p(\mathcal{M}; L^2(\Omega_0)_{rad})$.

For any angle $\omega \in (0,\pi)$, we define

$$H^\infty(\Sigma_{\omega+}) = \cup_{\theta>\omega} H^\infty(\Sigma_\theta) \qquad \text{and} \qquad H_0^\infty(\Sigma_{\omega+}) = \cup_{\theta>\omega} H_0^\infty(\Sigma_\theta). \tag{4.11}$$

Let A be a sectorial operator of type ω on $L^p(\mathcal{M})$. For any $F \in H_0^\infty(\Sigma_{\omega+})$ and any $t > 0$, let $F(tA) = F_t(A)$, where $F_t(z) = F(tz)$. Using Lebesgue's Theorem and (3.5), it is not hard to see that the function $t \mapsto F(tA)$ is continuous and bounded on Ω_0 (see also Lemma 6.5 below). Thus for any F_1, $F_2 \in H_0^\infty(\Sigma_{\omega+})$, the kernel $\kappa(s,t) = F_2(sA)F_1(tA)$ is continuous and bounded on $\Omega_0\times\Omega_0$. The study of operators associated with such kernels for sectorial operators on Hilbert space goes back to [**54**].

Theorem 4.14. — *Let A be a sectorial operator of type ω on $L^p(\mathcal{M})$, and let F_1, $F_2 \in H_0^\infty(\Sigma_{\omega+})$.*

(1) *If A is Col-sectorial of Col-type ω, then the operator with kernel $F_2(sA)F_1(tA)$ is bounded on $L^p(\mathcal{M}; L^2(\Omega_0)_c)$.*
(2) *If A is Row-sectorial of Row-type ω, then the operator with kernel $F_2(sA)F_1(tA)$ is bounded on $L^p(\mathcal{M}; L^2(\Omega_0)_r)$.*
(3) *If A is Rad-sectorial of Rad-type ω, then the operator with kernel $F_2(sA)F_1(tA)$ is bounded on $L^p(\mathcal{M}; L^2(\Omega_0)_{rad})$.*

Proof. — We shall only prove (1), the proofs of (2) and (3) being similar. We let $\theta > \omega$ be such that F_1, $F_2 \in H_0^\infty(\Sigma_\theta)$ and fix some $\gamma \in (\omega,\theta)$. Then applying (3.5) and the homomorphism property of the H^∞ functional calculus, we may write our kernel as

$$F_2(sA)F_1(tA) = \frac{1}{2\pi i}\int_{\Gamma_\gamma} F_2(sz)F_1(tz)R(z,A)\,dz\,, \qquad t>0,\ s>0. \tag{4.12}$$

We shall apply Proposition 4.4 on the measure space $(\Omega,\mu)=(\Gamma_\gamma,\left|\frac{dz}{z}\right|)$. Our assumption that A is Col-sectorial of Col-type ω implies that the set $\{zR(z,A)\,:\,z\in\Gamma_\gamma\}$ is Col-bounded. It therefore follows from Lemma 4.2 that the set

$$\left\{\frac{1}{\mu(I)}\int_I zR(z,A)\left|\frac{dz}{z}\right|\,:\,I\subset\Gamma_\gamma,\ 0<\mu(I)<\infty\right\}$$

is Col-bounded as well. Hence by Proposition 4.4, the function

$$\Phi(z)=\frac{zR(z,A)}{2\pi i}$$

induces a bounded multiplication operator

$$T_\Phi\colon L^p(\mathcal{M};L^2(\Omega)_c)\longrightarrow L^p(\mathcal{M};L^2(\Omega)_c). \tag{4.13}$$

Our next goal is to show that we may define bounded linear mappings $S_1\colon L^2(\Omega_0)\to L^2(\Omega)$ and $S_2\colon L^2(\Omega)\to L^2(\Omega_0)$ by letting

$$S_1a(z)=\int_0^\infty F_1(tz)a(t)\,\frac{dt}{t},\qquad a\in L^2(\Omega_0); \tag{4.14}$$

$$S_2b(s)=\int_{\Gamma_\gamma} F_2(sz)b(z)\,\frac{dz}{z},\qquad b\in L^2(\Omega). \tag{4.15}$$

First observe that

$$K=\sup_{t>0}\int_{\Gamma_\gamma}\bigl|F_1(tz)\bigr|\left|\frac{dz}{z}\right|<\infty\quad\text{and}\quad K'=\sup_{z\in\Gamma_\gamma}\int_0^\infty\bigl|F_1(tz)\bigr|\,\frac{dt}{t}<\infty. \tag{4.16}$$

Indeed, changing z into tz does not change $\int_{\Gamma_\gamma}\bigl|F_1(tz)\bigr|\left|\frac{dz}{z}\right|$, hence $K=\int_{\Gamma_\gamma}\bigl|F_1(z)\bigr|\left|\frac{dz}{z}\right|$, and this number is finite by (3.6). On the other hand, for any $z\in\Gamma_\gamma\setminus\{0\}$ we have

$$\int_0^\infty\bigl|F_1(tz)\bigr|\,\frac{dt}{t}+\int_0^\infty\bigl|F_1(t\overline{z})\bigr|\,\frac{dt}{t}=\int_{\Gamma_\gamma}\bigl|F_1(\lambda)\bigr|\left|\frac{d\lambda}{\lambda}\right|,$$

hence $K'\leq K<\infty$.

We let a be an arbitrary element of $L^2(\Omega_0)$. Then

$$\begin{aligned}
&\int_{\Gamma_\gamma}\left(\int_0^\infty\bigl|F_1(tz)a(t)\bigr|\,\frac{dt}{t}\right)^2\left|\frac{dz}{z}\right|\\
&\qquad\leq\int_{\Gamma_\gamma}\left(\int_0^\infty\bigl|F_1(tz)\bigr|\,\frac{dt}{t}\right)\left(\int_0^\infty\bigl|F_1(tz)\bigr|\bigl|a(t)\bigr|^2\,\frac{dt}{t}\right)\left|\frac{dz}{z}\right|\quad\text{by Cauchy-Schwarz,}\\
&\qquad\leq K'\int_{\Gamma_\gamma}\left(\int_0^\infty\bigl|F_1(tz)\bigr|\bigl|a(t)\bigr|^2\,\frac{dt}{t}\right)\left|\frac{dz}{z}\right|\leq KK'\int_0^\infty\bigl|a(t)\bigr|^2\,\frac{dt}{t}
\end{aligned}$$

by (4.16). This shows that (4.14) induces a bounded mapping with

$$\|S_1\colon L^2(\Omega_0)\longrightarrow L^2(\Omega)\|\leq\sqrt{KK'}.$$

The proof of the boundedness of S_2 is similar. Owing to Lemma 2.4, we may extend $I_{L^p}\otimes S_1$ and $I_{L^p}\otimes S_2$ to bounded mappings

$$\widehat{S_1}\colon L^p(\mathcal{M}; L^2(\Omega_0)_c) \longrightarrow L^p(\mathcal{M}; L^2(\Omega)_c)$$

and

$$\widehat{S_2}\colon L^p(\mathcal{M}; L^2(\Omega)_c) \longrightarrow L^p(\mathcal{M}; L^2(\Omega_0)_c).$$

The same computations as above show that $I_{L^p}\otimes S_1$ and $I_{L^p}\otimes S_2$ also extend to bounded operators from $L^2(\Omega_0; L^p(\mathcal{M}))$ into $L^2(\Omega; L^p(\mathcal{M}))$ and from $L^2(\Omega; L^p(\mathcal{M}))$ into $L^2(\Omega_0; L^p(\mathcal{M}))$ respectively. Moreover these tensor extensions are given by the integral representations (4.14) and (4.15). Thus we find that

$$\widehat{S_1}u(z) = \int_0^\infty F_1(tz)u(t)\,\frac{dt}{t}, \qquad u\in L^2(\Omega_0; L^p(\mathcal{M}))\cap L^p(\mathcal{M}; L^2(\Omega_0)_c); \tag{4.17}$$

$$\widehat{S_2}v(s) = \int_{\Gamma_\gamma} F_2(sz)v(z)\,\frac{dz}{z}, \qquad v\in L^2(\Omega; L^p(\mathcal{M}))\cap L^p(\mathcal{M}; L^2(\Omega)_c). \tag{4.18}$$

Now recall (4.13) and consider the composition operator

$$\widehat{S_2}\,T_\Phi\,\widehat{S_1}\colon L^p(\mathcal{M}; L^2(\Omega_0)_c) \longrightarrow L^p(\mathcal{M}; L^2(\Omega_0)_c).$$

We claim that $F_2(sA)F_1(tA)$ is a kernel for this operator, which will conclude the proof. To check this claim, we consider some $u\in (L^1(\Omega_0)\cap L^2(\Omega_0))\otimes L^p(\mathcal{M})$. It follows from (4.17) and (4.13) that $T_\Phi\widehat{S_1}u\in L^2(\Omega; L^p(\mathcal{M}))\cap L^p(\mathcal{M}; L^2(\Omega)_c)$ with

$$\big[T_\Phi\,\widehat{S_1}u\big](z) \;=\; \frac{1}{2\pi i}\int_0^\infty F_1(tz)zR(z,A)u(t)\,\frac{dt}{t}, \qquad z\in\Gamma_\gamma.$$

Hence applying (4.18) with $v = T_\Phi\widehat{S_1}u$, we obtain that

$$\begin{aligned}\big[\widehat{S_2}\,T_\Phi\,\widehat{S_1}u\big](s) &= \frac{1}{2\pi i}\int_{\Gamma_\gamma} F_2(sz)\Big(\int_0^\infty F_1(tz)zR(z,A)u(t)\,\frac{dt}{t}\Big)\,\frac{dz}{z}\\ &= \int_0^\infty F_2(sA)F_1(tA)u(t)\,\frac{dt}{t}\end{aligned}$$

by Fubini's Theorem and (4.12). □

CHAPTER 5

NONCOMMUTATIVE DIFFUSION SEMIGROUPS

In this chapter we will focus on a special class of semigroups acting on noncommutative L^p-spaces. Throughout we let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra.

Let $T\colon \mathcal{M} \to \mathcal{M}$ be a normal contraction. We say that T is selfadjoint if

$$\tau\bigl(T(x)y^*\bigr) = \tau\bigl(xT(y)^*\bigr), \qquad x,\, y \in \mathcal{M} \cap L^1(\mathcal{M}). \tag{5.1}$$

In this case, we have

$$\bigl|\tau\bigl(T(x)y\bigr)\bigr| \;=\; \bigl|\tau\bigl(xT(y^*)^*\bigr)\bigr| \;\leq\; \|x\|_1 \|T(y^*)^*\|_\infty \;\leq\; \|x\|_1\|y\|_\infty$$

for any x, y in $\mathcal{M} \cap L^1(\mathcal{M})$. Hence the restriction of T to $\mathcal{M} \cap L^1(\mathcal{M})$ (uniquely) extends to a contraction $T_1\colon L^1(\mathcal{M}) \to L^1(\mathcal{M})$. Then according to (2.4), it also extends by interpolation to a contraction $T_p\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ for any $1 \leq p < \infty$. We write $T_\infty = T$ for convenience. Then using the notation introduced in (2.5), we obtain that

$$T_p^* \;=\; T_{p'}^\circ, \qquad 1 \leq p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Indeed this follows from (5.1), and the hypothesis that T_∞ is normal. In particular, the operator $T_2\colon L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is selfadjoint.

It T is positive, then each T_p is positive, and hence $T_p^\circ = T_p$. Thus in this case, we have $T_p^* = T_{p'}$ for any $1 \leq p < \infty$.

If $T\colon \mathcal{M} \to \mathcal{M}$ is a normal selfadjoint contraction as above, we will usually use the same notation $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ instead of T_p, for all its L^p-realizations.

Let $(T_t)_{t\geq 0}$ be a semigroup of operators on $\mathcal{M}$. We say that $(T_t)_{t\geq 0}$ is a (noncommutative) diffusion semigroup if each $T_t\colon \mathcal{M} \to \mathcal{M}$ is a normal selfadjoint contraction and if for any $x \in \mathcal{M}$, $T_t(x) \to x$ in the w^*-topology of $\mathcal{M}$ when $t \to 0^+$. It follows from above that such a semigroup extends to a semigroup of contractions on $L^p(\mathcal{M})$ for any $1 \leq p < \infty$, and that $(T_t)_{t\geq 0}$ is a selfadjoint semigroup on $L^2(\mathcal{M})$. Moreover $(T_t)_{t\geq 0}$ is strongly continuous on $L^p(\mathcal{M})$ for any $1 \leq p < \infty$, by [**22**, Proposition 1.23].

In general we let $-A_p$ denote the infinitesimal generator of the realization of $(T_t)_{t\geq 0}$ on $L^p(\mathcal{M})$. If further each $T_t\colon \mathcal{M}\to\mathcal{M}$ is positive, then

$$A_p^* = A_{p'}, \qquad 1<p<\infty, \quad \frac{1}{p}+\frac{1}{p'}=1. \tag{5.2}$$

Indeed in this case, the dual semigroup of the realization of $(T_t)_{t\geq 0}$ on $L^p(\mathcal{M})$ is exactly the realization of $(T_t)_{t\geq 0}$ on $L^{p'}(\mathcal{M})$. Note that our terminology extends the one introduced by Stein in [**70**, Chapter 3] in the commutative setting.

Remark 5.1. — Let $T\colon \mathcal{M}\to\mathcal{M}$ be a normal complete contraction, and assume that T is selfadjoint. The tensor extension $I_{B(\ell^2)}\otimes T$ uniquely extends to a normal contraction $I\overline{\otimes}T\colon B(\ell^2)\overline{\otimes}\mathcal{M}\to B(\ell^2)\overline{\otimes}\mathcal{M}$, and it is easy to check that $I\overline{\otimes}T$ is selfadjoint. For any $1\leq p\leq\infty$, let $T_p\colon L^p(\mathcal{M})\to L^p(\mathcal{M})$ be the L^p-realization of T. Then T_p is completely contractive and $I\overline{\otimes}T_p\colon S^p[L^p(\mathcal{M})]\to S^p[L^p(\mathcal{M})]$ is the L^p-realization of $I\overline{\otimes}T_\infty$. This is proved by applying the above results to $I\overline{\otimes}T_\infty$. An alternative route is to apply (2.38) with $p=1$ to obtain that T_1 is a complete contraction, and then to deduce that $\|T_p\|_{cb}\leq 1$ for any $p\in(1,\infty)$ by interpolation.

Let $(T_t)_{t\geq 0}$ be a noncommutative diffusion semigroup on $\mathcal{M}$. We say that $(T_t)_{t\geq 0}$ is a completely contractive diffusion semigroup if $T_t\colon\mathcal{M}\to\mathcal{M}$ is a complete contraction for any $t\geq 0$. In this case, $(I\overline{\otimes}T_t)_{t\geq 0}$ is a noncommutative diffusion semigroup on $B(\ell^2)\overline{\otimes}\mathcal{M}$. We say that $(T_t)_{t\geq 0}$ is a completely positive diffusion semigroup if $T_t\colon\mathcal{M}\to\mathcal{M}$ is completely positive for any $t\geq 0$ (see paragraph 2.D). We recall that a completely positive contraction on a C^*-algebra is a complete contraction (see e.g. [**58**, Chapter 3]). Thus a completely positive diffusion semigroup is a completely contractive one.

Remark 5.2. — We can consider noncommutative diffusion semigroups from a slightly different point of view. Suppose that $(T_t)_{t\geq 0}$ is a selfadjoint semigroup of contractions on $L^2(\mathcal{M})$. Suppose further that for any $t\geq 0$, T_t extends to a contraction $T_{1,t}\colon L^1(\mathcal{M})\to L^1(\mathcal{M})$, and that $(T_{1,t})_{t\geq 0}$ is strongly continuous. Then $(T_t)_{t\geq 0}$ 'is' a noncommutative diffusion semigroup. Indeed, for any $t\geq 0$, $T_{1,t}^{*\circ}\colon\mathcal{M}\to\mathcal{M}$ is a normal selfadjoint contraction, $T_{1,t}^{*\circ}\to I_{\mathcal{M}}$ in the point w^*-topology, and the L^2-realization of $T_{1,t}^{*\circ}$ coincides with T_t for any $t\geq 0$.

We will need the following 'sectorial' form of Stein's interpolation principle (see e.g. [**70**, III. 2] or [**77**]). In that statement, we let

$$S(\theta) = \{z\in\mathbb{C}^* \,:\, 0\leq \operatorname{Arg}(z)\leq\theta\}$$

for any angle $\theta\in(0,\pi)$.

Lemma 5.3. — *Let (E_0,E_1) be any interpolation couple of Banach spaces, and for any $\alpha\in(0,1)$, let $E_\alpha=[E_0,E_1]_\alpha$ be the interpolation space obtained by the complex* *interpolation method. We consider a family of bounded operators* $U(z)\colon E_0 \cap E_1 \to E_0 + E_1$ *for* $z \in S(\theta)$. *Assume that:*

(a) *For any* $x \in E_0 \cap E_1$, *the function* $z \mapsto U(z)x$ *is continuous and bounded, and its restriction to the interior of* $S(\theta)$ *is analytic.*

(b) *For any* $x \in E_0 \cap E_1$, $U(t)x \in E_0$ *and* $U(te^{i\theta})x \in E_1$ *for any* $t > 0$, *and the resulting functions* $t \mapsto U(t)x$ *and* $t \mapsto U(te^{i\theta})x$ *are continuous from* $(0, \infty)$ *into* E_0 *and* E_1 *respectively.*

(c) *There exist nonnegative constants* C_0, C_1 *such that for any* $x \in E_0 \cap E_1$ *and any* $t > 0$, *we have*

$$\|U(t)x\|_{E_0} \leq C_0 \|x\|_{E_0} \qquad \text{and} \qquad \|U(te^{i\theta})x\|_{E_1} \leq C_1 \|x\|_{E_1}.$$

Then for any number $\alpha \in (0,1)$ *and any* $t > 0$, $U(te^{i\alpha\theta})$ *maps* $E_0 \cap E_1$ *into* E_α, *with*

$$\|U(te^{i\alpha\theta})x\|_{E_\alpha} \leq C_0^{1-\alpha} C_1^{\alpha} \|x\|_{E_\alpha}, \qquad x \in E_0 \cap E_1.$$

Throughout this chapter, we let

$$\omega_p = \pi \left| \frac{1}{p} - \frac{1}{2} \right|$$

for any $1 < p < \infty$.

Proposition 5.4. — *Let* $(T_t)_{t\geq 0}$ *be a noncommutative diffusion semigroup on* $\mathcal{M}$ *and for any* $1 < p < \infty$, *let* $-A_p$ *denote the generator of* $(T_t)_{t\geq 0}$ *on* $L^p(\mathcal{M})$. *Then* A_p *is a sectorial operator of type* ω_p.

Proof. — This result is well-known in the commutative case and we simply mimic its proof. By duality we may assume that $1 < p < 2$. We let p' denote the conjugate number of p. First we note that since $(T_t)_{t\geq 0}$ is a selfadjoint semigroup on $L^2(\mathcal{M})$, then A_2 is a positive selfadjoint operator. Hence A_2 is sectorial of type 0 and by spectral theory,

$$T_z = e^{-zA_2}\colon L^2(\mathcal{M}) \longrightarrow L^2(\mathcal{M})$$

is a well-defined contraction for any complex number z such that $\mathrm{Re}(z) \geq 0$. Let us apply Lemma 5.3 with $E_0 = L^1(\mathcal{M})$, $E_1 = L^2(\mathcal{M})$, $0 < \theta \leq \frac{\pi}{2}$, and $U(z)x = T_z x$. According to (2.4), we have $[L^1(\mathcal{M}), L^2(\mathcal{M})]_{\frac{2}{p'}} = L^p(\mathcal{M})$. Hence we obtain that

$$\|T_z\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M})\| \leq 1 \tag{5.3}$$

for any $z \in \mathbb{C}^*$ such that $0 \leq \mathrm{Arg}(z) \leq \frac{\pi}{p'}$. Likewise, (5.3) holds true if $-\frac{\pi}{p'} \leq \mathrm{Arg}(z) \leq 0$. Then Lemma 3.1 ensures that A_p is sectorial of type $\frac{\pi}{2} - \frac{\pi}{p'} = \omega_p$. □

Remark 5.5. — Let $(T_t)_{t\geq 0}$ be a noncommutative diffusion semigroup on $\mathcal{M}$, and consider two numbers $1 < p, q < \infty$. Let $\omega = \max\{\omega_p, \omega_q\}$, so that A_p and A_q are both sectorial operators of type ω. It easily follows from the Laplace formula (3.2) and from (3.5) that for any $\theta > \omega$ and any $f \in H_0^\infty(\Sigma_\theta)$, $f(A_p)$ and $f(A_q)$ are consistent operators, that is, they coincide on $L^p(\mathcal{M}) \cap L^q(\mathcal{M})$. Likewise if A_p and A_q both have dense ranges and admit a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta > \omega$, then $f(A_p)$ and $f(A_q)$ are consistent for any $f \in H^\infty(\Sigma_\theta)$. Indeed for $x \in L^p(\mathcal{M}) \cap L^q(\mathcal{M})$, $g_n(A_p)x = g_n(A_q)x$ is a common approximation of x in $L^p(\mathcal{M})$ and in $L^q(\mathcal{M})$, by Lemma 3.2. Hence

$$f(A_p)x \,=\, L^p - \lim_n (fg_n)(A_p)x \,=\, L^q - \lim_n (fg_n)(A_q)x \,=\, f(A_q)x.$$

The next theorem is the main result of this chapter. We refer to paragraph 2.D for the definition of 2-positivity.

Theorem 5.6. — *Let $(T_t)_{t\geq 0}$ be a noncommutative diffusion semigroup on $\mathcal{M}$ and for any $1 < p < \infty$, let $-A_p$ denote the generator of $(T_t)_{t\geq 0}$ on $L^p(\mathcal{M})$.*

(1) *If T_t is 2-positive for any $t \geq 0$, then A_p is Col-sectorial (resp. Row-sectorial) of Col-type (resp. Row-type) equal to ω_p.*
(2) *If T_t is positive for any $t \geq 0$, then A_p is Rad-sectorial of Rad-type ω_p.*

Proof. — (1): We assume that T_t is 2-positive for any $t \geq 0$. If $1 < p, p' < \infty$ are conjugate numbers, then $A_p^* = A_{p'}$ by (5.2). According to Lemma 4.7, we may therefore assume that $2 < p < \infty$ in our proof of (1). Our first step consists in showing that the set

$$\mathcal{F}_p \,=\, \{T_t \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}) \,:\, t \geq 0\}$$

is Col-bounded. Since T_t is 2-positive and contractive, we have

$$T_t(x)^* T_t(x) \,\leq\, T_t(x^* x), \qquad x \in L^p(\mathcal{M}). \tag{5.4}$$

Indeed if $x \in \mathcal{M}$, this is Choi's extension of the Kadison-Schwarz inequality for 2-positive maps on unital C^*-algebras (see [**18**] or [**58**, Ex. 3.4]). For an arbitrary $x \in L^p(\mathcal{M})$, let $(x_i)_{i\geq 1}$ be a sequence in $L^p(\mathcal{M}) \cap \mathcal{M}$ such that $\|x - x_i\|_p \to 0$ when $i \to \infty$. Then $\|x^*x - x_i^* x_i\|_{\frac{p}{2}} \to 0$. Since T_t is continuous on $L^{\frac{p}{2}}(\mathcal{M})$ we obtain that $T_t(x^*x)$ is the limit of $T_t(x_i^* x_i)$ in $L^{\frac{p}{2}}(\mathcal{M})$. Likewise since T_t is continuous on $L^p(\mathcal{M})$, we see that $T_t(x)^*T_t(x)$ is the limit of $T_t(x_i)^*T_t(x_i)$ in $L^{\frac{p}{2}}(\mathcal{M})$. Since (5.4) holds true for any x_i, it holds true for x as well.

Let $t_1, \ldots, t_n$ be nonnegative real numbers, and let $x_1, \ldots x_n$ in $L^p(\mathcal{M})$. We have

$$\Bigl\|\Bigl(\sum_{k=1}^n T_{t_k}(x_k)^* T_{t_k}(x_k)\Bigr)^{\frac{1}{2}}\Bigr\|_p^2 \,=\, \Bigl\|\sum_{k=1}^n T_{t_k}(x_k)^* T_{t_k}(x_k)\Bigr\|_{\frac{p}{2}} \,\leq\, \Bigl\|\sum_{k=1}^n T_{t_k}(x_k^* x_k)\Bigr\|_{\frac{p}{2}}$$

by (5.4). Let $1 < r < \infty$ be the conjugate number of $\frac{p}{2}$. Since $\sum_k T_{t_k}(x_k^* x_k)$ is positive, there exists some $y \in L^r(\mathcal{M})_+$ such that $\|y\|_r = 1$ and

$$\Bigl\|\sum_{k=1}^n T_{t_k}(x_k^* x_k)\Bigr\|_{\frac{p}{2}} = \Bigl\langle \sum_{k=1}^n T_{t_k}(x_k^* x_k), y\Bigr\rangle\,.$$

By the noncommutative maximal ergodic theorem for positive diffusion semigroups [**37**, Cor. 4 (iii)] (see also [**36**]), there exists some $\varphi \in L^r(\mathcal{M})_+$ such that $T_t(y) \leq \varphi$

for any $t > 0$ and $\|\varphi\|_r \leq K$, where $K \geq 1$ is an absolute constant not depending on y. By assumption the adjoint of the $L^{\frac{p}{2}}$-realization of T_t is equal to the L^r-realization of T_t for any $t \geq 0$, hence

$$\begin{aligned}\Big\|\sum_{k=1}^{n} T_{t_k}(x_k^* x_k)\Big\|_{\frac{p}{2}} &= \Big\langle \sum_{k=1}^{n} x_k^* x_k, T_{t_k}(y)\Big\rangle \\ &\leq \Big\langle \sum_{k=1}^{n} x_k^* x_k, \varphi\Big\rangle \\ &\leq \|\varphi\|_r \Big\|\sum_{k=1}^{n} x_k^* x_k\Big\|_{\frac{p}{2}} \\ &\leq K \Big\|\Big(\sum_{k=1}^{n} x_k^* x_k\Big)^{\frac{1}{2}}\Big\|_p^2.\end{aligned}$$

This shows that

$$\Big\|\Big(\sum_{k=1}^{n} T_{t_k}(x_k)^* T_{t_k}(x_k)\Big)^{\frac{1}{2}}\Big\|_p \leq \sqrt{K}\,\Big\|\Big(\sum_{k=1}^{n} x_k^* x_k\Big)^{\frac{1}{2}}\Big\|_p.$$

Thus $\mathcal{F}_p$ is a Col-bounded set, with $\mathrm{Col}(\mathcal{F}_p) \leq \sqrt{K}$.

We fix some $\beta \in (0, \frac{\pi}{p})$. Our second step consists in showing that the set

$$\mathcal{G}_p = \big\{T_{te^{i\beta}} \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}) \,:\, t \geq 0\big\}$$

is Col-bounded. For this, we define

$$q = p\Big(\frac{\pi - 2\beta}{\pi - p\beta}\Big) \qquad \text{and} \qquad \alpha = \frac{2\beta}{\pi}.$$

These numbers are chosen so that $\frac{1-\alpha}{q} + \frac{\alpha}{2} = \frac{1}{p}$. Thus we have $L^p(\mathcal{M}) = [L^q(\mathcal{M}), L^2(\mathcal{M})]_\alpha$ by (2.4). More generally, it follows from (2.13) that for any positive integer $n \geq 1$, we have

$$L^p(\mathcal{M}; (\ell_n^2)_c) = [L^q(\mathcal{M}; (\ell_n^2)_c), L^2(\mathcal{M}; (\ell_n^2)_c)]_\alpha. \tag{5.5}$$

We consider nonnegative real numbers $t_1, \ldots, t_n$, and apply Lemma 5.3 with the spaces $E_0 = L^q(\mathcal{M}; (\ell_n^2)_c)$, $E_1 = L^2(\mathcal{M}; (\ell_n^2)_c)$, the angle $\theta = \frac{\pi}{2}$, and the mappings $U(z)$ defined by letting

$$U(z)\Big(\sum_{k=1}^{n} x_k \otimes e_k\Big) = \sum_{k=1}^{n} T_{zt_k}(x_k) \otimes e_k,$$

for $x_1, \ldots, x_n \in L^2(\mathcal{M}) \cap L^q(\mathcal{M})$. We note that $\beta = \alpha\theta$. Since $L^2(\mathcal{M}; (\ell_n^2)_c) = \ell_n^2(L^2(\mathcal{M}))$ (see Remark 2.3 (1)), and since each $T_{te^{i\frac{\pi}{2}}} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is a contraction, we see that

$$\|U(te^{i\frac{\pi}{2}}) \colon E_1 \longrightarrow E_1\| \leq 1, \qquad t > 0.$$

On the other hand, using the fact that $\mathcal{F}_q$ is Col-bounded, we have

$$\|U(t)\colon E_0 \longrightarrow E_0\| \leq \mathrm{Col}(\mathcal{F}_q), \qquad t>0.$$

Hence $U(e^{i\beta})\colon E_\alpha \to E_\alpha$ has norm less than or equal to $\mathrm{Col}(\mathcal{F}_q)^{1-\alpha}$. Thus we find that

$$\Big\|\sum_{k=1}^n T_{t_k e^{i\beta}}(x_k)\otimes e_k\Big\|_{L^p(\mathcal{M};(\ell^2_n)_c)} \leq \mathrm{Col}(\mathcal{F}_q)^{1-\alpha}\,\Big\|\sum_{k=1}^n x_k\otimes e_k\Big\|_{L^p(\mathcal{M};(\ell^2_n)_c)}$$

for any $x_1,\ldots,x_n \in L^p(\mathcal{M})$. This shows that $\mathcal{G}_p$ is Col-bounded.

By symmetry, we have that $\{T_{te^{-i\beta}}\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})\,:\, t\geq 0\}$ also is a Col-bounded subset of $B(L^p(\mathcal{M}))$. Now appealing to Lemma 4.8, we deduce that the set

$$\{T_z\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M})\,:\, z\in\Sigma_\beta\}$$

is Col-bounded. Since this holds true for any $\beta<\frac{\pi}{p}$, this implies by Lemma 4.9 that A_p is Col-sectorial of Col-type $\frac{\pi}{2}-\frac{\pi}{p}$.

A similar proof shows that A_p is Row-sectorial of Row-type $\frac{\pi}{2}-\frac{\pi}{p}$.

(2): In this part, we only assume that T_t is positive for any $t\geq 0$, and aim at showing that A_p is Rad-sectorial of Rad-type ω_p. Again we may assume that $p>2$, and we follow a similar scheme of proof. The first step consists in showing that

$$\mathcal{F} \,=\, \{T_t\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M})\,:\, t\geq 0\}$$

is Rad-bounded. Since the T_t's are no longer assumed to be 2-positive, the inequality (5.4) is no longer available. However we have

$$T_t(x)^2 \,\leq\, T_t(x^2) \qquad \text{if } x=x^*\in L^p(\mathcal{M}). \tag{5.6}$$

If $x\in\mathcal{M}$ is selfadjoint, this is the Kadison-Schwarz inequality [**39**] for positive maps, and the case when $x\in L^p(\mathcal{M})$ is selfadjoint follows by approximation.

Let $t_1,\ldots,t_n$ be nonnegative real numbers, and let $x_1,\ldots x_n$ in $L^p(\mathcal{M})$ such that

$$\max\Big\{\Big\|\Big(\sum_k x_k^*x_k\Big)^{\frac12}\Big\|_p\,,\ \Big\|\Big(\sum_k x_kx_k^*\Big)^{\frac12}\Big\|_p\Big\} \,\leq\, 1.$$

According to (2.21), it suffices to show that we have

$$\Big\|\Big(\sum_k T_{t_k}(x_k)^*T_{t_k}(x_k)\Big)^{\frac12}\Big\|_p \,\leq\, K \qquad\text{and}\qquad \Big\|\Big(\sum_k T_{t_k}(x_k)T_{t_k}(x_k)^*\Big)^{\frac12}\Big\|_p \,\leq\, K, \tag{5.7}$$

for some constant $K>0$ not depending either on the t_k's or the $x_k's$. Arguing as in the proof of (1) and using (5.6) as a substitute for (5.4), we obtain an inequality (5.7) in the case when each x_k is selfadjoint.

For arbitrary x_k's, let us consider the real and imaginary parts $\mathrm{Re}(x_k)$ and $\mathrm{Im}(x_k)$, which are selfadjoint elements of $L^p(\mathcal{M})$. We have

$$\begin{aligned}
\Big\|\Big(\sum_k [\mathrm{Re}(x_k)]^2\Big)^{\frac{1}{2}}\Big\|_p &= \Big\|\sum_k \mathrm{Re}(x_k)\otimes e_k\Big\|_{L^p(\mathcal{M};(\ell^2_n)_c)}\\
&= \frac{1}{2}\Big\|\sum_k (x_k + x_k^*)\otimes e_k\Big\|_{L^p(\mathcal{M};(\ell^2_n)_c)}\\
&\le \frac{1}{2}\left(\Big\|\sum_k x_k\otimes e_k\Big\|_{L^p(\mathcal{M};(\ell^2_n)_c)} + \Big\|\sum_k x_k^*\otimes e_k\Big\|_{L^p(\mathcal{M};(\ell^2_n)_c)}\right)\\
&\le \frac{1}{2}\left(\Big\|\sum_k x_k\otimes e_k\Big\|_{L^p(\mathcal{M};(\ell^2_n)_c)} + \Big\|\sum_k x_k\otimes e_k\Big\|_{L^p(\mathcal{M};(\ell^2_n)_r)}\right)\\
&\le 1.
\end{aligned}$$

Hence

$$\Big\|\Big(\sum_k \big[T_{t_k}(\mathrm{Re}(x_k))\big]^2\Big)^{\frac{1}{2}}\Big\|_p \le K$$

by the preceding part of the proof. Likewise, we have

$$\Big\|\Big(\sum_k \big[T_{t_k}(\mathrm{Im}(x_k))\big]^2\Big)^{\frac{1}{2}}\Big\|_p \le K.$$

Since

$$\Big\|\Big(\sum_k T_{t_k}(x_k)^*T_{t_k}(x_k)\Big)^{\frac{1}{2}}\Big\|_p \le \Big\|\Big(\sum_k \big[T_{t_k}(\mathrm{Re}(x_k))\big]^2\Big)^{\frac{1}{2}}\Big\|_p + \Big\|\Big(\sum_k \big[T_{t_k}(\mathrm{Im}(x_k))\big]^2\Big)^{\frac{1}{2}}\Big\|_p,$$

we deduce that the first half (5.7) is fulfilled, up to doubling the constant. The second half holds true as well by the same arguments.

Now arguing as in the proof of (1), it suffices to show that for any $\beta \in (0, \frac{\pi}{p})$, the set

$$\mathcal{G} = \big\{T_{te^{i\beta}} : L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}) \ :\ t \ge 0\big\}$$

is Rad-bounded. The proof of this fact is essentially similar to the proof that the set $\mathcal{G}_p$ is Col-bounded in the proof of (1). The only significant change is that one has to use (2.33) with $r = 2$ in the place of (5.5). Details are left to the reader. □

Remark 5.7. — Let $T\colon \mathcal{M} \to \mathcal{M}$ be a selfadjoint normal contraction. Then arguing as in the proof of Theorem 5.6, we find that if T is positive, then the set

$$\{T^n \ :\ n \ge 0\} \subset B(L^p(\mathcal{M}))$$

is Rad-bounded for any $1 < p < \infty$. If further T is 2-positive, then this set is both Col-bounded and Row-bounded.

Our next statement is an angle reduction principle for noncommutative diffusion semigroups.

Proposition 5.8. — *Let $(T_t)_{t\geq 0}$ be a noncommutative diffusion semigroup on $\mathcal{M}$ and for any $1 < p < \infty$, let $-A_p$ denote the generator of $(T_t)_{t\geq 0}$ on $L^p(\mathcal{M})$. Assume further that for any $1 < p < \infty$ and for any $\theta > \frac{\pi}{2}$, A_p admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus. Then for any $1 < p < \infty$, A_p actually admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \omega_p$.*

Proof. — We may assume that $p > 2$, the proof for $p < 2$ being the same. As in the proof of Theorem 5.6, we need to choose some parameters allowing an efficient use of interpolation theory. We give ourselves two numbers $\theta > \delta > \omega_p$. Then we pick $\alpha \in (0, \frac{2}{p})$ such that $\frac{\pi}{2}(1-\alpha) < \delta$. Once α is fixed, we let $q \in (p, \infty)$ be the unique number such that $\frac{1}{p} = \frac{1-\alpha}{q} + \frac{\alpha}{2}$, so that we have

$$L^p(\mathcal{M}) = [L^q(\mathcal{M}), L^2(\mathcal{M})]_\alpha \tag{5.8}$$

by (2.4). Then we choose $\nu > \frac{\pi}{2}$ close enough to $\frac{\pi}{2}$ to ensure that $\nu(1-\alpha) \leq \delta$.

First assume that A_2, A_q and A_p are invertible, so that we can deal with their imaginary powers. Then for any real number $s \in \mathbb{R}$, the imaginary powers A_2^{is}, A_q^{is} and A_p^{is} are consistent operators, by Remark 5.5. Hence (5.8) yields

$$\|A_p^{is}\| \leq \|A_q^{is}\|^{1-\alpha} \|A_2^{is}\|^\alpha.$$

Since A_2 is a positive selfadjoint operator, we have $\|A_2^{is}\| = 1$, and hence

$$\|A_p^{is}\| \leq \|A_q^{is}\|^{1-\alpha}.$$

According to our assumption, the operator A_q admits a bounded $H^\infty(\Sigma_\nu)$ functional calculus. Hence applying (3.10) we deduce that there is a constant $K > 0$ such that $\|A_q^{is}\| \leq K e^{\nu|s|}$ for any $s \in \mathbb{R}$. Therefore,

$$\begin{aligned}\|A_p^{is}\| &\leq K^{1-\alpha} e^{\nu(1-\alpha)|s|}\\ &\leq K^{1-\alpha} e^{\delta|s|}, \qquad s \in \mathbb{R}.\end{aligned}$$

Since A_p admits a bounded $H^\infty(\Sigma_\nu)$ functional calculus, the above estimate and [**21**, Theorem 5.4] show that A_p actually admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus, which concludes the proof in the invertible case.

The general case can be deduced from above, using Lemma 3.5. Indeed, if $\varepsilon > 0$ is an arbitrary positive number, then $A_2 + \varepsilon$, $A_q + \varepsilon$ and $A_p + \varepsilon$ are both invertible, hence the preceding estimates apply to them. In fact the 'only if' part of Lemma 3.5 and Theorem 3.3 show that there is a constant $C > 0$ not depending on $\varepsilon > 0$ such that $\|(A_p + \varepsilon)^{is}\| \leq C e^{\delta|s|}$ for any $s \in \mathbb{R}$. Then the proof of [**21**, Theorem 5.4] shows that the operators $A_p + \varepsilon$ uniformly admit a bounded $H^\infty(\Sigma_\theta)$ functional calculus, and the result follows from the 'if' part of Lemma 3.5.

We note in passing that in the case when each T_t is positive, this proposition has a shorter proof. Indeed in that case it directly follows from Theorem 5.6 and [**41**, Proposition 5.1]. □

Remark 5.9. — Let $(T_t)_{t\geq 0}$ be a diffusion semigroup on a commutative von Neumann algebra $L^\infty(\Sigma)$.

(1) For any $1 < p < \infty$ and for any $\theta > \omega_p$, A_p admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus on $L^p(\Sigma)$. This result is due to Cowling [**20**]. The question whether this holds true for noncommutative diffusion semigroups is open.

A sketch of proof of Cowling's Theorem goes as follows. First one can show (see [**27**]) that for any $1 < p < \infty$, there exist a commutative L^p-space $L^p(\Sigma')$, a c_0-group $(U_t)_t$ of isometries on $L^p(\Sigma')$, and contractive maps $J\colon L^p(\Sigma) \to L^p(\Sigma')$ and $Q\colon L^p(\Sigma') \to L^p(\Sigma)$ such that

$$T_t = QU_tJ, \qquad t \geq 0. \tag{5.9}$$

Then by Proposition 3.12, A_p admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$. Applying the above Proposition 5.8 yields the result.

(2) For any $t \geq 0$, T_t is both a contraction on $L^\infty(\Sigma)$ and on $L^1(\Sigma)$. Hence for any $1 \leq p < \infty$, $T_t\colon L^p(\Sigma) \to L^p(\Sigma)$ is contractively regular in the sense of [**61**]. Thus for any Banach space X, $T_t \otimes I_X$ extends to contraction from $L^p(\Sigma; X)$ into itself.

Let $\mathcal{M}$ be any semifinite von Neumann algebra, and let $\mathcal{N} = L^\infty(\Sigma)\overline{\otimes}\mathcal{M}$. Then we have a canonical identification

$$L^p(\mathcal{N}) \; = \; L^p(\Sigma; L^p(\mathcal{M})).$$

Hence applying the above tensor extension property with $X = L^p(\mathcal{M})$, we deduce that for any $t \geq 0$, $T_t \otimes I_{\mathcal{M}}$ extends to a normal contraction

$$T_t\overline{\otimes}I_{\mathcal{M}}\colon \mathcal{N} \longrightarrow \mathcal{N},$$

and that $(T_t\overline{\otimes}I_{\mathcal{M}})_{t\geq 0}$ is a diffusion semigroup on $\mathcal{N}$. We claim that for any $1 < p < \infty$ and any $\theta > \omega_p$, the negative generator of its L^p realization admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus.

Indeed, let $1 < p < \infty$. According to [**27**], the dilation property (5.9) can be achieved with the additional property that J, Q, and U_t (for any t) are contractively regular. This gives rise to contractions

$$J\overline{\otimes}I_{L^p(\mathcal{M})}\colon L^p(\Sigma; L^p(\mathcal{M})) \longrightarrow L^p(\Sigma'; L^p(\mathcal{M}))$$

and

$$Q\overline{\otimes}I_{L^p(\mathcal{M})}\colon L^p(\Sigma'; L^p(\mathcal{M})) \longrightarrow L^p(\Sigma; L^p(\mathcal{M})).$$

Likewise, the $U_t \otimes I_{L^p(\mathcal{M})}$'s extend to a c_0-group $(U_t\overline{\otimes}I_{L^p(\mathcal{M})})_t$ of isometries on $L^p(\Sigma'; L^p(\mathcal{M}))$, and we have

$$T_t\overline{\otimes}I_{L^p(\mathcal{M})} = (Q\overline{\otimes}I_{L^p(\mathcal{M})})(U_t\overline{\otimes}I_{L^p(\mathcal{M})})(J\overline{\otimes}I_{L^p(\mathcal{M})}), \qquad t \geq 0.$$

We can therefore conclude as in (1) above.

We refer the reader to [**55**] for related results.

Remark 5.10. — We wish to record for further use an observation on the constants appearing in the proof of Proposition 5.8. If $(T_t)_{t\geq 0}$ is a noncommutative diffusion semigroup as in this proposition, if $1 < p < \infty$, and if $\theta > \omega_p$, let

$$\pi_{p,\theta} \colon H_0^\infty(\Sigma_\theta) \longrightarrow B(L^p(\mathcal{M}))$$

be the bounded homomorphism taking any $f \in H_0^\infty(\Sigma_\theta)$ to $f(A_p)$.

For any $1 < p < \infty$ and any $\theta > \omega_p$, let $q > p$ and $\nu > \frac{\pi}{2}$ be chosen as in the first lines of the proof of Proposition 5.8. Then it follows from the latter that for any constants $K, K' \geq 1$, there exists a constant $K'' \geq 1$ such that whenever $(T_t)_{t\geq 0}$ is a noncommutative diffusion semigroup on $\mathcal{M}$, then $\|\pi_{p,\theta}\| \leq K''$ provided that $\|\pi_{p,\nu}\| \leq K$ and $\|\pi_{q,\nu}\| \leq K'$.

CHAPTER 6

SQUARE FUNCTIONS ON NONCOMMUTATIVE L^p-SPACES

6.A. Square functions and their integral representations

In this chapter we introduce square functions associated to sectorial operators on noncommutative L^p-spaces, which generalize the ones considered in [**21**] in the commutative setting. Throughout we let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra. As in the previous chapter, we use the notation

$$\Omega_0 = \big(\mathbb{R}_+^*, \tfrac{dt}{t}\big).$$

We also recall the definition of $H_0^\infty(\Sigma_{\omega+})$ given by (4.11).

Let $1 \leq p < \infty$ and let A be a sectorial operator of type ω on $L^p(\mathcal{M})$. For any F in $H_0^\infty(\Sigma_{\omega+}) \setminus \{0\}$ and any x in $L^p(\mathcal{M})$, we define

(6.1)

$$\|x\|_{F,c} = \big\| t \mapsto F(tA)x \big\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)} \quad \text{and} \quad \|x\|_{F,r} = \big\| t \mapsto F(tA)x \big\|_{L^p(\mathcal{M};L^2(\Omega_0)_r)}.$$

We already noticed that the function $t \mapsto F(tA)$ is a continuous function from Ω_0 into $B(L^p(\mathcal{M}))$ (see paragraph 4.C). In particular, the function $t \mapsto F(tA)x$ is continuous hence measurable from Ω_0 into $L^p(\mathcal{M})$. Thus according to Definition 2.7 (1), it makes sense to wonder whether it belongs either to $L^p(\mathcal{M}; L^2(\Omega_0)_c)$ or to $L^p(\mathcal{M}; L^2(\Omega_0)_r)$. The proper meaning of (6.1) is therefore the following. If $t \mapsto F(tA)x$ belongs to $L^p(\mathcal{M}; L^2(\Omega_0)_c)$ (resp. $L^p(\mathcal{M}; L^2(\Omega_0)_r)$), then $\|x\|_{F,c}$ (resp. $\|x\|_{F,r}$) is the norm of that function in $L^p(\mathcal{M}; L^2(\Omega_0)_c)$ (resp. $L^p(\mathcal{M}; L^2(\Omega_0)_r)$). Otherwise, $\|x\|_{F,c}$ (resp. $\|x\|_{F,r}$) is equal to ∞.

It is easy to check that the set of all $x \in L^p(\mathcal{M})$ for which $\|x\|_{F,c} < \infty$ is a subspace of $L^p(\mathcal{M})$ on which $\| \ \|_{F,c}$ is a seminorm. The same comment applies to $\| \ \|_{F,r}$.

We now consider a 'symmetric' form of these seminorms. For F and x as above, we set

$$\|x\|_F = \big\| t \mapsto F(tA)x \big\|_{L^p(\mathcal{M};L^2(\Omega_0)_{rad})}.$$

Going back to the definition of $L^2(\Omega_0)_{rad}$ (see paragraph 2.B), we have more explicitly

(6.2)
$$\|x\|_F = \max\Big\{\big\|F(\cdot\, A)x\big\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)},\ \big\|F(\cdot\, A)x\big\|_{L^p(\mathcal{M};L^2(\Omega_0)_r)}\Big\} \ \text{ if } 2\leq p<\infty;$$

and

(6.3)
$$\|x\|_F = \inf\Big\{\|u_1\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)}+\|u_2\|_{L^p(\mathcal{M};L^2(\Omega_0)_r)}\,:\, u_1+u_2=F(\cdot\, A)x\Big\} \ \text{ if } 1\leq p\leq 2.$$

We call square functions associated with A the above functions $\|\ \|_{F,c}$, $\|\ \|_{F,r}$, and $\|\ \|_F$. It should be noticed that in general, column square functions $\|\ \|_{F,c}$ and row square functions $\|\ \|_{F,r}$ are not equivalent. See Appendix A for a concrete example.

It follows from Remark 2.13 that in the case when $\mathcal{M}\simeq L^\infty(\Sigma)$ is a commutative von Neumann algebra, the quantities $\|x\|_{F,c}$, $\|x\|_{F,r}$, and $\|x\|_F$ all coincide on $L^p(\Sigma)$. Indeed, they are equal to

$$\Big\|\Big(\int_0^\infty\Big|\big(F(tA)x\big)(\cdot\,)\Big|^2\,\frac{dt}{t}\Big)^{\frac12}\Big\|_{L^p(\Sigma)}, \tag{6.4}$$

and hence the square function $\|\ \|_F$ coincides with the one from [**21**] in this case.

In order to stick to (6.4) in the noncommutative case, it is desirable to have an integral representation of the norm on $L^p(\mathcal{M};L^2(\Omega_0)_c)$ and on $L^p(\mathcal{M};L^2(\Omega_0)_r)$. This is essentially provided by the next two propositions. In these statements, we shall only consider the column case, and the row case may be treated similarly. The results established below will be used later on for a function u of the form

$$u(t)=F(tA)x,\qquad t\in\Omega_0.$$

We recall Proposition 2.5 which is used in Lemma 6.1 and Proposition 6.2 below.

Lemma 6.1. — *Assume that* $2\leq p<\infty$ *and let* $u\in L^2(\Omega_0;L^p(\mathcal{M}))\subset L^p(\mathcal{M};L^2(\Omega_0)_c)$. *Then the function* $t\mapsto u(t)^*u(t)$ *belongs to* $L^1(\Omega_0,L^{\frac p2}(\mathcal{M}))$, *and we have*

$$u^*u=\int_0^\infty u(t)^*u(t)\,\frac{dt}{t}\,. \tag{6.5}$$

(Here, as explained in paragraph 2.B, we regard u^*u *as an element of* $L^{\frac p2}(\mathcal{M})$.*)*

Proof. — For any $t>0$ we have

$$\|u(t)^*u(t)\|_{L^{\frac p2}(\mathcal{M})} = \|u(t)\|^2_{L^p(\mathcal{M})},$$

hence the function $u(\cdot\,)^*u(\cdot\,)$ clearly belongs to the space $L^1(\Omega_0;L^{\frac p2}(\mathcal{M}))$. To prove (6.5), assume first that u belongs to $L^p(\mathcal{M})\otimes L^2(\Omega_0)$, and let $(a_k)_k$ and $(x_k)_k$ be finite families in $L^2(\Omega_0)$ and $L^p(\mathcal{M})$ respectively such that $u = \sum_k x_k \otimes a_k$. Then

$$u^*u = \sum_{i,j} \langle a_j, a_i\rangle\, x_i^* x_j = \sum_{i,j} \Big(\int_0^\infty \overline{a_i(t)} a_j(t)\, \frac{dt}{t}\Big) x_i^* x_j = \int_0^\infty u(t)^* u(t)\, \frac{dt}{t}\,.$$

For an arbitrary $u \in L^2(\Omega_0; L^p(\mathcal{M}))$, take a sequence u_n in $L^p(\mathcal{M}) \otimes L^2(\Omega_0)$ converging to u in $L^2(\Omega_0; L^p(\mathcal{M}))$. Then u_n also converges to u in $L^p(\mathcal{M}; L^2(\Omega_0)_c)$, hence $u_n^* u_n$ converges to u^*u in $L^{\frac{p}{2}}(\mathcal{M})$. Furthermore we know from above that each u_n satisfies (6.5) hence passing to the limit, we deduce (6.5) for u. □

Proposition 6.2. — *Assume that $2 \le p < \infty$ and let $u\colon \Omega_0 \to L^p(\mathcal{M})$ be any continuous function. The following two assertions are equivalent.*

(i) *u belongs to $L^p(\mathcal{M}; L^2(\Omega_0)_c)$.*

(ii) *There is a constant $K > 0$ such that for any $0 < \alpha < \beta < \infty$, we have*

$$\Big\| \int_\alpha^\beta u(t)^* u(t)\, \frac{dt}{t} \Big\|_{L^{\frac{p}{2}}(\mathcal{M})} \le K^2.$$

In that case, we have

$$u^*u = \lim_{\alpha \to 0;\, \beta \to \infty} \int_\alpha^\beta u(t)^* u(t)\, \frac{dt}{t} \tag{6.6}$$

and

$$\|u\|_{L^p(\mathcal{M}; L^2(\Omega_0)_c)} = \lim_{\alpha \to 0;\, \beta \to \infty} \Big\| \Big(\int_\alpha^\beta u(t)^* u(t)\, \frac{dt}{t}\Big)^{\frac{1}{2}} \Big\|_{L^p(\mathcal{M})}. \tag{6.7}$$

Proof. — We assume (i). For any $0 < \alpha < \beta < \infty$, we let $P_{\alpha,\beta}\colon L^2(\Omega_0) \to L^2(\Omega_0)$ be the orthogonal projection defined by letting $P_{\alpha,\beta}(a) = a\chi_{(\alpha,\beta)}$ for any $a \in L^2(\Omega_0)$. According to Lemma 2.4, $I_{L^p} \otimes P_{\alpha,\beta}$ extends to a contraction

$$\widehat{P_{\alpha,\beta}}\colon L^p(\mathcal{M}; L^2(\Omega_0)_c) \longrightarrow L^p(\mathcal{M}; L^2(\Omega_0)_c).$$

It is plain that $\widehat{P_{\alpha,\beta}}(u)$ is equal to the product function $u\chi_{(\alpha,\beta)}$.

Our hypothesis that u is continuous ensures that $\widehat{P_{\alpha,\beta}}(u)$ belongs to $L^2(\Omega_0; L^p(\mathcal{M}))$. Owing to Lemma 6.1, we then have

$$\big(\widehat{P_{\alpha,\beta}}(u)\big)^* \big(\widehat{P_{\alpha,\beta}}(u)\big) = \int_\alpha^\beta u(t)^* u(t)\, \frac{dt}{t}\,. \tag{6.8}$$

By (2.8), we have

$$\big\| \big(\widehat{P_{\alpha,\beta}}(u)\big)^* \big(\widehat{P_{\alpha,\beta}}(u)\big) \big\|_{\frac{p}{2}} = \big\| \widehat{P_{\alpha,\beta}}(u) \big\|^2_{L^p(\mathcal{M}; L^2(\Omega_0)_c)} \le \|u\|^2_{L^p(\mathcal{M}; L^2(\Omega_0)_c)}.$$

Hence (ii) holds true, with $K = \|u\|_{L^p(\mathcal{M}; L^2(\Omega_0)_c)}$.

Next we observe that $P_{\alpha,\beta}$ converges pointwise to the identity on $L^2(\Omega_0)$, when $\alpha \to 0$ and $\beta \to \infty$. Hence $I_{L^p} \otimes P_{\alpha,\beta}$ converges pointwise to the identity on $L^p(\mathcal{M}) \otimes$

$L^2(\Omega_0)$. Since $\|\widehat{P_{\alpha,\beta}}\| \leq 1$ for any α and β, we deduce that $\widehat{P_{\alpha,\beta}}$ converges pointwise to the identity on $L^p(\mathcal{M}; L^2(\Omega_0)_c)$. Thus $\widehat{P_{\alpha,\beta}}(u) \to u$, and (6.6) and (6.7) follow from (6.8).

For the converse direction, we assume (ii) and we let p' be the conjugate number of p. Let v be an arbitrary element of $L^{p'}(\mathcal{M}) \otimes L^2(\Omega_0)$, and consider $0 < \alpha < \beta < \infty$. The function $u\chi_{(\alpha,\beta)}$ belongs to $L^p(\mathcal{M}; L^2(\Omega_0)_c)$, and by (2.8) and Lemma 6.1, we have

$$\|u\chi_{(\alpha,\beta)}\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)} = \left\|\left(u\chi_{(\alpha,\beta)}\right)^*\left(u\chi_{(\alpha,\beta)}\right)\right\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\mathcal{M})} = \left\|\int_\alpha^\beta u(t)^*u(t)\,\frac{dt}{t}\right\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\mathcal{M})}.$$

Moreover we have

$$\int_\alpha^\beta \left|\langle v(t), u(t)\rangle\right| \frac{dt}{t} \;\leq\; \|u\chi_{(\alpha,\beta)}\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)}\,\|v\|_{L^{p'}(\mathcal{M};L^2(\Omega_0)_r)}$$

by Lemma 2.8. Applying (ii) we deduce that

$$\int_\alpha^\beta \left|\langle v(t), u(t)\rangle\right| \frac{dt}{t} \leq K\|v\|_{L^{p'}(N;L^2(\Omega_0)_r)}, \qquad v \in L^{p'}(\mathcal{M}) \otimes L^2(\Omega_0).$$

Letting $\alpha \to 0$ and $\beta \to \infty$ and using Lemma 2.10, this shows that the function u belongs to the space $L^p(\mathcal{M}; L^2(\Omega_0)_c)$. □

Proposition 6.2 does not extend to the range $1 \leq p < 2$. The obstacle here is that if we consider a measurable function $u\colon \Omega_0 \to L^p(\mathcal{M})$, then the function $t \mapsto u(t)^*u(t)$ is valued in $L^{\frac{p}{2}}(\mathcal{M})$ which is not a Banach space if $p < 2$. Thus in general we have no way to define a Bochner integral $\int_\alpha^\beta u(t)^*u(t)\,\frac{dt}{t}$. To circumvent this difficulty, we will consider approximation by simple functions provided by 'conditional expectations' associated with subpartitions. For the definition of a subpartition π and its associated mapping E_π, see (4.5) and the paragraph preceding Lemma 4.3.

We observe that if $w = \sum_k z_k \otimes c_k \in L^{\frac{p}{2}}(\mathcal{M}) \otimes L^1(\Omega_0)$, with $c_k \in L^1(\Omega_0)$ and $z_k \in L^{\frac{p}{2}}(\mathcal{M})$, then we may define

$$\int_0^\infty w(t)\,\frac{dt}{t} = \sum_k \left(\int_0^\infty c_k(t)\,\frac{dt}{t}\right) z_k \quad \in\; L^{\frac{p}{2}}(\mathcal{M}).$$

This yields a definition of $\int_0^\infty u(t)^*u(t)\,\frac{dt}{t}$ for any $u \in L^p(\mathcal{M}) \otimes L^2(\Omega_0)$.

Proposition 6.3. — *Assume that $1 \leq p < 2$ and let $u \in L^2(\Omega_0, L^p(\mathcal{M}))$. For any subpartition π of Ω_0, we let $u_\pi = E_\pi(u)$ be defined by (4.5) and we note that u_π belongs to $L^p(\mathcal{M}) \otimes L^2(\Omega_0)$. Then the following two assertions are equivalent.*

(i) *u belongs to $L^p(\mathcal{M}; L^2(\Omega_0)_c)$.*

(ii) *There is a constant $K > 0$ such that for any π, we have*

$$\left\|\int_0^\infty u_\pi(t)^*u_\pi(t)\,\frac{dt}{t}\right\|_{L^{\frac{p}{2}}(\mathcal{M})} \leq K^2.$$

In that case,

$$u^*u = \lim_\pi \int_0^\infty u_\pi(t)^* u_\pi(t)\,\frac{dt}{t} \tag{6.9}$$

and

$$\|u\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)} = \lim_\pi \Big\| \Big(\int_0^\infty u_\pi(t)^* u_\pi(t)\,\frac{dt}{t}\Big)^{\frac{1}{2}} \Big\|_{L^p(\mathcal{M})}. \tag{6.10}$$

Proof. — The proof is quite similar to the one of Proposition 6.2, hence we only outline it. If u satisfies (i), then $\|u_\pi\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)} \le \|u\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)}$ for any π, u_π converges to u in $L^p(\mathcal{M};L^2(\Omega_0)_c)$ by (4.6), and we have

$$u_\pi^* u_\pi = \int_0^\infty u_\pi(t)^* u_\pi(t)\,\frac{dt}{t}.$$

We deduce (ii) with $K = \|u\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)}$, as well as (6.9) and (6.10).

Conversely, (ii) implies (i) by using Lemma 2.10. □

Remark 6.4. — Here we give other substitutes of Proposition 6.2 in the case when $1<p<2$.

(1) Let $u\colon \Omega_0 \to L^p(\mathcal{M})$ be a continuous function. It is easy to deduce from the proof of Proposition 6.2 that u belongs to $L^p(\mathcal{M};L^2(\Omega_0)_c)$ if and only if for any $0<\alpha<\beta<\infty$, the restricted function $u\chi_{(\alpha,\beta)}$ belongs to $L^p(\mathcal{M};L^2(\Omega_0)_c)$, and there is a constant $K>0$ such that $\|u\chi_{(\alpha,\beta)}\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)} \le K$ for any $\alpha<\beta$. In that case, we have

$$\big\|u\big\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)} = \sup_{\alpha<\beta} \big\|u\chi_{(\alpha,\beta)}\big\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)}.$$

A similar result holds true with column norms replaced by row norms or Rademacher norms. Thus, u belongs to $L^p(\mathcal{M};L^2(\Omega_0)_{rad})$ if and only if there is a constant $K>0$ such that $\|u\chi_{(\alpha,\beta)}\|_{L^p(\mathcal{M};L^2(\Omega_0)_{rad})} \le K$ for any $0<\alpha<\beta<\infty$.

(2) Assume that $1<p<2$ and let $u\colon \Omega_0 \to L^p(\mathcal{M}) \cap L^2(\mathcal{M})$ be a continuous function. Then $t\mapsto u(t)^*u(t)$ is valued in $L^1(\mathcal{M})$ and for any $0<\alpha<\beta<\infty$, we may therefore define the integral

$$\int_\alpha^\beta u(t)^*u(t)\,\frac{dt}{t} \quad \in\; L^1(\mathcal{M}).$$

Then it follows from above that u belongs to $L^p(\mathcal{M};L^2(\Omega_0)_c)$ if and only if $\int_\alpha^\beta u(t)^*u(t)\,\frac{dt}{t}$ belongs to $L^{\frac{p}{2}}(\mathcal{M})$ for any $\alpha<\beta$, and there is a constant $K>0$ such that

$$\Big\|\int_\alpha^\beta u(t)^*u(t)\,\frac{dt}{t}\Big\|_{\frac{p}{2}} \le K^2 \qquad \text{for any } \ \alpha<\beta.$$

6.B. Equivalence of square functions

Square functions associated to a sectorial operator were first introduced on Hilbert spaces by McIntosh ([**53**]). Then it was shown in [**54**] that they are all equivalent and that any sectorial operator (on a Hilbert space) has a bounded H^∞ functional calculus with respect to $\| \ \|_F$. In [**49**], these results were extended to Rad-sectorial operators on classical (=commutative) L^p-spaces. Our objective (Theorem 6.7 below) is an extension of the latter results to noncommutative L^p-spaces.

We will need the following simple lemma.

Lemma 6.5. — *Let A be a sectorial operator of type ω on $L^p(\mathcal{M})$, with $1 \leq p < \infty$. Let $F \in H_0^\infty(\Sigma_{\omega+})$.*

(1) *For any $f \in H_0^\infty(\Sigma_{\omega+})$, the function $t \mapsto F(tA)f(A)$ from Ω_0 into $B(L^p(\mathcal{M}))$ is absolutely integrable and*
$$\int_0^\infty F(tA)f(A)\,\frac{dt}{t} = \Big(\int_0^\infty F(t)\,\frac{dt}{t}\Big)\, f(A).$$
(2) *For any $f \in H^\infty(\Sigma_{\omega+})$, $\sup_{t>0} \|F(tA)f(A)\| < \infty$.*

(3) *Assume that A is Col-sectorial of Col-type ω and let $\theta > \omega$ such that $F \in H_0^\infty(\Sigma_\theta)$. Then there is a constant $K > 0$ such that for any $f \in H^\infty(\Sigma_\theta)$, the set of all $F(tA)f(A)$ for $t > 0$ is Col-bounded, with*
$$Col\Big(\big\{F(tA)f(A) \,:\, t > 0\big\}\Big) \leq K\|f\|_{\infty,\theta}.$$
(4) *The result in (3) holds true with Row-boundedness or Rad-boundedness replacing Col-boundedness.*

Proof. — The first two assertions hold true in any Banach space and go back (at least implicitly) to McIntosh's earliest paper on H^∞ functional calculus [**53**]. To prove (3), we essentially repeat McIntosh's proof of (2). Given $\theta > \omega$ and $F \in H_0^\infty(\Sigma_\theta)$, let $\gamma \in (\omega, \theta)$ be an intermediate angle and recall from (4.16) that
$$K_0 = \sup_{t>0} \int_{\Gamma_\gamma} |F(tz)| \left|\frac{dz}{z}\right| \, < \, \infty\,.$$
Using (3.5), we now write
$$F(tA)f(A) = \frac{1}{2\pi i}\int_{\Gamma_\gamma} F(tz)f(z)R(z,A)\,dz$$
for any $f \in H^\infty(\Sigma_\theta)$ and any $t > 0$. Our assumption implies that the set $\{zR(z,A) : z \in \Gamma_\gamma\}$ is Col-bounded. Moreover
$$\int_{\Gamma_\gamma} |F(tz)f(z)| \left|\frac{dz}{z}\right| \, \leq \, K_0\|f\|_{\infty,\theta}$$
for any $f \in H^\infty(\Sigma_\theta)$ and any $t > 0$. We therefore deduce (3) from the second part of Lemma 4.2. The last assertion (4) can be proved in the same manner. □

Remark 6.6. — Let A be a sectorial opertor of type ω on $L^p(\mathcal{M})$ and let $F \in H_0^\infty(\Sigma_{\omega+}) \setminus \{0\}$. Then

$$\|x\|_{F,c} = 0 \Leftrightarrow \|x\|_{F,r} = 0 \Leftrightarrow \|x\|_F = 0 \Leftrightarrow x \in N(A).$$

Indeed each of the first three conditions means that $F(tA)x = 0$ for any $t > 0$, and this obviously holds if $x \in N(A)$. Assume conversely that $F(tA)x = 0$ for any $t > 0$ and let $\widetilde{F}$ be defined by $\widetilde{F}(z) = \overline{F(\overline{z})}$. Then $\widetilde{F}(tA)F(tA)x = 0$ for any $t > 0$. Since

$$\int_0^\infty \widetilde{F}(t)F(t)\,\frac{dt}{t} = \|F\|^2_{L^2(\Omega_0)} > 0,$$

the first part of Lemma 6.5 ensures that $f(A)x = 0$ for any $f \in H_0^\infty(\Sigma_{\omega+})$. Using e.g. $f(z) = g(z) = z(1+z)^{-2}$, this implies that $x \in N(A)$.

Thus if A is injective, $\|\ \|_{F,c}$, $\|\ \|_{F,r}$, and $\|\ \|_F$ are norms on the respective subspaces of $L^p(\mathcal{M})$ on which they are finite.

Theorem 6.7. — *Assume that $1 < p < \infty$. Let A be a sectorial operator of type ω on $L^p(\mathcal{M})$, and let $\theta \in (\omega, \pi)$. We consider two functions F and G in $H_0^\infty(\Sigma_\theta) \setminus \{0\}$.*

(1) *If A is Col-sectorial of Col-type ω, then there exists a constant $C > 0$ such that for any $f \in H_0^\infty(\Sigma_\theta)$ and any $x \in L^p(\mathcal{M})$, we have*

$$\|f(A)x\|_{F,c} \leq C\|f\|_{\infty,\theta}\|x\|_{G,c}.$$

Moreover we have an equivalence

$$\|x\|_{G,c} \asymp \|x\|_{F,c}, \qquad x \in L^p(\mathcal{M}).$$

(2) *If A is Row-sectorial of Row-type ω, then the same properties hold with $\|\ \|_{F,r}$ and $\|\ \|_{G,r}$ replacing $\|\ \|_{F,c}$ and $\|\ \|_{G,c}$.*

(3) *If A is Rad-sectorial of Rad-type ω, then the same properties hold with $\|\ \|_F$ and $\|\ \|_G$ replacing $\|\ \|_{F,c}$ and $\|\ \|_{G,c}$.*

Proof. — We shall only prove (1), the proofs of (2) and (3) being identical. Since $G \in H_0^\infty(\Sigma_\theta)$ is a non zero function, we can choose φ_1 and φ_2 in $H_0^\infty(\Sigma_\theta)$ with the property that

$$\int_0^\infty \varphi_1(t)\varphi_2(t)G(t)\frac{dt}{t} = 1.$$

We consider some $f \in H_0^\infty(\Sigma_\theta)$. According to Lemma 6.5 (1), the function mapping any $t > 0$ to $\varphi_1(tA)\varphi_2(tA)G(tA)f(A)$ is absolutely integrable on Ω_0, and

$$\int_0^\infty \varphi_1(tA)\varphi_2(tA)G(tA)f(A)\,\frac{dt}{t} = f(A). \tag{6.11}$$

On the other hand, it follows from Lemma 6.5 (3), and our hypothesis that A is Col-sectorial of Col-type ω, that the set of all operators $\varphi_2(tA)f(A)$ is Col-bounded and that we have an estimate

$$\mathrm{Col}\Big(\big\{\varphi_2(tA)f(A)\ :\ t > 0\big\}\Big) \leq K\|f\|_{\infty,\theta},$$

where $K > 0$ is a constant not depending on f. Let us now apply Proposition 4.4 and its subsequent Remark 4.5, with $\Omega = \Omega_0$, $d\mu(t) = \frac{dt}{t}$, and

$$\Phi(t) = \varphi_2(tA)f(A), \qquad t > 0.$$

By Lemma 4.2, we deduce from above that

$$\mathrm{Col}\bigg(\bigg\{\frac{1}{\mu(I)}\int_I \varphi_2(tA)f(A)\,\frac{dt}{t}\ :\ I \subset \Omega_0,\ 0 < \mu(I) < \infty\bigg\}\bigg) \le K\|f\|_{\infty,\theta}.$$

Then we obtain that the multiplication operator T_Φ is bounded on $L^p(\mathcal{M}; L^2(\Omega_0)_c)$, with

$$\big\|T_\Phi\colon L^p(\mathcal{M}; L^2(\Omega_0)_c) \longrightarrow L^p(\mathcal{M}; L^2(\Omega_0)_c)\big\| \ \le\ K\|f\|_{\infty,\theta}.$$

Assume that $\|x\|_{G,c} < \infty$, so that $G(\cdot\, A)x$ belongs to $L^p(\mathcal{M}; L^2(\Omega_0)_c)$. By Remark 4.6, $T_\Phi(G(\cdot\, A)x)$ is equal to the function $\varphi_2(\cdot\, A)G(\cdot\, A)f(A)x$. Hence we have proved that the latter function belongs to $L^p(\mathcal{M}; L^2(\Omega_0)_c)$, with

$$\big\|\varphi_2(\cdot\, A)G(\cdot\, A)f(A)x\big\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)} \le K\|f\|_{\infty,\theta}\|x\|_{G,c}. \tag{6.12}$$

We now apply Theorem 4.14 with $F_1 = \varphi_1$ and $F_2 = F$. According to our hypothesis that A is Col-sectorial, the operator T with kernel $F(sA)\varphi_1(tA)$ is bounded from $L^p(\mathcal{M}; L^2(\Omega_0)_c)$ into itself. Furthermore, Lemma 6.5 (1) ensures that the function $\varphi_2(\cdot\, A)G(\cdot\, A)f(A)x$ belongs to $L^1(\Omega_0; L^p(\mathcal{M}))$. Hence T maps this function to the function

$$s \longmapsto \int_0^\infty F(sA)\varphi_1(tA)\varphi_2(tA)G(tA)f(A)x\,\frac{dt}{t}\,.$$

By (6.11), the above integral is equal to $F(sA)f(A)x$. This shows that $F(\cdot\, A)f(A)x$ is a function which belongs to $L^p(\mathcal{M}; L^2(\Omega_0)_c)$ and using (6.12), we have the estimate

$$\big\|F(\cdot\, A)f(A)x\big\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)} \le K\|T\|\|f\|_{\infty,\theta}\|x\|_{G,c}.$$

This concludes the proof of the first part of (1), with $C = K\|T\|$.

To prove the second part, we will use the fact that $L^p(\mathcal{M})$ is reflexive (we assumed that $1 < p < \infty$). By Remark 3.4, we have a direct sum decomposition $L^p(\mathcal{M}) = N(A) \oplus \overline{R(A)}$. Moreover $\|x\|_{F,c} = \|x\|_{G,c} = 0$ for every $x \in N(A)$. Thus to prove that $\|\ \|_{F,c}$ and $\|\ \|_{G,c}$ are equivalent on $L^p(\mathcal{M})$, it suffices to prove that they are equivalent on $\overline{R(A)}$. Let $(g_n)_{n\ge 0}$ be the bounded sequence of $H_0^\infty(\Sigma_\theta)$ defined by (3.8) and let $C' = \sup_{n\ge 0}\{\|g_n\|_{\infty,\theta}\}$. The preceding estimate yields

$$\|g_n(A)x\|_{F,c} \le CC'\|x\|_{G,c}, \qquad n \ge 1,\ x \in L^p(\mathcal{M}).$$

Let $x \in \overline{R(A)}$ and let v be an arbitrary element of $L^{p'}(\mathcal{M}) \otimes L^2(\Omega_0)$, where p' is the conjugate number of p. For any $n \geq 1$, we have by Lemma 2.8 that

$$\begin{aligned}\int_0^\infty \big|\langle v(t), F(tA)g_n(A)x\rangle\big| \,\frac{dt}{t} &\leq \big\|F(\cdot\, A)g_n(A)x\big\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)} \|v\|_{L^{p'}(\mathcal{M};L^2(\Omega_0)_r)} \\ &\leq \|g_n(A)x\|_{F,c}\, \|v\|_{L^{p'}(\mathcal{M};L^2(\Omega_0)_r)} \\ &\leq CC'\|x\|_{G,c}\, \|v\|_{L^{p'}(\mathcal{M};L^2(\Omega_0)_r)}.\end{aligned}$$

Since $x \in \overline{R(A)}$, $g_n(A)x$ converges to x, by Lemma 3.2. Hence applying Fatou's Lemma immediately leads to

$$\int_0^\infty \big|\langle v(t), F(tA)x\rangle\big| \,\frac{dt}{t} \leq CC'\|x\|_{G,c}\, \|v\|_{L^{p'}(\mathcal{M};L^2(\Omega_0)_r)}.$$

Owing to Lemma 2.10, this shows that $F(\cdot\, A)x$ belongs to $L^p(\mathcal{M}; L^2(\Omega_0)_c)$, with

$$\|x\|_{F,c} \leq CC'\|x\|_{G,c}.$$

Switching the roles of F and G then shows that $\|\ \|_{G,c}$ and $\|\ \|_{F,c}$ are actually equivalent on $\overline{R(A)}$, which concludes the proof. □

CHAPTER 7

H^∞ FUNCTIONAL CALCULUS AND SQUARE FUNCTION ESTIMATES

In this chapter we investigate the interplay between bounded or completely bounded H^∞ functional calculus and square functions, for a sectorial operator on a noncommutative L^p-space. Our results should be regarded as noncommutative analogues of those proved by Cowling, Doust, McIntosh, and Yagi in [**21**, Sections 4 and 6]. For simplicity we will restrict to the case when $p > 1$, although some of the results of this chapter extend to the case when $p = 1$. We recall the notation $\Omega_0 = (\mathbb{R}_+^*, \frac{dt}{t})$.

Let A be a sectorial operator of type $\omega \in (0, \pi)$ on some noncommutative L^p-space $L^p(\mathcal{M})$, with $1 < p < \infty$. Let $F \in H_0^\infty(\Sigma_{\omega+}) \setminus \{0\}$. We say that A satisfies a square function estimate $(\mathcal{S}_F)$ if there is a constant $K > 0$ such that

$$(\mathcal{S}_\mathbf{F}) \qquad \|x\|_F \leq K\|x\|, \qquad x \in L^p(\mathcal{M}).$$

A straightforward application of the Closed Graph Theorem shows that $(\mathcal{S}_F)$ holds true if and only if $\|x\|_F$ is finite for any $x \in L^p(\mathcal{M})$.

Recall that the operator A^* is sectorial of type ω on $L^{p'}(\mathcal{M})$. Let $G \in H_0^\infty(\Sigma_{\omega+}) \setminus \{0\}$. We say that A satisfies a dual square function estimate $(\mathcal{S}_G^*)$ if A^* satisfies a square function estimate with respect to G, that is, there is a constant $K > 0$ such that

$$(\mathcal{S}_\mathbf{G}^*) \qquad \big\|G(\cdot\, A^*)y\big\|_{L^{p'}(\mathcal{M};L^2(\Omega_0)_{rad})} \leq K\|y\|_{p'}, \qquad y \in L^{p'}(\mathcal{M}).$$

We notice the following consequence of Theorem 6.7 (3).

Corollary 7.1. — *Assume that A is Rad-sectorial of Rad-type ω on $L^p(\mathcal{M})$. If A satisfies $(\mathcal{S}_F)$ for some $F \in H_0^\infty(\Sigma_{\omega+}) \setminus \{0\}$, then A satisfies $(\mathcal{S}_F)$ for all $F \in H_0^\infty(\Sigma_{\omega+}) \setminus \{0\}$.*

Our next statement extends some estimates from [**21**, Section 4 and 6] to the noncommutative setting. Keeping the notation from the latter paper we let $\varphi_e(t) =$

$\varphi(e^t)$ for any measurable function $\varphi\colon \Omega_0 \to \mathbb{C}$. Thus $\varphi \mapsto \varphi_e$ induces an isometric isomorphism from $L^p(\Omega_0)$ onto $L^p(\mathbb{R}; dt)$ for any $1 \leq p \leq \infty$. We let $\widehat{\varphi_e}$ be the Fourier transform of φ_e if φ belongs either to $L^1(\Omega_0)$ or to $L^2(\Omega_0)$.

Proposition 7.2. — *Let A be a sectorial operator of type ω on $L^p(\mathcal{M})$, with $1 < p < \infty$. Consider three numbers δ, ν, α such that $\omega < \delta < \alpha < 2\alpha - \delta < \nu < \pi$, and two functions $F, G \in H_0^\infty(\Sigma_\delta)$. Let $\varphi = GF$ and note that the restriction of φ to Ω_0 is integrable. Assume that there is a constant $C > 0$ such that*

$$\big|\widehat{\varphi_e}(s)\big| \geq Ce^{-\alpha|s|}, \qquad s \in \mathbb{R}. \tag{7.1}$$

If A satisfies a dual square function estimate $(\mathcal{S}_G^*)$, then there is a constant $K > 0$ such that for any $f \in H_0^\infty(\Sigma_\nu)$ and any $x \in L^p(\mathcal{M})$, we have

$$\|f(A)x\| \leq K\|x\|_F\|f\|_{\infty,\nu}.$$

Proof. — The assumption (7.1) ensures that there is a constant $C_1 > 0$ with the following property. For any $f \in H_0^\infty(\Sigma_\nu)$, there exists a function $b \in L^1(\Omega_0) \cap L^\infty(\Omega_0)$ such that

$$\|b\|_\infty \leq C_1\|f\|_{\infty,\nu} \tag{7.2}$$

and

$$f(z) = \int_0^\infty b(t)\varphi(tz)\,\frac{dt}{t}\,, \qquad z \in \Sigma_\delta.$$

Indeed this follows from the proof of [**21**, Theorem 4.4], see in particular (4.3) in that paper. Since $b \in L^1(\Omega_0)$, the second part of Lemma 6.5 ensures that $\int_0^\infty |b(t)|\|\varphi(tA)\|\,\frac{dt}{t}$ is finite. A simple computation using Fubini's Theorem then shows that

$$f(A) = \int_0^\infty b(t)\varphi(tA)\,\frac{dt}{t} = \int_0^\infty b(t)G(tA)F(tA)\,\frac{dt}{t}\,.$$

For any $x \in L^p(\mathcal{M})$ and any $y \in L^{p'}(\mathcal{M}) = L^p(\mathcal{M})^*$, we derive that

$$\langle f(A)x, y\rangle = \int_0^\infty \langle b(t)F(tA)x, G(tA^*)y\rangle\,\frac{dt}{t}\,.$$

Hence

$$\big|\langle f(A)x, y\rangle\big| \leq \|b\|_\infty \int_0^\infty \big|\langle F(tA)x, G(tA^*)y\rangle\big|\,\frac{dt}{t}\,.$$

Now assume that $\|x\|_F < \infty$, that is, $F(\cdot\, A)x$ belongs to $L^p(\mathcal{M}; L^2(\Omega_0)_{rad})$. We assumed that A satisfies $(\mathcal{S}_G^*)$, so that $G(\cdot\, A^*)y$ belongs to $L^{p'}(\mathcal{M}; L^2(\Omega_0)_{rad})$. Hence by Remark 2.9, there is a constant $C_2 > 0$ such that

$$\big|\langle f(A)x, y\rangle\big| \leq C_2\|b\|_\infty\,\|x\|_F\,\|y\|.$$

Applying (7.2) then yields

$$\big|\langle f(A)x, y\rangle\big| \leq C_1C_2\|x\|_F\|y\|\|f\|_{\infty,\nu}.$$

The result therefore follows by taking the supremum over all $y \in L^{p'}(\mathcal{M})$ with $\|y\| \leq 1$. $\square$

Remark 7.3. — Let A be a sectorial operator of type $\omega \in (0,\pi)$ on $L^p(\mathcal{M})$, with $1 < p < \infty$. Given any $\nu \in (\omega,\pi)$, choose δ and α such that $\omega < \delta < \alpha < 2\alpha - \delta < \nu$. According to [**21**, Example 4.7], there exists $F, G \in H_0^\infty(\Sigma_\delta)$ such that the product function $\varphi = \widetilde{G}F$ satisfies the assumption (7.1) in Proposition 7.2. For this specific pair (F,G) of non zero functions in $H_0^\infty(\Sigma_{\omega+})$, we obtain that if A satisfies $(\mathcal{S}_F)$ and $(\mathcal{S}_G^*)$, then it admits a bounded $H^\infty(\Sigma_\nu)$ functional calculus. Indeed this follows from Proposition 7.2.

Corollary 7.4. — *Let A be a Rad-sectorial operator of Rad-type $\omega \in (0,\pi)$ on $L^p(\mathcal{M})$, with $1 < p < \infty$. Assume that there exist two non zero functions $F, G \in H_0^\infty(\Sigma_{\omega+})$ such that A satisfies $(\mathcal{S}_F)$ and $(\mathcal{S}_G^*)$. Then A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta \in (\omega,\pi)$.*

Proof. — This follows from Corollary 7.1 and Remark 7.3 above. $\square$

Remark 7.5. — The assumptions that both A and A^* satisfy square function estimates are necessary in Corollary 7.4. Indeed there may exist A of type ω without any bounded H^∞ functional calculus such that A satisfies $(\mathcal{S}_F)$ for any $F \in H_0^\infty(\Sigma_{\omega+})$. See [**48**, Section 5] for an example on Hilbert space.

We now turn to the converse of Corollary 7.4 and an equivalence result.

Theorem 7.6. — *Let A be a sectorial operator of type $\omega \in (0,\pi)$ on $L^p(\mathcal{M})$, with $1 < p < \infty$. Assume that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (\omega,\pi)$.*

(1) *Then A satisfies a square function estimate $(\mathcal{S}_F)$ and a dual square function estimate $(\mathcal{S}_G^*)$ for any $F, G \in H_0^\infty(\Sigma_{\theta+}) \setminus \{0\}$.*

(2) *Let $P\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ be the projection onto $N(A)$ with kernel equal to $\overline{R(A)}$ (see Remark 3.4). Then for any $F \in H_0^\infty(\Sigma_{\theta+}) \setminus \{0\}$, we have an equivalence*

$$\|x\| \asymp \|x\|_F + \|P(x)\|, \qquad x \in L^p(\mathcal{M}).$$

Proof. — Consider $F \in H_0^\infty(\Sigma_\nu)\setminus\{0\}$ with $\nu > \theta$ and let us show the square function estimate $(\mathcal{S}_F)$ for A. Recall from Remark 6.6 that $\|x\|_F = 0$ if $x \in N(A)$. Hence according to Remark 3.4, we may assume that A has dense range. For any $z \in \Sigma_\nu$, we let

$$F^z(t) = F(tz), \qquad t > 0.$$

Clearly each F^z is both bounded and integrable on Ω_0. The starting point of the proof is the following construction extracted from [**21**]. Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be an infinitely many differentiable function with compact support included in $[-2,2]$ satisfying $\sum_{k=-\infty}^{\infty} \psi(s-k)^2 = 1$ for any $s \in \mathbb{R}$, and let $\psi_k = \psi(\cdot - k)$ for any integer

k. By definition, each ψ_k has support in $[k-2,k+2]$. Then for any $k,j\in\mathbb{Z}$, let $\tau_{jk}\colon [k-2,k+2]\to\mathbb{C}$ be defined by $\tau_{jk}(s)=\frac{1}{2}e^{\frac{\pi}{2}ijs}$. It is proved in [**21**, Lemma 6.5] that

$$\sum_k \sup_{z\in\Sigma_\theta}\sum_j \bigl|\bigl\langle \widehat{F_e^z}\,\psi_k,\tau_{jk}\bigr\rangle\bigr| < \infty.$$

Since $(\tau_{jk})_j$ is an orthonormal basis of $L^2([k-2,k+2];dt)$ for any $k\in\mathbb{Z}$, then we have

$$\begin{aligned}\|a\|^2_{L^2(\Omega_0)} = \|a_e\|^2_{L^2(\mathbb{R};dt)} &= \frac{1}{2\pi}\,\|\widehat{a_e}\|^2_{L^2(\mathbb{R};dt)}\\ &= \frac{1}{2\pi}\sum_k \|\widehat{a_e}\psi_k\|^2_{L^2([k-2,k+2];dt)}\\ &= \frac{1}{2\pi}\sum_{j,k}\bigl|\bigl\langle \widehat{a_e}\psi_k,\tau_{jk}\bigr\rangle\bigr|^2\end{aligned}$$

for any $a\in L^2(\Omega_0)$. Changing both the notation and the indexing, we deduce the existence of a sequence $(b_j)_{j\geq1}$ in $L^2(\Omega_0)$ with the following two properties. First,

$$\|a\|^2_{L^2(\Omega_0)} = \sum_{j\geq1}\bigl|\langle a,b_j\rangle\bigr|^2,\qquad a\in L^2(\Omega_0). \tag{7.3}$$

Second,

$$K=\sup_{z\in\Sigma_\theta}\sum_{j\geq1}\bigl|\langle F^z,b_j\rangle\bigr| < \infty. \tag{7.4}$$

For any $j\geq1$, we let $h_j\in H^\infty(\Sigma_\theta)$ be defined by

$$h_j(z)=\langle F^z,b_j\rangle = \int_0^\infty F(tz)\,\overline{b_j(t)}\,\frac{dt}{t}\,. \tag{7.5}$$

Let $(\varepsilon_j)_{1\leq j\leq N}$ be a finite sequence taking values in $\{-1,1\}$. For any $z\in\Sigma_\theta$, we have

$$\Bigl|\sum_{j=1}^N \varepsilon_j h_j(z)\Bigr|\leq K$$

by (7.4). Since A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus, we deduce that

$$\Bigl\|\sum_{j=1}^N \varepsilon_j h_j(A)\Bigr\|\leq K_1,$$

for some constant $K_1>0$ not depending either on N or on the ε_j's. Equivalently, we have that for every $x\in L^p(\mathcal{M})$, $\bigl\|\sum_j\varepsilon_j h_j(A)x\bigr\|\leq K_1\|x\|$. Hence averaging over all possible choices of $\varepsilon_j=\pm1$, we obtain that

$$\Bigl\|\sum_{j=1}^N \varepsilon_j\, h_j(A)x\Bigr\|_{\mathrm{Rad}(L^p(\mathcal{M}))}\leq K_1\|x\|,\qquad x\in L^p(\mathcal{M}).$$

By Corollary 2.12 and (2.34), this shows that for every $x \in L^p(\mathcal{M})$, the sequence $\big(h_j(A)x\big)_{j\geq 1}$ belongs to $L^p(\mathcal{M};\ell^2_{rad})$ and that

$$\Big\|\big(h_j(A)x\big)_{j\geq 1}\Big\|_{L^p(\mathcal{M};\ell^2_{rad})} \leq K_2\|x\|_{L^p(\mathcal{M})}, \tag{7.6}$$

for some constant $K_2 > 0$ not depending on x.

According to (7.3), we consider the linear isometry $V\colon L^2(\Omega_0) \to \ell^2$ defined by letting $V(a) = \big(\langle a, b_j\rangle\big)_{j\geq 1}$ for any $a \in L^2(\Omega_0)$. Its adjoint $V^*\colon \ell^2 \to L^2(\Omega_0)$ is a contraction, hence $I_{L^p} \otimes V^*$ extends to a contraction

$$\widehat{V^*}\colon L^p(\mathcal{M};\ell^2_{rad}) \longrightarrow L^p(\mathcal{M};L^2(\Omega_0)_{rad})$$

by Lemma 2.4. Let us show that for any $x \in D(A)\cap R(A)$, we have

$$\widehat{V^*}\Big(\big(h_j(A)x\big)_j\Big) = F(\cdot\, A)x. \tag{7.7}$$

Recall from Chapter 3 that with $g(z) = \frac{z}{(1+z)^2}$, we may write $x = g(A)x'$ for some (unique) $x' \in L^p(\mathcal{M})$. Moreover by the first two parts of Lemma 6.5, we have

$$\int_0^\infty \|F(tA)g(A)\|^2\,\frac{dt}{t} < \infty\,.$$

Hence $F(\cdot\, A)g(A)\overline{b_j(\cdot\,)}$ is integrable on Ω_0 for any $j \geq 1$. Then using Fubini's Theorem, it is easy to deduce from (7.5) that

$$h_j(A)g(A) = \int_0^\infty F(tA)g(A)\,\overline{b_j(t)}\,\frac{dt}{t}\,.$$

Now let $y \in L^{p'}(\mathcal{M})$ be an arbitrary functional on $L^p(\mathcal{M})$. We see from above that

$$\langle F(\cdot\, A)x, y\rangle = \langle F(\cdot\, A)g(A)x', y\rangle \ \in L^2(\Omega_0)$$

and that

$$\langle h_j(A)x, y\rangle = \int_0^\infty \langle F(tA)x, y\rangle\,\overline{b_j(t)}\,\frac{dt}{t}\,, \qquad j \geq 1.$$

Thus V maps the function $\langle F(\cdot\, A)x, y\rangle$ to the sequence $\big(\langle h_j(A)x, y\rangle\big)_{j\geq 1}$. Since V is an isometry, this implies that conversely, V^* maps the sequence $\big(\langle h_j(A)x, y\rangle\big)_{j\geq 1}$ to the function $\langle F(\cdot\, A)x, y\rangle$. Since this holds for any $y \in L^{p'}(\mathcal{M})$, this concludes the proof of (7.7). Owing to (7.6), this implies that

$$\|x\|_F \leq K_2\|x\|, \qquad x \in D(A)\cap R(A).$$

We now appeal to the approximating sequence $(g_n)_{n\geq 1}$ defined by (3.8). For any $x \in L^p(\mathcal{M})$ and any $n \geq 1$, $g_n(A)x$ belongs to $D(A)\cap R(A)$, hence $\|g_n(A)x\|_F \leq K_2\|g_n(A)x\|$. Since $(g_n(A))_{n\geq 1}$ is bounded, this shows that for an appropriate constant $K_3 > 0$, we have

$$\|g_n(A)x\|_F \leq K_3\|x\|, \qquad n \geq 1,\ x \in L^p(\mathcal{M}).$$

Arguing as at the end of the proof of Theorem 6.7, we deduce that $\|x\|_F \leq K_3\|x\|$ for any $x \in L^p(\mathcal{M})$. This concludes the proof that A satisfies $(\mathcal{S}_F)$. Applying this result for A^*, we obtain that A satisfies $(\mathcal{S}_G^*)$ as well.

We now turn to the second assertion. Since A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus, A is Rad-sectoriel of Rad-type θ by Theorem 4.11. Thus Theorem 6.7 ensures that $\|\ \|_{F_1}$ and $\|\ \|_{F_2}$ are equivalent for any two functions $F_1, F_2 \in H_0^\infty(\Sigma_{\theta+}) \setminus \{0\}$. It therefore suffices to prove the result for a particular function $F \in H_0^\infty(\Sigma_{\theta+})$. Furthermore, it clearly follows from the first part of this proof that we only need to show that $\|\ \|_F$ dominates the original norm on $\overline{R(A)}$. We fix numbers $0 < \omega < \theta < \delta < \alpha < 2\alpha - \delta < \nu < \pi$ and we recall from [**21**, Example 4.7] that there exist $F, G \in H_0^\infty(\Sigma_\delta)$ such that the product function $\varphi = GF$ satisfies (7.1). By the first part of this proof, A satisfies $(\mathcal{S}_G^*)$ hence applying Proposition 7.2, we find some constant $K > 0$ such that $\|f(A)x\| \leq K\|x\|_F\|f\|_{\infty,\nu}$ for any $f \in H_0^\infty(\Sigma_\nu)$. Let us apply this estimate with $f = g_n$. Since $(g_n)_{n\geq 1}$ is a bounded sequence of $H^\infty(\Sigma_\nu)$, we obtain an estimate

$$\|g_n(A)x\| \leq K'\|x\|_F, \qquad n \geq 1,\ x \in L^p(\mathcal{M}).$$

Now assume that $x \in \overline{R(A)}$. Then $g_n(A)x$ converges to x (see Lemma 3.2). This yields $\|x\| \leq K'\|x\|_F$ and completes the proof. $\square$

If A has dense range and has a bounded $H^\infty(\Sigma_\theta)$ functional calculus, the above theorem yields an equivalence

$$\|x\| \asymp \|x\|_F, \qquad x \in L^p(\mathcal{M}),$$

for any $F \in H_0^\infty(\Sigma_{\theta+}) \setminus \{0\}$.

This may be obviously combined with either Proposition 6.2 or Proposition 6.3. The resulting formula is easy to be written down when $p \geq 2$ and we give it explicitly in the next statement. The case when $p < 2$ is more involved and its statement is left to the reader. We will come back to this case in Corollary 7.10 below.

Corollary 7.7. — *Let A be a sectorial operator of type $\omega \in (0,\pi)$ on $L^p(\mathcal{M})$, with $2 \leq p < \infty$. Assume that A has dense range and admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (\omega,\pi)$. Then for any $F \in H_0^\infty(\Sigma_{\theta+})\setminus\{0\}$, we have an equivalence*

$$\|x\| \asymp \max\Biggl\{\lim_{\alpha\to 0;\,\beta\to\infty}\Bigl\|\Bigl(\int_\alpha^\beta \bigl(F(tA)x\bigr)^*\bigl(F(tA)x\bigr)\,\frac{dt}{t}\Bigr)^{\frac12}\Bigr\|_{L^p(\mathcal{M})},$$
$$\lim_{\alpha\to 0;\,\beta\to\infty}\Bigl\|\Bigl(\int_\alpha^\beta \bigl(F(tA)x\bigr)\bigl(F(tA)x\bigr)^*\,\frac{dt}{t}\Bigr)^{\frac12}\Bigr\|_{L^p(\mathcal{M})}\Biggr\},\quad x \in L^p(\mathcal{M}).$$

Let A be a sectorial operator of type $\omega \in (0,\pi)$ on $L^p(\mathcal{M})$, with $1 < p < 2$. For any $F \in H_0^\infty(\Sigma_{\omega+}) \setminus \{0\}$, we may consider an alternative square function by letting

$$[x]_F = \inf\bigl\{\|x_1\|_{F,c} + \|x_2\|_{F,r} \,:\, x = x_1 + x_2\bigr\}$$

for any $x \in L^p(\mathcal{M})$. It is clear that $\|x\|_F \leq [x]_F$. Indeed if $[x]_F$ is finite, and if we have a decomposition $x = x_1 + x_2$ with $\|x_1\|_{F,c} < \infty$ and $\|x_2\|_{F,r} < \infty$, then we have $F(\cdot\, A)x = u_1 + u_2$, with $u_1 = F(\cdot\, A)x_1$ and $u_2 = F(\cdot\, A)x_2$, and these functions belong to $L^p(\mathcal{M}; L^2(\Omega_0)_c)$ and $L^p(\mathcal{M}; L^2(\Omega_0)_r)$ respectively. We do not know if the two square functions $\|\ \|_F$ and $[\ \ ]_F$ are equivalent in general. In the next statement we give a sufficient condition for such an equivalence to hold true.

Theorem 7.8. — *Let A be a sectorial operator on $L^p(\mathcal{M})$, with $1 < p < 2$. Let $\omega \in (0,\pi)$ and assume that A is both Col-sectorial of Col-type ω and Row-sectorial of Row-type ω. Let F, G be two non zero functions in $H_0^\infty(\Sigma_{\omega+})$ and assume that A admits a dual square function estimate $(\mathcal{S}_G^*)$. Then $\|\ \|_F \asymp [\ \ ]_F$ on $L^p(\mathcal{M})$. Indeed, there is a constant $C \geq 1$ such that whenever $x \in L^p(\mathcal{M})$ satisfies $\|x\|_F < \infty$, then there exist $x_1, x_2 \in L^p(\mathcal{M})$ such that*

$$x = x_1 + x_2 \qquad \textit{and} \qquad \|x_1\|_{F,c} + \|x_2\|_{F,r} \leq C\|x\|_F.$$

Proof. — We may assume that A has dense range. We will use the function g defined by (3.9). The assumptions imply that A is Rad-sectorial of Rad-type ω. Thus all square functions $\|\ \|_F$ are pairwise equivalent, by Theorem 6.7. Hence it suffices to prove the result for a particular function $F \in H_0^\infty(\Sigma_{\theta+})$. Therefore we may assume that

$$\int_0^\infty G(t)F(t)\,\frac{dt}{t} = 1.$$

By the first part of Lemma 6.5, we have

$$\int_0^\infty G(tA)F(tA)g(A)\,\frac{dt}{t} = g(A). \tag{7.8}$$

Since A satisfies $(\mathcal{S}_G^*)$, we can introduce the bounded linear operator

$$W\colon L^{p'}(\mathcal{M}) \longrightarrow L^{p'}(\mathcal{M}; L^2(\Omega_0)_{rad}), \qquad W(y) = G(\cdot\, A)^* y.$$

Note that by (2.25), the adjoint of W maps $L^p(\mathcal{M}; L^2(\Omega_0)_{rad})$ into $L^p(\mathcal{M})$.

Let $x \in L^p(\mathcal{M})$ such that $\|x\|_F < \infty$. There exist two functions $u_1 \in L^p(\mathcal{M}; L^2(\Omega_0)_c)$ and $u_2 \in L^p(\mathcal{M}; L^2(\Omega_0)_r)$ such that $u_1 + u_2 = F(\cdot\, A)x$ and

$$\|u_1\|_{L^p(\mathcal{M};L^2(\Omega_0)_c)} + \|u_2\|_{L^p(\mathcal{M};L^2(\Omega_0)_r)} \leq 2\|x\|_F.$$

Since $L^p(\mathcal{M}; L^2(\Omega_0)_c) \subset L^p(\mathcal{M}; L^2(\Omega_0)_{rad})$ and $L^p(\mathcal{M}; L^2(\Omega_0)_r) \subset L^p(\mathcal{M}; L^2(\Omega_0)_{rad})$, we may introduce

$$x_1 = W^* u_1 \qquad \text{and} \qquad x_2 = W^* u_2.$$

Let $i=1,2$. By the first two parts of Lemma 6.5, we have $\int_0^\infty \|g(A)G(tA)\|^2\,\frac{dt}{t}<\infty$ and by Proposition 2.5 (1), we have $\int_0^\infty \|u_i(t)\|^2\,\frac{dt}{t}<\infty$. Hence $t\mapsto g(A)G(tA)u_i(t)$ is integrable on Ω_0, and we actually have

$$g(A)x_i=\int_0^\infty g(A)G(tA)u_i(t)\,\frac{dt}{t}\,. \tag{7.9}$$

Indeed, for any $y\in L^{p'}(\mathcal{M})$,

$$\begin{aligned}\langle g(A)x_i,y\rangle &= \langle u_i, Wg(A)^*y\rangle\\ &=\int_0^\infty \langle u_i(t), G(tA)^*g(A)^*y\rangle\,\frac{dt}{t} \qquad \text{by Remark 2.9}\\ &=\Big\langle \int_0^\infty g(A)G(tA)u_i(t)\,\frac{dt}{t}, y\Big\rangle.\end{aligned}$$

Since $u_1+u_2=F(\cdot\, A)$, it follows from (7.8) and (7.9) that $g(A)x=g(A)x_1+g(A)x_2$. We assumed that A has dense range, hence $g(A)$ is one-one. Thus $x=x_1+x_2$. It now remains to estimate $\|x_1\|_{F,c}$ and $\|x_2\|_{F,r}$.

By assumption, A is Col-sectorial of Col-type ω. According to Theorem 4.14, the operator with kernel $F(sA)G(tA)$ is therefore bounded on $L^p(\mathcal{M};L^2(\Omega_0)_c)$. Let

$$T_c\colon L^p(\mathcal{M};L^2(\Omega_0)_c)\longrightarrow L^p(\mathcal{M};L^2(\Omega_0)_c)$$

denote the resulting operator. Since A is also Row-sectorial of Row-type ω, we have a similar bounded operator

$$T_r\colon L^p(\mathcal{M};L^2(\Omega_0)_r)\longrightarrow L^p(\mathcal{M};L^2(\Omega_0)_r)$$

Consider $b\in L^1(\Omega_0)\cap L^2(\Omega_0)$ and $y\in L^{p'}(\mathcal{M})$. Suppose that y belongs to the range of $g(A)^*$, so that $y=g(A)^*y'$ for some y'. Then

$$\big[T_c^*(y\otimes b)\big](t)\;=\;G(tA)^*\bigg(\int_0^\infty F(sA)^*y\,b(s)\,\frac{ds}{s}\bigg).$$

Hence by Lemma 2.8, we have

$$\begin{aligned}\langle T_c(u_1),y\otimes b\rangle &=\int_0^\infty \Big\langle u_1(t), G(tA)^*\bigg(\int_0^\infty F(sA)^*y\,b(s)\,\frac{ds}{s}\bigg)\Big\rangle\,\frac{dt}{t}\\ &=\int_0^\infty\int_0^\infty \langle u_1(t), G(tA)^*F(sA)^*y\rangle\,b(s)\,\frac{ds}{s}\,\frac{dt}{t}\\ &=\int_0^\infty\int_0^\infty \langle F(sA)g(A)G(tA)u_1(t), y'\rangle\,b(s)\,\frac{ds}{s}\,\frac{dt}{t}\,.\end{aligned}$$

Since $b\in L^1(\Omega_0)$ and $\int_0^\infty \|g(A)G(tA)u_1(t)\|\,\frac{dt}{t}<\infty$, we may apply Fubini's Theorem in the last integral. Hence using (7.9), we deduce that

$$\begin{aligned}\langle T_c(u_1), y\otimes b\rangle &= \int_0^\infty \Big\langle F(sA)\int_0^\infty g(A)G(tA)u_1(t)\,\frac{dt}{t}\,, y'\Big\rangle\, b(s)\,\frac{ds}{s}\\ &= \int_0^\infty \langle F(sA)g(A)x_1, y'\rangle\, b(s)\,\frac{ds}{s}\\ &= \int_0^\infty \langle F(sA)x_1, y\rangle\, b(s)\,\frac{ds}{s}\,.\end{aligned}$$

Since $g(A)^*$ has dense range, this calculation shows that $T_c(u_1)=F(\cdot\, A)x_1$. Likewise we have $T_r(u_2)=F(\cdot\, A)x_2$. Consequently,

$$\|x_1\|_{F,c}+\|x_2\|_{F,r}\leq \|T_c\|\,\|u_1\|+\|T_r\|\,\|u_2\|\leq 2\max\big\{\|T_c\|,\|T_r\|\big\}\|x\|_F. \qquad \square$$

Corollary 7.9. — *Let A be a sectorial operator of type $\omega\in(0,\pi)$ on $L^p(\mathcal{M})$, with $1<p<2$. Assume that A admits a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta\in(\omega,\pi)$, and let $F\in H_0^\infty(\Sigma_{\theta+})\setminus\{0\}$. If further A has dense range, then*

$$\|x\|\asymp \inf\big\{\|x_1\|_{F,c}+\|x_2\|_{F,r}\ :\ x=x_1+x_2\big\},\qquad x\in L^p(\mathcal{M}).$$

Proof. — By Theorem 4.12, the operator A is both Col-sectorial and Row-sectorial of respective types θ. Moreover it satisfies dual square function estimates by Theorem 7.6 (1). Thus by Theorem 7.8 above, $\|\ \|_F$ and $[\ \ ]_F$ are equivalent. Furthermore $\|\ \|_F$ is equivalent to the usual norm, by Theorem 7.6 (2), which proves the result. $\square$

The next result is an immediate consequence of Corollary 7.9 and Proposition 6.3. For simplicity if $u\colon\Omega_0\to L^p(\mathcal{M})$ is defined by $u(t)=F(tA)z$ for some $z\in L^p(\mathcal{M})$ and if π is a subpartition of Ω_0, we write $\big(F(tA)z\big)_\pi$ instead of $u_\pi(t)$.

Corollary 7.10. — *Let A be a sectorial operator of type $\omega\in(0,\pi)$ on $L^p(\mathcal{M})$, with $1<p<2$. Assume that A has dense range and admits a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta\in(\omega,\pi)$. Then for any $F\in H_0^\infty(\Sigma_{\theta+})\setminus\{0\}$, we have an equivalence*

$$\begin{aligned}\|x\|\asymp \inf\Bigg\{ &\lim_\pi \Bigg\|\Bigg(\int_0^\infty \big(F(tA)x_1\big)_\pi^*\big(F(tA)x_1\big)_\pi\,\frac{dt}{t}\Bigg)^{\frac12}\Bigg\|_{L^p(\mathcal{M})} +\\ &\lim_\pi \Bigg\|\Bigg(\int_0^\infty \big(F(tA)x_2\big)_\pi\big(F(tA)x_2\big)_\pi^*\,\frac{dt}{t}\Bigg)^{\frac12}\Bigg\|_{L^p(\mathcal{M})}\Bigg\},\end{aligned}$$

where for any $x\in L^p(\mathcal{M})$, the infimum runs over all $x_1,x_2\in L^p(\mathcal{M})$ such that $x=x_1+x_2$.

Remark 7.11. — Let $(T_t)_{t\geq 0}$ be a bounded analytic semigroup on $L^p(\mathcal{M})$, with $1<p<\infty$, and let $-A$ denote its generator. Assume that A admits a bounded $H^\infty(\Sigma_\theta)$

functional calculus for some $\theta < \frac{\pi}{2}$. Assume for simplicity that A is one-one. The function $F(z) = ze^{-z}$ belongs to $H_0^\infty(\Sigma_\nu)$ for any $\nu < \frac{\pi}{2}$, and we have

$$F(tA)x = tAe^{-tA}x = -t\frac{\partial}{\partial t}\big(T_t(x)\big), \qquad x \in L^p(\mathcal{M}),\ t > 0.$$

Thus we deduce from Corollary 7.7 that if $p \geq 2$, we have an equivalence

$$\begin{aligned}\|x\| \asymp \max\Bigg\{ &\lim_{\alpha\to 0;\,\beta\to\infty} \left\| \left(\int_\alpha^\beta t \left| \frac{\partial}{\partial t}\big(T_t(x)\big)\right|^2 dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathcal{M})}, \\ &\lim_{\alpha\to 0;\,\beta\to\infty} \left\| \left(\int_\alpha^\beta t \left| \frac{\partial}{\partial t}\big(T_t(x)\big)^* \right|^2 dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathcal{M})} \Bigg\}, \qquad x \in L^p(\mathcal{M}).\end{aligned}$$

A similar result can be written down in the case $p \leq 2$, using Corollary 7.10.

CHAPTER 8

VARIOUS EXAMPLES OF MULTIPLIERS

8.A. Left and right multiplication operators

Let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra acting on some Hilbert space H, and let $1\leq p<\infty$. For any $a\in\mathcal{M}$, we define a bounded operator $\mathcal{L}_a\colon L^p(\mathcal{M})\to L^p(\mathcal{M})$ by letting

$$\mathcal{L}_a(x)=ax,\qquad x\in L^p(\mathcal{M}). \tag{8.1}$$

We will call $\mathcal{L}_a$ the left multiplication by a on $L^p(\mathcal{M})$. We aim at extending this definition to unbounded operators.

Thus we let $a\colon D(a)\subset H\to H$ be a closed and densely defined operator on H. We assume that $\rho(a)\neq\varnothing$ and that a is affiliated with $\mathcal{M}$. This means that $au=ua$ for any unitary u in the commutant $\mathcal{M}'\subset B(H)$. For any $z\in\rho(a)$, this implies that $(z-a)u=u(z-a)$, and hence $R(z,a)u=uR(z,a)$ for any unitary $u\in\mathcal{M}'$. Thus we have

$$R(z,a)\in\mathcal{M},\qquad z\in\rho(a). \tag{8.2}$$

We will not use (8.1) directly to define $\mathcal{L}_a$, because multiplying an unbounded operator a with some $x\in L^p(\mathcal{M})$ leads to technical difficulties. Instead we will use left multiplications by resolvents, see (8.3).

Lemma 8.1. — *Let $c\in\mathcal{M}\subset B(H)$ be a one-one operator, and let $x\in L^p(\mathcal{M})$. If $cx=0$, then $x=0$.*

Proof. — This is clear by regarding x and cx as unbounded operators on H in the usual way. Indeed if ζ belongs to the domain of x, then we have $cx(\zeta)=0$. Hence $x(\zeta)=0$. □

Lemma 8.2. — *Let $(b_t)_t\subset\mathcal{M}$ be any bounded net converging to 0 in the strong operator topology of $B(H)$. Then $\|b_tx\|_p\to 0$ for any $1\leq p<\infty$ and any $x\in L^p(\mathcal{M})$.*

Proof. — We start with $p=2$. Let $x\in L^2(\mathcal{M})$. Then $xx^*\in L^1(\mathcal{M})\simeq\mathcal{M}_*$ and we have

$$\|b_tx\|_2^2=\tau\big((b_tx)^*(b_tx)\big)=\tau(b_t^*b_txx^*)=\langle b_t^*b_t,xx^*\rangle_{\mathcal{M},\mathcal{M}_*}.$$

By Hahn-Banach, $xx^*\colon\mathcal{M}\to\mathbb{C}$ extends to some w^*-continuous functional on $B(H)$, and hence there exist two sequences $(\zeta_k)_{k\geq1}$ and $(\xi_k)_{k\geq1}$ belonging to $\ell^2(H)$ such that

$$\langle w,xx^*\rangle_{\mathcal{M},\mathcal{M}_*}=\sum_{k=1}^{\infty}\langle w(\zeta_k),\xi_k\rangle,\qquad w\in\mathcal{M}.$$

Thus we obtain that

$$\|b_tx\|_2^2=\sum_{k=1}^{\infty}\langle b_t(\zeta_k),b_t(\xi_k)\rangle\ \leq\ \Big(\sum_{k=1}^{\infty}\|b_t(\zeta_k)\|^2\Big)^{\frac12}\Big(\sum_{k=1}^{\infty}\|b_t(\xi_k)\|^2\Big)^{\frac12}.$$

Since $b_t\to0$ strongly and $(b_t)_t$ is bounded, we deduce that $\|b_tx\|_2\to0$.

Assume now that $p>2$ and take $x\in\mathcal{M}\cap L^2(\mathcal{M})$. We let $\alpha=\frac{2}{p}$, so that we have $[L^\infty(\mathcal{M}),L^2(\mathcal{M})]_\alpha=L^p(\mathcal{M})$ by (2.4). This implies that

$$\|b_tx\|_p\leq\|b_tx\|_2^{\alpha}\|b_tx\|_\infty^{1-\alpha}.$$

We know from the first part of this proof that $\|b_tx\|_2\to0$. Since $(b_t)_t$ is bounded, we deduce that $\|b_tx\|_p\to0$. Using again the boundedness of $(b_t)_t$, together with the density of $\mathcal{M}\cap L^2(\mathcal{M})$ in $L^p(\mathcal{M})$, we obtain that $\|b_tx\|_p\to0$ for any $x\in L^p(\mathcal{M})$.

Finally we assume that $1\leq p<2$, and we let $x\in L^p(\mathcal{M})$. By the converse of the noncommutative Hölder inequality (see paragraph 2.A), there exist x',x'' in $L^{2p}(\mathcal{M})$ such that $x=x'x''$. Then we have $\|b_tx\|_p\leq\|b_tx'\|_{2p}\|x''\|_{2p}$. However $\|b_tx'\|_{2p}\to0$ by the above paragraph, hence we obtain that $\|b_tx\|_p\to0$ □

Lemma 8.3. — *For any $z\in\rho(a)$, the left multiplication $\mathcal{L}_{R(z,a)}\colon L^p(\mathcal{M})\to L^p(\mathcal{M})$ is one-one and has dense range.*

Proof. — That $\mathcal{L}_{R(z,a)}$ is one-one follows from Lemma 8.1. Next let $y\in L^{p'}(\mathcal{M})$ be orthogonal to the range of $\mathcal{L}_{R(z,a)}$. Then

$$0=\tau\big(R(z,a)xy\big)=\tau\big(xyR(z,a)\big)$$

for any $x\in L^p(\mathcal{M})$, hence $yR(z,a)=0$. Thus $R(z,a)^*y^*=0$ and by Lemma 8.1, we deduce that $y=0$. This shows that $\mathcal{L}_{R(z,a)}$ has dense range. □

Let $z\in\rho(a)$. According to Lemma 8.3, we may consider the inverse of $\mathcal{L}_{R(z,a)}$, with domain $\mathcal{D}$ equal to the range of $\mathcal{L}_{R(z,a)}$. Then we define

$$\mathcal{L}_a:=z-\mathcal{L}_{R(z,a)}^{-1}\colon\mathcal{D}\longrightarrow L^p(\mathcal{M}).\tag{8.3}$$

Clearly $\mathcal{L}_a$ is a closed and densely defined operator. Using the resolvent equation

$$R(z_1,a)-R(z_2,a)=(z_2-z_1)R(z_1,a)R(z_2,a),$$

it is easy to see that this definition does not depend on z. Moreover $\rho(a) \subset \rho(\mathcal{L}_a)$ and

$$R(z, \mathcal{L}_a) = \mathcal{L}_{R(z,a)}, \qquad z \in \rho(a). \tag{8.4}$$

Details are left to the reader.

We now consider the specific case of sectorial operators.

Proposition 8.4

(1) *Assume that $a \colon D(a) \to H$ is a sectorial operator of type $\omega \in (0, \pi)$, which is affiliated with $\mathcal{M}$. Then $\mathcal{L}_a$ is sectorial of type ω on $L^p(\mathcal{M})$. Moreover $\rho(a) \subset \rho(\mathcal{L}_a)$ and for any $z \in \rho(a)$ and any $x \in L^p(\mathcal{M})$, we have $R(z, \mathcal{L}_a)(x) = R(z, a)x$.*

(2) *For any $f \in H_0^\infty(\Sigma_{\omega+})$, we have*

$$f(\mathcal{L}_a)(x) = f(a)x, \qquad x \in L^p(\mathcal{M}). \tag{8.5}$$

(3) *Let $\theta \in (\omega, \pi)$ be an angle. Then $\mathcal{L}_a$ has a bounded $H^\infty(\Sigma_\theta)$ functional calculus if and only if a has one. In that case, $\mathcal{L}_a$ actually has a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

(4) *If a has dense range, then $\mathcal{L}_a$ has dense range. If further a admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus, then (8.5) holds true for any $f \in H^\infty(\Sigma_\theta)$.*

Proof. — Part (1) clearly follows from (8.4). Next (2) follows (1) and (3.5), and (3) is a straightforward consequence of (2).

Let us turn to (4). We let $A = \mathcal{L}_a$. We assume that a has dense range. According to [**21**, Theorem 3.8], $a(t+a)^{-1} \to I_H$ strongly when $t \to 0^+$. Moreover the net $(a(t+a)^{-1})_{t>0}$ is bounded by sectoriality. Hence for any $x \in L^p(\mathcal{M})$, $\|a(t+a)^{-1}x - x\|_p \to 0$ when $t \to 0^+$, by Lemma 8.2. Using (1), we note that $a(t+a)^{-1}x = A(t+A)^{-1}(x)$. Consequently, $a(t+a)^{-1}x$ belongs to $R(A)$ for any $t > 0$. Thus $R(A)$ is a dense subspace of $L^p(\mathcal{M})$.

Assume moreover that a admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus and consider any $f \in H^\infty(\Sigma_\theta)$. Let $x \in L^p(\mathcal{M})$, and let g be defined by (3.9). Applying (2) twice, we see that

$$g(a)f(a)x = g(A)(f(A)(x)) = g(a)[f(A)(x)].$$

Since $g(a)$ in one-one, the identity $f(A)(x) = f(a)x$ now follows from Lemma 8.1. $\square$

Next we discuss left multiplications by c_0-semigroups.

Proposition 8.5

(1) *Let $(w_t)_{t\geq 0}$ be a bounded c_0-semigroup on H, with negative generator a, and assume that $w_t \in \mathcal{M}$ for each $t \geq 0$. Then a is affiliated with $\mathcal{M}$.*

(2) *For any $t \geq 0$ and $x \in L^p(\mathcal{M})$, we define $T_t(x) = w_t x$. Then $(T_t)_{t\geq 0}$ is a bounded c_0-semigroup on $L^p(\mathcal{M})$, with negative generator equal to $\mathcal{L}_a$.*

Proof. — According to the Laplace formula (3.2), we have

$$(1+a)^{-1}(\zeta) \;=\; \int_0^\infty e^{-t}\, w_t(\zeta)\, dt\,, \qquad \zeta\in H.$$

Since $\mathcal{M}\subset B(H)$ is strongly closed, this implies that $(1+a)^{-1}\in\mathcal{M}$. That a is affiliated with $\mathcal{M}$ follows at once.

It is clear that $(T_t)_{t\geq 0}$ is a bounded semigroup. Since $(w_t)_{t\geq 0}$ is bounded and strongly continuous, Lemma 8.2 ensures that $(T_t)_{t\geq 0}$ also is strongly continuous.

Let A be the negative generator of $(T_t)_{t\geq 0}$. To show that $A=\mathcal{L}_a$, it suffices to check that $(1+A)^{-1}=\mathcal{L}_{(1+a)^{-1}}$, by (8.4). We use the Laplace formula again. Let p' be the conjugate number of p. For $x\in L^p(\mathcal{M})$ and $y\in L^{p'}(\mathcal{M})$, we have

$$\big\langle (1+A)^{-1}(x), y\big\rangle_{L^p,L^{p'}} \;=\; \Big\langle \int_0^\infty e^{-t}\, T_t(x)\, dt\,, y\Big\rangle_{L^p,L^{p'}} \;=\; \int_0^\infty e^{-t}\, \tau(w_t x y)\, dt\,.$$

Arguing as in Lemma 8.2, we may find two sequences $(\zeta_k)_{k\geq 1}$ and $(\xi_k)_{k\geq 1}$ belonging to $\ell^2(H)$ such that

$$\tau(wxy) \;=\; \sum_{k=1}^{\infty} \langle w(\zeta_k),\xi_k\rangle, \qquad w\in\mathcal{M}.$$

Thus we obtain that

$$\begin{aligned}
\big\langle (1+A)^{-1}(x), y\big\rangle_{L^p,L^{p'}} \;&=\; \int_0^\infty e^{-t} \sum_{k=1}^{\infty} \langle w_t(\zeta_k),\xi_k\rangle\, dt\\
&=\; \sum_{k=1}^{\infty} \Big\langle \int_0^\infty e^{-t}\, w_t(\zeta_k)\, dt\,, \xi_k\Big\rangle\\
&=\; \sum_{k=1}^{\infty} \big\langle (1+a)^{-1}(\zeta_k),\xi_k\big\rangle\\
&=\; \tau((1+a)^{-1}xy)\\
&=\; \langle (1+a)^{-1}x, y\rangle_{L^p,L^{p'}}\,.
\end{aligned}$$

This proves the desired identity. □

Remark 8.6. — In order to apply Proposition 8.4 (3), one needs to know which sectorial operators on a Hilbert space have a bounded H^∞ functional calculus. This question was initiated in McIntosh's fundamental paper on H^∞ calculus [**53**]. We refer to [**54**], [**3**, Lecture 3], [**5**], and [**45**] for various results on this topic. Let $(w_t)_{t\geq 0}$ be a bounded c_0-semigroup on H, with negative generator a. We recall that if $(w_t)_{t\geq 0}$ is a contraction semigroup, then a has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta>\frac{\pi}{2}$. Furthermore, if a is sectorial of type $\omega<\frac{\pi}{2}$, then for any $\theta>\omega$, a has a bounded $H^\infty(\Sigma_\theta)$ functional calculus if and only if $(w_t)_{t\geq 0}$ is similar to a contraction semigroup [**45**].

Remark 8.7

(1) Let $a\colon D(a) \subset H \to H$ be a closed and densely defined operator affiliated with $\mathcal{M}$, and assume that $\rho(a) \neq \varnothing$. In (8.3) or (8.4) we defined the left multiplication by a on $L^p(\mathcal{M})$. By symmetry, one can clearly define the right multiplication by a on $L^p(\mathcal{M})$; we denote this operator by $\mathcal{R}_a$. Namely, if a is bounded, we let $\mathcal{R}_a(x) = xa$ for any $x \in L^p(\mathcal{M})$. Then if a is unbounded, we argue as in the 'left case' and for any $z \in \rho(a)$, the operator $\mathcal{R}_a$ is defined as $z - \mathcal{R}_{R(z,a)}^{-1}$. Equivalently, we have $R(z, \mathcal{R}_a) = \mathcal{R}_{R(z,a)}$. It is clear that Propositions 8.4 and 8.5 extend verbatim to right multiplications.

(2) If $1 < p, p' < \infty$ are two conjugate numbers, then the adjoint of the left multiplication by a on $L^p(\mathcal{M})$ coincides with the right multiplication by a on $L^{p'}(\mathcal{M})$. Indeed if a is bounded, we have

$$\langle ax, y\rangle = \tau(axy) = \tau(xya) = \langle x, ya\rangle, \qquad x \in L^p(\mathcal{M}),\ y \in L^{p'}(\mathcal{M}).$$

Then the general case follows from the bounded one, by using resolvents.

By a similar calculation, one has $\mathcal{L}_a^\circ = \mathcal{R}_{a^*}$ and $\mathcal{R}_a^\circ = \mathcal{L}_{a^*}$ (using the notation introduced in (2.5)). By (2.6) we deduce that if $p = 2$, we have

$$\mathcal{L}_a^\dagger = \mathcal{L}_{a^*} \qquad \text{and} \qquad \mathcal{R}_a^\dagger = \mathcal{R}_{a^*}.$$

Thus $\mathcal{L}_a$ (resp. $\mathcal{R}_a$) is selfadjoint on $L^2(\mathcal{M})$ if and only if a is selfadjoint.

(3) We recall that two (possibly unbounded) operators A and B with non empty resolvent sets are called commuting if for any $z_1 \in \rho(A)$ and $z_2 \in \rho(B)$, we have

$$R(z_1, A)R(z_2, B) = R(z_2, B)R(z_1, A).$$

It is clear that if a, b are two sectorial operators on H affiliated with $\mathcal{M}$, then the operators $A = \mathcal{L}_a$ and $B = \mathcal{R}_b$ on $L^p(\mathcal{M})$ commute in the above sense.

Left and right multiplications were used in the early days of H^∞ functional calculus to provide some examples involving pairs of commuting operators. Assume that $p \neq 2$, and consider the case when $\mathcal{M}$ is equal to $B(\ell^2)$ equipped with the usual trace. It was shown in [**44**] that there may exist positive selfadjoint operators a and b on ℓ^2 such that the pair $(\mathcal{L}_a, \mathcal{R}_b)$ does not have a bounded joint functional calculus on S^p (see the latter paper for a definition), although $\mathcal{L}_a$ and $\mathcal{R}_b$ each admit a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > 0$. On the other hand it follows from [**54**] and [**41**, Theorem 6.3] that there may exist a positive selfadjoint operator a on H such that the operator $\mathcal{L}_a$ on S^1 is not Rad-sectorial.

8.B. Hamiltonians

In this part we wish to consider a special class of quantum dynamical semigroups and their extensions to noncommutative L^p-spaces. For that purpose, we will need a few facts about bisectorial operators on general Banach spaces, their functional calculus, and relationships with sectorial operators. For any $\omega \in (0, \frac{\pi}{2})$, we let

$$\mathcal{S}_\omega = \{z \in \mathbb{C}^* \, : \, \frac{\pi}{2} - \omega < |\mathrm{Arg}(z)| < \frac{\pi}{2} + \omega\}$$

be the open cone of angle 2ω around the imaginary axis $i\mathbb{R}$. Then we let $H^\infty(\mathcal{S}_\omega)$ be the algebra of all bounded analytic functions on $\mathcal{S}_\omega$ equipped with the supremum norm, and we let $H_0^\infty(\mathcal{S}_\omega)$ be the subalgebra of all f for which there exists $s > 0$ such that $|f(z)| = O(|z|^{-s})$ as $|z| \to \infty$ for $z \in \mathcal{S}_\omega$, and $|f(z)| = O(|z|^s)$ as $|z| \to 0$ for $z \in \mathcal{S}_\omega$. We say that a closed and densely defined operator B on some Banach space X is bisectorial of type ω if its spectrum is contained in the closure of $\mathcal{S}_\omega$, and if for any $\theta \in (\omega, \frac{\pi}{2})$, $zR(z, B)$ is uniformly bounded outside $\overline{\mathcal{S}_\theta}$. This is the same as saying that B and $-B$ are both sectorial of type $\omega + \frac{\pi}{2}$.

Assume that $\omega + \frac{\pi}{2} < \gamma < \theta + \frac{\pi}{2}$, and let $g \in H_0^\infty(\mathcal{S}_\theta)$. By analogy with (3.5), we define

$$g(B) \, = \, \frac{1}{2\pi i} \int_{\Gamma_\gamma} g(z)R(z, B) + g(-z)R(z, -B)\, dz\,. \tag{8.6}$$

As in the sectorial case, this definition does not depend on γ, and $g \mapsto g(B)$ is an algebra homomorphism which is consistent with the functional calculus of rational functions. We say that B is bisectorial of type 0 if it is bisectorial of type ω for any $\omega \in (0, \frac{\pi}{2})$.

For any $0 < \omega < \frac{\pi}{2}$, the transformation $z \mapsto -z^2$ maps $\mathcal{S}_\omega$ onto $\Sigma_{2\omega}$. It is not hard to show that if B is bisectorial of type $\omega \in (0, \frac{\pi}{2})$, then $-B^2$ is a sectorial operator of type 2ω. Furthermore, the functional calculi of B and $-B^2$ are compatible in the following sense. Let $\theta \in (2\omega, \pi)$ and let $f \in H_0^\infty(\Sigma_\theta)$. Then the function $g\colon \mathcal{S}_{\theta/2} \to \mathbb{C}$ defined by $g(z) = f(-z^2)$ belongs to $H_0^\infty(\mathcal{S}_{\theta/2})$, and we have $g(B) = f(-B^2)$. This follows from (8.6) and (3.5), details are left to the reader.

We will apply the above construction to generators of bounded groups. Let X be a Banach space and let $(U_t)_{t\in\mathbb{R}}$ be a bounded c_0-group on X. We let iA denote its infinitesimal generator. It is clearly bisectorial of type 0, hence A^2 is a sectorial operator of type 0. The function f defined by

$$f(z) = e^{-\frac{z}{2}} \, - \, \tfrac{1}{1+z}$$

belongs to $H_0^\infty(\Sigma_\theta)$ for any $\theta < \frac{\pi}{2}$. Thus if we let $\gamma \in (\frac{\pi}{2}, \frac{3\pi}{4})$ and apply the above results to f, we find that

$$e^{-\frac{A^2}{2}} - (1 + A^2)^{-1} \, = \, \frac{1}{2\pi i} \int_{\Gamma_\gamma} \big(e^{\frac{z^2}{2}} \, - \, \tfrac{1}{1-z^2}\big) \big(R(z, iA) + R(z, -iA)\big)\, dz\,. \tag{8.7}$$

We claim that

$$e^{-\frac{A^2}{2}} \, = \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}}\, U_s\, ds \tag{8.8}$$

in the strong sense. To prove this identity we start from the following two standard identities. For any $s \geq 0$,

$$e^{-\frac{s^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist}\, e^{-\frac{t^2}{2}}\, dt \qquad \text{and} \qquad e^{-s} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-ist}\, \tfrac{1}{1+t^2}\, dt\,.$$

Using Cauchy's Theorem and the analyticity of the two functions

$$z \mapsto e^{sz}\, e^{\frac{z^2}{2}} \qquad \text{and} \qquad z \mapsto e^{sz}\, \frac{1}{1-z^2}\,,$$

we deduce that for any $s \geq 0$,

$$\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} = \frac{-1}{2\pi i} \int_{\Gamma_\gamma} e^{sz}\, e^{\frac{z^2}{2}}\, dz \qquad \text{and} \qquad \frac{e^{-s}}{2} = \frac{-1}{2\pi i} \int_{\Gamma_\gamma} e^{sz}\, \tfrac{1}{1-z^2}\, dz\,.$$

Next using Fubini's Theorem, we deduce that

$$\begin{aligned}
\int_{-\infty}^{\infty} \Big(\tfrac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} - \tfrac{e^{-|s|}}{2}\Big)\, U_s\, ds \;&=\; \int_{0}^{\infty} \Big(\tfrac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} - \tfrac{e^{-s}}{2}\Big)\, \big(U_s + U_{-s}\big)\, ds \\
&= \frac{-1}{2\pi i} \int_0^{\infty} \int_{\Gamma_\gamma} e^{sz}\, \big(e^{\frac{z^2}{2}} - \tfrac{1}{1-z^2}\big)\, \big(U_s + U_{-s}\big)\, dz\; ds \\
&= \frac{-1}{2\pi i} \int_{\Gamma_\gamma} \big(e^{\frac{z^2}{2}} - \tfrac{1}{1-z^2}\big) \Big[\int_0^{\infty} e^{sz}\, U_s\, ds + \int_0^{\infty} e^{sz}\, U_{-s}\, ds\Big]\, dz.
\end{aligned}$$

According to the Laplace formula (3.2), the two integrals in the above brackets are equal to $-R(z, -iA)$ and $-R(z, iA)$ respectively. Hence combining with (8.7), we have proved that

$$\int_{-\infty}^{\infty} \Big(\tfrac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} - \tfrac{e^{-|s|}}{2}\Big)\, U_s\, ds \;=\; e^{-\frac{A^2}{2}} \;-\; (1 + A^2)^{-1}.$$

To deduce (8.8), it remains to observe that

$$\int_{-\infty}^{\infty} e^{-|s|}\, U_s\, ds \;=\; 2(1 + A^2)^{-1}.$$

This is an easy consequence of the Laplace formula applied to the two semigroups $(U_s)_{s \geq 0}$ and $(U_{-s})_{s \geq 0}$.

Corollary 8.8. — *Let iA be the generator of a c_0-group of isometries on X. Then A^2 is a sectorial operator of type 0, and we have*

$$e^{-tA^2} \;=\; \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4}}\, U_{st^{\frac{1}{2}}}\, ds\,, \qquad t \geq 0. \tag{8.9}$$

Moreover, $\|e^{-tA^2}\| \leq 1$ for any $t \geq 0$.

Proof. — We already noticed that A^2 is sectorial of type 0. Formula (8.9) follows by applying (8.8) with A replaced by $\sqrt{2t}A$, and then changing s into $\sqrt{2}\, s$ in the resulting integral. Since each $U_{st^{\frac{1}{2}}}$ is a contraction and $\int e^{-\frac{s^2}{4}}\, ds \;=\; 2\sqrt{\pi}$, we deduce that $\|e^{-tA^2}\| \leq 1$. □

We give another general result which will be used later on in this chapter.

Lemma 8.9. — *Let $(T_t)_{t\geq 0}$ be a bounded c_0-semigroup on X, and let $-C$ denote its generator. We let*

$$h(s) \;=\; \frac{1}{2\sqrt{\pi}}\,\frac{e^{-\frac{1}{4s}}}{s^{\frac{3}{2}}} \qquad \textit{for any } s>0.$$

Then we have

$$e^{-tC^{\frac{1}{2}}} \;=\; \int_0^\infty h(s)\,T_{st^2}\,ds\,, \qquad t\geq 0. \tag{8.10}$$

If $(T_t)_{t\geq 0}$ is a contractive semigroup, then $\|e^{-tC^{\frac{1}{2}}}\|\leq 1$ for any $t\geq 0$.

Proof. — Formula (8.10) is well-known, see e.g. [**22**, Ex. 2.32]. Indeed a proof of (8.10) can be obtained by a computation similar to the one given for (8.8). Since $\int_0^\infty h(s)\,ds\,=1$, the last assertion is clear from (8.10). □

We will now apply the above results to a special class of quantum dynamical groups and their generators (see e.g. [**57**, III. 30]). Let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra, and let $1\leq p<\infty$ be any number. Let a and b be two selfadjoint operators affiliated with $\mathcal{M}$. If they are both bounded, we define an operator $\mathcal{A}d_{(a,b)}\colon L^p(\mathcal{M})\to L^p(\mathcal{M})$ by

$$\mathcal{A}d_{(a,b)}(x) = ax - xb, \qquad x\in L^p(\mathcal{M}).$$

We will extend this definition to the case when a or b is unbounded. Let $A=\mathcal{L}_a$ and $B=\mathcal{R}_b$ be the left and right multiplications on $L^p(\mathcal{M})$ by a and b respectively (see paragraph 8.A). We claim that the intersection $D(A)\cap D(B)$ is a dense subspace of $L^p(\mathcal{M})$, and that the difference operator

$$A-B\colon D(A)\cap D(B)\longrightarrow L^p(\mathcal{M})$$

taking x to $A(x)-B(x)$ is closable. To prove the density assertion, note that for any $x\in L^p(\mathcal{M})$, we have

$$inR(in,A)(x) = inR(in,a)x \longrightarrow x$$

when $n\to\infty$. Indeed this follows from (8.4) and Lemma 8.2. Likewise, $inR(in,B)(x)\to x$ when $n\to\infty$. We deduce that $n^2R(in,B)\,R(in,A)(x)\to -x$ when $n\to\infty$. Since $R(in,B)$ and $R(in,A)$ commute for any $n\geq 1$, each element $n^2R(in,B)\,R(in,A)(x)$ belongs to the subspace $D(A)\cap D(B)$. Hence x is the limit of a sequence of $D(A)\cap D(B)$.

To prove the closability of $A-B$, suppose that $(x_n)_{n\geq 1}$ is a sequence of $D(A)\cap D(B)$ converging to 0, such that $A(x_n)-B(x_n)$ converges to some $x\in L^p(\mathcal{M})$. The resolvent operators $R(i,A)$ and $R(i,B)$ commute, hence

$$R(i,A)R(i,B)(A-B) = R(i,B)\big[AR(i,A)\big] \;-\; R(i,A)\big[BR(i,B)\big]$$

on $D(A)\cap D(B)$. Thus

$$R(i,A)R(i,B)(x) = \lim_n R(i,B)\big[AR(i,A)\big](x_n) - \lim_n R(i,A)\big[BR(i,B)\big](x_n) = 0.$$

Since $R(i,A)R(i,B)$ is one-one, this shows that $x=0$. Hence $A-B$ is closable.

We can now define $\mathcal{A}d_{(a,b)}$ as the closure of $A - B$, that is,

$$\mathcal{A}d_{(a,b)} = \overline{\mathcal{L}_a - \mathcal{R}_b}. \tag{8.11}$$

Lemma 8.10. — *Let a and b be selfadjoint operators affiliated with $\mathcal{M}$, and let $1 \leq p < \infty$. For any $t \in \mathbb{R}$, we define $U_t\colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$ by*

$$U_t(x) = e^{ita}xe^{-itb}, \qquad x \in L^p(\mathcal{M}).$$

Then $(U_t)_{t\in\mathbb{R}}$ is a c_0-group of isometries on $L^p(\mathcal{M})$, with generator equal to $i\mathcal{A}d_{(a,b)}$.

Proof. — For any $t \in \mathbb{R}$, we define

$$T_t\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}) \qquad \text{and} \qquad S_t\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M})$$

by letting $T_t(x) = e^{ita}x$ and $S_t(x) = xe^{-itb}$ for any $x \in L^p(\mathcal{M})$. According to Proposition 8.5 and its 'right' version, $(T_t)_{t\in\mathbb{R}}$ and $(S_t)_{t\in\mathbb{R}}$ are both c_0-groups of isometries on $L^p(\mathcal{M})$, with generators equal to $i\mathcal{L}_a$ and $-i\mathcal{R}_b$ respectively. These two c_0-groups are commuting (that is, $S_sT_t = T_tS_s$ for any s,t), and $U_t = S_tT_t$ is defined as their product. Then it is easy to check that $(U_t)_{t\in\mathbb{R}}$ is a c_0-group of isometries. By e.g. [**56**, p. 24], its generator is the closure of the sum of the generators of $(T_t)_t$ and $(S_t)_t$. By (8.11), this operator is $i\mathcal{A}d_{(a,b)}$. □

Remark 8.11. — Let a, b and $(U_t)_t$ be as in Lemma 8.10 above. Let $\widetilde{a} = I\overline{\otimes}a$ be the closure of $I_{\ell^2} \otimes a$ on the Hilbertian tensor product $\ell^2 \otimes_2 H$, and let $\widetilde{b}$ be defined similarly. These are selfadjoint operators affiliated with $B(\ell^2)\overline{\otimes}\mathcal{M}$. Then it is clear that $(U_t)_t$ is a completely isometric c_0-group, with

$$I\overline{\otimes}U_t(y) = e^{it\widetilde{a}}ye^{-it\widetilde{b}}, \qquad t \in \mathbb{R},\ y \in S^p[L^p(\mathcal{M})].$$

By Lemma 3.9, $iI\overline{\otimes}\mathcal{A}d_{(a,b)} = i\mathcal{A}d_{(\widetilde{a},\widetilde{b})}$ is the generator of $(I\overline{\otimes}U_t)_t$.

Theorem 8.12. — *Consider two finite commuting families $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ of selfadjoint operators affiliated with $\mathcal{M}$. (Namely we asssume that $a_ia_j = a_ja_i$ and $b_ib_j = b_jb_i$ for any $1 \leq i,j \leq n$, but we do not assume that a_i commutes with b_j.) We assume that $1 < p < \infty$, and for any $1 \leq j \leq n$, we let $A_j = \mathcal{A}d_{(a_j,b_j)}$ be defined by (8.11) on $L^p(\mathcal{M})$.*

(1) *The sum operator*

$$C = A_1^2 + \cdots + A_n^2\colon \bigcap_{j=1}^{n} D\big(A_j^2\big) \longrightarrow L^p(\mathcal{M})$$

is closed and densely defined, and $-C$ generates a completely contractive semigroup on the space $L^p(\mathcal{M})$.

(2) *Furthermore for any $\theta > 0$, C admits a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus on $L^p(\mathcal{M})$.*

Proof. — It follows from Corollary 8.8 and Lemma 8.10 that for any $1 \leq j \leq n$, A_j^2 is a sectorial operator of type 0 and that $-A_j^2$ generates a contractive semigroup $(T_t^j)_{t\geq 0}$ on $L^p(\mathcal{M})$. Since $L^p(\mathcal{M})$ is UMD, it also follows from [**32**, Section 4] that A_j^2 admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > 0$.

Next the sectorial operators $A_1^2, \ldots, A_n^2$ are pairwise commuting, in the sense of Remark 8.7 (3). Indeed $A_1, \ldots, A_n$ are pairwise commuting, by our hypothesis that both families $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are commuting.

Since $L^p(\mathcal{M})$ is UMD, we deduce by [**41**, Proposition 3.2] and [**47**, Theorem 1.1] that the sum operator $C = A_1^2 + \cdots + A_n^2$ is a sectorial operator of type 0 (in particular, it is closed and densely defined), and that C admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > 0$. Further if we let

$$T_t = T_t^1 \cdots T_t^n \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M})$$

for any $t \geq 0$, then $(T_t)_{t\geq 0}$ is a c_0-semigroup of contractions. By [**56**, p. 24], its generator is $-C$. This proves the 'bounded' version of the theorem.

To prove the 'completely bounded' version, we let $\widetilde{a_j} = I\overline{\otimes} a_j$ and $\widetilde{b_j} = I\overline{\otimes} b_j$ be the closures of $I_{\ell^2} \otimes a_j$ and $I_{\ell^2} \otimes b_j$ on $\ell^2 \otimes_2 H$ respectively. According to Remark 8.11, $\mathcal{A}d_{(\widetilde{a_j},\widetilde{a_j})} = I\overline{\otimes} A_j$. Moreover $(\widetilde{a_1}, \ldots, \widetilde{a_n})$ and $(\widetilde{b_1}, \ldots, \widetilde{b_n})$ are commuting families. Hence applying the first part of this proof to these families, we obtain that $I\overline{\otimes} C$ generates a contractive semigroup and admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > 0$. □

Remark 8.13

(1) Consider $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ as in Theorem 8.12 above. Suppose that $p = 2$, and let $T_t = e^{-tC}$ be the semigroup generated by $-C$ on $L^2(\mathcal{M})$. It is clear that A_j^2 is selfadjoint for any $1 \leq j \leq n$. Indeed this follows either from Remark 8.7 (2), or from the fact that the generator of a group of isometries on Hilbert space is necessary skewadjoint. This implies that for any $t \geq 0$, $T_t \colon L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is selfadjoint. Furthermore applying Corollary 8.8 and Lemma 8.10 on $X = L^1(\mathcal{M})$, and arguing as in the proof of Theorem 8.12 (1), we see that T_t is contractive on $L^1(\mathcal{M})$ for any $t \geq 0$. Hence $(T_t)_{t\geq 0}$ is a diffusion semigroup on $\mathcal{M}$ (see Remark 5.2).

Later on in this chapter, we will consider the square root operator

$$A = C^{\frac{1}{2}} = \big(A_1^2 + \cdots + A_n^2\big)^{\frac{1}{2}}. \tag{8.12}$$

Applying Lemma 8.9 and the above paragraph, we see that $(e^{-tA})_{t\geq 0}$ also is a diffusion semigroup on $\mathcal{M}$.

(2) For a selfadjoint operator a affiliated with $\mathcal{M}$, we let

$$\mathcal{A}d_a = \mathcal{A}d_{(a,a)}.$$

For any $s \in \mathbb{R}$, the operator $U_s \colon L^2(\mathcal{M}) \to L^2(\mathcal{M})$ taking any $x \in L^2(\mathcal{M})$ to $e^{isa}xe^{-isa}$ is completely positive. Hence according to Lemma 8.10 and Corollary 8.8, $e^{-t\mathcal{A}d_a{}^2}$ is completely positive for any $t \geq 0$.

Next we consider a commuting family $(a_1,\dots,a_n)$ of selfadjoint operators affiliated with $\mathcal{M}$, we let $A_j=\mathcal{A}d_{a_j}$ for any $1\le j\le n$, and we let $C=A_1^2+\cdots+A_n^2$. (In other words, we consider the case when $a_j=b_j$ in Theorem 8.12 and in (1) above.) Since $T_t=e^{-tC}$ is the product of the $e^{-t\mathcal{A}d_{a_j}{}^2}$, we obtain from above that T_t is completely positive for any $t\ge 0$. Likewise, if A is defined by (8.12), then e^{-tA} is completely positive for any $t\ge 0$. Indeed this follows from Lemma 8.9. Thus $(e^{-tC})_{t\ge 0}$ and $(e^{-tA})_{t\ge 0}$ are completely positive diffusion semigroups.

These results apply in particular to the case when $A=\left|\mathcal{A}d_a\right| = \left((\mathcal{A}d_a)^2\right)^{\frac12}$ is the modulus of the operator $\mathcal{A}d_a$.

For a fixed $1<p<\infty$, let $A_1,\dots,A_n$ and C be as in Theorem 8.12, and let $\theta>0$ be a positive angle. The second part of the above theorem says that the homomorphism $\pi\colon H_0^\infty(\Sigma_\theta)\to B(L^p(\mathcal{M}))$ taking f to $f(C)$ is bounded. According to the methods we used for this result, the norm of that homomorphism can be dominated by a constant only depending on p, θ, and n, and not on the families $(a_1,\dots,a_n)$ and $(b_1,\dots,b_n)$ used to define $A_1,\dots,A_n$. For some applications (see paragraph 8.C below), the fact that $\|\pi\|$ may depend on n turns out to be a serious drawback. In the last part of this paragraph, we will show that this norm can be dominated by a constant which does not depend on n, provided that we insist that θ be large enough. As in Chapter 5, we let

$$\omega_p=\pi\left|\frac1p-\frac12\right|.$$

Theorem 8.14. — *Let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra, let $1<p<\infty$, and let $\theta>\omega_p$. There exists a constant $K_{\theta,p}$ satisfying the following property:*

If $(a_1,\dots,a_n)$ and $(b_1,\dots,b_n)$ are two commuting families of selfadjoint operators affiliated with $\mathcal{M}$, if $A_j=\mathcal{A}d_{(a_j,b_j)}$ on $L^p(\mathcal{M})$, and if we let $A=(A_1^2+\cdots+A_n^2)^{\frac12}$, then

$$\|f(A)\|\le K_{\theta,p}\,\|f\|_{\infty,\theta},\qquad f\in H_0^\infty(\Sigma_\theta).$$

Proof. — We noticed in Remark 8.13 (1) that $-A$ generates a diffusion semigroup on $\mathcal{M}$. Thus according to Proposition 5.8 and the subsequent Remark 5.10, it will suffice to prove the theorem for any $\theta>\frac{\pi}{2}$.

We write

$$b(y)=\frac{1}{2\sqrt{\pi}}\,e^{-\frac{y^2}{4}}\qquad\text{and}\qquad h(s)=\frac{1}{2\sqrt{\pi}}\,\frac{e^{-\frac{1}{4s}}}{s^{\frac32}}$$

for the two nonnegative functions appearing in (8.9) and (8.10) respectively.

Let $C=A_1^2+\cdots+A_n^2$ be the square of A, and let $(T_t)_{t\ge0}$ be the c_0-semigroup generated by $-C$. For any $1\le j\le n$, we let $(U_t^j)_t$ be the c_0-group on $L^p(\mathcal{M})$ generated by iA_j. We noticed in the proof of Theorem 8.12 that $(T_t)_{t\ge0}$ is the product of the semigroups generated by the A_j^2's. According to (8.9), we obtain that for any $t \geq 0$,

$$\begin{aligned} T_t &= \Big(\int_{-\infty}^{\infty} b(y_1)\, U^1_{y_1 t^{\frac{1}{2}}}\, dy_1\Big) \cdot \cdots \cdot \Big(\int_{-\infty}^{\infty} b(y_n)\, U^n_{y_n t^{\frac{1}{2}}}\, dy_n\Big) \\ &= \int_{\mathbb{R}^n} b(y_1)\cdots b(y_n)\, U^1_{y_1 t^{\frac{1}{2}}} \cdots U^n_{y_n t^{\frac{1}{2}}}\, dy_1 \cdots dy_n\,. \end{aligned}$$

Applying Lemma 8.10, we deduce that for any $x \in L^p(\mathcal{M})$,

$$T_t(x) = \int_{\mathbb{R}^n} b(y_1)\cdots b(y_n) \exp\big\{it^{\frac{1}{2}}(y_1 a_1 + \cdots + y_n a_n)\big\}\, x \\ \times \exp\big\{-it^{\frac{1}{2}}(y_1 b_1 + \cdots + y_n b_n)\big\}\, dy_1 \cdots dy_n\,.$$

If we change t into st^2 in the above identity and apply (8.10), we deduce that

$$e^{-tA}(x) = \int h(s)\, b(y_1)\cdots b(y_n) \exp\big\{it\, s^{\frac{1}{2}}(y_1 a_1 + \cdots + y_n a_n)\big\}\, x \\ \times \exp\big\{-it\, s^{\frac{1}{2}}(y_1 b_1 + \cdots + y_n b_n)\big\}\, ds\, dy_1 \cdots dy_n\,,$$

the latter integral being taken on $\mathbb{R}_+ \times \mathbb{R}^n$.

Thus e^{-tA} is an average of c_0-groups of isometries on $L^p(\mathcal{M})$. More precisely, for any $(s, y_1, \ldots, y_n)$ in the set $\mathbb{R}_+ \times \mathbb{R}^n$, let $B\{s, y_1, \ldots, y_n\}$ denote the operator $-i\mathcal{A}d_{(a,b)}$, where $a = s^{\frac{1}{2}}(y_1 a_1 + \cdots + y_n a_n)$ and $b = s^{\frac{1}{2}}(y_1 b_1 + \cdots + y_n b_n)$. With this notation, we have

$$\exp\{-tB\{s, y_1, \ldots, y_n\}\}(x) = \exp\big\{it\, s^{\frac{1}{2}}(y_1 a_1 + \cdots + y_n a_n)\big\}\, x \\ \times \exp\big\{-it\, s^{\frac{1}{2}}(y_1 b_1 + \cdots + y_n b_n)\big\}$$

for any $x \in L^p(\mathcal{M})$. Hence we actually have

$$e^{-tA} = \int h(s)\, b(y_1)\cdots b(y_n)\, \exp\{-tB\{s, y_1, \ldots, y_n\}\}\, ds\, dy_1 \cdots dy_n$$

in the strong sense.

We can now conclude by repeating the argument in the proof of Proposition 3.12 (it is actually possible to apply this proposition directly). Indeed by the Laplace formula we deduce from above that for any complex number z with $\mathrm{Re}(z) < 0$, we have

$$R(z, A) = \int h(s)\, b(y_1)\cdots b(y_n)\, R\big(z, B\{s, y_1, \ldots, y_n\}\big)\, ds\, dy_1 \cdots dy_n\,. \tag{8.13}$$

Then applying (3.5), we deduce that for any $\theta > \frac{\pi}{2}$ and any $f \in H_0^\infty(\Sigma_\theta)$, we have

$$f(A) = \int h(s)\, b(y_1)\cdots b(y_n)\, f\big(B\{s, y_1, \ldots, y_n\}\big)\, ds\, dy_1 \cdots dy_n\,.$$

According to Proposition 3.11 for $X = L^p(\mathcal{M})$, we have an estimate

$$\big\| f\big(B\{s, y_1, \ldots, y_n\}\big)\big\| \leq K_{\theta,p}\, \|f\|_{\infty,\theta},$$

for some uniform constant $K_{\theta,p}$ only depending on θ and p. Since h and b are nonnegative and have integrals equal to one, we deduce that for any $f\in H_0^\infty(\Sigma_\theta)$,

$$\begin{aligned}\|f(A)\| &\leq \int h(s)\,b(y_1)\cdots b(y_n)\,\big\|f\big(B\{s,y_1,\dots,y_n\}\big)\big\|\,ds\,dy_1\cdots dy_n\\ &\leq \int h(s)\,b(y_1)\cdots b(y_n)\,K_{\theta,p}\,\|f\|_{\infty,\theta}\,ds\,dy_1\cdots dy_n \;=\; K_{\theta,p}\,\|f\|_{\infty,\theta}.\end{aligned}$$ □

Remark 8.15. — Arguing as in the proof of Theorem 8.12, we obtain a completely bounded version of Theorem 8.14. Namely there is a constant $K_{\theta,p}$ such that if a_j, b_j and A are in this theorem, then $\|f(A)\|_{cb}\leq K_{\theta,p}\|f\|_{\infty,\theta}$ for any $f\in H_0^\infty(\Sigma_\theta)$.

Remark 8.16

(1) Let $A_1,\dots,A_n,C$, and A as above. Since $C=A^2$, it follows from Theorem 8.14 that for any $\theta>2\omega_p$ we have

$$\|f(C)\|\;\leq\;K_{\theta,p}\,\|f\|_{\infty,\theta},\qquad f\in H_0^\infty(\Sigma_\theta). \tag{8.14}$$

(2) Assume that $a_j=b_j$ for any $1\leq j\leq n$. In that case, the c_0-semigroup $(T_t)_{t\geq 0}$ generated by $-C$ is completely positive (see Remark 8.13 (2)). Hence C is Rad-sectoriel of Rad-type ω_p by Theorem 5.6. Therefore combining (8.14) and [**41**, Proposition 5.1], we deduce that for any $\theta>\omega_p$, there is a constant $K'_{\theta,p}$ only depending on p and θ such that $\|f(C)\|\;\leq\;K'_{\theta,p}\,\|f\|_{\infty,\theta}$ for any $f\in H_0^\infty(\Sigma_\theta)$. In turn this implies that

$$\|f(A)\|\;\leq\;K'_{\theta,p}\,\|f\|_{\infty,\theta},\qquad \theta>\frac{\omega_p}{2},\ f\in H_0^\infty(\Sigma_\theta).$$

8.C. Schur multipliers on S^p

Let $1\leq p\leq\infty$. As usual, we will regard the Schatten space S^p as a space of scalar valued infinite matrices, and we let E_{ij} denote the standard matrix units, for $i,j\geq 1$. Let $[a_{ij}]_{i,j\geq 1}$ be an infinite matrix of complex numbers. By definition, the Schur multiplier on S^p associated with this matrix is the linear operator A whose domain is the space of all $x=[x_{ij}]\in S^p$ such that $[a_{ij}x_{ij}]$ belongs to S^p, and whose action is given by

$$A(x)=\big[a_{ij}x_{ij}\big]_{i,j\geq 1},\qquad x=[x_{ij}]_{i,j\geq 1}\in D(A).$$

Each E_{ij} belongs to $D(A)$, hence A is densely defined. It is also easy to check that A is closed. Moreover the kernel of A is equal to

$$N(A)\;=\;\overline{\mathrm{Span}}\{E_{ij}\;:\;a_{ij}=0\}.$$

In particular, A is one-one if $a_{ij}\neq 0$ for any $i,j\geq 1$.

Let $z\in\mathbb{C}$ be a complex number. Clearly $z\in\rho(A)$ if and only if $a_{i,j}\neq z$ for any $i,j\geq 1$ and if the Schur multiplier associated with the matrix $[(z-a_{ij})^{-1}]$ is bounded. In that case, $R(z,A)$ coincides with that Schur multiplier.

For any $\theta \in (0,\pi)$ and any $f \in H^\infty(\Sigma_\theta)$, it will be convenient to let $\overset{\circ}{f} : \Sigma_\theta \cup \{0\} \to \mathbb{C}$ denote the prolongation of f obtained by letting $\overset{\circ}{f}(0) = 0$.

Using (3.5), we deduce from above that if A is sectorial of type $\omega \in (0,\pi)$, then $a_{ij} \in \overline{\Sigma_\omega}$ for any $i,j \geq 1$, and $f(A)$ is the Schur multiplier associated with the matrix $[\overset{\circ}{f}(a_{ij})]$ for any $f \in H_0^\infty(\Sigma_\theta)$ and any $\theta \in (\omega,\pi)$.

Furthermore, $-A$ generates a bounded c_0-semigroup $(T_t)_{t\geq 0}$ on S^p if and only if the Schur multipliers associated to the matrix $\left[e^{-ta_{ij}}\right]$ are uniformly bounded. In that case, T_t is indeed the Schur multiplier associated to the latter matrix.

The main result of this paragraph is the following.

Proposition 8.17. — *Let H be a real Hilbert space, and let $(\alpha_k)_{k\geq 1}$ and $(\beta_k)_{k\geq 1}$ be two sequences of H. In the next statements, $\|\cdot\|$ denotes the norm on H.*

1. *For any $1 < p < \infty$, the Schur multiplier on S^p associated with $\left[\|\alpha_i - \beta_j\|\right]$ is cb-sectorial of type $\omega_p = \pi|\frac{1}{2} - \frac{1}{p}|$ and admits a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \omega_p$.*
2. *For any $1 < p < \infty$, for any $\theta > \omega_p$, and for any $f \in H^\infty(\Sigma_\theta)$, the Schur multiplier associated with $\left[\overset{\circ}{f}\left(\|\alpha_i - \beta_j\|\right)\right]$ is completely bounded on S^p.*
3. *For any $t \geq 0$, the Schur product T_t associated with $\left[e^{-t(\|\alpha_i - \beta_j\|)}\right]$ is completely contractive on S^p for any $1 \leq p \leq \infty$, and $(T_t)_{t\geq 0}$ is a diffusion semigroup on $B(\ell^2)$.*

We will need the following approximation lemma. Its proof is elementary, using the facts given before Proposition 8.17. We leave it as an exercice for the reader.

Lemma 8.18. — *Let $1 \leq p \leq \infty$ and let $\omega \in (0,\pi)$ be an angle. For any $i,j \geq 1$, let $(a_{ij}^n)_{n\geq 1}$ be a sequence of $\overline{\Sigma_\omega}$, which admits a limit a_{ij} when $n \to \infty$. Let B_n (resp. A) be the Schur multiplier on S^p associated with $[a_{ij}^n]$ (resp. with $[a_{ij}]$).*

1. *Assume that $\sigma(B_n) \subset \overline{\Sigma_\omega}$ for any $n \geq 1$ and assume that for any $\theta > \omega$, there is a constant $K_\theta > 0$ such that $\|zR(z,B_n)\| \leq K_\theta$ for any $z \in \mathbb{C} \setminus \overline{\Sigma_\theta}$ and any $n \geq 1$. Then A is sectorial of type ω.*
2. *Assume further that for some $\theta > \omega$, there is a constant $K > 0$ such that $\|f(B_n)\| \leq K\|f\|_{\infty,\theta}$ for any $f \in H_0^\infty(\Sigma_\theta)$ and any $n \geq 1$. Then A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus.*
3. *Assume that $-B_n$ generates a bounded c_0-semigroup $(T_t^n)_{t\geq 0}$ for any $n \geq 0$, and that there is a constant $C \geq 1$ such that $\|T_t^n\| \leq C$ for any $t \geq 0$ and any $n \geq 1$. Then $-A$ generates a bounded c_0-semigroup $(T_t)_{t\geq 0}$, and $\|T_t\| \leq C$ for any $t \geq 0$.*

Proof of Proposition 8.17. — We fix some $1 < p < \infty$. Throughout this proof we let $a_{ij} = \|\alpha_i - \beta_j\|$, and we let A be the Schur multiplier on S^p associated with the matrix $[a_{ij}]_{i,j\geq 1}$. Replacing H by the closed linear span of the α_i's and β_j's if necessary, we may asssume that H is separable. Let $(e_k)_{k\geq 1}$ be an orthonormal basis of H. For any $k\geq 1$, we let

$$\alpha_{ik} = \langle \alpha_i, e_k\rangle \qquad \text{and} \qquad \beta_{jk} = \langle \beta_j, e_k\rangle\,.$$

Then for any $i,j\geq 1$, we have

$$a_{ij} \,=\, \lim_n a_{ij}^n, \qquad \text{with} \qquad a_{ij}^n \,=\, \Bigl(\sum_{k=1}^n \bigl|\alpha_{ik}-\beta_{jk}\bigr|^2\Bigr)^{\frac12}.$$

All numbers α_{ik} and β_{jk} are real, hence we may define selfadjoint operators a_k and b_k on ℓ^2 with diagonal matrices equal to $\mathrm{Diag}\{\alpha_{ik}\,:\, i\geq 1\}$ and $\mathrm{Diag}\{\beta_{jk}\,:\, j\geq 1\}$ respectively. Let A_k be the Schur multiplier associated to the matrix $[\alpha_{ik}-\beta_{jk}]$. Then

$$A_k(E_{ij}) = \bigl(\alpha_{ik}-\beta_{jk}\bigr)E_{ij} = a_kE_{ij} - E_{ij}b_k$$

for any $i,j\geq 1$. Hence $A_k = \mathcal{A}d_{(a_k,b_k)}$ in the notation of paragraph 8.B. For any integer $n\geq 1$, we let

$$B_n \,=\, \bigl(A_1^2+\cdots+A_n^2\bigr)^{\frac12}.$$

Thus B_n is the Schur multiplier associated with the matrix $[a_{ij}^n]$.

We fix some $\theta>\omega_p$. Then for any $\lambda\in\mathbb{C}\setminus\overline{\Sigma_\theta}$, we let

$$f_\lambda(z) = \frac{1}{1+z} - \frac{\lambda}{\lambda-z}\,.$$

Let $\theta'=(\omega_p+\theta)/2$ and note that f_λ belongs to $H_0^\infty(\Sigma_{\theta'})$, with $\sup_\lambda \|f_\lambda\|_{\infty,\theta'}<\infty$. Hence by Theorem 8.14 (applied with θ'), there is a constant K_θ not depending either on λ or n such that $\|f_\lambda(B_n)\|\leq K_\theta$.

On the other hand, by Theorem 8.12 and Lemma 8.10, $-B_n$ generates a contraction semigroup on S^p. Hence $\|(1+B_n)^{-1}\|\leq 1$ for any $n\geq 1$, by the Laplace formula. Since $f_\lambda(B_n) = (1+B_n)^{-1} - \lambda R(\lambda,B_n)$, we deduce that for any $n\geq 1$,

$$\|\lambda R(\lambda,B_n)\|\,\leq\, K_\theta+1, \qquad \lambda\in\mathbb{C}\setminus\overline{\Sigma_\theta}.$$

By Lemma 8.18 (1), this implies that A is sectorial of type ω_p.

Likewise using Lemma 8.18 (2) and Theorem 8.14, we obtain that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta>\omega_p$. This proves the 'bounded' part of (1). To obtain the 'completely bounded' part, it suffices to apply the same argument together with Remark 8.15 and a obvious completely bounded version of Lemma 8.18.

We now prove (2). Note that A may fail to have dense range. We let $f\in H^\infty(\Sigma_\theta)$. Multiplying f by the function g_n defined by (3.8), we find a bounded sequence $(f_n)_{n\geq 1}$ in $H_0^\infty(\Sigma_\theta)$ such that $\mathring{f}_n$ converges pointwise to $\mathring{f}$ on $\Sigma_\theta\cup\{0\}$. Since A admits a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus, there is a constant $C>0$ such that $\|f_n(A)\|_{cb}\leq C\|f_n\|_{\infty,\theta}$ for any $n\geq 1$. Thus the completely bounded norms of the

Schur multipliers associated with $[\overset{\circ}{f_n}(a_{ij})]$ are uniformly bounded. Passing to the limit, we deduce the result.

To prove (3), let $t \geq 0$ be any nonnegative real number. For any $n \geq 1$, the Schur product associated with $[e^{-ta_{ij}^n}]$ is e^{-tB_n}, and the latter is a complete contraction on S^p. In fact this is a complete contraction on S^q for any $1 \leq q \leq \infty$, by arguing as in Remark 8.13 (1). Passing to the limit, and using Lemma 8.18 (3), this shows that the the Schur product associated with $[e^{-ta_{ij}}]$ is a complete contraction on S^q for any $1 \leq q \leq \infty$. By Remark 5.2, this semigroup is a diffusion semigroup. $\square$

Remark 8.19. — Proposition 8.17 (1) is no longer true for $p \in \{1, \infty\}$. Indeed consider the following example. Take two sequences $(t_k)_{k\geq 1}$ and $(s_k)_{k\geq 1}$ of positive real numbers, and for any $i, j \geq 1$, define

$$\alpha_i = \sqrt{t_i}\, e_{2i} \qquad \text{and} \qquad \beta_j = \sqrt{s_j}\, e_{2j+1}$$

on ℓ^2 equipped with its canonical basis $(e_k)_{k\geq 1}$. Then $\|\alpha_i - \beta_j\| = t_i + s_j$ for any $i, j \geq 1$. Hence the operator A to be considered is the Schur multiplier associated with the matrix $[t_i + s_j]$. It we take e.g. $s_k = t_k = 2^k$, it was proved by Uijterdijk [**74**] that the latter Schur product does not have any bounded H^∞ functional calculus on S^1.

CHAPTER 9

SEMIGROUPS ON q-DEFORMED VON NEUMANN ALGEBRAS

9.A. The case $-1 < q < 1$

This chapter is devoted to semigroups derived from second quantization on von Neumann algebras of q-deformation $\Gamma_q(H)$ in the sense of Bozejko and Speicher (see [**14, 15**]). We start with a few definitions and some background, for which we refer the reader to the two latter papers and to [**13**].

If $\mathcal{H}$ is a complex Hilbert space and $n \geq 0$ is an integer, we let $\mathcal{H}^{\otimes n}$ be the algebraic n-fold tensor product $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ and we let $\langle\ ,\ \rangle_0$ be the standard inner product on $\mathcal{H}^{\otimes n}$. By convention, $\mathcal{H}^{\otimes 0} = \mathbb{C}$. We fix some $q \in (-1, 1)$. Then one defines

$$\langle \zeta, \zeta' \rangle_q \, = \, \langle Q_q \zeta, \zeta' \rangle_0, \qquad \zeta,\ \zeta' \in \mathcal{H}^{\otimes n}, \tag{9.1}$$

where $Q_q \colon \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ is a linear operator defined as follows. Let $\mathcal{S}_n$ denote the permutation group on the integers $\{1, \ldots, n\}$ and for any $\sigma \in \mathcal{S}_n$, let $\iota(\sigma)$ denote the number of inversions of σ. Then Q_q is defined by

$$Q_q\big(h_1 \otimes \cdots \otimes h_n\big) \, = \, \sum_{\sigma \in \mathcal{S}_n} q^{\iota(\sigma)}\, h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)}, \qquad h_1, \ldots, h_n \in \mathcal{H}. \tag{9.2}$$

According to [**14**], Q_p is a positive operator on $\mathcal{H}^{\otimes n}$, and $\zeta \mapsto \langle \zeta, \zeta \rangle_q^{\frac{1}{2}}$ is a norm on $\mathcal{H}^{\otimes n}$. We let $\mathcal{H}_q^{\otimes n}$ denote the resulting completion. Then by definition, the q-Fock space over $\mathcal{H}$ is the Hilbertian direct sum

$$\mathcal{F}_q(\mathcal{H}) \, = \, \mathop{\oplus}_{n \geq 0} \mathcal{H}_q^{\otimes n}.$$

In the sequel we will use $\langle\ ,\ \rangle_q$ to denote the inner product on the whole space $\mathcal{F}_q(\mathcal{H})$. Since its restriction to each $\mathcal{H}_q^{\otimes n}$ coincides with (9.1), there should be no confusion. Accordingly, $\|\ \|_q$ will stand for the norm on $\mathcal{F}_q(\mathcal{H})$.

We let Ω be the unit element in $\mathcal{H}^{\otimes 0} = \mathbb{C}$. This is usually called the vacuum. For any $h \in \mathcal{H}$, the creation operator $c(h)$ on $\mathcal{F}_q(\mathcal{H})$ is defined by letting $c(h)\Omega = h$,

$$c(h)\big(h_1 \otimes \cdots \otimes h_n\big) \, = \, h \otimes h_1 \otimes \cdots \otimes h_n, \qquad h_1, \ldots, h_n \in \mathcal{H},$$

and then extending by linearity and continuity. Indeed,

$$c(h)\colon \mathcal{F}_q(\mathcal{H}) \longrightarrow \mathcal{F}_q(\mathcal{H})$$

is a bounded operator taking $\mathcal{H}_q^{\otimes n}$ into $\mathcal{H}_q^{\otimes(n+1)}$ for any $n \geq 0$. Next the annihilation operator $a(h)\colon \mathcal{F}_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H})$ is defined by

$$a(h) = c(h)^*, \qquad h \in \mathcal{H}.$$

Throughout the rest of this chapter, we let H be a *real* Hilbert space, and we let $H_{\mathbb{C}}$ denote its complexification. We will use the above q-Fock space, as well as creation and annihilation operators, for $\mathcal{H} = H_{\mathbb{C}}$. For any $h \in H$, we let

$$w(h) = a(h) + c(h).$$

This is a selfadjoint operator on $\mathcal{F}_q(H_{\mathbb{C}})$, called a q-Gaussian operator. By definition, the von Neumann algebra of q-deformation associated with H is

$$\Gamma_q(H) \,=\, vN\big\{w(h)\,:\, h \in H\big\} \subset B\big(\mathcal{F}_q(H_{\mathbb{C}})\big),$$

the von Neumann algebra generated by all q-Gaussian operators.

We let

$$\tau(x) = \langle x\Omega, \Omega\rangle_q, \qquad x \in \Gamma_q(H). \tag{9.3}$$

It was proved in [**15**] that Ω is a cyclic and separating vector for the von Neumann algebra $\Gamma_q(H)$, so that the mapping

$$\Delta\colon \Gamma_q(H) \longrightarrow \mathcal{F}_q(H_{\mathbb{C}}), \quad \Delta(x) = x(\Omega),$$

is one-one and has dense range. Moreover τ is a normal, faithful, normalized trace on $\Gamma_q(H)$. In this chapter, we will consider the noncommutative L^p-spaces $L^p(\Gamma_q(H))$ associated with τ. Since

$$\|x\|_2^2 = \tau(x^*x) = \|x(\Omega)\|_q^2$$

for any $x \in \Gamma_q(H)$, we see that Δ extends to a unitary isomorphism

$$L^2(\Gamma_q(H)) \,\simeq\, \mathcal{F}_q(H_{\mathbb{C}}). \tag{9.4}$$

Following [**13**], we now consider the second quantization on q-Fock spaces and von Neumann algebras of q-deformation. Let H_1, H_2 be two real Hilbert spaces, with complexifications denoted by $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Let $a\colon H_1 \to H_2$ be a contraction, and let $\tilde{a}\colon \mathcal{H}_1 \to \mathcal{H}_2$ denote its complexification. Then there is a (necessarily unique) linear contraction

$$F_q(a)\colon \mathcal{F}_q(\mathcal{H}_1) \longrightarrow \mathcal{F}_q(\mathcal{H}_2)$$

such that $F_q(a)(\Omega) = \Omega$ and for any $n \geq 1$,

$$F_q(a)\big(h_1 \otimes \cdots \otimes h_n\big) \,=\, \tilde{a}(h_1) \otimes \cdots \otimes \tilde{a}(h_n), \qquad h_1, \ldots, h_n \in \mathcal{H}_1. \tag{9.5}$$

(See [**13**, Lemma 1.4].) Moreover we have

$$F_q(a)^* = F_q(a^*) \qquad \text{and} \qquad F(aa') = F(a)F(a')$$

for any contractions a, a'. Next, there is a (necessarily unique) normal unital completely positive map

$$\Gamma_q(a)\colon \Gamma_q(H_1) \longrightarrow \Gamma_q(H_2)$$

such that $\Delta \circ \Gamma_q(a) = F_q(a) \circ \Delta$. Equivalently,

$$[\Gamma_q(a)(x)]\Omega \;=\; F_q(a)(x\Omega), \qquad x \in \Gamma_q(H_1). \tag{9.6}$$

This is established in [**13**, Section 2]. According to that paper, or using (9.6), we see that

$$\Gamma_q(a)(x) = F_q(a)xF_q(a^*)$$

for any $x \in \Gamma_q(H_1)$. Hence we deduce that

$$\big(\Gamma_q(a)(x)\big)^* \;=\; \Gamma_q(a)(x^*), \qquad x \in \Gamma_q(H_1). \tag{9.7}$$

Lemma 9.1. — *For any contraction $a\colon H_1 \to H_2$, and any $1 \leq p < \infty$, the operator $\Gamma_q(a)$ (uniquely) extends to a complete contraction from $L^p(\Gamma_q(H_1))$ into $L^p(\Gamma_q(H_2))$.*

Proof. — The proof is similar to the one at the beginning of Chapter 5. Let $x \in \Gamma_q(H_1)$ and $y \in \Gamma_q(H_2)$. Using (9.3), (9.6), and (9.7), we have

$$\begin{aligned}\tau\big(y\,\Gamma_q(a)(x)\big) \;&=\; \langle y[\Gamma_q(a)(x)]\Omega, \Omega\rangle_q \;=\; \langle yF_q(a)(x\Omega), \Omega\rangle_q\\ &=\; \langle x\Omega, F_q(a^*)(y^*\Omega)\rangle_q \;=\; \langle x\Omega, [\Gamma_q(a^*)(y^*)]\Omega\rangle_q\\ &=\; \tau\big(\Gamma_q(a^*)(y)\,x\big).\end{aligned}$$

We deduce that

$$\big|\tau\big(y\,\Gamma_q(a)(x)\big)\big| \;\leq\; \|x\|_1\,\|\Gamma_q(a^*)(y)\|_\infty \;\leq\; \|x\|_1\,\|y\|_\infty.$$

Taking the supremum over y in the unit ball of $\Gamma_q(H_2)$, we obtain that $\|\Gamma_q(a)(x)\|_1 \leq \|x\|_1$. This shows that $\Gamma_q(a)$ extends to a contraction $\Gamma_q(a)\colon L^1(\Gamma_q(H_1)) \to L^1(\Gamma_q(H_2))$. By interpolation, we deduce that $\Gamma_q(a)\colon L^p(\Gamma_q(H_1)) \to L^p(\Gamma_q(H_2))$ is a contraction for any $p \geq 1$. Arguing as in Remark 5.1, we see that $\Gamma_q(a)\colon L^p(\Gamma_q(H_1)) \to L^p(\Gamma_q(H_2))$ is actually a complete contraction. □

Remark 9.2. — Under the identification (9.4), the extension of $\Gamma_q(a)$ to L^2 coincides with $F_q(a)$. It also follows from the above proof that $\Gamma_q(a)$ is selfadoint (in the sense of (5.1)) if $a\colon H_1 \to H_2$ is selfadjoint.

We now turn to semigroups of operators obtained from second quantization. We will silently use Lemma 9.1, which allows to consider these operators as contractions on noncommutative L^p-spaces.

Lemma 9.3. — *Let $q \in (-1, 1)$. Let H be a real Hilbert space and let $(a_t)_{t\geq 0}$ be a c_0-semigroup of contractions on H. For any $t \geq 0$, let $T_t = \Gamma_q(a_t)$ be defined by second quantization on $\Gamma_q(H)$.*

(1) *For any $1 \leq p < \infty$, $(T_t)_{t\geq 0}$ is a completely contractive c_0-semigroup on $L^p(\Gamma_q(H))$.*

(2) *If further $(a_t)_{t\geq 0}$ is selfadjoint, then $(T_t)_{t\geq 0}$ is a completely positive diffusion semigroup on $\Gamma_q(H)$ (in the sense of Chapter 5).*

Proof. — For simplicity, we write L^p instead of $L^p(\Gamma_q(H))$ along this proof. It is clear that $(T_t)_{t\geq 0}$ is a semigroup of complete contractions on each L^p. Since $(a_t)_{t\geq 0}$ is strongly continuous on H, $(\tilde{a}_t)_{t\geq 0}$ is strongly continuous on $H_{\mathbb{C}}$. Hence $(F_q(a_t))_{t\geq 0}$ is strongly continuous on each $H_{\mathbb{C}}^{\otimes n}$, by (9.5). By density, it is strongly continuous on $\mathcal{F}_q(\mathcal{H}_{\mathbb{C}})$. This implies that $(T_t)_{t\geq 0}$ is point w^*-continuous on the von Neumann algebra $\Gamma_q(H)$. In turn, arguing as in Chapter 5, this implies that $(T_t)_{t\geq 0}$ is strongly continuous on L^p for any $1\leq p<\infty$. This proves (1). The assertion (2) now follows from Remark 9.2. □

Theorem 9.4. — *Let H be a real Hilbert space and let $(a_t)_{t\geq 0}$ be a c_0-semigroup of contractions on H. For any $q\in(-1,1)$ and any $t\geq 0$, we let $T_t=\Gamma_q(a_t)$. Then for any $1<p<\infty$, we let $-A_p$ denote the generator of $(T_t)_{t\geq 0}$ on $L^p(\Gamma_q(H))$.*

(1) *For any $1<p<\infty$, and any $\theta>\frac{\pi}{2}$, the operator A_p has a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus.*
(2) *If further $(a_t)_{t\geq 0}$ is selfadjoint, then for any $1<p<\infty$, and any $\theta>\pi\bigl|\frac{1}{p}-\frac{1}{2}\bigr|$, the operator A_p has a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

Proof. — Clearly part (2) of this theorem follows from Lemma 9.3 (2), Proposition 5.8, and part (1). Thus we only have to prove (1). We fix some $1<p<\infty$ and write $A=A_p$ for simplicity. According to [**71**, Theorem 8.1], there exist a (real) Hilbert space K, a linear isometry $j\colon H\to K$, and a c_0-group $(u_t)_{t\in\mathbb{R}}$ of orthogonal operators on K such that

$$a_t \,=\, j^*u_tj, \qquad t\geq 0.$$

Applying second quantization, we have $\Gamma_q(a_t)=\Gamma_q(j^*)\Gamma_q(u_t)\Gamma_q(j)$, for any $t\geq 0$. Owing to Lemma 9.1, we consider the L^p-realizations of these quantized operators, which we denote by

$$J=\Gamma_q(j)\colon L^p(\Gamma_q(H))\longrightarrow L^p(\Gamma_q(K)), \qquad Q=\Gamma_q(j^*)\colon L^p(\Gamma_q(K))\longrightarrow L^p(\Gamma_q(H)),$$

and

$$U_t=\Gamma_q(u_t)\colon L^p(\Gamma_q(K))\longrightarrow L^p(\Gamma_q(K)), \qquad t\in\mathbb{R}.$$

Then J,Q are complete contractions. By Lemma 9.3, $(U_t)_t$ is a c_0-group of complete contractions on $L^p(\Gamma_q(K))$, or equivalently, a c_0-group of complete isometries. Moreover we have the following dilation property

$$T_t \,=\, QU_tJ, \qquad t\in\mathbb{R}.$$

The result therefore follows from Proposition 3.12. □

In the case when $a_t=e^{-t}I_H$, $(T_t)_{t\geq 0}$ is the so-called q-Ornstein-Uhlenbeck semigroup (see e.g. [**9, 11**]). This is a selfadjoint semigroup, and hence it satisfies the conclusion of Theorem 9.4 (2).

9.B. Clifford algebras

We now present an analogue of Theorem 9.4 on Clifford algebras. These algebras correspond to the ones considered in the previous paragraph for $q = -1$, up to some modifications due to the fact that the operator Q_q defined by (9.2) has a non trivial kernel if $q = -1$. Instead of formally using 9.A, we will consider the (equivalent) usual definition of Clifford algebras in terms of antisymmetric products. We refer the reader to [**16, 66**] for more information.

If $\mathcal{H}$ is a complex Hilbert space, we let $\Lambda^n(\mathcal{H})$ denote the n-fold antisymmetric product of $\mathcal{H}$, equipped with the canonical inner product given by

$$\big\langle h_1 \wedge \cdots \wedge h_n\, ,\, h'_1 \wedge \cdots \wedge h'_n \big\rangle \,=\, \det\big[\langle h_i, h'_j\rangle\big], \qquad h_i,\ h'_j \in \mathcal{H}.$$

By convention, $\Lambda^0(\mathcal{H}) = \mathbb{C}$. We let Ω be the unit element of $\Lambda^0(\mathcal{H})$. Then the antisymmetric Fock space over $\mathcal{H}$ is the Hilbertian direct sum

$$\Lambda(\mathcal{H}) \,=\, \mathop{\oplus}_{n\geq 0} \Lambda^n(\mathcal{H}).$$

For any $h \in \mathcal{H}$, the creation operator $c(h)$ on $\Lambda(\mathcal{H})$ is defined by letting $c(h)\Omega = h$,

$$c(h)\big(h_1 \wedge \cdots \wedge h_n\big) \,=\, h \wedge h_1 \wedge \cdots \wedge h_n, \qquad h_1, \ldots, h_n \in \mathcal{H},$$

and then extending by linearity and continuity. Its adjoint $c(h)^*$ is the annihilation operator, denoted by $a(h)$.

Next we consider a real Hilbert space H, we use the above construction on $\mathcal{H} = H_{\mathbb{C}}$, and we let $w(h) = a(h) + c(h)$ for any $h \in H$. These operators are called Fermions. The von Neumann Clifford algebra associated with H is

$$\mathcal{C}(H) \,=\, vN\big\{w(h)\,:\, h \in H\big\} \subset B\big(\Lambda(H_{\mathbb{C}})\big).$$

We equip it with the normal faithful normalized trace τ defined by $\tau(x) = \langle x\Omega, \Omega\rangle$, and we consider the associated noncommutative L^p-spaces $L^p(\mathcal{C}(H))$.

In the analogy with paragraph 9.A, we can think of $\Lambda(\mathcal{H})$ and $\mathcal{C}(H)$ as being equal to $\mathcal{F}_{-1}(\mathcal{H})$ and $\Gamma_{-1}(H)$ respectively. Then second quantization on these spaces can be defined as in 9.A. Namely if $a\colon H_1 \to H_2$ is a contraction between real Hilbert spaces and if $\tilde{a}$ denotes its complexification, the operator $F_{-1}(a)\colon \Lambda(\mathcal{H}_1) \longrightarrow \Lambda(\mathcal{H}_2)$ is the (necessarily unique) linear contraction defined by $F_{-1}(a)(\Omega) = \Omega$ and for any $n \geq 1$,

$$F_{-1}(a)\big(h_1 \wedge \cdots \wedge h_n\big) \,=\, \tilde{a}(h_1) \wedge \cdots \wedge \tilde{a}(h_n), \qquad h_1, \ldots, h_n \in \mathcal{H}_1. \tag{9.8}$$

Next, $\Gamma_{-1}(a)\colon \mathcal{C}(H_1) \longrightarrow \mathcal{C}(H_2)$ is the (necessarily unique) normal unital completely positive map such that

$$[\Gamma_{-1}(a)(x)]\Omega \,=\, F_{-1}(a)(x\Omega), \qquad x \in \mathcal{C}(H_1). \tag{9.9}$$

It is easy to see that Lemmas 9.1 and 9.3, as well as Remark 9.2 extend to the case $q = -1$. Likewise, Theorem 9.4 extends to that case with the same proof and we obtain the following statement.

Theorem 9.5. — *Let H be a real Hilbert space and let $(a_t)_{t\geq 0}$ be a c_0-semigroup of contractions on H. For any $t\geq 0$, we let $T_t=\Gamma_{-1}(a_t)$. Then for any $1<p<\infty$, we let $-A_p$ denote the generator of $(T_t)_{t\geq 0}$ on $L^p(\mathcal{C}(H))$.*

(1) *For any $1<p<\infty$, and any $\theta>\frac{\pi}{2}$, the operator A_p has a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus.*
(2) *If further $(a_t)_{t\geq 0}$ is selfadjoint, then for any $1<p<\infty$, and any $\theta>\pi\left|\frac{1}{p}-\frac{1}{2}\right|$, the operator A_p has a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

Assume now that H is infinite dimensional, and let $(e_i)_{i\geq 1}$ be an orthonormal family. We let $W_i=w(e_i)$ for any $i\geq 1$. It is well-known that these operators form a 'spin system'. Namely they are hermitian unitaries on $\Lambda(\mathcal{H})$ and

$$W_iW_j=-W_jW_i, \qquad \text{if } i\neq j.$$

We let $\mathcal{I}$ be the set of all increasing finite sequences $\{i_1<i_2<\cdots<i_m\}$ of positive integers. If F is such a sequence we let

$$V_F=W_{i_1}\cdots W_{i_m}.$$

By convention, the empty set belongs to $\mathcal{I}$, and we let $V_\varnothing=1$. Also we write $|F|$ for the cardinal of $F\in\mathcal{I}$. Since the W_i's form a spin system, the $*$-algebra they generate is equal to

$$\mathcal{P}\,=\,\mathrm{Span}\{\mathrm{V_F}\,:\,\mathrm{F}\in\mathcal{I}\}.$$

Thus $\mathcal{P}$ is w^*-dense in $\mathcal{C}(H)$, and it is dense in $L^p(\mathcal{C}(H))$ for any $1\leq p<\infty$.

It is easy to see that for any $F=\{i_1<i_2<\cdots<i_m\}$, we have

$$V_F\Omega\,=\,e_{i_1}\wedge\cdots\wedge e_{i_m}.$$

Hence the V_F's form an orthonormal basis of $L^2(\mathcal{C}(H))$.

We now focus on the completely positive noncommutative diffusion semigroup on $\mathcal{C}(H)$ defined by

$$T_t=\Gamma_{-1}(e^{-t}I_H), \qquad t\geq 0.$$

This is the Fermionic Ornstein-Uhlenbeck semigroup. According to the above discussion, we have

$$T_t(V_F)\,=\,e^{-t|F|}V_F, \qquad t\geq 0,\ F\in\mathcal{I}.$$

The operator $\mathcal{A}\colon\mathcal{P}\to\mathcal{P}$ defined by

$$\mathcal{A}(V_F)\,=\,|F|V_F, \qquad F\in\mathcal{I}, \tag{9.10}$$

is called the *number operator*. It follows from above that for any $1<p<\infty$, the negative generator A_p of $(T_t)_{t\geq 0}$ on $L^p(\mathcal{C}(H))$ is an extension of $\mathcal{A}$. Equivalently we can regard A_p as an L^p-realization of the number operator.

For convenience, we introduce $\overset{\circ}{\mathcal{I}}=\mathcal{I}\setminus\{\varnothing\}$.

Corollary 9.6. — *Let $1 < p < \infty$ and let $\theta > \pi\left|\frac{1}{p} - \frac{1}{2}\right|$ be an angle. Then for any function $f \in H^\infty(\Sigma_\theta)$ and for any finitely supported family of complex numbers $\{\alpha_F : F \in \mathring{\mathcal{I}}\}$, we have*

$$\Big\|\sum_F \alpha_F\, f(|F|)\, V_F\Big\|_p \leq K\|f\|_{\infty,\theta} \Big\|\sum_F \alpha_F\, V_F\Big\|_p, \tag{9.11}$$

where $K > 0$ is a constant not depending on f.

Proof. — Let $A = A_p$ be the negative generator of the Fermionic Ornstein-Uhlenbeck semigroup on $L^p(\mathcal{C}(H))$, and let $f \in H^\infty(\Sigma_\theta)$. We let

$$L^p(\mathring{\mathcal{C}}(H)) = \overline{\mathrm{Span}}\{V_F : F \in \mathring{\mathcal{I}}\}.$$

According to the above discussion and (9.10), we have $L^p(\mathring{\mathcal{C}}(H)) = \overline{R(A)}$. We let $\mathring{A}$ denote the restriction of A to that space. By Theorem 9.5 (2), A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus. Hence by Theorem 3.3 and Remark 3.4, we may define a bounded operator

$$f(\mathring{A})\colon\ L^p(\mathring{\mathcal{C}}(H)) \longrightarrow L^p(\mathring{\mathcal{C}}(H)),$$

and $\|f(\mathring{A})\| \leq K\|f\|_{\infty,\theta}$ for some constant K not depending on f. Now using (9.10), we see that $f(\mathring{A})$ takes V_F to $f(|F|)V_F$ for any $F \in \mathring{\mathcal{I}}$. Thus $f(\mathring{A})$ takes $\big(\sum_F \alpha_F\, V_F\big)$ to $\big(\sum_F \alpha_F\, f(|F|)\, V_F\big)$, which yields (9.11). □

The latter corollary can be regarded as a result on 'noncommutative Fourier multipliers' associated with a spin system. Indeed, Corollary 9.6 says that the family

$$\{f(|F|) : F \in \mathring{\mathcal{I}}\}$$

is a bounded multiplier on $L^p(\mathcal{C}(H))$ with respect to the basis $\{V_F : F \in \mathcal{I}\}$. In fact, $A = A_p$ has a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus on $L^p(\mathcal{C}(H))$. Hence $\{f(|F|) : F \in \mathring{\mathcal{I}}\}$ is a completely bounded multiplier. Namely, (9.11) remains true if $(\alpha_F)_F$ is a family lying in S^p, and multiplication is replaced by tensor products.

We noticed that results in this paragraph correspond to those in paragraph 9.A in the case $q = -1$. We can do the same in the case $q = 1$. However the results we may obtain in this case are not new. Indeed for a real Hilbert space H, the von Neumann algebra $\Gamma_1(H)$ is commutative, hence Cowling's Theorem (see Remark 5.9) apply to semigroups on $\Gamma_1(H)$ obtained from second quantization.

CHAPTER 10

A NONCOMMUTATIVE POISSON SEMIGROUP

10.A. Definitions

Let $n \geq 1$ be an integer, and let $G = \mathbb{F}_n$ be a free group with n generators denoted by $c_1, \ldots, c_n$. We let e be the unit element of G, and we let $(\delta_g)_{g \in G}$ denote the canonical basis of ℓ^2_G. Then we let $\lambda\colon G \to B(\ell^2_G)$ be the left regular representation of G, defined by

$$\lambda(g)\delta_h = \delta_{gh}, \qquad g,\, h \in G.$$

We recall that the group von Neumann algebra $VN(G) \subset B(\ell^2_G)$ is defined as the von Neumann algebra on ℓ^2_G generated by the $*$-algebra

$$\mathcal{P} \,=\, \mathrm{Span}\bigl\{\lambda(g)\,:\, g \in G\bigr\}.$$

We let τ be the normalized trace on $VN(G)$ defined by $\tau(x) = \langle x(\delta_e), \delta_e\rangle$ for any $x \in VN(G)$. We will consider the noncommutative L^p-spaces $L^p(VN(G))$ associated with this trace. For any $1 \leq p < \infty$, $\mathcal{P} \subset L^p(VN(G))$ is a dense subspace. Moreover for any finitely supported family $(\alpha_g)_g$ of complex numbers, we have

$$\Bigl\Vert \sum_g \alpha_g \lambda(g)\Bigr\Vert_2 = \Bigl(\sum_g |\alpha_g|^2\Bigr)^{\frac{1}{2}}.$$

Thus we have $L^2(VN(G)) = \ell^2_G$.

Since G is a free group, any $g \in G$ has a unique decomposition of the form

$$g = c_{i_1}^{k_1} c_{i_2}^{k_2} \cdots c_{i_p}^{k_p}, \tag{10.1}$$

where $p \geq 0$ is an integer, each i_j belongs to $\{1, \ldots, n\}$, each k_j is a non zero integer, and $i_j \neq i_{j+1}$ if $1 \leq j \leq p-1$. The case when $p = 0$ corresponds to the unit element $g = e$. By definition, the length of g is defined as

$$|g| \,=\, |k_1| + \cdots + |k_p|.$$

This is the number of factors in the above decomposition of g.

For any nonnegative real number $t \geq 0$, we let $T_t\colon \mathcal{P} \to \mathcal{P}$ be the linear mapping defined by letting

$$T_t\big(\lambda(g)\big) = e^{-t|g|}\,\lambda(g), \qquad g \in G. \tag{10.2}$$

It is proved in [**30**] that this operator uniquely extends to a normal unital completely positive map $T_t\colon VN(G) \to VN(G)$. It is easy to check that each T_t is selfadjoint (in the sense of (5.1)), and that $T_t(x) \to x$ as $t \to 0^+$ in the w^*-topology of $VN(G)$, for any $x \in VN(G)$. Thus $(T_t)_{t\geq 0}$ is a completely positive diffusion semigroup in the sense of Chapter 5 (see Remark 5.1).

Let $\mathbb{T}$ be the unit circle. If $n = 1$, then $G = \mathbb{Z}$, and $(T_t)_{t\geq 0}$ is the classical Poisson semigroup on $L^\infty(\mathbb{T})$.

Definition 10.1. — *The diffusion semigroup $(T_t)_{t\geq 0}$ on $VN(\mathbb{F}_n)$ defined by (10.2) is called the noncommutative Poisson semigroup.*

Following the notation in Chapter 5, we let $-A_p$ denote the infinitesimal generator of $(T_t)_{t\geq 0}$ on $L^p(VN(G))$ for any $1 < p < \infty$. It is clear from (10.2) that $\mathcal{P}$ is included in the domain of A_p, and that

$$A_p(\lambda(g)) \,=\, |g|\,\lambda(g), \qquad g \in G.$$

Our main objective is Theorem 10.12 below, which says that A_p has a (completely) bounded $H^\infty(\Sigma_\theta)$ functional calculus on $L^p(VN(G)))$ for any $\theta > \omega_p = \pi|p^{-1} - 2^{-1}|$. The proof will require several steps of independent interest. First we will show that each T_t can be 'dilated by a martingale', see Proposition 10.5. Then in the next paragraph, we will establish square function estimates for noncommutative martingales, which generalize well-known commutative results. In the final part of this chapter, we will combine these results to obtain square function estimates for the semigroup $(T_t)_{t\geq 0}$, and Theorem 10.12 will be deduced from these estimates. This scheme owes a lot to Stein's proof of square function estimates for commutative diffusion semigroups (see [**70**, Chapter IV]).

10.B. Dilation by martingales

If $\mathcal{M}$ and $\mathcal{M}'$ are two von Neumann algebras equipped with normalized normal faithful traces τ and τ', we say that an operator $T\colon \mathcal{M} \to \mathcal{M}'$ preserves traces (or is trace preserving) if $\tau' \circ T = \tau$ on $\mathcal{M}$.

If $\pi\colon (\mathcal{M}, \tau) \to (\mathcal{M}', \tau')$ is a normal unital faithful trace preserving $*$-representation, then it (uniquely) extends to an isometry from $L^p(\mathcal{M})$ into $L^p(\mathcal{M}')$ for any $1 \leq p < \infty$. In fact, these isometries are complete. We call the adjoint $Q\colon \mathcal{M}' \to \mathcal{M}$ of the embedding $L^1(\mathcal{M}) \hookrightarrow L^1(\mathcal{M}')$ induced by π the conditional expectation associated with π. This map is also trace preserving and extends to a complete contraction

$L^p(\mathcal{M}') \to L^p(\mathcal{M})$ for any $1 \leq p \leq \infty$. Moreover $Q\colon \mathcal{M}' \to \mathcal{M}$ is unital and completely positive.

In fact if $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{M}'$, and π is the canonical embedding, then Q is a conditional expectation in the usual sense. In this case, Q is actually the unique trace preserving conditional expectation $\mathcal{M}' \to \mathcal{M}$ and we call it the canonical conditional expectation onto $\mathcal{M}$.

Definition 10.2. — *Let $\mathcal{M}$ be a von Neumann algebra equipped with a normalized trace τ, and let $T\colon \mathcal{M} \to \mathcal{M}$ be a bounded operator. We say that T satisfies* Rota's dilation property *if there exist a von Neumann algebra $\mathcal{N}$ equipped with a normalized trace, a normal unital faithful $*$-representation $\pi\colon \mathcal{M} \to \mathcal{N}$ which preserves traces, and a decreasing sequence $(\mathcal{N}_m)_{m\geq 1}$ of von Neumann subalgebras of $\mathcal{N}$ such that*

$$T^m = Q \circ \mathcal{E}_m \circ \pi, \qquad m \geq 1, \tag{10.3}$$

where $\mathcal{E}_m\colon \mathcal{N} \to \mathcal{N}_m \subset \mathcal{N}$ is the canonical conditional expectation onto $\mathcal{N}_m$, and where $Q\colon \mathcal{N} \to \mathcal{M}$ is the conditional expectation associated with π.

Remark 10.3

(1) Assume that $T\colon \mathcal{M} \to \mathcal{M}$ satisfies Rota's dilation property. Then T is normal, unital, completely positive, and selfajoint. Indeed let σ be the trace on $\mathcal{N}$, then for any $x, y \in \mathcal{M}$ we have

$$\begin{aligned} \tau(T(x)y) &= \tau(Q\mathcal{E}_1\pi(x)y) \\ &= \sigma(\mathcal{E}_1\pi(x)\pi(y)) \\ &= \sigma(\pi(x)\mathcal{E}_1\pi(y)) \\ &= \tau(xQ\mathcal{E}_1\pi(y)) = \tau(xT(y)). \end{aligned}$$

Since T is positive, it therefore satisfies (5.1).

Thus in the sequel, we will mostly restrict our attention to operators T which are normal, unital, completely positive, and selfadjoint. Note that such an operator is necessarily trace preserving. Indeed,

$$\tau(T(x)) = \tau(T(x)1) = \tau(xT(1)) = \tau(x1) = \tau(x)$$

for any $x \in \mathcal{M}$.

(2) The above property is named after Rota's Theorem which asserts that if $\mathcal{M}$ is commutative, and if $T\colon \mathcal{M} \to \mathcal{M}$ is a normal unital positive selfadjoint operator, then T^2 satisfies Rota's dilation property (see e.g. [**70**, IV. 9]).

(3) Let $T\colon \mathcal{M} \to \mathcal{M}$ be a normal unital completely positive selfadjoint operator satisfying Definition 10.2. We noticed that $\pi, \mathcal{E}_m$ and Q all extend to associated L^p-spaces. In the sequel we will keep the same notation for these extensions. Then it is clear that (10.3) holds as well on $L^p(\mathcal{M})$ for any $1 \leq p < \infty$.

If $(\mathcal{M}_1, \tau_1), \ldots, (\mathcal{M}_n, \tau_n)$ is a finite family of von Neumann algebras equipped with distinguished normalized traces, we let

$$(\mathcal{M}, \tau) \;=\; \mathop{\bar{*}}_{1\leq i\leq n} (\mathcal{M}_i, \tau_i)$$

denote their reduced free product von Neumann algebra (in the sense of [**75, 76**]). On the other hand, we let $\mathop{*}\limits_{1\leq i\leq n} \mathcal{M}_i$ for the unital algebra free product of the $\mathcal{M}_i$'s, which is a w^*-dense $*$-subalgebra of $\mathcal{M}$. Then for any $1 \leq i \leq n$, we let $\overset{\circ}{\mathcal{M}}_i = \operatorname{Ker}(\tau_i) \subset \mathcal{M}_i$ denote the kernel of τ_i. Now suppose that we have a second family $(\mathcal{M}'_1, \tau'_1), \ldots, (\mathcal{M}'_n, \tau'_n)$ of von Neumann algebras with distinguished normalized traces, with reduced free product von Neumann algebra denoted by $(\mathcal{M}', \tau')$. Assume further that for each i, we have a normal unital completely positive map $T_i\colon \mathcal{M}_i \to \mathcal{M}'_i$ which preserves traces. According to [**10**, Theorem 3.8], there is a unique normal unital completely positive map $T\colon \mathcal{M} \to \mathcal{M}'$ such that

$$T(x_1 \cdots x_p) = T_{i_1}(x_1) \cdots T_{i_p}(x_p)$$

whenever $p \geq 1$ is an integer, $i_j \neq i_{j+1}$ for any $1 \leq j \leq p-1$, and $x_j \in \overset{\circ}{\mathcal{M}}_{i_j}$ for any $1 \leq j \leq p$. This map is called the 'free product' of the T_i's, and we will denote it by

$$T \;=\; T_1 \bar{*} \cdots \bar{*} T_n.$$

The above algebraic condition determines the free product on the algebra $\mathop{*}\limits_{1\leq i\leq n} \mathcal{M}_i$ (see [**10**] for details).

Lemma 10.4. — *For $1 \leq i \leq n$, let $T_i\colon (\mathcal{M}_i, \tau_i) \to (\mathcal{M}_i, \tau_i)$ be a normal unital completely positive map preserving traces. If each T_i satisfies Rota's dilation property, then their free product*

$$T_1 \bar{*} \cdots \bar{*} T_n \colon \mathop{\bar{*}}_{1\leq i\leq n} (\mathcal{M}_i, \tau_i) \longrightarrow \mathop{\bar{*}}_{1\leq i\leq n} (\mathcal{M}_i, \tau_i)$$

also satisfies Rota's dilation property.

Proof. — By assumption, there exist for any $i = 1, \ldots, n$ a von Neumann algebra $\mathcal{N}^i$ equipped with a normalized trace σ_i, a normal unital faithful $*$-representation $\pi_i\colon \mathcal{M}_i \to \mathcal{N}^i$ which preserves traces, and a decreasing sequence $(\mathcal{N}^i_m)_{m\geq 1}$ of von Neumann subalgebras of $\mathcal{N}^i$ such that $T_i^m = Q_i \circ \mathcal{E}^i_m \circ \pi_i$ for any integer $m \geq 1$, where $\mathcal{E}^i_m\colon \mathcal{N}^i \to \mathcal{N}^i_m \subset \mathcal{N}^i$ and $Q^i\colon \mathcal{N}^i \to \mathcal{M}_i$ are the conditional expectations given by Definition 10.2. We consider the free product

$$\pi = \pi_1 \bar{*} \cdots \bar{*} \pi_n \colon \mathop{\bar{*}}_{1\leq i\leq n} (\mathcal{M}_i, \tau_i) \longrightarrow \mathop{\bar{*}}_{1\leq i\leq n} (\mathcal{N}^i, \sigma_i).$$

According to [**10**, Theorem 3.7], the normal unital map π is a faithful trace preserving $*$-representation. Likewise for any $m \geq 1$, the product $\mathop{\bar{*}}\limits_{1\leq i\leq n} (\mathcal{N}^i_m, \sigma_i)$ can be regarded as a von Neumann subalgebra of $\underset{1\leq i\leq n}{\bar{*}}(\mathcal{N}^i,\sigma_i)$, and the sequence of these subalgebras is decreasing. We may also consider

$$\mathcal{E}_m = \mathcal{E}_m^1 \bar{*}\cdots\bar{*}\mathcal{E}_m^n \colon \underset{1\leq i\leq n}{\bar{*}}(\mathcal{N}^i,\sigma_i) \longrightarrow \underset{1\leq i\leq n}{\bar{*}}(\mathcal{N}^i_m,\sigma_i) \subset \underset{1\leq i\leq n}{\bar{*}}(\mathcal{N}^i,\sigma_i)$$

for any $m\geq 1$, and

$$Q = Q_1\bar{*}\cdots\bar{*}Q_n \colon \underset{1\leq i\leq n}{\bar{*}}(\mathcal{N}^i,\sigma_i) \longrightarrow \underset{1\leq i\leq n}{\bar{*}}(\mathcal{M}_i,\tau_i).$$

As one might expect, the mapping Q is the conditional expectation associated with π. Indeed Q is normal, unital, completely positive, and preserves traces. Hence it suffices to check that $Q\circ\pi$ is the identity mapping on $\underset{1\leq i\leq n}{\bar{*}}(\mathcal{M}_i,\tau_i)$. Since $Q_i\circ\pi_i = I$ on each $\mathcal{M}_i$, we easily see that $Q\circ\pi = I$ on the algebra free product $\underset{1\leq i\leq n}{*}\mathcal{M}_i$. Since Q is normal, this yields the result. Likewise, $\mathcal{E}_m$ is the canonical conditional expectation onto $\underset{1\leq i\leq n}{\bar{*}}(\mathcal{N}^i_m,\sigma_i)$. Thus it suffices to show that for any integer $m\geq 1$, we have

$$\big(T_{i_1}\bar{*}\cdots\bar{*}T_{i_n}\big)^m \;=\; Q\circ\mathcal{E}_m\circ\pi$$

on $\underset{1\leq i\leq n}{\bar{*}}(\mathcal{M}_i,\tau_i)$. Again it is easy to check that it holds true on $\underset{1\leq i\leq n}{*}\mathcal{M}_i$, and the result follows by normality. $\square$

We now come back to the von Neumann algebra $VN(\mathbb{F}_n)$ equipped with its standard trace τ (see paragraph 10.A). Here we assume that $n\geq 2$. We let τ_1 be the standard trace on $L^\infty(\mathbb{T})$, which is given by $\tau_1(f) = \int f(z)dm(z)$. For any integer $k\in\mathbb{Z}$, let e_k denote the function $z\mapsto z^k$ on $\mathbb{T}$. For any $r\in(0,1]$, we let $P_r\colon L^\infty(\mathbb{T})\to L^\infty(\mathbb{T})$ be the (unique) normal mapping taking e_k to $r^{|k|}e_k$ for any k. Equivalently, P_r is the convolution operator $f\mapsto p_r * f$, with p_r equal to the Poisson kernel. It is well-known that

$$(VN(\mathbb{F}_n),\tau) \;=\; \underset{1\leq i\leq n}{\bar{*}}(L^\infty(\mathbb{T}),\tau_1). \tag{10.4}$$

We note that P_r is unital, trace preserving, and positive (hence completely positive). According to our previous discussion we may therefore consider the free product (with n factors)

$$P_r\bar{*}\cdots\bar{*}P_r\colon (VN(\mathbb{F}_n),\tau)\longrightarrow (VN(\mathbb{F}_n),\tau). \tag{10.5}$$

It turns out that for any $g\in\mathbb{F}_n$,

$$(P_r\bar{*}\cdots\bar{*}P_r)(\lambda(g)) \;=\; r^{|g|}\lambda(g).$$

Indeed, let e_k^i denote the element e_k in the ith factor of $(L^\infty(\mathbb{T}),\tau_1)\bar{*}\cdots\bar{*}(L^\infty(\mathbb{T}),\tau_1)$. If $g\in G$ has a factorization (10.1), then $\lambda(g)$ corresponds to $e_{k_1}^{i_1}\cdots e_{k_p}^{i_p}$ through the identification (10.4). Each k_j is non zero, hence each $e_{k_j}^{i_j}$ belongs to the kernel of τ_1. Hence by the algebraic characterization of the free product operator, $P_r\bar{*}\cdots\bar{*}P_r$ takes $\lambda(g)$ to

$$P_r(e_{k_1}^{i_1})\cdots P_r(e_{k_p}^{i_p}) = (r^{|k_1|}\cdots r^{|k_p|})\,e_{k_1}^{i_1}\cdots e_{k_p}^{i_p} = r^{|g|}\lambda(g).$$

This shows that for any $t \geq 0$, the normal operator $T_t\colon VN(\mathbb{F}_n) \to VN(\mathbb{F}_n)$ defined by (10.2) coincides with the free product $P_{e^{-t}} \bar{*} \cdots \bar{*} P_{e^{-t}}$. Combining Rota's Theorem (see Remark 10.3 (2)) and Lemma 10.4, we deduce the following.

Proposition 10.5. — *Let $(T_t)_{t\geq 0}$ be the noncommutative Poisson semigroup on $VN(\mathbb{F}_n)$ (see Definition 10.2). For any $t \geq 0$, the operator T_t satisfies Rota's dilation property.*

10.C. Square function estimates for noncommutative martingales

Let $(\mathcal{N}, \sigma)$ be a von Neumann algebra equipped with a normalized normal faithful trace. Suppose that $(\mathcal{N}_m)_{m\geq 0}$ is an increasing sequence of von Neumann subalgebras of $\mathcal{N}$, and let $\mathcal{E}_m\colon \mathcal{N} \to \mathcal{N}_m$ be the canonical conditional expectations. A noncommutative martingale is defined as a sequence $(x_m)_{m\geq 0}$ in $L^1(\mathcal{N})$ such that $\mathcal{E}_m(x_{m+1}) = x_m$ for any $m \geq 0$. Clearly for any $x \in L^1(\mathcal{N})$, the sequence $(\mathcal{E}_m(x))_{m\geq 0}$ is a martingale.

Likewise if $(\mathcal{N}_m)_{m\geq 0}$ is a decreasing sequence, a reverse martingale is a sequence $(x_m)_{m\geq 0}$ in $L^1(\mathcal{N})$ such that $\mathcal{E}_{m+1}(x_m) = x_{m+1}$ for any $m \geq 0$. Then for any $x \in L^1(\mathcal{N})$, $(\mathcal{E}_m(x))_{m\geq 0}$ is a reverse martingale.

We refer the reader to [**64**], [**65**, Section 7] and the references therein for information on noncommutative martingales and related square functions, which play a crucial role in this topic. Proposition 10.8 below gives a square function estimate for noncommutative martingales, which generalizes an inequality due to Stein [**70**, p. 113].

We start from another noncommutative generalization of a result of Stein, due to Pisier and the third named author.

Proposition 10.6. — *([**64**]) Let $(\mathcal{E}_k)_{k\geq 0}$ be either an increasing or decreasing sequence of (canonical) conditional expectations on $\mathcal{N}$, and let $1 < p < \infty$. For any $k \geq 0$, we let*

$$I\overline{\otimes}\mathcal{E}_k\colon S^p[L^p(\mathcal{N})] \longrightarrow S^p[L^p(\mathcal{N})]$$

be the tensor extension of $\mathcal{E}_k$. Then the set $\{I\overline{\otimes}\mathcal{E}_k : k \geq 0\}$ is both Col-bounded and Row-bounded on $S^p[L^p(\mathcal{N})]$. Thus it is also Rad-bounded on $S^p[L^p(\mathcal{N})]$.

Proof. — In the case of a increasing sequence, this is essentially a restatement of [**64**, Theorem 2.3]. The proof in the decreasing case is identical. □

Lemma 10.7. — *Let $(\mathcal{E}_k)_{k\geq 0}$ be either an increasing or decreasing sequence of (canonical) conditional expectations on $\mathcal{N}$, let $1 < p < \infty$, and let $(x_j)_{j\geq 0}$ be a sequence of $L^p(\mathcal{N})$. For any integer $k \geq 2$, set $y_k = x_j$ if $2^j + 1 \leq k \leq 2^{j+1}$.*

(1) *If* $(x_j)_{j\geq 0}$ *belongs to the space* $L^p(\mathcal{N};\ell^2_c)$, *then the sequence* $\big(m^{-\frac{3}{2}}\sum_{k=2}^m \mathcal{E}_k(y_k)\big)_{m\geq 2}$ *belongs to* $L^p(\mathcal{N};\ell^2_c)$, *and*

$$\Big\|\Big(m^{-\frac{3}{2}}\sum_{k=2}^{m}\mathcal{E}_k(y_k)\Big)_{m\geq 2}\Big\|_{L^p(\mathcal{N};\ell^2_c)} \leq K_p\,\|(x_j)_{j\geq 0}\|_{L^p(\mathcal{N};\ell^2_c)}, \tag{10.6}$$

where $K_p>0$ *is a constant only depending on* p *(and not on either* $\mathcal{N}$ *or the* $\mathcal{E}_m$*'s).*

Moreover the same result holds true with $L^p(\mathcal{N};\ell^2_c)$ *replaced by* $L^p(\mathcal{N};\ell^2_r)$.

(2) *If* $(x_j)_{j\geq 0}$ *belongs to* $L^p(\mathcal{N};\ell^2_{rad})$, *then the sequence* $\big(m^{-\frac{3}{2}}\sum_{k=2}^m \mathcal{E}_k(y_k)\big)_{m\geq 2}$ *belongs to* $L^p(\mathcal{N};\ell^2_{rad})$, *and*

$$\Big\|\Big(m^{-\frac{3}{2}}\sum_{k=2}^{m}\mathcal{E}_k(y_k)\Big)_{m\geq 2}\Big\|_{L^p(\mathcal{N};\ell^2_{rad})} \leq K_p\,\|(x_j)_{j\geq 0}\|_{L^p(\mathcal{N};\ell^2_{rad})},$$

where $K_p>0$ *is a constant only depending on* p.

Proof. — According to Corollary 2.12, we may assume that $(x_j)_{j\geq 0}$ is a finite sequence. We define

$$z_{km}=\frac{1}{m}\,\mathcal{E}_k(y_k),\qquad m\geq k\geq 2.$$

Let $(e_k)_{k\geq 0}$ be the canonical basis of ℓ^2, and let E_{mn} be the standard matrix units on S^p. Using Remark 2.3 (3) twice and Proposition 10.6, we have an estimate

$$\begin{aligned}
\|(z_{km})_{m\geq k}\|_{L^p(\mathcal{N};(\ell^2\otimes\ell^2)_c)} &= \Big\|\sum_{m\geq k\geq 2}\frac{1}{m}\,e_k\otimes e_m\otimes\mathcal{E}_k(y_k)\Big\|_{L^p(\mathcal{N};(\ell^2\otimes\ell^2)_c)}\\
&= \Big\|\sum_{k\geq 2}e_k\otimes I\overline{\otimes}\mathcal{E}_k\Big(\sum_{m\geq k}\frac{1}{m}E_{m1}\otimes y_k\Big)\Big\|_{L^p(\mathcal{N}\overline{\otimes}B(\ell^2);\ell^2_c)}\\
&\leq C_p\Big\|\sum_{k\geq 2}e_k\otimes\Big(\sum_{m\geq k}\frac{1}{m}E_{m1}\otimes y_k\Big)\Big\|_{L^p(\mathcal{N}\overline{\otimes}B(\ell^2);\ell^2_c)}\\
&\leq C_p\Big\|\sum_{m\geq k\geq 2}\frac{1}{m}\,e_k\otimes e_m\otimes y_k\Big\|_{L^p(\mathcal{N};(\ell^2\otimes\ell^2)_c)}.
\end{aligned}$$

Next we have

$$\begin{aligned}
\Big\|\sum_{m\geq k\geq 2}\frac{1}{m}\,e_k\otimes e_m\otimes y_k\Big\|_{L^p(\mathcal{N};(\ell^2\otimes\ell^2)_c)} &= \Big\|\Big(\sum_{m\geq k\geq 2}\frac{1}{m^2}\,y_k^*y_k\Big)^{\frac{1}{2}}\Big\|_p\\
&= \Big\|\Big(\sum_{j\geq 0}x_j^*x_j\,\Big(\sum_{k=2^j+1}^{2^{j+1}}\sum_{m\geq k}\frac{1}{m^2}\Big)\Big)^{\frac{1}{2}}\Big\|_p\\
&\leq \Big\|\Big(\sum_{j\geq 0}x_j^*x_j\;2^j\sum_{m\geq 2^j+1}\frac{1}{m^2}\Big)^{\frac{1}{2}}\Big\|_p\\
&\leq c\,\Big\|\Big(\sum_{j\geq 0}x_j^*x_j\Big)^{\frac{1}{2}}\Big\|_p,
\end{aligned}$$

where $c = \big(\sup_j 2^j \sum_{m\geq 2^j+1} \frac{1}{m^2}\big)^{\frac{1}{2}}$ is a universal constant. Altogether we obtain that

$$\|(z_{km})_{m\geq k}\|_{L^p(\mathcal{N};(\ell^2\otimes\ell^2)_c)} \leq K_p \,\|(x_j)_{j\geq 0}\|_{L^p(\mathcal{N};\ell^2_c)},$$

with $K_p = c\, C_p$.

Now let $S\colon \ell^2 \otimes_2 \ell^2 \to \ell^2$ be defined by

$$S\big[(\alpha_{km})_{k,m\geq 1}\big] = \Big(\frac{1}{\sqrt{m}} \sum_{k\leq m} \alpha_{km}\Big)_{m\geq 1}$$

This mapping is a well-defined contraction. Indeed,

$$\big\|S\big[(\alpha_{km})_{k,m\geq 1}\big]\big\|_2^2 = \sum_{m\geq 1} \frac{1}{m} \Big|\sum_{k=1}^{m} \alpha_{km}\Big|^2 \leq \sum_k \sum_{m\geq k} |\alpha_{km}|^2 \leq \big\|(\alpha_{km})_{k,m}\big\|_2^2.$$

Let $\widehat{S}\colon L^p(\mathcal{N};(\ell^2\otimes\ell^2)_c) \to L^p(\mathcal{N};\ell^2_c)$ be the tensor extension given by Lemma 2.4. Then $\widehat{S}$ takes $(z_{km})_{m\geq k\geq 2}$ to $\Big(m^{-\frac{3}{2}} \sum_{k=2}^{m} \mathcal{E}_k(y_k)\Big)_{m\geq 2}$. Thus (10.6) follows from the above estimate.

The row counterpart of (10.6) has the same proof, and the second part of the lemma follows from the first one. □

Proposition 10.8. — *Let $(\mathcal{E}_m)_{m\geq 0}$ be either an increasing or decreasing sequence of (canonical) conditional expectations on $\mathcal{N}$, and let $1 < p < \infty$. For any $x \in L^p(\mathcal{N})$ and any $m \geq 0$, we let*

$$\Lambda_m(x) = \frac{1}{m+1} \sum_{k=0}^{m} \mathcal{E}_k(x) \qquad \textit{and} \qquad \Delta_m(x) = \Lambda_m(x) - \Lambda_{m-1}(x).$$

Then the sequence $\big(\sqrt{m}\,\Delta_m(x)\big)_{m\geq 1}$ belongs to the space $L^p(\mathcal{N};\ell^2_{rad})$ and satisfies

$$\big\|\big(\sqrt{m}\,\Delta_m(x)\big)_{m\geq 1}\big\|_{L^p(\mathcal{N};\ell^2_{rad})} \leq K_p \,\|x\|_p,$$

where $K_p > 0$ is a constant only depending on p.

Proof. — We shall prove this result in the case of an increasing sequence $(\mathcal{E}_m)_{m\geq 0}$, the proof for the decreasing case being similar. We adapt the arguments from [**70**, pp. 113-114] to the noncommutative setting.

Let $1 < p < \infty$ and let $x \in L^p(\mathcal{M})$. We set

$$d_0(x) = \mathcal{E}_0(x) \qquad \text{and} \qquad d_k(x) = \mathcal{E}_k(x) - \mathcal{E}_{k-1}(x) \;\text{ if }\; k \geq 1.$$

Given an integer $m \geq 1$, we let $N = [\log_2(m)]$, so that $2^N \leq m \leq 2^{N+1} - 1$. We have

$$\Lambda_m(x) = \sum_{k=0}^{m} \big(1 - \tfrac{k}{m+1}\big)\, d_k(x).$$

Hence

$$\begin{aligned}\Delta_m(x) &= \frac{1}{m(m+1)}\sum_{k=1}^{m} kd_k(x)\\ &= \frac{1}{m(m+1)}\Big(d_1(x) + \sum_{j=0}^{N-1}\Big(\sum_{k=2^j+1}^{2^{j+1}} kd_k(x)\Big) + \sum_{k=2^N+1}^{m} kd_k(x)\Big).\end{aligned}$$

For any integer $j \geq 0$, we set

$$x_j = \sum_{k=2^j+1}^{2^{j+1}} d_k(x), \tag{10.7}$$

and we note that for any integers $1 \leq q < r$, we have

$$\sum_{k=q+1}^{r} kd_k(x) = r\Big(\sum_{k=q+1}^{r} d_k(x)\Big) - \Big(\sum_{k=q+1}^{r-1} d_k(x)\Big) - \Big(\sum_{k=q+1}^{r-2} d_k(x)\Big) - \cdots - d_{q+1}(x).$$

Since $\mathcal{E}_{2^{j+1}}(x_j) = x_j$, we obtain that for any $j \geq 0$, we have

$$\begin{aligned}\sum_{k=2^j+1}^{2^{j+1}} kd_k(x) &= 2^{j+1}x_j - \sum_{k=2^j+1}^{2^{j+1}-1} \mathcal{E}_k(x_j)\\ &= (2^{j+1}+1)x_j - \sum_{k=2^j+1}^{2^{j+1}} \mathcal{E}_k(x_j).\end{aligned}$$

Likewise,

$$\begin{aligned}\sum_{k=2^N+1}^{m} kd_k(x) &= m\sum_{k=2^N+1}^{m} d_k(x) - \sum_{k=2^N+1}^{m-1} \mathcal{E}_k(x_N)\\ &= (m+1)\mathcal{E}_m(x_N) - \sum_{k=2^N+1}^{m} \mathcal{E}_k(x_N).\end{aligned}$$

For any integer $k \geq 2$, we set

$$y_k = x_j \quad \text{if } \ 2^j + 1 \leq k \leq 2^{j+1}.$$

Then we have obtained that

$$\Delta_m(x) = \frac{1}{m(m+1)}\Big(d_1(x) + \sum_{j=0}^{[\log_2(m)]-1} (2^{j+1}+1)x_j + (m+1)\mathcal{E}_m\big(x_{[\log_2(m)]}\big) - \sum_{k=2}^{m} \mathcal{E}_k(y_k)\Big).$$

The four terms in the above parenthesis provide a decomposition

$$\Delta_m(x) = \Delta_m^1(x) + \Delta_m^2(x) + \Delta_m^3(x) + \Delta_m^4(x),$$

and it now suffices to give an estimate for each of the four resulting sequences $\big(\sqrt{m}\,\Delta_m^i(x)\big)_{m\geq 1}$.

Obviously we have

$$\big\|\big(\sqrt{m}\,\Delta^1_m(x)\big)_{m\geq 1}\big\|_{L^p(\mathcal{N};\ell^2_{rad})} \leq \|d_1\|_{p\to p}\Big(\sum_{m\geq 1}\frac{1}{\sqrt{m}(m+1)}\Big)\,\|x\|_p.$$

Let $\mathcal{F}_j = \mathcal{E}_{2^j}$ for any integer $j\geq 0$. Then $x_j = \mathcal{F}_{j+1}(x) - \mathcal{F}_j(x)$, by (10.7). Thus $(x_j)_{j\geq 0}$ is a sequence of martingale differences. Hence according to the noncommutative Burkholder-Gundy inequalities [**64**], the sequence $(x_j)_{j\geq 0}$ belongs to $L^p(\mathcal{N};\ell^2_{rad})$ and we have an estimate

$$\big\|(x_j)_{j\geq 0}\big\|_{L^p(\mathcal{N};\ell^2_{rad})} \leq c_p\|x\|_p, \tag{10.8}$$

where c_p is a constant only depending on p. Using Lemma 10.7 (2), we immediately deduce an estimate for the fourth term,

$$\big\|\big(\sqrt{m}\,\Delta^4_m(x)\big)_{m\geq 1}\big\|_{L^p(\mathcal{N};\ell^2_{rad})} \leq K_p\,\|x\|_p.$$

Likewise, using a slight modification of Lemma 10.7 (2), with $\mathcal{E}_k = I$, and writing

$$\sum_{j=0}^{[\log_2(m)]-1} 2^{j+1}x_j = 2\sum_{k=2}^{2^{[\log_2(m)]}} y_k\,,$$

we obtain an estimate

$$\Big\|\Big(m^{-\frac{3}{2}}\sum_{j=0}^{[\log_2(m)]-1} 2^{j+1}x_j\Big)_{m\geq 2}\Big\|_{L^p(\mathcal{N};\ell^2_{rad})} \leq K_p\,\|x\|_p.$$

In turn, this implies an estimate

$$\big\|\big(\sqrt{m}\,\Delta^2_m(x)\big)_{m\geq 1}\big\|_{L^p(\mathcal{N};\ell^2_{rad})} \leq K'_p\,\|x\|_p.$$

We now turn to the third term of the decomposition, equal to $\Delta^3_m(x) = \frac{1}{m}\mathcal{E}_m(x_{[\log_2(m)]})$. By Proposition 10.6, we have an inequality

$$\big\|\big(\sqrt{m}\,\Delta^3_m(x)\big)_{m\geq 1}\big\|_{L^p(\mathcal{N};\ell^2_{rad})} \leq C_p\,\Big\|\Big(\frac{1}{\sqrt{m}}\,x_{[\log_2(m)]}\Big)_{m\geq 1}\Big\|_{L^p(\mathcal{N};\ell^2_{rad})}.$$

Then we introduce an operator $S\colon \ell^2\to\ell^2$ which maps any sequence $(\alpha_N)_{N\geq 0}$ to the sequence $(\beta_m)_{m\geq 1}$ defined by

$$\beta_m = \frac{1}{\sqrt{m}}\,\alpha_{[\log_2(m)]},\qquad m\geq 1.$$

Indeed, we have

$$\sum_{m\geq 1}|\beta_m|^2 = \sum_{N\geq 0}\Big(\sum_{m=2^N}^{2^{N+1}-1}\frac{1}{m}\Big)\,|\alpha_N|^2 \leq \sum_{N\geq 0}|\alpha_N|^2.$$

Let $\widehat{S}\colon L^p(\mathcal{N};\ell^2_{rad})\to L^p(\mathcal{N};\ell^2_{rad})$ be the tensor extension of S given by Lemma 2.4. Using (10.8), we see that

$$\Big(\frac{1}{\sqrt{m}}\,x_{[\log_2(m)]}\Big)_{m\geq 1} = \widehat{S}\Big[\big((x_N)_{N\geq 0}\big)\Big],$$

and that

$$\Big\| \Big(\frac{1}{\sqrt{m}}\, x_{[\log_2(m)]} \Big)_{m\geq 1} \Big\|_{L^p(\mathcal{N};\ell^2_{rad})} \,\leq\, \|(x_N)_{N\geq 0}\|_{L^p(\mathcal{N};\ell^2_{rad})} \,\leq\, c_p \|x\|_p.$$

Thus we obtain the last desired estimate,

$$\big\| \big(\sqrt{m}\, \Delta^3_m(x) \big)_{m\geq 1} \big\|_{L^p(\mathcal{N};\ell^2_{rad})} \,\leq\, K''_p \|x\|_p. \quad \square$$

Corollary 10.9. — *Let $\mathcal{M}$ be a von Neumann algebra equipped with a normalized normal faithful trace, let $T\colon \mathcal{M} \to \mathcal{M}$ be a normal unital completely positive selfadjoint operator, and assume that T satisfies Rota's dilation property. For any $x \in L^1(\mathcal{M})$ and any $m \geq 0$, we let*

$$S_m(x) = \frac{1}{m+1} \sum_{k=0}^{m} T^k(x) \qquad \textit{and} \qquad D_m(x) = S_m(x) - S_{m-1}(x).$$

Then for any $1 < p < \infty$ and any $x \in L^p(\mathcal{M})$, the sequence $\big(\sqrt{m}\, D_m(x)\big)_{m\geq 1}$ belongs to the space $L^p(\mathcal{M}, \ell^2_{rad})$ and satisfies

$$\big\| \big(\sqrt{m} D_m(x)\big)_{m\geq 1} \big\|_{L^p(\mathcal{M},\ell^2_{rad})} \,\leq\, K_p \,\|x\|_p,$$

where $K_p > 0$ is a constant only depending on p.

Proof. — Let $1 < p < \infty$. Let $\mathcal{N}, \pi, \mathcal{N}_m, \mathcal{E}_m$ and Q be as in Definition 10.2, and let $\mathcal{E}_0 = I_{\mathcal{N}}$. Then it follows from Remark 10.3 (3) that

$$D_m = Q \circ \Delta_m \circ \pi \quad \text{ on } L^p(\mathcal{M}),$$

where Δ_m is defined as in Proposition 10.8. Since the two mappings $\pi\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ and $Q\colon L^p(\mathcal{N}) \to L^p(\mathcal{M})$ are (completely) contractive, the result follows at once from the latter proposition. $\square$

10.D. Functional calculus for the noncommutative Poisson semigroup

According to Proposition 10.5 and Corollary 10.9, each Poisson operator T_t on $L^p(VN(\mathbb{F}))$ satisfies a certain 'discrete square function estimate', if $1 < p < \infty$. Later on in this chapter, we will deduce a 'continuous square function estimate', in the sense of Chapter 7, for the generator of the semigroup $(T_t)_{t\geq 0}$. Passing from discrete to continuous estimates will require the following approximation lemmas. In these statements, $\mathcal{M}$ is any semifinite von Neumann algebra.

Lemma 10.10. — *Suppose that $2 \leq p < \infty$, and let $0 < \alpha < \beta < \infty$ be two positive real numbers. We let $H = L^2([\alpha,\beta]; dt)$. Then for any continuous function $v\colon [\alpha,\beta] \to L^p(\mathcal{M})$, we have*

$$\|v\|_{L^p(\mathcal{M};H_{rad})} \,=\, \lim_{\varepsilon\to 0^+} \Big\| \big(\sqrt{\varepsilon}\, v(\varepsilon m) \big)_{\frac{\alpha}{\varepsilon} \leq m \leq \frac{\beta}{\varepsilon}} \Big\|_{L^p(\mathcal{M};\ell^2_{rad})}.$$

Proof. — It follows from (2.8) and Lemma 6.1 that

$$\|v\|_{L^p(\mathcal{M};H_c)} = \Big\| \int_\alpha^\beta v(s)^* v(s)\, ds \Big\|_{\frac{p}{2}}^{\frac12}.$$

Then by Riemann's approximation Theorem, we deduce that

$$\|v\|_{L^p(\mathcal{M};H_c)} = \lim_{\varepsilon\to 0^+} \Big\| \varepsilon \sum_{\frac{\alpha}{\varepsilon}\le m\le\frac{\beta}{\varepsilon}} v(\varepsilon m)^* v(\varepsilon m) \Big\|_{\frac{p}{2}}^{\frac12} = \lim_{\varepsilon\to 0^+} \Big\| \big(\sqrt{\varepsilon}\, v(\varepsilon m)\big)_{\frac{\alpha}{\varepsilon}\le m\le\frac{\beta}{\varepsilon}} \Big\|_{L^p(\mathcal{M};\ell^2_c)}.$$

Likewise, the norm of v in $L^p(\mathcal{M};H_r)$ is equal to the limit (when $\varepsilon \to 0^+$) of the norm of the finite sequence $\big(\sqrt{\varepsilon}\, v(\varepsilon m)\big)_{\frac{\alpha}{\varepsilon}\le m\le\frac{\beta}{\varepsilon}}$ in the space $L^p(\mathcal{M};\ell^2_r)$. The desired result follows from these two results, by (2.24). □

Lemma 10.11. — *We recall that $\Omega_0 = (\mathbb{R}_+^*, dt/t)$. Let $1 < p < \infty$, and let $\varphi\colon [0,\infty) \to L^p(\mathcal{M})$ be a continuous function which is continuously differentiable on $(0,\infty)$. We set*

$$\phi(t) = \frac{1}{t}\int_0^t \varphi(u)\, du\,, \qquad t > 0;$$

and

$$u_m^\varepsilon = \frac{1}{m+1}\sum_{k=0}^{m} \varphi(\varepsilon k), \qquad \varepsilon > 0,\ m \ge 0. \tag{10.9}$$

Assume that there is a constant $K > 0$ such that

$$\big\| \big(\sqrt{m}(u_m^\varepsilon - u_{m-1}^\varepsilon)\big)_{m\ge 1} \big\|_{L^p(\mathcal{M};\ell^2_{rad})} \le K \tag{10.10}$$

for any $\varepsilon > 0$. Then the function $t \mapsto t\phi'(t)$ from $(0,\infty)$ into $L^p(\mathcal{M})$ belongs to the space $L^p(\mathcal{M}; L^2(\Omega_0)_{rad})$, and we have

$$\big\| t \mapsto t\phi'(t) \big\|_{L^p(\mathcal{M};L^2(\Omega_0)_{rad})} \le K.$$

Proof. — Throughout this proof, we fix two constants $0 < \alpha < \beta < \infty$, and we consider the Hilbert space $H = L^2([\alpha,\beta]; dt)$. We set

$$\psi(t) = \sqrt{t}\,\phi'(t), \qquad t \in (\alpha,\beta).$$

According to Remark 6.4 (1), it will suffice to show that

$$\|\psi\|_{L^p(\mathcal{M};H_{rad})} \le K. \tag{10.11}$$

Since φ is continuously differentiable, we have the constant $c_\beta = \sup\big\{\|\varphi'(s)\| \,:\, 0 \le s \le \beta\big\}$ at our disposal. For any integer $m \ge 1$, we define

$$\phi_m(t) = \frac{1}{m}\sum_{k=0}^{m-1} \varphi\Big(\frac{tk}{m}\Big), \qquad t > 0.$$

For a fixed $t \in (0,\beta)$, and any integer $0 \leq k \leq m-1$, let I_k be the closed interval with endpoints $\frac{tk}{m}$ and $\frac{t(k+1)}{m}$. Then we may write

$$\phi_m(t) = \frac{1}{t}\int_0^t \sum_{k=0}^{m-1} \varphi\Big(\frac{tk}{m}\Big)\,\chi_{I_k}(u)\,du\,.$$

Hence we have

$$\|\phi(t) - \phi_m(t)\| \,\leq\, \frac{1}{t}\int_0^t \Big|\varphi(u) - \sum_{k=0}^{m-1} \varphi\Big(\frac{tk}{m}\Big)\,\chi_{I_k}(u)\Big|\,du\,.$$

On the other hand we have $\|\varphi(u) - \varphi\big(\frac{tk}{m}\big)\| \,\leq c_\beta\, t/m$ whenever $u \in I_k$. Letting $c'_\beta = \beta c_\beta$, we deduce that

$$\|\phi(t) - \phi_m(t)\| \,\leq\, \frac{c'_\beta}{m},\qquad 0 < t < \beta,\ m \geq 1. \tag{10.12}$$

The function ϕ is differentiable on $(0,\infty)$, and we have

$$\phi'(t) = \frac{1}{t}\,\big(\varphi(t) - \phi(t)\big),\qquad t > 0. \tag{10.13}$$

For any $\varepsilon > 0$ and any $m \geq 1$, we have

$$\varphi(\varepsilon m) = (m+1)u_m^\varepsilon - m u_{m-1}^\varepsilon \qquad\text{and}\qquad u_{m-1}^\varepsilon = \phi_m(\varepsilon m).$$

Hence we obtain

$$\varphi(\varepsilon m) - \phi_m(\varepsilon m) = (m+1)\big(u_m^\varepsilon - u_{m-1}^\varepsilon\big),$$

which is the discrete analogue of (10.13). Combining with the latter formula we deduce that

$$\begin{aligned}
\phi'(\varepsilon m) &= \frac{1}{\varepsilon m}\,\big(\varphi(\varepsilon m) - \phi(\varepsilon m)\big)\\
&= \frac{1}{\varepsilon m}\,\big(\varphi(\varepsilon m) - \phi_m(\varepsilon m)\big) \;+\; \frac{1}{\varepsilon m}\,\big(\phi_m(\varepsilon m) - \phi(\varepsilon m)\big)\\
&= \frac{m+1}{\varepsilon m}\,\big(u_m^\varepsilon - u_{m-1}^\varepsilon\big) \;+\; \frac{1}{\varepsilon m}\,\big(\phi_m(\varepsilon m) - \phi(\varepsilon m)\big).
\end{aligned}$$

Thus we finally have

$$\sqrt{\varepsilon}\,\psi(\varepsilon m) \;=\; \frac{m+1}{\sqrt{m}}\,\big(u_m^\varepsilon - u_{m-1}^\varepsilon\big) \;+\; \frac{1}{\sqrt{m}}\,\big(\phi_m(\varepsilon m) - \phi(\varepsilon m)\big).$$

The norm of the (finite) sequence $\big(\frac{1}{\sqrt{m}}\,\big(\phi_m(\varepsilon m) - \phi(\varepsilon m)\big)\big)_{\frac{\alpha}{\varepsilon}\leq m\leq\frac{\beta}{\varepsilon}}$ in $L^p(\mathcal{M};\ell^2_{rad})$ is less than or equal to

$$\sum_{\frac{\alpha}{\varepsilon}\leq m\leq\frac{\beta}{\varepsilon}} \frac{1}{\sqrt{m}}\,\big\|\phi(\varepsilon m) - \phi_m(\varepsilon m)\big\| \,\leq\, c'_\beta \sum_{m\geq\frac{\alpha}{\varepsilon}} \frac{1}{m^{\frac{3}{2}}}\,.$$

Indeed the latter inequality follows from (10.12). Hence

$$\limsup_{\varepsilon\to 0^+} \Big\|\Big(\tfrac{1}{\sqrt{m}}\,\big(\phi_m(\varepsilon m) - \phi(\varepsilon m)\big)\Big)_{\frac{\alpha}{\varepsilon}\leq m\leq\frac{\beta}{\varepsilon}}\Big\|_{L^p(\mathcal{M};\ell^2_{rad})} = 0.$$

On the other hand, our assumption on the u_m^ε's ensures that the limsup (for $\varepsilon \to 0^+$) of the norm of the sequence $\big(\frac{m+1}{\sqrt{m}}\,(u_m^\varepsilon - u_{m-1}^\varepsilon)\big)_{\frac{\alpha}{\varepsilon}\leq m\leq\frac{\beta}{\varepsilon}}$ in $L^p(\mathcal{M};H_{rad})$ is less than or equal to K. Thus we have proved that

$$\limsup_{\varepsilon\to 0^+}\Big\|\,\big(\sqrt{\varepsilon}\,\psi(\varepsilon m)\big)_{\frac{\alpha}{\varepsilon}\leq m\leq\frac{\beta}{\varepsilon}}\,\Big\|_{L^p(\mathcal{M};\ell^2_{rad})}\,\leq\, K. \tag{10.14}$$

If $p \geq 2$, we deduce (10.11) by applying Lemma 10.10 with $v = \psi$.

Now assume that $1 < p < 2$, and let p' be its conjugate number. We consider v in $L^{p'}(\mathcal{M}) \otimes C([\alpha,\beta])$ and we assume that $\|v\|_{L^{p'}(\mathcal{M};H_{rad})} \leq 1$. According to Remark 2.11, (10.11) will follow if we can show that $|\langle\psi, v\rangle| \leq K$. Since $t \mapsto \langle\psi(t), v(t)\rangle$ is continuous on $[\alpha,\beta]$, we have

$$\begin{aligned}\langle\psi, v\rangle \;&=\; \int_\alpha^\beta \langle\psi(t), v(t)\rangle\, dt\\ &=\; \lim_{\varepsilon\to 0^+}\,\varepsilon \sum_{\frac{\alpha}{\varepsilon}\leq m\frac{\alpha}{\varepsilon}} \langle\psi(\varepsilon m), v(\varepsilon m)\rangle.\end{aligned}$$

By the duality relation (2.25), $|\langle\psi, v\rangle|$ is therefore less than or equal to

$$\limsup_{\varepsilon\to 0^+}\Big\|\,\big(\sqrt{\varepsilon}\,\psi(\varepsilon m)\big)_{\frac{\alpha}{\varepsilon}\leq m\leq\frac{\beta}{\varepsilon}}\,\Big\|_{L^p(\mathcal{M};\ell^2_{rad})} \times \limsup_{\varepsilon\to 0^+}\Big\|\,\big(\sqrt{\varepsilon}\,v(\varepsilon m)\big)_{\frac{\alpha}{\varepsilon}\leq m\leq\frac{\beta}{\varepsilon}}\,\Big\|_{L^{p'}(\mathcal{M};\ell^2_{rad})}.$$

By Lemma 10.10 and (10.14), we obtain the inequality $|\langle\psi, v\rangle| \leq K$. □

Theorem 10.12. — *Let $(T_t)_{t\geq 0}$ be the noncommutative Poisson semigroup on $VN(\mathbb{F}_n)$ (see Definition 10.2). For any $1 < p < \infty$, we let $-A_p$ be the generator of $(T_t)_{t\geq 0}$ on $L^p(VN(\mathbb{F}_n))$, and we let $\omega_p = \pi\big|\frac{1}{p} - \frac{1}{2}\big|$. Then for $\theta > \omega_p$, the operator A_p has a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

Proof. — We will first show that A_p has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \omega_p$. We noticed in paragraph 10.A that $(T_t)_{t\geq 0}$ is a completely positive diffusion semigroup on $VN(\mathbb{F}_n)$. Hence for any $1 < p < \infty$, the operator A_p is Rad-sectorial of Rad-type ω_p by Theorem 5.6. Thus according to Corollary 7.4, it suffices to find $\theta > \omega_p$ and a non zero function $F \in H_0^\infty(\Sigma_\theta)$ such that A_p both satisfies the square function estimate $(\mathcal{S}_F)$, and the dual square function estimate $(\mathcal{S}_F^*)$ (in the sense of Chapter 7). Since $A_p^* = A_{p'}$, it actually suffices to prove $(\mathcal{S}_F)$ only.

We fix some $1 < p < \infty$. We let $x \in L^p(VN(\mathbb{F}_n))$ and apply Lemma 10.11 to the function $\varphi(t) = T_t(x)$. We let ϕ be the associated average function. Since $\varphi(\varepsilon k) = T_{\varepsilon k}(x) = T_\varepsilon^k(x)$, the averages defined by (10.9) are equal to

$$u_m^\varepsilon \;=\; \frac{1}{m+1}\sum_{k=0}^m T_\varepsilon^k(x)\,.$$

Then by Proposition 10.5 and Corollary 10.9, the uniform condition (10.10) holds true with $K = K_p\|x\|_p$, K_p being a universal constant. Thus Lemma 10.11 ensures that

$$\big\|t \mapsto t\phi'(t)\big\|_{L^p(\mathcal{M};L^2(\Omega_0)_{rad})} \;\leq\; K_p\|x\|_p. \tag{10.15}$$

We consider the holomorphic function

$$F(z) = e^{-z} - \frac{1-e^{-z}}{z}, \qquad z \in \mathbb{C}.$$

It is easy to check that $F \in H_0^\infty(\Sigma_\theta)$ for any $\theta < \frac{\pi}{2}$. We fix some $\theta \in (\omega_p, \frac{\pi}{2})$, and we will check that A_p satisfies $(\mathcal{S}_F)$. According to (10.15), it suffices to show that

$$t\phi'(t) \; = \; F(tA_p)x, \qquad t > 0. \tag{10.16}$$

Let us write $A = A_p$ for simplicity. We first observe that

$$zF'(z) + F(z) = [zF(z)]' = (1-e^{-z})' - (e^{-z})' = -ze^{-z}.$$

Hence the function $z \mapsto zF'(z)$ belongs to $H_0^\infty(\Sigma_\theta)$, the function $t \mapsto F(tA)$ is differentiable on $(0,\infty)$, and

$$t\frac{\partial}{\partial t}\big(F(tA)\big) \; = \; [zF'(z)](tA), \qquad t > 0.$$

Since $[ze^{-z}](tA) = -t\frac{\partial}{\partial t}(T_t)$, we deduce that

$$t\frac{\partial}{\partial t}\Big(F(tA)x\Big) \; + \; F(tA)x \; = \; t\frac{\partial}{\partial t}\big(\varphi(t)\big), \qquad t > 0.$$

Integrating this relation yields

$$tF(tA)x \; = \; t(\varphi(t) - \phi(t)), \qquad t > 0.$$

Indeed, $\frac{\partial}{\partial t}\big(t\phi(t)\big) = \varphi(t)$. Dividing the latter formula by t and applying (10.13), we obtain the desired identity (10.16).

It is not hard to check that the above arguments work as well with $I\overline{\otimes}A_p$ in the place of A_p. Thus A_p actually has a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \omega_p$. $\square$

We conclude this chapter by an application to 'noncommutative Fourier multipliers'. We recall that $G = VN(\mathbb{F}_n)$ and we let $\overset{\circ}{G} = G \setminus \{e\}$. Then arguing as in the proof of Corollary 9.6, we deduce the following.

Corollary 10.13. — *Let $1 < p < \infty$ and let $\theta > \pi\big|\frac{1}{p} - \frac{1}{2}\big|$ be an angle. Then for any function $f \in H^\infty(\Sigma_\theta)$ and for any finitely supported family of complex numbers $\{\alpha_g \, : \, g \in \overset{\circ}{G}\}$, we have*

$$\Big\|\sum_g \alpha_g \, f(|g|)\,\lambda(g)\Big\|_p \; \leq \; K\|f\|_{\infty,\theta}\Big\|\sum_g \alpha_g\,\lambda(g)\Big\|_p,$$

where $K > 0$ is a constant not depending on f.

For any integer $m \geq 0$, let

$$E_m \; = \; \mathrm{Span}\big\{\lambda(g) \, : \, |g| = m\big\}.$$

Thus $\mathcal{P}$ is the algebraic direct sum of the E_m's. In general, this direct sum does not induce a Schauder decomposition in $L^p(\mathcal{M})$. Namely let

$$P_m \colon \mathcal{P} \longrightarrow \mathop{\oplus}_{n=0}^{m} E_m$$

be the natural projection. Then it is shown in [**12**] that if $p < \frac{2}{3}$ or $p > 3$, we have

$$\sup_{m\geq 1} \big\| P_m \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}) \big\| \,=\, \infty.$$

In the opposite direction, the next statement says that the direct sum

$$\mathop{\oplus}_{k\geq 0} E_{2^k} \,\subset\, L^p(\mathcal{M})$$

induces an unconditional decomposition for any $1 < p < \infty$.

Corollary 10.14. — *Let $1 < p < \infty$. There is a constant $K > 0$ such that for any finite family $(x_k)_{k\geq 0}$ with $x_k \in E_{2^k}$, and for any $\varepsilon_k = \pm 1$, we have*

$$\Big\| \sum_{k\geq 0} \varepsilon_k x_k \Big\|_p \,\leq\, C \,\Big\| \sum_{k\geq 0} x_k \Big\|_p .$$

Proof. — According to Carleson's Theorem (see e.g. [**28**, Chapter 7]), $(2^k)_{k\geq 0}$ is an interpolating sequence for the open right half-plane $\Sigma_{\frac{\pi}{2}}$. This means that for any bounded sequence $(c_k)_{k\geq 0}$ of complex numbers, there exists a bounded analytic function $f \colon \Sigma_{\frac{\pi}{2}} \to \mathbb{C}$ such that $f(2^k) = c_k$ for any $k \geq 0$ and moreover

$$\|f\|_{\infty,\frac{\pi}{2}} \leq C \sup_k |c_k|$$

for some constant $C \geq 1$ not depending on $(c_k)_{k\geq 0}$. We apply this property with $c_k = \varepsilon_k$, and we let $f \in H^\infty(\Sigma_{\frac{\pi}{2}})$ be the resulting interpolating function.

Let us write

$$\sum_{k\geq 0} x_k \,=\, \sum_g \alpha_g \lambda(g),$$

where $\{\alpha_g \,:\, g \in \overset{\circ}{G}\}$ is a finite family of complex numbers. Then $\alpha_g = 0$ if $|g|$ is not a power of 2, and we have

$$\sum_{k\geq 0} \varepsilon_k x_k = \sum_{k\geq 0} \varepsilon_k \sum_{|g|=2^k} \alpha_g \lambda(g) = \sum_{k\geq 0} \sum_{|g|=2^k} \alpha_g f(2^k) \lambda(g) = \sum_g \alpha_g f(|g|) \lambda(g).$$

The result therefore follows from Corollary 10.13. $\square$

The above corollary may be combined with the noncommutative Khintchine inequalities (2.21) and (2.22). We obtain that if $2 \leq p < \infty$, we have an equivalence

$$\Big\| \sum_k x_k \Big\|_p \,\asymp\, \max\Big\{ \Big\| \Big(\sum_k x_k^* x_k \Big)^{\frac{1}{2}} \Big\|_p \,,\, \Big\| \Big(\sum_k x_k x_k^* \Big)^{\frac{1}{2}} \Big\|_p \Big\}$$

for finite families $(x_k)_{k\geq 0}$ such that $x_k \in E_{2^k}$ for any $k \geq 0$. Likewise if $1 < p < 2$, we obtain for these families that

$$\Big\|\sum_k x_k\Big\|_p \asymp \inf\Big\{\Big\|\Big(\sum_k y_k^* y_k\Big)^{\frac{1}{2}}\Big\|_p + \Big\|\Big(\sum_k z_k z_k^*\Big)^{\frac{1}{2}}\Big\|_p\Big\},$$

where the infimum runs over all $(y_k)_{k\geq 0}$ and $(z_k)_{k\geq 0}$ in $L^p(\mathcal{M})$ such that $x_k = y_k + z_k$ for any $k \geq 0$.

CHAPTER 11

THE NON TRACIAL CASE

In this short chapter, we briefly discuss extensions of the results established so far to the setting of noncommutative L^p-spaces associated with a non tracial state.

Let $\mathcal{M}$ be a von Neumann algebra and let φ be a distinguished normal faithful state on $\mathcal{M}$. We do not assume that φ is tracial. For any $1 \leq p \leq \infty$, we let $L^p(\mathcal{M}, \varphi)$ be the associated Haagerup noncommutative L^p-space, with norm denoted by $\| \ \|_p$. We refer the reader to [**73**] for a complete description of these spaces, and to [**65**] or [**38**] for a brief presentation. We merely recall that if $\mathcal{M} \subset B(H)$ acts on some Hilbert space H, then $L^p(\mathcal{M}, \varphi)$ is defined as a space of possibly unbounded operators on $L^2(\mathbb{R}; H)$ with the following properties. First, if $1 \leq p, q, r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then $xy \in L^r(\mathcal{M}, \varphi)$ whenever $x \in L^p(\mathcal{M}, \varphi)$ and $y \in L^q(\mathcal{M}, \varphi)$. Second, for any $1 \leq p, q < \infty$ and any $x \in L^p(\mathcal{M}, \varphi)$, the positive operator $|x|^{\frac{p}{q}}$ belongs to $L^q(\mathcal{M}, \varphi)$, with

$$\| |x|^{\frac{p}{q}} \|_q^q = \|x\|_p^p.$$

Third, there are two natural order-preserving isometric identifications

$$\mathcal{M} \simeq L^\infty(\mathcal{M}, \varphi) \qquad \text{and} \qquad \mathcal{M}_* \simeq L^1(\mathcal{M}, \varphi).$$

In particular, φ may be regarded as a positive element of $L^1(\mathcal{M}, \varphi)$. Consequently for any $1 \leq p < \infty$, we may regard the space

$$\varphi^{\frac{1}{2p}} \mathcal{M} \varphi^{\frac{1}{2p}} = \bigl\{ \varphi^{\frac{1}{2p}} x \varphi^{\frac{1}{2p}} \, : \, x \in \mathcal{M} \bigr\}$$

as a subspace of $L^p(\mathcal{M}, \varphi)$ and this subspace turns out to be dense. It should be noticed that if $p \neq q$, then $L^p(\mathcal{M}, \varphi) \cap L^q(\mathcal{M}, \varphi) = \{0\}$. This is in sharp contrast with the case of the noncommutative L^p-spaces considered so far in this paper (see paragraph 2.A).

As usual we let $tr \colon L^1(\mathcal{M}, \varphi) \to \mathbb{C}$ denote the functional corresponding to $1 \in \mathcal{M}$ in the above identification $\mathcal{M}_* \simeq L^1(\mathcal{M}, \varphi)$. It satisfies

$$tr(\varphi x) = \varphi(x), \qquad x \in \mathcal{M}.$$

We also recall that if $1 \leq p < \infty$ and $p^{-1} + {p'}^{-1} = 1$, then

$$tr(xy) = tr(yx), \qquad x \in L^p(\mathcal{M}, \varphi),\ y \in L^{p'}(\mathcal{M}, \varphi),$$

and the duality pairing $(x, y) \mapsto tr(xy)$ induces an isometric isomorphism

$$L^{p'}(\mathcal{M}, \varphi) = L^p(\mathcal{M}, \varphi)^*.$$

Furthermore $L^2(\mathcal{M}, \varphi)$ is a Hilbert space, with inner product given by $(x, y) \mapsto tr(y^*x)$.

Using [**38**, Section 1], one may naturally define spaces $L^p(\mathcal{M}, H_c)$ and $L^p(\mathcal{M}, H_r)$ for any Hilbert space H. Then as in Chapter 2, we define $L^p(\mathcal{M}, H_{rad})$ as the intersection $L^p(\mathcal{M}, H_c) \cap L^p(\mathcal{M}, H_r)$ if $2 \leq p < \infty$, and as the sum $L^p(\mathcal{M}, H_c) + L^p(\mathcal{M}, H_r)$ if $1 \leq p \leq 2$. Then it is not hard to check that all the results established in Chapters 3, 4, 6 and 7 for the tracial noncommutative L^p-spaces extend to the $L^p(\mathcal{M}, \varphi)$'s.

We now discuss analogs of the results obtained in Chapter 5. We need the following definition. Suppose that $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$ are two von Neumann algebras equipped with normal faithful states φ and ψ. Let $T\colon \mathcal{M} \to \mathcal{N}$ be a bounded operator, and let $1 \leq p < \infty$. Consider the linear mapping from

$$\varphi^{\frac{1}{2p}} \mathcal{M} \varphi^{\frac{1}{2p}} \longrightarrow \psi^{\frac{1}{2p}} \mathcal{N} \psi^{\frac{1}{2p}}$$

taking $\varphi^{\frac{1}{2p}} x \varphi^{\frac{1}{2p}}$ to $\psi^{\frac{1}{2p}} T(x) \psi^{\frac{1}{2p}}$ for any $x \in \mathcal{M}$. If this linear operator extends to a bounded operator from $L^p(\mathcal{M}, \varphi)$ into $L^p(\mathcal{N}, \psi)$, we say that T has an L^p extension and we let

$$T_p\colon L^p(\mathcal{M}, \varphi) \longrightarrow L^p(\mathcal{N}, \psi)$$

denote the resulting operator.

Let $(\mathcal{M}, \varphi)$ as above, and let $\sigma^\varphi = (\sigma^\varphi_s)_{s\in\mathbb{R}}$ denote the one parameter modular automorphism group of $\mathbb{R}$ on $\mathcal{M}$ associated with φ. Let $T\colon \mathcal{M} \to \mathcal{M}$ be a normal positive contraction such that

$$\varphi \circ T \leq \varphi \quad \text{on } \mathcal{M}_+.$$

According to [**35**, Theorem 5.1], T has an L^p extension $T_p\colon L^p(\mathcal{M}, \varphi) \to L^p(\mathcal{M}, \varphi)$ for any $1 \leq p < \infty$. Assume further that

$$\sigma^\varphi_s \circ T = T \circ \sigma^\varphi_s, \qquad s \in \mathbb{R},$$

and that T is φ-symmetric, that is,

$$\varphi\big(T(x)y\big) = \varphi\big(xT(y)\big), \qquad x, y \in \mathcal{M}.$$

Then $T_2\colon L^2(\mathcal{M}, \varphi) \to L^2(\mathcal{M}, \varphi)$ is a selfadjoint operator. Indeed, let $\mathcal{M}_a$ denote the family of elements of $\mathcal{M}$ which are analytic with respect to σ^φ. By [**35**, Proposition 5.5], we have $T_2(x\varphi^{\frac{1}{2}}) = T(x)\varphi^{\frac{1}{2}}$ for any $x \in \mathcal{M}_a$. Hence for any $x, y \in \mathcal{M}_a$, we have

$$\big\langle T_2(x\varphi^{\frac{1}{2}}), y\varphi^{\frac{1}{2}}\big\rangle_{L^2} = tr\big((y\varphi^{\frac{1}{2}})^* T(x)\varphi^{\frac{1}{2}}\big) = tr\big(\varphi y^* T(x)\big) = \varphi\big(y^* T(x)\big).$$

Likewise,

$$\langle x\varphi^{\frac{1}{2}}, T_2(y\varphi^{\frac{1}{2}})\rangle_{L^2} = \varphi\big(T(y^*)x\big).$$

Since $\mathcal{M}_a\varphi^{\frac{1}{2}}$ is dense in $L^2(\mathcal{M},\varphi)$ [**38**, Lemma 1.1], this proves the result.

Theorem 11.1. — *Let $(T_t)_{t\geq 0}$ be a w^*-continuous semigroup of operators on $(\mathcal{M},\varphi)$. Assume that for any $t\geq 0$, $T_t\colon \mathcal{M}\to\mathcal{M}$ is a normal positive φ-symmetric contraction, and that we both have*

$$\varphi\circ T_t \,\leq\, \varphi \quad \text{on } \mathcal{M}_+ \qquad \text{and} \qquad \sigma_s^\varphi\circ T_t = T_t\circ\sigma_s^\varphi, \quad s\in\mathbb{R}.$$

(1) *For any $t\geq 0$ and any $1\leq p<\infty$, the operator T_t admits an L^p extension $T_{p,t}$ on $L^p(\mathcal{M},\varphi)$, and $(T_{p,t})_{t\geq 0}$ is a c_0-semigroup of contractions on $L^p(\mathcal{M},\varphi)$. Moreover $(T_{2,t})_{t\geq 0}$ is a selfadjoint semigroup on $L^2(\mathcal{M},\varphi)$.*

(2) *Let A_p be the negative generator of $(T_{p,t})_{t\geq 0}$. Then for any $1<p<\infty$, A_p is a sectorial operator of type $\omega_p=\pi\big|\frac{1}{p}-\frac{1}{2}\big|$.*

(3) *Assume further that each $T_t\colon\mathcal{M}\to\mathcal{M}$ is completely positive. Then for any $1<p<\infty$, the operator A_p is Col-sectorial (resp. Row-sectorial) of Col-type (resp. Row-type) equal to ω_p.*

Proof. — Part (1) easily follows from the previous discussion. Part (2) is an analog of Lemma 5.4. Its proof relies on interpolation, using Kosaki's Theorem [**42**]. Part (3) is an analog of Theorem 5.6. Its proof is similar to the one of that theorem, using Kosaki's Theorem again and the noncommutative ergodic maximal theorem in the non tracial case (see [**36**, Section 7]). We skip the details. □

We now introduce an analogue of Rota's dilation property in the non tracial setting. We follow the scheme of paragraph 10.B. We consider a von Neumann algebra $(\mathcal{N},\psi)$ equipped with a normal faithful state ψ. Let $\mathcal{M}\subset\mathcal{N}$ be a von Neumann subalgebra and assume that it is invariant under σ^ψ, that is, $\sigma_s^\psi(\mathcal{M})\subset\mathcal{M}$ for any $s\in\mathbb{R}$. Let $\varphi\in\mathcal{M}_*$ be the restriction of ψ to $\mathcal{M}$. Then $\sigma_s^\varphi=\sigma^\psi_{s\,|\mathcal{M}}$ for any t. Let $1\leq p<\infty$. Then $L^p(\mathcal{M},\varphi)$ can be naturally identified with a subspace of $L^p(\mathcal{N},\psi)$. Indeed, the canonical embedding $\mathcal{M}\to\mathcal{N}$ has an L^p extension $L^p(\mathcal{M},\varphi)\to L^p(\mathcal{N},\psi)$ in the above sense, and this extension is an isometry (see [**38**, Section 2]). Furthermore there exists a unique normal conditional expectation $\mathcal{E}\colon\mathcal{N}\to\mathcal{M}$ such that $\psi=\varphi\circ\mathcal{E}$ [**72**]. We call it the canonical conditional expectation onto $\mathcal{M}$. This map also has an L^p extension $\mathcal{E}_p\colon L^p(\mathcal{N},\psi)\to L^p(\mathcal{M},\varphi)$, and this extension is a contraction [**38**, Lemma 2.2].

More generally, let $(\mathcal{M},\varphi)$ and $(\mathcal{M}',\varphi')$ be two von Neumann algebras equipped with normal faithful states, and let $\pi\colon\mathcal{M}\to\mathcal{M}'$ be a normal unital faithful $*$-representation such that

$$\varphi=\varphi'\circ\pi \qquad \text{and} \qquad \sigma_s^{\varphi'}\big(\pi(\mathcal{M})\big)\subset\pi(\mathcal{M}),\ s\in\mathbb{R}. \tag{11.1}$$

Then π admits an L^p extension for any $1 \leq p < \infty$, and

$$\pi_p \colon L^p(\mathcal{M}, \varphi) \longrightarrow L^p(\mathcal{M}', \varphi')$$

is an isometry. Let $\mathcal{E} \colon \mathcal{M}' \to \pi(\mathcal{M})$ be the canonical conditional expectation onto $\pi(\mathcal{M})$ and let $Q \colon \mathcal{M}' \to \mathcal{M}$ be defined by $\pi \circ Q = \mathcal{E}$. Then we say that Q is the conditional expectation associated with π. It is clear that Q has an L^p extension for any $1 \leq p < \infty$, with $\pi_p \circ Q_p = \mathcal{E}_p$. Moreover Q is the adjoint of π_1.

The non tracial analogue of Definition 10.2 is as follows. Let $(\mathcal{M}, \varphi)$ be a von Neumann algebra equipped with a normal faithful state, and let $T \colon \mathcal{M} \to \mathcal{M}$ be a bounded operator. We say that T satisfies Rota's dilation property if there exist a von Neumann algebra $(\mathcal{N}, \psi)$ equipped with a normal faithful state, a normal unital faithful $*$-representation $\pi \colon \mathcal{M} \to \mathcal{N}$ such that $\varphi = \psi \circ \pi$ and $\pi(M)$ is invariant under σ^ψ, and a decreasing sequence $(\mathcal{N}_m)_{m \geq 1}$ of von Neumann subalgebras of $\mathcal{N}$ which are invariant under σ^ψ, such that

$$T^m = Q \circ \mathcal{E}(m) \circ \pi, \qquad m \geq 1,$$

where $\mathcal{E}(m) \colon \mathcal{N} \to \mathcal{N}_m \subset \mathcal{N}$ is the canonical conditional expectation onto $\mathcal{N}_m$, and where $Q \colon \mathcal{N} \to \mathcal{M}$ is the conditional expectation associated with π.

Clearly such an operator is completely positive and for any $1 \leq p < \infty$, it admits an L^p extension $T_p \colon L^p(\mathcal{M}, \varphi) \to L^p(\mathcal{M}, \varphi)$, with $T_p = Q_p \circ \mathcal{E}(1)_p \circ \pi_p$. It is not hard to show that in addition, T is φ-symmetric.

With the above definition, Corollary 10.9 extends to the non tracial case. The proof is the same, using the noncommutative martingale inequalities from [**38**, Section 3].

Corollary 11.2. — *Let $(T_t)_{t \geq 0}$ be a w^*-continuous semigroup of operators on $(\mathcal{M}, \varphi)$. Assume that for any $t \geq 0$, $T_t \colon \mathcal{M} \to \mathcal{M}$ satisfies the above Rota's dilation property. Then it satisfies Theorem 11.1 and moreover, the operator A_p admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus on $L^p(\mathcal{M}, \varphi)$ for any $\theta > \omega_p$ and any $1 < p < \infty$.*

Proof. — The proof is similar to the one of Theorem 10.12, using Theorem 11.1 instead of Theorem 5.6. □

Following [**4**], we say that $T \colon (\mathcal{M}, \varphi) \to (\mathcal{M}, \varphi)$ is *factorizable* if there exist a von Neumann algebra $(\mathcal{M}', \varphi')$ equipped with a normal faithful state, and two normal unital faithful $*$-representations

$$\pi \colon \mathcal{M} \longrightarrow \mathcal{M}' \qquad \text{and} \qquad \widetilde{\pi} \colon \mathcal{M} \longrightarrow \mathcal{M}'$$

both satisfying (11.1), such that $T = \widetilde{Q} \circ \pi$, where $\widetilde{Q} \colon \mathcal{M}' \to \mathcal{M}$ is the conditional expectation associated with $\widetilde{\pi}$. According to [**4**, Theorem 6.5], $T^2 \colon \mathcal{M} \to \mathcal{M}$ satisfies Rota's dilation property if T is factorizable and φ-symmetric.

Consequently, if $(T_t)_{t\geq 0}$ is a w^*-continuous semigroup of operators on $\mathcal{M}$ such that each operator $T_t \colon \mathcal{M} \to \mathcal{M}$ is both factorizable and φ-symmetric, then it gives rise to a semigroup $(T_{p,t})_{t\geq 0}$ on $L^p(\mathcal{M}, \varphi)$ whose negative generator has a bounded $H^\infty(\Sigma_\theta)$ functional calculus, provided that $1 < p < \infty$ and $\theta > \omega_p$.

APPENDIX A

COMPARING ROW AND COLUMN SQUARE FUNCTIONS

We aim at showing that in general, row and column square functions as defined by (6.1) are not equivalent. We will provide an example on Schatten spaces $S^p = S^p(\ell^2)$. We let $(e_k)_{k\geq 1}$ denote the canonical basis of ℓ^2 and we let a be the unbounded positive selfadjoint operator on ℓ^2 such that

$$a\Bigl(\sum_k \alpha_k e_k\Bigr) = \sum_k \alpha_k 2^k e_k$$

for any finite family $(\alpha_k)_k$ of complex numbers.

We fix some $1 < p < \infty$, and we let $A_p = \mathcal{L}_a$ be the left multiplication by a on S^p (see paragraph 8.A). For any $\theta > 0$, the operator a has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on ℓ^2. Hence A_p also has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on S^p, by Proposition 8.4. Since a has dense range, it also follows from the latter proposition that A_p has dense range. Applying Theorem 7.6, we therefore obtain that for any $\theta > 0$, and any non zero $F \in H_0^\infty(\Sigma_\theta)$, we have an equivalence

$$\|x\| \asymp \|x\|_F, \qquad x \in S^p. \tag{A.1}$$

Lemma A.1. — *Let $F \in H_0^\infty(\Sigma_\theta)\setminus\{0\}$, and let $c_F = \bigl(\int_0^\infty |F(t)|^2 \,\frac{dt}{t}\,\bigr)^{\frac{1}{2}}$. Then we have*

$$\|x\|_{F,c} = c_F\|x\|, \qquad x \in S^p.$$

Proof. — Let $0 < \alpha < \beta < \infty$ be two positive numbers, and let $x \in S^p$. According to (8.5), we have

$$\bigl(F(tA_p)(x)\bigr)^*\bigl(F(tA_p)(x)\bigr) = \bigl(F(ta)x\bigr)^*\bigl(F(ta)x\bigr) = x^*F(ta)^*F(ta)x$$

for any $t > 0$. Hence

$$\int_\alpha^\beta \bigl(F(tA_p)(x)\bigr)^*\bigl(F(tA_p)(x)\bigr)\,\frac{dt}{t} = x^*\Bigl(\int_\alpha^\beta F(ta)^*F(ta)\,\frac{dt}{t}\Bigr)\,x. \tag{A.2}$$

For any $t > 0$ and any $k \geq 1$, $F(ta)e_k = F(t2^k)e_k$, hence

$$\bigl[F(ta)^*F(ta)\bigr]e_k \;=\; |F(t2^k)|^2 e_k.$$

Furthermore, we have

$$\int_\alpha^\beta |F(t2^k)|^2\,\frac{dt}{t}\ \nearrow\ \int_0^\infty |F(t2^k)|^2\,\frac{dt}{t}\ = c_F^2$$

when $\alpha \to 0^+$ and $\beta \to \infty$. Thus the operator in (A.2) converges to $c_F^2\, x^*x$ (in the $S^{\frac{p}{2}}$-norm, say), and we deduce from either Proposition 6.2 or Remark 6.4 (2) that the function $u\colon t \mapsto F(tA_p)(x)$ belongs to $S^p(L^2(\Omega_0)_c)$, and that $u^*u = c_F^2\, x^*x$. Consequently we have

$$\|x\|_{F,c} = \|(u^*u)^{\frac{1}{2}}\|_{S^p} = c_F\,\|(x^*x)^{\frac{1}{2}}\|_{S^p} = c_F\|x\|_{S^p}. \quad \square$$

Let F be any non zero function in $H_0^\infty(\Sigma_\theta)$, with $\theta \in (0,\pi)$. Combining the above lemma with (A.1), there exists a constant $K>0$ such that for any $x \in S^p$, we have

$$\|x\|_{F,r} \leq K\|x\|_{F,c} \qquad \text{for any } x \in S^p, \qquad \text{if } \ p>2;$$

$$\|x\|_{F,c} \leq K\|x\|_{F,r} \qquad \text{for any } x \in S^p, \qquad \text{if } \ p<2;$$

We shall now prove that except if $p=2$, these estimates cannot be reversed.

Proposition A.2

(1) *Assume that* $p>2$. *Then* $\sup\Big\{\frac{\|x\|_{F,c}}{\|x\|_{F,r}}\ :\ x \in S^p\Big\}\ =\infty\,.$

(2) *Assume that* $p<2$. *Then* $\sup\Big\{\frac{\|x\|_{F,r}}{\|x\|_{F,c}}\ :\ x \in S^p\Big\}\ =\infty\,.$

Proof. — By Proposition 8.4, A_p has a completely bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta>0$. Hence by Theorems 4.12 and 6.7, it suffices to prove the result for one specific function F. Throughout the proof, we will use the function

$$F(z) = z^{\frac{1}{2}}e^{-z}.$$

In the notation introduced in Lemma A.1, we have $c_F = \frac{1}{\sqrt{2}}$.

We first assume that $p>2$. For any integer $k\geq 0$, we let

$$d_k = \int_0^\infty F(t)F(t2^k)\,\frac{dt}{t}\,.$$

We will use the fact that for any $i,j\geq 1$, we have

$$\int_0^\infty F(t2^i)F(t2^j)\,\frac{dt}{t}\ =\ d_{|i-j|}.$$

Indeed this is obtained by changing t into $t2^j$ in this integral. Furthermore, we have

$$0 \leq d_k = \int_0^\infty e^{-t}\,2^{\frac{k}{2}}\,e^{-2^k t}\ dt\ =\ \frac{2^{\frac{k}{2}}}{1+2^k}\ \leq 2^{-\frac{k}{2}}.$$

Given an integer $n\geq 1$, we consider $e = e_1+\cdots+e_n$ and $x = \frac{e\otimes e}{\sqrt{n}}$. With $E_{ij} = e_i\otimes e_j$, we then have

$$xx^* = e\otimes e = \sum_{i,j=1}^n E_{ij}\,.$$

According to the definition of a, we have

$$\big(F(tA_p)(x)\big)\big(F(tA_p)(x)\big)^* = F(ta)xx^*F(ta)^* = \sum_{i,j=1}^{n} F(t2^i)F(t2^j)\, E_{ij}$$

for any positive real number $t > 0$. Taking the integral over Ω_0, and applying the row version of Proposition 6.2, we deduce that

$$\|x\|_{F,r}^2 \;=\; \Big\|\sum_{i,j=1}^{n} d_{|i-j|}\, E_{ij}\Big\|_{S^{\frac{p}{2}}}.$$

We let $\Delta = \big[d_{|i-j|}\big]$ be the $n \times n$ matrix in the right hand side of the above formula. Then we have

$$\|\Delta\|_{S^2}^2 \;=\; \sum_{i,j=1}^{n} \big|d_{|i-j|}\big|^2 \;\leq 2n\sum_{k=0}^{n} |d_k|^2 \;\leq 2n\sum_{k=0}^{n} 2^{-k} \;\leq 4n.$$

By construction, $\Delta \geq 0$, hence we have

$$\|\Delta\|_{S^1} \;=\; tr\Big(\sum_{i,j=1}^{n} d_{|i-j|}\, E_{ij}\Big) = nd_0 = nc_F^2 = \frac{n}{2}.$$

We need to divide our discussion into two cases.

If $2 < p \leq 4$, we let $\alpha \in (0,1]$ be such that $\frac{(1-\alpha)}{1} + \frac{\alpha}{2} = \frac{2}{p}$. Then

$$\|\Delta\|_{S^{\frac{p}{2}}} \;\leq\; \|\Delta\|_{S^1}^{1-\alpha}\,\|\Delta\|_{S^2}^{\alpha}.$$

This yields the estimate

$$\|x\|_{F,r}^2 \;\leq 2^{2\alpha-1}\, n^{1-\frac{\alpha}{2}}.$$

Since $x = \frac{e\otimes e}{\sqrt{n}}$ is rank one, its norm in S^p does not depend on p, and it is equal to $\frac{\|e\|^2}{\sqrt{n}} = \sqrt{n}$. Hence $\|x\|_{F,c}^2 = c_F^2\, n = \frac{n}{2}$ by Lemma A.1. We obtain that

$$\frac{\|x\|_{F,c}^2}{\|x\|_{F,r}^2} \;\geq 4^{-\alpha}\, n^{\frac{\alpha}{2}}.$$

Since n was arbitrary and $\alpha > 0$, we obtain (1) in this case.

If $p \geq 4$, we note that $\|\Delta\|_{S^{\frac{p}{2}}} \leq \|\Delta\|_{S^2}$. Hence $\|x\|_{F,r}^2 \leq 2\sqrt{n}$. Since $\|x\|_{F,c}^2 = \frac{n}{2}$, we also obtain (1) in that case.

We now turn to the proof of (2). We assume that $1 < p < 2$, and we let p' be its conjugate number. According to Remark 8.7 (2), A_p^* is the right multiplication by a on $S^{p'}$. For any $y \in S^{p'}$, we let $\|y\|_{F,c}$ and $\|y\|_{F,r}$ denote the column and row square functions corresponding to A_p^*. Of course Lemma A.1 has an analog for right multiplications, and the latter says that $\|y\|_{F,r} = c_F\|y\|$ for any $y \in S^{p'}$. Likewise, part (1) of Proposition A.2 has an analog for A_p^*, namely

$$\sup\Big\{\frac{\|y\|_{F,r}}{\|y\|_{F,c}} \;:\; y \in S^{p'}\Big\} \;=\; \infty\,. \tag{A.3}$$

To prove (2), assume on the contrary that there is a constant $K > 0$ such that

$$\|x\|_{F,r} \leq K \|x\|_{F,c}, \qquad x \in S^p. \tag{A.4}$$

Let $y \in S^{p'}$ and $x \in S^p$. We consider the approximating sequence $(g_n)_{n\geq 1}$ defined by (3.8) and we recall that $g_n(A_p)(x) = g_n(a)x \to x$ when $n \to \infty$. By the first part of Lemma 6.5, we have

$$\begin{aligned}\langle y, g_n(A_p)(x)\rangle \ &= \ \sqrt{2} \int_0^\infty \langle y, F(tA_p)^2(g_n(a)x)\rangle \, \frac{dt}{t} \\ &= \ \sqrt{2} \int_0^\infty \langle F(tA_p)^*(y), F(tA_p)(g_n(a)x)\rangle \, \frac{dt}{t} \, .\end{aligned}$$

According to Lemma 2.8, this implies that

$$\big|\langle y, g_n(A_p)(x)\rangle\big| \ \leq \ \sqrt{2} \, \|g_n(a)x\|_{F,r} \, \|y\|_{F,c}.$$

Now using (A.4) and Lemma A.1, we deduce that

$$\big|\langle y, g_n(A_p)(x)\rangle\big| \ \leq \ \sqrt{2}K \, c_F \, \|g_n(a)x\| \, \|y\|_{F,c} \ \leq \ K \, \|x\| \, \|y\|_{F,c}.$$

Passing to the limit when $n \to \infty$, this yields $|\langle y, x\rangle| \ \leq \ K \, \|x\| \|y\|_{F,c}$. Then taking the supremum over all $x \in S^p$ with $\|x\| = 1$, we obtain that $\|y\| \leq K\|y\|_{F,c}$ for any $y \in S^{p'}$. Since $\|y\|_{F,r} = c_F\|y\|$, this contradicts (A.3) and completes the proof of (2). □

APPENDIX B

MEASURABLE FUNCTIONS IN $L^p(L^2)$

Let $2 < p < \infty$. The Banach space $L^p(\mathbb{R}; L^2(\mathbb{R}))$ can be described as the space of all measurable functions $g\colon \mathbb{R}^2 \to \mathbb{C}$ such that

$$\|g\|^p_{L^p(L^2)} = \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} |g(s,t)|^2 \, dt \Big)^{\frac{p}{2}} \, ds < \infty \, ,$$

modulo the functions which vanish almost everywhere on $\mathbb{R}^2$. Then it is easy to check that a function $g \in L^p(\mathbb{R}; L^2(\mathbb{R}))$ is representable by a measurable function $u\colon \mathbb{R} \to L^p(\mathbb{R})$ in the sense of Definition 2.7 if and only if

$$\int_{-\infty}^{\infty} |g(s,t)|^p \, ds < \infty \qquad \text{for a.e. } \ t \in \mathbb{R}.$$

Indeed in that case we have $u(t) = g(\,\cdot\,, t)$ for almost every $t \in \mathbb{R}$. We will prove that not all elements of $L^p(\mathbb{R}; L^2(\mathbb{R}))$ are representable by a measurable function from $\mathbb{R}$ into $L^p(\mathbb{R})$ by exhibiting a function $g \in L^p(\mathbb{R}; L^2(\mathbb{R}))$ such that

$$\int_{-\infty}^{\infty} |g(s,t)|^p \, ds = \infty \qquad \text{for a.e. } \ t \in \mathbb{R}. \tag{B.1}$$

For any positive numbers a, b, m such that $b \geq a$, let $P_{a,b,m} \subset \mathbb{R}^2$ be the parallelogram with vertices equal to $(-a, 0)$, $(0,0)$, (b, mb), and $(b-a, mb)$. Thus this parallelogram has a pair of horizontal sides, and a pair of sides having slope equal to m. Next we let $g_{a,b,m}$ be the indicator function of $P_{a,b,m}$. It is clear that

$$\int_{-\infty}^{\infty} |g_{a,b,m}(s,t)|^p \, ds = a \qquad 0 \leq t \leq mb. \tag{B.2}$$

On the other hand,

$$
\begin{aligned}
\|g_{a,b,m}\|^p_{L^p(L^2)} &= \int_{-a}^{0} (m(s+a))^{\frac{p}{2}}\, ds \;+\; (b-a)\,(ma)^{\frac{p}{2}} \;+\; \int_{b-a}^{b} (m(b-s))^{\frac{p}{2}}\, ds \\
&= m^{\frac{p}{2}} \left(2 \int_0^a s^{\frac{p}{2}}\, ds \;+\; (b-a)\, a^{\frac{p}{2}}\right) \\
&\leq m^{\frac{p}{2}} \left(2a^{1+\frac{p}{2}} \;+\; (b-a)\, a^{\frac{p}{2}}\right) \;=\; m^{\frac{p}{2}} a^{\frac{p}{2}} (a+b).
\end{aligned}
$$

Since we assumed that $b \geq a$, this yields

$$\|g_{a,b,m}\|_{L^p(L^2)} \leq 2^{\frac{1}{p}}\, b^{\frac{1}{p}}\, m^{\frac{1}{2}}\, a^{\frac{1}{2}}. \tag{B.3}$$

We now make special choices for our parameters a, b, m. Let $n \geq 1$ be an integer. We let

$$a_n = 4^{np},$$

and then we choose b_n and m_n so that

$$b_n m_n = n \qquad \text{and} \qquad b_n^{\frac{1}{p}}\, m_n^{\frac{1}{2}}\, a_n^{\frac{1}{2}} \;=\; 1.$$

Writing $b_n^{\frac{1}{p}}\, m_n^{\frac{1}{2}}\, a_n^{\frac{1}{2}} \;=\; b_n^{\frac{1}{p}-\frac{1}{2}}\, (m_n b_n)^{\frac{1}{2}}\, 4^{\frac{np}{2}}$, this leads to the following choice:

$$b_n = n^{\frac{p}{p-2}}\, 4^{\frac{np^2}{p-2}} \qquad \text{and} \qquad m_n = n^{-\frac{2}{p-2}}\, 4^{-\frac{np^2}{p-2}}.$$

Note that since $p > 2$, we have $\frac{p^2}{p-2} = p\big(1 + \frac{2}{p-2}\big) \geq p$ and therefore we have $a_n \leq b_n$. Then we simply let g_n for the function g_{a_n,b_n,m_n} studied so far. According to (B.2) and (B.3), we both have $\|g_n\|_{L^p(L^2)} \leq 2^{\frac{1}{p}}$ and

$$\int_{-\infty}^{\infty} |g_n(s,t)|^p\, ds \;=\; 4^{np} \qquad 0 \leq t \leq n.$$

Therefore we can define

$$g = \sum_{n=1}^{\infty} 2^{-n}\, g_n \;\in\; L^p(\mathbb{R}; L^2(\mathbb{R})).$$

Moreover since each g_n is nonnegative, we have $g \geq 2^{-n} g_n \geq 0$ for any $n \geq 1$. Thus for $t \geq 0$, we have

$$\int_{-\infty}^{\infty} |g(s,t)|^p\, ds \;\geq 2^{-np} \int_{-\infty}^{\infty} |g_n(s,t)|^p\, ds \;\geq 2^{np}$$

provided that $n \geq t$. Hence $\int_{-\infty}^{\infty} |g(s,t)|^p\, ds \;=\; \infty$. This proves (B.1) for $t \geq 0$. An obvious modification yields a function g for which (B.1) holds for $t \in \mathbb{R}$.

BIBLIOGRAPHY

[1] M. Akcoglu – A pointwise ergodic theorem in L_p-spaces, *Canad. J. Math.* **27** (1975), p. 1075–1082.

[2] M. Akcoglu & L. Sucheston – Dilations of positive contractions on L_p spaces, *Canad. Math. Bull.* **20** (1977), p. 285–292.

[3] D. Albrecht, X. T. Duong & A. McIntosh – Operator theory and harmonic analysis, in *Instructional Workshop on Analysis and Geometry, Part III (Canberra, 1995)*, Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 34, Austral. Nat. Univ., 1996, p. 77–136.

[4] C. Anantharaman-Delaroche – On ergodic theorems for free group actions on noncommutative spaces, *Probab. Theory Related Fields* **135** (2006), p. 520–546.

[5] P. Auscher, A. McIntosh & A. Nahmod – Holomorphic functional calculi of operators, quadratic estimates and interpolation, *Indiana Univ. Math. J.* **46** (1997), p. 375–403.

[6] J. Bergh & J. Löfström – *Interpolation spaces*, Springer-Verlag, 1970.

[7] E. Berkson & T. A. Gillespie – Spectral decompositions and harmonic analysis on UMD Banach spaces, *Studia Math.* **112** (1994), p. 13–49.

[8] E. Berkson, T. A. Gillespie & P. S. Muhly – Abstract spectral decompositions guaranteed by the Hilbert transform, *Proc. London Math. Soc.* **53** (1986), p. 489–517.

[9] P. Biane – Free hypercontractivity, *Commun. Math. Phys.* **184** (1997), p. 457–474.

[10] E. Blanchard & K. Dykema – Embeddings of reduced free products of operator algebras, *Pacific J. Math.* **199** (2001), p. 1–19.

[11] M. Bożejko – Ultracontractivity and strong Sobolev inequality for q-Ornstein-Uhlenbeck semigroup, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **2** (1999), p. 203–220.

[12] M. Bożejko & G. Fendler – A note on certain partial sum operatos, 2005, Preprint.

[13] M. Bożejko, B. Kümmerer & R. Speicher – q-Gaussian processes: non-commutative and classical aspects, *Commun. Math. Phys.* **185** (1997), p. 129–154.

[14] M. Bożejko & R. Speicher – An example of a generalized brownian motion, *Commun. Math. Phys.* **137** (1991), p. 519–531.

[15] ______, Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces, *Math. Ann.* **300** (1994), p. 97–120.

[16] O. Bratelli & D. W. Robinson – *Operator algebras and quantum statistical mechanics II*, Springer-Verlag, 1981.

[17] D. Burkholder – Martingales and singular integrals in Banach spaces, in *Handbook of the Geometry of Banach Spaces, Vol. I* (W. B. Johnson & J. Lindenstrauss, eds.), Elsevier, 2001, p. 233–269.

[18] M.-D. Choi – A schwarz inequality for positive linear maps on C^*-algebras, *Illinois J. Math.* **18** (1974), p. 565–588.

[19] P. Clément, B. De Pagter, F. Sukochev & H. Witvliet – Schauder decompositions and multiplier theorems, *Studia Math.* **138** (2000), p. 135–163.

[20] M. Cowling – Harmonic analysis on semigroups, *Ann. of Math.* **117** (1983), p. 267–283.

[21] M. Cowling, I. Doust, A. McIntosh & A. Yagi – Banach space operators with a bounded H^∞ functional calculus, *J. Austral. Math. Soc. Ser. A* **60** (1996), p. 51–89.

[22] E. B. Davies – *One-parameter semigroups*, London Math. Soc. Monographs, vol. 15, Academic Press Inc., 1980.

[23] J. Diestel & J. J. Uhl – *Vector measures*, vol. 15, Amer. Math. Soc., 1977, préface de J. Pettis, Mathematical Surveys no. 15.

[24] X. T. Duong – H^∞ functional calculus of second order elliptic partial differential operators on L_p-spaces, Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 24, Austral. Nat. Univ., 1989, p. 91–102.

[25] T. Fack – Type and cotype inequalities for non-commutative L^p-spaces, *J. Operator Theory* **7** (1987), p. 255–279.

[26] T. Fack & H. Kosaki – Generalized s-numbers of τ-measurable operators, *Pacific J. Math.* **123** (1986), p. 269–300.

[27] G. Fendler – Dilations of one parameter semigroups of positive contractions on L_p-spaces, *Canad. J. Math.* **49** (1997), p. 736–748.

[28] J. B. Garnett – *Bounded analytic functions*, Pure Appl. Math., Academic Press Inc., 1981.

[29] J. A. Goldstein – *Semigroups of linear operators and applications*, Oxford University Press, 1985.

[30] U. Haagerup – An example of a non nuclear C^*-algebra, which has the metric approximation property, *Invent. Math.* **50** (1979), p. 279–293.

[31] A. Harcharras – Fourier analysis, Schur multipliers on S^p and non-commutative $\Lambda(p)$-sets, *Studia Math.* **137** (1999), p. 203–260.

[32] M. Hieber & J. Prüss – Functional calculi for linear operators in vector valued L^p-spaces via the transference principle, *Adv. Differential Equations* **3** (1998), p. 847–872.

[33] M. Junge – Doob's inequality for non-commutative martingales, *J. Reine Angew. Math.* **549** (2002), p. 149–190.

[34] M. Junge, C. Le Merdy & Q. Xu – Calcul fonctionnel et fonctions carrées dans les espaces L^p non commutatifs, *C. R. Acad. Sci. Paris Sér. I Math.* **337** (2003), p. 93–98.

[35] M. Junge & Q. Xu – Haagerup's reduction on noncommutative L^p-spaces and applications, in preparation.

[36] ______, Noncommutative maximal ergodic theorems, *J. Amer. Math. Soc.* to appear.

[37] ______, Théorèmes ergodiques maximaux dans les espaces L_p non commutatifs, *C. R. Acad. Sci. Paris, Sér. I Math.* **334** (2002), p. 773–778.

[38] ______, Noncommutative Burkholder/Rosenthal inequalities, *Ann. Probab.* **31** (2003), p. 948–995.

[39] R. Kadison – A generalized Schwarz inequality and algebraic invariants for operator algebras, *Ann. of Math.* **56** (1952), p. 494–503.

[40] R. Kadison & J. R. Ringrose – *Fundamentals of the theory of operator algebras, I and II*, Graduate Studies in Math., Amer. Math. Soc., 1997, **15** and **16**.

[41] N. Kalton & L. Weis – The H^∞ calculus and sums of closed operators, *Math. Ann.* **321** (2001), p. 319–345.

[42] H. Kosaki – Applications of the complex interpolation method to a von Neumann algebra: noncommutative L^p-spaces, *J. Funct. Anal.* **56** (1984), p. 29–78.

[43] P. C. Kunstmann & L. Weis – Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems, and H^∞-functional calculus, in *Functional analytic methods for evolution equations* (M. Iannelli, R. Nagel & S. Piazzera, eds.), Lect. Notes in Math., vol. 1855, Springer-Verlag, 2004, p. 65–311.

[44] F. Lancien, G. Lancien & C. Le Merdy – A joint functional calculus for sectorial operators with commuting resolvents, *Proc. London Math. Soc.* **77** (1998), p. 387–414.

[45] C. Le Merdy – The similarity problem for bounded analytic semigroups on Hilbert space, *Semigroup Forum* **56** (1998), p. 205–224.

[46] ———, H^∞ functional calculus and applications to maximal regularity, *Pub. Math. UFR Sci. Tech. Besançon* **16** (1999), p. 41–77.

[47] ———, Two results about H^∞ functional calculus on analytic UMD banach spaces, *J. Austr. Math. Soc.* **74** (2003), p. 351–378.

[48] ———, The Weiss conjecture for bounded analytic semigroups, *J. London Math. Soc.* **67** (2003), p. 715–738.

[49] ———, On square functions associated to sectorial operators, *Bull. Soc. Math. France* **132** (2004), p. 137–156.

[50] J. Lindenstrauss & L. Tzafriri – *Classical Banach spaces II*, Springer-Verlag, 1979.

[51] F. Lust-Piquard – Inégalités de Khintchine dans C_p, *C. R. Acad. Sci. Paris* **303** (1986), p. 289–292.

[52] F. Lust-Piquard & G. Pisier – Non-commutative Khintchine and Paley inequalities, *Arkiv för Math.* **29** (1991), p. 241–260.

[53] A. McIntosh – Operators which have an H^∞ functional calculus, in *Miniconference on operator theory and partial differential equations*, Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 14, Austral. Nat. Univ., 1986, p. 210–231.

[54] A. McIntosh & A. Yagi – Operators of type ω without a bounded H^∞ functional calculus, in *Miniconference on operators in analysis*, Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 24, Austral. Nat. Univ., 1989, p. 159–172.

[55] T. Mei – Operator valued Hardy spaces, Mem. Amer. Math. Soc., to appear.

[56] R. Nagel – *One parameter semigroups of positive operators*, Lect. Notes in Math., vol. 1184, Springer-Verlag, 1986.

[57] K. R. Parthasarathy – *An Introduction to Quantum Stochastic Calculus*, Monographs in Math., vol. 85, Birkhäuser Verlag, 1992.

[58] V. I. Paulsen – *Completely bounded maps and operator algebras*, Cambridge Studies in Adv. Math., vol. 78, Cambridge University Press, 2002.

[59] G. Pisier – Holomorphic semigroups and the geometry of Banach spaces, *Ann. of Math.* **115** (1982), p. 375–392.

[60] ______, *Factorization of linear operators and geometry of Banach spaces*, CBMS Regional Conf. Series in Math., vol. 60, Amer. Math. Soc., 1986.

[61] ______, Complex interpolation and regular operators between Banach lattices, *Arch. Math.* **62** (1994), p. 261–269.

[62] ______, *Non-commutative vector valued L_p-spaces and completely p-summing maps*, Astérisque, vol. 247, Soc. Math. France, 1998.

[63] ______, *Introduction to operator space theory*, London Math. Soc. Lect. Note Series, vol. 294, Cambridge University press, 2003.

[64] G. Pisier & Q. Xu – Non-commutative martingale inequalities, *Commun. Math. Phys.* **189** (1997), p. 667–698.

[65] ______, Non-commutative L^p-spaces, in *Handbook of the geometry of Banach spaces, Vol. II* (W. Johnson & J. Lindenstrauss, eds.), Elsevier, 2003, p. 1459–1517.

[66] R. J. Plymen & P. L. Robinson – *Spinors in Hilbert space*, Cambridge Tracts in Math., vol. 114, Cambridge University Press, 1994.

[67] J. Prüss – *Evolutionary integral equations and applications*, Monographs in Math., vol. 87, Birkhäuser Verlag, 1993.

[68] N. Randrianantoanina – Non-commutative martingale transforms, *J. Funct. Anal.* **194** (2002), p. 181–212.

[69] S. Sakai – *C^*-algebras and W^*-algebras*, Classics in Mathematics, Springer-Verlag, 1998.

[70] E. M. Stein – *Topics in harmonic analysis related to the Littlewood-Paley theory.*, Ann. Math. Studies, Princeton University Press, 1970.

[71] B. Sz-Nagy & C. Foias – *Harmonic analysis of operators on Hilbert space*, Akademiai Kiadó, 1970.

[72] M. Takesaki – Conditional expectations in von Neumann algebras, *J. Functional Analysis* **9** (1972), p. 306–321.

[73] M. Terp – L^p-spaces associated with von Neumann algeras, 1981, Notes, Math. Instititue, Copenhagen University.

[74] M. Uijterdijk – Functional calculi for closed sectorial operators, Ph.D. Thesis, Delft, 1988.

[75] D. Voiculescu – Symmetries of some reduced free product C^*-algebras, in *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, Lect. Notes in Math., vol. 1132, Springer-Verlag, 1985, p. 556–588.

[76] D. Voiculescu, K. Dykema & A. Nica – *Free random variables*, CRM Monograph Series, vol. 1, Amer. Math. Soc., 1992.

[77] J. Voigt – Abstract Stein interpolation, *Math. Nachr.* **157** (1992), p. 197–199.

[78] L. Weis – A new approach to maximal L_p-regularity, in *Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998)* (G. Lumer & L. Weis, eds.), Lect. Notes in Pure Appl. Math., vol. 215, Marcel Dekker, 2001, p. 195–214.

[79] ______, Operator-valued Fourier multiplier theorems and maximal L_p-regularity, *Math. Ann.* **319** (2001), p. 735–758.

[80] Q. Xu – Recent developments on non-commutative martingale inequalities, in *Functional space theory and its applications, Proc. of International Conference and 13th academic symposium in China*, Research information Ltd, 2003, p. 283–313.